Intro Prob Lecture Notes

William Sun

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Expected Value (Continued)

- Now interested in extending this concept to functions of more than one variable. Suppose X, Y are jointly distributed and $g : \mathbb{R}^2 \to \mathbb{R}$ is any function. Then

$$\mathbb{E}(g(X,Y)) = \sum_x \sum_y g(x,y) P_{X,Y}(x,y)$$

  if X, Y are jointly discrete. (Similar to Law of Unconscious Statistician) Otherwise,

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

  if jointly continuous. And, if one is discrete, and the other is continuous, then we'll have both a sum and integral.

- Example: Suppose $X, Y \sim f_{X,Y}(x,y) = xe^{-x(1+y)}$ for $x > 0, y > 0$. Compute

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$$\mathbb{E}\Big(\frac{X}{1+Y}\Big) = \int_0^\infty \int_0^\infty \frac{x}{1+y} x e^{-x(1+y)} dx dy$$
$$= \int_0^\infty \frac{1}{1+y} \Big(\int_0^\infty x^2 e^{-x(1+y)} dx \Big) dy$$
Content in parentheses is equal to $\frac{1}{(1+y)^3}\Gamma(3)$ because ... Gamma distribution
$$= \int_0^\infty \frac{1}{1+y} \cdot \frac{2}{(1+y)^3} dy$$
$$= \ldots$$

- Example: Suppose $X_1, X_2 \sim$ independent Uniform(0, 1)
  - Intuition: Arrival time of two people to lunch
  - Find the expected time that the first person to arrive has to wait for the second person to arrive
  - $Y_1 = min\{X_1, X_2\}, Y_2 = max\{X_1, X_2\}$
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$$\mathbb{E}(Y_2 - Y_1) = \int \int (y_2 - y_1) f_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2$$
$$\ldots$$
$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \leq y_1, X_2 \leq y_2) + P(X_2 \leq y_1, X_1 \leq y_2)$$
$$= 2P(X_1 \leq y_1)P(X_2 \leq y_2)$$
$$= 2y_1 y_2 \text{ for } 0 < y_1 < y_2 < 1$$
$$f_{Y_1,Y_2}(y_1, y_2) = \frac{d}{dy_2}\frac{d}{dy_1}$$
$$= 2 \text{ for } 0 < y_1 < y_2 < 1$$
$$\mathbb{E}(Y_2 - Y_1) = \int_0^1 \int_0^{\frac{y}{2}} (y_2 - y_1) 2 dy_1 dy_2$$
$$= \cdots = \frac{1}{3}$$

Linearity of Expectation

- If $X_1, X_2, \dots X_n$ are jointly distributed then

$$\mathbb{E}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

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$$\begin{aligned}
\mathbb{E}(X_1 + X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 + x_2 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{X_1,X_2}(x_1,x_2)dx_2dx_1 + \int_{-\infty}^{\infty} x_2 \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \mathbb{E}(X_1) + \mathbb{E}(X_2)
\end{aligned}$$

- Hypergeometric: N trials - M successes, $N - M$ failures.
  - $X_i = \begin{cases} 1 & \text{if ith is a success} \\ 0 & \text{otherwise} \end{cases}$
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  $$\sum_{i=1}^{n} X_i \sim \text{ Hypergeometric}$$
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  $$\begin{aligned}
  \mathbb{E}(\sum_{i=1}^{n} X_i) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\
  &= \sum_{i=1}^{n} \mathbb{E}(X_1) \\
  &= n\mathbb{E}(X_1) \\
  &= \frac{nM}{N}
  \end{aligned}$$

Expectation and Independence

- If X, Y are independent and g, h are any real-valued functions, then

$$\mathbb{E}\Big(g(X)h(Y)\Big) = \mathbb{E}\Big(g(X)\Big)\mathbb{E}\Big(h(Y)\Big)$$

  - Provided g, h are such that expected values exist!

- Proof: Assume X, Y jointly continuous.

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$$\begin{aligned}
\mathbb{E}\Big(g(X)h(Y)\Big) &= \int\limits_0^\infty \int\limits_0^\infty g(x)h(y)f_{X,Y}(x,y)dxdy \\
&= \int\limits_0^\infty \int\limits_0^\infty g(x)h(y)f_X(x)f_Y(y)dxdy \\
&= \int\limits_0^\infty h(y)f_Y(y)\Big(\int\limits_0^\infty g(x)f_X(x)dx\Big)dy \\
&= \int\limits_0^\infty h(y)f_Y(y)\Big(\mathbb{E}(g(x))dy \\
&= \mathbb{E}\Big(g(X)\Big)\mathbb{E}\Big(h(Y)\Big)
\end{aligned}$$

- In particular, if X, Y are independent and $\mathbb{E}(X), \mathbb{E}(Y)$ exist, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(XY)$$

  - Says X, Y are *uncorrelated*
    - Uncorrelated means there is no linear function relating them
    - Egregious example: Z, Z^2

- Define covariance:

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$$\begin{aligned} Cov(X,Y) &= \mathbb{E}\Big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\Big] \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \end{aligned}$$

  - *Bilinear form*