**550.420 Intro. to Probability - Spring 2017** **Homework #12**
**Due: in lecture, Friday, April 28.**

**From the textbook:**
Chapter 7 / Problems: 7.1, 7.6, 7.11, 7.14*, 7.30, 7.31, 7.41, 7.42 7.45
Theoretical exercises: 7.1, 7.4, 7.21
* Also, compute the variance. For this problem if we let $N$ represent the number of stages needed to eliminate all the black balls, then the idea is to write $N$ as a sum of random variables, i.e., $N = \sum_i X_i$, and apply linearity to compute the expected value.

**Additional problems:**
**A.12.1.** $X, Y$ are jointly continuous with joint pdf

$$f(x,y) = e^{-y} \quad \text{for } 0 < x < y < \infty.$$

(a) Compute the covariance between $X$ and $Y$.
(b) Compute $\rho_{X,Y}$, i.e., the correlation coefficient of $X$ and $Y$.
(c) You what you've already computed to find the value of $Var(X+Y)$. How about $Var(X-Y)$?
(d) Compute $E(X|Y=y)$ and $E(Y|X=x)$.

**A.12.2.** Suppose we have a (doubly infinite) sequence $\ldots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \ldots$ of independent normal rvs with mean 0 and variance $\sigma^2$, and let $\theta$ be a real number satisfying $0 < |\theta| < 1$. Consider the following sequence of random variables generated from the $Z$-process: for each $n$, $X_n = Z_n + \theta Z_{n-1}$.
(a) Compute the mean and variance of $X_n$. Notice that your answer doesn't depend on $n$.
(b) Compute $Cov(X_n, X_{n-1})$ and $\rho_{X_n, X_{n-1}}$. Again, your answer shouldn't depend on $n$.
(c) If $h \geq 2$ is an integer, compute $Cov(X_n, X_{n-h})$ and $\rho_{X_n, X_{n-h}}$.
In statistics, the process $(X_t)$ is called a *moving average process of lag 1.*

**A.12.3.** (a) If $X$ and $Y$ are independent with respective means $\mu_X$, $\mu_Y$ and variances $\sigma_X^2, \sigma_Y^2$, then
(i) find $Var(X+Y)$.
(ii) find $Var(X-Y)$. Why does this not contradict A.12.1(c)?
(b) Let $X$ and $Y$ be *any* random variables having the same variance $\sigma^2$ (but possibly different means). Compute $Cov(X+Y, X-Y)$.
(c) $X_1, X_2, \ldots, X_n$ are i.i.d. random variables each having mean $\mu$ and variance $\sigma^2$. Find $Var(\sum_{i=1}^n X_i)$ and find $Var(\overline{X})$, where the sample mean $\overline{X}$ is defined as $(X_1 + X_2 + \cdots + X_n)/n$.

**A.12.4.** Show that if $a$ and $b$ are constants, then $Cov(X+a, Y+b) = Cov(X,Y)$.

**A.12.5.** (The bivariate normal) Suppose $X, Y$ are bivariate normal with parameters $\mu_X, \mu_y, \sigma_X^2, \sigma_Y^2$, and $\rho$. I.e., the joint pdf of $X, Y$ is given by

$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

(a) Show that $E(X) = \mu_X$, $E(Y) = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$, $Cov(X,Y) = \rho\sigma_X\sigma_Y$ so that $\rho_{X,Y} = \rho$. There is an easy way without having to do any integrations.
(b) Identify the conditional distribution of $Y$ given $X = x$ (details) and then from it find $E(Y|X=x)$.
(c) Identify the conditional distribution of $X$ given $Y = y$ (details) and then from it find $E(X|Y=y)$.