Combinations

When we have n distinct objects and we wish to count the number of selections (order not important) of k of these objects, then there are

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

such selections.

The binomial coefficient $\binom{n}{k}$ is counting the number of subsets of an n-element set having k members.

Example We have a standard deck of 52 cards. A k-card hand is defined to be a selection of k cards from these 52 (ignoring order), i.e., a subset of size k from the 52 distinct objects.

So, there are $\binom{52}{5}$ 5-card hands

and $\binom{52}{13}$ 13-card hands.

A standard assumption usually made in this situation is that the deck of 52 is "well-shuffled" so that every possible k-card hand is equally-likely.

Continuing with this example....

- How many 5-card hands contain only face-cards?

  12 possible face-cards

  $\binom{12}{5}$ 5-card hands drawn from these 12.

  What is the probability of getting all face-cards in a 5-card hand? (Assume the deck is well-shuffled).

  ANSWER: $\binom{12}{5} \Big/ \binom{52}{5}$.

- How many 5-card hands have 3 hearts and 2 spades?

  We can think of a subset have 3 hearts and 2 spades as $\{h_1, h_2, h_3\} \cup \{s_1, s_2\}$ where $\{h_1, h_2, h_3\}$ is a subset of 3 hearts drawn from the 13 possible hearts, and $\{s_1, s_2\}$ is a subset of size 2 from the 13 possible spades.

  There are $\binom{13}{3} \cdot \binom{13}{2}$ 5-card hands with 3 hearts and 2 spades.

- How many 5-card hands have 3 of one number and 2 of some other number?

(such a hand is called a <u>full-house</u>)

Such a hand would look like

$$\{a_1, a_2, a_3\} \cup \{b_1, b_2\}$$

where a_1, a_2, a_3 are 3 cards of the same denomination drawn from the 4 possible suits; similarly b_1, b_2 are 2 cards of the same denomination (but different from a) drawn from the 4 possible suits.

We can apply the basic Counting principle.

1. First select a number from the 13 possible numbers (Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2)
2. Once a number has been selected in stage 1, draw 3 cards of this denomination from the 4 possible suits
3. Then, pick a denomination from the remaining 12 numbers (not selected in stage 1).
4. Select 2 cards of this denomination from the 4 suits.

ANSWER: $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

So the probability of getting a full-house is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{3{,}744}{2{,}598{,}960} \approx .00144$$

$\boxtimes$

When we have n distinct objects, we can think of drawing a subset of size k as partitioning the n objects into two sets:

$$\left(\{a_1, a_2, \ldots, a_k\}, \{a_{k+1}, a_{k+2}, \ldots, a_n\} \right)$$

i.e, an <u>ordered</u> pair of sets; the first is the subset of size k and the second is the subset of objects that weren't selected in the first subset (i.e., the complement of the set in the first)

Therefore, there are $\binom{n}{k}$ partitions of an n-element set with k elements in the first part and $n-k$ elements in the 2nd part.

We now want to generalize to partitions with more than two parts.

Counting the number of partitions of an n-element set.

We have an n-element set and integers $n_1, n_2, \ldots, n_r \geq 0$ that sum to n:

$$n_1 + n_2 + \cdots + n_r = n.$$

The number of partitions of an n-element set into r disjoint subsets where the i^{th} subset has n_i elements is *

$$\binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1! n_2! n_3! \ldots n_r!}.$$

called a Multinomial Coefficient. In this context $\binom{n}{k} = \binom{n}{k, n-k}$.

* it is important to understand that this multinomial coefficient is counting the # of ordered ~~sets~~ subsets with the i^{th} subset having n_i elements.

Example ~~There~~ We plan to deal out 13 cards to each of 4 players. Then there are

$$\binom{52}{13,13,13,13} = \frac{52!}{13!\,13!\,13!\,13!}$$

ways we can deal out the 52 cards to 4 distinct players.

Note here a partition will look like

$$\Big(\underbrace{\{\qquad\}}_{\substack{\text{a 13-card hand}\\ \text{given to Player 1}}},\ \underbrace{\{\qquad\}}_{\substack{\text{a 13-card}\\ \text{hand given}\\ \text{to player 2}}},\ \underbrace{\{\qquad\}}_{\substack{\text{a 13-card}\\ \text{hand given}\\ \text{to player 3}}},\ \underbrace{\{\qquad\}}_{\substack{\text{a 13-card hand}\\ \text{given to player 4.}}}\Big)$$

* If the exact same hands we given to different players then this would be a different partition.

Related Question We deal out all 52 cards to ~~each~~ ~~4~~ 4 players — each player receives 13 cards. What is the probability each player receives an Ace?

$\boxed{\text{Compare with Example 1.33}}$

in textbook.

There are $\binom{48}{12,12,12,12} = \dfrac{48!}{12!\,12!\,12!\,12!}$ ways

of dealing the 48 non-Aces to ~~each~~ the 4 players

At this point we have a partition

$$\text{A} \qquad \left(\underbrace{\{\qquad\}}_{\substack{\text{12 card-hand} \\ \text{to player 1}}},\ \underbrace{\{\qquad\}}_{\substack{\text{12 cards} \\ \text{to player 2}}},\ \underbrace{\{\qquad\}}_{\substack{\text{to player} \\ 3}},\ \underbrace{\{\qquad\}}_{\text{to player 4}} \right)$$

Now, for each of these partitions we need to deal out the 4 Aces in such a way that each player receives <u>one</u> Ace. For the particular deal given in A above, we can pass out the Aces as follows:

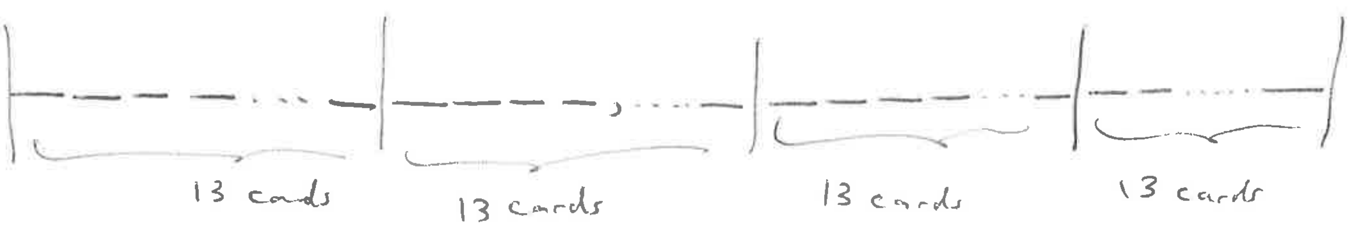
~~the~~ player 1 can receive any one of the 4 Aces. once player 1 receives their Ace, player 2 can receive any one of the 3 remaining Aces. Once players 1, 2 receives their Aces, player 3 can receive any one of the 2 remaining Aces. Finally, once players 1, 2, 3 receive their Aces, player 4 gets the one remaining Ace;

So $4 \cdot 3 \cdot 2 \cdot 1 \cdot \binom{48}{12,12,12,12}$ ways. So, the probability of each player receiving exactly one Ace is

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} = \frac{\frac{4!\;48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{13^4\cdot 4!}{52\cdot 51\cdot 50\cdot 49}$$

$$= \frac{\cancel{52}\cdot 39\cdot 26\cdot 13}{\cancel{52}\cdot 51\cdot 50\cdot 49} \approx .1055$$

Another approach. (where we keep track of order) pass out the cards in a line of 52 — the 1st 13 go to player 1 the next 13 to player 2, etc.



$$\frac{(4\cdot 13)\cdot(3\cdot 13)\cdot(2\cdot 13)\cdot(1\cdot 13)\cdot 48!}{52!} \approx .1055$$

Matching problem

Consider any permutation of the numbers 1 through n. We say a match occurs at i if the number i appears in the i^{th} position. Thus, we define

$$M_i = \text{event there is a match at } i$$

$P(M_i) = \frac{1}{n}$ for each $i = 1, \dots, n$

If $i \neq j$, then $P(M_i \cap M_j) = \frac{1}{n(n-1)}$

If $i < j < k$, then $P(M_i \cap M_j \cap M_k) = \frac{1}{n(n-1)(n-2)}$

etc.

The event $\bigcup_{i=1}^{n} M_i$ is the event that there is at least one match.

The Inclusion-exclusion formulas will help tremendously in computing $P\left(\bigcup_{i=1}^{n} M_i\right)$.

Let's first start with $n = 3$.

$$P\left(\bigcup_{i=1}^{3} M_i\right) = P(M_1) + P(M_2) + P(M_3)$$

$$- P(M_1 \cap M_2) - P(M_1 \cap M_3) - P(M_2 \cap M_3)$$

$$+ P(M_1 \cap M_2 \cap M_3)$$

$$= 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 1}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!}$$

How about $n = 4$?

$$P\left(\bigcup_{i=1}^{4} M_i\right) = P(M_1) + P(M_2) + P(M_3) + P(M_4)$$

$$- P(M_1 \cap M_2) - P(M_1 \cap M_3) - P(M_1 \cap M_4)$$

$$- P(M_2 \cap M_3) - P(M_2 \cap M_4) - P(M_3 \cap M_4)$$

$$+ P(M_1 \cap M_2 \cap M_3) + P(M_1 \cap M_2 \cap M_4)$$

$$+ P(M_1 \cap M_3 \cap M_4) + P(M_2 \cap M_3 \cap M_4)$$

$$- P(M_1 \cap M_2 \cap M_3 \cap M_4)$$

$$= 4 \cdot \frac{1}{4} - 6 \cdot \frac{1}{4 \cdot 3} + 4 \cdot \frac{1}{4 \cdot 3 \cdot 2} - \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$$

How about for a general n?

$$P\left(\bigcup_{i=1}^{n} M_i\right) = \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} - \binom{n}{4}\frac{1}{n(n-1)(n-2)(n-3)} + - \cdots$$

etc.

$$= n \cdot \frac{1}{n} + \frac{n!}{2!(n-2)!} \cdot \frac{1}{n(n-1)} + \frac{n!}{3!(n-3)!} \cdot \frac{1}{n(n-1)(n-2)} - + \cdots$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots + (-1)^{n+1}\frac{1}{n!}$$

$$\approx$$

Since $\bigcup_{i=1}^{n} M_i$ is the event of at least one match in n

$\left(\bigcup_{i=1}^{n} M_i\right)^c = \bigcap_{i=1}^{n} M_i^c$ is the event of <u>no</u> matches in n.

$$P\left(\bigcap_{i=1}^{n} M_i^c\right) = 1 - P\left(\bigcup_{i=1}^{n} M_i\right)$$

$$= 1 - \left\{1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + - \cdots + (-1)^{n+1}\frac{1}{n!}\right\}$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}$$

$\approx \frac{1}{e} = e^{-1}$ using the fact that

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - + \cdots$ when $x = 1$.

2m people comprised of m married male/female couples.

$$\binom{2m}{2,2,2,\ldots,2}\Big/ m! = \frac{(2m)!}{2^m\ m!}$$ possible pairings ~~#~~ of all 2m people.

How many pairings have married couple # 1 paired?

Same as the # of pairings of the remaining 2m-2 people

$$\binom{2m-2}{2,2,\ldots,2}\Big/(m-1)! = \frac{(2m-2)!}{2!2!\cdots2!\ (m-1)!} = \frac{(2m-2)!}{2^{m-1}(m-1)!}$$

How many pairings have at least one couple paired?

Let M_i be the event married couple i paired.

We want $P\left(\bigcup_{i=1}^{m} M_i\right)$.

Can use the inclusion exclusion principle paradigm.

We know

$$P(M_i) = \frac{\binom{2m-2}{2,2,\ldots,2}\Big/(m-1)!}{\binom{2m}{2,2,\ldots,2}\Big/m!} = \frac{\frac{(2m-2)!}{2^{m-1}(m-1)!}}{\frac{(2m)!}{2^m\ m!}} = \frac{2^m\ m!\ (2m-2)!}{2^{m-1}(m-1)!\ (2m)!}$$

$$= \frac{2m}{2m(2m-1)} = \frac{1}{2m-1}$$

$i<j$

$$P(M_i \cap M_j) = \frac{\binom{2m-4}{2,2,2,\dots,2}\Big/(m-2)!}{\binom{2m}{2,2,\dots,2}\Big/m!} = \frac{\dfrac{(2m-4)!}{2^{m-2}(m-2)!}}{\dfrac{(2m)!}{2^m\, m!}} = \frac{2^m\, m!\,(2m-4)!}{2^{m-2}(m-2)!\,(2m)!}$$

$$= \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{m}\cdot\cancel{(m-1)}}{\cancel{2m}(2m-1)\cancel{(2m-2)}(2m-3)} = \frac{1}{(2m-1)(2m-3)}$$

$i<j<k$

$$P(M_i \cap M_j \cap M_k) = \frac{1}{(2m-1)(2m-3)(2m-5)}$$

$\vdots$

etc.

So

$$P\left(\bigcup_{i=1}^m M_i\right) = \frac{m}{2m-1} - \binom{m}{2}\frac{1}{(2m-1)(2m-3)} + \binom{m}{3}\frac{1}{(2m-1)(2m-3)(2m-5)} - \binom{m}{4}\frac{1}{(2m-1)(2m-3)(2m-5)(2m-7)}$$

$$+ - + - \dots$$

Partitions of integers Fix an integer $n > 0$.

Consider lists of length r

$$(x_1, x_2, \dots, x_r)$$

where each x_i is a non-negative integer and

$$x_1 + x_2 + \dots + x_r = n$$

That is, $(x_1, x_2, \dots, x_r)$ is a vector of nonnegative integers that sums to n.

Question: How many vectors are there that sum to n?

Answer: $\binom{n+r-1}{n}$.

Here are some situations where this type of counting arises...

- the number of ways n people on an elevator can get off on r different floors.
- the numbers of ways n indistinguishable marbles can be put into r (distinguishable) boxes.

- roll an r-sided die n times. then the number of possible (distinct) rolls.

For instance if $r=4$ and $n=2$.

Then

$\{1,1\}$ $\{1,2\}$ $\{1,3\}$ $\{1,4\}$

$\{2,2\}$ $\{2,3\}$ $\{2,4\}$

$\{3,3\}$ $\{3,4\}$

$\{4,4\}$

are distinct rolls, there are 10 of them:

$$\binom{2+4-1}{2} = \binom{5}{2} = 10 \checkmark$$

Random Variables

Suppose we have an experiment and a Sample space Ω.
A real-valued function defined on Ω is called a random variable (abbreviated r.v.). That is,

$$X : \Omega \to \mathbb{R}$$

X is a function whose domain Ω and whose codomain $\mathbb{R}$.

is a random variable

If $\omega \in \Omega$, then $X(\omega) = x \in \mathbb{R}$.

So a random variable associates a real number to every $\omega \in \Omega$.

Random variables are classified by the set of its possible values

1. If the values of a r.v. form a finite set or a countably infinite set, the r.v. is called <u>Discrete</u>.
2. If the values of a r.v form a subinterval of the real-line, the r.v. is called <u>Continuous</u>.

Basic examples

1. The random variable X that counts the number of heads in 2 tosses of a coin.

$\Omega = \{HH, HT, TH, TT\}$

$X(TT) = 0$, $X(HT) = X(TH) = 1$, $X(HH) = 2$

So, $X \in \{0,1,2\}$ ← a finite set

and X is a discrete r.v.

2. The random variable Y that counts the number of trials needed to see the first head.

$\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$

$Y(H) = 1$, $Y(TH) = 2$, $Y(TTH) = 3$, ... etc.

So, $Y \in \{1,2,3,4,\ldots\} =: \mathbb{Z}_+$ ← set of positive integers – a countably infinite set

and Y is a discrete r.v.

3. The r.v. R that measures the distance from the bulls-eye of a dart thrown at a dartboard.

$\Omega = \mathbb{R}^2$ or $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le c^2\}$

Then $R((x,y)) = \sqrt{x^2+y^2}$

So $R \in \{r \in \mathbb{R} : r \geq 0\}$ ← the set of non-negative reals – a "continuum" sub interval of $\mathbb{R}$

and R is a continuous r.v.

We start with a full treatment of

Discrete r.v.s first.

If X is a discrete r.v. we will associate with it a probability mass function (pmf, for short) p_X where

1. $p_X(x) > 0$ for each $x \in \{\text{values of } X\}$

2. $\sum_x p_X(x) = 1$, where the sum is over all the possible values of X

Typically, pmfs are either given or modeled.

Example An experiment is to toss 2 balanced 6-sided dice. Then each of the 36 possible elementary outcomes (i,j) where $i,j = 1,2,3,4,5,6$. are equally-likely.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Ω

Consider the r.v.s

X = sum of the up-faces

Y_1 = minimum value of the up-face

The possible values of X are

$$\{2,3,4,5,6,7,8,9,10,11,12\}.$$

Then event

$(X=2)$ corresponds to the event $\{(1,1)\} \subseteq \Omega$

$(X=3)$ " " $\{(1,2),(2,1)\}$

$(X=4)$ " " $\{(1,3),(2,2),(3,1)\}$

⋮

$(X=12)$ " " $\{(6,6)\}$

Therefore,

$p_X(2) = P(X=2) = \frac{1}{36}$

$p_X(3) = P(X=3) = \frac{2}{36}$

$p_X(4) = P(X=4) = \frac{3}{36}$

⋮

$p_X(12) = P(X=12) = \frac{1}{36}$

We can write this in <u>functional form</u>:

$$p_X(x) = \frac{|7-x|}{36} \quad \text{for } x \in \{2,3,4,\ldots,12\}$$

or

in <u>tabular form</u>:

x	2	3	4	5	6	7	8	9	10	11	12
$p_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Exercise: you show the pmf for Y_1 is

$$p_{Y_1}(y) = \frac{13-2x}{36} \quad \text{for } y = 1,2,3,4,5,6 \qquad p_{Y_1}(y) = 0 \text{ for all other } y.$$

also,

y	1	2	3	4	5	6
$p_{Y_1}(y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Not all discrete r.v.s are integer-valued.

Example

A lottery ticket costs \$.50. You can win \$0, \$1, \$10.

with respective probabilities .9, .075, .025.

Let X represent your net winnings.

Then

x	$-.5$	$.5$	9.50
$p_X(x)$	$.90$	$.075$	$.025$

A probability mass function once known, given or constructed, allows for straight-forward probability computations.

To compute

$$P(X \in A) = \sum_{x \in A} p_X(x)$$

A is a subset of $\mathbb{R}$

To compute the probability that X takes values in the set A, we just sum the probability masses at each value x in the set A.

In the dice example...

$$P(X \leq 4) = \sum_{x=2}^{4} p_X(x) = p_X(2) + p_X(3) + p_X(4) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36}$$

$$= \frac{6}{36} = \frac{1}{6}.$$

$$P(3 < X < 6) = p_X(4) + p_X(5) = \frac{3}{36} + \frac{4}{36} = \frac{7}{36}.$$

Some important discrete r.v.s distributions that comes up often in probability.

- the Bernoulli(p)
- the Binomial(n, p)
- the Geometric(p)
- the Poisson(λ)
- the hypergeometric
- the negative binomial (Pascal)

The Bernoulli(p)

A r.v. X is said to have the Bernoulli(p) distribution, written

$$X \sim \text{Bernoulli}(p)$$

if X has the pmf

$$p(x) = \begin{cases} p & \text{when } x = 1 \\ 1-p & \text{when } x = 0. \end{cases}$$

Here, p is a number between 0 and 1 $(0 \leq p \leq 1)$.

This distribution arises in the following situations....

1. Toss a coin once where the probability the coin comes up heads is p and therefore tails is $1-p$.

2. Infinite population with a proportion p of successes (and, therefore) a proportion $1-p$ of failures. and we select one of these at random.

The Bernoulli(p) r.v. is uninteresting by itself, but it forms the building block of many very interesting and important discrete pmfs.

The Binomial(n,p)

A random variable X is said to have the binomial(n,p) distribution, written

$$X \sim \text{binomial}(n,p)$$

if X has the pmf

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \ldots, n$$

Here, $n > 0$ is a positive integer that represents either the number of trials or the sample size.

$0 < p < 1$ represents the probability of drawing a success.