ECE 5210 hw04

1. Sampling

a) If we have a continuous time domain signal

$$x_c(t) = \cos(40\pi t) + \sin(120\pi t)$$

and is sampled with a sampling period T to obtain a discrete-time signal

$$x[n] = \cos\left(\frac{2\pi n}{5}\right) - \sin\left(\frac{4\pi n}{5}\right).$$

What is T?

Solution: Matching the first term of the continuous signal to the sampled signal:

$$40\pi n T = \frac{2\pi n}{5} + 2\pi k n \implies 40T = \frac{2}{5} + 2k \implies T = \frac{1/5 + k}{20}$$

Matching the second term:

$$120\pi n T = -\frac{4\pi n}{5} + 2\pi l n \implies 60T = -\frac{2}{5} + l \implies T = \frac{-2/5 + l}{60}$$

Trying $k = 0$ in the first equation gives $T = \frac{1}{100}$. Checking if this satisfies the second equation: $60(1/100) = 0.6 = 3/5$. Using $-\frac{2}{5} + l = \frac{3}{5} \implies l = 1$. Since l is an integer for $k = 0$, $T = \frac{1}{100}$ seconds satisfies both conditions.

$$T = 0.01 \text{ s}$$

b) What is another value of T that will satisfy this C/D conversion?

Solution: Using the formula derived in part 1, we can try other integer values for k. If we try $k = 3$:

$$T = \frac{1/5 + 3}{20} = \frac{16/5}{20} = \frac{16}{100} = \frac{4}{25} = 0.16$$

Checking the second term condition:

$$60(0.16) = 9.6$$

$$-\frac{2}{5} + l = 9.6 \implies -0.4 + l = 9.6 \implies l = 10$$

Since l results in an integer, this value works.

$$T = \frac{4}{25} = 0.16 \text{ s}$$

c) If we were to sample $x_c(t)$ with $T = 12.5$ ms it would result in a single sinusoid in the form

$$x[n] = A\cos(\omega n + \phi).$$

What is the amplitude A, the frequency ω, and the phase ϕ?

Solution: With $T = 12.5$ ms $= 0.0125$ s:

$$x[n] = \cos(40\pi n(0.0125)) + \sin(120\pi n(0.0125))$$

$$x[n] = \cos(0.5\pi n) + \sin(1.5\pi n)$$

$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{3\pi}{2}n\right)$$

Note that $\sin\left(\frac{3\pi}{2}n\right) = \sin\left(2\pi n - \frac{\pi}{2}n\right) = \sin\left(-\frac{\pi}{2}n\right) = -\sin\left(\frac{\pi}{2}n\right)$. So,

$$x[n] = \cos\left(\frac{\pi}{2}n\right) - \sin\left(\frac{\pi}{2}n\right)$$

We want to match the form $A\cos(\omega n + \phi) = A(\cos(\omega n)\cos\phi - \sin(\omega n)\sin\phi)$. Comparing terms, we have $\omega = \frac{\pi}{2}$.

$$A\cos\phi = 1$$

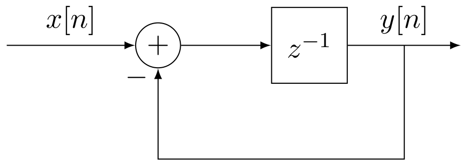
$$A\sin\phi = 1$$

Solving for phase: $\tan\phi = \frac{1}{1} = 1 \implies \phi = \frac{\pi}{4}$. Solving for amplitude: $A = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$$A = \sqrt{2}, \quad \omega = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}$$

2. Delay System

Consider a system H, which is depicted in the figure below.



a) Determine the transfer function of the system $H(z)$.

Solution: From the block diagram, we can write the difference equation. Let $w[n]$ be the signal after the summation and before the delay.

$$w[n] = x[n] - y[n]$$

The output $y[n]$ is a delayed version of $w[n]$:

$$y[n] = w[n-1]$$

Substituting for $w[n]$:

$$y[n] = x[n-1] - y[n-1]$$

taking the z-transform:

$$Y(z) = z^{-1}X(z) - z^{-1}Y(z)$$
$$Y(z)(1 + z^{-1}) = z^{-1}X(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 + z^{-1}} = \frac{1}{z+1}$$

b) Determine the impulse function of the system $h[n]$.

Solution: To find the impulse response, we can expand $H(z)$ into a power series in z^{-1}.

$$H(z) = z^{-1} \cdot \frac{1}{1 - (-z^{-1})}$$

Using the geometric series sum formula $\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k$ with $r = -z^{-1}$:

$$H(z) = z^{-1} \sum_{k=0}^{\infty} (-1)^k z^{-k} = \sum_{k=0}^{\infty} (-1)^k z^{-(k+1)}$$

Let $n = k + 1$. Then $k = n - 1$. The summation starts at $n = 1$.

$$H(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^{-n}$$

The impulse response is the coefficient of z^{-n}:

$$h[n] = \begin{cases} (-1)^{n-1} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Or using the unit step function:

$$h[n] = (-1)^{n-1}u[n-1]$$

c) Given some input $x[n] = u[n]$, determine the z-transform of the output $Y(z)$.

Solution: The z-transform of the unit step input $x[n] = u[n]$ is:

$$X(z) = \frac{1}{1 - z^{-1}}$$

The output is $Y(z) = H(z)X(z)$:

$$Y(z) = \left(\frac{z^{-1}}{1 + z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

$$Y(z) = \frac{z^{-1}}{(1 + z^{-1})(1 - z^{-1})} = \frac{z^{-1}}{1 - (z^{-1})^2} = \frac{z^{-1}}{1 - z^{-2}}$$

d) Find the time-domain output $y[n]$.

Solution: We have:

$$Y(z) = \frac{z^{-1}}{1 - z^{-2}}$$

We can expand this as a geometric series with ratio $r = z^{-2}$:

$$Y(z) = z^{-1}\sum_{k=0}^{\infty}(z^{-2})^k = z^{-1}\sum_{k=0}^{\infty} z^{-2k} = \sum_{k=0}^{\infty} z^{-(2k+1)}$$

The term $z^{-(2k+1)}$ corresponds to an impulse at time $n = 2k + 1$. This means $y[n]$ is non-zero only for odd values of $n \geq 1$, where it equals 1.

$$y[n] = \begin{cases} 1 & n \text{ is odd and } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Alternatively, doing partial fraction expansion:

$$Y(z) = \frac{z^{-1}}{(1 - z^{-1})(1 + z^{-1})} = \frac{A}{1 - z^{-1}} + \frac{B}{1 + z^{-1}}$$

$$z^{-1} = A(1 + z^{-1}) + B(1 - z^{-1})$$

Let $z^{-1} = 1$: $1 = 2A \implies A = 1/2$. Let $z^{-1} = -1$: $-1 = 2B \implies B = -1/2$.

$$Y(z) = \frac{1}{2}\frac{1}{1 - z^{-1}} - \frac{1}{2}\frac{1}{1 + z^{-1}}$$

Inverse z-transform:

$$y[n] = \frac{1}{2}u[n] - \frac{1}{2}(-1)^n u[n] = \frac{1}{2}u[n](1 - (-1)^n)$$

For $n < 0$, $y[n] = 0$. For $n \geq 0$: If n is even, $1 - (-1)^n = 1 - 1 = 0$. If n is odd, $1 - (-1)^n = 1 - (-1) = 2$, so $y[n] = 1$. This confirms the result:

$$y[n] = \begin{cases} 1 & n \text{ is positive and odd} \\ 0 & \text{otherwise} \end{cases}$$

3. Difference Equation

Consider a discrete LTI system which is described by the difference equation

$$y[n+2] - 5y[n+1] + 6y[n] = x[n+1]$$

a) Determine the transfer function $H(z)$.

Solution: Taking the Z-transform of the difference equation (assuming zero initial conditions):

$$z^2Y(z) - 5zY(z) + 6Y(z) = zX(z)$$

Divide by z^2:

$$Y(z) - 5z^{-1}Y(z) + 6z^{-2}Y(z) = z^{-1}X(z)$$
$$Y(z)(1 - 5z^{-1} + 6z^{-2}) = z^{-1}X(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 5z^{-1} + 6z^{-2}}$$

b) Determine the z-transform of $x[n]$ if $x[n] = u[n]$.

Solution: The Z-transform of the unit step function $u[n]$ is:

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

c) Find $Y(z)$.

Solution:

$$Y(z) = H(z)X(z) = \left(\frac{z^{-1}}{1 - 5z^{-1} + 6z^{-2}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

Factoring the denominator of $H(z)$:

$$1 - 5z^{-1} + 6z^{-2} = (1 - 2z^{-1})(1 - 3z^{-1})$$

So,

$$Y(z) = \frac{z^{-1}}{(1 - 2z^{-1})(1 - 3z^{-1})(1 - z^{-1})}$$

d) Find $y[n]$.

Solution: We perform partial fraction expansion on $Y(z)$.

$$Y(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-3z^{-1}} + \frac{C}{1-z^{-1}}$$

Solving for coefficients: For A (associated with pole at $z=2$):

$$A = Y(z)(1-2z^{-1})\Big|_{z^{-1}=1/2} = \frac{1/2}{(1-3(1/2))(1-(1/2))} = \frac{0.5}{(-0.5)(0.5)} = \frac{0.5}{-0.25} = -2$$

For B (associated with pole at $z=3$):

$$B = Y(z)(1-3z^{-1})\Big|_{z^{-1}=1/3} = \frac{1/3}{(1-2(1/3))(1-(1/3))} = \frac{1/3}{(1/3)(2/3)} = \frac{1/3}{2/9} = \frac{3}{2} = 1.5$$

For C (associated with pole at $z=1$):

$$C = Y(z)(1-z^{-1})\Big|_{z^{-1}=1} = \frac{1}{(1-2(1))(1-3(1))} = \frac{1}{(-1)(-2)} = \frac{1}{2} = 0.5$$

Thus,

$$Y(z) = \frac{-2}{1-2z^{-1}} + \frac{1.5}{1-3z^{-1}} + \frac{0.5}{1-z^{-1}}$$

Taking the inverse Z-transform:

$$y[n] = \left(-2(2)^n + 1.5(3)^n + 0.5\right) u[n]$$

4. Inverse z-transform

Part A

Consider $X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$.

a) Suppose $x[n]$ is causal. Find $x[n]$.

Solution: First, we factor the denominator:

$$1 - 2z^{-1} + z^{-2} = (1 - z^{-1})^2$$

So,

$$X(z) = \frac{1 + 2z^{-1}}{(1 - z^{-1})^2}$$

We can rewrite the numerator to separate terms into known transform pairs:

$$X(z) = \frac{1 - z^{-1} + 3z^{-1}}{(1 - z^{-1})^2} = \frac{1 - z^{-1}}{(1 - z^{-1})^2} + \frac{3z^{-1}}{(1 - z^{-1})^2} = \frac{1}{1 - z^{-1}} + \frac{3z^{-1}}{(1 - z^{-1})^2}$$

Using standard z-transform pairs for causal signals ($\mathcal{Z}^{-1}\{\frac{1}{1-az^{-1}}\} = a^n u[n]$ and $\mathcal{Z}^{-1}\{\frac{az^{-1}}{(1-az^{-1})^2}\} = na^n u[n]$ with $a = 1$):

$$x[n] = u[n] + 3nu[n] = (3n + 1)u[n]$$

b) Suppose $x[n]$ is anti-causal. Find $x[n]$.

Solution: For an anti-causal signal, the algebraic form is the same, but the region of convergence changes ($|z| < 1$), leading to time-reversed sequences. The inverse transform of $\frac{1}{1-z^{-1}}$ is $-u[-n-1]$. The inverse transform of $\frac{z^{-1}}{(1-z^{-1})^2}$ is $-nu[-n-1]$.

$$x[n] = -(1)u[-n - 1] - 3nu[-n - 1] = -(3n + 1)u[-n - 1]$$

Part B

Consider $X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}$.

a) Suppose $x[n]$ is causal. Find $x[n]$.

Solution: First, rewrite the denominator in standard form $(1 - az^{-1})$:

$$X(z) = \frac{5z^{-1}}{(1 - 2z^{-1}) \cdot 3(1 - \frac{1}{3}z^{-1})} = \frac{5z^{-1}}{3(1 - 2z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{\frac{5}{3}z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{3}z^{-1})}$$

Perform Partial Fraction Expansion:

$$X(z) = \frac{A}{1 - 2z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}$$

Solving for A (pole at $z = 2$):

$$A = X(z)(1 - 2z^{-1})\Big|_{z^{-1}=1/2} = \frac{\frac{5}{3}(\frac{1}{2})}{1 - \frac{1}{3}(\frac{1}{2})} = \frac{5/6}{1 - 1/6} = \frac{5/6}{5/6} = 1$$

Solving for B (pole at $z = 1/3$):

$$B = X(z)(1 - \frac{1}{3}z^{-1})\Big|_{z^{-1}=3} = \frac{\frac{5}{3}(3)}{1 - 2(3)} = \frac{5}{1 - 6} = \frac{5}{-5} = -1$$

So,

$$X(z) = \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - \frac{1}{3}z^{-1}}$$

For a causal signal ($|z| > 2$):

$$x[n] = (1)(2)^n u[n] - (1)(1/3)^n u[n] = (2^n - (1/3)^n)u[n]$$

b) Suppose $x[n]$ is anti-causal. Find $x[n]$.

Solution: For an anti-causal signal ($|z| < 1/3$), both poles correspond to left-sided sequences. Inverse of $\frac{1}{1-az^{-1}}$ is $-a^n u[-n-1]$.

$$x[n] = 1(-2^n u[-n-1]) - 1(-(1/3)^n u[-n-1])$$

$$x[n] = (-2^n + (1/3)^n)u[-n-1]$$

c) Suppose $x[n]$ is neither causal nor anti-causal. Find $x[n]$.

Solution: This corresponds to the annular ROC $1/3 < |z| < 2$. The pole at $z = 1/3$ is "inside" the ROC ($|z| > 1/3$), so it corresponds to a causal sequence. The pole at $z = 2$ is "outside" the ROC ($|z| < 2$), so it corresponds to an anti-causal sequence. From $X(z) = \frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}}$: Term 1 ($\frac{1}{1-2z^{-1}}$): Anti-causal part $\rightarrow$ $-2^n u[-n-1]$. Term 2 ($-\frac{1}{1-\frac{1}{3}z^{-1}}$): Causal part $\rightarrow$ $-(1/3)^n u[n]$.

$$x[n] = -2^n u[-n-1] - (1/3)^n u[n]$$

5. Notch Filtering

A significant problem in the recording of electrocardiograms (ECGs) is the appearance of unwanted 60 Hz interference in the output. The causes of this power line interference include magnetic induction, displacement currents in the leads on the patient's body, and equipment interconnections. Ultimately, we will want to get rid of this 60 Hz interference. Because this signal is analog, we could use an analog notch filter to remove it, but analog notch filters are a hassle to design. Instead, we are utilizing the C/D, DSP, and D/C signal processing approach discussed in class.

a) Assume that the bandwidth of the signal of interest is 240 Hz, that is (in radians per second),

$$X_a(j\Omega) = 0 \qquad |\Omega| > 2\pi \cdot 240\,\text{rad/s}$$

What should the minimum sampling period T_s be to satisfy Nyquist?

Solution: To satisfy the Nyquist criterion, the sampling frequency Ω_s must be at least twice the maximum frequency of the signal.

$$\Omega_{max} = 2\pi \cdot 240 \text{ rad/s} \implies f_{max} = 240 \text{ Hz}$$

$$f_s \geq 2f_{max} = 480 \text{ Hz}$$

The sampling period T_s is the inverse of the sampling frequency:

$$T_s = \frac{1}{f_s} = \frac{1}{480} \text{ s} \approx 2.083 \text{ ms}$$

b) The analog signal is converted into a discrete-time signal with an ideal C/D converter operating at the sampling frequency Ω_s based on the sampling period T_s in the previous part. The resulting signal $x[n] = x_a(nT_s)$ is then processed with a discrete-time system H that is described by the difference equation

$$y[n] = H\{x[n]\} = x[n] + ax[n-1] + bx[n-2]$$

where a and b are some constants. Please find the frequency response $H(e^{j\omega})$ in terms of a and b.

Solution: Taking the DTFT of the difference equation:

$$Y(e^{j\omega}) = X(e^{j\omega}) + ae^{-j\omega}X(e^{j\omega}) + be^{-j2\omega}X(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega})(1 + ae^{-j\omega} + be^{-j2\omega})$$

The frequency response is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + ae^{-j\omega} + be^{-j2\omega}$$

c) Solve for $h[n]$. Please solve for a and b such that the 60 Hz frequency is notched. We can assume that this unwanted interference can take the form

$$w_a(t) = A\sin(120\pi t)$$

in continuous time. If $h[n]$ is designed correctly, $w_a(t)$ should not appear in the output of the D/C converter.

Solution: We want to eliminate the frequency $f_0 = 60$ Hz. Given the sampling rate $f_s = 480$ Hz found in part 1, the corresponding digital frequency is:

$$\omega_0 = 2\pi\frac{f_0}{f_s} = 2\pi\frac{60}{480} = 2\pi\frac{1}{8} = \frac{\pi}{4}$$

For the filter to notch out this frequency, the frequency response must be zero at ω_0.

$$H(e^{j\pi/4}) = 0$$

$$1 + ae^{-j\pi/4} + be^{-j\pi/2} = 0$$

Using Euler's formula:

$$1 + a\left(\cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)\right) + b\left(\cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)\right) = 0$$

$$1 + a\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) + b(0 - j) = 0$$

$$\left(1 + a\frac{\sqrt{2}}{2}\right) - j\left(a\frac{\sqrt{2}}{2} + b\right) = 0$$

Equating the real and imaginary parts to zero: Real part: $1 + a\frac{\sqrt{2}}{2} = 0 \implies a = -\frac{2}{\sqrt{2}} = -\sqrt{2}$. Imaginary part: $a\frac{\sqrt{2}}{2} + b = 0 \implies b = -a\frac{\sqrt{2}}{2} = -(-\sqrt{2})\frac{\sqrt{2}}{2} = 1$. So the coefficients are $a = -\sqrt{2}$ and $b = 1$. The impulse response $h[n]$ corresponds to the coefficients of the difference equation (finite impulse response):

$$h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2]$$