550.420 Probability - SPRING 2016 **MIDTERM 1 SOLUTIONS**

1. Let $P[A \cap B] = .2$, $P[A] = .6$ and $P[B] = .5$. Find $P[A^c \cap B^c]$ and $P[A^c \cup B^c]$. Clearly label each.

$P[A^c \cap B^c] = P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - \{P[A] + P[B] - P[A \cap B]\} = 1 - \{.6 + .5 - .2\} = .1.$
$P[A^c \cup B^c] = P[(A \cap B)^c] = 1 - P[A \cap B] = 1 - P[A \cap B] = 1 - .2 = .8.$

2. Eleven (11) girls on a soccer team are randomly grouped into 4 positions comprised of 2 forwards, 4 midfielders, 4 defenders, and 1 goalie. *There's no need to simplify your answers.*
(a) How many different groupings are possible?

$\binom{11}{2,4,4,1}$.

(b) What is the probability that random grouping has Maddie as a midfielder and Denise as a defender?

$\frac{\binom{9}{2,3,3,1}}{\binom{11}{2,4,4,1}}$.

3. A coin with head probability $\frac{2}{3}$ is tossed. If a head occurs, 1 balanced 6-sided die is rolled; if a tail occurs, 2 balanced 6-sided dice are rolled. Compute the probability the sum of the dice is 6.

The law of total probability helps here:
$P[\{\text{sum is } 6\}] = P[H] \cdot P[\{\text{sum is } 6\}|H] + P[H^c] \cdot P[\{\text{sum is } 6\}|H^c] = \frac{2}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{5}{36} = \frac{17}{108}.$

4. Fred throws darts at a dartboard. On each trial, independently of other trials, he hits the bulls-eye with probability .05. How many darts should he throw so that the probability of hitting the bulls-eye (at least once) is .5?

Let H_i be the event that Fred hits the bulls-eye on the i-th throw.
We want $P[\bigcup_{i=1}^n H_i] = .5$. By complementary probability, DeMorgan's rule, and independence of the H_i:

$$P[\bigcup_{i=1}^n H_i] = 1 - P[\bigcap_{i=1}^n H_i^c] = 1 - P[H_1^c]P[H_2^c] \cdots P[H_n^c] = 1 - (.95)^n = .5$$

which implies $(.95)^n = .5$ and taking logarithms to solve for n we get $n = \frac{\ln(.5)}{\ln(.95)} \approx 14$ throws.

5. Two marbles are drawn without replacement from a box containing 3 red and 3 blue marbles. Let R be the event that at least one red marble is drawn, S the event that both marbles are the same color. Are the events R and S independent? Justify your assertion. *In your calculations ignore the order in which marbles are selected.*

$P[R] = \frac{\binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{0}}{\binom{6}{3}} = \frac{4}{5}.$
$P[S] = \frac{\binom{3}{2}\binom{3}{0} + \binom{3}{0}\binom{3}{2}}{\binom{6}{3}} = \frac{2}{5}.$
$P[R \cap S] = \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{1}{5} \neq P[R]P[S] = \frac{4}{5} \cdot \frac{2}{5} = \frac{8}{25}$, therefore, R nd S are NOT independent.

(over)

6. In 1992, Teen Talk Barbie dolls were introduced. Each Teen Talk Barbie doll could speak 4 of 270 possible sentences. One of the sentences was "Math class is tough!" – which caused a lot of protest, and production of dolls that had that sentence in their four-sentence repertoire were discontinued. Before the model was discontinued, what is the probability that a randomly-chosen Teen Talk Barbie doll could say the offending sentence? *Simplify completely.*

There are $\binom{270}{4}$ possible such dolls (one each having 4 sentences drawn from 270 possible sentences). The number of dolls having the offending sentence is $\binom{269}{3}$. Thus, assuming a doll picked at random can be any of the $\binom{270}{4}$ equally-likely, we would have the probability a doll can say the offending sentence as

$$\frac{\binom{269}{3}}{\binom{270}{4}} = \frac{\frac{269 \cdot 269 \cdot 267}{3 \cdot 2 \cdot 1}}{\frac{270 \cdot 269 \cdot 269 \cdot 267}{4 \cdot 3 \cdot 2 \cdot 1}} = \frac{4}{270} \approx .0148.$$

7. Suppose a box has 4 chips numbered 1 through 4. We draw chips one at a time without replacement and note its number. Let C_i be the event that there is a match at the i-th draw; i.e., the i-th chip drawn is numbered i. *Simplify completely.*
(a) Compute the probability of at least one match in the 4 draws, i.e., $P[\bigcup_{i=1}^4 C_i]$.

We note that $P[C_i] = \frac{1}{4}$ for each i. Also,
$P[C_i \cap C_j] = \frac{1}{4 \cdot 3}$ for all $1 \le i < j \le 4$,
$P[C_i \cap C_j \cap C_k] = \frac{1}{4 \cdot 3 \cdot 2}$ for all $1 \le i < j < k \le 4$, and
$P[C_1 \cap C_2 \cap C_3 \cap C_4] = \frac{1}{4!}$. So, by the inclusion-exclusion rule we get
$P[C_1 \cup C_2 \cup C_3 \cup C_4] = \binom{4}{1} P[C_1] - \binom{4}{2} P[C_1 \cap C_2] + \binom{4}{3} P[C_1 \cap C_2 \cap C_3] - \binom{4}{4} P[C_1 \cap C_2 \cap C_3 \cap C_4]$
$= 4 \cdot \frac{1}{4} - 6 \cdot \frac{1}{4 \cdot 3} + 4 \cdot \frac{1}{4 \cdot 3 \cdot 2} - \frac{1}{4!} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{15}{24} = \frac{5}{8}$.

(b) Given that there is at least one match, what is the probability there is a match on the 4th draw?

$$P[C_4 | \textstyle\bigcup_{i=1}^4 C_i] = \frac{P[C_4 \cap \bigcup_{i=1}^4 C_i]}{P[\bigcup_{i=1}^4 C_i]} = \frac{P[C_4]}{P[\bigcup_{i=1}^4 C_i]} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5} = .4.$$

8. A balanced coin is tossed 4 times. Let R be the random variable that determines the length of the longest run of consecutive heads. For example, $R(\texttt{HHHH}) = 4$, $R(\texttt{THHH}) = 3$ and $R(\texttt{HTHT}) = 1$, etc. Here are all the possible outcomes:

$$\Omega = \{ \quad \begin{array}{llll} \texttt{HHHH}(4), & \texttt{HHHT}(3), & \texttt{HHTH}(2), & \texttt{HHTT}(2), \\ \texttt{HTHH}(2), & \texttt{HTHT}(1), & \texttt{HTTH}(1), & \texttt{HTTT}(1), \\ \texttt{THHH}(3), & \texttt{THHT}(2), & \texttt{THTH}(1), & \texttt{THTT}(1), \\ \texttt{TTHH}(2), & \texttt{TTHT}(1), & \texttt{TTTH}(1), & \texttt{TTTT}(0) \quad \}. \end{array}$$

What are the possible of values of R? Also, construct the probability mass function of R.

The set of possible values of R is $\{0, 1, 2, 3, 4\}$.
Next to each sample point above in parentheses I put the value the random variable R returns.
Now, since each sample point is equally-likely, we just need to count the number of sample points in the events $(R = i)$ for $i = 0, 1, 2, 3, 4$ and divide by 16:
$P[R = 0] = \frac{1}{16}, \quad P[R = 1] = \frac{7}{16}, \quad P[R = 2] = \frac{5}{16}, \quad P[R = 3] = \frac{2}{16}, \quad P[R = 4] = \frac{1}{16}.$