Suppose that X, Y are jointly continuous with joint pdf $f_{X,Y}(x,y)$ for $(x,y) \in A \subseteq \mathbb{R}^2$.

Now suppose we have two other r.v.s defined in terms of X and Y, say

$$U = g_1(X, Y)$$

and

$$V = g_2(X, Y).$$

What is the joint pdf of U, V?

We will now present a technique (called the Jacobian method or the tranformation of variables method)

In what follows we will assume the transformation of $(X, Y) \to (U, V)$ is invertible (ie., one-to-one)

the result can be modified to handle the case where the transformation is not one-to-one but we deal with that later—

and we let

$$X = h_1(U, V) \quad \text{and}$$
$$Y = h_2(U, V)$$

represent the inverse transformation.

When $X = h_1(U,V)$ and $Y = h_2(U,V)$ ~~are~~ is the inverse transformation we have the following result:

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J|\ , \text{ where}$$

$$x = h_1(u,v)$$
$$y = h_2(u,v)$$

and

$|J|$ = is the absolute value of the Jacobian determinant

$$= \left| \det \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix} \right|$$

<u>Example</u> Suppose X, Y are independent $\exp(\lambda)$ r.v.s.

so that $f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$ for $(x,y) \in [0,\infty) \times [0,\infty)$

and Let $U = \dfrac{X}{Y}$ and $V = Y$.

Find the joint pdf of U, V.

To use the Jacobian method we 1st need to find the inverse transformation, that is, we must be able to solve for each of X and Y as a function of U, V. In this case this is not to hard...

$$\left.\begin{array}{l} u = \frac{x}{y} \Rightarrow x = u \cdot y \\ v = y \Rightarrow y = v \end{array}\right\} \Rightarrow \begin{array}{l} x = uv \\ y = v \end{array}$$

Next we compute the Jacobian determinant:

$$\left.\begin{array}{ll} \frac{\partial x}{\partial u} = v & \frac{\partial x}{\partial v} = u \\ \frac{\partial y}{\partial u} = 0 & \frac{\partial y}{\partial v} = 1 \end{array}\right\} \Rightarrow J = \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v$$

and note $|J| = v$ also since $v = y > 0$ for y in its essential domain.

Now we plug in:

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\cdot v$$

$$= f_{X,Y}(uv, v) = v$$

$$= \lambda^2 e^{-\lambda(uv+v)} \cdot v$$

$$= \lambda^2 v e^{-\lambda v(1+u)} \quad \text{for } u>0 \text{ and } v>0.$$

Interesting question: What is the marginal pdf of U?

$$f_U(u) = \int_0^\infty \lambda^2 v e^{-\lambda v(1+u)} dv$$

$$= \lambda^2 \int_0^\infty v e^{-\lambda v(1+u)} dv$$

Using the renormalization technique... this is almost the form of a Gamma pdf with
$\alpha = 2$
$\beta = [\lambda(1+u)]^{-1}$

$$= \lambda^2 \cdot \left([\lambda(1+u)]^{-1}\right)^2 \Gamma(2)$$

$$= \frac{1}{(1+u)^2} \quad \text{for } u>0.$$

$$= 0 \quad \text{for } u \le 0.$$

sometimes called a Pareto distribution.

<u>Example</u> Suppose X and Y are independent

Gammas, say

$$X \sim \text{Gamma}(\alpha, 1)$$
$$Y \sim \text{Gamma}(\beta, 1)$$

(having the same scale parameter)

So that

$$f_{X,Y}(x,y) = \frac{x^{\alpha-1} e^{-x} y^{\beta-1} e^{-y}}{\Gamma(\alpha)\Gamma(\beta)} \quad \text{for } x>0,\ y>0.$$

Let's find the pdf of

$$U = \frac{X}{X+Y}.$$

Notice that as $x>0$ any $y>0$ u is always between 0 and 1

In order to use the Jacobian method we need to introduce a second "dummy" variable to make the transformation one-to-one .. we have many choices to do this.

I propose introducing the variable

$$V = X+Y. \qquad \text{here } v>0.$$

$$\left.\begin{aligned} u &= \frac{x}{x+y} \\ v &= x+y \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} x &= uv \\ y &= v - x = v - uv \\ &= v(1-u). \end{aligned}$$

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= v \qquad \frac{\partial x}{\partial v} = u \\ \frac{\partial y}{\partial u} &= -v \qquad \frac{\partial y}{\partial v} = 1-u \end{aligned}\right\} \Rightarrow J = \det\begin{pmatrix} v & u \\ -v & 1-u \end{pmatrix}$$

$$= v(1-u) + uv = v.$$

So.

$$f_{U,V}(u,v) = f_{X,Y}(uv, v(1-u)) \cdot |v|$$

$$= \frac{(uv)^{\alpha-1} e^{-uv} [v(1-u)]^{\beta-1} e^{-v(1-u)}}{\Gamma(\alpha)\Gamma(\beta)} \cdot v$$

$$= \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot v^{\alpha+\beta-1} e^{-v} \quad \text{for } 0<u<1, v>0.$$

Now we can find the marginal pdf of U by integrating out v from 0 to ∞:

$$f_U(u) = \int_0^\infty \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot v^{\alpha+\beta-1} e^{-v}\, dv$$

Gamma with "$\alpha = \alpha+\beta$" and "$\beta = 1$"

$$= \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^\infty v^{\alpha+\beta-1} e^{-v}\, dv}_{\Gamma(\alpha+\beta)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot u^{\alpha-1}(1-u)^{\beta-1} \quad \text{for } 0 < u < 1$$

$$= 0 \quad \text{for other } u.$$

This pdf is called the Beta(α, β) pdf.

Very important pdf in Statistics (especially Bayesian Statistics). Notice that because this is a pdf we have for any $\alpha > 0$, $\beta > 0$

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

So, for instance, if $\alpha = 1/2$ and $\beta = 1/2$.

$$\int_0^1 u^{-1/2}(1-u)^{-1/2}\,du = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \frac{\sqrt{\pi}\,\sqrt{\pi}}{1} = \pi.$$

i.e.,

$$\int_0^1 \frac{1}{\sqrt{u(1-u)}}\,du = \pi$$

Suppose Z_1 and Z_2 are independent standard normals, so that

$$f_{Z_1, Z_2}(z_1, z_2) = \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

Let $U = \dfrac{Z_1 + Z_2}{\sqrt{2}}$ and $V = \dfrac{Z_1 - Z_2}{\sqrt{2}}$.

Then

$$\begin{aligned} z_1 + z_2 &= \sqrt{2}\, u \\ z_1 - z_2 &= \sqrt{2}\, v \end{aligned}$$

$$2 z_1 = \sqrt{2}(u+v) \implies z_1 = \frac{u+v}{\sqrt{2}}$$

$$2 z_2 = \sqrt{2}(u-v) \implies z_2 = \frac{u-v}{\sqrt{2}}.$$

$$J = \det\begin{bmatrix} \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial v} \\ \frac{\partial z_2}{\partial u} & \frac{\partial z_2}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = -1$$

$|J| = 1.$ (called a Unit transformation)

Now

$$f_{U,V}(u,v) = f_{Z_1, Z_2}\left(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}\right) \cdot 1 = \ldots$$

$$= \frac{e^{-\frac{\left(\frac{u+v}{\sqrt{2}}\right)^2 + \left(\frac{u-v}{\sqrt{2}}\right)^2}{2}}}{2\pi}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{u^2 + \cancel{2uv} + v^2 + \{u^2 - \cancel{2uv} + v^2\}}{2}\right)}}{2\pi}$$

$$= \frac{e^{-\frac{1}{2}(u^2+v^2)}}{2\pi} \quad \text{for } (u,v) \in \mathbb{R}^2$$

That is, (U, V) are again independent standard normals!