

# Intro Prob Lecture Notes

William Sun

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## Continuous Random Variables

- A random variable  $X$  is called *continuous* if it possesses a cumulative distribution function which is a continuous function on the real line.
  - Reminder: CDF is  $F_X(x) = P(X \leq x)$
- Let  $x \in \mathbb{R}$ . Then for  $h > 0$ ,

$$\begin{aligned}P(x - h < X \leq x) &= P(X \leq x) - P(X \leq x - h) \\&= F_X(x) - F_X(x - h)\end{aligned}$$

Approaches 0 as  $h$  approaches 0

- Therefore,  $P(X = x) = 0$ .
- But

$$\frac{P(\mathbf{x} - \mathbf{h} < \mathbf{X} \leq \mathbf{x})}{\mathbf{h}} = \frac{F_{\mathbf{X}}(\mathbf{x}) - F_{\mathbf{X}}(\mathbf{x} - \mathbf{h})}{\mathbf{h}} \rightarrow (\text{If this difference quotient converges}) F'_X(x) = f_X(x) \geq 0$$

- Bold value is probability mass per unit length, or a probability “density”

$$P(-\infty < X \leq x) = F_X(x) = \int_{-\infty}^x f(u) du$$

$$\text{As } x \rightarrow \infty, 1 = \int_{-\infty}^{\infty} f_X(u) du$$

- A probability density function (pdf) is any function  $f$  such that
  - $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) \geq 0 \forall x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f(x) dx = 1$
- To compute probability mass,  $P(a < X \leq b) := \int_a^b f(x) dx$
- Example:  $f(x) = x^2$  on  $0 \leq x \leq 2$ ,  $f(x) = 0$  elsewhere
  - Not a pdf.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 0 dx + \int_0^2 x^2 dx + \int_2^{\infty} 0 dx \\ &= \left. \frac{x^3}{3} \right|_0^2 \\ &= \frac{8}{3} \end{aligned}$$

- However, we can normalize the function to  $\tilde{f}$ , so that the value from 0 to 2 to  $\frac{3x^2}{8}$ .  $\tilde{f}$  would integrate to 1, so that would be a pdf
- Suppose  $X$  has pdf  $\tilde{f}$ . Compute

$$\begin{aligned} P(X > \frac{1}{2}) &= \int_{\frac{1}{2}}^{\infty} \tilde{f}(u) du \\ &= \int_{\frac{1}{2}}^2 \tilde{f}(u) du \\ &= \int_{\frac{1}{2}}^2 \frac{3u^2}{8} du \\ &= \left. \frac{u^3}{8} \right|_{\frac{1}{2}}^2 \\ &= 1 - \frac{1}{64} \\ &= \frac{63}{64} \end{aligned}$$

## Expected Value of Continuous Variables

- When  $f_X(x)$  is the pdf of  $X$ .

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- This will exist when  $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$
- The Cauchy pdf will *not* have an expected value

$$* f(x) \frac{1}{\pi(1+x^2)} \text{ for } -\infty < x < \infty$$

- Exercise:

$$\begin{aligned} E(X) &= \int_0^2 x \cdot \frac{3}{8} x^2 dx \\ &= \frac{3}{8} \cdot \frac{x^4}{4} \Big|_0^2 \\ &= \frac{3 \cdot 16}{32} \\ &= \frac{3}{2} \end{aligned}$$

- Example:  $X \sim \text{uniform}[0, 1]$ 
  - $f(x) = 1$  for  $0 \leq x \leq 1$
  - $f(x) = 0$  elsewhere.
  - Expectation is  $\frac{1}{2}$  by inspection

$$\begin{aligned} E(X) &= \int_0^1 x dx \\ &= \frac{x^2}{2} \Big|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

- And

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 dx \\
&= \frac{x^3}{3} \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

But we haven't proven this yet!

– Digression: Let's show the pdf for  $Y = X^2$ ,  $E(Y) = \int y f_Y(y) dy$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= (\text{Must assume } y \geq 0) \\
&= P(|X| \leq \sqrt{y}) \\
&= P(X \leq \sqrt{y}) \\
&= \int_0^{\sqrt{y}} f(x) dx
\end{aligned}$$

1 if  $\sqrt{y} > 1$ ,  $\sqrt{y}$  if  $0 \leq \sqrt{y} \leq 1$ , 0 if  $\sqrt{y} < 0$

– So,  $f_Y(y)$  is 0 if  $y > 1$ ,  $\frac{1}{2\sqrt{y}}$  if  $0 < y \leq 1$ , 0 if  $y \leq 0$ .

$$\begin{aligned}
E(X^2) &= E(Y) = \int_0^1 y \cdot \frac{1}{2y^{\frac{1}{2}}} dy \\
&= \frac{1}{2} \int_0^1 y^{\frac{1}{2}} dy \\
&= \frac{1}{3} y^{\frac{3}{2}} \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$