Intro Prob Lecture Notes

William Sun

May 5, 2017

The Chebyshev Inequalit(ies)

- For any random variable X with mean μ and variance σ^2, we have for any real number $k > 0$,

$$P(|X - \mu|) \geq k) \leq \frac{\sigma^2}{k^2}$$

- Equivalently (for the complement) ,

$$P(|X - \mu| < k) \geq 1 - \frac{\sigma^2}{k^2}$$

- Also equivalent to

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

  and

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

- Remember: Markov's Inequality is a corollary
- It also has a very important application to:

Weak Law of Large Numbers

- If $X_1, X_2, \ldots$ are independentrandom variables each having mean μ and variance σ^2, then

$$P\Big(\Big|\frac{X_1 + \cdots + X_n}{n} - \mu\Big| \geq \epsilon\Big) \to 0 \text{ as } n \to \infty$$

  - (Strong Law of Large Numbers):

$$P\Big(\lim_{n\to\infty}\Big|\frac{X_1 + \cdots + X_n}{n} - \mu\Big| = 0\Big) = 1$$

    * Too hard for us to prove, so we just prove the weak law

- Proof of Weak Law of Large Numbers:
  - $\overline{X} = \frac{X_1+\cdots+X_n}{n}$.
    * $E(\overline{X}_n) = \mu$
    * $Var(\overline{X}_n) = \frac{\sigma^2}{n}$
  - So apply Chebyshev's inequality
    *
$$P(|\overline{X}_n - \mu|) \geq k) \leq \frac{\sigma^2}{nk^2} \to 0 \text{ as } n \to \infty$$

- Remark: When (in WLLN) we also have identically distributed, the Central Limit Theorem $\implies$ WLLN.
  - We use the following fact (about the tail behavior of the standard normal distribution):

$$(\frac{1}{x} - \frac{1}{x^3})e^{-\frac{x^2}{2}} < \int_x^\infty e^{-\frac{z^2}{2}}dz < \frac{1}{x}e^{-\frac{x^2}{2}}$$

    for any x > 0. In other words, when $x \to \infty$, the ratio between the second two terms is 1.
  -
$$P\Big(\Big|\frac{\sum_{i=1}^{n} X_i - n\mu}{n}\Big| \geq \epsilon\Big) = P\Big(\Big|\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}}\Big| \geq \epsilon\sqrt{n}\Big) \approx \frac{2}{\sqrt{2\pi}}\int_{\epsilon\sqrt{n}}^{\infty} e^{-\frac{z^2}{2}}dz$$

  - As $n \to \infty$, we approach

$$\frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\epsilon\sqrt{n}}e^{-\frac{\epsilon^2 n}{2}}$$

Monte Carlo Method

- Ex:
  - $U \sim$ uniform(0, 1)
  - $h : [0,1] \to \mathbb{R}$
  - $X = h(U)$
  - Question: $E(X)$?
    - $E(X) = E(h(U)) = \int_0^1 h(u)du$
    - What if we can't integrate h in closed form?
  - Suppose sequence $U_1, U_2, U_3, \cdots \sim$ uniform(0, 1).
    - $X_i = h(U_i)$ $i = 1, 2, \ldots$
    - $E(X_i) = \int_0^1 h(u)du$
    - $Var(X_i) = E(X_i^2) - \Big(\int_0^1 h(u)du\Big)^2 = \int_0^1 h^2(u)du - \Big(\int_0^1 h(u)du\Big)^2$
    - $Var(X_i) \leq \int_0^1 h^2(u)du \leq 1$ if $\Big|h(x)\Big| \leq 1$
  - WLLN $\implies P\Big(\Big|\frac{\sum_{i=1}^n h(U_i)}{n} - \int_0^1 h(u)du\Big| \geq \epsilon\Big) \leq \frac{\sigma^2}{\epsilon^2 n}$
    - Given $\epsilon = .1$, $\frac{1}{(.1)^2 n} \leq .01 \implies n \geq 10^4$
  - Use randomness to solve deterministic problems

Mixture Distribution (Not on final)

- 2 coins: coin 0 has $\frac{1}{2}$ chance of head, coin 1 has $\frac{1}{3}$ chance of head
- Select mechanism: Pick coin 0 .4, coin 1 .6.
- Flip twice, let X count the number of heads.
- $X_0 \sim binom(2, \frac{1}{2})$, $X_1 \sim binom(2, \frac{1}{3})$
  - $X \sim .4\binom{(}{2}, \frac{1}{2}) + .6binom(2, \frac{1}{3})$
  - pmf of mixture: $.4\binom{2}{k}(\frac{1}{2})^2 + .6\binom{2}{k}(\frac{1}{3})^k(\frac{2}{3})^{2-k}$
    - $P(X = 2) = P(coin0)P(X = 2|coin0) + P(coin1)P(X = 2|coin1) = .4(\frac{1}{2})^2 + .6(\frac{1}{3})^2 = \frac{1}{6}$