Some important discrete r.v.s and their pmfs

1. A random variable X that takes only two values 1 or 0 with respective probabilities p and $1-p$:

x	0	1
$p_X(x)$	$1-p$	p

in tabular form

In functional form $p_X(x) = p^x (1-p)^{1-x}$ for $x=0,1$

This is a Bernoulli(p) r.v.; here $p \in (0,1)$.

A common way a Bernoulli r.v. arises is the following:

An experiment has sample space Ω. Let $A \subset \Omega$ be an event and suppose $P(A)=p$.

Then the function

(*) $$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

If $I_A = 1$ then the event A occurred
if $I_A = 0$ then the event A didn't occur

is a Bernoulli(p) r.v.

The function (*) is called the Indicator function of the set A

2. the binomial(n,p)

If an experiment consist of n INDEPENDENT Bernoulli(p) trials and. X counts the number of successes, then

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

is the binomial(n,p) pmf.

We will see later that if

meaning of independent rvs will come later → $X_1, X_2, \ldots, X_n$ are independent Bernoulli(p) rvs

then $X := X_1 + X_2 + \cdots + X_n$ has a binomial(n,p).

distribution.

If X has the binomial(n,p) pmf we will write

$$X \sim \text{binomial}(n,p).$$

Do the birthday example

3. the geometric(p)

An experiment consists of an infinite sequence of Bernoulli(p) trials. Let X be the random variable that gives the trial of the first success.

Then

$$p_X(x) = P(X=x) = (1-p)^{x-1} p \quad \text{for } x = 1, 2, 3, \ldots$$

(That is, the 1^{st} head occurs on trial x must be preceded by $x-1$ tails.)

I.e. the event $(X=x)$ corresponds to the sequences that start

$$(*) \qquad T_1 \cap T_2 \cap \cdots \cap T_{x-1} \cap H_x$$

where T_i is a tail on i^{th} toss
H_i is a head on i^{th} toss.

and since the events in $(*)$ are Independent it follows

$$P(X=x) = P(T_1 \cap T_2 \cap \cdots \cap T_{x-1} \cap H_x)$$

$$= P(T_1)\,P(T_2) \cdots P(T_{x-1})\,P(H_x)$$

$$= \underbrace{(1-p)(1-p)\cdots(1-p)}_{x-1 \text{ terms}} \cdot p = (1-p)^{x-1} p.$$

If X has a geometric(p) pmf we will write

$$X \sim \text{geometric}(p).$$

Let's show that this is indeed a pmf...

Since $(1-p) > 0$ and $p > 0$ we have

$$p_X(x) = (1-p)^x p > 0 \quad \text{for } x = 1, 2, 3, \ldots$$

Furthermore,

$$\sum_{x=1}^{\infty} p_X(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1.$$

Here we used the formula for the sum of a geometric series.

Digression on Geometric series.

if $|r| < 1$, then a series of the form $\sum_{n=n_0}^{\infty} ar^n$ is called a geometric series. (r is called the geometric ratio)

$$S = \sum_{n=n_0}^{\infty} ar^n = ar^{n_0} + ar^{n_0+1} + ar^{n_0+2} + ar^{n_0+3} + \cdots$$

$$rS = ar^{n_0+1} + ar^{n_0+2} + ar^{n_0+3} + \cdots$$

$$(1-r)S = ar^{n_0}$$

$$\Rightarrow \quad S = \frac{ar^{n_0}}{1-r} = \frac{1^{st} \text{ term in the series}}{1 - (\text{geometric ratio})}$$

4. The Poisson(λ)

the discrete r.v. X having the following pmf

$$(*) \qquad p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

is said to have the Poisson(λ) distribution

Here $\lambda > 0$ is a real number.

This is a ~~pdf~~ pmf since $p_X(x) > 0$ for $x = 0, 1, 2, \ldots$

and

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}(e^{\lambda}) = 1.$$

where we used the fact that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

is the McLaurin series expansion of e^y.

Where did (*) come from?

Answer: the Poisson(λ) distribution is a distribution that first arose as an approximation to the Binomial(n, p) distribution when n is large and p is small ...

Suppose $Y \sim$ binomial(n, p) where $p = \frac{\lambda}{n}$ and n is large. λ = rate $= \frac{\#\text{ of events}}{\text{time}}$.

Then for $y = 0, 1, 2, 3, \ldots$ fixed

$$P(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \frac{n!}{y!\,(n-y)!} \cdot \left(\frac{\lambda}{n}\right)^y \left(1 - \frac{\lambda}{n}\right)^{n-y}$$

$$= \frac{\lambda^y}{y!} \cdot \frac{n(n-1)(n-2)\cdots(n-(y-1))}{n \cdot n \cdot n \cdots n} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-y}$$

$$= \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \frac{\lambda^y}{y!} \cdot \underbrace{\left(1-\frac{1}{n}\right)}_{\to 1}\underbrace{\left(1-\frac{2}{n}\right)}_{\to 1}\cdots\underbrace{\left(1-\frac{y-1}{n}\right)}_{\to 1}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-y}}_{\to 1 \text{ as } n\to\infty}$$

and look at what happens to this expression as $n \to \infty$

$$\longrightarrow \frac{e^{-\lambda}\lambda^y}{y!} \quad \text{the Poisson pmf}$$

Here, we used a Calculus fact that $\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$.

We can therefore think of the Poisson(λ) r.v. as a r.v. that counts the # of rare events that happen in an interval of time.

Example: I walk into a room with 60 (other) people. What is the probability that at least one other person has my birthday?

If we let X count the # of people (in the 60) that have my birthday, then it seems reasonable to assume

$$X \sim \text{binomial}\left(60, \frac{1}{365}\right).$$

Therefore, on one hand

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - \binom{60}{0}\left(\frac{1}{365}\right)^0\left(\frac{364}{365}\right)^{60}$$

$$= 1 - \left(1 - \frac{1}{365}\right)^{60} \doteq .15177$$

Using a Poisson approximation $p = \frac{\lambda}{n} \Rightarrow \lambda = np = 60\left(\frac{1}{365}\right)$

$$P(X \geq 1) = 1 - P(X=0)$$

$$\approx 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda} = 1 - e^{-\frac{60}{365}} \approx .15158$$

How many people would need to be in the room to have this probability be .5?

$$1 - e^{-\frac{m}{365}} = .5 \Leftrightarrow e^{-\frac{m}{365}} = .5 \Rightarrow m \approx 253.$$

Functions of random variables

Once we have a r.v. we can have others by applying functions to it.

For example, if C is the r.v. that measures the degrees Celsius, then the r.v. $K = C + 273.15$ returns the measurements in degrees Kelvin.

If V is the velocity of a particle of mass m, then $E = \frac{1}{2} m V^2$ is the Kinetic energy of the particle.

A question one may ask is:

If we know the pmf of a discrete r.v. X, and g is a function, what is the pmf of $g(X)$?

The answer to this question is (sort of) easy when r.v.s are discrete. We demonstrate with some examples.

<u>Ex.1</u> Suppose X has pmf

x	-2	-1	0	1	2	3	4
$p_X(x)$	$.05$	$.10$	$.20$	$.30$	$.20$	$.05$	$.10$

What is the pmf of X^2?

the basic idea is : apply to function g to every possible value x of the rv X : $g(x)=x^2$

$\overbrace{g(-2)}^{(-2)^2=4}$	$\overbrace{g(-1)}^{(-1)^2=1}$	$\overbrace{g(0)}^{0}$	$\overbrace{g(1)}^{1}$	$\overbrace{g(2)}^{4}$	$\overbrace{g(3)}^{9}$	$\overbrace{g(4)}^{16}$
.05	.10	.20	.30	.20	.05	.10

there are only 5 distinct values of $g(x)$ — they are

0, 1, 4, 9, 16. So

y	0	1	4	9	16
$p_Y(y)$	.20	.40	.25	.05	.10

is the pmf of $Y=g(X)$.

Basically, what we did was for each value y we found

$$p_Y(y) = \sum_x p_X(x)$$

← where we sum over all x's s.t. $g(x)=y$.

$$= \sum_{\{x:\, g(x)=y\}} p_X(x).$$

Ex. Roll a balanced 6-sided die once.

Let X be the up-face. Then the pmf is the discrete uniform distribution:

x	1	2	3	4	5	6
$p_X(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now suppose you win \$2 if you roll a 1, 2, or 3 you win \$5 if you roll a 4, or 5 and you lose \$10 if roll a 6.

Then your winnings r.v. is $Y = g(X) = \begin{cases} -10 & \text{if } X=6 \\ 2 & \text{if } X=1,2,3 \\ 5 & \text{if } X=4,5. \end{cases}$

and

y	-10	2	5
$p_Y(y)$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{2}{6}$

Expected value of a discrete r.v.

If X is a discrete r.v., we define

$$E(X) = \sum_{x} x \, p_X(x)$$

as the expected value of X (also called expectation of X, also called the mean of X)

$E(X)$ is a weighted average of the values x of X where the weights are the probability masses.

Ex. Suppose Y is a discrete r.v. with pmf

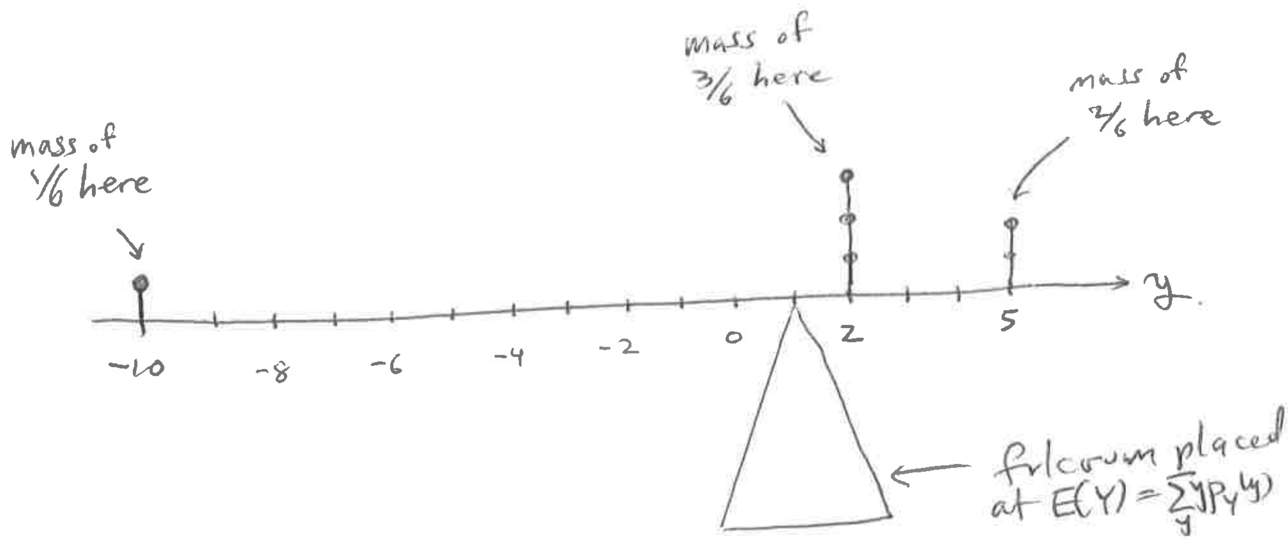
y	-10	2	5
$p_Y(y)$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{2}{6}$

then

$$E(Y) = -10\left(\tfrac{1}{6}\right) + 2\left(\tfrac{3}{6}\right) + 5\left(\tfrac{2}{6}\right) = \underline{1}.$$

What is the meaning of this number?

think of the values of the r.v. as on a number line;



The fulcrum of a see-saw with the masses above placed at the values of y needs to be placed at $E(Y)=1$ to "balance" the see-saw.

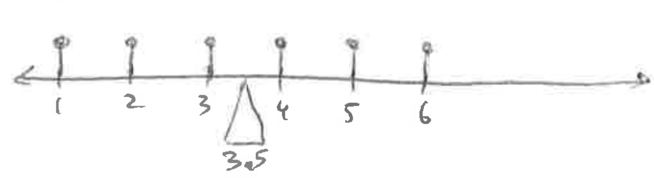
That is, $E(Y)=1$ represents the 1st moment of inertia.
"the center of mass" or more accurately
"the center of probability mass".

$E(Y)$ can be thought of as the "center" of the distribution.

Ex. 2 If you roll a 6-sided (balanced) die, and X is the up face, what is $E(X)$.

$$E(X) = 1\left(\tfrac{1}{6}\right)+2\left(\tfrac{1}{6}\right)+3\left(\tfrac{1}{6}\right)+4\left(\tfrac{1}{6}\right)+5\left(\tfrac{1}{6}\right)+6\left(\tfrac{1}{6}\right)$$

$$= \frac{21}{6} = \frac{7}{2} = 3.5$$



Ex. 3

If $X \sim \text{Poisson}(\lambda)$, i.e.,

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

what is $E(X)$?

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\cancel{x} \cdot \lambda^x}{\cancel{x}(x-1)!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda} \left\{ 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots \right\}$$

$$= \lambda e^{-\lambda} (e^{\lambda}) = \lambda.$$

So $E(X) = \lambda$.

In words, the mean (expected value) of a Poisson(λ) is its parameter λ.

$\boxtimes$

Ex. 4 Suppose $X \sim \text{Bernoulli}(p)$.
Find $E(X)$.

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = p.$$

In words, the mean of a Bernoulli(p) is the probability of success.

Ex. 5 Suppose $X \sim \text{binomial}(n, p)$.
Find $E(X)$. (We'll see a simpler way later)

$$E(X) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n-1} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

change of variable $k = x-1$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} p^k (1-p)^{(n-1)-k} = np.$$

So $E(X) = np$. The mean of a binomial(n,p) is just the product np.