550.420 Intro. to Probability - Spring 2017 **Homework #10**
Due by Friday, April 14 in lecture.

From the textbook:
Chapter 6 / Problems: 6.10, 6.19, 6.38, 6.40, 6.41, 6.43, 6.48; Theoretical exercises: 6.14

Additional problems:
A.10.1. If X_1 and X_2 are independent geometric(p) random variables, show that $X_1 + X_2$ has a negative-binomial($2, p$) distribution: recall for integer $r \geq 1$ a discrete rv W has the negative-binomial(r, p) provided the pmf of W is $p(w) = \binom{w-1}{r-1}p^r(1-p)^{w-r}$ for $w = r, r+1, r+2, \ldots$.

A.10.2. An amplifier uses a component whose time until failure has an exponential distribution with mean β, i.e, pdf $f(x) = \frac{1}{\beta}e^{-x/\beta}$ for $x > 0$. Once this component fails it is immediately (and instantaneously) replaced by an identical component with identical lifetime distribution. If X_i represents the lifetime of the ith component, and we have a total of n replacements (including the one already in the amplifier), what is the distribution of the total lifetime $\sum_{i=1}^{n} X_i$ of the amplifier? Hint: first find the distribution of $Y_1 = X_1 + X_2$. Then find the distribution of $Y_1 + X_3$, etc. Find a pattern.

A.10.3. (continued) A special case of **A.10.2** has if X, Y are independent exp(1), then $X + Y$ has a Gamma(2, 1). If $u > 0$ is given, find the conditional pdf $f_{X|X+Y}(x|u)$ of X given $X + Y = u$.
Hint: $f_{X|X+Y}(x|u) = \frac{f_{X,Y}(x,u-x)}{f_{X+Y}(u)}$... do you see why?

A.10.4. (The multinomial distribution) We generalize the binomial distribution: Suppose we have n independent and identical trials where each trial can result in any one of r possible outcomes, the probability a trial results in outcome i ($i = 1, 2, \ldots, r$) is p_i which is the same from trial to trial (think of rolling an r-sided die n times). If we let X_i count the number of trials that result in outcome i, then the vector $(X_1, X_2, \ldots, X_r)$ has the so-called multinomial distribution:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{x_1!x_2!\cdots x_r!}p_1^{x_1}p_2^{x_2}\cdots p_r^{x_r},$$

$x_i \geq 0$ are integer, $\sum_{i=1}^{r} x_i = n$ and $\sum_{i=1}^{r} p_i = 1$. Three important facts about the multinomial are that (1) the above is a pmf - this is a multinomial theorem, (2) the random variables X_i are dependent, and (3) for each i, X_i has a binomial(n, p_i) distribution. A hard way of showing fact (3) is to compute the marginal of X_i by summing out all possible values of x_j for $j \neq i$. Find a much easier explanation for why X_i is binomial(n, p_i).

A.10.5. (a) Suppose $X_1, X_2, \ldots, X_n$ are independent uniform(0, 1) random variables. Let $Y_n := \max\{X_1, X_2, \ldots, X_n\}$. Show that the cdf of Y_n is $F_n(y) = y^n$ for $0 \leq y \leq 1$.
(b) Suppose $X_1, X_2, \ldots, X_n$ are independent uniform(0, 1) rvs. Let $Y_1 := \min\{X_1, X_2, \ldots, X_n\}$. Show that the cdf of Y_1 is $F_1(y) = 1 - (1-y)^n$ for $0 \leq y \leq 1$.
(c) Compute $E(Y_n)$ and $E(Y_1)$.
(d)* Suppose $X_1, X_2, X_3, \ldots, X_n, \ldots$ is an i.i.d. sequence of uniform(0, 1) rvs. For each fixed n, find a formula for the cdf of $W_n := n\min\{X_1, \ldots, X_n\}$, that is, $G_n(w) = P(n\min\{X_1, \ldots, X_n\} \leq w)$ for $0 < w < n$. Then show that for any $w > 0$, that $\lim_{n\to\infty} G_n(w) = 1 - e^{-w}$ and conclude that as n gets large the random variables W_n have a distribution that is converging to a unit exponential.

A.10.6. Consider the jointly continuous rvs X, Y having joint pdf $f(x, y) = e^{-y}$ for $0 < x < y < \infty$.
(a) Compute the marginal pdf of X and the marginal pdf of Y.
(b) Determine whether X, Y are independent or not.
(c) Compute the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$.
(d) Evaluate $P(Y > 2|X = 1)$. Also evaluate $P(Y > 2|X > 1)$.