Intro Prob Lecture Notes

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Sum of Random Variables

- Suppose X, Y are jointly discrete with known join pmf. To find the pmf of the sum $X + Y$:
  - Naive approach: $P(X + Y = u) = \sum_{x+y=u} P(X = x, Y = y)$
  - Better approach: Partition it into all of the possible values of X:
  - $$\sum_x P(X = x, X + Y = u) \to \sum_x P(X = x, Y = u - x)$$
    - and, if X, Y are independent, then

      $$P(X + Y = u) = \sum_x P(X = x)P(Y = u - x)$$

      - This is the *Convolution formula*
    - If, further, $X \geq 0, Y \geq 0$ and each are integer-valued, then
    - $$P(X + Y = u) = \sum_{x=0}^{u} P(X = x)P(Y = u - x)$$
- Examples: Let $X \sim \text{binomial}(n, p)$, $Y \sim \text{binomial}(m, p)$ and X, Y independent. Let $u \in \{0, 1, 2, \ldots, m + n\}$
  - The distribution of the sum is:

$$
\begin{aligned}
P(X+Y=u) &= \sum_{x=0}^{u} \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{u-x} p^{u-x} (1-p)^{m-u+x} \\
&= p^u (1-p)^{n+m-u} \sum_{x=0}^{u} \binom{n}{x} \binom{m}{u-x} \\
&= \binom{n+m}{u} p^u (1-p)^{n+m-u} u = 0, 1, 2, \ldots, n+m \sim \text{ binomial}(n+m, p)
\end{aligned}
$$

- Now, as another example, suppose $X + Y = k$ for some known k
  * The distribution of $P(X = x)|X + Y = k)$ is

$$
\begin{aligned}
P(X = x | X + Y = k) &= \frac{P(X = x, X + Y = k)}{P(X + Y = k)} \\
&= \frac{P(X = x, Y = k - x)}{P(X + Y = k)} \\
&= \frac{P(X = x) P(k - x)}{P(X + Y = k)} \\
&= \frac{\binom{n}{x} p^x (1-p)^{n-x} \binom{m}{k-x} p^{k-x} (1-p)^{m-k+x}}{\binom{n+m}{u} p^u (1-p)^{n+m-k}} \\
&= \frac{\binom{n}{x} \binom{m}{k-x}}{\binom{n+m}{k}}
\end{aligned}
$$

- X, Y jointly continuous with joint pdf $F(x, y)$. Find pdf of $X + Y$.

$$
\begin{aligned}
F_{X+Y}(u) &= P(X + Y \leq u) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{u-y} f(x, y) dx dy \\
f_{X+Y}(u) &= \frac{d}{du} (F_{X+Y}(u)) \\
&= \int_{-\infty}^{\infty} f(u - y, y) dy \text{ or } \int_{-\infty}^{\infty} f(x, u - x) dx
\end{aligned}
$$

- If X, Y are independent
  -

$$
f_{X+Y}(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u - x) dx
$$

* *Convolution integral*

- If $X \geq 0, Y \geq 0$ then

$$f_{X+Y}(u) = \int_0^u f_X(x) f_Y(u-x) dx$$

- Suppose $X \sim \text{Gamma}(\alpha_1, \beta)$, $Y \sim \text{Gamma}(\alpha_2, \beta)$ independent.

$$\begin{aligned}
f_{X+Y}(u) &= \int_0^u \frac{x^{\alpha_1 - 1} e^{-\frac{x}{\beta}}}{\beta^{\alpha_1}\Gamma(\alpha_1)} \cdot \frac{(u-x)^{\alpha_2 - 1} e^{-\frac{(u-x)}{\beta}}}{\beta^{\alpha_2}\Gamma(\alpha_2)} \\
&= \frac{e^{-\frac{u}{\beta}}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^u x^{\alpha_1-1}(u-x)^{\alpha_2-1} dx \\
&\text{Let } x = uy, 0 < y < 1, dx = udy \\
&= \frac{e^{-\frac{u}{\beta}}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 (uy)^{\alpha_1-1}(u(1-y))^{\alpha_2-1} udy \\
&= \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 y^{\alpha_1-1}(1-y)^{\alpha_2-1} dy \\
&= \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} \sim \text{ Gamma}(\alpha_1+\alpha_2, \beta)
\end{aligned}$$

  - *Jacobian method*

Ordered Statistics

- $X_1, X_2 \ldots X_n \sim$ independent, continuous random variables all having the same distribution (iidf)
  - $X_{(1)}$ = smallest among $X_1, X_2 \ldots X_n$
  - $\vdots$
  - $X_{(j)}$ = jth smallest among $X_1, X_2 \ldots X_n$
  - $\vdots$
  - $X_{(n)}$ is the largest
  - Example: $n = 3$, $x_1 \approx .38, x_2 \approx .214, x_3 \approx .938)$ then $X_{(1)} = x_2, X_{(2)} = x_1, X_{(3)} = x_3$