# Homework #4

**A.4.1** $4 \times 4 \times 4 = \boxed{4^3}$ or $64$.

**A.4.2**

(a) $\underbrace{2 \times 2 \times \cdots \times 2}_{n\text{-times}} = \boxed{2^n}$

(b) $$\underbrace{2}_{\substack{\#\text{ coded} \\ \text{with 1} \\ \text{symbol}}} + \underbrace{2^2}_{\substack{\#\text{ coded} \\ \text{with 2} \\ \text{symbols}}} + \underbrace{2^3}_{\substack{\#\text{ coded} \\ \text{with 3} \\ \text{symbols}}} + \cdots + \underbrace{2^n}_{\substack{\#\text{ coded} \\ \text{with} \\ n\text{ symbols}}} = \frac{2^{n+1} - 2}{2 - 1} = \boxed{2(2^n - 1).}$$

**A.4.3** when $r = 1$ then probability correct key on $1^{st}$ try is $\frac{1}{n}$.

when $r = 2$, the prob. correct key on $2^{nd}$ try =

$$P(\text{wrong key on } 1^{st} \text{ try}) \times P(\text{correct key on } 2^{nd} \mid \text{wrong key on } 1^{st})$$

$$= \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}.$$

In general, the probability the correct key is the $r^{th}$ key tried is $\boxed{\frac{1}{n}}$.

$\boxed{A.4.4}$ The $1^{st}$ passenger can depart at any one of the 10 stops.

The $2^{nd}$ passenger can depart at any of the 9 remaining stops if she is not to leave at the same stop as the $1^{st}$ passenger.

Continuing in this manner...

$$P\left(\begin{array}{c}\text{no two passengers depart}\\ \text{at same stop}\end{array}\right) = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10} = \frac{(10)_6}{10^6}.$$

$\boxed{A.4.5}$ First of all $n \geq 2$ otherwise there cannot be 2 balls in any of the $r$ boxes. Secondly, $n \leq r$, otherwise we are certain some box will have at least 2 balls (i.e. the probability some box has 2 or more balls is 1). Therefore, $2 \leq n \leq r$.

For such an $n$, $n-1$ balls must be put into distinct boxes and the nth ball must be put into one of these ~~occu~~ $(n-1)$ occupied boxes.

Therefore, the probability is:

$$\frac{(n-1)\,(r)(r-1)(r-2)\cdots(r-(n-2))}{r^n} = \frac{(n-1)\,(r)_{n-1}}{r^n}.$$

**A.4.6** Suppose the $1^{st}$ replication of the experiments choses $N$ distinct numbers.

Let $A_j$ (for $j \geq 2$) be the event that the $j^{th}$ replication choses none of these $N$ numbers. $A_2, A_3, \ldots, A_r$ are independent events. and for any $2 \leq j \leq r$

$$P(A_j) = \frac{\binom{N}{0}\binom{r-N}{N}}{\binom{r}{N}} = \frac{\binom{r-N}{N}}{\binom{r}{N}}.$$

Therefore,

$$P\left(\bigcap_{j=2}^{r} A_j\right) = \prod_{j=2}^{r} \frac{\binom{r-N}{N}}{\binom{r}{N}} = \left(\frac{\binom{r-N}{N}}{\binom{r}{N}}\right)^{r-1}.$$

**A.4.7** When $k=0$

A B * * * * .... * * *
* A B * * * .... * * *
* * A B * * .... * * *
* * * A B * .... * * *
* * * * A B .... * * *
⋮
* * * * * * .... A B *
* * * * * * .... * A B

When $k=0$, there are $n-1$ positions in the line of $n$ that have A followed by B, and the same number of positions having B followed by A. Therefore, when $k=0$, there are $2(n-1)$ ~~lines~~ positions have A and B separated by 0 people.

Each of the $2(n-1)$ positions correspond the $(n-2)!$ line-ups, So the prob that $k=0$ is $\dfrac{2(n-1)(n-2)!}{n!} = \dfrac{2}{n}$.

A.4.7 (continued)

When $k=1$, $A$ and $B$ can be placed in $n-2$ distinct places in line. Each of the positions in line correspond to $(n-2)!$ lineups and we double this since we can swap $A$ and $B$ on any of these lineups. Thus the prob that $k=1$ is

$$\frac{2(n-2)(n-2)!}{n!} = \frac{2(n-2)}{n(n-1)}.$$

When $k=2$ a similar analysis shows the prob that $k=2$ is

$$\frac{2(n-3)(n-2)!}{n!} = \frac{2(n-3)}{n(n-1)}.$$

and for general $k$ with $0 \le k \le n-2$ we have the prob that $A$ and $B$ are separated by $k$ people is

$$\frac{2(n-k-1)(n-2)!}{n!} = \boxed{\frac{2(n-k-1)}{n(n-1)}}.$$

$\boxed{A.4.8}$

(a) Let $M_i$ be the event of a match at position $i$

Then

$$P(M_i \cap M_r^c) = P(M_i) - P(M_i \cap M_r)$$

$$= \frac{1}{n} - \frac{1}{n(n-1)} = \boxed{\frac{n-2}{n(n-1)}}.$$

(b) $$P(M_i \mid M_r^c) = \frac{P(M_i \cap M_r^c)}{P(M_r^c)} = \frac{\frac{n-2}{n(n-1)}}{1-\frac{1}{n}} = \boxed{\frac{n-2}{(n-1)^2}}.$$

$\boxed{A.4.9}$

(a) Box 1 is the only empty box if and only if 2 of the $n$ balls go into some box other than box 1 and the remaining $n-2$ balls are distributed among the remaining $n-2$ boxes, one in each box. This can happen in

$$\binom{n}{2} \cdot (n-1)(n-2)(n-3) \cdots (3) \cdot 2 \cdot 1 \quad \text{ways.}$$

Select which 2 balls will be doubled up.

select one of the $n-1$ positions to put the two balls.

there are any of $n-2$ balls that can go in the 1st unoccupied box

there are any of $n-3$ balls that can go in the 2nd unoccupied box

etc.

Therefore, there are $\binom{n}{2}(n-1)!$ ways that Box 1 is the only empty box. and the probability that this happens is

$$\frac{\binom{n}{2}(n-1)!}{n^n}$$

(b) If we let $B_i$ be the event that box $i$ is the only empty box, then $\bigcup_{i=1}^{n} B_i$ is the event of exactly one empty box

and moreover, $B_1, B_2, \dots, B_n$ are mutually exclusive (i.e., disjoint)

and $P(B_1) = P(B_2) = \cdots = P(B_n)$. Therefore,

$$P\left(\bigcup_{i=1}^{n} B_i\right) = P(B_1) + P(B_2) + \cdots + P(B_n) = n\,P(B_1) = \frac{\binom{n}{2} n!}{n^n}.$$

(c) ~~P(box 1 is empty) =~~

Box 1 is empty just means that the $n$ balls are distributed among the other $n-1$ boxes in any fashion. This can happen in

$$(n-1)^n \text{ ways.}$$

So the probability that Box 1 is empty is

$$\boxed{\frac{(n-1)^n}{n^n}} \text{ or } \left(1-\frac{1}{n}\right)^n.$$

Therefore,

$$P\left(\bigcup_{i=1}^{n} B_i \,\middle|\, \text{Box 1 is empty}\right) = \frac{P\left(\bigcup_{i=1}^{n} B_i \cap \text{box 1 is empty}\right)}{P(\text{Box 1 is empty})}$$

$$= \frac{P(B_1)}{P(\text{Box 1 is empty})} = \frac{\frac{\binom{n}{2}(n-1)!}{\cancel{n^n}}}{\frac{(n-1)^n}{\cancel{n^n}}} = \boxed{\frac{\binom{n}{2}(n-1)!}{(n-1)^n}}.$$

(d) $$P\left(\text{Box 1 is empty} \,\Big|\, \bigcup_{i=1}^{n} B_i\right) = \frac{P\left(\text{Box 1 is empty} \cap \bigcup_{i=1}^{n} B_i\right)}{P\left(\bigcup_{i=1}^{n} B_i\right)}$$

$$= \frac{P(B_i)}{P\left(\bigcup_{i=1}^{n} B_i\right)} = \frac{\dfrac{\cancel{\binom{n}{2}}(n-1)!}{\cancel{n^n}}}{\dfrac{\cancel{\binom{n}{2}}\, n!}{\cancel{n^n}}} = \frac{(n-1)!}{n!} = \boxed{\frac{1}{n}}.$$

$\boxed{\text{A.4.10}}$ Each ball is equally likely to be put in any of the $r$ boxes. Probability a ball is put in box 1 is $\frac{1}{r}$, and we have $n$ independent trials.

$$P(\text{exactly } j \text{ balls in box } 1) = \binom{n}{j}\left(\frac{1}{r}\right)^{j}\left(\frac{r-1}{r}\right)^{n-j}$$

for $0 \le j \le n$ integer.

$\boxed{\text{A.4.11}}$ If the first black ball occurs at trial $n$ then this happens if and only if the first $n-1$ selections were all red balls.

$$P(1^{st} \text{ black on trial } n) = \frac{r}{b+r} \cdot \frac{r-1}{b+r-1} \cdots \frac{r-(n-2)}{b+r-(n-2)} \cdot \frac{b}{b+r-(n-1)}$$

$$= \frac{(r)_{n-1} \cdot b}{(b+r)_n}.$$

$\boxed{A.4.12}$ $P(\text{balls 1 and 6 in our selection of 3}) = P\left(\begin{array}{c}\text{draw a number}\\ \text{that is not 1 or 6}\end{array}\right)$

$$= \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{120} = \frac{1}{15}.$$

$\boxed{A.4.13}$ $P(\text{all good}) = \dfrac{\binom{40}{10}}{\binom{50}{10}}$

$$P\left(\begin{array}{c}\text{at most one}\\ \text{defective}\end{array}\right) = \frac{\binom{40}{10} + \binom{40}{9}\binom{10}{1}}{\binom{50}{10}}.$$

$\boxed{\text{A.4.141}}$

(a) $$\frac{4}{\binom{52}{5}}$$

(b) $$\frac{10 \cdot 4 - 4}{\binom{52}{5}}$$

(c) $$\frac{13 \cdot 48}{\binom{52}{5}}$$

(d) $$\frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}}$$

(e) $$\frac{4\binom{13}{5} - 10 \cdot 4 - 4}{\binom{52}{5}}$$

(f) $$\frac{10 \cdot 4^5 - 10 \cdot 4}{\binom{52}{5}}$$

(g) $$\frac{13\binom{4}{3}\binom{12}{2}4^2}{\binom{52}{5}}$$

(h) $$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \, 44}{\binom{52}{5}}$$

(i) $$\frac{13\binom{4}{2} \cdot \binom{12}{3}4^3}{\binom{52}{5}}$$

## A.4.15

First king appears on $n^{th}$ draw iff $1^{st}$ $n-1$ are non-kings and $n^{th}$ is king. Therefore, prob $1^{st}$ king on $n^{th}$ draw is:

$$\frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\cdots\frac{48-(n-2)}{52-(n-2)}\cdot\frac{4}{52-(n-1)} = \frac{4\,(48)_{n-1}}{(52)_n}$$

## A.4.16

Prob getting distinct rolls. $\quad \dfrac{\cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}}{\cancel{6}\cdot \cancel{6}_3\cdot \cancel{6}_3\cdot \cancel{6}_2} = \dfrac{5}{18}$

Prob getting exactly one pair $\quad \dfrac{\cancel{6}\binom{\cancel{4}}{2}\cdot 5\cdot \cancel{4}}{\cancel{6}\cdot \cancel{6}\cdot \cancel{6}_3\cdot \cancel{6}_3} = \dfrac{5}{9}$

So getting exactly one pair is twice as likely as distinct numbers.

## A.4.17.

Suppose $\{x_1, x_2, \dots, x_n\}$ are the $n$ numbers selected from the first box. the prob. a random sample of size $n$ from the second box have a subset of size $k$ in common with the first sample is

$$\frac{\binom{n}{k}\binom{r-n}{n-k}}{\binom{r}{n}}.$$

## A.4.18

(a) $$\binom{100}{30,45,15,8,2} = \frac{100!}{30!\,45!\,15!\,8!\,2!}$$

(b) $$\frac{\binom{96}{28,43,15,8,2}}{\binom{100}{30,45,15,8,2}} = \frac{\frac{96!}{28!\,43!\,\cancel{15!}\,\cancel{8!}\,\cancel{2!}}}{\frac{100!}{30!\,45!\,\cancel{15!}\,\cancel{8!}\,\cancel{2!}}} = \frac{30\cdot 29\cdot 45\cdot 44}{100\cdot 99\cdot 98\cdot 97}.$$

## A.4.19

(a) $$\binom{40}{2,2,\ldots,2}\Big/20! = \frac{40!}{2^{20}(20!)}.$$

(b) $$\binom{38}{2,2,\ldots,2}\Big/19! = \frac{38!}{2^{19}(19!)}$$

(c) $$\frac{\frac{38!}{2^{19}(19!)}}{\frac{40!}{2^{20}(20!)}} = \frac{38!\,2^{20}(20!)}{40!\,2^{19}(19!)} = \frac{2\cdot 20}{40\cdot 39} = \frac{1}{39}.$$

Yes, intuitively 1 out of 39 will pair up with Fred.

$\boxed{/\!/}$