WEBER STATE UNIVERSITY
Engineering, Applied Science & Technology

DEPARTMENT OF
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ECE 3210

Signals and Systems

Course Notes

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Chapter 1

Review Material

1.1 Complex numbers

A complex number $z \in \mathbb{C}$ has a real part a and an imaginary part b, and is written as $z = a + jb$. The magnitude of a complex number is $|z| = \sqrt{a^2 + b^2}$, and the angle of a complex number is $\angle z = \tan^{-1}\left(\frac{b}{a}\right)$. Euler's formula states that the complex exponential is defined as

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Similarly, we can write a complex number in polar form as

$$z = |z|e^{j\angle z}.$$

We can expand Euler's formula to define a cosine function as

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

and a sine function as

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

The complex conjugate of a complex number $z = a + jb$ is denoted as $z^* = a - jb$, which essentially says we need to negate the imaginary part of z.

The product of a complex number and its conjugate is given by

$$zz^* = |z|^2.$$

The division of a complex number by its magnitude is given by

$$\frac{z}{|z|} = e^{j\angle z}.$$

The sum of two complex numbers is given by

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and the difference of two complex numbers is given by

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

The product of two complex numbers is given by

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

and the division of two complex numbers is given by

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}.$$

Generally, the addition and subtraction of two complex numbers is done in rectangular form, and the multiplication and division of two complex numbers is done in polar form.

1.2 Partial fraction expansion

Partial fraction expansion is a technique used to decompose a rational function (a ratio of polynomials) into a sum of simpler fractions. This is especially useful in signal processing and systems analysis for finding inverse transforms.

Suppose we have a rational function:

$$F(s) = \frac{P(s)}{Q(s)} \tag{1.1}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$.

The method of expansion depends on the nature of the roots of $Q(s)$:

1.2.1 Distinct real roots

If $Q(s)$ factors into distinct real roots, e.g.

$$F(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

for $Q(s) = (s - r_1)(s - r_2)$, then the coefficients A and B can be found by multiplying both sides by $Q(s)$ and solving for the unknowns.

Example (Heaviside Cover-Up Method):

$$F(s) = \frac{5}{(s + 1)(s + 2)}$$

Write as:

$$\frac{5}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

To find A, cover up $(s + 1)$ in the denominator and substitute $s = -1$

$$A = \left.\frac{5}{s + 2}\right|_{s=-1} = \frac{5}{-1 + 2} = 5$$

To find B, cover up $(s + 2)$ in the denominator and substitute $s = -2$

$$B = \left.\frac{5}{s + 1}\right|_{s=-2} = \frac{5}{-2 + 1} = -5$$

which yields

$$F(s) = \frac{5}{s + 1} - \frac{5}{s + 2}$$

1.2.2 Repeated Roots

If $Q(s)$ has repeated roots, e.g.

$$F(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2}$$

for $Q(s) = (s-r)^2$, then include terms for each power of the repeated factor.

There are a few ways to do this. We will focus on the Heaviside cover-up method.

Example:

Consider some rational function with a repeated root.

$$F(s) = \frac{3s+2}{(s+1)^2}$$

We can write this as

$$\frac{3s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

To use the Heaviside cover-up method for repeated roots, first find B by covering up $(s+1)^2$ and substituting $s = -1$, which is the "easy" coefficient

$$B = (3s+2)\Big|_{s=-1} = 3(-1) + 2 = -1$$

To find A, we will need to differentiate the denominator $(s+1)^2$ with respect to s to get $2(s+1)$, then multiply $F(s)$ by $(s+1)$ and substitute $s = -1$:

$$A = \frac{d}{ds}\left[(s+1)(3s+2)\right]\Bigg|_{s=-1}$$

Alternatively, for this simple case, plug in another value (e.g., $s = 0$):

$$3(0) + 2 = A(1) - 1 \implies 2 = A - 1 \implies A = 3.$$

Yet another approach we can use is multiply both sides by s

$$sF(s) = \frac{3s^2+2s}{(s+1)^2} = \frac{As}{s+1} + \frac{Bs}{(s+1)^2}.$$

We can take the limit as $s \to \infty$ (and using L'Hopitals's rule if needed)

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(\frac{3s^2+2s}{(s+1)^2}\right) = \lim_{s\to\infty}\left(\frac{As}{s+1} + \frac{Bs}{(s+1)^2}\right) = 3$$

$$3 = A + 0 \implies A = 3.$$

This last approach will only help get a single coefficient for the corresponding $s + r_n$ term. One strategy is to use the simply Heavyside approach to find the coefficient for the highest order $(s+r_n)^m$ term, then use this limit approach to find the $(s+r_n)$ coefficient. If there is another term, you could conider putting in a simple $s = 0$ or $s = 1$ value to get a another equation to solve for the remaining unknown.

In this specific case we have

$$F(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2}$$

1.2.3 Complex Roots

Because the systems we will working with are real-valued, any complex roots of $Q(s)$ will occur in complex conjugate pairs

$$Q(s) = (s - r)(s - r^*)$$

where $r = a + jb$, $r^* = a - jb$, then the expansion is:

$$F(s) = \frac{A}{s - r} + \frac{B}{s - r^*}$$

The coefficients A and B may be complex, but the sum will be real if $F(s)$ is real.

Example:

$$F(s) = \frac{2s + 3}{s^2 + 4s + 5}$$

Factor the denominator: $s^2 + 4s + 5 = (s + 2 + j1)(s + 2 - j1)$

You can expand $F(s)$

$$\frac{2s + 3}{(s + 2 + j1)(s + 2 - j1)} = \frac{A}{s + 2 + j1} + \frac{B}{s + 2 - j1}.$$

You can solve for A using the Heavyside coverup. If the coefficients in $F(s)$ are real, then $B = A^*$, so we don't need to explicitly solve for it.

1.3 Summations

We can compute the following summations of the form

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=N_1}^{N_2} r^n = \frac{r^{N_1} - r^{N_2+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1 - r} \quad \text{for } |r| < 1.$$

1.4 U-substitution integration

In this class we will often need to do a simple change of variables with the integrals we are evaluating. This technique is known as U-substitution. The basic idea is to substitute a new variable u for a function of t, which simplifies the integral.

The steps for U-substitution are as follows:

1. Choose a substitution $u = g(t)$ where $g(t)$ is a differentiable function.
2. Compute the differential $du = g'(t)dt$.

3. Convert the limits of integration: if the upper limit is originally $t = a$ then $u = g(a)$ and if the lower limit is originally $t = b$ then $u = g(b)$.
4. Rewrite the integral in terms of u and du and the new limits of integration.
5. Evaluate the integral with respect to u (or even leave it in terms of u).

Example:

Consider the integral

$$y(t) = \int_{-\infty}^{t} x(\tau - T)d\tau.$$

This is an accumulator integral, which integrates some function $x(\tau - T)$ from $-\infty$ to t. The value $y(t)$ is the area under the curve of $x(\tau - T)$ from $-\infty$ to t.

Suppose we wanted to do a quick change of variables here, we see that $u = \tau - T$, which gives $du = d\tau$. The upper limit of the integral in terms of τ was originally t, so the upper limit in terms of u is $t - T$. The lower limit of the integral in terms of τ was originally $-\infty$, so the lower limit in terms of u is also $-\infty$. Thus, we can rewrite the integral as

$$y(t) = \int_{-\infty}^{t-T} x(u)du.$$

This is a relatively simple example, but it shows the basic steps of U-substitution. In this course, we will generally stick to simple examples such as this.

Chapter 2

Signals

A signal is defined as a set of information corresponding to one or more independent variable(s) (often time or space).

2.1 Signal definitions

2.1.1 Classifications

A signal can be classified in a number of ways:

Continuous-time, analog Continuous-time signals are defined for every instant of time and are often represented by analog waveforms (range can take any value).

Discrete-time, analog Discrete-time signals are defined only at discrete intervals and are often represented by analog values (range can take any value).

Continuous-time, digital Signal is defined at every point in time, but takes on only a finite set of values (often quantized using a set of fixed levels).

Discrete-time, digital Discrete-time signals are defined only at discrete intervals and take on a finite set of values (often quantized using a set of fixed levels).

2.1.2 Periodic signals

A periodic signal is a signal that repeats itself at regular intervals over time. The smallest interval over which the signal repeats is called the period (T). Mathematically, a signal $x(t)$ is periodic if there exists a positive constant T such that:

$$x(t) = x(t + T)$$

for all values of t. Periodic signals can be classified as either continuous-time or discrete-time signals. The most common periodic signal is a sinusoidal signal, but it can also include square waves, triangular waves, and other waveforms.

2.1.3 Even and odd signals

A signal $x(t)$ is said to be even if it satisfies the following condition

$$x(t) = x(-t)$$

for all values of t. Even signals are symmetric about the vertical axis.

A signal $x(t)$ is said to be odd if it satisfies the following condition

$$x(t) = -x(-t)$$

for all values of t. Odd signals are antisymmetric about the vertical axis.

2.1.4 Causality

A signal is said to be causal if it is zero for all negative time values. In other words, a causal signal $x(t)$ satisfies the following condition:

$$x(t) = 0 \quad \text{for } t < 0$$

for all values of t. Causal signals are often used to model physical systems that cannot respond before an input is applied. Essentially, you cannot look into the future!

2.2 Measuring signals

We can measure signal strength in many ways (amplitude, RMS, etc.). The choice of measurement depends on the characteristics of the signal and the specific application. Two common metrics are signal energy and power.

2.2.1 Energy

The energy of a signal is a measure of the total power consumed by the signal over time. For a continuous-time signal $x(t)$, the energy E is defined as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For a discrete-time signal $x[n]$, the energy is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

2.2.2 Power

The power of a signal is a measure of the average energy consumed by the signal per unit time. For a continuous-time signal $x(t)$, the power P is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Similarly, you can find the power of a periodic signal by using the fundamental period T_0:

$$P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt.$$

For a discrete-time signal $x[n]$, the power is defined as

$$P = \lim_{N \to \infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} |x[n]|^2.$$

2.3 Common signals

2.3.1 Step function

The step function, also known as the Heaviside step function, is a mathematical function that is commonly used in signal processing and control systems. It is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

We can use the step function to model a signal "turning on" and "turning off" at specific points in time. For example, a single lobe of sinusoidal signal $\sin(\pi t)$ that turns on at $t = 0$ and off at $t = \pi$ can be represented as

$$x(t) = \sin(\pi t) \cdot (u(t) - u(t - \pi)).$$

2.3.2 Delta function

The delta function, also known as the Dirac delta function, is a mathematical function that is used to model an idealized impulse or point source. It is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

The delta function has the property that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

In practice, the delta function is often used to represent a signal that is concentrated at a single point in time. For example, a signal that consists of a single impulse at $t = 0$ can be represented as

$$x(t) = A \cdot \delta(t)$$

where A is the amplitude of the impulse.

An interesting case is when we have a signal $x(t)$ multiplied by a delta function $\delta(t-T)$. This has the effect of "sampling" the signal at $t = T$

$$x(t) \cdot \delta(t - T) = x(T) \cdot \delta(t - T).$$

We can extend this to the *sifting property* of the delta function, which states that for any function $x(t)$ and any constant T,

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t - T)\, dt = x(T).$$

Constructing $x(t)$ from delta functions

We can use the sifting property to express a signal $x(t)$ in terms of delta functions

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau)\, d\tau. \tag{2.1}$$

This idea is initially confusing, so let's walk through it step by step. Let's consider a case where $t = 0$. Eq. 2.1 becomes (and applying some sifting)

$$\begin{aligned} x(0) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(0 - \tau)\, d\tau \\ = x(0). \end{aligned}$$

Similarly, we can look at the case where $t = 1$

$$\begin{aligned} x(1) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(1 - \tau)\, d\tau \\ = x(1). \end{aligned}$$

We can keep applying this idea for any value of t, which gives us the signal $x(t)$. We can visualize this in Fig. 2.1.

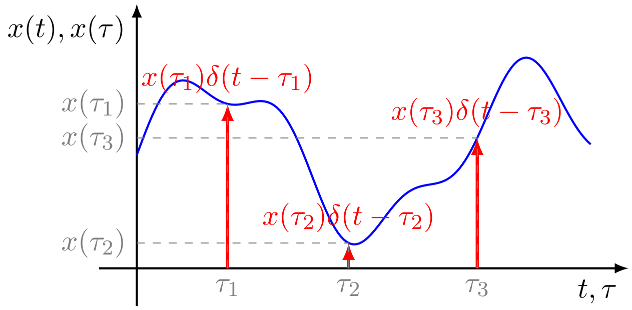


Figure 2.1: Visualizing the sifting property of the delta function. The signal $x(t)$ is constructed from a continuum (integral) of scaled and shifted impulses. Each impulse $x(\tau)\delta(t-\tau)$ is located at τ and has a weight of $x(\tau)$. Each time you integrate over one of these impulses, you get the value of $x(t)$ at that time.

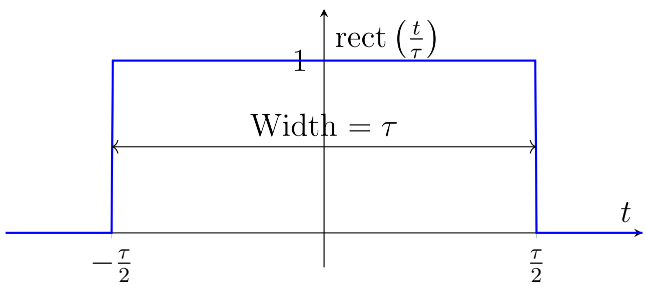


Figure 2.2: The rectangle function $\mathrm{rect}\left(\frac{t}{\tau}\right)$.

2.3.3 Rectangle function

A rectangle ("rect") function is a piecewise function that is defined as

$$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.2.

2.3.4 Triangle function

The triangle function $\Delta\left(\frac{t}{\tau}\right)$ is defined as

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau/2} & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.3.

2.3.5 Sinc function

The sinc function is defined[1] as

[1]There are different normalizations of the sinc function, but we will use this version in this course.

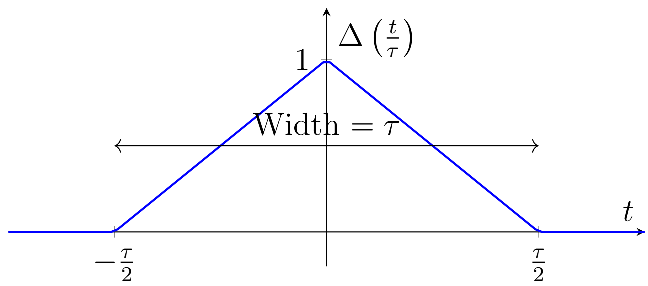


Figure 2.3: The triangle function $\Delta\left(\frac{t}{\tau}\right)$.

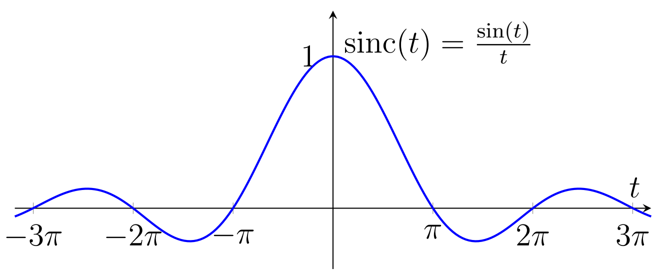


Figure 2.4: The sinc function $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$.

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The sinc function is often used in signal processing, particularly in the context of Fourier transforms and filtering. An example of a sinc function is seen in Fig. 2.4.

Chapter 3

Time Domain Systems

A system is a conceptual device that takes one or more inputs and produces one or more outputs. In the context of linear time-invariant (LTI) continuous-time systems, we can describe the relationship between the input and output using differential equations.

In this course we will focus on single-input, single-output (SISO) systems. A classic block diagram of this system behavior is seen in Fig. 3.1

3.1 System properties

3.1.1 Linearity

A system is linear if it satisfies the principles of superposition and scaling (homogeneity). That is, if an input $x_1(t)$ produces an output $y_1(t)$, and an input $x_2(t)$ produces an output $y_2(t)$, then for any constants a and b, the input $ax_1(t) + bx_2(t)$ produces the output $ay_1(t) + by_2(t)$. This is seen in Fig. 3.2.

Checking linearity

We can check if a system is linear by checking if it satisfies the superposition and scaling properties. For example, consider the system defined by

$$y(t) = 3x(t) + 5$$

Let $x_1(t)$ produce $y_1(t)$ and $x_2(t)$ produce $y_2(t)$ such that

$$y_1(t) = 3x_1(t) + 5$$

and

$$y_2(t) = 3x_2(t) + 5.$$

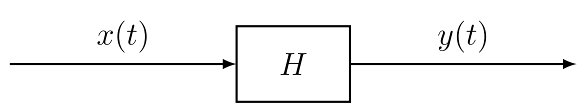


Figure 3.1: A generic SISO system block diagram for system H.

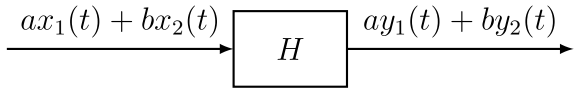


Figure 3.2: A linear SISO system block diagram for system H.

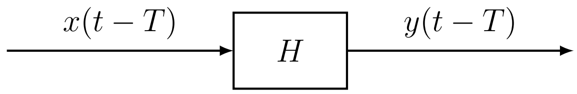


Figure 3.3: A time-invariant SISO system block diagram for system H.

We see that if we use an input that is the superposition and scaled inputs $ax_1(t) + bx_2(t)$, we can write

$$\begin{aligned} y(t) &= 3(ax_1(t) + bx_2(t)) + 5 \\ &= 3ax_1(t) + 3bx_2(t) + 5 \end{aligned}$$

which is not equal to $ay_1(t) + by_2(t)$ because of the constant term 5. Therefore, the system is not linear.

3.1.2 Time invariance

A system is time-invariant if its behavior and characteristics do not change over time. In other words, if we apply a time-shifted input to the system, the output will also be time-shifted by the same amount. Mathematically, if an input $x(t)$ produces an output $y(t)$, then for any time shift T, the input $x(t-T)$ will produce the output $y(t-T)$. A block diagram of this system behavior is seen in Fig. 3.3.

Similarly, we can visualize this behavior in Fig. 3.4. In this figure we observe a typical system input/output relationship. However, if we delay the input by a time T, the output is also delayed by the same amount, illustrating the time-invariance property.

Checking time invariance

To check system time invariance, apply a time-shifted input $x(t-T)$ to the system H and observe the output $\tilde{y}(t)$. Next, take the output for a typical $y(t) = H\{x(t)\}$ and shift it by the same amount to get $y(t-T)$. If $\tilde{y}(t) = y(t-T)$, then the system is time-invariant. If not, then the system is time-variant. This is best seen in example.

Example:

Consider the system

$$y(t) = x(t)\cos(t)$$

To check for time invariance, we apply a time-shifted input $x(t-T)$, which means putting a $-T$ term into the $x(t)$ function

$$\tilde{y}(t) = x(t-T)\cos(t)$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t-T$ in the output equation

$$y(t-T) = x(t-T)\cos(t-T)$$

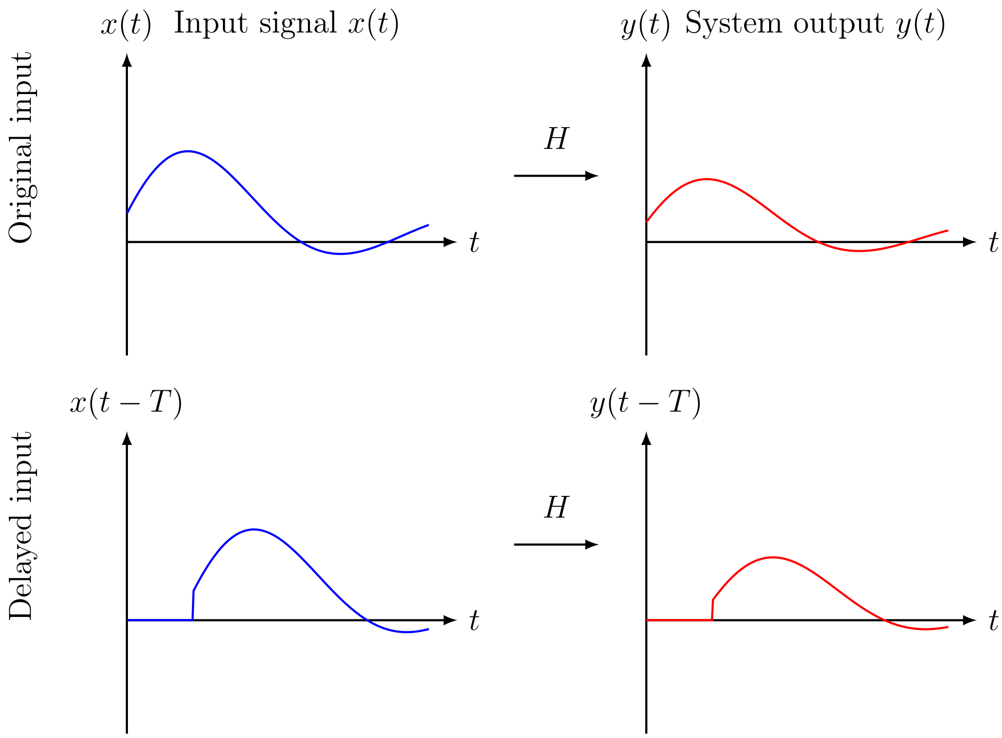


Figure 3.4: Time invariance illustrated with four plots in a grid.

Here, we clearly see that $\tilde{y}(t) \neq y(t-T)$ because of the $\cos(t)$ term, which introduces a time-dependent phase shift. Therefore, the system is time-variant. Typically, if some system H multiplies the input by a time-varying function, it will be time-variant.

Example:

Let's look at another example. Consider the integrator system

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

First, we apply a time-shifted input $x(t-T)$

$$\tilde{y}(t) = \int_{-\infty}^{t} x(\tau - T)d\tau$$

We can change the variable of integration to $\lambda = \tau - T$, which gives us $d\tau = d\lambda$. We also see that when $\tau == -\infty$, $\lambda = \infty$ and when $\tau = t$, $\lambda = t - T$. Thus, we can rewrite the integral as

$$\tilde{y}(t) = \int_{-\infty}^{t-T} x(\lambda)d\lambda$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t (not τ!) with $t - T$ in the output equation

$$y(t-T) = \int_{-\infty}^{t-T} x(\tau)d\tau$$

Here, we clearly see that $\tilde{y}(t) = y(t-T)$, which means the system is time-invariant.

Example:

Consider a compressor system defined by

$$y(t) = x(2t).$$

First we can shift the system input by T by adding a $-T$ to $x(t)$ to get

$$\tilde{y}(t) = x(2t - T).$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t - T$ in the output equation

$$\begin{aligned} y(t-T) &= x(2(t-T)) \\ &= x(2t - 2T). \end{aligned}$$

We see that $\tilde{y}(t) \neq y(t-T)$, so the system is time-variant.

3.2 System response

A general SISO LTI continuous-time system can be described by a linear constant-coefficient differential equation of the form

$$\begin{aligned} \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \dots \\ \dots b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t). \end{aligned} \tag{3.1}$$

We can also write this in a more compact form where each derivative term $\frac{d}{dt}$ can simply be represented with an operator D such that $D^n y(t) = \frac{d^n y(t)}{dt^n}$. Thus, we can rewrite (3.1) as

$$\begin{aligned} D^n y(t) + a_{n-1}D^{n-1}y(t) + \cdots + a_1 D y(t) + a_0 y(t) = \dots \\ \dots b_m D^m x(t) + b_{m-1}D^{m-1}x(t) + \cdots + b_1 D x(t) + b_0 x(t). \end{aligned} \tag{3.2}$$

Further, we can simplify this to

$$\underbrace{\left(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right)}_{Q(D)} y(t) = \underbrace{\left(b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0\right)}_{P(D)} x(t) \tag{3.3}$$

where $Q(D)$ and $P(D)$ are polynomials in the differential operator D.

If we were to solve the ODE described by (3.1), we would find the system's response has two components, the zero-input response and the zero-state response. Thus the solution is

$$y_{\text{tot}}(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

3.3 Zero-input response

In the case of the zero-input response, we are interested in how the system responds to initial conditions without any external input. This means we set the input $x(t)$ to zero and solve the homogeneous equation associated with the system. The zero-input response is determined solely by the system's characteristics and its initial conditions. Without the loss of generality, we will work with 2^{nd} order systems. Consider the following system

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_2\frac{d^2x(t)}{dt^2} + b_1\frac{dx(t)}{dt} + b_0x(t).$$

To find the zero-input response we can set $x(t) = 0$ to get

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = 0.$$

Assume the generic solution to this homogeneous equation is of the form

$$y_0(t) = Ce^{\lambda t}.$$

We can differentiate this expression to find the first and second derivatives:

$$\frac{dy_0(t)}{dt} = C\lambda e^{\lambda t},$$
$$\frac{d^2y_0(t)}{dt^2} = C\lambda^2 e^{\lambda t}.$$

Substituting this into the homogeneous equation gives us

$$C\lambda^2e^{\lambda t} + a_1C\lambda e^{\lambda t} + a_0Ce^{\lambda t} = 0.$$

Factoring out the common term $Ce^{\lambda t}$ gives us

$$Ce^{\lambda t}\left(\lambda^2 + a_1\lambda + a_0\right) = 0.$$

To make this true, we need to solve the characteristic equation (assuming $C \neq 0$ and $e^{\lambda t} \neq 0$ for all t)

$$Q(\lambda) = \lambda^2 + a_1\lambda + a_0 = 0.$$

which is called the *characteristic equation*. The solutions to this equation, λ_1 and λ_2, are called the *characteristic roots* or *eigenvalues* of the system. The nature of these roots (real or complex) will determine the form of the zero-input response. To summarize, the form of the zero-input response is determined by the characteristic roots:

- Real and distinct roots lead to two exponential terms.
- Real and repeated roots lead to an exponential term and a linear term.
- Complex conjugate roots lead to an exponential decay term and sinusoidal terms.

3.3.1 Real and distinct roots

If λ_1 and λ_2 are real and distinct ($\lambda_1 \neq \lambda_2$), then the homogeneous solution is

$$y_0(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}.$$

3.3.2 Real and repeated roots

If λ_1 is a real and repeated root ($\lambda_1 = \lambda_2$), then the homogeneous solution is

$$y_0(t) = (C_1 + C_2 t)e^{\lambda_1 t}.$$

3.3.3 Complex conjugate roots

If $\lambda = a+jb$ is a complex root of the characteristic equation, then its complex conjugate $\lambda^* = a-jb$ is also a root. The homogeneous solution for complex conjugate roots is also given by

$$\begin{aligned} y_0(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(a+jb)t} + C_2 e^{(a-jb)t}. \end{aligned}$$

The coefficients C_1 and C_2 may be complex, but they will be complex conjugates such that $C_1 = C_2^*$ and $C_1 = c + jd$ and $C_2 = c - jd$. We can rewrite $y_0(t)$ as

$$\begin{aligned} y_0(t) &= e^{at}\left[(c + jd)e^{jbt} + (c - jd)e^{-jbt}\right] \\ &= e^{at}\left[c(e^{jbt} + e^{-jbt}) + jd(e^{jbt} - e^{-jbt})\right] \\ &= e^{at}\left[c(2\cos(bt)) + jd(2j\sin(bt))\right] \\ &= e^{at}\left[2c\cos(bt) - 2d\sin(bt)\right]. \end{aligned}$$

To find the constants, we need to apply the initial conditions of the system, which are typically given as $y(0)$ and $y'(0)$. This usually turns into a 2×2 system of equations that can be solved with any linear algebra technique.

Application: circuit analysis

In circuit analysis, we often encounter second-order linear differential equations when analyzing RLC circuits. The zero-input response can be used to determine the natural response of the circuit. To find the coefficients of the zero-input response, we can use the initial conditions of the circuit, such as the initial voltage across a capacitor or the initial current through an inductor (or if it is given to you). Recall that we can represent voltages and currents in a circuit in the time-domain. Each of these quantities can be seen in Fig. 3.5.

Consider the circuit in Fig. 3.6 where the input to the system is some voltage $x(t) = 10e^{-3t}u(t)$ and the output is the current $y(t)$ flowing in the loop. We know that the initial voltage over the capacitor is $v_C(0^-) = 5\,\text{V}$.

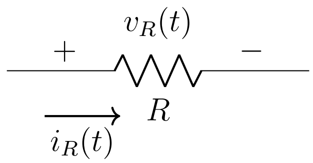
The first thing we want to do is derive an ODE for this expression, which is easiest by a simple voltage loop.

$$-x(t) + v_L(t) + v_R(t) + v_C(t) = 0.$$

From there substitute expressions relating the voltages to the current $y(t)$

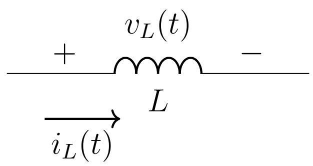
$$-x(t) + L\frac{dy(t)}{dt} + Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

Resistor voltage and current



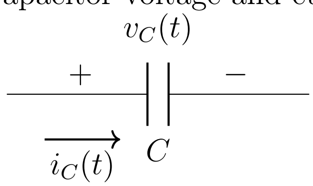
$$v_R(t) = R i_R(t)$$
$$i_R(t) = \frac{v_R(t)}{R}$$

Inductor voltage and current



$$v_L(t) = L\frac{di_L(t)}{dt}$$
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)\,d\tau$$

Capacitor voltage and current



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Figure 3.5: Time-domain representation of voltages and currents in resistors, inductors, and capacitors.

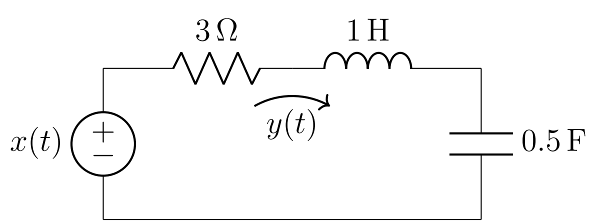


Figure 3.6: Zero-input response of an RLC circuit.

Since we know the component values we simplify to

$$-x(t) + \frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

We need to get rid of the integral term, so we differentiate both sides and simplify

$$\begin{aligned} -\frac{dx(t)}{dt} + \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= 0 \\ \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= \frac{dx(t)}{dt}. \end{aligned}$$

We see the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

which has roots $\lambda_1 = -1$ and $\lambda_2 = -2$ giving a general solution of

$$y_{\text{zir}}(t) = C_1e^{-t} + C_2e^{-2t}.$$

Next we need the initial conditions of the system. We see from the input voltage $x(t)$ that there is no initial voltage and so the current would be zero at $t = 0^-$ such that $y(0^-) = 0$. Looking at the system then at $t = 0^-$, we have

$$-x(0^-) + v_L(0^-) + v_R(0^-) + v_C(0^-) = 0$$

and since we know that $v_C(0^-) = -5\,\text{V}$, $y(0^-) = 0$, and $x(0^-) = 10\,\text{V}$ we can substitute this into the equation to get

$$-10 + v_L(0^-) + R\cdot 0 + 5 = 0.$$

We see that $v_L(0^-) = -5\,\text{V}$ and therefore (using the relationship $v_L = L\frac{dy}{dt}$) we can find the initial condition for the inductor current $y'(0^-) = -5\,\text{V}$, thus giving us our second initial condition. We can apply this to the general solution to find the coefficients C_1 and C_2 (and realizing that $y'_{\text{zir}}(t) = -C_1e^{-t} - 2C_2e^{-2t}$)

$$\begin{aligned} y_{\text{zir}}(0^-) &= C_1 + C_2 = 0, \\ y'_{\text{zir}}(0^-) &= -C_1 - 2C_2 = 5. \end{aligned}$$

and solving $C_1 = -5$ and $C_2 = 5$ giving us the final zero-input response

$$y_{\text{zir}}(t) = -5e^{-t} + 5e^{-2t}.$$

3.4 Zero-state response

The *zero-state response* is the part of the system's output that is solely due to the external input, assuming that all initial conditions are zero. In other words, we analyze how the system responds to an input signal when it starts from a state of rest (zero initial conditions).

Consider an LTI system seen in Fig. 3.7 with an input signal $x(t)$ and an output signal $y(t)$. The zero-state response can be determined using the system's impulse response $h(t)$, which is the output of the system when the input is an impulse function $\delta(t)$. The impulse response characterizes the

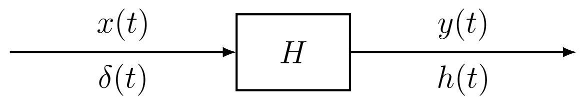


Figure 3.7: System response to an impulse function.

system's behavior and can be used to find the output for any arbitrary input signal through the convolution operation.

Consider some arbitrary signal $x(t)$, which can be written as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

which states that any signal can be represented as a weighted sum of impulse functions. Assume that some system is LTI and has an impulse function $h(t)$. If we pass $x(t)$ through this system, we see

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}. \end{aligned}$$

Applying superposition, we can pull out the integral

$$y(t) = \int_{-\infty}^{\infty} H\{x(\tau)\delta(t-\tau)\}d\tau.$$

Applying scaling, we can pull out the $x(\tau)$ term because in terms of the system H, there is no time dependence with $x(\tau)$. This would given

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau.$$

Because H is time-invariant, we notice a shift impulse function, will yield a shifted delta function such that

$$H\{\delta(t-\tau)\} = h(t-\tau).$$

We can rewrite the output of the system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

We see that the zero-state response can be expressed as a convolution between the input signal and the system's impulse response

$$y(t) = x(t) * h(t).$$

3.4.1 Convolution properties

The convolution integral has several properties that are going to be helpful in analyzing LTI systems.

Property	**Mathematical Expression**
Commutative	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
Associative	$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$
Distributive	$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
Shift Property	$f_1(t - t_0) * f_2(t) = [f_1(t) * f_2(t)]_{t \to t - t_0}$
Delta Function	$f_t(t) * \delta(t) = f_1(t)$, and $f_1(t) * \delta(t - t_0) = f_1(t - t_0)$
Width	If $f_1(t)$ and $f_2(t)$ are time-limited to T_1 and T_2, then $f_1(t) * f_2(t)$ is time-limited to $T_1 + T_2$

Table 3.1: Properties of convolution for signals $f_1(t)$, $f_2(t)$, and $f_3(t)$.

3.4.2 Determining the impulse function $h(t)$

To determine the zero-state response, we will need to find the impulse response $h(t)$ of the system. This can be done myriad ways:

- By applying an impulse input $x(t) = \delta(t)$ and measuring the output $y(t) = h(t)$.
- By applying a known input $x(t)$ and measuring the output $y(t)$, then using deconvolution techniques to extract $h(t)$.
- By solving the system's differential equation with the input $x(t) = \delta(t)$ and zero initial conditions.

Here, we will focus on the last method. To find the impulse response in the time-domain, we will use the formula

$$h(t) = b_n\delta(t) + P(D)y_n(t)u(t)$$

where n is the order of the system. The b_n coefficient is the coefficient from the system's differential equation, as described in Eq. 3.3. The $P(D)$ operator is the polynomial operator from the left-hand side of Eq. 3.3. The $y_n(t)$ term is the zero-input response given a specific set of initial conditions

$$\begin{aligned} y^{(n-1)}(0^-) &= 1 \\ y^{(n-2)}(0^-) &= b_{n-1} = \ldots = y(0^-) = 0. \end{aligned}$$

Example:

Consider the system defined by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt}.$$

To find the impulse response $h(t)$ for this system, we can follow a fairly procedural set of steps.

1. **Identify the system order n:** The highest derivative of $y(t)$ is 2, so $n = 2$.

2. **Determine the b_n coefficient:** From the right-hand side of the differential equation, we see that $b_n = b_2 = 0$ because there is no $\frac{d^2x(t)}{dt^2}$ term.

3. **Determine the $P(D)$ operator:** From the left-hand side of the differential equation, we have
$$P(D) = 2D$$

4. **Find the zero-input response $y_n(t)$:** We need to go through the steps to solve the zero-input response. First, we notice that the characteristic equation is
$$Q(\lambda) = \lambda^2 + 2\lambda + 2.$$
Solving this characteristic equation gives us the roots
$$\lambda = -1 \pm j$$
which gives a generic solution
$$y_n(t) = e^{-t}\left(C_1 \cos(t) + C_2 \sin(t)\right).$$
Applying the initial conditions $y(0^-) = 0$ and $y'(0^-) = 1$ gives us
$$0 = C_1 \quad \text{and} \quad 1 = -C_1 + C_2.$$
Solving this system gives us $C_1 = 0$ and $C_2 = 1$, so the zero-input response is
$$y_n(t) = e^{-t}\sin(t)$$

5. **Compute $P(D)y_n(t)$:** We need to apply the $P(D)$ operator to $y_n(t)$
$$\begin{aligned} P(D)y_n(t) &= 2Dy_n(t) \\ &= 2\frac{d}{dt}\left(e^{-t}\sin(t)\right) \\ &= 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right) \end{aligned}$$

6. **Combine to find $h(t)$:** Finally, we can combine all the pieces to find the impulse response
$$\begin{aligned} h(t) &= b_n\delta(t) + P(D)y_n(t)u(t) \\ &= 0 + 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right)u(t) \\ &= 2e^{-t}\left(\cos(t) - \sin(t)\right)u(t) \end{aligned}$$

3.4.3 Computing the convolution

The convolution integral of two mathematical functions $x_1(t)$ and $x_2(t)$ can be written as
$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau.$$

There several approaches to computing this integral, but we are going to focus on an approach that uses a graphical method. This method is best illustrated with an example.

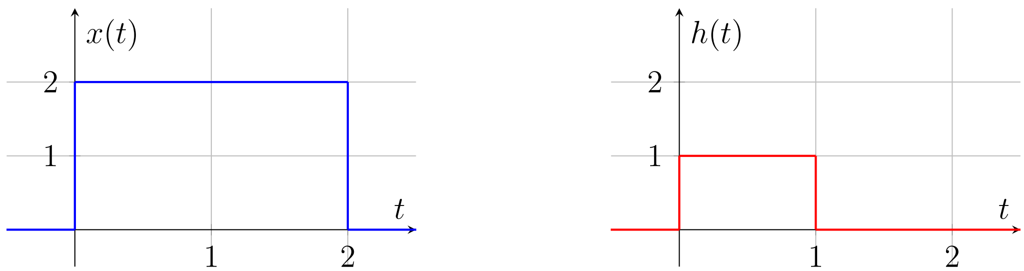


Figure 3.8: Signals $x(t)$ and $h(t)$ to be convolved.

Example:

Consider two functions that we want to convolve

$$x(t) = 2(u(t) - u(t-2))$$
$$h(t) = u(t) - u(t-1).$$

These two functions are shown in Fig. 3.8. To compute the convolution $y(t) = x(t) * h(t)$, we first write the integral in its basic form

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

Next, we need to graphically visualize the convolution process. In this equation, we see that we are integrating over some variable τ. We can rewrite $h(t-\tau)$ as $h(-(\tau - t))$ to see that we are flipping $h(\tau)$ around the vertical axis and then shifting it by t. This is shown in Fig. 3.9 for several values of t. In each subfigure, we see both $x(\tau)$ and $h(t-\tau)$, where the figures on the right show the resulting multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

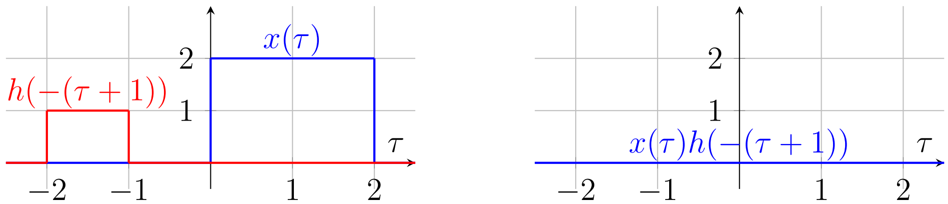
If plot these values for $y(t)$, we get the result shown in Fig. 3.10. In this figure the solid line is the exact result and the dots are the individual values computed in Fig. 3.9. As a quick remark, this result satisfies the width property of convolution because $x(t)$ is time-limited to 2 and $h(t)$ is time-limited to 1, so their convolution $y(t)$ is time-limited to 3. Whenever you perform a convolution, make sure the width property is satisfied as a sanity check (assuming both functions are time-limited).

Example:

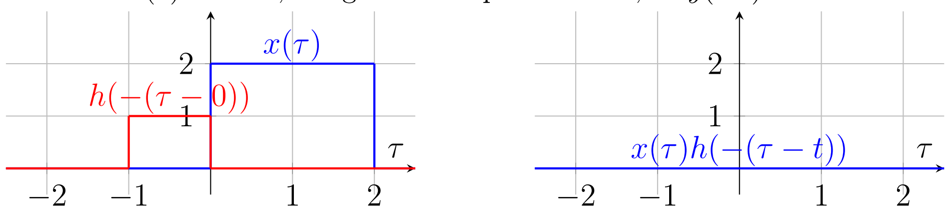
Let's look at a more complex example, visualized in Fig. 3.11. Consider the signals $x(t)$ and $h(t)$ defined in Fig. 3.11a. We will choose to flip and drag the "simpler" signal, so we will choose $x(t)$ to flip and drag since it is a simple box. We will need to approach this problem as a series of cases as we shift $x(-(\tau - t))$ from left to right.

Case $t < -1$: Fig. 3.11b shows the case where $t = -1$. We see that the two signals do not overlap. If we continue to shift to the left, the two signals will never overlap. For $t < -1$, the integral of the product will be zero. Thus, $y(t) = 0$ for $t < -1$.

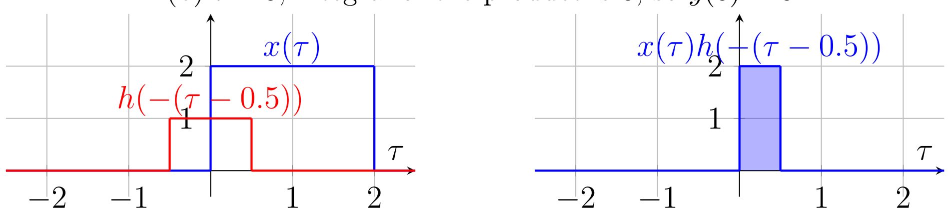
Case $-1 \leq t < 1$: Fig. 3.11c shows the case where $t = 0.5$. We see that the two signals overlap, though only partially. This case will remain from $-1 < t < 1$. The integral of the product triangle. As we shift $x(-(\tau - t))$ to the right, the lower limit of the integral remains fixed



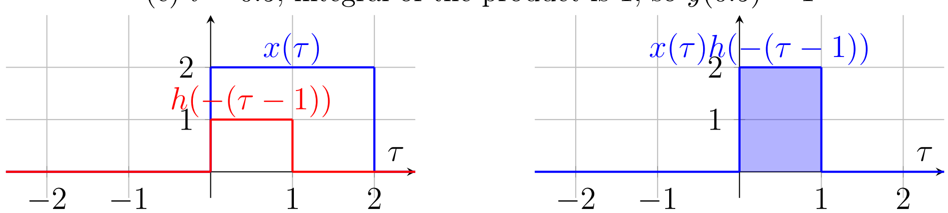
(a) $t = -1$, integral of the product is 0, so $y(-1) = 0$



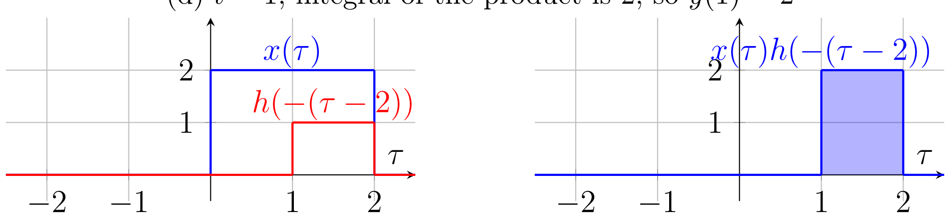
(b) $t = 0$, integral of the product is 0, so $y(0) = 0$



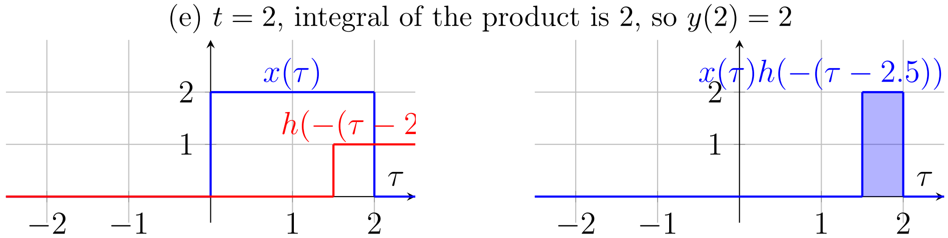
(c) $t = 0.5$, integral of the product is 1, so $y(0.5) = 1$



(d) $t = 1$, integral of the product is 2, so $y(1) = 2$



(e) $t = 2$, integral of the product is 2, so $y(2) = 2$



(f) $t = 2.5$, integral of the product is 1, so $y(2.5) = 1$

Figure 3.9: Graphical convolution of two box functions for different values of t. Each subfigure shows both $x(\tau)$ and $h(t-\tau) = h(-(\tau-t))$ at the specified t, where the figures on the right show the resulting the multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

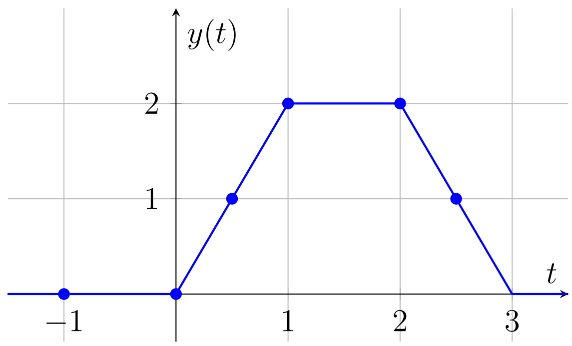


Figure 3.10: Result of the convolution $y(t) = x(t) * h(t)$. The solid line is the exact result and the dots are the individual values computed in Fig. 3.9.

at $\tau = 0$. However, the upper limit of the integral is constantly changing. Here, the upper limit of the integral is $\tau = t + 1$. The integral becomes

$$y(t) = \int_0^{t+1} \frac{1}{3}\tau \, d\tau = \frac{1}{6}(t+1)^2.$$

As a note, determining the limits of integration is the hardest part of convolution. One trick to help with this is to draw the shifted at $x(-(\tau - t))$ at some fixed t value (here, $t = 0.5$) and notice what value of τ that corresponds to. Typically, the moving limit will be some function $\tau = t + a$ where a is some constant. We can plug in our known $\tau = 1.5$ (see Fig. 3.11c) and $t = 0.5$ to find that $1.5 = 0.5 + a$ giving $a = 1$. The moving limit is $\tau = t + 1$.

Case $1 \leq t < 2$**:** Fig. 3.11d shows the case where $t = 1.5$. We see that the two signals fully overlap. This full overlap case remains from $1 \leq t < 2$. This will have *two* moving boundaries. We can find the lower limit similar to what we did in the previous case. In this instance, $t = 1.5$ and that corresponds to $\tau = 0.5$ on the lower bound such that $\tau = t + b$. We can apply our values such that $0.5 = 1.5 + b$ giving $b = -1$. The lower limit is $\tau = t - 1$. The upper limit remains $\tau = t + 1$. The integral becomes

$$y(t) = \int_{t-1}^{t+1} \frac{1}{3}\tau \, d\tau = \frac{2}{3}t.$$

Case $2 \leq t < 4$**:** Fig. 3.11e shows the case where $t = 3$. We see the two signals only partially overlap. This partial overlap case remains from $2 < t < 4$. The lower limit remains $\tau = t - 1$. However, the upper limit is now fixed at $\tau = 3$. The integral becomes

$$y(t) = \int_{t-1}^{3} \frac{1}{3}\tau \, d\tau = \frac{1}{6}(4 - (t-1)^2) = \frac{1}{6}(-t^2 + 2t + 8).$$

Case $t \geq 4$**:** After $t = 4$, the integral will go to zero and remain zero as we continue to shift $x(-(\tau - t))$ to the right. Thus, $y(t) = 0$ for $t \geq 4$.

Putting all these cases together, we can define $y(t)$ piecewise

$$y(t) = \begin{cases} 0 & t < -1, \\ \frac{(t+1)^2}{6} & -1 \leq t < 1, \\ \frac{2}{3}t & 1 \leq t < 2, \\ \frac{1}{6}(-t^2 + 2t + 8) & 2 \leq t < 4, \\ 0 & t \geq 4. \end{cases}$$

We see this result plotted in Fig. 3.12. The solid line is the exact result and the dots are the transition points between each case. When we compute the convolution integral, we should do two sanity checks:

- Check that the width property is satisfied. Here, $x(t)$ is time-limited to 2 and $h(t)$ is time-limited to 3, so their convolution $y(t)$ is time-limited to 5. We see that this is satisfied.
- Check that the function is continuous. Here, we see that $y(t)$ is continuous at each transition point.

3.5 Total response

The *total response* of a system is the sum of the zero-input response and the zero-state response. In other words, the total response $y(t)$ can be expressed as

$$y(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

where $y_{\text{zir}}(t)$ is the zero-input response and $y_{\text{zsr}}(t)$ is the zero-state response. This total response accounts for both the effects of the system's initial conditions and the external input signal.

If you have the differential equation of the system, you can find the total response by solving the differential equation with the given initial conditions and input signal. This involves finding the homogeneous solution (zero-input response) and a particular solution (zero-state response). The zero-input response is found by solving the homogeneous equation with the initial conditions.

To find the zero-state response, you can first find the system's impulse response $h(t)$ and then convolve it with the input signal $x(t)$. Finally, you can sum the zero-input response and the zero-state response to obtain the total response of the system.

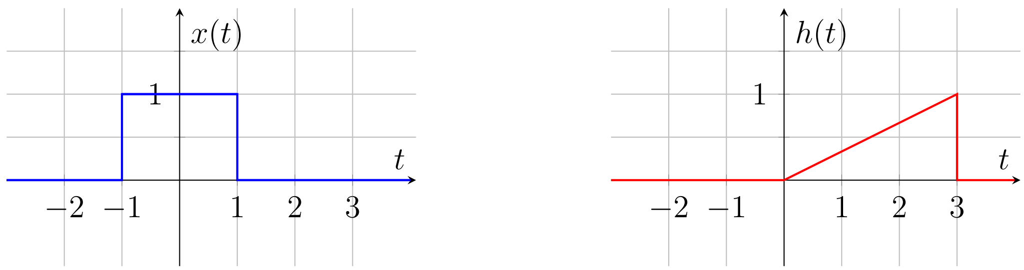
3.6 System stability

An LTI system is considered *BIBO stable* (bounded-input, bounded-output stable) if every bounded input produces a bounded output. In other words, if the input signal $x(t)$ is bounded such that there exists some finite constant M_x where $|x(t)| \leq M_x < \infty$ for all t, then the output signal $y(t)$ must also be bounded such that there exists some finite constant M_y where $|y(t)| \leq M_y < \infty$ for all t. In other words, if we put a finite signal into the system, we should get a finite signal out of the system. In general, we want our systems to be BIBO stable.

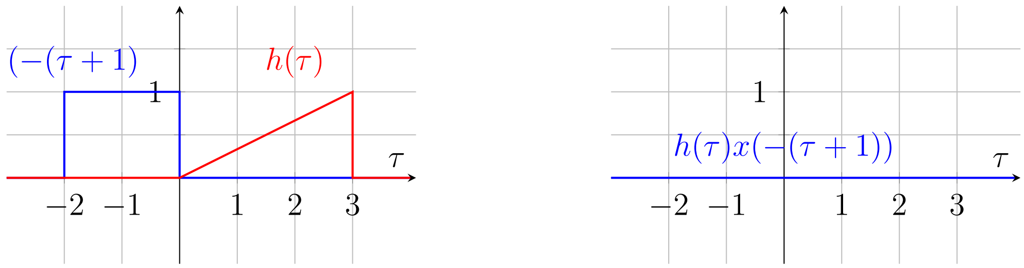
For an LTI system, the BIBO stability can be determined by examining the system's impulse response $h(t)$. Specifically, an LTI system is BIBO stable if and only if the impulse response is absolutely integrable, which means that the integral of the absolute value of the impulse response over all time is finite

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

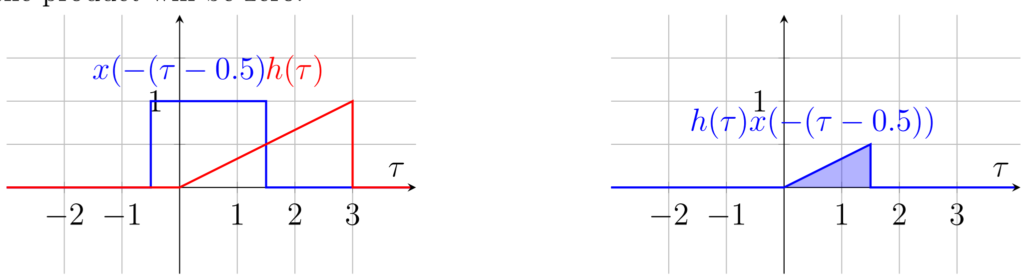
We can prove this by assuming that the input signal $x(t)$ is bounded such that $|x(t)| \leq M_x < \infty$ for all t. The output signal $y(t)$ can be expressed as the convolution of the input signal and the



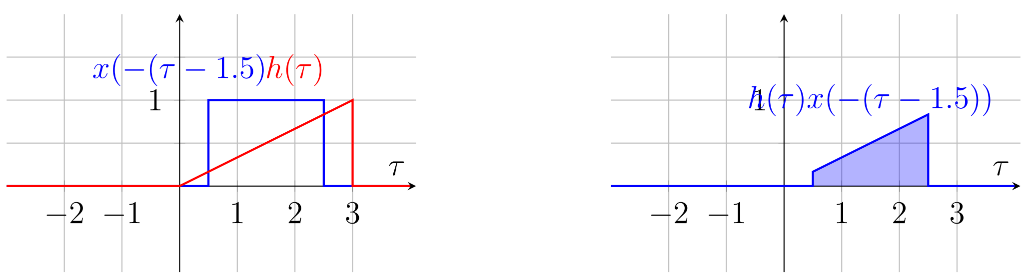
(a) Signals $x(t)$ and $h(t)$ to be convolved.



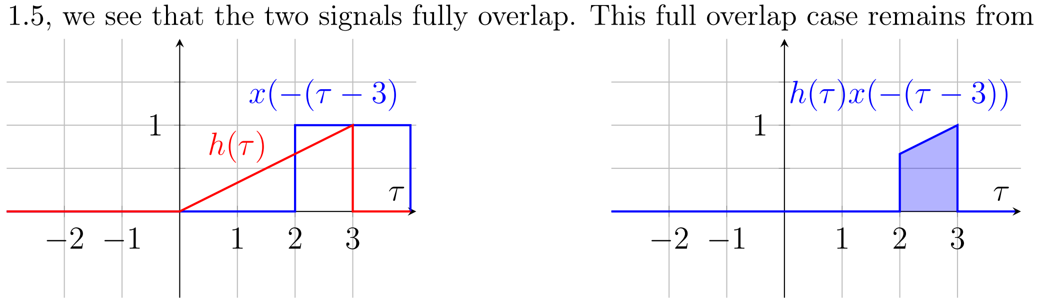
(b) Signal $h(\tau)$ remains in place. The signal $x(-(\tau-t))$ is shifted by $t=-1$. We see that the two signals do not overlap. If we continue to shift to the left, the two signals will never overlap. Thus, for $t<-1$, the integral of the product will be zero.



(c) At $t=0.5$, we see that the two signals overlap, though only partially. This case will remain from $-1<t<1$.



(d) At $t=1.5$, we see that the two signals fully overlap. This full overlap case remains from $1<t<2$.



(e) At $t=3$, we see the two signals only partially overlap. This partial overlap case remains from $2<t<4$. After $t=4$, the integral will go to zero and remain zero as we continue to shift $x(-(\tau-t))$ to the right.

Figure 3.11: Convolution process for $x(t)$ and $h(t)$ at different values of t. Each subfigure shows both $x(\tau)$ and $h(t-\tau)=h(-(\tau-t))$ at the specified t, where the figures on the right show the resulting the multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

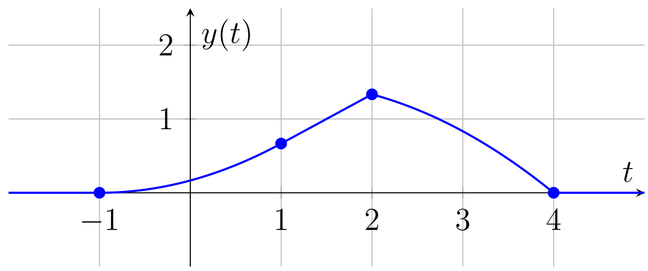


Figure 3.12: Result of the convolution $y(t) = x(t) * h(t)$ in Fig. 3.11a. The solid line is the exact result and the dots are the transition points between each case.

impulse response

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \end{aligned}$$

Taking the absolute value of both sides gives

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \right| \\ &\le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\,d\tau \quad \text{(by the triangle inequality)} \\ &\le M_x \int_{-\infty}^{\infty} |h(t-\tau)|\,d\tau \quad (\text{since } |x(\tau)| \le M_x) \\ &= M_x \int_{-\infty}^{\infty} |h(u)|\,du \quad (\text{by substituting } u = t-\tau) \\ &= M_x M_y \quad \left(\text{where } M_y = \int_{-\infty}^{\infty} |h(u)|\,du \text{ is a finite constant}\right) \end{aligned}$$

Thus, we see that $|y(t)| \le M_x M_y < \infty$ for all t, which means that the output signal $y(t)$ is also bounded. Therefore, if the impulse response $h(t)$ is absolutely integrable, then the LTI system is BIBO stable.

What does this mean in practical terms? Consider the impulse response $h(t) = e^t u(t)$. We see this is clearly not absolutely integrable. If we were to put a bounded input (say, $x(t) = u(t)$) into this system, the convolution integral would explode, and the output would be unbounded. Therefore, this system would not be BIBO stable.

We can also determine BIBO stability by examining the system's characteristic equation. If we look at the characteristic roots λ_i of the system

- If any root has a positive real part, the system is unstable.
- If all roots have negative real parts, the system is BIBO stable.
- If any root has a zero real part (and all others have negative real parts), the system is marginally stable. This means that the system is not BIBO stable, but it is not unstable either. In this case, a bounded input *may* produce an unbounded output, depending on the

specific input signal. In this instance, the characteristic root would lie on the imaginary axis of the complex plane. If a characteristic root lies on the imaginary axis and has a multiplicity greater than one (i.e., it is repeated), the system is unstable.

This can be understood by examining the impulse response of the system. If any characteristic root has a positive real part, the impulse response will contain terms that grow exponentially over time, leading to an unbounded output for a bounded input. Conversely, if all characteristic roots have negative real parts, the impulse response will decay exponentially, ensuring that the output remains bounded for any bounded input. If a characteristic root lies on the imaginary axis, the impulse response will contain oscillatory terms that do not decay, which can lead to unbounded outputs for certain bounded inputs.

Example:

Consider the system defined by the differential equation

$$(D+1)(D^2+4D+8)y(t) = (D-3)x(t).$$

To determine if this system is BIBO stable, we can find the characteristic roots

$$\lambda_1 = -1, \quad \lambda_2 = -2+2j, \quad \lambda_3 = -2-2j.$$

We see that all the characteristic roots have negative real parts, so we can conclude that this system is BIBO stable.

3.7 Transfer functions

Consider an LTI system that has as an input $x(t)$ and an output $y(t)$. Suppose the input is some everlasting complex sinusoid

$$x(t) = e^{st}$$

where $s = \sigma + j\omega$ is a complex number. If we put this input into the system, we will get some output defined by the convolution of $x(t)$ and the system's impulse response $h(t)$

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\ &= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}_{H(s)} \\ &= e^{st}H(s) \end{aligned}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$, which is called the system's transfer function. The transfer function $H(s)$ is a complex function that describes how the system modifies the amplitude and phase of the input signal at different frequencies. Essentially, given some input $x(t)$ which is a sinusoid at some frequency ω, the output $y(t)$ will be a sinusoid at the same frequency ω but with a different amplitude and phase (assuming that $\sigma = 0$ and $s = j\omega$) such that

$$\begin{aligned} y(t) &= H(j\omega)e^{j\omega t} \\ &= |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}. \end{aligned}$$

We saw this type of behavior in steady-state AC circuit analysis in the introductory circuits course.

We can derive the transfer function $H(s)$ directly from the system's differential equation. Consider a general LTI system defined by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \ldots + b_1\frac{dx(t)}{dt} + b_0 x(t).$$

We can rewrite this equation using the differential operator D such that

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y(t) = (b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0)x(t).$$

We can apply the input $x(t) = e^{st}$ and the output $y(t) = H(s)e^{st}$ to this equation

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)H(s)e^{st} = (b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0)e^{st}.$$

We can apply the differential operator D to the exponential function e^{st}

$$D^k e^{st} = s^k e^{st}$$

for any integer $k \geq 0$. Applying this to the previous equation gives

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)H(s)e^{st} = (b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0)e^{st}.$$

We can divide both sides by e^{st} (which is never zero) and solve for $H(s)$

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}.$$

This is a rational function in s where the numerator is a polynomial of degree m and the denominator is a polynomial of degree n.

In shorthand, from a differential equation of the form of derivative operators D, we can find the transfer function $H(s)$ by replacing each D with s and taking the ratio of the input polynomial $P(s)$ to the output polynomial $Q(s)$.

Chapter 4

Fourier Analysis

4.1 General Fourier series

4.1.1 Signal inner product

We define the inner product of two signals $x_1(t)$ and $x_2(t)$ as

$$\langle x_1, x_2 \rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt,$$

where $x_2^*(t)$ is the complex conjugate of $x_2(t)$. If the signals are real-valued, then the complex conjugate is not necessary.

This inner product is similar to the dot product of vectors in Euclidean space. Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Their dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

which is analogous to multiplying two signals pointwise and integrating (summing) over all time.

Two signals $x_1(t)$ and $x_2(t)$ are *orthogonal* if their inner product is zero

$$\langle x_1, x_2 \rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt = 0.$$

4.1.2 General Fourier series representation

We can represent a signal $x(t)$ over a finite interval $[t_1, t_2]$ as a linear combination of orthogonal basis functions $\{x_n(t)\}$

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t) \tag{4.1}$$

where the coefficients c_n are given by the inner product

$$c_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)x_n^*(t)\,dt$$

where E_n is the energy of the basis function. For this to work correctly, the basis functions must be orthogonal over the interval $[t_1, t_2]$ such that

$$\int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt = \begin{cases} E_n, & n = m \\ 0, & n \neq m \end{cases}.$$

Consider Eq. 4.1, and we can multiply both sides by $x_m^*(t)$ and integrate over $[t_1, t_2]$:

$$\begin{aligned}\int_{t_1}^{t_2} f(t)x_m^*(t)\,dt &= \int_{t_1}^{t_2} \left(\sum_{n=1}^{\infty} c_n x_n(t)\right) x_m^*(t)\,dt \\ &= \sum_{n=1}^{\infty} c_n \int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt\end{aligned}$$

We can use the orthogonality property of the basis functions to simplify the right-hand side. The integral $\int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt$ is zero for $n \neq m$ and equals E_n for $n = m$. Therefore, all terms in the sum vanish except for the term where $n = m$

$$\int_{t_1}^{t_2} f(t)x_m^*(t)\,dt = c_m E_m$$

which we can rearrange to solve for the coefficient c_m

$$c_m = \frac{1}{E_m}\int_{t_1}^{t_2} f(t)x_m^*(t)\,dt.$$

In earlier courses you might have seen Fourier series represented using sine and cosine functions as basis functions. The sine and cosine functions are orthogonal over the period of the sinusoid, so they can be used as basis functions in Eq. 4.1. However, we can use any set of orthogonal functions as basis functions, not just sines and cosines. For example, we could use the rectangular function $\mathrm{rect}(t/\tau)$ or the triangular function $\Delta(t/\tau)$ as basis functions.

4.2 Fourier series using trigometric basis functions

4.2.1 Sines and cosines as basis functions

We can represent a periodic signal $f(t)$ with a period T_0 as a linear combination of sines and cosines

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \tag{4.2}$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients a_0, a_n, and b_n are given by

$$\begin{aligned}a_0 &= \frac{1}{T_0}\int_{T_0} f(t)\,dt \\ a_n &= \frac{2}{T_0}\int_{T_0} f(t)\cos(n\omega_0 t)\ dt \\ b_n &= \frac{2}{T_0}\int_{T_0} f(t)\sin(n\omega_0 t)\ dt\end{aligned}$$

where the integrals are taken over one period of the signal. When we compute these coefficients, we are projecting the signal $f(t)$ onto the orthogonal basis functions $\{1, \cos(n\omega_0 t), \sin(n\omega_0 t)\}$. The basis functions are orthogonal over the interval $[t_0, t_0 + T_0]$ for any t_0. When we compute the coefficients, you choose any period T_0 of the signal to integrate over. Choose a period that makes the integrals easiest to compute.

Using symmetry to simplify coefficient calculations

We can simplify computing a_0, a_n, b_n by leveraging symmetry properties of the signal $f(t)$.

Even symmetry: A signal $f(t)$ is even if $f(t) = f(-t)$. The cosine function is even, and the sine function is odd. If $f(t)$ is even, then all b_n coefficients are zero because the product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned} a_0 &= \frac{2}{T_0}\int_0^{T_0/2} f(t)\,dt \\ a_n &= \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos\left(n\omega_0 t\right)\,dt \\ b_n &= 0 \end{aligned}$$

Odd symmetry: A signal $f(t)$ is odd if $f(t) = -f(-t)$. If $f(t)$ is odd, then all a_n coefficients are zero because the product of two odd functions is even, and the integral of an odd function over a symmetric interval is zero. Additionally, a_0 is zero because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin\left(n\omega_0 t\right)\,dt \end{aligned}$$

Half-wave symmetry: A signal $f(t)$ has half-wave symmetry if $f(t) = -f(t - T_0/2)$. A function with half-wave symmetry is periodic with period T_0, and it is also anti-symmetric about the midpoint of the period. A signal can have half-wave symmetry without being purely odd or even, but if it has half-wave symmetry *and* odd or even symmetry, it will simplify the computation of the Fourier coefficients. An illustration of this is shown in Figure 4.1. This means that the positive and negative halves of the waveform are mirror images of each other, but inverted. The coefficients for a signal with half-wave symmetry are

$$\begin{aligned} a_0 &= 0 \\ a_n &= \begin{cases} \frac{4}{T_0}\int_{T_0/2} f(t)\cos\left(n\omega_0 t\right)\,dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \\ b_n &= \begin{cases} \frac{4}{T_0}\int_{T_0/2} f(t)\sin\left(n\omega_0 t\right)\,dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

We see that you can have a $f(t)$ that has half-wave symmetry as well as even or odd symmetry. If $f(t)$ has both half-wave symmetry and even symmetry, then only the odd a_n coefficients

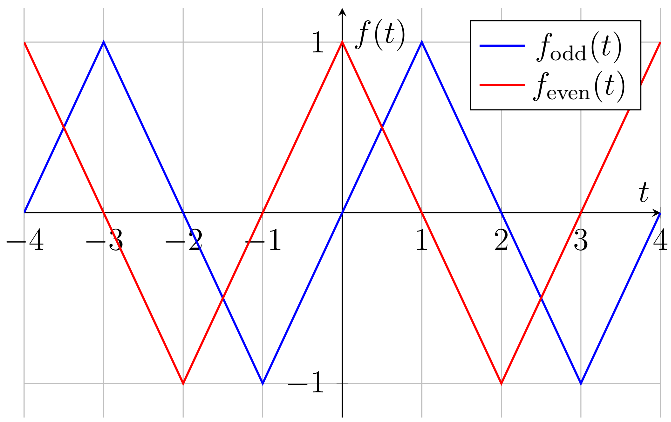


Figure 4.1: A signal with half-wave and odd symmetry (blue) and a signal with half-wave and even symmetry (red).

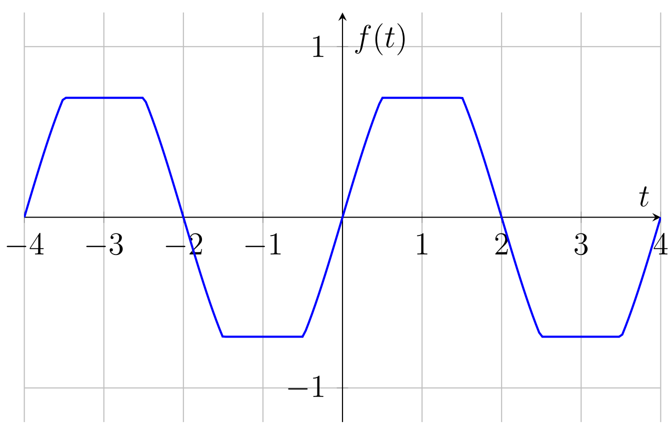


Figure 4.2: A signal with quarter-wave symmetry.

are non-zero. If $f(t)$ has both half-wave symmetry and odd symmetry, then only the odd b_n coefficients are non-zero.

Quarter-wave symmetry: A signal $f(t)$ has quarter-wave symmetry if $f(t) = f(T_0/4 - t)$. A function with quarter-wave symmetry is periodic with period T_0, and it is also symmetric about the quarter points of the period. This means that the first and last quarters of the waveform are mirror images of each other, and the second and third quarters of the waveform are mirror images of each other. Alternatively, you can think of a signal $f(t)$ with quarter-wave symmetry as having half-wave symmetry and even symmetry about the quarter points. An illustration of this is shown in Figure 4.2. Coincidentally, the signals shown in Fig. 4.1 also have quarter-wave symmetry.

The coefficients for a signal with quarter-wave symmetry *and* even symmetry are

$$\begin{aligned}
a_0 &= 0 \\
a_n &= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{8}{T_0}\int_0^{T_0/4} f(t)\cos(n\omega_0 t)\,dt, & \text{for } n \text{ odd} \end{cases} \\
b_n &= 0
\end{aligned}$$

The coefficients for a signal with quarter-wave symmetry *and* odd symmetry are

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= 0 \\
b_n &= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{8}{T_0} \int_0^{T_0/4} f(t) \sin(n\omega_0 t)\, dt, & \text{for } n \text{ odd} \end{cases}
\end{aligned}
$$

4.2.2 Fourier series with shifted cosines

We can also represent a periodic signal $f(t)$ with a period T_0 as a linear combination of cosines with phase shifts

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n).$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients c_0, c_n, and ϕ_n are given by

$$
\begin{aligned}
c_n &= \sqrt{a_n^2 + b_n^2} \\
\phi_n &= \tan^{-1}\left(-\frac{b_n}{a_n}\right)
\end{aligned}
$$

where a_n and b_n are the coefficients from the trigonometric Fourier series representation in Eq. 4.2. The coefficient c_0 is the same as a_0.

The advantage of this representation is that it uses only cosine functions. If we are interested in determining the frequency composition of signal, we could plot the c_n as a stem plot and easily identify the dominant frequencies. We could also plot the phase shifts ϕ_n to see how the different frequency components are shifted in time. The disadvantage is that you need to compute both a_n and b_n to find c_n and ϕ_n.

4.2.3 Fourier series using complex exponential basis functions

We can represent a periodic signal $f(t)$ with a period T_0 as a linear combination of complex exponentials

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{4.3}$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients D_n are given by

$$D_n = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jn\omega_0 t}\, dt.$$

Typically, the D_n coefficients are complex-valued. The magnitude $|D_n|$ indicates the amplitude of the frequency component at $n\omega_0$, and the angle $\angle D_n$ indicates the phase shift of that frequency component. Similar to the shifted cosine representation in Sec. 4.2.2, we can plot the magnitudes $|D_n|$ as a stem plot to identify the dominant frequencies in the signal, and we can plot the angles $\angle D_n$ to see how the different frequency components are shifted in time.

There are multiple ways to derive how the Fourier series works. One approach is realizing that the complex exponentials used as basis functions are all mutually orthogonal over the interval $[0, T_0]$. This means that

$$\int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt = \begin{cases} T_0, & n = m \\ 0, & n \neq m \end{cases}.$$

We can use this orthogonality property to derive the coefficients D_n. Consider Eq. 4.3, and we can multiply both sides by $e^{-jm\omega_0 t}$ and integrate over $[0, T_0]$:

$$\begin{aligned}\int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt &= \int_0^{T_0} \left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \right) e^{-jm\omega_0 t}\, dt \\ &= \sum_{n=-\infty}^{\infty} D_n \int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt\end{aligned}$$

We can use the orthogonality property of the basis functions to simplify the right-hand side. The integral $\int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt$ is zero for $n \neq m$ and equals T_0 for $n = m$. Therefore, all terms in the sum vanish except for the term where $n = m$

$$\int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt = D_m T_0$$

which we can rearrange to solve for the coefficient D_m

$$D_m = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt.$$

Alternatively, we can derive the complex exponential Fourier series from the trigonometric Fourier series using Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta).$$

Using Euler's formula, we can express the cosine and sine functions in terms of complex exponentials

$$\begin{aligned}\cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j}\end{aligned}$$

We can substitute these expressions into the trigonometric Fourier series in Eq. 4.2 to convert it to the complex exponential form. Starting with Eq. 4.2, we substitute the expressions for cosine and sine

$$\begin{aligned}f(t) &= a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right] \\ &= a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{2} + \frac{b_n}{2j} \right) e^{jn\omega_0 t} + \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) e^{-jn\omega_0 t} \right]\end{aligned}$$

We can rewrite the constant term a_0 as $\frac{a_0}{2}e^{j0\omega_0 t} + \frac{a_0}{2}e^{-j0\omega_0 t}$ to match the form of the other terms

$$\begin{aligned}
f(t) &= \frac{a_0}{2}e^{j0\omega_0 t} + \frac{a_0}{2}e^{-j0\omega_0 t} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \left(\frac{a_n}{2} - \frac{b_n}{2j}\right)e^{-jn\omega_0 t}\right] \\
&= \sum_{n=0}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \sum_{n=0}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2j}\right)e^{-jn\omega_0 t}
\end{aligned}$$

We can change the index of the second sum to run from $-\infty$ to -1 by substituting $m = -n$

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \sum_{m=-\infty}^{-1}\left(\frac{a_{-m}}{2} - \frac{b_{-m}}{2j}\right)e^{jm\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}
\end{aligned}$$

where

$$D_n = \begin{cases} \frac{a_n}{2} + \frac{b_n}{2j}, & n \geq 0 \\ \frac{a_{-n}}{2} - \frac{b_{-n}}{2j}, & n < 0 \end{cases}.$$

We can simplify this further by noting that a_n is an even function and b_n is an odd function, so $a_{-n} = a_n$ and $b_{-n} = -b_n$. Therefore, we can express D_n as

$$D_n = \frac{a_n}{2} + \frac{b_n}{2j}$$

for all integer n (positive, negative, and zero).

4.3 Periodic signals and LTI systems

If we have some periodic signal $f(t)$ with a period T_0 and a fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$, we can represent it using the complex exponential Fourier series in Eq. 4.3. Now suppose we are working with a LTI system H that has an impulse response $h(t)$. If we put a single complex exponential $e^{jn\omega_0 t}$ into the system, we will get some output $y(t)$ that is defined by the input sinusoid multiplied by a complex constant $H(n\omega_0)$ defined by the transfer funciton (see Sec. 3.7)

$$y(t) = H(n\omega_0)e^{jn\omega_0 t}.$$

Because this system is LTI, superpostion and scaling apply, therefore if we put the entire periodic signal $f(t)$—which we can define as a sum of scaled complex exponentials—into the system, we will get an output that is the sum of the outputs for each individual complex exponential that are scaled. In other words, if our input is

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

then the output will be

$$\begin{aligned}
y(t) &= H\left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}\right) \\
&= \sum_{n=-\infty}^{\infty} D_n H(n\omega_0) e^{jn\omega_0 t}.
\end{aligned}$$

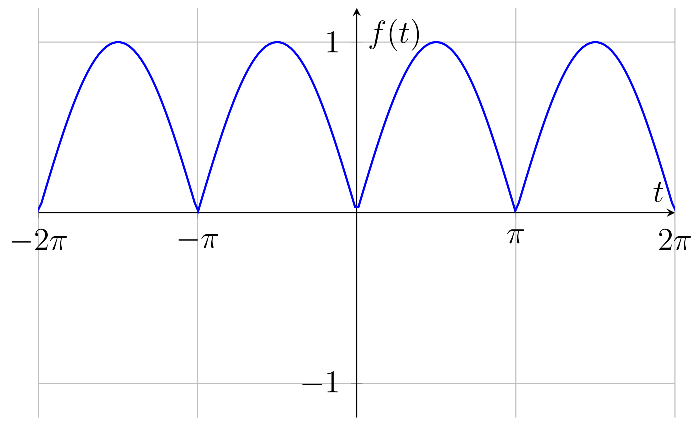


Figure 4.3: A rectified sine wave with period $T_0 = \pi$.

This means that the output $y(t)$ is also periodic with the same period T_0 as the input $f(t)$. The Fourier series coefficients of the output $y(t)$ are simply the Fourier series coefficients of the input $f(t)$ multiplied by the transfer function evaluated at the corresponding frequencies $n\omega_0$.

Example:

Suppose we have as an input a rectified sine wave defined by $f(t) = |\sin(t)|$ as seen in Fig. 4.3. This function is periodic with a period of $T_0 = \pi$. We can compute the Fourier series using complex exponentials which is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{j2nt}.$$

Now suppose we have a LTI system that is defined by the differential equation

$$3\frac{dy}{dt} + y(t) = f(t).$$

Using the approach in Sec. 3.7, we can find the transfer function of this system is

$$H(s) = \frac{1}{3s+1}$$

and evaluating it at $s = jn\omega_0 = jn2$ (because $\omega_0 = 2$ in this instance) gives us

$$H(j2n) = \frac{1}{1+j6n}.$$

Therefore, the output of the system will be

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} D_n H(jn2) e^{j2nt} \\ &= \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)(1+j6n)} e^{j2nt}. \end{aligned}$$