Midterm 2 Study Guide

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Random Variables

- $X : \Omega \to \mathbb{R}$ can be discrete or continuous
- If X is a discrete random variable, we will associate it with a *probability mass function (pmf)* P_X
  - $P_X(x) > 0 \; \forall \; x \in \{\text{values of } X\}$
  - $\sum_x P_X(x) = 1$, where the sum is over all the possible values of X
  - Used to calculate some probabilities: $P(X \in A) = \sum_{x \in A} P_X(x)$

Functions of Random variables

- With pmf of X and $Y = g(X)$:

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

Expected Value and Variance of a Discrete Random Variable

- $\mathbb{E}$: also the mean, weighted average, or center of mass

$$\mathbb{E}(X) = \sum_x x P_X(x)$$

- The variance of an random variable: $Var(x) = (x - \mu)^2$, where $\mu = \mathbb{E}(X)$
  - So $\mathbb{E}(\{X - \mu\}^2) := Var(X)$
- A form of the $Var(X)$ more amenable to calculations: $\mathbf{Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2}$

Cumulative Distribution Function (CDF)

- $F(X) = P(X \leq x)$
    1) $F : \mathbb{R} \to [0, 1]$
    2) If $x < y$, $F(x) \leq F(y)$
- Notation: Left-limit notation $F(c-) = lim_{x \to c-} F(x)$
- If we know the CDF,
    - $P(a < x \leq b) = F(b) - F(a)$
    - $P(a \leq x \leq b) = F(b) - F(a-)$
    - $P(a \leq x < b) = F(b-) - F(a-)$
    - $P(a < x < b) = F(b-) - F(a)$
    - General rule: If near "$<$", $a \to -F(a)$, $b \to F(b-)$
        * "a $<$ b" ... so if "$<$" is near a, the "-" is on the left; "-" is on the right for b

Law of the Unconscious Statistician

- If X is discrete and $G : \mathbb{R} \to \mathbb{R}$, then
    - $\mathbb{E}(\mathbf{g}(\mathbf{X})) = \sum_{\mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{P}(\mathbf{X} = \mathbf{x})$ when the expectation exists
- Used when we know $G(X)$ and the distribution of X but not the distribution of $G(X)$
- Some consequences:
    - Linearity of Expectation #1
        * $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
    - Linearity of Expectation #2
        * $\mathbb{E}(X_1 + X_2 + \cdots + X_n) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}X_n)$
        * Expectation of a sum is the sum of the individual expected values
        * For any random variables for which $\mathbb{E}(X_i)$ exists for all i
        * *To be proved later*