HW#8 Additional Problems solutions

$\boxed{A.8.1}$ If $X \sim$ binomial$(n,p)$, then $X = \sum_{i=1}^{n} X_i$

where $X_1, X_2, \dots, X_n$ are (independent) Bernoulli $(p)$ random variables. Moreover, since $X_i \sim$ Bernoulli$(p)$

$$X_i = \begin{cases} 1 & \text{with pr. } p \\ 0 & \text{with pr. } 1-p \end{cases}$$

and $E(X_i) = 1 \cdot p + 0 \cdot (1-p) = p$.

Therefore, by linearity,

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np.$$

$\boxed{A.8.2}$

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x}$$

$$= n(n-1)p^2 \underbrace{\sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x}}_{=1}$$

$$= n(n-1)p^2.$$

So $E(X^2) = E(X(X-1)+X) = E(X(X-1)) + E(X) = n(n-1)p^2 + np$

Finally, $Var(X) = E(X^2) - \{E(X)\}^2 = n(n-1)p^2 + np - \{np\}^2$

$$= n^2p^2 - np^2 + np - n^2p^2 = np - np^2$$

$$= np(1-p).$$

$\boxed{A.8.3}$

(i) $Var(X-b) = E\left[(X-b-E(X-b))^2\right]$

$= E\left[(X-b-\{E(X)-b\})^2\right] = E\left[(X-E(X))^2\right]$

$= Var(X)$.

(ii) $Var(aX) = E\left[(aX - E(aX))^2\right]$

$= E\left[(aX - aE(X))^2\right]$

$= E\left[a^2 (X-E(X))^2\right]$

$= a^2 E\left[(X-E(X))^2\right] = a^2 Var(X)$.

$\boxed{A.8.4}$

(a) $E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx = \lambda \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\lambda x} dx}_{dv}$

$= \lambda \left\{ x \frac{e^{-\lambda x}}{-\lambda} \Big]_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right\}$

$= \lambda \left\{ \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx \right\} = \int_0^\infty e^{-\lambda x} dx = \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty = \frac{1}{\lambda}$.

Thus, the mean of an $\exp(\lambda)$ is $\frac{1}{\lambda}$.

$\boxed{\text{A.8.4}}$ (continued)

(b) If $x>0$, then

$$F_X(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big]_0^x = 1 - e^{-\lambda u}$$

If $x \le 0$,

$F_X(x) = 0$.

Therefore, $F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0. \end{cases}$

(c) $X \sim \exp(\frac{1}{2})$ means $f(x) = \frac{1}{2} e^{-x/2}$.

$$F_V(v) = P(V \le v) = P(\sqrt{X} \le v) = P(X \le v^2) = F_X(v^2)$$

$$= \begin{cases} 0 & \text{if } v^2 \le 0 \\ 1 - e^{-\frac{v^2}{2}} & \text{if } v^2 > 0. \end{cases}$$

Therefore, the pdf of $V$ is $\frac{d}{dv} F_V(v)$:

$$f_V(v) = \frac{d}{dv}\left(1 - e^{-\frac{v^2}{2}}\right) = v e^{-\frac{v^2}{2}} \quad \text{for } v > 0.$$

is the pdf of the Rayleigh distribution.

$\boxed{A.8.5}$ (treating $s$ as if it were a constant....)

$$E(e^{sX}) = \int_0^\infty e^{sx}\,\lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - s)x}\,dx$$

(This integral will be finite when $\lambda - s > 0$, i.e., when $s < \lambda$.)

$$= -\frac{\lambda e^{-(\lambda-s)x}}{\lambda - s}\Bigg|_0^\infty$$

$$= \frac{\lambda}{\lambda - s} \quad \text{. or} \quad \frac{1}{1 - s/\lambda} = \left(1 - \frac{s}{\lambda}\right)^{-1} \quad \text{for } s < \lambda.$$

$\boxed{A.8.6}$

$$E(U) = \int_\alpha^\beta u \cdot \frac{1}{\beta - \alpha}\,du = \frac{1}{\beta-\alpha} \cdot \frac{u^2}{2}\Bigg|_\alpha^\beta = \frac{1}{\beta-\alpha} \cdot \left(\frac{\beta^2 - \alpha^2}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{\cancel{\beta-\alpha}} \cdot \cancel{(\beta-\alpha)}(\beta+\alpha) = \frac{\alpha+\beta}{2}$$

(the midpoint which agrees with our intuition.)

$$E(U^2) = \int_\alpha^\beta u^2 \frac{1}{\beta-\alpha}\,du = \frac{1}{\beta-\alpha}\,\frac{\beta^3 - \alpha^3}{3} = \frac{1}{\beta-\alpha} \cdot \frac{(\beta-\alpha)(\beta^2 + \alpha\beta + \alpha^2)}{3}$$

$$= \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

$$Var(U) = \frac{\beta^2 - \alpha\beta + \alpha^2}{3} - \frac{(\alpha+\beta)^2}{4} = \frac{4\beta^2 - 4\alpha\beta + 4\alpha^2 - 3(\alpha^2 + 2\alpha\beta + \beta^2)}{12}$$

$$= \frac{\beta^2 - 2\alpha\beta + \alpha^2}{12} = \frac{(\beta - \alpha)^2}{12}.$$

A.8.7 We will show a r.v. has the uniform $(0,1)$ distribution by showing it has the cdf of a uniform$(0,1)$ distribution which is

(*) $$F(u) = \begin{cases} 0 & \text{if } u < 0 \\ u & \text{if } 0 \le u \le 1 \\ 1 & \text{if } u > 1. \end{cases}$$

(a) Suppose $0 \le u \le 1$

$F_U(u) = P(U \le u)$

plug in definition of $U$

$= P(F_X(X) \le u)$

apply the function $F_X^{-1}$ to both sides

$= P\left(F_X^{-1}(F_X(X)) \le F_X^{-1}(u)\right)$

$= P(X \le F_X^{-1}(u))$ ← this is the cdf of $X$ evaluated at $F_X^{-1}(u)$

$= F_X(F_X^{-1}(u))$

$= u$ which agrees with (*) when $0 \le u \le 1$.

Now, since $F_X(X)$ is always $\le 1$, if $u > 1$ we must have

$P(F_X(X) \le u) = 1.$ which agrees with (*) when $u > 1$.

Finally, since $F_X(X) \ge 0$ always, if $u < 0$, then

$P(F_X(X) \le u) = 0$ which agrees with (*) when $u < 0$

Thus. $U$ and a uniform$(0,1)$ have the same cdf and therefore $U \sim$ uniform$(0,1)$.

(continued)

A.8.7 (b) Since $F_X(x) = 1 - e^{-\lambda x}$ for $x > 0$,

$$U = F_X(X) = 1 - e^{-\lambda X} \quad \text{when } X > 0 \ (\Rightarrow U \in (0,1))$$

$$e^{-\lambda X} = 1 - U$$

$$-\lambda X = \ln(1-U)$$

$$\boxed{X = -\frac{1}{\lambda} \cdot \ln(1-U).}$$

So if $U \sim \text{uniform}(0,1)$, then

$$-\frac{1}{\lambda} \cdot \ln(1-U) \sim \exp(\lambda).$$