Intro Prob Lecture Notes

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- If X is a continuous random variable with pdf $f_X(x)$ and $Y = g(X)$ where g is a monotone (increasing or decreasing) then the pdf of Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

- Or since $x = g^{-1}(y)$,

$$f_Y(y) = f_X(x)\frac{dx}{dy}.$$

- Example: (The log-normal distribution). If $X \sim \text{Normal}(\mu, \sigma^2)$ then $Y = e^X$ has the normal distribution (i.e. $ln(Y) \sim \text{Normal}(\mu, \sigma^2)$ and $g(x) = e^x$)

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$$\begin{aligned} f_Y(y) &= f_X(ln(y)) \cdot \frac{dln(y)}{dy} \\ &= \frac{e^{-\frac{1}{2}(\frac{ln(y)-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} \cdot \frac{1}{y} \\ &\text{For } y > 0 \end{aligned}$$

- Aside: Kolmogorov's Law of Fragmentation
  - The size of an individual particle drawn from a large collection of particles resulting from fragmentation will have a log-normal distirbution.
- Example: (Cauchy distribution)
  - Unit height "pole" at 0, flashlight on pole placed it angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

- Let $\theta \sim \text{uniform}(-\frac{\pi}{2}, \frac{\pi}{2})$
- Let $x = tan(\theta) \rightarrow \theta = tan^{-1}(x)$
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$$\begin{aligned} f_X(x) &= f(\theta)\frac{d\theta}{dx} \\ &= \frac{1}{2\pi} \cdot \frac{1}{1+x^2} \text{For } -\infty < x < \infty \end{aligned}$$

  - This is the Cauchy pdf.

Chapter 6 - Joint Distributions

- Experiment has sample space Ω
- Can possibly define "many" random variables
  - $X : \Omega \rightarrow \mathbb{R}$
  - $Y : \Omega \rightarrow \mathbb{R}$
  - etc.
- X, Y can be any combination of discrete and continuous
- When X, Y are jointly discrete, we define the *joint pmf*

$$P_{X,Y}(x, y) := P(X = x, Y = y)$$

  - Which is shorthand for $P(\{X = x\} \cap \{Y = y\})$
- Example ($\approx$ 1(a) in chapter 6 of the textbook)
  - Box of socks
    - 3 White Socks
    - 4 Black Socks
    - 5 Red Socks
  - Draw two socks at random, without replacement (and without caring about order)
  - Let X count the number of white socks, and Y count the number of red socks.
    - $X \in \{0, 1, 2\}$
    - $Y \in \{0, 1, 2\}$

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$$P(X=1,Y=1)=\frac{\binom{3}{1}\binom{4}{1}\binom{5}{0}}{\binom{12}{2}}=\frac{12}{66}$$

– Full table of ways to choose certain numbers of black and white socks; divide by 66 for probability

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	x = 0	x = 1	x = 2	
y = 0	$\binom{5}{2}=10$	$\binom{3}{1}\ \binom{5}{1}=15$	$\binom{3}{2}=3$	28
y = 1	$\binom{4}{1}\binom{5}{1}=20$	$\binom{3}{1}\binom{4}{1}=12$	0	32
y = 2	$\binom{4}{2}=6$	0	0	6
	36	27	3	

– $P(X\le 1, Y\le 1)=\frac{57}{66}$

– $P(X<Y)=\frac{26}{66}$

– Can reconstruct the pmf of X from the joint distribution!

  ∗ $P(X=0)=\frac{36}{66}$

  ∗ $P(X=1)=\frac{27}{66}$

  ∗ $P(X=2)=\frac{3}{66}$

- When X, Y jointly continuous we introduce a *joint pdf* where
  - $f_{X,Y}\ge 0$ for all $x,y\in\mathbb{R}$
  - $\int\int f(x,y)dA=1$
- Example: Dart board that is a unit circle, with an equal chance of hitting each point
  - $P(X>\frac{1}{2})=?$
  - $P(X>\frac{1}{2},Y>\frac{1}{2})=?$
  - $\pi r^2=\pi\to$
$$f_{X,Y}(x,y)=\frac{1}{\pi}\text{ for }x^2+y^2\le 1, 0\text{ otherwise}$$