Intro Prob Lecture Notes

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Continuous Random Variables

- A random variable X is called *continuous* if it possesses a cumulative distribution function which is a continuous function on the real line.
  - Reminder: CDF is $F_X(x) = P(X \leq x)$
- Let $x \in \mathbb{R}$. Then for $h > 0$,

$$\begin{aligned} P(x - h < X \leq x) &= P(X \leq x) - P(X \leq x - h) \\ &= F_X(x) - F_X(x - h) \\ &\text{Approaches 0 as h approaches 0} \end{aligned}$$

- Therefore, $P(X = x) = 0$.
- But

$$\frac{\mathbf{P(x - h < X \leq x)}}{\mathbf{h}} = \frac{\mathbf{F_X(x) - F_X(x - h)}}{\mathbf{h}} \rightarrow \text{(If this difference quotient converges) } F'_X(x) = f_X(x) \geq 0$$

- Bold value is probability mass per unit length, or a probability “density”

$$P(-\infty < X \leq x) = F_X(x) = \int_{-\infty}^{x} f(u)du$$

$$\text{As } x \to \infty, 1 = \int_{-\infty}^{\infty} f_X(u)du$$

- A probability density function (pdf) is any function f such that
  - $f : \mathbb{R} \to \mathbb{R}, f(x) \geq 0 \forall x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f(x)dx = 1$
- To compute probability mass, $P(a < X \leq b) := \int_a^b f(x)dx$
- Example: $f(x) = x^2$ on $0 \leq x \leq 2$, $f(x) = 0$ elsewhere
  - Not a pdf.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0dx + \int_0^2 x^2dx + \int_2^{\infty} 0dx$$
$$= \frac{x^3}{3}\Big|_0^2$$
$$= \frac{8}{3}$$

  - However, we can normalize the function to $\tilde{f}$, so that the value from 0 to 2 to $\frac{3x^2}{8}$. $\tilde{f}$ would integrate to 1, so that would be a pdf
- Suppose X has pdf $\tilde{f}$. Compute

$$P(X > \frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} \tilde{f}(u)du$$
$$= \int_{\frac{1}{2}}^{2} \tilde{f}(u)du$$
$$= \int_{\frac{1}{2}}^{2} \frac{3u^2}{8}du$$
$$= \frac{u^3}{8}\Big|_{\frac{1}{2}}^{2}$$
$$= 1 - \frac{1}{64}$$
$$= \frac{63}{64}$$

Expected Value of Continuous Variables

- When $f_X(x)$ is the pdf of X.

$$E(X) = \int\limits_{-\infty}^{\infty} x f_X(x) dx$$

  - This will exist when $\int\limits_{-\infty}^{\infty} |x| f_X(x) dx < \infty$
  - The Cauchy pdf will *not* have an expected value
    - $f(x) \frac{1}{\pi(1x^2)}$ for $-\infty < x < \infty$
- Exercise:

$$\begin{aligned} E(X) &= \int\limits_{0}^{2} x \cdot \frac{3}{8} x^2 dx \\ &= \frac{3}{8} \cdot \frac{x^4}{4} \Big|_0^2 \\ &= \frac{3 \cdot 16}{32} \\ &= \frac{3}{2} \end{aligned}$$

- Example: $X \sim$ uniform[0, 1]
  - $f(x) = 1$ for $0 \leq x \leq 1$
  - $f(x) = 0$ elsewhere.
  - Expectation is $\frac{1}{2}$ by inspection

$$\begin{aligned} E(X) &= \int\limits_{0}^{1} x dx \\ &= \frac{x^2}{2} \Big|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

  - And

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2 dx \\
&= \frac{x^3}{3}\Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

But we haven't proven this yet!

– Digression: Let's show the pdf for $Y = X^2$, $E(Y) = \int y f_Y(y) dy$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= (\text{Must assume } y \geq 0) \\
&= P(|X| \leq \sqrt{y}) \\
&= P(X \leq \sqrt{y}) \\
&= \int_0^{\sqrt{y}} f(x) dx
\end{aligned}
$$

1 if $\sqrt{y} > 1$, $\sqrt{y}$ if $0 \leq \sqrt{y} \leq 1$, 0 if $\sqrt{y} < 0$

– So, $f_Y(y)$ is 0 if $y > 1$, $\frac{1}{2\sqrt{y}}$ if $0 < y \leq 1$, 0 if $y \leq 0$.

$$
\begin{aligned}
E(X^2) = E(Y) &= \int_0^1 y \cdot \frac{1}{2y^{\frac{1}{2}}} dy \\
&= \frac{1}{2}\int_0^1 y^{\frac{1}{2}} dy \\
&= \frac{1}{3} y^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$