1. Suppose that X is a continuous random variable having pdf $f(x) = \begin{cases} \frac{3x^2}{2} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

(a) Compute the cdf $F_X(x)$ of X. Recall that (just as a pdf) a cdf should be defined on the entire real line.

For $x < -1$, $f(x) = 0$ which implies $F_X(x) = 0$.

For $-1 \le x \le 1$, $f(x) = \frac{3}{2}x^2$ which implies $F_X(x) = \int_{-1}^{x} \frac{3}{2}u^2\,du = \frac{u^3}{2}\Big|_{u=-1}^{u=x} = \frac{1}{2} + \frac{x^3}{2}$.

For $x > 1$, $F_X(x) = 1$ since all the probability mass under this pdf would have been accumulated already.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1+x^3}{2} & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$.

(b) Compute $P(0 < X \le \frac{1}{2})$. $P(0 < X \le \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(0) = (\frac{1+(\frac{1}{2})^3}{2}) - (\frac{1+(0)^3}{2}) = \frac{1}{16}$.

2. Suppose that X has a Gamma(α, β) distribution where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

(a) By performing an appropriate integration, *clearly* show why $E(X) = \alpha\beta$.

$E(X) = \int_0^\infty x \cdot \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}\,dx = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{(\alpha+1)-1}e^{-x/\beta}\,dx = \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \frac{x^{(\alpha+1)-1}e^{-x/\beta}}{\beta^{\alpha+1}\Gamma(\alpha+1)}\,dx = \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)}$

since the last integral is 1 because $\frac{x^{(\alpha+1)-1}e^{-x/\beta}}{\beta^{\alpha+1}\Gamma(\alpha+1)}$ is a pdf on $0 < x < \infty$. But then by the reduction property of the Euler Gamma function: $E(X) = \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$.

(b) If we further assume $\alpha > 1$, compute $E(\frac{1}{X})$. *For an extra bonus point*, why assume $\alpha > 1$?

In a very similar way

$E(\frac{1}{X}) = \int_0^\infty \frac{1}{x} \cdot \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}\,dx = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{(\alpha-1)-1}e^{-x/\beta}\,dx = \frac{\beta^{\alpha-1}\Gamma(\alpha-1)}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \frac{x^{(\alpha-1)-1}e^{-x/\beta}}{\beta^{\alpha-1}\Gamma(\alpha-1)}\,dx = \frac{\beta^{\alpha-1}\Gamma(\alpha-1)}{\beta^\alpha\Gamma(\alpha)}$

$= \frac{\Gamma(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha-1)} = \frac{1}{(\alpha-1)\beta}$.

Since $\Gamma(\alpha - 1)$ is only defined when $\alpha - 1 > 0$, we see this is equivalent to $\alpha > 1$.

3. Suppose that X and Y are jointly discrete random variables having the following joint pmf:

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$	
$x=1$	.30	.18	.12	$p_X(1) = .6$
$x=2$	.20	.12	.08	$p_X(2) = .4$
	$p_Y(1) = .5$	$p_Y(2) = .3$	$p_Y(3) = .2$	

(a) Clearly *verify* whether or not X and Y are independent. Be sure to state your conclusion.

X and Y are independent since $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for every $x = 1, 2$ and $y = 1, 2, 3$. Check:

$.3 = .5 \times .6,\ .18 = .3 \times .6,\ .12 = .2 \times .6,\ .2 = .5 \times .4,\ .12 = .3 \times .4,\ .08 = .2 \times .4.$

(b) Compute

(i) $P(Y > X) = p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(2,3) = .18 + .12 + .08 = .38$.

(ii) $P(Y = 1) = p_Y(1) = .5$.

(iii) $P(X = 1|Y <= 2) = P(X = 1) = .6$ since X and Y are independent.

(c) Compute

(i) $E(X) = 1 \times .6 + 2 \times .4 = 1.4$.

(ii) $E(Y|X = 1) = E(Y) = 1 \times .5 + 2 \times .3 + 3 \times .2 = 1.7$.

4. (a) If Z is a standard normal random variable, compute $P(-2.1 < Z < 2.1)$.
From the standard normal table: $P(-2.1 < Z < 2.1) = \Phi(2.1) = \Phi(-2.1) = \Phi(2.1) - (1 - \Phi(2.1))$
$= 2\Phi(2.1) - 1 = 2(.9821) - 1 = .9642$.
(b) Suppose W represents the score on a certain exam. Assume that W is normally distributed having a mean $\mu = 50$ points and variance $\sigma^2 = 400$ points^2 (i.e., $\sigma = 20$ points). Compute the probability that W will be between 8 and 92 inclusive.
$P(8 \le W \le 92) = P(\frac{8-50}{20} \le \frac{W-\mu}{\sigma} \le \frac{92-50}{20}) = P(-2.1 \le Z \le 2.1) = .9642$ (from part (a) since Z is a continuous random variable the two probabilities are the same.

5. Suppose $U \sim \text{uniform}(0, \frac{1}{2})$, i.e., U is a continuous random variable with pdf $f(x) = 2$ for $0 < x < \frac{1}{2}$. Compute the n-th moment of U, i.e., $E(U^n)$, where $n \ge 0$ is an integer.
For any $n \ge 0$, $E(U^n) = \int_0^{1/2} u^n \cdot 2\, du = \frac{2u^{n+1}}{n+1}\Big|_{u=0}^{u=1/2} = \frac{1}{2^n(n+1)}$.

(*bonus question*) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n(n+1)} = 2\ln(2)$ by computing $E(\frac{1}{1-U})$ two different ways.
On the one hand, $E(\frac{1}{1-U}) = \int_0^{1/2} \frac{1}{1-u} \cdot 2\, du = -2\ln(1-u)|_{u=0}^{u=1/2} = 2\ln(2)$.
On the other hand, since $|U| \le \frac{1}{2} < 1$, we have $\sum_{n=0}^{\infty} U^n = \frac{1}{1-U}$ so $E(\frac{1}{1-U}) = E(\sum_{n=1}^{\infty} U^n)$
$= \sum_{n=0}^{\infty} E(U^n) = \sum_{n=0}^{\infty} \frac{1}{2^n(n+1)}$.
Therefore, $\sum_{n=0}^{\infty} \frac{1}{2^n(n+1)} = 2\ln(2)$.

6. Suppose $X|Y = y \sim \text{uniform}(0, y)$ and $Y \sim \text{Gamma}(2, 1)$. That is,

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} ye^{-y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the marginal pdf of X. *Be careful with your ranges of integration!*
First we find the joint pdf of X and Y: $f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) = \frac{1}{y} \cdot ye^{-y} = e^{-y}$ for $0 < x < y, y > 0$.
Therefore, the marginal of X is $f_X(x) = \int_x^{\infty} e^{-y}\, dy = e^{-x}$ for $x > 0$; and, $f_X(x) = 0$ for $x \le 0$.
(b) Compute *only one* of the following (you choose): $P(3 < X < 7|Y = 10)$ *or* $P(3 < X < 7|Y \le 10)$. *Do not compute both!*
The easiest one to compute is the first one: $P(3 < X < 7|Y = 10) = \int_3^7 f_{X|Y}(x|10)\, dx = \int_3^7 \frac{1}{10}\, dx = .4$.

7. Suppose X_1 and X_2 are independent random variables with $X_i \sim \text{Poisson}(\lambda_i)$ for $i = 1, 2$. Suppose we observe $X_1 + X_2 = n$. Find the probability that $X_1 = x$, that is, compute $P(X_1 = x|X_1 + X_2 = n)$ for appropriate x.
Feel free to use any results you may recall from homework about sums of independent Poisson random variables.
I will use the fact that a sum of independent Poisson random variables is again a Poisson with the parameters adding; i.e., $X_1 + X_2$ has a $\text{Poisson}(\lambda_1 + \lambda_2)$ distribution. In this case, for any $x = 0, 1, \ldots, n$ (x cannot be bigger than n)

$$\begin{aligned} P(X_1 = x|X_1 + X_2 = n) &= \frac{P(X_1 = x, X_1 + X_2 = n)}{P(X_1 + X_2 = n)} = \frac{P(X_1 = x, X_2 = n - x)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\ &= \frac{P(X_1 = x)P(X_2 = n - x)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} = \frac{\frac{e^{-\lambda_1}\lambda_1^x}{x!} \frac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\ &= \frac{n!}{x!(n-x)!} \cdot (\frac{\lambda_1}{\lambda_1 + \lambda_2})^x (\frac{\lambda_2}{\lambda_1 + \lambda_2})^{n-x} \end{aligned}$$

i.e., the conditional distribution is binomial with parameters n and $p = \lambda_1/(\lambda_1 + \lambda_2)$.