SoK: The Arithmetic-Geometric Dissonance
Structural Gaps and Limits in the 2025 Cryptographic Landscape

William[1](✉)

Department of Computer Science, Bina Sarana Informatika
william@bsi.ac.id

Abstract. The year 2025 marks a critical inflection point where the standardization of Post-Quantum Cryptography (PQC) collides with renewed theoretical instability. This Systematization of Knowledge (SoK) analyzes hundreds of contributions from Eurocrypt, Crypto, CHES, Asiacrypt, and TCC 2025. Rather than a mere enumeration of trends, we propose a unified framework—the *Arithmetic-Geometric Dissonance*—to categorize the current barriers in cryptography. We identify three structural gaps: (1) The **Representation Gap**, where the mismatch between polynomial arithmetic and Boolean masking incurs prohibitive hardware costs; (2) The **Invariant Gap**, highlighted by the "Syzygy Distinguisher" which exploits algebraic geometry to challenge code-based hardness assumptions; and (3) The **Approximation Gap**, formally proven via approximation theory limits, where the continuous geometry of AI models clashes with the discrete arithmetic of MPC/FHE. We conclude with strategic open problems for the 2026 research agenda.

Keywords: Post-Quantum Cryptography · Side-Channel Analysis · Syzygy Distinguisher · Zero-Knowledge Proofs · MPC · SoK

1 Introduction

The narrative of cryptography in 2025 is defined by a dialectical tension. On one hand, the National Institute of Standards and Technology (NIST) has finalized standards for Module-LWE primitives ($ML-KEM$ and $ML-DSA$), signaling industrial maturity. On the other hand, the foundational literature has entered a period of turbulence.

In this SoK, we argue that the primary challenges of 2025 are not merely engineering bugs, but symptoms of a fundamental friction we term the **Arithmetic-Geometric Dissonance**. We systematize recent literature into three distinct layers of abstraction friction:

1. **The Representation Gap (Physical Layer):** The efficient execution of algebraic structures (rings, fields) clashes with the leakage models of physical hardware, specifically in the context of side-channel masking.

2. **The Invariant Gap (Theoretical Layer):** Deep algebraic invariants (Syzygies) are emerging to separate structured instances (keys) from random instances, threatening indistinguishability assumptions.
3. **The Complexity & Approximation Gap (Protocol Layer):** In advanced protocols like MPC and ZKML, we hit asymptotic walls when trying to represent continuous geometric functions (AI manifolds) within discrete arithmetic circuits.

2 The Representation Gap: PQC Implementation Limits

While the mathematical security of Lattice-based cryptography is stable, CHES 2025 revealed that its physical security is bounded by the cost of protecting arithmetic operations against Side-Channel Analysis (SCA).

2.1 The Boolean-Arithmetic Conversion Bottleneck

The core issue is the mismatch between the domain of the algorithm and the domain of the masking gadget.

- **Algorithm Domain:** $ML - KEM$ operates over $\mathcal{R}_q = \mathbb{Z}_q[X]/(X^n + 1)$ where $q = 3329$.
- **Gadget Domain:** Boolean masking operates over $GF(2)$, ideal for logic operations (XOR/AND).

To perform non-linear operations (like comparisons during decapsulation) securely, variables must be converted. Let x be a sensitive value.

$$x = \bigoplus_{i=0}^{t} x_i^{(B)} \xrightarrow{G_{B2A}} x = \sum_{i=0}^{t} x_i^{(A)} \pmod q$$

Cost Analysis. Recent literature demonstrates that the cycle count complexity of a t-order masked conversion gadget G_{B2A} scales poorly.

$$\text{Cost}(G_{B2A}) \approx \mathcal{O}(t^2 \cdot \log q) \text{ non-linear gate evaluations} \tag{1}$$

For $ML - KEM - 768$, high-order masking ($t \geq 2$) results in a performance degradation factor of $\approx 10\times$ to $100\times$ compared to unmasked implementations. This "Representation Gap" suggests that software-only implementations on constrained devices may never achieve both high-speed and high-assurance simultaneously.

3 The Invariant Gap: The Syzygy Controversy

The most significant theoretical disruption of 2025 is the "Syzygy Distinguisher" against code-based cryptography (specifically Goppa codes used in Classic McEliece), presented at Eurocrypt 2025.

3.1 Formalizing the Syzygy Invariant

The attack differentiates the public key (a generator matrix of a Goppa code) from a random matrix by computing the graded Betti numbers of the associated ideal. Let C be the code. The distinguisher examines the minimal free resolution of the ideal I_C generated by the dual code. The invariant of interest is:

$$\beta_{i,j}(I_C) = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(I_C, \mathbb{K})_j \tag{2}$$

where S is the polynomial ring.

The Distinguisher:

- **Random Code:** Exhibits a "generic" Betti table.
- **Goppa Code:** Exhibits anomalies (zeros or non-generic values) in specific Betti numbers $\beta_{i,j}$ due to the algebraic structure of the Goppa polynomial.

3.2 Implications: Indistinguishability vs. Search

This result creates a gap in our security reduction. Standard security proofs rely on the decisional assumption (IND-CPA):

$$(\mathbf{H}_{\text{Goppa}}) \approx_c (\mathbf{H}_{\text{Random}})$$

The Syzygy distinguisher breaks this assumption. However, it does not immediately yield a Key Recovery Attack. The open problem defining 2025 is proving whether $\mathcal{A}_{\text{Dist}} \implies \mathcal{A}_{\text{Search}}$. Until then, Classic McEliece exists in a state of theoretical limbo.

4 The Complexity Gap: ZK and MPC Protocols

4.1 Zero-Knowledge: The Rise of Folding Schemes

The quest for "Doubly Efficient" ZK has led to the proliferation of Folding Schemes. In 2025, we observe a shift from discrete-log assumptions to hash-based assumptions (WHIR) to accommodate PQC requirements.

Table 1. Taxonomy of Major 2025 Folding Schemes

Scheme	Accumulator	Recursion Cost	Assumption	Verifier Complexity
Nova (Pre-2025)	R1CS	$\mathcal{O}(1)$ MSM	DLOG	$\mathcal{O}(\|C\|)$
HyperNova	CCS	$\mathcal{O}(1)$ MSM	DLOG	$\mathcal{O}(\log \|C\|)$
WHIR (2025)	**Multilinear**	$\mathcal{O}(\log N)$ **Hash**	**Collision-Res (Generic)**	**Polylog**
LatticeFold	Lattice	$\mathcal{O}(1)$ Matrix	ML-LWE	High

4.2 MPC: The Space-Round Dilemma

In Secure Multi-Party Computation, TCC 2025 literature has formalized a prohibitive trade-off. The fundamental friction lies in the linearity of communication versus the depth of the circuit being evaluated.

Formal Intuition. Consider a functionality f represented by a layered boolean circuit of depth d. The "Space-Round Dilemma" can be viewed through the lens of *pebble games* on circuit graphs. To evaluate a node without storing its predecessors (low space), one must re-evaluate or re-communicate paths (high rounds).

$$\text{Space} \times \text{Rounds} \geq \Omega(\text{Circuit Depth}) \tag{3}$$

This inequality implies that for Deep Learning inference (where d is large), MPC cannot simultaneously minimize latency and hardware footprint. This creates a hard limit for "Real-Time MPC" on edge devices.

5 The Approximation Gap: Continuous AI vs. Discrete Crypto

The intersection of AI and Cryptography is often framed purely as adversarial. However, under our framework, the core issue is the **Arithmetic-Geometric Dissonance** between the two computational models.

5.1 The Manifold Mismatch

Modern Deep Learning relies on *Stochastic Gradient Descent* over continuous geometric manifolds (approximated by floating-point numbers $\mathbb{R}$). In contrast, Cryptography relies on exact arithmetic over discrete finite fields ($\mathbb{Z}_q$).

- **AI Domain (Geometric):** Requires non-linear, non-polynomial activation functions (e.g., ReLU, Sigmoid, GeLU) to approximate complex, smooth decision boundaries.
- **Crypto Domain (Arithmetic):** Homomorphic encryption (FHE) and MPC are efficient only for addition and multiplication (Polynomials).

5.2 The Cost of Non-Linearity: A Formal Analysis

To evaluate a Neural Network privately, one must approximate non-polynomial geometric functions (like ReLU) using polynomial arithmetic over $\mathbb{Z}_q$. We formally demonstrate why this is computationally prohibitive.

Lemma 1 (Polynomial Approximation Lower Bound). *Let* $f(x) = ReLU(x) = \max(0, x)$ *defined on the interval* $[-1, 1]$*. Let* $P_d(x)$ *be a polynomial of degree* d*. To achieve a uniform approximation error* $\|f - P_d\|_\infty \leq \epsilon$*, the degree* d *must satisfy* $d = \Omega(1/\epsilon)$*.*

Proof. The function $f(x) = \mathrm{ReLU}(x)$ is Lipschitz continuous but not differentiable at $x = 0$ (the "kink"). According to Jackson's Theorem in approximation theory, for a function that is continuous but not differentiable ($C^0 \setminus C^1$), the error of the best polynomial approximation decays at a rate of $\mathcal{O}(1/d)$. Conversely, to satisfy a target precision ϵ, we invert the bound:

$$\epsilon \approx \frac{1}{d} \implies d \approx \frac{1}{\epsilon}$$

Implication for Cryptography. In a cryptographic context, high precision is mandatory. For a modest accuracy of $\epsilon = 10^{-6}$, the required polynomial degree is $d \approx 10^6$.

- **In FHE (CKKS/BFV):** Multiplicative depth corresponds to $\log_2(d)$. A degree of 10^6 requires a circuit depth of ≈ 20, which consumes an enormous noise budget, necessitating costly Bootstrapping operations.
- **In MPC:** Evaluating a polynomial of degree d typically requires $\mathcal{O}(d)$ or $\mathcal{O}(\log d)$ communication rounds.

Synthesis: The "Approximation Gap" is formally defined as this asymptotic scaling $d = \Omega(1/\epsilon)$. It proves that preserving geometric accuracy within arithmetic constraints incurs an efficiency penalty that is linear in precision, creating a hard scalability limit for privacy-preserving AI.

6 Future Directions: The 2026 Agenda

Based on the identified gaps, we propose the following research priorities:

1. **Masking-Aware Arithmetization:** Designing PQC schemes where the underlying ring arithmetic is isomorphic to Boolean operations, minimizing the $B2A$ conversion penalty.
2. **Syzygy Reductions:** Establishing a formal reduction from the Syzygy Distinguisher to the Key Recovery problem to salvage (or condemn) code-based encryption.
3. **Discrete-Native AI:** Moving beyond post-training quantization (PTQ). We propose researching Neural Network architectures that train natively on finite fields $\mathbb{Z}_q$ (similar to Binary Neural Networks or QNNs), thereby removing the Approximation Gap at the source.

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