550.420 Intro to Probability - Spring 2017 **Homework #05**
Will not be collected.

A.5.1. (a) Roll a balanced 6-sided die 3 times. Compute the probability that outcomes are in increasing order; i.e., (x, y, z) where $x < y < z$.
(b) $n > 3$ chips labeled 1 through n are in a box. Three (3) chips are drawn without replacement. Compute the probability that the chips you draw are in increasing order.
(c) $n > k$ chips labeled 1 through n are in a box. In experiment 1, k chips are drawn without replacement. Compute the probability that the chips are in increasing order. In experiment 2 the k chips are drawn *with* replacement. Compute the probability the chips are in increasing order. Under which experiment is it more likely to observe the chips drawn in increasing order?

A.5.2. Suppose we have 4 people from each of 4 different groups. Call them

$$\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4\},$$

where the letters a, b, c, d designate the group and the number 1,2,3,4 designates the person within the group. These 16 people are to be randomly divided into 4 groups of 4 each for the purpose of solving a problem. Let's agree to call a subset of 4 people a *team* so that the experiment is to divide the 16 people randomly into 4 teams.
(a) In how many ways can these 16 people be divided into 4 teams?
(b) What is the probability that all the people from group a form a team?
(c) What is the probability that at least one team is comprised of all members from the same group? Intuitively, should this probability be greater than the answer to part (b)?
(d) What is the probability that all the teams are comprised of exactly one member from each group?

A.5.3. Consider n distinct objects comprised of r groups, say, n_i are from group i $(i = 1, 2, \ldots, r)$ and $n_1 + n_2 + \cdots + n_r = n$.
(a) If we draw k of these *without* replacement after each draw, what is the probability that we observe k_i from group i $(i = 1, 2, \ldots, r)$ where $k_1 + k_2 + \cdots + k_r = k$?
(b) We deal a person 10 cards from a standard deck of 52 cards. What is the probability we observe one club, two diamonds, three hearts, and four spades?

A.5.4. We play the following game: We have 10 one dollar bills. There are 4 slots marked 1, 2, 3, and 4. We are to divvy up all 10 dollars amongst the 4 slots in *any* manner. (No ripping of the bills are allowed!).
(a) In how many ways can the 10 one dollars bills be divvied up amongst the 4 slots?
(b) In how many ways can the 10 one dollars bills be divvied up amongst the 4 slots if we are required to put at least one dollar in each slot?
(c) In how many ways can the 10 one dollars bills be divvied up amongst the 4 slots if we are required to put at least two dollars in each slot?
(d) (continued from part (a)) Suppose that someone has divvied up their 10 dollars randomly amongst the 4 slots in such a way that all possibilities from part (a) are equally likely. What is the probability that some slot(s) receives no dollars?
(e) (continued from part (d)) Given that each slot has at least one dollar, compute the probability that slot 1 has exactly one dollar?

A.5.5. Let $n > 1$ be an integer and consider the disc $D = \{(x, y) : x^2 + y^2 \leq n^2\}$ of radius n centered at the origin. We define the events $R_1 = \{(x, y) : x^2 + y^2 \leq 1\}$, $R_2 = \{(x, y) : 1 < x^2 + y^2 \leq 2^2\}$, $R_3 = \{(x, y) : 2^2 < x^2 + y^2 \leq 3^2\}$, and so forth, up to $R_n = \{(x, y) : (n-1)^2 < x^2 + y^2 \leq n\}$.

The area of the disc D is πn^2. So, if we toss a dart at random to the disc in such a way that every point in D has the same chance as any other to be hit, then it seems natural that the probability that the dart lands in region R_1 should be $\frac{\pi 1^2}{\pi n^2} = \frac{1}{n^2}$, and in region R_2 should be $\frac{\pi 2^2 - \pi 1^2}{\pi n^2} = \frac{3}{n^2}$, and generally the probability that the dart lands in region R_i should be $\frac{\pi i^2 - \pi (i-1)^2}{\pi n^2} = \frac{2i-1}{n^2}$ for $i = 1, 2, \ldots, n$.

Let's now define a random variable X to be the region the dart lands, that is, if for some $i = 1, 2, 3, \ldots, n$ the dart lands in region R_i, then $X = i$. The above shows the pmf of X is

$$P(X = i) = \frac{2i - 1}{n^2} \quad \text{for } i = 1, 2, \ldots, n.$$

(a) Show that the pmf described above is indeed a pmf; i.e. show $P(X = i) \geq 0$ for each $i = 1, 2, \ldots n$ and $\sum_x P(X = x) = 1$.
(b) If $n = 10$, compute $P(X \leq 5)$, $P(X \geq 5)$, $P(X = 5)$.
(c) In general show that $P(X \leq i) = \frac{i^2}{n^2}$ for $i = 1, 2, \ldots, n$.
(d)* Suppose $n > 1$ is even. Compute $P(X \text{ is odd})$. *a little challenging...may require induction.

A.5.6. Suppose an experiment is to toss a balanced 6-sided die twice. We define the random variable Y to be the maximum up-face value.
(a) Find the pmf of Y.
(b) Compute $P(Y \leq y)$ for $y = 1, 2, 3, 4, 5, 6$.
(c) Compute $P((Y - 3)^2 - Y \geq 1)$.