WEBER STATE UNIVERSITY
Engineering, Applied Science & Technology

DEPARTMENT OF
ELECTRICAL & COMPUTER ENGINEERING

ECE 3210

Signals and Systems

Course Notes

Author
Eric Gibbons

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Review Material

1.1 Complex numbers

A complex number $z \in \mathbb{C}$ has a real part a and an imaginary part b, and is written as $z = a + jb$. The magnitude of a complex number is $|z| = \sqrt{a^2 + b^2}$, and the angle of a complex number is $\angle z = \tan^{-1}\left(\frac{b}{a}\right)$. Euler's formula states that the complex exponential is defined as

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Similarly, we can write a complex number in polar form as

$$z = |z|e^{j\angle z}.$$

We can expand Euler's formula to define a cosine function as

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

and a sine function as

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

The complex conjugate of a complex number $z = a + jb$ is denoted as $z^* = a - jb$, which essentially says we need to negate the imaginary part of z.

The product of a complex number and its conjugate is given by

$$zz^* = |z|^2.$$

The division of a complex number by its magnitude is given by

$$\frac{z}{|z|} = e^{j\angle z}.$$

The sum of two complex numbers is given by

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and the difference of two complex numbers is given by

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

The product of two complex numbers is given by

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

and the division of two complex numbers is given by

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}.$$

Generally, the addition and subtraction of two complex numbers is done in rectangular form, and the multiplication and division of two complex numbers is done in polar form.

1.2 Trig identities

The following table are a few common trigonometric identities that are useful to know for this course.

1.3 Partial fraction expansion

Partial fraction expansion is a technique used to decompose a rational function (a ratio of polynomials) into a sum of simpler fractions. This is especially useful in signal processing and systems analysis for finding inverse transforms.

Suppose we have a rational function:

$$F(s) = \frac{P(s)}{Q(s)} \tag{1.1}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$.

The method of expansion depends on the nature of the roots of $Q(s)$:

1.3.1 Distinct real roots

If $Q(s)$ factors into distinct real roots, e.g.

$$F(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

for $Q(s) = (s - r_1)(s - r_2)$, then the coefficients A and B can be found by multiplying both sides by $Q(s)$ and solving for the unknowns.

Example (Heaviside Cover-Up Method):

$$F(s) = \frac{5}{(s + 1)(s + 2)}$$

Write as:

$$\frac{5}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

To find A, cover up $(s + 1)$ in the denominator and substitute $s = -1$

$$A = \left.\frac{5}{s + 2}\right|_{s=-1} = \frac{5}{-1 + 2} = 5$$

Table 1.1: Basic trigonometric identities

Identity Category	**Formula(s)**
Pythagorean Identity	$\sin^2(\theta) + \cos^2(\theta) = 1$
Angle Sum & Difference	$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$ $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
Double-Angle	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$
Power-Reduction	$\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$
Product-to-Sum	$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$ $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$ $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$
Sum-to-Product	$\cos(A) + \cos(B) = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$ $\cos(A) - \cos(B) = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
Phase Shift & Periodicity	$\sin\left(\theta \pm \frac{\pi}{2}\right) = \pm\cos(\theta)$ $\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp\sin(\theta)$ $\sin(\theta+\pi) = -\sin(\theta)$ $\cos(\theta+\pi) = -\cos(\theta)$

To find B, cover up $(s+2)$ in the denominator and substitute $s = -2$

$$B = \left.\frac{5}{s+1}\right|_{s=-2} = \frac{5}{-2+1} = -5$$

which yields

$$F(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

1.3.2 Repeated Roots

If $Q(s)$ has repeated roots, e.g.

$$F(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2}$$

for $Q(s) = (s-r)^2$, then include terms for each power of the repeated factor.

There are a few ways to do this. We will focus on the Heaviside cover-up method.

Example:

Consider some rational function with a repeated root.

$$F(s) = \frac{3s+2}{(s+1)^2}$$

We can write this as

$$\frac{3s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

To use the Heaviside cover-up method for repeated roots, first find B by covering up $(s+1)^2$ and substituting $s=-1$, which is the "easy" coefficient

$$B = (3s+2)\Big|_{s=-1} = 3(-1)+2 = -1$$

To find A, we will need to differentiate the denominator $(s+1)^2$ with respect to s to get $2(s+1)$, then multiply $F(s)$ by $(s+1)$ and substitute $s=-1$:

$$A = \frac{d}{ds}\left[(s+1)(3s+2)\right]\Big|_{s=-1}$$

Alternatively, for this simple case, plug in another value (e.g., $s=0$):

$$3(0)+2 = A(1) - 1 \implies 2 = A - 1 \implies A = 3.$$

Yet another approach we can use is multiply both sides by s

$$sF(s) = \frac{3s^2+2s}{(s+1)^2} = \frac{As}{s+1} + \frac{Bs}{(s+1)^2}.$$

We can take the limit as $s \to \infty$ (and using L'Hopitals's rule if needed)

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(\frac{3s^2+2s}{(s+1)^2}\right) = \lim_{s\to\infty}\left(\frac{As}{s+1} + \frac{Bs}{(s+1)^2}\right) = 3$$
$$3 = A + 0 \implies A = 3.$$

This last approach will only help get a single coefficient for the corresponding $s+r_n$ term. One strategy is to use the simply Heavyside approach to find the coefficient for the highest order $(s+r_n)^m$ term, then use this limit approach to find the $(s+r_n)$ coefficient. If there is another term, you could conider putting in a simple $s=0$ or $s=1$ value to get a another equation to solve for the remaining unknown.

In this specific case we have

$$F(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2}$$

1.3.3 Complex Roots

Because the systems we will working with are real-valued, any complex roots of $Q(s)$ will occur in complex conjugate pairs

$$Q(s) = (s - r)(s - r^*)$$

where $r = a + jb$, $r^* = a - jb$, then the expansion is:

$$F(s) = \frac{A}{s - r} + \frac{B}{s - r^*}$$

The coefficients A and B may be complex, but the sum will be real if $F(s)$ is real.

Example:

$$F(s) = \frac{2s + 3}{s^2 + 4s + 5}$$

Factor the denominator: $s^2 + 4s + 5 = (s + 2 + j1)(s + 2 - j1)$

You can expand $F(s)$

$$\frac{2s + 3}{(s + 2 + j1)(s + 2 - j1)} = \frac{A}{s + 2 + j1} + \frac{B}{s + 2 - j1}.$$

You can solve for A using the Heavyside coverup. If the coefficients in $F(s)$ are real, then $B = A^*$, so we don't need to explicitly solve for it.

1.4 Summations

We can compute the following summations of the form

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=N_1}^{N_2} r^n = \frac{r^{N_1} - r^{N_2+1}}{1 - r} \quad \text{for } r \neq 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1 - r} \quad \text{for } |r| < 1.$$

1.5 Integration

1.5.1 U-substitution integration

In this class we will often need to do a simple change of variables with the integrals we are evaluating. This technique is known as U-substitution. The basic idea is to substitute a new variable u for a function of t, which simplifies the integral.

The steps for U-substitution are as follows:

1. Choose a substitution $u = g(t)$ where $g(t)$ is a differentiable function.
2. Compute the differential $du = g'(t)dt$.
3. Convert the limits of integration: if the upper limit is originally $t = a$ then $u = g(a)$ and if the lower limit is originally $t = b$ then $u = g(b)$.
4. Rewrite the integral in terms of u and du and the new limits of integration.
5. Evaluate the integral with respect to u (or even leave it in terms of u).

Example:

Consider the integral

$$y(t) = \int_{-\infty}^{t} x(\tau - T)d\tau.$$

This is an accumulator integral, which integrates some function $x(\tau - T)$ from $-\infty$ to t. The value $y(t)$ is the area under the curve of $x(\tau - T)$ from $-\infty$ to t.

Suppose we wanted to do a quick change of variables here, we see that $u = \tau - T$, which gives $du = d\tau$. The upper limit of the integral in terms of τ was originally t, so the upper limit in terms of u is $t - T$. The lower limit of the integral in terms of τ was originally $-\infty$, so the lower limit in terms of u is also $-\infty$. Thus, we can rewrite the integral as

$$y(t) = \int_{-\infty}^{t-T} x(u)du.$$

This is a relatively simple example, but it shows the basic steps of U-substitution. In this course, we will generally stick to simple examples such as this.

1.5.2 Basic integrals

Table 1.2 provides a list of some common integrals that are useful to know for this course. This is not an exhaustive list, but it should cover many of the integrals you will encounter.

Table 1.2: An Abbreviated Table of Integrals

1. $\displaystyle\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$
2. $\displaystyle\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$
3. $\displaystyle\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$
4. $\displaystyle\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$
5. $\displaystyle\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx)-b\cos(bx))$
6. $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx)+b\sin(bx))$
7. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$
8. $\displaystyle\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\left(\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right)$
9. $\displaystyle\int \sin(ax)\sin(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)}, \quad a^2\neq b^2$
10. $\displaystyle\int \cos(ax)\cos(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)}, \quad a^2\neq b^2$
11. $\displaystyle\int \sin(ax)\cos(bx)\,dx = -\frac{\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)}, \quad a^2\neq b^2$
12. $\displaystyle\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$
13. $\displaystyle\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$
14. $\displaystyle\int_0^\infty \frac{a}{a^2+x^2}\,dx = \begin{cases} \pi/2, & a>0 \\ 0, & a=0 \\ -\pi/2, & a<0 \end{cases}$
15. $\displaystyle\int_0^\infty \frac{\sin(ax)}{x}\,dx = \begin{cases} \pi/2, & a>0 \\ -\pi/2, & a<0 \end{cases}$
16. $\displaystyle\int x^2\sin(ax)\,dx = \frac{2x}{a^2}\sin(ax) - \frac{a^2x^2-2}{a^3}\cos(ax)$
17. $\displaystyle\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax)$
18. $\displaystyle\int e^{ax}\sin^2(bx)\,dx = \frac{e^{ax}}{a^2+4b^2}\left[(a\sin(bx)-2b\cos(bx))\sin(bx)+\frac{2b^2}{a}\right]$
19. $\displaystyle\int e^{ax}\cos^2(bx)\,dx = \frac{e^{ax}}{a^2+4b^2}\left[(a\cos(bx)+2b\sin(bx))\cos(bx)+\frac{2b^2}{a}\right]$

Chapter 2

Signals

A signal is defined as a set of information corresponding to one or more independent variable(s) (often time or space).

2.1 Signal definitions

2.1.1 Classifications

A signal can be classified in a number of ways:

Continuous-time, analog Continuous-time signals are defined for every instant of time and are often represented by analog waveforms (range can take any value).

Discrete-time, analog Discrete-time signals are defined only at discrete intervals and are often represented by analog values (range can take any value).

Continuous-time, digital Signal is defined at every point in time, but takes on only a finite set of values (often quantized using a set of fixed levels).

Discrete-time, digital Discrete-time signals are defined only at discrete intervals and take on a finite set of values (often quantized using a set of fixed levels).

2.1.2 Periodic signals

A periodic signal is a signal that repeats itself at regular intervals over time. The smallest interval over which the signal repeats is called the period (T). Mathematically, a signal $x(t)$ is periodic if there exists a positive constant T such that:

$$x(t) = x(t + T)$$

for all values of t. Periodic signals can be classified as either continuous-time or discrete-time signals. The most common periodic signal is a sinusoidal signal, but it can also include square waves, triangular waves, and other waveforms.

2.1.3 Even and odd signals

A signal $x(t)$ is said to be even if it satisfies the following condition

$$x(t) = x(-t)$$

for all values of t. Even signals are symmetric about the vertical axis.

A signal $x(t)$ is said to be odd if it satisfies the following condition

$$x(t) = -x(-t)$$

for all values of t. Odd signals are antisymmetric about the vertical axis.

2.1.4 Causality

A signal is said to be causal if it is zero for all negative time values. In other words, a causal signal $x(t)$ satisfies the following condition:

$$x(t) = 0 \quad \text{for } t < 0$$

for all values of t. Causal signals are often used to model physical systems that cannot respond before an input is applied. Essentially, you cannot look into the future!

2.2 Measuring signals

We can measure signal strength in many ways (amplitude, RMS, etc.). The choice of measurement depends on the characteristics of the signal and the specific application. Two common metrics are signal energy and power.

2.2.1 Energy

The energy of a signal is a measure of the total power consumed by the signal over time. For a continuous-time signal $x(t)$, the energy E is defined as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For a discrete-time signal $x[n]$, the energy is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

2.2.2 Power

The power of a signal is a measure of the average energy consumed by the signal per unit time. For a continuous-time signal $x(t)$, the power P is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Similarly, you can find the power of a periodic signal by using the fundamental period T_0:

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt.$$

For a discrete-time signal $x[n]$, the power is defined as

$$P = \lim_{N \to \infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} |x[n]|^2.$$

2.3 Common signals

2.3.1 Step function

The step function, also known as the Heaviside step function, is a mathematical function that is commonly used in signal processing and control systems. It is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

We can use the step function to model a signal "turning on" and "turning off" at specific points in time. For example, a single lobe of sinusoidal signal $\sin(\pi t)$ that turns on at $t = 0$ and off at $t = \pi$ can be represented as

$$x(t) = \sin(\pi t) \cdot (u(t) - u(t - \pi)).$$

2.3.2 Delta function

The delta function, also known as the Dirac delta function, is a mathematical function that is used to model an idealized impulse or point source. It is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

The delta function has the property that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

In practice, the delta function is often used to represent a signal that is concentrated at a single point in time. For example, a signal that consists of a single impulse at $t = 0$ can be represented as

$$x(t) = A \cdot \delta(t)$$

where A is the amplitude of the impulse.

An interesting case is when we have a signal $x(t)$ multiplied by a delta function $\delta(t-T)$. This has the effect of "sampling" the signal at $t = T$

$$x(t) \cdot \delta(t - T) = x(T) \cdot \delta(t - T).$$

We can extend this to the *sifting property* of the delta function, which states that for any function $x(t)$ and any constant T,

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t - T)\, dt = x(T).$$

We see that the delta function is related to the step function through differentiation

$$\frac{d}{dt} u(t) = \delta(t)$$

and integration

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t).$$

Constructing $x(t)$ from delta functions

We can use the sifting property to express a signal $x(t)$ in terms of delta functions

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau)\, d\tau. \tag{2.1}$$

This idea is initially confusing, so let's walk through it step by step. Let's consider a case where $t = 0$. Eq. 2.1 becomes (and applying some sifting)

$$\begin{aligned} x(0) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(0 - \tau)\, d\tau \\ = x(0). \end{aligned}$$

Similarly, we can look at the case where $t = 1$

$$\begin{aligned} x(1) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(1 - \tau)\, d\tau \\ = x(1). \end{aligned}$$

We can keep applying this idea for any value of t, which gives us the signal $x(t)$. We can visualize this in Fig. 2.1.

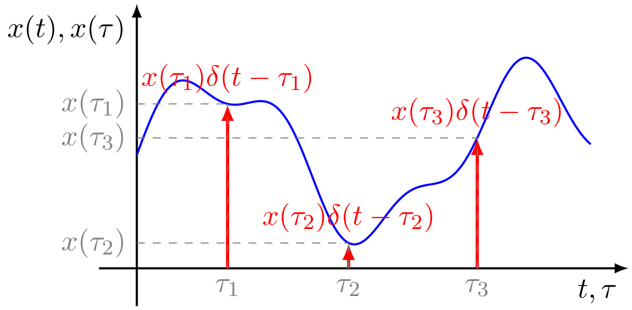


Figure 2.1: Visualizing the sifting property of the delta function. The signal $x(t)$ is constructed from a continuum (integral) of scaled and shifted impulses. Each impulse $x(\tau)\delta(t-\tau)$ is located at τ and has a weight of $x(\tau)$. Each time you integrate over one of these impulses, you get the value of $x(t)$ at that time.

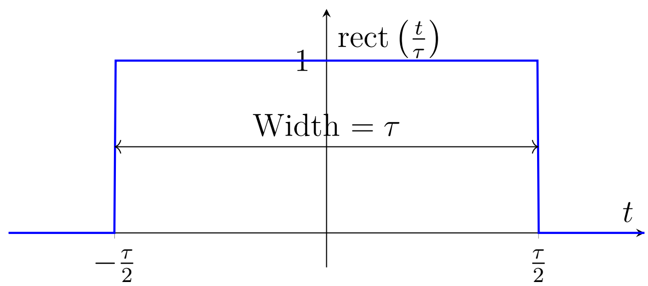


Figure 2.2: The rectangle function $\mathrm{rect}\left(\frac{t}{\tau}\right)$.

2.3.3 Rectangle function

A rectangle ("rect") function is a piecewise function that is defined as

$$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.2.

2.3.4 Triangle function

The triangle function $\Delta\left(\frac{t}{\tau}\right)$ is defined as

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau/2} & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{for } |t| > \frac{\tau}{2} \end{cases}.$$

This is seen in Fig. 2.3.

2.3.5 Sinc function

The sinc function is defined[1] as

[1]There are different normalizations of the sinc function, but we will use this version in this course.

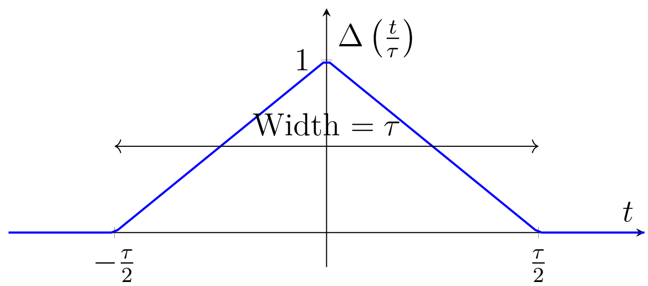


Figure 2.3: The triangle function $\Delta\left(\frac{t}{\tau}\right)$.

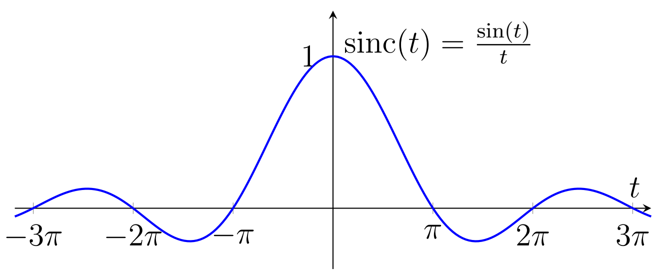


Figure 2.4: The sinc function $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$.

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The sinc function is often used in signal processing, particularly in the context of Fourier transforms and filtering. An example of a sinc function is seen in Fig. 2.4.

Chapter 3

Time Domain Systems

A system is a conceptual device that takes one or more inputs and produces one or more outputs. In the context of linear time-invariant (LTI) continuous-time systems, we can describe the relationship between the input and output using differential equations.

In this course we will focus on single-input, single-output (SISO) systems. A classic block diagram of this system behavior is seen in Fig. 3.1

3.1 System properties

3.1.1 Linearity

A system is linear if it satisfies the principles of superposition and scaling (homogeneity). That is, if an input $x_1(t)$ produces an output $y_1(t)$, and an input $x_2(t)$ produces an output $y_2(t)$, then for any constants a and b, the input $ax_1(t) + bx_2(t)$ produces the output $ay_1(t) + by_2(t)$. This is seen in Fig. 3.2.

Checking linearity

We can check if a system is linear by checking if it satisfies the superposition and scaling properties. For example, consider the system defined by

$$y(t) = 3x(t) + 5$$

Let $x_1(t)$ produce $y_1(t)$ and $x_2(t)$ produce $y_2(t)$ such that

$$y_1(t) = 3x_1(t) + 5$$

and

$$y_2(t) = 3x_2(t) + 5.$$

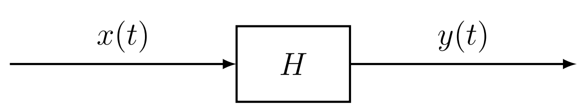


Figure 3.1: A generic SISO system block diagram for system H.

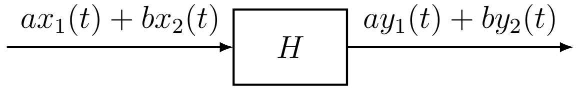


Figure 3.2: A linear SISO system block diagram for system H.

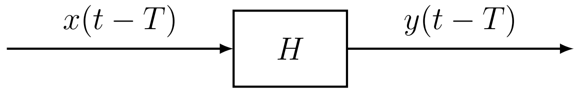


Figure 3.3: A time-invariant SISO system block diagram for system H.

We see that if we use an input that is the superposition and scaled inputs $ax_1(t) + bx_2(t)$, we can write

$$\begin{aligned} y(t) &= 3(ax_1(t) + bx_2(t)) + 5 \\ &= 3ax_1(t) + 3bx_2(t) + 5 \end{aligned}$$

which is not equal to $ay_1(t) + by_2(t)$ because of the constant term 5. Therefore, the system is not linear.

3.1.2 Time invariance

A system is time-invariant if its behavior and characteristics do not change over time. In other words, if we apply a time-shifted input to the system, the output will also be time-shifted by the same amount. Mathematically, if an input $x(t)$ produces an output $y(t)$, then for any time shift T, the input $x(t-T)$ will produce the output $y(t-T)$. A block diagram of this system behavior is seen in Fig. 3.3.

Similarly, we can visualize this behavior in Fig. 3.4. In this figure we observe a typical system input/output relationship. However, if we delay the input by a time T, the output is also delayed by the same amount, illustrating the time-invariance property.

Checking time invariance

To check system time invariance, apply a time-shifted input $x(t-T)$ to the system H and observe the output $\tilde{y}(t)$. Next, take the output for a typical $y(t) = H\{x(t)\}$ and shift it by the same amount to get $y(t-T)$. If $\tilde{y}(t) = y(t-T)$, then the system is time-invariant. If not, then the system is time-variant. This is best seen in example.

Example:

Consider the system

$$y(t) = x(t)\cos(t)$$

To check for time invariance, we apply a time-shifted input $x(t-T)$, which means putting a $-T$ term into the $x(t)$ function

$$\tilde{y}(t) = x(t-T)\cos(t)$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t-T$ in the output equation

$$y(t-T) = x(t-T)\cos(t-T)$$

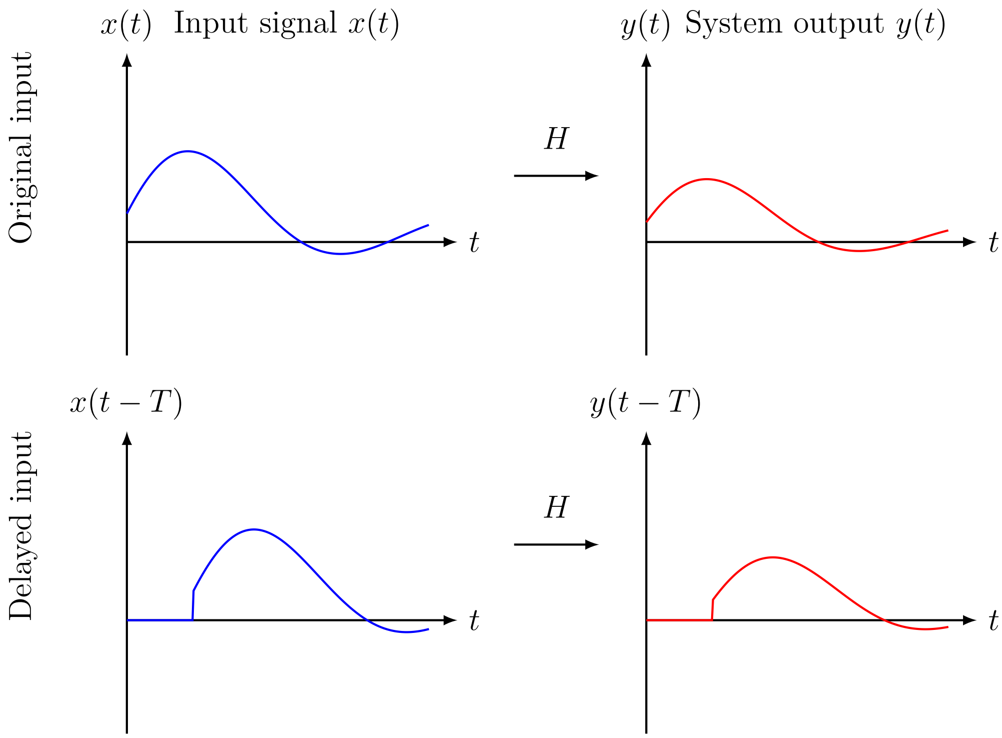


Figure 3.4: Time invariance illustrated with four plots in a grid.

Here, we clearly see that $\tilde{y}(t) \neq y(t-T)$ because of the $\cos(t)$ term, which introduces a time-dependent phase shift. Therefore, the system is time-variant. Typically, if some system H multiplies the input by a time-varying function, it will be time-variant.

Example:

Let's look at another example. Consider the integrator system

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

First, we apply a time-shifted input $x(t-T)$

$$\tilde{y}(t) = \int_{-\infty}^{t} x(\tau - T) d\tau$$

We can change the variable of integration to $\lambda = \tau - T$, which gives us $d\tau = d\lambda$. We also see that when $\tau == -\infty$, $\lambda = \infty$ and when $\tau = t$, $\lambda = t - T$. Thus, we can rewrite the integral as

$$\tilde{y}(t) = \int_{-\infty}^{t-T} x(\lambda) d\lambda$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t (not τ!) with $t - T$ in the output equation

$$y(t-T) = \int_{-\infty}^{t-T} x(\tau) d\tau$$

Here, we clearly see that $\tilde{y}(t) = y(t-T)$, which means the system is time-invariant.

Example:

Consider a compressor system defined by

$$y(t) = x(2t).$$

First we can shift the system input by T by adding a $-T$ to $x(t)$ to get

$$\tilde{y}(t) = x(2t - T).$$

Next, we find the output for the original input and shift it, which means we need to replace every instance of t with $t - T$ in the output equation

$$\begin{aligned} y(t-T) &= x(2(t-T)) \\ &= x(2t - 2T). \end{aligned}$$

We see that $\tilde{y}(t) \neq y(t-T)$, so the system is time-variant.

3.2 System response

A general SISO LTI continuous-time system can be described by a linear constant-coefficient differential equation of the form

$$\begin{aligned} \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \ldots \\ \ldots b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t). \end{aligned} \tag{3.1}$$

We can also write this in a more compact form where each derivative term $\frac{d}{dt}$ can simply be represented with an operator D such that $D^n y(t) = \frac{d^n y(t)}{dt^n}$. Thus, we can rewrite (3.1) as

$$\begin{aligned} D^n y(t) + a_{n-1}D^{n-1}y(t) + \cdots + a_1 Dy(t) + a_0 y(t) = \ldots \\ \ldots b_m D^m x(t) + b_{m-1}D^{m-1}x(t) + \cdots + b_1 Dx(t) + b_0 x(t). \end{aligned} \tag{3.2}$$

Further, we can simplify this to

$$\underbrace{\left(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right)}_{Q(D)} y(t) = \underbrace{\left(b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0\right)}_{P(D)} x(t) \tag{3.3}$$

where $Q(D)$ and $P(D)$ are polynomials in the differential operator D.

If we were to solve the ODE described by (3.1), we would find the system's response has two components, the zero-input response and the zero-state response. Thus the solution is

$$y_{\text{tot}}(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

3.3 Zero-input response

In the case of the zero-input response, we are interested in how the system responds to initial conditions without any external input. This means we set the input $x(t)$ to zero and solve the homogeneous equation associated with the system. The zero-input response is determined solely by the system's characteristics and its initial conditions. Without the loss of generality, we will work with 2^{nd} order systems. Consider the following system

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_2\frac{d^2x(t)}{dt^2} + b_1\frac{dx(t)}{dt} + b_0x(t).$$

To find the zero-input response we can set $x(t) = 0$ to get

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = 0.$$

Assume the generic solution to this homogeneous equation is of the form

$$y_0(t) = Ce^{\lambda t}.$$

We can differentiate this expression to find the first and second derivatives:

$$\frac{dy_0(t)}{dt} = C\lambda e^{\lambda t},$$
$$\frac{d^2y_0(t)}{dt^2} = C\lambda^2 e^{\lambda t}.$$

Substituting this into the homogeneous equation gives us

$$C\lambda^2 e^{\lambda t} + a_1C\lambda e^{\lambda t} + a_0Ce^{\lambda t} = 0.$$

Factoring out the common term $Ce^{\lambda t}$ gives us

$$Ce^{\lambda t}\left(\lambda^2 + a_1\lambda + a_0\right) = 0.$$

To make this true, we need to solve the characteristic equation (assuming $C \neq 0$ and $e^{\lambda t} \neq 0$ for all t)

$$Q(\lambda) = \lambda^2 + a_1\lambda + a_0 = 0.$$

which is called the *characteristic equation.* The solutions to this equation, λ_1 and λ_2, are called the *characteristic roots* or *eigenvalues* of the system. The nature of these roots (real or complex) will determine the form of the zero-input response. To summarize, the form of the zero-input response is determined by the characteristic roots:

- Real and distinct roots lead to two exponential terms.
- Real and repeated roots lead to an exponential term and a linear term.
- Complex conjugate roots lead to an exponential decay term and sinusoidal terms.

3.3.1 Real and distinct roots

If λ_1 and λ_2 are real and distinct ($\lambda_1 \neq \lambda_2$), then the homogeneous solution is

$$y_0(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}.$$

3.3.2 Real and repeated roots

If λ_1 is a real and repeated root ($\lambda_1 = \lambda_2$), then the homogeneous solution is

$$y_0(t) = (C_1 + C_2 t)e^{\lambda_1 t}.$$

3.3.3 Complex conjugate roots

If $\lambda = a + jb$ is a complex root of the characteristic equation, then its complex conjugate $\lambda^* = a - jb$ is also a root. The homogeneous solution for complex conjugate roots is also given by

$$\begin{aligned} y_0(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(a+jb)t} + C_2 e^{(a-jb)t}. \end{aligned}$$

The coefficients C_1 and C_2 may be complex, but they will be complex conjugates such that $C_1 = C_2^*$ and $C_1 = c + jd$ and $C_2 = c - jd$. We can rewrite $y_0(t)$ as

$$\begin{aligned} y_0(t) &= e^{at}\left[(c + jd)e^{jbt} + (c - jd)e^{-jbt}\right] \\ &= e^{at}\left[c(e^{jbt} + e^{-jbt}) + jd(e^{jbt} - e^{-jbt})\right] \\ &= e^{at}\left[c(2\cos(bt)) + jd(2j\sin(bt))\right] \\ &= e^{at}\left[2c\cos(bt) - 2d\sin(bt)\right]. \end{aligned}$$

To find the constants, we need to apply the initial conditions of the system, which are typically given as $y(0)$ and $y'(0)$. This usually turns into a 2×2 system of equations that can be solved with any linear algebra technique.

Application: circuit analysis

In circuit analysis, we often encounter second-order linear differential equations when analyzing RLC circuits. The zero-input response can be used to determine the natural response of the circuit. To find the coefficients of the zero-input response, we can use the initial conditions of the circuit, such as the initial voltage across a capacitor or the initial current through an inductor (or if it is given to you). Recall that we can represent voltages and currents in a circuit in the time-domain. Each of these quantities can be seen in Fig. 3.5.

Consider the circuit in Fig. 3.6 where the input to the system is some voltage $x(t) = 10e^{-3t}u(t)$ and the output is the current $y(t)$ flowing in the loop. We know that the initial voltage over the capacitor is $v_C(0^-) = 5\,\mathrm{V}$.

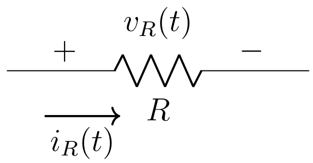
The first thing we want to do is derive an ODE for this expression, which is easiest by a simple voltage loop.

$$-x(t) + v_L(t) + v_R(t) + v_C(t) = 0.$$

From there substitute expressions relating the voltages to the current $y(t)$

$$-x(t) + L\frac{dy(t)}{dt} + Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

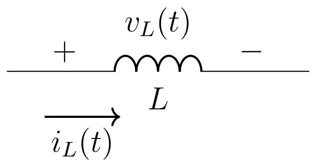
Resistor voltage and current



$$v_R(t) = R i_R(t)$$

$$i_R(t) = \frac{v_R(t)}{R}$$

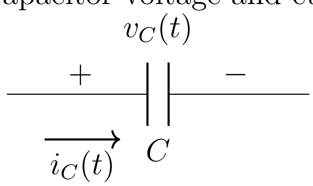
Inductor voltage and current



$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)\,d\tau$$

Capacitor voltage and current



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau$$

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Figure 3.5: Time-domain representation of voltages and currents in resistors, inductors, and capacitors.

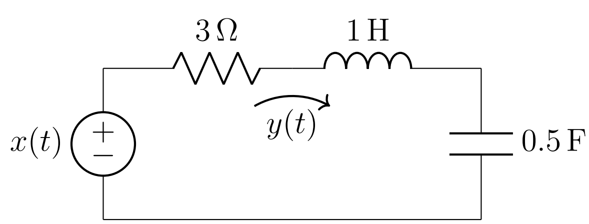


Figure 3.6: Zero-input response of an RLC circuit.

Since we know the component values we simplify to

$$-x(t) + \frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = 0.$$

We need to get rid of the integral term, so we differentiate both sides and simplify

$$\begin{aligned} -\frac{dx(t)}{dt} + \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= 0 \\ \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) &= \frac{dx(t)}{dt}. \end{aligned}$$

We see the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

which has roots $\lambda_1 = -1$ and $\lambda_2 = -2$ giving a general solution of

$$y_{\text{zir}}(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

Next we need the initial conditions of the system. We see from the input voltage $x(t)$ that there is no initial voltage and so the current would be zero at $t = 0^-$ such that $y(0^-) = 0$. Looking at the system then at $t = 0^-$, we have

$$-x(0^-) + v_L(0^-) + v_R(0^-) + v_C(0^-) = 0$$

and since we know that $v_C(0^-) = -5\,\text{V}$, $y(0^-) = 0$, and $x(0^-) = 10\,\text{V}$ we can substitute this into the equation to get

$$-10 + v_L(0^-) + R \cdot 0 + 5 = 0.$$

We see that $v_L(0^-) = -5\,\text{V}$ and therefore (using the relationship $v_L = L\frac{dy}{dt}$) we can find the initial condition for the inductor current $y'(0^-) = -5\,\text{V}$, thus giving us our second initial condition. We can apply this to the general solution to find the coefficients C_1 and C_2 (and realizing that $y'_{\text{zir}}(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$)

$$\begin{aligned} y_{\text{zir}}(0^-) &= C_1 + C_2 = 0, \\ y'_{\text{zir}}(0^-) &= -C_1 - 2C_2 = 5. \end{aligned}$$

and solving $C_1 = -5$ and $C_2 = 5$ giving us the final zero-input response

$$y_{\text{zir}}(t) = -5e^{-t} + 5e^{-2t}.$$

3.4 Zero-state response

The *zero-state response* is the part of the system's output that is solely due to the external input, assuming that all initial conditions are zero. In other words, we analyze how the system responds to an input signal when it starts from a state of rest (zero initial conditions).

Consider an LTI system seen in Fig. 3.7 with an input signal $x(t)$ and an output signal $y(t)$. The zero-state response can be determined using the system's impulse response $h(t)$, which is the output of the system when the input is an impulse function $\delta(t)$. The impulse response characterizes the

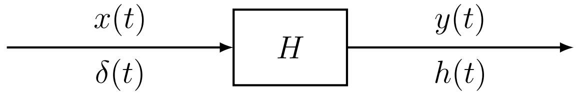


Figure 3.7: System response to an impulse function.

system's behavior and can be used to find the output for any arbitrary input signal through the convolution operation.

Consider some arbitrary signal $x(t)$, which can be written as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

which states that any signal can be represented as a weighted sum of impulse functions. Assume that some system is LTI and has an impulse function $h(t)$. If we pass $x(t)$ through this system, we see

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}. \end{aligned}$$

Applying superposition, we can pull out the integral

$$y(t) = \int_{-\infty}^{\infty} H\{x(\tau)\delta(t-\tau)\}d\tau.$$

Applying scaling, we can pull out the $x(\tau)$ term because in terms of the system H, there is no time dependence with $x(\tau)$. This would given

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau.$$

Because H is time-invariant, we notice a shift impulse function, will yield a shifted delta function such that

$$H\{\delta(t-\tau)\} = h(t-\tau).$$

We can rewrite the output of the system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

We see that the zero-state response can be expressed as a convolution between the input signal and the system's impulse response

$$y(t) = x(t) * h(t).$$

3.4.1 Convolution properties

The convolution integral has several properties that are going to be helpful in analyzing LTI systems.

Table 3.1: Properties of convolution for signals $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Property	**Mathematical Expression**
Commutative	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
Associative	$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$
Distributive	$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
Shift Property	$f_1(t-t_0) * f_2(t) = [f_1(t) * f_2(t)]_{t\to t-t_0}$
Delta Function	$f_t(t) * \delta(t) = f_1(t)$, and $f_1(t) * \delta(t-t_0) = f_1(t-t_0)$
Width	If $f_1(t)$ and $f_2(t)$ are time-limited to T_1 and T_2, then $f_1(t) * f_2(t)$ is time-limited to $T_1 + T_2$

3.4.2 Determining the impulse function $h(t)$

To determine the zero-state response, we will need to find the impulse response $h(t)$ of the system. This can be done myriad ways:

- By applying an impulse input $x(t) = \delta(t)$ and measuring the output $y(t) = h(t)$.
- By applying a known input $x(t)$ and measuring the output $y(t)$, then using deconvolution techniques to extract $h(t)$.
- By solving the system's differential equation with the input $x(t) = \delta(t)$ and zero initial conditions.

Here, we will focus on the last method. To find the impulse response in the time-domain, we will use the formula

$$h(t) = b_n\delta(t) + P(D)y_n(t)u(t)$$

where n is the order of the system. The b_n coefficient is the coefficient from the system's differential equation, as described in Eq. 3.3. The $P(D)$ operator is the polynomial operator from the left-hand side of Eq. 3.3. The $y_n(t)$ term is the zero-input response given a specific set of initial conditions

$$\begin{aligned} &y^{(n-1)}(0^-) = 1 \\ &y^{(n-2)}(0^-) = b_{n-1} = \ldots = y(0^-) = 0. \end{aligned}$$

Example:

Consider the system defined by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt}.$$

To find the impulse response $h(t)$ for this system, we can follow a fairly procedural set of steps.

1. **Identify the system order n:** The highest derivative of $y(t)$ is 2, so $n = 2$.

2. **Determine the b_n coefficient:** From the right-hand side of the differential equation, we see that $b_n = b_2 = 0$ because there is no $\frac{d^2x(t)}{dt^2}$ term.

3. **Determine the $P(D)$ operator:** From the left-hand side of the differential equation, we have
$$P(D) = 2D$$

4. **Find the zero-input response $y_n(t)$:** We need to go through the steps to solve the zero-input response. First, we notice that the characteristic equation is
$$Q(\lambda) = \lambda^2 + 2\lambda + 2.$$
Solving this characteristic equation gives us the roots
$$\lambda = -1 \pm j$$
which gives a generic solution
$$y_n(t) = e^{-t}\left(C_1\cos(t) + C_2\sin(t)\right).$$
Applying the initial conditions $y(0^-) = 0$ and $y'(0^-) = 1$ gives us
$$0 = C_1 \quad \text{and} \quad 1 = -C_1 + C_2.$$
Solving this system gives us $C_1 = 0$ and $C_2 = 1$, so the zero-input response is
$$y_n(t) = e^{-t}\sin(t)$$

5. **Compute $P(D)y_n(t)$:** We need to apply the $P(D)$ operator to $y_n(t)$
$$\begin{aligned} P(D)y_n(t) &= 2Dy_n(t) \\ &= 2\frac{d}{dt}\left(e^{-t}\sin(t)\right) \\ &= 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right) \end{aligned}$$

6. **Combine to find $h(t)$:** Finally, we can combine all the pieces to find the impulse response
$$\begin{aligned} h(t) &= b_n\delta(t) + P(D)y_n(t)u(t) \\ &= 0 + 2\left(-e^{-t}\sin(t) + e^{-t}\cos(t)\right)u(t) \\ &= 2e^{-t}\left(\cos(t) - \sin(t)\right)u(t) \end{aligned}$$

3.4.3 Computing the convolution

The convolution integral of two mathematical functions $x_1(t)$ and $x_2(t)$ can be written as
$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau.$$

There several approaches to computing this integral, but we are going to focus on an approach that uses a graphical method. This method is best illustrated with an example.

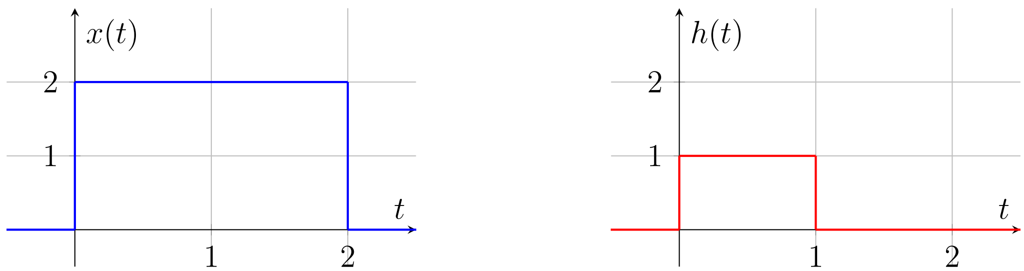


Figure 3.8: Signals $x(t)$ and $h(t)$ to be convolved.

Example:

Consider two functions that we want to convolve

$$\begin{aligned} x(t) &= 2(u(t) - u(t-2)) \\ h(t) &= u(t) - u(t-1). \end{aligned}$$

These two functions are shown in Fig. 3.8. To compute the convolution $y(t) = x(t) * h(t)$, we first write the integral in its basic form

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

Next, we need to graphically visualize the convolution process. In this equation, we see that we are integrating over some variable τ. We can rewrite $h(t-\tau)$ as $h(-(\tau-t))$ to see that we are flipping $h(\tau)$ around the vertical axis and then shifting it by t. This is shown in Fig. 3.9 for several values of t. In each subfigure, we see both $x(\tau)$ and $h(t-\tau)$, where the figures on the right show the resulting multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

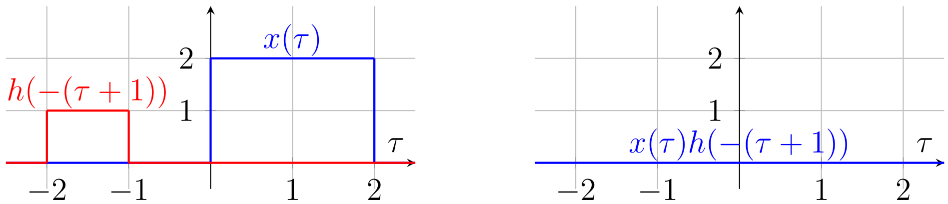
If plot these values for $y(t)$, we get the result shown in Fig. 3.10. In this figure the solid line is the exact result and the dots are the individual values computed in Fig. 3.9. As a quick remark, this result satisfies the width property of convolution because $x(t)$ is time-limited to 2 and $h(t)$ is time-limited to 1, so their convolution $y(t)$ is time-limited to 3. Whenever you perform a convolution, make sure the width property is satisfied as a sanity check (assuming both functions are time-limited).

Example:

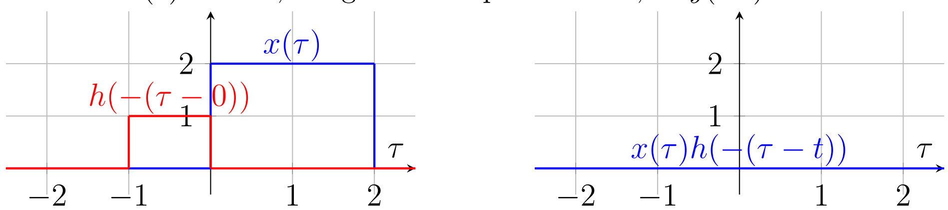
Let's look at a more complex example, visualized in Fig. 3.11. Consider the signals $x(t)$ and $h(t)$ defined in Fig. 3.11a. We will choose to flip and drag the "simpler" signal, so we will choose $x(t)$ to flip and drag since it is a simple box. We will need to approach this problem as a series of cases as we shift $x(-(\tau-t))$ from left to right.

Case $t < -1$: Fig. 3.11b shows the case where $t = -1$. We see that the two signals do not overlap. If we continue to shift to the left, the two signals will never overlap. For $t < -1$, the integral of the product will be zero. Thus, $y(t) = 0$ for $t < -1$.

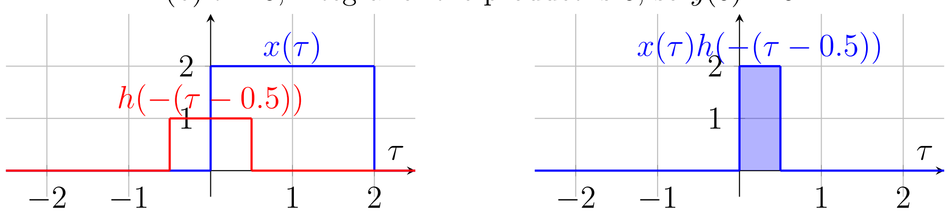
Case $-1 \le t < 1$: Fig. 3.11c shows the case where $t = 0.5$. We see that the two signals overlap, though only partially. This case will remain from $-1 < t < 1$. The integral of the product triangle. As we shift $x(-(\tau-t))$ to the right, the lower limit of the integral remains fixed



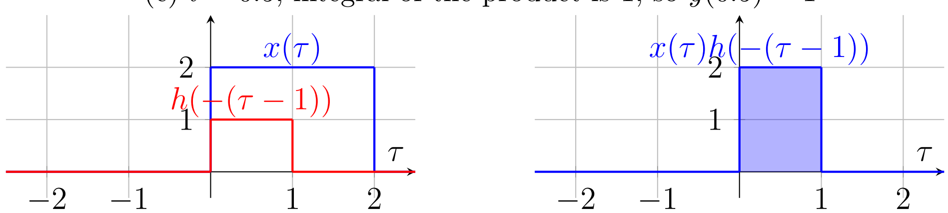
(a) $t=-1$, integral of the product is 0, so $y(-1)=0$



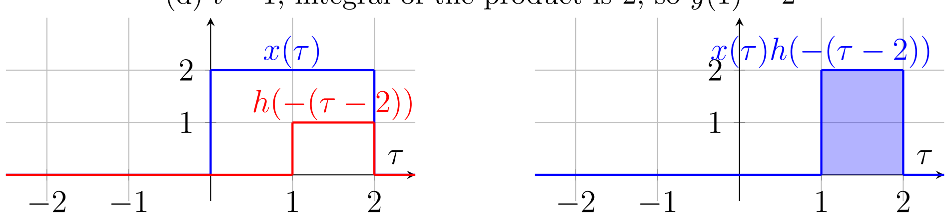
(b) $t=0$, integral of the product is 0, so $y(0)=0$



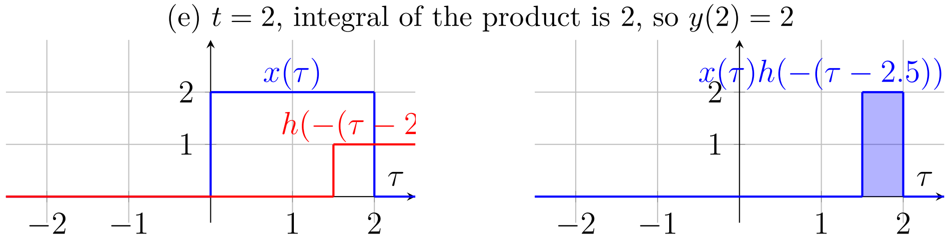
(c) $t=0.5$, integral of the product is 1, so $y(0.5)=1$



(d) $t=1$, integral of the product is 2, so $y(1)=2$



(e) $t=2$, integral of the product is 2, so $y(2)=2$



(f) $t=2.5$, integral of the product is 1, so $y(2.5)=1$

Figure 3.9: Graphical convolution of two box functions for different values of t. Each subfigure shows both $x(\tau)$ and $h(t-\tau)=h(-(\tau-t))$ at the specified t, where the figures on the right show the resulting the multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

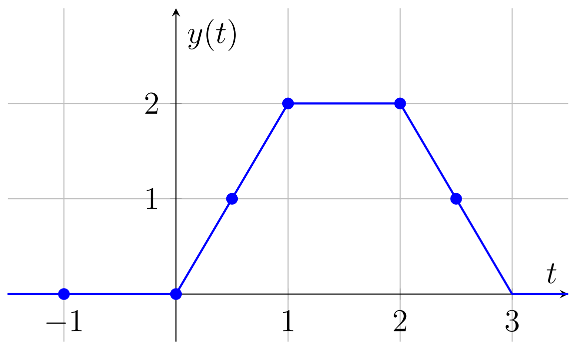


Figure 3.10: Result of the convolution $y(t) = x(t) * h(t)$. The solid line is the exact result and the dots are the individual values computed in Fig. 3.9.

at $\tau = 0$. However, the upper limit of the integral is constantly changing. Here, the upper limit of the integral is $\tau = t + 1$. The integral becomes

$$y(t) = \int_0^{t+1} \frac{1}{3}\tau \, d\tau = \frac{1}{6}(t+1)^2.$$

As a note, determining the limits of integration is the hardest part of convolution. One trick to help with this is to draw the shifted at $x(-(\tau - t))$ at some fixed t value (here, $t = 0.5$) and notice what value of τ that corresponds to. Typically, the moving limit will be some function $\tau = t + a$ where a is some constant. We can plug in our known $\tau = 1.5$ (see Fig. 3.11c) and $t = 0.5$ to find that $1.5 = 0.5 + a$ giving $a = 1$. The moving limit is $\tau = t + 1$.

Case $1 \leq t < 2$**:** Fig. 3.11d shows the case where $t = 1.5$. We see that the two signals fully overlap. This full overlap case remains from $1 \leq t < 2$. This will have *two* moving boundaries. We can find the lower limit similar to what we did in the previous case. In this instance, $t = 1.5$ and that corresponds to $\tau = 0.5$ on the lower bound such that $\tau = t + b$. We can apply our values such that $0.5 = 1.5 + b$ giving $b = -1$. The lower limit is $\tau = t - 1$. The upper limit remains $\tau = t + 1$. The integral becomes

$$y(t) = \int_{t-1}^{t+1} \frac{1}{3}\tau \, d\tau = \frac{2}{3}t.$$

Case $2 \leq t < 4$**:** Fig. 3.11e shows the case where $t = 3$. We see the two signals only partially overlap. This partial overlap case remains from $2 < t < 4$. The lower limit remains $\tau = t-1$. However, the upper limit is now fixed at $\tau = 3$. The integral becomes

$$y(t) = \int_{t-1}^{3} \frac{1}{3}\tau \, d\tau = \frac{1}{6}(4 - (t-1)^2) = \frac{1}{6}(-t^2 + 2t + 8).$$

Case $t \geq 4$**:** After $t = 4$, the integral will go to zero and remain zero as we continue to shift $x(-(\tau - t))$ to the right. Thus, $y(t) = 0$ for $t \geq 4$.

Putting all these cases together, we can define $y(t)$ piecewise

$$y(t) = \begin{cases} 0 & t < -1, \\ \frac{(t+1)^2}{6} & -1 \leq t < 1, \\ \frac{2}{3}t & 1 \leq t < 2, \\ \frac{1}{6}(-t^2 + 2t + 8) & 2 \leq t < 4, \\ 0 & t \geq 4. \end{cases}$$

We see this result plotted in Fig. 3.12. The solid line is the exact result and the dots are the transition points between each case. When we compute the convolution integral, we should do two sanity checks:

- Check that the width property is satisfied. Here, $x(t)$ is time-limited to 2 and $h(t)$ is time-limited to 3, so their convolution $y(t)$ is time-limited to 5. We see that this is satisfied.
- Check that the function is continuous. Here, we see that $y(t)$ is continuous at each transition point.

3.5 Total response

The *total response* of a system is the sum of the zero-input response and the zero-state response. In other words, the total response $y(t)$ can be expressed as

$$y(t) = y_{\text{zir}}(t) + y_{\text{zsr}}(t)$$

where $y_{\text{zir}}(t)$ is the zero-input response and $y_{\text{zsr}}(t)$ is the zero-state response. This total response accounts for both the effects of the system's initial conditions and the external input signal.

If you have the differential equation of the system, you can find the total response by solving the differential equation with the given initial conditions and input signal. This involves finding the homogeneous solution (zero-input response) and a particular solution (zero-state response). The zero-input response is found by solving the homogeneous equation with the initial conditions.

To find the zero-state response, you can first find the system's impulse response $h(t)$ and then convolve it with the input signal $x(t)$. Finally, you can sum the zero-input response and the zero-state response to obtain the total response of the system.

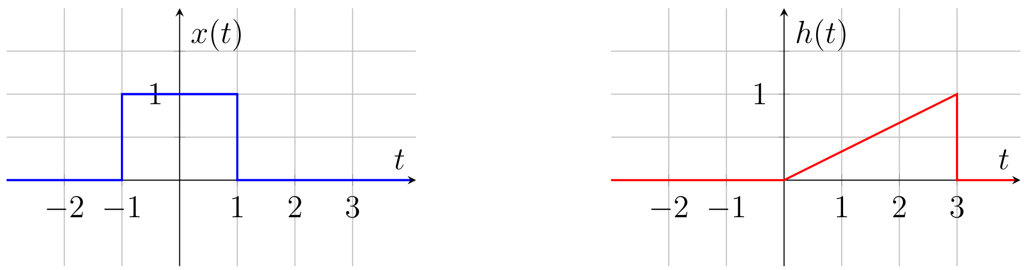
3.6 System stability

An LTI system is considered *BIBO stable* (bounded-input, bounded-output stable) if every bounded input produces a bounded output. In other words, if the input signal $x(t)$ is bounded such that there exists some finite constant M_x where $|x(t)| \leq M_x < \infty$ for all t, then the output signal $y(t)$ must also be bounded such that there exists some finite constant M_y where $|y(t)| \leq M_y < \infty$ for all t. In other words, if we put a finite signal into the system, we should get a finite signal out of the system. In general, we want our systems to be BIBO stable.

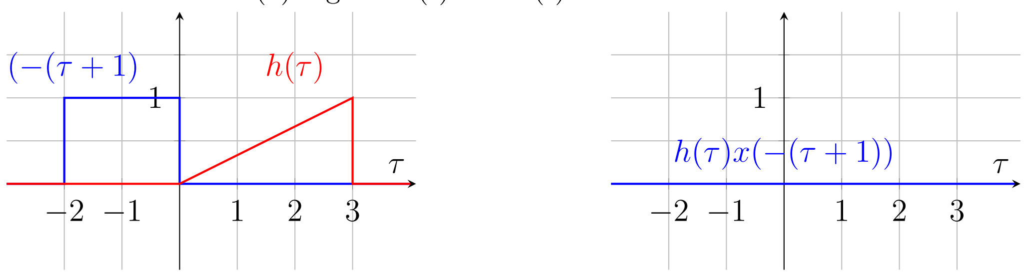
For an LTI system, the BIBO stability can be determined by examining the system's impulse response $h(t)$. Specifically, an LTI system is BIBO stable if and only if the impulse response is absolutely integrable, which means that the integral of the absolute value of the impulse response over all time is finite

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty.$$

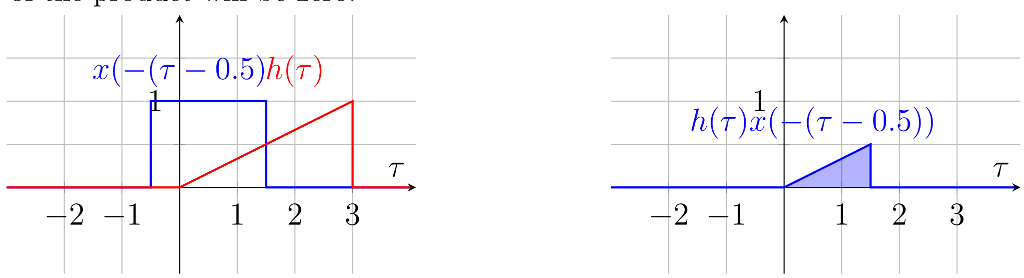
We can prove this by assuming that the input signal $x(t)$ is bounded such that $|x(t)| \leq M_x < \infty$ for all t. The output signal $y(t)$ can be expressed as the convolution of the input signal and the



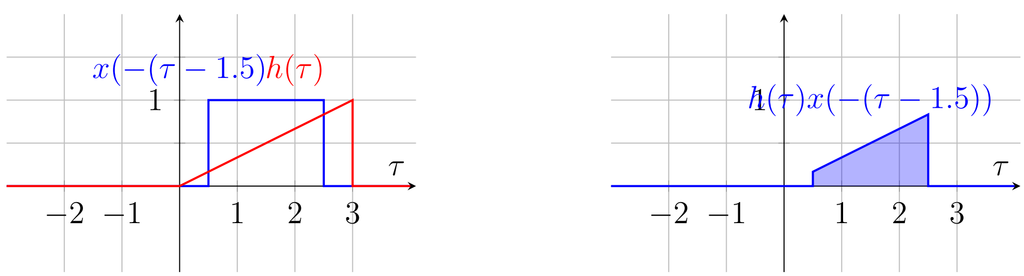
(a) Signals $x(t)$ and $h(t)$ to be convolved.



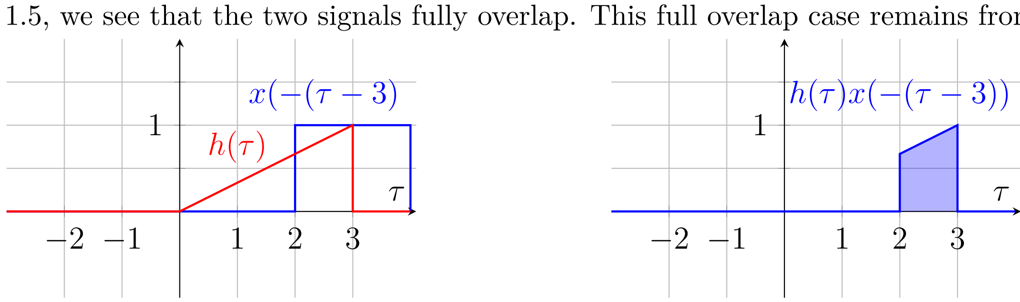
(b) Signal $h(\tau)$ remains in place. The signal $x(-(\tau - t))$ is shifted by $t = -1$. We see that the two signals do not overlap. If we continue to shift to the left, the two signals will never overlap. Thus, for $t < -1$, the integral of the product will be zero.



(c) At $t = 0.5$, we see that the two signals overlap, though only partially. This case will remain from $-1 < t < 1$.



(d) At $t = 1.5$, we see that the two signals fully overlap. This full overlap case remains from $1 < t < 2$.



(e) At $t = 3$, we see the two signals only partially overlap. This partial overlap case remains from $2 < t < 4$. After $t = 4$, the integral will go to zero and remain zero as we continue to shift $x(-(\tau - t))$ to the right.

Figure 3.11: Convolution process for $x(t)$ and $h(t)$ at different values of t. Each subfigure shows both $x(\tau)$ and $h(t-\tau) = h(-(\tau - t))$ at the specified t, where the figures on the right show the resulting the multiplication of $x(\tau)$ and $h(t-\tau)$. The shaded area represents the integral of the product, which gives the value of $y(t)$ at that specific time.

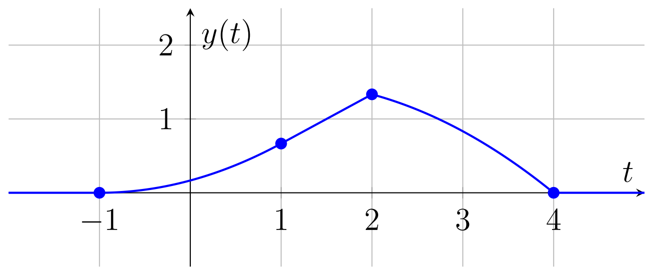


Figure 3.12: Result of the convolution $y(t) = x(t) * h(t)$ in Fig. 3.11a. The solid line is the exact result and the dots are the transition points between each case.

impulse response

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \end{aligned}$$

Taking the absolute value of both sides gives

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\, d\tau \quad \text{(by the triangle inequality)} \\ &\leq M_x \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau \quad (\text{since } |x(\tau)| \leq M_x) \\ &= M_x \int_{-\infty}^{\infty} |h(u)|\, du \quad (\text{by substituting } u = t-\tau) \\ &= M_x M_y \quad \left(\text{where } M_y = \int_{-\infty}^{\infty} |h(u)|\, du \text{ is a finite constant}\right) \end{aligned}$$

Thus, we see that $|y(t)| \leq M_x M_y < \infty$ for all t, which means that the output signal $y(t)$ is also bounded. Therefore, if the impulse response $h(t)$ is absolutely integrable, then the LTI system is BIBO stable.

What does this mean in practical terms? Consider the impulse response $h(t) = e^t u(t)$. We see this is clearly not absolutely integrable. If we were to put a bounded input (say, $x(t) = u(t)$) into this system, the convolution integral would explode, and the output would be unbounded. Therefore, this system would not be BIBO stable.

We can also determine BIBO stability by examining the system's characteristic equation. If we look at the characteristic roots λ_i of the system

- If any root has a positive real part, the system is unstable.
- If all roots have negative real parts, the system is BIBO stable.
- If any root has a zero real part (and all others have negative real parts), the system is marginally stable. This means that the system is not BIBO stable, but it is not unstable either. In this case, a bounded input *may* produce an unbounded output, depending on the

specific input signal. In this instance, the characteristic root would lie on the imaginary axis of the complex plane. If a characteristic root lies on the imaginary axis and has a multiplicity greater than one (i.e., it is repeated), the system is unstable.

This can be understood by examining the impulse response of the system. If any characteristic root has a positive real part, the impulse response will contain terms that grow exponentially over time, leading to an unbounded output for a bounded input. Conversely, if all characteristic roots have negative real parts, the impulse response will decay exponentially, ensuring that the output remains bounded for any bounded input. If a characteristic root lies on the imaginary axis, the impulse response will contain oscillatory terms that do not decay, which can lead to unbounded outputs for certain bounded inputs.

Example:

Consider the system defined by the differential equation

$$(D+1)(D^2+4D+8)y(t) = (D-3)x(t).$$

To determine if this system is BIBO stable, we can find the characteristic roots

$$\lambda_1 = -1, \quad \lambda_2 = -2+2j, \quad \lambda_3 = -2-2j.$$

We see that all the characteristic roots have negative real parts, so we can conclude that this system is BIBO stable.

3.7 Transfer functions

Consider an LTI system that has as an input $x(t)$ and an output $y(t)$. Suppose the input is some everlasting complex sinusoid

$$x(t) = e^{st}$$

where $s = \sigma + j\omega$ is a complex number. If we put this input into the system, we will get some output defined by the convolution of $x(t)$ and the system's impulse response $h(t)$

$$\begin{aligned}
y(t) &= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\
&= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}_{H(s)} \\
&= e^{st}H(s)
\end{aligned}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$, which is called the system's transfer function. The transfer function $H(s)$ is a complex function that describes how the system modifies the amplitude and phase of the input signal at different frequencies. Essentially, given some input $x(t)$ which is a sinusoid at some frequency ω, the output $y(t)$ will be a sinusoid at the same frequency ω but with a different amplitude and phase (assuming that $\sigma = 0$ and $s = j\omega$) such that

$$\begin{aligned} y(t) &= H(j\omega)e^{j\omega t} \\ &= |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}. \end{aligned}$$

We saw this type of behavior in steady-state AC circuit analysis in the introductory circuits course.

We can derive the transfer function $H(s)$ directly from the system's differential equation. Consider a general LTI system defined by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \ldots + b_1\frac{dx(t)}{dt} + b_0 x(t).$$

We can rewrite this equation using the differential operator D such that

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y(t) = (b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0)x(t).$$

We can apply the input $x(t) = e^{st}$ and the output $y(t) = H(s)e^{st}$ to this equation

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)H(s)e^{st} = (b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0)e^{st}.$$

We can apply the differential operator D to the exponential function e^{st}

$$D^k e^{st} = s^k e^{st}$$

for any integer $k \geq 0$. Applying this to the previous equation gives

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)H(s)e^{st} = (b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0)e^{st}.$$

We can divide both sides by e^{st} (which is never zero) and solve for $H(s)$

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}.$$

This is a rational function in s where the numerator is a polynomial of degree m and the denominator is a polynomial of degree n.

In shorthand, from a differential equation of the form of derivative operators D, we can find the transfer function $H(s)$ by replacing each D with s and taking the ratio of the input polynomial $P(s)$ to the output polynomial $Q(s)$.

Chapter 4

Fourier Analysis

4.1 General Fourier series

4.1.1 Signal inner product

We define the inner product of two signals $x_1(t)$ and $x_2(t)$ as

$$\langle x_1, x_2 \rangle = \int_{-\infty}^{\infty} x_1(t) x_2^*(t)\, dt,$$

where $x_2^*(t)$ is the complex conjugate of $x_2(t)$. If the signals are real-valued, then the complex conjugate is not necessary.

This inner product is similar to the dot product of vectors in Euclidean space. Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Their dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

which is analogous to multiplying two signals pointwise and integrating (summing) over all time.

Two signals $x_1(t)$ and $x_2(t)$ are *orthogonal* if their inner product is zero

$$\langle x_1, x_2 \rangle = \int_{-\infty}^{\infty} x_1(t) x_2^*(t)\, dt = 0.$$

4.1.2 General Fourier series representation

We can represent a signal $x(t)$ over a finite interval $[t_1, t_2]$ as a linear combination of orthogonal basis functions $\{x_n(t)\}$

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t) \tag{4.1}$$

where the coefficients c_n are given by the inner product

$$c_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n^*(t)\, dt$$

where E_n is the energy of the basis function. For this to work correctly, the basis functions must be orthogonal over the interval $[t_1, t_2]$ such that

$$\int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt = \begin{cases} E_n, & n = m \\ 0, & n \neq m \end{cases}.$$

Consider Eq. 4.1, and we can multiply both sides by $x_m^*(t)$ and integrate over $[t_1, t_2]$:

$$\begin{aligned} \int_{t_1}^{t_2} f(t)x_m^*(t)\,dt &= \int_{t_1}^{t_2} \left(\sum_{n=1}^{\infty} c_n x_n(t) \right) x_m^*(t)\,dt \\ &= \sum_{n=1}^{\infty} c_n \int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt \end{aligned}$$

We can use the orthogonality property of the basis functions to simplify the right-hand side. The integral $\int_{t_1}^{t_2} x_n(t)x_m^*(t)\,dt$ is zero for $n \neq m$ and equals E_n for $n = m$. Therefore, all terms in the sum vanish except for the term where $n = m$

$$\int_{t_1}^{t_2} f(t)x_m^*(t)\,dt = c_m E_m$$

which we can rearrange to solve for the coefficient c_m

$$c_m = \frac{1}{E_m} \int_{t_1}^{t_2} f(t)x_m^*(t)\,dt.$$

In earlier courses you might have seen Fourier series represented using sine and cosine functions as basis functions. The sine and cosine functions are orthogonal over the period of the sinusoid, so they can be used as basis functions in Eq. 4.1. However, we can use any set of orthogonal functions as basis functions, not just sines and cosines. For example, we could use the rectangular function $\text{rect}(t/\tau)$ or the triangular function $\Delta(t/\tau)$ as basis functions.

4.2 Fourier series using trigometric basis functions

4.2.1 Sines and cosines as basis functions

We can represent a periodic signal $f(t)$ with a period T_0 as a linear combination of sines and cosines

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \tag{4.2}$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients a_0, a_n, and b_n are given by

$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_{T_0} f(t)\,dt \\ a_n &= \frac{2}{T_0} \int_{T_0} f(t)\cos(n\omega_0 t)\;dt \\ b_n &= \frac{2}{T_0} \int_{T_0} f(t)\sin(n\omega_0 t)\;dt \end{aligned}$$

where the integrals are taken over one period of the signal. When we compute these coefficients, we are projecting the signal $f(t)$ onto the orthogonal basis functions $\{1, \cos(n\omega_0 t), \sin(n\omega_0 t)\}$. The basis functions are orthogonal over the interval $[t_0, t_0 + T_0]$ for any t_0. When we compute the coefficients, you choose any period T_0 of the signal to integrate over. Choose a period that makes the integrals easiest to compute.

Using symmetry to simplify coefficient calculations

We can simplify computing a_0, a_n, b_n by leveraging symmetry properties of the signal $f(t)$.

Even symmetry: A signal $f(t)$ is even if $f(t) = f(-t)$. The cosine function is even, and the sine function is odd. If $f(t)$ is even, then all b_n coefficients are zero because the product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned} a_0 &= \frac{2}{T_0} \int_0^{T_0/2} f(t)\, dt \\ a_n &= \frac{4}{T_0} \int_0^{T_0/2} f(t) \cos\left(n\omega_0 t\right)\, dt \\ b_n &= 0 \end{aligned}$$

Odd symmetry: A signal $f(t)$ is odd if $f(t) = -f(-t)$. If $f(t)$ is odd, then all a_n coefficients are zero because the product of two odd functions is even, and the integral of an odd function over a symmetric interval is zero. Additionally, a_0 is zero because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{4}{T_0} \int_0^{T_0/2} f(t) \sin\left(n\omega_0 t\right)\, dt \end{aligned}$$

Half-wave symmetry: A signal $f(t)$ has half-wave symmetry if $f(t) = -f(t - T_0/2)$. A function with half-wave symmetry is periodic with period T_0, and it is also anti-symmetric about the midpoint of the period. A signal can have half-wave symmetry without being purely odd or even, but if it has half-wave symmetry *and* odd or even symmetry, it will simplify the computation of the Fourier coefficients. An illustration of this is shown in Figure 4.1. This means that the positive and negative halves of the waveform are mirror images of each other, but inverted. The coefficients for a signal with half-wave symmetry are

$$\begin{aligned} a_0 &= 0 \\ a_n &= \begin{cases} \frac{4}{T_0} \int_{T_0/2} f(t) \cos\left(n\omega_0 t\right)\, dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \\ b_n &= \begin{cases} \frac{4}{T_0} \int_{T_0/2} f(t) \sin\left(n\omega_0 t\right)\, dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

We see that you can have a $f(t)$ that has half-wave symmetry as well as even or odd symmetry. If $f(t)$ has both half-wave symmetry and even symmetry, then only the odd a_n coefficients

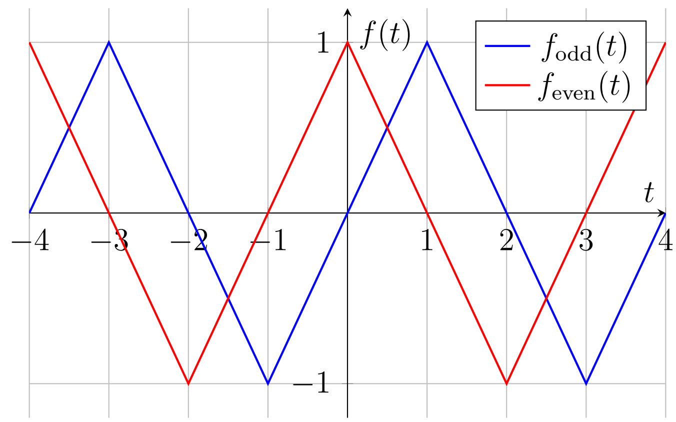


Figure 4.1: A signal with half-wave and odd symmetry (blue) and a signal with half-wave and even symmetry (red).

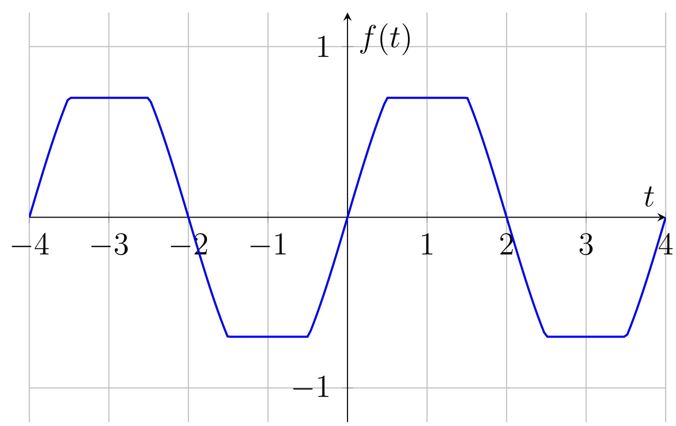


Figure 4.2: A signal with quarter-wave symmetry.

are non-zero. If $f(t)$ has both half-wave symmetry and odd symmetry, then only the odd b_n coefficients are non-zero.

Quarter-wave symmetry: A signal $f(t)$ has quarter-wave symmetry if $f(t) = f(T_0/4 - t)$. A function with quarter-wave symmetry is periodic with period T_0, and it is also symmetric about the quarter points of the period. This means that the first and last quarters of the waveform are mirror images of each other, and the second and third quarters of the waveform are mirror images of each other. Alternatively, you can think of a signal $f(t)$ with quarter-wave symmetry as having half-wave symmetry and even symmetry about the quarter points. An illustration of this is shown in Figure 4.2. Coincidentally, the signals shown in Fig. 4.1 also have quarter-wave symmetry.

The coefficients for a signal with quarter-wave symmetry *and* even symmetry are

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{8}{T_0}\int_0^{T_0/4} f(t)\cos\left(n\omega_0 t\right)\,dt, & \text{for } n \text{ odd} \end{cases} \\
b_n &= 0
\end{aligned}
$$

The coefficients for a signal with quarter-wave symmetry *and* odd symmetry are

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= 0 \\
b_n &= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{8}{T_0}\int_0^{T_0/4} f(t)\sin\left(n\omega_0 t\right)\, dt, & \text{for } n \text{ odd} \end{cases}
\end{aligned}
$$

4.2.2 Fourier series with shifted cosines

We can also represent a periodic signal $f(t)$ with a period T_0 as a linear combination of cosines with phase shifts

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(n\omega_0 t + \phi_n\right).$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients c_0, c_n, and ϕ_n are given by

$$
\begin{aligned}
c_n &= \sqrt{a_n^2 + b_n^2} \\
\phi_n &= \tan^{-1}\left(-\frac{b_n}{a_n}\right)
\end{aligned}
$$

where a_n and b_n are the coefficients from the trigonometric Fourier series representation in Eq. 4.2. The coefficient c_0 is the same as a_0.

The advantage of this representation is that it uses only cosine functions. If we are interested in determining the frequency composition of signal, we could plot the c_n as a stem plot and easily identify the dominant frequencies. We could also plot the phase shifts ϕ_n to see how the different frequency components are shifted in time. The disadvantage is that you need to compute both a_n and b_n to find c_n and ϕ_n.

4.2.3 Fourier series using complex exponential basis functions

We can represent a periodic signal $f(t)$ with a period T_0 as a linear combination of complex exponentials

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{4.3}$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency (in radians per second), and the coefficients D_n are given by

$$D_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jn\omega_0 t}\, dt.$$

Typically, the D_n coefficients are complex-valued. The magnitude $|D_n|$ indicates the amplitude of the frequency component at $n\omega_0$, and the angle $\angle D_n$ indicates the phase shift of that frequency component. Similar to the shifted cosine representation in Sec. 4.2.2, we can plot the magnitudes $|D_n|$ as a stem plot to identify the dominant frequencies in the signal, and we can plot the angles $\angle D_n$ to see how the different frequency components are shifted in time.

There are multiple ways to derive how the Fourier series works. One approach is realizing that the complex exponentials used as basis functions are all mutually orthogonal over the interval $[0, T_0]$. This means that

$$\int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt = \begin{cases} T_0, & n = m \\ 0, & n \neq m \end{cases}.$$

We can use this orthogonality property to derive the coefficients D_n. Consider Eq. 4.3, and we can multiply both sides by $e^{-jm\omega_0 t}$ and integrate over $[0, T_0]$:

$$\begin{aligned}\int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt &= \int_0^{T_0} \left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \right) e^{-jm\omega_0 t}\, dt \\ &= \sum_{n=-\infty}^{\infty} D_n \int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt\end{aligned}$$

We can use the orthogonality property of the basis functions to simplify the right-hand side. The integral $\int_0^{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt$ is zero for $n \neq m$ and equals T_0 for $n = m$. Therefore, all terms in the sum vanish except for the term where $n = m$

$$\int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt = D_m T_0$$

which we can rearrange to solve for the coefficient D_m

$$D_m = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jm\omega_0 t}\, dt.$$

Alternatively, we can derive the complex exponential Fourier series from the trigonometric Fourier series using Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Using Euler's formula, we can express the cosine and sine functions in terms of complex exponentials

$$\begin{aligned}\cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j}\end{aligned}$$

We can substitute these expressions into the trigonometric Fourier series in Eq. 4.2 to convert it to the complex exponential form. Starting with Eq. 4.2, we substitute the expressions for cosine and sine

$$\begin{aligned}f(t) &= a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right] \\ &= a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{2} + \frac{b_n}{2j} \right) e^{jn\omega_0 t} + \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) e^{-jn\omega_0 t} \right]\end{aligned}$$

We can rewrite the constant term a_0 as $\frac{a_0}{2}e^{j0\omega_0 t} + \frac{a_0}{2}e^{-j0\omega_0 t}$ to match the form of the other terms

$$\begin{aligned} f(t) &= \frac{a_0}{2}e^{j0\omega_0 t} + \frac{a_0}{2}e^{-j0\omega_0 t} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \left(\frac{a_n}{2} - \frac{b_n}{2j}\right)e^{-jn\omega_0 t}\right] \\ &= \sum_{n=0}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \sum_{n=0}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2j}\right)e^{-jn\omega_0 t} \end{aligned}$$

We can change the index of the second sum to run from $-\infty$ to -1 by substituting $m = -n$

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \sum_{m=-\infty}^{-1}\left(\frac{a_{-m}}{2} - \frac{b_{-m}}{2j}\right)e^{jm\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{aligned}$$

where

$$D_n = \begin{cases} \frac{a_n}{2} + \frac{b_n}{2j}, & n \geq 0 \\ \frac{a_{-n}}{2} - \frac{b_{-n}}{2j}, & n < 0 \end{cases}.$$

We can simplify this further by noting that a_n is an even function and b_n is an odd function, so $a_{-n} = a_n$ and $b_{-n} = -b_n$. Therefore, we can express D_n as

$$D_n = \frac{a_n}{2} + \frac{b_n}{2j}$$

for all integer n (positive, negative, and zero).

4.3 Periodic signals and LTI systems

If we have some periodic signal $f(t)$ with a period T_0 and a fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$, we can represent it using the complex exponential Fourier series in Eq. 4.3. Now suppose we are working with an LTI system H that has an impulse response $h(t)$. If we put a single complex exponential $e^{jn\omega_0 t}$ into the system, we will get some output $y(t)$ that is defined by the input sinusoid multiplied by a complex constant $H(jn\omega_0)$[1] defined by the transfer function (see Sec. 3.7)

$$y(t) = H(jn\omega_0)e^{jn\omega_0 t}.$$

Because this system is LTI, superpostion and scaling apply, therefore if we put the entire periodic signal $f(t)$—which we can define as a sum of scaled complex exponentials—into the system, we will get an output that is the sum of the outputs for each individual complex exponential that are scaled. In other words, if our input is

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

[1] The notation here can be ambiguous. Some people will include the j to denote that $s \to j\omega$, while others might just drop the j term and write $H(n\omega_0)$. Both conventions are appropriate.

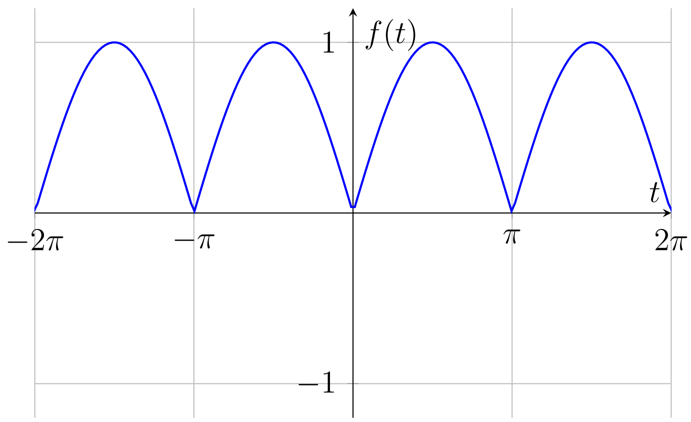


Figure 4.3: A rectified sine wave with period $T_0 = \pi$.

then the output will be

$$\begin{aligned} y(t) &= H\left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}\right) \\ &= \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0) e^{jn\omega_0 t}. \end{aligned}$$

This means that the output $y(t)$ is also periodic with the same period T_0 as the input $f(t)$. The Fourier series coefficients of the output $y(t)$ are simply the Fourier series coefficients of the input $f(t)$ multiplied by the transfer function evaluated at the corresponding frequencies $n\omega_0$.

Example:

Suppose we have as an input a rectified sine wave defined by $f(t) = |\sin(t)|$ as seen in Fig. 4.3. This function is periodic with a period of $T_0 = \pi$. We can compute the Fourier series using complex exponentials which is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{j2nt}.$$

Now suppose we have a LTI system that is defined by the differential equation

$$3\frac{dy}{dt} + y(t) = f(t).$$

Using the approach in Sec. 3.7, we can find the transfer function of this system is

$$H(s) = \frac{1}{3s+1}$$

and evaluating it at $s = jn\omega_0 = jn2$ (because $\omega_0 = 2$ in this instance) gives us

$$H(j2n) = \frac{1}{1+j6n}.$$

Therefore, the output of the system will be

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} D_n H(jn2) e^{j2nt} \\ &= \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)(1+j6n)} e^{j2nt}. \end{aligned}$$

4.4 Fourier transform

Thus far we have only considered periodic signals. However, many signals we encounter in practice are not periodic. We can extend the Fourier series to non-periodic signals by considering the limit as the period T_0 approaches infinity. Let's first consider some periodic signal $f_{T_0}(t)$ with period T_0. We can represent this signal using the complex exponential Fourier series in Eq. 4.3

$$f_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T_0}$, and the coefficients D_n are given by

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_0}(t) e^{-jn\omega_0 t}\, dt.$$

Let's define a function of frequency ω as

$$F(\omega) = \int_{-T_0/2}^{T_0/2} f_{T_0}(t) e^{-j\omega t}\, dt. \tag{4.4}$$

We see this can be related to the Fourier series coefficients D_n by noting that $\omega = n\omega_0$, so $n = \frac{\omega}{\omega_0} = \frac{\omega T_0}{2\pi}$. Therefore, we can express D_n in terms of $F(\omega)$ as

$$D_n = \frac{1}{T_0} F(n\omega_0).$$

We can substitute this expression for D_n back into the Fourier series representation of $f_{T_0}(t)$

$$f_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} F(n\omega_0) e^{jn\omega_0 t}$$

Now we can make $f_{T_0}(t)$ non-periodic by letting the period T_0 approach infinity. As $T_0 \to \infty$, the fundamental frequency $\omega_0 = \frac{2\pi}{T_0} \to 0$. Because ω_0 becomes very small, we can change the notation from $\omega_0 \to \Delta\omega$ to indicate that it is a small increment in frequency. We can take a limit as $T_0 \to \infty$ and $\Delta\omega \to 0$

$$\begin{aligned} f(t) &= \lim_{T_0\to\infty} f_{T_0}(t) \\ &= \lim_{\substack{T_0\to\infty \\ \Delta\omega\to 0}} \sum_{n=-\infty}^{\infty} \frac{1}{T_0} F(n\Delta\omega) e^{jn\Delta\omega t} \\ &= \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} F(n\Delta\omega) e^{jn\Delta\omega t} \qquad = \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega. \end{aligned}$$

We can recognize the right-hand side as a Riemann sum, which we can convert to an integral

$$f(t) = \frac{1}{2\pi} \int_{-infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \tag{4.5}$$

where $F(\omega)$ is defined in Eq. 4.4. We can rewrite Eq. 4.4 as $T_0 \to \infty$ to get the final form of the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt. \tag{4.6}$$

The function $F(\omega)$ in Eq. 4.6 is called the Fourier transform of $f(t)$, and $f(t)$ in Eq. 4.5 is called the inverse Fourier transform of $F(\omega)$. The Fourier transform decomposes a signal into its frequency components, and the inverse Fourier transform reconstructs the signal from its frequency components.

4.4.1 Nature of the Fourier transform

The Fourier transform $F(\omega)$ is generally complex-valued. The magnitude $|F(\omega)|$ indicates the amplitude of the frequency component at frequency ω, and the angle $\angle F(\omega)$ indicates the phase shift of that frequency component. We can plot the magnitude $|F(\omega)|$ as a function of frequency to see which frequencies are present in the signal $f(t)$. This plot is called the *magnitude spectrum* of the signal. We can also plot the angle $\angle F(\omega)$ as a function of frequency to see how the different frequency components are shifted in time. This plot is called the *phase spectrum* of the signal.

Symmetry plays a large role in the Fourier transform.

$f(t)$ **real:** If $f(t)$ is real-valued, then $F(-\omega) = F^*(\omega)$, which means that the magnitude spectrum is even $|F(\omega)| = |F(-\omega)|$ and the phase spectrum is odd $\angle F(\omega) = -\angle F(-\omega)$. This type of symmetry is called *conjugate symmetry* or *Hermitian symmetry*.

$f(t)$ **imaginary:** If $f(t)$ is imaginary-valued, then $F(-\omega) = -F^*(\omega)$, which means that the magnitude spectrum is even and the phase spectrum is odd, but shifted by π radians.

$f(t)$ **real and even:** If $f(t)$ is real-valued and even, then $F(\omega)$ is real-valued and even. This means that the magnitude spectrum is even and the phase spectrum is zero.

$f(t)$ **real and odd:** If $f(t)$ is real-valued and odd, then $F(\omega)$ is imaginary-valued and odd. This means that the magnitude spectrum is even and the phase spectrum is $\pm\frac{\pi}{2}$ radians.

$f(t)$ **imaginary and even:** If $f(t)$ is imaginary-valued and even, then $F(\omega)$ is imaginary-valued and even. This means that the magnitude spectrum is even and the phase spectrum is $\pm\frac{\pi}{2}$ radians.

$f(t)$ **imaginary and odd:** If $f(t)$ is imaginary-valued and odd, then $F(\omega)$ is real-valued and odd. This means that the magnitude spectrum is even and the phase spectrum is zero.

Generally we will focus on real-valued signals, so the Fourier transform will exhibit conjugate symmetry.

4.4.2 LTI systems and the Fourier transform

From Sec. 4.3, we saw that if we have a periodic signal $f(t)$ with a Fourier series representation and we put it into a LTI system, the output $y(t)$ will also be periodic with the same period, and the Fourier series coefficients of the output are simply the Fourier series coefficients of the input multiplied by the transfer function evaluated at the corresponding frequencies

$$y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) \cdot D_n e^{jn\omega_0 t}.$$

If we let the period $T_0 \to \infty$, then the fundamental frequency $\omega_0 \to 0$, and we can change the notation from $\omega_0 \to \Delta\omega$ to indicate that it is a small increment in frequency. We can take a limit as $T_0 \to \infty$ and $\Delta\omega \to 0$

$$\begin{aligned} y(t) &= \lim_{T_0 \to \infty} y_{T_0}(t) \\ &= \lim_{\substack{T_0 \to \infty \\ \Delta\omega \to 0}} \sum_{n=-\infty}^{\infty} H(n\Delta\omega) \cdot D_n e^{jn\Delta\omega t} \\ &= \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} \frac{H(n\Delta\omega) \cdot F(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t} \\ &= \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} H(n\Delta\omega) \cdot F(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega. \end{aligned}$$

Converting the right-hand side to an integral, we get

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \cdot F(\omega) e^{j\omega t} \, d\omega.$$

We can recognize this as the inverse Fourier transform of $H(\omega) \cdot F(\omega)$. Therefore, if we put a non-periodic signal $f(t)$ with a Fourier transform $F(\omega)$ into a LTI system with a transfer function $H(\omega)$, the output $y(t)$ will have a Fourier transform

$$Y(\omega) = H(\omega) \cdot F(\omega).$$

This means that the output $y(t)$ is generally not periodic, and its frequency components are the frequency components of the input $f(t)$ scaled by the transfer function $H(\omega)$.

This is an important feature of LTI systems: they scale the frequency components of the input signal by the transfer function. This means that if a frequency component is not present in the input signal (i.e., $F(\omega) = 0$), it will not be present in the output signal regardless of the value of the transfer function at that frequency. Conversely, if a frequency component is present in the input signal, its amplitude and phase in the output signal will be determined by the transfer function at that frequency. LTI systems can scale existing frequency components, but they cannot create new frequency components that were not present in the input signal.

4.4.3 Fourier transforms of common signals

A somewhat comprehensive table of common Fourier transforms can be found in Table **??**. Here we will compute the Fourier transforms of a couple common signals to illustrate how the Fourier transform works.

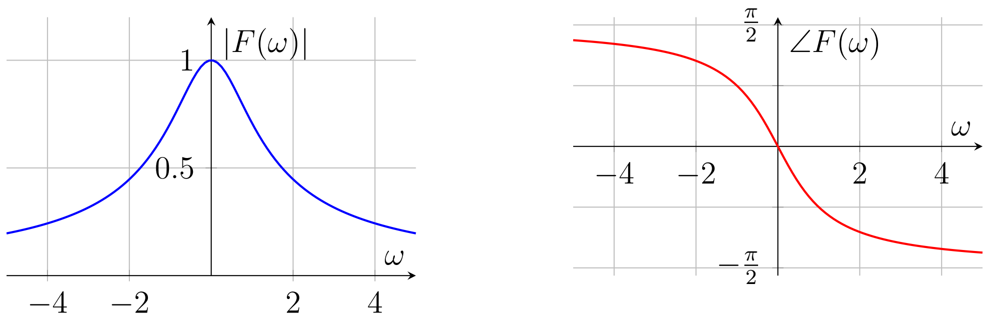


Figure 4.4: Magnitude and phase spectra of $F(\omega) = \frac{1}{a+j\omega}$ for $a = 1$.

Fourier transform of $f(t) = e^{-at}u(t)$

Let's compute the Fourier transform of the signal $f(t) = e^{-at}u(t)$ where $a > 0$ and $u(t)$ is the unit step function. We can start with the definition of the Fourier transform in Eq. 4.6

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt \\
&= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} \\
&= 0 - \frac{1}{-(a+j\omega)} \quad \left(\text{since } e^{-(a+j\omega)t} \to 0 \text{ as } t \to \infty\right) \\
&= \frac{1}{a+j\omega}.
\end{aligned}$$

We notice $f(t)$ is real-valued, so $F(\omega)$ exhibits conjugate symmetry. The magnitude and phase of $F(\omega)$ are

$$\begin{aligned}
|F(\omega)| &= \frac{1}{\sqrt{a^2+\omega^2}} \\
\angle F(\omega) &= -\tan^{-1}\left(\frac{\omega}{a}\right).
\end{aligned}$$

The magnitude and phase spectra are shown in Fig. 4.4. We notice a few things. First, $f(t)$ is real-valued, so the magnitude spectrum is even and the phase spectrum is odd. Second, the magnitude spectrum has a low-pass characteristic, meaning that low frequencies have higher amplitudes than high frequencies. This makes sense because $f(t)$ is a decaying exponential, which is a smooth signal that does not change rapidly. Therefore, it does not contain many high-frequency components. Finally, the phase spectrum indicates that low frequencies have little phase shift, while high frequencies have a phase shift approaching $-\frac{\pi}{2}$ radians (or $-90°$).

Fourier transform of $f(t) = \mathbf{rect}\left(\frac{t}{\tau}\right)$

Let's compute the Fourier transform of the signal $f(t) = \text{rect}\left(\frac{t}{\tau}\right)$ where $\tau > 0$ and

$$\text{rect}(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}.$$

We can start with the definition of the Fourier transform in Eq. 4.6

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\,dt \\ &= \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt \\ &= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2} \\ &= \frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega} \\ &= \frac{2\sin(\omega\tau/2)}{\omega} \quad \text{(by using Euler's formula)} \\ &= \frac{2\sin(\omega\tau/2)}{\omega} \\ &= \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

where $\text{sinc}(x) = \frac{\sin(x)}{x}$. We notice $f(t)$ is real-valued and even, so $F(\omega)$ is real-valued and even. We see that the $f(t)$ is a rectangular pulse in the time domain and is time-limited. However, it's frequency representation is a sinc function which is not frequency-limited. This is a common theme in signal processing: time-limited signals are not frequency-limited, and frequency-limited signals are not time-limited.

Fourier transform of $f(t) = \delta(t)$

Let's compute the Fourier transform of the signal $f(t) = \delta(t)$ where $\delta(t)$ is the Dirac delta function. We can start with the definition of the Fourier transform in Eq. 4.6

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt \\ &= e^{-j\omega \cdot 0} \quad \text{(sifting property)} \\ &= 1. \end{aligned}$$

The Fourier transform of the delta function is a constant function equal to 1 for all frequencies. This means that the delta function contains all frequencies with equal amplitude and zero phase shift. This makes sense because the delta function is an impulse in the time domain, which is a very short and sharp signal that contains all frequencies.

Inverse Fourier transform of $F(\omega) = \delta(\omega - \omega_0)$

Let's compute the inverse Fourier transform of the signal $F(\omega) = \delta(\omega - \omega_0)$ where $\delta(\omega)$ is the Dirac delta function and ω_0 is some constant frequency. We can start with the definition of the inverse Fourier transform in Eq. 4.5

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}\,d\omega \\
&= \frac{1}{2\pi}e^{j\omega_0 t} \quad \text{(sifting property)} \\
&= \frac{1}{2\pi}e^{j\omega_0 t}.
\end{aligned}$$

The inverse Fourier transform of a delta function in the frequency domain is a complex exponential in the time domain. This means that a delta function in the frequency domain represents a complex-valued sinusoid at a single frequency component in the time domain.

Similarly, if $F(\omega) = \delta(\omega + \omega_0)$, then the inverse Fourier transform is

$$f(t) = \frac{1}{2\pi}e^{-j\omega_0 t}.$$

Fourier transform of $f(t) = \cos(\omega_0 t)$

To compute the Fourier transform of the signal $f(t) = \cos(\omega_0 t)$ where ω_0 is some constant frequency, we can use Euler's formula to express the cosine function in terms of complex exponentials

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}.$$

We can then use the linearity property of the Fourier transform to compute the Fourier transform

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega_0 t}e^{-j\omega t}\,dt
\end{aligned}$$

We can recognize these integrals as the Fourier transforms of delta functions

$$\begin{aligned}
F(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}\,dt \\
&= \frac{1}{2}\cdot 2\pi\delta(\omega-\omega_0) + \frac{1}{2}\cdot 2\pi\delta(\omega+\omega_0) \\
&= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0).
\end{aligned}$$

The Fourier transform of a cosine function is the sum of two delta functions in the frequency domain, one at the positive frequency ω_0 and one at the negative frequency $-\omega_0$. This means that a cosine function in the time domain represents two frequency components in the frequency domain, one at ω_0 and one at $-\omega_0$.

Fourier transform of $f(t) = \mathbf{sgn}(t)$

The signum function is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

which essentially just indicates the sign of t. We can try to compute the Fourier transform of the signal $f(t) = \operatorname{sgn}(t)$ using the definition of the Fourier transform in Eq. 4.6. However, we will run into issues with convergence because the signum function does not decay to zero as $t \to \pm\infty$, so the Fourier integral will not converge (because $e^{-j\omega t}$ does not decay to zero). To get around this, we can multiply the signum function by a decaying exponential $e^{-\alpha|t|}$ where $\alpha > 0$ is a small positive constant. Using the definition in Eq. 4.6, we can compute the Fourier transform of the modified signal

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \lim_{\alpha\to 0^+} \int_{-\infty}^{\infty} \operatorname{sgn}(t)e^{-\alpha|t|}e^{-j\omega t}\,dt \\
&= \lim_{\alpha\to 0^+} \left(\int_{-\infty}^{0} -e^{\alpha t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-\alpha t}e^{-j\omega t}\,dt \right) \\
&= \lim_{\alpha\to 0^+} \left(-\int_{-\infty}^{0} e^{(\alpha-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(\alpha+j\omega)t}\,dt \right) \\
&= \lim_{\alpha\to 0^+} \left(-\left[\frac{e^{(\alpha-j\omega)t}}{\alpha-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right]_{0}^{\infty} \right) \\
&= \lim_{\alpha\to 0^+} \left(-\frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} \right) \\
&= \lim_{\alpha\to 0^+} \left(\frac{-(\alpha+j\omega)+(\alpha-j\omega)}{(\alpha-j\omega)(\alpha+j\omega)} \right) \\
&= \lim_{\alpha\to 0^+} \left(\frac{-2j\omega}{\alpha^2+\omega^2} \right) \\
&= -j \cdot \frac{2\omega}{\omega^2} \\
&= -j\frac{2}{\omega}.
\end{aligned}$$

The Fourier transform of the signum function is $F(\omega) = -j\frac{2}{\omega} = \frac{2}{j\omega}$. We notice $f(t)$ is real-valued and odd, so $F(\omega)$ is imaginary-valued and odd.

Fourier transform of $f(t) = u(t)$

If we try to compute the Fourier transform of the unit step function $f(t) = u(t)$ using the definition of the Fourier transform in Eq. 4.6, we will run into the same issues with convergence as the signum function because the unit step function does not decay to zero as $t \to \infty$, so the Fourier integral will not converge (because $e^{-j\omega t}$ does not decay to zero). To get around this, we can define the unit step function in terms of the signum function

$$u(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(t)\right).$$

Using the linearity property of the Fourier transform, we can compute the Fourier transform of the unit step function

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(1 + \operatorname{sgn}(t)\right)e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sgn}(t)e^{-j\omega t}\,dt \\
&= \frac{1}{2}\cdot 2\pi\delta(\omega) + \frac{1}{2}\cdot\frac{2}{j\omega} \\
&= \pi\delta(\omega) + \frac{1}{j\omega}.
\end{aligned}$$

The Fourier transform of the unit step function is $F(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$. We notice $f(t)$ is real-valued, so $F(\omega)$ exhibits conjugate symmetry.

4.5 Properties of the Fourier transform

There are several important properties of the Fourier transform that are useful to know. These properties can be used to simplify the computation of Fourier transforms and to understand the behavior of signals in the frequency domain.

4.5.1 Linearity

The Fourier transform is a linear operator. This means that if we have two signals $f_1(t)$ and $f_2(t)$ with Fourier transforms $F_1(\omega)$ and $F_2(\omega)$, respectively, then the Fourier transform of a linear combination of these signals is the same linear combination of their Fourier transforms

$$\mathcal{F}\{af_1(t) + bf_2(t)\} = aF_1(\omega) + bF_2(\omega)$$

where a and b are constants.

4.5.2 Duality

The duality property of the Fourier transform states that if $f(t)$ has a Fourier transform $F(\omega)$, then $F(t)$ has a Fourier transform $2\pi f(-\omega)$. In other words, the Fourier transform of a function in the time domain corresponds to a scaled version of the original function in the frequency domain.

Mathematically, this can be expressed as:

$$\mathcal{F}\{f(t)\} = F(\omega) \quad \Rightarrow \quad \mathcal{F}\{F(t)\} = 2\pi f(-\omega)$$

The basic idea with this property is that if we solved for the Fourier transform of $f(t)$ to get $F(\omega)$, we can swap the roles of time and frequency to get the Fourier transform of $F(t)$ to get $2\pi f(-\omega)$.

The negative sign in the argument of $f(-\omega)$ indicates a time-reversal operation. This is essentially a "buy one get one free" property: if you know the Fourier transform of one function, you can immediately get the Fourier transform of its dual function.

Example: Suppose we know the Fourier transform of the rectangular function $f(t) = \text{rect}\left(\frac{t}{\tau}\right)$ is given by $F(\omega) = \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$. Using the duality property, we can find the Fourier transform of $F(t) = \tau \cdot \text{sinc}\left(\frac{t\tau}{2}\right)$ as follows:

$$\begin{aligned}
\mathcal{F}\{F(t)\} &= 2\pi f(-\omega) \\
&= 2\pi \cdot \text{rect}\left(\frac{-\omega}{\tau}\right) \quad \text{(by duality property)} \\
&= 2\pi \cdot \text{rect}\left(\frac{\omega}{\tau}\right) \quad \text{(since rect is even).}
\end{aligned}$$

4.5.3 Time scaling

If we scale a signal in time, its Fourier transform is scaled in frequency. Specifically, if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of the time-scaled signal $f(at)$ is given by

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

where a is a scaling factor. This property indicates that a time scaling in the time domain corresponds to a frequency scaling in the frequency domain.

Intuitively, we see that if we compress a signal in time (i.e., $|a| > 1$), it will contain higher frequency components, so its Fourier transform will be stretched out in frequency. Conversely, if we stretch a signal in time (i.e., $|a| < 1$), it will contain lower frequency components, so its Fourier transform will be compressed in frequency. The factor of $\frac{1}{|a|}$ ensures that the energy of the signal is preserved during the scaling process.

4.5.4 Time-reversal

If we reverse a signal in time, its Fourier transform is also reversed in frequency. Specifically, if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of the time-reversed signal $f(-t)$ is given by

$$\mathcal{F}\{f(-t)\} = F(-\omega).$$

This property is a special case of the time scaling property with $a = -1$.

Example: Suppose we want to find the Fourier transform of $f(t) = e^{at}u(-t)$ where $a > 0$. We can use the time-reversal property to find the Fourier transform. First, we note that $f(t)$ can be expressed as the time-reversed version of $g(t) = e^{-at}u(t)$. We know the Fourier transform of $g(t)$ is given by $G(\omega) = \frac{1}{a+j\omega}$. Using the time-reversal property, we can find the Fourier transform of $f(t)$ as follows

$$\begin{aligned}
\mathcal{F}\{f(t)\} &= \mathcal{F}\{\}(-\sqcup)\} \quad \text{(by definition of } f(t)) \\
&= G(-\omega) \quad \text{(by time-reversal property)} \\
&= \frac{1}{a - j\omega}.
\end{aligned}$$

Example: If we want to find the Fourier transform of $h(t) = e^{-a|t|}$ where $a > 0$, we can use the time-reversal property. We can express $f(t)$ as the sum of two time-reversed functions

$$\begin{aligned} h(t) &= e^{-at}u(t) + e^{at}u(-t) \\ &= g(t) + f(t) \end{aligned}$$

where $g(t) = e^{-at}u(t)$ and $f(t) = e^{at}u(-t)$. We can apply the linearity property such that $H(\omega) = G(\omega) + F(\omega)$. We know the Fourier transform of $g(t)$ is given by $G(\omega) = \frac{1}{a+j\omega}$, and we can use the time-reversal property to find the Fourier transform of $f(t)$ as $F(\omega) = \frac{1}{a-j\omega}$. Therefore, we can find the Fourier transform of $h(t)$ as follows

$$\begin{aligned} H(\omega) &= G(\omega) + F(\omega) \\ &= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} \\ &= \frac{(a-j\omega)+(a+j\omega)}{(a+j\omega)(a-j\omega)} \\ &= \frac{2a}{a^2+\omega^2}. \end{aligned}$$

We notice that $h(t)$ is real-valued and even, so $H(\omega)$ is real-valued and even, which is consistent with Sec. 4.5.2.

4.5.5 Time shifting

If we shift a signal in time, its Fourier transform is multiplied by a complex exponential. Specifically, if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of the time-shifted signal $f(t - t_0)$ is given by

$$\mathcal{F}\{f(t - t_0)\} = F(\omega)e^{-j\omega t_0}$$

where t_0 is the amount of time shift. This property indicates that a time shift in the time domain corresponds to a phase shift in the frequency domain.

Intuitively, we see that if we delay a signal in time (i.e., $t_0 > 0$), it will introduce a negative phase shift in the frequency domain. Conversely, if we advance a signal in time (i.e., $t_0 < 0$), it will introduce a positive phase shift in the frequency domain. The magnitude of the Fourier transform remains unchanged during the time shifting process.

4.5.6 Frequency shifting

If we shift a signal in frequency, its Fourier transform is multiplied by a complex exponential. Specifically, if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of the frequency-shifted signal $f(t)e^{j\omega_0 t}$ is given by

$$\mathcal{F}\{f(t)e^{j\omega_0 t}\} = F(\omega - \omega_0)$$

where ω_0 is the amount of frequency shift. This property indicates that a frequency shift in the frequency domain corresponds to a time shift in the time domain.

Conversely, if we multiply a signal by a complex exponential $e^{-j\omega_0 t}$, its Fourier transform is given by

$$\mathcal{F}\{f(t)e^{-j\omega_0 t}\} = F(\omega + \omega_0).$$

4.5.7 Modulation

The modulation property of the Fourier transform states that if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of the product of $f(t)$ with a cosine function is given by

$$\begin{aligned}
\mathcal{F}\{f(t)\cos(\omega_0 t)\} &= \mathcal{F}\{f(t)\cdot\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\} \quad \text{(by Euler's formula)} \\
&= \frac{1}{2}\left[\mathcal{F}\{f(t)e^{j\omega_0 t}\}+\mathcal{F}\{f(t)e^{-j\omega_0 t}\}\right] \quad \text{(by linearity property)} \\
&= \frac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]
\end{aligned}$$

where ω_0 is the frequency of the cosine function. Intuitively, this property indicates that multiplying a signal by a cosine function in the time domain corresponds to shifting its Fourier transform in frequency by $\pm\omega_0$ and averaging the two shifted versions. This is a fundamental concept in communication systems, where signals are often modulated by cosine functions to shift their frequency content for transmission.

4.5.8 Convolution in time

The convolution property of the Fourier transform states that if $f(t)$ and $g(t)$ have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively, then the Fourier transform of their convolution $y(t) = (f*g)(t)$ is given by

$$\mathcal{F}\{y(t)\} = Y(\omega) = F(\omega)G(\omega).$$

This property indicates that convolution in the time domain corresponds to multiplication in the frequency domain.

This has major implications in analyzing linear time-invariant (LTI) systems. If we have an input signal $x(t)$ with Fourier transform $X(\omega)$ and an LTI system with impulse response $h(t)$ and Fourier transform $H(\omega)$, then the output signal $y(t)$ is given by the convolution of the input signal and the impulse response

$$y(t) = x(t) * h(t).$$

Using the convolution property of the Fourier transform, we can find the Fourier transform of the output signal as follows

$$Y(\omega) = X(\omega)H(\omega).$$

This means that the frequency response of the system $H(\omega)$ acts as a filter on the input signal's frequency content $X(\omega)$ to produce the output signal's frequency content $Y(\omega)$. This also implies that if the input signal contains frequency components that are not present in the system's frequency response, those components will be attenuated or eliminated in the output signal. Conversely, if the input signal contains frequency components that are amplified by the system's frequency response, those components will be enhanced in the output signal. Additionally, if a frequency component is not present in the input signal, it will not be present in the output signal regardless of the system's frequency response. An LTI system cannot create new frequency components that were not present in the input signal.

Example: Suppose we have to functions $f(t) = e^{-at}u(t)$ and $g(t) = e^{-bt}u(t)$ where $a, b > 0$. We can compute the convolution of these two functions in the time domain as follows

$$y(t) = f(t) * g(t).$$

We notice that if $t < 0$, then there is no overlap between $f(\tau)$ and $g(t-\tau)$, so $y(t) = 0$ for $t < 0$. If $t \geq 0$, then there is some overlap between $f(\tau)$ and $g(t-\tau)$, so we can compute the convolution integral as follows

$$y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$$

Because $f(\tau) = 0$ for $\tau < 0$ and $g(t-\tau) = 0$ for $\tau > t$, we can change the limits of integration to 0 to t

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau}e^{-b(t-\tau)}\,d\tau \\ &= e^{-bt}\int_0^t e^{(b-a)\tau}\,d\tau \\ &= e^{-bt}\left[\frac{e^{(b-a)\tau}}{b-a}\right]_0^t \\ &= e^{-bt}\left(\frac{e^{(b-a)t}-1}{b-a}\right) \\ &= \frac{e^{-at}-e^{-bt}}{b-a}. \end{aligned}$$

Putting this together, we have

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{e^{-at}-e^{-bt}}{b-a}, & t \geq 0 \end{cases}.$$

We can also compute the convolution using the convolution property of the Fourier transform. We know the Fourier transforms of $f(t)$ and $g(t)$ are given by

$$F(\omega) = \frac{1}{a+j\omega} \tag{4.7}$$

$$G(\omega) = \frac{1}{b+j\omega}. \tag{4.8}$$

Using the convolution property of the Fourier transform, we can find the Fourier transform of $y(t)$ as follows

$$\begin{aligned} Y(\omega) &= F(\omega)G(\omega) \\ &= \frac{1}{(a+j\omega)(b+j\omega)}. \end{aligned}$$

To find $y(t)$, we can compute the inverse Fourier transform of $Y(\omega)$

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(a+j\omega)(b+j\omega)}e^{j\omega t}\,d\omega. \end{aligned}$$

Evaluating this integral would be onerous, but we can use partial fraction decomposition to find obtain a more manageable form that matches entries on Table **??**. We can express $Y(\omega)$ as

$$Y(\omega) = \frac{c}{a + j\omega} + \frac{d}{b + j\omega}.$$

We can solve for c and d using the cover-up method. To solve for c, we multiply both sides by $a + j\omega$ and then set $\omega = -ja$

$$\begin{aligned} c &= (a + j\omega)Y(\omega)\big|_{\omega=-ja} \\ &= (a + j\omega)\frac{1}{(a + j\omega)(b + j\omega)}\bigg|_{\omega=-ja} \\ &= \frac{1}{b - a}. \end{aligned}$$

To solve for d, we multiply both sides by $b + j\omega$ and then set $\omega = -jb$

$$\begin{aligned} d &= (b + j\omega)Y(\omega)\big|_{\omega=-jb} \\ &= (b + j\omega)\frac{1}{(a + j\omega)(b + j\omega)}\bigg|_{\omega=-jb} \\ &= \frac{-1}{b - a}. \end{aligned}$$

Putting this together, we have

$$Y(\omega) = \frac{1}{b - a} \cdot \frac{1}{a + j\omega} - \frac{1}{b - a} \cdot \frac{1}{b + j\omega}.$$

Using Table **??**, we can find the inverse Fourier transform of $Y(\omega)$ as follows

$$\begin{aligned} y(t) &= \frac{1}{b - a}e^{-at}u(t) - \frac{1}{b - a}e^{-bt}u(t) \\ &= \frac{e^{-at} - e^{-bt}}{b - a}u(t). \end{aligned}$$

This matches the result we obtained by directly computing the convolution integral in the time domain.

While this matches the result we obtained by directly computing the convolution integral in the time domain, the Fourier transform method is not the best transform method. The Laplace transform is a more robust approach for this type of problem because it is defined for a wider class of signals and can handle convergence issues more effectively.

4.5.9 Multiplication in time

The multiplication property of the Fourier transform states that if $f(t)$ and $g(t)$ have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively, then the Fourier transform of their product $y(t) = f(t)g(t)$ is given by

$$\mathcal{F}\{y(t)\} = Y(\omega) = \frac{1}{2\pi}F(\omega) * G(\omega).$$

This property indicates that multiplication in the time domain corresponds to convolution in the frequency domain. This has major implications in signal processing, particularly in the context of windowing and filtering.

4.5.10 Differentiation in time

The differentiation property of the Fourier transform states that if $f(t)$ has a Fourier transform $F(\omega)$, then the Fourier transform of its derivative $f'(t)$ is given by

$$\mathcal{F}\{\frac{df}{dt}\} = j\omega F(\omega).$$

Extending this yields

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}\right\} = (j\omega)^n F(\omega).$$

This property indicates that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. This has major implications in analyzing systems that involve differentiation, such as high-pass filters.

Example: Look at the triangle function $f(t) = \Delta\left(\frac{t}{\tau}\right)$ where $\tau > 0$. There are several approaches we could take to compute the Fourier transform of this function. One approach is to use the definition of the Fourier transform in Eq. 4.6. However, this would require evaluating a piecewise integral, which can be cumbersome. Another approach is to use the differentiation property of the Fourier tranform. If we take the first derivative of $f(t)$, we get

$$f'(t) = \frac{2}{\tau}\text{rect}\left(\frac{2(t+\tau/4)}{\tau}\right) - \frac{2}{\tau}\text{rect}\left(\frac{2(t-\tau/4)}{\tau}\right).$$

If we take the second derivative of $f(t)$, we get

$$f''(t) = \frac{2}{\tau}\delta\left(t + {}^{\tau}\!/_2\right) - \frac{4}{\tau}\delta(t) + \frac{2}{\tau}\delta\left(t - {}^{\tau}\!/_2\right).$$

We can use the linearity property of the Fourier transform to find the Fourier transform of $f''(t)$ as follows

$$\begin{aligned}
\mathcal{F}\{f''(t)\} &= \frac{2}{\tau}\mathcal{F}\left\{\delta\left(t + {}^{\tau}\!/_2\right)\right\} - \frac{4}{\tau}\mathcal{F}\{\delta(t)\} + \frac{2}{\tau}\mathcal{F}\left\{\delta\left(t - {}^{\tau}\!/_2\right)\right\} \\
&= \frac{2}{\tau}e^{j\omega\tau/2} - \frac{4}{\tau} + \frac{2}{\tau}e^{-j\omega\tau/2}
\end{aligned}$$

We recall that $\mathcal{F}\{f''(t)\} = (j\omega)^2 F(\omega) = -\omega^2 F(\omega)$, so we can solve for $F(\omega)$ as follows

$$\begin{aligned}
-\omega^2 F(\omega) &= \frac{2}{\tau}e^{j\omega\tau/2} - \frac{4}{\tau} + \frac{2}{\tau}e^{-j\omega\tau/2} \\
F(\omega) &= -\frac{2}{\tau\omega^2}\left(e^{j\omega\tau/2} - 2 + e^{-j\omega\tau/2}\right) \\
&= -\frac{2}{\tau\omega^2}\left(2\cos(\omega\tau/2) - 2\right) \\
&= -\frac{4}{\tau\omega^2}\left(\cos(\omega\tau/2) - 1\right) \\
&= -\frac{4}{\tau\omega^2}\left(-2\sin^2(\omega\tau/4)\right) \\
&= \frac{8}{\tau\omega^2}\sin^2(\omega\tau/4) \\
&= \tau\left(\frac{\sin(\omega\tau/4)}{\omega\tau/4}\right)^2 \\
&= \tau\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right).
\end{aligned}$$

This matches the entry for the triangle function in Table 4.1. We notice $f(t)$ is real-valued and even, so $F(\omega)$ is real-valued and even, which is consistent with Sec. ??.

We could approach this derivation using myriad approaches. One such approach would be to stop at the first derivative of $f(t)$ and use the Fourier transform of the rectangular function from Table ?? along with the time-shifting property of the Fourier transform to find the Fourier transform of $f'(t)$. Then, we could use the differentiation property of the Fourier transform to find the Fourier transform of $f(t)$. Another approach is to recognize that the triangle function can be expressed as the convolution of two rectangular functions

$$f(t) = \Delta\left(\frac{t}{\tau}\right) = \frac{2}{\tau}\text{rect}\left(\frac{t}{\tau/2}\right) * \text{rect}\left(\frac{t}{\tau/2}\right).$$

We could use the Fourier transform of the rectangular function from Table ?? along with the convolution property of the Fourier transform to find the Fourier transform of $f(t)$.

4.5.11 Parseval's theorem

The Parseval's theorem states that the total energy of a signal in the time domain is equal to the total energy of its Fourier transform in the frequency domain. Mathematically, this can be expressed as

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega.$$

This theorem has important implications in signal processing, particularly in the context of energy conservation and signal analysis.

We can extend this idea to the *generalized* Parseval's theorem, which states that the inner product of two signals in the time domain is equal to the inner product of their Fourier transforms in the frequency domain. Mathematically, this can be expressed as

$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) F_2^*(\omega) \, d\omega$$

where $f_1(t)$ and $f_2(t)$ have Fourier transforms $F_1(\omega)$ and $F_2(\omega)$, respectively, and the asterisk denotes the complex conjugate. This generalized theorem has important implications in signal processing, particularly in the context of correlation and signal similarity. It also allows us to compute some integrals that would be impossible in the time-domain alone.

4.5.12 Fourier transform property table

A summary of the properties of the Fourier transform is provided in Table 4.2.

4.6 Fourier transform from a system perspective

4.6.1 LTI systems and frequency response

Suppose we have a linear and time-invariant (LTI) system with an input output relationship such that

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= x(t) * h(t) \end{aligned}$$

where $h(t)$ is the impulse response of the system. If we take the Fourier transform of both sides, we get

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(t) * h(t)\} \\ &= X(\omega)H(\omega) \end{aligned}$$

where $H(\omega) = \mathcal{F}\{h(t)\}$ is the *frequency response* of the system. As mentioned earlier, this multiplication in the frequency domain implies that the system can attenuate or amplify existing frequency components in the input signal $X(\omega)$, but it cannot create new frequency components that were not present in the input signal.

Because the frequency response, input signal, and output signal are all complex-valued functions, we can express them in terms of their magnitude and phase as follows

$$\begin{aligned} H(\omega) &= |H(\omega)|e^{j\angle H(\omega)} \\ X(\omega) &= |X(\omega)|e^{j\angle X(\omega)} \\ Y(\omega) &= |Y(\omega)|e^{j\angle Y(\omega)}. \end{aligned}$$

Using these expressions, we can express the output signal in terms of its magnitude and phase as follows

$$\begin{aligned} |Y(\omega)|e^{j\angle Y(\omega)} &= |X(\omega)|e^{j\angle X(\omega)}|H(\omega)|e^{j\angle H(\omega)} \\ |Y(\omega)| &= |X(\omega)||H(\omega)| \\ \angle Y(\omega) &= \angle X(\omega) + \angle H(\omega). \end{aligned}$$

This shows that the magnitude of the output signal is the product of the magnitudes of the input signal and the frequency response, and the phase of the output signal is the sum of the phases of the input signal and the frequency response.

4.6.2 Distortionless system

We will classify a distortionless system as a system that does not alter the shape of the input signal. This means that the output signal is a scaled and time-shifted version of the input signal. Mathematically, we can express this as

$$y(t) = Kx(t - t_d)$$

where K is a scaling factor and t_d is a time delay. If we take the Fourier transform of both sides, we get

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{Kx(t - t_d)\} \\ &= KX(\omega)e^{-j\omega t_d}. \end{aligned}$$

This shows that the frequency response of a distortionless system is given by

$$H(\omega) = Ke^{-j\omega t_d}.$$

We notice that the magnitude of the frequency response is constant across all frequencies, and the phase of the frequency response is a linear function of frequency. This means that a distortionless system does not alter the amplitude of any frequency component in the input signal, and it introduces a constant time delay to all frequency components in the input signal.

The phase linearity is less intuitive than the constant magnitude response. Essentially, if we were to plot $\angle F(\omega)$ versus ω, we would get a straight line with a slope of $-t_d$. The slope of this line is known as the *group delay* of the system, which is defined as

$$\tau_g(\omega) = -\frac{d}{d\omega}\angle H(\omega).$$

Generally, the group delay is a function of frequency and indicates how different frequency components in the input signal are delayed by the system. For a distortionless system, the group delay is constant across all frequencies and is equal to the time delay t_d. This means that all frequency components in the input signal experience the same time delay when passing through the system, which is a key characteristic of a distortionless system.

4.6.3 Ideal low-pass filter

An ideal low-pass filter is a theoretical system that allows all frequency components below a certain cutoff frequency ω_c to pass through without attenuation, while completely attenuating all frequency components above the cutoff frequency. An example of this frequency response is seen in Fig.

The frequency response of an ideal low-pass filter is given by Fig. 4.5 and can be mathematically expressed as

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}.$$

To find the impulse response of the ideal low-pass filter, we can compute the inverse Fourier transform of its frequency response as

$$h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right).$$

The impulse response of the ideal low-pass filter is a sinc function, which is non-causal and extends infinitely in both directions in time. This means that the ideal low-pass filter cannot be implemented in practice, as it would require knowledge of future input values.

We can delay this impulse response by a time t_d to make it *more* causal, resulting in

$$h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c(t-t_d)}{\pi}\right).$$

This delay will add linear phase, which is often desirable in filtering applications to avoid phase distortion. However, even with this delay, the ideal low-pass filter remains non-causal and cannot be implemented in practice.

We can force this to be causal by multiplying it by the unit step function $u(t)$ or some windowing function, resulting in

$$h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c(t-t_d)}{\pi}\right)u(t).$$

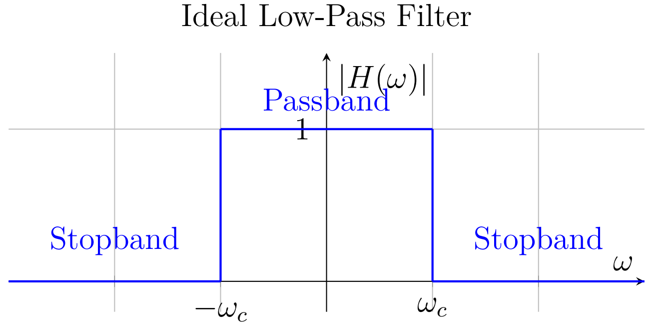


Figure 4.5: Frequency response of an ideal low-pass filter with cutoff frequency ω_c.

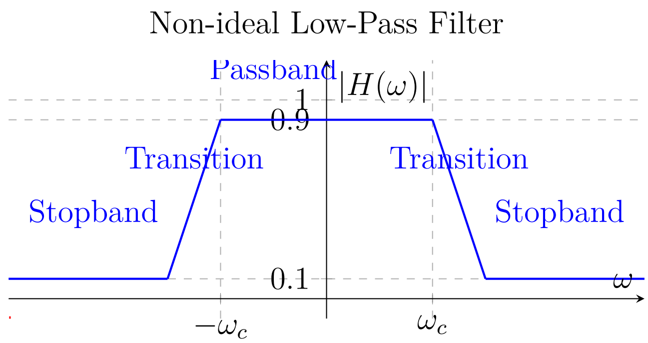


Figure 4.6: Frequency response of a causal low-pass filter with cutoff frequency ω_c.

However, this modification alters the frequency response of the filter, introducing ripples in the passband and a gradual roll-off in the stopband, which deviates from the ideal low-pass filter characteristics. This frequency response is seen in Fig. 4.6.

This is explained theoretically in the *Paley-Wiener theorem*, which states that a signal cannot be both time-limited and band-limited. In this case, by forcing the impulse response to be causal (time-limited), we have introduced non-ideal characteristics in the frequency response (not band-limited).

The formal Paley-Wiener theorem and its proof are provided in the next section. You do *not* need to know this proof for the purposes of this course, but it is provided for completeness. You should, however, understand the implications of the Paley-Wiener theorem in the context of signal processing and system design.

4.6.4 The Paley-Wiener Theorem

The Paley-Wiener theorem is a fundamental result in harmonic analysis that establishes a fundamental limitation on the simultaneous localization of signals in both time and frequency domains.

Theorem 1 (Paley-Wiener Theorem). Let $f(t)$ be a function such that:

1. $f(t) = 0$ for $|t| > T$ (time-limited to $[-T, T]$)
2. $\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$ (finite energy)

Then the Fourier transform $F(\omega) = \mathcal{F}\{f(t)\}$ cannot be band-limited. Specifically, if $F(\omega) = 0$ for $|\omega| > \Omega$ for some finite Ω, then $f(t) \equiv 0$.

Conversely, if $F(\omega) = 0$ for $|\omega| > \Omega$ (band-limited) and $\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega < \infty$, then $f(t)$ cannot be time-limited.

Proof. We prove this theorem using the properties of analytic functions and the theory of entire functions.

Step 1: Extension to the complex plane

For a time-limited function $f(t)$ with $f(t) = 0$ for $|t| > T$, we can extend the Fourier transform to the complex plane by considering

$$F(s) = \int_{-T}^{T} f(t)e^{-st} \, dt$$

where $s = \sigma + j\omega$ is a complex variable. Since $f(t)$ is time-limited, this integral converges for all finite values of s, making $F(s)$ an *entire function* (analytic everywhere in the complex plane).

Step 2: Growth condition

For the entire function $F(s)$, we can bound its growth. Since $f(t)$ has finite energy, we have

$$\begin{aligned}
|F(\sigma + j\omega)| &= \left| \int_{-T}^{T} f(t)e^{-(\sigma + j\omega)t} \, dt \right| \\
&= \left| \int_{-T}^{T} f(t)e^{-\sigma t}e^{-j\omega t} \, dt \right| \\
&\leq \int_{-T}^{T} |f(t)|e^{-\sigma t} \, dt \\
&\leq e^{|\sigma| T} \int_{-T}^{T} |f(t)| \, dt
\end{aligned}$$

By the Cauchy-Schwarz inequality:

$$\int_{-T}^{T} |f(t)| \, dt \leq \sqrt{2T} \left(\int_{-T}^{T} |f(t)|^2 \, dt \right)^{1/2}$$

Therefore, $F(s)$ grows at most exponentially with $|s|$, i.e., $|F(s)| \leq Ae^{B|s|}$ for some constants A and B.

Step 3: Application of Hadamard's theorem

Now suppose $F(\omega) = 0$ for $|\omega| > \Omega$. This means that $F(s)$ vanishes on the real axis outside the interval $[-\Omega, \Omega]$.

Since $F(s)$ is an entire function of exponential type (grows at most exponentially), and it vanishes on an infinite set of points on the real axis (namely, $|\omega| > \Omega$), Hadamard's theorem on the factorization of entire functions applies.

For an entire function of exponential type that vanishes on an infinite set with an accumulation point, the only possibility is that the function is identically zero.

Step 4: Conclusion

Since $F(s) \equiv 0$ for all s, we have $F(\omega) \equiv 0$ for all ω. By the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega = 0 \tag{4.9}$$

Therefore, $f(t) \equiv 0$, which proves that a non-trivial function cannot be both time-limited and band-limited.

The converse statement follows by a similar argument, considering the dual relationship between time and frequency domains in Fourier analysis. ■

Physical Interpretation:

The Paley-Wiener theorem has profound implications for signal processing:

- **Ideal filters are unrealizable:** An ideal low-pass filter with perfect brick-wall frequency response would require an infinite-duration impulse response (sinc function).
- **Causality and bandwidth trade-off:** Making a filter causal (time-limited to $t \geq 0$) necessarily introduces frequency response distortions.
- **Windowing effects:** Truncating a signal in time (windowing) spreads its spectrum, introducing spectral leakage.
- **Uncertainty principle:** The theorem is closely related to the time-frequency uncertainty principle, stating that a signal cannot be simultaneously localized in both domains.

This fundamental limitation explains why practical filters must compromise between sharp frequency selectivity and time-domain behavior, leading to the design of various filter approximations (Butterworth, Chebyshev, Elliptic, etc.) that balance these competing requirements.

4.6.5 Application: Dual sideband, suppressed carrier (DSB-SC) modulation

Dual sideband, suppressed carrier (DSB-SC) modulation is a technique used in communication systems to transmit information by modulating a carrier signal with a baseband signal ($m(t)$). In DSB-SC modulation, the carrier signal is multiplied by the baseband signal, resulting in a modulated signal that contains both upper and lower sidebands, but the carrier itself is suppressed.

We can see this from a system's perspective in Fig. 4.7. The input signal $x(t)$ is multiplied by a carrier signal $\cos(\omega_c t)$ to produce the modulated signal $m(t)\cos(\omega_c t)$. Suppose we have a baseband signal $m(t)$ with a Fourier transform $M(\omega)$. The carrier signal $\cos(\omega_c t)$ has a Fourier transform given by

$$\mathcal{F}\{\cos(\omega_c t)\} = \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right].$$

We see this in Fig. 4.8. In Fig. 4.8a, we see the time-domain representation of DSB-SC modulation, where the baseband signal $m(t)$ is multiplied by the carrier signal $\cos(\omega_c t)$. In Fig. 4.8b, we see the frequency-domain representation of DSB-SC modulation, where the Fourier transform of the modulated signal consists of two shifted versions of the baseband signal's Fourier transform $M(\omega)$, centered at $\pm\omega_c$.

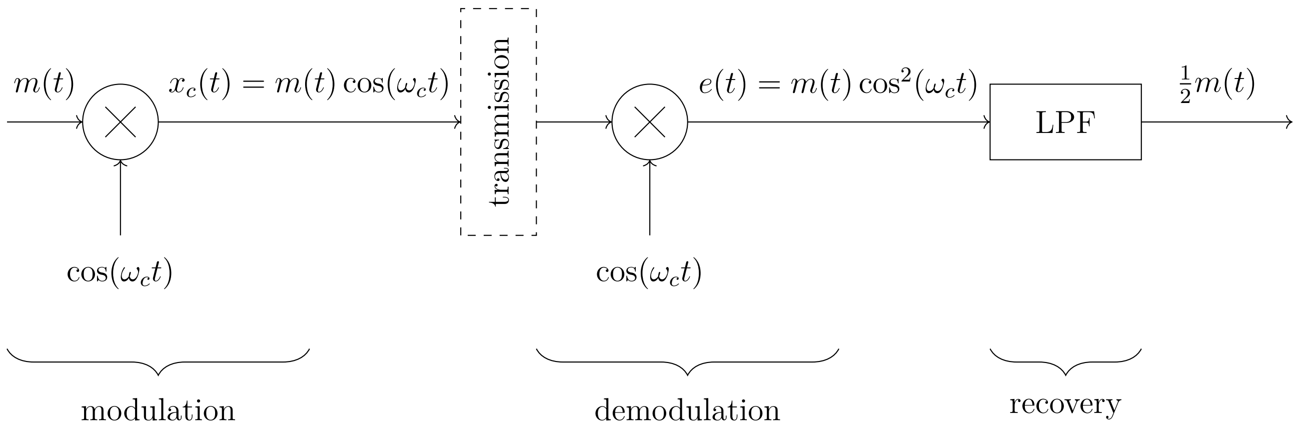
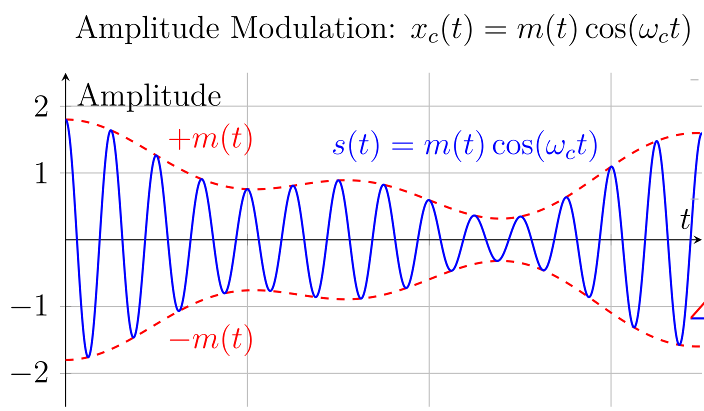
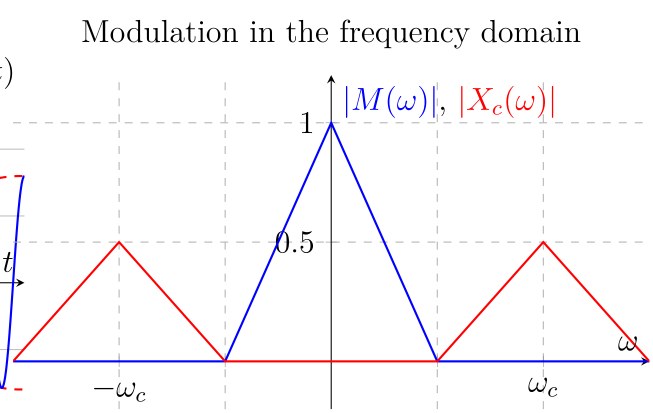


Figure 4.7: System diagram for dual sideband, suppressed carrier (DSB-SC) modulation.

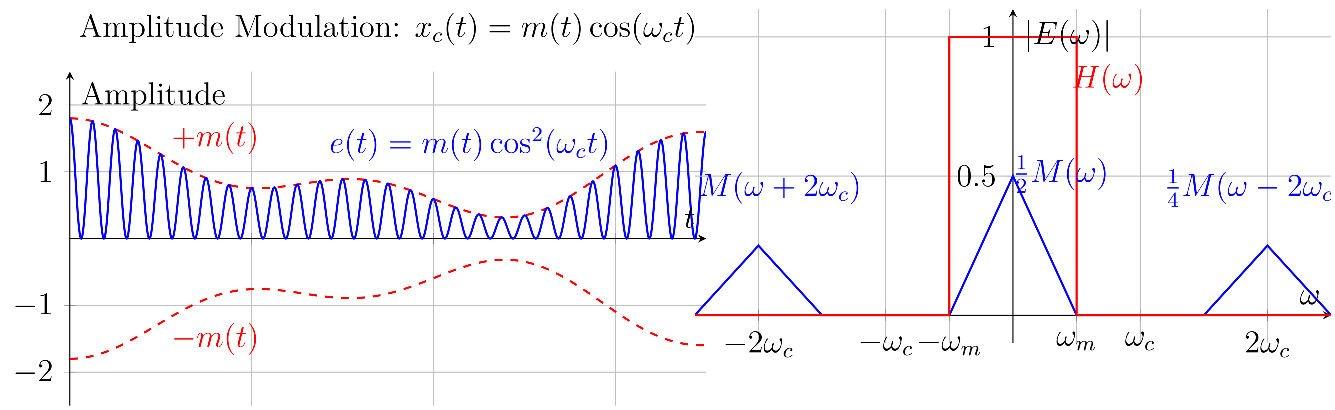


(a) Time domain representation of DSB-SC modulation.



(b) Frequency domain representation of DSB-SC modulation.

Figure 4.8: Dual sideband, suppressed carrier (DSB-SC) modulation in both frequency and time domains.



(a) Time domain representation of DSB-SC demodulation.

(b) Frequency domain representation of DSB-SC demodulation.

Figure 4.9: Dual sideband, suppressed carrier (DSB-SC) demodulation in both frequency and time domains.

After transmission, we recover the modulated signal, but we need to demodulate it to retrieve the original baseband signal $m(t)$. To do this, we can multiply the received signal by the same carrier signal $\cos(\omega_c t)$ used during modulation. This process is known as coherent demodulation. The demodulated signal $y(t)$ can be expressed as

$$\begin{aligned} y(t) &= m(t)\cos(\omega_c t)\cdot\cos(\omega_c t) \\ &= m(t)\frac{1+\cos(2\omega_c t)}{2} \\ &= \frac{m(t)}{2} + \frac{m(t)\cos(2\omega_c t)}{2}. \end{aligned}$$

In the frequency domain, this corresponds to

$$Y(\omega) = \frac{1}{2}M(\omega) + \frac{1}{4}\left[M(\omega - 2\omega_c) + M(\omega + 2\omega_c)\right].$$

The first term $\frac{1}{2}M(\omega)$ represents the original baseband signal scaled by a factor of $\frac{1}{2}$, while the second term contains frequency components centered around $\pm 2\omega_c$. To recover the original baseband signal $m(t)$, we can apply a low-pass filter with a cutoff frequency slightly above the highest frequency component of $m(t)$. This filter will remove the high-frequency components centered around $\pm 2\omega_c$, leaving us with the desired baseband signal scaled by $\frac{1}{2}$. We can then multiply the output of the low-pass filter by 2 to recover the original baseband signal $m(t)$. We can see this in both time and frequency domain in Fig. 4.9.

Issue: Phase synchronization

We have described DSB-SC in terms of a transmitter and reciever that are perfectly in sync with each other. In practice, this is difficult to achieve, and any phase mismatch between the transmitter and receiver can lead to degredation of the recovered signal. Let's look at it mathematically.

Suppose we have a phase mismatch of θ between the transmitter and receiver. This means that the receiver uses a carrier signal of $\cos(\omega_c t + \theta)$ instead of $\cos(\omega_c t)$. The demodulated signal $y(t)$

can be expressed as

$$\begin{aligned} y(t) &= m(t)\cos(\omega_c t)\cdot\cos(\omega_c t+\theta) \\ &= m(t) \\ frac\cos(\theta)+\cos(2\omega_c t+\theta)2 \\ &= \frac{m(t)\cos(\theta)}{2}+\frac{m(t)\cos(2\omega_c t+\theta)}{2}. \end{aligned}$$

In the frequency domain, this corresponds to

$$Y(\omega)=\frac{\cos(\theta)}{2}M(\omega)+\frac{1}{4}\left[M(\omega-2\omega_c)+M(\omega+2\omega_c)\right].$$

The first term $\frac{\cos(\theta)}{2}M(\omega)$ represents the original baseband signal scaled by a factor of $\frac{\cos(\theta)}{2}$. We notice that if $\theta=\frac{\pi}{2}$, then $\cos(\theta)=0$, and the original baseband signal is completely lost during demodulation. The modulated signal is also still present, and we can remove it as before with a low-pass filter. While we are able to recover the original baseband signal $m(t)$ for other values of θ, the amplitude of the recovered signal is attenuated by a factor of $\cos(\theta)$.

Issue: Frequency synchronization

More critical is frequency synchronization between the transmitter and receiver. Suppose there is a frequency offset of $\Delta\omega$ between the transmitter and receiver. This means that the receiver uses a carrier signal of $\cos((\omega_c+\Delta\omega)t)$ instead of $\cos(\omega_c t)$. The demodulated signal $y(t)$ can be expressed as

$$\begin{aligned} y(t) &= m(t)\cos(\omega_c t)\cdot\cos((\omega_c+\Delta\omega)t) \\ &= m(t)\frac{\cos(\Delta\omega t)+\cos((2\omega_c+\Delta\omega)t)}{2} \\ &= \frac{m(t)\cos(\Delta\omega t)}{2}+\frac{m(t)\cos((2\omega_c+\Delta\omega)t)}{2}. \end{aligned}$$

In the frequency domain, this corresponds to

$$Y(\omega)=\frac{1}{4}\left[M(\omega-\Delta\omega)+M(\omega+\Delta\omega)\right]+\frac{1}{4}\left[M(\omega-(2\omega_c+\Delta\omega))+M(\omega+(2\omega_c+\Delta\omega))\right].$$

The first term $\frac{1}{4}\left[M(\omega-\Delta\omega)+M(\omega+\Delta\omega)\right]$ represents two shifted versions of the original baseband signal. We notice that even a small frequency offset $\Delta\omega$ can lead to significant distortion of the recovered baseband signal, as the frequency components are now shifted away from their original positions. The modulated signal is also still present, and we can remove it as before with a low-pass filter. However, the recovered baseband signal is now distorted due to the frequency offset, leading to potential loss of information.

DSB-SC systems can avoid this frequency matching issue through the use of a pilot carrier, which is a small portion of the carrier signal that is transmitted along with the modulated signal. The receiver can use this pilot carrier to estimate and correct for any frequency offset, ensuring proper demodulation of the baseband signal. Alternatively the receiver can use phase lock loops (PLLs) to lock onto the carrier frequency and phase, allowing for accurate demodulation even in the presence of frequency and phase offsets. However, these techniques are beyond the scope of this course.

Table 4.1: A short table of Fourier transforms

	$f(t)$	$F(\omega)$	**Notes**
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Table 4.2: Properties of the Continuous-Time Fourier Transform

Operation	**Time Domain:** $f(t)$	**Frequency Domain:** $F(\omega)$
Addition	$f_1(t) + f_2(t)$	$F_1(\omega) + F_2(\omega)$
Scalar Multiplication	$kf(t)$	$kF(\omega)$
Symmetry	$F(t)$	$2\pi f(-\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time Shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Frequency Shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Time Convolution	$f_1(t) * f_2(t)$	$F_1(\omega)F_2(\omega)$
Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$
Time Differentiation	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time Integration	$\int_{-\infty}^{t} f(x)dx$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
Cosine Modulation	$f(t)\cos(\omega_0 t)$	$\frac{1}{2}\left[F(\omega - \omega_0) + F(\omega + \omega_0)\right]$
Sine Modulation	$f(t)\sin(\omega_0 t)$	$\frac{1}{2j}\left[F(\omega - \omega_0) - F(\omega + \omega_0)\right]$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$
Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} f_1(t)f_2^*(t)dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2^*(\omega)d\omega$

Chapter 5

Sampling and the Discrete Fourier Transform

Signal sampling is the bridge between continuous-time and discrete-time signal processing. It allows us to convert continuous signals into discrete sequences that can be processed using digital systems. This chapter explores the fundamental concepts of sampling, the implications of the sampling process, and the Discrete Fourier Transform (DFT), which is a powerful tool for analyzing discrete signals in the frequency domain.

5.1 Nyquist-Shannon sampling theorem

The Nyquist-Shannon sampling theorem states that a continuous-time signal can be completely reconstructed from its samples if it is bandlimited and the sampling frequency is at least twice the highest frequency present in the signal. This minimum sampling rate is known as the Nyquist rate.

5.1.1 Bandwidth definitions

A continuous-time signal $x(t)$ is said to be *bandlimited* if its Fourier transform $X(\omega)$ is zero for all frequencies $|\omega|$ greater than some finite value ω_N. The smallest such ω_N is called the *bandwidth* of the signal. In other words, a bandlimited signal has no frequency components above a certain cutoff frequency.

Baseband bandwidth The range of frequencies from $-\omega_N$ to ω_N where the signal's Fourier transform is non-zero. A signal is considered bandlimited if there exists a finite ω_N such that $X(\omega) = 0$ for all $|\omega| > \omega_N$. Thus, the baseband bandwidth is ω_N or $B = \omega_N/(2\pi)$ in Hz.

Bandpass bandwidth The width of a frequency band where the signal has non-zero frequency components, typically centered around a carrier frequency ω_c. For a bandpass signal, the bandwidth is defined as the difference between the upper and lower cutoff frequencies of the band. A bandpass signal is said to be bandlimited if its Fourier transform is zero outside this specified frequency band.

5.1.2 Formal statement and proof

We give a precise statement of the Nyquist–Shannon sampling theorem and a standard proof based on the Fourier transform and impulse-train sampling.

Theorem 2 (Nyquist–Shannon sampling theorem). Let $x(t)$ be a continuous-time signal with Fourier transform $X(\omega)$ and suppose $x(t)$ is bandlimited to $|\omega| \leq \omega_N$ (i.e. $X(\omega) = 0$ for $|\omega| > \omega_N$). Let the sampling period be T_s and the sampling frequency be $\omega_s = \dfrac{2\pi}{T_s}$. If

$$\omega_s > 2\omega_N, \qquad (\text{equivalently } T_s < \pi/\omega_N),$$

then $x(t)$ is uniquely determined by its samples $x[n] = x(nT_s)$ and can be reconstructed exactly from those samples by the interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \,\mathrm{sinc}(2\pi B(t - nT_s)) \,. \tag{5.1}$$

If $\omega_s \leq 2\omega_N$ then perfect reconstruction in general is impossible because spectral replicas overlap (aliasing).

Proof. We prove the theorem using the frequency-domain properties of sampling. Define the impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s),$$

and form the sampled signal

$$\bar{x}(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s).$$

An example of sampling a continuous-time signal is illustrated in Figure 5.1.

Using the convolution-multiplication duality and the known transform of the impulse train, we obtain

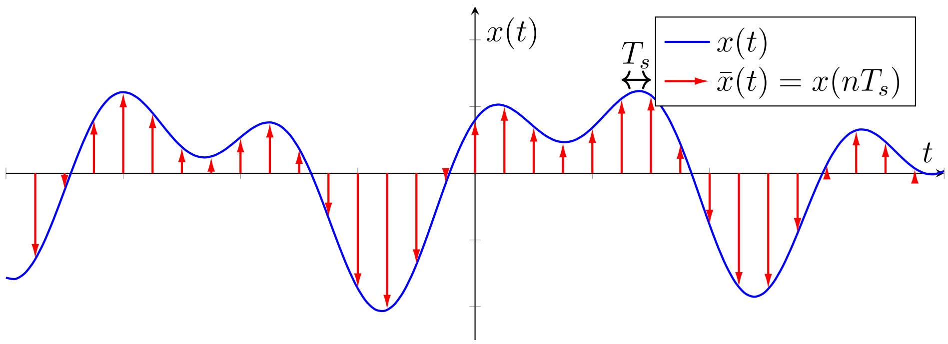
$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s), \tag{5.2}$$

i.e. sampling produces a superposition of frequency-shifted replicas of $X(\omega)$ separated by the sampling frequency ω_s. See Figure 5.2 for an illustration.

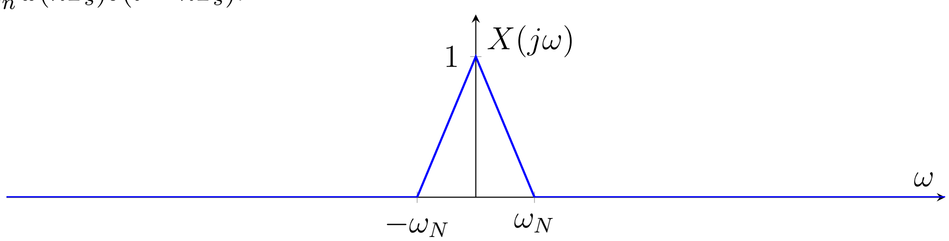
Because $X(\omega)$ is supported only on $|\omega| \leq \omega_N$, the terms in (5.2) are nonzero only in intervals of width $2\omega_N$ centered at $k\omega_s$. If the sampling rate satisfies $\omega_s > 2\omega_N$, these replicas do not overlap: the interval $[-\omega_N, \omega_N]$ around the origin is disjoint from the intervals around $\pm k\omega_s$ for nonzero integer k.

Under the non-overlap condition we can recover $X(\omega)$ by applying an ideal low-pass filter that retains the baseband (the copy centered at zero) and removes all shifted copies. Concretely, define the ideal reconstruction filter with frequency response

$$H(\omega) = \begin{cases} T_s, & |\omega| \leq \omega_c, \\ 0, & |\omega| > \omega_c, \end{cases}$$



(a) Sampling a continuous-time signal $x(t)$ at times $t = nT_s$ produces a weighted impulse train $\bar{x}(t) = \sum_n x(nT_s)\delta(t - nT_s)$.



(b) Frequency spectrum $X(\omega)$ of the original continuous-time signal $x(t)$.

Figure 5.1: (Top) Sampling a continuous-time signal in the time domain. (Bottom) The corresponding frequency spectrum $X(\omega)$ of the original signal.

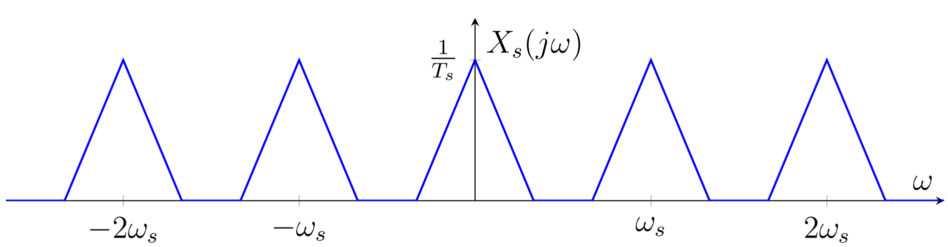


Figure 5.2: Spectrum of the sampled signal $X_s(\omega)$ consists of shifted replicas of $X(\omega)$ separated by the sampling frequency ω_s. If $\omega_s > 2\omega_N$ the replicas do not overlap and perfect reconstruction is possible.

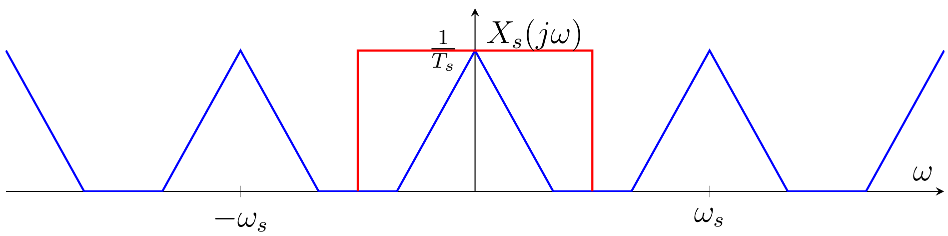


Figure 5.3: Reconstruction of the original signal $x(t)$ from the sampled signal $\bar{x}(t)$ by low-pass filtering. The filter passes the baseband replica of $X(\omega)$ and removes all other replicas, allowing perfect reconstruction when $\omega_s > 2\omega_N$.

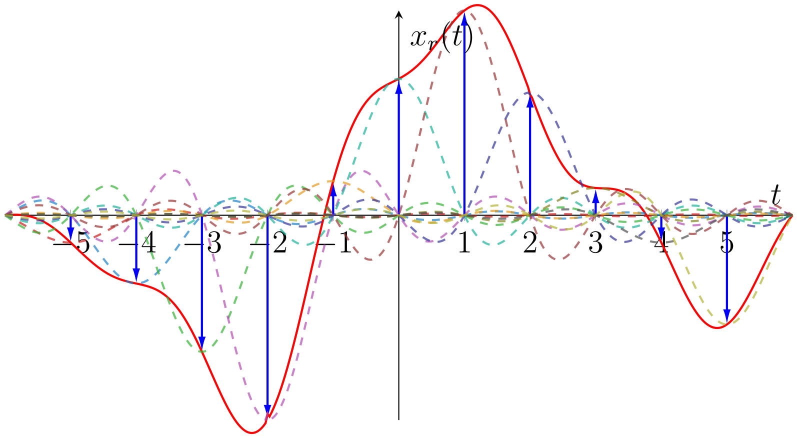


Figure 5.4: Sinc interpolation of discrete samples $x(nT_s)$ (where $T_s = 1$ for simplicity) to reconstruct the continuous-time signal $x(t)$. Each sample is weighted by a shifted sinc function, and the sum of these sinc functions produces the reconstructed signal.

where $\omega_N \leq \omega_c < \frac{\omega_s - \omega_N}{2}$ may be chosen so that the passband contains $[-\omega_N, \omega_N]$ and excludes neighboring replicas. (The factor T_s compensates for the $1/T_s$ in (5.2).) Applying $H(\omega)$ to $X_s(\omega)$ yields

$$X_r(\omega) = H(\omega)X_s(\omega) = X(\omega) \text{ for } |\omega| \leq \omega_N,$$

and zero elsewhere. Fig. 5.3 shows an example of the reconstruction process. Taking the inverse Fourier transform gives the reconstructed signal

$$x_r(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} X(\omega)e^{j\omega t}\, d\omega = x(t),$$

so the reconstruction is exact.

To obtain the explicit interpolation formula (5.1) one may insert the expression for $X_s(\omega)$ from (5.2), multiply by $H(\omega)$ and compute the inverse Fourier transform. A standard calculation (or applying the Poisson summation formula) yields the sinc-interpolation series above. (An illustration of sinc interpolation is shown in Figure 5.4.)

If instead $\omega_s \leq 2\omega_N$, the shifted replicas in (5.2) overlap: there exist frequencies where more than one shifted copy contributes. In this case the original $X(\omega)$ cannot be uniquely recovered from $X_s(\omega)$ because information from different parts of the spectrum has been irreversibly superposed (aliasing). Hence, perfect reconstruction is in general impossible. ■

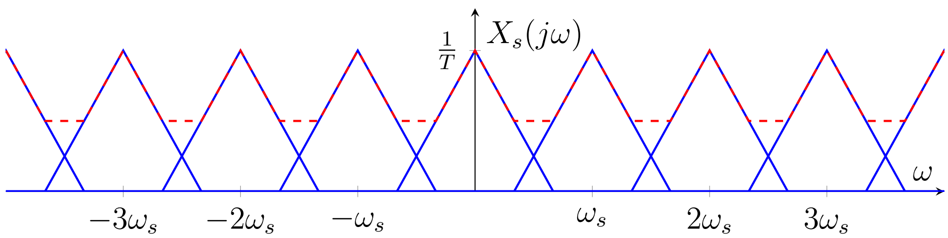


Figure 5.5: Aliasing occurs when the sampling frequency ω_s is less than or equal to twice the highest frequency component ω_N of the signal. The overlapping spectral replicas cause distortion and loss of information in the sampled signal.

Remarks

- The strict inequality $\omega_s > 2\omega_N$ can be relaxed to $\omega_s \geq 2\omega_N$ when $X(\omega)$ vanishes at the band edges and when one is willing to consider perfect reconstruction up to boundary values; in practice one usually samples strictly above the Nyquist rate to avoid numerical and practical issues.
- When $\omega_s \leq 2\omega_N$ one may still recover a bandlimited signal if additional a priori constraints are known (e.g. finite rate of innovation, sparsity) but not by linear, time-invariant reconstruction alone (e.g., compressed sensing).
- If the signal $x(t)$ is not bandlimited, perfect reconstruction is impossible for any finite sampling rate. However, if $X(\omega)$ decays sufficiently fast as $|\omega| \to \infty$, approximate reconstruction with small error is possible by sampling above a certain rate and applying suitable low-pass filtering (anti-aliasing filter) before sampling.
- If the signal $x(t)$ is complex-valued and is sampled in quadrature (i.e. both real and imaginary parts are sampled), the Nyquist rate is halved to ω_N because negative frequencies can be represented using positive frequencies.
- The sinc interpolation formula (5.1) is of theoretical interest but not practical because it requires infinitely many samples and the sinc function decays slowly. In practice, one uses finite-length approximations (e.g., windowed sinc functions) or other interpolation methods (e.g., polynomial, spline). Two very simple reconstruction filter would be $h(t) = \frac{1}{T_s}\text{rect}(t/T_s)$ (sample-hold) and $h(t) = \frac{1}{T_s}\Delta(t/T_s)$ (linear interpolation).

5.1.3 Issues with ideal sampling and reconstruction

The Nyquist-Shannon sampling theorem relies on idealized assumptions that are not fully realizable in practice.

Aliasing

In the situation where the sampling frequency ω_s is less than or equal to twice the highest frequency component ω_N of the signal, aliasing occurs. Aliasing is a phenomenon where different frequency components of the original signal become indistinguishable in the sampled signal, leading to distortion and loss of information. This happens because the spectral replicas created by sampling overlap, causing high-frequency components to be misrepresented as lower frequencies. An illustration of aliasing is shown in Figure 5.5.

Non-ideal filters

In practice, ideal low-pass filters with a perfect cutoff frequency do not exist. Real-world filters have a transition band where the filter response gradually changes from passband to stopband (which is never quite $H(\omega) = 0$, see Theorem **??**), and they may introduce phase distortion and other artifacts. This means that even if the sampling rate is above the Nyquist rate, the reconstruction may not be perfect due to the non-ideal characteristics of practical filters.

Quantization

Quantization is the process of mapping a continuous range of values into a finite set of discrete levels. In digital systems, after sampling a continuous-time signal, the sampled values must be quantized to be represented in a digital format (e.g., binary). This introduces quantization error, which is the difference between the actual sampled value and the quantized value. The precision of quantization depends on the number of bits used; more bits allow for finer resolution and lower quantization error. However, quantization inherently introduces noise and distortion, which can affect the quality of the reconstructed signal.

Non-ideal sampling

In Theorem 2, we assumed ideal impulse sampling, which is not physically realizable. In practice, sampling is often performed using sample-and-hold circuits or other methods that do not produce perfect impulses. One approach is to model impulses as narrow pulses of finite width and height (i.e., rectangular pulses) rather than ideal delta functions. We will model this sampling function $x(t)$ as a train of rectangular pulses with a cosine Fourier series

$$s(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n \cos(n\omega_s t),$$

where the coefficients C_n depend on the pulse width and height. Multiplying $x(t)$ by $s(t)$ in the time domain gives

$$\begin{aligned} x_s(t) &= x(t)s(t) \\ &= C_0 x(t) + \sum_{n=-\infty}^{\infty} C_n x(t) \cos(n\omega_s t). \end{aligned}$$

Thus, we have a similar result to what we saw in Theorem 2, but now the sampled signal consists of a scaled version of the original signal plus a series of modulated versions of the original signal at multiples of the sampling frequency. We will still have repeating copies of the original $X(\omega)$ at integer multiples of ω_s, but each copy will be scaled by some C_n. This is not the end of the world since ultimately we will not care much about these higher-frequency components during reconstruction. The reconstruction process will be similar as we still need to remove the modulated high frequency components using a low-pass filter.

5.2 The Discrete Fourier Transform (DFT)

The Discrete Fourier Transform (DFT) is a mathematical technique used to analyze the frequency information in some sampled and finite-length signal. An example would be some $x[n]$ that is represented by some Python array `x`.

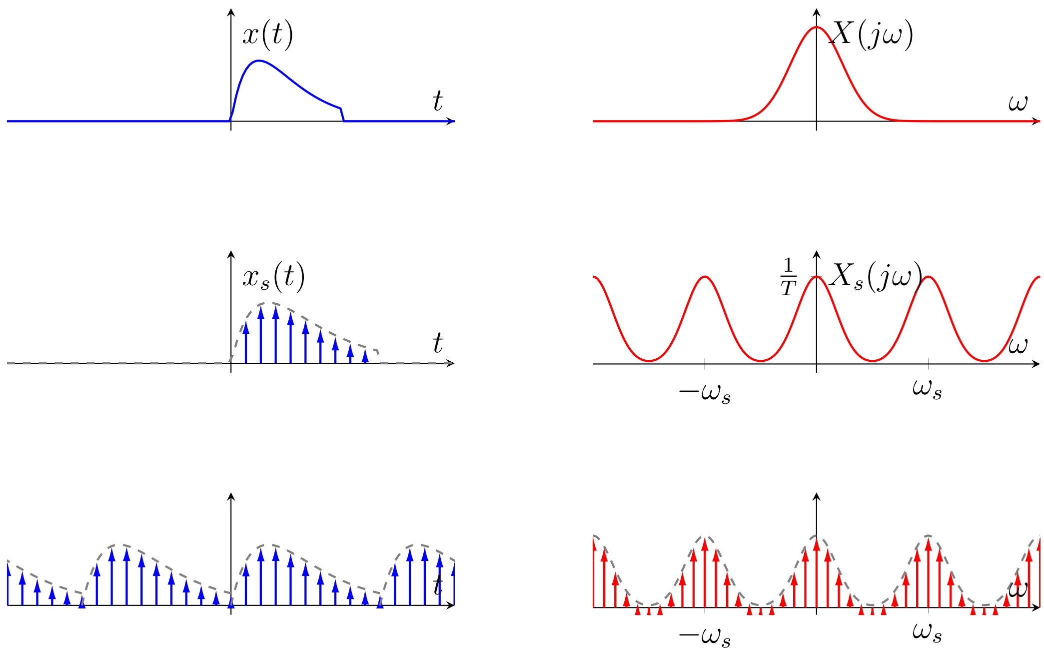


Figure 5.6: Sampling a continuous-time signal $x(t)$ produces a sampled signal $x_s(t)$ and a periodic spectrum $X_s(\omega)$. Sampling the spectrum at discrete frequencies produces a periodic time-domain signal.

5.2.1 Spectral sampling and periodic spectra

When a continuous-time signal is sampled periodically the resulting Fourier transform is a periodic function consisting of shifted replicas of the original spectrum (see (5.2)). Similarly, if you sample *in frequency* (i.e. take periodic samples of a continuous spectrum), the resulting function in the time domain is a periodic signal. This idea is not new. We saw from Fourier series analysis that a periodic signal can be represented as a linear combination of discrete frequency components. Specifically, the coefficients D_n in the Fourier series expansion were just samples of the continuous-time Fourier transform $X(\omega)$ taken at discrete frequencies $\omega_n = n\omega_0$—though it was initially presented that way. Thus, sampling in frequency produces a periodic signal in time.

Suppose we have some continuous-time signal $x(t)$ with Fourier transform $X(\omega)$. If we were sample this signal with a sampling rate $f_s = 1/T_s$ giving some sampled signal $x(nT_s) = \bar{x}(t)$ and a Fourier transform $\bar{X}(\omega)$. Now sample $\bar{X}(\omega)$ at discrete frequencies $k\omega_0$ for integer k. We obtain the sampled spectrum

$$X_s(\omega) = \sum_{k=-\infty}^{\infty} \bar{X}(k\omega_0)\delta(\omega - k\omega_0).$$

We sampled $\bar{X}(\omega)$ periodically with period $\omega_0 = \omega_s/N$ such that each period has N samples, the resulting time-domain signal is going to still be sampled, but now will *also* be periodic with a period of $T_0 = 2\pi/\omega_0$ and with each period having N samples ($N = T_0/T_s$). The Fourier series coefficients of this periodic signal are given by the samples $X(\omega_k)$. An illustration of this process is shown in Figure 5.6.

5.2.2 Periodic signals and the Discrete-Time Fourier Series (DTFS)

If we sample some periodic signal $x(t)$ to be discrete-time x_k such that it is also periodic with period N (i.e. $x_k = x_{k+N}$ for all k), it can be expanded to be a Fourier series (just like we did in Chapter 4), but we will call this the discrete-time Fourier series (DTFS) as

$$x_k = \sum_{n=0}^{N-1} X_n e^{j2\pi nk/N}, \tag{5.3}$$

where the DTFS coefficients X_n are given by the analysis formula

$$X_r = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-j2\pi rk/N}, \qquad r = 0, 1, \ldots, N-1. \tag{5.4}$$

We see that the DTFS represents the discrete-time periodic sequence as a finite linear combination of complex exponentials with discrete frequencies $2\pi n/N$ (you can verify this numerically in Python). More importantly, we also see that the DTFS coefficients are just samples of $\bar{X}(\omega)$ (the continuous-time spectrum of the sampled signal $\bar{x}(t)$) at frequencies $\omega_0 = 2\pi n/N$.

5.2.3 From DTFS to the DFT

The Discrete Fourier Transform (DFT) is just the N DTFS coefficients X_r. It is defined for a finite sequence x_k, $k = 0, \ldots, N-1$, by

$$X_r = \sum_{k=0}^{N-1} x_k e^{-j2\pi rk/N}, \qquad r = 0, 1, \ldots, N-1. \tag{5.5}$$

We can interpret this as viewing the finite sequence x_k as one period of an N-periodic sequence. With that interpretation the DFT coefficients are (up to the $1/N$ scaling convention) exactly the DTFS coefficients of the periodic extension of x_k. The inverse DFT recovers the finite-length sequence from its frequency samples:

$$x_k = \frac{1}{N}\sum_{r=0}^{N-1} X_r e^{j2\pi rk/N}, \qquad k = 0, 1, \ldots, N-1. \tag{5.6}$$

In practice, most signals x_k are not periodic, but we can still compute the DFT of a finite-length signal (by truncating a longer sequence). We can pretend x_k is "periodic", which allows us to compute the DFT.

5.2.4 Properties of the DFT

Linearity

The DFT is a linear transformation. If f_k and g_k are two sequences of length N with DFTs F_r and G_r, respectively, then for any scalars a and b, the DFT of the linear combination $af_k + bg_k$ is given by $aF_r + bG_r$.

Circular shift

Because some signal f_k is finite length, there is not a straightforward to shift it. Instead, we define a circular shift of f_k by m samples as

$$f_{(k-m) \mod N},$$

where mod is the modulo operation. Circular shifting essentially assumes that the signal is periodic with period N and shifts the samples accordingly, and then takes the first N samples. Alternatively, you can think of circular shifting as shifting the signal and wrapping around any samples that fall off the end back to the beginning. As an example, if $f_k = [1, 2, 3, 4]$ then a circular shift by $m = 1$ gives $f_{(k-1) \mod 4} = [4, 1, 2, 3]$.

The DFT of a circularly shifted sequence $f_{(k-m) \mod N}$ is given by

$$F_{(r-m) \mod N} = e^{-j2\pi mk/N} F_r.$$

5.2.5 Time-domain modulation and frequency-domain shift

Modulating a discrete-time signal f_k by a complex exponential $e^{j2\pi mk/N}$ in the time domain corresponds to a circular shift of its DFT F_r by m samples in the frequency domain:

$$\text{If } g_k = f_k e^{j2\pi mk/N}, \text{ then } G_r = F_{(r-m) \mod N}.$$

5.2.6 Circular convolution

Convolution for finite-length sequences is defined as

$$f_k \circledast g_k = \sum_{m=0}^{N-1} f_m g_{(k-m) \mod N}, \tag{5.7}$$

which is known as circular convolution. Circular convolution is different from linear convolution. First, we need to make sure that both sequences are the same length N (by zero-padding if necessary). One simple approach is to choose one signal, say f_k, and add periodic extensions, such that it has period N. Leave this signal in place. Then take the other signal and flip and drag it across the first signal, multiplying and summing as you go (shifting N times). Suppose we have two sequences $f_k = [1, 2, 3, 4]$ and $g_k = [1, 1, 1, 1, 1]$. To compute the 7-point circular convolution, we first make f_k periodic with period 7 by adding zeros: $f_k = [1, 2, 3, 4, 0, 0, 0]$. Next, we zero-pad g_k to length 7: $g_k = [1, 1, 1, 1, 1, 0, 0]$. Fig. 5.7 shows f_m with periodic extensions and $g_{-(m-k)}$ being flipped and dragged across f_m to compute the circular convolution. Based on (5.7), for $k = 0$, we have $y_k = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 4 + 0 \cdot 3 + 0 \cdot 2 = 5$. Continuing this process for $k = 1, 2, \ldots, 6$ gives the final result of the circular convolution as $y_k = [5, 3, 6, 10, 10, 9, 7]$. The circular convolution operation is related to the DFT. Given some time-domain sequences f_k and g_k with DFTs F_r and G_r, respectively, the DFT of their circular convolution is given by the pointwise product of their DFTs

$$\text{If } y_k = f_k \circledast g_k, \text{ then } Y_r = F_r G_r.$$

Linear convolution of discrete-time, finite-length sequences is defined as the convolution of the two sequences without any periodic extension. Specifically, if f_k has support for $k = 0, \ldots, L-1$

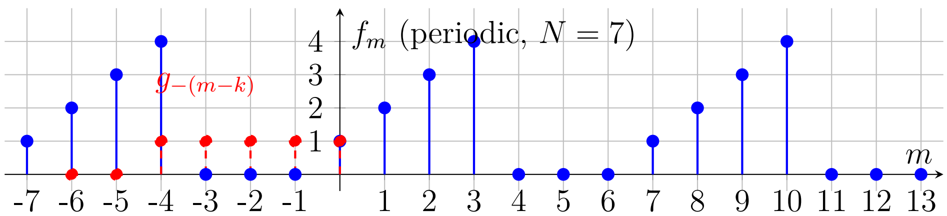


Figure 5.7: Computing the circular convolution of two finite-length sequences f_k and g_k by making one sequence periodic and dragging the flipped version of the other sequence across it.

(length L) and g_k has support for $k = 0, \ldots, M-1$ (length M), then their linear convolution $h_k = f_k * g_k$ has length $L + M - 1$. The linear convolution operation is given by

$$y_k = (f * g)_k = \sum_{m=-\infty}^{\infty} f_m g_{k-m}.$$

Linear convolution is related to circular convolution. Specifically, if we zero-pad both f_k and g_k to length $N \geq L + M - 1$ and then compute their N-point circular convolution, the result is equivalent to their linear convolution. Going back to our previous example, if we wanted to compute the linear convolution of $f_k = [1, 2, 3, 4]$ and $g_k = [1, 1, 1, 1, 1]$, we would first zero-pad both sequences to length $N = 8$ (since $4 + 5 - 1 = 8$). Thus, we have $f_k = [1, 2, 3, 4, 0, 0, 0, 0]$ and $g_k = [1, 1, 1, 1, 1, 0, 0, 0]$. Computing the 8-point circular convolution of these zero-padded sequences gives the same result as their linear convolution: $y_k = [1, 3, 6, 10, 10, 9, 7, 4]$. If we know the linear convolution of two sequences, we can compute any N-point circular convolution by simply taking the linear convolution result and folding it modulo N.

5.3 The Fast Fourier Transform (FFT)

The DFT is an invaluable tool for analyzing discrete signals in the frequency domain, but computing it directly from the definition can be computationally expensive for large sequences. The Fast Fourier Transform (FFT) is an efficient algorithm for computing the DFT and its inverse, significantly reducing the computational complexity from $O(N^2)$ to $O(N \log N)$.

The FFT exploits the symmetry and periodicity properties of the DFT to reduce the number of computations required. The most common FFT algorithm is the Cooley-Tukey algorithm, which recursively divides the DFT computation into smaller DFTs, ultimately reducing the overall complexity.

Essentially, the FFT works by breaking down a DFT of size N into two smaller DFTs of size $N/2$ (one for the even-indexed samples and one for the odd-indexed samples). This process is repeated recursively until the DFTs are reduced to size 1, which can be computed directly. The results of these smaller DFTs are then combined to produce the final DFT.

Suppose we have some sequence f_k of length N where N is a power of 2 (i.e. $N = 2^m$ for some integer m). The FFT algorithm can be summarized as follows:

1. If $N = 1$, return f_0 as the DFT.
2. Split the sequence f_k into two subsequences: one containing the even-indexed samples f_{2m} and the other containing the odd-indexed samples f_{2m+1}.

3. Recursively compute the DFTs of the even and odd subsequences, denoted as F_{even} and F_{odd}.

4. Combine the results of the two DFTs to obtain the DFT of the original sequence using the formula:
$$F_r = F_{\text{even},r} + e^{-j2\pi r/N} F_{\text{odd},r}, \quad r = 0, 1, \ldots, N/2 - 1,$$
and
$$F_{r+N/2} = F_{\text{even},r} - e^{-j2\pi r/N} F_{\text{odd},r}, \quad r = 0, 1, \ldots, N/2 - 1.$$
What is critical is that we only need to compute $e^{-j2\pi r/N} \cdot F_{\text{odd},r}$ once for each r and reuse it.

Chapter 6

Laplace Transform

6.1 Laplace Transform from a mathematical perspective

One shortcoming of the Fourier transform is that it is only defined for signals that are absolutely integrable (i.e., signals that decay sufficiently quickly as time goes to infinity). Many signals of interest in engineering, such as exponentially growing signals, do not satisfy this condition. The Laplace transform extends the Fourier transform by introducing a complex frequency variable, allowing it to handle a broader class of signals, including those that grow exponentially.

Consider some continuous-time function $f(t)$ defined for all values of t. Suppose we were to multilply $f(t)$ by an exponentially decaying term $e^{-\sigma t}$, where σ is a positive real number. The product $f(t)e^{-\sigma t}$ will decay more quickly than $f(t)$ alone, and may be absolutely integrable even if $f(t)$ is not. We can then take the Fourier transform of this product

$$\begin{aligned}\mathcal{F}\left\{f(t)e^{-\sigma t}\right\} &= \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt.\end{aligned}$$

By defining a new complex variable $s = \sigma + j\omega$, we can express the integral as

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

This integral is known as the **Laplace transform** of $f(t)$, and is typically denoted by $\mathcal{L}\{f(t)\}$. The variable s is called the **complex frequency variable**, and can be interpreted as a frequency variable ω shifted by a real component σ that controls the exponential decay (or growth) of the signal.

The Laplace transform can be viewed as a generalization of the Fourier transform. When $\sigma = 0$, the complex variable s reduces to the purely imaginary variable $j\omega$, and the Laplace transform reduces to the Fourier transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \mathcal{F}\{f(t)\}.$$

The Laplace transform is not always guaranteed to exist for all values of s. The set of values of s for which the Laplace transform converges is called the **region of convergence** (ROC). The ROC depends on the behavior of the original function $f(t)$, and is an important aspect of the Laplace transform. Typically, if you derive the forward Laplace transform of a function, you should also specify its ROC. See Section 6.1.1 for examples of Laplace transforms and their corresponding regions of convergence.

We can use similar reasoning to define the inverse Laplace transform. Starting with a function $F(s)$ in the complex frequency domain, we can multiply it by an exponentially growing term $e^{\sigma t}$ and then take the inverse Fourier transform

$$\begin{aligned}\mathcal{F}^{-1}\left\{F(s)e^{\sigma t}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\sigma t}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{(\sigma+j\omega)t}d\omega.\end{aligned}$$

By substituting $s = \sigma + j\omega$, we can express the integral as

$$f(t) = \frac{1}{2\pi j}\int_{c'-j\infty}^{c'+j\infty} F(s)e^{st}ds.$$

This integral is known as the **inverse Laplace transform** of $F(s)$, and is typically denoted by $\mathcal{L}^{-1}\{F(s)\}$. This essentially a line integral in the complex plane. The constant c' is a real number that must be chosen such that the integration path lies within the region of convergence of $F(s)$.

Bilateral versus unilateral Laplace transform

The Laplace transform can be defined in two forms: the bilateral (or two-sided) Laplace transform and the unilateral (or one-sided) Laplace transform. The bilateral Laplace transform is defined as

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt,$$

while the unilateral Laplace transform is defined as

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt.$$

The unilateral Laplace transform is often used in engineering applications, particularly in control systems and signal processing, where the focus is on causal systems (i.e., systems that do not respond before an input is applied). In such cases, the function $f(t)$ is typically defined to be zero for $t < 0$, making the unilateral Laplace transform more appropriate. The bilateral Laplace transform is more general and can be used for functions defined over the entire real line. In this class, we will primarily focus on the unilateral Laplace transform, as it is more commonly used in engineering applications.

The convergence properties of the bilateral and unilateral Laplace transforms differ in important ways. For the **bilateral Laplace transform**, the region of convergence (ROC) is typically a vertical strip in the complex plane of the form $\sigma_1 < \mathrm{Re}\{s\} < \sigma_2$, where σ_1 and σ_2 are real constants. This strip arises because the bilateral transform integrates over both positive and negative time. For the integral to converge, we need:

- The negative-time portion $\int_{-\infty}^{0} f(t)e^{-st}dt$ to converge, which typically requires $\mathrm{Re}\{s\} < \sigma_2$

- The positive-time portion $\int_0^\infty f(t)e^{-st}dt$ to converge, which typically requires $\text{Re}\{s\} > \sigma_1$

If these two constraints are incompatible (i.e., if $\sigma_1 \geq \sigma_2$), then the bilateral Laplace transform does not exist for any value of s.

For the **unilateral Laplace transform**, the ROC is always a right half-plane of the form $\text{Re}(s) > \sigma_0$ for some real constant σ_0. This simpler form occurs because the integration only covers $t \geq 0$, eliminating the constraint from negative time. The value of σ_0 depends on the growth rate of $f(t)$ as $t \to \infty$. For example:

- If $f(t)$ grows like e^{at}, then the ROC is $\text{Re}\{s\} > a$
- If $f(t)$ is bounded, then the ROC is $\text{Re}\{s\} > 0$
- If $f(t)$ decays exponentially, then the ROC may include $\text{Re}\{s\} > \sigma_0$ for some negative σ_0

Region of convergence (ROC) and the Fourier transform

The region of convergence (ROC) of the Laplace transform is closely related to the existence of the Fourier transform of a signal. Specifically, the Fourier transform of a signal exists if and only if the Laplace transform converges along the imaginary axis (i.e., when $s = j\omega$). This means that for the Fourier transform to exist, the ROC of the Laplace transform must include the line $\text{Re}\{\text{s}\} = 0$. If the ROC does not include this line, then the Fourier transform does not exist.

6.1.1 Laplace transform of common functions

In this section, we will derive the Laplace transforms of several common functions, along with their regions of convergence (ROC).

$\mathcal{L}\{e^{-at}u(t)\}$

Consider the function $f(t) = e^{-at}u(t)$, where $u(t)$ is the unit step function. The Laplace transform of this function is given by

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_0^\infty e^{-(a+s)t}dt \\ &= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^\infty \\ &= \frac{1}{s+a}, \quad \text{for Re}(s) > -a. \end{aligned}$$

To ensure convergence of this integral, we see the terms $e^{-(a+s)t}$ must decay as $t \to \infty$, which requires that $\text{Re}\{s\} > -a$, defining the region of convergence.

$\mathcal{L}\{u(t)\}$

Consider the function $f(t) = u(t)$, where $u(t)$ is the unit step function. The Laplace transform of this function is given by

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-st}dt. \end{aligned}$$

To evaluate this integral, we find

$$F(s) = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}, \quad \text{for } \mathrm{Re}(s) > 0.$$

The region of convergence is $\mathrm{Re}(s) > 0$ to ensure that the integral converges.

$\mathcal{L}\{\delta(t)\}$

Consider the function $f(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function. The Laplace transform of this function is given by

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} \delta(t)e^{-st}dt \\ &= e^{-s\cdot 0} = 1. \end{aligned}$$

The Laplace transform of the delta function is simply 1, and it converges for all values of s. Therefore, the region of convergence is the entire complex plane.

$\mathcal{L}\{t^n u(t)\}$

Consider the function $f(t) = t^n u(t)$, where n is a non-negative integer. The Laplace transform of this function is given by

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} t^n u(t)e^{-st}dt \\ &= \int_{0}^{\infty} t^n e^{-st}dt. \end{aligned}$$

To evaluate this integral, we can use integration by parts or recognize it as a standard integral. The result is

$$F(s) = \frac{n!}{s^{n+1}}, \quad \text{for } \mathrm{Re}\{s\} > 0.$$

The region of convergence is $\mathrm{Re}\{s\} > 0$ to ensure that the integral converges.

$\mathcal{L}\{\cos(\omega_0 t)u(t)\}$

Consider the function $f(t) = \cos(\omega_0 t)u(t)$, where ω_0 is a constant. To evaluate the Laplace transform of this function, we recognize that the Laplace transform is a linear operation (similar to the Fourier transform). Using Euler's formula, we can express the cosine function in terms of complex exponentials:

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}.$$

Thus, the Laplace transform becomes

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) u(t) e^{-st} dt \\ &= \int_{0}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-st} dt \\ &= \frac{1}{2} \left(\int_{0}^{\infty} e^{(j\omega_0 - s)t} dt + \int_{0}^{\infty} e^{(-j\omega_0 - s)t} dt \right). \end{aligned}$$

Evaluating these integrals, we find

$$F(s) = \frac{s}{s^2 + \omega_0^2}, \quad \text{for } \mathrm{Re}\{s\} > 0.$$

The region of convergence is $\mathrm{Re}\{s\} > 0$ to ensure that the integrals converge.

6.1.2 Table of common Laplace transforms

A table of common Laplace transforms is provided in Table 6.1.

6.1.3 Properties of the Laplace transform

The Laplace transform has several important properties that can be used to simplify the computation of transforms and inverse transforms. A summary of these properties is provided in Table 6.2.

6.1.4 Computing the inverse Laplace transform

Generally, if we are wanting to compute the inverse Laplace transform of some function $F(s)$, we will *not* compute the line integral directly. Instead, we will typically rely on a table of common Laplace transforms (such as Table 6.1) and properties of the Laplace transform to find the inverse transform.

The Laplace transform $F(s)$ is often expressed as a rational function (i.e., a ratio of two polynomials in s). In such cases, we can use **partial fraction expansion** to decompose $F(s)$ into a sum of simpler fractions, each of which corresponds to a known Laplace transform from the table. By taking the inverse Laplace transform of each term separately and then summing the results, we can obtain the inverse Laplace transform of the original function.

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{7s - 6}{s^2 - s - 6}.$$

First, we perform partial fraction expansion on $F(s)$. We start by factoring the denominator

$$F(s) = \frac{7s - 6}{(s - 3)(s + 2)}.$$

Table 6.1: A Short Table of (Unilateral) Laplace Transforms

	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}, \quad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{c-a^2}}$	
	$b=\sqrt{c-a^2}$	
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$	$\frac{As+B}{s^2+2as+c}$
	$b=\sqrt{c-a^2}$	

Table 6.2: The Laplace Transform Properties

Operation	$f(t)$	$F(s)$
Addition	$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$
Scalar multiplication	$kf(t)$	$kF(s)$
Time differentiation	$\dfrac{df}{dt}$	$sF(s) - f(0^-)$
	$\dfrac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - \dot{f}(0^-)$
	$\dfrac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - s\dot{f}(0^-) - \ddot{f}(0^-)$
Time integration	$\displaystyle\int_{0^-}^{t} f(\tau)\,d\tau$	$\dfrac{1}{s}F(s)$
	$\displaystyle\int_{-\infty}^{t} f(\tau)\,d\tau$	$\dfrac{1}{s}F(s) + \dfrac{1}{s}\displaystyle\int_{-\infty}^{0^-} f(t)\,dt$
Time shift	$f(t-t_0)u(t-t_0)$	$F(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shift	$f(t)e^{s_0t}$	$F(s-s_0)$
Frequency differentiation	$-tf(t)$	$\dfrac{dF(s)}{ds}$
Frequency integration	$\dfrac{f(t)}{t}$	$\displaystyle\int_{s}^{\infty} F(z)\,dz$
Scaling	$f(at), \quad a \geq 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Frequency convolution	$f_1(t)f_2(t)$	$\dfrac{1}{2\pi j}F_1(s) * F_2(s)$
Initial value	$f(0^+)$	$\displaystyle\lim_{s\to\infty} sF(s) \quad (n > m)$
Final value	$f(\infty)$	$\displaystyle\lim_{s\to 0} sF(s) \quad$ (poles of $sF(s)$ in LHP)

Next, we express $F(s)$ as a sum of simpler fractions

$$F(s) = \frac{a}{s-3} + \frac{b}{s+2}.$$

Next, we can solve for the constants a and b through different approaches. One approach is the Heavyside coverup method. To find a, we multiply both sides by $(s-3)$ and then evaluate at $s=3$

$$\begin{aligned} a = F(s)(s-3)\Big|_{s=3} &= \frac{7(3)-6}{(3+2)} \\ &= \frac{21-6}{5} \\ &= \frac{15}{5} \\ &= 3. \end{aligned}$$

To find b, we can use a similar approach. We multiply both sides by $(s+2)$ and then evaluate at $s=-2$

$$\begin{aligned} b = F(s)(s+2)\Big|_{s=-2} &= \frac{7(-2)-6}{(-2-3)} \\ &= \frac{-14-6}{-5} \\ &= \frac{-20}{-5} \\ &= 4. \end{aligned}$$

Thus, we have

$$F(s) = \frac{3}{s-3} + \frac{4}{s+2}.$$

Using the table of common Laplace transforms (Table 6.1), we can identify the inverse Laplace transform of each term. From entry 5 in the table, we see that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-\lambda}\right\} = e^{\lambda t}u(t).$$

Using this result, we find

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{3}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+2}\right\} \\ &= 3e^{3t}u(t) + 4e^{-2t}u(t) \\ &= \left(3e^{3t} + 4e^{-2t}\right)u(t). \end{aligned}$$

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{2s^2+5}{s^2+3s+2}.$$

To perform partial fraction expansion, we will need to recognize that the degree of the numerator is not less than the degree of the denominator. Thus, we will first need to perform polynomial long division to rewrite $F(s)$ as a sum of a polynomial and a proper rational function. Performing polynomial long division, we find

$$F(s) = 2 + \frac{-6s+1}{s^2+3s+2}.$$

Next, we can perform partial fraction expansion on the proper rational function. We start by factoring the denominator

$$F(s) = 2 + \frac{-6s+1}{(s+1)(s+2)}.$$

Next, we express the proper rational function as a sum of simpler fractions

$$\begin{aligned} F_1(s) &= \frac{-6s+1}{(s+1)(s+2)} \\ &= \frac{a}{s+1} + \frac{b}{s+2} \\ &= \frac{7}{s+1} - \frac{13}{s+2}. \end{aligned}$$

We can put this all together to get

$$F(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}.$$

Using the table of common Laplace transforms (Table 6.1), we can identify the inverse Laplace transform of each term

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\{2\} + \mathcal{L}^{-1}\left\{\frac{7}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{13}{s+2}\right\} \\ &= 2\delta(t) + 7e^{-t}u(t) - 13e^{-2t}u(t) \\ &= 2\delta(t) + \left(7e^{-t} - 13e^{-2t}\right)u(t). \end{aligned}$$

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)}.$$

When we factor the denominator, we see we have complex conjugate roots, thus the partial fraction will take the form

$$F(s) = \frac{a}{s} + \frac{b}{s+5-j3} + \frac{b^*}{s+5+j3}$$

where b^* is the complex conjugate of b. Using the Heavyside coverup method, we can find a by multiplying both sides by s and evaluating at $s = 0$

$$a = F(s)s\Big|_{s=0} = \frac{6(0+34)}{0^2+10(0)+34} = 6$$

To find b, we can multiply both sides by $(s+5-j3)$ and then evaluate at $s=-5+j3$

$$b = F(s)(s+5-j3)\Big|_{s=-5+j3} = -3+j4.$$

Thus, we have

$$F(s) = \frac{6}{s} + \frac{-3+j4}{s+5-j3} + \frac{-3-j4}{s+5+j3}.$$

We can use the standard Laplace transform pair $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$ to find the inverse Laplace transform of each term. For the complex conjugate terms, we can combine them using Euler's formula to express the result in terms of real-valued functions (sines and cosines)

$$\begin{aligned}
f(t) =& \mathcal{L}^{-1}\{F(s)\} \\
=& \mathcal{L}^{-1}\left\{\frac{6}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-3+j4}{s+5-j3}\right\} + \mathcal{L}^{-1}\left\{\frac{-3-j4}{s+5+j3}\right\} \\
=& 6u(t) + (-3+j4)e^{(-5+j3)t}u(t) + (-3-j4)e^{(-5-j3)t}u(t) \\
=& 6u(t) + e^{-5t}\left[(-3+j4)e^{j3t} + (-3-j4)e^{-j3t}\right]u(t) \\
=& 6u(t) + e^{-5t}\left[-3e^{j3t} - 3e^{-j3t} + 4je^{j3t} - 4je^{-j3t}\right]u(t) \\
=& \left[6 - 2e^{-5t}\left(3\cos(3t) + 4\sin(3t)\right)\right]u(t).
\end{aligned}$$

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{8s+10}{(s+1)(s+2)^3}.$$

To perform partial fraction expansion, we express $F(s)$ as a sum of simpler fractions

$$F(s) = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{(s+2)^2} + \frac{d}{(s+2)^3}.$$

We can find a and d directly using the traditional Heavyside coverup method. To find a, we multiply both sides by $(s+1)$ and then evaluate at $s=-1$

$$a = F(s)(s+1)\Big|_{s=-1} = 2$$

To find d, we multiply both sides by $(s+2)^3$ and then evaluate at $s=-2$

$$d = F(s)(s+2)^3\Big|_{s=-2} = 6.$$

To find b and c, we will need to be more thoughtful. One of the easiest approaches is to multiply both sides by s and then take the limit as $s \to \infty$. Doing so, we find

$$\begin{aligned}
\lim_{s\to\infty} sF(s) &= \lim_{s\to\infty} s\left(\frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{(s+2)^2} + \frac{d}{(s+2)^3}\right) \\
&= \lim_{s\to\infty}\left(a\frac{s}{s+1} + b\frac{s}{s+2} + c\frac{s}{(s+2)^2} + d\frac{s}{(s+2)^3}\right) \\
&= a+b.
\end{aligned}$$

Evaluating the left-hand side, we find

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{8s^2+10s}{(s+1)(s+2)^3} = 0.$$

Thus, we have

$$a + b = 0 \implies b = -a = -2.$$

Now we know a, b, and d. To find c, we can simply let $s = 0$ or some other convenient value. Letting $s = 0$, on the left-hand side we find

$$\begin{aligned} F(0) &= \frac{8(0)+10}{(0+1)(0+2)^3} \\ &= \frac{10}{8} \\ &= \frac{5}{4} \end{aligned}$$

On the right-hand side, we have

$$\begin{aligned} F(0) &= \frac{a}{0+1} + \frac{b}{0+2} + \frac{c}{(0+2)^2} + \frac{d}{(0+2)^3} \\ &= a + \frac{b}{2} + \frac{c}{4} + \frac{d}{8} \end{aligned}$$

Setting these equal, we have

$$\begin{aligned} \frac{5}{4} &= 2 + \frac{-2}{2} + \frac{c}{4} + \frac{6}{8} \\ \frac{5}{4} &= 2 - 1 + \frac{c}{4} + \frac{3}{4} \\ 5 &= 8 - 4 + c + 3 \\ \implies c &= -2. \end{aligned}$$

Thus, we have

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{6}{(s+2)^3}.$$

Using the table of common Laplace transforms (Table 6.1), we can identify the inverse Laplace transform of each term

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s+2)^3}\right\} \\ &= 2e^{-t}u(t) - 2e^{-2t}u(t) - 2te^{-2t}u(t) + 3t^2e^{-2t}u(t) \\ &= \left[2e^{-t} - \left(3t^2 - 2t - 2\right)e^{-2t}\right]u(t). \end{aligned}$$

6.2 Properties of the Laplace transform

The Laplace transform possesses several important properties that make it a powerful tool for analyzing and solving linear time-invariant (LTI) systems. Many of these are similar to those in Sec. **??** for the Fourier transform, but there are some key differences.

6.2.1 Linearity

The Laplace transform is a linear operation. This means that for any two functions $f_1(t)$ and $f_2(t)$, and any two constants a_1 and a_2, the Laplace transform of their linear combination is given by

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathcal{L}\{f_1(t)\} + a_2 \mathcal{L}\{f_2(t)\}.$$

6.2.2 Time shifting

If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of the time-shifted function $f(t-t_0)u(t-t_0)$ is given by

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s).$$

We need to be careful when applying this property. The time shift must be applied to a causal function (i.e., multiplied by the unit step function $u(t-t_0)$) to ensure that the Laplace transform exists. In other words, all values of t need to be delayed by t_0, for both the function itself and the unit step function.

Example:

Suppose we have a ramp and a box function defined as

$$f(t) = (t-1)\left[u(t-1) - u(t-2)\right] + \left[u(t-2) - u(t-4)\right].$$

We can expand this out such that

$$\begin{aligned} f(t) &= tu(t-1) - tu(t-2) - u(t-1) + u(t-2) + u(t-2) - u(t-4) \\ &= tu(t-1) - tu(t-2) - u(t-1) + 2u(t-2) - u(t-4) \\ &= (t-1)u(t-1) - (t-2)u(t-2) - u(t-4) \end{aligned}$$

Notice that both ramp functions $(t-1)u(t-1)$ and $(t-2)u(t-2)$ are time-shifted versions of the basic ramp function $tu(t)$ (in the sense that both t and $u(t)$ are shifted by the same amount). We can apply the time-shifting property to find the Laplace transform of each term

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \mathcal{L}\{(t-1)u(t-1)\} - \mathcal{L}\{(t-2)u(t-2)\} - \mathcal{L}\{u(t-4)\} \\ &= \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s} \\ &= \frac{e^{-s} - e^{-2s}}{s^2} - \frac{e^{-4s}}{s}. \end{aligned}$$

6.2.3 Frequency shifting

If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of the function $e^{at}f(t)$ is given by

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

This property is useful for analyzing systems with exponential growth or decay, as it allows us to easily account for these effects in the Laplace domain.

Example:

Suppose we have a function defined as

$$f(t) = e^{-at}\cos(bt)u(t).$$

To find the Laplace transform of this function, we can use the frequency-shifting property. From Table 6.1, we know that the Laplace transform of $\cos(bt)u(t)$ is given by

$$\mathcal{L}\{\cos(bt)u(t)\} = \frac{s}{s^2+b^2}, \quad \text{for } \mathrm{Re}\{s\} > 0.$$

Using the frequency-shifting property, we find

$$F(s) = \mathcal{L}\{e^{-at}\cos(bt)u(t)\} = \frac{s+a}{(s+a)^2+b^2}, \quad \text{for } \mathrm{Re}\{s\} > -a.$$

6.2.4 Differentiation in the time domain

If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of its derivative $f'(t)$ is given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-).$$

This property is particularly useful for solving differential equations, as it allows us to convert derivatives in the time domain into algebraic expressions in the Laplace domain. We can extend this property to higher-order derivatives as well. Consider the second derivative of $f(t)$, denoted as $f''(t)$. The Laplace transform is given by

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0^-) - f'(0^-).$$

Continuing this pattern, we can generalize to the n-th derivative. For the n-th derivative of $f(t)$, denoted as $f^{(n)}(t)$, the Laplace transform is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \ldots - f^{(n-1)}(0^-).$$

This property is extremely useful for solving higher-order differential equations, as it allows us to systematically convert derivatives into algebraic terms in the Laplace domain, making it easier to manipulate and solve for the unknown function.

Example:

Consider the second derivative of a function $f(t)$, denoted as $f''(t)$,

$$\frac{d^2f(t)}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3).$$

We can the Laplace transform of both sides of this equation. Using the property of differentiation in the time domain, we have

$$s^2F(s) - sf(0^-) - f'(0^-) = 1 - 3e^{-2s} + 2e^{-3s}.$$

Assuming that the initial conditions are zero (i.e., $f(0^-) = 0$ and $f'(0^-) = 0$), we can simplify this to

$$s^2F(s) = 1 - 3e^{-2s} + 2e^{-3s}.$$

Solving for $F(s)$, we find

$$F(s) = \frac{1 - 3e^{-2s} + 2e^{-3s}}{s^2}.$$

To find the inverse Laplace transform of $F(s)$, we can use the linearity property

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} \\ &= tu(t) - 3(t-2)u(t-2) + 2(t-3)u(t-3). \end{aligned}$$

6.2.5 Integration in the time-domain

If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of its integral $\int_0^t f(\tau)d\tau$ is given by

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}.$$

Alternatively, we can express this property as

$$\mathcal{L}\left\{\int_{-\infty}^t f(\tau)d\tau\right\} = \frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^0 f(\tau)d\tau,$$

which accounts for the case where the function $f(t)$ is not causal. This property is useful for analyzing systems where integration plays a role.

6.2.6 Time-domain scaling

If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of the time-scaled function $f(at)$, where a is a positive constant, is given by

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

This property is useful for analyzing systems that involve time-scaling operations, such as compression or expansion of signals. Notice there is a factor of $\frac{1}{a}$ that appears in front of the Laplace transform. This is different from the Fourier transform time-scaling property (see Sec. **??**), where the scaling factor is $\frac{1}{|a|}$. This difference arises because the Laplace transform is typically defined for causal functions (i.e., functions that are zero for $t < 0$), whereas the Fourier transform is defined for functions that can be non-zero for all time.

6.2.7 Convolution in the time-domain

If two functions $f_1(t)$ and $f_2(t)$ have Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, then the Laplace transform of their convolution $f_1(t) * f_2(t)$ is given by

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s).$$

This property is particularly useful for analyzing linear time-invariant (LTI) systems, as it allows us to easily compute the output of a system given its impulse response and an input signal.

6.2.8 Multiplication in the time-domain

If two functions $f_1(t)$ and $f_2(t)$ have Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, then the Laplace transform of their product $f_1(t)f_2(t)$ is given by

$$\mathcal{L}\{f_1(t)f_2(t)\} = \frac{1}{2\pi j}\int_{c'-j\infty}^{c'+j\infty} F_1(\sigma)F_2(s-\sigma)d\sigma,$$

where c' is a real constant chosen such that the contour of integration lies within the region of convergence of both $F_1(\sigma)$ and $F_2(s-\sigma)$.

6.2.9 Table of common Laplace transform properties

A table of common Laplace transform properties is provided in Table 6.2.

6.3 Laplace transforms and LTI systems

Suppose we have an LTI system characterized by its impulse response $h(t)$. If we apply an input signal $f(t)$ to this system we will get some output signal $y(t)$. Thus the input-ouptut relationship of the system can be expressed as

$$y(t) = H\{f(t)\}$$

where $H\{\cdot\}$ is the operator that represents the action of the LTI system on the input signal. Suppose the input to our system is represented by the inverse Laplace transform of some function $F(s)$, i.e.,

$$\begin{aligned} f(t) &= \frac{1}{2\pi j}\int_{c'-j\infty}^{c'+j\infty} F(s)e^{st}ds \\ &= \lim_{\Delta s\to 0}\sum_{n=-\infty}^{\infty}\frac{F(n\Delta s)\Delta s}{2\pi j}e^{(n\Delta s)t} \end{aligned}$$

The system output of a single exponential input $e^{(n\Delta s)t}$ is given by

$$H\{e^{(n\Delta s)t}\} = H(n\Delta s)e^{(n\Delta s)t}.$$

The output of the system can be expressed as

$$\begin{aligned} y(t) &= \lim_{\Delta s\to 0}\sum_{n=-\infty}^{\infty}\frac{F(n\Delta s)\Delta s}{2\pi j}H(n\Delta s)e^{(n\Delta s)t} \\ &= \frac{1}{2\pi j}\int_{c'-j\infty}^{c'+j\infty} F(s)H(s)e^{st}ds \end{aligned}$$

Thus, we see that the Laplace transform of the output signal is given by

$$Y(s) = F(s)H(s).$$

In other words, the Laplace transform of the output signal is simply the product of the Laplace transform of the input signal and the system's transfer function $H(s)$. Additionally, in the time-domain, the system output is $y(t) = f(t) * h(t)$, i.e., the convolution of the input signal and the system's impulse response. We also see that the system's transfer function is the Laplace transform of its impulse response, i.e., $H(s) = \mathcal{L}\{h(t)\}$.

6.4 System response analysis using the Laplace transform

Up until now, we used time-domain methods to analyze the response of LTI systems represented as linear constant-coefficient differential equations. Consider some second order LTI system represented by the differential equation

$$\frac{d^(n)y}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_n\frac{d^n f(t)}{dt^n} + b_{n-1}\frac{d^{n-1}f(t)}{dt^{n-1}} + \ldots + b_1\frac{df(t)}{dt} + b_0 f(t).$$

We could first solve for the system's zero-input response $y_{zi}(t)$ by solving the corresponding homogeneous differential equation. Then we would need to determine the impulse function $h(t)$ of the system using $h(t) = b_n\delta(t) + P(D)y_{zi}(t)u(t)$, where $P(D)$ is the differential operator on the right-hand side of the differential equation. Finally, we could compute the zero-state response $y_{zs}(t)$ by convolving the impulse response with the input signal, i.e., $y_{zs}(t) = h(t) * f(t)$. The overall system response would then be given by $y(t) = y_{zi}(t) + y_{zs}(t)$. While this approach works, it can be quite tedious and time-consuming. Instead, we can use the Laplace transform to analyze the system response in a more straightforward manner.

Consider some LTI represented by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{df(t)}{dt} + f(t)$$

where the system's input is given as $f(t) = e^{-4t}u(t)$ and initial conditions $y(0^-) = 2$ and $y'(0^-) = 1$. We can take the Laplace transform of both sides of the differential equation. Using the differentiation property of the Laplace transform, we can find the Laplace transform of each term. Taking the Laplace transform of the left-hand side, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} &= s^2Y(s) - sy(0^-) - y'(0^-) = s^2Y(s) - 2s - 1 \\
\mathcal{L}\left\{5\frac{dy(t)}{dt}\right\} &= 5\left(sY(s) - y(0^-)\right) = 5sY(s) - 10 \\
\mathcal{L}\left\{6y(t)\right\} &= 6Y(s)
\end{aligned}$$

On the right-hand side, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= sF(s) - f(0^-) = sF(s) \\
\mathcal{L}\left\{f(t)\right\} &= F(s)
\end{aligned}$$

We can find the Laplace transform of the input signal $f(t)$ as

$$F(s) = \mathcal{L}\{e^{-4t}u(t)\} = \frac{1}{s+4}, \quad \text{for } \mathrm{Re}\{s\} > -4.$$

Putting this all together, we have

$$s^2Y(s) - 2s - 1 + 5sY(s) - 10 + 6Y(s) = s\left(\frac{1}{s+4}\right) + \frac{1}{s+4}.$$

Combining like terms, we find

$$Y(s)\left(s^2 + 5s + 6\right) \underbrace{-2s - 11}_{\text{initial conditions}} = \underbrace{\frac{s+1}{s+4}}_{\text{input}}.$$

Solving for $Y(s)$, we have

$$Y(s) = \underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state}} + \underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{zero-input}}.$$

We see here that both the zero-state and zero-input responses are clearly identified in the Laplace domain. To find the overall system response $y(t)$, we can take the inverse Laplace transform of $Y(s)$. We can perform partial fraction expansion on each term separately to make finding the inverse Laplace transform easier. Starting with the zero-state response, we have

$$\begin{aligned}
\frac{s+1}{(s+4)(s^2+5s+6)} &= \frac{s+1}{(s+4)(s+2)(s+3)} \\
&= \frac{a}{s+4} + \frac{b}{s+3} + \frac{c}{s+2} \\
&= \frac{-1/2}{s+2} + \frac{2}{s+3} + \frac{-3/2}{s+4}.
\end{aligned}$$

Using this result, we find

$$\begin{aligned}
y_{zs}(t) &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+4)(s^2+5s+6)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-1/2}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{-3/2}{s+4}\right\} \\
&= -\frac{1}{2}e^{-2t}u(t) + 2e^{-3t}u(t) - \frac{3}{2}e^{-4t}u(t) \\
&= \left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t).
\end{aligned}$$

Next, we can perform partial fraction expansion on the zero-input response

$$\begin{aligned}
\frac{2s+11}{s^2+5s+6} &= \frac{2s+11}{(s+2)(s+3)} \\
&= \frac{a}{s+3} + \frac{b}{s+2} \\
&= \frac{7}{s+2} - \frac{5}{s+3}.
\end{aligned}$$

Using this result, we find

$$\begin{aligned}
y_{zi}(t) &= \mathcal{L}^{-1}\left\{\frac{2s+11}{s^2+5s+6}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{7}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\} \\
&= 7e^{-2t}u(t) - 5e^{-3t}u(t) \\
&= \left(7e^{-2t} - 5e^{-3t}\right)u(t).
\end{aligned}$$

Thus, the overall system response is given by

$$\begin{aligned} y(t) &= y_{zs}(t) + y_{zi}(t) \\ &= \left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t) + \left(7e^{-2t} - 5e^{-3t}\right)u(t) \\ &= \left(\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t). \end{aligned}$$

6.4.1 The transfer function $H(s)$

From the previous example, we saw that the system's zero-state response in the Laplace domain could be represented as

$$Y_{zs}(s) = \frac{s+1}{s^2+5s+6} \cdot \frac{1}{s+4}$$

We recall that the $\frac{1}{s+4}$ term is the Laplace transform of the input signal $f(t) = e^{-4t}u(t)$. The remaining term

$$H(s) = \frac{s+1}{s^2+5s+6}$$

is known as the system's **transfer function**. The transfer function is the Laplace transform of the system's impulse response $h(t)$, i.e., $H(s) = \mathcal{L}\{h(t)\}$. The transfer function completely characterizes the LTI system in the Laplace domain. Given the transfer function, we can easily compute the system's output for any input signal by multiplying the Laplace transform of the input signal with the transfer function.

To quickly identify the transfer function from a linear constant-coefficient differential equation, we can take the Laplace transform of both sides of the differential equation, assuming zero initial conditions. The transfer function is then given by the ratio of the Laplace transform of the output to the Laplace transform of the input. Given some generic differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_n\frac{d^n f(t)}{dt^n} + b_{n-1}\frac{d^{n-1}f(t)}{dt^{n-1}} + \ldots + b_1\frac{df(t)}{dt} + b_0 f(t),$$

we can take the Laplace transform of both sides to obtain

$$s^nY(s)+a_{n-1}s^{n-1}Y(s)+\ldots+a_1sY(s)+a_0Y(s) = b_ns^nF(s)+b_{n-1}s^{n-1}F(s)+\ldots+b_1sF(s)+b_0F(s).$$

We can factor out $Y(s)$ and $F(s)$ to find

$$Y(s)\left(s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0\right) = F(s)\left(b_ns^n + b_{n-1}s^{n-1} + \ldots + b_1s + b_0\right).$$

Thus, the transfer function is given by

$$H(s) = \frac{Y(s)}{F(s)} = \frac{b_ns^n + b_{n-1}s^{n-1} + \ldots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0}.$$

6.4.2 Application: circuit analysis using the Laplace transform

A classic application of the Laplace transform is in the analysis of electrical circuits, particularly those involving resistors, inductors, and capacitors (RLC circuits). The Laplace transform allows us to convert the differential equations governing the circuit's behavior into algebraic equations

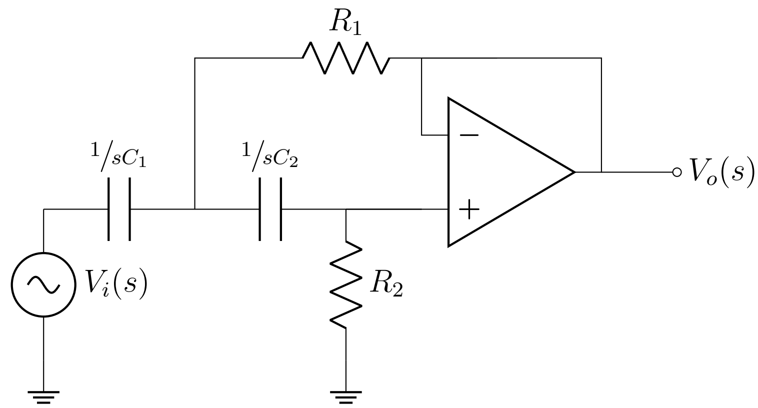


Figure 6.1: High-pass Sallen-Key filter circuit.

in the Laplace domain, making it easier to analyze and solve for the circuit's response to various inputs.

Earlier, we described each circuit in the time-domain, and that is represented in Fig. **??**. Using the Laplace transform, we can represent each circuit element in the Laplace domain, as shown in Fig. 3.5. We can take the Laplace transform of the voltage-current relationships for each circuit element to obtain their corresponding representations in the Laplace domain.

Table 6.3: Laplace-domain representations of R, L, and C

Element	Time-domain relation	Laplace-domain (with ICs)	Impedance $Z(s)$ (ICs = 0)
Resistor R	$v(t) = R\,i(t)$	$V(s) = R\,I(s)$	R
Inductor L	$v(t) = L\,\dfrac{di(t)}{dt}$	$V(s) = L[sI(s) - i(0^-)]$	sL
Capacitor C	$i(t) = C\,\dfrac{dv(t)}{dt}$	$I(s) = C[sV(s) - v(0^-)]$	$\dfrac{1}{sC}$

In this class, we will primarily focus on the zero-state response of circuits. We will often assume zero initial conditions when applying the Laplace transform to circuit equations, which simplifies the analysis. Essentially, we can treat inductors and capacitors as having impedances of sL and $\frac{1}{sC}$, respectively, when analyzing circuits in the Laplace domain. This allows us to use familiar circuit analysis techniques, such as mesh analysis and nodal analysis, to solve for voltages and currents in the circuit.

This is best understood with an example. Consider a high-pass Sallen-Key filter circuit shown in Fig. 6.1 (already drawn in the s-domain). We can analyze this circuit using the Laplace transform to find its transfer function $H(s) = \frac{V_o(s)}{V_i(s)}$.

This procedure is mostly some node-voltage analysis and some tedious algebra. First, we can write a node-voltage equation at node $V_x(s)$ which is the node between C_1 and C_2. We have

$$\frac{V_x(s) - V_i(s)}{^1/_{sC_1}} + \frac{V_x(s) - V_o(s)}{^1/_{sC_2}} + \frac{V_x(s) - V_o(s)}{R_1} = 0.$$

We will need a second equation to eliminate the $V_x(s)$ variable, so we can spot a voltage divider at the positive input of the op-amp, which is $V_o(s)$ based on the properties of op-amps. Thus, we

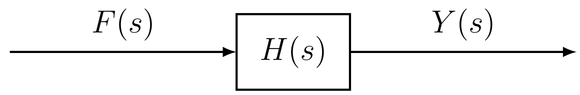


Figure 6.2: High-level block diagram representation of an LTI system in the s-domain.

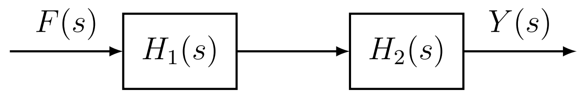


Figure 6.3: Cascade block diagram representation of an LTI system in the s-domain.

have

$$V_o(s) = V_x(s) \cdot \frac{R_2}{R_2 + {}^1/_{sC_2}}.$$

Solving for $V_x(s)$, and inserting this into the first equation, we can solve for the transfer function $H(s) = \frac{V_o(s)}{V_i(s)}$. After some algebraic manipulation, we find

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s^2}{s^2 + s\left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right) + \frac{1}{R_1R_2C_1C_2}}.$$

6.5 Block diagrams

It is often useful to represent LTI systems using block diagrams. A block diagram is a graphical representation of a system that shows the relationships between its components and how they interact with each other. In a block diagram, each component of the system is represented by a block, and the connections between the blocks represent the flow of signals through the system.

6.5.1 Higher level block diagrams

We consistently seen the input/output representation of LTI systems, where the system is represented as a single block with an input and output. This is a high-level block diagram representation of the system, as shown in Fig. 6.2 (though this figure is now represented in the Laplace/s-domain compared to the time-domain).

We can decompose this high-level block diagram into lower-level block diagrams that represent the internal structure of the system. There are three varieties that we should be familiar with.

Cascade: In a cascade block diagram, the system is represented as a series of interconnected blocks, where the output of one block serves as the input to the next block. This representation is useful for systems that can be decomposed into a series of simpler subsystems. This is seen in Fig. 6.3 where two smaller transfer functions $H_1(s)$ and $H_2(s)$ are cascaded to form the overall system transfer function $H(s) = H_2(s)H_1(s)$. It is worth noting that the commutative property is valid here as well, i.e., $H(s) = H_2(s)H_1(s) = H_1(s)H_2(s)$, so the order of the blocks can be interchanged without affecting the overall system transfer function.

Parallel: In a parallel block diagram, the system is represented as a set of interconnected blocks that operate simultaneously on the input signal. The outputs of each block are then summed together to produce the overall output of the system. This representation is useful for systems

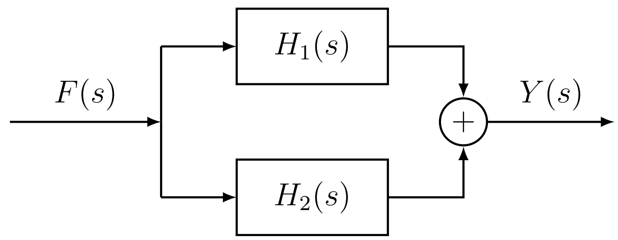


Figure 6.4: Parallel block diagram representation of an LTI system in the s-domain.

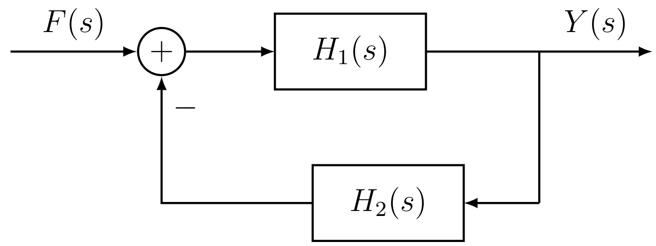


Figure 6.5: Feedback block diagram representation of an LTI system in the s-domain.

that can be decomposed into parallel subsystems. This is seen in Fig. 6.4 where two smaller transfer functions $H_1(s)$ and $H_2(s)$ operate in parallel to form the overall system transfer function $H(s) = H_1(s) + H_2(s)$.

Feedback: In a feedback block diagram, the system is represented as a set of interconnected blocks that include a feedback loop. The output of the system is fed back into the input through a feedback block, which modifies the input signal based on the output signal. This representation is useful for systems that require feedback control to maintain stability or achieve desired performance. This is seen in Fig. 6.5 where a transfer function $H_1(s)$ is in the forward path and a transfer function $H_2(s)$ is in the feedback path. The overall system transfer function is given by $H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$.

6.5.2 Lower level block diagrams (Canonical/Direct Form II representation)

In addition to the higher-level block diagrams discussed earlier, we can also represent LTI systems using lower-level block diagrams that show the internal structure of the system in more detail. This is particularly useful for systems that can be represented as linear constant-coefficient differential equations. Without the loss of generality, we can consider some third order LTI system represented by the differential equation

$$\frac{d^3y(t)}{dt^3} + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_3\frac{d^3f(t)}{dt^3} + b_2\frac{d^2f(t)}{dt^2} + b_1\frac{df(t)}{dt} + b_0f(t).$$

Solving for the input/output relationship in the s-domain, we have

$$Y(s) = \underbrace{\frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}}_{H(s)} F(s).$$

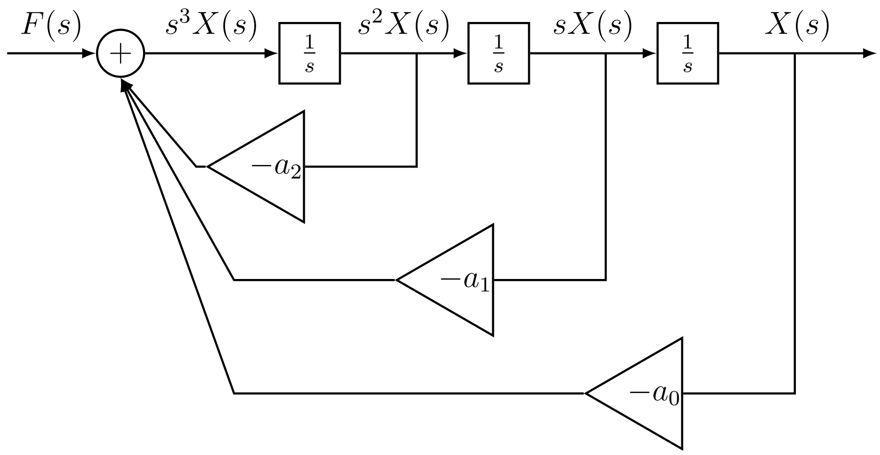


Figure 6.6: Lower-level block diagram representation of an LTI system in the s-domain (first half).

We can further break this down into a cascade of two blocks: one block representing the numerator and another block representing the denominator

$$Y(s) = \underbrace{\left(b_3s^3 + b_2s^2 + b_1s + b_0\right)}_{H_1(s)} \cdot \underbrace{\underbrace{\frac{1}{s^3 + a_2s^2 + a_1s + a_0}}_{H_2(s)} F(s)}_{X(s)}$$

where the intermediate variable $X(s)$ represents the output of the first block and the input to the second block such that

$$X(s) = \frac{1}{s^3 + a_2s^2 + a_1s + a_0} F(s).$$

. We can manipulate this further to express it as a feedback system such that

$$\begin{aligned} &X(s)\left(s^3 + a_2s^2 + a_1s + a_0\right) = F(s) \\ \implies\ &s^3X(s) = F(s) - \left(a_2s^2X(s) + a_1sX(s) + a_0X(s)\right). \end{aligned}$$

We can visualize this in the block diagram shown in Fig. 6.6. In this figure, the $1/s$ blocks represent integrators in the time-domain. The triangle blocks represent some scaling (amplification) operation by the constant value inside the block.

Now consider the first block representing the numerator. We can express this as

$$Y(s) = \left(b_3s^3 + b_2s^2 + b_1s + b_0\right) X(s).$$

We can express this further as

$$Y(s) = b_3s^3X(s) + b_2s^2X(s) + b_1sX(s) + b_0X(s).$$

We can add these terms to the block diagram in Fig. 6.7 to complete the lower-level block diagram representation of the LTI system.

The procedure to create lower-level block diagrams from linear constant-coefficient differential equations is as follows:

1. Have the system input $F(s)$ enter a summer block.

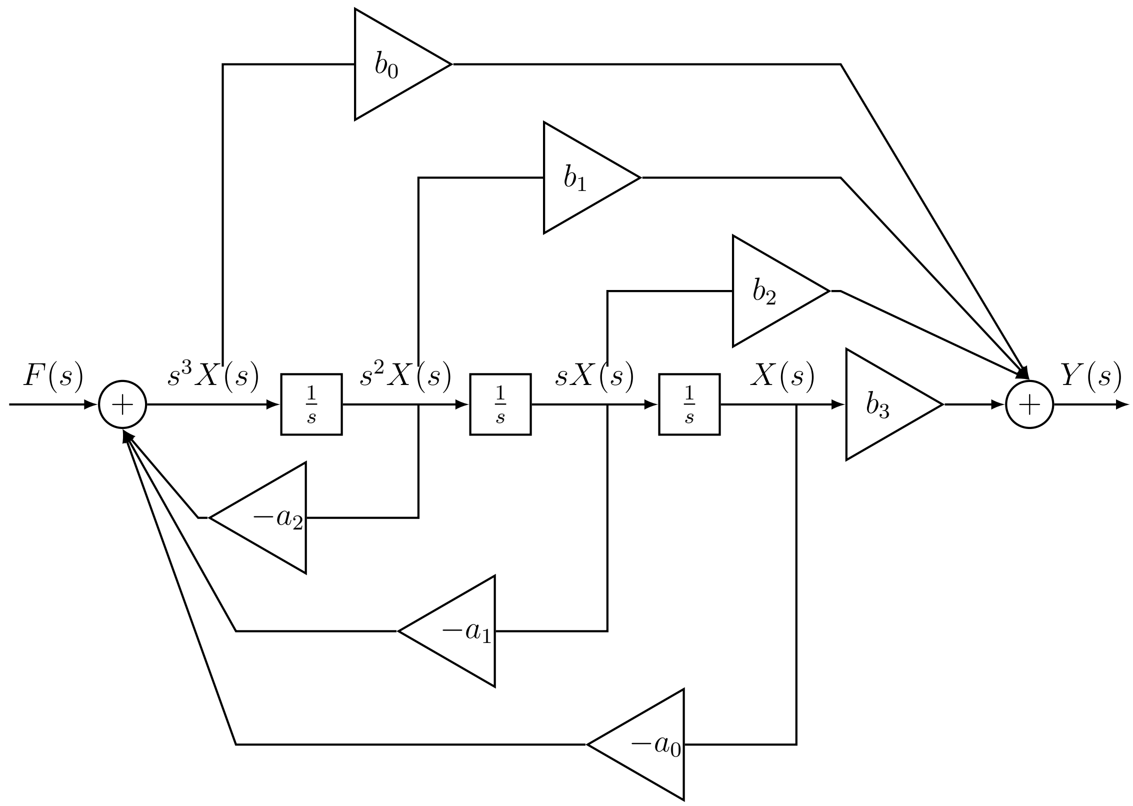


Figure 6.7: Lower-level block diagram representation of an LTI system in the s-domain.

2. The output of the summer block enters a series of integrator blocks equal to the order of the system.

3. The output of each integrator block is fed back into the summer block through a scaling block representing the coefficients of the denominator of the transfer function (with negative signs).

4. The output of each integrator block is also fed into scaling blocks representing the coefficients of the numerator of the transfer function into an output summer.

5. The outputs of the scaling blocks are summed together to produce the system output $Y(s)$.

Chapter 7

Analog Filter Design

7.1 System frequency response

The frequency response of a system describes how the system modifies the amplitude and phase of input signals at different frequencies. It is typically represented as a complex function $H(j\omega)$, where ω is the angular frequency in radians per second.

If you recall from Sec. 3.7, if some LTI system has an input $f(t) = e^{st}$, where $s = \sigma + j\omega$ is a complex frequency, then the output is given by

$$y(t) = H(s)e^{st},$$

where $H(s)$ is the system's transfer function evaluated at s. Essentially, the system itself acts as a complex gain $H(s)$ that scales the sinusoids magnitude and shifts the phase of the input signal.

Suppose we were to let $s \to j\omega$, which corresponds to a sinusoidal input with no exponential growth or decay. In this case, the output becomes

$$y(t) = H(j\omega)e^{j\omega t}$$

where $H(j\omega)$ is the system's frequency response. Similarly, if we were to let $s \to -j\omega$, we would have

$$y(t) = H(-j\omega)e^{-j\omega t}.$$

Because the system is LTI, if you were let $f(t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) = \cos(\omega t)$, then the output would be

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right) \\ &= |H(j\omega)| \cos\left(\omega t + \angle H(j\omega)\right). \end{aligned}$$

Note, we are able to combine the two $H(j\omega)$ and $H(-j\omega)$ terms because they are complex conjugates of each other for real-valued systems. Thus, the system scales the amplitude of the cosine wave by $|H(j\omega)|$ and shifts its phase by $\angle H(j\omega)$. We have seen this before in your introduction to circuits course when analyzing steady-state AC circuits using phasors. In that context, we were using a single frequency ω. By contrast, here we are interested in how the system behaves over a range of frequencies.

Example:

Suppose we have some system with transfer function

$$H(s) = \frac{s+0.1}{s+5}.$$

If we had an input signal $f(t) = \cos(2t)$, then the output would be

$$y(t) = |H(j2)| \cos\left(2t + \angle H(j2)\right)$$

where

$$|H(j2)| = \left|\frac{j2+0.1}{j2+5}\right| = 0.372,$$
$$\angle H(j2) = \angle(j2+0.1) - \angle(j2+5) = 65.3°.$$

Thus, the output is

$$y(t) = 0.372\cos(2t + 65.3°).$$

If we were to use another input signal $f(t) = \cos(10t)$, then the output would

$$y(t) = |H(j10)| \cos\left(10t + \angle H(j10)\right)$$

where

$$|H(j10)| = \left|\frac{j10+0.1}{j10+5}\right| = 0.894,$$
$$\angle H(j10) = \angle(j10+0.1) - \angle(j10+5) = 25.99°.$$

Thus, the output is

$$y(t) = 0.894\cos(10t + 25.99°).$$

We notice that for the two inputs, the output is a cosine wave at the same frequency as the input, but with different amplitudes and phase shifts. We can plot the frequency response to see how the system behaves over a range of frequencies. Often this is plotted in terms of frequency ω (radians per second) or f (Hertz), where $\omega = 2\pi f$ on the x-axis. The magnitude $|H(j\omega)|$ is often plotted in decibels (dB), where

$$|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)|.$$

The phase $\angle H(j\omega)$ is typically plotted in degrees or radians on the y-axis. Figure 7.1 shows the frequency response for the example system.

7.2 Bode plots

Bode plots are a common way to represent the frequency response of a system. They consist of two plots: one for the magnitude (in dB) and one for the phase (in degrees or radians) versus frequency (usually on a logarithmic scale). Bode plots are particularly useful for analyzing the behavior of filters and control systems.

To construct Bode plots, we typically break down the transfer function $H(s)$ into its constituent parts, such as poles, zeros, and gain factors. Each part contributes to the overall frequency response predictably, allowing us to sketch the Bode plots by combining these contributions.

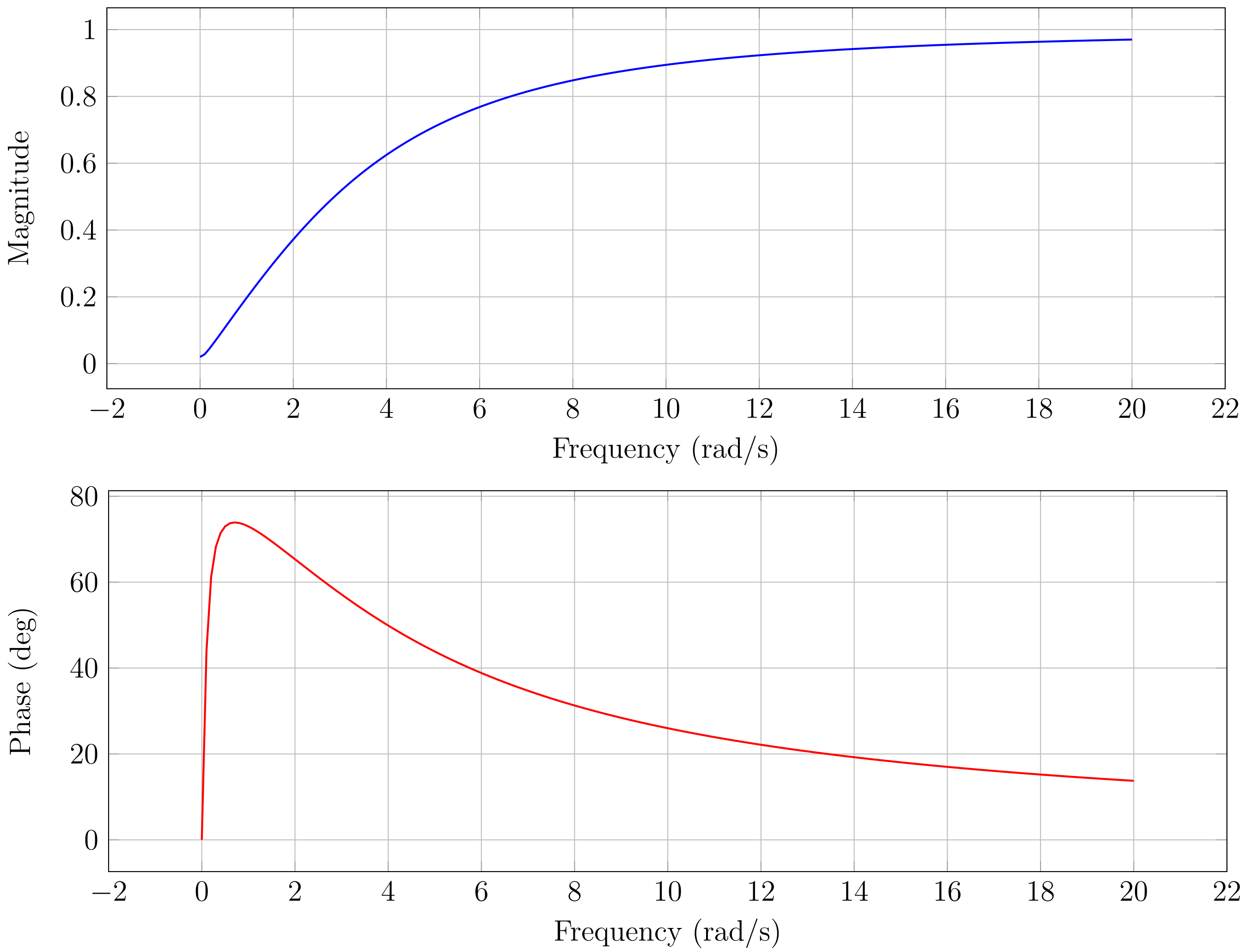


Figure 7.1: Frequency response of the system with transfer function $H(s) = \frac{s+0.1}{s+5}$. The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted.

Consider a transfer function of the form

$$H(s) = K\frac{(s+a_1)(s+a_2)}{s(s+b_1)(s^2+b_2s+b_3)},$$

where K is a constant gain, a_1 and a_2 are the locations of the zeros, b_1 is a real pole, and $s^2+b_2s+b_3$ represents a pair of complex conjugate poles. Each of these components affects the Bode plots in specific ways. First, we need to normalize the transfer function such that each term is expressed in terms of (s/a_n+1) for zeros and (s/b_n+1) for poles. For the example above this would be

$$H(s) = K\frac{a_1a_2}{b_1b_3}\frac{\left(\frac{s}{a_1}+1\right)\left(\frac{s}{a_2}+1\right)}{\frac{s}{b_1}\left(\frac{s}{b_1}+1\right)\left(\frac{s^2}{b_3}+\frac{b_2}{b_3}s+1\right)}.$$

If we let $s \to j\omega$, then we can rewrite this as

$$H(j\omega) = K\frac{a_1a_2}{b_1b_3}\frac{\left(j\frac{\omega}{a_1}+1\right)\left(j\frac{\omega}{a_2}+1\right)}{j\frac{\omega}{b_1}\left(j\frac{\omega}{b_1}+1\right)\left(-\frac{\omega^2}{b_3}+j\frac{b_2}{b_3}\omega+1\right)}.$$

If we take the magnitude and phase of the transfer function we have

$$|H(j\omega)| = |K|\frac{a_1a_2}{b_1b_3}\frac{\sqrt{\left(\frac{\omega}{a_1}\right)^2+1}\sqrt{\left(\frac{\omega}{a_2}\right)^2+1}}{\frac{\omega}{b_1}\sqrt{\left(\frac{\omega}{b_1}\right)^2+1}\sqrt{\left(-\frac{\omega^2}{b_3}+1\right)^2+\left(\frac{b_2}{b_3}\omega\right)^2}},$$

$$\angle H(j\omega) = \angle K + \tan^{-1}\left(\frac{\omega}{a_1}\right) + \tan^{-1}\left(\frac{\omega}{a_2}\right) - 90^\circ - \tan^{-1}\left(\frac{\omega}{b_1}\right) - \tan^{-1}\left(\frac{\frac{b_2}{b_3}\omega}{1-\frac{\omega^2}{b_3}}\right).$$

We can convert the magnitude to decibels by taking $20\log_{10}$ of the magnitude expression

$$\begin{aligned}|H(j\omega)|_{dB} = 20\log_{10}|K| + 20\log_{10}\left(\frac{a_1a_2}{b_1b_3}\right) + 20\log_{10}\left(\sqrt{\left(\frac{\omega}{a_1}\right)^2+1}\right) + \dots \\ \dots + 20\log_{10}\left(\sqrt{\left(\frac{\omega}{a_2}\right)^2+1}\right) - 20\log_{10}\left(\frac{\omega}{b_1}\right) - 20\log_{10}\left(\sqrt{\left(\frac{\omega}{b_1}\right)^2+1}\right) - \dots \\ \dots - 20\log_{10}\left(\sqrt{\left(-\frac{\omega^2}{b_3}+1\right)^2+\left(\frac{b_2}{b_3}\omega\right)^2}\right).\end{aligned}$$

We notice in this example transfer function, we have two first-order zeros at $\omega = a_1$ and $\omega = a_2$, a pole at the origin, a first-order pole at $\omega = b_1$, and a pair of complex conjugate poles characterized by the quadratic term. Each of these components contributes to the overall Bode plots in specific ways, which we can summarize as follows:

Constant The constant gain $\frac{Ka_1a_2}{b_1b_3}$ contributes a flat line in the magnitude plot at $20\log_{10}\left|\frac{Ka_1a_2}{b_1b_3}\right|$ dB and a constant phase shift of $\angle\frac{Ka_1a_2}{b_1b_3}$ degrees. In this class, we will typically assume the K is real-valued, so the phase contribution will be either 0° (for positive $\frac{Ka_1a_2}{b_1b_3}$) or 180° (for negative $\frac{Ka_1a_2}{b_1b_3}$).

Zero at origin A zero at the origin contributes a +20 dB/decade slope in the magnitude plot and a constant +90 degree phase shift. We notice that if $\omega = 1\,\text{rad/s}$, then the magnitude contribution is $20\log_{10}(1) = 0\,\text{dB}$. The phase contribution is always +90 degrees regardless of frequency.

Pole at origin A pole at the origin contributes a -20 dB/decade slope in the magnitude plot and a constant -90 degree phase shift. We notice that if $\omega = 1\,\text{rad/s}$, then the magnitude contribution is $20\log_{10}(1) = 0\,\text{dB}$. The phase contribution is always -90 degrees regardless of frequency.

Zero at $s = -a$ A first-order zero contributes a +20 dB/decade slope starting at $\omega = a$. At frequencies $\omega < a$, the magnitude plot is flat. At $\omega = a$, the magnitude is increased by $20\log_{10}(\sqrt{2}) \approx 3\,\text{dB}$. For frequencies $\omega > a$, the slope increases to +20 dB/decade. The phase shift starts at 0 degrees for $\omega \ll a$, reaches +45 degrees at $\omega = a$, and approaches +90 degrees for $\omega \gg a$.

Pole at $s = -b$ A first-order pole contributes a -20 dB/decade slope starting at $\omega = b$. At frequencies $\omega < b$, the magnitude plot is flat. At $\omega = b$, the magnitude is decreased by $20\log_{10}(\sqrt{2}) \approx -3\,\text{dB}$. For frequencies $\omega > b$, the slope decreases to -20 dB/decade. The phase shift starts at 0 degrees for $\omega \ll b$, reaches -45 degrees at $\omega = b$, and approaches -90 degrees for $\omega \gg b$.

Complex conjugate poles A pair of complex conjugate poles characterized by the quadratic term $s^2 + b_2s + b_3$ contributes a -40 dB/decade slope starting at the natural frequency $\omega_n = \sqrt{b_3}$. The damping ratio $\zeta = \frac{b_2}{2\sqrt{b_3}}$ affects the peak and phase response. At frequencies $\omega < \omega_n$, the magnitude plot is flat. At $\omega = \omega_n$, the magnitude may exhibit a peak depending on the damping ratio. For frequencies $\omega > \omega_n$, the slope decreases to -40 dB/decade. The phase shift starts at 0 degrees for $\omega \ll \omega_n$, reaches -90 degrees at $\omega = \omega_n$, and approaches -180 degrees for $\omega \gg \omega_n$.

To make Bode plots, we can sketch each of these contributions separately and then combine (add) them to get the overall Bode plots for the system.

7.3 Pole-zero placement and plots

The placement of poles and zeros in the complex plane has a significant impact on the frequency response of a system. By strategically placing poles and zeros, we can design filters that meet specific performance criteria, such as cutoff frequency, roll-off rate, and phase characteristics.

For example, consider a simple first-order system with a transfer function

$$H(s) = s + z_1$$

which has a frequency response

$$H(j\omega) = j\omega + z_1.$$

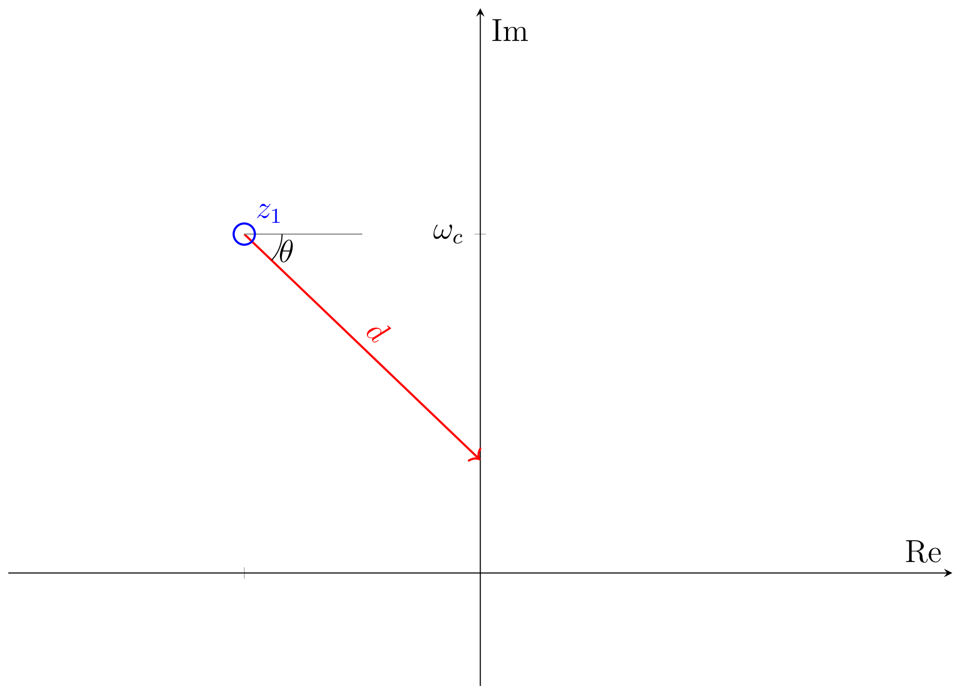


Figure 7.2: Pole-zero plot showing a zero at z_1 in the complex plane. The distance d from the zero to a point on the imaginary axis at $j\omega_c$ determines the contribution of the zero to the frequency response at that frequency. The angle θ indicates the phase contribution from the zero.

The magnitude and phase of the frequency response are given by

$$|H(j\omega)| = \sqrt{\omega^2 + z_1^2},$$
$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{z_1}\right).$$

You can see the impact of pole placement in Fig. 7.2, which shows a pole-zero plot with a zero at z_1 in the complex plane. The distance d from the zero to a point on the imaginary axis at $j\omega$ determines the contribution of the zero to the frequency response at that frequency ω. The angle θ indicates the phase contribution from the zero. We also notice that d is minimized when $\omega = \omega_c$, where ω_c is the imaginary part of the zero's location. Thus, the zero has the most significant impact on the frequency response at that frequency. Fig. 7.3 shows the frequency response of a system with a single zero at $s = -2 + j3$. The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the magnitude d is minimized at the frequency corresponding to the imaginary part of the zero's location, and the phase shift θ is $0°$ at that frequency, which is based on the geometry of the pole-zero plot.

Similarly, poles can be analyzed in the same way. A pole at p_1 contributes to the frequency response based on its distance and angle to points on the imaginary axis. However, there is a key difference: the transfer function of single pole is

$$H(s) = \frac{1}{s + p_1}$$

which has a frequency response

$$H(j\omega) = \frac{1}{j\omega + p_1}.$$

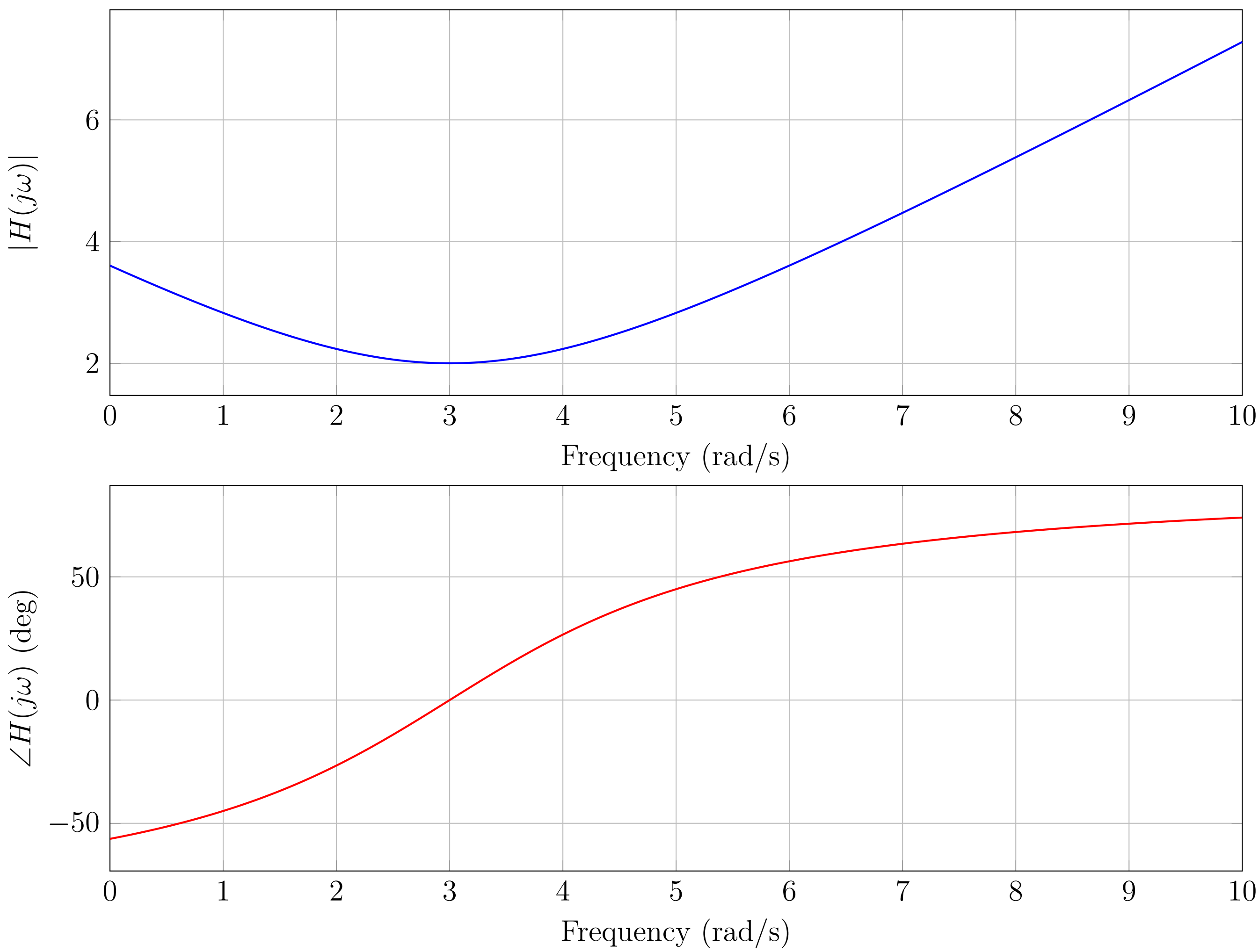


Figure 7.3: Frequency response of a system with a single zero (in this case $s = -2 + j3$). The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the magnitude d is minimized at the frequency corresponding to the imaginary part of the zero's location, and the phase shift θ is $0°$ at that frequency.

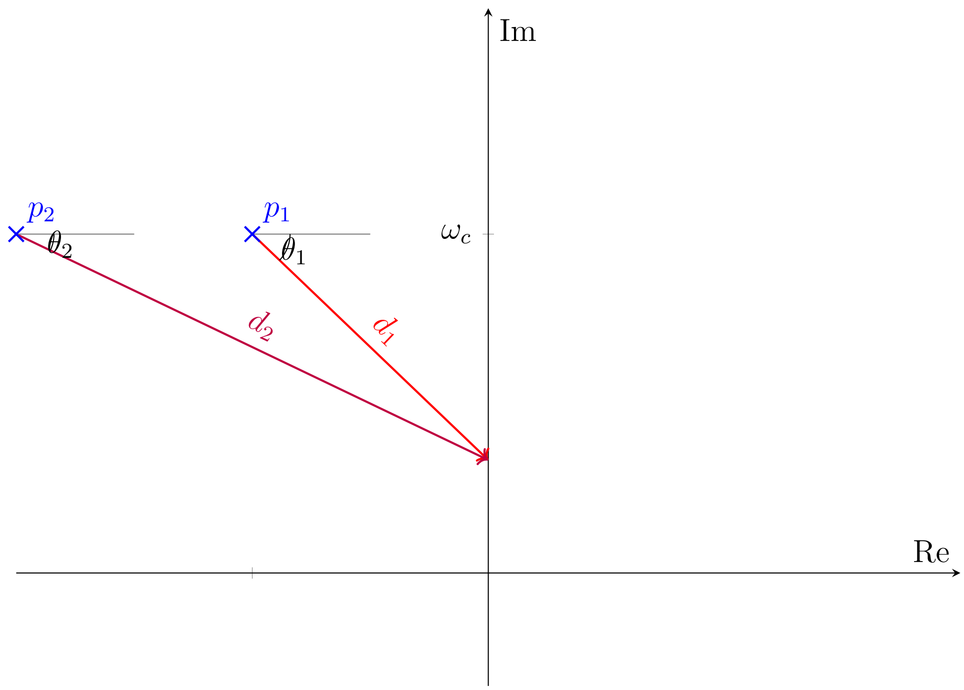


Figure 7.4: Pole-zero plot showing poles at p_1 and p_2 in the complex plane. The distances d_1 and d_2 from the poles to a point on the imaginary axis at $j\omega_c$ determine the contribution of the poles to the frequency response at that frequency. The angles θ_1 and θ_2 indicate the phase contributions from the poles. Notice that the even though both poles have the same imaginary part, the pole closer to the imaginary axis (i.e., p_1) will have a larger impact on the frequency response at that frequency.

The magnitude and phase of the frequency response are given by

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2 + p_1^2}},$$
$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{p_1}\right).$$

Thus, poles attenuate signals at frequencies near their imaginary part, and introduce phase shifts in the opposite direction compared to zeros. We can look at the geometry of the pole-zero plot to understand this behavior. Fig. 7.4 shows a pole-zero plot with two poles at p_1 and p_2 in the complex plane. The distances d_1 and d_2 from the poles to a point on the imaginary axis at $j\omega_c$ determine the contribution of the poles to the frequency response at that frequency. The angles θ_1 and θ_2 indicate the phase contributions from the poles. Notice that even though both poles have the same imaginary part, the pole closer to the imaginary axis (i.e., p_1) will have a larger impact on the frequency response at that frequency. This is seen in Fig. 7.5, which shows the frequency response of a system with two poles (in this case $p_1 = -2 + j3$ and $p_2 = -4 + j3$). The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the pole closer to the imaginary axis (i.e., p_1) has a larger impact on the frequency response at that frequency.

In practice, we avoid systems with complex-valued coefficients, so poles and zeros will always come in complex conjugate pairs. Thus, the frequency response contributions from complex conjugate poles and zeros can be analyzed by considering both members of the pair together. The overall

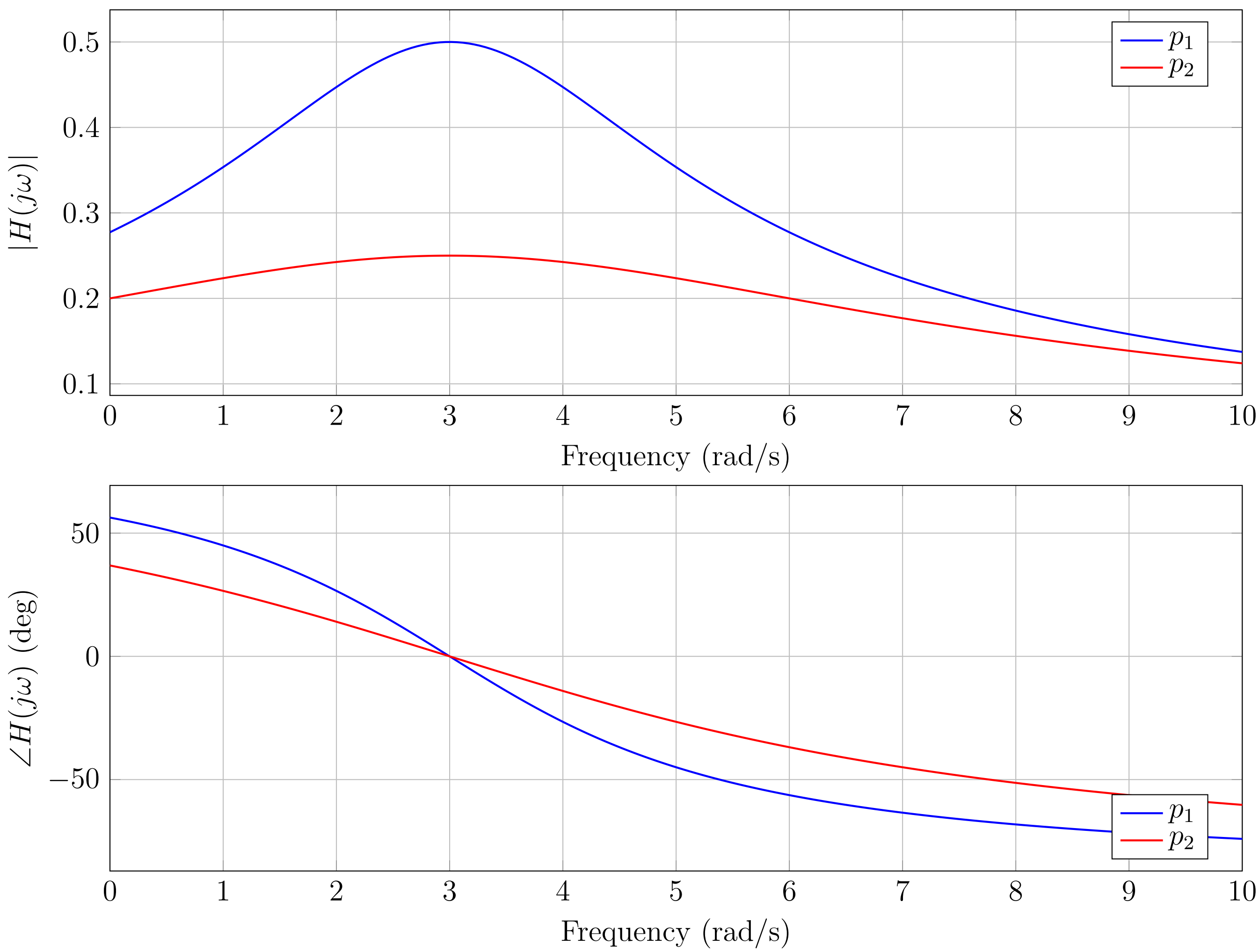


Figure 7.5: Frequency response of a system with two poles (in this case $p_1 = -2 + j3$ and $p_2 = -4 + j3$). The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the pole closer to the imaginary axis (i.e., p_1) has a larger impact on the frequency response at that frequency.

frequency response is determined by the combined effects of all poles and zeros in the system.

For example, consider the system with two poles which are complex conjugates of each other

$$H(s) = \frac{1}{(s+p)(s+p^*)}$$

where $p = \sigma + j\omega_c$ and $p^* = \sigma - j\omega_c$. The frequency response is given by

$$H(j\omega) = \frac{1}{(j\omega + p)(j\omega + p^*)}.$$

The magnitude and phase of the frequency response are given by

$$\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{(\sigma^2 + (\omega - \omega_c)^2)(\sigma^2 + (\omega + \omega_c)^2)}}, \\ &= \frac{1}{d_1 \cdot d_2}, \\ \angle H(j\omega) &= -\tan^{-1}\left(\frac{\omega - \omega_n}{\sigma}\right) - \tan^{-1}\left(\frac{\omega + \omega_n}{\sigma}\right). \end{aligned}$$

The geometry of the pole-zero plot helps us understand how complex conjugate poles affect the frequency response. Fig. 7.6 shows a pole-zero plot with two complex conjugate poles at p_1 and p_2 in the complex plane. The distances d_1 and d_2 from the poles to a point on the imaginary axis at $j\omega$ determine the contribution of the poles to the frequency response at that frequency. The angles θ_1 and θ_2 indicate the phase contributions from the poles. We notice that the distances d_1 and d_2 are minimized when $\omega = \omega_c$, where ω_c is the imaginary part of the poles' locations. Thus, the complex conjugate poles have the most significant impact on the frequency response at that frequency. Fig. 7.7 shows the frequency response of a system with complex conjugate poles at p_1 and $p_1^* = p_2$. The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the magnitude $d_1 \cdot d_2$ is minimized at the frequency corresponding to the imaginary part of the poles' locations.

In summary, the placement of poles and zeros on the complex plane directly influences the frequency response of a system. By analyzing the geometry of the pole-zero plot, we can predict how the system will behave at different frequencies and design filters that meet specific performance criteria.

7.3.1 Filter types

Analog filters can be classified into several types based on their frequency response characteristics. The most common types of filters are:

Low-pass filters Low-pass filters allow signals with frequencies below a certain cutoff frequency to pass through while attenuating higher frequencies. They are commonly used in audio processing to remove high-frequency noise.

High-pass filters High-pass filters allow signals with frequencies above a certain cutoff frequency to pass through while attenuating lower frequencies. They are often used in applications such as audio equalization and image processing.

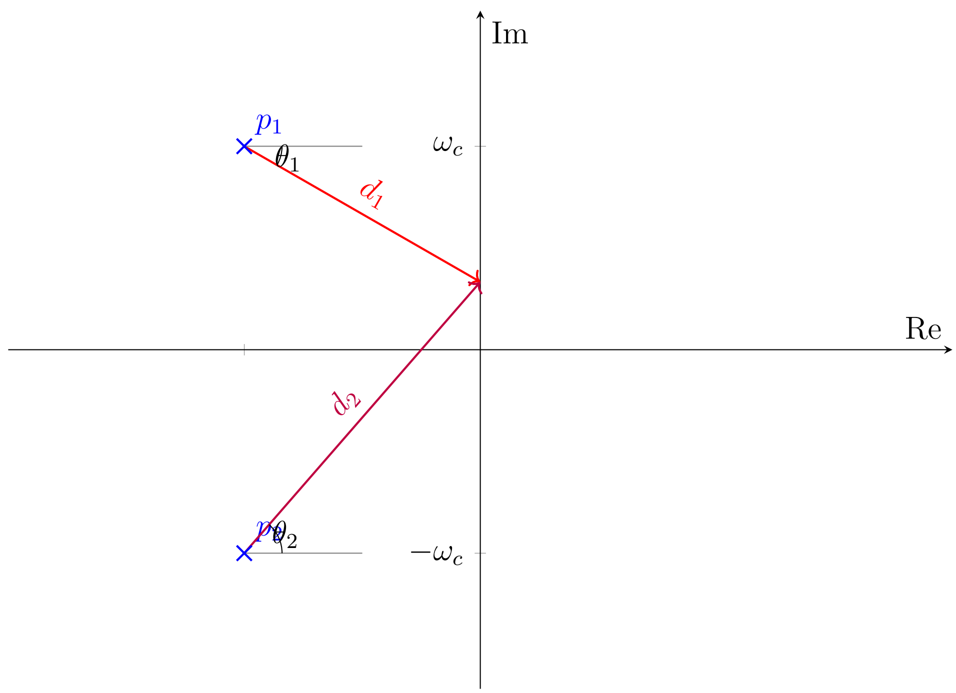


Figure 7.6: Pole-zero plot showing complex conjugate poles at p_1 and p_2 in the complex plane. The distances d_1 and d_2 from the poles to a point on the imaginary axis at $j\omega_c$ determine the contribution of the poles to the frequency response at that frequency. The angles θ_1 and θ_2 indicate the phase contributions from the poles. We notice that the distances d_1 and d_2 are minimized when $\omega = \omega_c$, where ω_c is the imaginary part of the poles' locations. Thus, the complex conjugate poles have the most significant impact on the frequency response at that frequency.

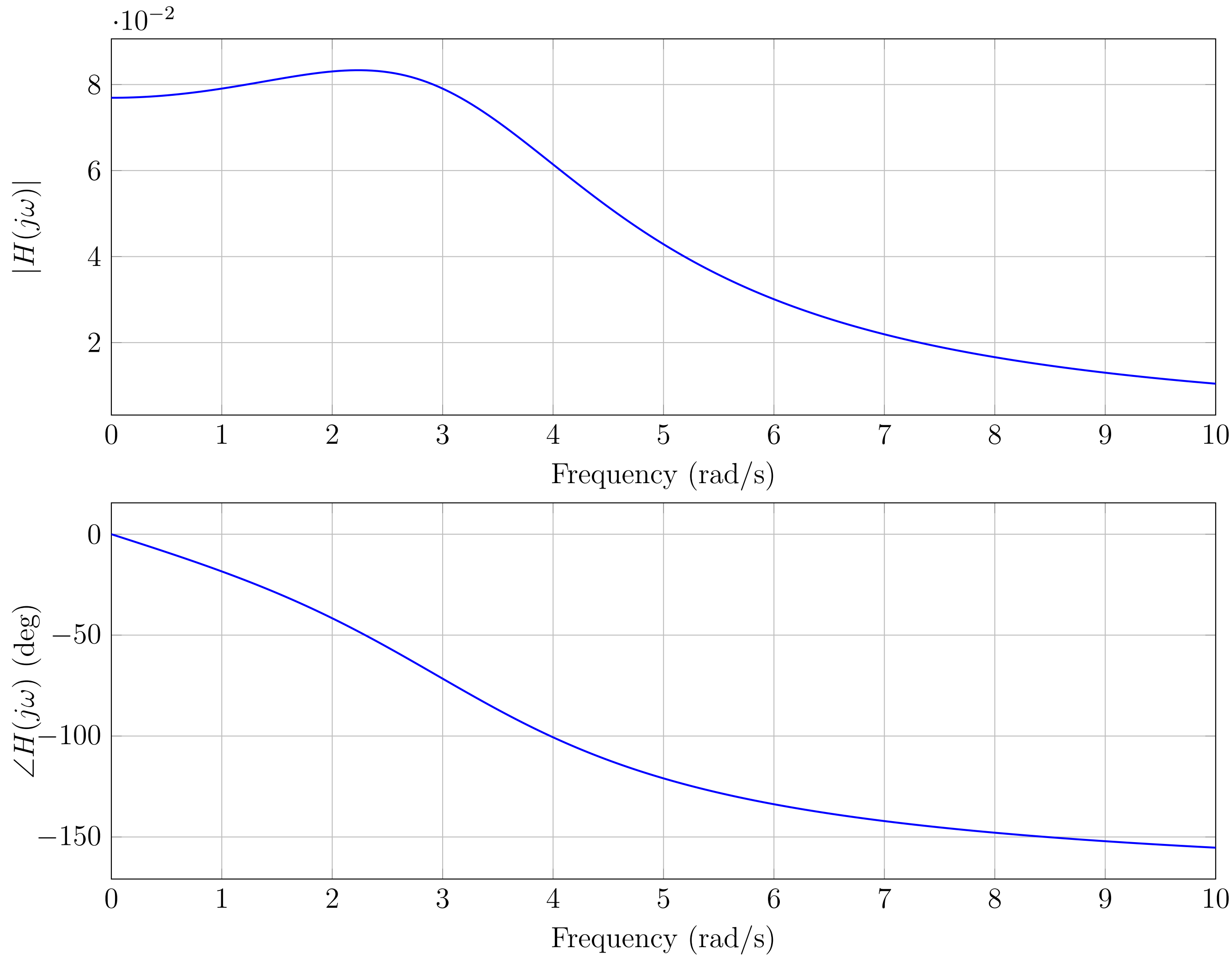


Figure 7.7: Frequency response of a system with complex conjugate poles at p_1 and $p_1^* = p_2$. The magnitude plot shows how the amplitude of input signals at different frequencies are scaled, while the phase plot shows how the phase of input signals are shifted. Notice that the magnitude $d_1 \cdot d_2$ is minimized at the frequency corresponding to the imaginary part of the poles' locations.

Band-pass filters Band-pass filters allow signals within a specific frequency range (band) to pass through while attenuating frequencies outside that range. They are used in applications such as wireless communication and audio processing.

Band-stop filters Band-stop filters (also known as notch filters) attenuate signals within a specific frequency range while allowing frequencies outside that range to pass through. They are used to eliminate unwanted frequencies, such as power line interference in audio signals.

To better understand these filters, we need to introduce the concept of quality factor Q. The quality factor is a dimensionless parameter that describes the selectivity or sharpness of a filter's frequency response. It is defined as

$$Q = \frac{\omega_0}{B}$$

where ω_0 is the center frequency of the filter and B is the bandwidth of the filter. A higher Q value indicates a narrower bandwidth and a more selective filter, while a lower Q value indicates a wider bandwidth and a less selective filter. An alterntive description of quality factor is in terms of the damping ratio ζ, where

$$Q = \frac{1}{2\zeta}.$$

If $Q < 1/2$, the system is overdamped, if $Q = 1/2$, the system is critically damped, and if $Q > 1/2$, the system is underdamped.

Low-pass filters

A simple first-order low-pass filter can be implemented using a resistor and capacitor in series, with the output taken across the capacitor. The transfer function of this filter is given by

$$H(s) = \frac{1}{RCs + 1}$$

where R is the resistance, C is the capacitance, and s is the complex frequency variable. The cutoff frequency ω_c is given by

$$\omega_c = \frac{1}{RC}.$$

This system has a single pole at $s = -\frac{1}{RC}$, which determines the filter's frequency response. The Bode plot for this low-pass filter shows a flat magnitude response at low frequencies, with a -20 dB/decade slope starting at the cutoff frequency ω_c. The phase response starts at 0 degrees for low frequencies and approaches -90 degrees as the frequency increases.

An ideal low-pass filter would have a perfectly flat response up to the cutoff frequency and then drop to zero immediately after. To achieve this, we would need a transfer function with an infinite number of poles placed along the left half of the complex plane along a circle centered at the origin with radius ω_c. This is not physically realizable.

High-pass filters

High-pass filters can be implemented using a resistor and capacitor in series, with the output taken across the resistor. The transfer function of this filter is given by

$$H(s) = \frac{RCs}{RCs + 1}$$

where R is the resistance, C is the capacitance, and s is the complex frequency variable. The cutoff frequency ω_c is given by

$$\omega_c = \frac{1}{RC}.$$

This system has a single zero at the origin and a single pole at $s = -\frac{1}{RC}$, which determines the filter's frequency response. The single zero at the origin indicates that low-frequency signals are attenuated, while the pole determines the rate of attenuation beyond the cutoff frequency.

The Bode plot for this high-pass filter shows a -20 dB/decade slope at low frequencies, with a flat magnitude response starting at the cutoff frequency ω_c. The phase response starts at -90 degrees for low frequencies and approaches 0 degrees as the frequency increases. An ideal high-pass filter would have a perfectly flat response above the cutoff frequency and then drop to zero immediately before. To achieve this, we would need a transfer function with an infinite number of zeros placed along the imaginary axis. This is not physically realizable.

Band-pass filters

Band-pass filters can be implemented using a combination of resistors, capacitors, and inductors. A simple second-order band-pass filter can be realized using an RLC circuit in series where the output is measured across the resistor. The transfer function of this filter is given by

$$H(s) = \frac{RCs}{LCs^2 + RCs + 1}$$

where R is the resistance, C is the capacitance, L is the inductance, and s is the complex frequency variable. The center frequency ω_0 and bandwidth B of the filter are given by

$$\omega_0 = \frac{1}{\sqrt{LC}},$$
$$B = \frac{R}{L}.$$

This system has a zero at the origin and two complex conjugate poles, which determine the filter's frequency response. The zero at the origin indicates that low-frequency signals are attenuated, while the complex conjugate poles determine the passband characteristics around the center frequency. The Bode plot for this band-pass filter shows attenuation at low and high frequencies, with a peak in the magnitude response around the center frequency ω_0. The phase response varies significantly around the center frequency, reflecting the resonant behavior of the filter.

Oftentimes bandpass filters can be also be designed by cascading a low-pass filter and a high-pass filter. The overall transfer function is the product of the individual transfer functions, and the frequency response is determined by the combined effects of both filters. Alternatively, band-pass filters can be designed using multiple feedback amplifier configurations to achieve desired bandwidth and center frequency characteristics.

The quality factor Q of the band-pass filter is defined as

$$Q = \frac{\omega_0}{B} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

A higher Q value indicates a narrower bandwidth and a more selective filter, while a lower Q value indicates a wider bandwidth and a less selective filter.

Band-stop (and notch) filters

Band-stop filters can be implemented using a high and low-pass filter in parallel, or by using RLC circuits in parallel. A simple second-order band-stop filter can be realized using an RLC circuit in parallel where the output is measured across the input. The transfer function of this filter is given by

$$H(s) = \frac{LCs^2 + 1}{LCs^2 + RCs + 1}$$

where R is the resistance, C is the capacitance, L is the inductance, and s is the complex frequency variable. The center frequency ω_0 and bandwidth B of the filter are given by

$$\omega_0 = \frac{1}{\sqrt{LC}},$$
$$B = \frac{1}{RC}.$$

This system has two complex conjugate zeros and two complex conjugate poles, which determine the filter's frequency response. The complex conjugate zeros indicate that signals around the center frequency are attenuated, and are placed on the imaginary axis at $s = \pm j\omega_0$. The complex conjugate poles determine the passband characteristics around the center frequency. The Bode plot for this band-stop filter shows a dip in the magnitude response around the center frequency ω_0, with attenuation at that frequency and relatively flat response at low and high frequencies. The phase response varies significantly around the center frequency, reflecting the resonant behavior of the filter.

Some physical implementations will use a notch filter, which is a specific type of band-stop filter designed to attenuate a very narrow frequency band. Notch filters are commonly used to eliminate unwanted frequencies, such as power line interference in audio signals. The transfer function of a simple notch filter can be expressed as

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

where ω_0 is the notch frequency and Q is the quality factor that determines the width of the notch. A higher Q value results in a narrower notch, while a lower Q value results in a wider notch. This is often implemented using a twin-T network or other active filter configurations.

7.4 Filter design process

7.4.1 Butterworth filters

We discussed a simple low-pass filter implemented using a single pole at $-\omega_c$. The downside to this filter is the roll-off rate of -20 dB/decade, which may not be sufficient for some applications. To achieve a steeper roll-off rate, we can use higher-order filters, which increases the roll-off rate by -20 dB/decade for each additional pole. For example, a second-order low-pass filter with two poles would have a roll-off rate of -40 dB/decade, while a third-order low-pass filter with three poles would have a roll-off rate of -60 dB/decade. The issue here is that if you simply cascade multiple first-order low-pass filters with the same ω_c, the magnitude response at the corner frequency will decrease by -3 dB for each filter stage, resulting in a significant attenuation at the cutoff frequency.

To address this, we can design higher-order filters with specific pole placements to achieve the desired frequency response characteristics.

The Butterworth filter is one such higher-order filter that provides a maximally flat frequency response in the passband. The Butterworth filter is characterized by its frequency response, which is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

where n is the order of the filter and ω_c is the cutoff frequency. If we set $\omega = \omega_c$, we find that

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}$$

which corresponds to a -3 dB point, regardless of the filter order n. To solve for the poles of the Butterworth filter, we start with the squared magnitude response, we first assume that $\omega_c = 1\,\mathrm{rad/s}$ for simplicity, and then we have

$$|H(j\omega)|^2 = \frac{1}{1 + (j\omega)^{2n}}.$$

We can replace $\omega_c \to {}^{s}/_{j}$ to get

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}}.$$

The poles of the Butterworth filter are found by solving the equation

$$1 + \left(\frac{s}{j}\right)^{2n} = 0$$

which can be rearranged to

$$\left(\frac{s}{j}\right)^{2n} = -1.$$

The solutions to this equation are given by

$$\frac{s}{j} = e^{j\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots, 2n-1.$$

Multiplying both sides by j, we find the poles of the Butterworth filter

$$s_k = je^{j\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right)} = e^{j\left(\frac{\pi}{2} + \frac{\pi}{2n} + \frac{k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots, 2n-1.$$

These poles are evenly spaced around a circle. However, we are only interested in the poles that lie in the left half of the complex plane, since those are the ones that contribute to a stable filter. Thus, we take the n poles on the left-hand side of the complex plane. The transfer function of the Butterworth filter can then be expressed as

$$H(s) = \frac{1}{B_n(s)}$$

where $B_n(s)$ is the Butterworth polynomial of order n, defined as

$$B_n(s) = \prod_{k=1}^{n} (s - s_k)$$

with s_k being the poles in the left half of the complex plane. This polynomial can be found in tables, such as the Tab. 7.1 and Tab. 7.2.

Suppose we want to design a 3rd-order Butterworth low-pass filter with a cutoff frequency of $\omega_c = 3\,\text{rad/s}$. From Tab. 7.1, we find that the Butterworth polynomial for $n = 3$ is

$$B_3(s) = s^3 + 2s^2 + 2s + 1.$$

We can construct the transfer function

$$H(s) = \frac{1}{B_3(s)} = \frac{1}{s^3 + 2s^2 + 2s + 1}.$$

This transfer function assumes a cutoff frequency of $\omega_c = 1\,\text{rad/s}$. To adjust the cutoff frequency to $\omega_c = 3\,\text{rad/s}$, we perform a frequency scaling by substituting s with $\frac{s}{\omega_c}$:

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s}{3}\right)^3 + 2\left(\frac{s}{3}\right)^2 + 2\left(\frac{s}{3}\right) + 1} \\ &= \frac{27}{s^3 + 6s^2 + 18s + 27}. \end{aligned}$$

Design procedure

In practice, you are likely to be given parameters such as desired passband gain, passband frequencies, stopband attenuation, and stopband frequencies. The general procedure for designing a Butterworth filter is as follows:

1. **Calculate the filter order:** Use the filter specifications to calculate the minimum order n of the Butterworth filter needed to meet the design criteria. The formula is

$$n \geq \frac{\log\left(\frac{10^{-\hat{G}_s/10} - 1}{10^{-\hat{G}_p/10} - 1}\right)}{2\log\left(\frac{\omega_s}{\omega_p}\right)}$$

   where $\hat{G}_p$ is the passband gain in dB, $\hat{G}_s$ is the stopband gain in dB, ω_p is the passband edge frequency, and ω_s is the stopband edge frequency. Round up to the nearest integer if necessary.

2. **Find the Butterworth polynomial:** Use Tab. 7.1 or Tab. 7.2 to find the Butterworth polynomial $B_n(s)$ for the calculated order n.

3. **Construct the transfer function:** Formulate the transfer function $H(s) = \frac{1}{B_n(s)}$.

4. **Apply frequency scaling:** You will need to solve for the cutoff frequency ω_c. There are two possible formulas:

$$\omega_c = \omega_p \left(10^{\frac{-\hat{G}_p}{10}} - 1\right)^{-\frac{1}{2n}}$$

   or

$$\omega_c = \omega_s \left(10^{\frac{-\hat{G}_s}{10}} - 1\right)^{-\frac{1}{2n}}.$$

   We have two options here; either use the passband specifications (which makes the filter surpass the stopband requirements) or the stopband specifications (which makes the filter surpass the passband requirements). Choose one based on which specifications are more critical for your design. Once you have ω_c, perform frequency scaling by substituting s with $\frac{s}{\omega_c}$ in the transfer function.

5. **Implement the filter:** Usually using active filter topologies such as Sallen-Key or multiple feedback configurations to realize the desired transfer function.

7.4.2 Chebyshev filters

Chebyshev filters are another class of analog filters that provide a steeper roll-off rate compared to Butterworth filters for a given filter order. They achieve this by allowing ripples in the passband (Type I) or stopband (Type II) of the frequency response. The Chebyshev filter is characterized by its frequency response, which is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2 C_n^2\left(\frac{\omega}{\omega_c}\right)}}$$

where ϵ is the ripple factor, $C_n(x)$ is the Chebyshev polynomial of order n, and ω_c is the cutoff frequency. The Chebyshev polynomials are defined as

$$C_n(x) = \cos\left(n\cos^{-1}(x)\right)$$

for $|x| \leq 1$ and

$$C_n(x) = \cosh\left(n\cosh^{-1}(x)\right)$$

for $|x| > 1$. The ripple factor ϵ determines the amount of ripple in the passband, with larger values of ϵ resulting in more ripple. The poles of the Chebyshev filter can be found using the following formula:

$$s_k = \omega_c\left[-\sin\left(\frac{\pi(2k-1)}{2n}\right)\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) + j\cos\left(\frac{\pi(2k-1)}{2n}\right)\cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right)\right]$$

for $k = 1, 2, \ldots, n$. The transfer function of the Chebyshev filter can then be expressed as

$$H(s) = \frac{1}{C_n(s)}$$

where $C_n(s)$ is the Chebyshev polynomial of order n, defined as

$$C_n(s) = \prod_{k=1}^{n}(s - s_k)$$

with s_k being the poles in the left half of the complex plane. The design procedure for Chebyshev filters is similar to that of Butterworth filters, with the addition of selecting the ripple factor ϵ based on the desired passband ripple.

Type II Chebyshev filters, also known as inverse Chebyshev filters, have ripples in the stopband instead of the passband. In practice Type I Chebyshev filters are more commonly used due to their better performance in the passband and simpler implementation compared to Type II filters.

7.4.3 Elliptic filters

Elliptic filters, also known as Cauer filters, are a class of analog filters that provide the steepest roll-off rate for a given filter order compared to Butterworth and Chebyshev filters. They achieve this by allowing ripples in both the passband and stopband of the frequency response. The elliptic filter is characterized by its frequency response, which is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2\left(\frac{\omega}{\omega_c}, k\right)}}$$

where ϵ is the ripple factor, $R_n(x, k)$ is the elliptic rational function of order n with modulus k, and ω_c is the cutoff frequency. The elliptic rational functions are defined using Jacobi elliptic functions, which are more complex than Chebyshev polynomials. The ripple factor ϵ determines the amount of ripple in the passband, with larger values of ϵ resulting in more ripple. The poles and zeros of the elliptic filter can be found using elliptic integrals and Jacobi elliptic functions, which are beyond the scope of this discussion. The transfer function of the elliptic filter can then be expressed as

$$H(s) = \frac{N_n(s)}{D_n(s)}$$

where $N_n(s)$ and $D_n(s)$ are the numerator and denominator polynomials of order n, respectively, defined based on the poles and zeros of the filter. The design procedure for elliptic filters is similar to that of Butterworth and Chebyshev filters, with the addition of selecting the ripple factor ϵ and modulus k based on the desired passband and stopband ripple.

7.4.4 Pros and cons of different filter types

When choosing a filter type for a specific application, it is important to consider the trade-offs between different filter characteristics. Table 7.3 summarizes the pros and cons of the different filter types.

In summary, the choice of filter type depends on the specific requirements of the application, including acceptable levels of ripple, desired roll-off rate, transient response, and implementation complexity. Butterworth filters are suitable for applications requiring a flat passband, while Chebyshev and elliptic filters are preferred when a steeper roll-off is necessary, albeit at the cost of ripple and transient performance.

7.5 Conversion to high-pass, band-pass, and band-stop filters

Once a low-pass filter has been designed using one of the methods described above, it can be converted to a high-pass, band-pass, or band-stop filter using frequency transformation techniques. These transformations involve substituting the complex frequency variable s in the low-pass transfer function with a new expression that corresponds to the desired filter type.

7.5.1 High-pass filter transformation

To convert a low-pass filter to a high-pass filter, we use the following substitution:

$$s \to \frac{\omega_p}{s}$$

where ω_p is the passband. You will need to determine the parameters to design the prototype low-pass filter $H_{lp}(s)$, which we will use $\hat{\omega}_p = 1\,\mathrm{rad/s}$ and $\hat{\omega}_s = \omega_p/\omega_s$. This substitution effectively inverts the frequency response, allowing high frequencies to pass while attenuating low frequencies. The resulting transfer function for the high-pass filter is given by

$$H_{hp}(s) = H_{lp}\left(\frac{\omega_p}{s}\right)$$

where $H_{lp}(s)$ is the transfer function of the original low-pass filter (which had a cutoff frequency of $\omega_c = 1\,\mathrm{rad/s}$).

7.5.2 Band-pass filter transformation

To convert a low-pass filter to a band-pass filter, we use the following substitution:

$$s \to \frac{s^2 + \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})s}$$

where ω_0 is the center frequency and B is the bandwidth of the desired band-pass filter. This substitution creates a frequency response that allows signals within a specific frequency range ($\omega_{p1} \leq \omega \leq \omega_{p2}$) to pass while attenuating frequencies outside that range. The resulting transfer function for the band-pass filter is given by

$$H_{bp}(s) = H_{lp}\left(\frac{s^2 + \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})s}\right)$$

where $H_{lp}(s)$ is the transfer function of the original low-pass filter (which had a cutoff frequency of $\omega_c = 1\,\mathrm{rad/s}$).

7.5.3 Band-stop filter transformation

To convert a low-pass filter to a band-stop filter, we use the following substitution:

$$s \to \frac{(\omega_{p2} - \omega_{p1})s}{s^2 + \omega_{p1}\omega_{p2}}$$

where ω_{p1} and ω_{p2} are the lower and upper passband edge frequencies of the desired band-stop filter. This substitution creates a frequency response that attenuates signals within a specific frequency range ($\omega_{p1} \leq \omega \leq \omega_{p2}$) while allowing frequencies outside that range to pass through. The resulting transfer function for the band-stop filter is given by

$$H_{bs}(s) = H_{lp}\left(\frac{(\omega_{p2} - \omega_{p1})s}{s^2 + \omega_{p1}\omega_{p2}}\right)$$

where $H_{lp}(s)$ is the transfer function of the original low-pass filter (which had a cutoff frequency of $\omega_c = 1\,\mathrm{rad/s}$).

7.6 Real-life filter design

While the theoretical design of analog filters provides a solid foundation, real-life implementations often require additional considerations to account for non-ideal components, environmental factors, and specific application requirements. Given these challenges, it is common practice to use software tools such as SciPy libraries or the Analog Devices Filter Wizard to design the filters. These tools can help automate the design process, optimize component values, and simulate the filter's performance before physical implementation.

Table 7.1: Coefficients of Butterworth Polynomial $B_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Table 7.2: Butterworth Polynomials in Factorized Form

n	$B_n(s)$
1	$s+1$
2	$s^2+1.41421356s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76536686s+1)(s^2+1.84775907s+1)$
5	$(s+1)(s^2+0.61803399s+1)(s^2+1.931803399s+1)$
6	$(s^2+0.51763809s+1)(s^2+1.41421356s+1)(s^2+1.93185165s+1)$
7	$(s+1)(s^2+0.44504187s+1)(s^2+1.24697960s+1)(s^2+1.80193774s+1)$
8	$(s^2+0.39018064s+1)(s^2+1.11114047s+1)(s^2+1.66293922s+1)(s^2+1.96157056s+1)$
9	$(s+1)(s^2+0.34729636s+1)(s^2+s+1)(s^2+1.53208889s+1)(s^2+1.87938524s+1)$
10	$(s^2+0.31286893s+1)(s^2+0.90798100s+1)(s^2+1.41421356s+1)(s^2+1.78201305s+1)(s^2+1.97537668s+1)$

Table 7.3: Comparison of different filter types.

Filter Type	**Pros**	**Cons**
Butterworth	• Maximally flat passband (no ripple). • Good transient response. • Simple design.	• Slowest roll-off rate for a given order. • Requires a higher order for sharp attenuation.
Chebyshev (Type I)	• Steeper roll-off than Butterworth. • More efficient order for a given attenuation.	• Ripples in the passband. • Poorer transient response (ringing). • Non-linear phase in the passband.
Chebyshev (Type II)	• Maximally flat passband. • Steeper roll-off than Butterworth.	• Ripples in the stopband. • Slower roll-off than Type I. • Complex hardware design.
Elliptic (Cauer)	• Steepest roll-off rate (most efficient). • Lowest order for a given specification.	• Ripples in both passband and stopband. • Worst transient response. • Highly non-linear phase response. • Most complex design.

Chapter 8

Discrete-Time Systems and Signals

Modern signal processing systems often operate on discrete-time signals, which are sequences of values defined at specific time intervals. In this chapter, we will explore the fundamental concepts of discrete-time signals and systems, including their representation, analysis, and processing techniques.

8.1 Discrete-time signals

We will describe a discrete-time signal as a sequence of numbers, usually denoted as $f[k]$ or $f[k]$ where k is an integer index representing the time step. Often times n is used as the integer index, though this simply a notational difference. Discrete-time signals are typically obtained by sampling continuous-time signals at regular intervals T, which we described at length in Chap. 5. We will explore various types of common discrete-time signals, such as unit impulses, step functions, and sinusoidal sequences.

8.1.1 Common discrete-time signals

Unit impulse

In continuous-time, we defined the unit impulse function $\delta(t)$ as a function that is zero everywhere except at $t = 0$, where it has an infinite value such that its integral over all time is equal to one. A rigorous treatment of this function requires limits, ϵ-δ definitions, and the theory of distribution. In discrete-time the unit impulse function is considerably simpler. It is defined as

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

This function is also known as the Kronecker delta function.

Unit step

In continuous time, the step function is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

It should be no surprise then that the step function is defined in discrete-time as

$$u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}.$$

We can relate the unit step and unit impulse functions in discrete-time through the following relationship

$$u[k] = \sum_{n=-\infty}^{k} \delta[n].$$

as well as the inverse relationship

$$\delta[k] = u[k] - u[k-1].$$

This is analogous to the relationship between the continuous-time unit step and unit impulse functions discussed in Sec. 2.3.2.

Real exponential

In continuous time we defined the real exponential signal as $x(t) = e^{\lambda t}$ where λ is a real-valued constants. In discrete-time, the real exponential signal is defined as

$$x[k] = \gamma^k.$$

This is similar to the continuous-time case, where $\gamma = e^{\lambda}$. The behavior of the discrete-time real exponential signal depends on the value of γ:

- If $|\gamma| < 1$, the signal decays to zero as k increases.
- If $|\gamma| = 1$, the signal remains constant in magnitude.
- If $|\gamma| > 1$, the signal grows without bound as k increases.

Often times $\gamma < 0$, which results in an alternating signal. For example, if $\gamma = -0.5$, the signal will alternate between positive and negative values while decaying in magnitude.

Sinusoidal sequence

A typical continuous-time sinusoidal signal is defined as $x(t) = A\cos(\omega t + \phi)$, where A is the amplitude, ω is the angular frequency, and ϕ is the phase shift. In discrete-time, the sinusoidal sequence is defined as

$$x[k] = A\cos(\Omega k + \phi),$$

where Ω is the discrete-time angular frequency in radians per sample. This frequency Ω is the digital frequency and is *different* than ω, the continuous-time angular frequency in radians per second. The relationship between these two frequencies is given by

$$\Omega = \omega T,$$

where T is the sampling period.

Discrete-time sinusoidal sequences have some non-intuitive properties compared to their continuous-time counterparts:

Periodicity: For example, discrete-time sinusoids are periodic such that $f[k] = f[k+N]$ only if the frequency Ω is a rational multiple of 2π. If we were to delay $f[k]$ by N samples, we have

$$\begin{aligned} f[k+N] &= \cos(\Omega(k+N)) \\ &= \cos(\Omega k + \Omega N) \\ &= \cos(\Omega k + 2\pi m) \\ &= \cos(\Omega k) \\ &= f[k]. \end{aligned}$$

If $\Omega N \neq 2\pi m$ for some integer m, then the sinusoid will not be periodic. Thus, we require

$$\Omega N = 2\pi m.$$

We can use this relationship to solve for the period N where we choose the smallest positive integer value of N and m.

Uniqueness: Unlike continuous-time sinusoids, discrete-time sinusoids are not unique for frequencies that differ by integer multiples of 2π. That is

$$\begin{aligned} f[k] &= \cos(\Omega k) \\ &= \cos(\Omega k + 2\pi m k) \quad \text{for any integer } m \\ &= \cos((\Omega k + 2\pi m)k). \end{aligned}$$

We see that if we add $2\pi m$ to the frequency Ω, the resulting discrete-time sinusoid remains unchanged. Thus, discrete-time sinusoids are only unique over the interval $\Omega \in [-\pi, \pi]$.

For example, consider the discrete-time sinusoid $f[k] = \cos(9.6\pi k)$. We can subtract $2\pi \cdot 5k$ such that

$$\begin{aligned} f[k] &= \cos(9.6\pi k - 10\pi k) \\ &= \cos(-0.4\pi k) \\ &= \cos(0.4\pi k). \end{aligned}$$

Even though $f[k]$ was defined with a frequency of 9.6π, it is equivalent to a sinusoid with a frequency of 0.4π. In practice, you can find the original frequency modulo 2π to find an equivalent frequency in the range $[-\pi, \pi]$.

This is related to aliasing. Consider the example from lab where we had a 7 kHz sinusoid sampled at 10 kHz. The digital frequency is

$$\begin{aligned} \Omega &= 2\pi \frac{f}{f_s} \\ &= 2\pi \frac{7\,\text{kHz}}{10\,\text{kHz}} \\ &= 1.4\pi. \end{aligned}$$

However, this is equivalent to

$$\begin{aligned} \Omega &= 1.4\pi - 2\pi \\ &= -0.6\pi \\ &= 0.6\pi. \end{aligned}$$

Thus, the 7 kHz sinusoid sampled at 10 kHz is indistinguishable from a 3 kHz sinusoid sampled at 10 kHz, which is what we observed in lab.

Further, as a rule of thumb, to sample any frequency, we need two samples per cycle (period). Thus, the maximum frequency we can sample without aliasing is half the sampling frequency, which is the Nyquist frequency.

An example of this phenomenon is shown in Fig. 8.1. The blue curve shows a continuous-time sinusoid at $\omega = 1.7\pi$. The green dots show samples taken at $T = 1$. The orange curve shows the discrete-time sinusoid reconstructed from the samples, which is indistinguishable from a $\omega = 0.3\pi$ sinusoid sampled at the same rate.

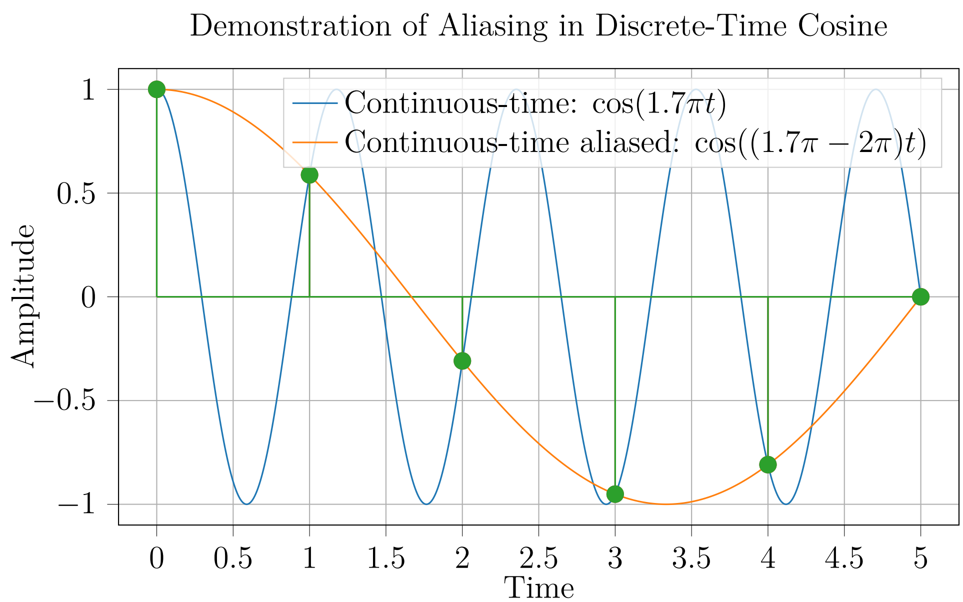


Figure 8.1: Demonstration of aliasing in discrete-time sinusoids. The blue curve shows a continuous-time sinusoid at $\omega = 1.7\pi$. The green dots show samples taken at $T = 1$. The orange curve shows the discrete-time sinusoid reconstructed from the samples, which is indistinguishable from a $\omega = 0.3\pi$ sinusoid sampled at the same rate.

Damped sinusoids

A damped sinusoid in continuous-time is typically defined as

$$x(t) = Ae^{\lambda t}\cos(\omega t + \phi),$$

where λ is a negative real-valued constant that controls the rate of decay. In discrete-time, the damped sinusoid is defined as

$$x[k] = A\gamma^k \cos(\Omega k + \phi),$$

where γ is a real-valued constant with $|\gamma| < 1$ that controls the rate of decay.

8.1.2 Operations on discrete-time signals

Similar to continuous-time signals, we can perform various operations on discrete-time signals, including time shifting, time scaling, and amplitude scaling.

Time shifting

We can shift a discrete-time signal $f[k]$ by an integer number of samples N using the following operation

$$f[k-N],$$

where a positive value of N shifts the signal to the right (delays it) and a negative value of N shifts the signal to the left (advances it). This is analogous to the time shifting operation for continuous-time signals discussed in Sec. **??**.

Time scaling

Time scaling in discrete-time is not as straightforward as in continuous-time. In continuous-time, we can scale the time variable by a factor a using the operation

$$f(at).$$

However, in discrete-time, we can only scale the time variable by integer factors. The time scaling operation for discrete-time signals is defined as

$$f[Mk],$$

where M is a positive integer. If $M > 1$, the signal is compressed in time. This is also known as downsampling or decimation. Consider the signal $f[k]$ which downsampled by a factor $M = 2$ such that $f_d[k] = f[2k]$. The resulting signal would be

$$\begin{aligned} f_d[0] &= f[0] \\ f_d[1] &= f[2] \\ f_d[2] &= f[4]. \end{aligned}$$

This suggests that we are keeping every even sample and discarding every odd sample. This is in contrast to the continuous-time case where scaling by a factor $a > 1$ results in a stretched signal, but does not necessarily discard any information.

If $0 < M < 1$, the signal is expanded in time. This is also known as upsampling or expansion. However, since M must be a positive integer. Consider the signal $f[k]$ which is upsampled by a factor of $L = 2$ such that $f_e[k] = f\left[\frac{k}{2}\right]$. The resulting signal would be

$$\begin{aligned} f_e[0] &= f[0] \\ f_e[1] &= f\left[\frac{1}{2}\right] = 0 \\ f_e[2] &= f[1] \\ f_e[3] &= f\left[\frac{3}{2}\right] = 0 \\ f_e[4] &= f[2]. \end{aligned}$$

This suggests that we are inserting zeros between each sample of the original signal. This is in contrast to the continuous-time case where scaling by a factor $0 < a < 1$ results in a compressed signal, but does not necessarily insert any new values. In practice, we will expand the signal and interpolate the missing values using various techniques such as zero-order hold, linear interpolation, or more advanced methods.

Amplitude scaling

Amplitude scaling in discrete-time is similar to continuous-time. We can scale the amplitude of a discrete-time signal $f[k]$ by a constant factor A using the operation

$$Af[k],$$

where A is a real-valued constant.

8.1.3 Signal size

In discrete-time, we can also define the energy and power of a signal. The energy E of a discrete-time signal $f[k]$ is defined as

$$E = \sum_{k=-\infty}^{\infty} |f[k]|^2.$$

A signal is considered an energy signal if its energy is finite ($E < \infty$) and its power is zero. The power P of a discrete-time signal $f[k]$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |f[k]|^2.$$

A signal is considered a power signal if its power is finite ($P < \infty$) and its energy is infinite.

8.2 Difference equations

In continuous-time systems, we often describe linear time-invariant (LTI) systems using differential equations with constant coefficients. The general form of such a differential equation is

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \ldots + b_1 \frac{df(t)}{dt} + b_0 f(t)$$

where $y(t)$ is the output signal, $f(t)$ is the input signal, and a_i and b_j are constant coefficients.

In discrete-time systems, we cannot take a continuous derivative, so we use *difference equations* to describe LTI systems. A difference equation relates the current output sample to past output samples and current and past input samples. The general form of a linear constant-coefficient difference equation is

$$y[k]+a_{n-1}y[k-1]+a_{n-2}y[k-2]+\ldots+a_0 y[k-n] = b_n f[k]+b_{n-1}f[k-1]+b_{n-2}f[k-2]+\ldots+b_0 f[k-m] \tag{8.1}$$

where $y[k]$ is the output signal, $f[k]$ is the input signal, and a_i and b_j are constant coefficients. The order of the difference equation is determined by the highest index of the output samples on the left-hand side. The form in (8.1) uses only delays (i.e., $y[k-1]$, $y[k-2]$, etc.). Alternatively, we can express the difference equation using advances (i.e., $y[k+1]$, $y[k+2]$, etc.) as

$$y[k+n]+a_{n-1}y[k+n-1]+a_{n-2}y[k+n-2]+\ldots+a_0 y[k] = b_n f[k+n]+b_{n-1}f[k+n-1]+b_{n-2}f[k+n-2]+\ldots+b_0 f[k+ \tag{8.2}$$

Both forms are equivalent and can be used interchangeably depending on the context.

In continuous-time, we used the Euler derivative notation of $\frac{d^n y(t)}{dt^n} = D^n y(t)$ to represent the nth derivative of $y(t)$. In discrete-time, we are going to use the shift operator E to represent a unit advance in the time index. That is,

$$E^n y[k] = y[k+n].$$

Using this notation, we can rewrite the difference equation in (8.2) as

$$(E^n + a_{n-1}E^{n-1} + a_{n-2}E^{n-2} + \ldots + a_0)y[k] = (b_n E^n + b_{n-1}E^{n-1} + b_2 E^{n-2} + \ldots + b_0)f[k].$$

We can define the polynomials

$$Q(E) = E^n + a_{n-1}E^{n-1} + a_{n-2}E^{n-2} + \ldots + a_0$$

and

$$P(E) = b_n E^n + b_{n-1}E^{n-1} + b_2 E^{n-2} + \ldots + b_0$$

such that the difference equation can be compactly expressed as

$$Q(E)y[k] = P(E)f[k].$$

Just like differential equations, difference equations can be solved using various methods, including time-domain techniques (e.g., iteration, z-transform) and frequency-domain techniques (e.g., discrete-time Fourier transform). We will explore these methods in subsequent sections. Here, we will focus on time-domain approaches.

8.2.1 Solving difference equations using iteration

One method to solve difference equations is through iteration. This involves expressing the output signal $y[k]$ in terms of the input signal $f[k]$ and the previous output samples. We will illustrate this method using a simple first-order difference equation. Consider the first-order difference equation

$$y[k+2] - y[k+1] + 0.24y[k] = f[k+2] - 2f[k+1]$$

with initial conditions $y[-1] = 2$, $y[-2] = 1$, and input $f[k] = ku[k]$. We can rearrange the equation to express $y[k+2]$ in terms of $y[k+1]$, $y[k]$, and $f[k]$ as

$$y[k+2] = y[k+1] - 0.24y[k] + f[k+2] - 2f[k+1].$$

We can then iteratively compute the output samples starting from the initial conditions. The first few iterations are as follows:

$k = -2$

$$\begin{aligned} y[0] &= y[-1] - 0.24y[-2] + f[0] - 2f[-1] \\ &= 2 - 0.24(1) + 0 - 0 \\ &= 1.76 \end{aligned}$$

$k = -1$

$$\begin{aligned} y[1] &= y[0] - 0.24y[-1] + f[1] - 2f[0] \\ &= 1.76 - 0.24(2) + 1 - 0 \\ &= 2.28 \end{aligned}$$

$k = 0$

$$
\begin{aligned}
y[2] &= y[1] - 0.24y[0] + f[2] - 2f[1] \\
&= 2.28 - 0.24(1.76) + 2 - 2(1) \\
&= 1.91
\end{aligned}
$$

Continuing this process, we can compute as many output samples as needed. This iterative method is straightforward for low-order difference equations but can become cumbersome for higher-order equations or longer sequences. In such cases, other methods could be more efficient.

8.2.2 Zero-input response

The zero-input response of a discrete-time system refers to the behavior of the system when the input signal is zero for all time steps (i.e., $f[k] = 0$ for all k). In this case, the output signal $y[k]$ is determined solely by the initial conditions of the system. To find the zero-input response, we can set the input terms in the difference equation to zero and solve for the output signal using the initial conditions. This is best understood through example.

Consider the first-order difference equation