WEBER STATE UNIVERSITY
Engineering, Applied Science & Technology

DEPARTMENT OF
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ENGINEERING

ECE 3210

Signals and Systems

Course Notes

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Chapter 1

Preliminary Mathematics

1.1 Complex numbers

A complex number $z \in \mathbb{C}$ has a real part a and an imaginary part b, and is written as $z = a + jb$. The magnitude of a complex number is $|z| = \sqrt{a^2 + b^2}$, and the angle of a complex number is $\angle z = \tan^{-1}\left(\frac{b}{a}\right)$. Euler's formula states that the complex exponential is defined as

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Similarly, we can write a complex number in polar form as

$$z = |z|e^{j\angle z}.$$

We can expand Euler's formula to define a cosine function as

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

and a sine function as

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

The complex conjugate of a complex number $z = a + jb$ is denoted as $z^* = a - jb$, which essentially says we need to negate the imaginary part of z.

The product of a complex number and its conjugate is given by

$$zz^* = |z|^2.$$

The division of a complex number by its magnitude is given by

$$\frac{z}{|z|} = e^{j\angle z}.$$

The sum of two complex numbers is given by

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and the difference of two complex numbers is given by

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

The product of two complex numbers is given by

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

and the division of two complex numbers is given by

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}.$$

Generally, the addition and subtraction of two complex numbers is done in rectangular form, and the multiplication and division of two complex numbers is done in polar form.

1.2 Partial fraction expansion

Partial fraction expansion is a technique used to decompose a rational function (a ratio of polynomials) into a sum of simpler fractions. This is especially useful in signal processing and systems analysis for finding inverse transforms.

Suppose we have a rational function:

$$F(s) = \frac{P(s)}{Q(s)} \tag{1.1}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$.

The method of expansion depends on the nature of the roots of $Q(s)$:

1.2.1 Distinct real roots

If $Q(s)$ factors into distinct real roots, e.g.

$$F(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

for $Q(s) = (s - r_1)(s - r_2)$, then the coefficients A and B can be found by multiplying both sides by $Q(s)$ and solving for the unknowns.

Example (Heaviside Cover-Up Method):

$$F(s) = \frac{5}{(s + 1)(s + 2)}$$

Write as:

$$\frac{5}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

To find A, cover up $(s + 1)$ in the denominator and substitute $s = -1$:

$$A = \left.\frac{5}{s + 2}\right|_{s=-1} = \frac{5}{-1 + 2} = 5$$

To find B, cover up $(s + 2)$ in the denominator and substitute $s = -2$:

$$B = \left.\frac{5}{s + 1}\right|_{s=-2} = \frac{5}{-2 + 1} = -5$$

So:

$$F(s) = \frac{5}{s + 1} - \frac{5}{s + 2}$$

1.2.2 Repeated Roots

If $Q(s)$ has repeated roots, e.g.

$$F(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2}$$

for $Q(s) = (s-r)^2$, then include terms for each power of the repeated factor.

There are a few ways to do this. We will focus on the Heaviside cover-up method.

Example:

Consider some rational function with a repeated root.

$$F(s) = \frac{3s+2}{(s+1)^2}$$

We can write this as

$$\frac{3s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

To use the Heaviside cover-up method for repeated roots, first find B by covering up $(s+1)^2$ and substituting $s=-1$, which is the "easy" coefficient

$$B = (3s+2)\Big|_{s=-1} = 3(-1) + 2 = -1$$

To find A, we will need to differentiate the denominator $(s+1)^2$ with respect to s to get $2(s+1)$, then multiply $F(s)$ by $(s+1)$ and substitute $s=-1$:

$$A = \frac{d}{ds}\left[(s+1)(3s+2)\right]\Big|_{s=-1}$$

Alternatively, for this simple case, plug in another value (e.g., $s=0$):

$$3(0) + 2 = A(1) - 1 \implies 2 = A - 1 \implies A = 3.$$

Yet another approach we can use is multiply both sides by s

$$sF(s) = \frac{3s^2+2s}{(s+1)^2} = \frac{As}{s+1} + \frac{Bs}{(s+1)^2}.$$

We can take the limit as $s \to \infty$ (and using L'Hopitals's rule if needed)

$$\begin{aligned} &\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(\frac{3s^2+2s}{(s+1)^2}\right) = \lim_{s\to\infty}\left(\frac{As}{s+1} + \frac{Bs}{(s+1)^2}\right) = 3 \\ &3 = A + 0 \implies A = 3. \end{aligned}$$

This last approach will only help get a single coefficient for the corresponding $s + r_n$ term. One strategy is to use the simply Heavyside approach to find the coefficient for the highest order $(s+r_n)^m$ term, then use this limit approach to find the $(s+r_n)$ coefficient. If there is another term, you could conider putting in a simple $s=0$ or $s=1$ value to get a another equation to solve for the remaining unknown.

In this specific case we have

$$F(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2}$$

1.2.3 Complex Roots

Because the systems we will working with are real-valued, any complex roots of $Q(s)$ will occur in complex conjugate pairs

$$Q(s) = (s - r)(s - r^*)$$

where $r = a + jb$, $r^* = a - jb$, then the expansion is:

$$F(s) = \frac{A}{s - r} + \frac{B}{s - r^*}$$

The coefficients A and B may be complex, but the sum will be real if $F(s)$ is real.

Example:

$$F(s) = \frac{2s + 3}{s^2 + 4s + 5}$$

Factor the denominator: $s^2 + 4s + 5 = (s + 2 + j1)(s + 2 - j1)$

You can expand $F(s)$

$$\frac{2s + 3}{(s + 2 + j1)(s + 2 - j1)} = \frac{A}{s + 2 + j1} + \frac{B}{s + 2 - j1}.$$

You can solve for A using the Heavyside coverup. If the coefficients in $F(s)$ are real, then $B = A^*$, so we don't need to explicitly solve for it.

1.3 Summations

We can compute the following summations of the form

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=N_1}^{N_2} r^n = \frac{r^{N_1} - r^{N_2+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1 - r} \quad \text{for } |r| < 1.$$

Chapter 2

Signals

A signal is defined as a set of information corresponding to one or more independent variable(s) (often time or space).

2.1 Signal definitions

2.1.1 Classifications

A signal can be classified in a number of ways:

Continuous-time, analog Continuous-time signals are defined for every instant of time and are often represented by analog waveforms (range can take any value).

Discrete-time, analog Discrete-time signals are defined only at discrete intervals and are often represented by analog values (range can take any value).

Continuous-time, digital Signal is defined at every point in time, but takes on only a finite set of values (often quantized using a set of fixed levels).

Discrete-time, digital Discrete-time signals are defined only at discrete intervals and take on a finite set of values (often quantized using a set of fixed levels).

2.1.2 Periodic signals

A periodic signal is a signal that repeats itself at regular intervals over time. The smallest interval over which the signal repeats is called the period (T). Mathematically, a signal $x(t)$ is periodic if there exists a positive constant T such that:

$$x(t) = x(t + T)$$

for all values of t. Periodic signals can be classified as either continuous-time or discrete-time signals. The most common periodic signal is a sinusoidal signal, but it can also include square waves, triangular waves, and other waveforms.

2.1.3 Even and odd signals

A signal $x(t)$ is said to be even if it satisfies the following condition

$$x(t) = x(-t)$$

for all values of t. Even signals are symmetric about the vertical axis.

A signal $x(t)$ is said to be odd if it satisfies the following condition

$$x(t) = -x(-t)$$

for all values of t. Odd signals are antisymmetric about the vertical axis.

2.1.4 Causality

A signal is said to be causal if it is zero for all negative time values. In other words, a causal signal $x(t)$ satisfies the following condition:

$$x(t) = 0 \quad \text{for } t < 0$$

for all values of t. Causal signals are often used to model physical systems that cannot respond before an input is applied. Essentially, you cannot look into the future!

2.2 Measuring signals

We can measure signal strength in many ways (amplitude, RMS, etc.). The choice of measurement depends on the characteristics of the signal and the specific application. Two common metrics are signal energy and power.

2.2.1 Energy

The energy of a signal is a measure of the total power consumed by the signal over time. For a continuous-time signal $x(t)$, the energy E is defined as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For a discrete-time signal $x[n]$, the energy is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

2.2.2 Power

The power of a signal is a measure of the average energy consumed by the signal per unit time. For a continuous-time signal $x(t)$, the power P is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Similarly, you can find the power of a periodic signal by using the fundamental period T_0:

$$P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt.$$

For a discrete-time signal $x[n]$, the power is defined as

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} |x[n]|^2.$$

2.3 Common signals

2.3.1 Step function

The step function, also known as the Heaviside step function, is a mathematical function that is commonly used in signal processing and control systems. It is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

We can use the step function to model a signal "turning on" and "turning off" at specific points in time. For example, a single lobe of sinusoidal signal $\sin(\pi t)$ that turns on at $t = 0$ and off at $t = \pi$ can be represented as

$$x(t) = \sin(\pi t) \cdot (u(t) - u(t - \pi)).$$

2.3.2 Delta function

The delta function, also known as the Dirac delta function, is a mathematical function that is used to model an idealized impulse or point source. It is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

The delta function has the property that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

In practice, the delta function is often used to represent a signal that is concentrated at a single point in time. For example, a signal that consists of a single impulse at $t = 0$ can be represented as

$$x(t) = A \cdot \delta(t)$$

where A is the amplitude of the impulse.

An interesting case is when we have a signal $x(t)$ multiplied by a delta function $\delta(t-T)$. This has the effect of "sampling" the signal at $t = T$

$$x(t) \cdot \delta(t-T) = x(T) \cdot \delta(t-T).$$

We can extend this to the *sifting property* of the delta function, which states that for any function $x(t)$ and any constant T,

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t-T)\, dt = x(T).$$

Constructing $x(t)$ from delta functions

We can use the sifting property to express a signal $x(t)$ in terms of delta functions:

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau.$$

This essentially says that if we take the value of $x(\tau)$ at every point in time τ and multiply it by a delta function centered at that point, then integrate over all time, we reconstruct the original signal $x(t)$.

2.3.3 Rectangle function

A rectangle ("rect") function is a piecewise function that is defined as

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.1.

2.3.4 Triangle function

The triangle function $\Delta\left(\frac{t}{\tau}\right)$ is defined as

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau/2} & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.2.

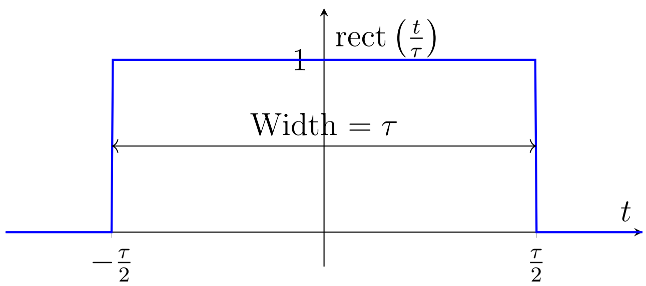

Figure 2.1: The rectangle function rect $\left(\frac{t}{\tau}\right)$.

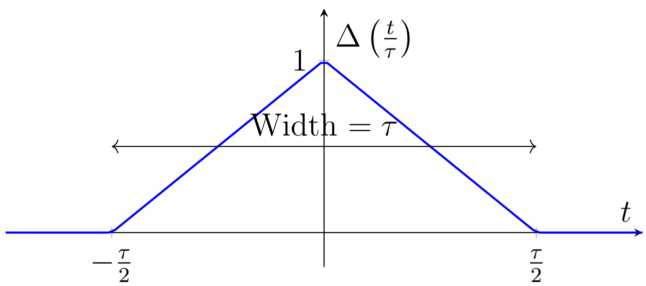

Figure 2.2: The triangle function $\Delta\left(\frac{t}{\tau}\right)$.

2.3.5 Sinc function

The sinc function is defined[1] as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The sinc function is often used in signal processing, particularly in the context of Fourier transforms and filtering. An example of a sinc function is seen in Fig. 2.3.

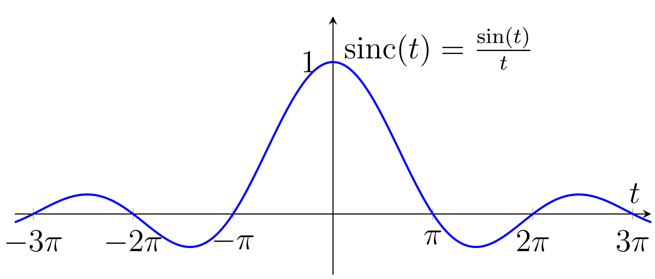

Figure 2.3: The sinc function $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$.

[1]There are different normalizations of the sinc function, but we will use this version in this course.

Chapter 3

Time Domain Systems

A system is a conceptual device that takes one or more inputs and produces one or more outputs. In the context of linear time-invariant (LTI) continuous-time systems, we can describe the relationship between the input and output using differential equations.

In this course we will focus on single-input, single-output (SISO) systems. A classic block diagram of this system behavior is seen in Fig. 3.1

3.1 System properties

3.1.1 Linearity

A system is linear if it satisfies the principles of superposition and scaling (homogeneity). That is, if an input $x_1(t)$ produces an output $y_1(t)$, and an input $x_2(t)$ produces an output $y_2(t)$, then for any constants a and b, the input $ax_1(t) + bx_2(t)$ produces the output $ay_1(t) + by_2(t)$. This is seen in Fig. 3.2.

Checking linearity

We can check if a system is linear by checking if it satisfies the superposition and scaling properties. For example, consider the system defined by

$$y(t) = 3x(t) + 5$$

Let $x_1(t)$ produce $y_1(t)$ and $x_2(t)$ produce $y_2(t)$ such that

$$y_1(t) = 3x_1(t) + 5$$

and

$$y_2(t) = 3x_2(t) + 5.$$

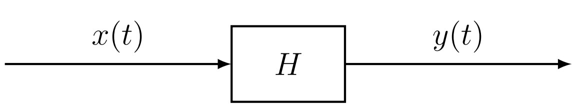

Figure 3.1: A generic SISO system block diagram for system H.

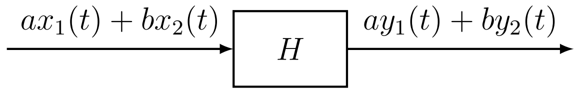

Figure 3.2: A linear SISO system block diagram for system H.

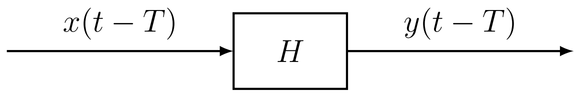

Figure 3.3: A time-invariant SISO system block diagram for system H.

We see that if we use an input that is the superposition and scaled inputs $ax_1(t) + bx_2(t)$, we can write

$$\begin{aligned} y(t) &= 3(ax_1(t) + bx_2(t)) + 5 \\ &= 3ax_1(t) + 3bx_2(t) + 5 \end{aligned}$$

which is not equal to $ay_1(t) + by_2(t)$ because of the constant term 5. Therefore, the system is not linear.

3.1.2 Time invariance

A system is time-invariant if its behavior and characteristics do not change over time. In other words, if we apply a time-shifted input to the system, the output will also be time-shifted by the same amount. Mathematically, if an input $x(t)$ produces an output $y(t)$, then for any time shift T, the input $x(t-T)$ will produce the output $y(t-T)$. A block diagram of this system behavior is seen in Fig. 3.3.

Similarly, we can visualize this behavior in Fig. 3.4. In this figure we observe a typical system input/output relationship. However, if we delay the input by a time T, the output is also delayed by the same amount, illustrating the time-invariance property.

Checking time invariance

To check system time invariance, apply a time-shifted input $x(t-T)$ to the system H and observe the output $\tilde{y}(t)$. Next, take the output for a typical $y(t) = H\{x(t)\}$ and shift it by the same amount to get $y(t-T)$. If $\tilde{y}(t) = y(t-T)$, then the system is time-invariant. If not, then the system is time-variant. This is best seen in example.

Example:

Consider the system

$$y(t) = x(t)\cos(t)$$

To check for time invariance, we apply a time-shifted input $x(t-T)$, which means putting a $-T$ term into the $x(t)$ function

$$\tilde{y}(t) = x(t-T)\cos(t)$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t-T$ in the output equation

$$y(t-T) = x(t-T)\cos(t-T)$$

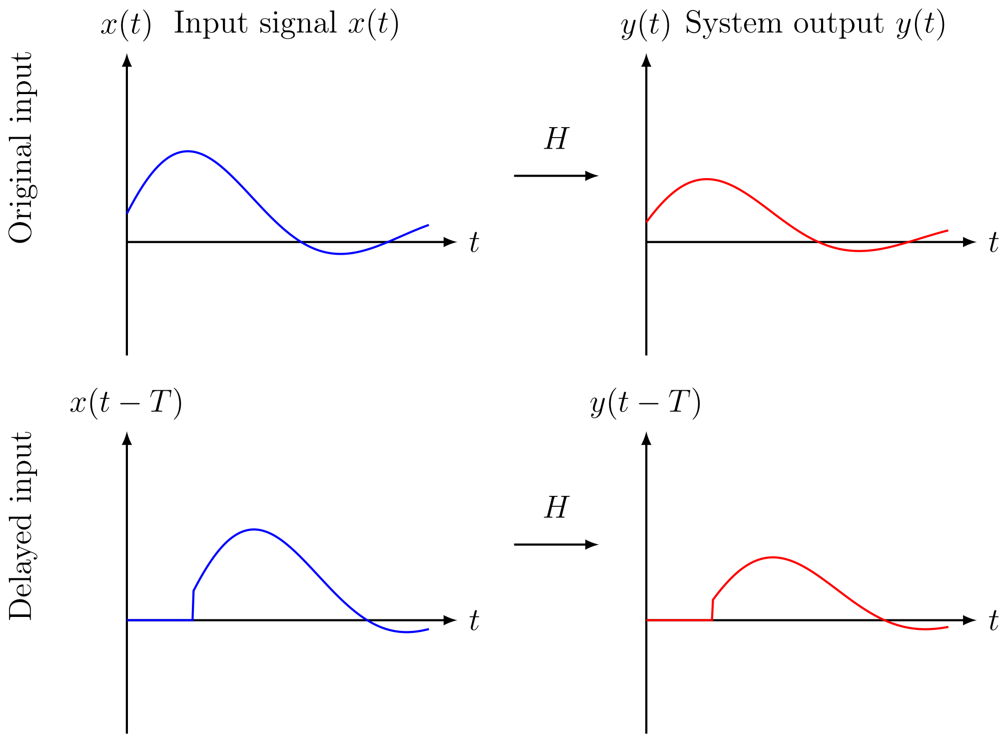

Figure 3.4: Time invariance illustrated with four plots in a grid.

Here, we clearly see that $\tilde{y}(t) \neq y(t-T)$ because of the $\cos(t)$ term, which introduces a time-dependent phase shift. Therefore, the system is time-variant. Typically, if some system H multiplies the input by a time-varying function, it will be time-variant.

Example:

Let's look at another example. Consider the integrator system

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

First, we apply a time-shifted input $x(t-T)$

$$\tilde{y}(t) = \int_{-\infty}^{t} x(\tau - T)d\tau$$

We can change the variable of integration to $\lambda = \tau - T$, which gives us $d\tau = d\lambda$. We also see that when $\tau == -\infty$, $\lambda = \infty$ and when $\tau = t$, $\lambda = t - T$. Thus, we can rewrite the integral as

$$\tilde{y}(t) = \int_{-\infty}^{t-T} x(\lambda)d\lambda$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t (not τ!) with $t - T$ in the output equation

$$y(t-T) = \int_{-\infty}^{t-T} x(\tau)d\tau$$

Here, we clearly see that $\tilde{y}(t) = y(t-T)$, which means the system is time-invariant.

Example:

Consider a compressor system defined by

$$y(t) = x(2t).$$

First we can shift the system input by T by adding a $-T$ to $x(t)$ to get

$$\tilde{y}(t) = x(2t - T).$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t - T$ in the output equation

$$\begin{aligned} y(t-T) &= x(2(t-T)) \\ &= x(2t - 2T). \end{aligned}$$

We see that $\tilde{y}(t) \neq y(t - T)$, so the system is time-variant.

3.2 System response

A general SISO LTI continuous-time system can be described by a linear constant-coefficient differential equation of the form

$$\begin{aligned} \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \ldots \\ \ldots b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t) \end{aligned} \tag{3.1}$$

If we were to solve the ODE described by (3.1), we would find the system's response has two components, the zero-input response and the zero-state response. Thus the solution is

$$y_{\text{tot}}(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

3.3 Zero-input response