Additional problems solutions HW #11.

A.11.1

$$f_{X,Y}(x,y) = \frac{1}{\pi} \quad \text{for } x^2+y^2 \le 1.$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy \quad \text{for } -1 \le x \le 1.$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \quad \text{for } -1 \le x \le 1.$$

Therefore, the ~~pd~~ conditional pdf of Y given $X=x$ is

$$f_{Y|X}(y|x) = \frac{1/\pi}{\frac{2\sqrt{1-x^2}}{\pi}} = \frac{1}{2\sqrt{1-x^2}} \quad \text{for } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

i.e., $Y|X=x \sim \text{uniform}\left(-\sqrt{1-x^2}, \sqrt{1-x^2}\right)$.

Now, $f_{Y|X}(y|\tfrac{1}{2}) = \dfrac{1}{2\sqrt{1-(\frac{1}{2})^2}} = \dfrac{1}{\sqrt{3}}$

$$P(0 \le Y \le \tfrac{1}{2} \mid X=\tfrac{1}{2}) = \int_0^{1/2} \frac{1}{\sqrt{3}}\, dy = \frac{1}{2\sqrt{3}}.$$

$\boxed{A.11.21}$

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{e^{-\frac{z_1^2+z_2^2}{2}}}{2\pi} \quad \text{for } -\infty < z_1 < \infty, \; -\infty < z_2 < \infty.$$

The transformation is

$$u = \frac{1}{\sqrt{2}} z_1 + \frac{1}{\sqrt{2}} z_2$$

$$v = \frac{1}{\sqrt{2}} z_1 - \frac{1}{\sqrt{2}} z_2$$

whose inverse transformation is :

$$\left.\begin{aligned} \sqrt{2}\, u &= z_1 + z_2 \\ \sqrt{2}\, v &= z_1 - z_2 \end{aligned}\right\} \Rightarrow \sqrt{2}(u+v) = 2z_1 \Rightarrow z_1 = \frac{u+v}{\sqrt{2}}.$$

and $\sqrt{2}(u-v) = 2z_2 \Rightarrow z_2 = \frac{u-v}{\sqrt{2}}$.

$$J = \det\begin{pmatrix} \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial v} \\ \frac{\partial z_2}{\partial u} & \frac{\partial z_2}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = 1 \quad \text{and } |J| = 1$$

implies that this is a unit transformation. Finally,

$$f_{U,V}(u,v) = f_{Z_1,Z_2}\left(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}\right) = \cdots = \frac{e^{-\frac{u^2+v^2}{2}}}{2\pi} \quad \text{for } \begin{matrix} -\infty < u < \infty \\ -\infty < v < \infty \end{matrix}$$

and we see (the remarkable fact) that U, V are also <u>independent</u> standard Normals.

A.11.3

(a) $p_{N,W}(n,w) = p_{N|W}(n|w)\, f_W(w)$

$$= \frac{e^{-w}\cdot w^n}{n!}\cdot\frac{w^{\alpha-1}e^{-w/\beta}}{\beta^\alpha\,\Gamma(\alpha)} \quad \text{for } w>0 \text{ and } n=0,1,2,\dots$$

(b) First we find $p_N(n)$ the marginal pmf of N:

$$p_N(n) = \int_0^\infty \frac{w^{n+\alpha-1}e^{-w(1+\frac{1}{\beta})}}{n!\,\beta^\alpha\,\Gamma(\alpha)}\,dw$$

$$= \frac{1}{n!\,\beta^\alpha\,\Gamma(\alpha)}\cdot\int_0^\infty w^{n+\alpha-1}e^{-w/(1+\frac{1}{\beta})^{-1}}\,dw$$

$$= \frac{1}{n!\,\beta^\alpha\,\Gamma(\alpha)}\cdot\left(1+\frac{1}{\beta}\right)^{-(\alpha+n)}\Gamma(\alpha+n) \quad \text{for } n=0,1,2,\dots$$

and, therefore, the conditional pdf of W given $N=n$ is

$$f_{W|N}(w|n) = \frac{p_{N,W}(n,w)}{p_N(n)} = \dots$$

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$$\cdots = \frac{w^{n+\alpha-1}\, e^{-w(1+\frac{1}{\beta})}}{\cancel{n!\,\beta^{\alpha}\,\Gamma(\alpha)} \cdot \Gamma(\alpha+n)} \cdot \cancel{n!\,\beta^{\alpha}\,\Gamma(\alpha)}\left(1+\frac{1}{\beta}\right)^{\alpha+n}$$

$$= \frac{w^{n+\alpha-1}\, e^{-w/\left(\frac{\beta}{\beta+1}\right)}}{\left(\frac{\beta}{\beta+1}\right)^{\alpha+n} \Gamma(\alpha+n)} \quad \text{for } w > 0$$

which is the pdf of a $\text{Gamma}\left(\alpha+n, \frac{\beta}{\beta+1}\right)$.