Intro Prob Lecture Notes

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March 8, 2017

$\mathbb{E}, Var$ of Poisson

- $\mathbb{E}(X) = \lambda$
- Example: 38 category 5 hurricanes made landfall over the last 100 years
  - $\lambda = .38$
  - What is the probability a category 5 hurricane makes landfall in the next two years?
  - $X \sim$ Poisson(.76) (.38 * 2)
  - $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-.76} \approx .5323$

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \\
&\text{With } y = x - 1 = 0, 1, 2, \ldots \\
&= \lambda e^{-y} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}
$$

- To show $Var(X)\ lambda$
  - The textbook computes $E(X^2)$ directly.
- Aside: Moments
  - Second Moment $\mathbb{E}(X^2)$
  - Factorial Moment $\mathbb{E}(X(X-1))$
  - Second Central Moment $\mathbb{E}((X-\mu)^2)$
- Let's instead compute $\mathbb{E}(X(X-1))$
  - Strategy: this solution is equal to $\mathbb{E}(X^2) - \mathbb{E}(X)$

$$
\begin{aligned}
\mathbb{E}(X(X-1)) &= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x!} \\
&= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \\
&= \lambda^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X) \\
&\implies \mathbb{E}(X^2) = \lambda^2 + \lambda \\
&\implies Var(X) = (\lambda^2 + \lambda) - (\lambda)^2 = \lambda \\
&\implies \sigma = \sqrt{\lambda}
\end{aligned}
$$

- Independent Bernoulli(p) trials
  - Let $Y \sim$ Pascal(r, p)
    - Y answers: When does the rth success happen?

$$
\begin{aligned}
P(Y = y) &= \binom{y-1}{r-1} p^{r-1}(1-p)^{y-r} * p \\
&= \binom{y-1}{r-1} p^{r}(1-p)^{y-r}
\end{aligned}
$$

$$
y = r, r+1, r+2, \ldots
$$

    - pmf of a Pascal $(, p)$

* Can be seen as many consecutive geometric variables
* $\sum_{i=1}^{r} X_i = Y$
  * Where X_i is the first time the event occurs after the $i-1$th time.

$$
\begin{aligned}
E(Y) &= E(\sum_{i=1}^{r} X_i) \\
&= \sum_{i=1}^{r} \mathbb{E}(X_i) \\
&= \sum_{i=1}^{r} \frac{1}{p} \\
&= \frac{r}{p}
\end{aligned}
$$