Intro Prob Lecture Notes

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Last Time: Gamma PDF

- $X \sim \text{Gamma}(\alpha, \beta)$ where $\alpha > 0, \beta > 0$ has pdf
- $$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$
  - For $x > 0$
  - The denominator is the normalizing constant.
  - We usually aren't interested in finding the area under the curve, but we may later come back to compute tail probabilities.
- Remark: By recognizing this pdf in one form another and using the fact
- $$\int_0^{\infty} x^{\alpha-1}e^{-\frac{x}{\beta}}\,dx = \beta^{\alpha}\Gamma(\alpha)$$
- Will allow us to compute $\mathbb{E}(X^n)\ \forall\ n$.

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$$\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty x \cdot \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}dx \\
&= \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\frac{x}{\beta}}dx \\
&= \frac{1}{\beta^\alpha\Gamma(\alpha)}\beta^{\alpha+1}\Gamma(\alpha+1) \\
&\text{This substitution is the "Normalization trick" - you don't always have to integrate it out} \\
&= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)} \\
&= \frac{\beta^{\alpha+1}\alpha\Gamma(\alpha)}{\beta^\alpha\Gamma(\alpha)} \\
&= \alpha\beta
\end{aligned}$$

- Exercise: Show $\mathbb{E}(X^2) = \alpha(\alpha+1)\beta^2$ by using the normalization trick

Beta-family of pdf's

- Important for ordered statistics, Bayesian statistics
- We say $X \sim \text{Beta}(\alpha, \beta)$ if X has the pdf with $\alpha > 0, \beta > 0$
-
$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}$$
  - for $0 < x < 1$.
- Where, in fact, $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ (show this for extra credit)
- Therefore the pdf is
-
$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$
  - for $0 < x < 1$.
- Exercise: Show $\mathbb{E}(x) = \frac{\alpha}{\alpha+\beta}$

CDF technique

- Suppose $X \sim f_X(x)$
  - "X is a random variable that has $f_X(x)$ as its pdf"
- And $Y = g(X)$. What is the pdf of Y?
- Example 1: Suppose $Z \sim$ Normal(0, 1)
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$$\phi = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \text{ for } -\infty < x < \infty$$
  - Find the pdf of $Y = Z^2$
  - Step 1: Compute the cdf of Y in terms of the cdf of X
  - Step 2: Take a derivative and use the chain rule
  - Step 1
    * Assume $y \geq 0$
    *
$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(Z^2 \leq y) \\
&= P(|Z| \leq \sqrt{y}) \\
&= P(-\sqrt{y} \leq Z \leq \sqrt{y}) \\
&= P(-\sqrt{y} < Z \leq \sqrt{y}) \\
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})
\end{aligned}$$
  -

Step 2

$$
\begin{aligned}
\frac{d}{dy}F_Y(y) &= f_Y(y) \\
&= \frac{d}{dy}(\Phi(\sqrt{y}) - \Phi(-\sqrt{y})) \\
&= \phi(\sqrt{y})\frac{d}{dy}(\sqrt{y}) - \phi(-\sqrt{y})\frac{d}{dy}(-\sqrt{y}) \\
&= \phi(\sqrt{y})\frac{d}{dy}(\sqrt{y}) + \phi(\sqrt{y})\frac{d}{dy}(\sqrt{y}) \\
&= \dots \\
&= 2 \cdot \phi(\sqrt{y})\frac{1}{2\sqrt{y}} \\
&= \frac{1}{\sqrt{y}}\phi(\sqrt{y}) \\
&= \frac{1}{y^{\frac{1}{2}}} \cdot \frac{e^{-\frac{y}{2}}}{\sqrt{2\pi}} \\
&= \frac{y^{-\frac{1}{2}}e^{-\frac{y}{2}}}{\sqrt{2\pi}} \text{ for } y > 0 \\
&= \text{Gamma}(\frac{1}{2}, 2) \sim \chi_1^2
\end{aligned}
$$

- Example 2: Suppose $Y = g(X)$ and g is strictly monotone
  - Claim
  - $$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$$
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Case 1: g strictly increasing

$$
\begin{aligned}
F_Y(y) &= P(g(X) \leq y) \\
&= P(X \leq g^{-1}(y)) \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

    * $f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y))$
  -

Case 2: G decreasing

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= P(X \le g^{-1}(y)) \\ &= F_X(g^{-1}(y)) \\ &= 1 - F_X(g^{-1}(y)) \end{aligned}$$

$*$ $f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{d}{dy}(-g^{-1}(y)) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}(-g^{-1}(y))\right|$

Weibull distribution

- $X \sim \exp(1)$ $f_X(x) = e^{-x}$ for $x > 0$
- Find pdf of $Y = \nu + \alpha X^{\frac{1}{\beta}}$ $(\nu \in \mathbb{R}, \alpha > 0, \beta > 0)$