550.420 Probability - FALL 2015 **MIDTERM 2 solutions**

1. X is continuous with pdf $f(x) = \frac{2}{x^3}$ for $x > 1$; $= 0$ otherwise.
(a) Compute the cdf $F(x)$ of X. *Be sure to show domains of definition.*
(b) The *median* of the continuous distribution is the value m such that $P(X \leq m) = \frac{1}{2}$. Find the median of this distribution.
(c) Compute $P(2 < X \leq 3)$.
(d) Compute $E(X)$.

(a) $F(x) = \int_{-\infty}^{x} f(u)\,du$. If $u \leq 1$, then $f(u) = 0$ and $F(x) = \int_{-\infty}^{x} 0\,du = 0$ for any $x \leq 1$. On the other hand, if $x > 1$, then $F(x) = \int_{-\infty}^{1} 0\,du + \int_{1}^{x} 2u^{-3}\,du = -u^{-2}\Big|_{u=1}^{u=x} = 1 - \frac{1}{x^2}$. Thus, $F(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{if } x > 1 \\ 0 & \text{elsewhere} \end{cases}$.
(b) Solve for m in $F(m) = \frac{1}{2}$: $1 - \frac{1}{m^2} = \frac{1}{2} \implies \frac{1}{m^2} = \frac{1}{2} \implies m = \pm\sqrt{2} \implies m = \sqrt{2}$ since m must be greater than 1.
(c) $P(2 < X \leq 3) = F(3) - F(2) = (1 - \frac{1}{9}) - (1 - \frac{1}{4}) = \frac{8}{9} - \frac{3}{4} = \frac{32-27}{36} = \frac{5}{36}$.
(d) $E(X) = \int_{-\infty}^{\infty} uf(u)\,du = \int_{1}^{\infty} u(2u^{-3})\,du = \int_{1}^{\infty} 2u^{-2}\,du = -2u^{-1}\Big|_{u=1}^{\infty} = 2$.

2. Let X and Y be *independent* with respective means μ_X, μ_Y and respective variances σ_X^2, σ_Y^2. Show that $var(XY) = \sigma_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2$.

$$\begin{aligned} var(XY) &= E((XY)^2) - [E(XY)]^2 \\ &= E(X^2Y^2) - [E(X)E(Y)]^2 \quad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= E(X^2)E(Y^2) - [E(X)]^2[E(Y)]^2 \quad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= [var(X) + E(X)^2][var(Y) + E(Y)^2] - [E(X)]^2[E(Y)]^2 \\ &= [\sigma_X^2 + \mu_X^2][\sigma_Y^2 + \mu_Y^2] - [\mu_X]^2[\mu_Y]^2 \\ &= \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2 + \mu_X^2\mu_Y^2 - \mu_X^2\mu_Y^2 \\ &= \sigma_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2. \end{aligned}$$

3. Recall the Gamma(α, β) distribution has pdf $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}$ for $x > 0$ and has moment generating function $M(s) = (1 - \beta s)^{-\alpha}$. Also, for positive integers n, when $\alpha = \frac{n}{2}$ and $\beta = 2$, the Gamma distribution is sometimes called the *chi-square distribution with n degrees of freedom.*
(a) If X has a chi-square distribution with 1 degree of freedom, show that $E(X) = 1$ and $var(X) = 2$ in any way you wish.
(b) If $X_1, X_2, \ldots, X_n$ are independent each distributed as a chi-square with 1 degree of freedom, identify the distribution of the sum $S = X_1 + X_2 + \cdots + X_n$.
(c) Find the mean and variance of S.

(a) For the chi-square with 1 degree of freedom, the mgf is $M(s) = (1 - 2s)^{-1/2}$. Therefore, $M'(s) = -\frac{1}{2}(1-2s)^{-3/2}\cdot(-2) = (1-2s)^{-3/2}$ and $M''(s) = -\frac{3}{2}(1-2s)^{-5/2}\cdot(-2) = 3(1-2s)^{-5/2}$. Consequently, $E(X) = M'(0) = 1$ and $E(X^2) = M''(0) = 3$ which implies $var(X) = 3 - 1^2 = 2$.
(b) The mgf of S is then $[M(s)]^n = [(1-2s)^{-1/2}]^n = (1-2s)^{-n/2}$,which is the mgf of a chi-square with n degrees of freedom. Therefore, S must be distributed as a chi-square with n degrees of freedom.
(c) $E(S) = \sum_{i=1}^{n} E(X_i) = n$ and since the X_i's are independent, $var(S) = var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} var(X_i) = 2n$.

4. The joint pdf of X and Y is $f(x,y) = xe^{-x(1+y)}$ for $x > 0$ and $y > 0$.
(a) Find the marginal pdfs of X and Y. Clearly label each.
(b) Compute $P(Y > 1 | X = \frac{1}{2})$.

(a) if $x > 0$, $f_X(x) = \int_0^\infty xe^{-x(1+y)}\,dy = xe^{-x}\int_0^\infty e^{-xy}\,dy = xe^{-x}\left\{-\frac{e^{-xy}}{x}\Big|_{y=0}^{\infty}\right\} = e^{-x}$; $f_X(x) = 0$ for $x \le 0$.

If $y > 0$, $f_Y(y) = \int_0^\infty xe^{-x(1+y)}\,dx = \int_0^\infty x^{2-1}e^{-\frac{x}{(1+y)^{-1}}}\,dx = [(1+y)^{-1}]^2\Gamma(2) = \frac{1}{(1+y)^2}$; $f_Y(y) = 0$ for $y \le 0$.

(b) Now, $f_{Y|X}(y|x) = \frac{xe^{-x(1+y)}}{e^{-x}} = xe^{-xy}$, and when $x = \frac{1}{2}$, $f_{Y|X}(y|\frac{1}{2}) = \frac{1}{2}e^{-\frac{y}{2}}$ for $y > 0$. Therefore,
$P(Y > 1|X = \frac{1}{2}) = \int_1^\infty f_{Y|X}(y|\frac{1}{2})\,dy = \int_1^\infty \frac{1}{2}e^{-\frac{y}{2}}\,dy = -e^{-\frac{y}{2}}\Big|_{y=1}^{\infty} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

5. There are two bank tellers waiting on customers. The time T_i (in minutes) it takes teller $\#i$ to service a customer is an exponential random variable with parameter $\lambda = i$ (for $i = 1, 2$). Assume the tellers operate independently so that T_1 and T_2 are independent. Two customers enter the bank simultaneously and are immediately serviced by tellers 1 and 2. Compute the probability that both customers are still being serviced after 1 minute. Hint: in the context of this problem what does the event $(\min\{T_1, T_2\} > t)$ mean?

Since $(\min\{T_1,T_2\} > t) = (T_1 > t, T_2 > t)$ this is the event that it takes more than t minutes to service each customer. $P(\min\{T_1,T_2\} > 1) = P(T_1 > 1, T_2 > 1) = P(T_1 > 1)P(T_2 > 1) = \int_1^\infty e^{-x}\,dx \int_1^\infty 2e^{-2y}\,dy = e^{-1}e^{-2} = e^{-3}$.

6. A random rectangle is constructed as follows: the length X and the width Y are independent uniform$(0,1)$ random variables, i.e., each have pdf $f(x) = 1$ for $0 < x < 1$. Find the pdf of the area $A = XY$ of this rectangle.

Using the cdf method: First of all, $0 < a < 1$ so $F_A(a) = P(XY \le a) = 1 - P(XY > a) = 1 - \int_a^1\int_{a/y}^1 dx\,dy = 1 - \int_a^1 (1 - \frac{a}{y})\,dy = 1 - \left\{y - a\ln(y)|_{y=a}^{y=1}\right\} = 1 - \{(1-0) - (a - a\ln(a))\} = a - a\ln(a)$.
Therefore, $f_A(a) = \frac{d}{da}F_A(a) = -\ln(a)$ for $0 < a < 1$.

Using the transformation (Jacobian) method: the transformation $a = xy$, $b = y$ has inverse transformation $x = \frac{a}{b}$, $y = b$ with $|J| = \left|\det\begin{bmatrix}\frac{1}{b} & -\frac{a}{b^2}\\ 0 & 1\end{bmatrix}\right| = \frac{1}{b}$ since $b > 0$.

Now since $0 < x = \frac{a}{b} < 1$ we must have $0 < a < b < 1$.
So, $f_{A,B}(a,b) = f_{X,Y}(x,y)|J| = f_{X,Y}(\frac{a}{b}, b)\cdot\frac{1}{b} = \frac{1}{b}$ for $0 < a < b < 1$. Finally, $f_A(a) = \int_a^1 \frac{1}{b}\,db = -\ln(a)$ for $0 < a < 1$.

7. $X \sim$ normal(μ, σ^2). Recall that the moment generating function of X is $M(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$. Let $Y = e^X$ so that Y has the so-called *log-normal* distribution. Compute the mean and variance of Y.

$E(Y) = E(e^X) = M(1) = e^{\mu + \frac{\sigma^2}{2}}$.
$E(Y^2) = E((e^X)^2) = E(e^{2X}) = M(2) = e^{2\mu + 2\sigma^2}$.
$var(Y) = e^{2\mu+2\sigma^2} - \{e^{\mu+\frac{\sigma^2}{2}}\}^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$