Intro Prob Lecture Notes

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Conditional Expectation

- Law of total expectation $E(X) = E(E(X|Y))$
  - Example: random variable N $p(N = n) = p_n$ for $n = 0, 1, 2, 3, \ldots$ $x_1, x_2, x_3, \ldots$ i.i.d. and independent of N
    - $\rightarrow E(S) = \mu_x \mu_N$
  - $Var(S) = E(S^2) - (E(S))^2 = E(S^2) - \mu_X^2 \mu_N^2$
    - $(\sum\limits_{i=1}^{N} x_i)(\sum\limits_{j=1}^{N} x_j) = \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} x_i x_j = \sum\limits_{i=1}^{N} x_i^2 \sum\limits_{i \neq j}^{N} x_i x_j$
  -

$$\begin{aligned}
E(S^2) &= E(E(S^2|N)) \\
&= E(\sum_{i=1}^{N} x_i^2 \sum_{i \neq j}^{N} x_i x_j | N = n) \\
&= E(\sum_{i=1}^{N} x_i^2 \sum_{i \neq j}^{N} x_i x_j) \\
&= nE(x_1^2) + n(n-1)E(x_1 x_2) \\
&= n(\sigma_X^2 + \mu_X^2) + n(n-1)E(x_1)E(x_2) \\
&= n\sigma_X^2 + n\mu_X^2 + n(n-1)\mu_x^2 \\
&= n\sigma_X^2 + n^2\mu_X^2
\end{aligned}$$

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\begin{aligned}
E(S^2) &= E(E(S^2|N)) \\
&= E(n\sigma_X^2 + n^2\mu_X^2) \\
&= \sum_{n=0}^{\infty}(n\sigma_X^2 + n^2\mu_X^2)P_N(n) \qquad = \sigma_X^2 \sum_{n=0}^{\infty} nP_N(n) + \mu_X^2 \sum_{n=0}^{\infty} n^2 P_N(n) \\
&= \sigma_X^2\mu_N + \sigma_X^2 E(N^2)
\end{aligned}
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Var(S) = \sigma_X^2\mu_N + \mu_X^2 E(N^2) - \mu_X^2\mu_N^2 = \sigma^2\mu_N + \mu_X^2\sigma_N^2
$$

Function with another variable

- Let X, Y r.v.'s and h is any function. Then

$$
E(Xh(Y)|Y) = h(Y)E(X|Y)
$$

  or

$$
E(Xh(Y)|Y = y) = h(y)E(X|Y = y)
$$

- Proof:

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\begin{aligned}
E(Xh(Y)|Y = y) &= \int_{-\infty}^{\infty} xh(y)f_{X|Y}(x, y)dx \\
&= h(y)\int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx \\
&= h(y)E(X|Y = y)
\end{aligned}
$$

- $$
\begin{aligned}
E(g(X)h(Y)) &= E(E(g(X)h(Y)Y)) \\
&= E(h(Y)E(g(X)|Y))
\end{aligned}
$$

- Ex: $X \sim \text{geometric}(p)$
  - Let Y be the outcome on the first toss. $f_Y(y) = \begin{cases} p & y = 1 \\ 1-p & y = 0 \end{cases}$
  - $E(X) = E(X|Y=0) + E(X|Y=1) = (1+E(X))(1-p) + 1(p) = 1 \cdot p + (1-p)(1+\mu) = \frac{1}{p}$
  - $$Var(X) = E(X^2) - (E(X))^2$$

$$
\begin{aligned}
E(X^2) &= E(E(X^2|Y)) \\
&= E(X^2Y = 1)p + E(X^2|Y = 0)(1-p) \\
&= 1 \cdot p + E((1+X)^2)(1-p) \\
&= p + E(1 + 2X + X^2)(1-p) \\
&= p + (1-p) + (1-p)\frac{2}{p} + E(X^2)(!-p) \\
pE(X^2) &= 1 + \frac{2(1-p)}{p} \\
E(X^2) &= \frac{1}{p} + \frac{2(1-p)}{p^2}
\end{aligned}
$$

    * So
$$Var(X) = \frac{1}{p} + \frac{2(1-p)}{p^2} - (\frac{1}{p})^2 = \frac{1-p}{p^2}$$

Conditional Variance

- $$Var(X|Y = y) = E([X - E(X|Y = y)]^2|Y = y)$$

- $$f(x,y) = xe^{-x(1+y)} \text{ for } x > 0, y > 0$$

- $$f_Y(y) = \frac{1}{(1+y)^2} for y > 0$$

- $$f_{X|Y}(x|y) = (1+y)^2 xe^{-x(1+y)} \text{ for } x > 0$$

- Gamma(2, $\frac{1}{1+y}$)
- $X|Y = y \sim$ Gamma(α, β) with $\alpha = 2, \beta = \frac{1}{1+y}$
  - Means $= \alpha\beta = \frac{2}{1+y}$
  - Variance $= \alpha\beta^2 = \frac{2}{(1+y)^2}$

- $$\begin{aligned}
Var(X|Y = y) &= E([X - E(X|Y = y)]^2 | Y = y) \\
&= E(\left[X - \frac{2}{1+y}\right]^2 | Y = y) \\
&= E(X^2 - \frac{4}{1+y} X + \frac{4}{(1+y)^2} | Y = y) \\
&= \int\limits_0^\infty x^2 f_{X|Y}(x,y) dx - \frac{4}{1+y} \int\limits_0^\infty x f_{X|Y}(x,y) dx + \frac{4}{(1+y)^2} \\
&= (1+y^2) \int\limits_0^\infty x^2 xe^{-x(1+y)} dx - \frac{4}{1+y} \frac{2}{1+y} + \frac{4}{(1+y)^2} \\
&= \frac{(1+y)^2 \cdot 3!}{(1+y)^4} - \frac{8}{(1+y)^2} + \frac{4}{(1+y)^2} \\
&= \frac{2}{(1+y)^2}
\end{aligned}$$

- Example: Z_1, Z_2 independent standard normals.
  - $X = \mu_X + \sigma_X Z_1$
  - $Y = \mu_Y + \sigma_Y \rho Z_1 + \sigma_{X(?)} \sqrt{1-\rho^2} Z_2$
  - $Z_1 = \frac{X - \mu_X}{\sigma_X}$

- $Y = \mu_Y + \sigma_Y \rho \left(\frac{X - \mu_X}{\sigma_X} \right) + \sigma_Y \sqrt{1 - \rho^2} Z_2$
- $Y|X \sim \text{Normal}(\mu_Y + \frac{\sigma_Y}{\sigma_X} \rho(x - \mu_X), \sigma_Y^2(1 - \rho^2))$
- Then:
- $E(Y|X) = \mu_Y + \frac{\sigma_Y \rho(X - \mu_X)}{\sigma_Y^2(1 - \rho^2)}$
- $Var(Y|X) = \sigma_Y^2(1 - \rho^2)$
- $E(Y|X) = \mu_Y - \frac{\sigma_Y \mu_X \rho}{\sigma_X} + \frac{\sigma_{Y(?)} \rho}{\sigma_X} X$
- $Var(E(Y|X)) = \frac{\sigma_Y^2 \rho^2}{\sigma_X^2} Var(X) = \sigma_Y^2 \rho^2$
  - $\implies \rho^2 = \frac{Var(E(Y|X))}{Var(Y)}$

Moment Generating Functions

- For a random variable X, $M_X(s) = E(e^{sX})$ (when this function is finite/exists in an open neighborhood of s = 0)
- $$\begin{aligned} e^y &= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots \\ e^{sX} &= 1 + sX + \frac{sX^2}{2!} + \frac{sX^3}{3!} + \dots \\ E(e^{sX}) &= 1 + E(X)s + E(X^2)\frac{s^2}{2!} + \dots \\ &= \sum_{n=0} \infty E(X^n) \frac{s^n}{n!} = M_X(s) \end{aligned}$$
- Suppose $X \sim$ uniform(0, 1). Compute $M_X(s)$

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\begin{aligned}
M_X(s) &= E(e^{sX}) \\
&= \int_0^1 e^{sX} 1 dx \\
&= \frac{e^{sX}}{s}\Big|_{x=0}^{1} \\
&= \frac{e^s - 1}{s}
\end{aligned}
$$

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\begin{aligned}
e^s &= 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \dots \\
e^s - 1 &= s + \frac{s^2}{2!} + \frac{s^3}{3!} + \dots \\
\frac{e^s - 1}{s} &= 1 + \frac{s}{2!} + \frac{s^2}{3!} + \dots + \frac{s^{n-1}}{n!} + \frac{s^{n(?)}}{(n+1)!} \\
&= 1 + (\frac{1}{2})s + (\frac{1}{3})\frac{s^2}{2} + \dots + (\frac{1}{n})\frac{s^{n-1}}{(n-1)!} + (\frac{1}{n+1})\frac{s^n}{n!} + \dots \\
E(X^n) &= \frac{1}{n+1}
\end{aligned}
$$

- Application of mgf:
  - $M'_X(s) = \frac{d}{ds}(M_X(s)) = \frac{d}{ds}E(e^{sX}) = E(\frac{d}{ds}(e^{sX})) = E(Xe^{sX})$
  - $\mathbf{M'_X(0) = E(X)}$
  - $\frac{d^n}{ds^n} M'_X(s) = E(X^n e^{sX})$
  - $\mathbf{\frac{d^n}{ds^n}(M_X(0)) = E(X^n)}$