550.420 Probability - FALL 2015 **FINAL EXAM**

PRINT NAME HERE______________________________

I agree to complete this examination without unauthorized assistance from any person, materials, or device.

Signature: ______________________________________

Instructions: This is a closed book examination. No notes are permitted. No cell phone use is permitted. Answers are to be written clearly and concisely, and clearly labeled (for example, with a box drawn around the intended answer). Please answer each question on the page on which it is stated, using the back of the page if necessary. It is important to show and explain your work; your answer must be properly justified (with at least key words and phrases) to receive full credit.

Problem 01 [10,10]
Problem 02 [10,10]
Problem 03 [10]
Problem 04 [5,5,5,5]
Problem 05 [10]
Problem 06 [5,5]
Problem 07 [10,10,10]
Problem 08 [5,5,10*]
Problem 09 [10]
Problem 10 [10,10]
Problem 11 [10]
Problem 12 [10,10]
Problem 13 [10]
Problem 14 [10,10]
Total = 220 pts + 10 pt bonus*

1. These are separate questions. Do not simplify.

(a) A classroom consists of 12 female and 8 male students. Six (6) students are randomly selected for a class exercise. Compute the probability a selection will have an equal number of females and males.

(b) A deck of 52 cards is shuffled thoroughly and then turned over one after the other in a line. Compute the probability that the 4 Aces are next to each other.

2. A father has three (3) sons: Fred, Greg, and Henry. Only one of his son's can go to work with him for Father-Son Day. The father plans to pick his son in one of two ways:

(Plan 1) The father picks a number from the set $\{1, 2, 3\}$. He starts with Fred and tells him to pick a number from the set $\{1, 2, 3\}$. If Fred correctly guesses the number he is thinking, Fred makes the trip; otherwise, the father asks Greg to pick one of the two remaining numbers. If Greg correctly guesses, Greg goes; otherwise, Henry goes.

(Plan 2) The father picks a number from the set $\{1, 2, 3\}$. The father asks each son to secretly write a number from the set $\{1, 2, 3\}$ on a piece of paper. The father then looks at Fred's paper first. If the number agrees with his, Fred goes. Otherwise, the father looks at Greg's paper. If Greg wrote his father's number, Greg goes; otherwise, the father looks at Henry's paper, and if Henry picked his father's number, Henry goes.

Let F be the event that Fred goes, G the event that Greg goes, and H the event that Henry goes. Compute $P(F), P(G)$ and $P(H)$ under each plan.

3. Four (4) men put their hats in a room, they are mixed and each man selects a hat at random.

(a) Compute the probability that at least one man receives his own hat. *Simplify.*

(b) Given that the first man selects his own hat, compute the conditional probability that at least one other man selects his own hat. *Simplify.*

4. Do not simplify unless noted otherwise.

A coin having head probability p with $0 < p < 1$ is tossed independently and repeatedly. Let

H_n be the number of heads in the first n tosses, and

N_j be the trial of the jth head.

(a) Compute $P(H_3 = 2)$.

(b) Compute $P(N_1 = 3)$.

(c) Compute $P(H_4 = 3, N_1 = 1)$

(d) Compute $P(H_3 = 2 | N_3 = 5)$. *Simplify.*

5. Let A, B, and C be events with $P(A) = 0.3, P(B) = 0.4, P(C) = 0.5$, where A and B are mutually exclusive, A and C are independent, and $P(B|C) = 0.1$. Find the value of $P(A \cup B \cup C)$. Hint: *Perhaps drawing a Venn diagram of this situation may help.*

6. On the Ph.D. Candidacy examination in given department, each student is allowed two attempts to pass. Experience shows that 60% of the students pass on the first try, and for those who don't, 80% pass on the second try.

(1) What is the probability that a student passes the Candidacy exam?

(2) If a student passed the Candidacy exam, what is the probability that he or she passed it on the first attempt?

7. You may use the result of a part (even if you can't do it) to aid in other parts.
The Poisson(λ) distribution has pmf $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$ and mgf $M(s) = e^{\lambda(e^s - 1)}$.

(a) Show that if $X \sim$ Poisson(λ), then $E(X) = var(X) = \lambda$.

(b) Let $X_1, X_2, X_3, \ldots$ be i.i.d. Poisson(λ) random variables. Show that $\sum_{j=1}^{n} X_j$ has a Poisson($n\lambda$) distribution.

(c) Suppose X_j represents the number of category-5 hurricanes in a year. Assume $X_j \sim$ Poisson(.36) are independent. Estimate the probability that in 100 years we will observe *at most* 49 category-5 hurricanes, i.e., estimate $P(\sum_{j=1}^{100} X_j \leq 49)$. *Leave in terms of* $\Phi(z)$ *for appropriate* z.

8. Person A tosses a balanced coin n times. Let X be the number of heads that person A tosses. Then person B tosses the same coin X times. Let Y be the number of heads person B tosses.
(a) Find the joint pmf of X and Y, and specify the domain of this function.
(b) Compute $E(Y)$. Hint: $Y|X \sim \text{binomial}(X, \frac{1}{2})$.
(c) [BONUS] Compute the marginal pmf of Y. *This part is challenging.*

9. Suppose X has an exponential distribution with parameter $\lambda = 1$ (pdf: $f(x) = e^{-x}$ for $x > 0$), and let $a > 0$ be a constant. Find the pdf of $Y = \sqrt{X/a}$ by any method you wish.

10. Suppose X and Y are jointly continuous random variables having joint pdf

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Show that $f(x, y)$ is not the product of its marginal pdfs. What can we conclude from this?
(b) Consider the following transformation: $U = XY,\ V = 1/Y$. Find the joint pdf of U, V.

11. If X has a normal distribution with mean μ and variance σ^2, then show that $Z = \frac{X-\mu}{\sigma}$ has a normal distribution with mean 0 and variance 1.

12. The parts of this problem are separate.

(a) X takes on the values 1, 2, 3, 4 with respective probabilities $c, 2c, 3c, 4c$.
Find c and compute $E(X)$.

(b) If $X \sim \text{Gamma}(\alpha, \beta)$, compute $E(\frac{1}{X})$. You may assume $\alpha > 1$. Then show it is not $\frac{1}{E(X)}$.

Pdf of $\text{Gamma}(\alpha, \beta)$ is $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$ for $x > 0$.

13. Suppose $X_1, X_2, X_3, \ldots, X_{2n}$ are i.i.d. Bernoulli(p). Let W_m be the sample mean of the first m X's, i.e., $W_m = \frac{1}{m}\sum_{i=1}^{m} X_i$. Find the covariance between W_n and W_{2n}.

14. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. having pdf $f(x)$ and cdf $F(x)$. In class we showed that if $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_n = \max\{X_1, X_2, \ldots, X_n\}$, then the marginal pdf of Y_1, Y_n is

$$f_{Y_1,Y_n}(y_1, y_n) = n(n-1)f(y_1)f(y_n)[F(y_n) - F(y_1)]^{n-2}, \text{ for } y_1 < y_n$$

and the marginal pdf of Y_j is

$$f_{Y_j}(y_j) = \frac{n!}{(j-1)!(n-j)!} f(y_j)[F(y_j)]^{j-1}[1 - F(y_j)]^{n-j}.$$

(a) Write down what each of these pdfs look like in the case where $n = 3$, $X_1, X_2, X_3 \sim \text{uniform}(0, 1)$. Specifically, write down the joint pdf of Y_1, Y_3, and the pdfs for Y_1, Y_2 and Y_3 separately. *Simplify as much as you can, and be sure to write the domains.*

(b) (continued from part (a)) Compute $E(Y_2)$ and $Var(Y_2)$.