Intro Prob Lecture Notes

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- Suppose X, Y are jointly continuous with joint pdf
- $f(x, y) = xy$ for $0 \leq x \leq 2, 0 \leq y \leq 1$, 0 elsewhere
  - Show that this is a joint pdf
    * When $(x, y) \in (\mathbf{0}, \mathbf{2}) \times (\mathbf{0}, \mathbf{1})$ we have, in particular, that $x > 0, y > 0$ Therefore, $f(x, y) = xy > 0$ in this part of the domain.
      · The bolded is the essential domain, or $supp(f)$
    * Furthermore, $f(x, y) = 0$ for $(x, y) \notin (0, 2) \times (0, 1)$.
    * Therefore, $f(x, y) \geq 0 \; \forall \; x, y \in \mathbb{R}$
    *

$$\begin{aligned}
\int_0^1 \int_0^2 xydxdy &= \int_0^1 \left(\frac{x^2 y}{2}\Big|_{x=0}^{x=2}\right)dy \\
&= \int_0^1 2ydy \\
&= y^2\Big|_0^1 \\
&= 1
\end{aligned}$$

    * So, the total area is also 1
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Compute $P(X \le Y)$

$$\begin{aligned}
P(X \le Y) &= \int\int_R f(x,y)dA \\
&= \int_0^1 \int_0^y xydxdy \\
&= \int_0^1 \left(\frac{x^2y}{2}\Big|_{x=0}^{x=y}\right)dy \\
&= \int_0^1 \frac{y^3}{2}dy \\
&= \frac{y^4}{8}\Big|_{y=0}^{y=1} \\
&= \frac{1}{8}
\end{aligned}$$

- The marginal pdf of X
  - $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$
  - and the marginal pdf of Y
  - $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$
- $f_X(x) = 0$ when $x \notin (0,2)$
- When $x \in (0,2)$
-

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{0} 0dy + \int_0^1 xydy + \int_1^{\infty} 0dy \\
&= \frac{x}{2}
\end{aligned}$$

- Exercise: Show $f_Y(y) = 2y$ for $0 < y < 1$, 0 otherwise
- Aside: comments on *marginal pdf*

  - If function is $f(x_1, x_2, x_3, x_4, x_5)$, $f_{x_1}(x_1) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x_1, x_2, x_3, x_4, x_5)dx_2dx_3dx_4dx_5$
    * Also extends to multivariate e.g. $f_{x_2,x_4}(x_2, x_4)$
  - Integrate out everything that's not the thing you're interested in.

Jointly Distributed Random Variables with One Discrete and One Continuous

- Ex: $P_{N,X}(n, x) = n(\frac{1}{2})^n e^{-nx}$ for $n = 1, 2, 3, \ldots$, $x > 0$
  - Intuition: Flipping a balanced coin n times. If N = n, win $X \sim$ exp(n).
  - pmf in first variable
  - pdf in second variable
  - Just called a "joint distribution"
  - Find the marginal of X.
  -

$$\begin{aligned} f_X(x) &= \sum_{n=1}^{\infty} n(\frac{e^{-x}}{2})^n \\ &= \ldots \\ &= \frac{e^{-\frac{x}{2}}}{(1 - e^{-\frac{x}{2}})^2} \text{ For } x > 0 \end{aligned}$$

  - $P_N(n) = (\frac{1}{2})^n$ for $n = 1, 2, \ldots$

Independence

- $X_1, X_2, \ldots X_n$ jointly distributed random variables
- We'll say they are independent if joint distribution $= \prod_{i=1}^{n}$ marginal distribution
  - $p(x_1, x_2, \ldots x_n) = P_{X_1}(x_1)P_{X_2}(x_2) \ldots P_{X_n}(x_n) \ \forall \ x_1, x_2, \ldots x_n$
- Back to the beginning. Ex:
- $f(x, y) = xy$ for $0 < x < 2, 0 < y < 1$. $f(x, y) = 0$ otherwise.
- $f_X(x) = \frac{x}{2}$ for $0 < x < 2$, 0 otherwise.

- $f_Y(y) = 2y$ for $0 < y < 1$, 0 otherwise.
- $f_X(x)f_Y(y) = f_{X,Y}(x,y) = \frac{x}{2}2y = xy$ if $0 < x < 2, 0 < y < 1$. 0 otherwise