Intro Prob Lecture Notes

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Conditional Distributions

- Suppose X, Y are jointly continuous
- Define $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$ assuming $f_X(x) > 0$.
  - Conditional pdf of Y given $X = x$
- Notice that

  $$f(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

  which will later allow us to construct joint pdfs via conditional pdfs
- Ex: Suppose X, Y joint pdf
  -
    $$f(x,y) = \begin{cases} x + y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 \text{ otherwise} & \end{cases}$$
  - Suppose we are told $Y = \frac{3}{4}$. $\mathrm{P}(X \leq \frac{1}{2}|Y = \frac{3}{4}) = ?$
    * Note: We can't do (*) $\frac{P(X\leq\frac{1}{2}|Y=\frac{3}{4})}{P(Y=\frac{3}{4})}$ because the denominator is 0.
  - Compute

$$
\begin{aligned}
f_{X|Y}(x|\frac{3}{4}) &= \frac{f(x,\frac{3}{4})}{f_Y(\frac{3}{4})} \\
&= \frac{x+\frac{3}{4}}{\frac{1}{2}+\frac{3}{4}} \\
&= \frac{x+\frac{3}{4}}{\frac{5}{4}} \\
&= \frac{4}{5}x + \frac{3}{5}
\end{aligned}
$$

for $0 < x < 1$

- Ex: $f_Y(y) = \int_0^1 (x+y)dx = \frac{x^2}{2} + xy\Big|_0^1 = \frac{1}{2} + y$ for $0 < y < 1$
  - $P(X \le \frac{1}{2}|Y = \frac{3}{4}) = \int_0^{\frac{1}{2}} f_{X|Y}(x|\frac{3}{4})dx = \cdots = \frac{4}{10}$
    - This technique needed when conditioning $P = 0$. If asked $P(X \le \frac{1}{2}|Y > \frac{3}{4})$, we can use above (*)
  - Check if $P(X \le \frac{1}{2}|Y > \frac{3}{4}) = \frac{P(X \le \frac{1}{2}|Y > \frac{3}{4})}{P(Y > \frac{3}{4})}$
    - In general, $\neq \int_{\frac{3}{4}}^{1} \int_0^{\frac{1}{2}} f_{X|Y}(x|y)dxdy$
      - Try it!
- Ex: Compound Poisson. N claims made. N $\sim$ Poisson(λ)
  - X_i payout for ith claim, $i = 1, \ldots, n$.
  - $T = \sum_{i=1}^{N} X_i$, $X_i \sim$ independent $\exp(\frac{1}{\beta})$, $T|N = n \sim$ Gamma(n, β)
    - Incomplete discussion. Later.
- Ex: X $\sim \exp(\Lambda)$ and suppose Λ is another random variable
  - $\Lambda \sim$ Gamma(α, β)
  - $(X|\Lambda = \lambda) \sim \exp(\lambda)$
  - $f_{X|\Lambda}(x|\lambda) = \lambda e^{-\lambda x}$ for $x > 0$
  - $f_\Lambda(\lambda) = \frac{\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}}{\beta^\alpha \Gamma(\alpha)}$
  -

$$
\begin{aligned}
f_{X,\Lambda}(x,\lambda) &= [f_{X|\Lambda}(x|\lambda)] \times [f_\Lambda(\lambda)] \\
&= \frac{\lambda^\alpha e^{-\lambda(x+\frac{1}{\beta})}}{\beta^\alpha \Gamma(\alpha)}
\end{aligned}
$$

for $x > 0, \lambda > 0$

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$$
\begin{aligned}
f_X(x) &= \int_0^\infty \frac{\lambda^\alpha e^{-\lambda(x+\frac{1}{\beta})}}{\beta^\alpha \Gamma(\alpha)} d\lambda \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-\lambda/(x+\frac{1}{\beta})^{-1}} d\lambda \\
&\text{Integral is a gamma distribution} \\
&= \frac{\alpha}{\beta^\alpha (x+\frac{1}{\beta})^{\alpha+1}} \text{ for } x > 0 \sim \text{ Pareto Distribution} \\
&= \frac{\alpha\beta}{(1+\beta x)^{\alpha+1}}
\end{aligned}
$$

- Ex: Bayes' Statistics
  - $X|p \sim$ Binomial(n, p)
  - $p \sim$ Uniform(0, 1) $\sim$ "prior distribution"
  - $$P_{X,p}(x,p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \cdot 1 \text{ for } x = 0, 1, 2, \ldots, n, 0 < p < 1$$
  - Exercises: $(x = 0, 1, 2, \ldots n)$
    * $P_X(x) \sim$ discrete uniform $= \frac{1}{n+1}$ $x = 0, 1, \ldots n+1$
    * $P|X \sim$ Beta$(x+1, n-x+1) \sim$ "Posterior distribution"