Intro Prob Lecture Notes

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- Some stuff I missed...
- Two important properties:

1) If X, Y are independent... etc
2) If X, Y have the same MGF, then X, Y have to have the same probability distribution.
   - When they exist, they uniquely identify the distribution of a random variable. That is, two different distributions cannot lead to the same MGF.

- Ex: $X \sim$ Poisson(λ). Find the MGF if it exists

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^2}{x!} \\
&= e^{-\lambda} e^{\lambda e^t} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}
$$

- Homework (expectation): Compute MGF of Gamma($\alpha, beta$) to be $(1-\beta t)^{-\alpha}$
- Ex: $Z \sim$ Normal(0, 1)

$$
\begin{aligned}
E(e^{tZ}) &= \int_{-\infty}^{\infty} e^{tz} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z^2-2tz)}}{\sqrt{2\pi}} dz \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z^2-2tz+t^2-t^2)}}{\sqrt{2\pi}} dz \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}((z-t)^2-t^2)}}{\sqrt{2\pi}} dz \\
&= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(z-t)^2}{2}}}{\sqrt{2\pi}} dz \\
&= e^{\frac{t^2}{2}}
\end{aligned}
$$

- Recall if $X \sim \mathrm{N}(\mu, \sigma^2)$ then

$$X = \mu + \sigma Z$$

  So,

$$E(e^{tX}) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \text{ (Details omitted)}$$

- Ex: Suppose $X_1, X_2 \sim$ independent Poisson(λ)
  - Find the distribution of $X_1 + X_2$ using MGFs
  -

$$
\begin{aligned}
M_{X_1+X_2}(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \\
&= e^{\lambda(e^t-1)} \cdot e^{\lambda(e^t-1)} \\
&= e^{2\lambda(e^t-1)}
\end{aligned}
$$

  - Same MGF as MGF of Poisson(2λ). Therefore the sum IS the Poisson(2λ)
- Remark: If $X_1, X_2, \ldots, X_n$ are independent then

$$M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

A Central Limit Theorem

- $X_1, X_2, X_3, \cdots \sim$ independent Poisson(λ)
- Define $S_n = \sum_{i=1}^{n} X_i \sim$ Poisson($n\lambda$)
- Standarize S_n by subtracting the mean and dividing by the variance $Y_n := \frac{S_n - n\lambda}{\sqrt{n\lambda}}$
- Find the MGFA of Y_n
-

$$\begin{aligned}
E(e^{t(\frac{S_n - n\lambda}{\sqrt{n\lambda}})}) &= E(e^{\frac{t}{\sqrt{n\lambda}\cdot S_n}} e^{-\sqrt{n\lambda}t}) \\
&= e^{-\sqrt{n\lambda}t} E(e^{\frac{t}{\sqrt{n\lambda}}\cdot S_n}) \text{ Note: the expectation is MGF } M_{S_n}(\frac{t}{\sqrt{n\lambda}}) \\
&= e^{-\sqrt{n\lambda}t} e^{n\lambda(e^{\frac{t}{\sqrt{n\lambda}}} - 1)} \\
&\text{Note: } e^{\frac{t}{\sqrt{n\lambda}}} \approx 1 + \frac{t}{\sqrt{n\lambda}} + \frac{t^2}{2n\lambda} + \dots \\
&= e^{\frac{t^2}{2} + \text{ terms with n in denominator}} \to e^{\frac{t^2}{2}}
\end{aligned}$$

- Therefore, it converges to the standard normal distribution