550.420 Intro to Probability - Spring 2017 **Homework #06**

Also will not be collected, but answers are on the next page!.

A.6.1. Toss 3 balanced six-sided dice repeatedly. Consider the following random variables:

X the trial we observe the first two-of-a-kind,

Y the trial we observe the second two-of-a-kind.

W the number of two-of-a-kinds we observe in 10 trials.

(a) If we agree to call a success the event that we roll a two-of-a-kind (using 3 dice), compute the probability p of success on a trial. ANS: $p = 5/12$.

(b) What type of probability distribution does the random variable X have? How about Y? and W?

(c) Please write down each of the pmfs for X, Y and W in functional form. Eg., $p_X(x) = P(X = x) = \dots$ for $x = \cdots$; $p_Y(y) = P(Y = y) = \dots$ for $y = \cdots$, $p_W(w) = P(W = w) = \dots$ for $w = \cdots$.

(d) Consider the event that it takes at least 3 trials to see a two-of-a-kind for the first time. Write this event by using one of the random variables above, and then compute the probability of this event.

(e) Compute $P(W \leq 1)$.

(f) Compute $P(W \leq 9)$.

(g) Compute $P(Y = 3)$.

(h) Compute $P(Y \leq 3)$.

A.6.2. Fred throws darts at a dartboard repeatedly. The probability he hits the bulls-eye on any trial is .4 independent of other trials. Let B be the random variable that counts the number of times Fred hits the bulls-eye in 4 consecutive throws.

(a) Write down the pmf of B.

(b) Compute the cdf of B.

(c) (separate question) If Fred throws the dart 4 times, what is the probability that he will hit the bulls-eye in at least one of his last two throws?

A.6.3. Suppose the cdf of a random variable X is given by $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{8} & \text{if } 0 \leq x < 2 \\ \frac{1}{2} & \text{if } 2 \leq x < 2.5 \\ \frac{1}{4}x + .125 & \text{if } 2.5 \leq x < 3.5 \\ 1 & \text{if } x \geq 3.5 \end{cases}$.

(a) Use what you know about the distribution function to compute $P(0 \leq X \leq 1)$, $P(1 \leq X < 2.5)$ and $P(1 < X \leq 2.5)$.

(b) Is this the distribution function of a discrete random variable?

A.6.4. Suppose the velocity of a particle of mass $m = 1$ is a (discrete) random variable V having the pmf $P(V = v) = \frac{|v|}{20}$ for $v = -4, -3, -2, -1, 1, 2, 3, 4$.

(a) Compute the expected velocity $E(V)$ of the particle.

(b) Compute the expected kinetic energy $K = \frac{mV^2}{2} = \frac{V^2}{2}$ of our particle.

A.6.5. Toss a balanced coin repeatedly and let Y be the random variable that returns the trial of the second head.

(a) Show that the pmf $p_Y(y) = P(Y = y) = \frac{y-1}{2^y}$ for $y = 2, 3, 4, \dots$.

(b) Compute the $P(Y \geq 4)$, that is, compute the probability you will have to toss the coin at least 4 times to see the second head. *Try to do this two ways if you can.*

(c)* Compute $E(Y)$.

answers:

A.6.1. (a) $p = \frac{\binom{3}{2}\cdot 6\cdot 5}{6^3} = \frac{5}{12}$.

(b) X has the geometric$(5/12)$ distribution; Y has the Pascal$(2, 5/12)$ distribution (also the neg.binom$(2, 5/12)$ distribution); and W has the binomial$(10, 5/12)$ distribution.

(c) $p_X(x) = \frac{5}{12}\left(\frac{7}{12}\right)^{x-1}$ for $x = 1, 2, 3, \ldots$.

$p_Y(y) = (y-1)\left(\frac{5}{12}\right)^2\left(\frac{7}{12}\right)^{y-2}$ for $y = 2, 3, 4, \ldots$.

$p_W(w) = \binom{10}{w}\left(\frac{5}{12}\right)^w\left(\frac{7}{12}\right)^{10-w}$ for $w = 0, 1, 2, \ldots, 9, 10$.

(d) This event is $(X \geq 3)$. Furthermore, $P(X \geq 3) = \sum_{x=3}^{\infty} \frac{5}{12}\left(\frac{7}{12}\right)^{x-1} = \frac{5}{12}\sum_{x=3}^{\infty}\left(\frac{7}{12}\right)^{x-1} = \left(\frac{7}{12}\right)^3$.

(e) $P(W \leq 1) = P(W = 0) + P(W = 1) = \binom{10}{0}(\frac{5}{12})^0(\frac{7}{12})^{10} + \binom{10}{1}(\frac{5}{12})^1(\frac{7}{12})^9$.

(f) It is easier to compute $P(W \leq 9)$ as $1 - P(W > 9) = 1 - P(W = 10) = 1 - (\frac{5}{12})^{10}$.

(g) $P(Y = 3) = (3-1)\frac{(}{5}12)^2(\frac{7}{12})^{3-2} = \frac{175}{864}$

(h) $P(Y \leq 3) = P(Y = 2) + P(Y = 3) = (\frac{5}{12})^2 + \frac{175}{864} = \frac{325}{864}$

A.6.2. B has a binomial$(4, .4)$ distribution: $p_B(b) = \binom{4}{b}(.4)^b(.6)4 - b$ for $b = 01, 2, 3, 4$.

(b) $F_B(x) = \begin{cases} 0 & \text{if } x < 0 \\ .1296 & \text{if } 0 \leq x < 1 \\ .4752 & \text{if } 1 \leq x < 2 \\ .8208 & \text{if } 2 \leq x < 3 \\ .9744 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$.

(c) Let S_i be the event that throw number i hits the bulls-eye. In particular, S_3 and S_4 are independent. $P(S_3 \cup S_4) = P(S_3) + P(S_4) - P(S_3 \cap S_4) = .4 + .4 - P(S_3)P(S_4)$ by independence; therefore, $P(S_3 \cup S_4) = .64$.

A.6.3. (a) $P(0 \leq X \leq 1) = F(1) - F(0-) = \frac{1^2}{9} - 0 = \frac{1}{8}$. $P(1 \leq X < 2.5) = F(2.5-) - F(1-) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$. $P(1 < X \leq 2.5) = F(2.5) - F(1) = \frac{3}{4} - \frac{1}{8} = \frac{5}{8}$.

(b) This is not the cdf of a discrete random variable since it is not a step function, that is, it doesn't have the property that the only points of increase in F occur at discrete points.

A.6.4. (a) $E(V) = 0$.

(b) Using the Law of the unconscious Statistician $E(K) = \frac{(-4)^2}{2}\frac{4}{20} + \frac{(-3)^2}{2}\frac{3}{20} + \frac{(-2)^2}{2}\frac{2}{20} + \frac{(-1)^2}{2}\frac{1}{20} + \frac{(1)^2}{2}\frac{1}{20} + \frac{(2)^2}{2}\frac{2}{20} + \frac{(3)^2}{2}\frac{3}{20} + \frac{(4)^2}{2}\frac{4}{20} = \frac{100}{20} = 5$.

A.6.5. (a) The event $(Y = y)$ occurs if and only if exactly one head occurs in trials 1 through $y - 1$ *and* the y toss is a head. Therefore, we need at least two tosses to see the second head and $P(Y = y) = \binom{y-1}{1}(\frac{1}{2})^{y-1}\cdot\frac{1}{2} = \frac{y-1}{2^y}$ for $y = 2, 3, 4, \ldots$.

(b) $P(Y \geq 4) = 1 - P(Y < 4) = 1 - \Big(P(Y = 2) + P(Y = 3)\Big) = 1 - (\frac{1}{4} + \frac{2}{8}) = \frac{1}{2}$. Also,

$$P(Y \geq 4) = \frac{3}{2^4} + \frac{4}{2^5} + \frac{5}{2^6} + \frac{6}{2^7}\cdots.$$

Multiplying both sides above by $\frac{1}{2}$ gives:

$$\frac{1}{2}P(Y \geq 4) = \qquad \frac{3}{2^5} + \frac{4}{2^6} + \frac{5}{2^7} + \cdots$$

and then subtracting the latter equation from the previous one gives the series: $\frac{1}{2}P(Y \geq 4) = \frac{3}{2^4} + \{\frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots\} = \frac{3}{16} + \frac{\frac{1}{32}}{1-\frac{1}{2}} = \frac{1}{4} \implies P(Y \geq 4) = \frac{1}{2}$.

(c) $E(Y) = 4$. Hint: use the very first technique I introduced to compute the expected value of a geometric$(1/2)$ ...see where this takes you.