Hw #9 Additional problems (solutions)

A.9.1

(a) $\displaystyle\int_0^\infty u^3 e^{-2u}\,du = \int_0^\infty u^{4-1} e^{-u/\frac{1}{2}}\,du$

$\alpha=4,\ \beta=\frac{1}{2}$ $\displaystyle = \beta^\alpha \Gamma(\alpha) = \left(\tfrac{1}{2}\right)^4 \Gamma(4)$

$\displaystyle = \frac{1}{16}\cdot 3! = \frac{3}{8}.$

(b) $\displaystyle\int_0^\infty \sqrt{x}\, e^{-x/2}\,dx = \int_0^\infty x^{\frac{3}{2}-1} e^{-x/2} \overset{\alpha=3/2,\ \beta=2}{=} 2^{3/2}\,\Gamma(3/2)$

$\displaystyle = 2^{3/2}\cdot\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = 2^{3/2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

$= \sqrt{2\pi}.$

(c) $\displaystyle\int_{-\infty}^\infty e^{-\frac{1}{2}\left(\frac{x-7}{1}\right)^2} dx \overset{\sigma=1\ \mu=7}{=} \sigma\sqrt{2\pi} = \sqrt{2\pi}$

(d) $\displaystyle\int_{-\infty}^\infty e^{-\frac{(x^2-2x)^2}{2}} dx = \int_{-\infty}^\infty e^{-\frac{1}{2}\{x^2-2x+1-1\}} dx = \int_{-\infty}^\infty e^{-\frac{1}{2}\{(x-1)^2-1\}} dx$

$\displaystyle = \int_{-\infty}^\infty e^{-\frac{1}{2}(x-1)^2}\cdot e^{\frac{1}{2}}\,dx = \sqrt{e}\int_{-\infty}^\infty e^{-\frac{1}{2}(x-1)^2} dx \overset{\sigma=1,\ \mu=1}{=} \sqrt{e}\cdot\sigma\sqrt{2\pi} = \sqrt{2\pi e}$

(e) $\int_0^1 \sqrt{x}\,(1-x)^{3/2}\,dx = \int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{5}{2}-1}$

$\alpha = \frac{3}{2},\ \beta = 5/2$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3!}$$

$$= \frac{3\pi}{8\cdot 6} = \boxed{\frac{\pi}{16}}$$

A 9.2

(a) $2n\cdot(2n-2)\cdot(2n-4)\cdots 6\cdot 4\cdot 2$

$$= 2\cdot n\cdot 2(n-1)\cdot 2(n-2)\cdots 2\cdot 3\cdot 2\cdot 2\cdot 2\cdot 1$$

$$= \underbrace{2\cdot 2\cdot 2\cdots 2\cdot 2}_{n\text{-times}}\cdot n(n-1)(n-2)\cdots(3)\cdot 2\cdot 1$$

$$= 2^n\cdot n!$$

(b) $(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(2n)\qquad(2n-2)\qquad(2n-4)\cdots 6\cdot 4\cdot 2}$$

$$= \frac{(2n)!}{2^n\cdot n!}$$ from part (a).

$\boxed{A.4.3}$

(a) $$E(X) = \int_0^\infty x \cdot \frac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{\frac{n}{2}-1} \cdot \Gamma(\frac{n}{2})} dx = \frac{\int_0^\infty x^{\frac{n}{2}} e^{-x/2} dx}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})}$$

$\alpha = \frac{n}{2}+1 \quad \beta = 2$

$$= \frac{2^{\frac{n}{2}+1} \Gamma(\frac{n}{2}+1)}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})} = \frac{2 \cdot \frac{n}{2} \cdot \cancel{\Gamma(\frac{n}{2})}}{\cancel{\Gamma(\frac{n}{2})}} = \boxed{n}$$

$$E(X^2) = \int_0^\infty \frac{x^2 \cdot x^{\frac{n}{2}-1} e^{-x/2}}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})} dx = \frac{\int_0^\infty x^{\frac{n}{2}+1} e^{-x/2} dx}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})}$$

$\alpha = \frac{n}{2}+2, \ \beta = 2$

$$= \frac{2^{\frac{n}{2}+2} \Gamma(\frac{n}{2}+2)}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})} = \frac{2^2 \cdot (\frac{n}{2}+1)(\frac{n}{2}) \cancel{\Gamma(\frac{n}{2})}}{\cancel{\Gamma(\frac{n}{2})}} = (n+2)n$$

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$= (n+2)n - \{n\}^2 = \boxed{2n}$$

A.9.4 $y = g(x) := \nu + \alpha x^{\frac{1}{\beta}} \Leftrightarrow g^{-1}(y) = \left(\frac{y-\nu}{\alpha}\right)^{\beta}$

if $x > 0$ $\qquad$ if $y > \nu$.

and g is monotone (increasing) for $x > 0$ and therefore

$$f_Y(y) = f_X\left(\left(\frac{y-\nu}{\alpha}\right)^{\beta}\right) \cdot \frac{d}{dy}\left(\left(\frac{y-\nu}{\alpha}\right)^{\beta}\right) = e^{-\left(\frac{y-\nu}{\alpha}\right)^{\beta}} \cdot \beta\left(\frac{y-\nu}{\alpha}\right)^{\beta-1} \cdot \frac{1}{\alpha}$$

$$f_Y(y) = \frac{\beta}{\alpha}\left(\frac{y-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{y-\nu}{\alpha}\right)^{\beta}} \quad \text{for } y > \nu$$

is the Weibull pdf.

A.9.5 $A = L^2$ and $L \sim \text{uniform}\left(1-\frac{h}{2}, 1+\frac{h}{2}\right)$.

$$F_A(a) = P(A \le a) = P(L^2 \le a) = 0 \text{ if } a \le \left(1-\frac{h}{2}\right)^2$$
$$= 1 \text{ if } a \ge \left(1+\frac{h}{2}\right)^2$$

So we assume $\left(1-\frac{h}{2}\right)^2 < a < \left(1+\frac{h}{2}\right)^2$. In this case,

$F_A(a) = P(L^2 \le a) = P(L \le \sqrt{a}) = \int_{1-\frac{h}{2}}^{\sqrt{a}} \frac{1}{h}\,dx$ and taking a deriv. in a:

$$f_A(a) = \frac{1}{h}\cdot\frac{d}{da}\left(a^{1/2}\right) = \frac{1}{h}\cdot\frac{1}{2\sqrt{a}} = \frac{1}{2h\sqrt{a}} \quad \text{for } \left(1-\frac{h}{2}\right)^2 < a < \left(1+\frac{h}{2}\right)^2$$

Thus,

$$f_A(a) = \begin{cases} \dfrac{1}{2h\sqrt{a}} & \text{for } \left(1-\frac{h}{2}\right)^2 < a < \left(1+\frac{h}{2}\right)^2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Recall that if $U \sim \text{uniform}(\alpha, \beta)$ then

$$E(U) = \frac{\alpha+\beta}{2} \quad \text{and} \quad Var(U) = \frac{\beta-\alpha}{12}.$$

Thus

$$E(A) = E(L^2) = Var(L) + \{E(L)\}^2$$

and $L \sim \text{uniform}\left(1-\frac{h}{2}, 1+\frac{h}{2}\right)$

$$E(L) = 1 \qquad Var(L) = \frac{h}{12}.$$

So, $E(A) = \frac{h}{12} + 1.$

Notice the mean area of the square is larger than 1!

$\boxed{A\ 9.6}$ Suppose $X \sim \text{Normal}(10, 2^2)$.

(a) $P(X > 12) = P\left(\frac{X-\mu}{\sigma} > \frac{12-10}{2}\right) = P(Z > 1) \overset{\text{using } \Phi \text{ tables}}{=} .1587$

(b) Binomial probability

$$\underbrace{\binom{5}{0}(.1587)^0(.8413)^5}_{\text{prob. none exceed 12 mm}} + \underbrace{\binom{5}{1}(.1587)(.8413)^4}_{\text{prob. exactly 1 exceeds 12 mm}}$$

$$\approx .42146 + .39751 = \boxed{.81897}$$

(c) Geometric probability

$$\underbrace{.1587}_{\text{captured on } 1^{st} \text{ draw}} + \underbrace{(.8413)(.1587)}_{\text{captured on } 2^{nd} \text{ draw (but not first)}} \approx \boxed{.2922}$$

(d) $$\sum_{j=5}^{\infty} (.8413)^{j-1}(.1587) = \frac{(.1587)(.8413)^4}{1 - .8413} = \boxed{(.8413)^4.}$$