HW #10 Solutions to Additional problems

$\boxed{A.10.1}$ $X_1, X_2 \sim$ independent geometric(p) Let $u \geq 2$.

$$P_{X_1+X_2}(u) = \sum_{x=1}^{u} p(1-p)^{x-1} \cdot p(1-p)^{u-x-1}$$

$$= p^2(1-p)^{u-2} \sum_{x=1}^{u} 1 = (u-1)p^2(1-p)^{u-2}$$

That is,

$$P_{X_1+X_2}(u) = \binom{u-1}{1} p^2 (1-p)^{u-2} \quad \text{for } u \geq 2.$$

Therefore, $X_1 + X_2 \sim$ neg. binom$(2, p)$.

$\boxed{A.10.2}$ $X_1, X_2 \sim \exp(\frac{1}{\beta})$ and are independent.

$$f_{X_1+X_2}(u) = \int_0^u \frac{1}{\beta} e^{-x/\beta} \cdot \frac{1}{\beta} e^{-\frac{(u-x)}{\beta}} dx$$

$$= \frac{1}{\beta^2} \cdot e^{-u/\beta} \int_0^u dx = \frac{u e^{-u/\beta}}{\beta^2} \sim \text{Gamma}(2, \beta)$$

Suppose X_3 is independent of X_1 and X_2. Then X_3 is independent of $X_1 + X_2$, and

$$f_{X_1+X_2+X_3}(u) = \int_0^u f_{X_1+X_2}(x) f_{X_3}(u-x) dx = \int_0^u \frac{1}{\beta^2} x e^{-x/\beta} \cdot \frac{1}{\beta} e^{-\frac{u-x}{\beta}} dx$$

$$= \frac{1}{\beta^3} e^{-u/\beta} \int_0^u x \, dx = \frac{u^2 e^{-u/\beta}}{\beta^3 \cdot 2} \sim \text{Gamma}(3, \beta).$$

Suppose

$$X_1 + X_2 + \dots + X_k \sim \text{Gamma}(k, \beta) \text{ for some } k \geq 3.$$

Then

$$f_{X_1+X_2+\dots+X_{k+1}}(u) = \int_0^u f_{X_1+\dots+X_k}(x)\, f_{X_{k+1}}(u-x)\, dx$$

$$= \int_0^u \frac{x^{k-1} e^{-x/\beta}}{\beta^k (k-1)!} \cdot \frac{1}{\beta} e^{-\frac{(u-x)}{\beta}}\, dx$$

$$= \frac{1}{\beta^{k+1}(k-1)!} e^{-u/\beta} \int_0^u x^{k-1}\, dx = \frac{u^k e^{-u/\beta}}{\beta^{k+1}\, k!} \sim \text{Gamma}(k, \beta)$$

Therefore,

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta).$$

$\boxed{A.10.3}$

$$f_{X|X+Y}(x|u) = \frac{f(x, u-x)}{f_{X+Y}(u)} = \frac{f(x)\, f(u-x)}{f_{X+Y}(u)}$$

$$= \frac{e^{-x} \cdot e^{-(u-x)}}{u e^{-u}} = \frac{1}{u} \text{ for } 0 < x < u.$$

That is, $X | X+Y = u \sim \text{uniform}(0, u)$.

A-10.4

~~trial~~ trial j results in outcome i with prob p_i or not i with prob. $1-p_i$

Y_i: counts the number of times outcome i occurs in n independent trials.

Thus, $Y_i \sim$ binomial (n, p_i).

A-10.5

(a) $P(Y_n \le y) = P(\max\{X_1, \dots, X_n\} \le y)$

$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$

$= P(X_1 \le y)\, P(X_2 \le y) \cdots P(X_n \le y)$

if $0 \le y \le 1$ $= y \cdot y \cdots y = y^n$.

(b) $P(Y_1 \le y) = 1 - P(Y_1 > y)$

$= 1 - P(\min\{X_1, \dots, X_n\} > y)$

$= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y)$

$= 1 - P(X_1 > y)\, P(X_2 > y) \cdots P(X_n > y)$

if $0 \le y \le 1$ $= 1 - (1-y)\cdot(1-y)\cdots(1-y) = 1 - (1-y)^n$.

(c) $f_n(y) = n y^{n-1}$ for $0 \le y \le 1$ is pdf of Y_n

$f_1(y) = n(1-y)^{n-1}$ for $0 \le y \le 1$ is pdf of Y_1

$\boxed{A.10.5}$

part(c) (continued)

$$E(Y_n) = \int_0^1 y\, f_n(y)\, dy = \int_0^1 y \cdot n y^{n-1}\, dy = n \int_0^1 y^n\, dy$$

$$= \frac{n}{n+1}.$$

$$E(Y_1) = \int_0^1 y \cdot n(1-y)^{n-1}\, dy = n \int_0^1 y(1-y)^{n-1}\, dy$$

recognize a normalizing constant of a Beta(2,n) distribution

$$= n \cdot \frac{\Gamma(2)\Gamma(n)}{\Gamma(2+n)} = n \cdot \frac{(n-1)!}{(n+1)!} = \frac{1}{n+1}.$$

(d) $G_n(w) = P(n \cdot \min\{X_1, \ldots, X_n\} \le w)$

$$= P\left(\min\{X_1, \ldots, X_n\} \le \frac{w}{n}\right) = 1 - P\left(\min\{X_1, \ldots, X_n\} > \frac{w}{n}\right)$$

$$= 1 - P\left(X_1 > \frac{w}{n}, X_2 > \frac{w}{n}, \ldots, X_n \geq \frac{w}{n}\right)$$

$$= 1 - \left(1 - \frac{w}{n}\right)^n \text{ for } 0 \le w \le n.$$

If $w > 0$, then some n will exceed w, i.e., $w \le n$ for some n eventually and

$$G_m(w) = 1 - \left(1 - \frac{w}{m}\right)^m \text{ for all } m \geq n.$$

Therefore,

$$\lim_{m\to\infty} G_m(w) = \lim_{m\to\infty}\left(1 - \left(1 - \frac{w}{m}\right)^m\right) = 1 - \lim_{m\to\infty}\left(1 - \frac{w}{m}\right)^m$$

$$= 1 - e^{-w}.$$

which is the cdf of an exp(1).

$\boxed{A.10.6}$ For $x > 0$.

(a) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} e^{-y}\,dy = -e^{-y}\Big]_{x}^{\infty} = e^{-x}$$

for $y > 0$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{y} e^{-y}\,dx = y e^{-y}.$$

(b) Since $f_{X,Y}(x,y) = e^{-y}$ ~~and~~ for $0 < x < y < \infty$

and $f_X(x) f_Y(y) = e^{-x} \cdot y e^{-y} = y e^{-(x+y)}$ for $0 < x < y < \infty$

and these functions disagree, for instance, when $x = 1$ and $y = 2$, these rvs are <u>dependent</u>.

(c) $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-y}}{y e^{-y}} \text{ for } 0 < x < y.$$

$$= \frac{1}{y} \text{ for } 0 < x < y,$$ i.e. $X|Y=y \sim \text{Uniform}(0,y)$.

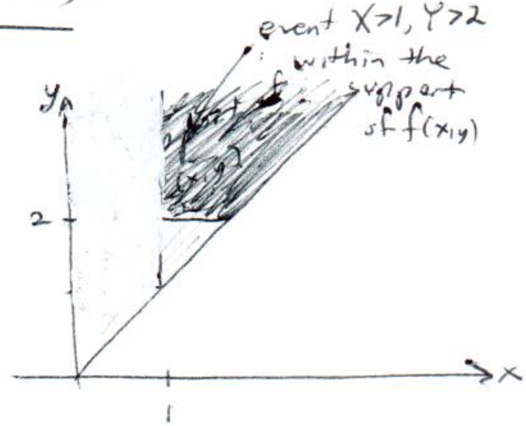
$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \text{ for } y > x.$$

A. 10.61 (continued)

(d) $$P(Y>2 \mid X=1) = \int_2^{\infty} f_{Y|X}(y|1)\,dy$$

$$= \int_2^{\infty} e^{-(y-1)}\,dy = -e^{-(y-1)}\Big]_2^{\infty} = e^{-1}.$$

$$P(Y>2 \mid X>1) = \frac{P(X>1, Y>2)}{P(X>1)}$$



$$= \frac{\int_2^{\infty}\int_1^{y} e^{-y}\,dx\,dy}{\int_1^{\infty} e^{-x}\,dx}$$

$$= \frac{\int_2^{\infty}(y-1)e^{-y}\,dy}{e^{-1}} = \frac{-(y-1)e^{-y}\Big|_2^{\infty} + \int_2^{\infty} e^{-y}\,dy}{e^{-1}} = \frac{e^{-2}+e^{-2}}{e^{-1}}$$

$$= 2e^{-1} = \frac{2}{e}.$$