ECE 5210 hw03

1. DTFT

a) Suppose you were to implement a three sample moving average filter. The impulse response is

$$h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$

What is the frequency response of this filter?

Solution: The DTFT is linear and the DTFT of a shifted delta function is given by

$$\mathcal{F}\{\delta[n-k]\} = e^{-j\omega k}$$

Therefore, we have

$$H(e^{j\omega}) = \frac{1}{3}(1 + e^{-j\omega} + e^{-j2\omega})$$

We can simplify this further by recognizing that

$$1 + e^{-j2\omega} = 2\cos(\omega)e^{-j\omega}$$

Thus, we have

$$H(e^{j\omega}) = \frac{1}{3}e^{-j\omega}(1 + 2\cos(\omega)).$$

Similarly, we can use the table entry for a rectangular window to write

$$H(e^{j\omega}) = \frac{1}{3} \cdot \frac{\sin\left(\frac{3\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega}.$$

It is also worth mentioning that $h[n]$ is real-valued but not even symmatric. If it did have symmetry (i.e., if it were defined from $n = -1$ to $n = 1$), then the frequency response would be purely real-valued and the answer would be the same but without the linear phase component $e^{-j\omega}$.

b) Suppose we were to send a signal $x[n]$ into this filter that is some cosine $x[n] = \cos\left(\frac{2\pi}{3}n\right)$. What is the output $y[n]$?

Solution: Here we can use the frequency response to find the output. We see that here the input has a frequency of $\omega_0 = \frac{2\pi}{3}$. We can evaluate the frequency response at this frequency to find the gain and phase shift introduced by the filter. We have

$$H\left(e^{j\frac{2\pi}{3}}\right) = \frac{1}{3}\left(1 + e^{-j\frac{2\pi}{3}} + e^{-j\frac{4\pi}{3}}\right)$$

Evaluating the exponentials, we find

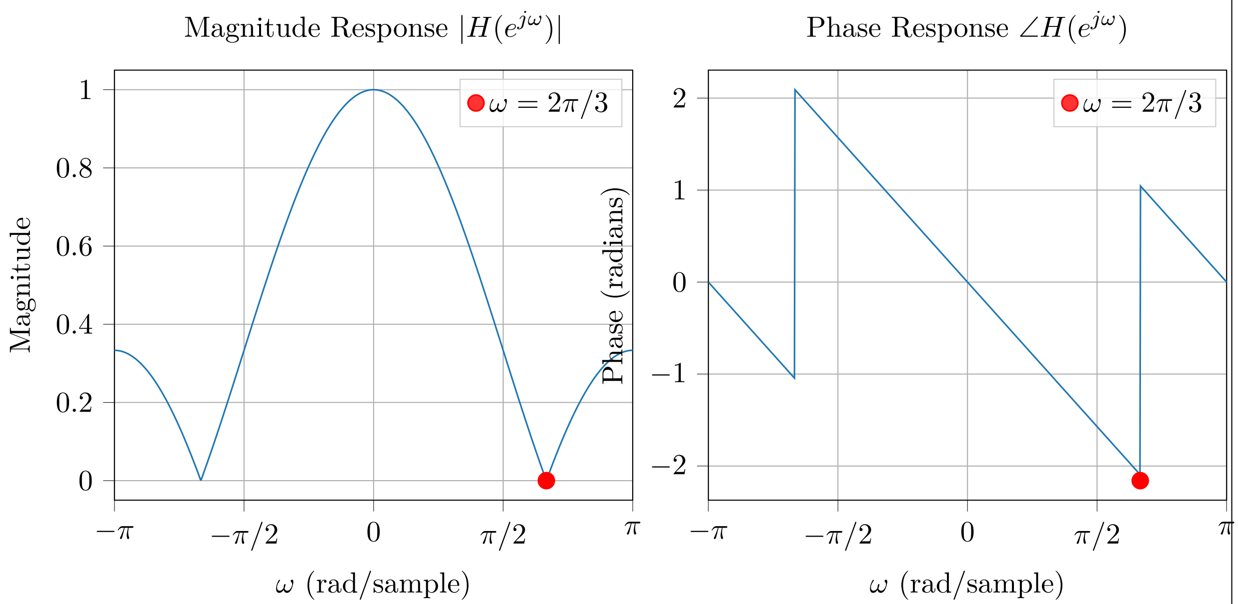
$$H\left(e^{j\frac{2\pi}{3}}\right) = \frac{1}{3}\left(1 - \frac{1}{2} - j\frac{\sqrt{3}}{2} - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 0$$

Thus, the output of the system is

$$y[n] = 0.$$

Essentially, with a input with a pure sinusoid, the frequency ω happens to be exactly at the null point of the moving average filter's frequency response, so the output is completely attenuated.

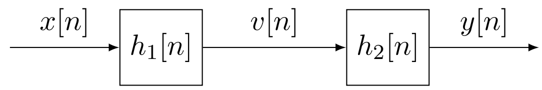
The frequency response magnitude and phase are plotted below, with the input frequency $\omega = 2\pi/3$ marked.



Note that the magnitude is zero at this frequency, confirming our result.

2. System response of cascade of LTI systems

Consider the system below with input $x[n]$ and output $y[n]$.



Subsystems $h_1[n]$ and $h_2[n]$ are known to be LTI. The output of $h_1[n]$ is $v[n]$ and can be described by the difference equation

$$v[n] = x[n] - x[n-1].$$

Additionally, subsystem $h_2[n]$ is described by

$$h_2[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}.$$

a) Find the impulse response $h[n]$.

Solution: To find the impulse response, we need to start with an input $x[n] = \delta[n]$. The intermediate output is then

$$v[n] = \delta[n] - \delta[n-1].$$

We can take this and put it into the next subsystem to find the overall impulse response. We have

$$y[n] = h_2[n] * v[n] = h_2[n] * (\delta[n] - \delta[n-1]).$$

Using the sifting property of the delta function, we have

$$y[n] = h_2[n] - h_2[n-1].$$

Therefore, the overall impulse response is

$$h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} - \frac{\sin(\frac{\pi}{2}(n-1))}{\pi(n-1)}.$$

b) Consider some input

$$x[n] = \cos(0.4\pi n) + \sin(0.6\pi n) + u[n].$$

What is the output $y[n]$? Note: The cosine and sine terms are everlasting in that they are not multiplied by $u[n]$.

Solution: Here, we are going to look at each term of the input separately. First, we can look at the cosine term. The frequency is $\omega_1 = 0.4\pi$. We can see that the frequency response of $h[n]$ is given by

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = (1 - e^{-j\omega}) \cdot \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Evaluating this at ω_1, we have

$$H(e^{j0.4\pi}) = (1 - e^{-j0.4\pi}) \cdot 1 = 1 - e^{-j0.4\pi}.$$

Therefore, the output due to the cosine term is

$$y_1[n] = |H(e^{j0.4\pi})| \cos(0.4\pi n + \angle H(e^{j0.4\pi})).$$

Next, we can look at the sine term. The frequency is $\omega_2 = 0.6\pi$. Evaluating the frequency response at this frequency, we have

$$H(e^{j0.6\pi}) = (1 - e^{-j0.6\pi}) \cdot 0 = 0.$$

Therefore, the output due to the sine term is

$$y_2[n] = 0.$$

Finally, we can look at the unit step term. If $x[n] = u[n]$, then the intermediate output is

$$v[n] = u[n] - u[n-1] = \delta[n].$$

Putting this into the next subsystem, we have

$$y_3[n] = h_2[n] * \delta[n] = h_2[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}.$$

Therefore, the overall output is given by

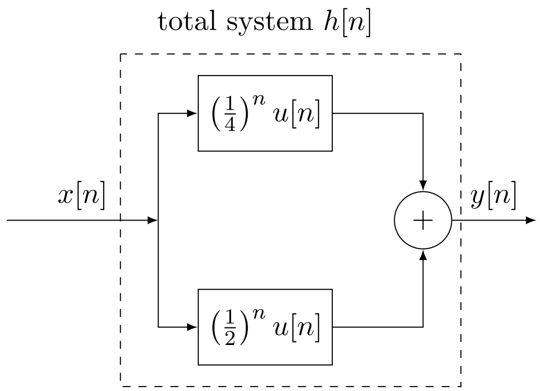
$$y[n] = y_1[n] + y_2[n] + y_3[n] = |H(e^{j0.4\pi})| \cos(0.4\pi n + \angle H(e^{j0.4\pi})) + \frac{\sin(\frac{\pi}{2}n)}{\pi n}.$$

Using actual values, we have

$$y[n] = 1.175 \cos(0.4\pi n + 54.0^\circ) + \frac{\sin(\frac{\pi}{2}n)}{\pi n}.$$

3. Parallel system

Consider the system below.



a) Find the impulse response for the entire system $h[n]$.

Solution: Since the systems are in parallel, the total impulse response is the sum of the individual impulse responses:

$$h[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n]$$

b) Find the DTFT of the entire system $H(e^{j\omega})$.

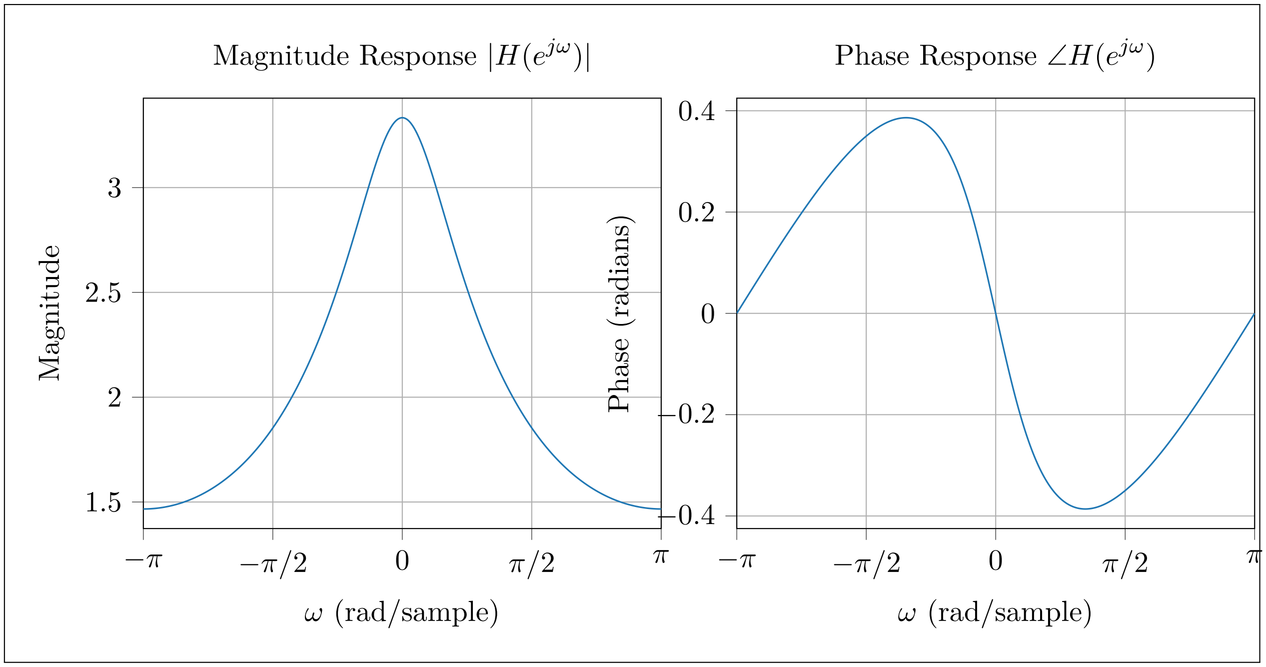
Solution: Using the linearity property and the known transform pair $a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}$:

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}} + \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Combining the terms:

$$H(e^{j\omega}) = \frac{(1 - \frac{1}{2}e^{-j\omega}) + (1 - \frac{1}{4}e^{-j\omega})}{(1 - \frac{1}{4}e^{-j\omega})(1 - \frac{1}{2}e^{-j\omega})} = \frac{2 - \frac{3}{4}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

The frequency response magnitude and phase are plotted below:



c) Find the difference equation for this system.

Solution: From the transfer function:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - \frac{3}{4}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

Cross-multiplying:

$$Y(e^{j\omega})\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\right) = X(e^{j\omega})\left(2 - \frac{3}{4}e^{-j\omega}\right)$$

Taking the inverse DTFT:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] - \frac{3}{4}x[n-1]$$

d) Suppose we were to send an input signal $x[n] = \cos(n)$ into this system. What is the output $y[n]$?

Solution: For an input $x[n] = \cos(\omega_0 n)$, the output is $y[n] = |H(e^{j\omega_0})|\cos(\omega_0 n + \angle H(e^{j\omega_0}))$. Here $\omega_0 = 1$ rad/sample.

$$H(e^{j1}) = \frac{2 - 0.75e^{-j1}}{1 - 0.75e^{-j1} + 0.125e^{-j2}}$$

Let $A = |H(e^{j1})|$ and $\phi = \angle H(e^{j1})$.

$$y[n] = 2.287\cos(n - 22.041^\circ)$$

4. Deconvolution

Oftentimes we know some signal $y[n]$, which is the output of some LTI system H. We might even be lucky enough to know the system's impulse function $h[n]$. However, we are interested in the input signal $x[n]$, which is unknown. This process is called *deconvolution* and is very common in image processing (e.g., removing camera shake blurring from photos).

Suppose we have a system with an impulse function

$$h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

and we have some output

$$y[n] = 3\delta[n-1] - \delta[n-2] + 5\delta[n-3] + 3\delta[n-4] + 2\delta[n-5].$$

Please determine the input $x[n]$.

Solution: We know that $y[n] = x[n] * h[n]$. Since $h[n]$ is causal and starts at $n = 0$, and $y[n]$ starts at $n = 1$, we can infer that $x[n]$ starts at $n = 1$. Let's perform the deconvolution step-by-step (or by polynomial division).

$$y[n] = 3x[n] + 2x[n-1] + 1x[n-2]$$

For $n = 1$:

$$y[1] = 3x[1] \implies 3 = 3x[1] \implies x[1] = 1$$

For $n = 2$:

$$y[2] = 3x[2] + 2x[1] \implies -1 = 3x[2] + 2(1) \implies 3x[2] = -3 \implies x[2] = -1$$

For $n = 3$:

$$y[3] = 3x[3] + 2x[2] + x[1] \implies 5 = 3x[3] + 2(-1) + 1 \implies 5 = 3x[3] - 1 \implies 3x[3] = 6 \implies x[3] = 2$$

For $n = 4$:

$$y[4] = 3x[4] + 2x[3] + x[2] \implies 3 = 3x[4] + 2(2) + (-1) \implies 3 = 3x[4] + 3 \implies x[4] = 0$$

Checking remaining terms: For $n = 5$:

$$y[5] = 3x[5] + 2x[4] + x[3] \implies 2 = 3(0) + 2(0) + 2 \implies 2 = 2 \quad \text{(Consistent)}$$

Thus, the input signal is:

$$x[n] = \delta[n-1] - \delta[n-2] + 2\delta[n-3]$$

You could also approach this problem using the z-transform or frequency domain methods, but the step-by-step method is often more intuitive for discrete signals. First, we recognize that the convolution in the time domain corresponds to multiplication in the z-domain:

$$Y(z) = H(z)X(z)$$

where

$$H(z) = 3 + 2z^{-1} + z^{-2}$$

and

$$Y(z) = 3z^{-1} - z^{-2} + 5z^{-3} + 3z^{-4} + 2z^{-5}.$$

Therefore, we can solve for $X(z)$:

$$X(z) = \frac{Y(z)}{H(z)}.$$

Performing the polynomial division, we find:

$$X(z) = z^{-1} - z^{-2} + 2z^{-3}.$$

Taking the inverse z-transform, we arrive at the same result:

$$x[n] = \delta[n-1] - \delta[n-2] + 2\delta[n-3].$$

5. Inverse DTFT

An LTI discrete-time system has a frequency response given by

$$H(e^{j\omega}) = \frac{(1 - je^{-j\omega})(1 + je^{-j\omega})}{1 - 0.8e^{-j\omega}}$$

a) Find the impulse response $h[n]$ of the system.

Solution: Here we can multiply out the numerator to get

$$H(e^{j\omega}) = \frac{1 + e^{-j2\omega}}{1 - 0.8e^{-j\omega}}$$

We can then split this into two separate fractions

$$H(e^{j\omega}) = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1 - 0.8e^{-j\omega}}$$

We can now recognize each of these as the DTFT of a scaled and shifted exponential sequence. The first term corresponds to

$$h_1[n] = (0.8)^n u[n]$$

and the second term corresponds to

$$h_2[n] = (0.8)^{n-2} u[n-2]$$

Therefore, the overall impulse response is given by

$$h[n] = (0.8)^n u[n] + (0.8)^{n-2} u[n-2]$$

b) From the frequency response, determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$.

Solution: From the frequency response, we can write

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{(1 - je^{-j\omega})(1 + je^{-j\omega})}{1 - 0.8e^{-j\omega}} X(e^{j\omega})$$

Multiplying both sides by the denominator, we have

$$(1 - 0.8e^{-j\omega})Y(e^{j\omega}) = (1 - je^{-j\omega})(1 + je^{-j\omega})X(e^{j\omega})$$

Multiplying out the right hand side, we get

$$(1 - 0.8e^{-j\omega})Y(e^{j\omega}) = (1 + e^{-j2\omega})X(e^{j\omega})$$

Now, taking the inverse DTFT of both sides, we have the difference equation

$$y[n] - 0.8y[n-1] = x[n] + x[n-2]$$

c) If the input to this system is

$$x[n] = 4 + 2\cos(\omega_0 n)$$

for $-\infty < n < \infty$, for what value of ω_0 will the output be of the form $y[n] = A$ where A is a constant for all $-\infty < n < \infty$? What is the constant A? Do not use ω_0 as multiples of 2π.

Solution: Essentially, we need to find where $H(e^{j\omega})$ is zero for the frequency component ω_0. We can see that the numerator of $H(e^{j\omega})$ is zero when

$$(1 - je^{-j\omega})(1 + je^{-j\omega}) = 0$$

This occurs when either factor is zero. Setting the first factor to zero, we have

$$1 - je^{-j\omega} = 0 \implies e^{-j\omega} = j \implies \omega = -\frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

Setting the second factor to zero, we have

$$1 + je^{-j\omega} = 0 \implies e^{-j\omega} = -j \implies \omega = \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$$

Therefore, we can choose $\omega_0 = \frac{\pi}{2}$ (or $\omega_0 = -\frac{\pi}{2}$) to eliminate the cosine term from the output. To find the constant A, we can evaluate the frequency response at $\omega = 0$ to find the gain for the DC component:

$$H(e^{j0}) = \frac{(1-j)(1+j)}{1-0.8} = \frac{1+1}{0.2} = 10$$

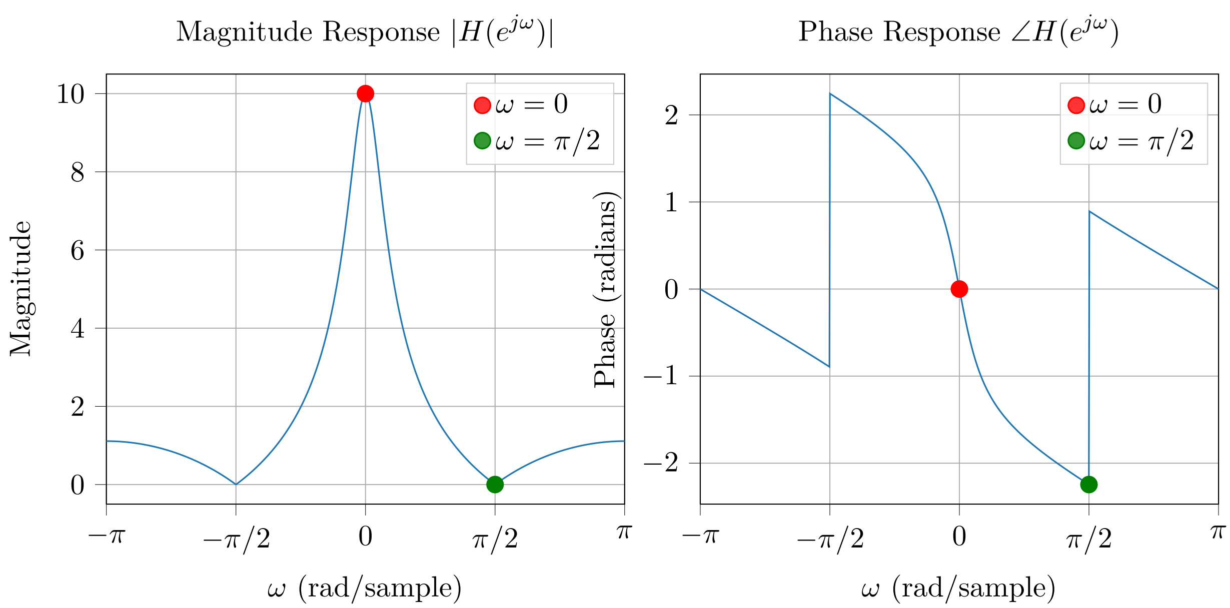
Therefore, the output will be

$$y[n] = 10 \cdot 4 = 40$$

Thus, we have

$$\omega_0 = \frac{\pi}{2}, \quad A = 40$$

We can see this in the figure below, where the frequency response magnitude is plotted and the zero at $\omega = \frac{\pi}{2}$ is indicated.



6. z-Transform

a) Determine the unilateral z-transform for: $\delta[n-1]$. Find the corresponding ROC.

Solution: We can start with the definition of the unilateral z-transform:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

For the given signal, we have:

$$X(z) = \sum_{n=0}^{\infty} \delta[n-1] z^{-n}$$

The delta function $\delta[n-1]$ is zero for all n except when $n = 1$. Therefore, the sum reduces to:

$$X(z) = z^{-1}$$

The ROC is all values of z except $z = 0$, so:

$$X(z) = z^{-1}, \quad \text{ROC: } |z| > 0$$

b) Determine the unilateral z-transform for: $\left(\frac{1}{2}\right)^{n-1} u[n-1]$. Find the corresponding ROC.

Solution: Let's approach this problem from the definition of the unilateral z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

For the given signal, we have

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} u[n-1] z^{-n}$$

The unit step function $u[n-1]$ is zero for $n < 1$, so we can change the lower limit of the sum to $n = 1$

$$X(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} z^{-n}$$

We can factor out a z^{-1} term

$$X(z) = z^{-1} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} z^{-(n-1)}$$

Now, we can change the index of summation by letting $m = n - 1$, which gives

$$X(z) = z^{-1} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} z^{-m}$$

This is a geometric series with common ratio $\frac{1}{2}z^{-1}$. The series converges when the magnitude of the common ratio is less than 1, i.e., when $|z| > \frac{1}{2}$. The sum of the geometric series is given by

$$\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$$

where $r = \frac{1}{2}z^{-1}$. Therefore, we have

$$X(z) = z^{-1} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z^{-1}}{1 - 0.5z^{-1}}$$

Thus, the unilateral z-transform is:

$$X(z) = \frac{z^{-1}}{1 - 0.5z^{-1}} = \frac{1}{z - 0.5}, \quad \text{ROC: } |z| > 0.5$$