Independence is a very important and central notion in probability (and statistics).

- To show to events, say A and B, are independent we need to verify if

$$P(A \cap B) = P(A) \cdot P(B).$$

and if this is the case we declare A and B independent.

Let's do a few straight-forward examples.

Example 1 A box has 4 blue and 6 green marbles. The experiment is to draw two marbles — one at a time — note the colors on each draw and do Not replace the marbles after each draw.

Let B_1 and B_2 be the events that you draw a blue marble on the 1^{st} and 2^{nd} draws, respectively.

Are B_1 and B_2 independent?

$$P(B_1 \cap B_2) = P(B_1)P(B_2 \mid B_1) = \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{6}{45}$$

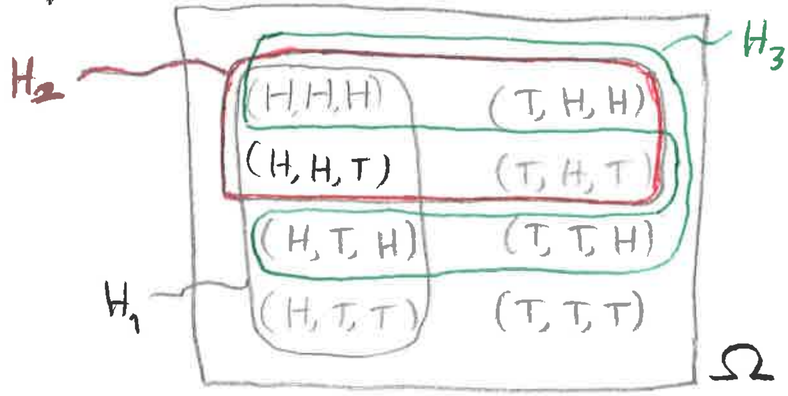
$$P(B_1) = \frac{4}{10} = \frac{2}{5}$$

$$P(B_2) = P(B_1)P(B_2 \mid B_1) + P(B_1^c)P(B_2 \mid B_1^c)$$

$$= \frac{4}{10} \cdot \frac{3}{9} + \frac{6}{10} \cdot \frac{4}{9} = \frac{36}{90} = \frac{2}{5}$$

and Notice $\frac{6}{45} = P(B_1 \cap B_2) \neq \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$ and B_1 & B_2 are Dependent

Example 3 Toss a balanced coin 3 times in a row and note the sequence of heads and tails observed.



Ω consists of $2^3 = 8$ equally-likely outcomes

Let H_1, H_2, H_3 be the events that a head is showing on toss i $(i = 1, 2, 3)$.

It's plain that $P(H_1) = \frac{1}{2} = P(H_2) = P(H_3)$.

Moreover,

$$P(H_1 \cap H_2) = \frac{2}{8} = \frac{1}{4} = P(H_1)P(H_2)$$

$$P(H_1 \cap H_3) = \frac{2}{8} = \frac{1}{4} = P(H_1)P(H_3)$$

$$P(H_2 \cap H_3) = \frac{2}{8} = \frac{1}{4} = P(H_2)P(H_3)$$

and also,

$$P(H_1 \cap H_2 \cap H_3) = \frac{1}{8} = P(H_1)P(H_2)P(H_3).$$

which shows all the events H_1, H_2, H_3 are independent.

$\blacksquare$

Example 2 Same box as in example 1 but now the experiment replaces the marble after each draw.

Now, $P(B_1) = \frac{4}{10}$ and $P(B_2) = \frac{4}{10}$

But,

$$P(B_1 \cap B_2) = P(B_1)\,P(B_2 \mid B_1) = \frac{4}{10} \cdot \frac{4}{10} = P(B_1)P(B_2)$$

and in this experiment B_1 & B_2 are independent.

In order to do the next example, I want to extend the notion of independence of event to more than 2 events...

We say events $A_1, A_2, A_3, \ldots, A_n$ are independent to mean that for every possible subcollection of these events, the probability of their intersection is equal to the product of the probabilities of the individual events in the subcollection:

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})\,P(A_{i_2}) \cdots P(A_{i_k}).$$

for any subset $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$.

So, for example A_1, A_2, A_3 are independent means

$P(A_1 \cap A_2) = P(A_1)P(A_2)$, $P(A_1 \cap A_3) = P(A_1)P(A_3)$

$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$, AND $P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$.

All of these conditions must be true for A_1, A_2, A_3 to be independent.

Remark

As we saw in the last example, in order to verify that 3 event are independent, there we 4 conditions to check.

If we need to verify 4 events are independent then there turns out to be 11 conditions to check.

and as the number of event to verify increases, the number of conditions to check for independence grows enormous!

Fortunately, independence will typically be an assumption that is made (and not usually one we will need to verify when we have many events). This is because the condition of independence is something that can often be guaranteed by designing the experiment appropriately. For example, if an experiment is designed to observe the outcomes of two (or more) distinct and non-interacting systems, then the events produced by these systems can be assumed independent

For instance, if Anna and Bob each toss coins and the experiment observes the number of heads each tosses, then the event A that Anna tosses x heads and the event B that Bob tosses y heads can be considered independent.

Example Anna and Bob separately to a balanced coin 3 times. Whoever ends up with more heads in their 3 tosses wins. Compute the probability Anna and Bob end up tied.

If we let A_i (respectively B_i) be the event that Anna (resp., Bob) tosses i heads, then the probability Anna and Bob are tied is:

$$P(A_0 \cap B_0) + P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3)$$

But A_i, B_i are all independent! So this equals

$$P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) + P(A_3)P(B_3)$$

$$= \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{3}{8} + \frac{1}{8} \cdot \frac{1}{8}$$

$$= \frac{18}{64} = \frac{9}{32}$$

$\blacksquare$

Suppose we know A and B are independent

Then $P(A\cap B) = P(A)P(B)$.

Now I claim that the pairs of events

$$A^c, B \qquad A, B^c \qquad \text{and} \qquad A^c, B^c$$

inherit independence from A and B...

Let's see:

$$\begin{aligned} P(A^c\cap B) &= P(B) - P(A\cap B) \\ &= P(B) - P(A)P(B) = P(B)[1-P(A)] \\ &= P(B)P(A^c) \implies A^c \text{ and } B \text{ are independent!} \end{aligned}$$

Similarly,

$$\begin{aligned} P(B^c\cap A) &= P(A) - P(A\cap B) \\ &= P(A) - P(A)P(B) = P(A)P(B^c) \end{aligned}$$

and A and B^c are independent

Finally,

$$\begin{aligned} P(A^c\cap B^c) &= 1 - P(A\cup B) \\ &= 1 - \{P(A) + P(B) - P(A\cap B)\} \quad \text{by the inclusion-exclusion rule} \\ &= 1 - P(A) - P(B) + P(A\cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \quad \text{using independence of } A, B \\ &= 1 - P(A) - P(B)[1-P(A)] \\ &= [1-P(A)][1-P(B)] = P(A^c)P(B^c). \end{aligned}$$

$\blacksquare$

The Counting Principle (on page 45 of text)

Consider a process consisting of r stages.

Suppose

(a) There are n_1 possible results ~~of~~ at the 1^{st} stage

(b) For every possible result at the 1^{st} stage, there are n_2 possible results at stage 2

(c) and, more generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the i^{th} stage.

Then the total # of possible results of the r stage process is

$$n_1 \times n_2 \times \dots \times n_r .$$

Example 0

A person has 4 pairs of pants, 6 shirts, 8 pairs of socks, and 3 pairs of shoes. In how many ways can this person get dressed?

$r = 4$ stages: Stage 1 = select pants, stage 2 = select shirt, etc.

We are trying to count the total # of possible results in the r stages. Answer is

$$4 \times 6 \times 8 \times 3 = 576 \text{ ways}$$

$\boxed{2}$

In our course we will primarily be concerned with the next few examples.

Example 1 A box has m distinct marbles labeled $1, 2, \ldots, m$. A ball is drawn from the box, its number is noted and the marble is returned. This procedure is repeated k times. How many outcomes are possible?

$$\underbrace{m \times m \times m \times \cdots \times m}_{k\text{-times}} = m^k$$

This example is called.
Sampling with replacement here.

Example 2 (Sampling without replacement)
A box has m marbles labeled $1, 2, \ldots, m$. A ball is chosen, its number is noted but now the marble is not returned. This procedure is repeated k times (we assume $k \le m$). How many outcomes are possible?

$$m \times (m-1) \times (m-2) \times \cdots \times (m-(k-1))$$

any one of the m marbles can be selected first.

Once the marble in first position is selected, the second marble is chosen from a set of $m-1$ marbles

Once first two marbles have been chosen, the third marble is chosen from a set of $m-2$ marbles, etc

The outcomes in this example are called k-permutations

$$\frac{m!}{(m-k)!}$$

<u>Subexample</u> 3 marbles labeled 1, 2, 3. Sample without replacement twice. Possible outcomes:

$(1,2)$ $\quad$ 3×2 possible outcomes.
$(1,3)$
$(2,1)$
$(2,3)$
$(3,1)$
$(3,2)$

<u>Example 3</u> m distinct marbles labeled $1, 2, \dots, m$ are in a box. Select k of these marbles <u>with</u> replacement. What is the probability that no marble is selected again? (I.e., all marbles drawn are distinct). Assume $k \leq m$

Since each of the m^k possible outcomes are equally-likely. Then this probability is equal to

$$\frac{m(m-1)(m-2)\cdots(m-(k-1))}{m^k} =$$

$$\frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-(k-1)}{m} = \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{k-1}{m}\right).$$

$\boxtimes$

The Birthday problem

Assume that peoples' birthdays are equally-likely to be any of the 365 days of the year. (We ignore Leap years and the fact that that birth-rates are not uniform over the year.) Find the probability that no two people in a group of n people will have a common birthday.

There are 365 "marbles" and we draw n with replacement. The probability that all n marbles drawn will have distinct labels is

$$p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

The numerical consequences are quite unexpected: when $n = 23$, $p < 1/2$ and when, for instance,

$n = 56$, $p = .01$.

In a group of 23 people there is a better than 50% chance that at least two people will share a common birthday, and in a group of 56 people there's a 99% chance at least two people will share a common birthday.

Example How many 4-letter words (including non-sense words) consist of 4 distinct letters?

$$26 \times 25 \times 24 \times 23 = \frac{26!}{22!}$$

How many of these start and end in a vowel?
(a vowel is in the set $\{a, e, i, o, u\}$)

$$5 \times 4 \times 24 \times 23.$$

What's the probability that a randomly selected 4-letter word with distinct letters starts and ends in a vowel?

$$\frac{5 \times 4 \times \cancel{24} \times \cancel{23}}{26 \times 25 \times \cancel{24} \times \cancel{23}} = \frac{2}{65} \approx .03$$

The Basic Counting principle gives us a way to count objects that have a natural order to them.

<u>Examples</u>

- The number of arrangements of k books from a set of n books. Here, an arrangement means an ordering of k books, say from left-to-right, and the same k books in a different order is considered a different arrangement

sub-Example: $n = 15$ books, $k = 6$ on a shelf

has

$$15 \times 14 \times 13 \times 12 \times 11 \times 10 = \frac{15!}{(15-6)!}$$

possible (distinct) arrangements.

- Another situation is rolling a 6-sided dice many times

Say, $k = 4$ times

If the 4 dice we all different colors, then we might consider the result of

$(1, 6, 3, 4)$ distinct from $(6, 1, 4, 3)$

one on red, 6 on green, 3 on yellow, 4 on blue

6 on red, one on green, 4 on yellow, 3 on blue

There are $6 \times 6 \times 6 \times 6 = 6^4$ such rolls of these 4 dice.

In some situations, the objects we wish to count do not have a natural order.

For example, select 3 people from a group of 5 to be on a committee. Then if A, B, C are on the committee it doesn't matter which order they were selected.

This is the case where we have n distinct objects and we select k of them: we wish to count the number of combinations of k objects taken from a group of n distinct objects.

called a binomial coefficient and is pronounced "n choose k"

Answer is there are $\binom{n}{k}$ such combinations.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \text{where } 0! = 1 \text{ by convention}$$

$\binom{n}{k}$ counts the number of subsets of size k that can be formed from a set of n objects.

Example: 4 people a, b, c, d. we want to select 2.

possible combinations:

$$\{a,b\},\ \{a,c\},\ \{a,d\},\ \{b,c\},\ \{b,d\},\ \{c,d\}.$$

there are 6.

(Check) $\binom{4}{2} = \dfrac{4!}{2!(4-2)!} = \dfrac{4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{2\cdot 1(\cancel{2}\cdot \cancel{1})} = 6.$ Yes!

Example:
How many selections of 10 people from 60 are possible?
Ans:

$$\binom{60}{10} = \frac{60!}{10!\ 50!} = \frac{60\cdot 59\cdot 58\cdot 57\cdot 56\cdot 55\cdot 54\cdot 53\cdot 52\cdot 51}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= 75{,}394{,}027{,}566.$$

i.e., there are this many subsets of 10 people that can be formed from a group of 60 (distinct) people.

We can now combine some ideas of counting ....

Suppose that the 60 people are comprised of 20 males and 40 females.

How many subsets of size 10 have exactly 5 of each gender?

To answer this question, we look at a subsets of size 10 that has 5 males and 5 females.

They all look like the union of two sets:

$$\underbrace{\{m_1, m_2, m_3, m_4, m_5\}}_{\text{a set of 5 males}} \cup \underbrace{\{f_1, f_2, f_3, f_4, f_5\}}_{\text{a set of 5 females.}}$$

a set of 5 males and a set of 5 females.

So, if we want to count the # of such sets we can construct a 2-stage procedure.

(1) Select a subset of 5 males from the 20 possible males.

(2) Once the selection of males has been made select a subset of 5 females from the 40 possible.

the result of this procedure identifies a distinct set of 10 people — 5 of which are male and 5 which are female.

Thus, there are

$$\binom{20}{5}\cdot\binom{40}{5}$$

possible subsets of size 10 that have 5 of each gender

Continuing in this manner... there are

$\binom{20}{10}\binom{40}{0}$ subset having no females

$\binom{20}{9}\binom{40}{1}$ subset having exactly one female

$\binom{20}{8}\binom{40}{2}$ subsets having exactly two females, etc.

therefore, there are

$$\underbrace{\binom{20}{10}\binom{40}{0}}_{184{,}756}+\underbrace{\binom{20}{9}\binom{40}{1}}_{6{,}718{,}400}+\underbrace{\binom{20}{8}\binom{40}{2}}_{98{,}256{,}600} = 105{,}159{,}756$$

subset having 2 or less females.

Now if we assume an experiment selects 10 people "at random" → so that every possible subset of size 10 from the 60 people are equally-likely, then the probability a subset will having 2 or fewer females is

$$\approx .0013948 \approx \frac{\binom{20}{10}\binom{40}{0}+\binom{20}{9}\binom{40}{1}+\binom{20}{8}\binom{40}{2}}{\binom{60}{10}}$$

An important example of how the binomial coefficient arises is the following:

$\binom{n}{k}$ represent the # of sequences of heads and tails that have exactly k heads.

To see why take $n=7$ and $k=3$ (for example)

then a sequence of 7 coin tosses can be represented as an ordered 7-tuple

$$(\_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_)$$

There are 7 distinct positions in which heads and tails can occupy. Each subset of ~~the~~ size 3 from these 7 corresponds to a particular sequence of heads and tails having heads in the positions in this subset.

For example,

$$\{1, 3, 5\} \longrightarrow (H, T, H, T, H, T, T)$$
$$\{2, 3, 7\} \longrightarrow (T, H, H, T, T, T, H)$$

In fact each subset of size 3 is in one-to-one correspondence to a sequence of heads and tails with exactly 3 heads.

So there are $\binom{7}{3} = \frac{7!}{4!\,3!}$ sequences with exactly 3 heads.

<u>Example</u> Now we toss a coin having success probability p of getting heads on a trial independently n times.

What is the probability of getting exactly k heads?

Each sequence with exactly k heads (and, therefore, exactly $n-k$ tails) has a probability

$$p^k (1-p)^{n-k}$$

and we just showed there are $\binom{n}{k}$ such sequences.

Therefore, the probability of exactly k heads in n tosses is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

$\boxed{G}$

Since when we toss a coin n times the only possible number of heads we can observe is between 0 and n inclusive, we must have

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

This is the binomial theorem from Calculus.

The Binomial theorem

Suppose $n \geq 1$ is an integer and x and y are any real numbers. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} .$$

Special case: take $x = p$ and $y = 1-p$, where p is the success probability

$$1 = [p + (1-p)]^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} . \quad ▨$$