Intro Prob Lecture Notes

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The Normal(μ, σ^2) pdf

- $$f(x) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} for - \infty < x < \infty$$
- When $\mu = 0$ and $\sigma = 1$ we call this the *standard normal pdf* and use the letter ϕ (Lowercase Greek 'phi')
  - Notation: In this class, $\mathbb{Z} \sim$ Normal(0, 1) *ALWAYS*
- $$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} for - \infty < x < \infty$$
- The corresponding cdf is given the symbol Φ (Uppercase Greek 'Phi')
- $$\Phi(x) = \int\limits_{-\infty}^{\infty} \phi(u)du$$
  - This cdf is tabulated for values of x between 0.00 and 3.49 in increments of .01 on page 190 of the textbook
  - Check:
    - $P(\mathbb{Z} \leq 2.17) = \Phi(2.17) = .9850$
    - $P(\mathbb{Z} \leq 1.00) = \Phi(1.00) = .8413$
    - $P(\mathbb{Z} \leq 0.00) = \Phi(0.00) = .5$
  - Since this is a normal distribution, $P(\mathbb{Z} \leq -2.17) = 1 - P(\mathbb{Z} \leq 2.17)$
  - And in general, $\Phi(x) = 1 - \Phi(-x)$

- Last time: If $X \sim \text{Normal}(\mu, \sigma^2)$, any $aX + b \sim \text{Normal}(a\mu + b, a^2\sigma^2)$
  - Example: $\mathbb{Z} \sim \text{Normal}(0,\ 1)$, $X = \sigma\mathbb{Z} + \mu \sim \text{Normal}(\mu, \sigma^2)$
  - Therefore, every random variable with a normal distribution is a linear transformation of a standard normal distribution and vice versa.

$$\implies \mathbb{Z} = \frac{X - \mu}{\sigma} \sim \text{Normal}(0,\ 1)$$

    * Z-score transformation (standardizing)

- Exercise: Show that both
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$$\mathbb{E}(\mathbb{Z}) = \int_{-\infty}^{\infty} z \cdot \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = 0$$

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$$\mathbb{E}(\mathbb{Z}^2) = \int_{-\infty}^{\infty} z \cdot \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = 1$$

The Euler Gamma Function

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$$\Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy$$

  - Not defined at 0, negative even integers. However, it is defined for everywhere else in the complex plane.
  - When $\alpha > 0$, it is defined in finite.
  - Note: Wikipedia has a beautiful drawing of this function
  - Generalizes the factorial
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$$\Gamma(1) = \int_{0}^{\infty} y^{0} e^{-y} dy = 1$$

    * Suppose $\alpha > 1$

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$$\begin{aligned}\Gamma(\alpha) &= y^{\alpha-1}e^{-y}dy \\ &\text{Let } u = y^{\alpha-1}, dv = e^{-y}dy \\ &= y^{\alpha-1}(-e^{-y})\Big|_0^\infty - \int_0^\infty -e^{-y}\cdot(\alpha-1)y^{\alpha-2}dy \\ &= (\alpha-1)\int_0^\infty y^{\alpha-2}e^{-y}dy \\ &= (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

  - This is known as the reduction property of $\Gamma(\alpha)$.
  - Ex: suppose $\alpha = 4$
    - $\Gamma(4) = 3 * \Gamma(3) = 3 * 2 * \Gamma(2) = 3 * 2 * 1 * \Gamma(1) = 3!$
  - Therefore, when $n > 0, n$ is integer, $\Gamma(n) = (n-1)!$
  - Tricky fact: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (see notes)
  - $\Gamma(\frac{7}{2}) = \frac{5\cdot3\cdot1}{2\cdot2\cdot2}\Gamma(\frac{1}{2}) = \frac{15\sqrt{\pi}}{8}$
  - $\Gamma(7.3) = (6.3)(5.3)(4.3)(3.3)(2.3)(1.3)(.3)\Gamma(.3)$

– Suppose $\alpha > 0$ and $\beta > 0$. Integrate

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$$\begin{aligned}\int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta}}dx &= \\ &\text{Let } u = \frac{x}{\beta} \to x = \beta u, dx = \beta du \\ &= \int_0^\infty (\beta u)^{\alpha-1}e^{-u}\beta du \\ &= \beta^\alpha\Gamma(\alpha)\end{aligned}$$

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$$\int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta}}dx = \beta^\alpha\Gamma(\alpha)$$

– Be smart:

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$$\int_0^\infty x^3 e^{-\frac{x}{2}}\,dx = w^4\Gamma(4) = 16(6) = 96$$

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$$\int_0^\infty x^2 e^{-2x} = \frac{1}{2}^3\Gamma(3) = \frac{1}{4}$$

- We define

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$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}\text{for } x > 0$$

- As the Gamma(α, β) pdf
  * α is the shape parameter
  * β is the scale parameter

- Plug in $\alpha = 1$ in the Gamma(α, β) pdf

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$$f(x) = \frac{e^{-\frac{x}{\beta}}}{\beta} = \frac{1}{\beta}e^{-\frac{x}{\beta}}\text{for } x > 0$$

- Suppose $n > 1$ integer takes $\alpha = \frac{n}{2}$ and $\beta = 2$

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$$f(x) = \frac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \sim \chi^2_n$$

  * Chi-square with n degrees of freedom