WEBER STATE UNIVERSITY
Engineering, Applied Science & Technology

DEPARTMENT OF
ELECTRICAL & COMPUTER ENGINEERING

ECE 3210

Signals and Systems

Course Notes

Author
Eric Gibbons

Version 1.0.0, Fall 2025

September 4, 2025

Contents

Chapter 1

Review Material

1.1 Complex numbers

A complex number $z \in \mathbb{C}$ has a real part a and an imaginary part b, and is written as $z = a + jb$. The magnitude of a complex number is $|z| = \sqrt{a^2 + b^2}$, and the angle of a complex number is $\angle z = \tan^{-1}\left(\frac{b}{a}\right)$. Euler's formula states that the complex exponential is defined as

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Similarly, we can write a complex number in polar form as

$$z = |z|e^{j\angle z}.$$

We can expand Euler's formula to define a cosine function as

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

and a sine function as

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

The complex conjugate of a complex number $z = a + jb$ is denoted as $z^* = a - jb$, which essentially says we need to negate the imaginary part of z.

The product of a complex number and its conjugate is given by

$$zz^* = |z|^2.$$

The division of a complex number by its magnitude is given by

$$\frac{z}{|z|} = e^{j\angle z}.$$

The sum of two complex numbers is given by

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and the difference of two complex numbers is given by

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

The product of two complex numbers is given by

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

and the division of two complex numbers is given by

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}.$$

Generally, the addition and subtraction of two complex numbers is done in rectangular form, and the multiplication and division of two complex numbers is done in polar form.

1.2 Partial fraction expansion

Partial fraction expansion is a technique used to decompose a rational function (a ratio of polynomials) into a sum of simpler fractions. This is especially useful in signal processing and systems analysis for finding inverse transforms.

Suppose we have a rational function:

$$F(s) = \frac{P(s)}{Q(s)} \tag{1.1}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$.

The method of expansion depends on the nature of the roots of $Q(s)$:

1.2.1 Distinct real roots

If $Q(s)$ factors into distinct real roots, e.g.

$$F(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

for $Q(s) = (s - r_1)(s - r_2)$, then the coefficients A and B can be found by multiplying both sides by $Q(s)$ and solving for the unknowns.

Example (Heaviside Cover-Up Method):

$$F(s) = \frac{5}{(s+1)(s+2)}$$

Write as:

$$\frac{5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

To find A, cover up $(s + 1)$ in the denominator and substitute $s = -1$

$$A = \left.\frac{5}{s+2}\right|_{s=-1} = \frac{5}{-1+2} = 5$$

To find B, cover up $(s + 2)$ in the denominator and substitute $s = -2$

$$B = \left.\frac{5}{s+1}\right|_{s=-2} = \frac{5}{-2+1} = -5$$

which yields

$$F(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

1.2.2 Repeated Roots

If $Q(s)$ has repeated roots, e.g.

$$F(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2}$$

for $Q(s) = (s-r)^2$, then include terms for each power of the repeated factor.

There are a few ways to do this. We will focus on the Heaviside cover-up method.

Example:

Consider some rational function with a repeated root.

$$F(s) = \frac{3s+2}{(s+1)^2}$$

We can write this as

$$\frac{3s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

To use the Heaviside cover-up method for repeated roots, first find B by covering up $(s+1)^2$ and substituting $s = -1$, which is the "easy" coefficient

$$B = (3s+2)\Big|_{s=-1} = 3(-1) + 2 = -1$$

To find A, we will need to differentiate the denominator $(s+1)^2$ with respect to s to get $2(s+1)$, then multiply $F(s)$ by $(s+1)$ and substitute $s = -1$:

$$A = \frac{d}{ds}\left[(s+1)(3s+2)\right]\Bigg|_{s=-1}$$

Alternatively, for this simple case, plug in another value (e.g., $s = 0$):

$$3(0) + 2 = A(1) - 1 \implies 2 = A - 1 \implies A = 3.$$

Yet another approach we can use is multiply both sides by s

$$sF(s) = \frac{3s^2+2s}{(s+1)^2} = \frac{As}{s+1} + \frac{Bs}{(s+1)^2}.$$

We can take the limit as $s \to \infty$ (and using L'Hopitals's rule if needed)

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(\frac{3s^2+2s}{(s+1)^2}\right) = \lim_{s\to\infty}\left(\frac{As}{s+1} + \frac{Bs}{(s+1)^2}\right) = 3$$
$$3 = A + 0 \implies A = 3.$$

This last approach will only help get a single coefficient for the corresponding $s + r_n$ term. One strategy is to use the simply Heavyside approach to find the coefficient for the highest order $(s+r_n)^m$ term, then use this limit approach to find the $(s+r_n)$ coefficient. If there is another term, you could conider putting in a simple $s = 0$ or $s = 1$ value to get a another equation to solve for the remaining unknown.

In this specific case we have

$$F(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2}$$

1.2.3 Complex Roots

Because the systems we will working with are real-valued, any complex roots of $Q(s)$ will occur in complex conjugate pairs

$$Q(s) = (s - r)(s - r^*)$$

where $r = a + jb$, $r^* = a - jb$, then the expansion is:

$$F(s) = \frac{A}{s - r} + \frac{B}{s - r^*}$$

The coefficients A and B may be complex, but the sum will be real if $F(s)$ is real.

Example:

$$F(s) = \frac{2s + 3}{s^2 + 4s + 5}$$

Factor the denominator: $s^2 + 4s + 5 = (s + 2 + j1)(s + 2 - j1)$

You can expand $F(s)$

$$\frac{2s + 3}{(s + 2 + j1)(s + 2 - j1)} = \frac{A}{s + 2 + j1} + \frac{B}{s + 2 - j1}.$$

You can solve for A using the Heavyside coverup. If the coefficients in $F(s)$ are real, then $B = A^*$, so we don't need to explicitly solve for it.

1.3 Summations

We can compute the following summations of the form

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=N_1}^{N_2} r^n = \frac{r^{N_1} - r^{N_2+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1 - r} \quad \text{for } |r| < 1.$$

1.4 U-substitution integration

In this class we will often need to do a simple change of variables with the integrals we are evaluating. This technique is known as U-substitution. The basic idea is to substitute a new variable u for a function of t, which simplifies the integral.

The steps for U-substitution are as follows:

1. Choose a substitution $u = g(t)$ where $g(t)$ is a differentiable function.
2. Compute the differential $du = g'(t)dt$.

3. Convert the limits of integration: if the upper limit is originally $t = a$ then $u = g(a)$ and if the lower limit is originally $t = b$ then $u = g(b)$.
4. Rewrite the integral in terms of u and du and the new limits of integration.
5. Evaluate the integral with respect to u (or even leave it in terms of u).

Example:

Consider the integral

$$y(t) = \int_{-\infty}^{t} x(\tau - T)d\tau.$$

This is an accumulator integral, which integrates some function $x(\tau - T)$ from $-\infty$ to t. The value $y(t)$ is the area under the curve of $x(\tau - T)$ from $-\infty$ to t.

Suppose we wanted to do a quick change of variables here, we see that $u = \tau - T$, which gives $du = d\tau$. The upper limit of the integral in terms of τ was originally t, so the upper limit in terms of u is $t - T$. The lower limit of the integral in terms of τ was originally $-\infty$, so the lower limit in terms of u is also $-\infty$. Thus, we can rewrite the integral as

$$y(t) = \int_{-\infty}^{t-T} x(u)du.$$

This is a relatively simple example, but it shows the basic steps of U-substitution. In this course, we will generally stick to simple examples such as this.

Chapter 2

Signals

A signal is defined as a set of information corresponding to one or more independent variable(s) (often time or space).

2.1 Signal definitions

2.1.1 Classifications

A signal can be classified in a number of ways:

Continuous-time, analog Continuous-time signals are defined for every instant of time and are often represented by analog waveforms (range can take any value).

Discrete-time, analog Discrete-time signals are defined only at discrete intervals and are often represented by analog values (range can take any value).

Continuous-time, digital Signal is defined at every point in time, but takes on only a finite set of values (often quantized using a set of fixed levels).

Discrete-time, digital Discrete-time signals are defined only at discrete intervals and take on a finite set of values (often quantized using a set of fixed levels).

2.1.2 Periodic signals

A periodic signal is a signal that repeats itself at regular intervals over time. The smallest interval over which the signal repeats is called the period (T). Mathematically, a signal $x(t)$ is periodic if there exists a positive constant T such that:

$$x(t) = x(t + T)$$

for all values of t. Periodic signals can be classified as either continuous-time or discrete-time signals. The most common periodic signal is a sinusoidal signal, but it can also include square waves, triangular waves, and other waveforms.

2.1.3 Even and odd signals

A signal $x(t)$ is said to be even if it satisfies the following condition

$$x(t) = x(-t)$$

for all values of t. Even signals are symmetric about the vertical axis.

A signal $x(t)$ is said to be odd if it satisfies the following condition

$$x(t) = -x(-t)$$

for all values of t. Odd signals are antisymmetric about the vertical axis.

2.1.4 Causality

A signal is said to be causal if it is zero for all negative time values. In other words, a causal signal $x(t)$ satisfies the following condition:

$$x(t) = 0 \quad \text{for } t < 0$$

for all values of t. Causal signals are often used to model physical systems that cannot respond before an input is applied. Essentially, you cannot look into the future!

2.2 Measuring signals

We can measure signal strength in many ways (amplitude, RMS, etc.). The choice of measurement depends on the characteristics of the signal and the specific application. Two common metrics are signal energy and power.

2.2.1 Energy

The energy of a signal is a measure of the total power consumed by the signal over time. For a continuous-time signal $x(t)$, the energy E is defined as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For a discrete-time signal $x[n]$, the energy is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

2.2.2 Power

The power of a signal is a measure of the average energy consumed by the signal per unit time. For a continuous-time signal $x(t)$, the power P is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Similarly, you can find the power of a periodic signal by using the fundamental period T_0:

$$P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt.$$

For a discrete-time signal $x[n]$, the power is defined as

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} |x[n]|^2.$$

2.3 Common signals

2.3.1 Step function

The step function, also known as the Heaviside step function, is a mathematical function that is commonly used in signal processing and control systems. It is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

We can use the step function to model a signal "turning on" and "turning off" at specific points in time. For example, a single lobe of sinusoidal signal $\sin(\pi t)$ that turns on at $t = 0$ and off at $t = \pi$ can be represented as

$$x(t) = \sin(\pi t) \cdot (u(t) - u(t - \pi)).$$

2.3.2 Delta function

The delta function, also known as the Dirac delta function, is a mathematical function that is used to model an idealized impulse or point source. It is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

The delta function has the property that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

In practice, the delta function is often used to represent a signal that is concentrated at a single point in time. For example, a signal that consists of a single impulse at $t = 0$ can be represented as

$$x(t) = A \cdot \delta(t)$$

where A is the amplitude of the impulse.

An interesting case is when we have a signal $x(t)$ multiplied by a delta function $\delta(t-T)$. This has the effect of "sampling" the signal at $t = T$

$$x(t) \cdot \delta(t - T) = x(T) \cdot \delta(t - T).$$

We can extend this to the *sifting property* of the delta function, which states that for any function $x(t)$ and any constant T,

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t - T)\, dt = x(T).$$

Constructing $x(t)$ from delta functions

We can use the sifting property to express a signal $x(t)$ in terms of delta functions

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau)\, d\tau. \tag{2.1}$$

This idea is initially confusing, so let's walk through it step by step. Let's consider a case where $t = 0$. Eq. 2.1 becomes (and applying some sifting)

$$\begin{aligned} x(0) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(0 - \tau)\, d\tau \\ = x(0). \end{aligned}$$

Similarly, we can look at the case where $t = 1$

$$\begin{aligned} x(1) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(1 - \tau)\, d\tau \\ = x(1). \end{aligned}$$

We can keep applying this idea for any value of t, which gives us the signal $x(t)$. We can visualize this in Fig. 2.1.

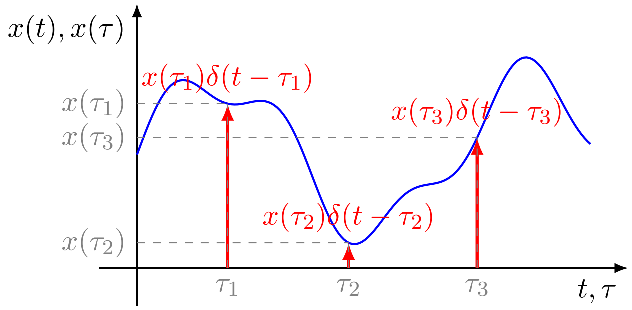


Figure 2.1: Visualizing the sifting property of the delta function. The signal $x(t)$ is constructed from a continuum (integral) of scaled and shifted impulses. Each impulse $x(\tau)\delta(t-\tau)$ is located at τ and has a weight of $x(\tau)$. Each time you integrate over one of these impulses, you get the value of $x(t)$ at that time.

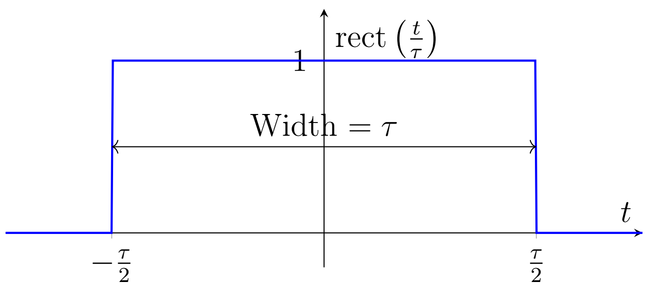


Figure 2.2: The rectangle function rect $\left(\frac{t}{\tau}\right)$.

2.3.3 Rectangle function

A rectangle ("rect") function is a piecewise function that is defined as

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.2.

2.3.4 Triangle function

The triangle function $\Delta\left(\frac{t}{\tau}\right)$ is defined as

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau/2} & \text{for } |t| \le \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.3.

2.3.5 Sinc function

The sinc function is defined[1] as

[1]There are different normalizations of the sinc function, but we will use this version in this course.

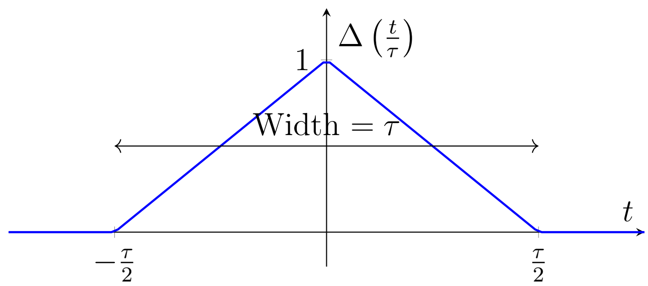


Figure 2.3: The triangle function $\Delta\left(\frac{t}{\tau}\right)$.

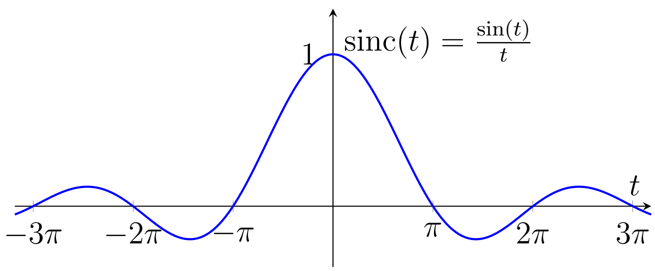


Figure 2.4: The sinc function $\text{sinc}(t) = \frac{\sin(t)}{t}$.

$$\text{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The sinc function is often used in signal processing, particularly in the context of Fourier transforms and filtering. An example of a sinc function is seen in Fig. 2.4.

Chapter 3

Time Domain Systems

A system is a conceptual device that takes one or more inputs and produces one or more outputs. In the context of linear time-invariant (LTI) continuous-time systems, we can describe the relationship between the input and output using differential equations.

In this course we will focus on single-input, single-output (SISO) systems. A classic block diagram of this system behavior is seen in Fig. 3.1

3.1 System properties

3.1.1 Linearity

A system is linear if it satisfies the principles of superposition and scaling (homogeneity). That is, if an input $x_1(t)$ produces an output $y_1(t)$, and an input $x_2(t)$ produces an output $y_2(t)$, then for any constants a and b, the input $ax_1(t) + bx_2(t)$ produces the output $ay_1(t) + by_2(t)$. This is seen in Fig. 3.2.

Checking linearity

We can check if a system is linear by checking if it satisfies the superposition and scaling properties. For example, consider the system defined by

$$y(t) = 3x(t) + 5$$

Let $x_1(t)$ produce $y_1(t)$ and $x_2(t)$ produce $y_2(t)$ such that

$$y_1(t) = 3x_1(t) + 5$$

and

$$y_2(t) = 3x_2(t) + 5.$$

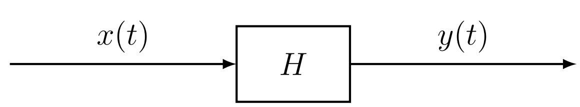


Figure 3.1: A generic SISO system block diagram for system H.

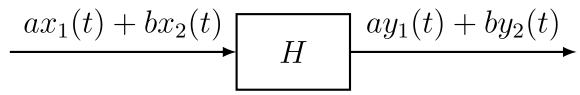


Figure 3.2: A linear SISO system block diagram for system H.

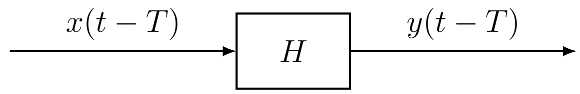


Figure 3.3: A time-invariant SISO system block diagram for system H.

We see that if we use an input that is the superposition and scaled inputs $ax_1(t) + bx_2(t)$, we can write

$$\begin{aligned} y(t) &= 3(ax_1(t) + bx_2(t)) + 5 \\ &= 3ax_1(t) + 3bx_2(t) + 5 \end{aligned}$$

which is not equal to $ay_1(t) + by_2(t)$ because of the constant term 5. Therefore, the system is not linear.

3.1.2 Time invariance

A system is time-invariant if its behavior and characteristics do not change over time. In other words, if we apply a time-shifted input to the system, the output will also be time-shifted by the same amount. Mathematically, if an input $x(t)$ produces an output $y(t)$, then for any time shift T, the input $x(t-T)$ will produce the output $y(t-T)$. A block diagram of this system behavior is seen in Fig. 3.3.

Similarly, we can visualize this behavior in Fig. 3.4. In this figure we observe a typical system input/output relationship. However, if we delay the input by a time T, the output is also delayed by the same amount, illustrating the time-invariance property.

Checking time invariance

To check system time invariance, apply a time-shifted input $x(t-T)$ to the system H and observe the output $\tilde{y}(t)$. Next, take the output for a typical $y(t) = H\{x(t)\}$ and shift it by the same amount to get $y(t-T)$. If $\tilde{y}(t) = y(t-T)$, then the system is time-invariant. If not, then the system is time-variant. This is best seen in example.

Example:

Consider the system

$$y(t) = x(t)\cos(t)$$

To check for time invariance, we apply a time-shifted input $x(t-T)$, which means putting a $-T$ term into the $x(t)$ function

$$\tilde{y}(t) = x(t-T)\cos(t)$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t-T$ in the output equation

$$y(t-T) = x(t-T)\cos(t-T)$$

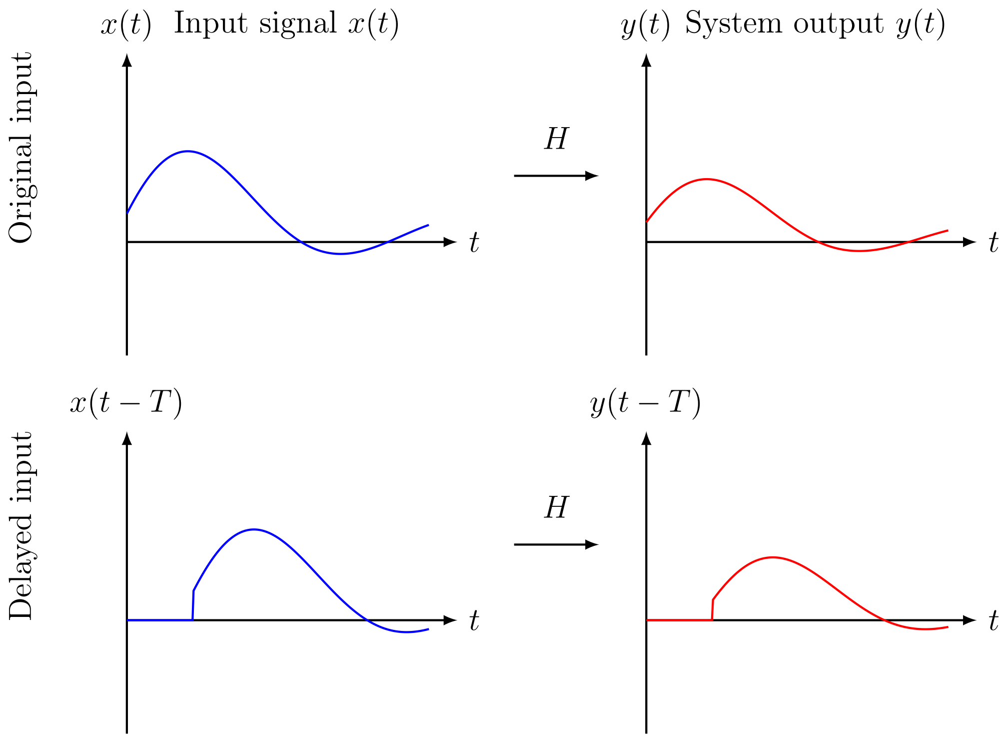


Figure 3.4: Time invariance illustrated with four plots in a grid.

Here, we clearly see that $\tilde{y}(t) \neq y(t-T)$ because of the $\cos(t)$ term, which introduces a time-dependent phase shift. Therefore, the system is time-variant. Typically, if some system H multiplies the input by a time-varying function, it will be time-variant.

Example:

Let's look at another example. Consider the integrator system

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

First, we apply a time-shifted input $x(t-T)$

$$\tilde{y}(t) = \int_{-\infty}^{t} x(\tau - T)d\tau$$

We can change the variable of integration to $\lambda = \tau - T$, which gives us $d\tau = d\lambda$. We also see that when $\tau == -\infty$, $\lambda = \infty$ and when $\tau = t$, $\lambda = t - T$. Thus, we can rewrite the integral as

$$\tilde{y}(t) = \int_{-\infty}^{t-T} x(\lambda)d\lambda$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t (not τ!) with $t-T$ in the output equation

$$y(t-T) = \int_{-\infty}^{t-T} x(\tau)d\tau$$

Here, we clearly see that $\tilde{y}(t) = y(t-T)$, which means the system is time-invariant.

Example:

Consider a compressor system defined by

$$y(t) = x(2t).$$

First we can shift the system input by T by adding a $-T$ to $x(t)$ to get

$$\tilde{y}(t) = x(2t-T).$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t-T$ in the output equation

$$\begin{aligned} y(t-T) &= x(2(t-T)) \\ &= x(2t-2T). \end{aligned}$$

We see that $\tilde{y}(t) \neq y(t-T)$, so the system is time-variant.

3.2 System response

A general SISO LTI continuous-time system can be described by a linear constant-coefficient differential equation of the form

$$\begin{aligned} \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \dots + a_1\frac{dy(t)}{dt} + a_0 y(t) &= \dots \\ \dots b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \dots + b_1\frac{dx(t)}{dt} + b_0 x(t). \end{aligned} \tag{3.1}$$

We can also write this in a more compact form where each derivative term $\frac{d}{dt}$ can simply be represented with an operator D such that $D^n y(t) = \frac{d^n y(t)}{dt^n}$. Thus, we can rewrite (3.1) as

$$\begin{aligned} D^n y(t) + a_{n-1}D^{n-1}y(t) + \dots + a_1 Dy(t) + a_0 y(t) &= \dots \\ \dots b_m D^m x(t) + b_{m-1}D^{m-1}x(t) + \dots + b_1 Dx(t) + b_0 x(t). \end{aligned} \tag{3.2}$$

Further, we can simplify this to

$$\underbrace{\left(D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0\right)}_{Q(D)} y(t) = \underbrace{\left(b_m D^m + b_{m-1}D^{m-1} + \dots + b_1 D + b_0\right)}_{P(D)} x(t) \tag{3.3}$$

where $Q(D)$ and $P(D)$ are polynomials in the differential operator D.

If we were to solve the ODE described by (3.1), we would find the system's response has two components, the zero-input response and the zero-state response. Thus the solution is

$$y_{\text{tot}}(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

3.3 Zero-input response

In the case of the zero-input response, we are interested in how the system responds to initial conditions without any external input. This means we set the input $x(t)$ to zero and solve the homogeneous equation associated with the system. The zero-input response is determined solely by the system's characteristics and its initial conditions. Without the loss of generality, we will work with 2^{nd} order systems. Consider the following system

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_2\frac{d^2x(t)}{dt^2} + b_1\frac{dx(t)}{dt} + b_0x(t).$$

To find the zero-input response we can set $x(t) = 0$ to get

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = 0.$$

Assume the generic solution to this homogeneous equation is of the form

$$y_0(t) = Ce^{\lambda t}.$$

We can differentiate this expression to find the first and second derivatives:

$$\begin{aligned}\frac{dy_0(t)}{dt} &= C\lambda e^{\lambda t},\\ \frac{d^2y_0(t)}{dt^2} &= C\lambda^2e^{\lambda t}.\end{aligned}$$

Substituting this into the homogeneous equation gives us

$$C\lambda^2e^{\lambda t} + a_1C\lambda e^{\lambda t} + a_0Ce^{\lambda t} = 0.$$

Factoring out the common term $Ce^{\lambda t}$ gives us

$$Ce^{\lambda t}\left(\lambda^2 + a_1\lambda + a_0\right) = 0.$$

To make this true, we need to solve the characteristic equation (assuming $C \neq 0$ and $e^{\lambda t} \neq 0$ for all t)

$$Q(\lambda) = \lambda^2 + a_1\lambda + a_0 = 0.$$

which is called the *characteristic equation*. The solutions to this equation, λ_1 and λ_2, are called the *characteristic roots* or *eigenvalues* of the system. The nature of these roots (real or complex) will determine the form of the zero-input response. To summarize, the form of the zero-input response is determined by the characteristic roots:

- Real and distinct roots lead to two exponential terms.
- Real and repeated roots lead to an exponential term and a linear term.
- Complex conjugate roots lead to an exponential decay term and sinusoidal terms.

3.3.1 Real and distinct roots

If λ_1 and λ_2 are real and distinct ($\lambda_1 \neq \lambda_2$), then the homogeneous solution is

$$y_0(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}.$$

3.3.2 Real and repeated roots

If λ_1 is a real and repeated root ($\lambda_1 = \lambda_2$), then the homogeneous solution is

$$y_0(t) = (C_1 + C_2 t)e^{\lambda_1 t}.$$

3.3.3 Complex conjugate roots

If $\lambda = a+jb$ is a complex root of the characteristic equation, then its complex conjugate $\lambda^* = a-jb$ is also a root. The homogeneous solution for complex conjugate roots is also given by

$$\begin{aligned} y_0(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(a+jb)t} + C_2 e^{(a-jb)t}. \end{aligned}$$

The coefficients C_1 and C_2 may be complex, but they will be complex conjugates such that $C_1 = C_2^*$ and $C_1 = c + jd$ and $C_2 = c - jd$. We can rewrite $y_0(t)$ as

$$\begin{aligned} y_0(t) &= e^{at}\left[(c+jd)e^{jbt} + (c-jd)e^{-jbt}\right] \\ &= e^{at}\left[c(e^{jbt} + e^{-jbt}) + jd(e^{jbt} - e^{-jbt})\right] \\ &= e^{at}\left[c(2\cos(bt)) + jd(2j\sin(bt))\right] \\ &= e^{at}\left[2c\cos(bt) - 2d\sin(bt)\right]. \end{aligned}$$

To find the constants, we need to apply the initial conditions of the system, which are typically given as $y(0)$ and $y'(0)$. This usually turns into a 2×2 system of equations that can be solved with any linear algebra technique.

Application: circuit analysis

In circuit analysis, we often encounter second-order linear differential equations when analyzing RLC circuits. The zero-input response can be used to determine the natural response of the circuit. To find the coefficients of the zero-input response, we can use the initial conditions of the circuit, such as the initial voltage across a capacitor or the initial current through an inductor (or if it is given to you). Recall that we can represent voltages and currents in a circuit in the time-domain. Each of these quantities can be seen in Fig. 3.5.

Consider the circuit in Fig. 3.6 where the input to the system is some voltage $x(t) = 10e^{-3t}u(t)$ and the output is the current $y(t)$ flowing in the loop. We know that the initial voltage over the capacitor is $v_C(0^-) = 5\,\mathrm{V}$.

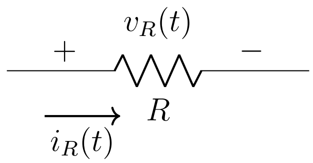
The first thing we want to do is derive an ODE for this expression, which is easiest by a simple voltage loop.

$$-x(t) + v_L(t) + v_R(t) + v_C(t) = 0.$$

From there substitute expressions relating the voltages to the current $y(t)$

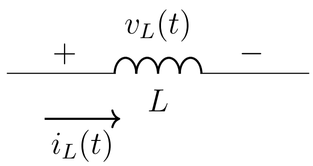
$$-x(t) + L\frac{dy(t)}{dt} + Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

Resistor voltage and current



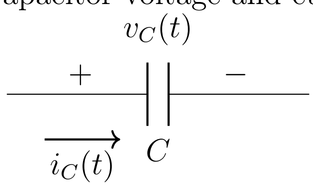
$$v_R(t) = R i_R(t)$$
$$i_R(t) = \frac{v_R(t)}{R}$$

Inductor voltage and current



$$v_L(t) = L\frac{di_L(t)}{dt}$$
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)\,d\tau$$

Capacitor voltage and current



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Figure 3.5: Time-domain representation of voltages and currents in resistors, inductors, and capacitors.

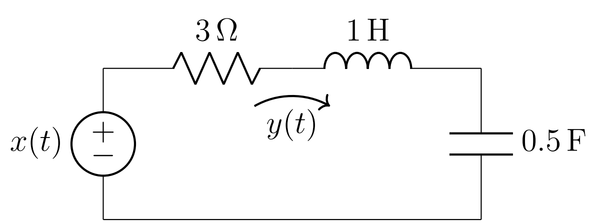


Figure 3.6: Zero-input response of an RLC circuit.

Since we know the component values we simplify to

$$-x(t) + \frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

We need to get rid of the integral term, so we differentiate both sides and simplify

$$\begin{aligned} -\frac{dx(t)}{dt} + \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= 0 \\ \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= \frac{dx(t)}{dt}. \end{aligned}$$

We see the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

which has roots $\lambda_1 = -1$ and $\lambda_2 = -2$ giving a general solution of

$$y_{\text{zir}}(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

Next we need the initial conditions of the system. We see from the input voltage $x(t)$ that there is no initial voltage and so the current would be zero at $t = 0^-$ such that $y(0^-) = 0$. Looking at the system then at $t = 0^-$, we have

$$-x(0^-) + v_L(0^-) + v_R(0^-) + v_C(0^-) = 0$$

and since we know that $v_C(0^-) = -5\,\text{V}$, $y(0^-) = 0$, and $x(0^-) = 10\,\text{V}$ we can substitute this into the equation to get

$$-10 + v_L(0^-) + R \cdot 0 + 5 = 0.$$

We see that $v_L(0^-) = -5\,\text{V}$ and therefore (using the relationship $v_L = L\frac{dy}{dt}$) we can find the initial condition for the inductor current $y'(0^-) = -5\,\text{V}$, thus giving us our second initial condition. We can apply this to the general solution to find the coefficients C_1 and C_2 (and realizing that $y'_{\text{zir}}(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$)

$$\begin{aligned} y_{\text{zir}}(0^-) &= C_1 + C_2 = 0, \\ y'_{\text{zir}}(0^-) &= -C_1 - 2C_2 = 5. \end{aligned}$$

and solving $C_1 = -5$ and $C_2 = 5$ giving us the final zero-input response

$$y_{\text{zir}}(t) = -5e^{-t} + 5e^{-2t}.$$

3.4 Zero-state response

The *zero-state response* is the part of the system's output that is solely due to the external input, assuming that all initial conditions are zero. In other words, we analyze how the system responds to an input signal when it starts from a state of rest (zero initial conditions).

Consider an LTI system seen in Fig. 3.7 with an input signal $x(t)$ and an output signal $y(t)$. The zero-state response can be determined using the system's impulse response $h(t)$, which is the output of the system when the input is an impulse function $\delta(t)$. The impulse response characterizes the

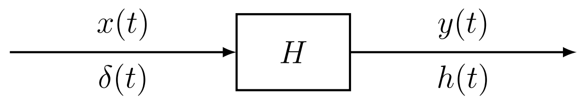


Figure 3.7: System response to an impulse function.

system's behavior and can be used to find the output for any arbitrary input signal through the convolution operation.

Consider some arbitrary signal $x(t)$, which can be written as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

which states that any signal can be represented as a weighted sum of impulse functions. Assume that some system is LTI and has an impulse function $h(t)$. If we pass $x(t)$ through this system, we see

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}. \end{aligned}$$

Applying superposition, we can pull out the integral

$$y(t) = \int_{-\infty}^{\infty} H\{x(\tau)\delta(t-\tau)\}d\tau.$$

Applying scaling, we can pull out the $x(\tau)$ term because in terms of the system H, there is no time dependence with $x(\tau)$. This would given

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau.$$

Because H is time-invariant, we notice a shift impulse function, will yield a shifted delta function such that

$$H\{\delta(t-\tau)\} = h(t-\tau).$$

We can rewrite the output of the system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

We see that the zero-state response can be expressed as a convolution between the input signal and the system's impulse response

$$y(t) = x(t) * h(t).$$

3.4.1 Convolution properties

The convolution integral has several properties that are going to be helpful in analyzing LTI systems.

Property	**Mathematical Expression**
Commutative	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
Associative	$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$
Distributive	$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
Shift Property	$f_1(t - t_0) * f_2(t) = [f_1(t) * f_2(t)]_{t \to t - t_0}$
Delta Function	$f_t(t) * \delta(t) = f_1(t)$, and $f_1(t) * \delta(t - t_0) = f_1(t - t_0)$
Width	If $f_1(t)$ and $f_2(t)$ are time-limited to T_1 and T_2, then $f_1(t) * f_2(t)$ is time-limited to $T_1 + T_2$

Table 3.1: Properties of convolution for signals $f_1(t)$, $f_2(t)$, and $f_3(t)$.

3.4.2 Determining the impulse function $h(t)$

To determine the zero-state response, we will need to find the impulse response $h(t)$ of the system. This can be done myriad ways:

- By applying an impulse input $x(t) = \delta(t)$ and measuring the output $y(t) = h(t)$.
- By applying a known input $x(t)$ and measuring the output $y(t)$, then using deconvolution techniques to extract $h(t)$.
- By solving the system's differential equation with the input $x(t) = \delta(t)$ and zero initial conditions.

Here, we will focus on the last method. To find the impulse response in the time-domain, we will use the formula

$$h(t) = b_n\delta(t) + P(D)y_n(t)u(t)$$

where n is the order of the system. The b_n coefficient is the coefficient from the system's differential equation, as described in Eq. 3.3. The $P(D)$ operator is the polynomial operator from the left-hand side of Eq. 3.3. The $y_n(t)$ term is the zero-input response given a specific set of initial conditions

$$\begin{aligned} y^{(n-1)}(0^-) &= 1 \\ y^{(n-2)}(0^-) &= b_{n-1} = \ldots = y(0^-) = 0. \end{aligned}$$

Example:

Consider the system defined by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt}.$$

To find the impulse response $h(t)$ for this system, we can follow a fairly procedural set of steps.

1. **Identify the system order n:** The highest derivative of $y(t)$ is 2, so $n = 2$.

2. **Determine the b_n coefficient:** From the right-hand side of the differential equation, we see that $b_n = b_2 = 0$ because there is no $\frac{d^2x(t)}{dt^2}$ term.

3. **Determine the $P(D)$ operator:** From the left-hand side of the differential equation, we have
$$P(D) = 2D$$

4. **Find the zero-input response $y_n(t)$:** We need to go through the steps to solve the zero-input response. First, we notice that the characteristic equation is
$$Q(\lambda) = \lambda^2 + 2\lambda + 2.$$
Solving this characteristic equation gives us the roots
$$\lambda = -1 \pm j$$
which gives a generic solution
$$y_n(t) = e^{-t}\left(C_1 \cos(t) + C_2 \sin(t)\right).$$
Applying the initial conditions $y(0^-) = 0$ and $y'(0^-) = 1$ gives us
$$0 = C_1 \quad \text{and} \quad 1 = -C_1 + C_2.$$
Solving this system gives us $C_1 = 0$ and $C_2 = 1$, so the zero-input response is
$$y_n(t) = e^{-t}\sin(t)$$

5. **Compute $P(D)y_n(t)$:** We need to apply the $P(D)$ operator to $y_n(t)$
$$\begin{aligned} P(D)y_n(t) &= 2Dy_n(t) \\ &= 2\frac{d}{dt}\left(e^{-t}\sin(t)\right) \\ &= 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right) \end{aligned}$$

6. **Combine to find $h(t)$:** Finally, we can combine all the pieces to find the impulse response
$$\begin{aligned} h(t) &= b_n\delta(t) + P(D)y_n(t)u(t) \\ &= 0 + 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right)u(t) \\ &= 2e^{-t}\left(\cos(t) - \sin(t)\right)u(t) \end{aligned}$$