Intro Prob Lecture Notes

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Transformation Method (or method of Jacobians)

- For finding distributions of functions of continuous random variables
- (2-d) Theorem: Suppose X, Y are jointly continuous with joint pdf $f_{X, Y}(x, y) with support A and $u = g_1(x, y)$ and $v = g_2(x, y)$ is a one-to-one transformation of A into B. Then the inverse transformation is

$$x = h_1(u, v) \text{ and } y = h_2(u, v)$$

- and the joint pdf of U, V is of the form

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J|$$

- where J is the determinant of $\begin{bmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{bmatrix} = \frac{dx}{du} \cdot \frac{dy}{dv} - \frac{dx}{dv} \cdot \frac{dy}{du}$ (the Jacobian determinant)
- Why do we care?
  - Ex: V, R independent random variables $f_V(v), f_R(r)$, and $I = \frac{V}{R}$ (electricity)
  - Ex: $X_1, X_2, \dots, X_{100}$ and $\overline{X} = \frac{X_1 + X_2 + \cdots + X_{100}}{100}$
    - Find $S^2 = \frac{\sum^n (X_i - \overline{X})^2}{n-1}$
  - Ex: Suppose X, Y are independent, $X \sim \text{Gamma}(\alpha, 1)$ and $Y \sim \text{Gamma}(\beta, 1)$
    - $$\to f_{X,Y}(x, y) = \frac{x^{\alpha-1} e^{-x} y^{\beta-1} e^{-y}}{\Gamma(\alpha)\Gamma(\beta)} \text{ for } A = \{x > 0, y > 0\}$$
    - Define $U = \frac{X}{X+Y}$, $V = X + Y$, find the joint pdf of U, V
      - $x = uv, y = v(1 - u) = y$

- * $J = det \begin{bmatrix} v & u \\ -v & 1-u \end{bmatrix} = v(1-u) - u(-v) = v$
  * $$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, v(1-u)) \cdot |v| \\ &= \frac{(uv)^{\alpha-1}e^{-uv} \cdot (v(1-u))^{\beta-1}e^{-v(1-u)} \cdot v}{\Gamma(\alpha)\Gamma(\beta)} \\ &= \frac{v^{\alpha+\beta-1}e^{-v} \cdot \Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$
  * Note: check! $$f_U(u) = \frac{\Gamma(\alpha+\beta)u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$
  * $$f_V(v) = \frac{v^{\alpha+\beta-1}e^{-v}}{\Gamma(\alpha+\beta)}$$
  * In particular, U, V are independent and $U \sim$ Beta(α, β)
  * Remark: If allwe cared about was finding the distribution of $U = \frac{X}{X+Y}$, then we have a choice for V. For instance $U = \frac{X}{X+Y}$ and $V = X$ is one-to-one, as is $V = Y$.

- Quick remark about homework: $f_{X|X+Y}(x|u) = \frac{f_{X,Y}(x,u-x)}{f_{X+Y}(u)}$
  - $f_{X,X+Y}(x,u) = f_{X,Y}(x, u-x)$. Let $x = u, y = u - x, |J| = 1$ and see this was an example of the Transformation Method!
- Ex: $Z \sim \Phi(z)$, $U = \Phi(Z) \sim$ Uniform(0, 1). $\Phi^{-1}(U) = Z$
  - $Z_1, Z_2 \sim$ i.i.d. N(0, 1)$

$$f_{Z1,Z_2}(z_1, z_2) = \frac{e^{-\frac{1}{2}(z_1^2+z_2^2)}}{2\pi}(-\infty < z_1, z_2 < \infty)$$

  - Let $R = \sqrt{Z_1^2 + Z_2^2}$, $\Theta = arctan(\frac{Z_2}{Z_1}) \in (0, 2\pi]$
    * $\rightarrow Z_1 = Rcos(\Theta), z_2 = Rsin(\Theta), |J| = r$
  - $$\begin{aligned} f_{R,\Theta}(r,\theta) &= f_{Z_1,Z_2}(rcos(\theta), rsin(\theta)) \cdot r \\ &= \frac{re^{-\frac{1}{2}((rcos(\theta))^2+(rsin(\theta))^2)}}{2\pi} \\ &= re^{-\frac{r^2}{2}} \cdot \frac{1}{2\pi} \end{aligned}$$

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$$f_\Theta(\theta) = \frac{1}{2\pi} \text{ for } 0 < \theta \leq 2\pi$$
$$f_R(r) = re^{-\frac{r^2}{2}} \text{ Rayleigh distribution}$$

  * $F_R(r) = 1 - e^{-\frac{r^2}{2}}$
- Since $f_{R,\Theta} = f_R \cdot f_\Theta$, R, Θ independent
- Geometry: Uniform$(0, 1)$ is easy. 2π Uniform$(0, 1)$ = Uniform$(0, 2\pi)$
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Let $u \sim$ uniform(0, 1)

$$u = 1 - e^{-\frac{r^2}{2}} \rightarrow r$$
$$= (-2ln(1-u))^{\frac{1}{2}}$$
$$= (-2ln(U_2))^{\frac{1}{2}} cos(2\pi U_1)$$
$$= (-2ln(U_2))^{\frac{1}{2}} sin(2\pi U_1)$$