550.420 Intro. to Probability - Spring 2017 **Homework #13**
Will NOT be collected.

From the textbook:
Chapter 8 / Problems: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.15; Theoretical exercises: 8.7.

Additional problems:
A.13.1. Suppose that a random variable satisfies

$$E(X) = 0, \quad E(X^2) = 1, \quad E(X^3) = 0, \quad E(X^4) = 3,$$

and let $Y = a + bX + cX^2$. Find the correlation coefficient $\rho_{X,Y}$.

A.13.2. A retired professor comes to the office at a time which is uniformly distributed between 9 AM and 1 PM, performs a single task, and leaves when the task is completed. The duration of the task is exponentially distributed with parameter $\lambda(y) = 1/(5 - y)$, where y is the length of the time interval between 9 AM and the time of their arrival.
(a) What is the expected amount of time the professor devotes to the task?
(b) What is the expected time at which the task is completed?

A.13.3. Suppose that $N \sim$ geometric(q) and $X_1, X_2, \cdots \sim$ i.i.d. geometric(p) which are all independent of N. Consider the compound random variable

$$S = \sum_{i=1}^{N} X_i.$$

Derive the mgf of S. Hint: use the law of total expectation: $E(e^{tS}) = E\Big(E(e^{tS}|N)\Big)$. Please identify the distribution of S from what you've found.

A.13.4. (a) Roll a balanced die 3 times and let X_1, X_2, and X_3 represent the up-faces. Compute the probability the sum $X_1 + X_2 + X_3 = 9$ exactly. (Old question, nothing new here!) Answer: $\frac{25}{216} \approx .1157$.
(b) Use the Central limit theorem to estimate $P(8.5 \le X_1 + X_2 + X_3 \le 9.5)$. Answer $\approx .1182$.

A.13.5. It has been hypothesized that the proportion of allotted study time students taking 550.420 spend studying probability is a random variable having pdf $f(x) = 2x^3 - 2x + 1$ for $0 \le x \le 1$. In a class of 90 students taking probability let $X_1, X_2, \ldots, X_{90}$ represent the proportion of this time each student spends studying probability. Let $\overline{X} = \frac{X_1 + \cdots + X_{90}}{90}$ represent the class average.
Estimate $P(\overline{X} > \frac{1}{3})$. Answer $\approx .0023$.

A.13.6. Show that the mgf of a Exp(1) is $M(t) = (1 - t)^{-1}$. Then, mimic the proof of the Central limit theorem give in class to show that if $X_1, X_2, \ldots$ are independent Exp(1), then $Y_n := \frac{S_n - n}{\sqrt{n}}$ (where $S_n = \sum_{i=1}^{n} X_i$) has a distribution which converges to a standard normal distribution. Use this result to estimate $P(S_{100} > 120)$.