# Addition problems HW#12 (solutions)

$\boxed{A.12.1.1}$

(a) $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$.

$$E(XY) = \int_0^\infty \int_0^y xy\, f(x,y)\, dx\, dy$$

$$= \int_0^\infty \int_0^y xy\, e^{-y}\, dx\, dy = \int_0^\infty \frac{y^3}{2} \cdot e^{-y}\, dy$$

$$= \frac{1}{2}\Gamma(4) = \frac{3!}{2} = 3.$$

$$E(X) = \int_0^\infty \int_0^y x \cdot e^{-y}\, dx\, dy = \int_0^\infty \frac{y^2}{2} e^{-y}\, dy = \frac{1}{2}\Gamma(3) = 1.$$

$$E(Y) = \int_0^\infty \int_0^y y\, e^{-y}\, dx\, dy = \int_0^\infty y^2 e^{-y}\, dy = \Gamma(3) = 2.$$

Therefore, $\text{Cov}(X,Y) = 3 - 1(2) = 1$.

(b) To compute $\rho_{X,Y}$ in addition to $\text{Cov}(X,Y)$ we need $\sigma_X, \sigma_Y$. We compute these now...

$$E(X^2) = \int_0^\infty \int_0^y x^2 e^{-y}\, dx\, dy = \int_0^\infty \frac{y^3}{3} e^{-y}\, dy = \frac{1}{3}\Gamma(4) = 2.$$

$$E(Y^2) = \int_0^\infty \int_0^y y^2 e^{-y}\, dx\, dy = \int_0^\infty y^3 e^{-y}\, dy = \Gamma(4) = 6.$$

So $\sigma_X^2 = E(X^2) - \{E(X)\}^2 = 2 - \{1\}^2 = 1$, $\sigma_Y^2 = E(Y^2) - \{E(Y)\}^2 = 6 - \{2\}^2 = 2$.

Therefore,

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \frac{1}{\sqrt{1\cdot 2}} = \frac{1}{\sqrt{2}}.$$

(c) $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

$= 1 + 2 + 2(1)$

$= 5$

As for $Var(X-Y)$ we can write this as

$Var(X-Y) = Var(X+(-Y)) = Var(X) + Var(-Y) + 2\,Cov(X,-Y)$

$= Var(X) + Var(Y) - 2\,Cov(X,Y)$

$= 1 + 2 - 2(1) = 1.$

Notice, in this example, that $Var(X+Y) \neq Var(X-Y)$.

(d) We find $f_{X|Y}(x|y)$ to compute $E(X|Y=y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y e^{-y}\,dx = ye^{-y} \text{ for } y>0.$$

Therefore, $f_{X|Y}(x|y) = \dfrac{e^{-y}}{ye^{-y}} = \dfrac{1}{y}$ for $0<x<y$

Ie., $X|Y=y \sim uniform(0,y)$.

Now $E(X|Y=y) = y/2$ by appealing to the fact that the mean of a uniform is the midpoint of the interval.

Also,

$$E(X|Y=y) = \int_0^y x\cdot\frac{1}{y}\,dx = \frac{x^2}{2y}\Big|_0^y = \frac{y}{2}.$$

Also,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{\infty} e^{-y}\,dy = e^{-x} \text{ for } x > 0.$$

i.e., $X \sim \exp(1)$.

$$f_{Y|X}(y|x) = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \text{ for } y > x$$

$$E(Y|X=x) = \int_x^{\infty} y \cdot e^{-(y-x)}\,dy \overset{\substack{u=y-x\\du=dy}}{=} \int_0^{\infty} (u+x)\,e^{-u}\,du$$

$$= \int_0^{\infty} u\,e^{-u}\,du + x\int_0^{\infty} e^{-u}\,du = 1 + x.$$

$\boxed{A.12.2}$

(a) $E(X_n) = E(Z_n + \theta Z_{n-1}) = E(Z_n) + \theta E(Z_{n-1})$

$= 0 + \theta \cdot 0 = 0$ for any $n$.

$Var(X_n) = Var(Z_n + \theta Z_{n-1}) = Var(Z_n) + Var(\theta Z_{n-1}) + 2Cov(Z_n, \theta Z_{n-1})$

$= \sigma^2 + \theta^2\sigma^2 + \theta \underbrace{Cov(Z_n, Z_{n-1})}_{=0}$ $=0$ since $Z_n, Z_{n-1}$ are independent

$= \sigma^2(1+\theta^2).$

and these characteristics do not depend on which $X_n$ was chosen, i.e., the mean and variance do not depend on $n$.

(b) $Cov(X_n, X_{n-1}) = Cov(Z_n + \theta Z_{n-1}, Z_{n-1} + \theta Z_{n-2})$

$$= Cov(Z_n, Z_{n-1}) + \theta\, Cov(Z_n, Z_{n-2}) + \theta\, Cov(Z_{n-1}, Z_{n-1}) + \theta^2 Cov(Z_{n-1}, Z_{n-2})$$

$$= 0 + \theta \cdot 0 + \theta \cdot \sigma^2 + \theta^2 \cdot 0$$

$$= \theta \sigma^2.$$

$$\rho_{X_n, X_{n-1}} = \frac{Cov(X_n, X_{n-1})}{\sqrt{Var(X_n) Var(X_{n-1})}} = \frac{\theta \sigma^2}{\sqrt{\sigma^2(1+\theta^2)\sigma^2(1+\theta^2)}} = \frac{\theta}{1+\theta^2}.$$

and the correlation between two successive X' values is $\frac{\theta}{1+\theta^2}$ and it doesn't depend on $n$.

(c) Now suppose $h \geq 2$ is an integer

$$Cov(X_n, X_{n-h}) = Cov(Z_n + \theta Z_{n-1}, Z_{n-h} + \theta Z_{n-h-1})$$

$$= \underbrace{Cov(Z_n, Z_{n-h})}_{=0} + \theta \underbrace{Cov(Z_n, Z_{n-h-1})}_{=0} + \theta \underbrace{Cov(Z_{n-1}, Z_{n-h})}_{=0} + \theta^2 \underbrace{Cov(Z_{n-1}, Z_{n-h-1})}_{=0}$$

$= 0$. and clearly, $\rho_{X_n, X_{n-h}} = 0$.

$\boxed{A.12.3}$

(a) (i) $Var(X+Y) = Var(X) + Var(Y) = \sigma_X^2 + \sigma_Y^2$.

(ii) $Var(X-Y) = Var(X+(-Y)) = Var(X) + Var(Y) = \sigma_X^2 + \sigma_Y^2$

So that when $X, Y$ independent,

$$Var(X+Y) = Var(X-Y).$$

This doesn't contradict A.12.1(c) since those r.v.s were not independent.

(b) $Cov(X+Y, X-Y) = Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y)$

symm.

$$= Cov(X,X) \underbrace{- Cov(X,Y) + Cov(X,Y)}_{=0} - cov(Y,Y)$$

$= var(X) - var(Y) = 0$ since $X, Y$ have same variance.

(c) $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$ since $X_i$'s are independent

$$= \sum_{i=1}^{n} \sigma^2 = n\sigma^2.$$

$$Var(\overline{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}.$$

$\boxed{A.12.4.}$

$$Cov(X+a, Y+b) = Cov(X,Y) + Cov(X,b) + Cov(a,Y) + cov(a,b)$$

Then the result will follow if we can show that for any constant $c$,

$$Cov(X,c) = 0.$$

To this end,

$$\begin{aligned} Cov(X,c) &= E(X \cdot c) - E(X)E(c) \\ &= cE(X) - E(X) \cdot c = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} Cov(X+a, Y+b) &= Cov(X,Y) + 0 + \underbrace{Cov(Y,a)}_{=0} + \underbrace{Cov(a,b)}_{=0} \\ &= Cov(X,Y). \end{aligned}$$

$\boxed{A.12.5}$ We use the fact that $X, Y$ can be represented as

$$X = \mu_X + \sigma_X Z_1 \qquad \text{and}$$

$$Y = \mu_Y + \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2}\, Z_2$$

where $Z_1$ and $Z_2$ are independent Normal(0,1).

Then...

(a) $E(X) = E(\mu_X + \sigma_X Z_1) = \mu_X + \sigma_X \underbrace{E(Z_1)}_{=0} = \mu_X.$

$$E(Y) = E(\mu_Y + \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2}\, Z_2)$$

$$= \mu_Y + \sigma_Y \rho \underbrace{E(Z_1)}_{=0} + \sigma_Y \sqrt{1-\rho^2} \underbrace{E(Z_2)}_{=0}$$

$$= \mu_Y.$$

$Var(X) = Var(\mu_X + \sigma_X Z_1) = Var(\sigma_X Z_1) = \sigma_X^2 \underbrace{Var(Z_1)}_{=1} = \sigma_X^2.$

$$Var(Y) = Var(\mu_Y + \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2}\, Z_2)$$

$$= Var(\sigma_Y \rho Z_1) + Var(\sigma_Y \sqrt{1-\rho^2}\, Z_2)$$ since $Z_1, Z_2$ are independent

$$= \sigma_Y^2 \rho^2 \underbrace{Var(Z_1)}_{=1} + \sigma_Y^2 (1-\rho^2) \underbrace{Var(Z_2)}_{=1}$$

$$= \sigma_Y^2 (\rho^2 + (1-\rho^2)) = \sigma_Y^2.$$

$$Cov(X, Y) = Cov(\mu_X + \sigma_X Z_1,\ \mu_Y + \sigma_Y \rho Z_1 + \sigma_Y\sqrt{1-\rho^2}\, Z_2)$$

$$= Cov(\sigma_X Z_1,\ \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2}\, Z_2)$$ using the result of problem A.12.4.

$$= \sigma_X \sigma_Y \rho \underbrace{Cov(Z_1, Z_1)}_{=Var(Z_1)=1} + \sigma_X \sigma_Y \sqrt{1-\rho^2} \underbrace{Cov(Z_1, Z_2)}_{=0 \ Z_1, Z_2 \text{ indep}}$$

$$= \sigma_X \sigma_Y \rho$$

Finally,

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \frac{\sigma_X \sigma_Y \rho}{\sigma_X \sigma_Y} = \rho.$$

$\blacksquare$

(b) Given $X = x$, i.e. $X = x = \mu_X + \sigma_X Z_1 \Rightarrow Z_1 = \frac{x - \mu_X}{\sigma_X}$.

Then

$$Y = \mu_Y + \sigma_Y \rho \left(\frac{x-\mu_X}{\sigma_X}\right) + \sigma_Y \sqrt{1-\rho^2}\, Z_2$$

$$\Rightarrow Y | X = x \sim Normal\left(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X),\ \sigma_Y^2(1-\rho^2)\right).$$

~~Also~~ So $E(Y|X=x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$.

(c) If given $Y=y$, i.e.

$$Y = y = \mu_Y + \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2}\, Z_2$$

Then (solving for $Z_1$) we have

$$\frac{y-\mu_Y}{\sigma_Y} = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2 .$$

and

$$\frac{y-\mu_Y}{\sigma_Y \rho} - \frac{\sqrt{1-\rho^2}}{\rho} Z_2 = Z_1 .$$

Consequently, given $Y=y$,

$$X = \mu_X + \sigma_X Z_1 = \mu_X + \sigma_X \left( \frac{y-\mu_Y}{\sigma_Y \rho} - \frac{\sqrt{1-\rho^2}}{\rho} Z_2 \right)$$

$$= \mu_X + \frac{\sigma_X}{\sigma_Y \rho}(y-\mu_Y) - \frac{\sigma_X \sqrt{1-\rho^2}}{\rho} Z_2 .$$

So that

$$X \mid Y=y \sim \text{Normal}\left( \mu_X + \frac{\sigma_X (y-\mu_Y)}{\sigma_Y \rho},\ \frac{\sigma_X^2 (1-\rho^2)}{\rho^2} \right)$$

and

$$E(X \mid Y=y) = \mu_X + \frac{\sigma_X (y-\mu_Y)}{\sigma_Y \rho} .$$