Intro Prob Lecture Notes

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Independence of Random Variables

- If $X_1, X_2, \dots X_n$ are a collection of jointly distributed random variables, then we say that they are (mutually) independent if their joint distribution factors as the product of the individual marginal distributions
  - For example, if $X_1, X_2, \dots X_n$ are jointly discrete then they will be independent if
    - $$P_{X_1, X_2, \dots X_n}(x_1, x_2, \dots x_n) = P_{X_1}(x_1)P_{X_2}(x_2)\dots P_{X_n}(x_n)$$
    - for *all* possible values of $x_1, x_2, \dots x_n$
  - Also, if $X_1, X_2, \dots X_n$ are jointly continuous then they will be independent if
    - $$f(x_1, x_2, \dots x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\dots f_{X_n}(x_n)$$
    - for all $x_1, x_2, \dots x_n$
- Example: Dart board
  - $f(x_1, x_2) = \frac{1}{\pi}$ if $x_1^2 + x_2^2 =\leq 1$, 0 otherwise.
  - Compute the marginals
    - If $x < -1$ or $x > 1$, marginal is 0. Otherwise:
    - $$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) d_{x_2} = \int_{-\sqrt{1-x_1^2}}^{+\sqrt{1-x_1^2}} \frac{1}{\pi} d_{x_2} = \frac{2\sqrt{1-x_1^2}}{\pi}$$

    - Similarly, when in range,

$$f_{X_2}(x_2) = \frac{2\sqrt{1-x_2^2}}{\pi}$$

  - Multiply the marginals
    - $f(\frac{1}{2},\frac{1}{2}) = \frac{1}{\pi}$
    - $$f_{X_1}(\frac{1}{2})f_{X_2}(\frac{1}{2}) = \frac{2\sqrt{1-\frac{1}{4}}}{\pi} \cdot \frac{2\sqrt{1-\frac{1}{4}}}{\pi} = \frac{3}{\pi^2}$$
    - $\frac{1}{\pi} \neq \frac{3}{\pi^2}$, so X_1, X_2 are *dependent.*

- Usually independence is an assumption that is made
  - $X_1, \ldots X_n$ are independent $\mathrm{N}(\mu, \sigma^2)$
  - $$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}$$
  - $$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n-1}$$
  - are, in fact, independent (to be proved later)

Sums of random variables

- If X_1, X_2 are jointly distributed random variables, then what is the distribution (pmf) of $X_1 + X_2$?
- Case 1: Suppose X_1, X_2 jointly discrete, $P_{X_1,X_2}(x_1,x_2) = P(X_1 = x_1, X_2 = x_2)$ given
  - $P_{X_1,X_2}(u) = P(X_1 + X_2 = u) = \sum_{x_1} P(X_1 + X_2, X_1 = x_1) = \sum_{x_1} P(X_1 = x_1, X_2 = u - x) = \sum_{x_1} P_{X_1,X_2}(x_1, u - x_1)$
    - (Law of total probability)
  - Formula:

$$P_{X_1+X_2}(u) = \sum_{x_1} P_{X_1,X_2}(x_1, u - x_1)$$

  - A common assumption is that X_1, X_2 independent. In this case:
    - $$P_{X_1+X_2}(u) = \sum_{x_1} P_{X_1}(x_1) P_{X_2}(u - x_1)$$

* Convolution $(P_{X_1} * P_{X_2})(u)$

- Binomial theorem $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

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Example: $X_1 \sim$ Poisson(λ_1), $X_2 \sim$ Poisson(λ_2), and they are independent

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\begin{aligned}
P_{X_1+X_2}(u) &= \sum_{x_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}\lambda_2^{u-x_1}}{(u-x_1)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{u!} \sum_{x_1=0}^{u} \frac{u!}{x!(u-x_1)!} \cdot \lambda_1^{x_1}\lambda_2^{u-x_1} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^u}{u!}
\end{aligned}
$$

* The pdf of a Poisson$(\lambda_1 + \lambda_2)$
* Note: For formula to be valid, upper limit must be ∞

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Suppose $X_1 + X_2 = n$. $P(X_1 = k | X_1 + X_2 = n) =?$

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\begin{aligned}
P(X_1 = k|X_1+X_2=n) &= \frac{P(X_1=k, X_2=n-k)}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!}\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^k \Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big)^{n-k} \sim \text{ Binomial}\Big(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\Big)
\end{aligned}
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