Suppose X is a continuous r.v. having pdf $f(x)$.
Then for any possible value of X, say, x and any $h>0$

$$P(x \le X \le x+h) = \int_x^{x+h} f(u)\,du \approx f(x)h \text{ when } h \text{ is small.}$$

Notice that as $h \to 0$, $P(x \le X \le x+h) \to 0$, too.

But it goes to 0 in such a way that

$$\frac{P(x \le X \le x+h)}{h} \longrightarrow f(x) \quad \text{as } h \to 0.$$

Important to note that $f(x)$ can be bigger than 1 - see example 3.3 on page 143 of text

also, $f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$ is a pdf but is bigger than 1

There are important named distributions for continuous r.v.'s

- the exponential(λ).
- the uniform(a, b)
- the Gamma(α, β)
- * the $\chi^2(n)$ ← pronounced the Chi-square distribution with n degrees of freedom
- * the Normal(μ, σ^2)
- * the F-distribution or $F(m,n)$ distribution
- * the Student's t-distribution with n degrees of freedom.

The ones marked with an asterisk (*) are especially important in normal random sampling theory in statistics.

We already introduced the exponential(λ) and the uniform

As with discrete r.v.s, Expected values (expectations) are important; however, when the r.v.s are continuous having pdf $f(x)$ we define it slightly differently:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Recall for discrete r.v.s $E(X) = \sum x \, p_X(x)$.

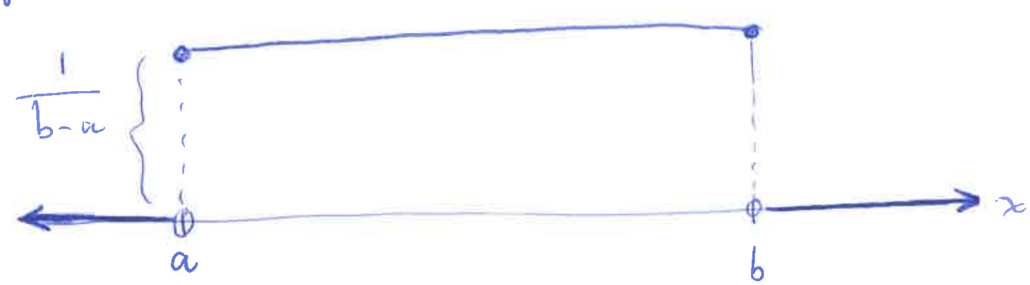
This definition is very much analogous to its discrete counterpart but now we weight the values of X against its density.

The Uniform(a,b).

$X \sim$ uniform(a,b) means X is continuous having the pdf

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

whose graph is...



Let's compute the mean and variance of X:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \left.\frac{x^2}{2}\cdot\frac{1}{b-a}\right]_{x=a}^{x=b}$$

since $f(x) = 0$
when $x \notin (a,b)$

$$= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{b+a}{2} \longleftarrow \text{the midpoint of the interval } (a,b)!$$

How about the second moment?

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \left.\frac{x^3}{3}\cdot\frac{1}{b-a}\right]_{x=a}^{x=b}$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \frac{b^2+ab+a^2}{3}$$

Therefore $Var(X) = E(X^2) - \{E(X)\}^2$

$$Var(X) = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} = \frac{4b^2+4ab+4a^2}{12} - \frac{3b^2+6ab+3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}. \quad \therefore \sigma_X = \frac{|b-a|}{\sqrt{12}}.$$

Ex. Compute the mean and variance of $X \sim \exp(\lambda)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underset{u}{x} \underbrace{(\lambda e^{-\lambda x})\,dx}_{dv}$$ and use integration by parts...

$$= -xe^{-\lambda x}\Big]_{x=0}^{x=\infty} - \int_0^{\infty} -e^{-\lambda x}\,dx$$

$$= 0 + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{e^{-\lambda x}}{\lambda}\Big]_{x=0}^{x=\infty} = 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}.$$

So $\mu = E(X) = \frac{1}{\lambda}$ for the $\exp(\lambda)$ distribution.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} \underset{u}{x^2} \underbrace{(\lambda e^{-\lambda x})\,dx}_{dv}$$

$$= -x^2 e^{-\lambda x}\Big]_{x=0}^{\infty} + 2\int_0^{\infty} x e^{-\lambda x}\,dx$$

$$= 0 + \frac{2}{\lambda}\int_0^{\infty} x(\lambda e^{-\lambda x})\,dx = \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}.$$

Therefore,

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

$\sigma_X = \frac{1}{\lambda}$ is the standard deviation.

Prelude to the Gamma(α, β) distribution

The function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \qquad \text{for } \alpha > 0$$

is called the Euler's Gamma function.

We will see that this function generalizes the notion of factorial to arbitrary positive real numbers.

Easy to show: $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1.$

Suppose $\alpha > 1$. Then

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x}\, dx \\
&= -x^{\alpha-1} e^{-x} \Big]_0^\infty + \int_0^\infty (\alpha-1)\, x^{(\alpha-1)-1} e^{-x}\, dx \\
&= 0 + (\alpha-1) \int_0^\infty x^{(\alpha-1)-1} e^{-x}\, dx
\end{aligned}$$

So that we have a "reduction" formula of the Euler Gamma function:

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \qquad \text{whenever } \alpha - 1 > 0.$$

Combining the reduction formula with the fact that

$\Gamma(1) = 1$ we have . . .

when $n > 1$ is an INTEGER.

$$
\begin{aligned}
\Gamma(n) &= (n-1)\,\Gamma(n-1) \\
&= (n-1)(n-2)\,\Gamma(n-2) \\
&= (n-1)(n-2)(n-3)\,\Gamma(n-3) \\
&= \ldots \text{ etc.} \\
&= (n-1)(n-2)(n-3)\cdots(3)(2)(1)\underbrace{\Gamma(1)}_{=1} \\
&= (n-1)!
\end{aligned}
$$

So that the Gamma function has the property that

$\Gamma(n) = (n-1)!$ for positive integers n.

Not so easy to show: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (may show this in a bit -)

But if we believe for now that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ then the reduction formula says, for instance, that

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\,\underset{\substack{\uparrow \\ \text{bigger than } 1}}{\Gamma\left(\tfrac{5}{2}\right)} = \tfrac{5}{2}\cdot\tfrac{3}{2}\,\underset{\substack{\uparrow \\ \text{bigger than } 1}}{\Gamma\left(\tfrac{3}{2}\right)} = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\underset{\substack{\uparrow \\ \text{not bigger than } 1 \to \text{stop.}}}{\Gamma\left(\tfrac{1}{2}\right)} = \tfrac{15}{8}\sqrt{\pi}.$$

Remark

If one can tabulate $\Gamma(\alpha)$ for all values of α between 0 and 1 then we can compute $\Gamma(u)$ for any positive real number u through reduction:

$$\Gamma(u) = (u-1)(u-2)(u-3)\cdots(u-K)\,\Gamma(\underbrace{u-K})$$

where $u-K$ is strictly between 0 and 1.

So, for instance,

$$\Gamma(4.7) = (3.7)(2.7)(1.7)(.7)\,\underbrace{\Gamma(.7)}$$

and we would just need to "look-up" the value $\Gamma(.7)$ to compute $\Gamma(4.7)$.

$\blacksquare$

The point now is that for any $\alpha > 0$ and $\beta > 0$

$$\int_0^\infty x^{\alpha-1} e^{-x/\beta}\,dx = \int_0^\infty (u\beta)^{\alpha-1} e^{-u}\,\beta\,du$$

change of variable
$x = u\beta$
$dx = \beta\,du.$

$$= \int_0^\infty \beta^{\alpha-1} \cdot \beta \cdot u^{\alpha-1} e^{-u} \, du$$

$$= \beta^\alpha \int_0^\infty u^{\alpha-1} e^{-u} \, du = \beta^\alpha \Gamma(\alpha).$$

Thus we just showed that

$$\int_0^\infty x^{\alpha-1} e^{-x/\beta} \, dx = \beta^\alpha \Gamma(\alpha).$$

or that

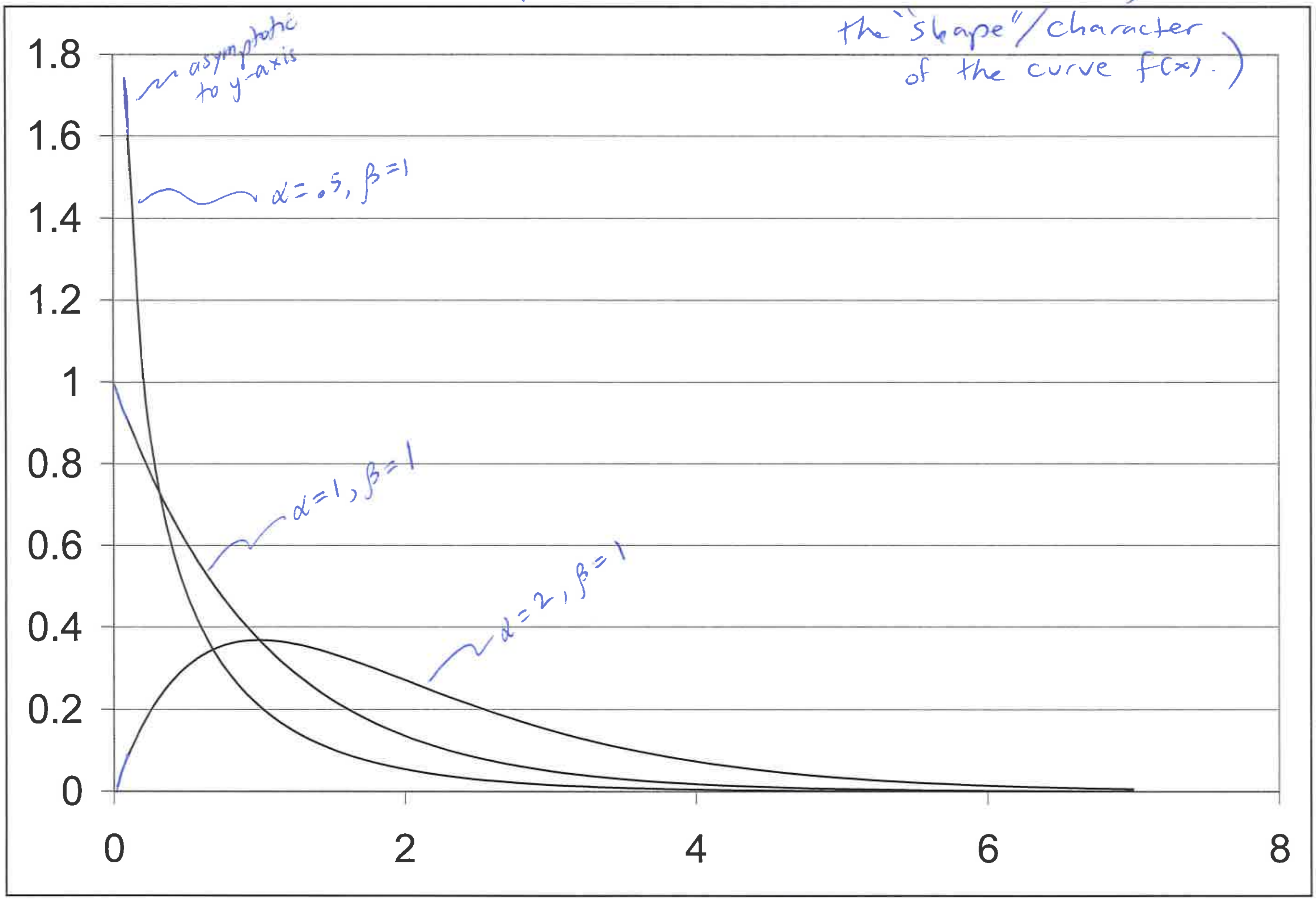
$$(\star) \quad \int_0^\infty \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \, dx = 1.$$

when we realize that

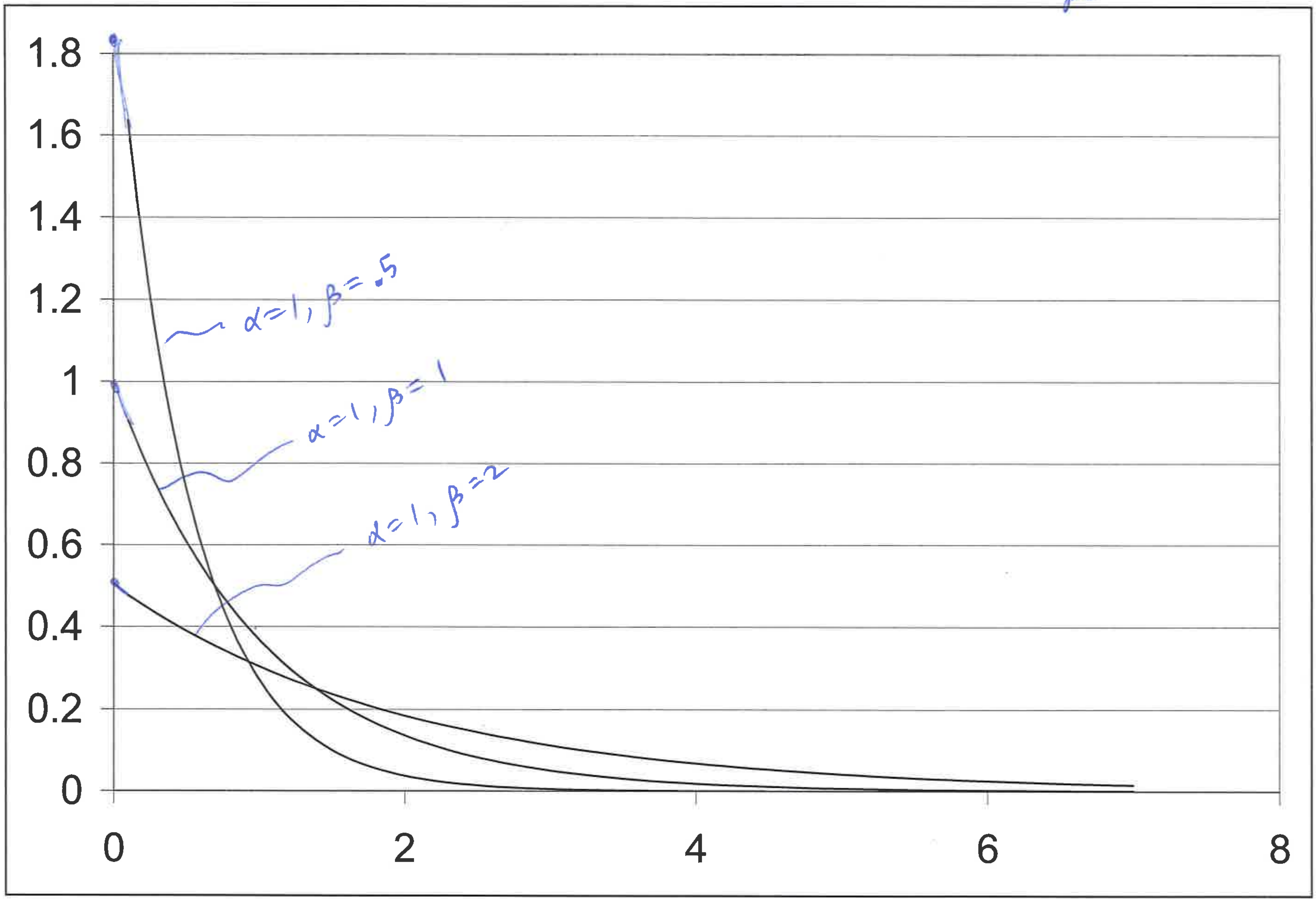
$$* \qquad f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & \text{for } x > 0. \\ 0 & \text{for } x \le 0 \end{cases}$$

is a non-negative function., we see that $(\star)$ tells us $f(x)$ is a pdf. We call this the pdf of Gamma(α, β) distribution.

The Graph of the pdf $f(x)$ of a Gamma(α,β) for various choices of α keeping β fixed at 1. (Notice how changing α changes the "shape"/character of the curve $f(x)$.)



The Graph of the pdf $f(x)$ of a Gamma(α, β) for various choices of β keeping α fixed at 1 (Notice how the scale changes but not the "shape"



In the Gamma(α, β) distribution α is called the shape parameter and β is called the scale parameter.

The power of knowing that $f(x)$ given by * is a pdf allows us to compute integrals of this type efficiently. That is, once we know

$$\int_0^\infty \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \, dx = 1$$

we also know

$$\int_0^\infty x^{\alpha-1} e^{-x/\beta} \, dx = \beta^\alpha \Gamma(\alpha)$$

Therefore if we are faced with an integration on left — if we can recognize the α and β — we'll know the value of the integral is the expression on the right.

For example,

$$\int_0^\infty x^2 e^{-x/2}\,dx = \int_0^\infty \overbrace{x^{3-1} e^{-x/2}}^{\alpha=3,\ \beta=2} = \beta^\alpha \Gamma(\alpha) = 2^3\,\Gamma(3)$$

$$= 8 \cdot 2! = 16.$$

Also,

$$\int_0^\infty x^3 e^{-2x}\,dx \overset{\alpha=4,\ \beta=\frac{1}{2}}{=} \left(\tfrac{1}{2}\right)^4 \Gamma(4) = \frac{1}{16}\cdot 3! = \frac{6}{16} = \frac{3}{8}.$$

$\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ but $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

$$\int_0^\infty x^{\frac{1}{2}} e^{-x/2}\,dx \overset{\alpha=\frac{3}{2},\ \beta=2}{=} 2^{3/2}\,\Gamma\left(\tfrac{3}{2}\right) = \sqrt{8}\left(\tfrac{1}{2}\sqrt{\pi}\right) = \sqrt{2\pi}.$$

etc.

So recognizing the Gamma pdf can save us lots of calculation.

Unfortunately the above needs the integral bounds to be 0 to ∞. So that if we were faced with the integral

$$\int_0^{1.8} x^{\frac{1}{2}} e^{-x/2}\,dx$$

then the formula above will not help. This integral would need to be computed numerically.

Remark

When $\alpha = \frac{\nu}{2}$ (with $\nu > 0$ an integer)

and $\beta = 2$.

the Gamma$(\frac{\nu}{2}, 2)$ distribution is called the Chi-square distribution with ν degrees of freedom. Specifically, the pdf is

$$f(x) = \begin{cases} \dfrac{x^{\frac{\nu}{2}-1} e^{-x/2}}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} & \text{for } x > 0 \\ 0 & \text{for } x \le 0. \end{cases}$$

More on this distribution later.

Lets compute the Mean and Variance of a Gamma(α,β).

$$E(X) = \int_0^\infty x \cdot \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x/\beta}\, dx = \frac{\beta^{\alpha+1}\,\Gamma(\alpha+1)}{\beta^\alpha\,\Gamma(\alpha)}$$

$$= \beta\left(\frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}\right) = \alpha\beta$$

where we used the reduction formula of the Euler Gamma fct. in the last step.

Let's compute the 2^{nd} moment for a Gamma(α, β):

$$E(X^2) = \int_0^\infty x^2 \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x/\beta}\, dx$$

$$= \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} = \beta^{\alpha+2}\frac{(\alpha+1)\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)}$$

$$= \frac{\beta^{\alpha+2}(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\beta^\alpha}\cancel{\Gamma(\alpha)}} = \alpha(\alpha+1)\beta^2$$

So that

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \alpha(\alpha+1)\beta^2 - \{\alpha\beta\}^2 = \alpha\beta^2.$$

$\boxed{\text{Aside}}$ Why does $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$?

$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$ make the change of variable $x = u^2$ $dx = 2u\,du$.

$$= \int_0^\infty u^{-1} e^{-u^2} \cdot 2u\,du = 2\int_0^\infty e^{-u^2}\, du. \quad \text{So } \Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\, du.$$

Now, use a trick due to F. Gauss — compute $\{\Gamma(\tfrac{1}{2})\}^2$ instead:

$$\{\Gamma(\tfrac{1}{2})\}^2 = 2\int_0^\infty e^{-u^2}\, du \cdot 2\int_0^\infty e^{-v^2}\, dv = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\, du\, dv = 4\int_0^{\pi/2}\int_0^\infty r e^{-r^2}\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \left\{\tfrac{1}{2}\right\} d\theta = \pi \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

used Polar coordinates. $r^2 = u^2 + v^2$ $du\,dv = r\,dr\,d\theta$.

The Cumulative Distribution function CDF

Let X be a r.v. we defined the cdf of X as the Function

$$(\star) \quad F(x) = F_X(x) = P(X \le x)$$

This definition is valid for both types of r.vs discrete and continuous.
— in the continuous r.v. case $F(x)$ will be a continuous function

This function returns the probability that $X \in (-\infty, x]$.

Let's say we know $F_X(x)$ for a r.v. X. Then by $(\star)$

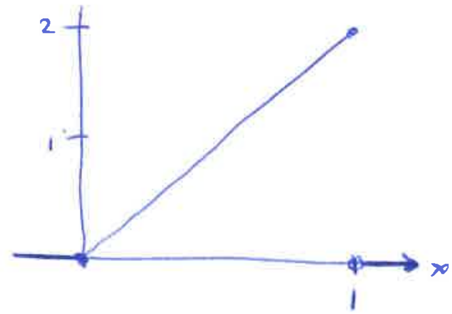
$$P(a < X \le b) = P(X \le b) - P(X \le a)$$
$$= F_X(b) - F_X(a)$$

Example Consider the continuous r.v. X having pdf

$$f(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1 \\ 0 & \text{for other } x. \end{cases}$$

Then

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$



and

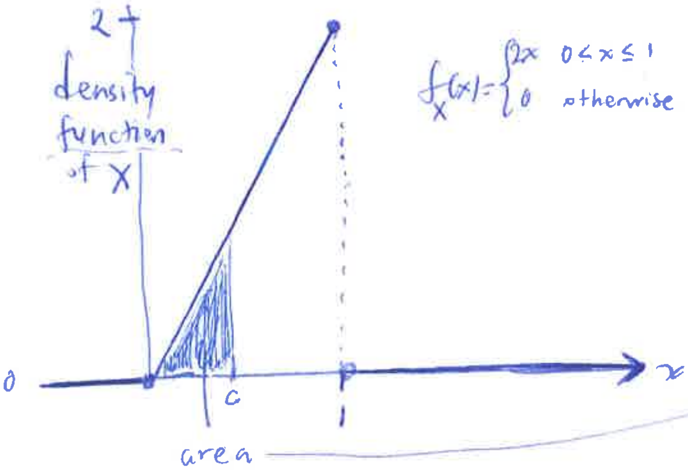
$$P\left(\tfrac{1}{4} < X \le \tfrac{3}{4}\right) = F\left(\tfrac{3}{4}\right) - F\left(\tfrac{1}{4}\right) = \left(\tfrac{3}{4}\right)^2 - \left(\tfrac{1}{4}\right)^2 = \tfrac{9}{16} - \tfrac{1}{16} = \tfrac{1}{2}.$$

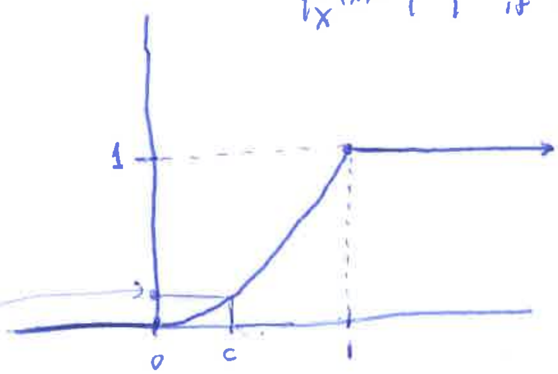
The Cumulative distribution function (cdf) of an r.v. X.

$$F_X(x) = \begin{cases} \sum\limits_{k \le x} p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{if } X \text{ is continuous} \end{cases}$$

In either case $F_X(x) = P(X \le x)$ and the cdf accumulates all the probability mass up through x.

$$f_X(x) = \begin{cases} 2x & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 & \text{if } 0 < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$



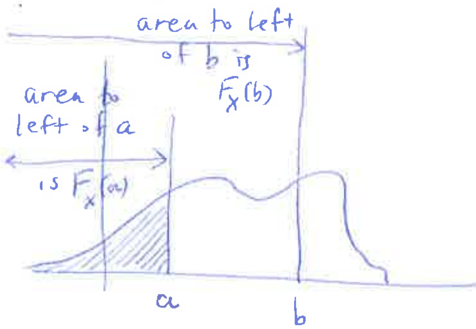


Notice the cdf in this example is a continuous function.

Using the cdf one can find Probabilities of the form

$P(a < X \le b)$ as $\underbrace{F(b) - F(a)}_{\text{area under } f_X(x) \text{ between } a \text{ and } b}$

$=$ prob. of $a < X \le b$.

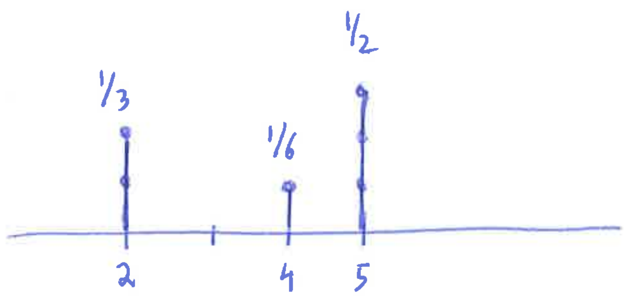


shorthand for continuous

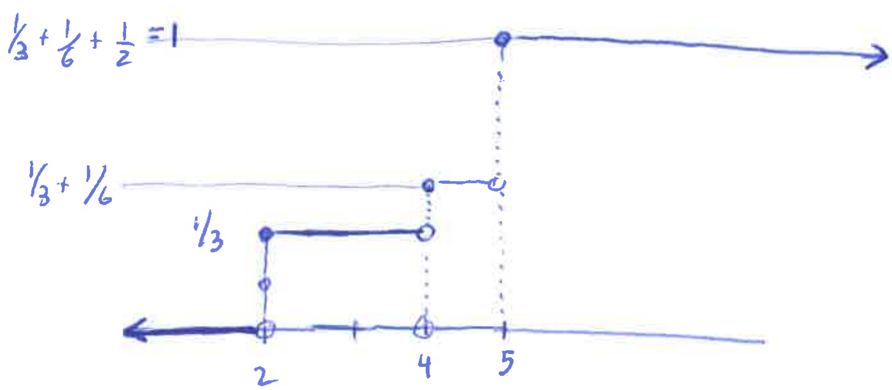
Contrast the cts r.v. situation with the discrete r.v. situation

Consider the pmf

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{for } x = 2 \\ \frac{1}{6} & \text{for } x = 4 \\ \frac{1}{2} & \text{for } x = 5 \end{cases}$$



graph of pmf



graph of cdf. Notice for discrete rvs the cdf is a pure step function.

$$P(2.5 < X \le 4.5) = F(4.5) - F(2.5)$$

$$= \left(\tfrac{1}{3} + \tfrac{1}{6}\right) - \left(\tfrac{1}{3}\right) = \tfrac{1}{6}$$

$$P(1.7 < X \le 3.2) = F(3.2) - F(1.7) = \tfrac{1}{3} - 0 = 1/3.$$

<u>Remark</u> if we know the pmf of a discrete r.v then we also know it's cdf <u>and</u> <u>vice-versa</u>.

if we know the pdf of a continuous r.v. then we also know its cdf <u>and</u> <u>vice-versa</u>.

Eg.

To recover the pdf from a cdf just differentiate:

Since $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$ we can get $f_X(x)$ by

taking a derivative:

$$\frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{x} f_X(u)\,du = f_X(x)$$

This is the 2nd fundamental theorem of calculus.

<u>Example</u> Suppose the cdf of an r.v. X is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

then the pdf is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$ i.e. $X \sim \exp(\lambda)$.

The distribution of the maximum of several independent cts. r.v.s.

Suppose X_1, X_2, X_3 are independent $\exp(\lambda)$ r.v.s.
Find the cdf of $Y = \max\{X_1, X_2, X_3\}$ and the pdf of Y also.

Solution.

$$F_Y(y) = P(Y \le y) = P(\max\{X_1, X_2, X_3\} \le y)$$

$$= P(X_1 \le y, X_2 \le y, X_3 \le y)$$

This step we used the fact that the $\max\{X_1, X_2, X_3\} \le y$ iff each $X_i \le y$.

$$= P(X_1 \le y) P(X_2 \le y) P(X_3 \le y)$$ by independence

$$= F_X(y) F_X(y) F_X(y)$$ since X_1, X_2, X_3 have the same cdf.

$$= \{F_X(y)\}^3 \begin{cases} = \{1 - e^{-\lambda y}\}^3 & \text{if } y > 0 \\ = 0 & \text{if } y \le 0. \end{cases}$$

from formula on previous page.

the cdf of Y

the pdf of Y we get by taking the derivative of $F_Y(y)$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 3(1 - e^{-\lambda y})^2 \lambda e^{-\lambda y} = 3\lambda e^{-\lambda y}(1 - e^{-\lambda y})^2 & \text{for } y > 0 \\ 0 & \text{for } y \le 0. \end{cases}$$

The pdf of Y.

The distribution of the Minimum of several independent cts r.v.s

Here we use the fact that

$$\min\{X_1, X_2, X_3, \dots, X_n\} > y \text{ iff}$$

$$X_1 > y,\ X_2 > y,\ X_3 > y, \dots, X_n > y.$$

Suppose $X_1, X_2, X_3 \sim$ independent $\exp(\lambda)$ r.v.s.
Find the cdf and pdf of $W = \min\{X_1, X_2, X_3\}$.

Solution:

$$F_W(w) = P(\min\{X_1, X_2, X_3\} \le w)$$

$$= 1 - P(\min\{X_1, X_2, X_3\} > w\}$$

$$= 1 - P(X_1 > w,\ X_2 > w,\ X_3 > w)$$

$$= 1 - P(X_1 > w)P(X_2 > w)P(X_3 > w)$$

$$= 1 - (1 - P(X_1 \le w))(1 - P(X_2 \le w))(1 - P(X_3 \le w))$$

$$= 1 - (1 - F_X(w))^3 \quad \text{where } F_X(w) = \begin{cases} 1 - e^{-\lambda w} & \text{for } w > 0 \\ 0 & \text{for } w \le 0. \end{cases}$$

$$= \begin{cases} 1 - (1 - [1 - e^{-\lambda w}])^3 & \text{for } w > 0 \\ 0 & \text{if } w \le 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-3\lambda w} & \text{for } w > 0 \\ 0 & \text{for } w \le 0 \end{cases}$$

pdf of W is the pdf of $\exp(3\lambda)$.

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 3\lambda e^{-3\lambda w} & \text{for } w > 0 \\ 0 & \text{for } w \le 0 \end{cases}$$

The Normal (a.k.a. Gaussian) distribution

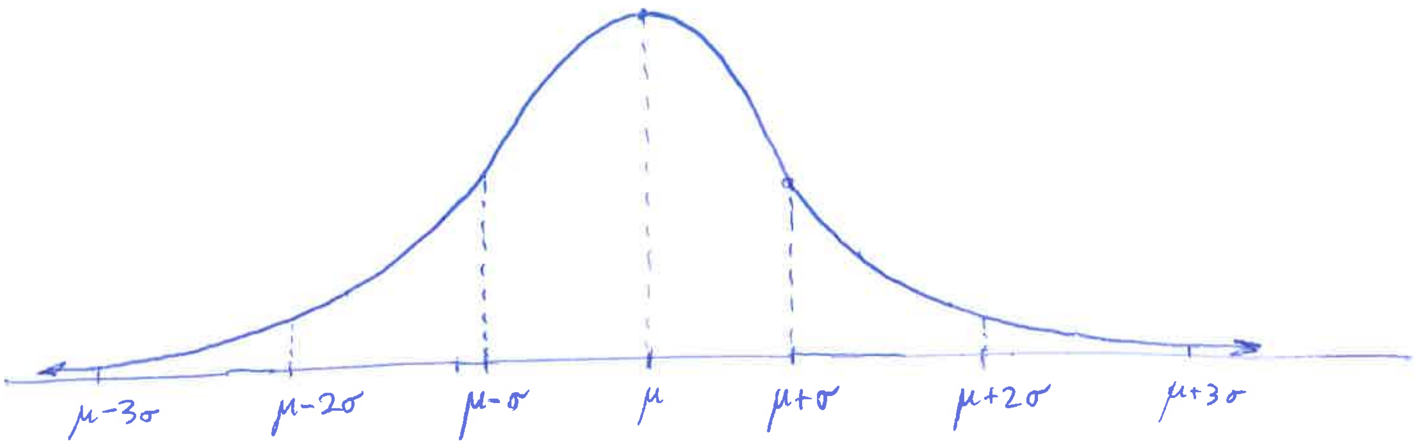
is the distribution of a continuous r.v. X having pdf given by

$$(*) \qquad f_X(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} \qquad \text{for } -\infty < x < \infty.$$

Facts about this pdf.

1. the functional form is defined on all reals.
2. μ and σ are > 0 and fixed parameters. (they will correspond, respectively, to the mean and standard deviation of X)
3. $f_X(x)$ has a maximum at $x = \mu$ and is symmetric about the line $x = \mu$.
4. $f_X(x)$ has pts of inflection at $x = \sigma$ and $x = -\sigma$
5. $f_X(x)$ is "concave down" on $(\mu-\sigma, \mu+\sigma)$ and "concave up" on $(-\infty, \mu-\sigma)$ and $(\mu+\sigma, \infty)$
6. $\lim_{|x|\to\infty} f_X(x) = 0$.
7. Has ~~a~~ "bell-shape"
8. We will write $X \sim \text{Normal}(\mu, \sigma^2)$ to mean X has pdf $(*)$

Graph of the Normal(μ, σ^2) pdf.



Notice that μ really is the mean since it is the center of the symmetric graph. Also notice changing μ up or down will slide this curve up or down by the same amount.

(Although not yet apparent) σ is the standard deviation and increasing σ makes the pdf more "spread out" and decreasing σ makes the pdf less "spread out" (or more "peaked").

Important Special Case of the Normal(μ, σ^2) distribution is the case $\mu = 0$ and $\sigma = 1$ $(\sigma^2 = 1)$ and is called the Standard Normal.