Intro Prob Lecture Notes

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Law of the Unconscious Statistician (For Continuous Random Variables)

- If X is a continuous random variable with pdf $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ is any function such that

  $$\int_{-\infty}^{\infty} |g(x)| f(x) dx < \infty$$

  (This is a condition which will guarantee that the expected value exists and is finite.) then

- $$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- Example: Suppose $X \sim \exp(1)$
  - $f_X(x) = e^{-x}$ for x > 0, $f_X(x) = 0$ otherwise
  - Let's compute $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.
  - $\text{Supp}(X) := \{x : F_X(x) > 0\}$

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_0^{\infty} x e^{-x} dx \\ &= x(-e^{-x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-x} dx \\ &= \int_0^{\infty} e^{-x} dx \\ &= 1.\end{aligned}$$

(Remember integration by parts!)

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$$\begin{aligned}\mathbb{E}(X^2) &= \int_0^{\infty} x^2 e^{-x} dx \\ &= x^2(-e^{-x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-x} \cdot 2x dx \\ &= 2\int_0^{\infty} x e^{-x} dx \\ &= 2\end{aligned}$$

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$$\begin{aligned}Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= 2 - 1^2 \\ &= 1\end{aligned}$$

- Interesting and useful way to compute expectations when the random variable is non-negative: (Generalized)

- If $X \geq 0$ then

$$\mathbb{E}(X) = \int_0^\infty P(X > u)du$$

(proof later)

- For example, if $X \sim \exp(1)$ then first:

$$\begin{aligned} P(X > u) &= \int_u^\infty e^{-x}dx \\ &= e^{-x}\Big|_u^\infty \\ &= e^{-u} \end{aligned}$$

- Then:

$$\begin{aligned} \mathbb{E}(X) &= \int_0^\infty P(X > u)du = \mathbf{1 - F_X(u)} \\ &= \int_0^\infty e^{u}du \\ &= 1 \end{aligned}$$

- And

$$\begin{aligned} \mathbb{E}(X^2) &= \int_0^\infty P(X^2 > u)du \\ &= \int_0^\infty P(X^2 > \sqrt{u})du \\ &= \int_0^\infty e^{-\sqrt{u}}du \\ &= \int_0^\infty e^{-w} \cdot 2wdw \\ &= 2 \end{aligned}$$

The Normal Distribution (The Gaussian Distribution (In France, the Laplace Distribution))

- $X \sim$ Normal (μ, σ^2) if the pdf (for $-\infty < x < \infty$) is

$$f(x) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}}$$

  - (Trivially) μ is the average/$\mathbb{E}$, σ is the standard deviation / $\sqrt{Var}$
- If p is fixed, then as $n \to \infty$, Binomial$(n, p) \to$ Normal
  - Also Poisson(λ) as $\lambda \to \infty$
  - Central Limit Theorem
- Totals 1

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} dx = 1 \to \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} = 1$$

- $$I^2 = \int \frac{e^{-\frac{u^2}{2}}}{\sigma\sqrt{2\pi}} du \int \frac{e^{-\frac{v^2}{2}}}{\sigma\sqrt{2\pi}} dv = \int\int \frac{e^{-\frac{u^2+v^2}{2}}}{\sigma\sqrt{2\pi}} dudv = \cdots = 1$$

  (See Text)
- Theorem: If $X \sim$ Normal(μ, σ^2) and a, b are any constants, then
  - $Y = aX + b \sim$ Normal$(a\mu + b, a^2\sigma^2)$
  - If you have a normal random variable, any linear transformation on it is also a normal random variable.
  - *Consequence:* $X \sim$ Normal(μ, σ^2), then $\frac{X-\mu}{\sigma} \sim$ Normal(0, 1)
    * Any normal random variable can be converted into a *standard normal distribution* with a mean of 0 and a standard deviation of 1
  - Proof:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) \\ &= P(X \leq \frac{y-b}{a})(\text{Assuming } a > 0) \\ &= F_X(\frac{y-b}{a}) \end{aligned}$$

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\begin{aligned}
f_Y(y) &= \frac{d}{dy}(F_X(\frac{y-b}{a})) \\
&= f_X(\frac{y-b}{a}) \cdot \frac{1}{a} \\
&= \frac{1}{a}(\frac{e^{-\frac{1}{2}(\frac{\frac{y-b}{a}-u}{\sigma})^2}}{\sigma\sqrt{2\pi}}) \\
&= \frac{e^{-\frac{1}{2}(\frac{\frac{y-b}{a}-u}{\sigma})^2}}{a\sigma\sqrt{2\pi}}
\end{aligned}
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