Vector Graphics

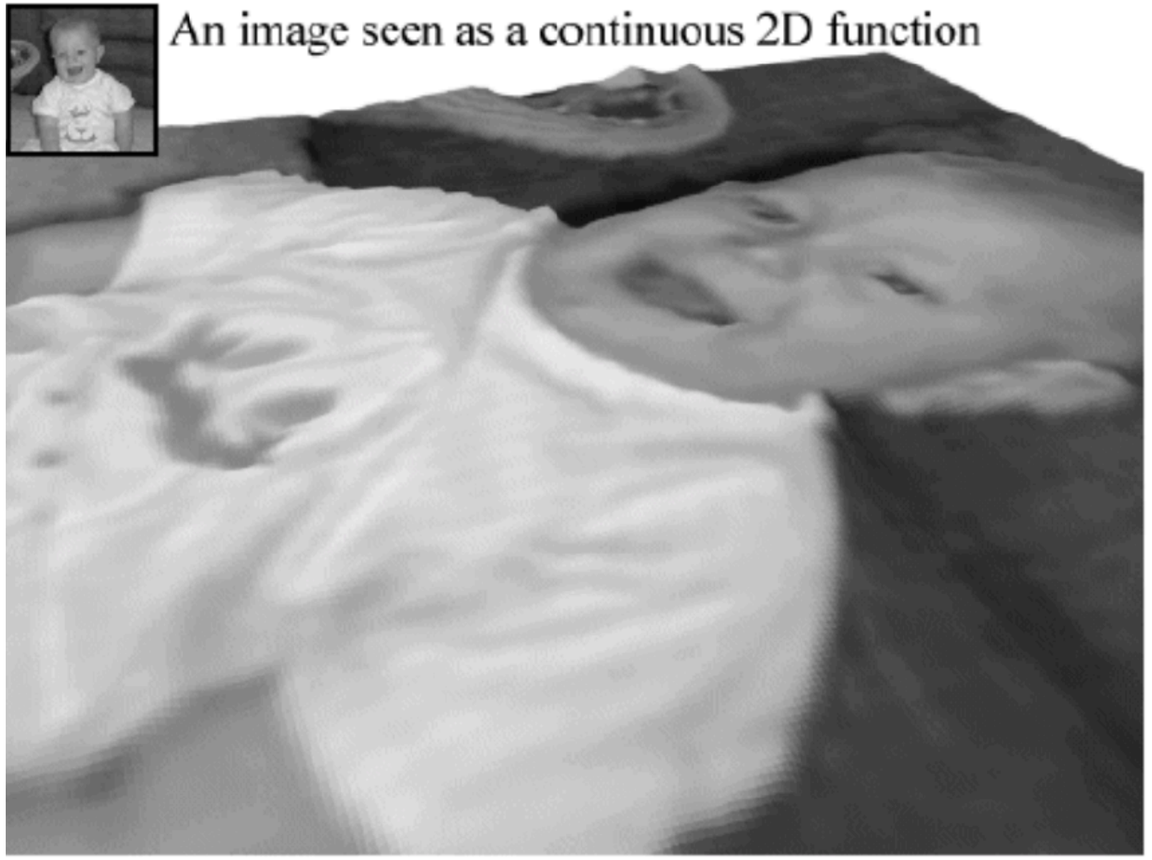
UCSD CSE 167

Tzu-Mao Li

Q: What are images?

An image is a 2D function

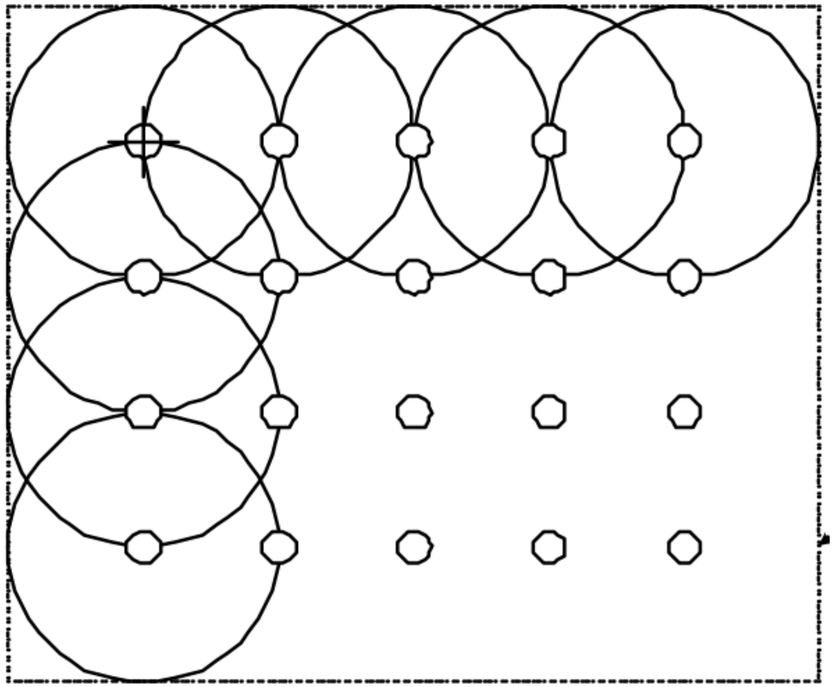
- an *image* is a function $I(x, y)$ of colors with $x, y \in \mathbb{R}$

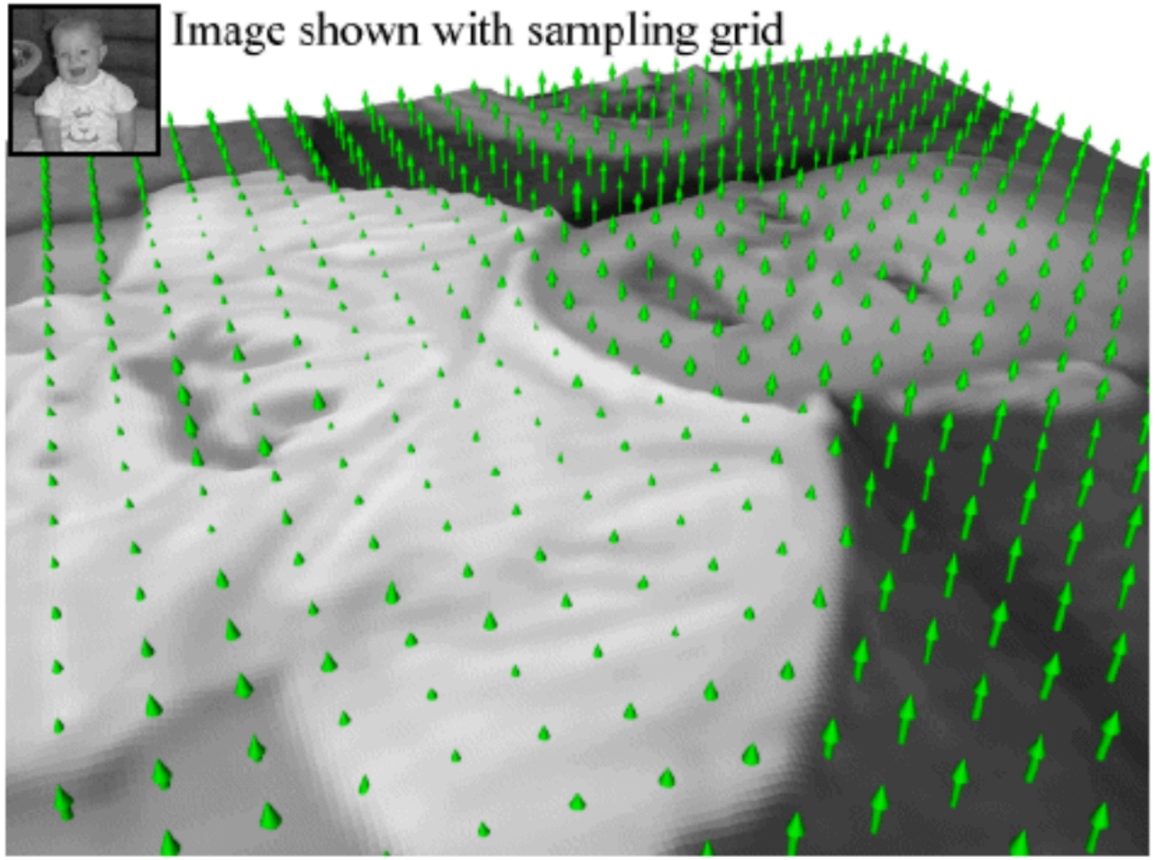


https://groups.csail.mit.edu/graphics/classes/6.837/F03/lectures/23_aliasing.pdf

A popular representation of image: raster images

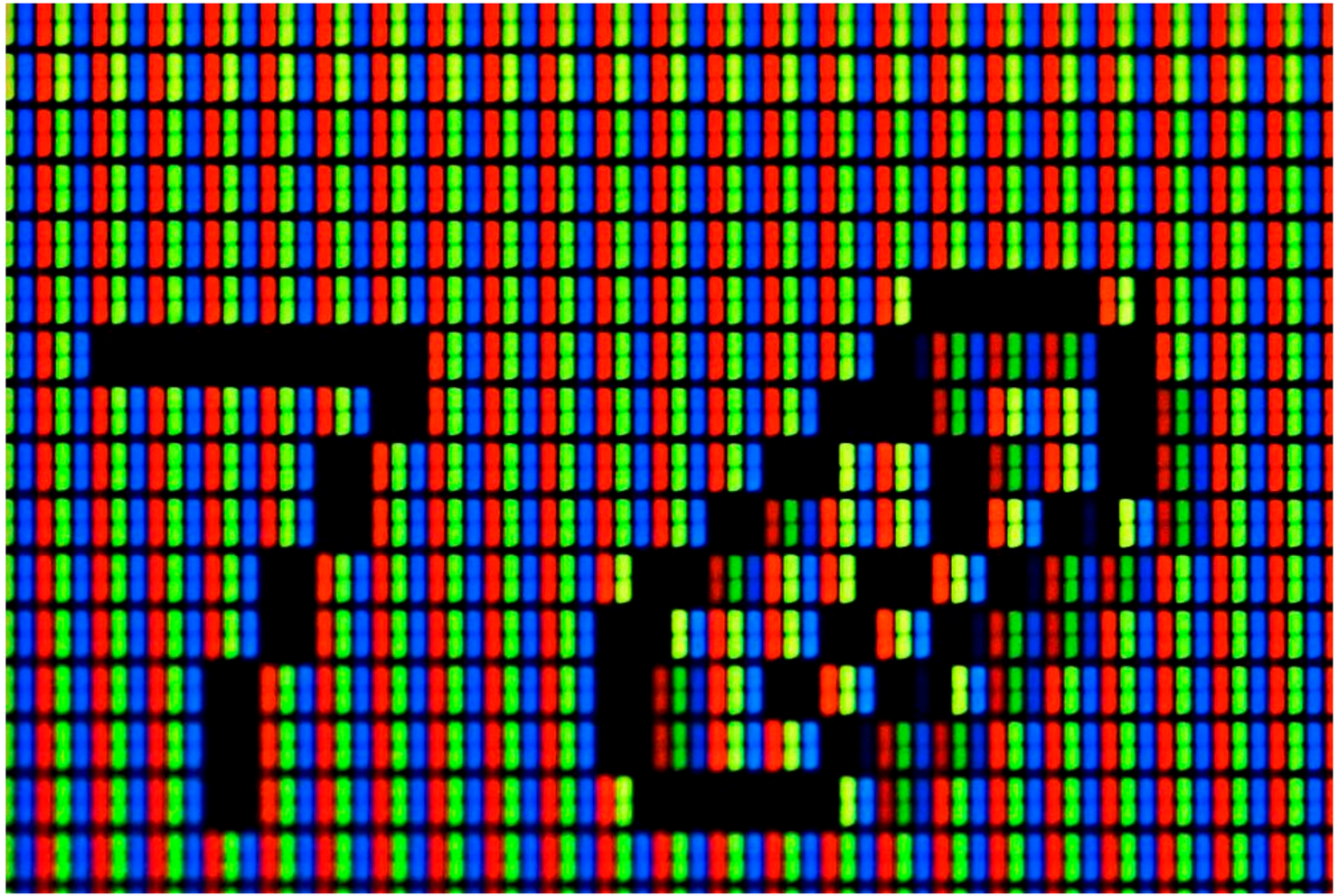
a “pixel” stores the average color around the pixel center





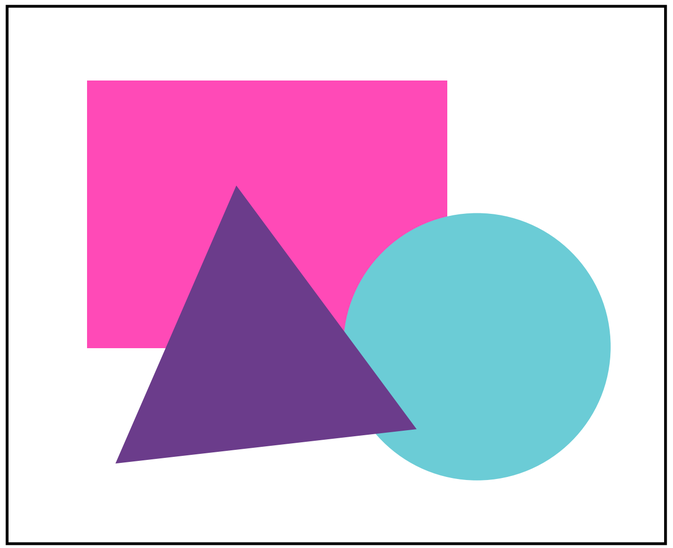
https://groups.csail.mit.edu/graphics/classes/6.837/F03/lectures/23_aliasing.pdf

A close-up look of an LCD screen



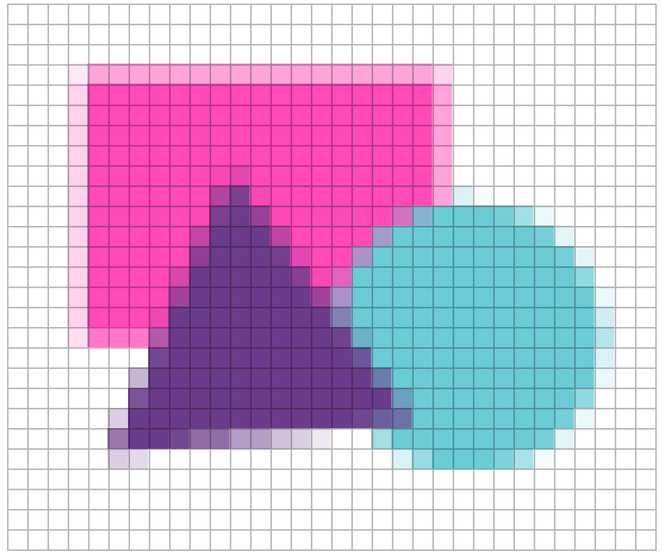
https://commons.wikimedia.org/wiki/File:LCD_TFT_Screen_Closeup.jpg

Rendering =
painting on the raster image



shape description

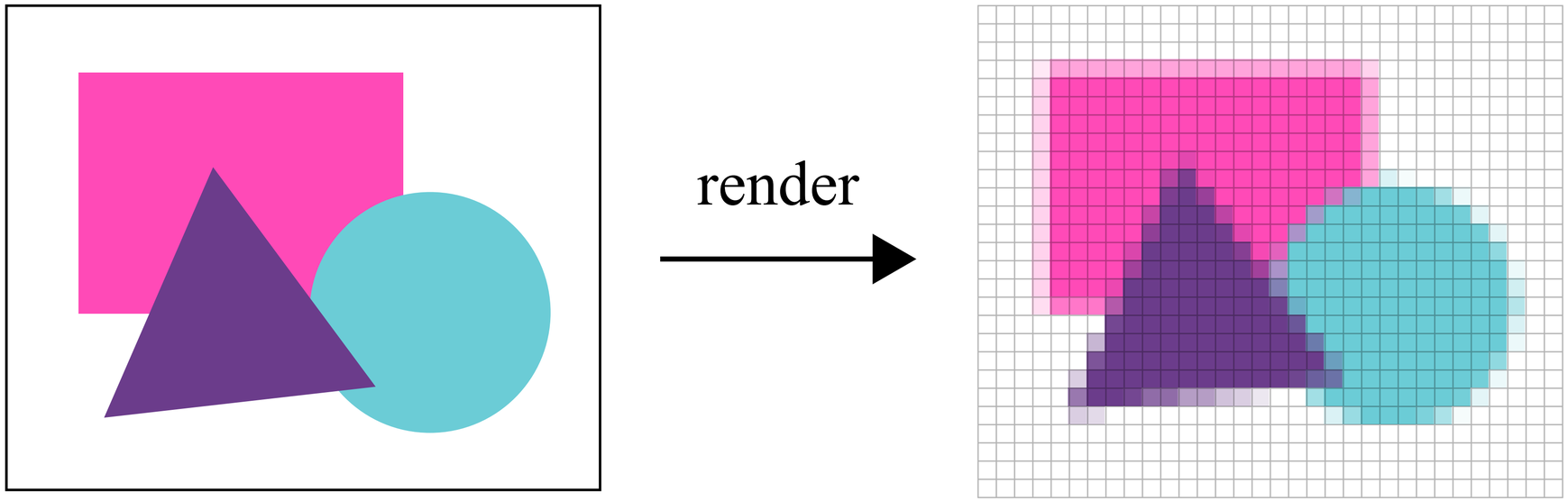
render



raster image

Rendering = painting on the raster image

in 2D, this is also called a “vector image”



shape description

raster image

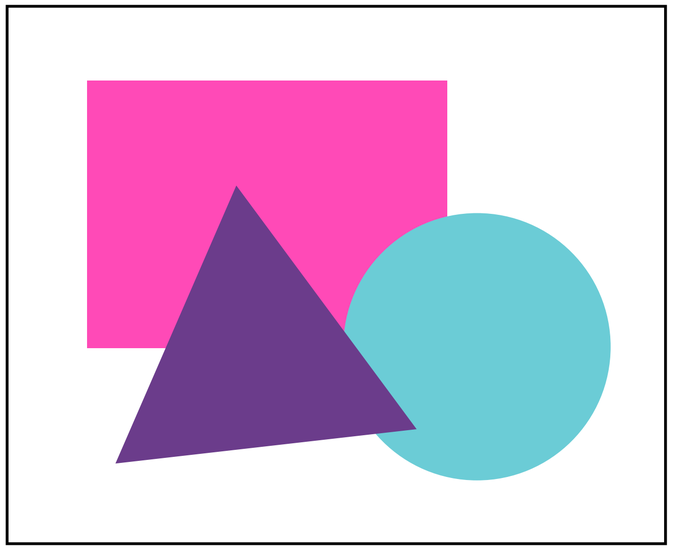
Rendering =
painting on the raster image

in 3D, this is called a 3D “scene”

render

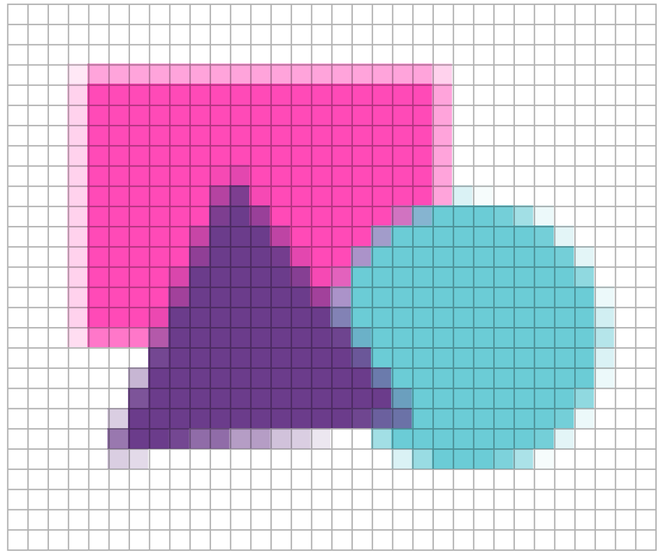
Rendering =
painting on the raster image

we will focus on 2D for now



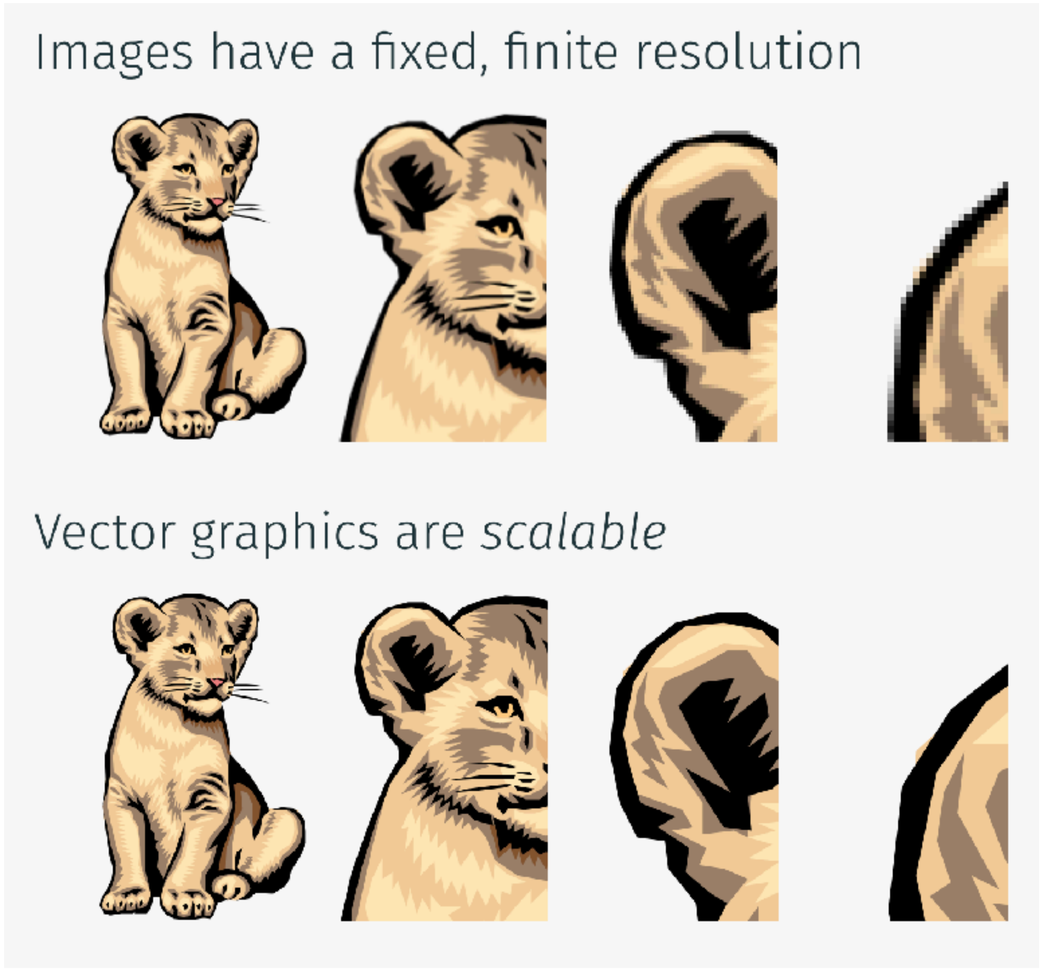
shape description

render



raster image

It is often more convenient to store vector images than raster images



from Diego Nehab
https://w3.impa.br/~diego/teaching/2021.0/slides-1.pdf

Vector graphics are everywhere

GUI

web design

logo

animation

The quick brown fox jumps over the lazy dog

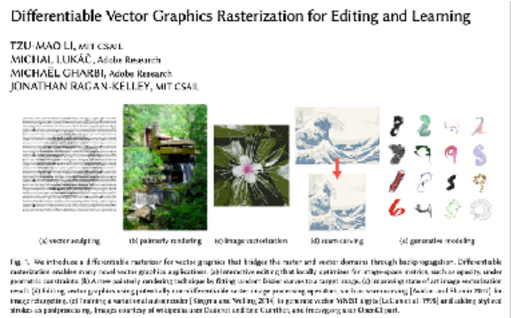
The quick brown fox jumps over the lazy dog

The quick brown fox jumps over the lazy dog

font

vector art

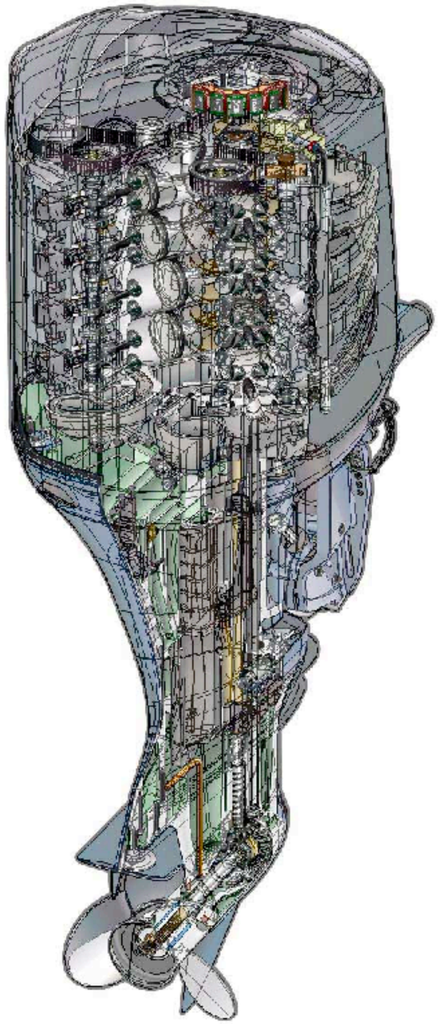
map

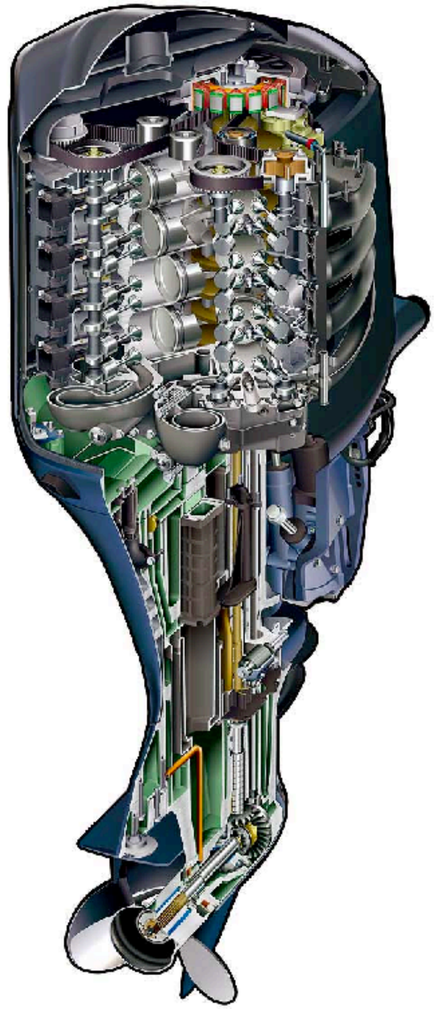


academic paper

Amazing vector art

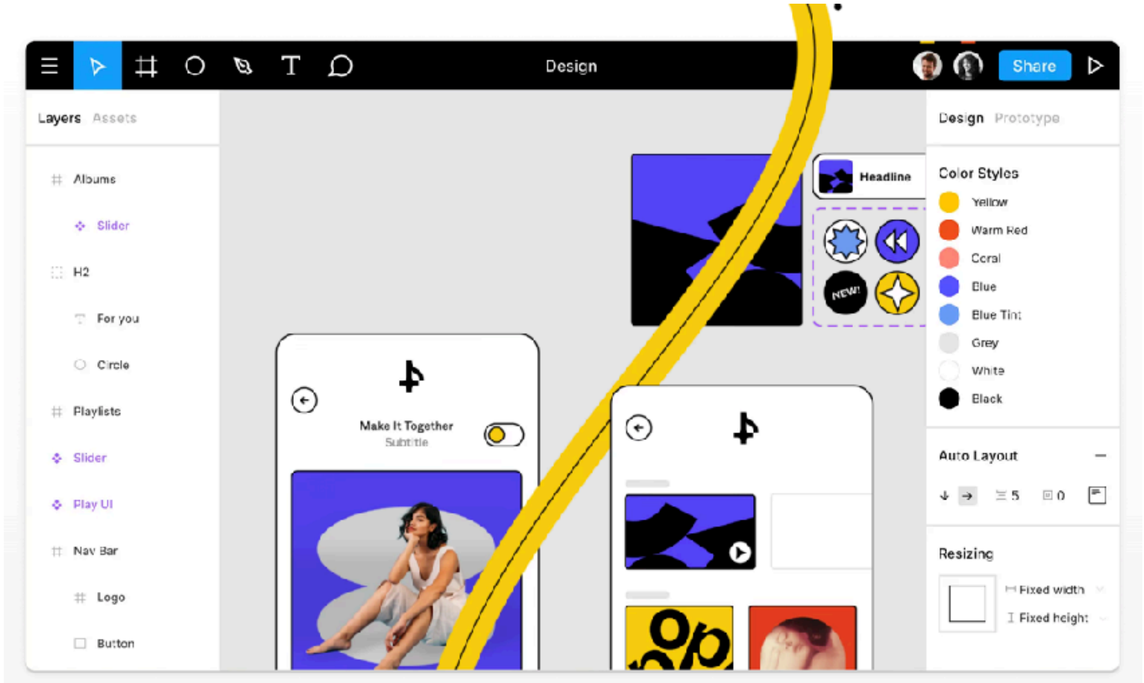
from Yukio Miyamoto



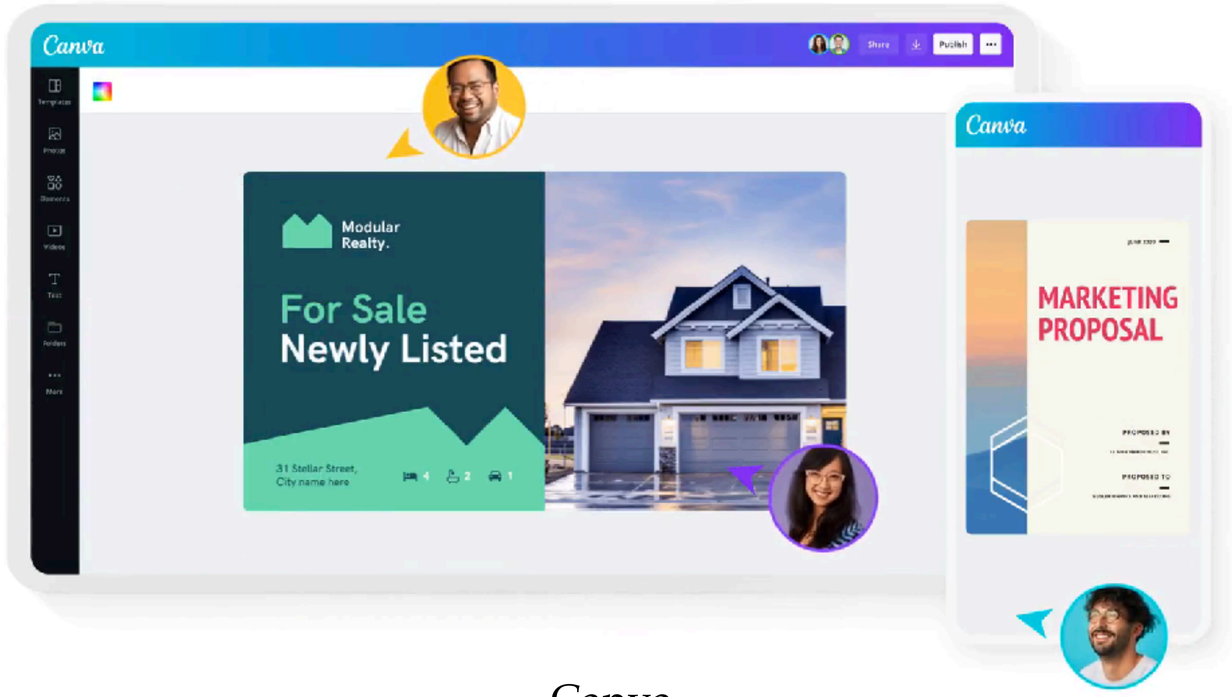


https://yukiomiyamoto.web.fc2.com/

Vector graphics startups



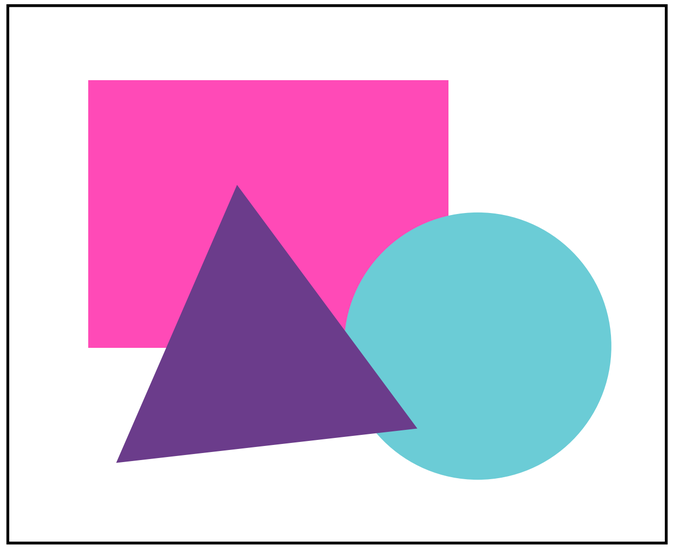
Figma



Canva

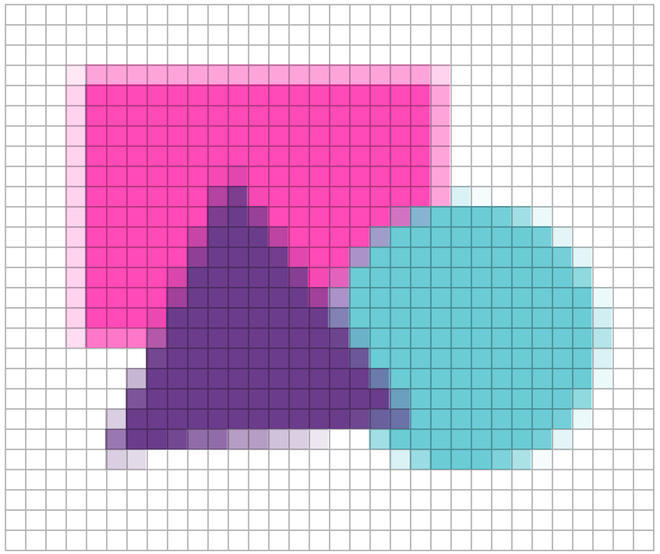
Today: rasterizing 2D primitives

(what you need to do in Homework 1!)



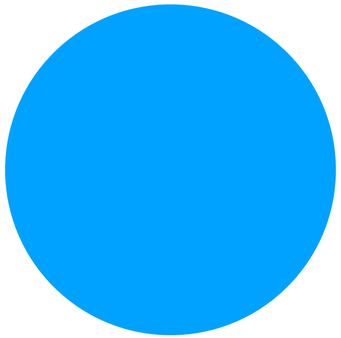
shape description

render

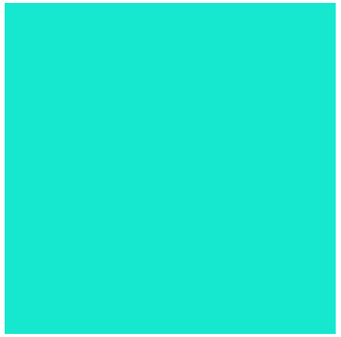


raster image

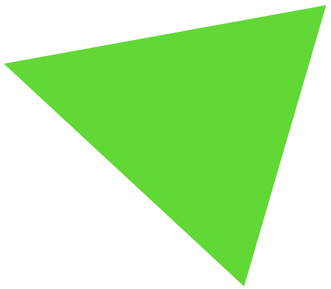
Some common primitives



circles



rectangles

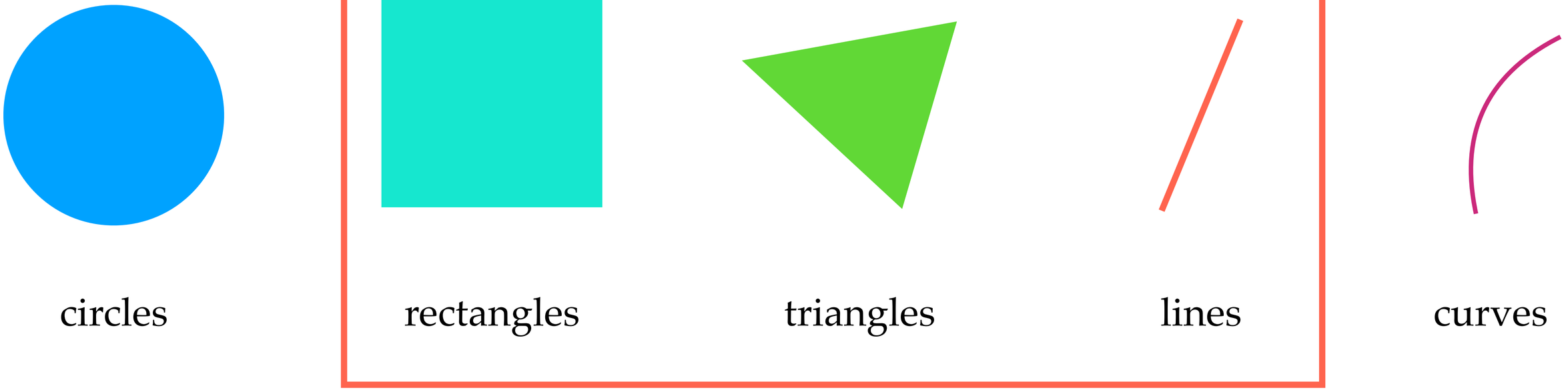


triangles

lines

curves

Some common primitives



circles

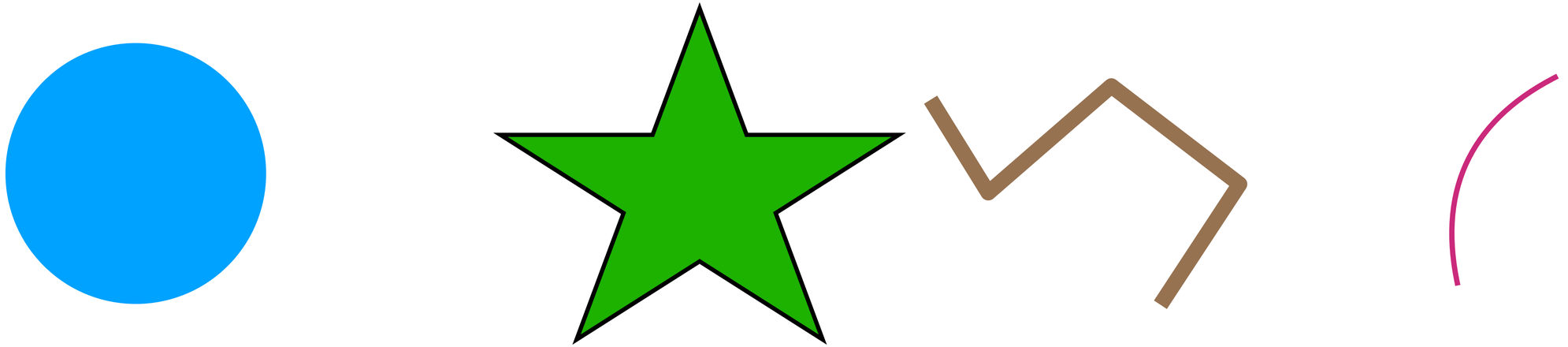
rectangles

triangles

lines

curves

Some common primitives

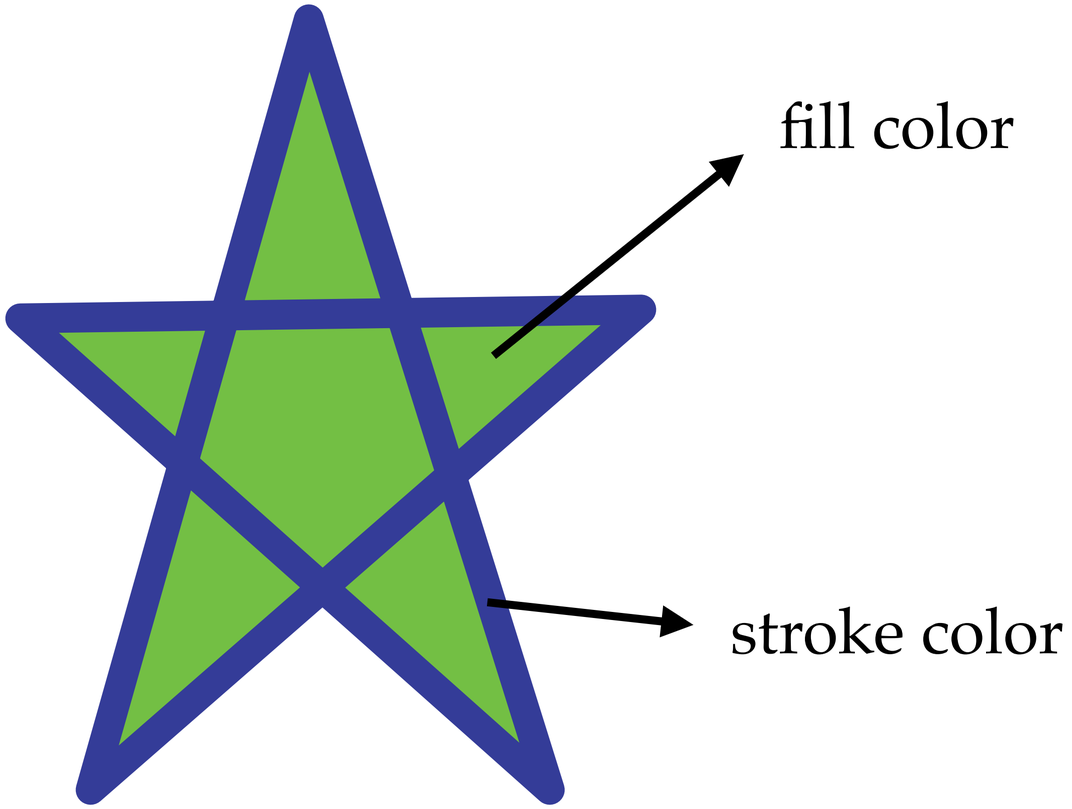


circles

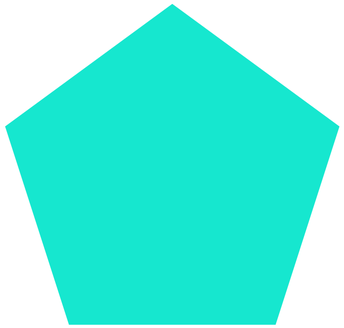
“polylines”

curves

A shape can have “fill color” and “stroke color”

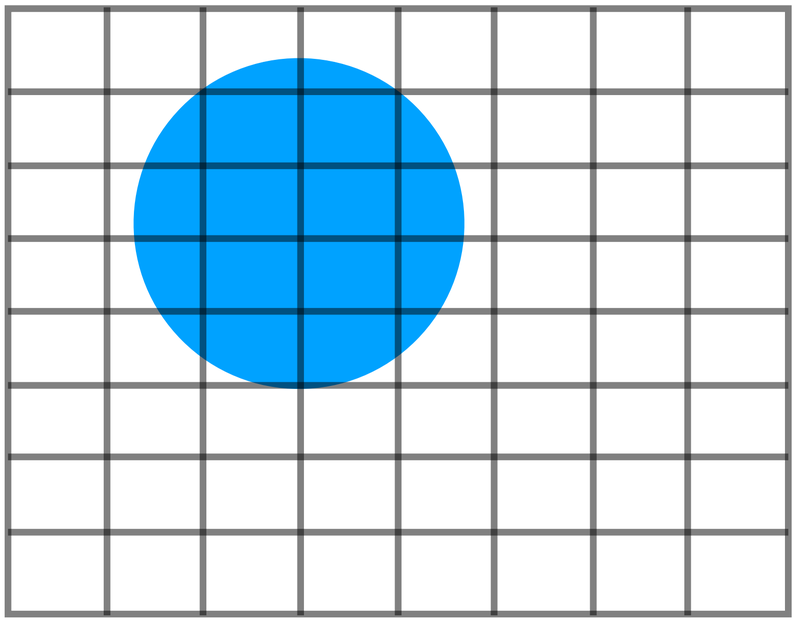



stroke only


fill only

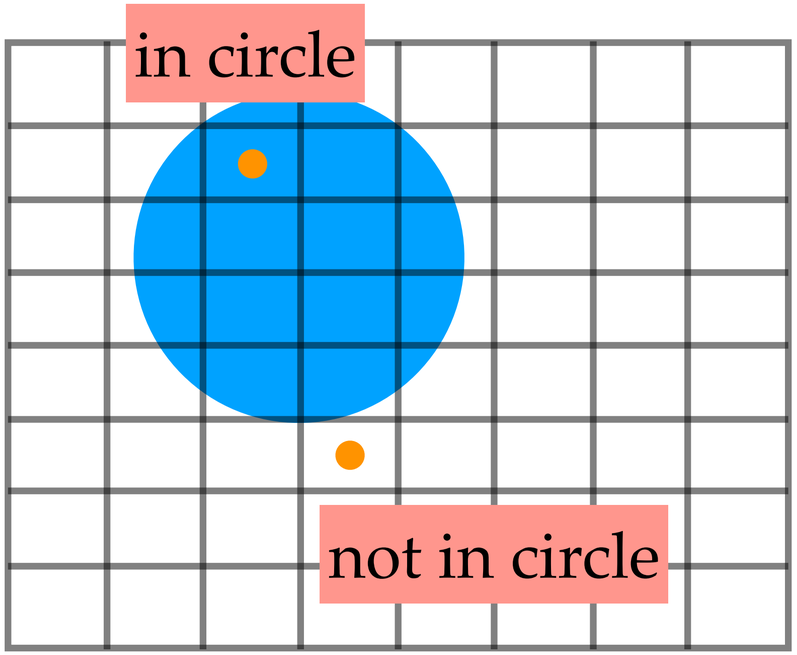
from the Scalable Vector Graphics standard
https://developer.mozilla.org/en-US/docs/Web/SVG/Tutorials/SVG_from_scratch/Fills_and_strokes

Let’s first assume we only have one fill-only circle



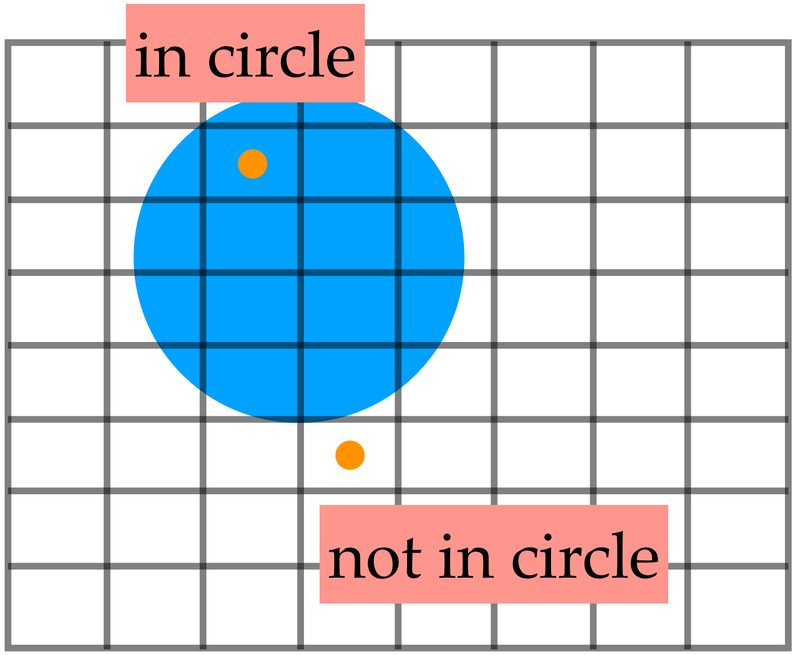
Q: how would you render it?

Let's first assume we only have one fill-only circle




```
for each pixel
  if pixel center hits the primitive:
    color = primitive.fill_color
  else:
    color = background
```

Let's first assume we only have one fill-only circle

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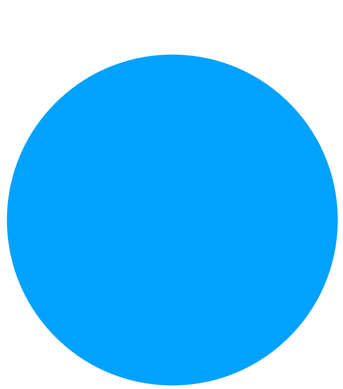
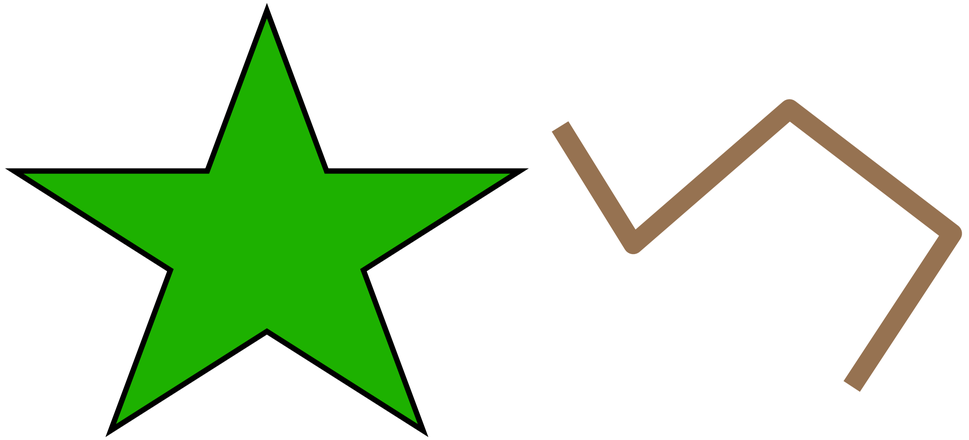
```

Point in primitive test



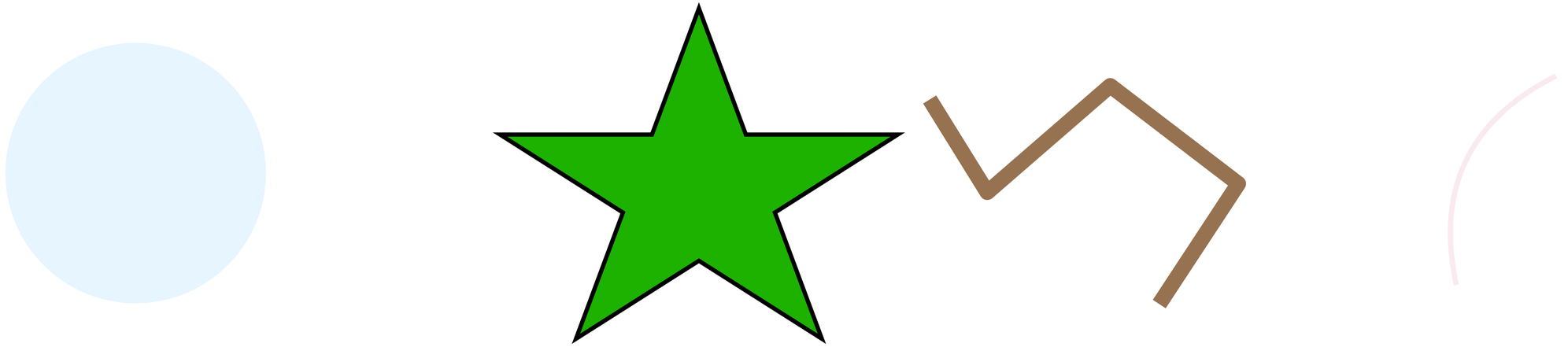
circles



“polylines”

curves

Point in primitive test

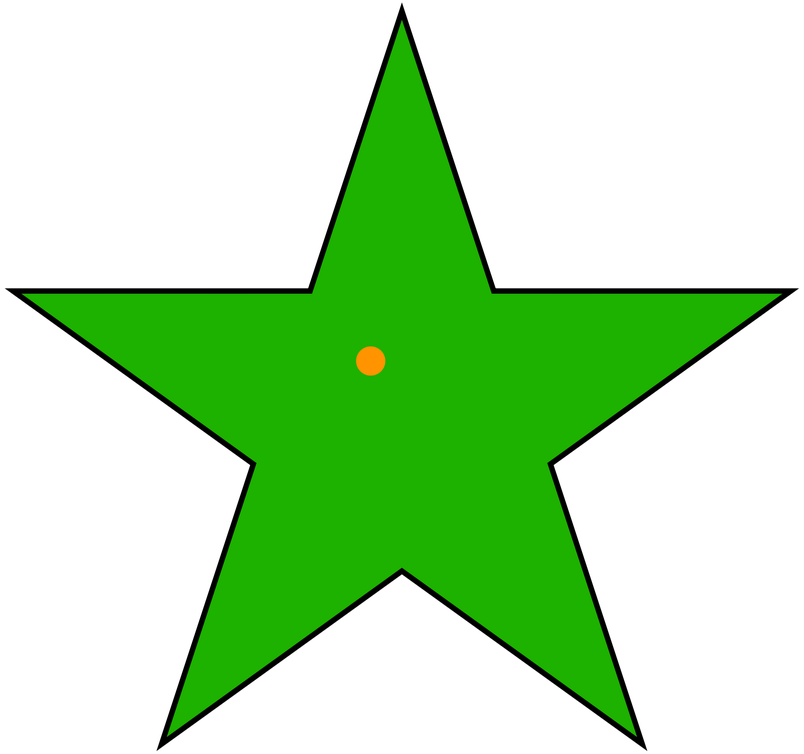


"polylines"

You figure it out!

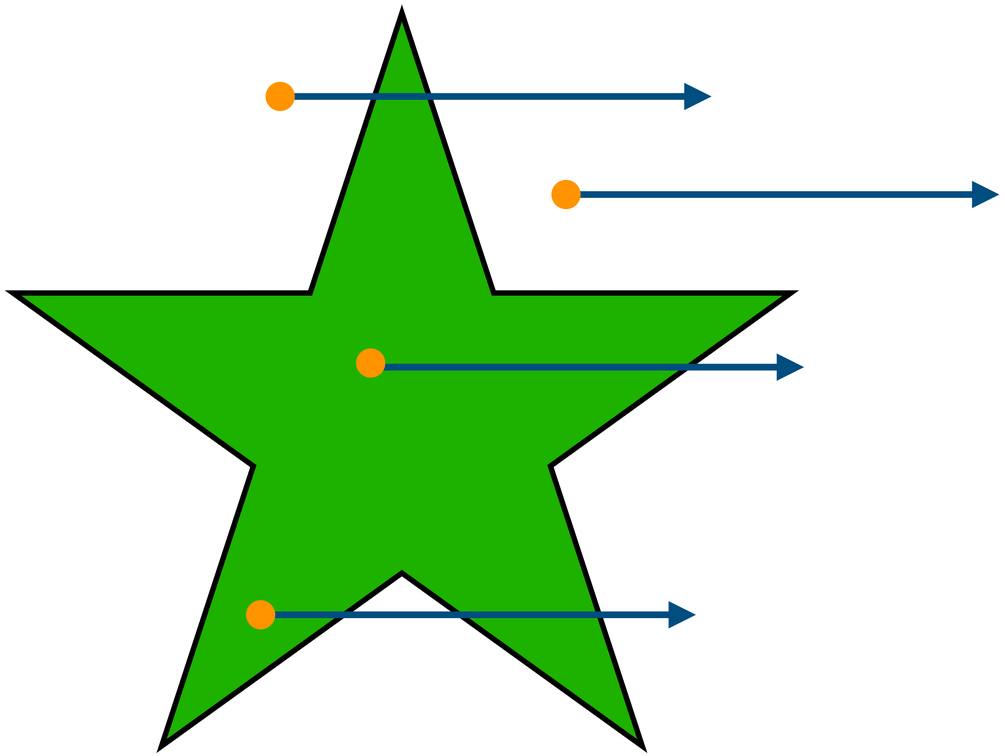
More about them later

Point in polyline test



How do we know if a point is inside a closed polyline or not?

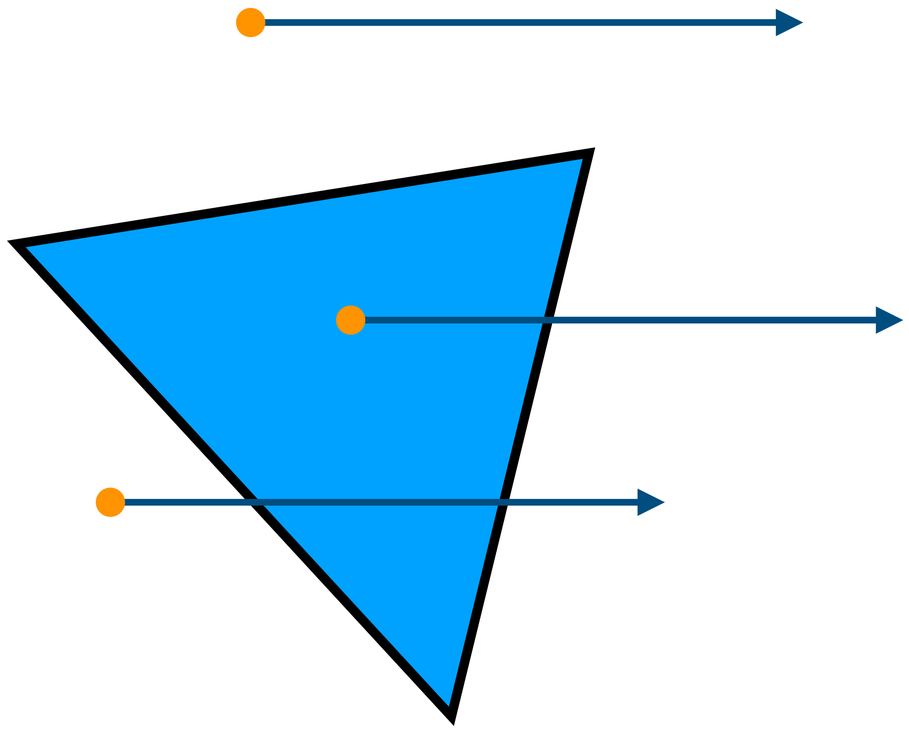
Point in polyline test



Idea: cast a ray from the point, and count the intersections with the shape

Q: what's the criteria?

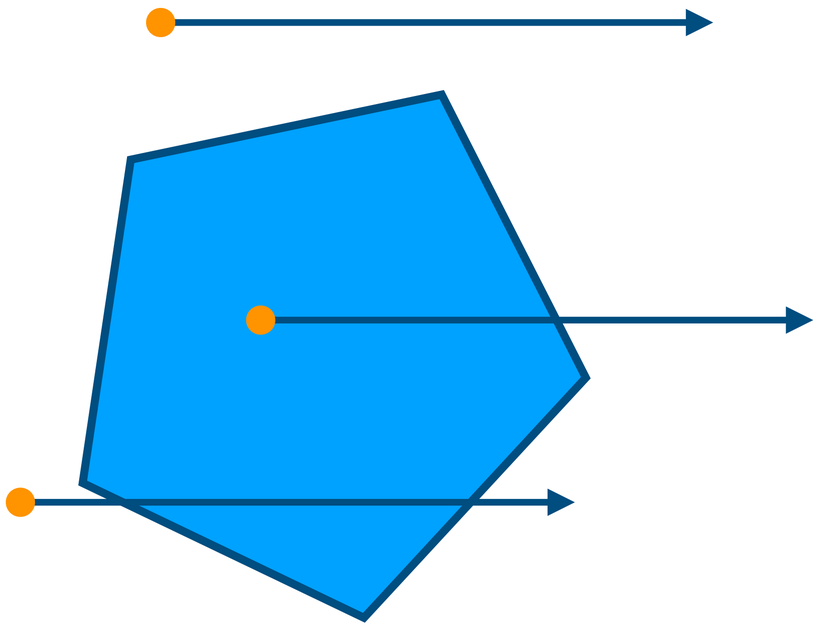
For a triangle…



rays from points outside intersect with the triangle edges either zero times or two times

rays from points inside intersect with the triangle edges exactly one time

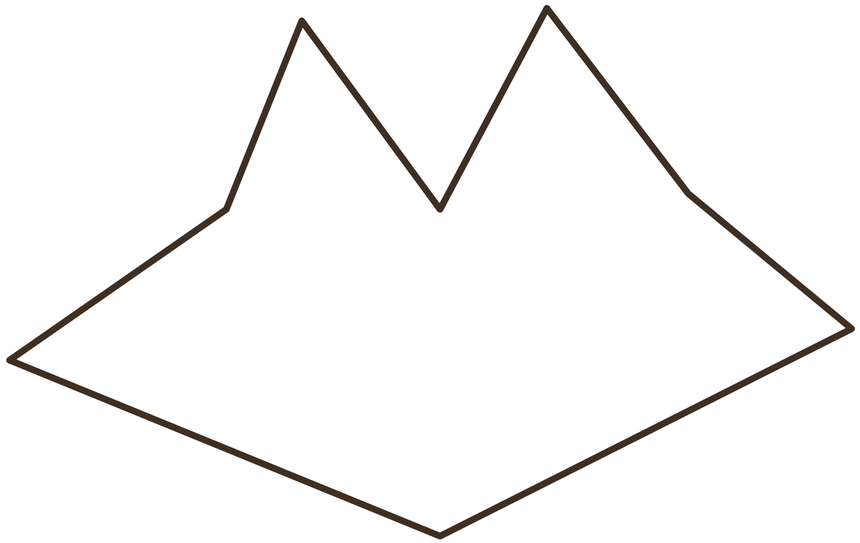
For convex polygons…

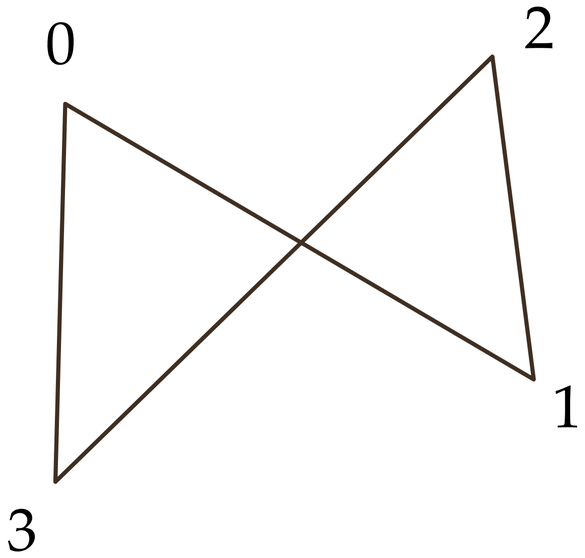


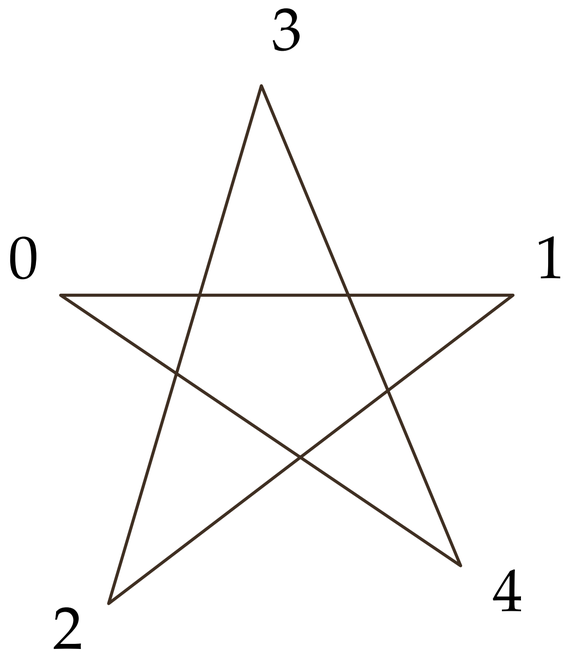
rays from points outside intersect with the polygon edges either zero times or two times

rays from points inside intersect with the polygon edges exactly one time

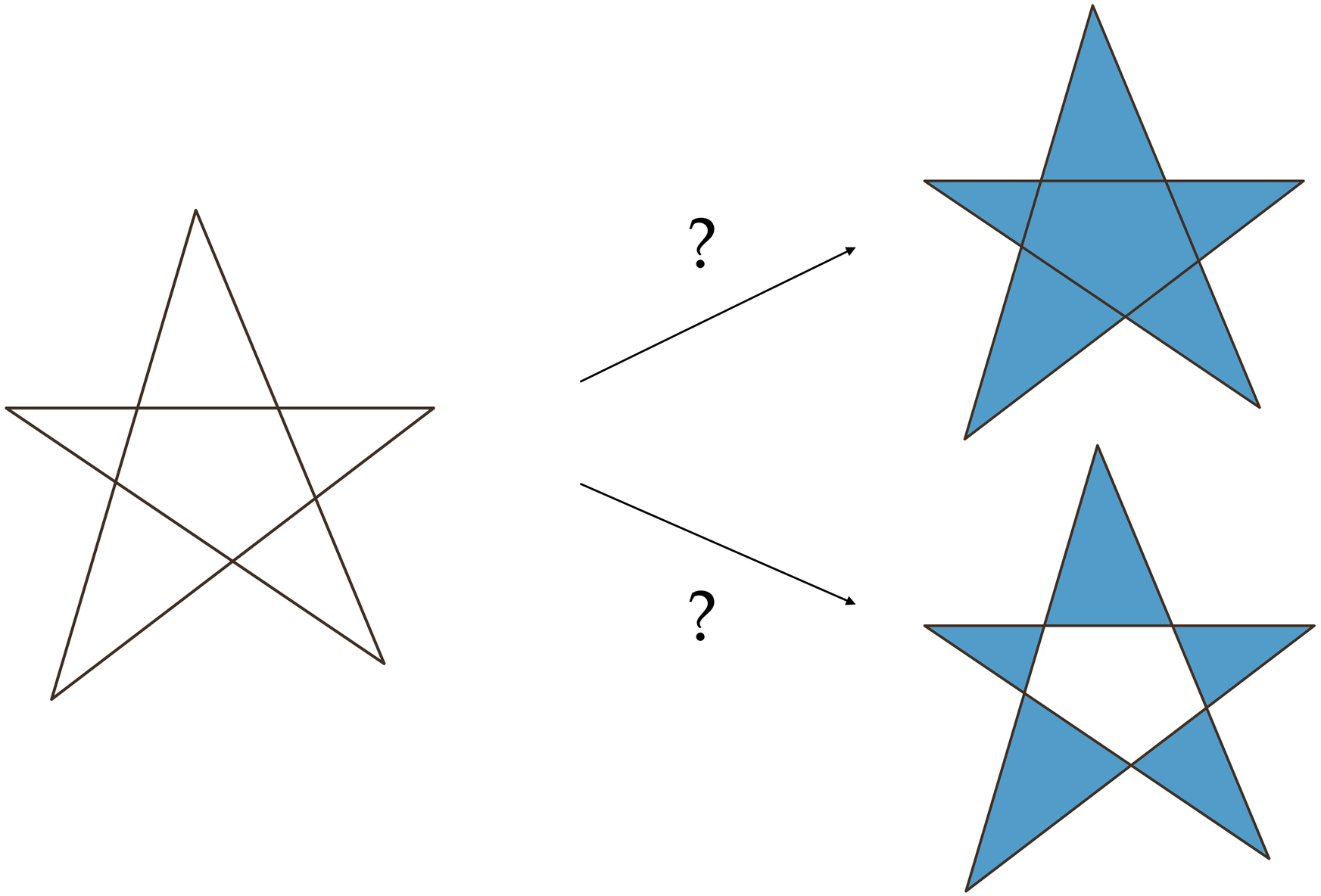
For general closed polylines, the rules are more complicated



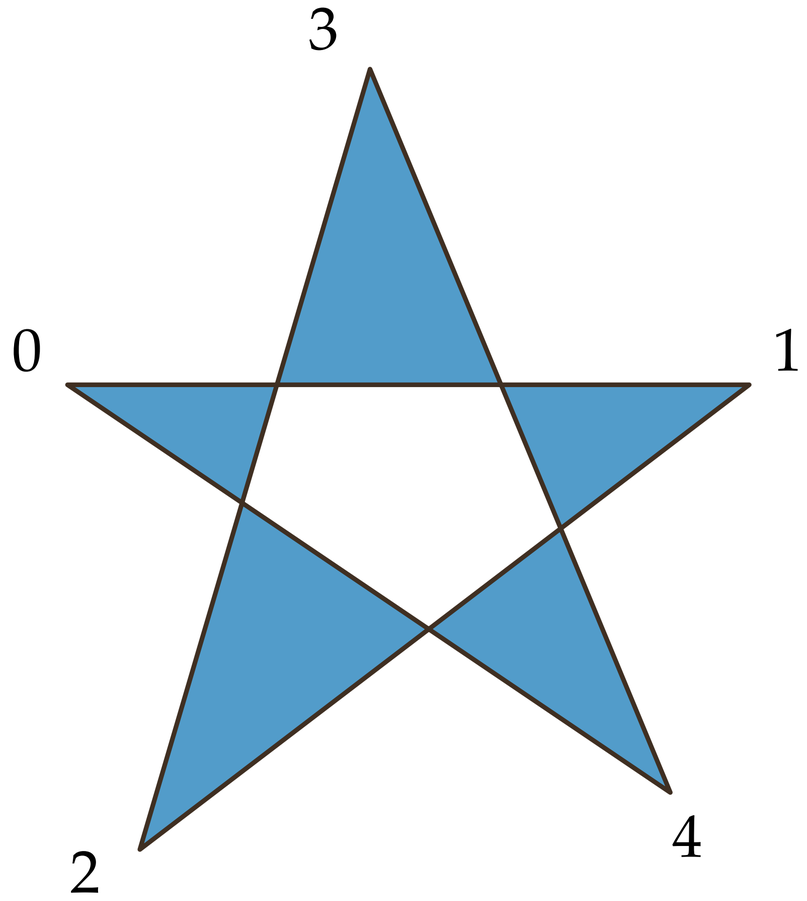




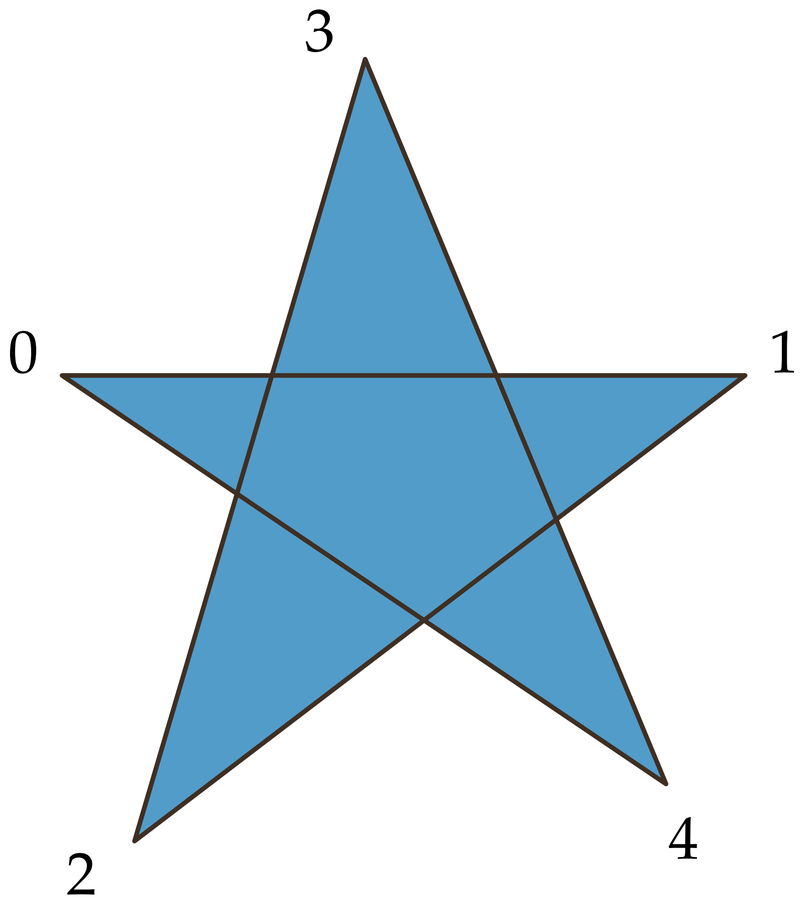
Should we fill the center of the star?



Two common different rules for point-in-polyline test



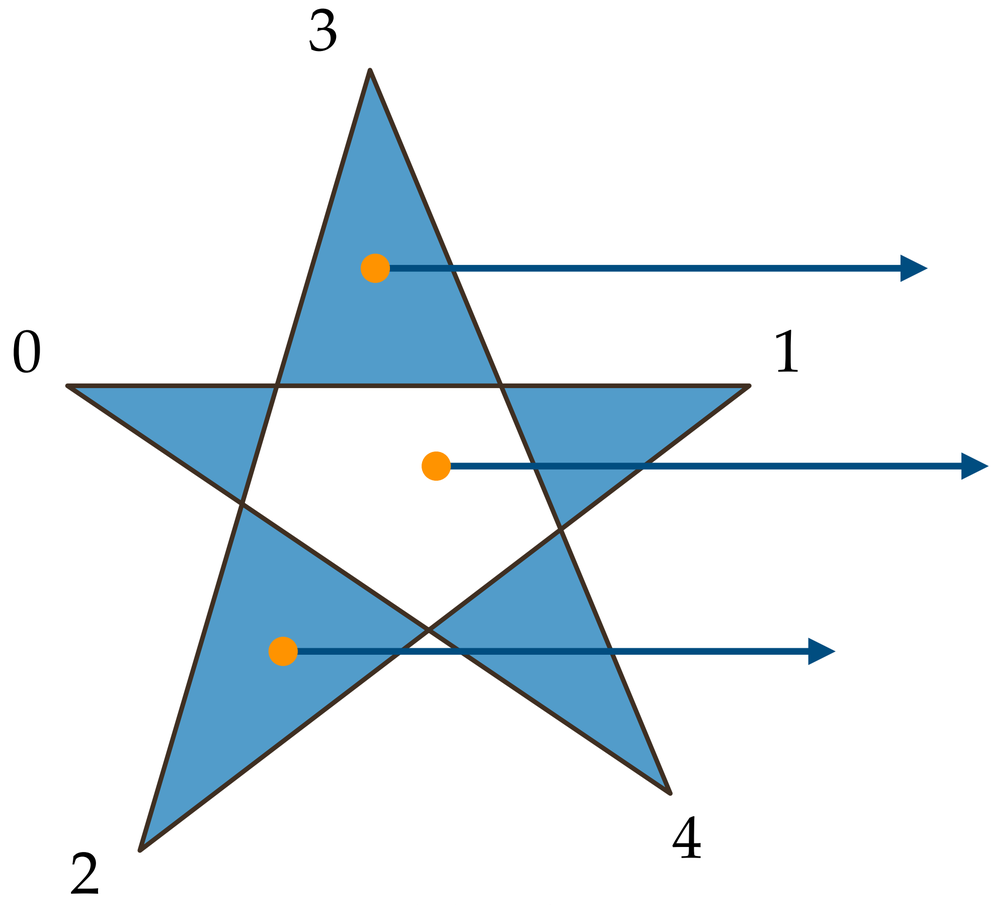
even-odd rule



nonzero rule

from SVG standard https://developer.mozilla.org/en-US/docs/Web/SVG/Reference/Attribute/fill-rule

Even-odd rule

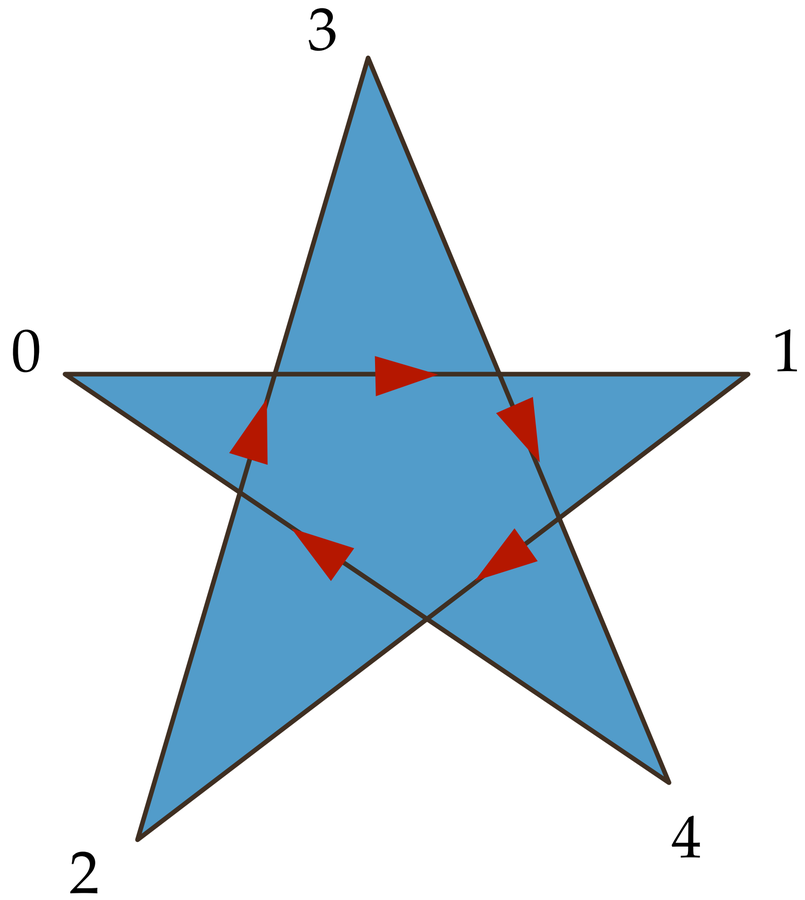


even-odd rule

if a ray intersect the shape **even** times,
the origin is **outside** the shape

if a ray intersect the shape **odd** times,
the origin is **inside** the shape

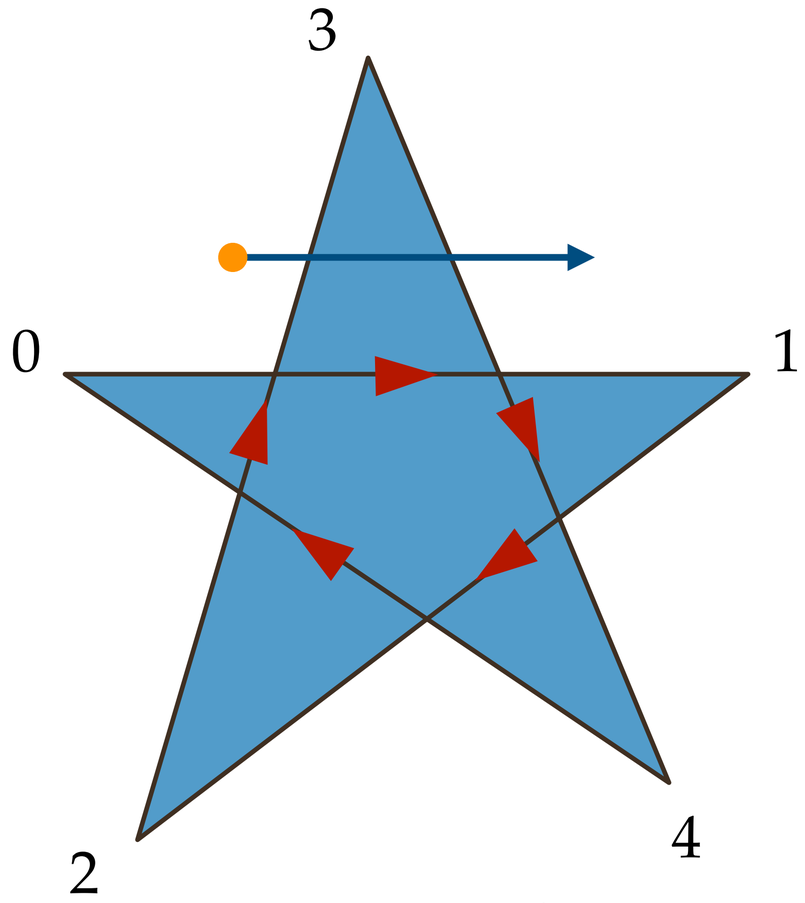
Non-zero rule



nonzero rule

need to account for the orientation of the lines

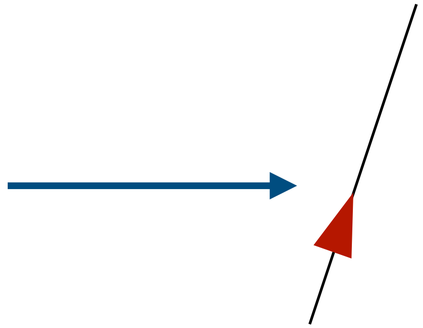
Non-zero rule



nonzero rule

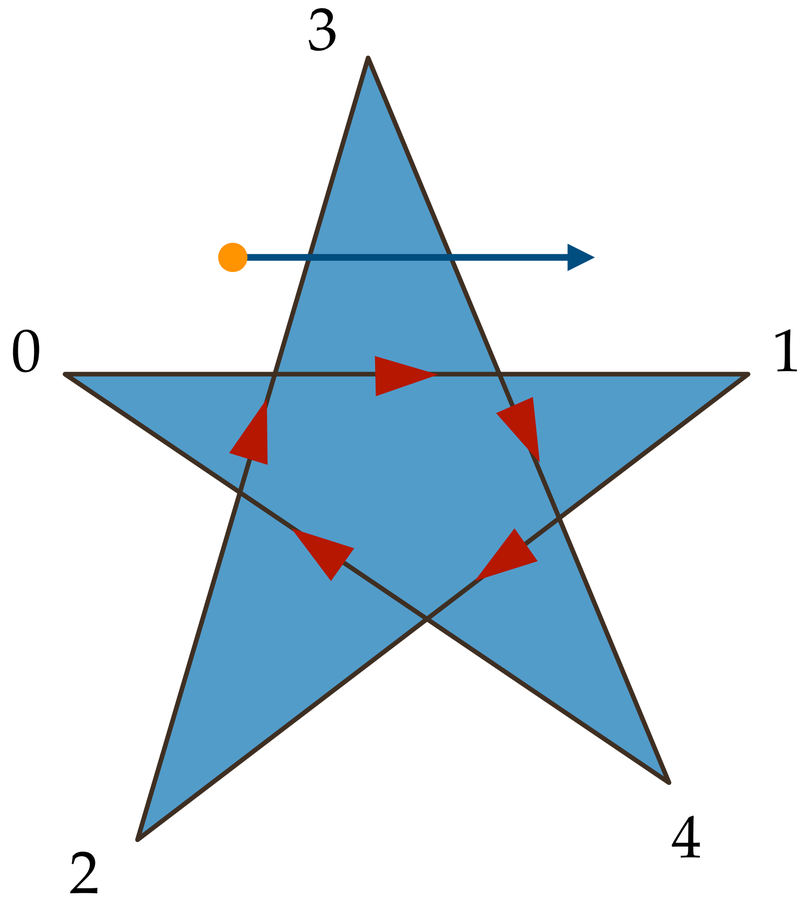
need to account for the orientation of the lines

every time a ray hits a line…



if the line is going up,
count += 1

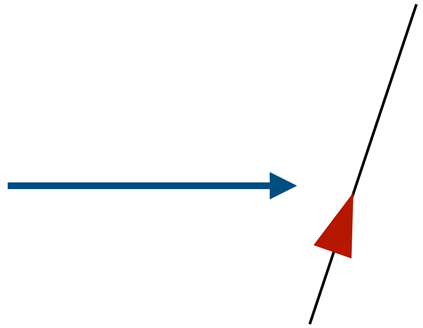
Non-zero rule



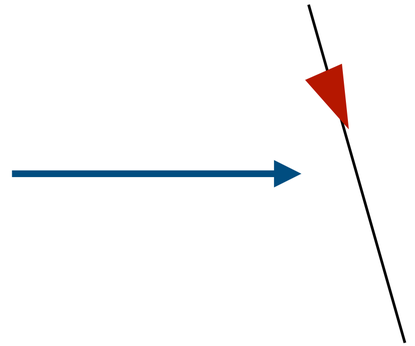
nonzero rule

need to account for the orientation of the lines

every time a ray hits a line…

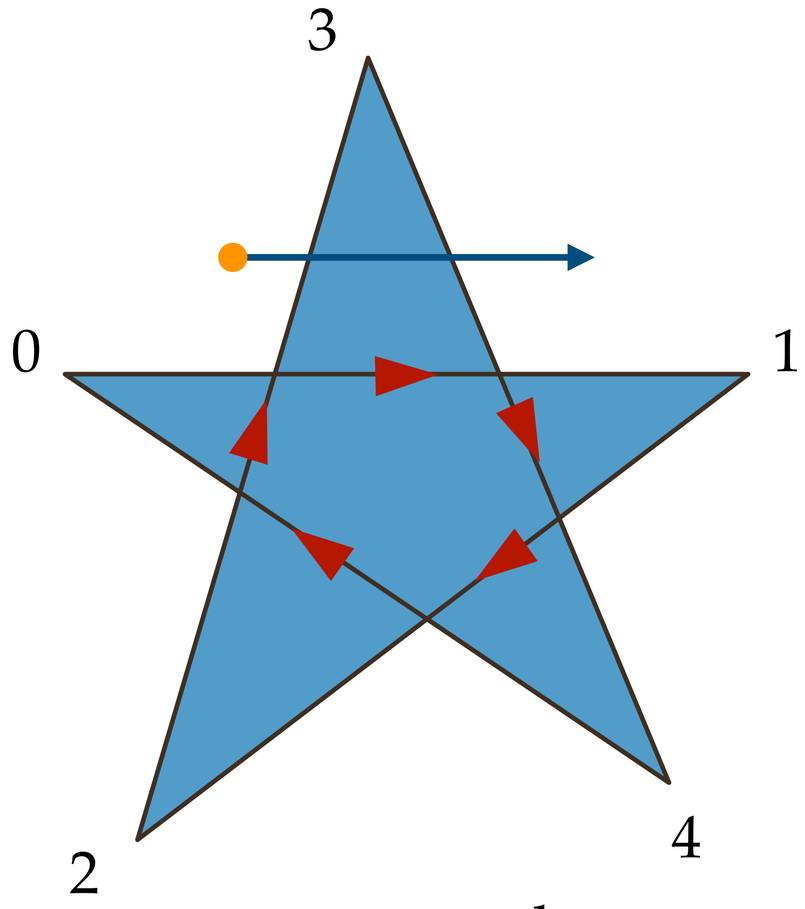


if the line is going up,
count += 1



if the line is going down,
count -= 1

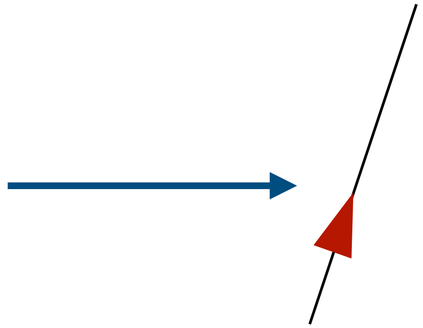
Non-zero rule



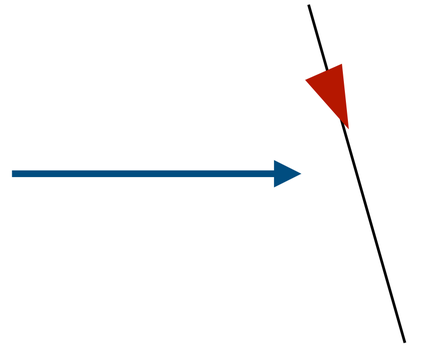
nonzero rule

need to account for the orientation of the lines

every time a ray hits a line…



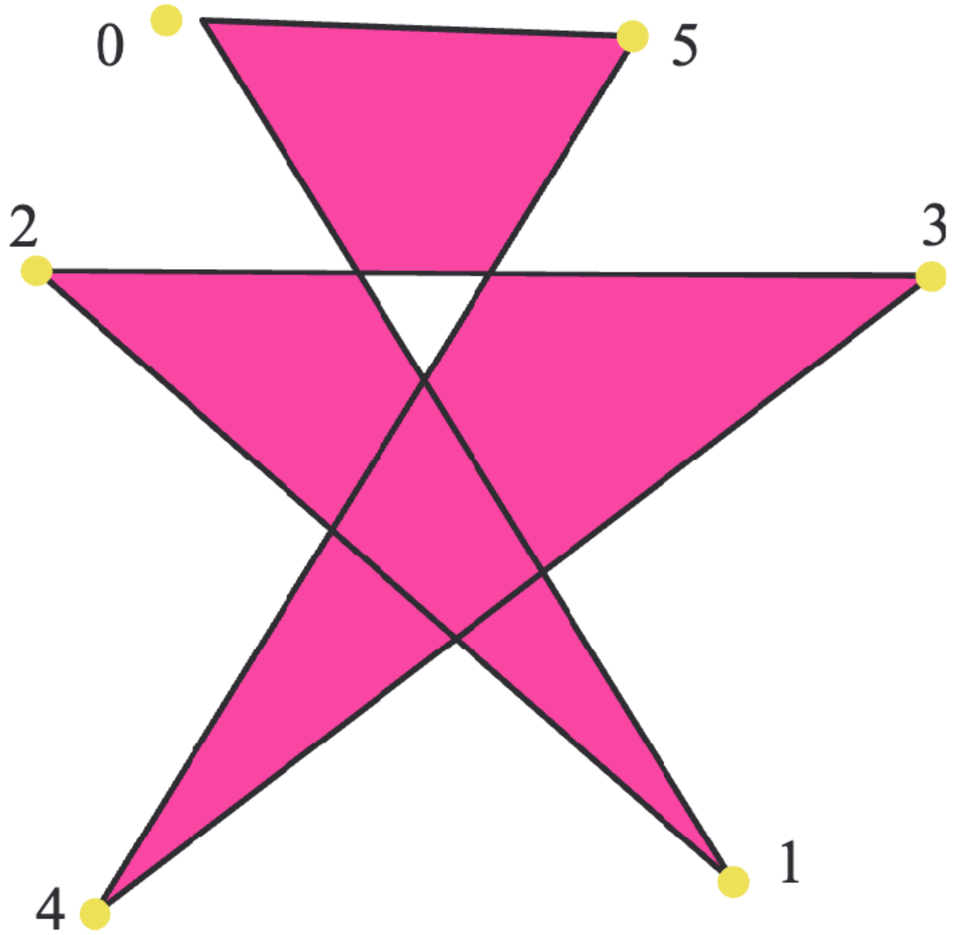
if the line is going up,
count += 1



if the line is going down,
count -= 1

the point is inside if count != 0
(this count is called “winding number” in math)

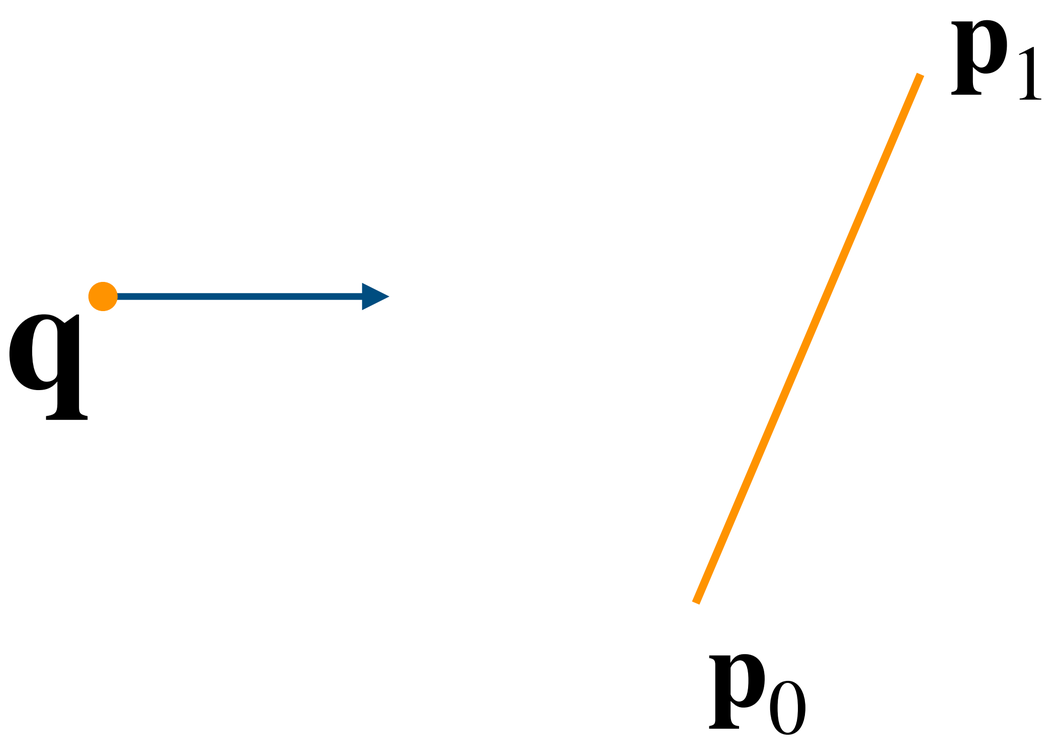
In homework, you need to guess which rule we used!



More resources

- https://en.wikipedia.org/wiki/Point_in_polygon
- https://en.wikipedia.org/wiki/Even%E2%80%93odd_rule
- https://en.wikipedia.org/wiki/Nonzero-rule

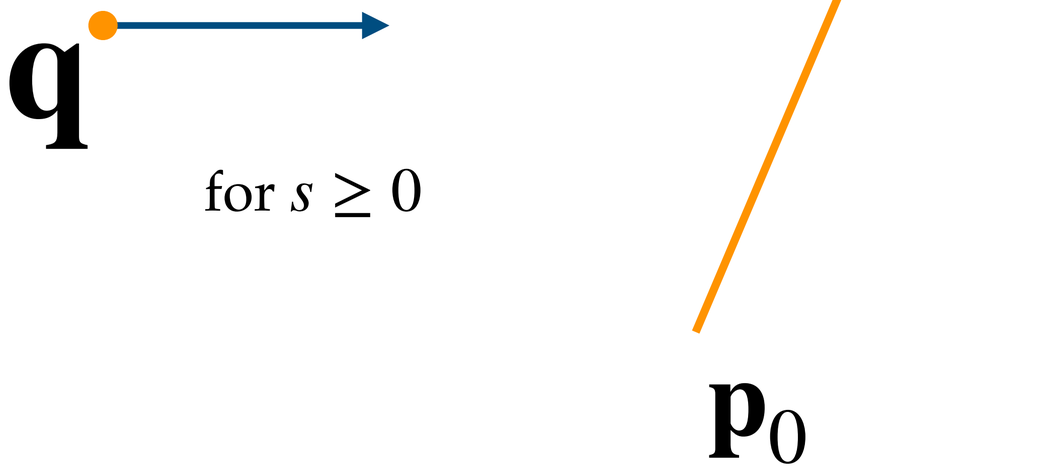
Ray-line intersection



Ray-line intersection

every point on the ray can be represented as

$$\mathbf{q} + (1,0) \cdot s$$



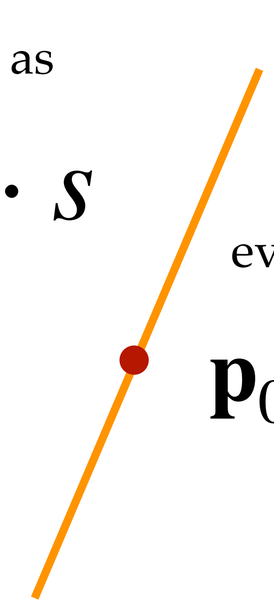
for $s \geq 0$

Ray-line intersection

every point on the ray can be represented as

$$\mathbf{q} + (1,0) \cdot s$$

for $s \geq 0$



every point on the line can be represented as

$$\mathbf{p}_0 + t \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$

for $0 \leq t \leq 1$

Ray-line intersection

$$s \geq 0 \qquad\qquad 0 \leq t \leq 1$$

$$\mathbf{q} + (1,0) \cdot s = \mathbf{p}_0 + t \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right)$$

Ray-line intersection

$$s \geq 0 \qquad\qquad 0 \leq t \leq 1$$

$$\mathbf{q}_x + (1,0) \cdot s = \mathbf{p}_{0_x} + t \cdot \left(\mathbf{p}_{1_x} - \mathbf{p}_{0_x} \right)$$

$$\mathbf{q}_y + (1,0) \cdot s = \mathbf{p}_{0_y} + t \cdot \left(\mathbf{p}_{1_y} - \mathbf{p}_{0_y} \right)$$

Ray-line intersection

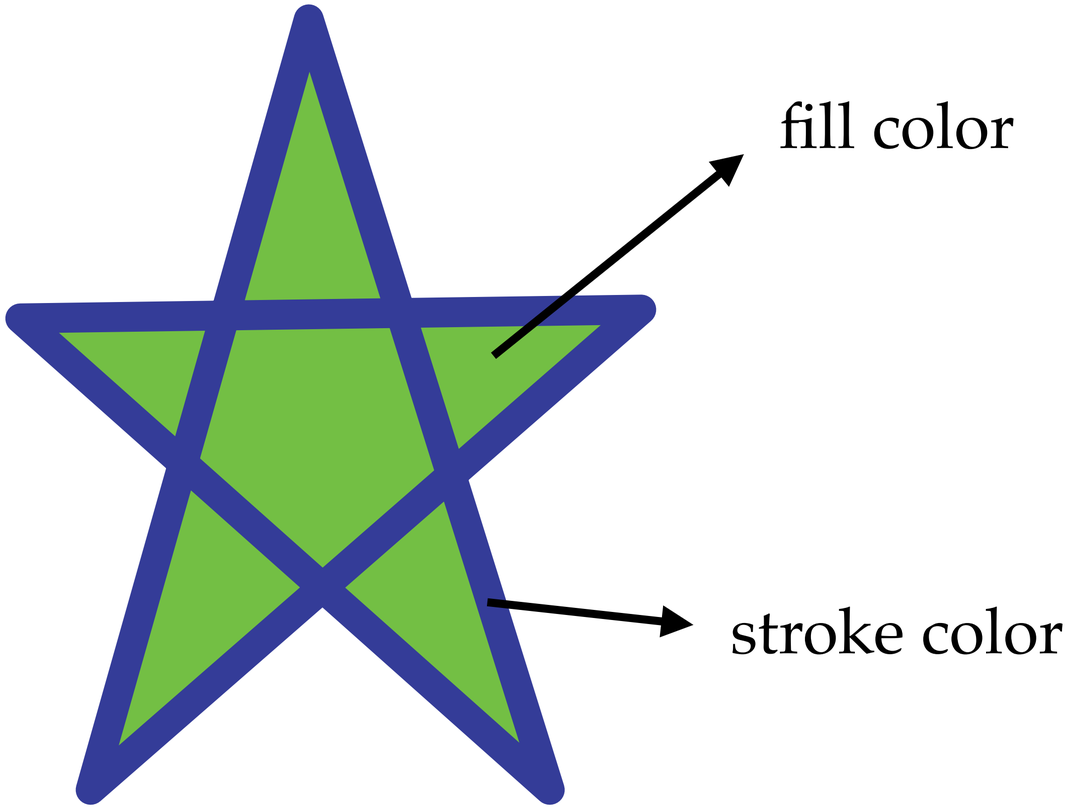
$s \geq 0$ $\qquad\qquad$ $0 \leq t \leq 1$

$$\mathbf{q}_x + (1,0) \cdot s = \mathbf{p}_{0_x} + t \cdot \left(\mathbf{p}_{1_x} - \mathbf{p}_{0_x} \right)$$

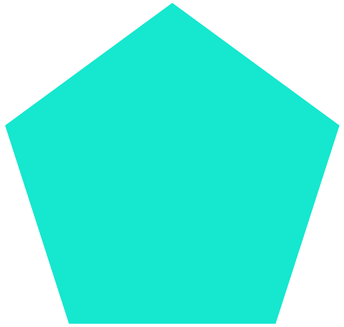
$$\mathbf{q}_y + (1,0) \cdot s = \mathbf{p}_{0_y} + t \cdot \left(\mathbf{p}_{1_y} - \mathbf{p}_{0_y} \right)$$

solve for s and t, if $s \geq 0$ and $0 \leq t \leq 1$, then the ray intersects with the line

What about “stroke color”?

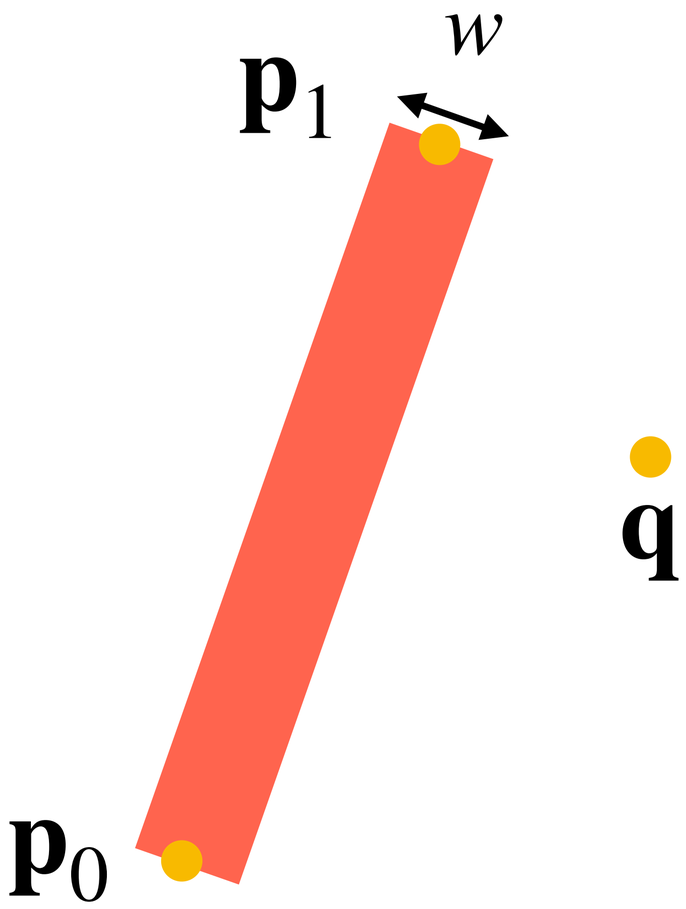


stroke only

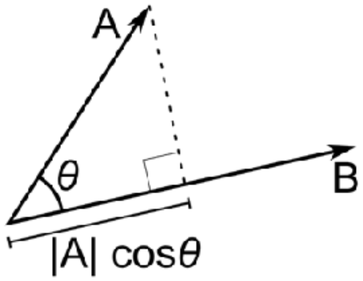


fill only

Goal: determine whether the point is within a width of the line



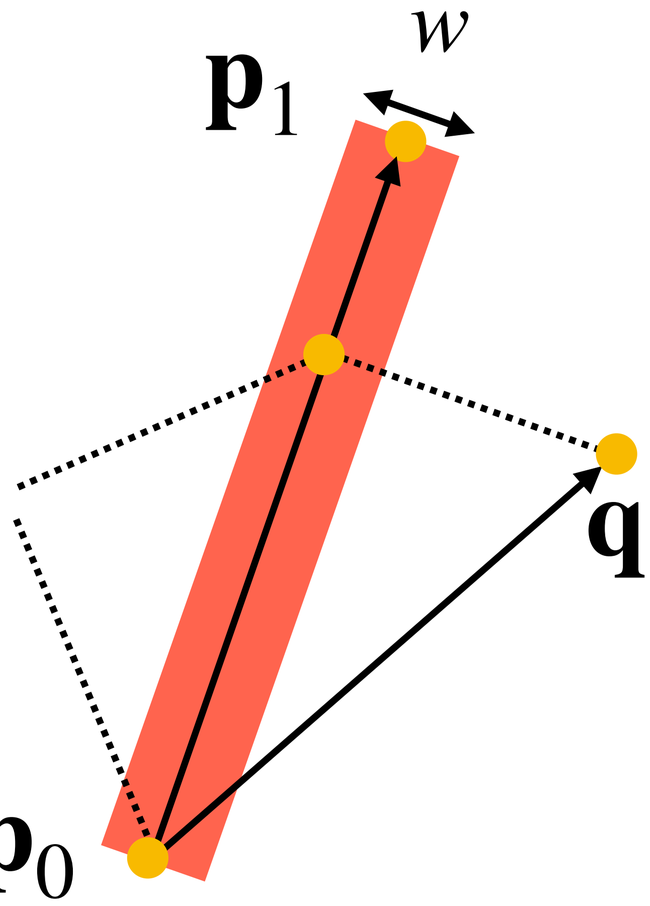
Point-in-line test



dot product = projected length * length of the projected vector

$$A \cdot B = \| A \| \ \| B \| \cos\theta$$

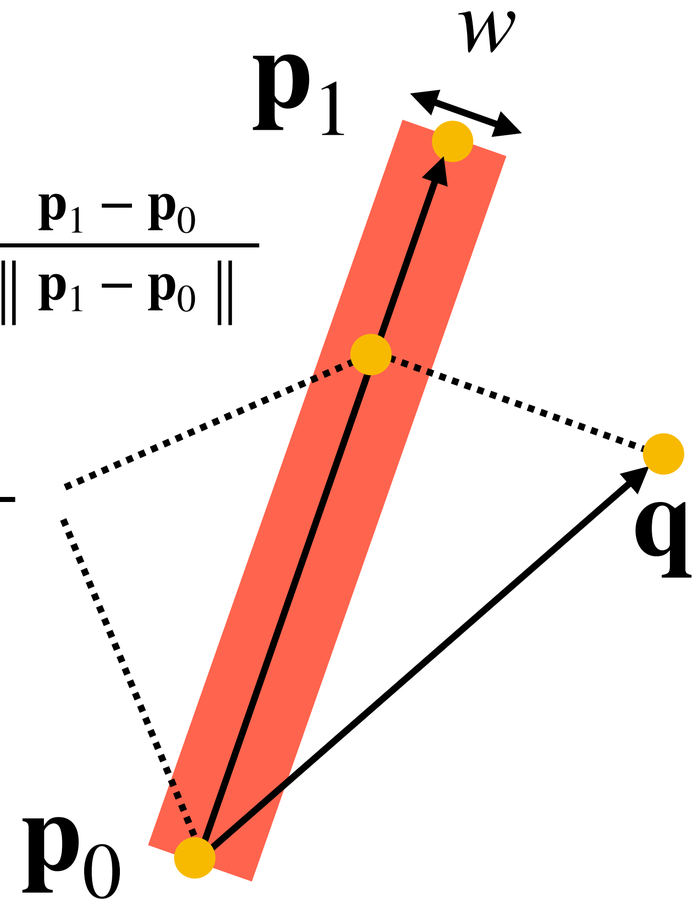
$$l = \frac{(\mathbf{p}_1 - \mathbf{p}_0) \cdot (\mathbf{q} - \mathbf{p}_0)}{\| \mathbf{p}_1 - \mathbf{p}_0 \|}$$



Point-in-line test

$$\mathbf{q}' = \mathbf{p}_0 + l \cdot \frac{\mathbf{p}_1 - \mathbf{p}_0}{\| \mathbf{p}_1 - \mathbf{p}_0 \|}$$

$$l = \frac{(\mathbf{p}_1 - \mathbf{p}_0) \cdot (\mathbf{q} - \mathbf{p}_0)}{\| \mathbf{p}_1 - \mathbf{p}_0 \|}$$

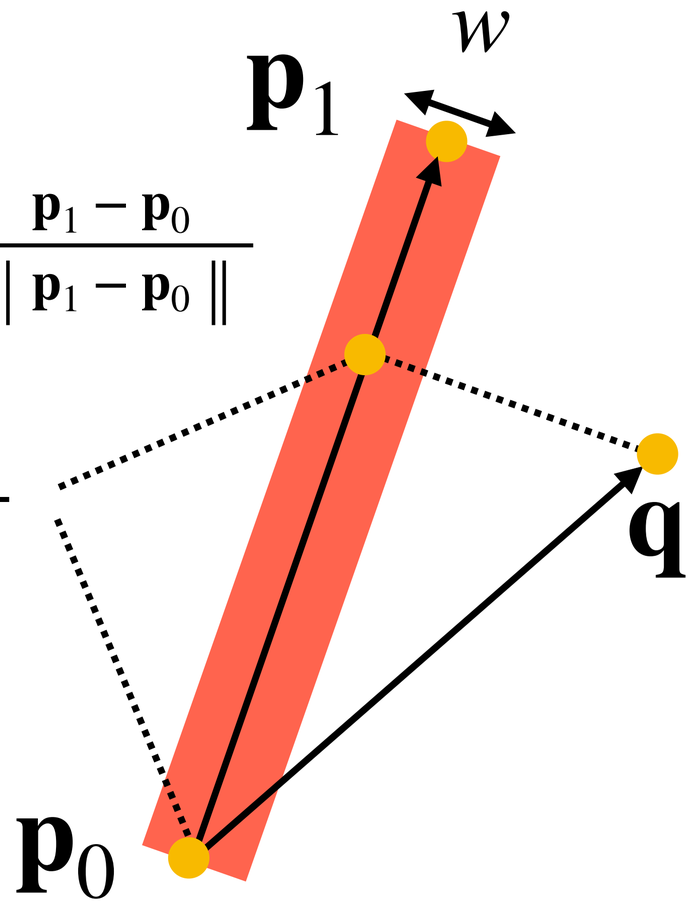


Point-in-line test

$$\mathbf{q}' = \mathbf{p}_0 + l \cdot \frac{\mathbf{p}_1 - \mathbf{p}_0}{\| \mathbf{p}_1 - \mathbf{p}_0 \|}$$

$$l = \frac{(\mathbf{p}_1 - \mathbf{p}_0) \cdot (\mathbf{q} - \mathbf{p}_0)}{\| \mathbf{p}_1 - \mathbf{p}_0 \|}$$

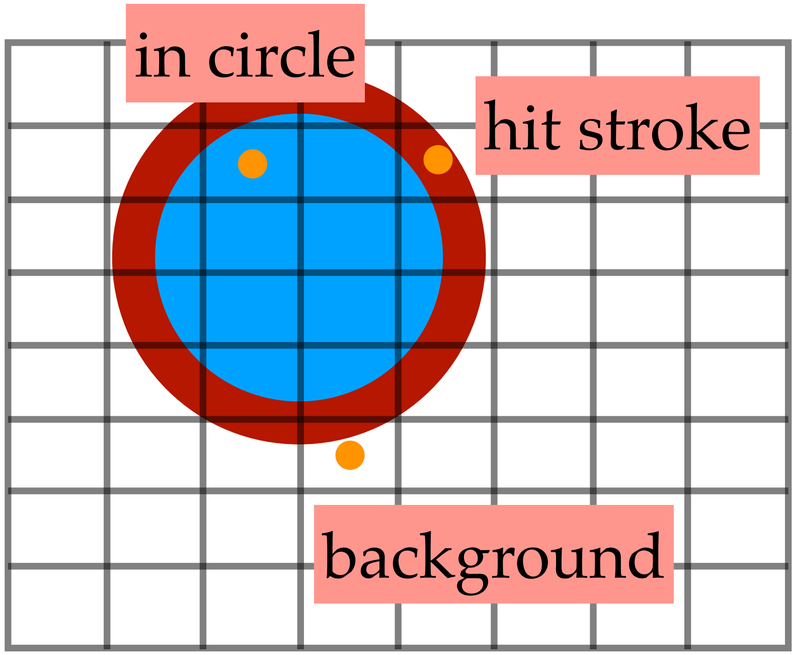
if $l > 0$, $l < \| \mathbf{p}_1 - \mathbf{p}_0 \|$, and $\| \mathbf{q} - \mathbf{q}' \| < \frac{w}{2}$, then $\mathbf{q}$ is in the line



“Joins” and “caps” of lines

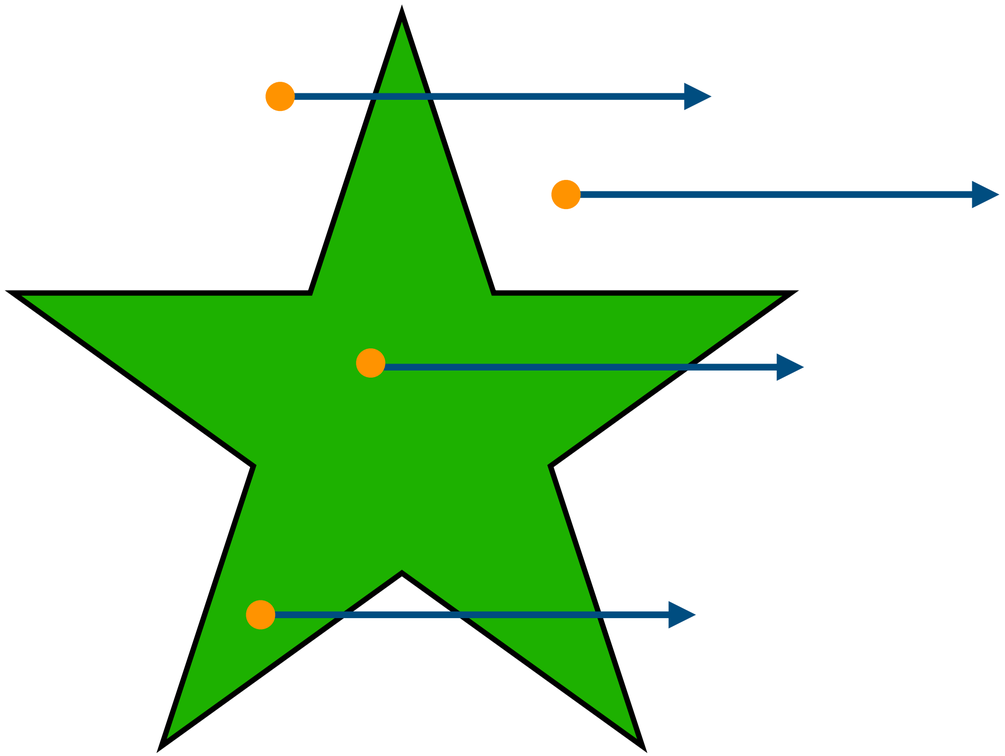
in homework, we ask you to have flat caps, and round joins
— try to figure out how to do it!

Recap




```
for each pixel
  if pixel center hits the stroke:
    color = primitive.stroke_color
  else if pixel center is inside primitive:
    color = primitive.stroke_color
  else:
    color = background
```

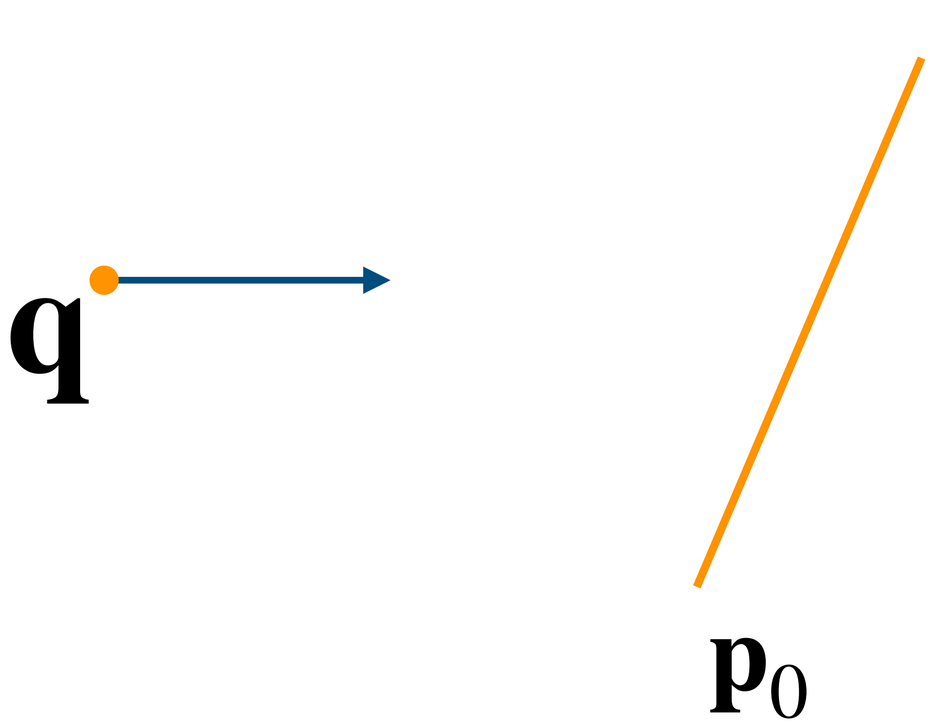
Recap



for polylines fill color
cast ray to the right to determine inside-outside

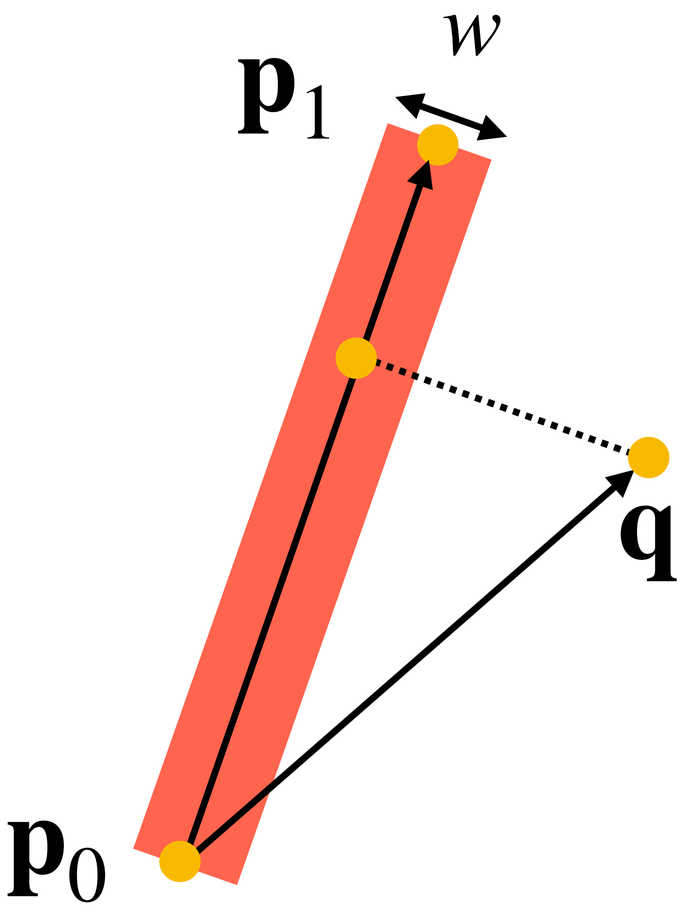
Recap

$$\mathbf{q} + (1,0) \cdot t = \mathbf{p}_0 + s \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$



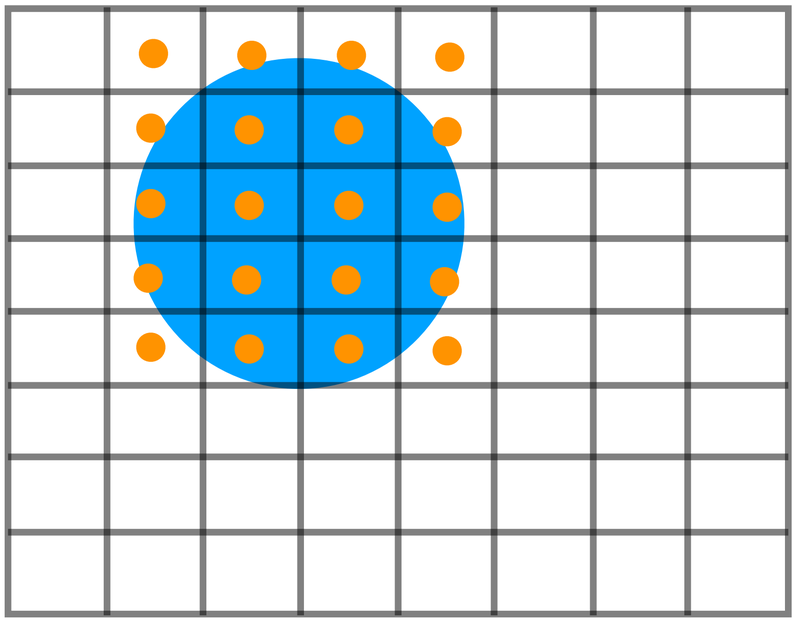
solve a linear system with two variables (s, t)
to determine intersection

Recap

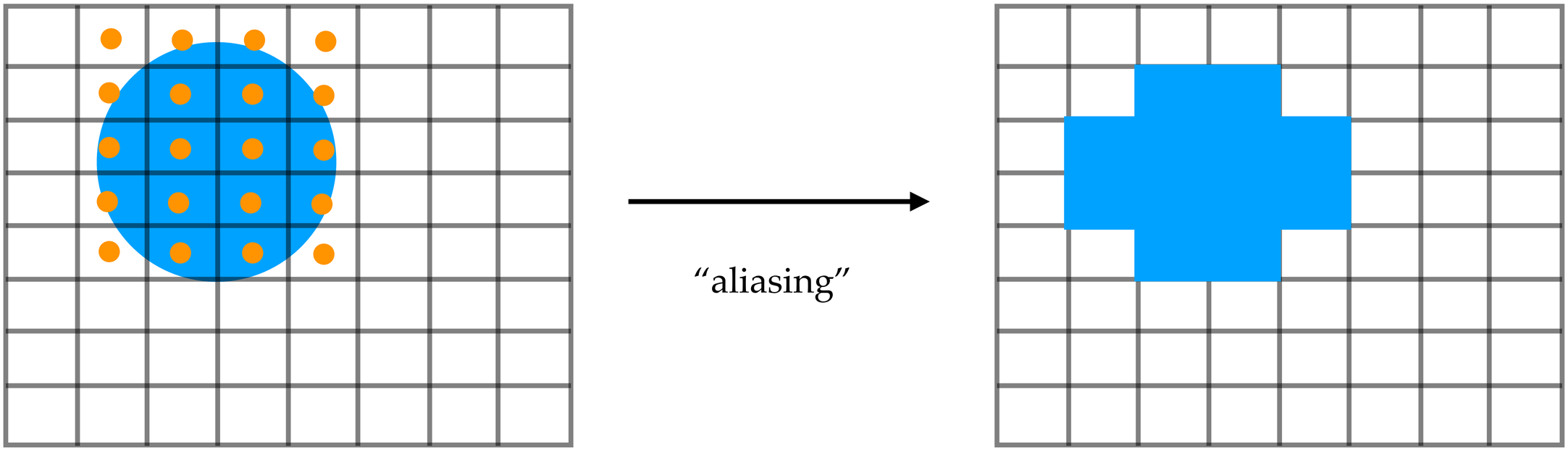


for polylines stroke color, measure the distance between the pixel center and the line

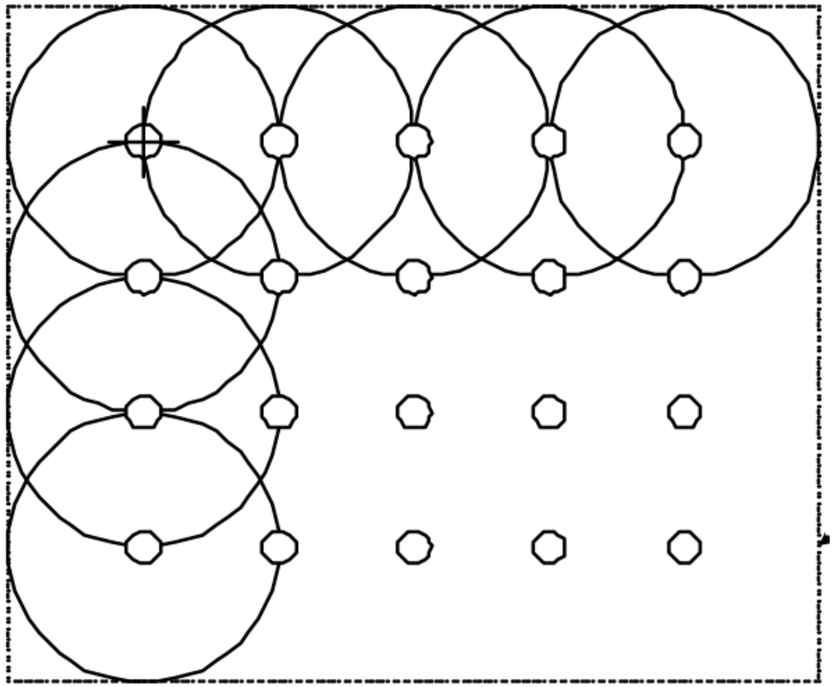
Is testing pixel center alone sufficient?

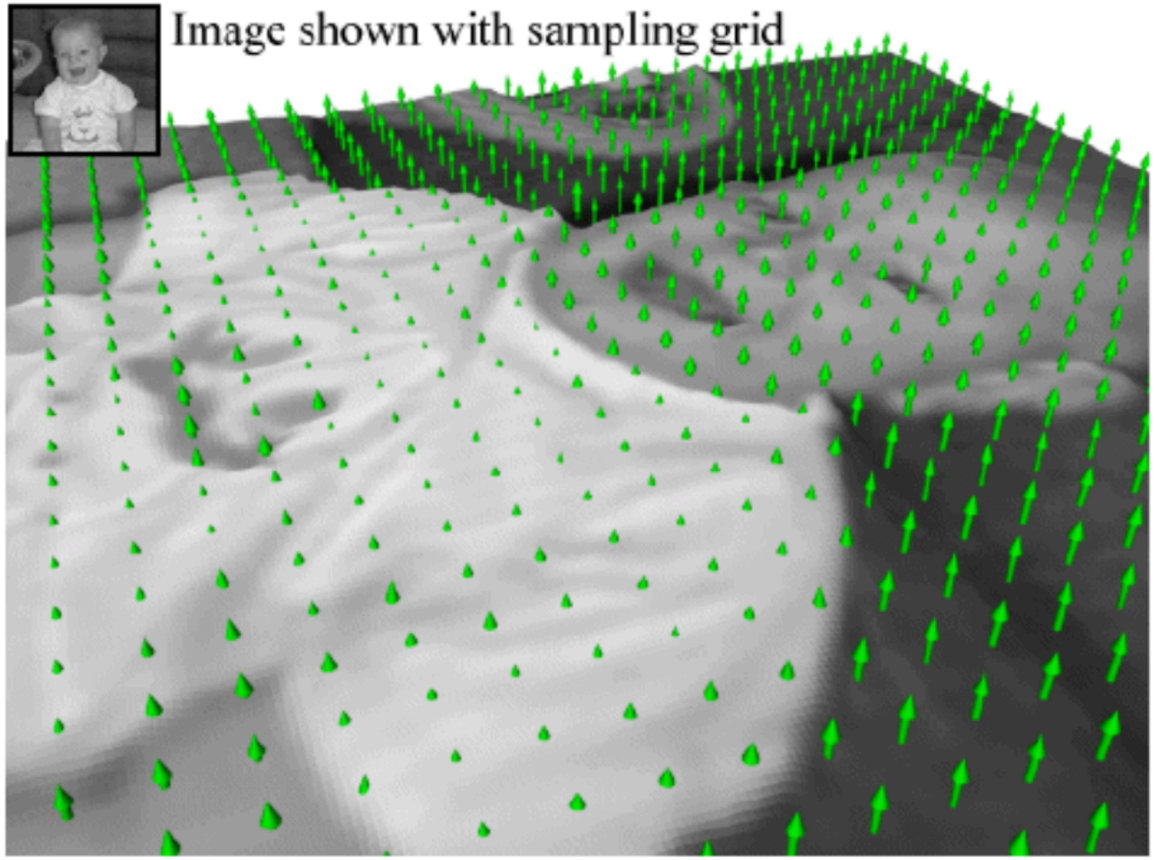


Is testing pixel center alone sufficient?



Recall: a pixel should store the average color over a region

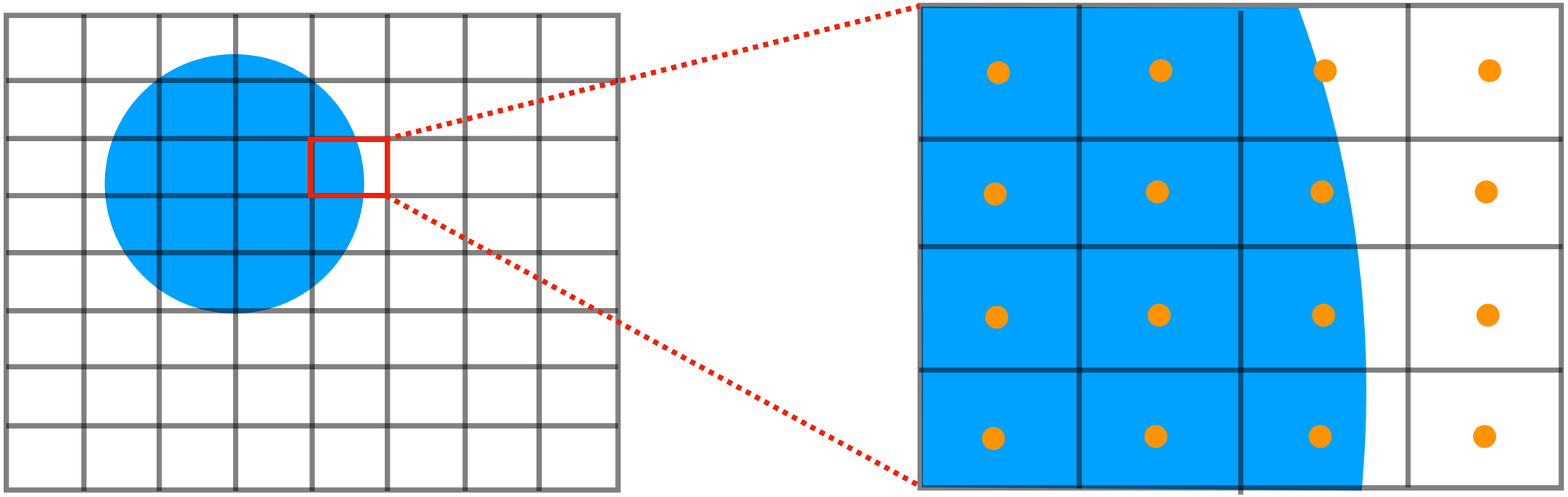




https://groups.csail.mit.edu/graphics/classes/6.837/F03/lectures/23_aliasing.pdf

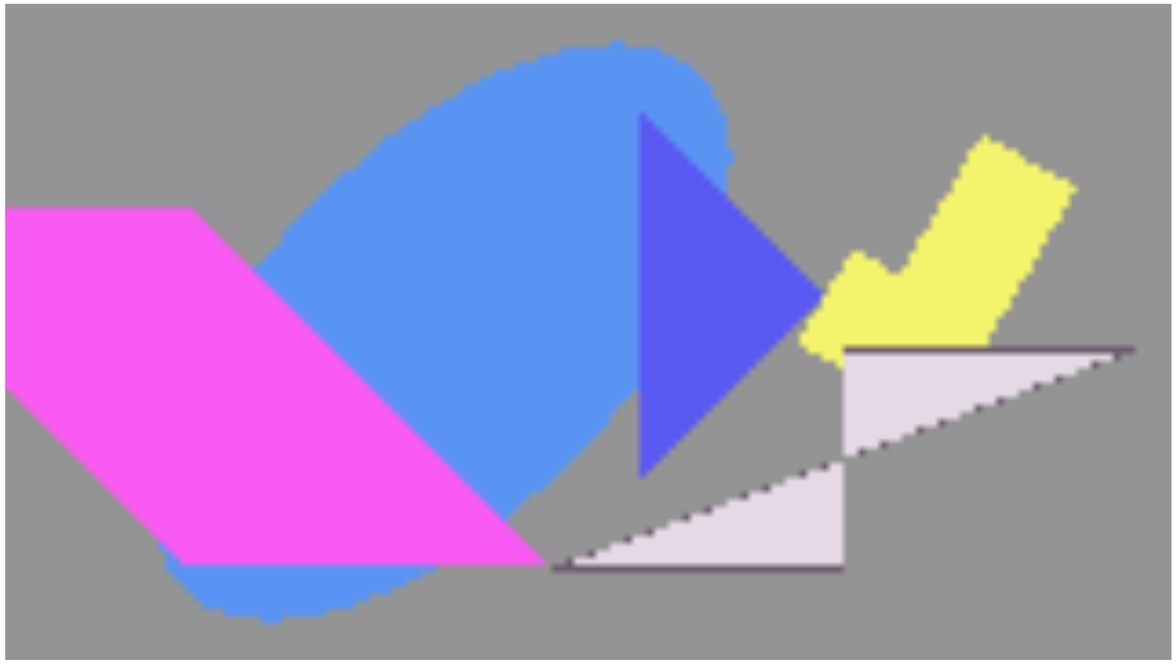
Super-sampling Antialiasing

combat aliasing artifacts by taking average of color inside a pixel

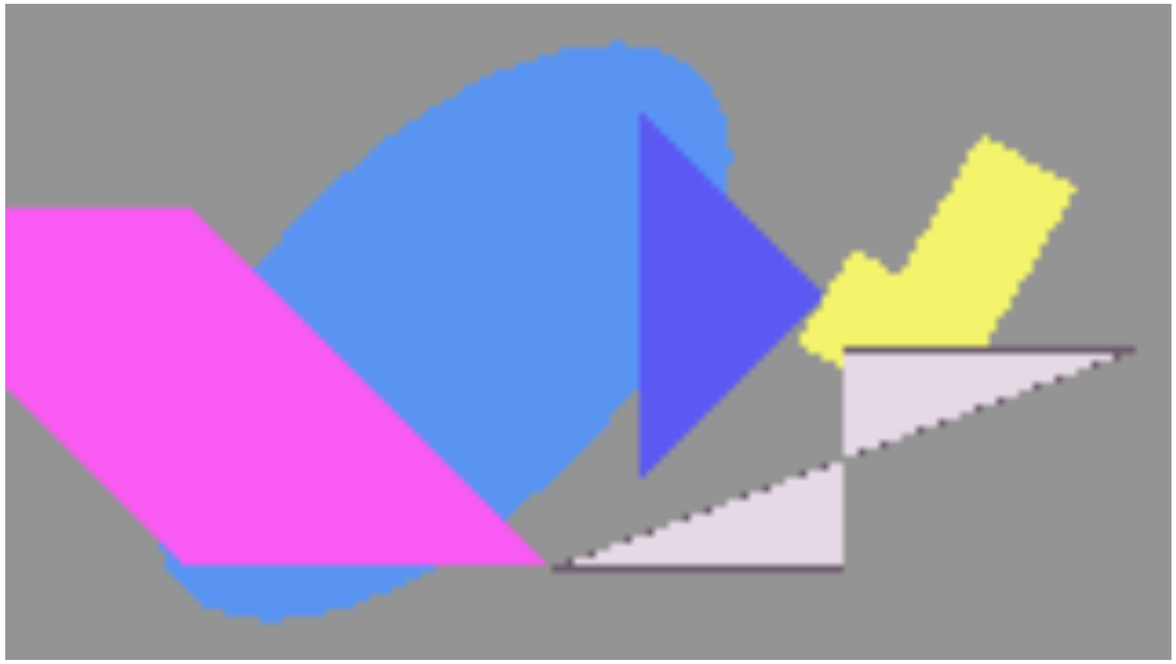


Aliased vs antialiased

which one is antialiased?

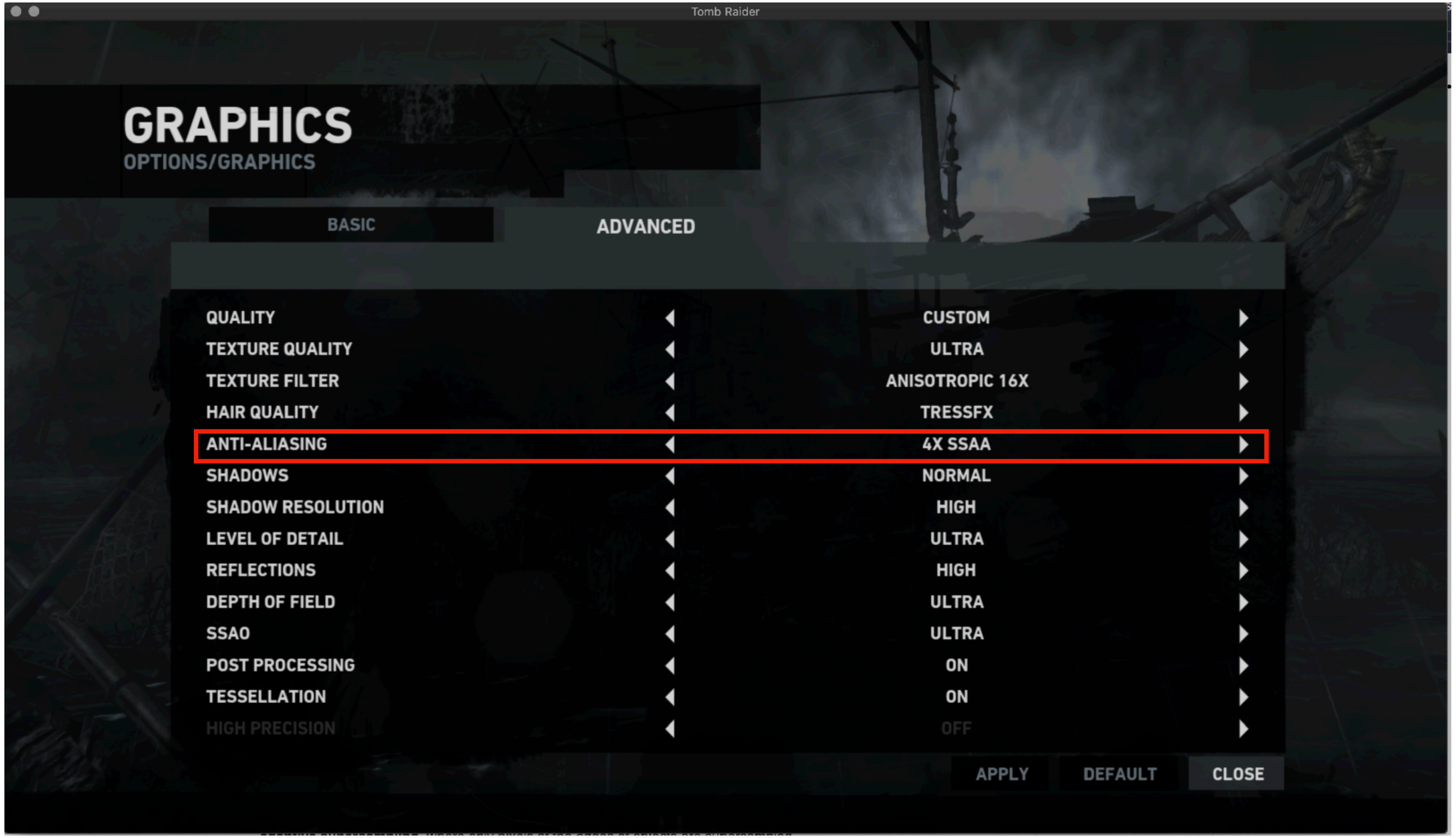


Aliased vs antialiased



aliased

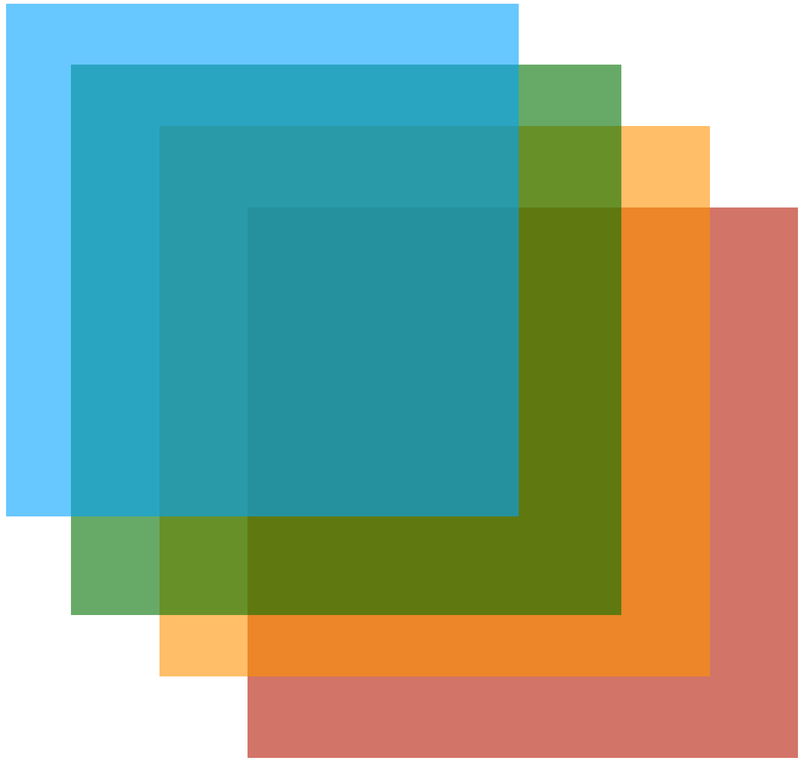
antialiased



"SSAA" = supersampling anti-aliasing

Transparency

How do we render transparent primitives?



Rendering a single transparent object

Q: how do we compute the color here?

color C

100% opacity

60% opacity

Rendering a single transparent object

color C

100% opacity

Q: how do we compute the color here?

$0.6 * C + 0.4 *$ background

60% opacity

Rendering two transparent objects

quiz 1: which one is at the front?

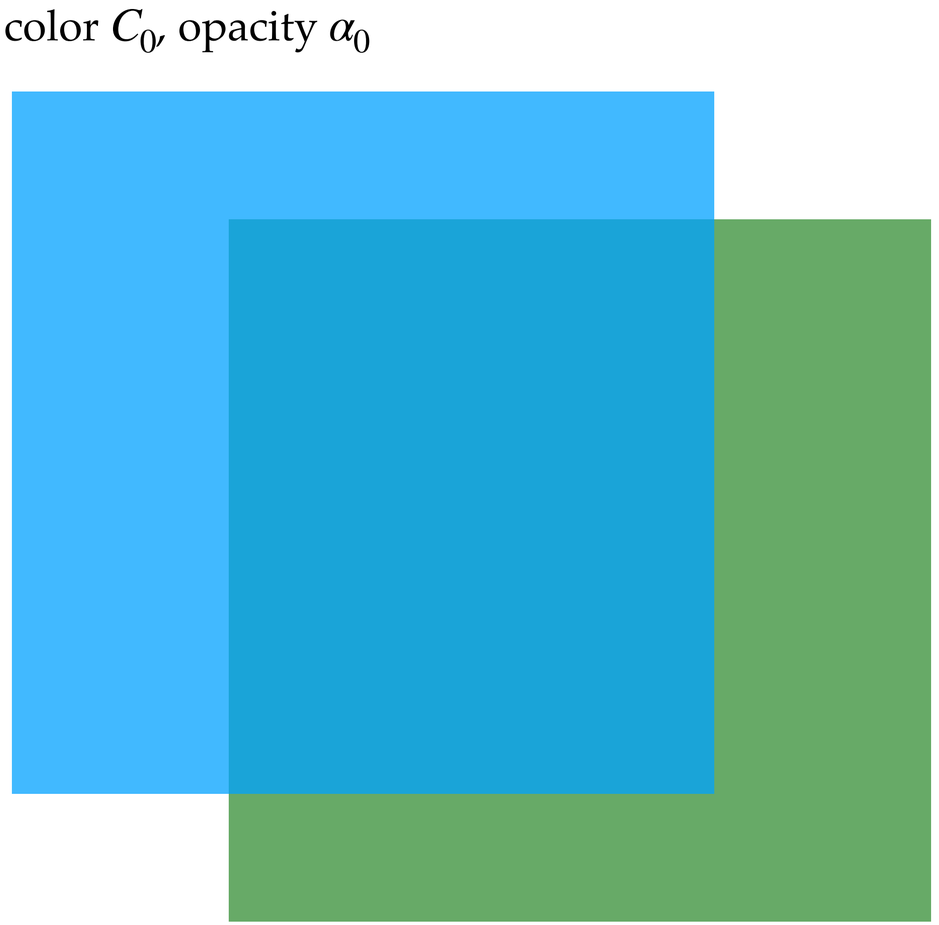
color C_0, opacity α_0

color C_1, opacity α_1

Rendering two transparent objects

quiz 1: which one is at the front?

quiz 2: how do we compute the color in the middle?



Rendering two transparent objects

quiz 1: which one is at the front?

quiz 2: how do we compute the color in the middle?

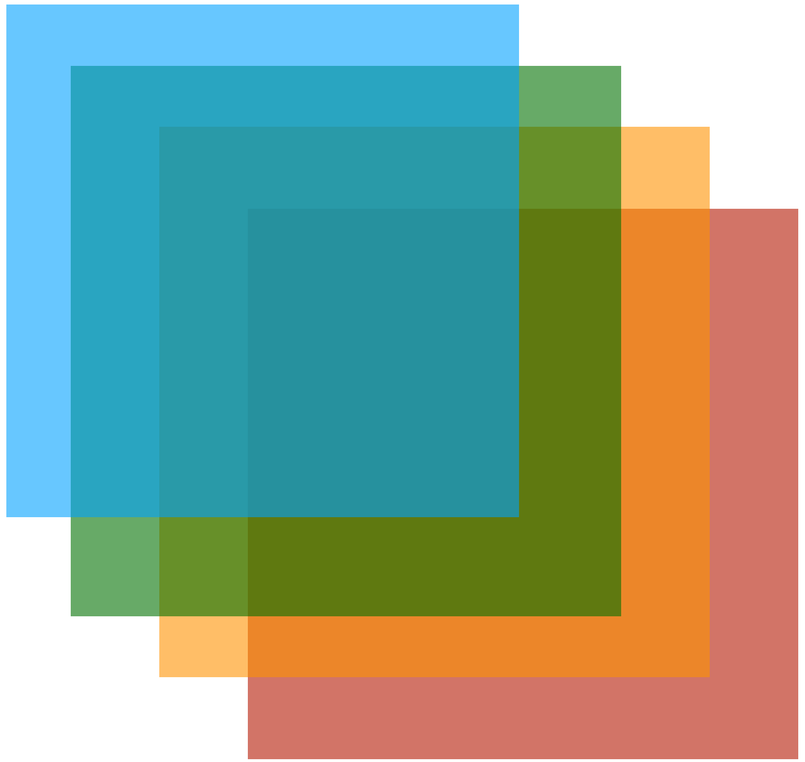
$C_0 * \alpha_0 + C_1 * (1 - \alpha_0)\alpha_1 + \text{background} * (1 - \alpha_0) * (1 - \alpha_1)$

color C_0, opacity α_0

color C_1, opacity α_1

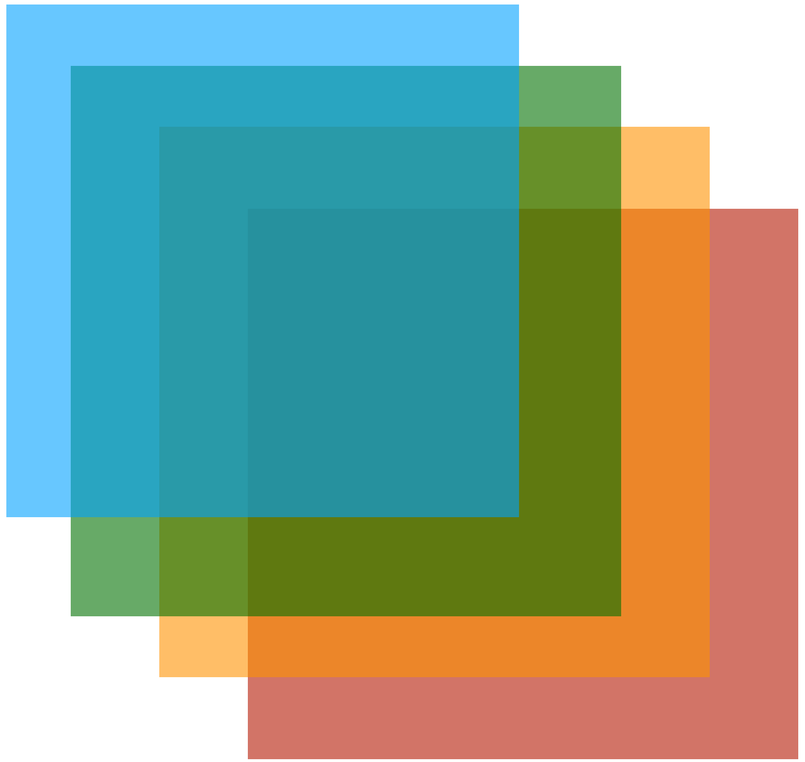
Alpha blending

$$
\begin{aligned}
C_0 * \alpha_0 + \\
C_1 * (1 - \alpha_0)\alpha_1 + \\
C_2 * (1 - \alpha_0)(1 - \alpha_1)\alpha_2 + \\
\cdots + \\
\text{background} * (1 - \alpha_0)(1 - \alpha_1)\cdots
\end{aligned}
$$



Alpha blending

$$
\begin{aligned}
C_0 * \alpha_0 + \\
C_1 * (1 - \alpha_0)\alpha_1 + \\
C_2 * (1 - \alpha_0)(1 - \alpha_1)\alpha_2 + \\
\cdots + \\
\text{background} * (1 - \alpha_0)(1 - \alpha_1)\cdots
\end{aligned}
$$

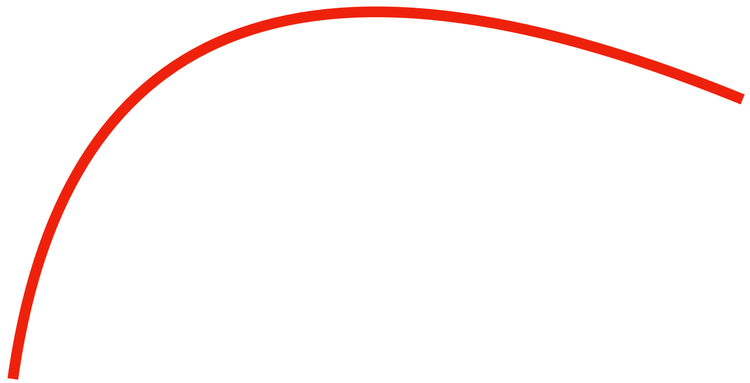


$C_i * a_i$ is often called the “premultiplied alpha”

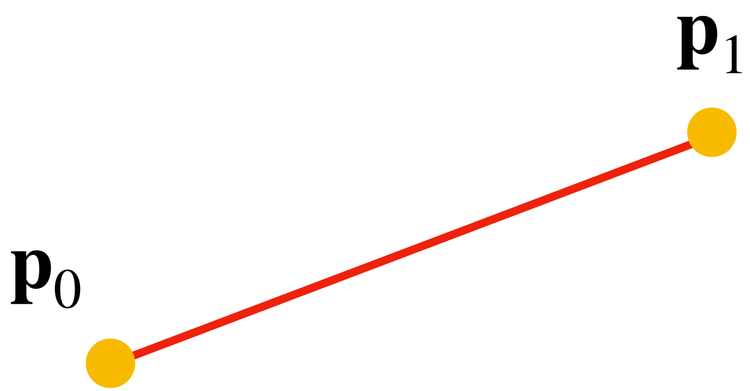
What about curves?

curves

How do we represent curves?



How do we represent curves?



let's start from a straight line

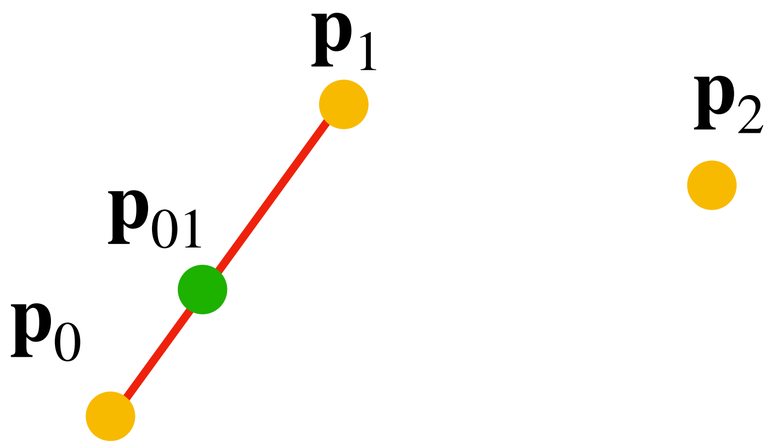
points on the line can be represented as

$$\mathbf{p}(t) = \mathbf{p}_0 (1 - t) + \mathbf{p}_1 t$$

$$t \in [0,1]$$

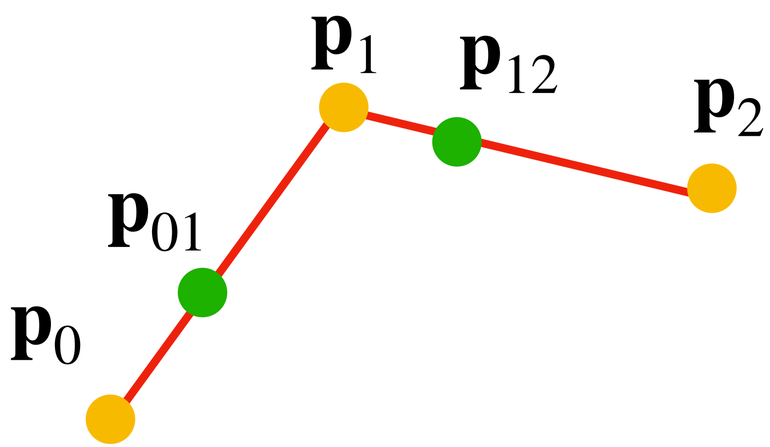
$\mathbf{p}(t)$ is a linear interpolation between $\mathbf{p}_0$ and $\mathbf{p}_1$

Bézier curve [de Casteljau 1959]



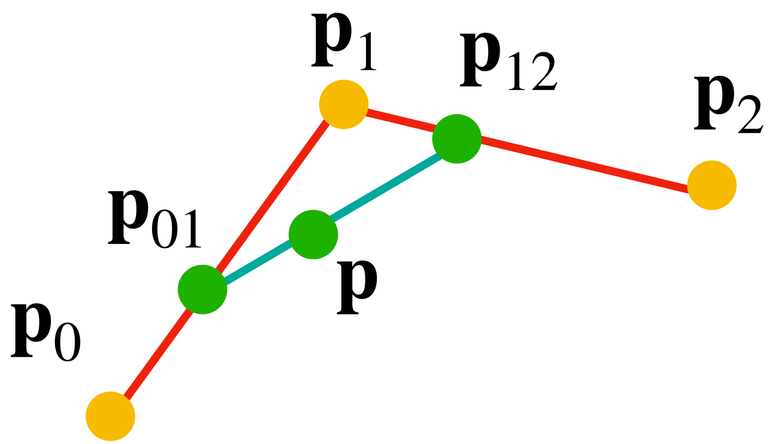
1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$

Bézier curve [de Casteljau 1959]



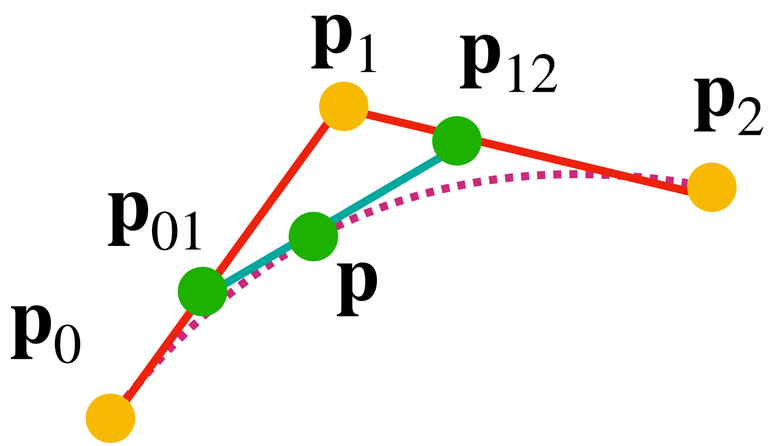
1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$
2. let $\mathbf{p}_{12}(t) = \mathbf{p}_1(1-t) + \mathbf{p}_2 t$

Bézier curve [de Casteljau 1959]



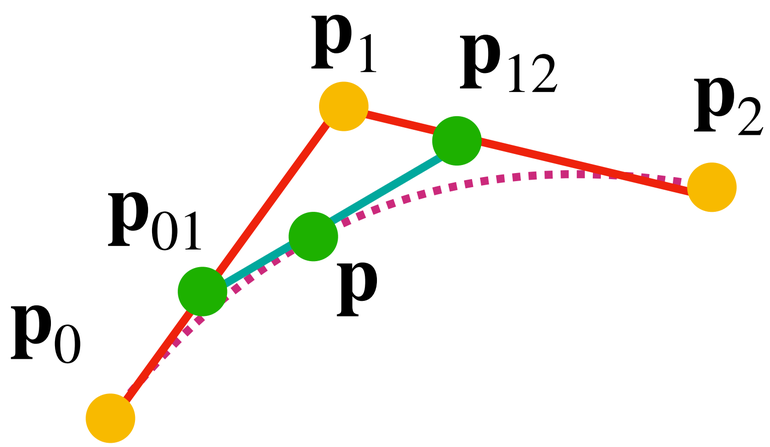
1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$
2. let $\mathbf{p}_{12}(t) = \mathbf{p}_1(1-t) + \mathbf{p}_2 t$
3. let $\mathbf{p}(t) = \mathbf{p}_{01}\,(1-t) + \mathbf{p}_{12} t$

Bézier curve [de Casteljau 1959]



1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$
2. let $\mathbf{p}_{12}(t) = \mathbf{p}_1(1-t) + \mathbf{p}_2 t$
3. let $\mathbf{p}(t) = \mathbf{p}_{01}\,(1-t) + \mathbf{p}_{12} t$

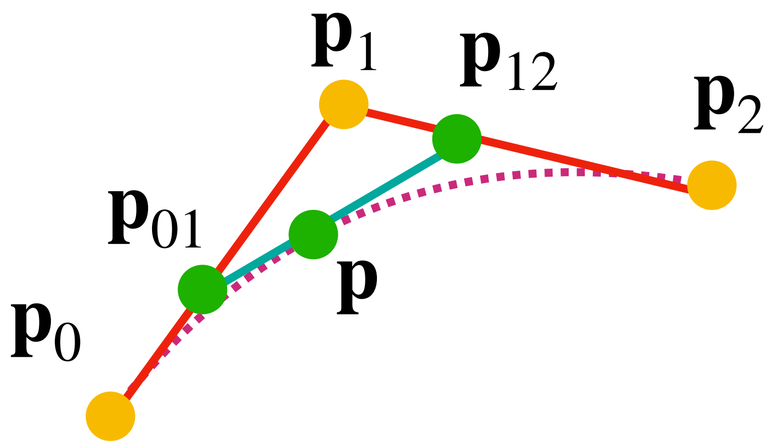
Bézier curve [de Casteljau 1959]



1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$
2. let $\mathbf{p}_{12}(t) = \mathbf{p}_1(1-t) + \mathbf{p}_2 t$
3. let $\mathbf{p}(t) = \mathbf{p}_{01}(1-t) + \mathbf{p}_{12} t$

cool animation at
https://ciechanow.ski/curves-and-surfaces/

Bézier curve [de Casteljau 1959]



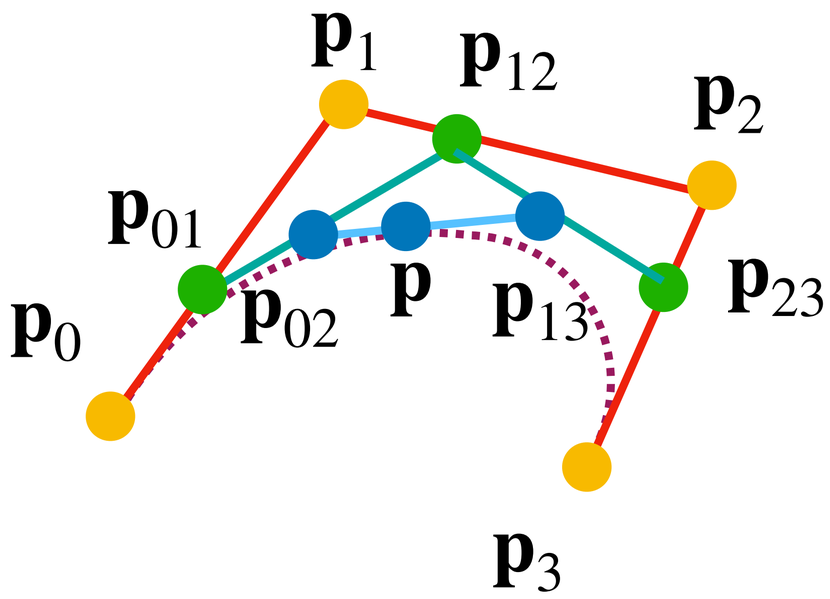
1. let $\mathbf{p}_{01}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$
2. let $\mathbf{p}_{12}(t) = \mathbf{p}_1(1-t) + \mathbf{p}_2 t$
3. let $\mathbf{p}(t) = \mathbf{p}_{01}(1-t) + \mathbf{p}_{12} t$

expanding, we get

$$\mathbf{p}(t) = (1-t)^2 \mathbf{p}_0 + 2(1-t)t\mathbf{p}_1 + t^2\mathbf{p}_2$$

this is called a “quadratic Bézier curve”

Bézier curve [de Casteljau 1959]



we can add more control points, 4 control points form a "cubic Bézier curve"

$$\mathbf{p}(t) = (1-t)^3\,\mathbf{p}_0 + 3\,(1-t)^2\,t\mathbf{p}_1 + 3\,(1-t)\,t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

in practice, usually quadratic and cubic Bézier curves are used

Bézier curve [de Casteljau 1959]

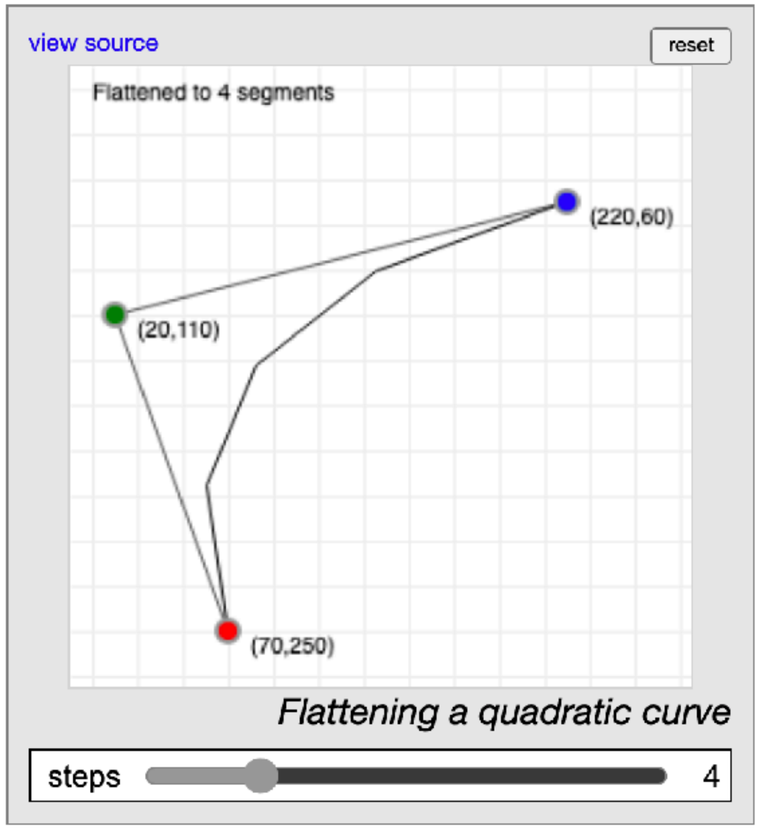
Excellent resources about Bezier curves:

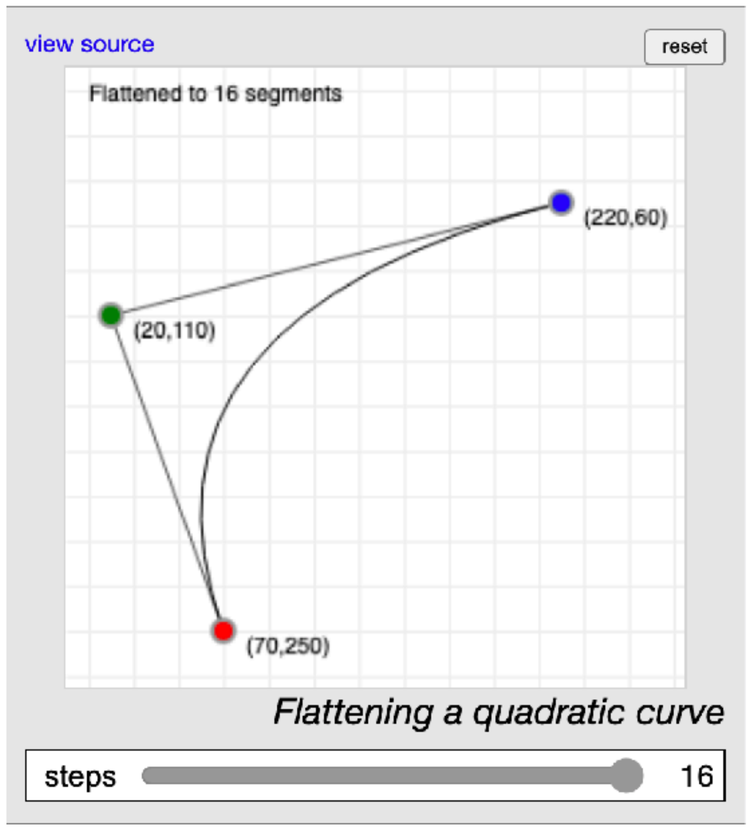
- Bartosz Ciechanowski — Curves and Surfaces (https://ciechanow.ski/curves-and-surfaces/)
- Pomax — A Primer on Bézier Curves (https://pomax.github.io/bezierinfo/)

we will talk more about curves and surfaces later in the class

Rendering Bézier curve

Method 1: flatten the Bézier curve into line segments



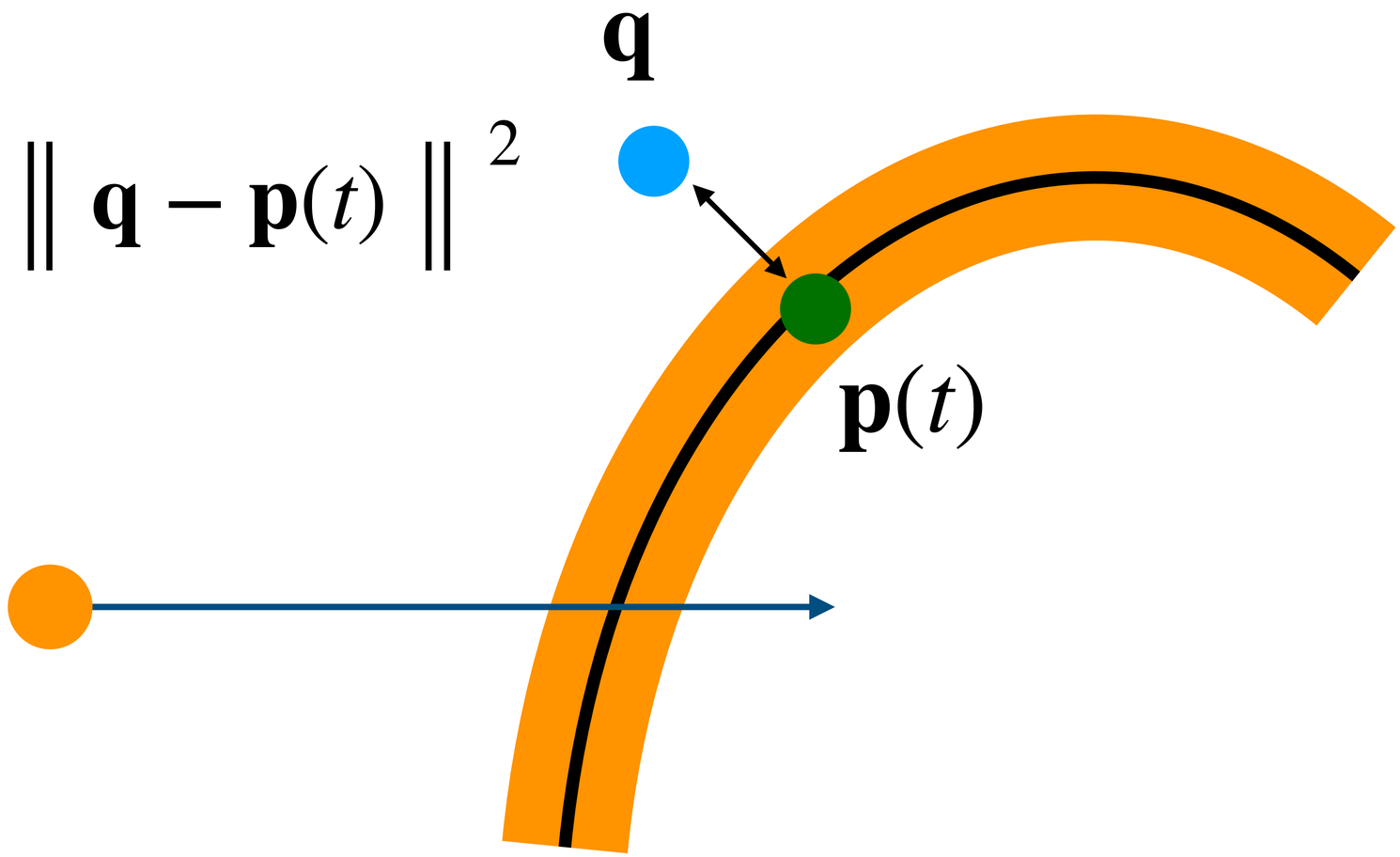


https://pomax.github.io/bezierinfo/#flattening

Rendering Bézier curve

Method 2: directly compute ray-curve intersection & distance between point and curve

$$t^* = \arg\min_t \left\| \mathbf{q} - \mathbf{p}(t) \right\|^2$$



SVG (Scalable Vector Graphics)

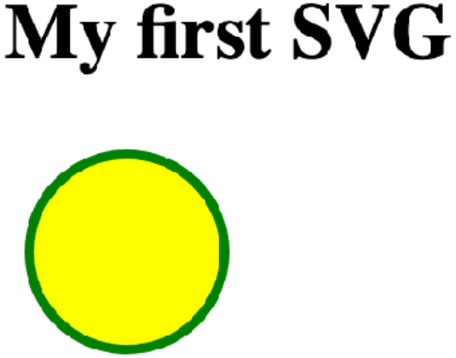
https://www.w3schools.com/graphics/svg_intro.asp

```
<html>
<body>

<h1>My first SVG</h1>

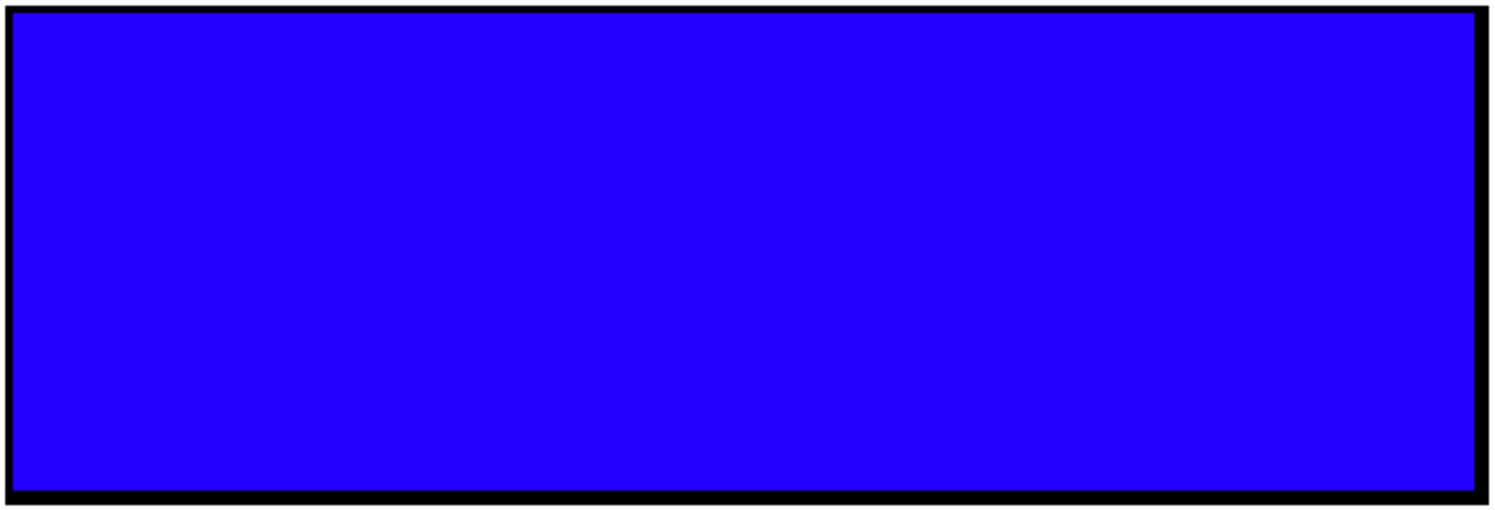
<svg width="100" height="100">
  <circle cx="50" cy="50" r="40" stroke="green" stroke-width="4" fill="yellow" />
</svg>

</body>
</html>
```



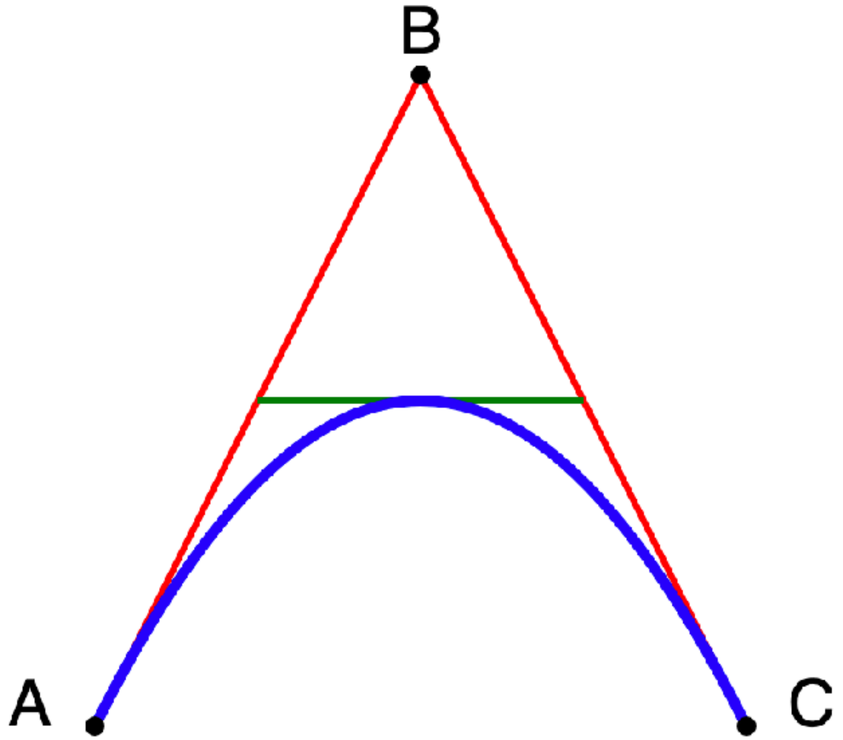
SVG (Scalable Vector Graphics)

```
<svg width="400" height="110">
  <rect width="300" height="100" style="fill:rgb(0,0,255);stroke-width:3;stroke:rgb(0,0,0)" />
</svg>
```

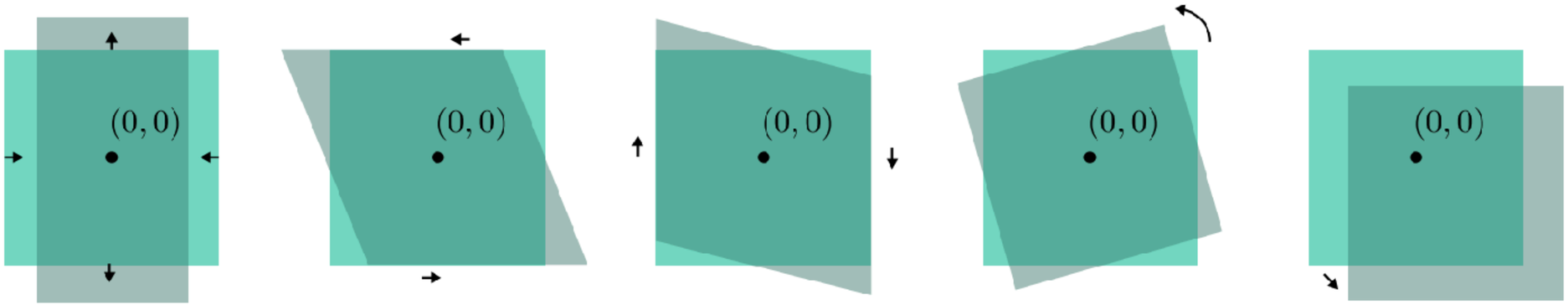


SVG (Scalable Vector Graphics)

```
<svg height="400" width="450">
  <path id="lineAB" d="M 100 350 l 150 -300" stroke="red"
  stroke-width="3" fill="none" />
  <path id="lineBC" d="M 250 50 l 150 300" stroke="red"
  stroke-width="3" fill="none" />
  <path d="M 175 200 l 150 0" stroke="green" stroke-width="3"
  fill="none" />
  <path d="M 100 350 q 150 -300 300 0" stroke="blue"
  stroke-width="5" fill="none" />
  <!-- Mark relevant points -->
  <g stroke="black" stroke-width="3" fill="black">
    <circle id="pointA" cx="100" cy="350" r="3" />
    <circle id="pointB" cx="250" cy="50" r="3" />
    <circle id="pointC" cx="400" cy="350" r="3" />
  </g>
  <!-- Label the points -->
  <g font-size="30" font-family="sans-serif" fill="black" stroke="none"
  text-anchor="middle">
    <text x="100" y="350" dx="-30">A</text>
    <text x="250" y="50" dy="-10">B</text>
    <text x="400" y="350" dx="30">C</text>
  </g>
</svg>
```



Next: (2D) linear transformations



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) scale

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \lambda_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) shear x

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) shear y

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(d) rotate

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(e) translate