Homework #5

A.5.1

(a) $\dfrac{\binom{6}{3}}{6^3} = \dfrac{5}{54}$ This is because subsets of size 3 from 6 are in one-to-one correspondence with 3 increasing (distinct) values from 6.

(b) $\dfrac{\binom{n}{3}}{n(n-1)(n-2)} = \dfrac{n(n-1)(n-2)}{3!\, n(n-1)(n-2)} = \dfrac{1}{3!}$

(c) Experiment #1: $\dfrac{\binom{n}{k}}{(n)_k} = \dfrac{1}{k!}$

Experiment #2: $\dfrac{\binom{n}{k}}{n^k} = \dfrac{1}{k!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)$

and it is more likely to experience increasing values in experiment #1 (the case of sampling without replacement)

A.5.2

(a) $\binom{16}{4,4,4,4}\Big/4!$

(b) $\dfrac{\binom{12}{4,4,4}\Big/3!}{\binom{16}{4,4,4,4}\Big/4!} \approx .002198$

(c) If we let G_i be the event that all members of group i form a team, then

$$P(G_i) = \frac{\binom{12}{4,4,4}/3!}{\binom{16}{4,4,4,4}/4!} \quad \text{for each } i = 1, 2, 3, 4.$$

$i<j$

$$P(G_i \cap G_j) = \frac{\binom{8}{4,4}/2!}{\binom{16}{4,4,4,4}/4!} \quad \text{for } 1 \le i < j \le 4.$$

and

$$P(G_i \cap G_j \cap G_k) = \frac{\binom{4}{4}/1!}{\binom{16}{4,4,4,4}/4!} \quad \text{for } 1 \le i < j < k \le 4$$

and

$$P(G_1 \cap G_2 \cap G_3 \cap G_4) = \frac{1}{\binom{16}{4,4,4,4}/4!} .$$

Now, by the inclusion exclusion principle

$$P\left(\bigcup_{i=1}^{4} G_i\right) = \binom{4}{1} \cdot \frac{\binom{12}{4,4,4}/3!}{\binom{16}{4,4,4,4}/4!} - \binom{4}{2} \frac{\binom{8}{4,4}/2!}{\binom{16}{4,4,4,4}/4!} + \binom{4}{3} \frac{1}{\binom{16}{4,4,4,4}/4!} - \binom{4}{4} \frac{1}{\binom{16}{4,4,4,4}/4!}$$

$$\approx .008712$$

(d) $4\cdot4\cdot4\cdot4\cdot3\cdot3\cdot3\cdot3\cdot2\cdot2\cdot2\cdot2\cdot1\cdot1\cdot1\cdot1 = (4!)^4$ is the number of ways we can have all the teams ~~have~~ be ~~exactly~~ comprised of exactly one person from each group where the teams are ordered. To remove the ordering of the teams we divide the above by $4!$. Thus, there are

$$(4!)^3$$

ways to divide 16 people into 4 teams where each team has exactly one member from each group.

Therefore, the desired probability is

$$\frac{(4!)^3}{\binom{16}{4,4,4,4}/4!} \approx .005261$$

$\boxed{A.5.3}$

(a) $$\frac{\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}}{\binom{n}{k}}.$$

(b) $$\frac{\binom{13}{1}\binom{13}{2}\binom{13}{3}\binom{13}{4}}{\binom{52}{10}} \approx .0131.$$

$\boxed{A.5.4}$ This is a partition of integers problem!

(a) $$\binom{10+4-1}{4-1} = \binom{13}{3} = 286.$$

(b) If we put \$1 into each slot we have \$6 one dollar bills left to divvy up into 4 slots

$$\binom{6+4-1}{4-1} = \binom{9}{3} = 84$$

(c) If we put two \$1 bills into each slot we have 2 one dollar bills remaining to be divvied up into 4 slots:

$$\binom{2+4-1}{4-1} = \binom{5}{3} = 10$$

(d) $P(\text{some slot receives 0 dollars})$

$$= 1 - P(\text{all slots receive} \geq \$1.00)$$

$$= 1 - \frac{\binom{9}{3}}{\binom{13}{3}} = 1 - \frac{84}{286} \approx .7063$$

(e) $P\left(\text{slot 1 has } \$1 \,\middle|\, \begin{array}{l}\text{all slots have}\\ \text{at least } \$1\end{array}\right)$

$$= \frac{P(\text{slot 1 has } \$1)}{P\left(\begin{array}{l}\text{all slots have at}\\ \text{least } \$1\end{array}\right)} = \frac{\dfrac{\binom{11}{2}}{\binom{13}{3}}}{\dfrac{\binom{9}{3}}{\binom{13}{3}}}$$

slot 1 has \$1 means 9 one dollar bills to be distributed among 3 slots. This can be done in $\binom{9+3-1}{3-1} = \binom{11}{2}$ ways

$$= \frac{\binom{11}{2}}{\binom{9}{3}} \approx .65476.$$

$\boxed{A.5.1}$

(a) For $i=1,2,\ldots,n$ it is clear that $1 \le 2i-1 \le 2n-1$ and therefore, $\frac{2i-1}{n^2} > 0$.

Furthermore,

$$\sum_{i=1}^{n} \frac{(2i-1)}{n^2} = \frac{1}{n^2}\left(1+3+5+\cdots+(2n-1)\right)$$

$$= \frac{1}{n^2}(n^2) = 1.$$

$i=1 \quad 1 = 1^2$

$i=2 \quad 1+3=4=2^2$

$i=3 \quad 1+3+5=9=3^2$

$i=4 \quad 1+3+5+7=16=4^2$

$\vdots$

$i=n \quad 1+3^2+5^2+\cdots+2n-1 = n^2$

So $p(x) = P(X=x)$ for $x=1,2,\ldots,n$ is a pmf.

(b) $P(X \le 5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$$= \frac{1}{100}+\frac{3}{100}+\frac{5}{100}+\frac{7}{100}+\frac{9}{100}$$

$$= \frac{25}{100} = \frac{1}{4}.$$

$P(X \ge 5) = P(X=5)+P(X=6)+P(X=7)+P(X=8)+P(X=9)+P(X=10)$

$$= \frac{9}{100}+\frac{11}{100}+\frac{13}{100}+\frac{15}{100}+\frac{17}{100}+\frac{19}{100} = \frac{84}{100}$$

$= .84$

$P(X=5) = \frac{9}{100} = .09$

(c) $P(X \le i) = \dfrac{1 + 3 + 5 + \cdots + (2i-1)}{n^2}$

$= \dfrac{i^2}{n^2}$

(d) $P(X \text{ is odd}) = P(X=1) + P(X=3) + P(X=5) + \cdots + P(X=n-1)$

$= \dfrac{1}{n^2} + \dfrac{5}{n^2} + \dfrac{9}{n^2} + \cdots + \dfrac{4n-3}{n^2}$

$= \dfrac{1 + 5 + 9 + \cdots + (4n-3)}{n^2} = \dfrac{\binom{n}{2}}{n^2}$

$n=2 \quad 1 = 1$

$n=4 \quad 1+5 = 6 = 2 \cdot 3 = \frac{n}{2} \cdot (n-1)$

$n=6 \quad 1+5+9 = 15 = 3 \cdot 5 = \frac{n}{2}(n-1)$

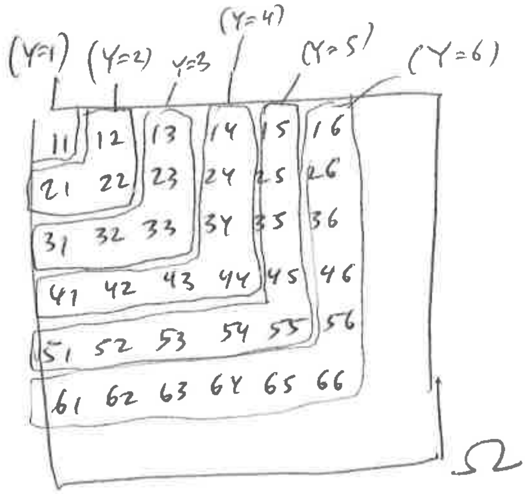
$n=8 \quad 1+5+9+13 = 28 = 4 \cdot 7 = \frac{n}{2}(n-1)$

$n=10 \quad 1+5+9+13+17 = 45 = 5 \cdot 9 = \frac{n}{2}(n-1)$

generally,

$1+5+9+\cdots+(4n-3) = \frac{n}{2} \cdot (n-1) = \binom{n}{2}$

A.5.6



(a)

$P(Y=1) = \frac{1}{36}$

$P(Y=2) = \frac{3}{36}$

$P(Y=3) = \frac{5}{36}$

$P(Y=4) = \frac{7}{36}$

$P(Y=5) = \frac{9}{36}$

$P(Y=6) = \frac{11}{36}$

Thus in tabular form the pmf of Y is

y	1	2	3	4	5	6
$p(y)=P(Y=y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

in functional form

$$p_Y(y) = P(Y=y) = \frac{2y-1}{36} \text{ for } y = 1,2,3,4,5,6.$$

(b) $P(Y \le 1) = \frac{1}{36}$

$P(Y \le 2) = \frac{1}{36} + \frac{3}{36} = \frac{4}{36}$

$P(Y \le 3) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} = \frac{9}{36}$

$P(Y \le 4) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{7}{36} = \frac{16}{36}$

$P(Y \le 5) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{7}{36} + \frac{9}{36} = \frac{25}{36}$

$P(Y \le 6) = 1$

(c) $P((Y-3)^2 - Y \ge 1)$

$= P(Y=1)$

$+ P(Y=6) = \frac{1}{36} + \frac{11}{36}$

$= \frac{12}{36}$