Intro Prob Lecture Notes

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A few applications of the Central Limit Theorem

- Example: A professor plans to grade 91 exams sequentially. The time sto finish grading a single exam are independent *(continuous)* random variables all having the same distribution with mean $\mu = .25$ hours and standard deviation $\sigma = .1$ hours.
  - Estimate the probability it takes at least 24 hours to finish grading. $S_{91} = T_1 + T_2 + \cdots + T_{91}$ is the total completion time.
  - $$\begin{aligned} P\Big(S_{91} \geq 24) &= P(\frac{S_{91} = 91(.25)}{.1\sqrt{91}} \geq \frac{24 - 91(.25)}{.1\sqrt{91}}\Big) \\ &\approx 1 - \Phi(1.31) \\ &\approx .0951 \end{aligned}$$
  - Now, estimate the probability that at least 20 exams are graded in the first 4 hours. N_4 is the number of exams graded by time 4
  - Note: 20 might not be a large enough sample for the Central Limit Theorem to be accurate, but we don't hav eany other approximation methods right now.
  - $$P(N_4 \geq 20)...???$$

    but we can't recognize N_4 as the sum of i.i.d. random variables, so how would we apply the Central Limit?

∗ There's a duality principle between S_n and N_t:

$$(N_t \geq n) = (S_n \leq t)$$

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$$P(N_4 \geq 20) = P(S_{20} \leq 4) = P\Big(\frac{S_{20} - 20(.25)}{.1\sqrt{20}} \leq \frac{4 - 20(.25)}{.1\sqrt{20}}\Big)$$
$$= \Phi(-2.24)$$

Stirling's Approximation for n!

$$n! \approx \sqrt{2\pi n} \cdot (\frac{n}{e})^n$$

- This means that the two sides converge as $n \to \infty$ - Ex: 10! = 3,628,800. Stirling Approximation: 3,598,695.61874 - In formulas with factorials, this approximation makes it easier to understand large-number behavior - Let $X_1, X_2, \cdots \sim$ i.i.d. Poisson(1). Let $S_n = \sum_{i=1}^{n} X_i$. We saw before that S_n is a ??? (fill in later) - $P(S_n = n) = \frac{e^{-n} n^n}{n!}$ - On the other hand, -

$$P(S_n = n) = P(\frac{S_n - n}{\sqrt{n}} = \frac{n - n}{\sqrt{n}})$$
$$\approx 0$$

- Not a great approximation. Let's apply a *continuity correction*

- $$\begin{aligned} P(S_n = n) &= P(n - \frac{1}{2} \leq S_n \leq n + \frac{1}{2}) \\ &= P(-\frac{1}{2\sqrt{n}} \leq \frac{S_n - n}{\sqrt{n}} \leq \frac{1}{2\sqrt{n}}) \\ &\approx \int_{-\frac{1}{2\sqrt{n}}}^{\frac{1}{2\sqrt{n}}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\ &\approx \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{n}} \\ \rightarrow \frac{e^{-n}n^n}{n!} &\approx \frac{1}{\sqrt{2\pi n}} \end{aligned}$$

- Interesting application of the Stirling formula: partitions of integer n
  - We know the answer is $\binom{n-1+r}{r}$. Fix r, let $n \rightarrow \infty$. $= \frac{(n-1+r)!}{r!(n-1)!} \approx c(r)n^r$.

Markov Inequality

- If $X \geq 0$ random variable, for any constant $a \geq -$ the probability $P(X \geq a) \leq \frac{E(X)}{a}$
- Chebyshev inequality is a corollary
- Important concept, subtracting a positive quantity to produce a lower bound
- $$\begin{aligned} E(X) &= \int_0^\infty xf(x)dx \\ &= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx \\ E(X) &\geq \int_a^\infty xf(x)dx \geq \int_a^\infty af(x)dx = aP(X \geq a) \\ \rightarrow P(X \geq a) &\leq \frac{E(X)}{a} \end{aligned}$$

- Chebyshev Inequality: Now suppose X is any random variable with a finite mean μ and (not necessarily finite) variance σ^2

- $$P(|X-\mu| \geq k) = P((X-\mu)^2 \geq k^2) \leq \frac{E((X-\mu)^2)}{k^2} = \frac{\sigma^2}{k^2}$$
  $$\rightarrow P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Weak Law of Large Numbers