$\boxed{A.13.1.}$

$$\mathrm{Cov}(X, a+bX+cX^2) = \mathrm{Cov}(X,a) + \mathrm{cov}(X,bX) + \mathrm{Cov}(X, cX^2)$$

$$= 0 + b\,\mathrm{Var}(X) + c\,\mathrm{Cov}(X, X^2)$$

$$= b\left(E(X^2) - \underbrace{\{E(X)\}^2}_{=0}\right) + c\left(\underbrace{E(X^3)}_{=0} - \underbrace{E(X)E(X^2)}_{=0}\right)$$

$$= b$$

$\mathrm{Var}(X) = 1.$

$$\mathrm{Var}(Y) = \mathrm{Var}(a+bX+cX^2) = \mathrm{Var}(bX+cX^2)$$

$$= b^2\,\mathrm{Var}(X) + c^2\,\mathrm{Var}(X^2) + 2bc\,\underbrace{\mathrm{Cov}(X,X^2)}_{=0 \text{ from above calculation}}$$

$$= b^2 + c^2\left(E(X^4) - \{E(X^2)\}^2\right)$$

$$= b^2 + c^2(3-1) = b^2 + 2c^2.$$

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{b}{\sqrt{b^2+2c^2}}.$$

$\boxed{A.13.2}$ Let Y = arrival time of the professor. We are told

(a) $Y \sim \mathrm{uniform}(0,4)$ where $Y=0$ defines 9 am. and $Y=4$ defines 1 pm.. Given $Y=y$, the duration of the task is X where $X|Y=y \sim \mathrm{exponential}(\frac{1}{5-y})$. Consequently, $E(X|Y=y) = 5-y$ since the mean of an exponential$(\lambda) = \frac{1}{\lambda}$. Therefore, the expected duation is (by the law of total expectation)

$$E(X) = E(E(X|Y)) = E(5-Y) = 5 - E(Y) = 5-2 = 3,$$

since the mean of a uniform(a,b) is the midpoint of the interval.

A.13.21 (continued)

(b) $E(X+Y) = E(X) + E(Y) = 3 + 2 = 5.$

A.13.31

$$M_S(t) = E(e^{tS}) = E(E(e^{tS} | N))$$

$$= \sum_{n=1}^{\infty} E(e^{tS} | N=n) P(N=n) \quad \text{by the Law of total expectation.}$$

Now, given $N=n$, $S = \sum_{i=1}^{n} X_i$ is independent of N and, therefore,

$$M_S(t) = \sum_{n=1}^{\infty} E\left(e^{t\sum_{i=1}^{n} X_i}\right) \cdot (1-q)^{n-1} q = \sum_{n=1}^{\infty} \left(M_{X_1}(t)\right)^n (1-q)^{n-1} q$$

$$= \sum_{n=1}^{\infty} \left(\frac{pe^t}{1-(1-p)e^t}\right)^n (1-q)^{n-1} q = \frac{q}{1-q} \sum_{n=1}^{\infty} \left(\frac{(1-q)pe^t}{1-(1-p)e^t}\right)^n$$

$$= \frac{q}{1-q} \cdot \frac{\frac{(1-q)pe^t}{1-(1-p)e^t}}{1 - \frac{(1-q)pe^t}{1-(1-p)e^t}} = \frac{q}{\cancel{1-q}} \cdot \frac{\cancel{(1-q)}pe^t}{(1-(1-p)e^t) - (1-q)pe^t}$$

$$= \frac{qpe^t}{1-(1-qp)e^t}$$ is the mgf of S, which is the mgf of a geometric(pq) and therefore, the (unconditional) distribution of S is geometric(pq).

A.13.4

(a) We know $P(X_1 = i) = \frac{1}{6}$ for each $i = 1, 2, 3, 4, 5, 6$.

and $P(X_2 + X_3 = j) = \frac{6 - |7-j|}{36}$ for each $j = 2, 3, 4, \ldots, 12$.

Therefore, by the law of total probability

$$P(X_1 + X_2 + X_3 = 9) = \sum_{i=1}^{6} P(X_1 + X_2 + X_3 = 9 \mid X_1 = i)\, P(X_1 = i)$$

$$= \sum_{i=1}^{6} P(X_2 + X_3 = 9 - i \mid X_1 = i) \cdot \frac{1}{6} = \frac{1}{6} \sum_{i=1}^{6} P(X_2 + X_3 = 9 - i)$$

since $X_2 + X_3$ is indep of X_1.

$$= \frac{1}{6}\Big(P(X_2 + X_3 = 8) + P(X_2 + X_3 = 7) + P(X_2 + X_3 = 6) + P(X_2 + X_3 = 5) + P(X_2 + X_3 = 4) + P(X_2 + X_3 = 3)\Big)$$

$$= \frac{1}{6}\left(\frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36}\right) = \frac{25}{216} \approx .1157.$$

(b) If we naively apply the Central Limit theorem using $\mu = E(X_1) = 3.5$ and $\sigma^2 = Var(X_1) = \frac{35}{12} \Rightarrow \sigma = 1.707825\ldots$ then

$$P(8.5 \le X_1 + X_2 + X_3 \le 9.5) = P\left(\frac{8.5 - 3(3.5)}{1.707825\sqrt{3}} \le \frac{S_3 - n\mu}{\sigma\sqrt{n}} \le \frac{9.5 - 3(3.5)}{1.707825\sqrt{3}}\right)$$

$$= P\left(-.68 \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le -.34\right) \approx \Phi(-.34) - \Phi(-.68) =$$

$$\Phi(.68) - \Phi(.34) = .1186.$$

Note that this is a fairly good approximation!

$\boxed{A.13.5}$ First of all

$$\mu = E(X_1) = \int_0^1 x(2x^3-2x+1)\,dx = \int_0^1 2x^4-2x^2+x\,dx = \frac{2}{5}-\frac{2}{3}+\frac{1}{2} = \frac{7}{30}.$$

$$E(X_1^2) = \int_0^1 x^2(2x^3-2x+1)\,dx = \int_0^1 2x^5-2x^3+x^2\,dx = \frac{2}{6}-\frac{1}{2}+\frac{1}{3} = \frac{1}{6}.$$

$$\sigma^2 = E(X_1^2) - \{E(X_1)\}^2 = \frac{1}{6} - \left(\frac{7}{30}\right)^2 = \frac{101}{900} \Rightarrow \sigma = .334996\ldots$$

$$P(\bar{X} > \tfrac{1}{3}) = P\left(\frac{S_n}{n} > \frac{1}{3}\right) = P\left(\underbrace{\frac{\frac{S_n}{n}-\mu}{\frac{\sigma}{\sqrt{n}}}}_{=\frac{S_n-n\mu}{\sigma\sqrt{n}}} > \frac{\frac{1}{3}-\frac{7}{30}}{\frac{.334996}{\sqrt{90}}}\right)$$

$$= P\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} > 2.83\right) \approx 1 - \Phi(2.83) = .0023.$$

$\boxed{A.13.6.}$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}e^{-x}\,dx = \int_0^\infty e^{-x(1-t)}\,dx < \infty \text{ when } t<1.$$

$$= -\frac{e^{-x(1-t)}}{1-t}\Bigg]_{x=0}^{\infty} = \frac{1}{1-t} = (1-t)^{-1}.$$

Let $X_1, X_2, \ldots$ be independent Exp(1), so that $\mu = 1$ and $\sigma = 1$.

$$M_{Y_n}(t) = E(e^{tY_n}) = E\left(e^{t\left(\frac{S_n-n\mu}{\sigma\sqrt{n}}\right)}\right) = E\left(e^{t\left(\frac{S_n-n}{\sqrt{n}}\right)}\right) = e^{-\frac{tn}{\sqrt{n}}}E\left(e^{\frac{t}{\sqrt{n}}S_n}\right)$$

$$= e^{-\frac{tn}{\sqrt{n}}}\left(1-\frac{t}{\sqrt{n}}\right)^{-n} = \left(e^{\frac{t}{\sqrt{n}}}\left(1-\frac{t}{\sqrt{n}}\right)\right)^{-n} = \left(\left(1+\frac{t}{\sqrt{n}}+\frac{t^2}{2n}+\cdots\right)\left(1-\frac{t}{\sqrt{n}}\right)\right)^{-n}$$

$$= \left(1+\cancel{\frac{t}{\sqrt{n}}}+\frac{t^2}{2n}+\cdots\cancel{-\frac{t}{\sqrt{n}}}-\frac{t^2}{n}-\cdots\right)^{-n} = \left(1-\frac{t^2}{2n}+\underbrace{\cdots}_{\text{terms having } n^x \text{ in denominator with } x>1.}\right)^{-n}$$

$$= \left\{\left(1-\frac{t^2}{2n}+\cdots\right)^n\right\}^{-1} \text{ So that } \lim_{n\to\infty}\left\{\left(1-\frac{t^2}{2n}+\cdots\right)^n\right\}^{-1} = \left\{e^{-t^2/2}\right\}^{-1} = e^{t^2/2}$$

thus, Y_n has a distribution converging to that of a standard normal.

$\boxed{A.13.6}$ (continued)

$$P(S_{100} > 120) = P\left(\frac{S_{100}-100}{\sqrt{100}} > \frac{120-100}{10}\right)$$

$$= P\left(\frac{S_{100}-100}{10} > 2\right) \approx 1 - \Phi(2) = .0228.$$