550.420 Intro. to Probability - Spring 2017 **Homework #08**
Due by Friday, March 31 in lecture.

From the textbook:
Chapter 5 / Problems 5.1, 5.4, 5.13, 5.14, 5.32

Additional problems:
A.8.1. In class I mentioned that expectation is a linear operation and one consequence of this is the following: if $X_1, X_2, \ldots, X_n$ are any random variables each of whose expectation exists, then $E\Big(\sum_{i=1}^n X_i\Big) = \sum_{i=1}^n E(X_i)$. Use this property to show that if $X \sim$ binomial(n, p), then $E(X) = np$. As a hint recall that a binomial random variable is the sum of independent Bernoulli(p) random variables.

A.8.2. If $X \sim$ binomial(n, p), then compute the second moment $E(X^2)$ by using the idea developed in lecture that $E(X^2) = E\Big(X(X-1)\Big) + E(X)$. (You should get $E\Big(X(X-1)\Big) = n(n-1)p^2$.) Then use this to show that $Var(X) = np(1-p)$.

A.8.3. Use the definition of variance to show that for any (real) constants a and b,
(i) $Var(X - b) = Var(X)$, i.e., variance is unchanged by shifting by a fixed constant;
(ii) $Var(aX) = a^2 Var(X)$, i.e., changing the scale of X by a factor a implies its variance will scale by a factor a^2.

A.8.4. Suppose X is a continuous random variable having pdf $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ (and $f(x) = 0$ elsewhere). Here, the parameter (constant) λ is strictly positive. A random variable having this pdf is called the *exponential* distribution and we will abbreviate this as $X \sim \exp(\lambda)$.
(a) Compute $E(X)$. You might need to use the integration by parts formula to perform this task.
(b) Compute the cdf $F_X(x)$ of X.
(c) Suppose $X \sim \exp(\frac{1}{2})$. Use an idea developed in class to find the pdf of $V = X^{1/2}$. The distribution of V is an example of the *Rayleigh distribution* used, for example, in scattering theory and understanding wind speed.

A.8.5. Suppose $X \sim \exp(\lambda)$, where as stated above $\lambda > 0$. Use the Law of the Unconscious Statistician to compute $E(e^{sX})$. In probability, the function $M_X(s) = E(e^{sX})$ is called the *moment-generating function of* X. As part of this problem note the values of s for which this expectation is defined.

A.8.6. Suppose $\alpha < \beta$ and let U be a continuous random variable having pdf

$$f_U(u) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } \alpha < u < \beta \\ 0 & \text{otherwise} \end{cases} .$$

We will abbreviate this as $U \sim$ uniform(α, β) and say that U has the uniform distribution on the interval (α, β). Compute the mean and variance of U.

A.8.7. Let Y be a continuous random variable whose cdf $F(y)$ is a strictly increasing function (so that, in particular, F^{-1} exists).
(a) Show that the random variable $U = F(Y) \sim$ uniform$(0, 1)$. Note: $F(y)$ is a function and U is the random variable we get by plugging in the random variable Y into $F(y)$ for y.
(b) You should have found from A.8.4(b) that $F_X(x) = 1 - e^{-\lambda x}$ and this function is strictly increasing on $(0, \infty)$. From part (a) we must have that $U = F_X(X)$ has a uniform distribution on the interval $(0, 1)$. Solve for X in terms of U in the equation $U = F_X(X)$. *Thus, if we can simulate uniform(0,1) values, we can use these to simulate* $\exp(\lambda)$ *values.*