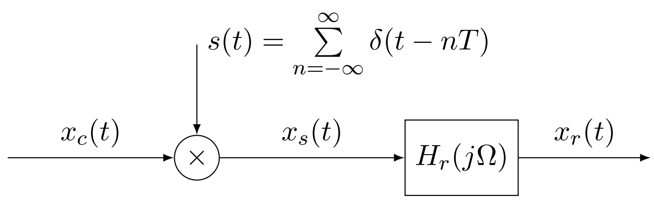
ECE 5210 hw05

1. Sampling and Reconstruction

Consider the representation of the process of sampling followed by reconstruction shown below.



Assume that the input signal is

$$x_c(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3)$$

The frequency response of the reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \leq \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

a) Assume that $f_s = 1/T = 500\,\text{samples/sec}$, what is the output $x_r(t)$?

Solution: The sampling frequency is $\Omega_s = 2\pi f_s = 1000\pi$ rad/s. The Nyquist frequency is $\Omega_N = \Omega_s/2 = 500\pi$ rad/s. The input frequencies are $\Omega_1 = 100\pi$ rad/s and $\Omega_2 = 300\pi$ rad/s. Since both $\Omega_1 < 500\pi$ and $\Omega_2 < 500\pi$, no aliasing occurs. The reconstruction filter passes frequencies up to $\pi/T = \Omega_N = 500\pi$. Therefore, the reconstruction is perfect:

$$x_r(t) = x_c(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3)$$

b) Assume that $f_s = 1/T = 250\,\text{samples/sec}$, what is the output $x_r(t)$?

Solution: The sampling frequency is $\Omega_s = 2\pi f_s = 500\pi$ rad/s. The Nyquist frequency is $\Omega_N = 250\pi$ rad/s. The input frequencies are:

- $\Omega_1 = 100\pi$. Since $100\pi < 250\pi$, this term is not aliased.
- $\Omega_2 = 300\pi$. Since $300\pi > 250\pi$, this term is aliased.

The alias of Ω_2 is found by shifting by integer multiples of Ω_s into the range $[-\Omega_N, \Omega_N]$.

$$\Omega_{2,alias} = 300\pi - 500\pi = -200\pi$$

The cosine term becomes:

$$\cos(300\pi t + \pi/3) \to \cos(-200\pi t + \pi/3) = \cos(200\pi t - \pi/3)$$

The reconstructed signal is:

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(200\pi t - \pi/3)$$

c) What if you wanted the output to look like

$$x_r(t) = A + 2\cos(100\pi t - \pi/4)$$

where A is a constant. What is the sampling rate f_s and what is the numerical value of A?

Solution: We require the first term $2\cos(100\pi t - \pi/4)$ to be preserved, and the second term $\cos(300\pi t + \pi/3)$ to become a DC constant A. For the second term to alias to DC ($\Omega = 0$), we must have

$$300\pi = k\Omega_s = k(2\pi f_s)$$

for some integer k. Choosing $k = 1$, we get $\Omega_s = 300\pi$, so $f_s = 150$ Hz. Let's verify this sampling rate for the first term. With $f_s = 150$, $\Omega_s = 300\pi$ and $\Omega_N = 150\pi$. The first term has frequency 100π. Since $|100\pi| < 150\pi$, it is not aliased and is preserved. The second term samples as:

$$x_2[n] = \cos(300\pi(nT) + \pi/3) = \cos\left(300\pi n \frac{1}{150} + \pi/3\right) = \cos(2\pi n + \pi/3)$$

Since n is an integer, $2\pi n$ is a multiple of 2π, so

$$x_2[n] = \cos(\pi/3) = 0.5$$

The reconstruction filter $H_r(j\Omega)$ is an ideal lowpass filter with cutoff $\pi/T = \Omega_N = 150\pi$. The constant sequence 0.5 samples corresponds to a DC impulse at $\omega = 0$ in DTFT, which maps to analog DC. So the output is

$$x_r(t) = 2\cos(100\pi t - \pi/4) + 0.5$$

Thus,

$$f_s = 150 \text{ Hz}, \quad A = 0.5$$

2. Sampling and Reconstruction (Midterm 1 2024)

Consider the continuous-time signal $x_c(t) = \text{sinc}^2(\pi t)$.
Recall the Fourier transform pairs $\frac{W}{\pi}\text{sinc}\,(Wt) \Longleftrightarrow \text{rect}\left(\frac{\Omega}{2W}\right)$ and $\frac{W}{2\pi}\text{sinc}^2\left(\frac{Wt}{2}\right) \Longleftrightarrow \Delta\left(\frac{\Omega}{2W}\right)$.

a) What is the bandwidth of this signal and the minimum sampling rate to satisfy Nyquist in hertz?

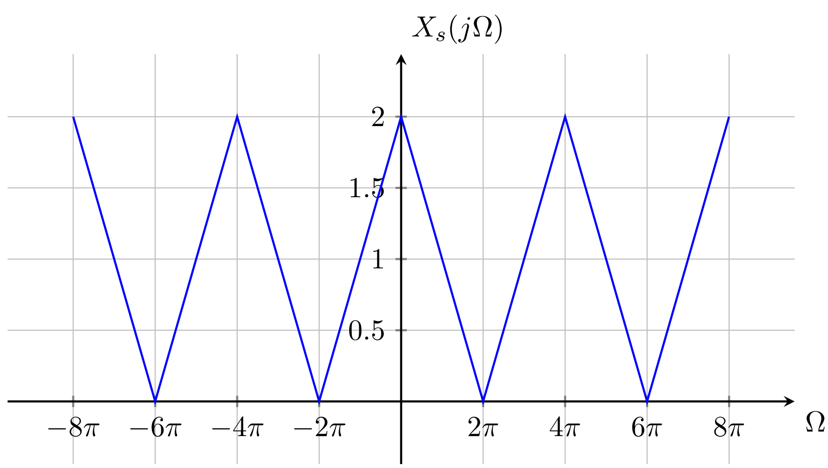
Solution: If we take the Fourier transform of $x_c(t)$, we get

$$X_c(j\Omega) = \Delta\left(\frac{\Omega}{4\pi}\right)$$

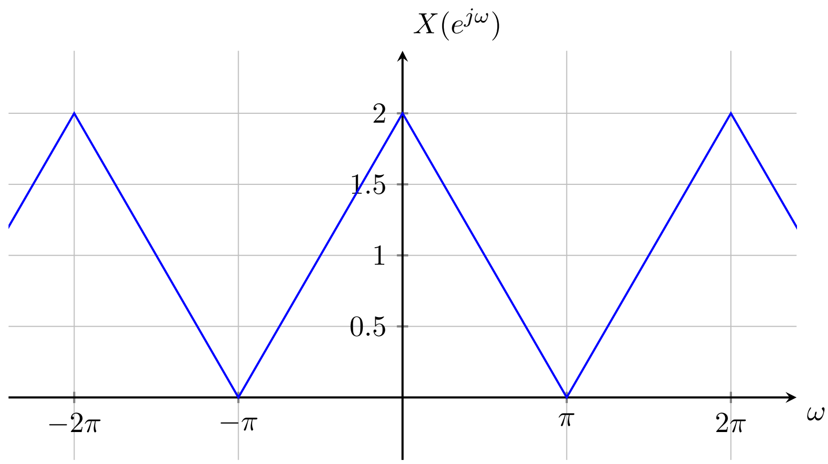
The triangle function is nonzero for $|\Omega| < 2\pi$ rad/s, so the bandwidth is 2π rad/s or 1 Hz. The Nyquist sampling rate is twice the bandwidth, so $f_s = 2\,\text{Hz}$.

b) Suppose we sampled this signal right at Nyquist, sketch the resulting DTFT $X(e^{j\omega})$ from $-2\pi \leq \omega \leq 2\pi$.

Solution: If we sampled at Nyquist, then $T = \frac{1}{2}$ seconds. The CTFT $X_c(j\Omega)$ is a triangle with base from -2π to 2π rad/s and peak at $\Omega = 0$ of height 1. The CTFT $X_s(j\Omega)$ of the sampled signal is a periodic repetition of $X_c(j\Omega)$ every $\Omega_s = \frac{2\pi}{T} = 4\pi$ rad/s. The amplitude of each repetition is scaled by $\frac{1}{T} = 2$. This sketched below.



The DTFT $X(e^{j\omega})$ is obtained by replacing Ω with $\omega/T = 2\omega$. Thus, $X(e^{j\omega}) = X_s(j2\omega)$. The resulting DTFT is a triangle spanning $[-\pi, \pi]$ with peak at $\omega = 0$ of height 2, as sketched below.



$X(e^{j\omega}) = 2\Delta\left(\frac{\omega}{\pi}\right)$ spanning $[-\pi, \pi]$.

c) What if decide to discard half of the samples such that

$$y[n] = \begin{cases} x[n] & n \, \text{even} \\ 0 & n \, \text{odd}. \end{cases}$$

Sketch the resulting DTFT $Y(e^{j\omega})$.

Solution: $Y(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X(e^{j(\omega-\pi)})]$. $X(e^{j\omega})$ is a triangle $2\Delta\left(\frac{\omega}{2\pi}\right)$. $X(e^{j(\omega-\pi)})$ is the shifted triangle. The sum of these two aliased triangles is a constant 2, but it will need to be divided by 2 which gives $Y(e^{j\omega}) = 1$ for $|\omega| < \pi$. Alternatively, $x[n] = \text{sinc}^2(n\pi/2)$. $x[0] = 1$. For n even $\neq 0$, $x[n] = 0$. $y[n]$ retains even samples (which are $\delta[n]$) and discards odd samples. So $y[n] = \delta[n]$ and the Fourier transform is $Y(e^{j\omega}) = 1$.

d) Suppose we were able to reconstruct the continuous-time signal using an ideal reconstruction filter

$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \pi/T \\ 0 & \text{else} \end{cases}.$$

Determine $y_c(t)$.

Solution:

Since $y[n] = \delta[n]$, then the reconstructed signal is simply

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] h_r(t - nT).$$

Because $y[n] = \delta[n]$, only the $n = 0$ term contributes, so

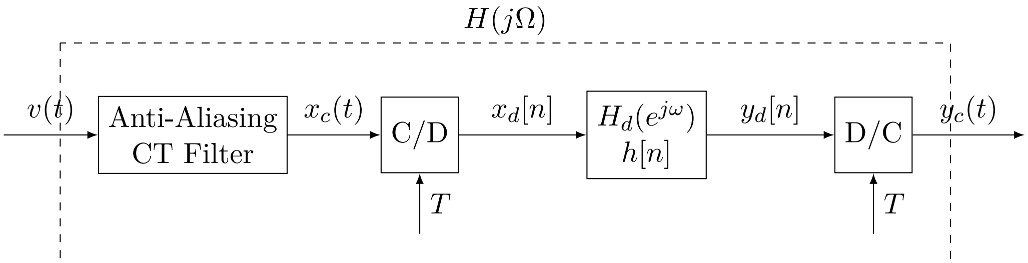
$$y_c(t) = h_r(t) = \text{sinc}(\pi t/T).$$

With $T = \frac{1}{2}$ seconds, we have

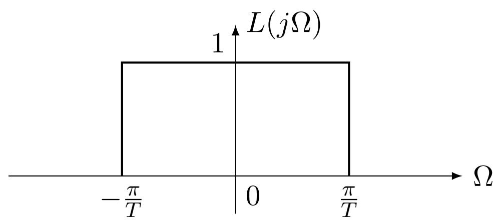
$$y_c(t) = \text{sinc}(2\pi t).$$

3. Impulse Invariance

Consider the system shown below.



The frequency response of the anti-aliasing filter is seen below.



The frequency response of the LTI discrete-time system between the converters is given by

$$H_d(e^{j\omega}) = e^{-j\omega/3}, \qquad |\omega| < \pi$$

a) What is the effective continuous-time frequency response of the overall system $H(j\Omega)$?

Solution: The overall system consists of an anti-aliasing filter $L(j\Omega)$ followed by a C/D converter, a discrete-time system $H_d(e^{j\omega})$ and a D/C converter. The combination of C/D, $H_d(e^{j\omega})$, and ideal D/C (reconstruction) acts as a continuous-time system with frequency response $H_{\text{eff}}(j\Omega)$ if input is bandlimited to $\pm\pi/T$:

$$H_{\text{eff}}(j\Omega) = \begin{cases} H_d(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

Since the anti-aliasing filter $L(j\Omega)$ strictly bandlimits the input $v(t)$ to $|\Omega| < \pi/T$, the aliasing is avoided, and the overall response is the product:

$$H(j\Omega) = L(j\Omega)H_{\text{eff}}(j\Omega)$$

Within the passband $|\Omega| < \pi/T$

$$H(j\Omega) = (1) \cdot H_d(e^{j\Omega T}) = e^{-j(\Omega T)/3}$$

Outside the passband, $L(j\Omega) = 0$, so $H(j\Omega) = 0$. Thus,

$$H(j\Omega) = \begin{cases} e^{-j\Omega T/3} & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

b) Determine the impulse response $h[n]$ of the discrete-time LTI system.

Solution: The impulse response is the inverse DTFT of $H_d(e^{j\omega})$.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

Substituting $H_d(e^{j\omega}) = e^{-j\omega/3}$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/3}e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1/3)}d\omega$$

$$h[n] = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1/3)}}{j(n-1/3)}\right]_{-\pi}^{\pi}$$

$$h[n] = \frac{1}{2\pi j(n-1/3)}\left(e^{j\pi(n-1/3)} - e^{-j\pi(n-1/3)}\right)$$

Using Euler's formula $2j\sin(\theta) = e^{j\theta} - e^{-j\theta}$

$$h[n] = \frac{\sin(\pi(n-1/3))}{\pi(n-1/3)}$$

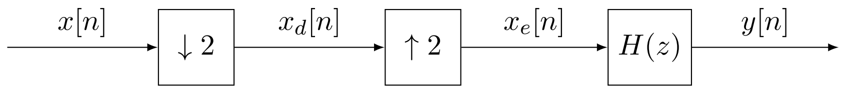
This can be also written as

$$h[n] = \mathrm{sinc}(\pi(n-1/3))$$

depending on the definition of sinc used.

4. Decimation and Interpolation 1

Consider the system below. For each of the following input signals $x[n]$ determine $y[n]$.



The system $H(z)$ has a frequency response from $-\pi < \omega \leq \pi$

$$H(e^{j\omega}) = \begin{cases} 2 & |\omega| < \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

a) Find $y[n]$ given $x[n] = \cos\left(\frac{\pi n}{4}\right)$.

Solution: First, the signal is downsampled by 2.

$$x_d[n] = x[2n] = \cos\left(\frac{\pi(2n)}{4}\right) = \cos\left(\frac{\pi n}{2}\right)$$

Next, the signal is upsampled by 2. This inserts zeros between samples.

$$x_e[n] = \begin{cases} x_d[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} = \begin{cases} \cos(\frac{\pi n}{4}) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

We can rewrite the zero-insertion as multiplication by $\frac{1+(-1)^n}{2}$:

$$x_e[n] = \cos\left(\frac{\pi n}{4}\right)\frac{1+\cos(\pi n)}{2} = \frac{1}{2}\cos\left(\frac{\pi n}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi n}{4}\right)\cos(\pi n)$$

Using the product-to-sum formula:

$$x_e[n] = \frac{1}{2}\cos\left(\frac{\pi n}{4}\right) + \frac{1}{4}\left(\cos\left(\frac{5\pi n}{4}\right) + \cos\left(-\frac{3\pi n}{4}\right)\right)$$

The frequencies present are $\omega = \pm\pi/4$ and $\omega = \pm 3\pi/4$ (since $5\pi/4 \equiv -3\pi/4$). The filter $H(e^{j\omega})$ is an ideal lowpass filter with cutoff $\pi/2$ and gain 2. It passes $\pi/4$ and rejects $3\pi/4$. The output is:

$$y[n] = 2 \cdot \frac{1}{2}\cos\left(\frac{\pi n}{4}\right) = \cos\left(\frac{\pi n}{4}\right)$$

b) Find $y[n]$ given $x[n] = \cos\left(\frac{5\pi n}{4}\right)$.

Solution: First, downsample by 2:

$$x_d[n] = x[2n] = \cos\left(\frac{5\pi(2n)}{4}\right) = \cos\left(\frac{5\pi n}{2}\right)$$

Since discrete-time cosine frequencies are periodic modulo 2π:

$$\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$$

So,

$$x_d[n] = \cos\left(\frac{\pi n}{2}\right)$$

This is the same intermediate signal $x_d[n]$ as in part 1. Therefore, the calculations for the upsampling and filtering stages are identical.

$$y[n] = \cos\left(\frac{\pi n}{4}\right)$$

Note that the original frequency $5\pi/4$ aliased to $\pi/4$ in the output.

c) Find $y[n]$ given $x[n] = \frac{\sin(\frac{\pi n}{3})}{\pi n}$.

Solution: The input $x[n]$ is a sinc function corresponding to an ideal lowpass signal with bandwidth $\omega_c = \pi/3$. Since $\pi/3 < \pi/2$, the signal is bandlimited within the Nyquist range of the downsampler (conceptually, if we view downsampling as reducing sampling rate, we need input bandlimited to π/M). Here, the decimation generally causes aliasing if bandwidth $> \pi/2$. Since $\pi/3 < \pi/2$, no aliasing occurs in the downsampling step. The ideal reconstruction (interpolation) filter $H(z)$ has cutoff $\pi/2$ and gain 2. Since the original signal was within $|\omega| < \pi/2$, and the downsampling/upsampling process with ideal filtering acts as an identity system for bandlimited signals (bandwidth $< \pi/2$), the output is perfect reconstruction.

Mathematically in frequency domain: $X(e^{j\omega}) = 1$ for $|\omega| \leq \pi/3$. After downsampling:

$$X_d(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2})\right)$$

The spectrum stretches by 2. New width $2\pi/3 < \pi$. No overlap. After upsampling:

$$X_e(e^{j\omega}) = X_d(e^{j2\omega})$$

This compresses the spectrum back to width $\pi/3$, but creates images at multiples of 2π (centered at π in the fundamental interval).

$$X_e(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \text{images centered at } \pi$$

The filter $H(e^{j\omega})$ (gain 2, cutoff $\pi/2$) removes the images (which start at $\pi - \pi/3 = 2\pi/3 > \pi/2$) and scales the passband by 2.

$$Y(e^{j\omega}) = 2 \cdot \frac{1}{2}X(e^{j\omega}) = X(e^{j\omega})$$

Thus,

$$y[n] = x[n] = \frac{\sin\left(\frac{\pi n}{3}\right)}{\pi n}$$

5. Decimation and Interpolation 2

Consider the system below.

$$x[n] \longrightarrow \boxed{\uparrow 5} \longrightarrow \boxed{\downarrow 3} \longrightarrow \boxed{\downarrow 5} \longrightarrow \boxed{\uparrow 3} \xrightarrow{y[n]}$$

Suppose we had some input $x[n] = \cos(0.3n)$, what is $y[n]$?

Solution: Let's analyze the sequence of operations. Let U_L denote the upsampling by L operator, and D_M denote the downsampling by M operator. The system outputs y from input x as:

$$y[n] = U_3\{D_5\{D_3\{U_5\{x[n]\}\}\}\}$$

First, observe that downsampling operations commute with each other: $D_5 D_3 = D_{15} = D_3 D_5$. So we can rewrite the system as:

$$y[n] = U_3\{D_3\{D_5\{U_5\{x[n]\}\}\}\}$$

We know that downsampling by M following upsampling by M recovers the original signal exactly: $D_M\{U_M\{x[n]\}\} = x[n]$. Therefore, the inner part $D_5\{U_5\{x[n]\}\}$ simplifies to $x[n]$. The system effectively reduces to just the outer operations:

$$y[n] = U_3\{D_3\{x[n]\}\}$$

Let $w[n] = D_3\{x[n]\}$. Then $w[n] = x[3n]$. Then $y[n] = U_3\{w[n]\}$. This inserts two zeros between each sample of w:

$$y[n] = \begin{cases} w[n/3] & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

Substituting $w[k] = x[3k]$:

$$y[n] = \begin{cases} x[3(n/3)] & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \begin{cases} x[n] & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

Given $x[n] = \cos(0.3n)$, the output is:

$$y[n] = \begin{cases} \cos(0.3n) & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

6. Decimation and Interpolation Similarities

Consider the two systems below.

System A: $\xrightarrow{x[n]} \boxed{\downarrow M} \xrightarrow{w_A[n]} \boxed{\uparrow L} \xrightarrow{y_A[n]}$

System B: $\xrightarrow{x[n]} \boxed{\uparrow L} \xrightarrow{w_B[n]} \boxed{\downarrow M} \xrightarrow{y_B[n]}$

a) For $M = 2$, $L = 3$, and any arbitrary $x[n]$, will $y_A[n] = y_B[n]$?

Solution: Let's analyze the mathematical expressions for both systems. For System A:

$$w_A[n] = x[Mn]$$

$$y_A[n] = \begin{cases} w_A[n/L] & n/L \text{ is integer} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x[M(n/L)] & n \text{ is multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

For System B:

$$w_B[n] = \begin{cases} x[n/L] & n \text{ is multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

$$y_B[n] = w_B[Mn] = \begin{cases} x[(Mn)/L] & Mn \text{ is multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

For equality, we need the values to match and the conditions for non-zero values to match. The value inside the argument of $x[\cdot]$ is Mn/L in both cases. The conditions differ:

- System A is non-zero if n is a multiple of L.
- System B is non-zero if Mn is a multiple of L.

For $M = 2, L = 3$:

- A: n is multiple of 3.
- B: $2n$ is multiple of 3. Since $\gcd(2, 3) = 1$, $2n$ is a multiple of 3 if and only if n is a multiple of 3.

The conditions match, so $y_A[n] = y_B[n]$. **Answer: True.**

(General rule: Upsampling by L and downsampling by M commute if and only if $\gcd(M, L) = 1$).

b) For $M = 4$, $L = 2$, and any arbitrary $x[n]$, will $y_A[n] = y_B[n]$?

Solution: Using the logic from the previous part:

- A: Non-zero if n is multiple of 2.
- B: Non-zero if $4n$ is multiple of 2. Since $4n = 2(2n)$, it is always a multiple of 2 for any integer n.

System B produces a non-zero output for every sample (it preserves all samples of x but at different rate/indices), whereas System A inserts zeros at odd indices. Thus, $y_A[n] \neq y_B[n]$. **Answer: False.**

c) For $M = 2$, $L = 4$, and any arbitrary $x[n]$, will $y_A[n] = y_B[n]$?

Solution: Using the logic from the first part:

- A: Non-zero if n is multiple of 4.
- B: Non-zero if $2n$ is multiple of 4. This implies $2n = 4k \implies n = 2k$. So B is non-zero for any even n.

Consider $n = 2$. System A: $n = 2$ is not a multiple of 4, so $y_A[2] = 0$. System B: $2n = 4$, which is a multiple of 4, so $y_B[2] = x[4/4] = x[1]$ (which is generally non-zero). Thus, $y_A[n] \neq y_B[n]$. **Answer: False.**