550.420 Probability - FALL 2015 **MIDTERM 1 solutions**

1. A box contains 2 red, 4 green marbles. You randomly select a marble, note its color, and put it back, and repeat.
You do not need to simplify your answers below.
(a) Find the probability that in 4 draws you will see 2 red marbles. ANSWER: $\binom{4}{2}(\frac{1}{3})^2(\frac{2}{3})^2$.
(b) Find the probability that the first red marble occurs on the 4th draw. ANSWER: $(\frac{2}{3})^3 \cdot \frac{1}{3}$
(c) Find the probability that the third red marble occurs on the 8th draw. ANSWER: $\binom{7}{2}(\frac{1}{3})^3(\frac{2}{3})^5$
(d) If the first red marble occurred on the 4th draw, find the probability that the third red marble occurs on the 8th draw. ANSWER: $\binom{3}{1}(\frac{1}{3})^2(\frac{2}{3})^2$

2. Nine children are randomly seated in three rows of three desks each.
Let A be the event that Alan and Betty are in the same row.
Let B be the event that Alan and Betty are each seated in one of the (four) corner desks.
Are A and B independent? *Use the definition of independence to justify your assertion.*

Solution #1:

Since Alan will be seated somewhere in the 9 seats, there are 2 seats of the remaining 8 that will seat Betty in the same row as Alan, therefore, $P(A) = 2/8 = 1/4$.

If Alan and Betty occupy corner seats, there is only 1 seat of the 3 available that will keep Betty in the same row as Alan, therefore, $P(A|B) = 1/3$.

Since $P(A) \neq P(A|B)$, A and B are *not* independent.

Solution #2:

$P(A) = \frac{9 \cdot 2 \cdot 7!}{9!} = \frac{1}{4}$: explanation: the 9! arrangement of the children are all equally-likely. Alan can occupy any of the 9 seats but to keep Betty in the same row she can only occupy one of the 2 other seats in the row, the remaining 7 children can occupy the remaining seats in 7! ways.

$P(B) = \frac{4 \cdot 3 \cdot 7!}{9!} = \frac{1}{6}$: explanation: Alan occupies one of the 4 possible corner seats, Betty occupies any of the remaining 3 corner seats, the 7 other children are placed in the remaining seats in 7! ways.

$P(A \cap B) = \frac{4 \cdot 1 \cdot 7!}{9!} = \frac{1}{18}$: explanation: Alan occupies one of the 4 possible corner seats, there is only 1 choice of corner seat for Betty to keep her in the same row as Alan, the remaining 7 children are placed in the remaining seats in 7! ways.

Since $P(A \cap B) \neq P(A) \cdot P(B)$ these event are *not* independent.

3. Each day you buy one lottery ticket even though the probability you will win money is only 1/1000 independent from day-to-day. *Do not simplify your answers below.*
(a) You play for 500 straight days. Find the probability you win at least once.
ANSWER: $1 - \binom{500}{0}(\frac{1}{1000})^0(\frac{999}{1000})^{500} = 1 - (\frac{999}{1000})^{500}$.
(b) *Approximate* the probability in part (a). ANSWER: Since p is small and n is large, setting $\lambda = 500(\frac{1}{1000}) = \frac{1}{2}$, a Poisson approximation gives $1 - e^{-\frac{1}{2}}$.

4. Box A has 2 red and 1 green marble in it, Box B has 1 red and 1 green marble in it. A marble is selected at random from box A and transferred to box B. Then a marble is drawn from box B. If a red marble is drawn from box B, find the probability a red marble was transferred.
To fix notation: let R be the event a red marble is drawn from box B and let R_T be the event that a red marble is transferred.

Using Bayes' rule, $P(R_T|R) = \frac{P(R|R_T)P(R_T)}{P(R|R_T)P(R_T)+P(R|R_T^c)P(R_T^c)} = \frac{\frac{2}{3}\cdot\frac{2}{3}}{\frac{2}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}} = \frac{4}{5}$.

5. Consider the sample space of all possible 30-long sequences of **1**'s, **2**'s, and **3**'s. Assume that each possible sequence is equally-likely. We define A_i to be the event that there are exactly 10 i's for $i = 1, 2, 3$. Compute the probability that at least one of the numbers **1**, **2**, or **3** appears exactly 10 times in a sequence. *You do not need to simplify.*

Be sure to write down the event of interest in terms of the A_i.

There are 3^{30} possible equally-likely sequences. TO find $|A_1|$ we place the 10 **1**'s: there are $\binom{30}{10}$ ways to do this, the remaining 20 slots in the sequence can be occupied by any of the 2 remaining numbers, i.e., in 2^{20} ways. Thus, $|A| = \binom{30}{10}2^{20}$. The same argument shows $|A_1| = |A_2| = |A_3|$.

To find $|A_1 \cap A_2|$ we pick the 10 slots to put the **1**'s down, there are $\binom{30}{10}$ ways to do this; once this has been done, we pick 10 of the remaining 20 slots to put the **2**'s down, there are $\binom{20}{10}$ ways to do this. The remaining 10 slots will be occupied by **3**'s. Thus, $|A_1 \cap A_2| = \binom{30}{10}\binom{20}{10} = \binom{30}{10,10,10}$. The same argument shows $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3|$.

Finally, $|A_1 \cap A_2 \cap A_3| = \binom{30}{10,10,10}$ as any sequence that has exactly 10 **1**'s and exactly 10 **2**'s must also have 10 **3**'s.

By the inclusion-exclusion principle,

$$P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{\binom{30}{10}2^{20}}{3^{30}} - 3 \cdot \frac{\binom{30}{10,10,10}}{3^{30}} + \frac{\binom{30}{10,10,10}}{3^{30}} = \frac{3 \cdot \binom{30}{10}2^{20} - 2 \cdot \binom{30}{10,10,10}}{3^{30}}.$$

6. (a) The facts $P(A \cup B) = P(A) + P(A^c \cap B)$ and $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$ can be easily verified by a Venn diagram. Do not verify these facts but use them to show that

$$P(A \cup B \cup C \cup D) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C) + P(A^c \cap B^c \cap C^c \cap D).$$

Solution: $P([A \cup B \cup C] \cup D) = P(A \cup B \cup C) + P([A \cup B \cup C]^c \cap D)$ by the first fact. By the second fact and DeMorgan's law:

$$P([A \cup B \cup C] \cup D) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C) + P(A^c \cap B^c \cap C^c \cap D).$$

(b) Compute the probability of at least 1 spade in a hand of 3 cards in *two* ways - one of which uses part (a). *Do not simplify either.*

Way #1: Define A_i to be the ith card is a spade. $P(A_1) = \frac{13}{52}$, $P(A_1^c \cap A_2) = P(A_1^c)P(A_2|A_1^c) = \frac{39}{52} \cdot \frac{13}{51}$, $P(A_1^c \cap A_2^c \cap A_3) = P(A_1^c)P(A_2^c|A_1^c)P(A_3|A_1^c \cap A_2^c) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50}$. Putting this together, $P(A_1 \cup A_2 \cup A_3) = \frac{13}{52} + \frac{39}{52} \cdot \frac{13}{51} + \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50}$.

Also, by letting A_i be the hand has exactly i spades, $P(A_1) = \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}}$, $P(A_1^c \cap A_2) = P(A_2) = \frac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}}$, and $P(A_1^c \cap A_2^c \cap A_3) = P(A_3) = \frac{\binom{13}{3}\binom{39}{0}}{\binom{52}{3}}$.

Way #2: probability of at least one spade $= 1 - P(\text{no spades}) = 1 - \frac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}}$.

Also, $1 - \frac{39 \cdot 38 \cdot 37}{52 \cdot 51 \cdot 50}$.

7. This problem is a slight modification of the game `YAHTZEE`©. We wish to compute the probability of 3-of-a-kind rolling 3 dice in at most two stages: in the first stage, you roll the 3 dice. If you roll 3-of-a-kind, your done! Otherwise, we move to the next stage, where you get to re-roll at most 2 of the dice to get your 3-of-a-kind as follows: if you rolled a pair in stage 1, then re-roll the one die that is different; if you rolled 3 distinct numbers in stage 1, then grab any two and re-roll to try to match the other number. What is the probability of getting 3-of-a-kind in this scenario?
You do not have to simplify.

You roll a 3-of-a-kind in the first stage with probability $\frac{6}{6^3}$. If you don't roll a 3-of-a-kind, then you either rolled a pair or you rolled distinct numbers.

You roll a pair with probability $\frac{6 \cdot \binom{3}{2} \cdot 5}{6^3}$. Re-rolling the die that is not in the pair, you will have a probability of $\frac{1}{6}$ of getting the 3-of-a-kind.

You roll distinct numbers with probability $\frac{6 \cdot 5 \cdot 4}{6^3}$. Re-rolling any two, you will have a probability $\frac{1}{36}$ of matching the other die.

Thus, the probability of rolling 3-of-a-kind in this game is $\frac{6}{6^3} + \frac{6 \cdot \binom{3}{2} \cdot 5}{6^3} \cdot \frac{1}{6} + \frac{6 \cdot 5 \cdot 4}{6^3} \cdot \frac{1}{36}$.

8. A room contains m married couples. The $2m$ people are randomly into m pairs. Let M be the random variable that counts the number of married couples paired. Compute $E(M)$.
Simplify completely.

Let $M_i = \begin{cases} 1 & \text{if married couple } i \text{ is paired} \\ 0 & \text{if not} \end{cases}$ so that $M = \sum_{i=1}^{m} M_i$.

$E(M_i) = P(\text{married couple } i \text{ is paired})$. Now,

$P(\text{married couple } i \text{ is paired}) = \frac{1}{2m-1}$ since the male must be matched with his 1 spouse out of the $2m-1$ remaining people. Also, the number of (unordered) pairings that have couple i paired is in one-to-one correspondence with the number of (unordered) pairings of the remaining $2m-2$ people, namely, $\binom{2m-2}{2,2,\ldots,2}/(m-1)!$ and therefore,

$P(\text{if married couple } i \text{ is paired}) = \frac{\binom{2m-2}{2,2,\ldots,2}/(m-1)!}{\binom{2m}{2,2,\ldots,2}/m!} = \frac{1}{2m-1}$.

Finally, $E(M) = \sum_{i=1}^{m} E(M_i) = \sum_{i=1}^{m} \frac{1}{2m-1} = \frac{m}{2m-1}$. Interestingly, as the number of couples increases to infinity, this expectation converges to $\frac{1}{2}$.