Intro Prob Lecture Notes

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- Deal 5 cards from a deck of 52. Let X count the number of different suits in the hand. Compute $E(X)$.
  - One way: try to first find $P(X=1)\left(\frac{4\binom{13}{5}}{\binom{52}{5}}\right)$, $P(X=2)$ etc
  - Other way: Define $X_i = \begin{cases} 1 & \text{if at least one card of suit i belongs to the hand} \\ 0 & \text{otherwise} \end{cases}$
    * so that $X = \sum_{i=1}^{4} X_i$, then
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$$\begin{aligned} E(X) &= \sum_{i=1}^{4} E(X_i) \\ &= \sum_{i=1}^{4} P(\geq 1 \text{ card of suit } i) \\ &= \sum_{i=1}^{4} 1 - P(\text{no card of suit } i) \\ &= \sum_{i=1}^{4} (1 - \frac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}}) \\ &= 4(1 - \frac{\binom{39}{5}}{\binom{52}{5}}) \\ &\approx 3.11 \end{aligned}$$

- n people with n hats. Each randomly selects a hat. Let X = number of people that select their own hat. Compute $E(X)$ and $Var(X)$
  - Define $X_i = \begin{cases} 1 & \text{if person i selects their own hat} \\ 0 & \text{otherwise} \end{cases}$

- $X = \sum_{i=1}^{n} X_i$
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$$\begin{aligned} E(X) &= \sum_{i=1}^{n} P(X_i = 1) \\ &= \sum_{i=1}^{n} \frac{1}{n} \\ &= 1 \end{aligned}$$

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$$\begin{aligned} Cov(X_i, X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\ &= P(X_i = 1, X_j = 1) - \frac{1}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n(n-1)} - \frac{1}{n^2} \\ &= \frac{1}{n^2(n-1)} \\ Var(X) &= Var(\sum_{i=1}^{n} X_i) \\ &= \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j) \\ &= n \cdot (\frac{1}{n} \cdot (1 - \frac{1}{n})) + 2\Big(\frac{n(n-1)}{2}\Big) \cdot \frac{1}{n^2(n-1)} \\ &= 1 \end{aligned}$$

* Note: Covariance is positive. Makes a little sense since, if person i gets their own hat, person j is slightly more likely to get their own hat

Conditional Expectation

- Discrete

$$E(X|Y = y) = \sum_{x} xP(X = x|Y = y)$$

- Continuous

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

- Remember:

  1) $E(X|Y=y)$ is a function of y

  2) If Y is independent of X, $E(X|Y=y) = E(X)$

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Ex:

	x = 1	x = 2	x = 3	
y = 1	.1	0	.2	.3
y = 2	.1	.1	0	.2
y = 3	.1	.2	.1	.4
y = 4	0	.1	0	.1
	.3	.4	.3	

  - Compute:
    * $E(X|Y=1) = 1(1/3) + 2(0/3) + 3(2/3) = 7/3$
    * $E(X|Y=2) = 1(1/2) + 2(1/2) + 3(0/2) = 3/2$
    * $E(X|Y=3) = 1(1/4) + 2(2/4) + 3(1/4) = 2$
    * $E(X|Y=4) = 1(0) + 2(1) + 3(0) = 2$
      · X "degenerates" when $Y = 4$ since X can only have one value

- Notation:

  - $E(X|Y) = g(Y)$ so that when $Y = y$, $E(X|Y) = E(X|Y = y)$

- Law of Total Expectation (a.k.a. Law of Total Probability):

  - $E(E(X|Y)) = E(X)$ for any Y

- Ex: N = number of customers that enter store, X_i = amount of money spent by customer i

  - Assumption: $N, X_1, X_2, \ldots X_n$ independent

  - $S = \sum_{i=1}^{N} X_i$ represents the total sales

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$E(S)$?

$$E(\sum_{i=1}^{N} X_i | N = n) = E(\sum_{i=1}^{n} X_i | N = n)$$
$$= E(\sum_{i=1}^{N} X_i)$$
$$= \sum_{i=1}^{N} E(X_i)$$
$$= n\mu_X$$
$$E(S) = E(\sum_{i=1}^{N} X_i)$$

Using the above property, and Y = N

$$= E\Big(E(\sum_{i=1}^{N} X_i | N)\Big)$$
$$= E(N\mu_x)$$
$$= \mu_x E(N)$$
$$= \mu_x \mu_N$$