Intro Prob Lecture Notes

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Conditional Expectation

- Law of total expectation $E(X) = E(E(X|Y))$
  - Example: random variable N $p(N = n) = p_n$ for $n = 0, 1, 2, 3, \ldots$ $x_1, x_2, x_3, \ldots$ i.i.d. and independent of N
    - $\rightarrow E(S) = \mu_x \mu_N$
  - $Var(S) = E(S^2) - (E(S))^2 = E(S^2) - \mu_X^2 \mu_N^2$
    - $(\sum_{i=1}^{N} x_i)(\sum_{j=1}^{N} x_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j = \sum_{i=1}^{N} x_i^2 \sum_{i \neq j}^{N} x_i x_j$
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$$
\begin{aligned}
E(S^2) &= E(E(S^2|N)) \\
&= E(\sum_{i=1}^{N} x_i^2 \sum_{i \neq j}^{N} x_i x_j | N = n) \\
&= E(\sum_{i=1}^{N} x_i^2 \sum_{i \neq j}^{N} x_i x_j) \\
&= nE(x_1^2) + n(n-1)E(x_1 x_2) \\
&= n(\sigma_X^2 + \mu_X^2) + n(n-1)E(x_1)E(x_2) \\
&= n\sigma_X^2 + n\mu_X^2 + n(n-1)\mu_x^2 \\
&= n\sigma_X^2 + n^2\mu_X^2
\end{aligned}
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\begin{aligned}
E(S^2) &= E(E(S^2|N)) \\
&= E(n\sigma_X^2 + n^2\mu_X^2) \\
&= \sum_{n=0}^{\infty}(n\sigma_X^2 + n^2\mu_X^2)P_N(n) \qquad = \sigma_X^2 \sum_{n=0}^{\infty} nP_N(n) + \mu_X^2 \sum_{n=0}^{\infty} n^2 P_N(n) \\
&= \sigma_X^2\mu_N + \sigma_X^2 E(N^2)
\end{aligned}
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Var(S) = \sigma_X^2\mu_N + \mu_X^2 E(N^2) - \mu_X^2\mu_N^2 = \sigma^2\mu_N + \mu_X^2\sigma_N^2
$$

Function with another variable

- Let X, Y r.v.'s and h is any function. Then

$$
E(Xh(Y)|Y) = h(Y)E(X|Y)
$$

  or

$$
E(Xh(Y)|Y = y) = h(y)E(X|Y = y)
$$

- Proof:

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$$
\begin{aligned}
E(Xh(Y)|Y = y) &= \int_{-\infty}^{\infty} xh(y)f_{X|Y}(x, y)dx \\
&= h(y)\int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx \\
&= h(y)E(X|Y = y)
\end{aligned}
$$

- $$\begin{aligned} E(g(X)h(Y)) &= E(E(g(X)h(Y)Y)) \\ &= E(h(Y)E(g(X)|Y)) \end{aligned}$$

- Ex: $X \sim \text{geometric}(p)$
  - Let Y be the outcome on the first toss. $f_Y(y) = \begin{cases} p & y = 1 \\ 1-p & y = 0 \end{cases}$
  - $E(X) = E(X|Y=0) + E(X|Y=1) = (1+E(X))(1-p) + 1(p) = 1 \cdot p + (1-p)(1+\mu) = \frac{1}{p}$
  - $$Var(X) = E(X^2) - (E(X))^2$$

$$\begin{aligned} E(X^2) &= E(E(X^2|Y)) \\ &= E(X^2 Y = 1)p + E(X^2|Y=0)(1-p) \\ &= 1 \cdot p + E((1+X)^2)(1-p) \\ &= p + E(1 + 2X + X^2)(1-p) \\ &= p + (1-p) + (1-p)\frac{2}{p} + E(X^2)(! - p) \\ pE(X^2) &= 1 + \frac{2(1-p)}{p} \\ E(X^2) &= \frac{1}{p} + \frac{2(1-p)}{p^2} \end{aligned}$$

    * So
$$Var(X) = \frac{1}{p} + \frac{2(1-p)}{p^2} - (\frac{1}{p})^2 = \frac{1-p}{p^2}$$

Conditional Variance

Moment Generating Functions