Intro Prob Lecture Notes

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- Suppose X, Y are jointly continuous with joint pdf
- $f(x,y) = xy$ for $0 \leq x \leq 2, 0 \leq y \leq 1$, 0 elsewhere
  - Show that this is a joint pdf
    - When $(x,y) \in (\mathbf{0},\mathbf{2}) \times (\mathbf{0},\mathbf{1})$ we have, in particular, that $x > 0, y > 0$ Therefore, $f(x,y) = xy > 0$ in this part of the domain.
      - The bolded is the essential domain, or $supp(f)$
    - Furthermore, $f(x,y) = 0$ for $(x,y) \notin (0,2) \times (0,1)$.
    - Therefore, $f(x,y) \geq 0 \; \forall \; x, y \in \mathbb{R}$
    - $$\begin{aligned} \int_0^1 \int_0^2 xydxdy &= \int_0^1 (\frac{x^2y}{2}\Big|_{x=0}^{x=2})dy \\ &= \int_0^1 2ydy \\ &= y^2\Big|_0^1 \\ &= 1 \end{aligned}$$
    - So, the total area is also 1
  -

Compute $P(X \leq Y)$

$$\begin{aligned}
P(X \leq Y) &= \int \int_R f(x,y)dA \\
&= \int_0^1 \int_0^y xydxdy \\
&= \int_0^1 (\frac{x^2y}{2}\Big|_{x=0}^{x=y})dy \\
&= \int_0^1 \frac{y^3}{2}dy \\
&= \frac{y^4}{8}\Big|_{y=0}^{y=1} \\
&= \frac{1}{8}
\end{aligned}$$

- The marginal pdf of X
  - $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$
  - and the marginal pdf of Y
  - $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$
- $f_X(x) = 0$ when $x \notin (0,2)$
- When $x \in (0,2)$
-

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{0} 0dy + \int_0^1 xydy + \int_1^{\infty} 0dy \\
&= \frac{x}{2}
\end{aligned}$$

- Exercise: Show $f_Y(y) = 2y$ for $0 < y < 1$, 0 otherwise
- Aside: comments on *marginal pdf*

- If function is $f(x_1,x_2,x_3,x_4,x_5)$, $f_{x_1}(x_1) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x_1,x_2,x_3,x_4,x_5)dx_2dx_3dx_4dx_5$
    * Also extends to multivariate e.g. $f_{x_2,x_4}(x_2,x_4)$
- Integrate out everything that's not the thing you're interested in.

Jointly Distributed Random Variables with One Discrete and One Continuous

- Ex: $P_{N,X}(n,x) = n(\frac{1}{2})^n e^{-nx}$ for $n = 1,2,3,\ldots$, $x > 0$
    - Intuition: Flipping a balanced coin n times. If N = n, win $X \sim$ exp(n).
    - pmf in first variable
    - pdf in second variable
    - Just called a "joint distribution"
    - Find the marginal of X.
    -

$$\begin{aligned} f_X(x) &= \sum_{n=1}^{\infty} n(\frac{e^{-x}}{2})^n \\ &= \ldots \\ &= \frac{e^{-\frac{x}{2}}}{(1-e^{-\frac{x}{2}})^2} \text{ For } x > 0 \end{aligned}$$

- $P_N(n) = (\frac{1}{2})^n$ for $n = 1,2,\ldots$

Independence

- $X_1, X_2, \ldots X_n$ jointly distributed random variables
- We'll say they are independent if joint distribution = $\prod\limits_{i=1}^{n}$ marginal distribution
    - $p(x_1,x_2,\ldots x_n) = P_{X_1}(x_1)P_{X_2}(x_2)\ldots P_{X_n}(x_n) \; \forall \; x_1,x_2,\ldots x_n$
- the joint distribution of the collection is equal to the product of the marginal distributions in the collection