Intro Prob Lecture Notes

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Applications of the Transformation Theorem

Bivariate Normal

- Recall: If $Z \sim$ Normal(0, 1) and $X := \mu + \sigma Z$ where $\mu \in \mathbb{R}$, $\sigma > 0$ then $X \sim \text{Normal}(\mu, \sigma^2)$
  - Also, as said before, $X \sim \text{Normal}(\mu, \sigma^2) \to Z = \frac{X-\mu}{\sigma}$
  - "Normality is closed"
- Bivariate Normal X, Y with
  - $\mu_X, \mu_Y \in \mathbb{R}$ means
  - σ_X, σ_Y 0 standard deviations
  - $-1 < \rho < 1$ correlation coefficient
  - Let Z_1, Z_2 be independent standard normal

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{exp(-\frac{z_1^2+z_2^2}{2})}{2\pi}$$

    * See handouts from April 14 for a picture of this distribution
- *Define:*
  -
$$X = \mu_x + \sigma_X Z_1$$
  -
$$Y = \mu_Y + \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2} Z_2$$
- Inverse transformation:

-
$$Z_1 = \frac{X - \mu_X}{\sigma_X}$$

-
$$Z_2 = \frac{Y - \mu_Y - \sigma_Y \rho(\frac{X-\mu_X}{\sigma_X})}{\sigma_Y \sqrt{1-\rho^2}}$$

•
$$J = \begin{bmatrix} \frac{1}{\sigma_X} & 0 \\ * & \frac{1}{\sigma_Y \sqrt{1-\rho^2}} \end{bmatrix} = \frac{1}{\sigma_X \sigma_Y \sqrt{1-\rho^2}} > 0$$

• From the transformation theorem,

•
$$f_{X,Y}(x,y) = f_{Z_1,Z_2}(\frac{x-\mu_X}{\sigma_X}, \frac{y - \mu_Y - \sigma_Y \rho(\frac{x-\mu_X}{\sigma_X})}{\sigma_Y \sqrt{1-\rho^2}}) \cdot \frac{1}{\sigma_X \sigma_Y \sqrt{1-\rho^2}}$$

- Exercise: Carry through the algebra, and end up with

-
$$f_{X,Y}(x,y) = \frac{exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_X})^2 - 2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) + (\frac{y-\mu_Y}{\sigma_Y})^2]\}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

- Statisticians don't want to do math with this! Better to go back to the above *definitions*

• Suppose we want to find $f_{Y|X}(y|x) = \frac{fX,Y(x,y)}{f_X(x)}$. $\leftarrow$ But how can we identify the numerator for $Y|X = x$? We saw that its full form is very messy

• Here's an easy way to identify $Y|X = x$:
  - $Y = \{\mu_Y + \sigma_Y \rho(\frac{x-\mu_X}{\sigma_X}\} + \sigma_Y \sqrt{1-\rho^2} Z_2)$
  - The term within curly braces is a constant. So, this is a normal distribution with the constant as a mean, and the term after it as a standard deviation.
  - So, we see $Y|X = x \sim \text{Normal}(\mu_Y + \sigma_Y \rho(\frac{x-\mu_X}{\sigma_X}, \sigma_Y^2(1-\rho^2))$

Fisher's F-distribution

• $X \sim \chi^2_n, Y \sim \chi^2_m$ and they are independent

• Then define
$$U = \frac{X/n}{Y/m}, V = Y$$

• Plan: Find the joint distribution, then the marginal of U. We claim that U has Fisher's F-distribution.

- $u = \frac{m}{n}\frac{x}{y}, v = y \rightarrow \mathbf{y = v}$
- $\frac{\mathbf{nuv}}{\mathbf{m}} = \mathbf{x}$
- $$J = det\Big(\begin{bmatrix} \frac{nv}{m} & \frac{nu}{m} \\ 0 & 1 \end{bmatrix}\Big) = \frac{nv}{m} > 0$$
- So $|J| = \frac{nv}{m}$
- $$f_{U,V}(u, v) = f_{X,Y}(\frac{nuv}{m}, v)|J|$$
- $$f_{X,Y}(x, y) = \frac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \cdot \frac{y^{\frac{m}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}$$
- So,

$$\begin{aligned} f_{U,V}(u, v) &= \frac{\frac{nuv}{m}^{\frac{n}{2}-1} \cdot e^{-\frac{nuv}{2m}} \cdot v\frac{m}{2} - 1 \cdot e^{-\frac{v}{2}} \cdot \frac{n}{m} \cdot v}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \\ &= \frac{(\frac{n}{m})^{\frac{n}{2}} v^{\frac{n+m}{2}-1} e^{-v(\frac{nu}{2m}+\frac{1}{2})} u^{\frac{n}{2}-1}}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \end{aligned}$$

- For $u > 0$, (show as an exercise)

$$f_U(u) = \frac{n\Gamma(\frac{n+m}{2})}{m\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \cdot \frac{(\frac{nu}{m})^{\frac{n}{2}-1}}{(1 + \frac{nu}{m})^{\frac{n+m}{2}}}$$

  - The F-distribution with n Numerator d.f. (degrees of freedom), m Denominator d.f.
  - Remember: If $W \sim$ Cauchy, $W^2 = F_{1, 1}
- If $Z \sim$ Normal(0, 1) and $X \sim \chi^2_m$ are independent,
  - $$T = \frac{Z}{\sqrt{\chi/m}} \sim \text{ Student's t-distribution with m d.f.}$$