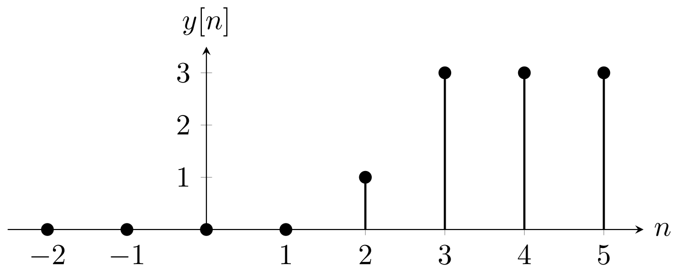
ECE 5210 hw02

1. Convolution

Compute the following convolutions.

a) Compute the convolution $y[n] = n(u[n-1] - u[n-3]) * u[n-1]$ for $n = -2, \ldots, 5$.

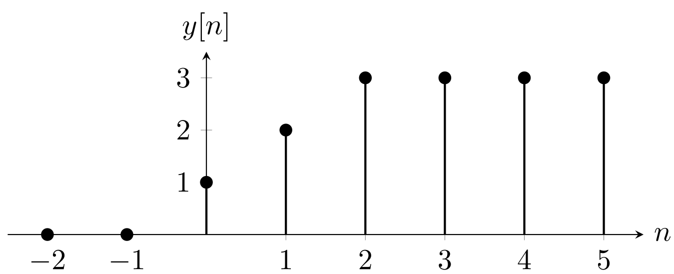
Solution:



Values: $y[n] = \{0, 0, 0, 0, 1, 3, 3, 3\}$ for $n = -2, \ldots, 5$.

b) Compute the convolution $y[n] = (u[n-1] - u[n-4]) * u[n+1]$ for $n = -2, \ldots, 5$.

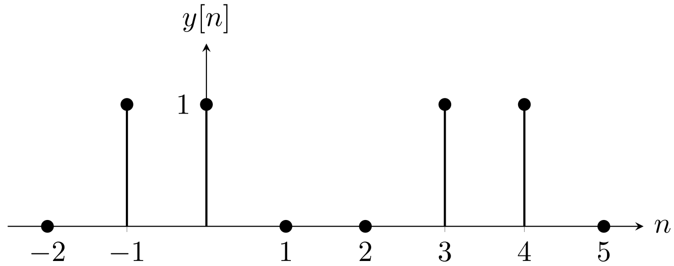
Solution:



Values: $y[n] = \{0, 0, 1, 2, 3, 3, 3, 3\}$ for $n = -2, \ldots, 5$.

c) Compute the convolution $y[n] = \sin\left(\frac{\pi n}{2}\right) u[n] * u[n+2]$ for $n = -2, \ldots, 5$.

Solution:



Values: $y[n] = \{0, 1, 1, 0, 0, 1, 1, 0\}$ for $n = -2, \dots, 5$.

d) Compute the convolution $y[n] = (u[n-1] - u[n-5]) * 0.5^n(u[n] - u[n-8])$. Express your answer as a piecewise function $y[n]$.

Solution:

$$y[n] = \begin{cases} 2(1 - 0.5^n) & 1 \le n \le 4 \\ 0.5^{n-5} - 0.5^{n-1} & 4 < n \le 8 \\ 0.5^{n-5} - 0.5^7 & 8 < n \le 11 \\ 0 & \text{otherwise} \end{cases}$$

2. Difference Equation

An engineer is asked to evaluate a simple signal processing system with a single digital filter. The input $x[n]$ is obtained a continuous-time signal at a sampling rate of $1/T$. The goal for $H(e^{j\omega})$ is to be a linear-phase FIR filter, and ideally it should have the following amplitude response such that it acts as a bandlimited differentiator

$$|H_{id}(e^{j\omega})| = \begin{cases} -\omega/T & \omega < 0 \\ \omega/T & \omega > 0. \end{cases}$$

For one implementation of $H(e^{j\omega})$, referred to as $H_1(e^{j\omega})$, the designer, motivated by the definition

$$\frac{d[x(t)]}{dt} = \lim_{\Delta t \to 0} \frac{x(t) - x(t - \Delta t)}{\Delta t},$$

chooses the system impulse response $h_1[n]$ so the input-output relationship is

$$y[n] = \frac{x[n] - x[n-1]}{T}.$$

a) Find $H_1(e^{j\omega})$.

> **Solution:** Taking the DTFT of the difference equation:
>
> $$Y(e^{j\omega}) = \frac{1}{T}(X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})) = \frac{1}{T}(1 - e^{-j\omega})X(e^{j\omega})$$
>
> $$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{T}(1 - e^{-j\omega})$$

b) We are interested in how well this approximation matches against the ideal response $H_{id}(e^{j\omega})$. Find the difference in the squared magnitudes between the two responses, i.e.,

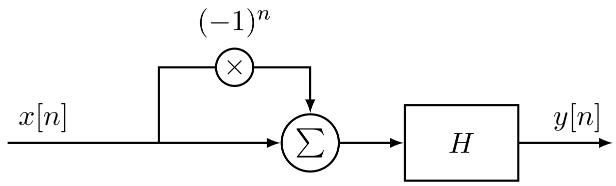
$$\text{error } = |H_1(e^{j\omega})|^2 - |H_{id}(e^{j\omega})|^2.$$

Represent your answer as 6th order polynomial function of ω.

> **Solution:**
>
> $$|H_{id}(e^{j\omega})|^2 = \frac{\omega^2}{T^2}$$
>
> $$|H_1(e^{j\omega})|^2 = \left|\frac{1}{T}(1 - e^{-j\omega})\right|^2 = \frac{1}{T^2}|1 - \cos\omega + j\sin\omega|^2 = \frac{1}{T^2}((1-\cos\omega)^2 + \sin^2\omega)$$
>
> $$= \frac{1}{T^2}(1 - 2\cos\omega + \cos^2\omega + \sin^2\omega) = \frac{1}{T^2}(2 - 2\cos\omega) = \frac{2}{T^2}(1 - \cos\omega)$$
>
> Using Taylor series expansion for $\cos\omega \approx 1 - \frac{\omega^2}{2} + \frac{\omega^4}{24} - \frac{\omega^6}{720}$:
>
> $$|H_1(e^{j\omega})|^2 \approx \frac{2}{T^2}\left(1 - \left(1 - \frac{\omega^2}{2} + \frac{\omega^4}{24} - \frac{\omega^6}{720}\right)\right) = \frac{2}{T^2}\left(\frac{\omega^2}{2} - \frac{\omega^4}{24} + \frac{\omega^6}{720}\right)$$
>
> $$= \frac{\omega^2}{T^2} - \frac{\omega^4}{12T^2} + \frac{\omega^6}{360T^2}$$
>
> The error is:
>
> $$\text{error} = \left(\frac{\omega^2}{T^2} - \frac{\omega^4}{12T^2} + \frac{\omega^6}{360T^2}\right) - \frac{\omega^2}{T^2} = -\frac{\omega^4}{12T^2} + \frac{\omega^6}{360T^2}$$

3. DTFT Systems

Consider the discrete-time system below where the subsystem H is an ideal low-pass filter with a passband gain of 1 and a cutoff frequency of $\omega_c = \frac{\pi}{4}$.



If we have an input $x[n]$ that has a DTFT of

$$X(e^{j\omega}) = 2\Delta\left(\frac{\omega}{2\pi}\right)$$

from $-\pi < \omega \leq \pi$.

a) Find an expression for the output $Y(e^{j\omega})$ for $-\pi < \omega \leq \pi$.

Solution: The input spectrum is flat with amplitude 2 over the entire frequency range (since $\Delta(\omega/2\pi)$ is 1 for $|\omega| < \pi$). The filter $H(e^{j\omega})$ is an ideal LPF with cutoff $\pi/4$.

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \begin{cases} 2 & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \pi \end{cases}$$

b) Find $y[n]$ in the time domain.

Solution: Using the inverse DTFT or recognizing the sinc function form:

$$y[n] = \frac{1}{\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/4}^{\pi/4} = \frac{1}{\pi n j}(e^{j\pi n/4} - e^{-j\pi n/4}) = \frac{2}{\pi n}\sin(\frac{\pi n}{4})$$

For $n = 0$, using L'Hopital's rule or the integral directly:

$$y[0] = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} 2d\omega = \frac{1}{2\pi}(2)(\frac{\pi}{2}) = \frac{1}{2}$$

Using this, we can write the complete expression:

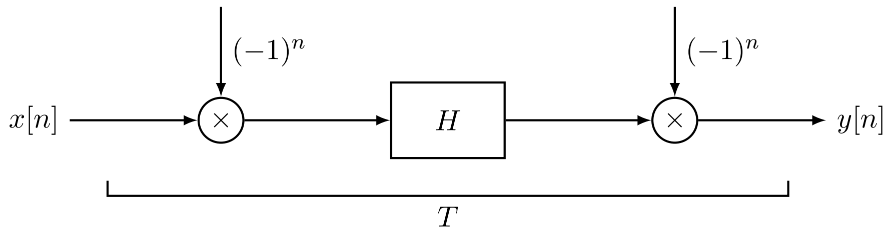
$$y[n] = \frac{2\sin(\frac{\pi n}{4})}{\pi n}$$

with $y[0] = \frac{1}{2}$. Or we could write it as as sinc function:

$$y[n] = \frac{1}{2}\text{sinc}\left(\frac{\pi n}{4}\right)$$

4. More DTFT Systems

Consider the system below. The entire system $y[n] = T\{x[n]\}$ will have the effect of being an ideal high-pass filter with a cutoff frequency of $\omega_c = \frac{\pi}{2}$. H is a subsystem within the larger system T.



a) Derive an expression for $H(e^{j\omega})$ such that the entire system will behave as an ideal high-pass filter as described above.

Solution: We that the first multiplication by $(-1)^n$ in time domain corresponds to a frequency shift of π in the frequency domain. Thus, the DTFT after the first multiplication is:

$$X_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

The subsystem H has a frequency response of $H(e^{j\omega})$. Therefore, the output of the subsystem H is:

$$X_2(e^{j\omega}) = X_1(e^{j\omega})H(e^{j\omega}) = X(e^{j(\omega-\pi)})H(e^{j\omega})$$

The second multiplication by $(-1)^n$ corresponds to another frequency shift of π. Thus, the final output DTFT is:

$$Y(e^{j\omega}) = X_2(e^{j(\omega-\pi)}) = X(e^{j(\omega-2\pi)})H(e^{j(\omega-\pi)}) = X(e^{j\omega})H(e^{j(\omega-\pi)})$$

Since we want the entire system to behave as an ideal high-pass filter with cutoff frequency $\omega_c = \frac{\pi}{2}$, we need:

$$Y(e^{j\omega}) = \begin{cases} 0 & |\omega| \le \frac{\pi}{2} \\ X(e^{j\omega}) & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

This implies that:

$$H(e^{j(\omega-\pi)}) = \begin{cases} 0 & |\omega| \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

By substituting $\theta = \omega - \pi$, we find:

$$H(e^{j\theta}) = \begin{cases} 1 & |\theta| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\theta| \le \pi \end{cases}$$

Therefore, $H(e^{j\omega})$ is an ideal low-pass filter with cutoff frequency $\frac{\pi}{2}$:

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

b) Find $h[n]$ that would satisfy such a system.

Solution: This inverse DTFT of an ideal low-pass filter with cutoff frequency $\frac{\pi}{2}$ is given by:

$$h[n] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi n j}(e^{j\pi n/2} - e^{-j\pi n/2}) = \frac{2}{\pi n}\sin\left(\frac{\pi n}{2}\right)$$

We can also represent this as a sinc function:

$$h[n] = \frac{1}{2}\text{sinc}\left(\frac{n}{4}\right)$$

5. DTFT Pairs

The DTFT pair

$$a^n u[n] \Longleftrightarrow \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

is given.

a) Determine the DTFT, $X(e^{j\omega})$, of the sequence

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n & n \le -1 \\ 0 & n \ge 0 \end{cases}.$$

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} -b^n e^{-j\omega n}$$

Letting $m = -n$, we have

$$-\sum_{m=1}^{\infty} (b^{-1}e^{j\omega})^m = -\frac{b^{-1}e^{j\omega}}{1 - b^{-1}e^{j\omega}} = \frac{1}{1 - be^{-j\omega}}$$

b) What restrictions must you put on b to make this a valid DTFT?

Solution: For the geometric series to converge, we need $|b^{-1}e^{j\omega}| < 1$, which implies $|b^{-1}| < 1$, or $|b| > 1$.

c) Determine the sequence $y[n]$ whose DTFT is

$$Y(e^{j\omega}) = \frac{2e^{-j\omega}}{1 + 2e^{-j\omega}}.$$

Solution: We can start by rewriting $Y(e^{j\omega})$ to match a known DTFT pair.

$$Y(e^{j\omega}) = \underbrace{\frac{2}{1 + 2e^{-j\omega}}}_{Y_1(e^{j\omega})} \cdot e^{-j\omega}.$$

We can solve for $y_1[n]$ first, which is the inverse DTFT of $Y_1(e^{j\omega})$

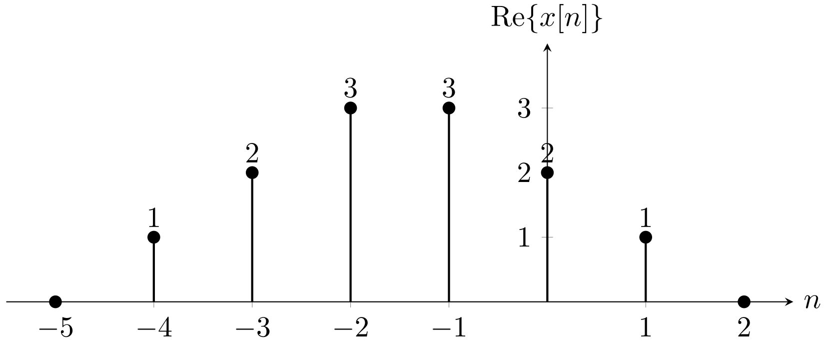
$$y_1[n] = 2(-2)^n u[-n-1] \quad \text{(from part (a) with } b = -2\text{)}.$$

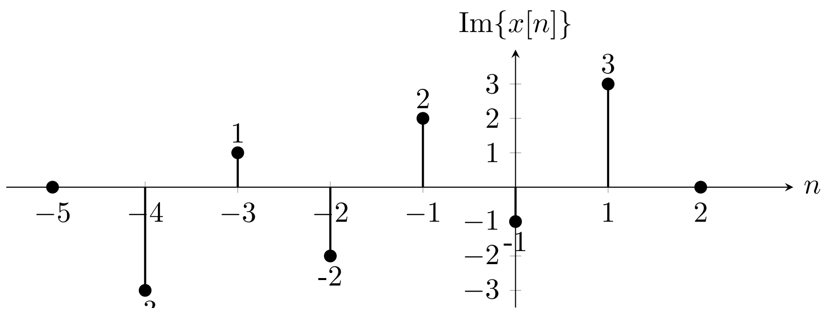
We realize that $Y_1(e^{j\omega})$ is multiplied by $e^{-j\omega}$, which corresponds to a time shift of 1 in the time domain. Therefore,

$$y[n] = y_1[n-1] = 2(-2)^{n-1}u[-(n-1)-1] = (-2)^n u[-n].$$

6. DTFT Values

$X(e^{j\omega})$ denotes the Fourier transform of the complex-valued signal $x[n]$, where the real and imaginary parts of $x[n]$ are given below. (The sequence is zero outside the interval shown.)





a) Evaluate $X(e^{j\omega})|_{\omega=0}$.

Solution: $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$.

Real part sum: $1 + 2 + 3 + 3 + 2 + 1 = 12$.

Imaginary part sum: $-3 + 1 - 2 + 2 - 1 + 3 = 0$.

So, $X(e^{j0}) = 12$.

b) Evaluate $X(e^{j\omega})|_{\omega=\pi}$.

Solution: $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x[n](-1)^n$.

Real part alternating sum: $1(-1)^{-4} + 2(-1)^{-3} + 3(-1)^{-2} + 3(-1)^{-1} + 2(-1)^0 + 1(-1)^1 = 1 - 2 + 3 - 3 + 2 - 1 = 0$.

Imaginary part alternating sum: $-3(-1)^{-4} + 1(-1)^{-3} - 2(-1)^{-2} + 2(-1)^{-1} - 1(-1)^0 + 3(-1)^1 = -3 - 1 - 2 - 2 - 1 - 3 = -12$.

So, $X(e^{j\pi}) = -12j$.

c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

Solution:

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega\bigg|_{n=0} = 2\pi x[0]$$

From the graph, $x[0] = 2 - 1j$, so the integral is:

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi(2 - 1j)$$

7. Determining $h[n]$

Consider some input signal $x[n] = -\delta[n] + \delta[n-1]$ which passes through some causal system H which gives some output signal

$$y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3].$$

a) Find the impulse response $h[n]$ for this LTI system.

Solution: We notice that if the input is $x[n] = -\delta[n]+\delta[n-1]$, then the output can be expressed as the convolution of the input with the impulse response:

$$y[n] = x[n] * h[n] = (-\delta[n] + \delta[n-1]) * h[n] = -h[n] + h[n-1]$$

Given the length property of discrete convolution, the length of $h[n]$ must be 3 or less since the length of $y[n]$ is 4 and the length of $x[n]$ is 2. Therefore, we can express $h[n]$ generically as

$$h[n] = h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2]$$

. Because the system is causal, we know that output $y[n]$ can be written as

$$y[n] = -h[0]\delta[n] - h[1]\delta[n-1] - h[2]\delta[n-2] + h[0]\delta[n-1] + h[1]\delta[n-2] + h[2]\delta[n-3]$$

Grouping the delta functions, we have

$$y[n] = -h[0]\delta[n] + (-h[1] + h[0])\delta[n-1] + (-h[2] + h[1])\delta[n-2] + h[2]\delta[n-3]$$

Equating coefficients with the given output $y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$, we have the following system of equations:

$$-h[0] = 1$$

$$-h[1] + h[0] = 1$$

$$-h[2] + h[1] = -1$$

$$h[2] = -1.$$

Solving this system, we find $h[0] = -1$, $h[1] = -2$, and $h[2] = -1$. Therefore, the impulse response is

$$h[n] = -\delta[n] - 2\delta[n-1] - \delta[n-2].$$

b) Find $y_2[n]$, which is the system response to the signal $x_2[n] = \delta[n] - \delta[n-5]$.

Solution: The output $y_2[n]$ can be found by convolving the input $x_2[n]$ with the impulse response $h[n]$:

$$y_2[n] = x_2[n] * h[n] = (\delta[n] - \delta[n-5]) * (-\delta[n] - 2\delta[n-1] - \delta[n-2]).$$

If we evaluate the convolution, we get

$$y_2[n] = -\delta[n] - 2\delta[n-1] - \delta[n-2] + \delta[n-5] + 2\delta[n-6] + \delta[n-7].$$