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# Zero-sum Games in Normal Form and Matrix Games

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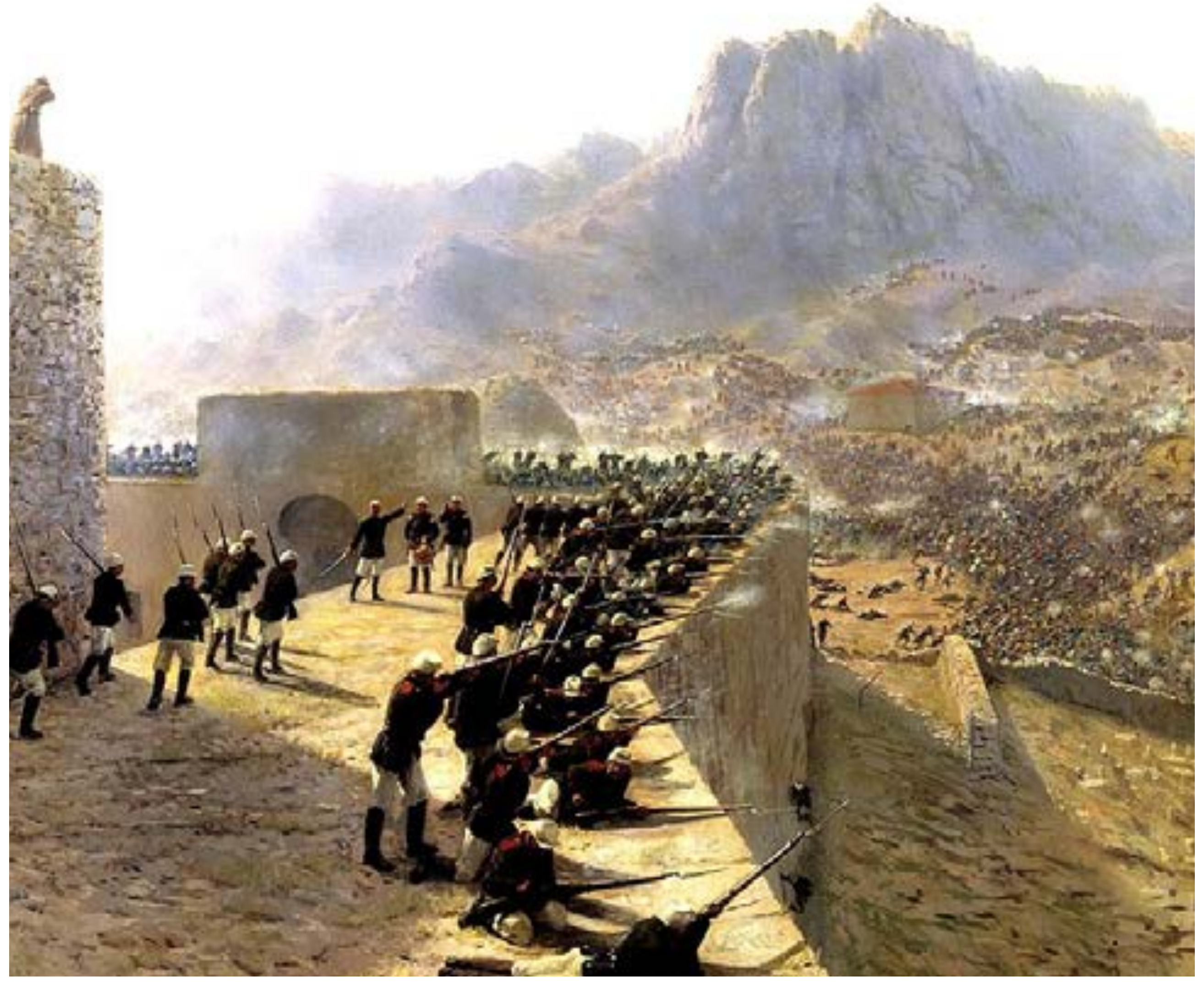
PhD

# Colonel Blotto Game

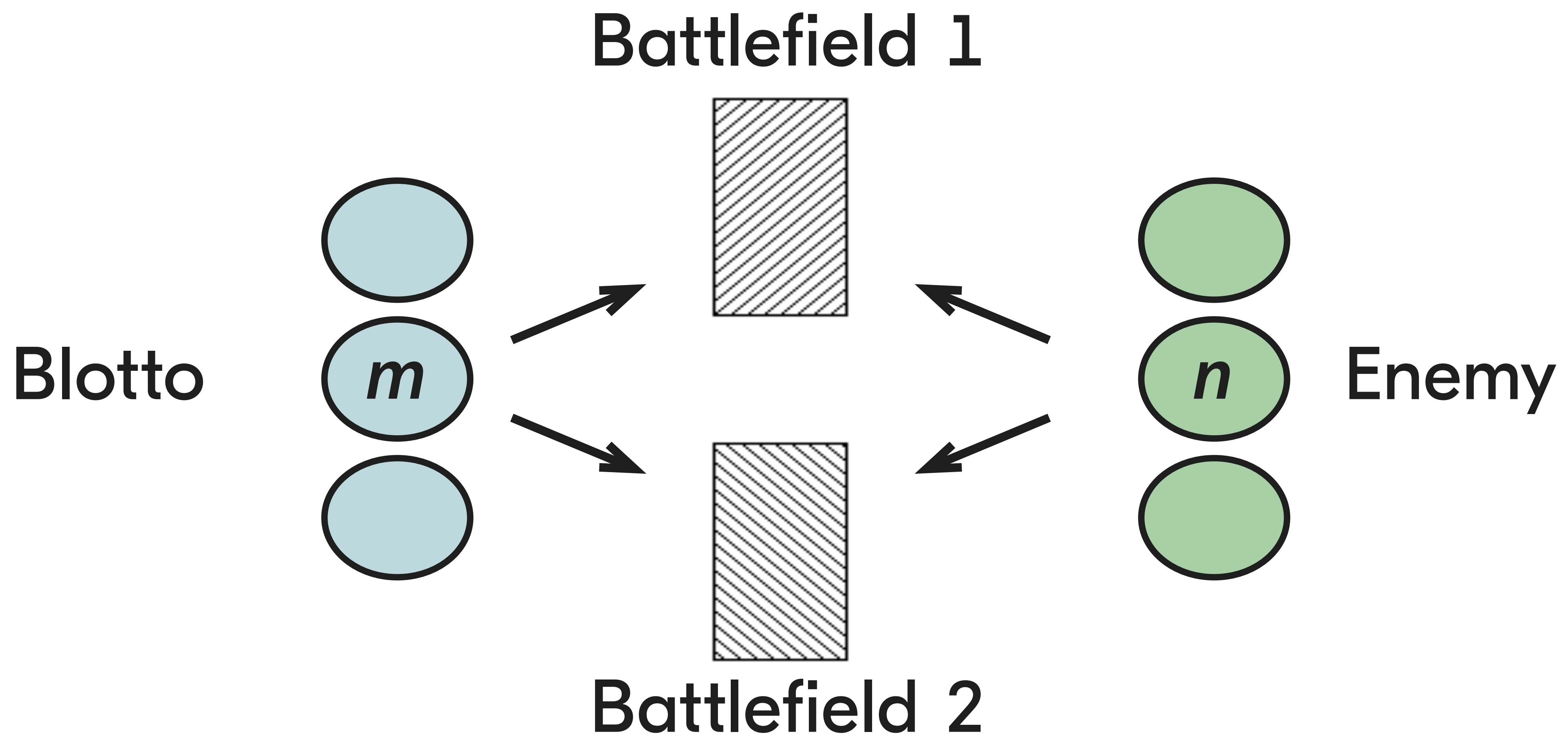


**“Defence of Bayezet during the Russo-Turkish War”,**
L. Lagorio, 1891

- Colonel Blotto has $m$ regiments.
- His enemy has $n$ regiments.

Colonel Blotto has to find optimal allocation of regiments for 2 battlefields.

# Colonel Blotto Game




- On each battlefield the side that allocates more regiments wins.
- Neither side knows how many regiments the opposing side allocates on each battlefield.

# Zero-sum Games in Normal Form

**Definition.**
The system

$$\Gamma = (X, Y, K),$$

where $X$ and $Y$ are strategy sets of players 1 and 2 correspondingly and the function $K\colon X \times Y \longrightarrow R^1$, is called two-person zero-sum game in normal form.

- $x \in X$ is the strategy of player 1,
  $y \in Y$ is the strategy of player 2.
- $(x, y) \in X \times Y$ is the strategy profile the game $\Gamma$.
- $K(x, y)$ is the payoff function of player 1,
  $[-K(x, y)]$ is the payoff function of player 2.

# Zero-sum Games in Normal Form

**Definition.**
The system

$$\Gamma = (X, Y, K),$$

where $X$ and $Y$ are strategy sets of players 1 and 2 correspondingly, and the function $K: X \times Y \longrightarrow R^1$, is called a two-person zero-sum game in normal form.

**Colonel Blotto game.**

- $x = (x_1, x_2) \in X$, where $x_1 + x_2 = m$, $x_i \geq 0$, $i = 1, 2$.
  $y = (y_1, y_2) \in Y$, where $y_1 + y_2 = n$, $y_i \geq 0$, $i = 1, 2$.
- $K(x, y) = h_1(x, y) + h_2(x, y)$, $h_i(x, y) = \begin{cases} y_i + 1, \text{ if } x_i > y_i \text{ (Blotto's victory)}, \\ 0, \text{ if } x_i = y_i \text{ (drow)}, \\ -(x_i + 1), \text{ if } x_i < y_i \text{ (Blotto's defeat)}, \end{cases}$

  $[-K(x, y)]$ — payoff function of player 2.

# Matrix Games

**Definition.**
Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

**Notations.**

- $\Gamma_A = (X, Y, K)$ is the matrix game.
- $x_i \in X$, where $i \in \{0, 1, \dots, m\}$ is the strategy of player 1,
  $y_j \in Y$, where $j \in \{0, 1, \dots, n\}$ is the strategy of player 2.
- $(x_i, y_j) \in X \times Y$ is the strategy profile in game $\Gamma_A = (X, Y, K)$.
- $K(x_i, y_j) = a_{i,j}$ is the payoff function of player 1,
  $[-K(x_i, y_j)] = -a_{i,j}$ is the payoff function of player 2.

# Matrix Games

**Definition.**
Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

**Strategies and payoffs.**
$x_i = (m - i, i)$,
$y_j = (n - j, j)$.

$$a_{ij} = \begin{cases} n + 2, & \text{if } m - i > n - j,\ i > j, \\ n - j + 1, & \text{if } m - i > n - j,\ i = j, \\ n - j - i, & \text{if } m - i > n - j,\ i < j, \\ -m + i + j, & \text{if } m - i < n - j,\ i > j, \\ j + 1, & \text{if } m - i = n - j,\ i > j, \\ -m - 2, & \text{if } m - i < n - j,\ i < j, \\ -i - 1, & \text{if } m - i = n - j,\ i < j, \\ -m + i - 1, & \text{if } m - i < n - j,\ i = j, \\ 0, & \text{if } m - i = n - j,\ i = j. \end{cases}$$

Colonel Blotto game ($m = 4$, $n = 3$):

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

# References


1. Owen, G. (1982). Game Theory. London: Academic Press.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag
3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.

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# Saddle Point

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# Maximin and Minimax Strategies

**Definition.**
Maximin strategy of player 1 is the strategy $x_{i_0}$ which satisfies:

$$\max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = \min_{y_i \in Y} K(x_{i_0}, y_j) = \underline{v}$$

here $\underline{v}$ is called the lower value of the game.

$$\left.\begin{array}{c} \left[\begin{array}{cccc} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{array}\right] \end{array}\begin{array}{l} \min\limits_j a_{0j} \\ \min\limits_j a_{1j} \\ \dots \\ \min\limits_j a_{mj} \end{array}\right\} \max_i \min_j a_{ij}$$

# Maximin and Minimax Strategies

**Definition.**
Minimax strategy of player 2 is the strategy $y_{j_0}$ which satisfies:

$$\min_{y_i \in Y} \max_{x_j \in X} K(x_i, y_j) = \max_{x_i \in X} K(x_i, y_{j_0}) = \overline{v}$$

here $\overline{v}$ is called the upper value of the game.

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m0} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\underbrace{\max_i a_{i0} \quad \max_i a_{i1} \quad \cdots \quad \max_i a_{in}}_{\min\limits_j \max\limits_i a_{ij}}$$

# Colonel Blotto Game

$$
A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}
\begin{array}{c}
\begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}
\end{array}
$$

How Colonel Blotto should behave,
what strategy should he choose?

# Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Suppose, the enemy chooses strategy $y_1$,
then Colonel Blotto has to choose strategy $x_1$:

$$\max_{x_i \in X} K(x_i, y_1) = K(x_1, y_1).$$

# Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Colonel Blotto does not know in advance what strategy the enemy will choose!

# Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Colonel Blotto can ensure himself the payoff:

$$\underline{v} = \max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = a_{0,3} = a_{4,0} = 0.$$

Whatever the behavior of the enemy, Colonel Blotto will receive not less than 0 choosing the maxmin strategies $x_0$ or $x_4$!

# Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Similarly for the enemy,
he can be sure that he will loose not more than:

$$\overline{v} = \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = a_{1,1} = a_{3,2} = 3.$$

Whatever the behavior of Colonel Blotto, the enemy will not lose more than 3 choosing the minmax strategies $\boldsymbol{y_1}$ or $\boldsymbol{y_2}$!



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# Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

In this game

$$\underline{v} = \max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) \neq \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = \overline{v},$$

$$0 \neq 3.$$

# Another Example

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_0 & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 4 \\ 0 & -2 & 7 \end{pmatrix} \end{array}$$

In this game

$$\max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = K(x_1, y_0) = 2.$$

# Saddle Point

**Definition.**
In the two-person zero-sum game $\mathbf{\Gamma} = (X, Y, K)$ strategy profile $(x^*, y^*)$ is called saddle point, if

$$K(x, y^*) \leq K(x^*, y^*), \ \forall x \in X,$$

$$K(x^*, y) \geq K(x^*, y^*), \ \forall y \in Y.$$

# Existence of Saddle Point

**Theorem (necessary and sufficient conditions for the existence of saddle point).**
Saddle point in the game **Γ** exists, if and only if

$$\underline{v} = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y) = \overline{v}.$$

# References


1. Fudenberg, D. & Tirole. (2000). J. Game Theory. Cambridge: MIT-press.

2. Kolokoltsov, V. N. & Malafeyev, O. A. (2010). Understanding Game Theory: Introduction to the Analysis of Many Agent Systems with Competition and Cooperation. Singapore: World Scientific.

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4. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.

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# Mixed Extension

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# Mixed Strategies

What Colonel Blotto needs to do to avoid disclosing information about his strategy?

$$
\begin{array}{c c}
 & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

# Mixed Strategies

**Definition.**
Mixed strategy of the player is a probability distribution defined over the set of pure strategies.

Mixed strategies of players 1 and 2 have the following form:

$$x = (\xi_0, \dots, \xi_m),\ \sum_{i=0}^{m} \xi_i = 1,\ \xi_i \geq 0,\ i = 0, \dots, m.$$

$$y = (\eta_0, \dots, \eta_n),\ \sum_{j=0}^{n} \eta_j = 1,\ \eta_j \geq 0,\ j = 0, \dots, n.$$

where $\xi_i$ и $\eta_j \geq 0$ are the probability of choosing pure strategies $i$ and $j$ by the first and the second player respectively.
In what follows, by $X$, $Y$ we will denote sets of mixed strategies.

# Mixed Strategies

**Definition.**
Mixed strategy of the player is a probability distribution defined over the set of pure strategies.

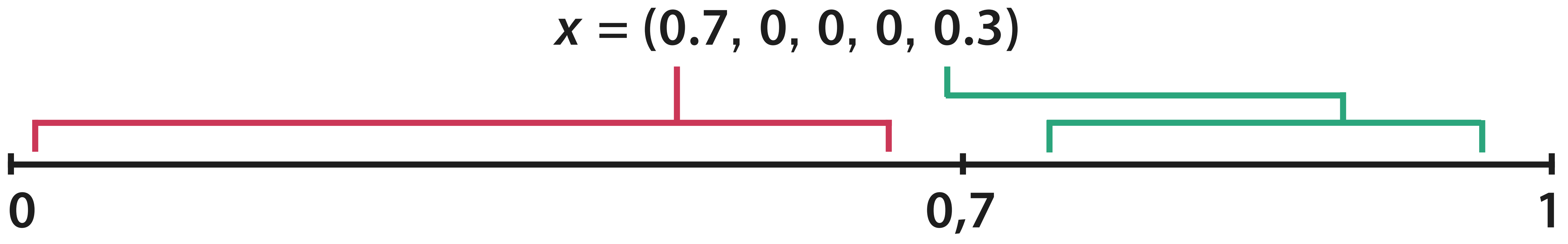
**Colonel Blotto Game.**
Suppose $x = (0.2, 0.2, 0.2, 0.2, 0.2)$, $y = (0.25, 0.25, 0.25, 0.25)$.

$$
\begin{array}{c}
 \\
0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0.25 & 0.25 & 0.25 & 0.25 \end{array} \\
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

# Mixed Strategies

**Suppose $x = (0.7, 0, 0, 0, 0.3)$.**

How does Colonel Blotto realize this strategy?




Random number table

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.83 | 0.48 | 0.88 | 0.81 | 0.37 | 0.09 | 0.56 | 0.88 | 0.29 | 0.37 |
| 0.26 | 0.43 | 0.65 | 0.08 | 0.97 | 0.26 | 0.28 | 0.53 | 0.61 | 0.42 |
| 0.41 | 0.63 | 0.84 | 0.04 | 0.42 | 0.61 | 0.05 | 0.50 | 0.67 | 0.75 |
| 0.45 | 0.53 | 0.33 | 0.19 | 0.10 | 0.39 | 0.53 | 0.04 | 0.24 | 0.79 |
| 0.22 | 0.98 | 0.54 | 0.77 | 0.04 | 0.55 | 0.76 | 0.13 | 0.32 | 0.46 |
| 0.15 | 0.42 | 0.86 | 0.35 | 0.24 | 0.83 | 0.85 | 0.36 | 0.49 | 0.11 |

# References


1. Fudenberg, D. & Tirole. (2000). J. Game Theory. Cambridge: MIT-press.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.
3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
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# Saddle Point in Mixed Strategies

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# Payoff in Mixed Strategies

**Definition.**
Pair $(x, y)$ of mixed strategies in the matrix game $\Gamma_A$ is called the strategy profile in mixed strategies.

**Definition.**
Payoff in mixed strategies $x = (\xi_0, \dots, \xi_m)$, $y = (\eta_0, \dots, \eta_n)$ is defined as mathematical expectation of payoff in pure strategies:

$$K(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \xi_i a_{ij} \eta_j = (xA)y = x(Ay).$$

# Payoff in Mixed Strategies

Suppose $x = (0.7, 0, 0, 0, 0.3)$, $y = (0.1, 0.9, 0, 0)$:

$$\begin{array}{cc} & \begin{array}{cccc} 0.1 & 0.9 & 0 & 0 \end{array} \\ \begin{array}{c} 0.7 \\ 0 \\ 0 \\ 0 \\ 0.3 \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Payoff in strategy profile $(x, y)$:

$$K(x, y) = \sum_{i=0}^{4} \sum_{j=0}^{3} \xi_i a_{ij} \eta_j = 0.7(0.1 \cdot 4 + 0.9 \cdot 2 + 0 \cdot 1 + 0 \cdot 0) +$$

$$+ 0(\ldots) + 0(\ldots) + 0(\ldots) + 0.3(0.1 \cdot 0 + 0.9 \cdot 1 + 0 \cdot 2 + 0 \cdot 4) = 1.81.$$

# Saddle Point in Mixed Strategies

**Theorem (main theorem of matrix games).**
Any matrix game has a saddle point in mixed strategies.

There always exists strategy profile in mixed strategies $(x^*, y^*)$, such that:

$$K(x, y^*) \leq K(x^*, y^*), \forall x \in X,$$
$$K(x^*, y) \geq K(x^*, y^*), \forall y \in Y.$$

Also the following equality is satisfied:

$$K(x^*, y^*) = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y).$$

Payoff in saddle point is called value of the game and denoted by $v$.

# Saddle Point in Mixed Strategies

Suppose $x^* = \left(\frac{4}{9}, 0, \frac{1}{9}, 0, \frac{4}{9}\right)$, $y^* = \left(\frac{7}{90}, \frac{32}{90}, \frac{48}{90}, \frac{3}{90}\right)$:

$$\begin{array}{cc} & \begin{array}{cccc} \frac{7}{90} & \frac{32}{90} & \frac{48}{90} & \frac{3}{90} \end{array} \\ \begin{array}{c} \frac{4}{9} \\ 0 \\ \frac{1}{9} \\ 0 \\ \frac{4}{9} \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

For strategy profile $(x^*, y^*)$ the following holds:

$$K(x^*, y^*) = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y) = \frac{14}{9}.$$

Therefore, $(x^*, y^*)$ is a saddle point in the game $\Gamma_A$.

# Properties of Optimal Strategies and Game Value

**Theorem.**
Strategy profile $(x^*, y^*)$ in mixed strategies is a saddle point in the game $\Gamma_A$, if and only if the following equality holds:

$$\min_{y_j \in Y} K(x^*, y_j) = \max_{x_i \in X} K(x_i, y^*).$$

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# Example

Suppose $x = (\xi, 1-\xi)$, $y = (\eta, 1-\eta)$:

$$\begin{array}{cc} & \begin{array}{cc} \eta & 1-\eta \end{array} \\ \begin{array}{c} \xi \\ 1-\xi \end{array} & \begin{pmatrix} 6 & 5 \\ 3 & 7 \end{pmatrix} \end{array}$$

$K(x_1, y^*) = 6\eta^* + 5(1-\eta^*)$ $\qquad \eta^* + 5 = 7 - 4\eta^*$

$K(x_2, y^*) = 3\eta^* + 7(1-\eta^*)$ $\qquad \eta^* = 0.4$

$K(x_1, y^*) = K(x_2, y^*)$ $\qquad y^* = (\eta^*, 1-\eta^*) = (0.4, 0.6)$

$K(x^*, y_1) = 6\xi^* + 3(1-\xi^*)$ $\qquad 3\xi^* + 3 = 7 - 2\xi^*$

$K(x^*, y_2) = 5\xi + 7(1-\xi^*)$ $\qquad \xi^* = 0.8$

$K(x^*, y_1) = K(x^*, y_2)$ $\qquad x^* = (\xi^*, 1-\xi^*) = (0.8, 0.2)$

$$K(x^*, y^*) = 5.4$$

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# References


1. Fudenberg, D. & Tirole, J. (2000). Game Theory. Cambridge: MIT-press.

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# Dominance of Strategies

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# Dominance of Strategies

$$A = \begin{pmatrix} 4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\ 0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\ 3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\ 2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\ 4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\ 0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0 \end{pmatrix}$$

How to find a saddle point in this game?

# Dominance of Strategies

**Definition.**
Strategy $x'$ ($y'$) of player 1 (2) dominates the strategy $x''$ ($y''$), if the following inequalities hold:

$$x'a^j \geq x''a^j,\ j \in \{1, \dots, n\}$$
$$(y'\,a_i \leq y''a_i,\ i \in \{1, \dots, m\}).$$

**Definition.**
Strategy $x''$ ($y''$) of player 1 (2) is dominated if there exists a strategy $x' \neq x''$ ($y' \neq y''$) of player 1 (2), which dominates $x''$ ($y''$).

Strategy $x''$ ($y''$) of player 1 (2) is strictly dominated if there exists a strategy $x'$ ($y'$) for which the inequalities above are strict.

# Dominance of Strategies

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{array} &
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

- Strategies $x_6$, $x_7$, $x_5$, $x_7$, $x_4$, $x_7$, $x_5$, $x_4$, $y_4$, $y_6$ strictly dominate strategies $x_0$, $x_1$, $x_2$, $x_3$, $x_8$, $x_9$, $x_{10}$, $x_{11}$, $y_0$, $y_8$ correspondingly.
- Therefore, strategies $x_0$, $x_1$, $x_2$, $x_3$, $x_8$, $x_9$, $x_{10}$, $x_{11}$, $y_0$, $y_8$ are strictly dominated.

# Dominance of Strategies

**Theorem.**
If strategy $x'$ dominates an optimal strategy $x^*$, then strategy $x'$ is also optimal*.

**Theorem.**
If strategy $x^*$ is optimal, then it is not strictly dominated.

* — optimal strategies are the strategies from the saddle point.

# Dominance of Strategies

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{array} &
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

Strategies $x_0$, $x_1$, $x_2$, $x_3$, $x_8$, $x_9$, $x_{10}$, $x_{11}$, $y_0$, $y_8$ are not included with positive probabilities in optimal strategies.

# Dominance of Strategies

Denote by $A'$ matrix obtained from $A$ by deleting the $i$-th row.

By $\overline{x_i^*}$ denote the extension of strategy $x^*$ at the $i$-th place $\overline{x_i^*} = (x_1^*, \dots, x_{i-1}^*, 0, x_i^*, x_{i+1}^*, \dots, x_n^*)$.

**Theorem.**
Suppose that the $i$-th row of matrix $A$ is dominated, then:

- $v_A = v_{A'}$.
- Any optimal strategy $y^*$ of player 2 in the game $\Gamma_{A'}$ is optimal in the game $\Gamma_A$.
- If $x^*$ is optimal strategy of player 1 in the game $\Gamma_{A'}$, then $\overline{x_i^*}$ is optimal in the game $\Gamma_A$.

# Dominance of Strategies

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_3 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{array} &
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}$$

Optimal strategies in this game are

$x^* = y^* = (0, 0, 0, 0, 0, 0, \frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0)$,

game value is $v = \frac{3}{2}$.

# Dominance of Strategies

$$
\begin{array}{c|cccccccccccc}
 & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \\
\begin{matrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{matrix} & \multicolumn{12}{c}{\begin{pmatrix}
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6
\end{pmatrix}}
\end{array}
$$

Optimal strategies in this game:

$x^* = (0, 0, \frac{3}{4}, \frac{1}{4}),$

$y^* = (0, 0, 0, 0, 0, 0, \frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0).$

# Dominance of Strategies

Denote by $A'$ matrix obtained from $A$ by deleting the $j$-th column.
By $\overline{y_j^*}$ denote the extension of strategy $y^*$ at the $j$-th place.

**Theorem.**
Suppose that the $j$-th column of matrix $A$ of the game $\Gamma_A$ is dominated, then:

- $v_A = v_{A'}$.
- Any optimal strategy $x^*$ of player 1 in the game $\Gamma_{A'}$ is optimal in the game $\Gamma_A$.
- If $y^*$ is optimal strategy of player 2 in the game $\Gamma_{A'}$, then $\overline{y_j^*}$ is optimal in the game $\Gamma_A$.

# References


1. Vorob'ov, N. N. (1994). Foundations of Game Theory: Noncooperative Games. Basel: Birkhäuser.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.
3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
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# Dominance of Strategies

**O. Petrosian**

PhD

# Dominance of Strategies

$$
\begin{array}{c|cccccccccccc}
 & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \\
\hline
x_4 & 2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
x_5 & 3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
x_6 & 4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
x_7 & 0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6
\end{array}
$$

- Strategies $y_4$, $y_4$, $y_5$, $y_7$, $y_6$, $y_5$, $y_6$, $y_7$ strictly dominate strategies $y_0$, $y_1$, $y_2$, $y_3$, $y_8$, $y_9$, $y_{10}$, $y_{11}$ correspondingly.
- Therefore, strategies $y_0$, $y_1$, $y_2$, $y_3$, $y_8$, $y_9$, $y_{10}$, $y_{11}$ are strictly dominated.

# Dominance of Strategies

Strategy $x_5$ dominates the strategy $x_4$:

$$\begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\ \begin{pmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix} \end{array}$$

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# Dominance of Strategies

Strategy $y_6$ dominates the strategy $y_4$:

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\ \begin{pmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccc} y_5 & y_6 & y_7 \end{array} \\ \begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{pmatrix} \end{array}$$

# Dominance of Strategies

Strategy $y = (0, 1/2, 1/2)$ dominates the strategy $y_5$:

$$\begin{array}{cc} & \begin{array}{ccc} y_5 & y_6 & y_7 \end{array} \\ \begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

# Dominance of Strategies

Strategy $x = (0, 1/2, 1/2)$ dominates the strategy $x_5$:

$$\begin{array}{c c} & \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{c c} & \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{array}{c} x_6 \\ x_7 \end{array} & \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

Optimal strategies are $x^* = y^* = \left(\frac{3}{4}, \frac{1}{4}\right)$, game value is $v = \frac{3}{2}$.

# Colonel Blotto Game

Suppose $m = 3$, $n = 1$. Pure strategies of players $x_i = (m - i, i)$, $y_j = (n - j, j)$.

$$\begin{array}{cc} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 1 & 3 \\ 0 & 2 \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \end{array}$$

Optimal strategies are $x^* = y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, game value is $v = 2$.

# References


1. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
2. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
3. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.
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# Iterative Solution Method for Matrix Games

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# Iterative Method

Iterative Brown-Robinson method is an iterative procedure
for constructing a sequence of $(\underline{v}^k, \overline{v}^k)$ converging to the game value.

- On each iteration, players 1 and 2 use pure strategies.
- Choice of pure strategy on the current iteration is based on the accumulated payoff.

# Iterative Method

**Algorithm:**

- **Iteration** 0:
  $x_{i_0}$, $y_{i_0}$ are the arbitrary initial pure strategies.

- **Iteration** 1:
  $x_{i_1}$: $\max\limits_i a_{i,j_0} = \overline{v}^1$; $y_{i_1}$: $\min\limits_j a_{i_0,j} = \underline{v}^1$.
  …

- **Iteration** $k + 1$:
  $x_{i_{k+1}}$: $\max\limits_i \sum_j a_{i,j}\,\eta_j^k / k = \overline{v}^k$; $y_{j_{k+1}}$: $\min\limits_j \sum_i a_{ij}\,\xi^k / k = \underline{v}^k$,
  where $\xi_i^k$ and $\eta_j^k$ is the number of chosing pure strategies $x_i$, $y_j$ correspondingly in $k$ iterations.
  …

Accuracy of the algorithm is defined by $\varepsilon = \max\limits_k \overline{v}^k - \min\limits_k \underline{v}^k$.

# Iterative method

$x^k = (\xi_1^k / k, \ldots, \xi_m^k / k)$ and $y^k = (\eta_1^k / k, \ldots, \eta_n^k / k)$ are the frequencies of pure strategies.

**Interval for the game value:**

$$v \in \left[\max_k \overline{v}^k / k, \min_k \underline{v}^k / k\right].$$

**Theorem (convergence of algorithm).**

$$\lim_{k \to +\infty}\left(\min_k \underline{v}^k / k\right) = \lim_{k \to +\infty}\left(\max_k \overline{v}^k / k\right) = v.$$

# Colonel Blotto Game

Solve Colonel Blotto game using iterative method:

$$
\begin{array}{c}
 \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4
\end{array}
\begin{array}{c}
\begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

# Colonel Blotto Game

| № | Choice of 1 player | Choice of 1 player | Payoff of player 1 | | | | | Payoff of player 2 | | | | vk dash below | vk dash above |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | x0 | x1 | x2 | x3 | x4 | y0 | y1 | y2 | y3 | | |
| 1 | x0 | y2 | 4 | 1 | -2 | -1 | 0 | 4 | 2 | 1 | 0 | 0,00 | 4,00 |
| 2 | x0 | y3 | 4 | 0 | -4 | 0 | 4 | 8 | 4 | 2 | 0 | 0,00 | 2,00 |
| 3 | x4 | y3 | 4 | -1 | -2 | 1 | 8 | 8 | 5 | 4 | 4 | 1,33 | 2,67 |
| 4 | x4 | y3 | 4 | -2 | -4 | 2 | 12 | 8 | 6 | 6 | 8 | 1,50 | 3,00 |
| 5 | x4 | y2 | 5 | -2 | -2 | 5 | 16 | 8 | 7 | 8 | 12 | 1,40 | 3,20 |
| 6 | x4 | y1 | 7 | 1 | 0 | 5 | 17 | 8 | 8 | 10 | 16 | 1,33 | 2,83 |
| 7 | x4 | y1 | 9 | 4 | 2 | 5 | 17 | 8 | 9 | 12 | 20 | 1,14 | 2,43 |
| 8 | x4 | y0 | 13 | 5 | 0 | 4 | 17 | 8 | 10 | 14 | 24 | 1,00 | 2,13 |
| 9 | x4 | y0 | 17 | 6 | -2 | 3 | 17 | 8 | 11 | 16 | 28 | 0,89 | 1,89 |
| 10 | x4 | y0 | 21 | 7 | -2 | 2 | 17 | 8 | 12 | 18 | 32 | 0,80 | 2,10 |
| 11 | x0 | y0 | 25 | 8 | -4 | 1 | 17 | 12 | 14 | 19 | 32 | 1,09 | 2,27 |
| 12 | x0 | y0 | 29 | 9 | -6 | 0 | 17 | 16 | 16 | 20 | 32 | 1,33 | 2,42 |
| | | | | | | | | | | | | | |
| 789 | x0 | y0 | 1234 | 1033 | 1208 | 1052 | 1228 | 1184 | 1202 | 1243 | 1348 | 1,50 | 1,56 |
| 790 | x0 | y0 | 1238 | 1034 | 1206 | 1051 | 1228 | 1188 | 1204 | 1244 | 1348 | 1,50 | 1,57 |
| 791 | x0 | y0 | 1242 | 1035 | 1204 | 1050 | 1228 | 1192 | 1206 | 1245 | 1348 | 1,51 | 1,57 |
| 792 | x0 | y0 | 1246 | 1036 | 1202 | 1049 | 1228 | 1196 | 1208 | 1246 | 1348 | 1,51 | 1,57 |
| 793 | x0 | y0 | 1250 | 1037 | 1200 | 1048 | 1228 | 1200 | 1210 | 1247 | 1348 | 1,51 | 1,58 |
| 794 | x0 | y0 | 1254 | 1038 | 1198 | 1047 | 1228 | 1204 | 1212 | 1248 | 1348 | 1,52 | 1,58 |
| 795 | x0 | y0 | 1258 | 1039 | 1196 | 1046 | 1228 | 1208 | 1214 | 1249 | 1348 | 1,52 | 1,58 |
| 796 | x0 | y0 | 1262 | 1040 | 1194 | 1045 | 1228 | 1212 | 1216 | 1250 | 1348 | 1,52 | 1,59 |
| 797 | x0 | y0 | 1266 | 1041 | 1192 | 1044 | 1228 | 1216 | 1218 | 1251 | 1348 | 1,53 | 1,59 |
| 798 | x0 | y0 | 1270 | 1042 | 1190 | 1043 | 1228 | 1220 | 1220 | 1252 | 1348 | 1,53 | 1,59 |
| 799 | x0 | y1 | 1272 | 1045 | 1192 | 1043 | 1229 | 1224 | 1222 | 1253 | 1348 | 1,53 | 1,59 |
| 800 | x0 | y1 | 1274 | 1048 | 1194 | 1043 | 1230 | 1228 | 1224 | 1254 | 1348 | 1,53 | 1,59 |

## Colonel Blotto Game

$$
\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}
\begin{array}{c}
\begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}
\end{array}
$$

Solution on the iteration № 800:

- $x^{800} = (0.433, 0, 0.098, 0, 0.470)$, $y^{800} = (0.073, 0.429, 0.443, 0.056)$.
- $v^{800} = 1.555675$.
- [1.53, 1.59] is the interval for game value on the interation 800.

# Other Methods for Solving Matrix Games

Other methods for solving matrix games:

- Graphic-analytical method (for $[2 \times n]$, $[m \times 2]$ games).
- Linear programming method.

# References

1. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
2. Vorob'ov, N. N. (1994). Foundations of Game Theory: Noncooperative Games. Basel: Birkhäuser.

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# Noncooperative Games in Normal Form and Bimatrix Games

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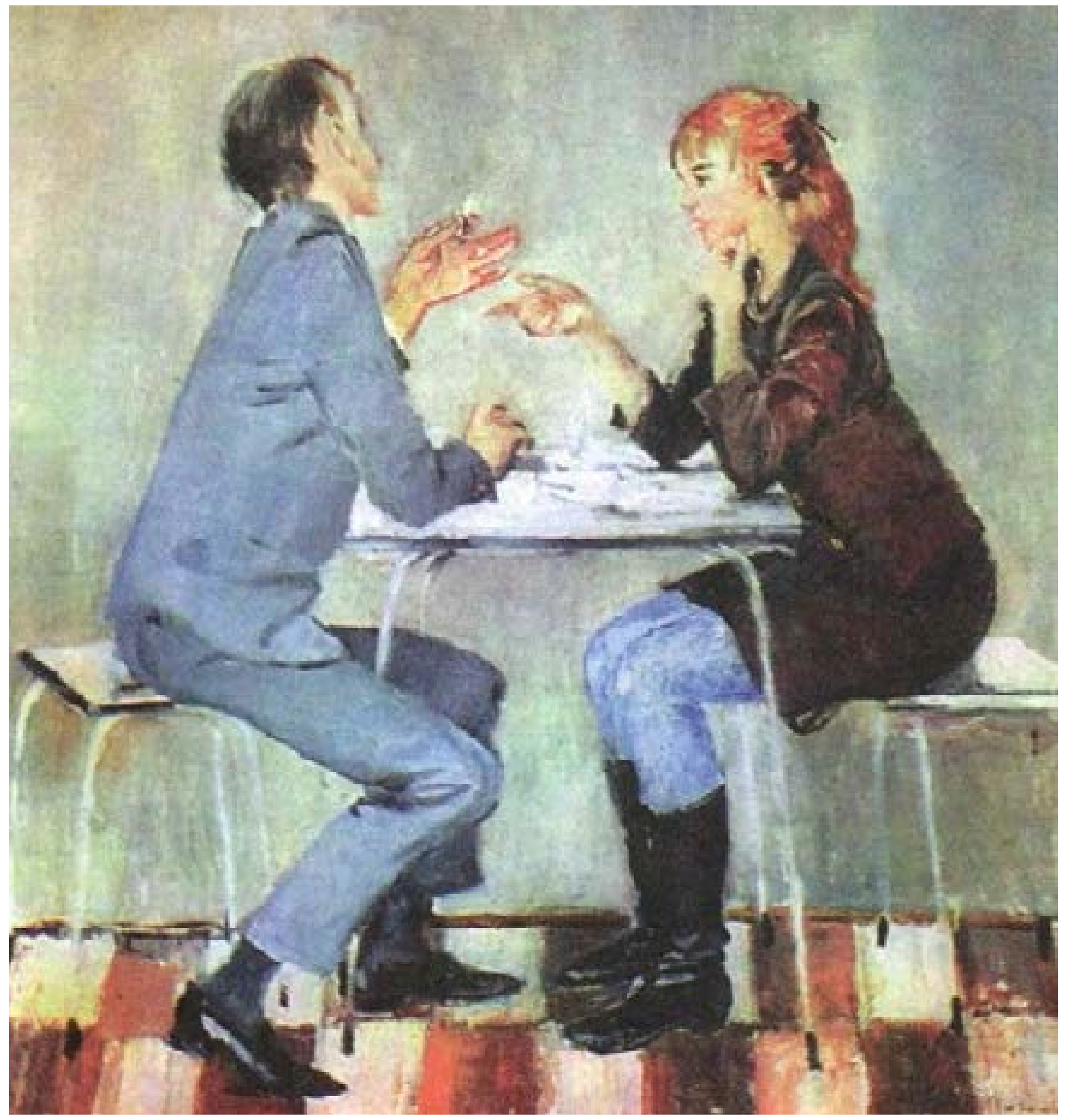
# Prisoner’s Dilemma

**“Illustrations to the novels of M. Lermontov”,**
V. Polyakov, 1900

Two criminals have commited a crime together and are interrogated separately. Each one of them can betray the other or remain silent:

- both remain silent — both of them will only serve six months in prison,
- each betray the other — both serve 2 years in prison,
- one betrays, another remains silent — first will be set free and second will serve 10 years in prison.

# Battle of Sexes



“Argument”,
Yu. Pimenov, 1968

Husband and his wife may choose one of two evening entertainments:

- football match,
- theatre.

If they choose different entertainments, then they stay at home. For both of them it is important to spend the evening together.

# Noncooperative Games in Normal Form

**Definition.**
System $\Gamma = (N, \{X_i\}_{i \in N}, \{K_i\}_{i \in N})$, where $X_i$ is a set of strategies of player $i$, $K_i : X_1 \times \ldots \times X_n \longrightarrow R^1$ is a payoff function of player $i \in N$ is called noncooperative game in normal form.

**Notation.**

- $N = \{1, 2, \ldots, n\}$ is a set of players.
- $x_i \in X_i$ is a strategy of player $i$.
- $x = (x_1, \ldots, x_n)$ is a strategy profile in the game $\Gamma$.
- $K_i(x_1, \ldots, x_n)$ is a payoff of player $i$ in the strategy profile $x = (x_1, \ldots, x_n)$.

# Noncooperative Games in Normal Form

**Definition.**
System $\Gamma = (N, \{X_i\}_{i \in N}, \{K_i\}_{i \in N})$, where $X_i$ is a set of strategies of player $i$, $K_i : X_1 \times \ldots \times X_n \longrightarrow R^1$ is a payoff function of player $i \in N$ is called noncooperative game in normal form.

**Prisoner's dilemma.**

- $N = \{1, 2\}$.
- $X_1 = (x_1, x_2)$, where $x_1$ — remain silent, $x_2$ — betray.
  $X_2 = (y_1, y_2)$, where $y_1$ — remain silent, $y_2$ — betray.
- $K_1(x_1, y_1) = K_2(x_1, y_1) = -0.5$ — both remain silent,
  $K_1(x_2, y_2) = K_2(x_2, y_2) = -2$ — each betray the other,
  $K_1(x_2, y_1) = K_2(x_1, y_2) = 0$ — one betrays, another remains silent,
  $K_2(x_2, y_1) = K_1(x_1, y_2) = -10$.

# Noncooperative Games in Normal Form

**Definition.**
System $\Gamma = (N, \{X_i\}_{i \in N}, \{K_i\}_{i \in N})$, where $X_i$ is a set of strategies of player $i$, $K_i : X_1 \times \ldots \times X_n \longrightarrow R^1$ is a payoff function of player $i \in N$ is called noncooperative game in normal form.

**Battle of sexes:**

- $N = \{1, 2\}$.
- $X_1 = (x_1, x_2)$, where $x_1$ — football, $x_2$ — theater,
  $X_2 = (y_1, y_2)$, where $y_1$ — football, $y_2$ — theater.
- $K_1(x_1, y_2) = K_2(x_2, y_1) = 0$ — players choose different entertainments,
  $K_1(x_1, y_1) = K_2(x_2, y_2) = 4$ — players choose the same entertainments,
  $K_2(x_1, y_1) = K_1(x_2, y_2) = 1$.

# Bimatrix game

**Definition.**
Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ with finite sets of strategies is called bimatrix game.

**Notation.**

- $N = \{1, 2\}$.
- $X_1$ ($X_2$) is the strategy set of player 1(2),
  $X_1 \times X_2$ is the set of all strategy profiles in the game $\Gamma$.
- $K_1$ ($K_2$): $X_1 \times X_2 \longrightarrow R^1$ is the payoff function of player 1(2)
  ($A$ and $B$ are the payoff matrixes of players 1 and 2).

# Bimatrix game

**Definition.**
Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ with finite sets of strategies is called bimatrix game.

**Prisoner's dilemma.**

$$(A, B) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} (-0.5, -0.5) & (-10, 0) \\ (0, -10) & (-2, -2) \end{pmatrix} \end{array}$$

- $(x_1, y_1)$ — both remain silent.
- $(x_2, y_2)$ — each betray the other.
- $(x_2, y_1), (x_1, y_2)$ — one betrays, another remains silent.

# Bimatrix game

**Definition.**
Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ with finite sets of strategies is called bimatrix game.

**Battle of sexes.**

$$(A, B) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & \quad y_2 \end{array} \\ \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

- $(x_1, y_1)$ — both choose to go to the football.
- $(x_2, y_2)$ — both choose to go to the theater.
- $(x_2, y_1)$, $(x_1, y_2)$ — both choose different entertainments.

# References


1. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MMA notes.
2. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
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# Optimality Principles in Bimatrix Games

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# Optimality Principles in Bimatrix Games

**Definition (for n-person game).**

Strategy profile $x^{*} = (x_1^{*}, \dots, x_i^{*}, \dots, x_n^{*})$ is called Nash equilibrium, if

$$K_i(x^{*}) \geq K_i(x^{*} \,\|\, x_i), \; \forall x_i \in X_i, \; i = 1, \dots, n,$$

where

$$(x^{*} \,\|\, x_i) = (x_1^{*}, \dots, x_{i-1}^{*}, x_i, x_{i+1}^{*}, \dots, x_n^{*}).$$

# Principles of Optimality in Bimatrix Games

**Definition.**
Strategy profile $(x^*, y^*)$ is called Nash equilibrium, if

$$K_1(x^*, y^*) \geq K_1(x, y^*), \ \forall x \in X_1,$$
$$K_2(x^*, y^*) \geq K_2(x^*, y), \ \forall y \in X_2.$$

**Prisoner's dilemma.**

$$(A, B) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} (-0.5, -0.5) & (-10, 0) \\ (0, -10) & (-2, -2) \end{pmatrix} \end{array}$$

Nash equilibrium: $(x_2, y_2)$

- $K_1(x_2, y_2) = -2 \geq -10 = K_1(x_1, y_2)$.
- $K_2(x_2, y_2) = -2 \geq -10 = K_2(x_2, y_1)$.

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# Optimality Principles in Bimatrix Games

**Definition.**
Strategy profile $(x^*, y^*)$ is called Nash equilibrium, if

$$K_1(x^*, y^*) \geq K_1(x, y^*),\ \forall x \in X_1,$$
$$K_2(x^*, y^*) \geq K_2(x^*, y),\ \forall y \in X_2.$$

**Battle of sexes.**

$$(A, B) = \begin{array}{c c} & \begin{array}{cc} y_1 & \quad\quad y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

Nash equilibrium: $(x_1, y_1)$, $(x_2, y_2)$

- $K_1(x_1, y_1) = 4 \geq 0 = K_1(x_2, y_1)$, $\quad K_1(x_2, y_2) = 1 \geq 0 = K_1(x_1, y_2)$.
- $K_2(x_1, y_1) = 1 \geq 0 = K_2(x_1, y_2)$, $\quad K_2(x_2, y_2) = 4 \geq 0 = K_2(x_2, y_1)$.

# Optimality Principles in Bimatrix Games

**Definition.**
Strategy profile $\overline{x}$ is called Pareto-optimal, if there is no strategy profile $x \in X$, for which the following inequalities hold:

- $K_i(x) \geq K_i(\overline{x})$ for all $i \in N$,
- $K_{i_0}(x) > K_{i_0}(\overline{x})$ for at least one $i_0 \in N$.

**Prisoner's dilemma.**

$$(A, B) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} \quad y_1 \quad & \quad\quad y_2 \quad \end{array} \\ \begin{pmatrix} (-0.5, -0.5) & (-10, 0) \\ (0, -10) & (-2, -2) \end{pmatrix} \end{array}$$

Pareto-optimal strategy profiles: $(x_1, y_1)$, $(x_1, y_2)$, $(x_2, y_1)$

Nash equilibrium $(\boldsymbol{x_2}, \boldsymbol{y_2})$ is not Pareto-optimal!

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# Mixed Strategies and Nash equilibrium

**O. Petrosian**

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# Mixed Strategies

**Definition.**
Mixed strategy is the probability distribution over the set of strategies (pure strategies).

Mixed strategies of players 1 and 2 have the following form:

$$x = (\xi_1, \dots, \xi_m), \sum_{i=1}^{m} \xi_i = 1, \xi_i \geq 0, i = 1, \dots, m.$$

$$y = (\eta_1, \dots, \eta_n), \sum_{i=1}^{n} \eta_j = 1, \eta_j \geq 0, j = 1, \dots, n.$$

where $\xi_i$ and $\eta_j \geq 0$ are the probabilities of choosing pure strategies $i$ and $j$ of players 1 and 2 correspondingly, $|X_1| = m$, $|X_2| = n$.

# Mixed Strategies

**Definition.**
Mixed strategy is the probability distribution over the set of strategies (pure strategies).

**Prisoner's dilemma.**
Suppose $x = (0.8, 0.2)$, $y = (0.5, 0.5)$.

$$(A, B) = \begin{array}{c} \\ 0.8 \\ 0.2 \end{array} \begin{array}{c} \begin{array}{cc} 0.5 & 0.5 \end{array} \\ \begin{pmatrix} (-0.5, -0.5) & (-10, 0) \\ (0, -10) & (-2, -2) \end{pmatrix} \end{array}$$

# Mixed Strategies

**Definition.**
Mixed strategy is the probability distribution over the set of strategies (pure strategies).

**Battle of sexes.**
Suppose $x = (0.8, 0.2)$, $y = (0.2, 0.8)$.

$$(A, B) = \begin{array}{cc} & \begin{array}{cc} 0.2 & \quad 0.8 \end{array} \\ \begin{array}{c} 0.8 \\ 0.2 \end{array} & \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

# Payoff in Mixed Strategies

**Definition.**
Pair $(x, y)$ of mixed strategies in bimatrix game is called the strategy profile in mixed strategies.

**Definition.**
Payoffs in mixed strategies $K_1(x, y)$ and $K_2(x, y)$ are the mathematical expectations of payoffs in pure strategies:

$$K_1(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_i a_{ij} \eta_j = (xA)y = x(Ay),$$

$$K_2(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_i b_{ij} \eta_j = (xB)y = x(By).$$

# Payoffs in Mixed Strategies

**Battle of sexes.**

Suppose $x = (0.8, 0.2)$, $y = (0.2, 0.8)$.

$$(A, B) = \begin{array}{cc} & \begin{array}{cc} 0.2 & 0.8 \end{array} \\ \begin{array}{c} 0.8 \\ 0.2 \end{array} & \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

Payoffs in strategy profile $(x, y)$:

- $K_1(x, y) = xAy = 0.8$.
- $K_2(x, y) = xBy = 0.8$.

# Existence of Nash Equilibrium

**Theorem.**
Given any finite $n$-person game $\Gamma$ in normal form, there exists at least one Nash equilibrium in mixed strategies.

The proof is based on the following theorem:

**Theorem.**
Let $S$ be any nonempty, convex, bounded, and closed subset of a finite-dimensional vector space $R^m$. Let $F$: $S \longrightarrow S$ be any upper-semicontinuous point-to-set correspondence such that, for every $x$ in $S$, $F(x)$ is a nonempty, bounded and closed convex subset of $S$. Then there exists $\bar{x}$ in $S$ such that $\bar{x} \in F(x)$.

# Properties of Optimal Solutions

**Theorem.**
Strategy profile $(x^*, y^*)$ in mixed strategies in the game $\Gamma = (X_1, X_2, K_1, K_2)$ is Nash equilibrium, if and only if the following inequalities are satisfied:

$$K_1(x_i, y^*) \leq K_1(x^*, y^*), \forall x_i \in X_1,$$

$$K_2(x^*, y_j) \leq K_2(x^*, y^*), \forall y_j \in X_2.$$

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# Properties of Optimal Solutions

**Theorem.**
Let $\Gamma(A, B)$ be a bimatrix $m \times n$ game and let $(x^*, y^*)$ be a Nash equilibrium in mixed strategies. Then the following equations hold:

$$K_1(x_i, y^*) = K_1(x^*, y^*), \ \forall x_i \in M_{x^*},$$

$$K_2(x^*, y_j) = K_2(x^*, y^*), \ \forall y_j \in N_{y^*},$$

where $M_{x^*}$ ($N_{y^*}$) is the spectrum of a mixed strategy $x^*(y^*)$.

# Battle of Sexes

Suppose $x = (\xi, 1-\xi)$, $y = (\eta, 1-\eta)$:

$$\begin{array}{cc} & \begin{array}{cc} \eta & 1-\eta \end{array} \\ \begin{array}{c} \xi \\ 1-\xi \end{array} & \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

$$K_2(x^*, y_1) = 1\xi^* + 0(1-\xi^*) \qquad \xi^* = 4(1-\xi^*)$$
$$K_2(x^*, y_2) = 0\xi^* + 4(1-\xi^*) \qquad \xi^* = 0.8$$

$$K_2(x^*, y_1) = K_2(x^*, y_2) \qquad x^* = (\xi^*, 1-\xi^*) = (0.8, 0.2)$$

$$K_1(x_1, y^*) = 4\eta^* + 0(1-\eta^*) \qquad 4\eta^* = 1-\eta^*$$
$$K_1(x_2, y^*) = 0\eta^* + 1(1-\eta^*) \qquad \eta^* = 0.2$$

$$K_1(x_1, y^*) = K_1(x_2, y^*) \qquad y^* = (\eta^*, 1-\eta^*) = (0.2, 0.8)$$

$$K_1(x^*, y^*) = K_2(x^*, y^*) = 0.8$$

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# Games in Strategic Form

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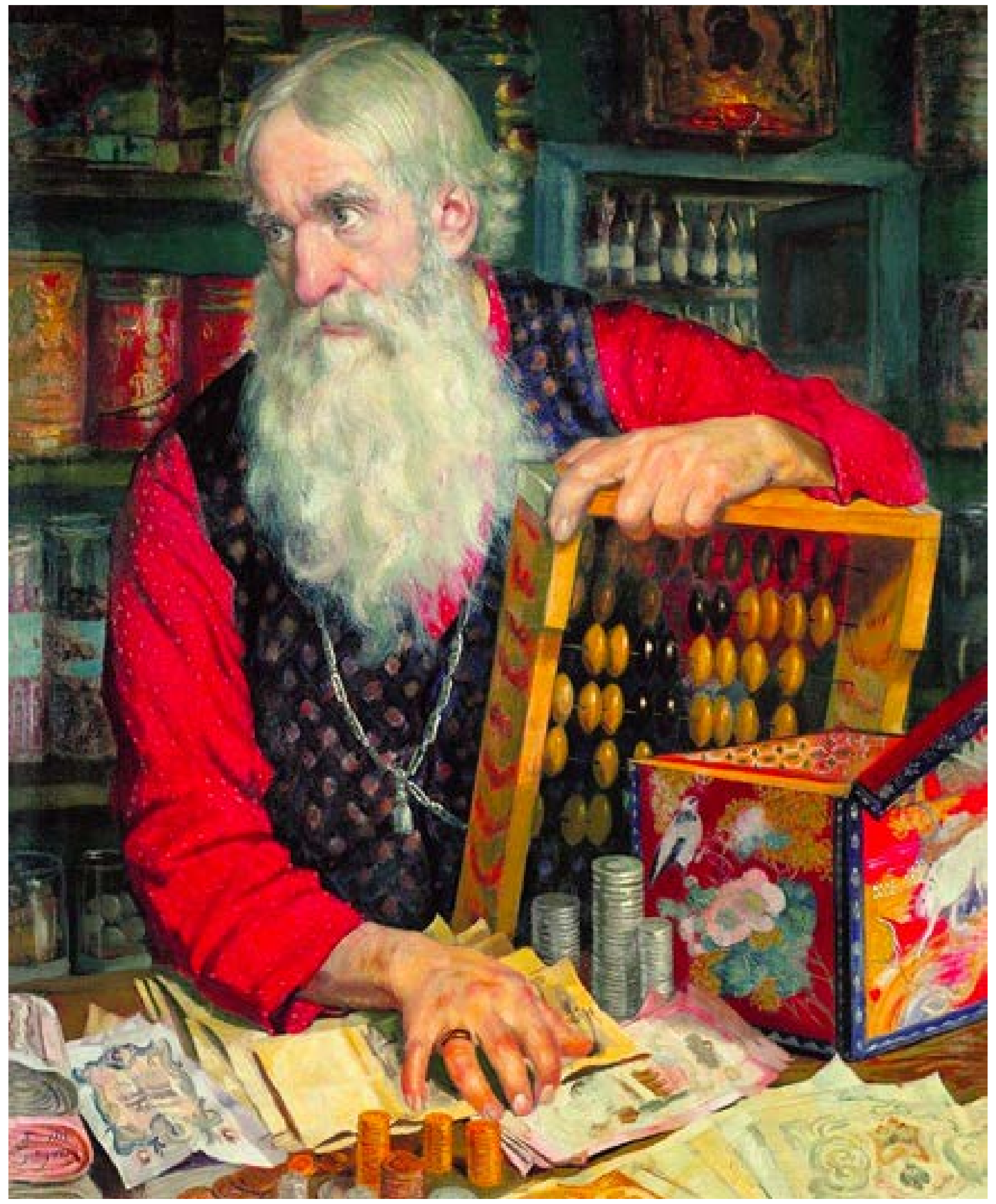
# Cournot duopoly

**“Cabby in the tavern”,**
B. Kustodiev, 1920

On the market, two firms compete, producing a homogeneous product and deciding on the volumes of production:

- Each firm knows the influence of supply on the market price.
- Both firms make decisions about production simultaneously and independently of each other.

# Bertrand duopoly

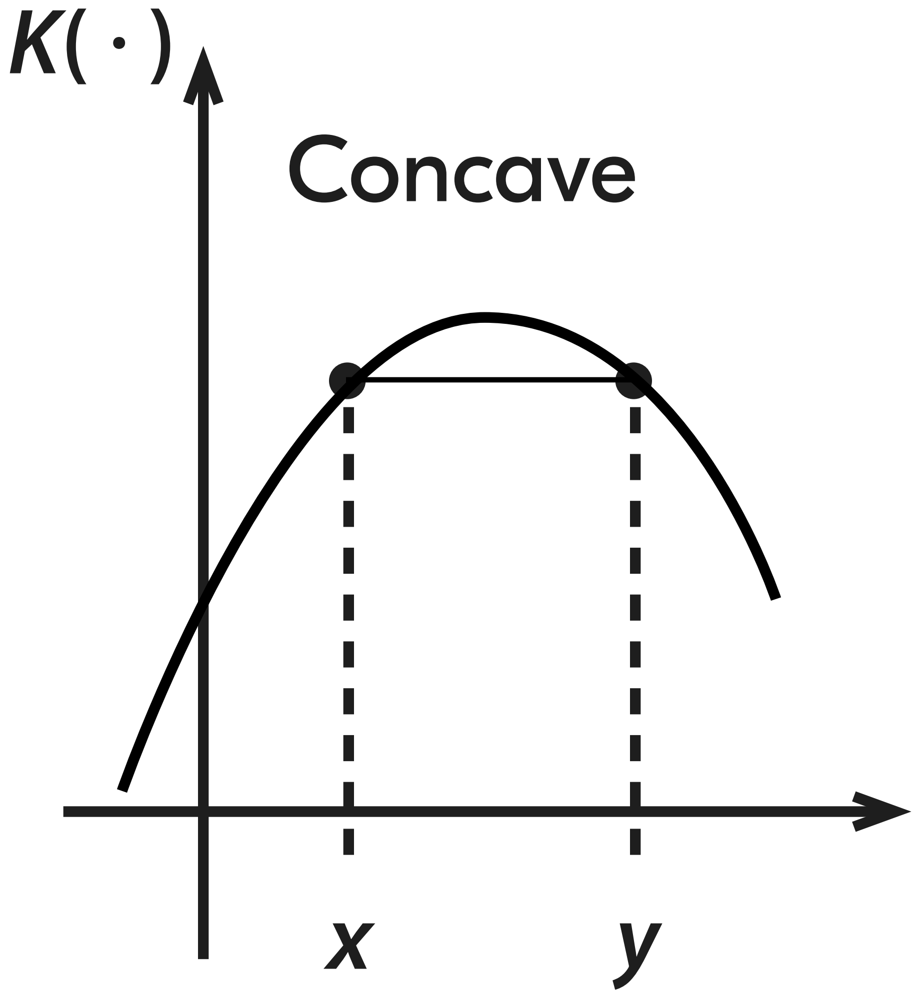


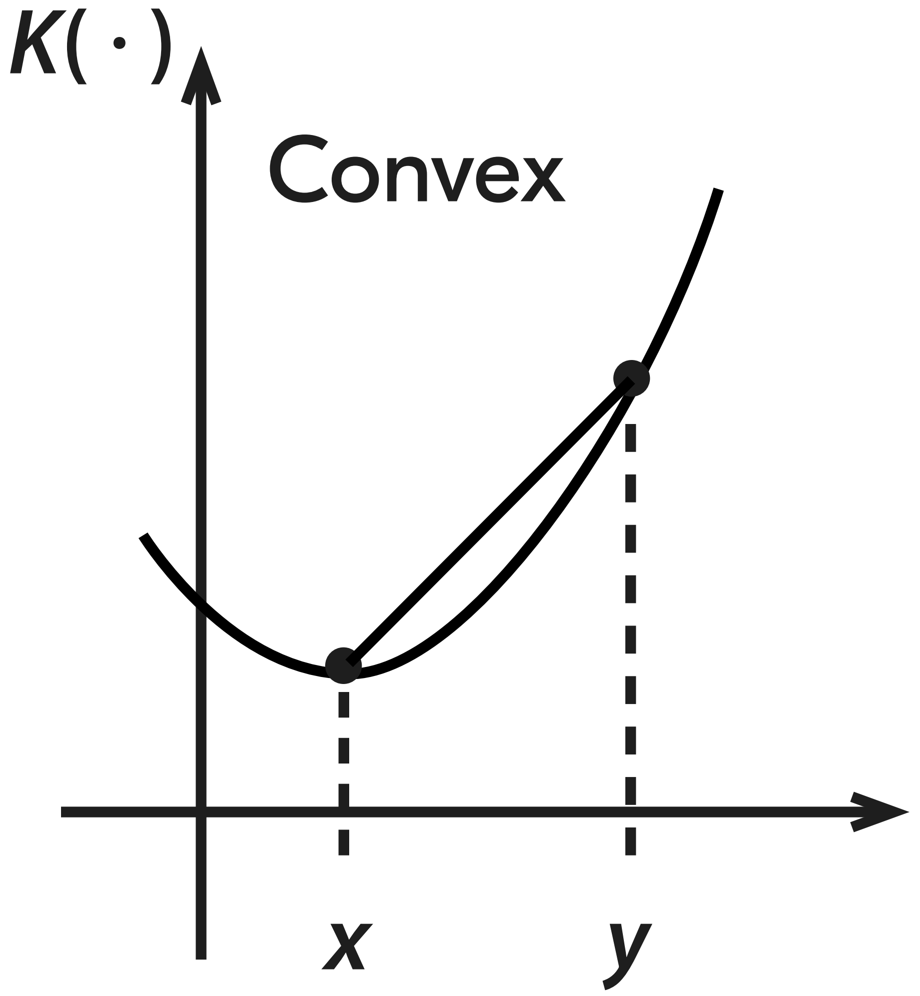
**“The merchant”,**
B. Kustodiev, 1918

On the market, two firms compete, producing two similar products and deciding on the prices:

- Each firm knows influence of price on the demand.
- Both firms make decisions about production simultaneously and independently of each other.

# Preliminary information




**Definition.**
Function $K(x)$ is called concave (convex) on a set $X \subset R^n$, if for any $x, y \in X$ and $a \in [0, 1]$ holds:

$$K(a\,x + (1 - a)y) \geq a\,K(x) + (1 - a)K(y)$$
$$(K(a\,x + (1 - a)y) \leq a\,K(x) + (1 - a)K(y)).$$

# Convex games

**Definition.**

Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ is called convex, if

- Sets of pure strategies $X_1$ and $X_2$ of players 1 and 2 are convex.
- Payoff functions $K_1(x, y)$ and $K_2(x, y)$ are concave for $x$ and $y$ correspondingly.

**Notations.**

- $N = \{1, 2\}$.
- $X_1$ ($X_2$) is the set of strategies of player 1 (2).
- $X_1 \times X_2$ is the set of strategy profiles in the game $\Gamma$.
- $K_1$ ($K_2$) : $X_1 \times X_2 \longrightarrow R^1$ is the payoff function of player 1 (2).

# Convex games

**Definition.**

Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ is called convex, if

- Sets of pure strategies $X_1$ and $X_2$ of players 1 and 2 are convex.
- Payoff functions $K_1(x, y)$ and $K_2(x, y)$ are concave for $x$ and $y$ correspondingly.

**Cournot duopoly.**

- $N = \{1, 2\}$.
- $X_1 = X_2 = [0, +\infty)$, $q_1 \in X_1$, $q_2 \in X_2$ are the volumes of production of players 1 and 2.
- $K_1(q_1, q_2) = (p - q_1 - q_2)q_1 - cq_1$,
  $K_2(q_1, q_2) = (p - q_1 - q_2)q_2 - cq_2$,
  where $(p - q_1 - q_2)$ is the price of the product, $p$ is the initial price, $c$ is the unit cost.

# Convex games

**Definition.**

Noncooperative two-person game $\Gamma = (N, X_1, X_2, K_1, K_2)$ is called convex, if

- Sets of pure strategies $X_1$ and $X_2$ of players 1 and 2 are convex.
- Payoff functions $K_1(x, y)$ and $K_2(x, y)$ are concave for $x$ and $y$ correspondingly.

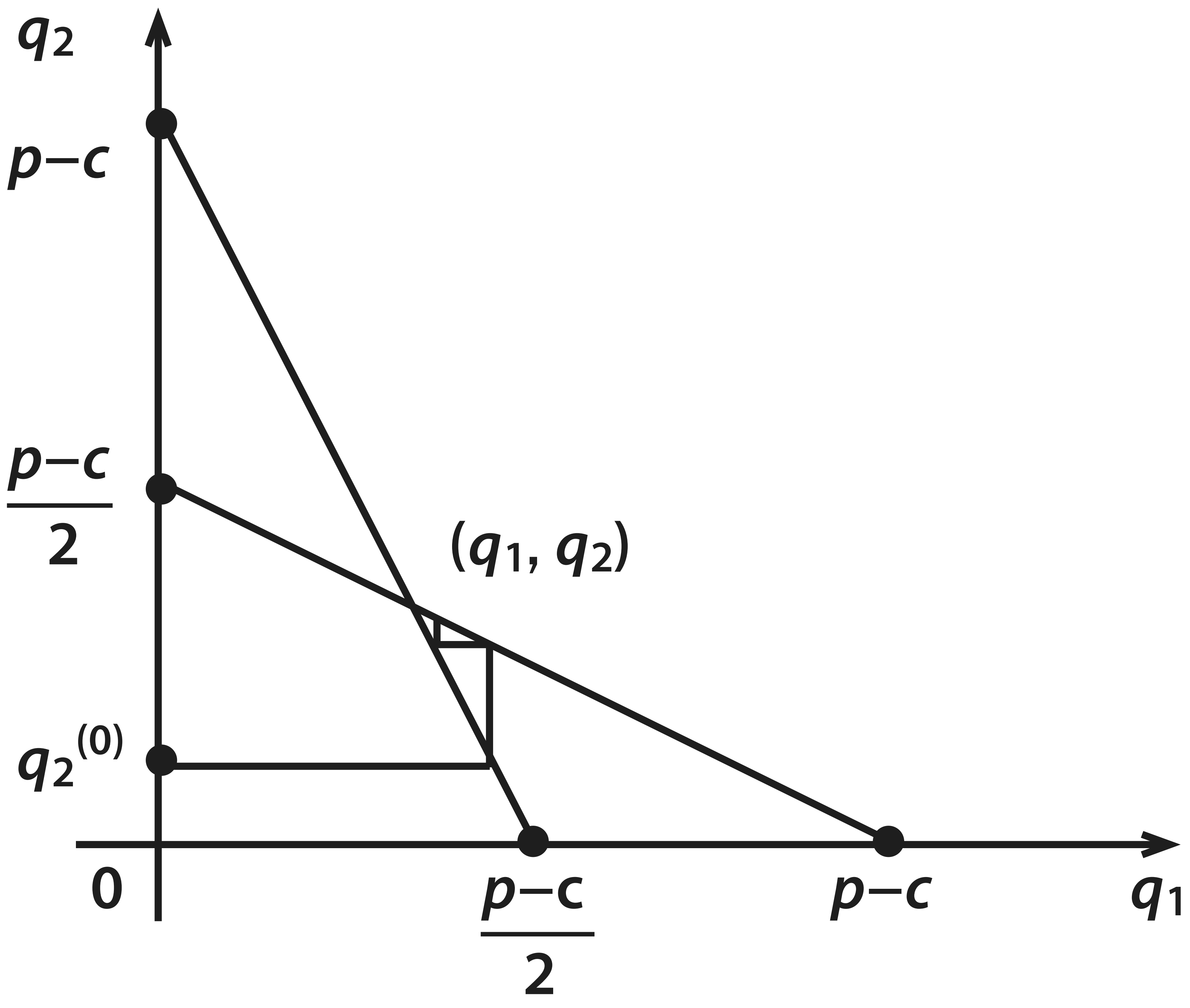
**Bertrand duopoly.**

- $N = \{1, 2\}$.
- $X_1 = X_2 = [0, +\infty)$, $c_1 \in X_1$, $c_2 \in X_2$ are the prices for products of players 1 and 2.
- $K_1(c_1, c_2) = (q - c_1 + kc_2)(c_1 - c)$,
  $K_2(c_1, c_2) = (q - c_2 + kc_1)(c_2 - c)$, where $(q - c_1 + kc_2)$, $(q - c_2 + kc_1)$ are the demands for products of players 1 and 2, $q$ is the initial demand, $k$ is the interchangeability parameter of products, $c$ is the unit cost.

# Convex games

**Theorem.**
Consider two-player game $\Gamma = (N, X_1, X_2, K_1, K_2)$. Suppose that the strategy sets $X_1$, $X_2$ are compact convex sets in the space $R^n$, and payoffs $K_1(x, y)$, $K_2(x, y)$ are continuous convex functions in $x$ and $y$ respectively, then in the game exists Nash equilibrium in pure strategies.

# Cournot duopoly



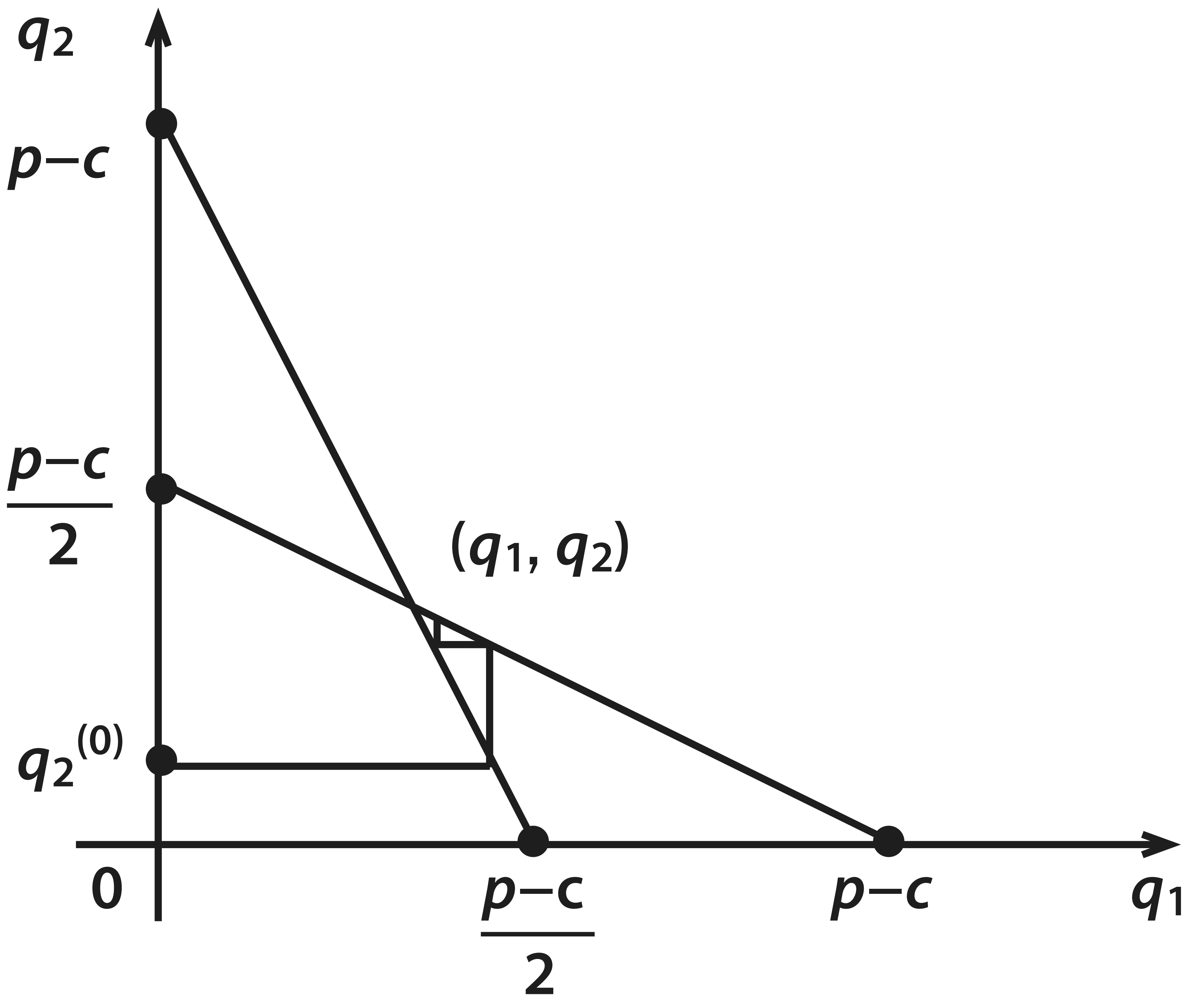

In order to find Nash equilibrium, it is necessary to solve the system of optimization problems:

$$\max_{q_1} K_1(q_1, q_2^*),$$
$$\max_{q_2} K_2(q_1^*, q_2).$$

We obtain system for $q_1^*$ and $q_2^*$:

$$\begin{cases} q_1^* = \frac{1}{2}(p - c - q_2^*), \\ q_2^* = \frac{1}{2}(p - c - q_1^*). \end{cases}$$

# Cournot duopoly



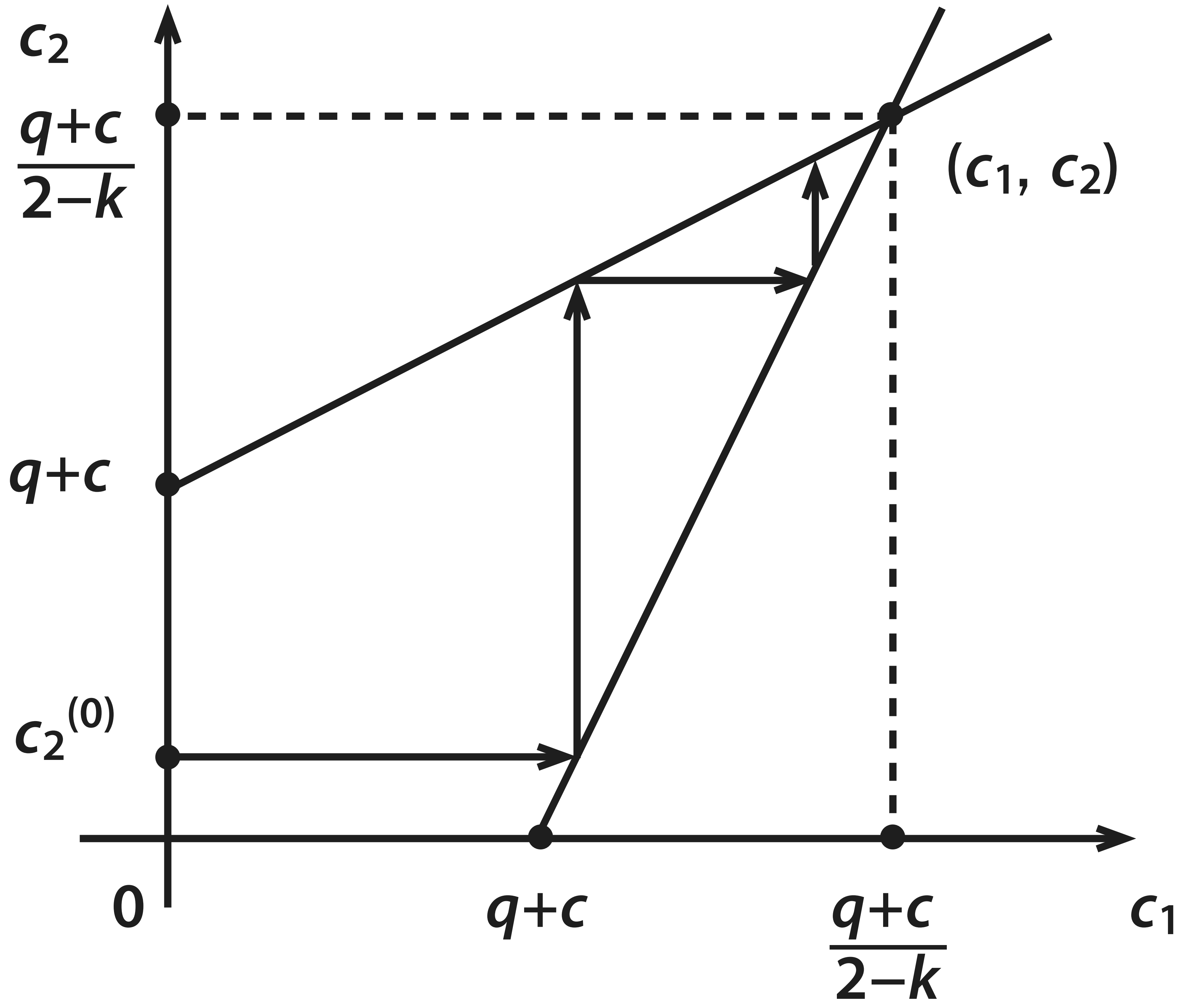

Solution of system:

$$q_1^* = q_2^* = \frac{p-c}{3}.$$

Payoffs in Nash equilibrium $(q_1^*, q_2^*)$:

$$K_1(q_1^*, q_2^*) = K_2(q_1^*, q_2^*) = \frac{(p-c)^2}{9}.$$

# Bertrand duopoly



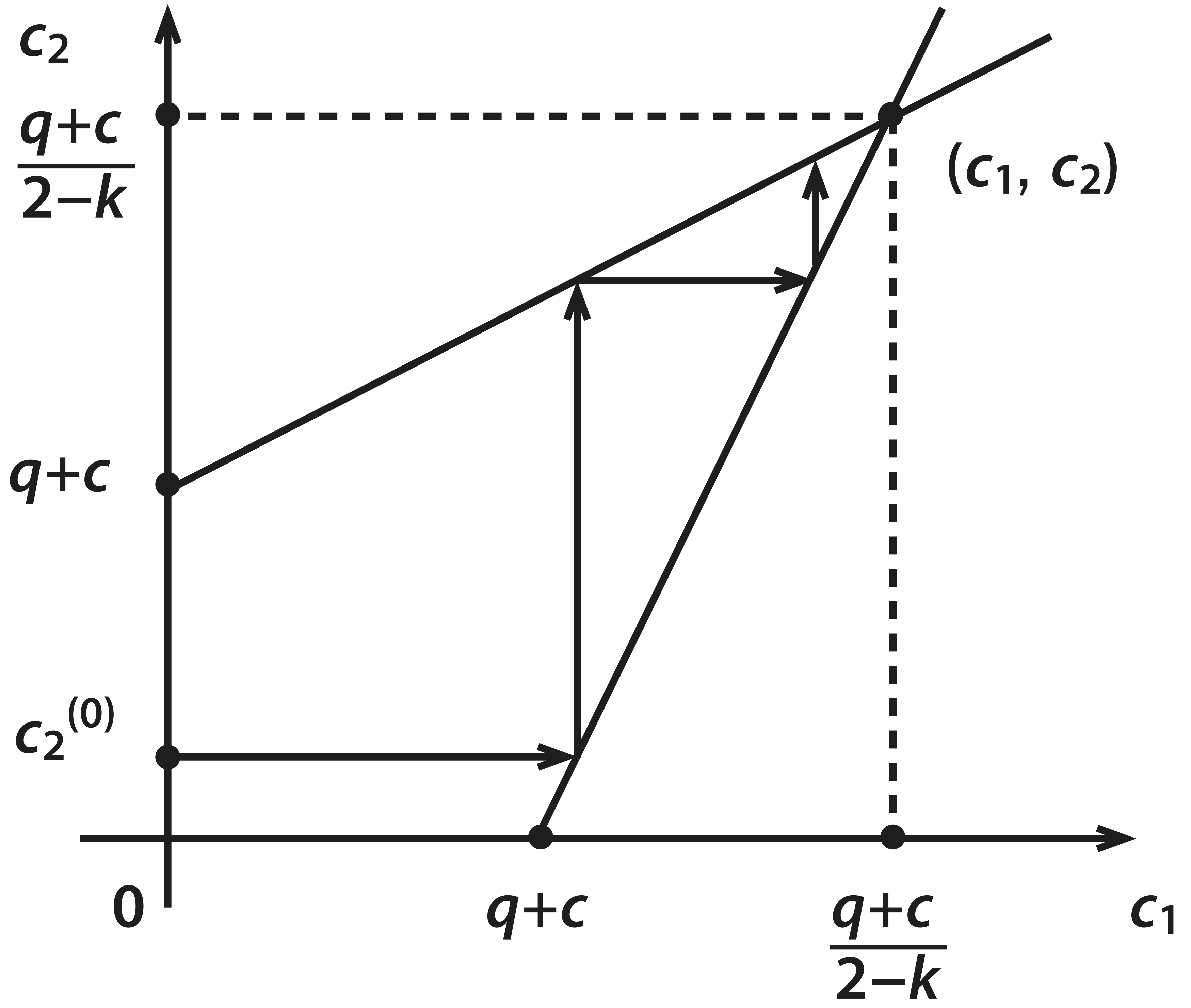

In order to find Nash equilibrium, it is necessary to solve the system of optimization problems:

$$\max_{c_1} K_1(c_1, c_2^*),$$
$$\max_{c_2} K_2(c_1^*, c_2).$$

We obtain system for $c_1^*$ and $c_2^*$:

$$\begin{cases} c_1^* = \frac{1}{2}(q + kc_2^* + c), \\ c_2^* = \frac{1}{2}(q + kc_1^* + c). \end{cases}$$

# Bertrand duopoly




Solution of system:

$$c_1^* = c_2^* = \frac{q+c}{2-k} \quad .$$

Payoffs in Nash equilibrium $(c_1^*, c_2^*)$:

$$K_1(c_1^*, c_2^*) = K_2(c_1^*, c_2^*) = \left(\frac{q - c(1-k)}{2-k}\right)^2 .$$

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# Transportation Games and Price of Anarchy

**O. Petrosian**

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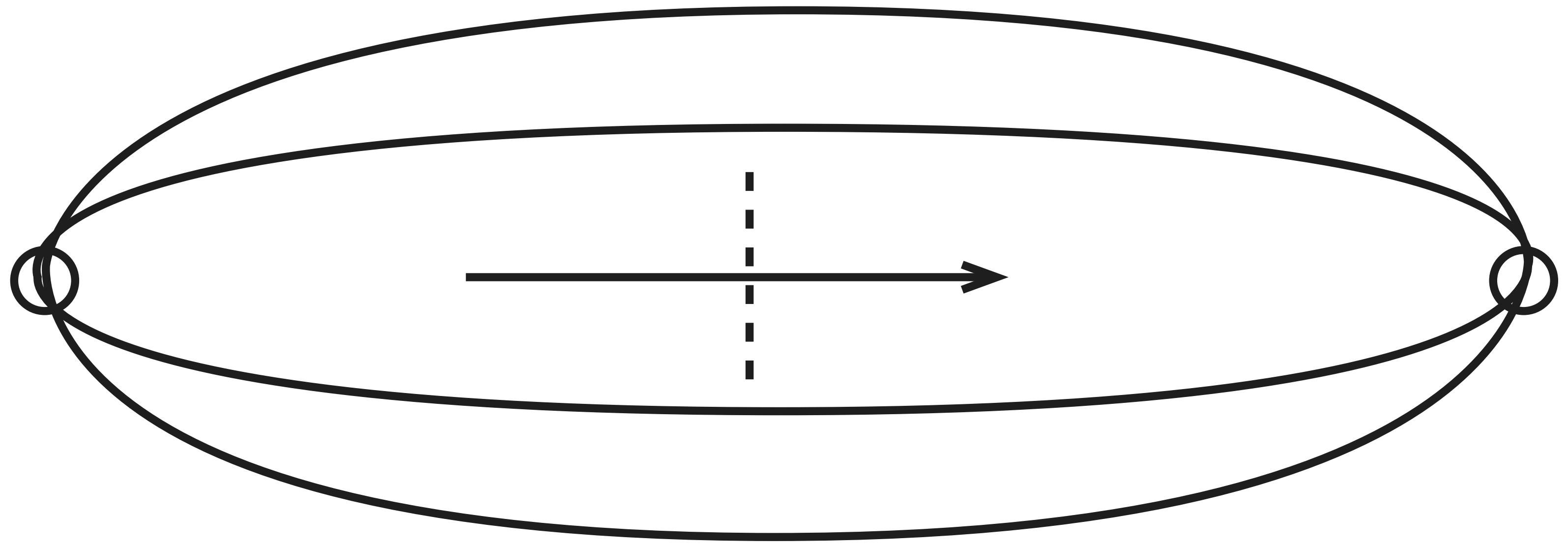
# Transportation Games

**“Coachman”,**
P. Kovalevskiy, 1902

Drivers decide on routes to the destination:

- Routes consist of $m$ parallel roads.
- Each route has a capacity $c_j$.
- The car of each driver has a certain volume $w_i$.

# KP-model of Optimal Routing with Indivisible Traffic



**Game description** $\Gamma = (N, \{L_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$:

- $N = \{1, \dots, n\}$.
- $L_i = L = \{1, \dots, m\}$ is the strategy set of player $i$ (set of routes).
- $l_i \in L$ is the strategy of player $i$ (choice of route for driver with volume $w_i$).
- $l = (l_1, \dots, l_n)$ is the strategy profile.
- $\lambda_i = \dfrac{\sum_{k:\ l_k = l_i} w_k}{c_{l_i}}$ is the loss function of player $i$ (time delay on the route).

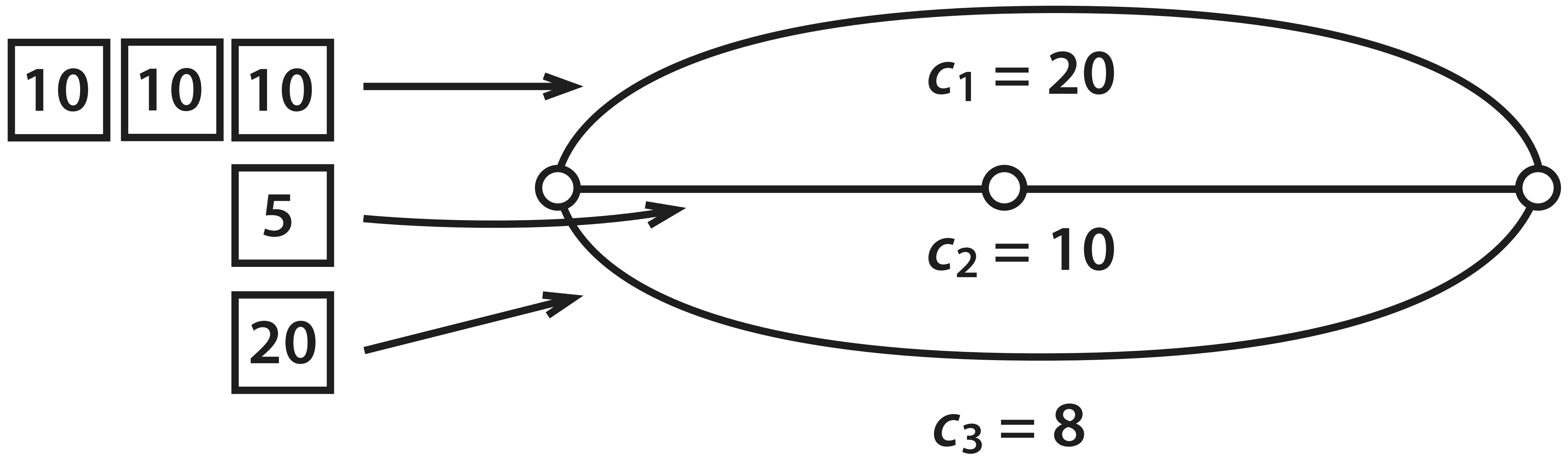
# KP-model of Optimal Routing with Indivisible Traffic

**Theorem.**
In the game $\Gamma = (N, \{L_i\}_{i \in N}, \{\lambda_i\}_{i \in N})$ exists Nash equilibrium in pure strategies.

Nash equilibrium in pure strategies $l^* = (l_1^*, \dots, l_n^*)$ can be obtained using the following system:

$$\lambda_i^* = \min_j \frac{W_i + \sum_{k \neq i \,:\, l_k^* = j} W_k}{c_j}, \quad i = 1, \dots, n.$$

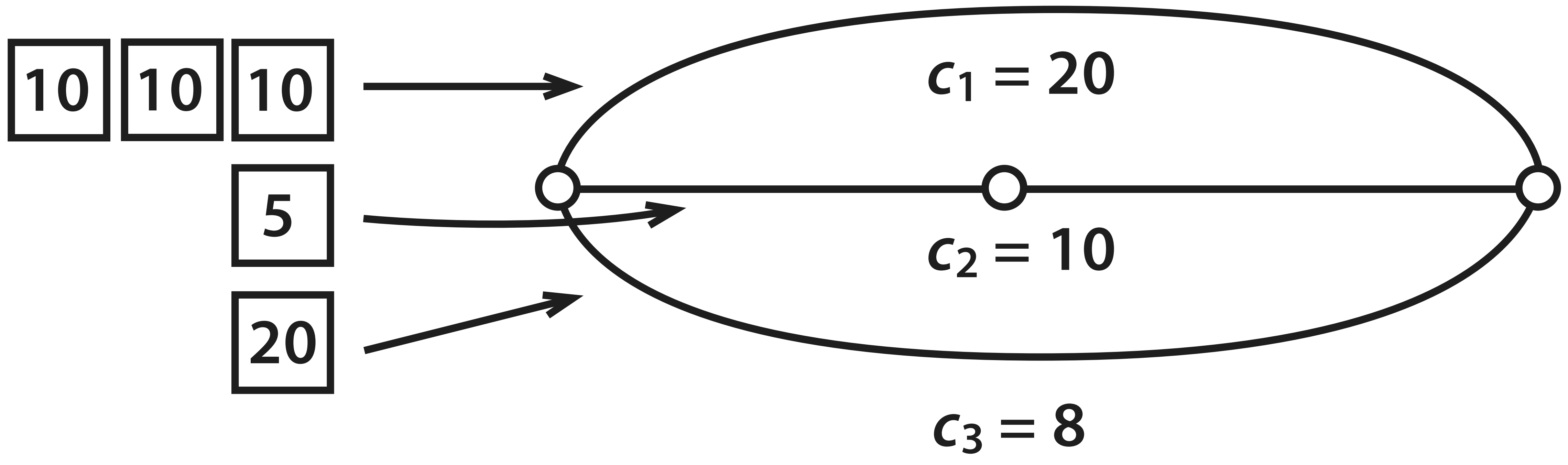
# KP-model of Optimal Routing with Indivisible Traffic




Consider the following road network:

- three parallel roads, $m = 3$,
- five drivers, $n = 5$,
- volumes of cars, $w = (20, 10, 10, 10, 5)$,
- capacities, $c = (20, 10, 8)$.

# KP-model of Optimal Routing with Indivisible Traffic




Consider the strategy profile $I^* = (3, 1, 1, 1, 2)$:

$$\{w \longrightarrow c : 20 \longrightarrow 8, (10, 10, 10) \longrightarrow 20, 5 \longrightarrow 10\}.$$

Losses in $I^* = (3, 1, 1, 1, 2)$:

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (2.5, 1.5, 1.5, 1.5, 0.5).$$

Strategy profile $I^* = (3, 1, 1, 1, 2)$ is a Nash equilibrium.

# Price of Anarchy

**Definition.**
Social costs characterize the maximum losses of system:

$$SC(w, c, l) = \max_j \frac{\sum_{k\,:\,l_k = j} w_k}{c_j}.$$

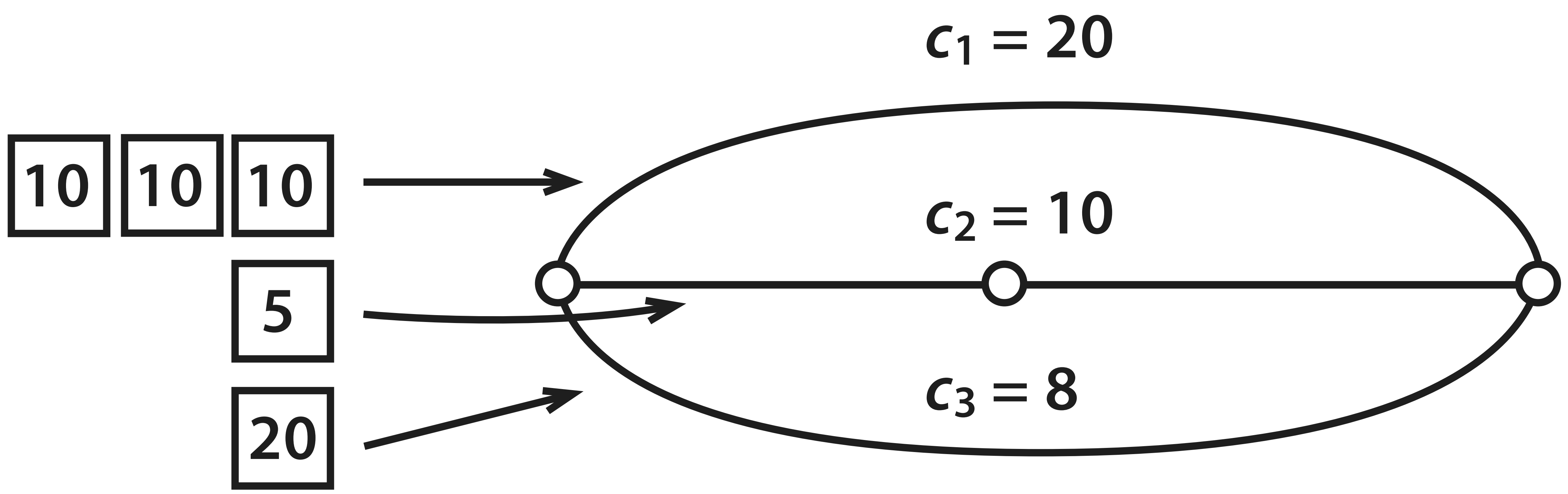
**Definition.**
Optimal social costs are the minimum costs of system:

$$opt = \min_{l\,:\,l_i \in L} SC(w, c, l).$$

**Definition.**
Price of anarchy is the ratio of social costs in the worst-case Nash equilibrium and optimal social costs:

$$PA = \max_{l^*} \frac{SC(w, c, l^*)}{opt}.$$

# Price of Anarchy




Consider the Nash equilibrium $l^* = (3, 1, 1, 1, 2)$:

$$\{w \longrightarrow c : 20 \longrightarrow 8, (10, 10, 10) \longrightarrow 20, 5 \longrightarrow 10\}.$$

Social costs:

$$SC(w, c, l^*) = \max(2.5, 1.5, 1.5, 1.5, 0.5) = 2.5.$$

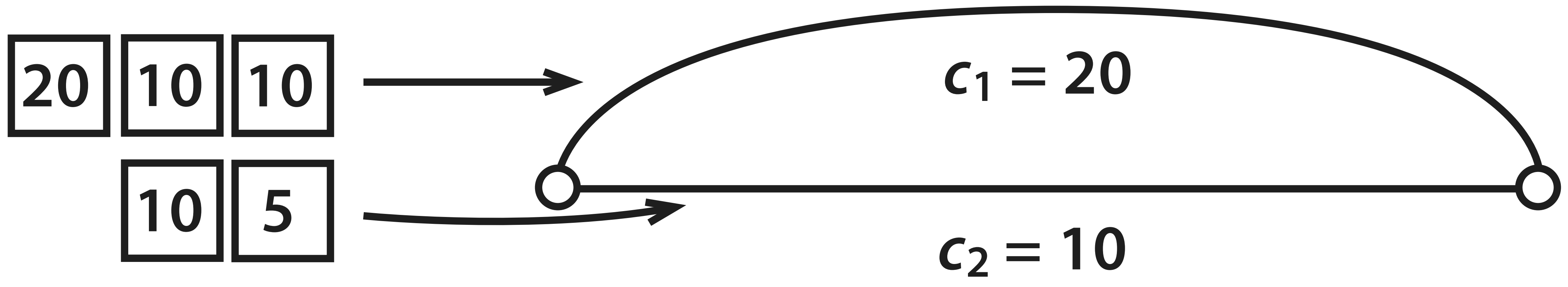
Optimal social costs:

$$opt = \min_{l\,:\,l_i \in L} SC(w, c, l) = 1.5.$$

Price of anarchy: $PA = \frac{2.5}{1.5} = \frac{5}{3}$.

# Price of Anarchy

Consider a road network without the route 3:




Consider the Nash equilibrium $I^* = (1, 1, 1, 2, 2)$:

$$\{w \longrightarrow c : (20, 10, 10) \longrightarrow 20, (10, 5) \longrightarrow 10\}.$$

Social costs:

$$SC(w, c, I^*) = \max(2, 2, 2, 1.5, 1.5) = 2.$$

Optimal social costs:

$$opt = \min_{I\,:\,I_i \in L} SC(w, c, I) = 2.$$

Price of anarchy: $PA = \frac{2}{2} = 1.$

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# Braess's Paradox and Wardrop equilibrium

**O. Petrosian**

PhD

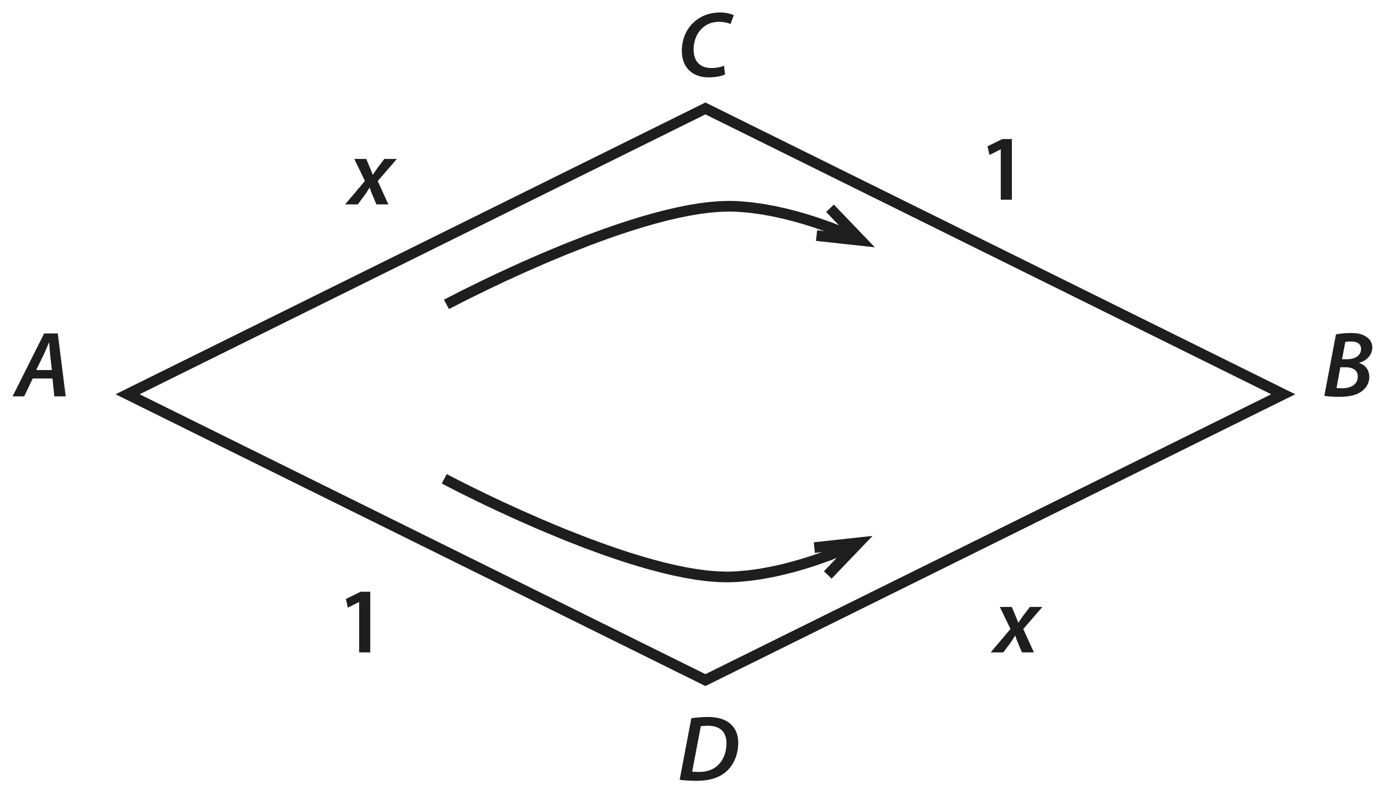
# Braess's Paradox

"**Nicholas I on a walk**",
N. Sverchkov, 1902

Consider a road network
with 4 routes:

- On two routes delay is proportional to the number of moving cars.
- On the other two delay is equal to one hour.

# Braess's Paradox



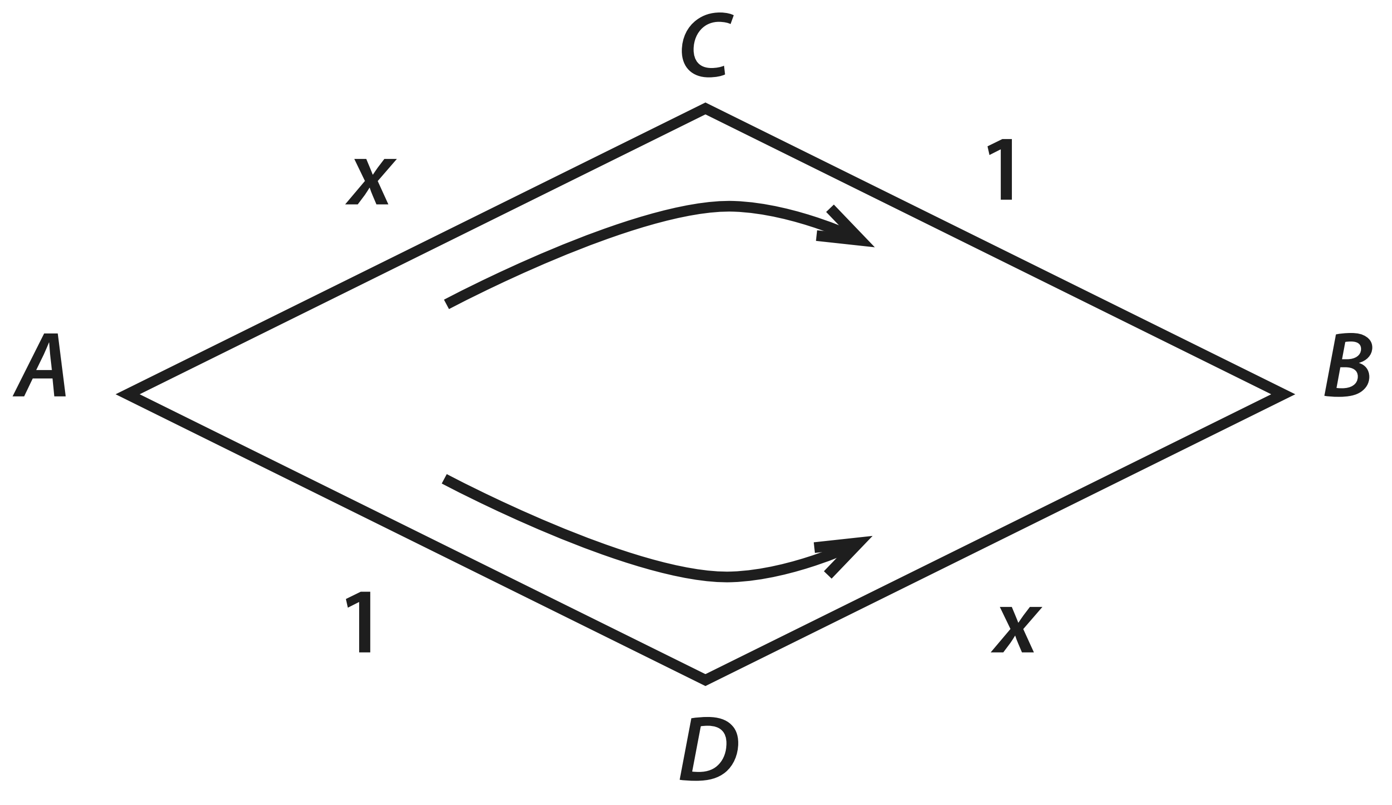

**Game description.**

- $N = \{1, \ldots, 60\}$ — 60 cars moving from $A$ to $B$.
- $L = \{ACB, ADB\}$ — set of routes from $A$ to $B$.
- Delay on route $ACB$ (the same for $ADB$):

$$\lambda(ACB) = \lambda(AC) + \lambda(CB) = x + 1 = \Big(\sum_{k\,:\,l_k = ACB} w_k\Big)/60 + 1,$$

where $w_i = 1,\ \forall i \in N$.

# Braess's Paradox




Consider strategy profile:

$$I^* = (I_i^* = ACB,\ i = \overline{1,30};\ I_i^* = ADB,\ i = \overline{31,60}).$$

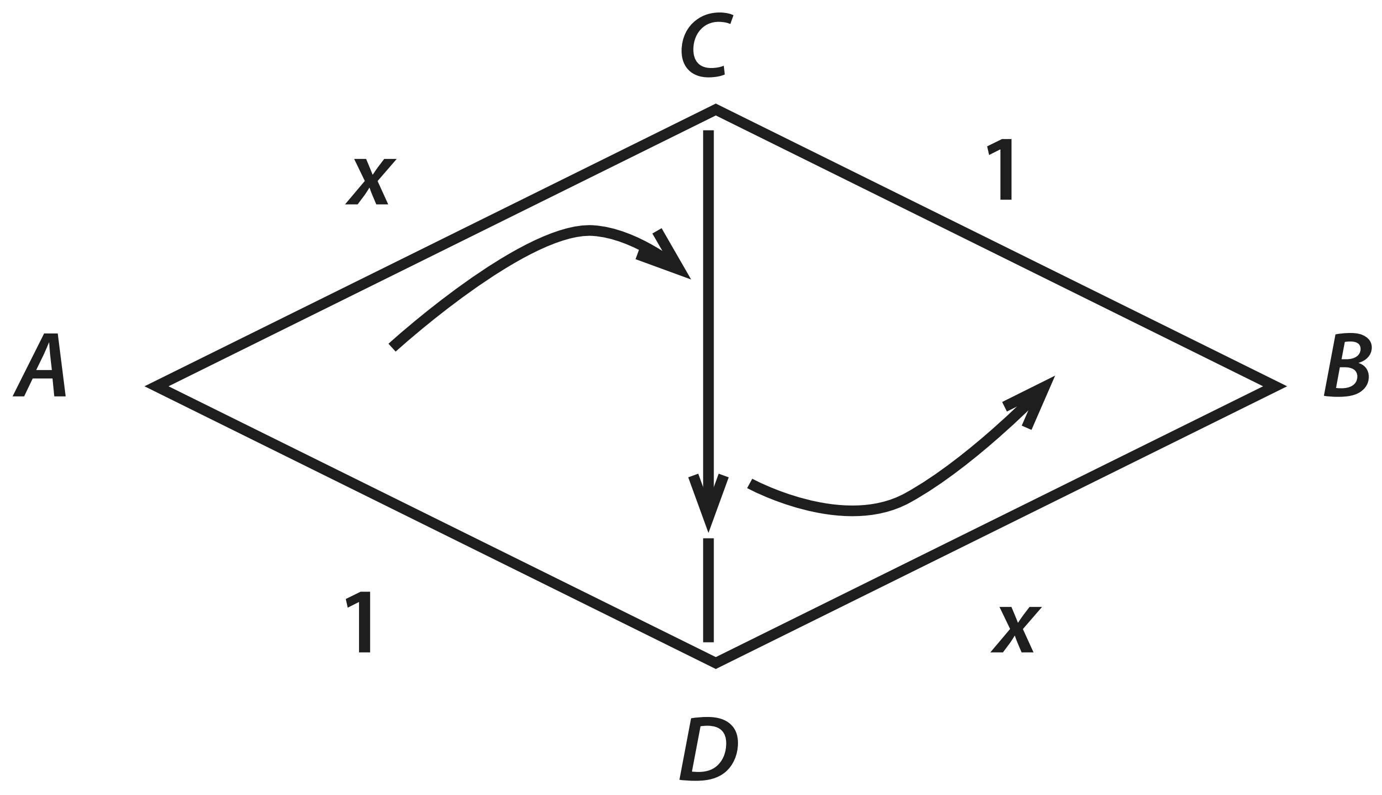
Losses in strategy profile $I^*$:

$$\lambda_i = 1.5,\ i = \overline{1,60}.$$

Strategy profile $I^*$ is Nash equilibrium.

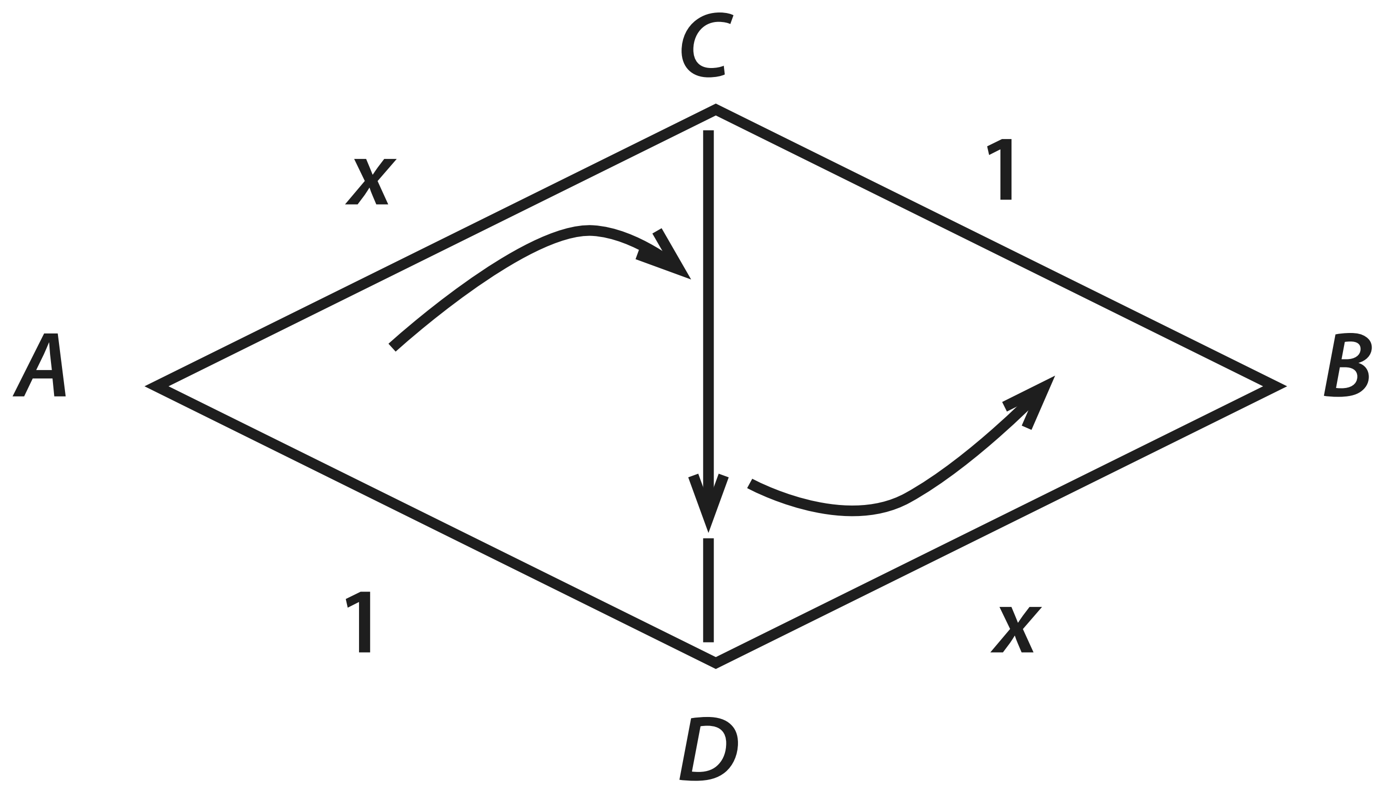
# Braess's Paradox

Suppose $CD$ is connected by the highway with delay $c(CD) = 0$:




- $L = \{ACB, ADB, ACDB, ADCB\}$.
- Delay on route $ACDB$ (for others the same):
$$\lambda(ACDB) = \lambda(AC) + \lambda(CD) + \lambda(DB) = x + 0 + x = 2\Big(\sum_{k\,:\,l_k = ACDB} w_k\Big)/60,$$
where $w_i = 1$, $\forall i \in N$.

# Braess's Paradox




Consider strategy profile:

$$I^* = (I_i^* = ACDB,\ i = \overline{1,60}).$$

Losses in strategy profile $I^*$:

$$\lambda_i = 2,\ i = \overline{1,60}.$$

Strategy profile $I^*$ is Nash equilibrium.

Braess's paradox: New highway increased the time delay for all participants!

# Wardrop Optimal Routing Model with Divisible Traffic

**Game description** $\Gamma = (N, \{X_i\}_{i=\overline{1,n}}, \{PC_i(x)\}_{i=\overline{1,n}})$:

- $N = \{1, \dots, n\}$.
- $X_i = \{x_i = (x_1^i, \dots, x_m^i) : \sum_{j=1}^{m} x_j^i = w_i,\ x_j^i \geq 0,\ j = \overline{1, m}\}$, where $w_i$ is the traffic volume of player $i$,

  $x_j^i$ is the traffic of player $i$ on the route $j$.
- $x_i = (x_1^i, \dots, x_m^i) \in X_i$ is the strategy of player $i$ (allocation of traffic $w_i$ by player $i$).
- $PC_i(x) = \max\limits_{j\,:\,x_i^j > 0} f_j(x)$ are the losses of player $i$

  (maximum delay of player i on all routes),

  where $f_j(x) = f_j(\delta(x))$ is the delay function on the route $j$,

  $\delta(x) = (\delta_1(x), \dots, \delta_m(x))$, where $\sum_{i=1}^{n} x_j^i = \delta_j(x)$ is the traffic load vector.

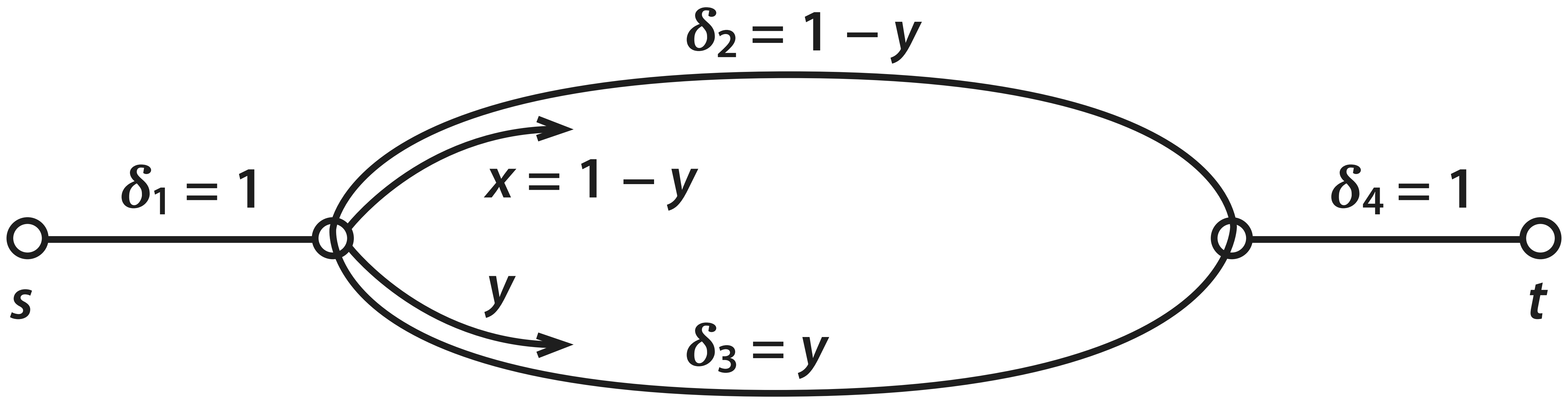
# Wardrop Equilibrium

**Definition.**
Strategy profile $x^* = (x_1^*, \dots, x_n^*)$ is called Wardrop equilibrium, if for each i the following hold:

- if $x_k^{*i} > 0$, then $f_k(x^*) = \min_j f_j(x^*)$,
- if $x_k^{*i} = 0$, then $f_k(x^*) > \min_j f_j(x^*)$.

**Theorem.**
If strategy profile $x^* = (x_1^*, \dots, x_n^*)$ is Wardrop equilibrium, then $x^*$ is Nash equilibrium.

# Wardrop Equilibrium

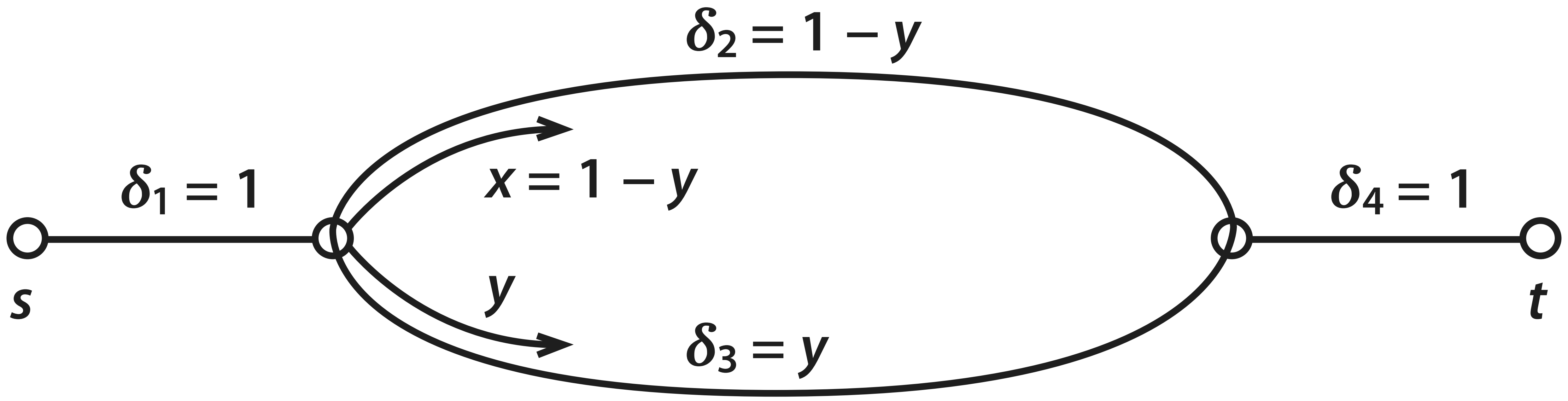



**Description of the game $\Gamma$** $= (N, \{Z_i\}_{i=1}, \{PC_i(z)\}_{i=1})$:

- $N = \{1\}$ — one player.
- $z_1 = (x, y)$ is the strategy of player, where $x, y \geq 0$, $x + y = w_1 = 1$.
- $PC_1(z) = \max\limits_{j:\, z_j^1 > 0} f_j(z)$ are the losses of player,

  where $f_1(z) = \max\{1, x, 1\}$, $f_2(z) = \min\{1, y, 1\}$ are delay functions.

# Wardrop Equilibrium



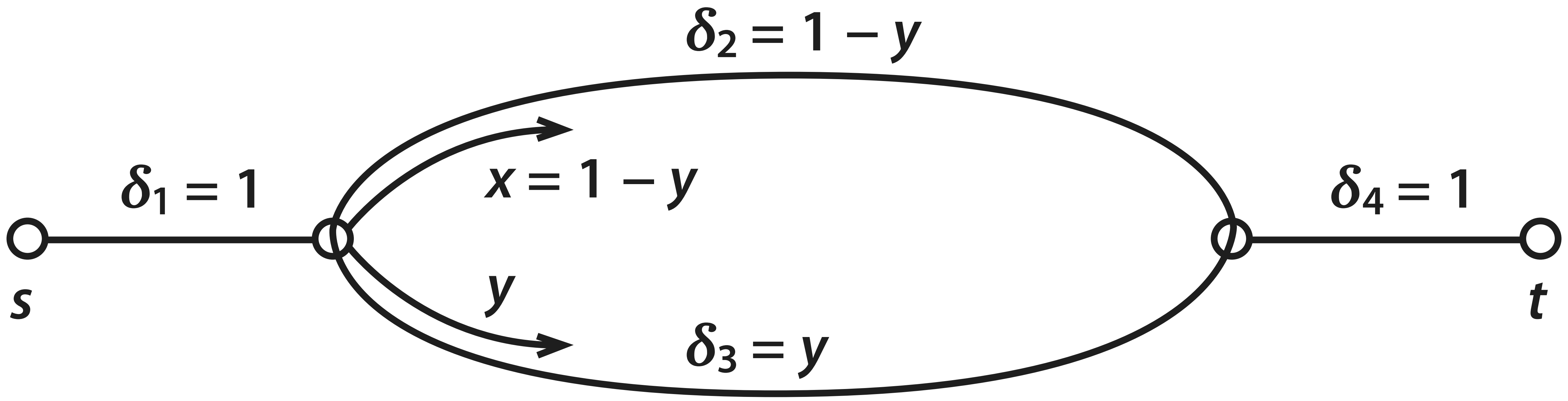

Nash equilibrium $z^{*} = (x,\ 1 - x),\ 0 \le x \le 1$:

since $f_1(z) = \max\{1, x, 1\} = 1$,

$f_2(z) = \min\{1, y, 1\} = y$,

therefore, $PC_1(z) = \max\limits_{j:\ z_j^1 > 0} f_j(z) = 1,\ \forall z.$

# Wardrop Equilibrium




Wardrop equilibrium $z^* = (0, 1)$:

- $f_1(z) = \max\{1, x, 1\} = 1 > y = \min\{1, y, 1\} = f_2(z)$, for $\forall z$: $x > 0$,
- $f_1(z) = \max\{1, x, 1\} = 1 = y = \min\{1, y, 1\} = f_2(z)$, for $z$: $x = 0$.

# References

1. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.
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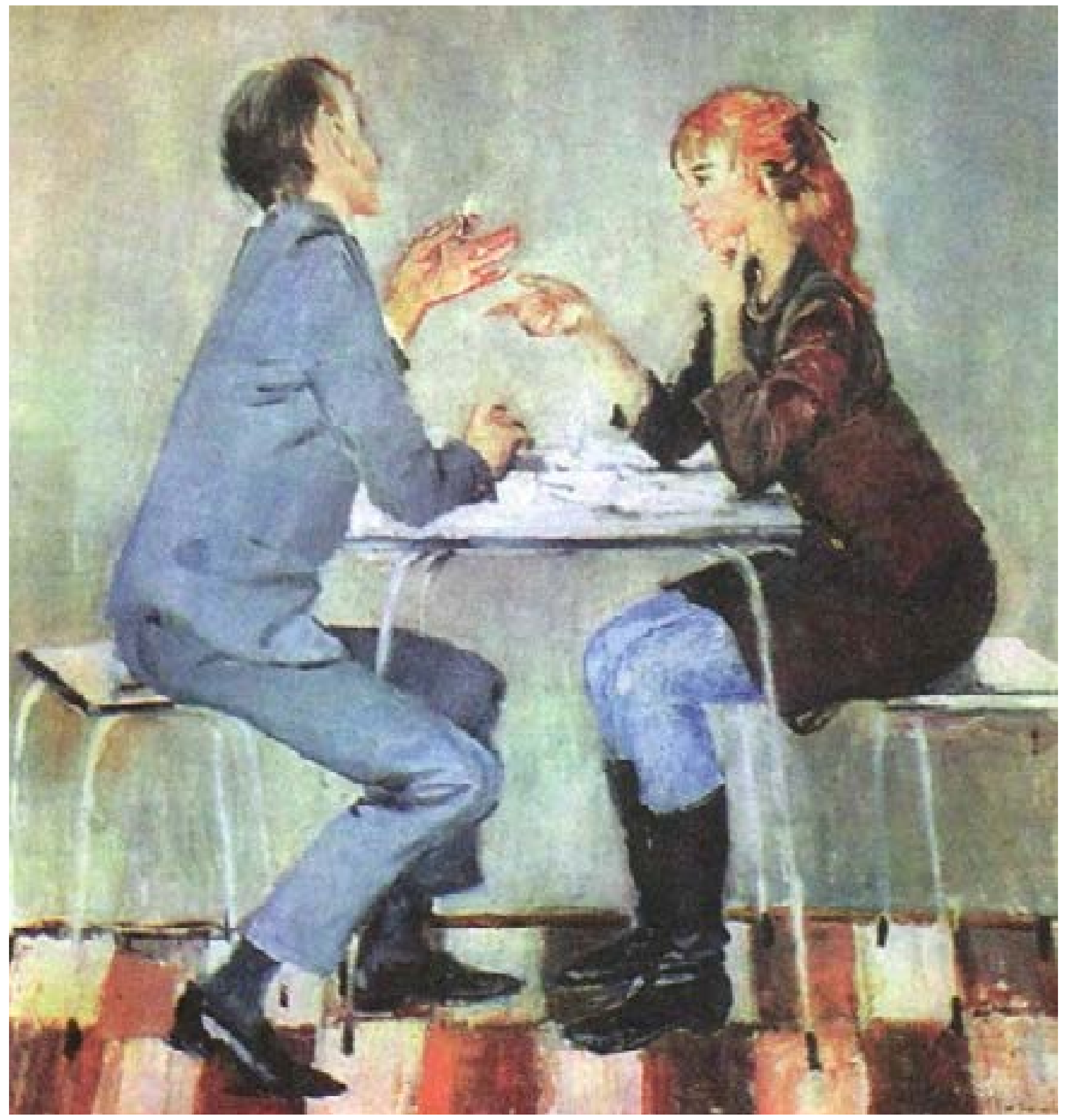
# Bargaining Problems and Nash Bargaining Solution

**O. Petrosian**

PhD

# Battle of Sexes



**“Argument”,**
Pimenov Yu. I., 1968

Husband and his wife may choose one of two evening entertainments:

- football match,
- theatre.

If they choose different entertainments, then they stay at home. For both of them it is important to spend the evening together.

# Battle of Sexes

$$(A, B) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \end{array}$$

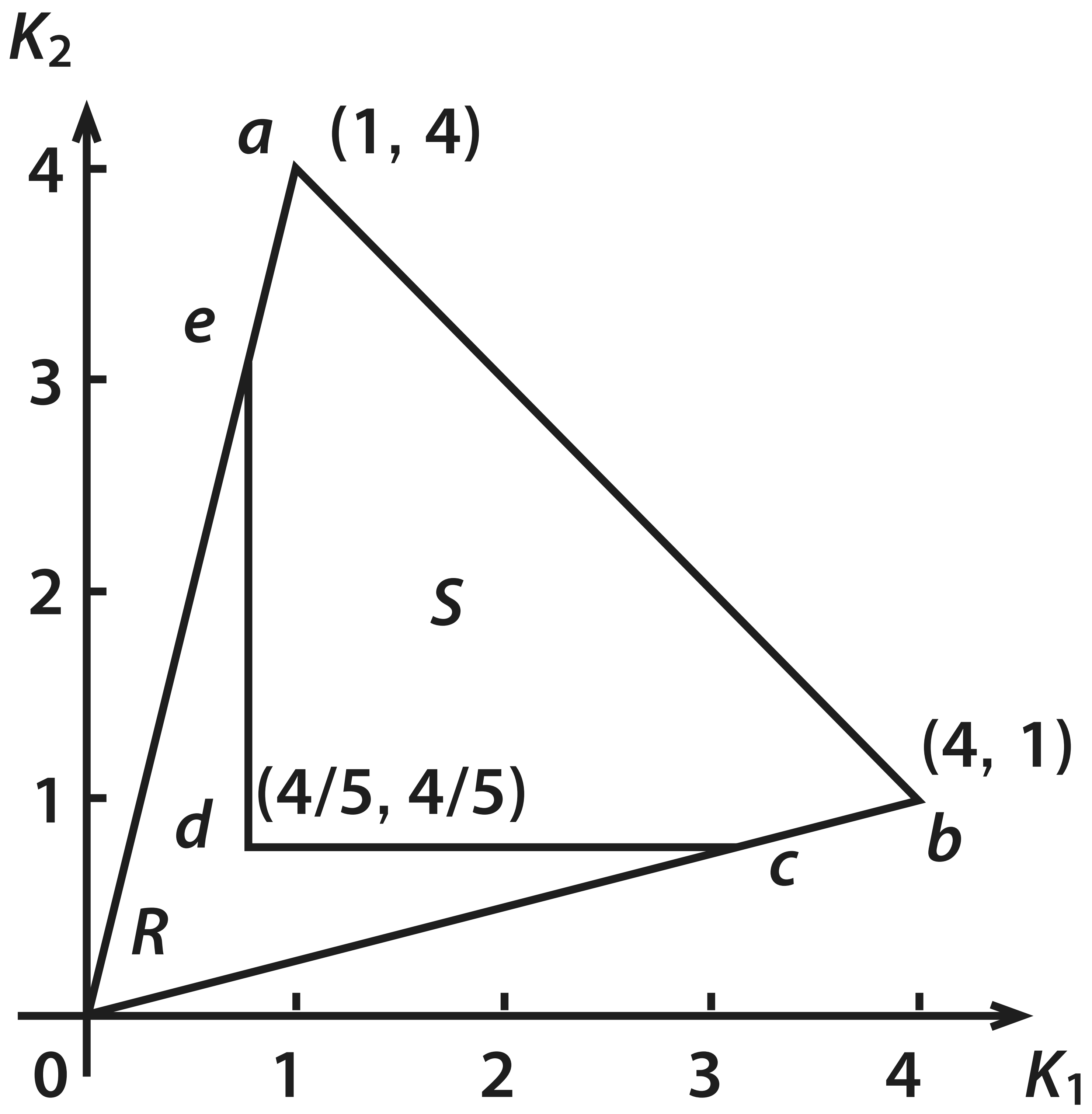
Nash equilibrium:

- $(x_1, y_1)$ — both choose to go to the football.
- $(x_2, y_2)$ — both choose to go to the theater.
- $x^* = \left(\frac{4}{5}, \frac{1}{5}\right)$, $y^* = \left(\frac{1}{5}, \frac{4}{5}\right)$ – Nash equilibrium in mixed strategies.

If players could make an agreement on outcomes,
what outcome would they choose?

# Battle of Sexes

Set of possible payoffs over the joint probability distributions of strategy profiles:



# Bargaining Problem

**Bargaining problem.**
It is necessary to determine function $\varphi$, such that

$$\varphi(S, d) = \bar{v} \in S,$$

where $S \in \sum^n$ is a bargaining set, $d = (d_1, \dots, d_n) \in S$ is a disagreement point.

**Notations.**

- $d = (d_1, \dots, d_n) \in S$ are payoffs that the players obtain, if they do not reach an agreement.
- $S$ is a set of all outcomes satisfying the conditions:
  $S \in \sum^n \Leftrightarrow S = \{v\colon v \geq d;\ \exists v' \in S\colon v' > d\} \subset R^n$.

# Bargaining Problem

**Bargaining problem.**
It is necessary to determine function $\varphi$, such that

$$\varphi(S, d) = \overline{v} \in S,$$

where $S \in \sum^n$ is a bargaining set, $d = (d_1, \dots, d_n) \in S$ is a disagreement point.

**Battle of Sexes.**

- $N = \{1, 2\}$ is a set of players.
- $S$ is a set of possible payoffs $(K_1(x_i, y_j), K_2(x_i, y_j))$ defined over the joint probability distributions of strategy profiles $(x_i, y_j)$, for which $(K_1, K_2) \geq d = (d_1, d_2)$.
- $\varphi(S, d) = v \in S \subseteq R^2$.
- $d = (d_1, d_2) = \left(\frac{4}{5}, \frac{4}{5}\right)$ is a disagreement point.

# Nash Bargaining Solution for $n$-player game

**Axioms.**

1. **Pareto-optimality:**
if $v \geq \varphi(S, d)$ and $v \neq \varphi(S, d)$, then $v \notin S$ for $S \in \sum^n$, $v \in R^n$.

2. **Symmetry:**
suppose $S \in \sum^n$, if for $\pi \in \Pi^n$, $\pi(S) = S$, then $\varphi_i(S, d) = \varphi_j(S, d)$ for $i, j$ (wherein $\pi(S) \in \sum^n$).

3. **Scale invariance:**
for $S \in \sum^n$ and $l \in L^n$, holds $\varphi(l(S), d) = l(\varphi(S, d))$ ($l(S) \in \sum^n$).

4. **Independence of irrelevant alternatives:**
if $S' \subset S$ and $\varphi(S, d) \in S'$, then $\varphi(S', d) = \varphi(S, d)$ for $S, S' \in \sum^n$.

# Nash Bargaining Solution for *n*-player game

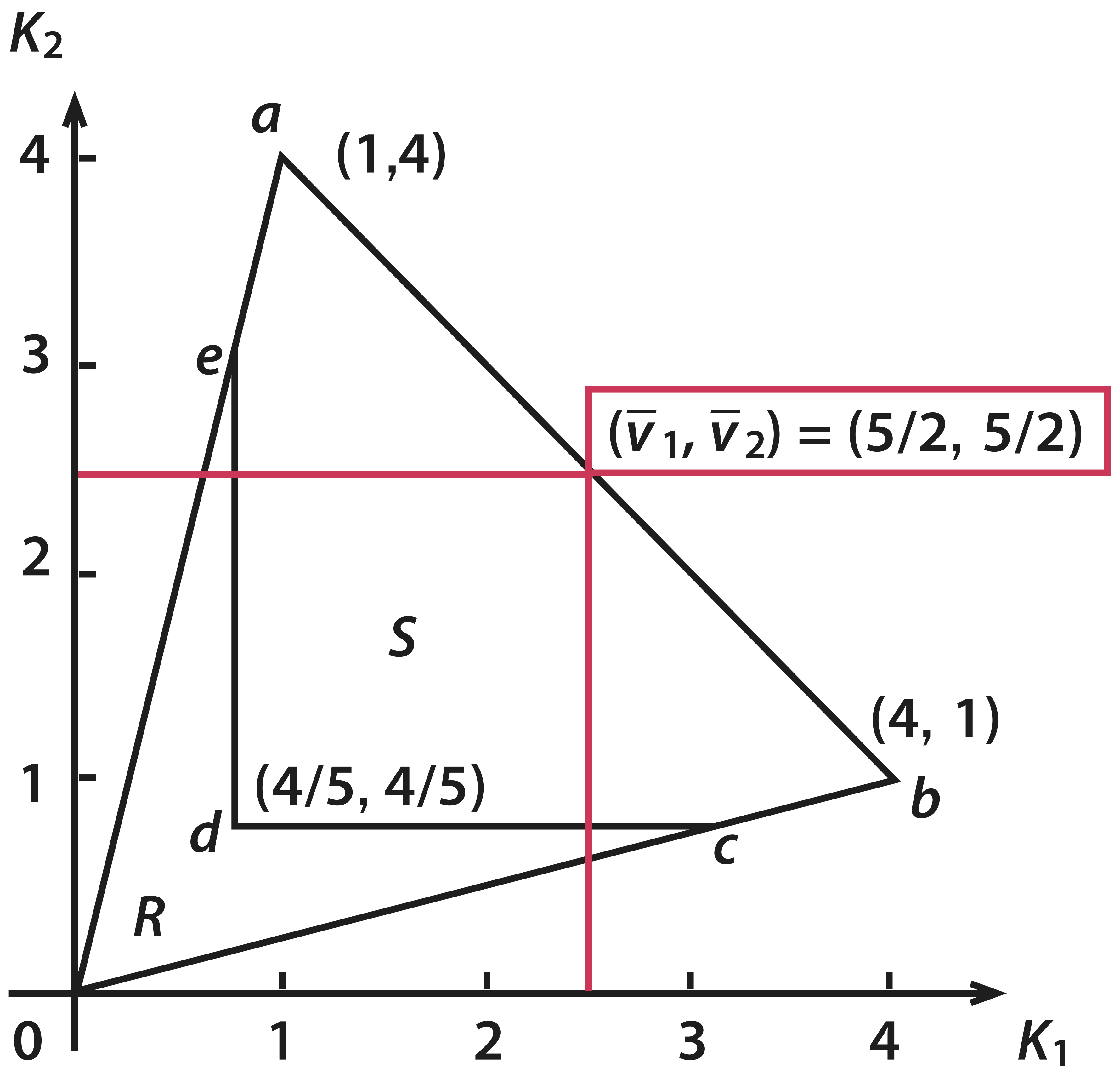
**Theorem.**

There exists unique function $\varphi$: $\varphi(S, d) = \bar{v} \in S$,
satisfing axioms 1-4 and it is calculated according to the formula:

$$\varphi(S, d) = arg\max_{v \in S} \prod_{i=1}^{n} (v_i - d_i).$$

Function $\varphi(S, d) = N(S, d)$ is called the Nash bargaining solution.

# Nash Bargaining Solution for *n*-player game




**Battle of Sexes.**

$N(S, d) = \left(\frac{5}{2}, \frac{5}{2}\right),$

where $d = \left(\frac{4}{5}, \frac{4}{5}\right).$

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# References


1. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory.
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# Other Bargaining Solutions

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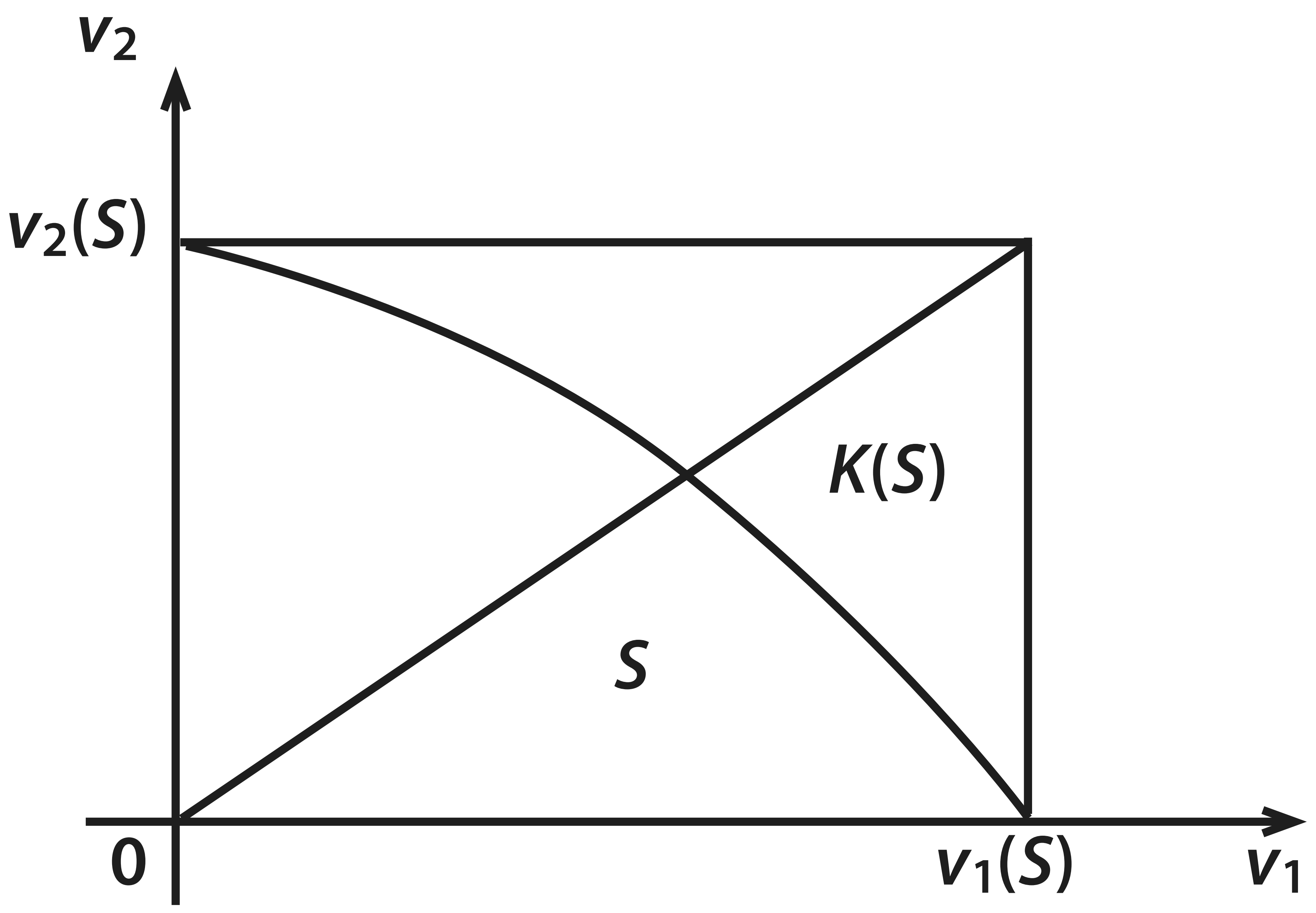
# Kalai-Smorodinsky Solution for 2-player game




Suppose $d = 0$.

**Definition.**

Kalai-Smorodinsky solution $K(S, d)$ is the maximal point of set $S \in \sum^n$ on the interval connecting $d$ with point $v(S) = (v_1(S), v_2(S))$: $v_i(S) = \max\{v_i - d_i \mid v \in S\}$, $i = 1, 2$.

# Kalai-Smorodinsky Solution for 2-player game

**Theorem.**
Solution $\varphi(S, d)$ satisfies the axioms 1–3, 5, if it is a Kalai-Smorodinsky solution.

**Axioms.**

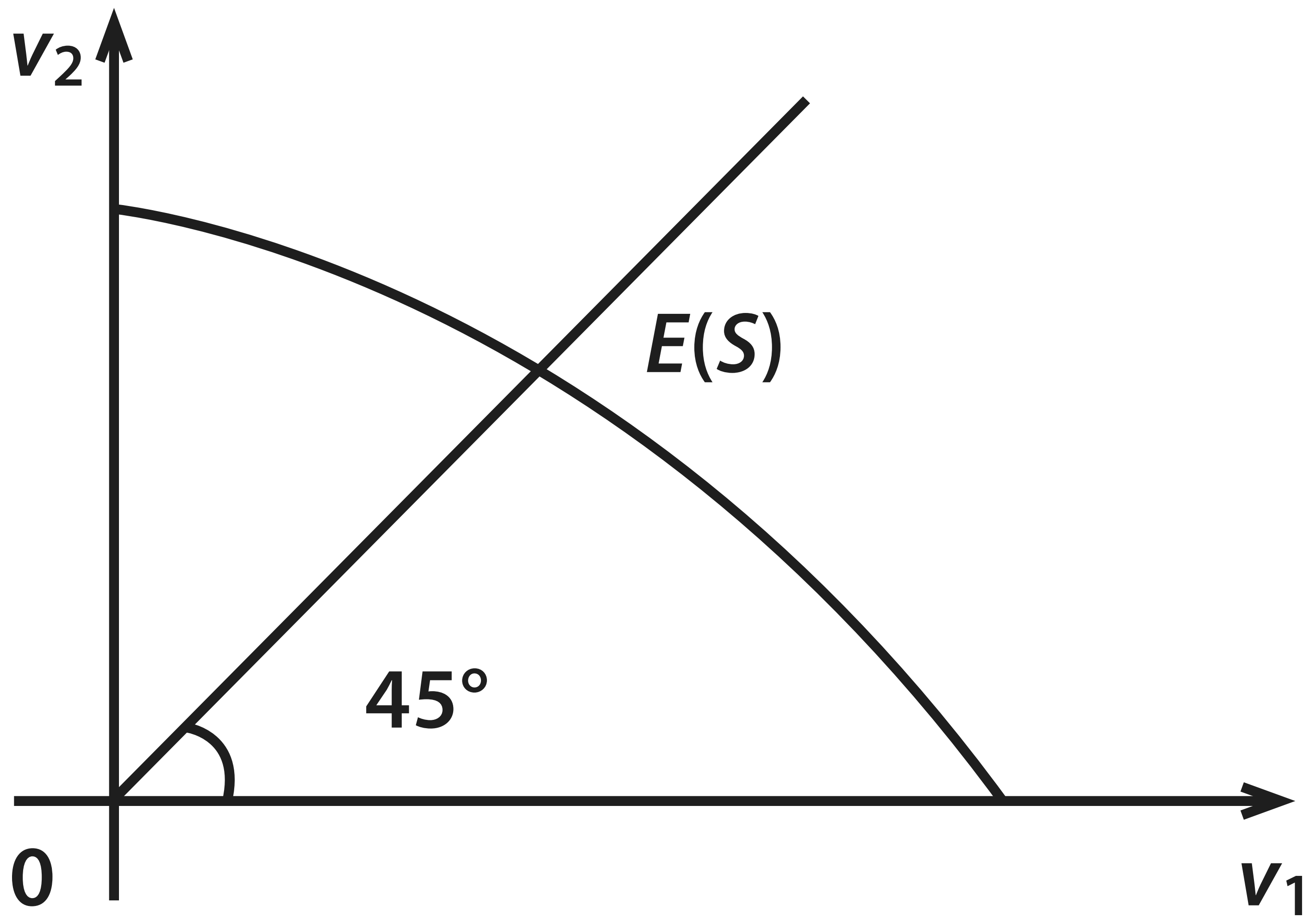
1. **Pareto-optimality:**
if $v \geq \varphi(S, d)$ and $v \neq \varphi(S, d)$, then $v \notin S$ for all $S \in \sum^2$, $v \in R^2$.

2. **Symmetry:**
suppose $S \in \sum^2$, if for all $\pi \in \Pi^2$, $\pi(S) = S$, then $\varphi_i(S, d) = \varphi_j(S, d)$ for all $i, j$.

3. **Scale invariance:**
for all $S \in \sum^2$ and $l \in L^2$, hold $\varphi(l(S, d)) = l(\varphi(S, d))$ $(l(S) \in \sum^2)$.

5. **Individual monotonicity:**
if $S \subset S'$, $v_i(S) = v_i(S')$ for $i = 1, 2$, then $\varphi(S, d) \leq \varphi(S', d)$ for all $S, S' \in \sum^2$.

# Egalitarian Solution




Suppose $d = 0$.

**Definition.**
Egalitarian solution $E(S, d)$ for all $S \in \sum^n$ is the maximal point in $S$ with equal coordinates.

# Egalitarian Solution

**Theorem.**
Solution $\varphi(S)$, $S \in \sum^n$ satisfies the axioms 1′, 2′, 5′, if and only if it is an egalitarian solution.

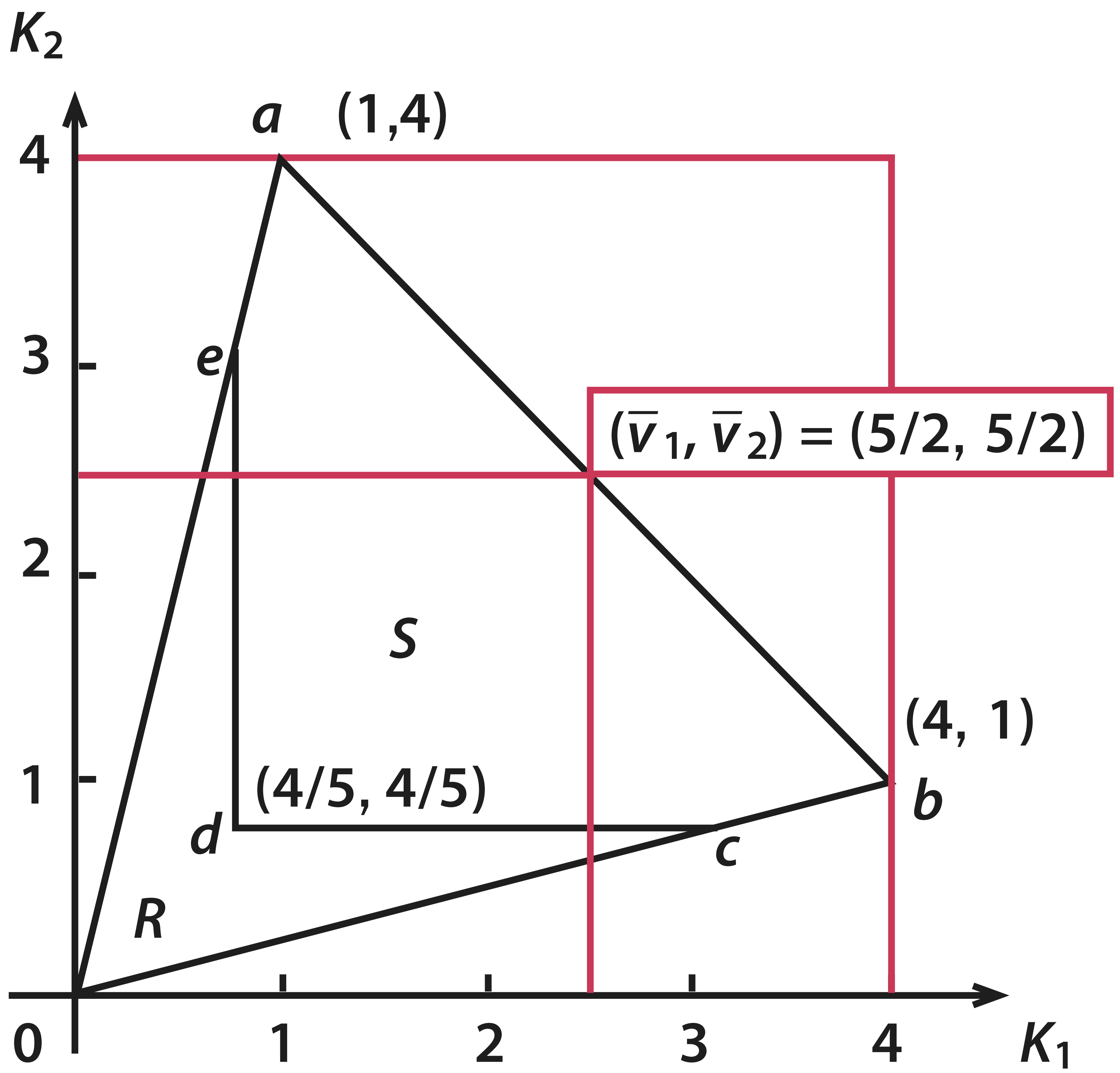
**Axioms.**

1′ **Weak pareto-optimality**:
if $v > \varphi(S, d)$, then $v \notin S$ for all $S \in \sum^n$, $v \in R^n$.

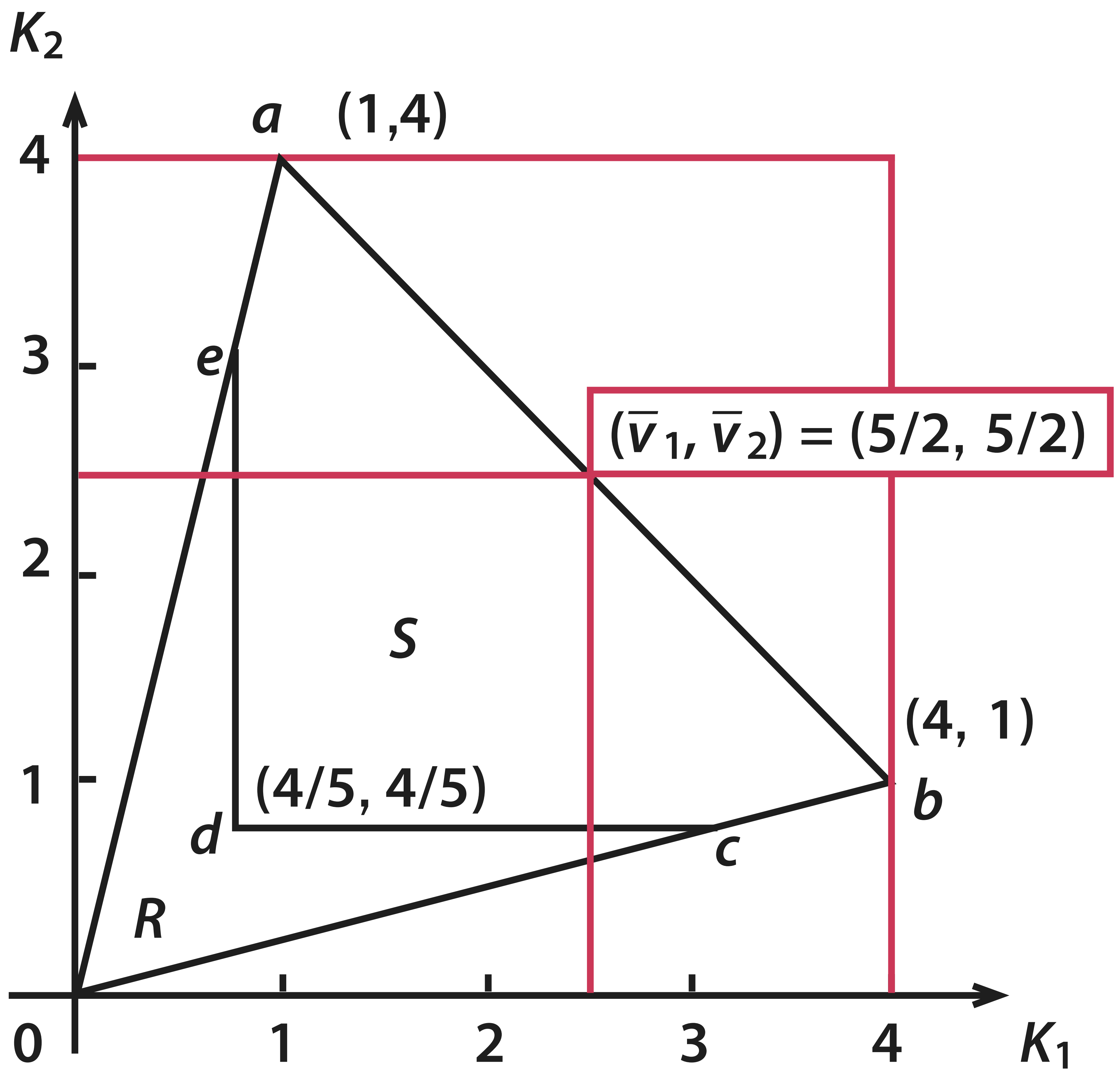
2 **Symmetry:**
suppose $S \in \sum^n$, if for all $\pi \in \Pi^n$, $\pi(S) = S$, then $\varphi_i(S, d) = \varphi_j(S, d)$ for all $i, j$ (wherein $\pi(S) \in \sum^n$).

5′ **Strong monotonicity:**
if $S \subset S'$, then $\varphi(S, d) \leq \varphi(S', d)$ for all $S, S' \in \sum^n$.

# Battle of Sexes



# Battle of Sexes




**Bargaining solution.**

- $N(S, d) = \left(\frac{5}{2}, \frac{5}{2}\right)$ — Nash bargaining solution.
- $K(S, d) = \left(\frac{5}{2}, \frac{5}{2}\right)$ — Kalai-Smorodinsky solution.
- $E(S, d) = \left(\frac{5}{2}, \frac{5}{2}\right)$ — egalitarian solution.

Bargaining solutions coincide!

# Another Example

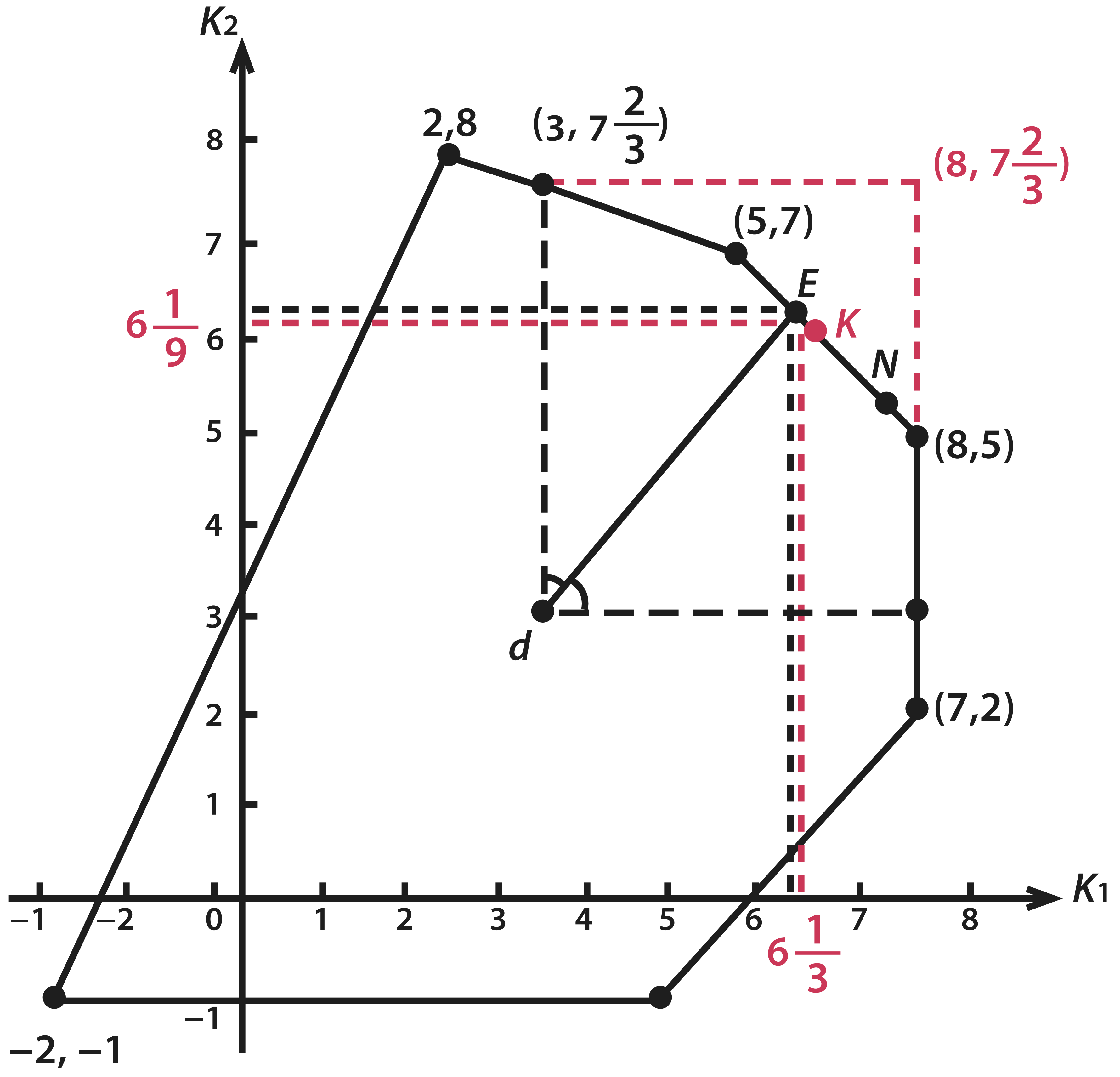
$$(A, B) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & (2, 8) & (7, 2) & (-2, -1) \\ x_2 & (5, -1) & (5, 7) & (1, 5) \\ x_3 & (3, 4) & (6, 4) & (8, 5) \end{array}$$

- Nash equilibrium: $(x^*, y^*) = (x_3, y_3)$,

$$(K_1(x^*, y^*), K_2(x^*, y^*)) = (8, 5).$$

- Suppose the disagreement point: $(d_1, d_2) = (3, 3)$.

# Another Example




**Bargaining solution.**

- $N(S, d) = \left(7, 5\frac{2}{3}\right)$.
- $K(S, d) = \left(6\frac{1}{3}, 6\frac{1}{9}\right)$.
- $E(S, d) = \left(6\frac{1}{5}, 6\frac{1}{5}\right)$.

Bargaining solutions are different!

# References


1. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory.
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# Games in Characteristic Function Form and Imputations

**O. Petrosian**

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# Jazz Band


**“Orchestra of Duke Ellington”**,
Los Angeles, California, 1943

The club owner promises 100$ to the singer, pianist and drummer for a joint performance.

How should performers share the joint profit?

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# Jazz Band

Performing individually or as duets, they can earn:

- Coalition singer-pianist — 80$.
- Coalition drummer-pianist — 65$.
- Coalition singer-drummer — 50$.
- Pianist alone — 30$.
- Singer — 20$.
- Drummer alone cannot earn anything.

# Games in Characteristic Function Form

**Definition.**
Real-valued function $v: 2^N \longrightarrow R$, $v(\varnothing) = 0$, defined over the set of coalitions $S \subseteq N$ is called a characteristic function $v(S)$.

**Classical requirement.**

- **Superadditivity:** $v(A \cup B) \geq v(A) + v(B)$, $A \cap B = \varnothing$, $A, B \subset N$.

# Games in Characteristic Function Form

**Definition.**
Real-valued function $v : 2^N \longrightarrow R$, $v(\varnothing) = 0$, defined over the set of coalitions $S \subseteq N$ is called a characteristic function $v(S)$.

**Definition.**
Cooperative game is a pair $(N, v)$, where $N$ is the set of players, $v$ is the characteristic function.

# Games in Characteristic Function Form

**Definition.**
Cooperative game is a pair $(N, v)$, where $N$ is the set of players, $v$ is the characteristic function.

**Jazz band.**

- $N = \{1, 2, 3\} = \{$singer, drummer, pianist$\}$,
- $v(\{1, 2, 3\}) = 100$,
  $v(\{1, 2\}) = 50$, $v(\{1, 3\}) = 80$, $v(\{2, 3\}) = 65$,
  $v(\{1\}) = 20$, $v(\{2\}) = 0$, $v(\{3\}) = 30$.

# Games in Characteristic Function Form

**Classical requirement.**

- **Superadditivity:** $v(A \cup B) \geq v(A) + v(B)$, $A \cap B = \varnothing$.

**Jazz band.**

- $v(\{1, 2, 3\}) = 100 \quad \geq \quad 80 = 50 + 30 = v(\{1, 2\}) + v(\{3\})$,
  $v(\{1, 2, 3\}) = 100 \quad \geq \quad 85 = 20 + 65 = v(\{1\}) + v(\{2, 3\})$,
  $v(\{1, 2, 3\}) = 100 \quad \geq \quad 80 = 80 + 0 = v(\{1, 3\}) + v(\{2\})$,
- $v(\{1, 2\}) = 50 \geq 20 \quad = \quad 20 + 0 = v(\{1\}) + v(\{2\})$,
- $v(\{1, 3\}) = 80 \geq 50 \quad = \quad 20 + 30 = v(\{1\}) + v(\{3\})$,
- $v(\{2, 3\}) = 65 \geq 30 \quad = \quad 0 + 30 = v(\{2\}) + v(\{3\})$.

# References


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# Shapley Value and Core

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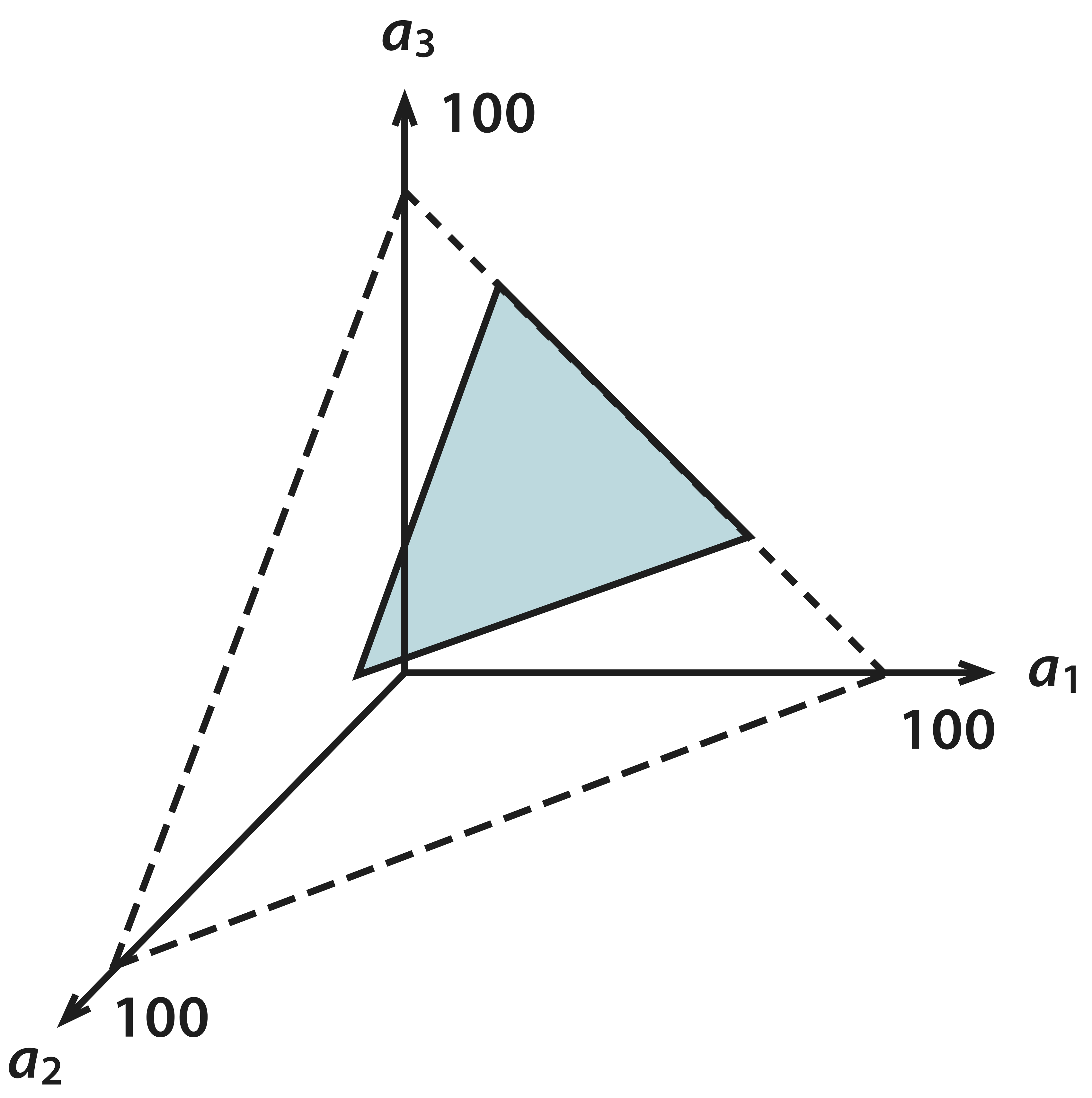
# Imputation

**Definition.**
Vector $a = (a_1, \dots, a_n)$, satisfying the conditions

- **Individual rationality:** $a_i \geq v(\{i\}),\ i \in N$,
- **Group rationality:** $\sum_{i=1}^{n} a_i = v(N)$,

is called an imputation.

# Imputation




**Jazz band.**

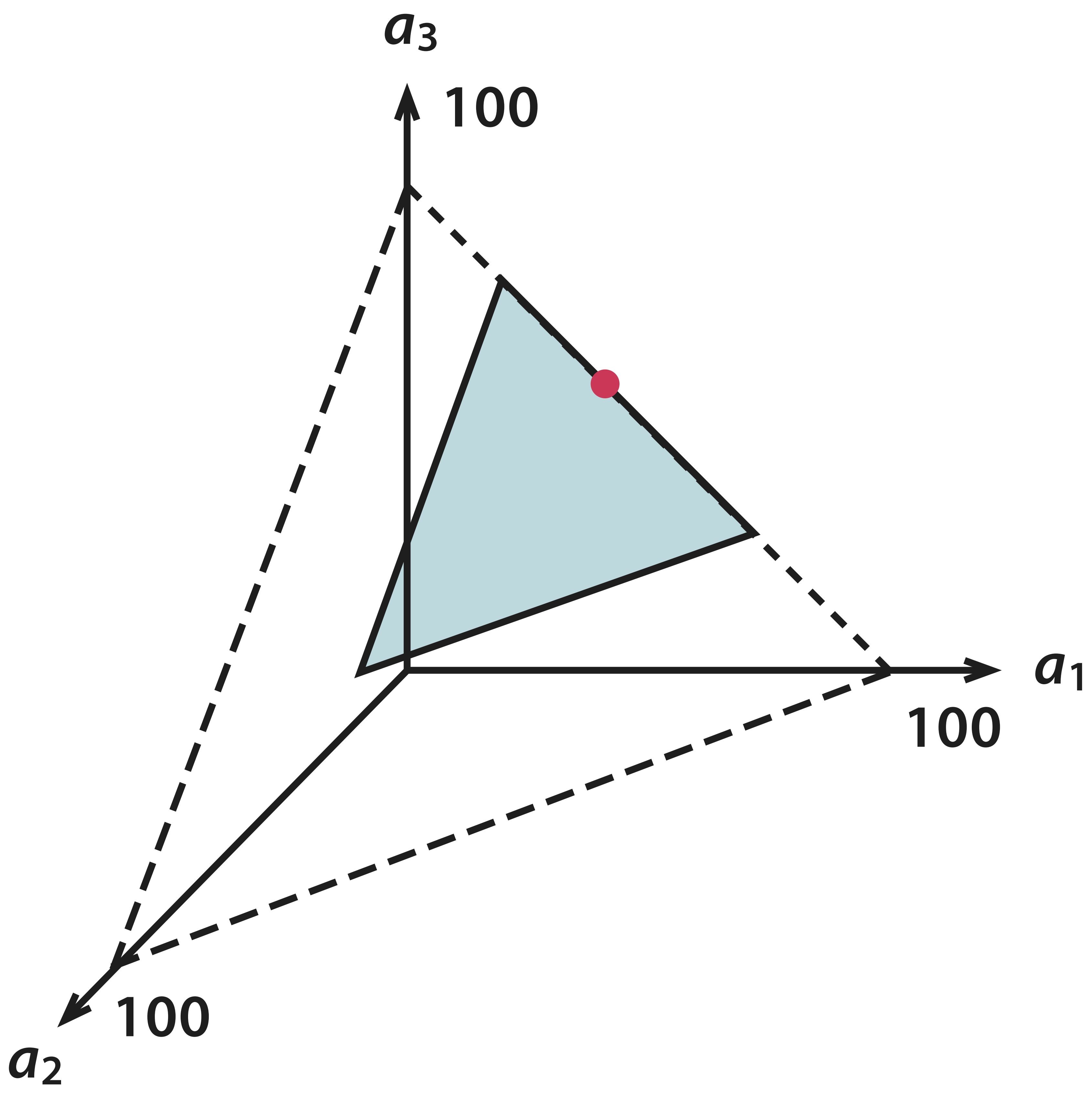
$a_1 \geq 20$,

$a_2 \geq 0$,

$a_3 \geq 30$,

$a_1 + a_2 + a_3 = 100$.

# Imputation




Proportional solution $prop = (prop_1, \dots, prop_n)$:

$$prop_i = \frac{v(i)}{\sum_{i=1}^{3} v(i)} v(N), i \in N.$$

**Jazz band.**

$$prop = (40, 0, 60).$$

# Shapley Value

**Definition.**
Value operator is a mapping $\varphi$, which for each game $(N, v)$ assignes the unique imputation $\varphi[v]$.

**Axioms.**

1 **Anonymity:**
for any permutation $\pi$ and $i \in N$, $\varphi_{\pi(i)}[\pi v] = \varphi_i[v]$.

2 **Additivity:**
if $(N, u)$ and $(N, v)$ are any two cooperative games,
then $\varphi_i[u + v] = \varphi_i[u] + \varphi_i[v]$.

3 **Dummy axiom:**
if for any cooperative game $(N, v)$
$v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$, then $\varphi_i[v] = v(\{i\})$.

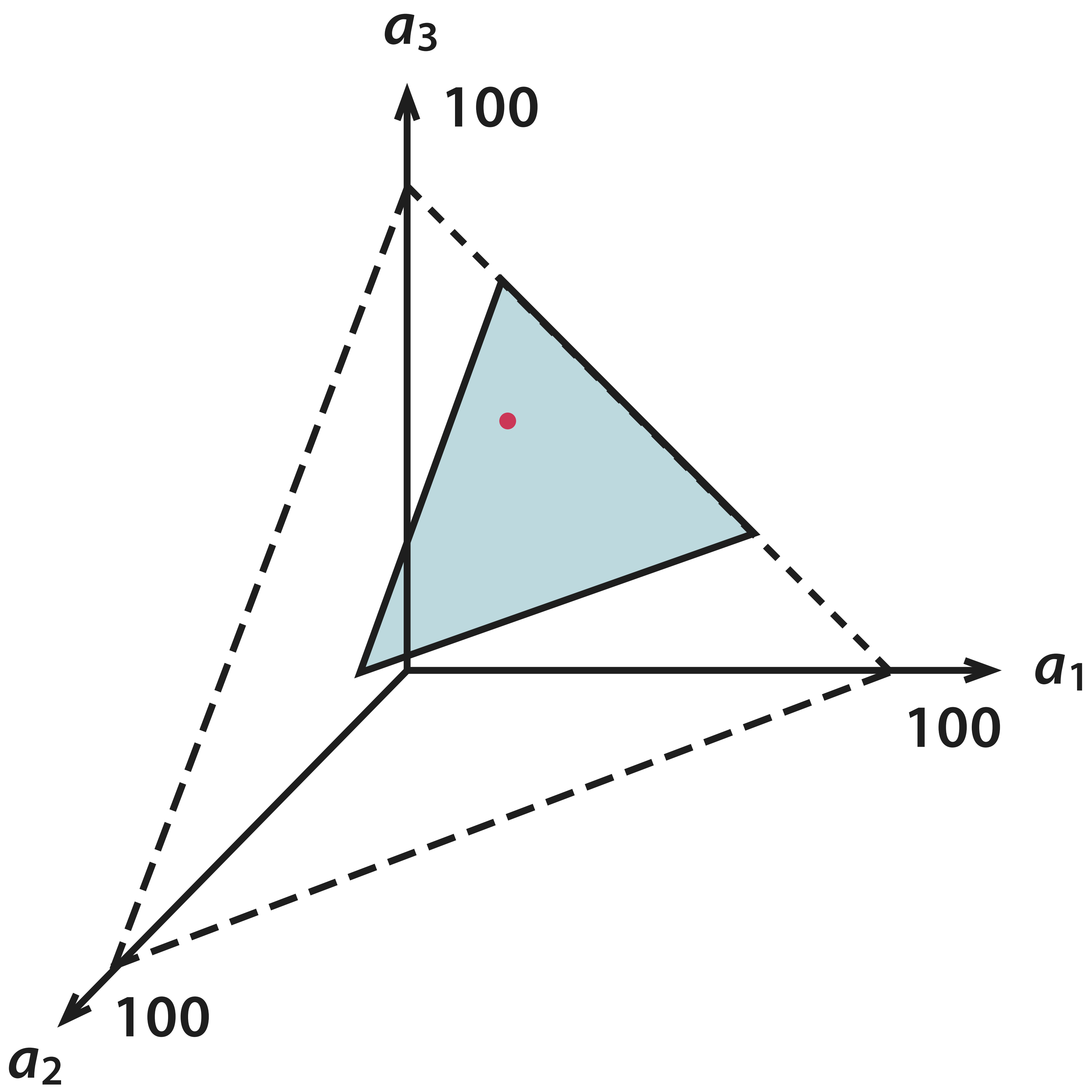
# Shapley Value

**Theorem.**
There exists only one value operator $\varphi$, satisfing the axioms 1–3, it is the Shapley value:

$$\varphi_i[v] = \sum_{S \mid i \in S \subseteq N} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!}\,[v(S) - v(S \setminus \{i\})],$$

where $|S|$ is the number of elements of set $S$.

# Shapley Value




**Jazz band.**

$\varphi_i[v] = (35,\ 17.5,\ 47.5).$

# Dominance of Imputations

**Definition.**
Imputation $\alpha$ dominates the imputation $\beta$ by coalition $S$ (notation $\alpha \overset{S}{\succ} \beta$), if

$$\alpha_i > \beta_i,\ i \in S,$$

$$\sum_{i \in S} \alpha_i \le v(S).$$

**Definition.**
Imputation $\alpha$ dominates the imputation $\beta$ ($\alpha \succ \beta$), if there exists coalition $S$, for which

$$\alpha \overset{S}{\succ} \beta.$$

# Dominance of Imputations

**Jazz band.**

Consider two imputations: $\alpha = (35, 30, 35)$, $\beta = (45, 25, 30)$.

- $\alpha \overset{\{2,\,3\}}{\succ} \beta$:

$$\alpha_2 = 30 > 25 = \beta_2,$$
$$\alpha_3 = 35 > 30 = \beta_3,$$
$$\alpha_2 + \alpha_3 = 65 \leq 65 = v(S).$$

- $\alpha \succ \beta$:
  exists coalition {2, 3}: $\alpha \overset{\{2,\,3\}}{\succ} \beta$.

# Core

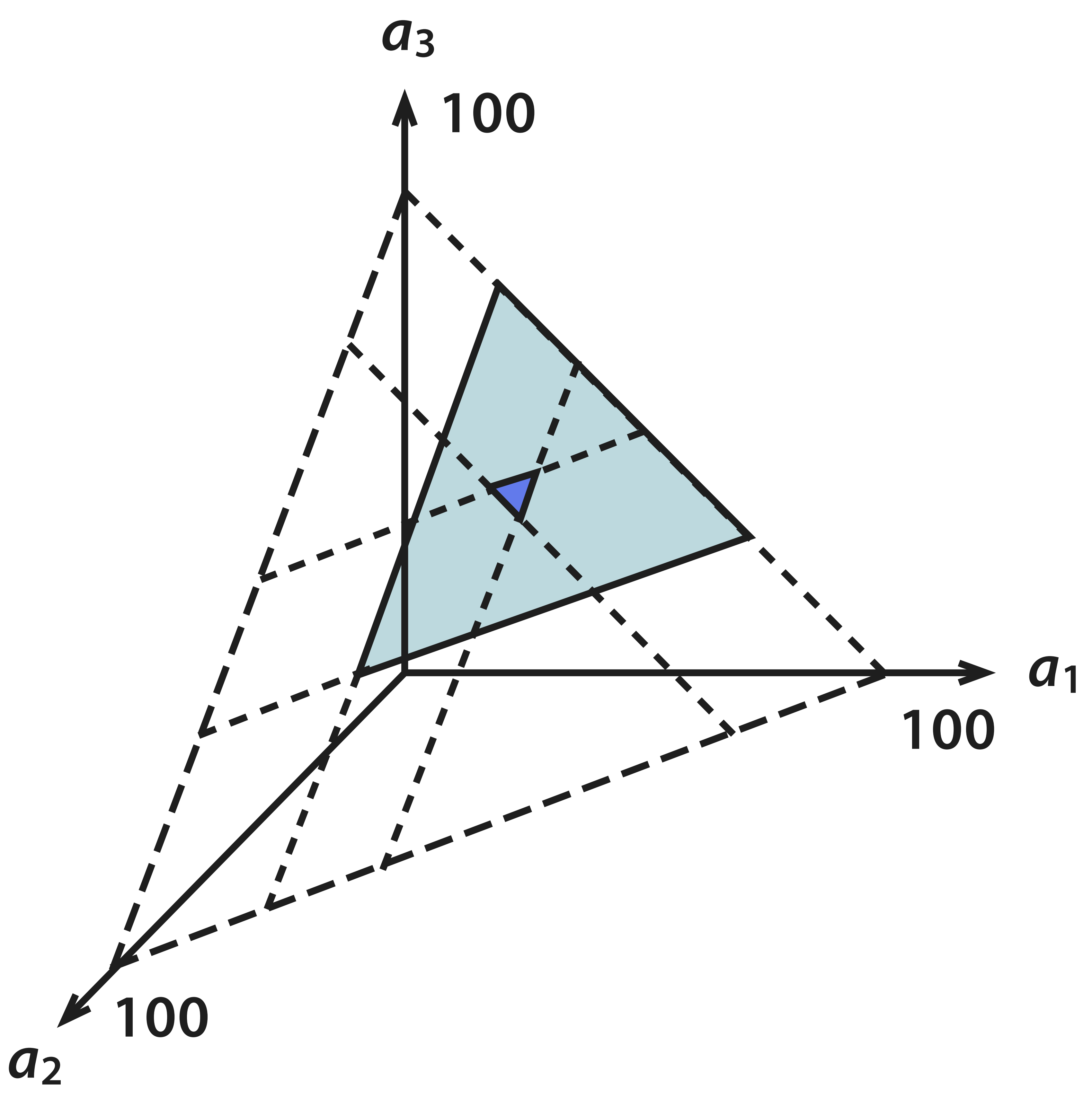
**Definition.**
Set of all nondominated imputations of cooperative game $(N, v)$ is called the Core.

**Theorem.**
Imputation $\alpha$ belongs to the Core, if and only if:

$$\sum_{i \in S} \alpha_i \geq v(S),\ S \subset N,$$

$$\sum_{i \in N} \alpha_i = v(N).$$

# Core




**Jazz band.**

$a_1 \geq 20,$

$a_2 \geq 0,$

$a_3 \geq 30,$

$a_1 + a_2 \geq 50,$

$a_1 + a_3 \geq 80,$

$a_2 + a_3 \geq 65,$

$a_1 + a_2 + a_3 = 100.$

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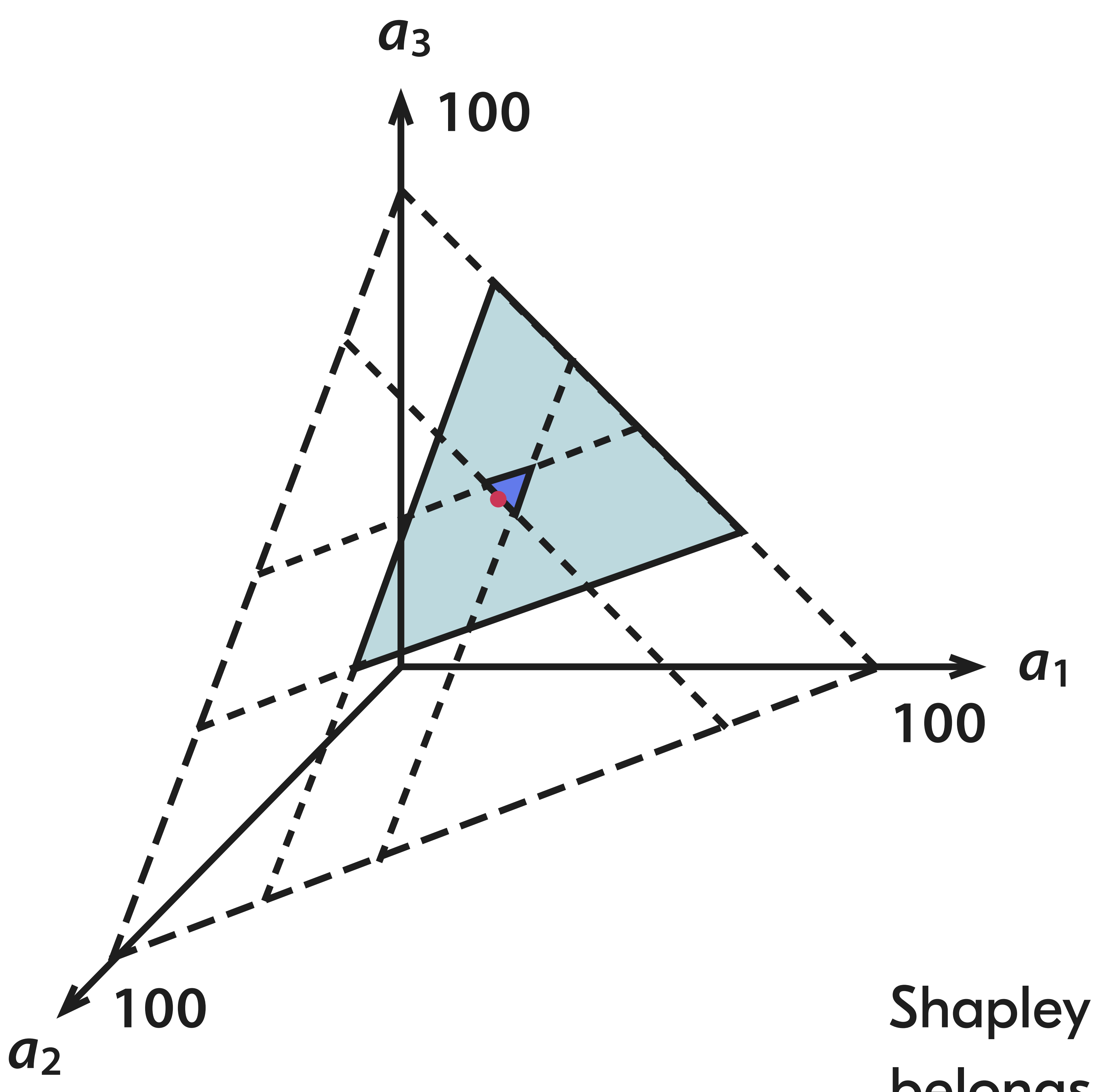
# Core




**Jazz band.**
$20 \le a_1 \le 35$,
$0 \le a_2 \le 20$,
$30 \le a_3 \le 50$,
$a_1 + a_2 + a_3 = 100$.

Shapley value $\varphi_i[v] = (35, 17.5, 47.5)$ belongs to the Core.

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# Voting Games

**O. Petrosian**

PhD

# Voting Games

**“Meeting of the State Council on May 7, 1901”,**
I.Repin, 1903

# Voting Games

**Definition.**
Suppose $w_i$ is a number of votes of player $i$, $i \in N$, $q$ is a threshold of votes. Game $(N, v)$ is called the voting game, if for all $S \subseteq N$:

- $$v(S) = 0 \Longrightarrow \sum_{i \in S} w_i < q,$$
- $$v(S) = 1 \Longrightarrow \sum_{i \in S} w_i \geq q.$$

# Voting Games

**Council of the European Economic Community in 1973.**

- $\sum_{i \in N} w_i = 58$ is the total number of votes in the parliament.
- $q = 41$ is the threshold of votes for the affirmative decision.

| Country | $W_i$ | Country | $W_i$ |
|---|---|---|---|
| France | 10 | Netherlands | 5 |
| Germany | 10 | Luxembourg | 2 |
| Italy | 10 | Denmark | 3 |
| United Kingdom | 10 | Ireland | 3 |
| Belgium | 5 | | |

# Shapley–Shubik Power Index

**Definition.**
Shapley–Shubik power index for voting game $(N, v)$
is the vector $\varphi(v) = (\varphi_1(v), \dots, \varphi_n(v))$:

$$\varphi_i(v) = \sum_{S \notin W,\, S \cup \{i\} \in W} \frac{(|S|)!\,(n - |S| - 1)!}{n!}, \quad i \in N,$$

where $W = \{S \subseteq N : v(S) = 1\}$ is the set of winning coalitions $S$.

# Shapley–Shubik Power Index

**Council of the European Economic Community in 1973.**

- $\sum_{i \in N} w_i = 58$ is the total number of votes in the parliament.
- $q = 41$ is the threshold of votes for the affirmative decision.

| Country | $\varphi_i$ | Country | $\varphi_i$ |
|---|---|---|---|
| France | 0.179 | Netherlands | 0.081 |
| Germany | 0.179 | Denmark | 0.057 |
| Italy | 0.179 | Ireland | 0.057 |
| United Kingdom | 0.179 | Luxembourg | 0.010 |
| Belgium | 0.081 | | |

# Banzhaf Power Index

**Definition.**
Banzhaf power index in the voting game $(N, v)$ is vector $\beta(v) = (\beta_1(v), \ldots, \beta_n(v))$:

$$\beta_i(v) = \frac{\eta_i(v)}{\sum_{i \in N} \eta_i(v)}, \quad i \in N,$$

where $\eta_i(v) = |\{S \subseteq N : i \in N, v(S) = 1, v(S \setminus \{i\}) = 0\}|$ is the number of coalitions $S$, where player $i$ is a veto-player.

# Banzhaf Power Index

**Council of the European Economic Community in 1973.**

- $\sum_{i \in N} w_i = 58$ is the total number of votes in the parliament.
- $q = 41$ is the threshold of votes for the affirmative decision.

| Country | $\beta_i$ | Country | $\beta_i$ |
|---|---|---|---|
| France | 0.167 | Netherlands | 0.091 |
| Germany | 0.167 | Denmark | 0.066 |
| Italy | 0.167 | Ireland | 0.066 |
| United Kingdom | 0.167 | Luxembourg | 0.016 |
| Belgium | 0.091 | | |

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# Preliminary Information

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# Board Games

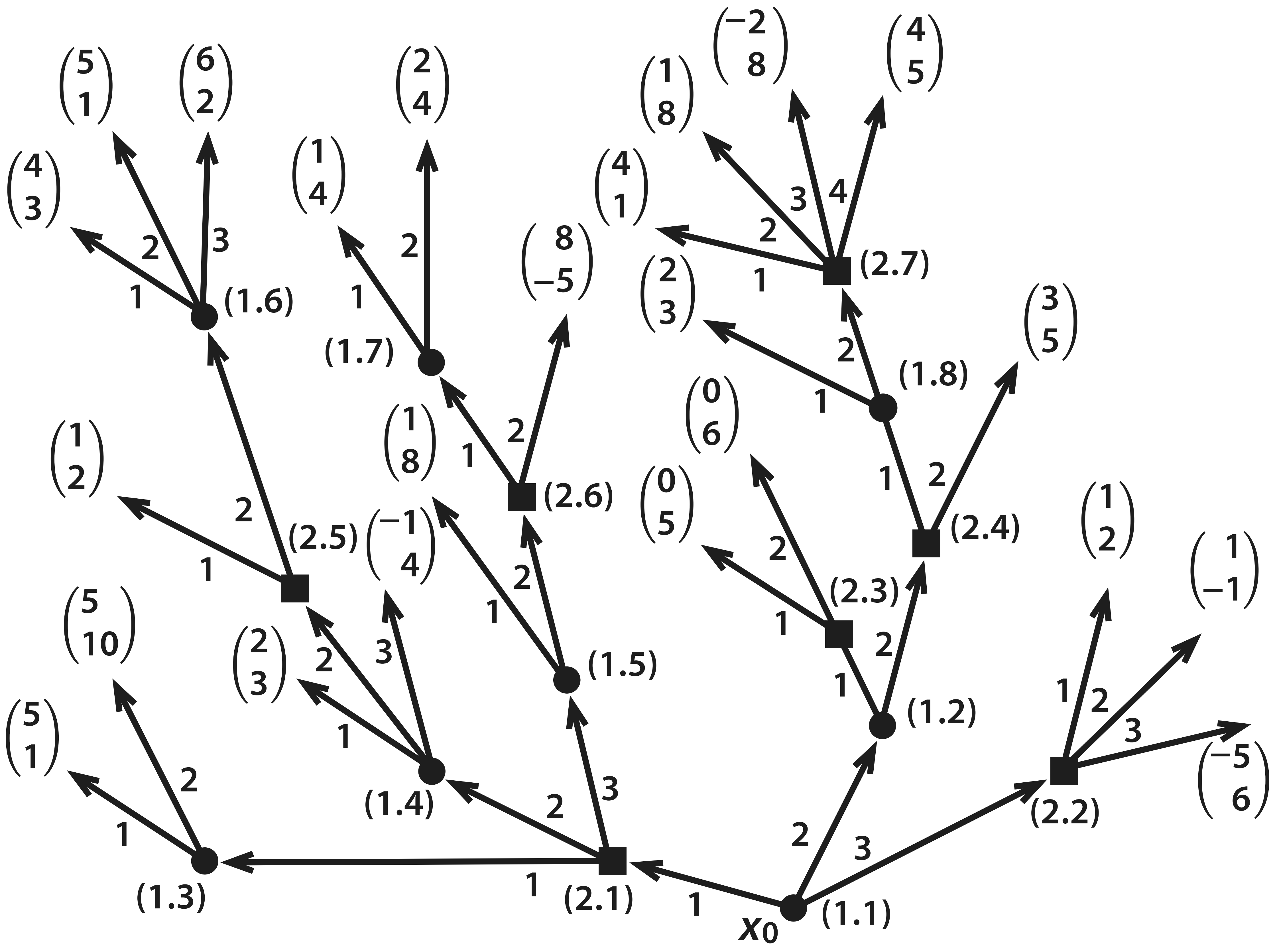
**“Chess players”,**
Lucas van Leyden, 1508

Consider games, such as chess or checkers:

- Players make moves one after another.
- At every stage players know moves and capabilities of the opponents.

In this type of games theoretically it is possible to calculate all possible moves of the enemy, but since the problem has large dimension, then practically it is impossible.

# Board Games




**Game description.**

- Game evolves in time and is determined by the moves of players.
- Order of moves is defined by the game rules.
- At every stage of the game players know current state of the game, remember their previous moves and previous moves of other players.

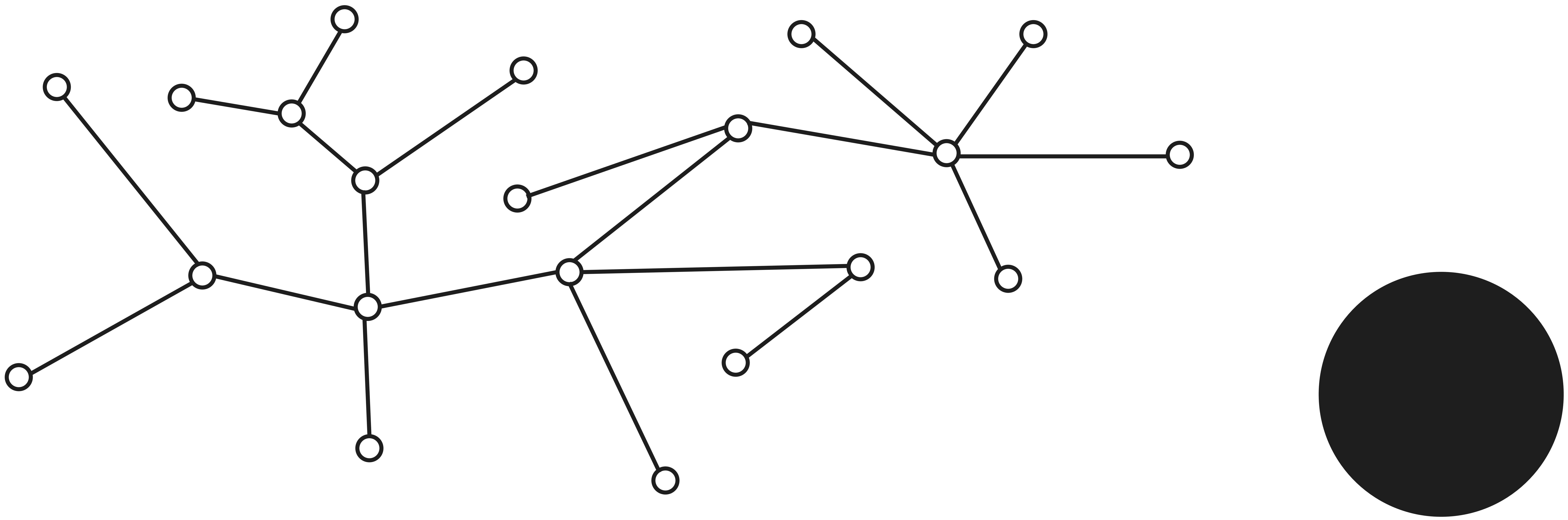
# Preliminary Information

**Definition.**
Pair $(X, F)$ is called a graph if $X$ is a finite set and $F$ is a point-to-set mapping from $X$ to $X$.

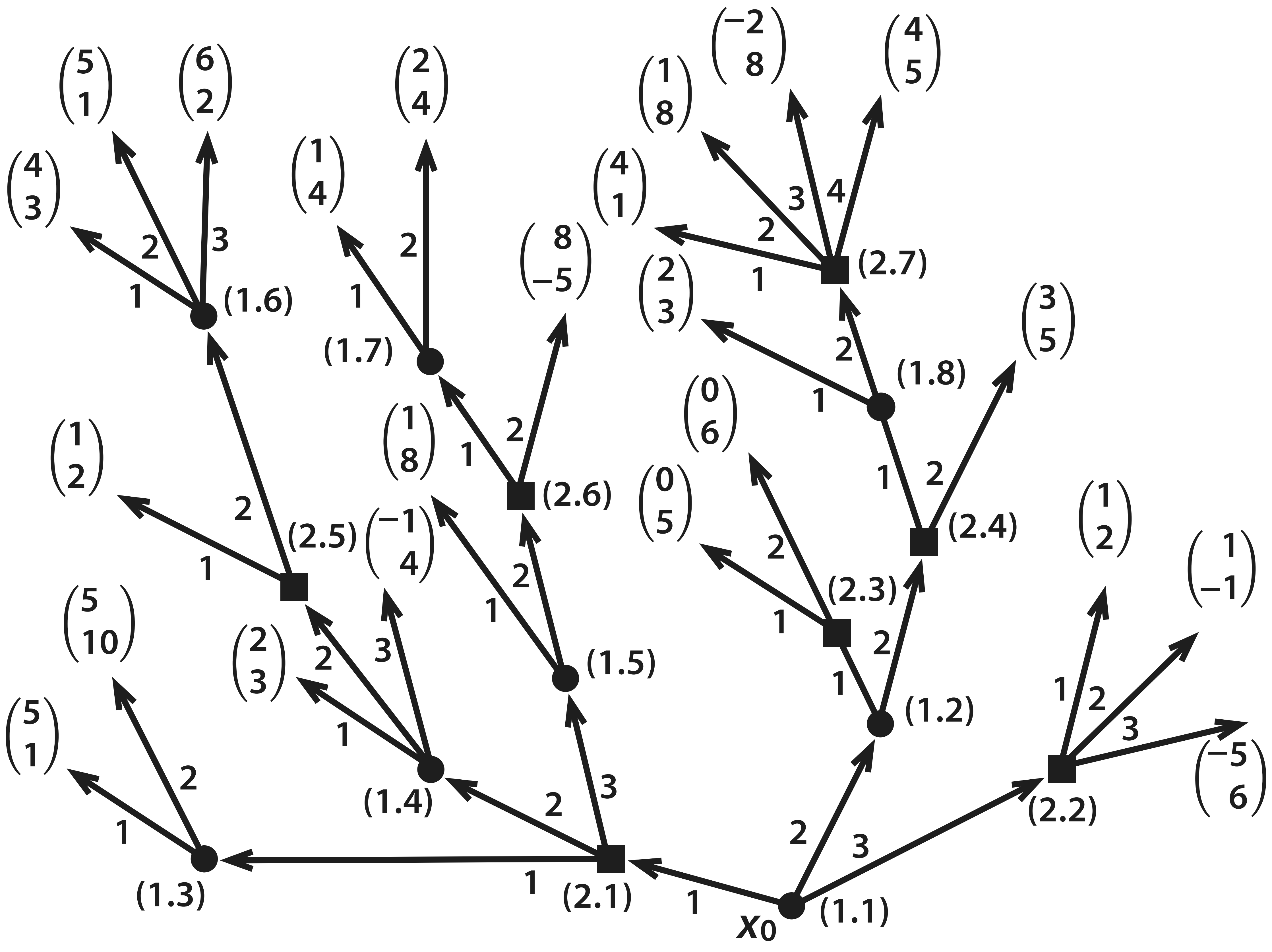
- **Arc** is any pair $(x, y)$, where $x \in X, y \in F_x$. The set of arcs in the graph is denoted by $P$.
- **Path** is a sequence of arcs, $p = (p_1, p_2, \dots, p_l)$ such that the end of the preceding arc coincides with the origin of the next one.
- **Edge** is a pair $x, y \in X$, for which either $(x, y) \in P$ or $(y, x) \in P$.
- **Chain** is a sequence of non-repeating edges, in which each node of an edge coincides with one of nodes from neighboring edges.

## Preliminary Information

- **Cycle** is a finite chain starting in a node and terminating in the same node.
- Graph is called **connected** if any two nodes can be connected by a chain.



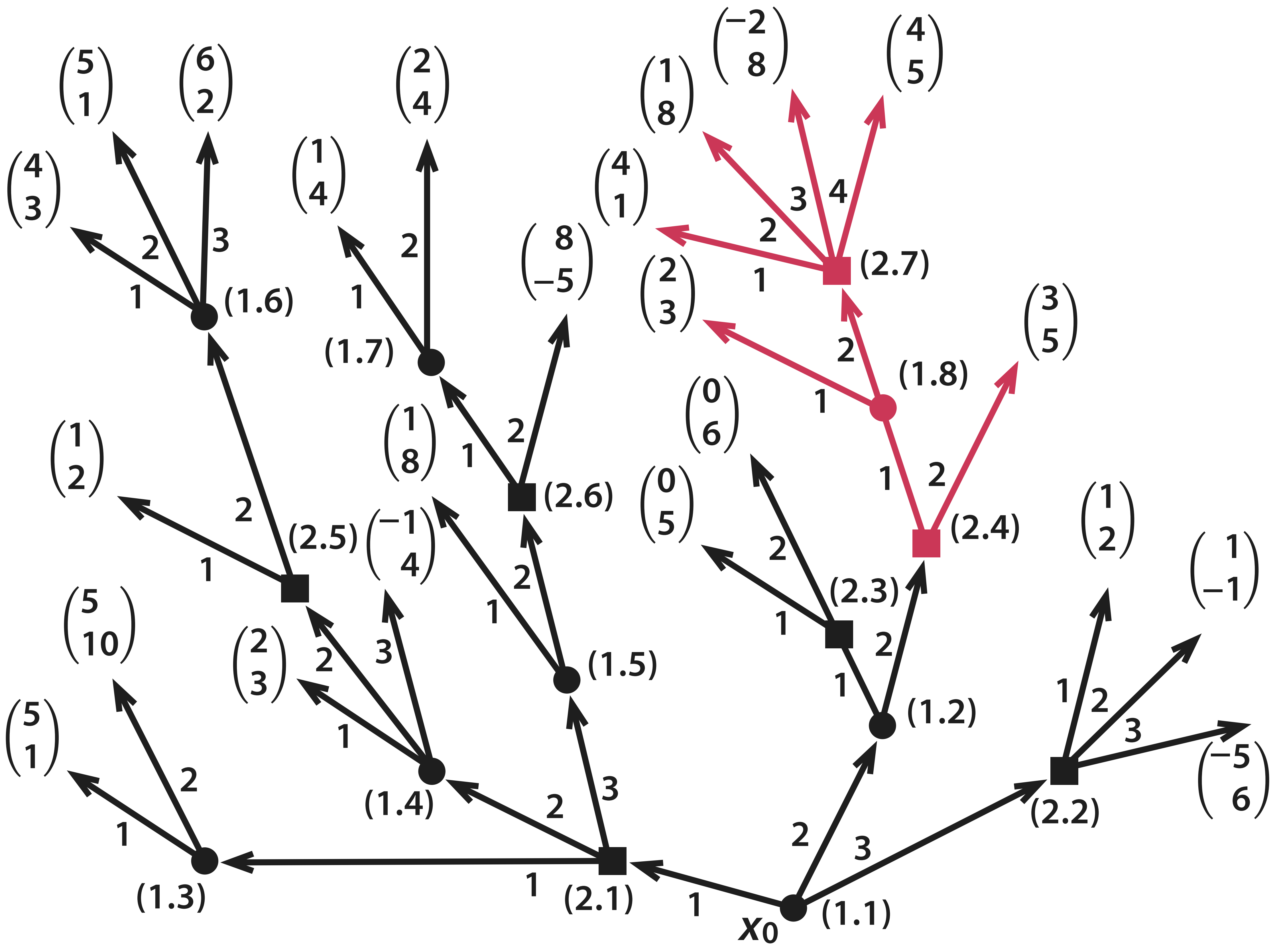
# Preliminary Information




**Definition.**
Tree is a finite connected graph without cycles, which has a unique node $x_0$ such that, $\hat{F}_{x_0} = \{x_0\} \cup F_{x_0} \cup F^2_{x_0} \cup \ldots \cup F^k_{x_0} \ldots = X$, where $F^k_x = F(F^{k\text{-}1}_x)$.

**Example.**
Tree $G = (X, F)$ of multistage game with perfect information.

# Preliminary Information




**Definition.**
Let $z \in X$. Subgraph $G_z$ of tree $G = (X, F)$ is a graph of the form $(X_z, F)$, where $X_z = \hat{F}_z$.

**Example.**
Subgraph $G_{x_{(2.4)}} = (X_{x_{(2.4)}}, F)$ of the multistage game with perfect information on a tree starting in $x_{(2.4)}$.

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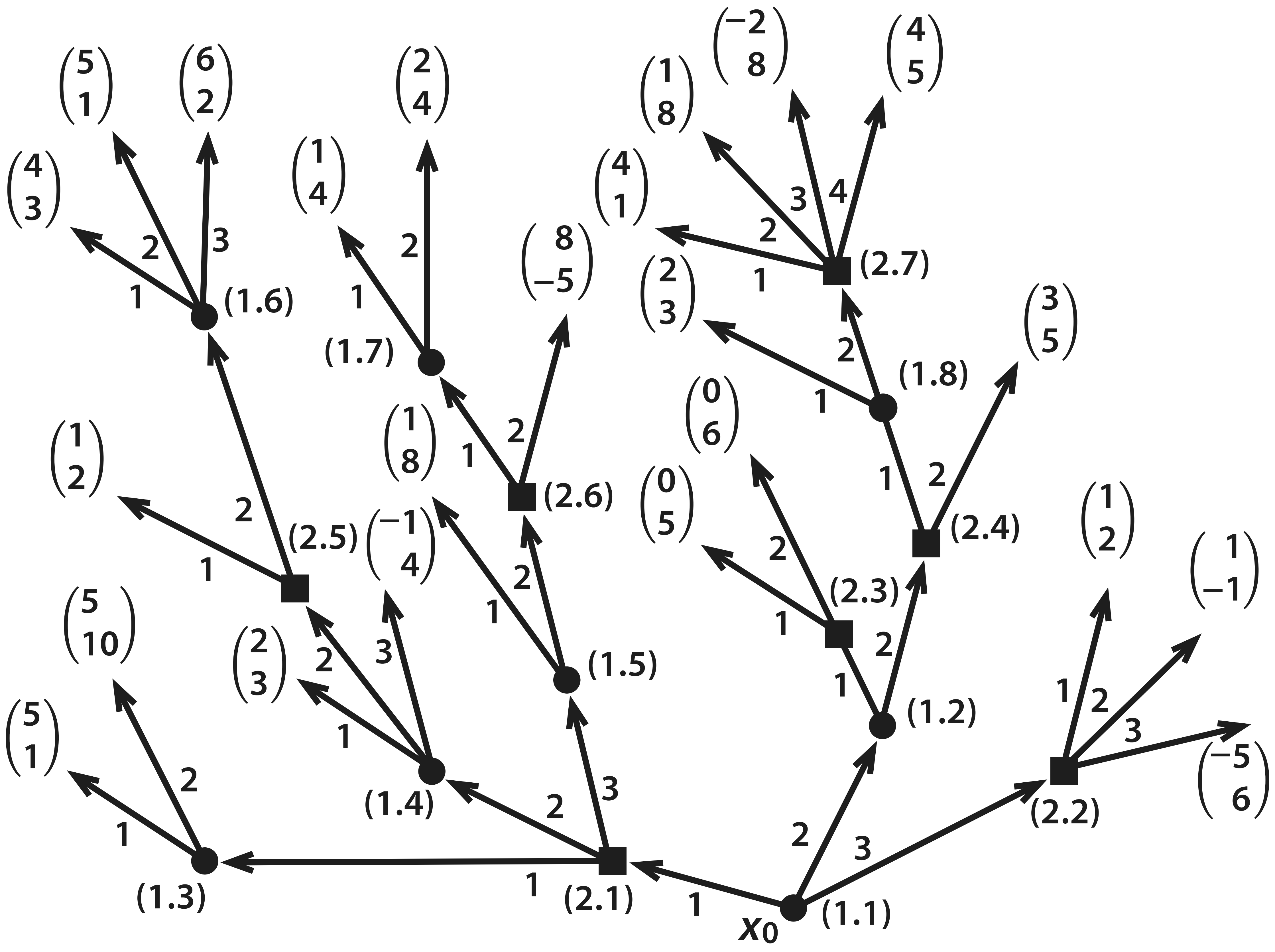
# Multistage Games with Perfect Information

**O. Petrosian**

PhD

# Multistage Games with Perfect Information

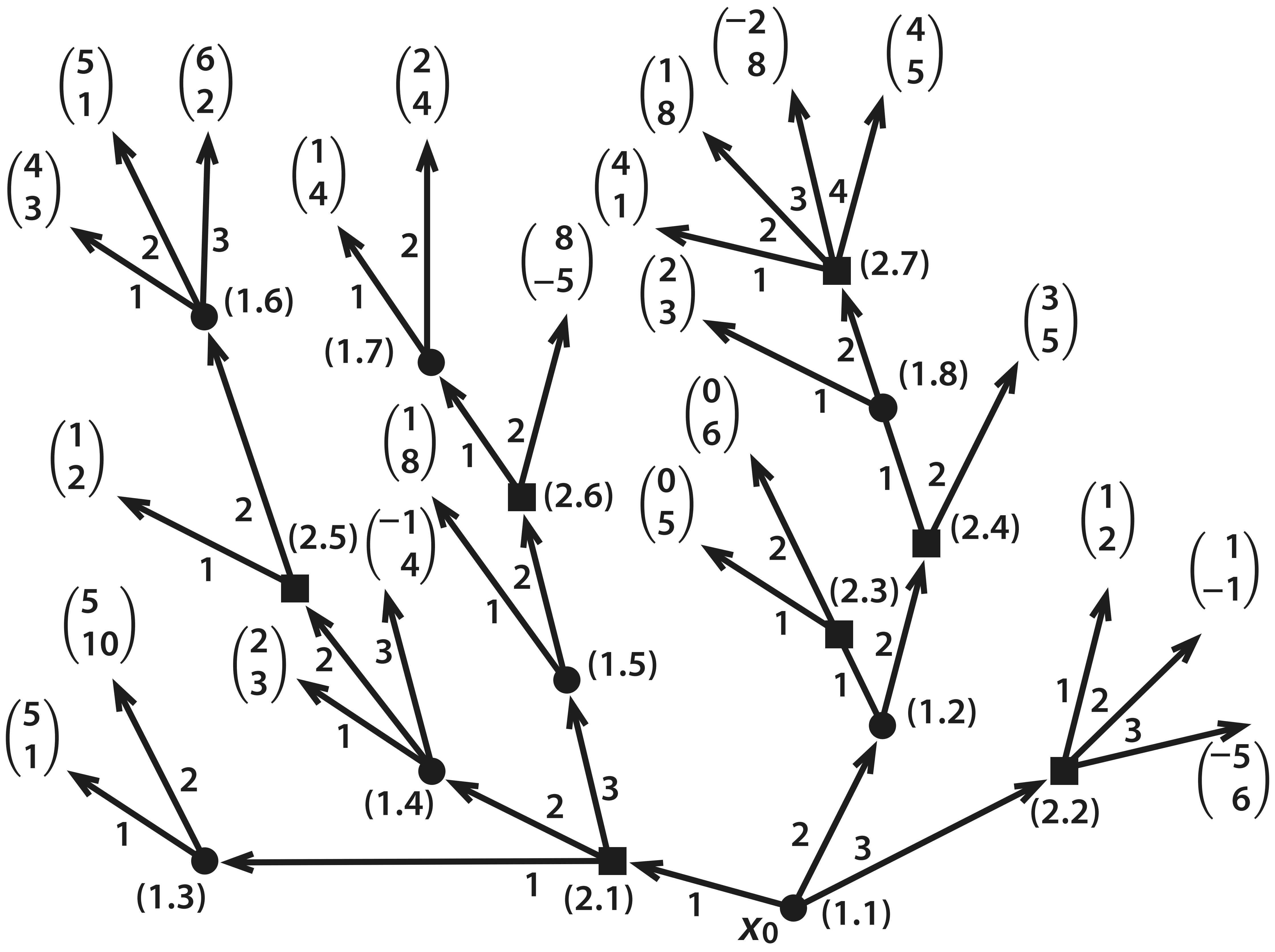



**Definition.**

Suppose $X = \bigcup_{i=1}^{n+1} X_i$, where $X_k \cap X_l = \emptyset$, $k \neq l$:

- $N = \{1, \dots, n\}$ is a set of players.
- $X_i$, $i = 1, \dots, n$ is a set of personal positions of player $i$.
- $X_{n+1} = \{x : F_x = \emptyset\}$ is a set of final or terminal positions.

# Multistage Games with Perfect Information

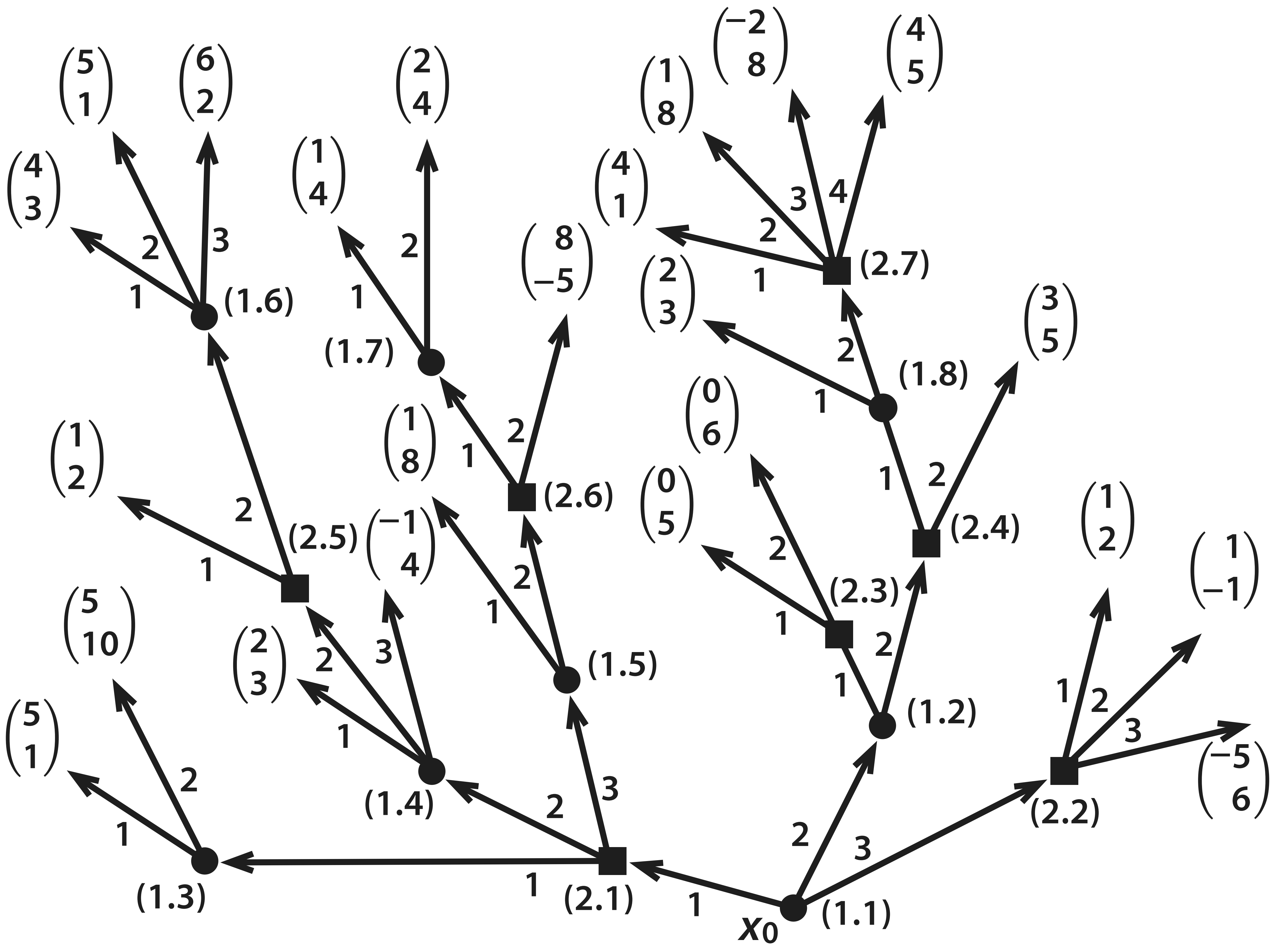



**Example.**

Here $X = X_1 \bigcup X_2 \bigcup X_3$, where $X_k \cap X_l = \emptyset$, $k \neq l$:

- N = {1, 2} is a set of players.
- $X_1$ = {● –circles},
  $X_2$ = {■ –squares}
  are the sets of personal positions of players.
- $X_3 = X \setminus (X_1 \cap X_2)$ is a set of final or terminal positions.

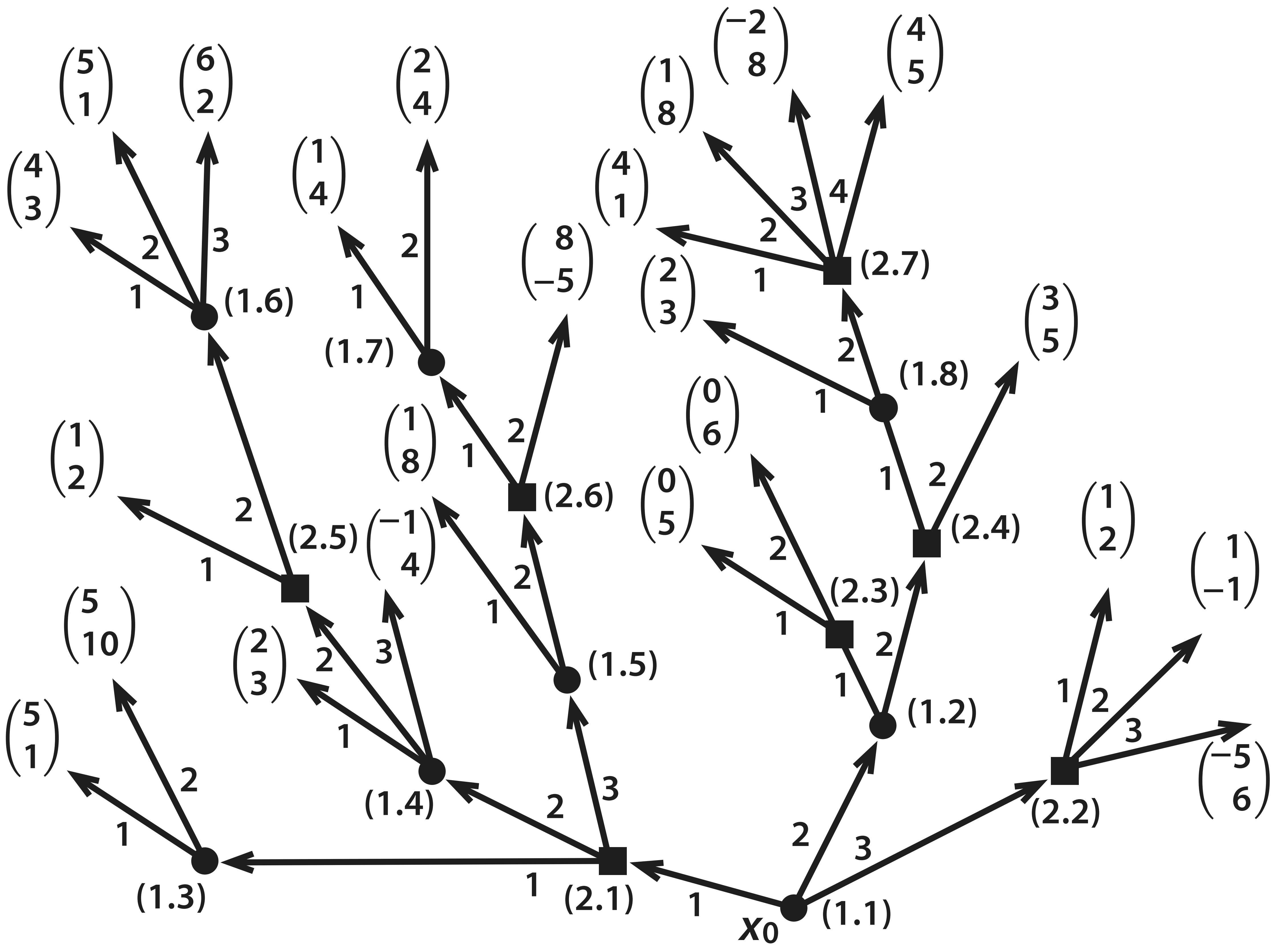
# Multistage Games with Perfect Information




**Definition.**
Payoff of player $i$ is a real-valued function $H_i(x)$ defined on the set of final positions $x \in X_{n+1}$.

**Definition.**
Single-valued mapping $u_i : u_i(x) = y$, which sets up a correspondence between each node $x \in X_i$ (position) and some unique node $y \in F_x$ (position) is called a strategy of player $i$.

# Multistage Games with Perfect Information

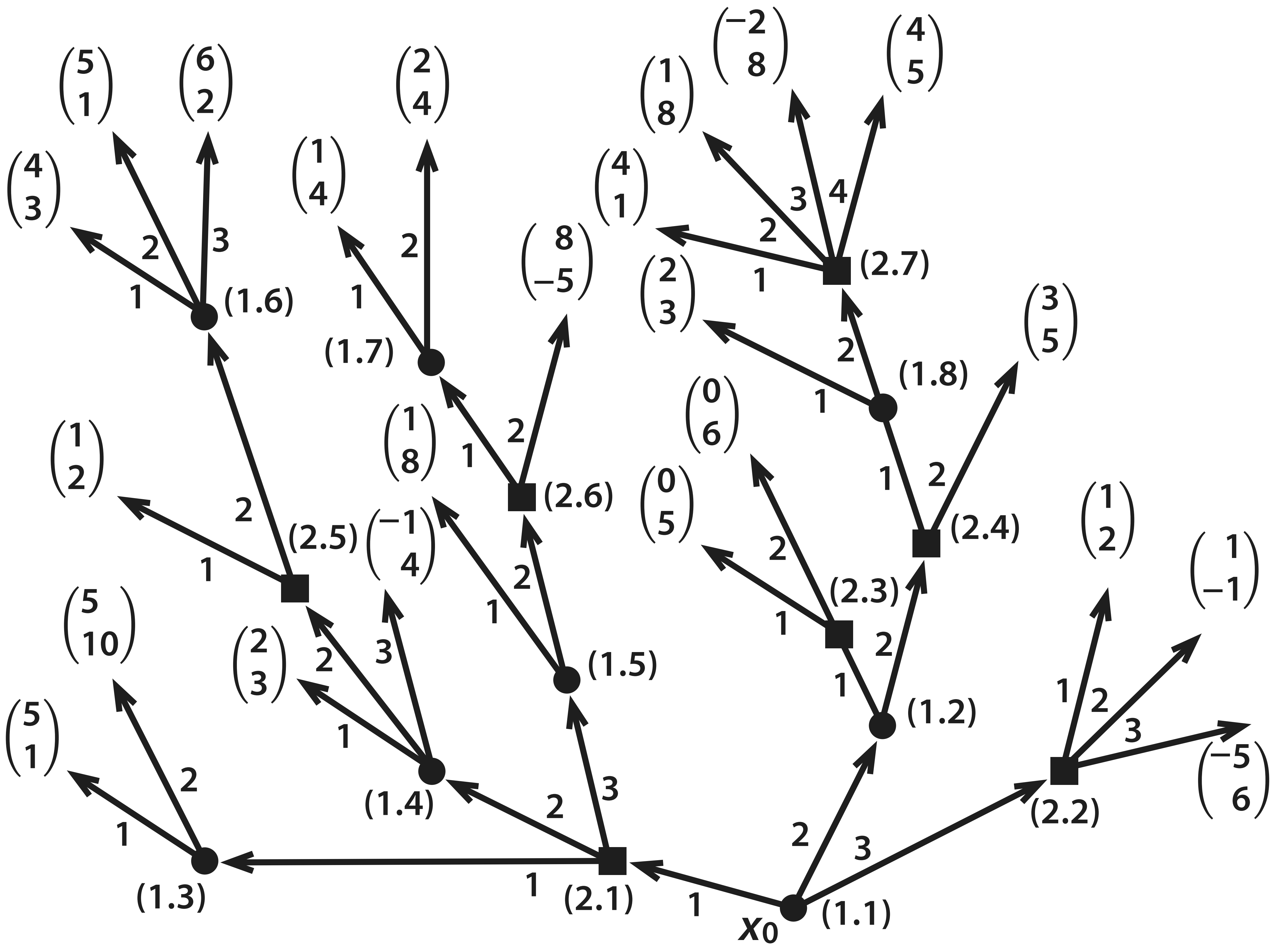



**Example.**

$H_1(x)$ and $H_2(x)$ are the payoffs of players 1 and 2, $x \in X_3$.

$\begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix}$ are the payoffs in the terminal position $x \in X_3$.

# Multistage Games with Perfect Information




**Game description.**

- **Step 1:** At the node $x_0 \in X_{i_1}$ : $u_{i_1}(x_0) = x_1 \in F_{x_0}$.
- ...
- **Step k:** If $x_{k-1} \in X_{i_k}$, then $u_{i_k}(x_{k-1}) = x_k \in F_{x_{k-1}}$.
- ...
- **Step l:** If $x_{l-1} \in X_{i_l}$ and $F_{x_{k-1}} \subseteq X_{n+1}$, then $u_{i_l}(x_{l-1}) = x_l \in F_{x_{l-1}}$ and game terminates.

# Multistage Games with Perfect Information

**Definition.**
Ordered set $u = (u_1, \dots, u_i, \dots, u_n)$, where $u_i \in U_i$, is called strategy profile, while $U = \prod_{i=1}^{n} U_i$ is the set of strategy profiles.

**Definition.**
Payoff function of player $i \in N$, $K_i(u_1, \dots, u_i, \dots, u_n) = H_i(x_l)$, $i = 1, \dots, n$, $x_l \in X_{n+1}$.

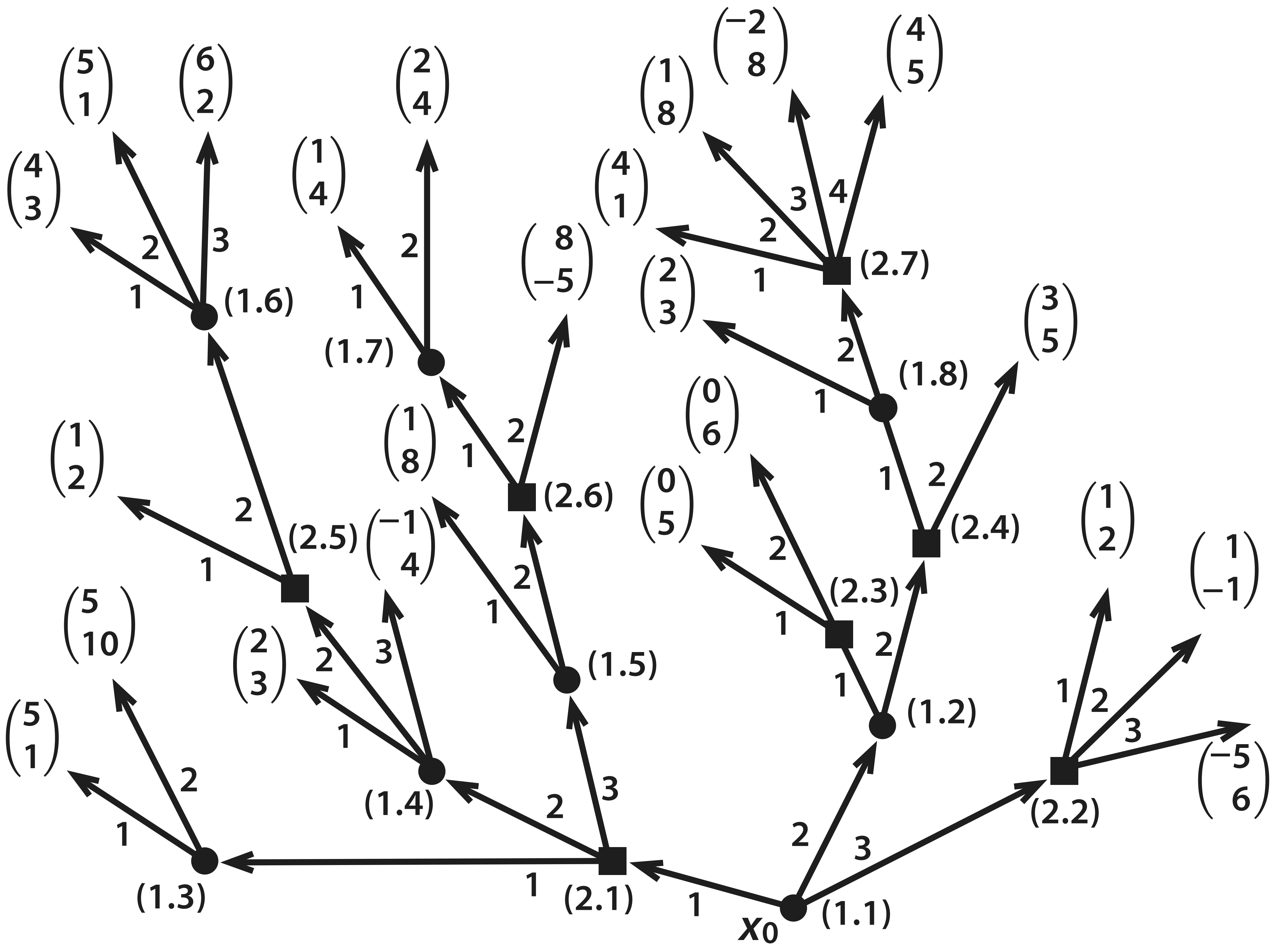
Multistage game and game in normal form:

- Strategy profile $(u_1, \dots, u_i, \dots, u_n) \longrightarrow$
- $\longrightarrow$ path $(x_1, \dots, x_k, \dots, x_l) \longrightarrow$
- $\longrightarrow$ terminal position $x_l \in \mathrm{X}_{n+1} \longrightarrow$
- $\longrightarrow$ payoffs $H_i(x_l)$, $i \in N$.

# Multistage Games with Perfect Information

Game with perfect information can be considered also as a game in normal form $\mathbf{\Gamma} = (N, \{U_i\}_{i \in N}, \{K_i\}_{i \in N})$.

# Multistage Games with Perfect Information




**Example.**

- Player 1: 864 strategies.
- Player 2: 576 strategies.

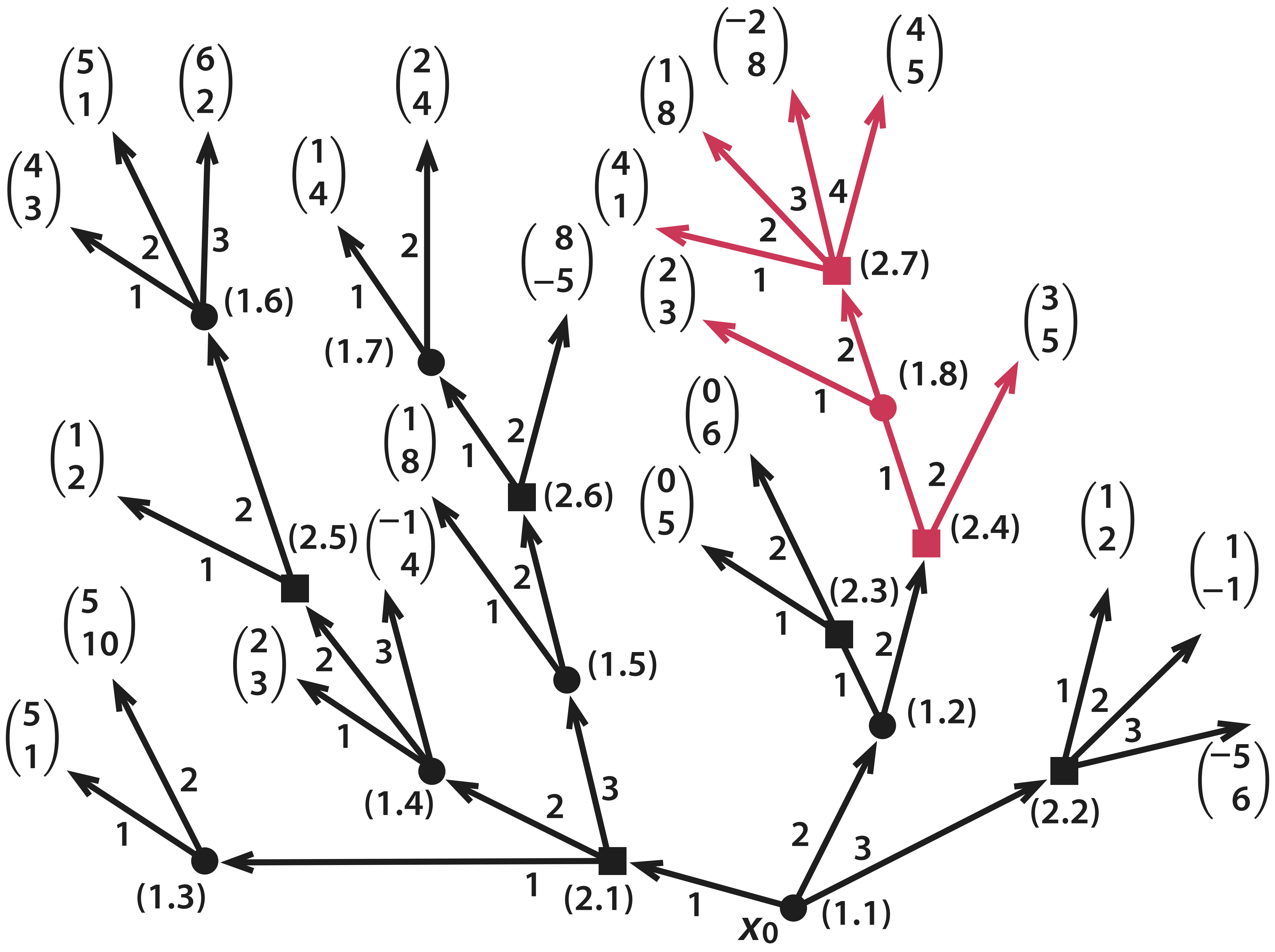
# Multistage Games with Perfect Information

**Definition.**
$\Gamma_z = (N, \{U_i^z\}_{i \in N}, \{K^z_i\}_{i \in N})$ is a subgame, defined on a subgraph $G_z = (X_z, F)$.

**Notations.**

- $N = \{1, \dots, n\}$ is the set of players.
- $Y_i^z = X_i \cap X_z$ is the set of personal positions.
- $Y_{n+1}^z = X_{n+1} \cap X_z$ is the set of final or terminal positions.
- $K_i^z(x) = K_i(x)$, $x \in Y_{n+1}^z$, $i \in N$ is the payoff of player $i$.

# Multistage Games with Perfect Information




**Definition.**
$\Gamma_z = (N, \{U_i^z\}_{i\in N}, \{K_i^z\}_{i\in N})$ is a subgame, defined on a subgraph $G_z = (X_z, F)$.

**Example.**
Subgame $\Gamma_{(2.4)}$ with the initial position $x_{(2.4)}$.

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# Subgame-perfect Nash Equilibrium

**O. Petrosian**

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# Nash Equilibrium

**Definition.**

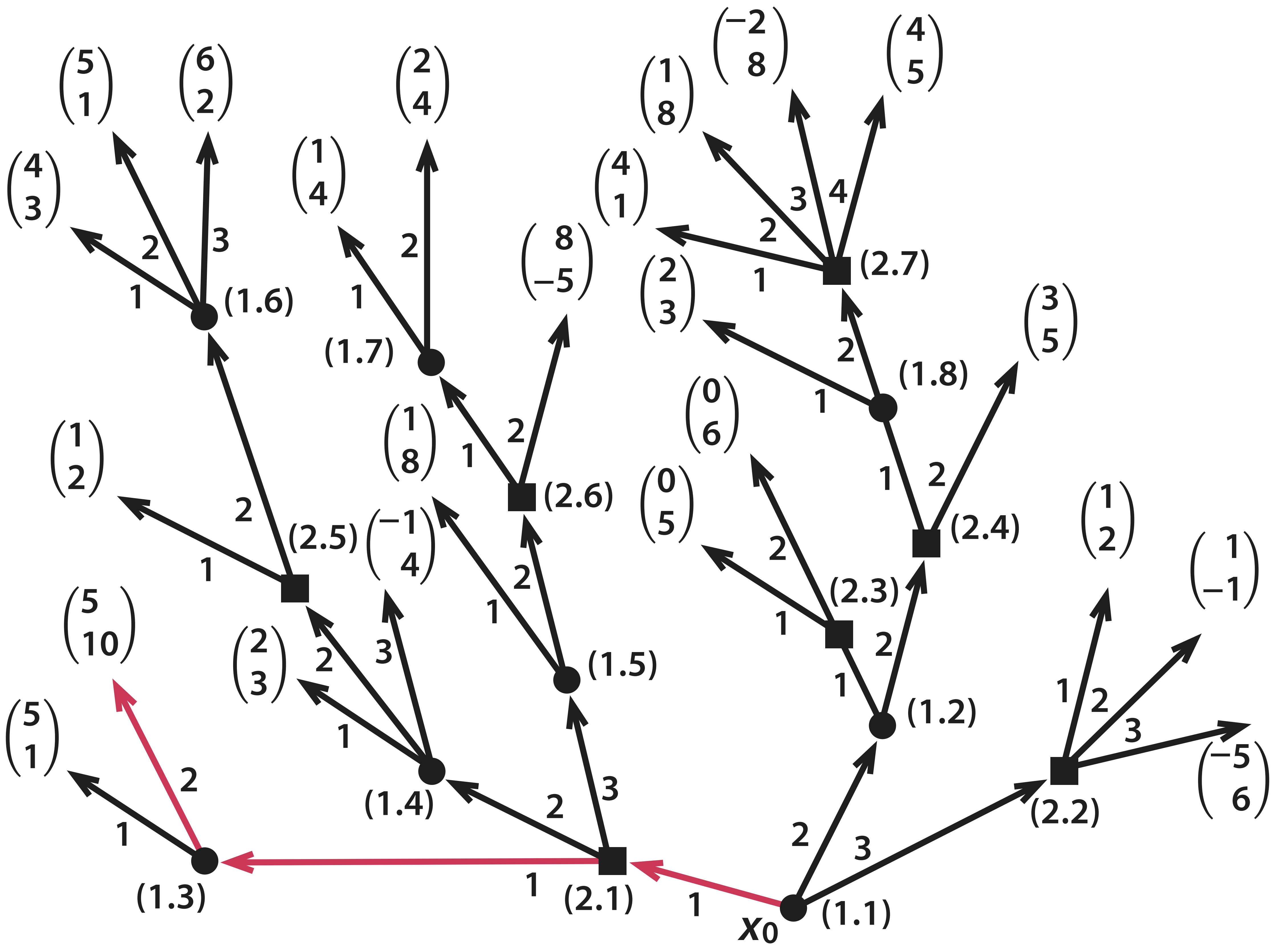
Strategy profile $u^* = (u_1^*, \dots, u_i^*, \dots, u_n^*)$ is called Nash equilibrium, if

$$K_i\,(u_1^*, \dots, u_{i-1}^*, u_i^*, u_{i+1}^*, \dots, u_n^*) \geq K_i(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*) \text{ for all } u_i \in U_i,\ i \in N.$$

**Note.**

In any finite game in normal form, there exists Nash equilibrium in mixed strategies.

# Multistage Games with Perfect Information




**Nash equilibrium** $u^* = (u_1^*, u_2^*)$:

$u_1^* = (1, 2, 2, 2, 2, 3, 2, 1)$,

$u_2^* = (1, 3, 2, 2, 2, 1, 2)$.

Payoff of players $u^* = (u_1^*, u_2^*)$:

$K_1(u_1^*, u_2^*) = 5$,

$K_2(u_1^*, u_2^*) = 10$.

# Subgame-perfect Nash Equilibrium

**Definition.**
Truncation of player $i$ strategy $u_i$ in the subgame $\Gamma_z$ is a strategy:

$$u_i^z = u_i(x),\ x \in Y_i^z = X_i \cap X_z,\ i \in N.$$

**Definition.**
Nash equilibrium $u^* = (u_1^*, \dots, u_n^*)$ in the game $\Gamma$ is subgame-perfect, if for any $z \in X$ strategy profile $(u^*)^z = ((u_1^*)^z, \dots, (u_n^*)^z)$ is a Nash equilibrium in the subgame $\Gamma_z$.

# Subgame-perfect Nash Equilibrium

**Definition.**
Nash equilibrium $u^* = (u_1^*, \dots, u_n^*)$ in the game **Γ** is subgame-perfect Nash equilibrium, if for any $z \in X$ strategy profile $(u^*)^z = ((u_1^*)^z, \dots, (u_n^*)^z)$ is Nash Equilibrium in the subgame **Γ**z.

**Theorem.**
In any multistage game with perfect information defined on a finite tree graph there exists subgame-perfect Nash equilibrium in pure strategies.

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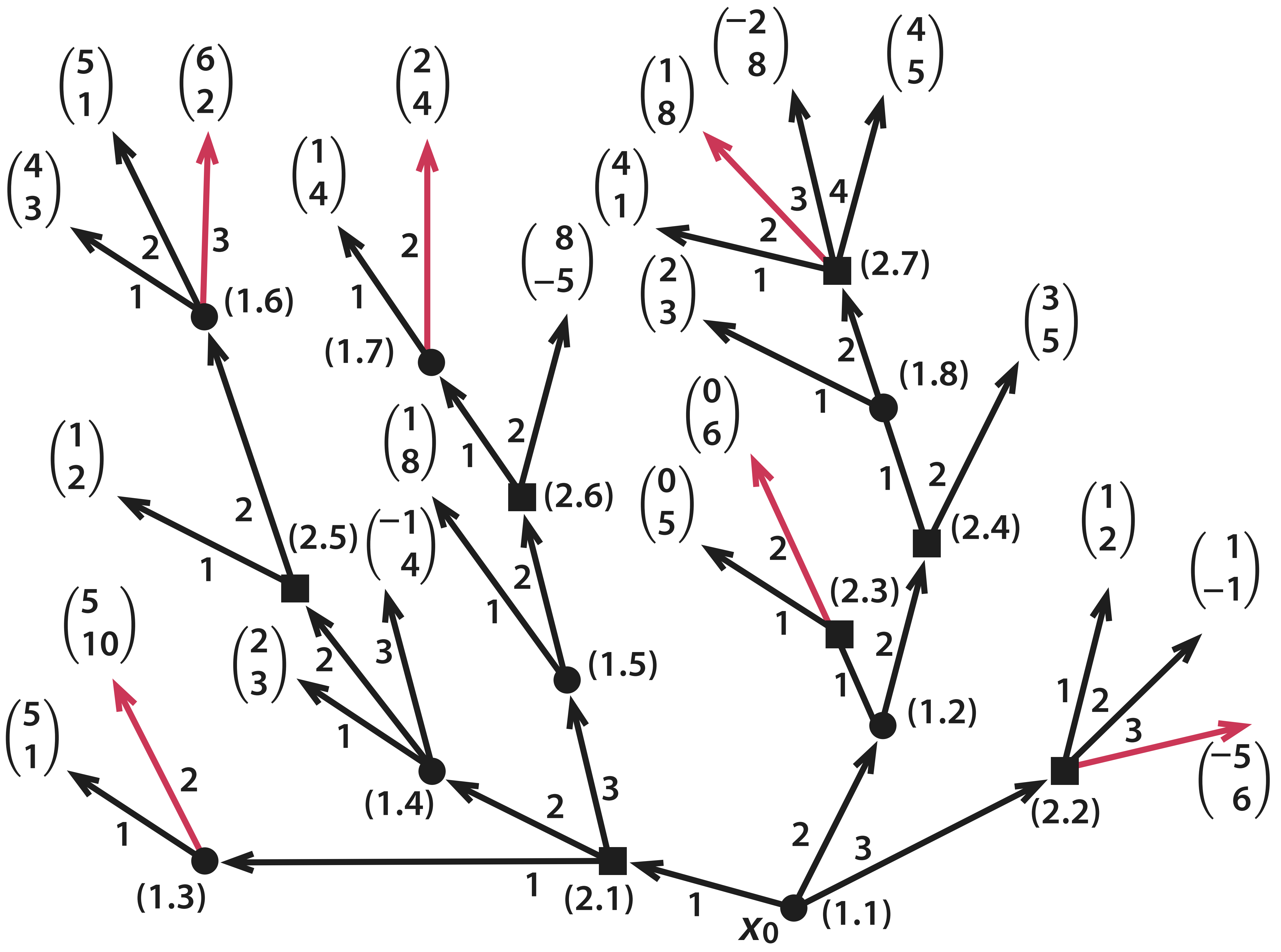
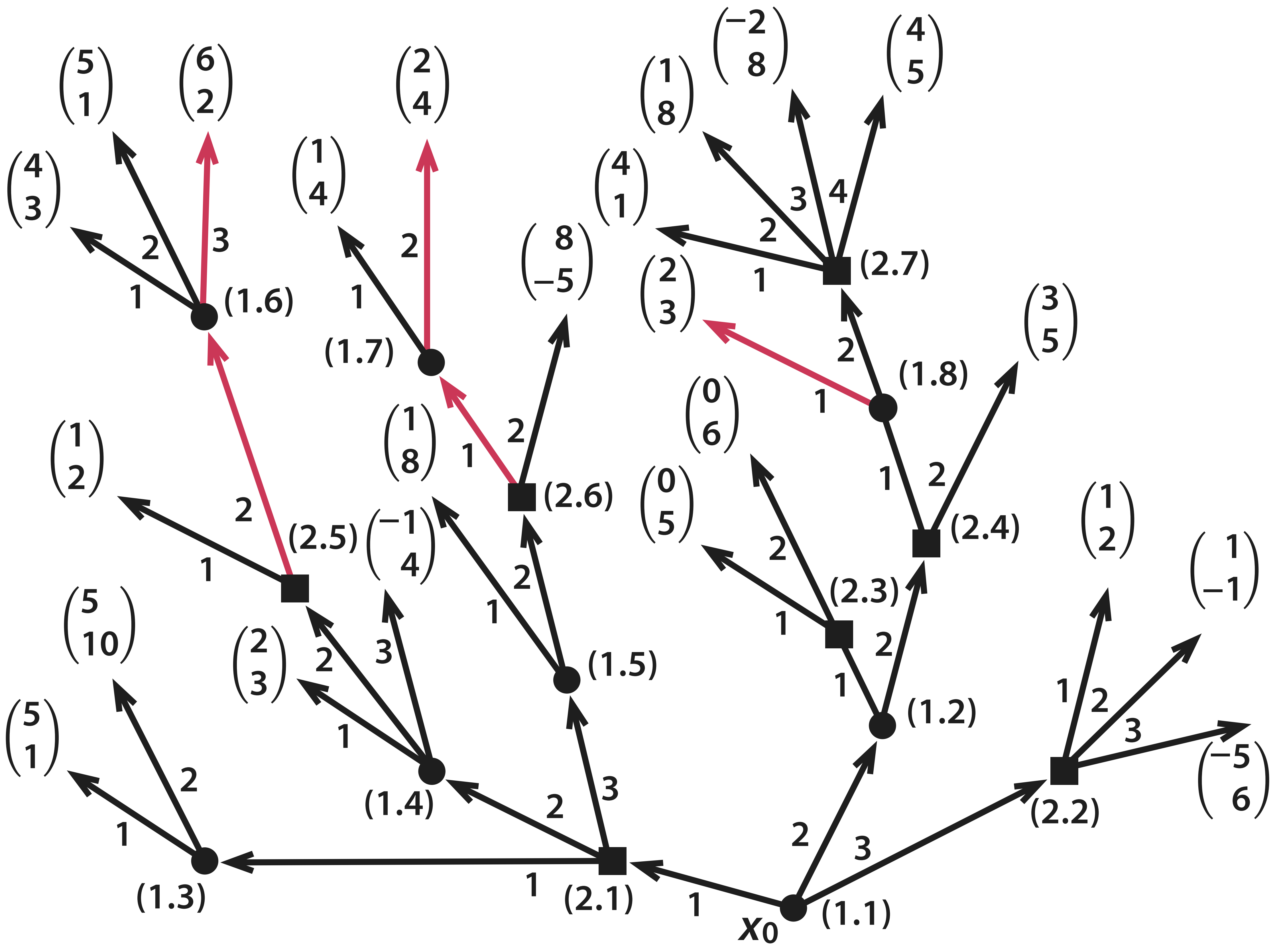
# Subgame-perfect Nash Equilibrium




**Subgame-perfect Nash equilibrium** $u^* = (u_1^*, u_2^*)$:

- $\Gamma_{(1.3)} : (u_1^*(x_{(1.3)}))^{x(1.3)} = 2,\ (K_1^{x(1.3)}, K_2^{x(1.3)}) = (5,\ 10).$
- $\Gamma_{(1.6)} : (u_1^*(x_{(1.6)}))^{x(1.6)} = 3,\ (K_1^{x(1.6)}, K_2^{x(1.6)}) = (6,\ 2).$
- $\Gamma_{(1.7)} : (u_1^*(x_{(1.7)}))^{x(1.7)} = 2,\ (K_1^{x(1.7)}, K_2^{x(1.7)}) = (2,\ 4).$
- $\Gamma_{(2.2)} : (u_2^*(x_{(2.2)}))^{x(2.2)} = 3,\ (K_1^{x(2.2)}, K_2^{x(2.2)}) = (-5,\ 6).$
- $\Gamma_{(2.3)} : (u_2^*(x_{(2.3)}))^{x(2.3)} = 2,\ (K_1^{x(2.3)}, K_2^{x(2.3)}) = (0,\ 6).$
- $\Gamma_{(2.7)} : (u_2^*(x_{(2.7)}))^{x(2.7)} = 2,\ (K_1^{x(2.7)}, K_2^{x(2.7)}) = (1,\ 8).$

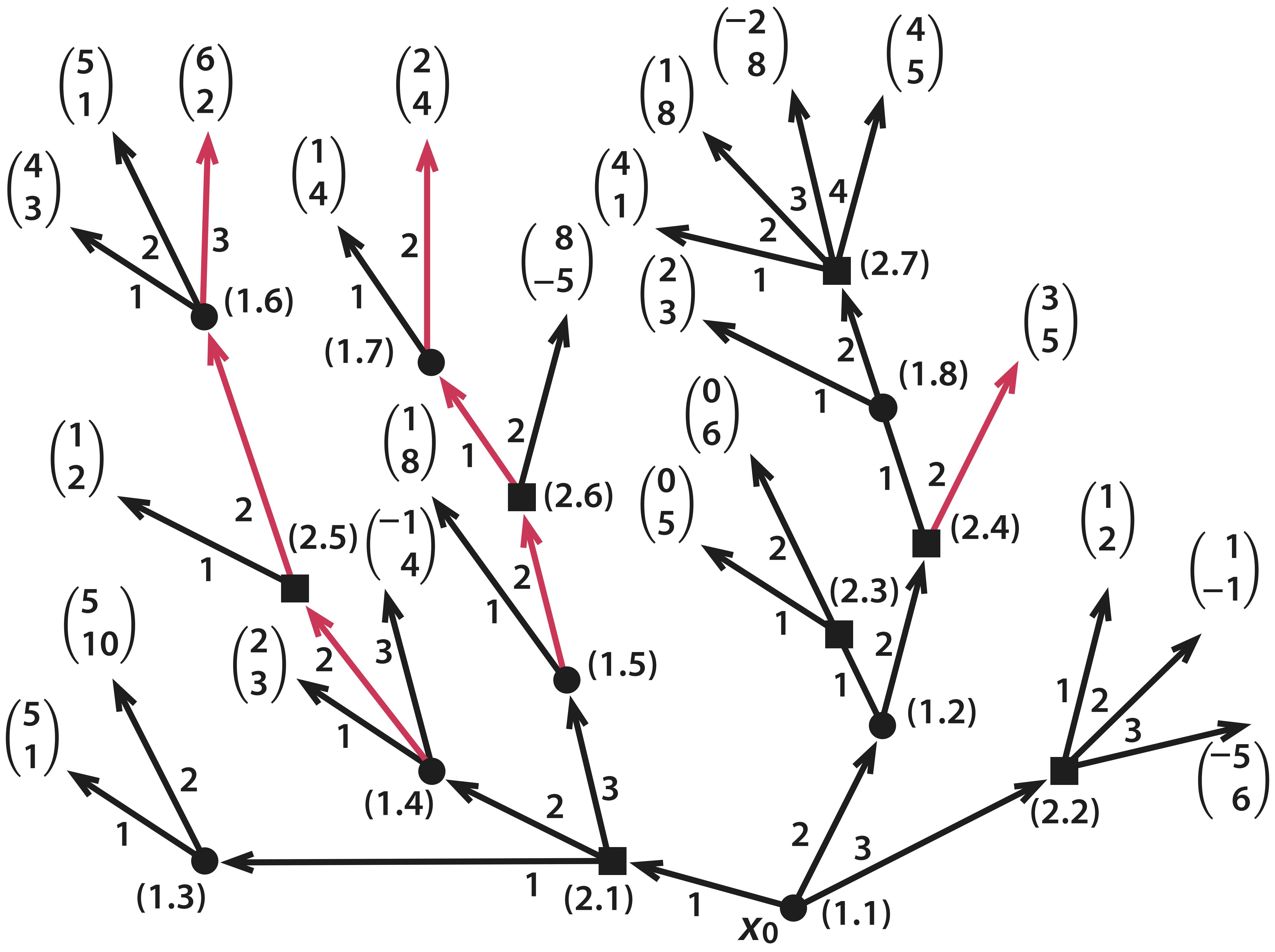
# Subgame-perfect Nash Equilibrium




**Subgame-perfect Nash equilibrium** $u^* = (u_1^*, u_2^*)$:

- $\Gamma_{(1.8)} : (u_1^*(x_{(1.8)}))^{x(1.8)} = 1,\ (K_1^{x(1.8)}, K_2^{x(1.8)}) = (2, 3).$
- $\Gamma_{(2.5)} : (u_2^*(x_{(2.5)}))^{x(2.5)} = 2,\ (K_1^{x(2.5)}, K_2^{x(2.5)}) = (6, 2).$
- $\Gamma_{(2.6)} : (u_2^*(x_{(2.6)}))^{x(2.6)} = 1,\ (K_1^{x(2.6)}, K_2^{x(2.6)}) = (2, 4).$

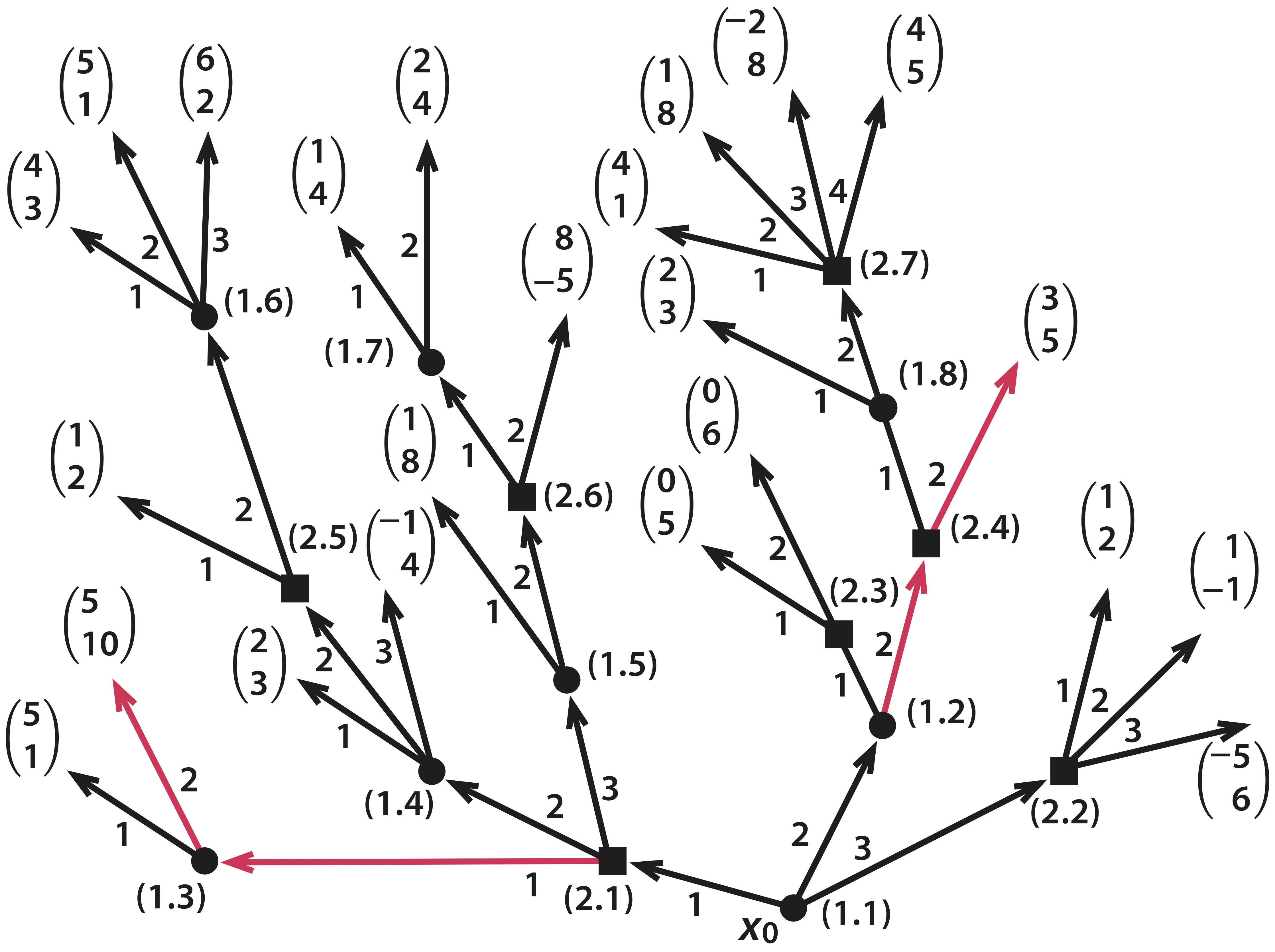
# Subgame-perfect Nash Equilibrium




**Subgame-perfect Nash equilibrium** $u^* = (u_1^*, u_2^*)$**:**

- $\Gamma_{(1.4)} : (u_1^*(x_{(1.4)}))^{x_{(1.4)}} = 2,\ (K_1^{x_{(1.4)}}, K_2^{x_{(1.4)}}) = (6, 2).$
- $\Gamma_{(1.5)} : (u_1^*(x_{(1.5)}))^{x_{(1.5)}} = 2,\ (K_1^{x_{(1.5)}}, K_2^{x_{(1.5)}}) = (2, 4).$
- $\Gamma_{(2.4)} : (u_2^*(x_{(2.4)}))^{x_{(2.4)}} = 2,\ (K_1^{x_{(2.4)}}, K_2^{x_{(2.4)}}) = (3, 5).$

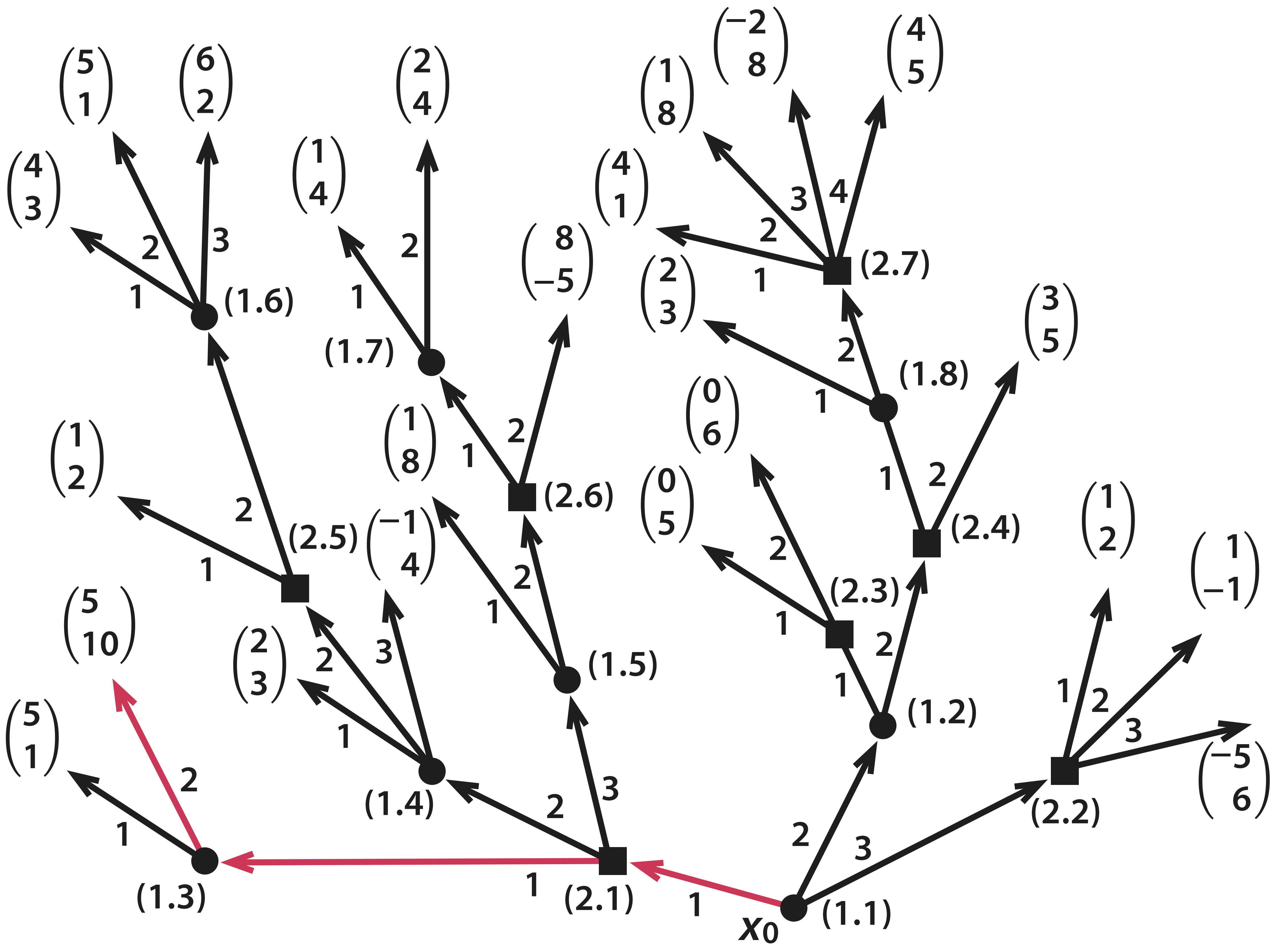
# Subgame-perfect Nash Equilibrium




**Subgame-perfect Nash equilibrium $u^* = (u_1^*, u_2^*)$:**

- $\Gamma_{(1.2)}: (u_1^*(x_{(1.2)}))^{x_{(1.2)}} = 2,\ (K_1^{x_{(1.2)}}, K_2^{x_{(1.2)}}) = (3, 5).$
- $\Gamma_{(2.1)}: (u_2^*(x_{(2.1)}))^{x_{(2.1)}} = 1,\ (K_1^{x_{(2.1)}}, K_2^{x_{(2.1)}}) = (5, 10).$

# Subgame-perfect Nash Equilibrium




**Subgame-perfect Nash equilibrium** $u^* = (u_1^*, u_2^*)$:

- $\Gamma_{(1.1)} : (u_1^*(x_{(1.1)}))^{x_{(1.1)}} = 1,\ (K_1^{x_{(1.1)}}, K_2^{x_{(1.1)}}) = (5, 10).$

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# Special Class of Strategies

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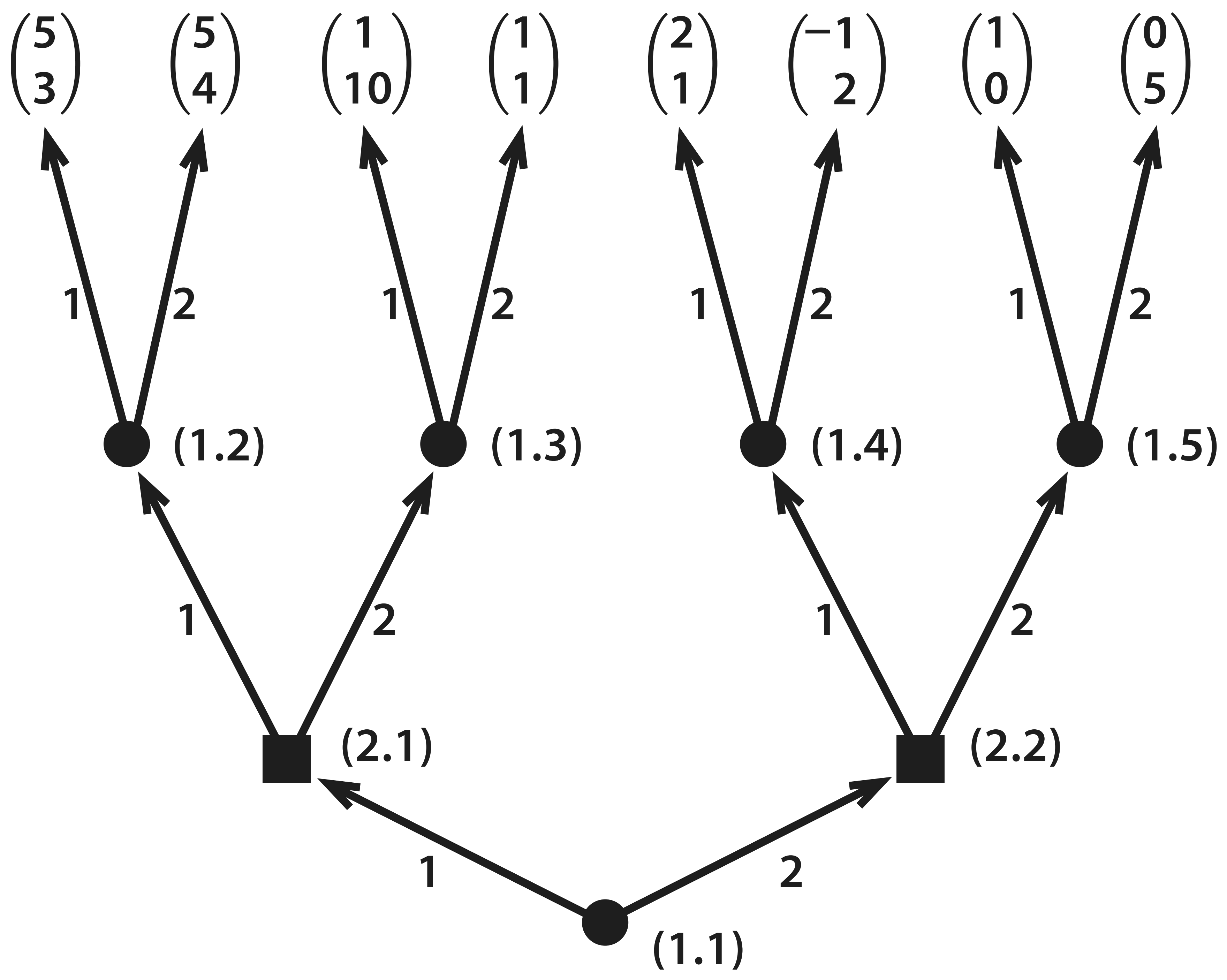
# Favorable and Unfavorable Behavior

**"The Harpsichord Lesson",**
Jan Steen, 1660

At the certain stage of the game one of players may use favorable behavior:

- His payoff does not decrease in favorable behavior.
- Payoff of the second player increases from the favorable behavior of first player.

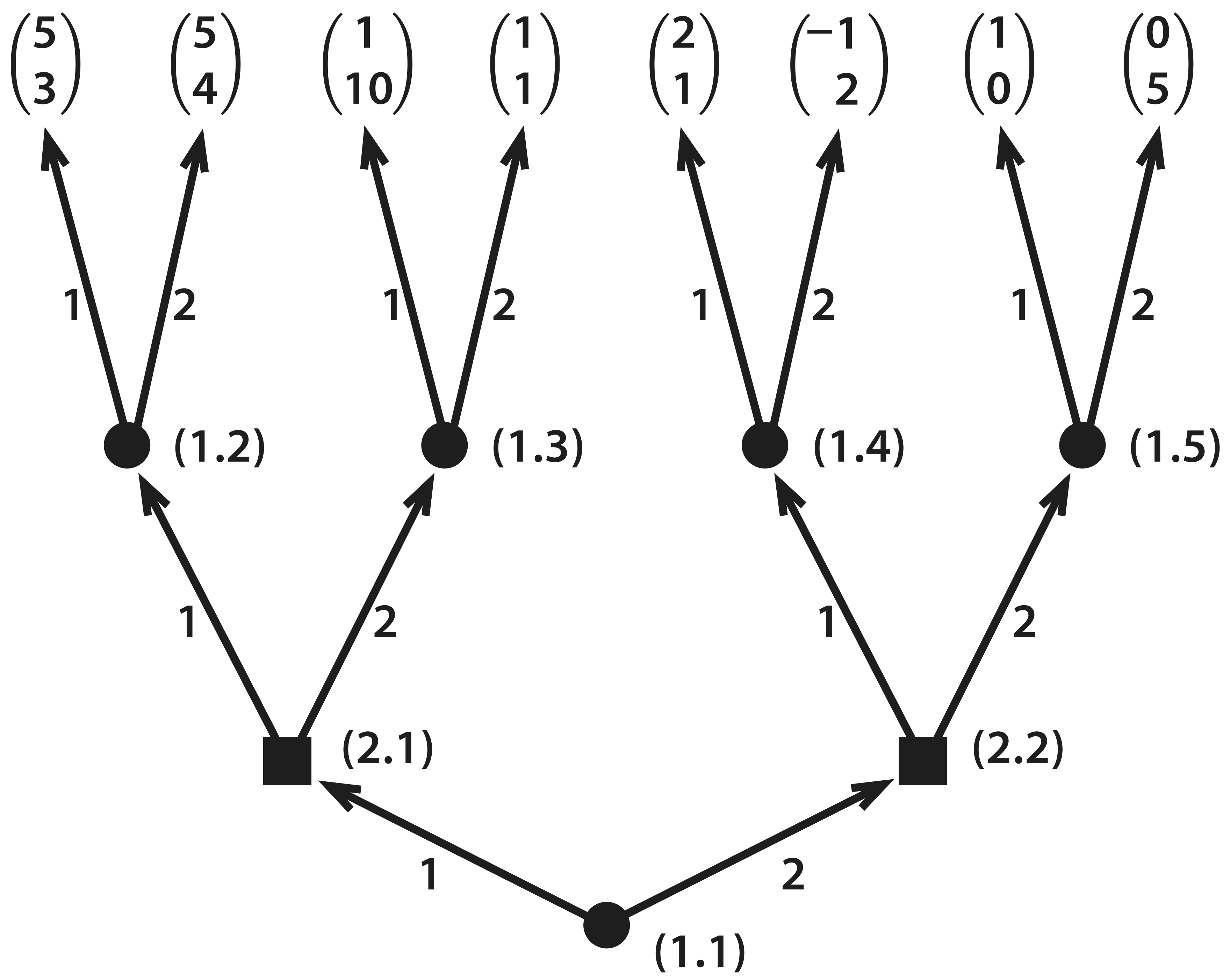
# Favorable and Unfavorable Behavior




**Description.**

- If player 2 chooses the alternative that leads to a greater payoff of player 1, then the behavior of player 2 is called favorable.
- Otherwise, player 2's behavior is called unfavorable.

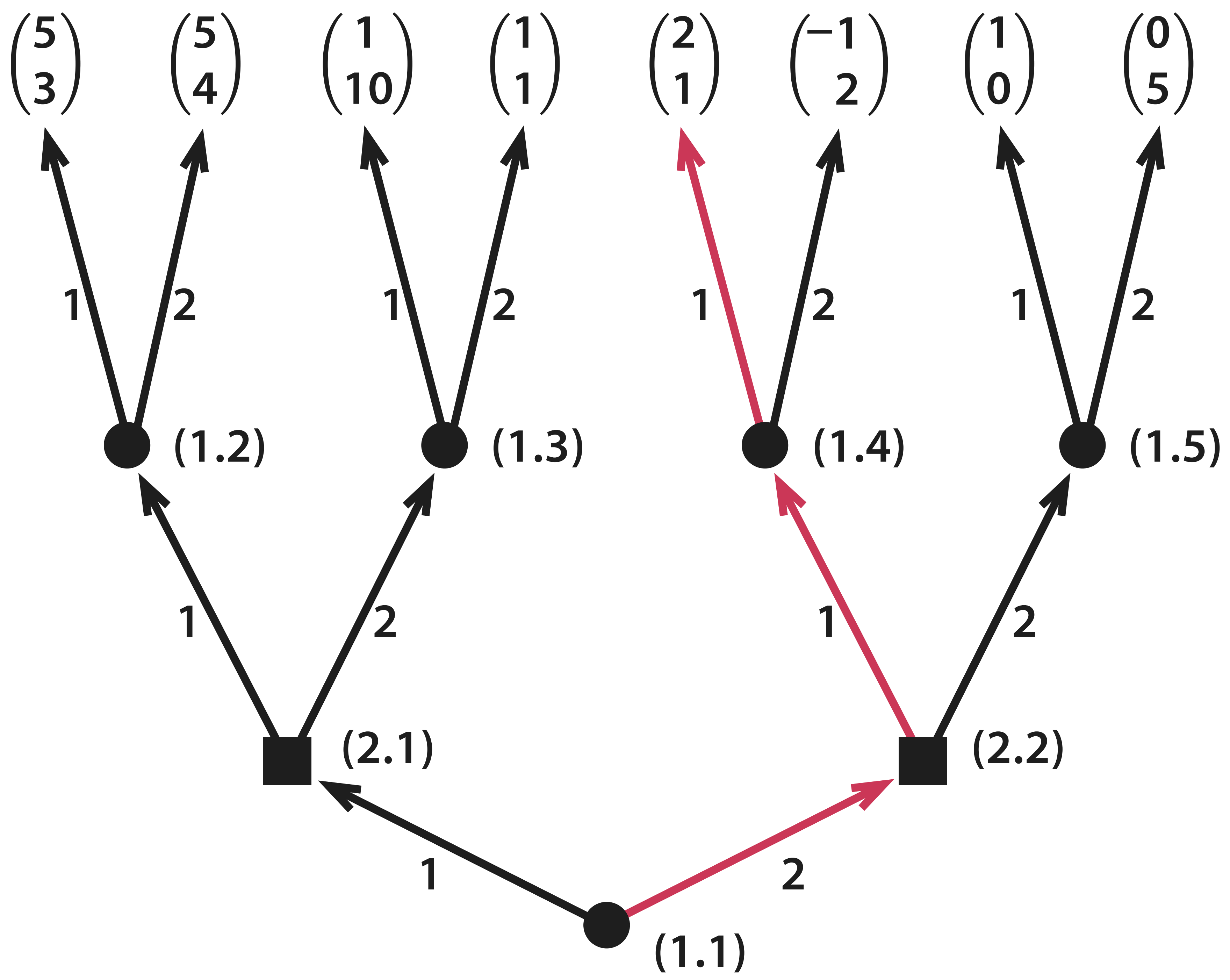
# Favorable Behavior




**Subgame-perfect Nash equilibrium** $u^* = (u_1^*, u_2^*)$:

- **Node (1.2):** Player 1 chooses the alternative 2, $u_1(x_{(1.2)}) = 2$ is a favorable behavior.
- **Node (1.3)**: Player 1 chooses the alternative 1, $u_1(x_{(1.3)}) = 1$ is a favorable behavior.
- **Node (1.4):** $u_1(x_{(1.4)}) = 1$.
- **Node (1.5):** $u_1(x_{(1.5)}) = 1$.

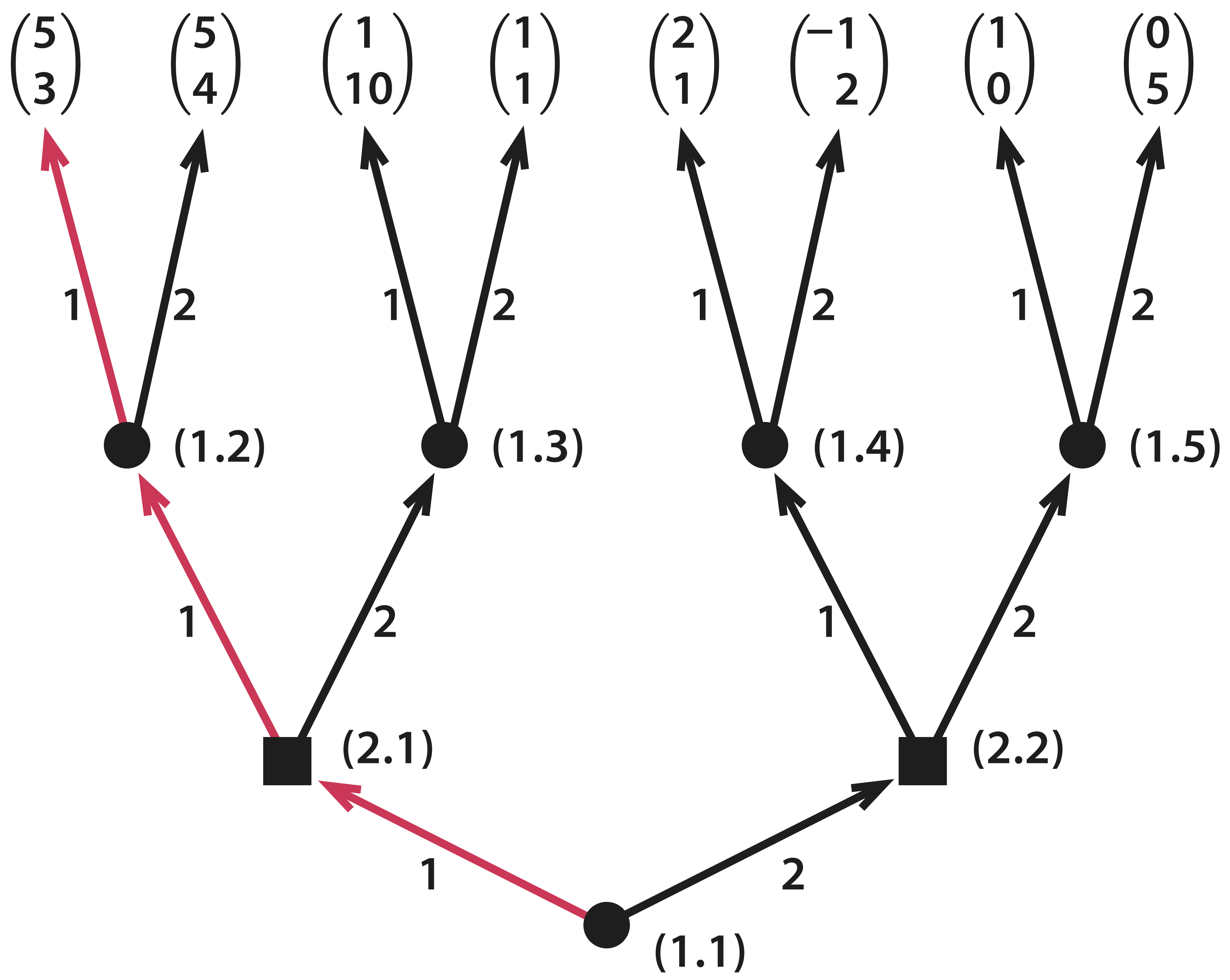
# Favorable Behavior




**Subgame-perfect Nash equilibrium $u^* = (u_1^*, u_2^*)$:**

- **(2.1):** Player 2 chooses the alternative 2, $u_2(x_{(2.1)}) = 2$.
- **(2.2):** Player 2 chooses the alternative 1, $u_2(x_{(2.2)}) = 1$.
- **(1.1):** Player 1 chooses the alternative 2, $u_1(x_{(1.1)}) = 2$.
- Here $u^* = ((2, 2, 1, 1, 1), (2, 1))$.
  Payoffs are $K_1(u_1^*, u_2^*) = 2$, $K_2(u_1^*, u_2^*) = 1$.

# Unfavorable Behavior




**Another subgame-perfect Nash equilibriumm**

$u^* = (u_1^*, u_2^*)$:

- **(1.2):** $u_1(x_{(1.2)}) = 1$ is unfavorable behavior.
- **(1.3):** $u_1(x_{(1.3)}) = 2$ is unfavorable behavior.
- **(1.4):** $u_1(x_{(1.4)}) = 1$.
- **(1.5):** $u_1(x_{(1.5)}) = 1$.
- **(2.1):** $u_2(x_{(2.1)}) = 1$.
- **(2.2):** $u_2(x_{(2.2)}) = 1$.
- **(1.1):** $u_1(x_{(1.1)}) = 1$.

Here $u^* = ((1, 1, 2, 1, 1), (1, 1))$.

Payoffs are $K_1(u_1^*, u_2^*) = 5$, $K_2(u_1^*, u_2^*) = 3$.

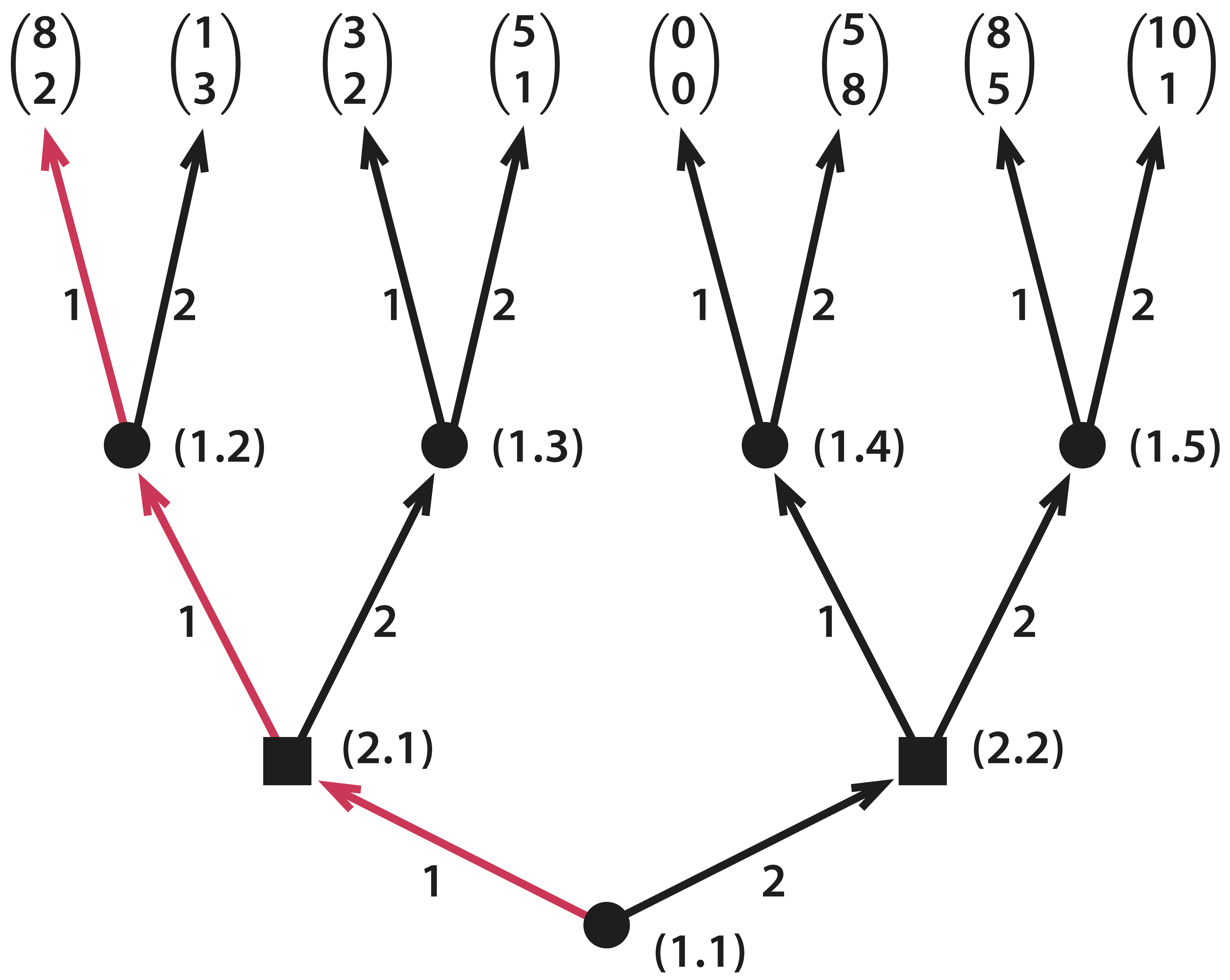
Unfavorable strategies are more profitable for both players!

# Penalty Strategies

**“Punishment of a clumsy maid”,**
Panin Sergey, 2000

- To ensure the desired development of process, one of players may use penalties, namely, in the case of second player deviation from the desired behavior, first player may play against the second player.
- We will refer to the corresponding strategies as penalty strategies.

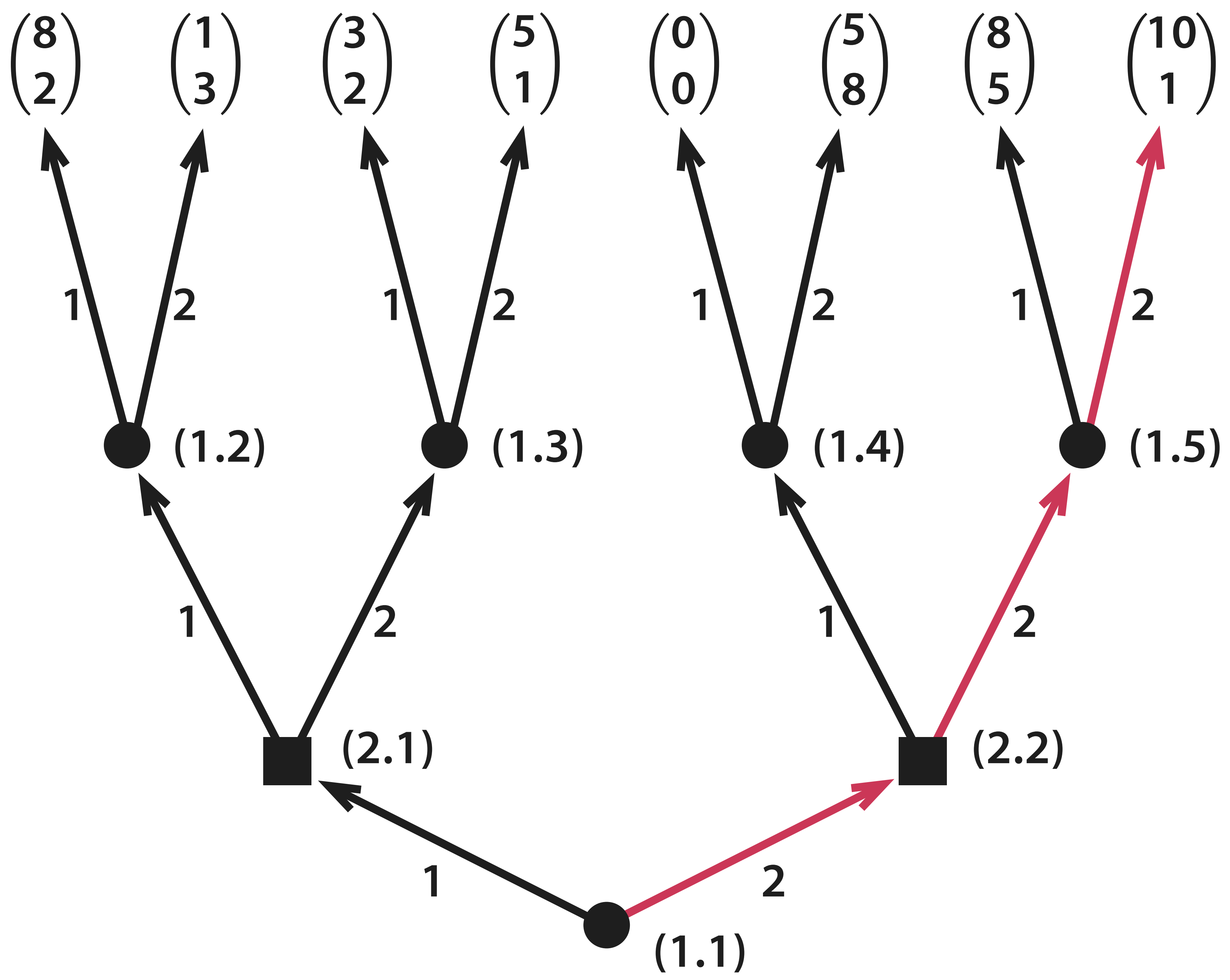
# Penalty Strategies




**Subgame-perfect Nash equilibrium** $(u_1^*, u_2^*)$:

- $u_1^* = (1, 1, 2, 2, 2)$,
  $u_2^* = (1, 1)$.
- $K_1(u_1^*, u_2^*) = 8$,
  $K_2(u_1^*, u_2^*) = 2$.

# Penalty Strategies

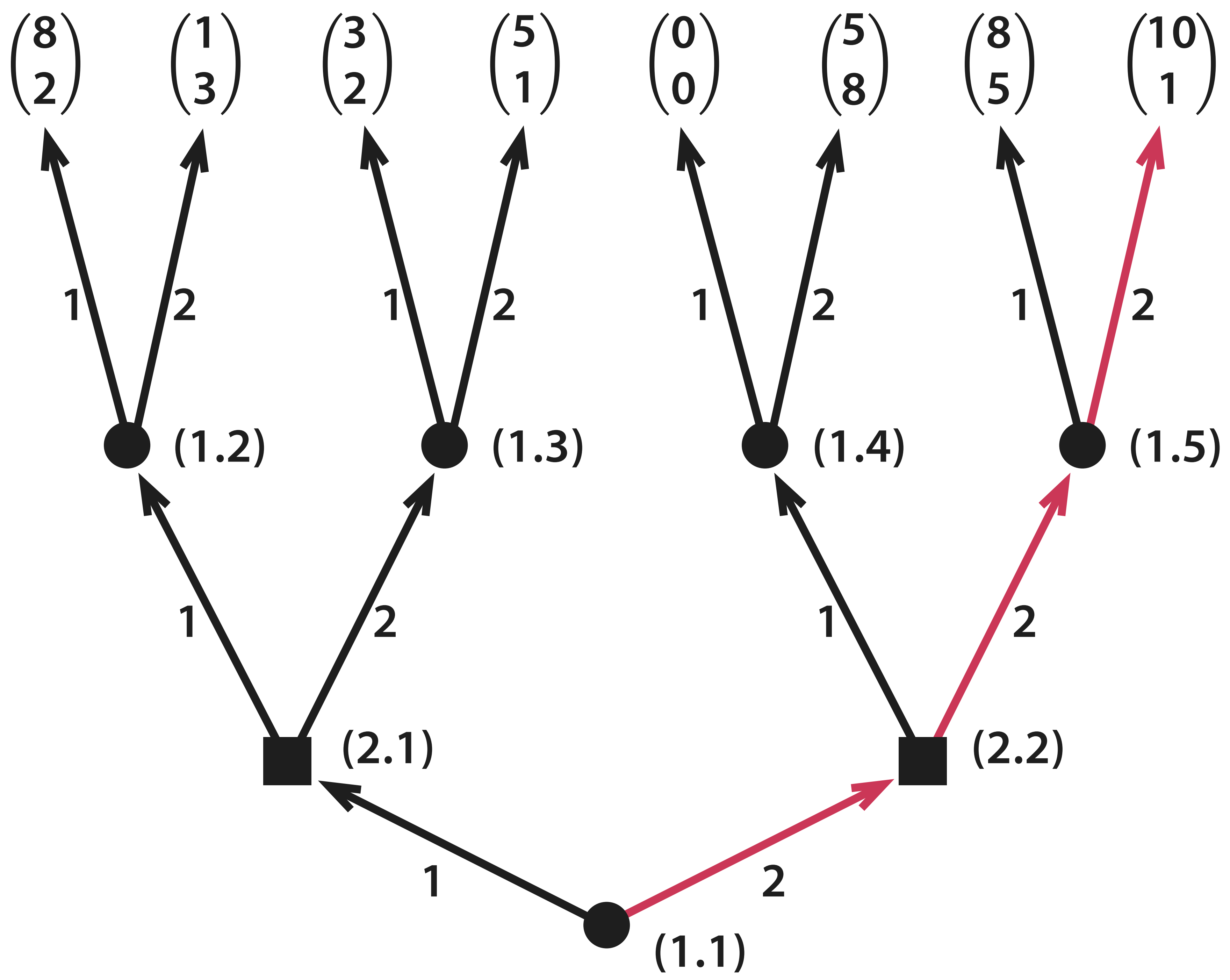



**Another Nash equilibrium $(u_1, u_2)$:**

- $u_1 = (2, 1, 2, 1, 2)$,
  $u_2 = (2, 2)$.
- $K_1(u_1, u_2) = 10$,
  $K_2(u_1, u_2) = 1$.

Strategy $\boldsymbol{u_1}$ is called a penalty strategy!

# Penalty Strategies

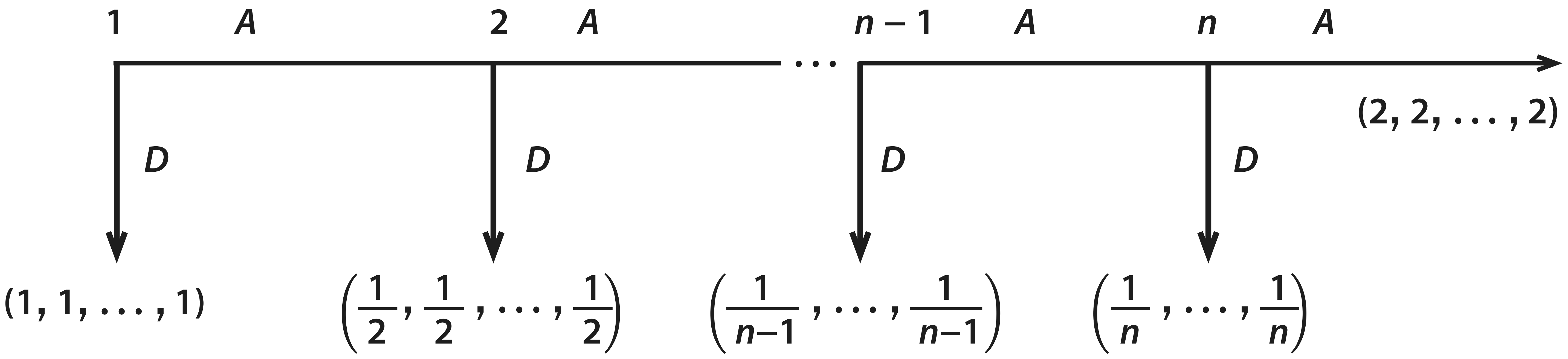



**Consider a subgame $\Gamma_{(1.4)}$:**

- $u_1^{x(1.4)} = (1)$.
- $K_1^{x(1.4)}(u_1^{x(1.4)}, u_2^{x(1.4)}) = 0$,
  $K_2^{x(1.4)}(u_1^{x(1.4)}, u_2^{x(1.4)}) = 0$.

Strategy profile $(u_1, u_2)$ is not subgame-perfect Nash equilibrium!

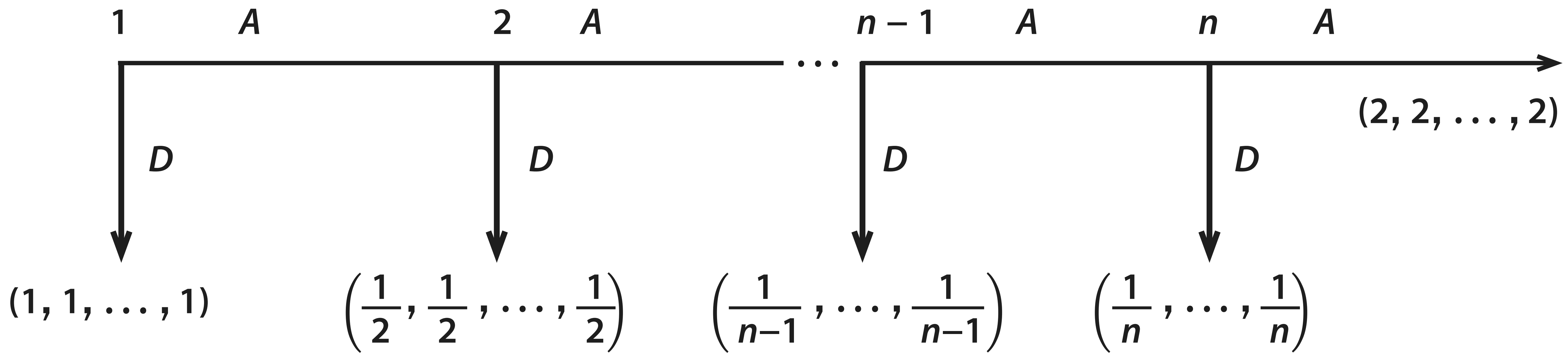
# Another Example




**Subgame-perfect Nash equilibrium:**

- $u^* = (A, A, \dots, A)$: $u_i^* = A$, $i = 1, \dots, n$,
  $(K_1(u^*), K_2(u^*), \dots, K_n(u^*)) = (2, 2, \dots, 2)$.
- $2 = K_i(u^*) > K_i(u^* \mid u_i = D) = \frac{1}{i}$, $i = 1, \dots, n$.

# Another Example




**Another Nash equilibrium.**

- $u = (D, A, \dots, D, \dots, A)$,
  $(K_1(u), K_2(u), \dots, K_n(u)) = (1, 1, \dots, 1)$.
- Nash equilibrium is any strategy profile
  $u = (u_1 = D, u_i = A, u_j = D, i \neq j = 3, \dots, n)$.

Strategy $u_j$ is a penalty strategy!

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1. Basar, T. & Olsder, G. J. (1998). Dynamic Noncooperative Game Theory. (2nd ed.). New York: Academic Press.
2. Fudenberg, D. & Tirole, J. (2000). Game Theory. Cambridge: MIT-press.
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# Multistage Games with Imperfect Information

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# Multistage Games with Imperfect Information

**"Assassination of Julius Caesar",**
Karl Theodor von Pilotyи, 1865

In games with imperfect information players at any given time do not know (or do not know all) the information about the previously chosen alternatives.

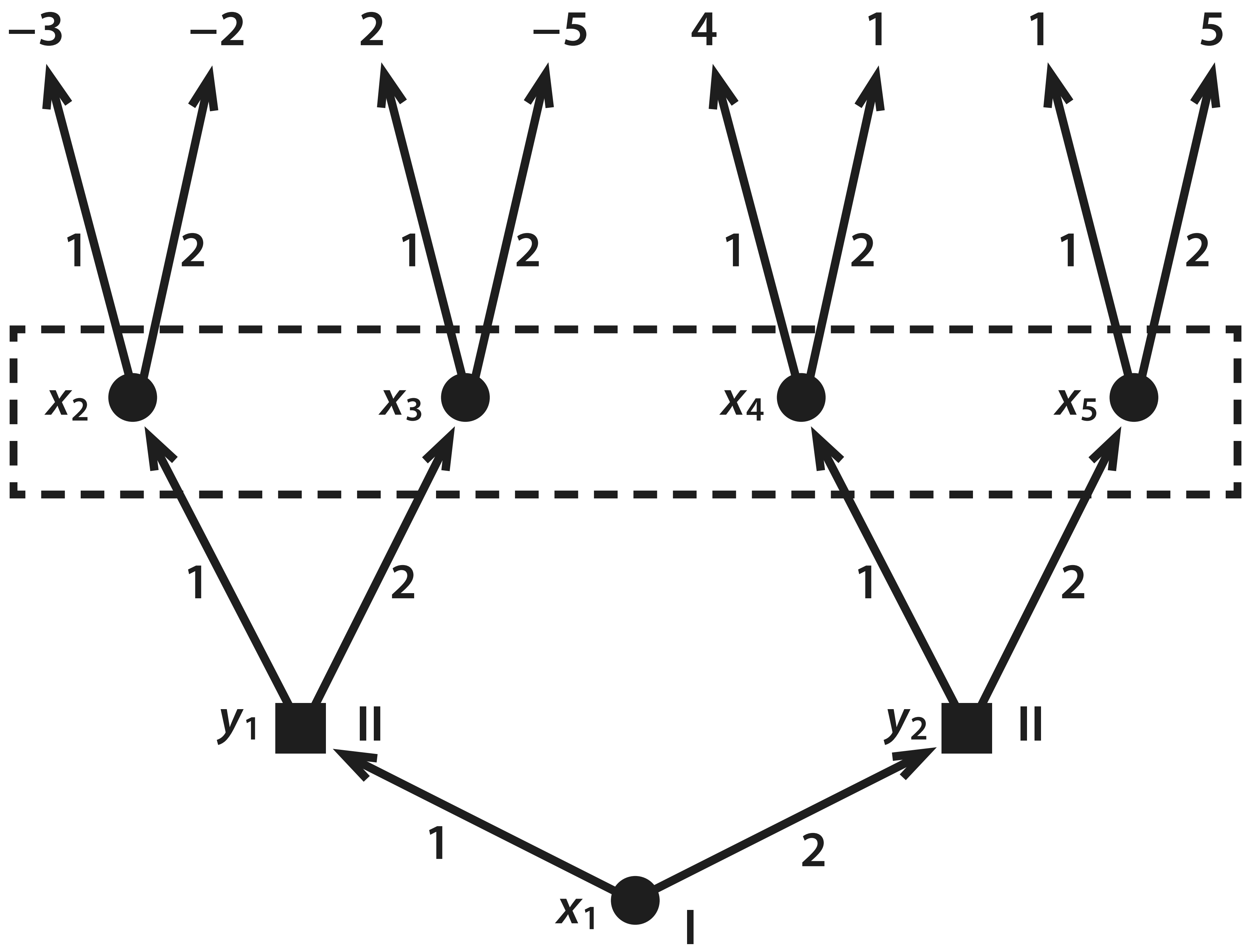
# Multistage Games with Imperfect Information

**Definition.**

Multistage n-person game with imperfect information (game in extensive form):

- $G = (X, F)$ is the tree graph starting at $x_0$.
- $X = \bigcup\limits_{i=1}^{n+1} X_i$, where $X_k \cap X_e = \emptyset$, $k \neq l$ is a partition nodes set.
- $K_1(x), \dots, K_n(x)$, $x \in X_{n+1}$ payoffs of players.
- $X_i = \bigcup\limits_{j} X_i^j$, where $X_i^k \cap X_i^l = \emptyset$, $k \neq l$ is the information set of player $i$ and $\forall x, y \in X_i^j : |F_x| = |F_y|$.

# Multistage Games with Imperfect Information

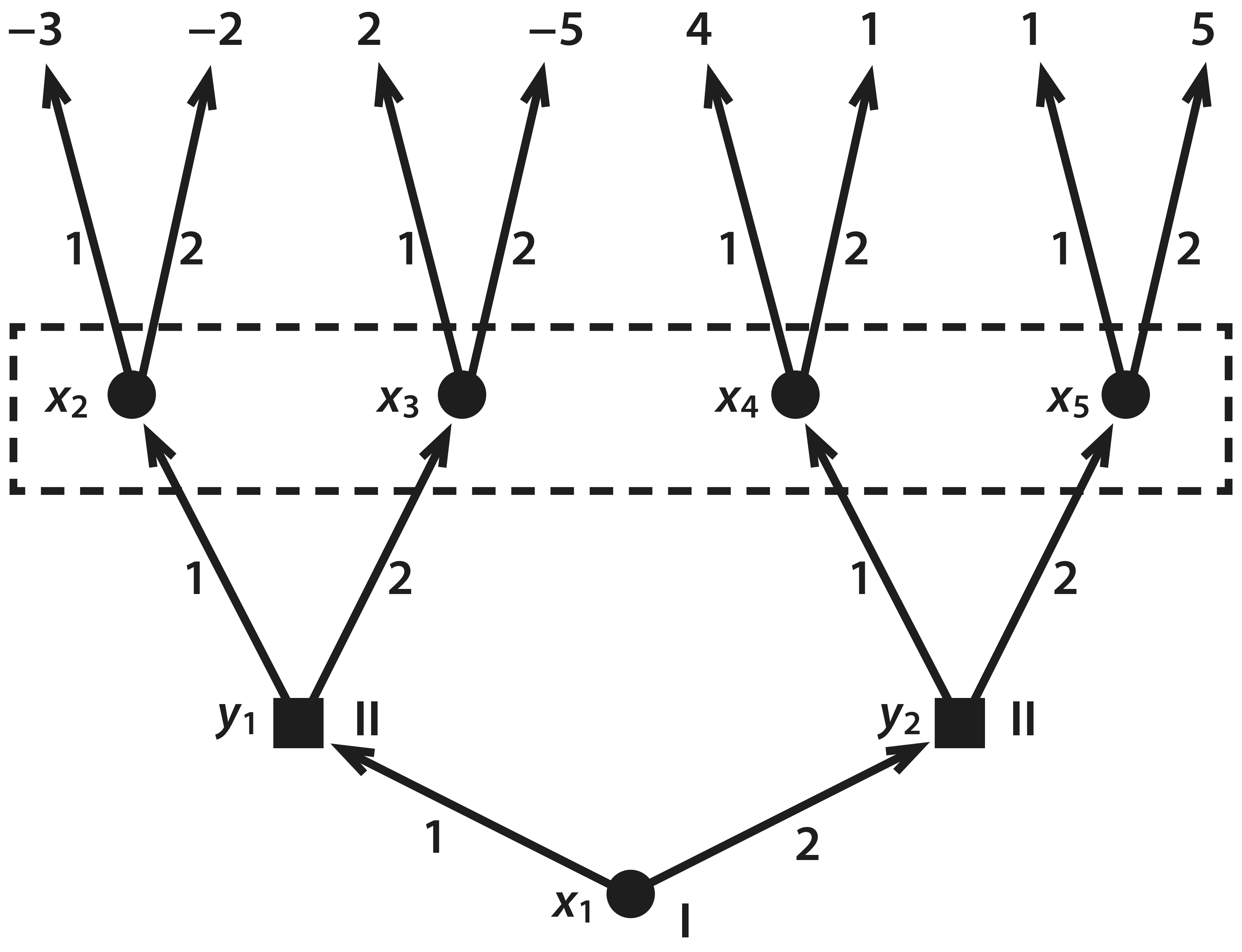



**Example.**

Here $X = X_1 \bigcup X_2 \bigcup X_3$, where $X_k \cap X_l = \emptyset$, $k \neq l$, then

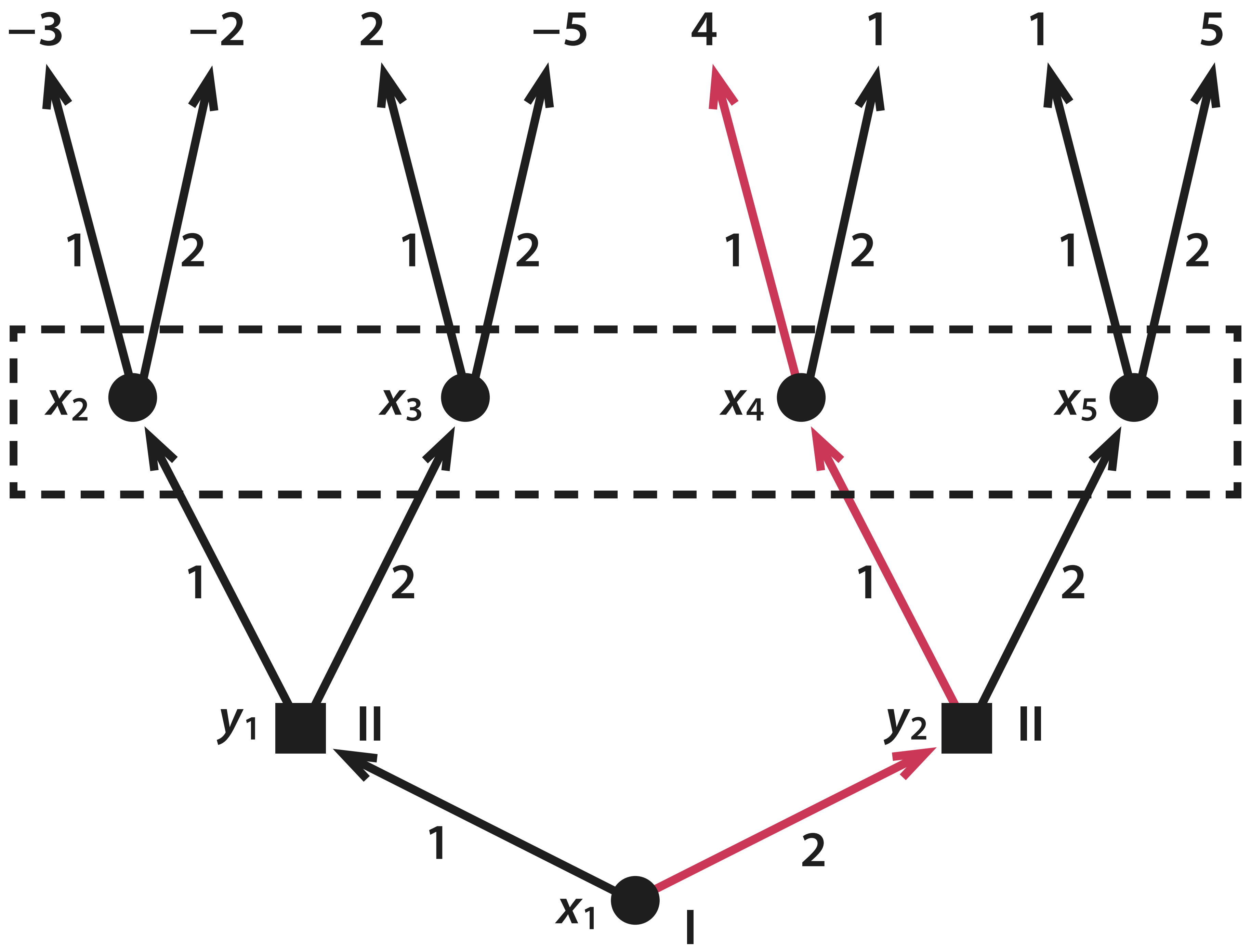
- $N = \{1, 2\}$.
- $\{X_1^1, X_1^2\} = \{\{x_1\}, \{x_2, x_3, x_4, x_5\}\}$,
  $\{X_2^1, X_2^2\} = \{\{y_1\}, \{y_2\}\}$, where $X_1^1$, $X_1^2$ ($X_2^1$, $X_2^2$) are the information sets of player 1 (2).
- $K_1(u_1, u_2) = -K_2(u_1, u_2)$: zero-sum game.

# Multistage Games with Imperfect Information




$$\begin{array}{c} \\ (1.1) \\ (1.2) \\ (2.1) \\ (2.2) \end{array}
\begin{array}{c} \begin{array}{cccc} (1.1) & (1.2) & (2.1) & (2.2) \end{array} \\ \begin{bmatrix} -3 & -3 & 2 & 2 \\ -2 & -2 & -5 & -5 \\ 4 & 1 & 4 & 1 \\ 1 & 5 & 1 & 5 \end{bmatrix} \end{array}$$

# Multistage Games with Imperfect Information

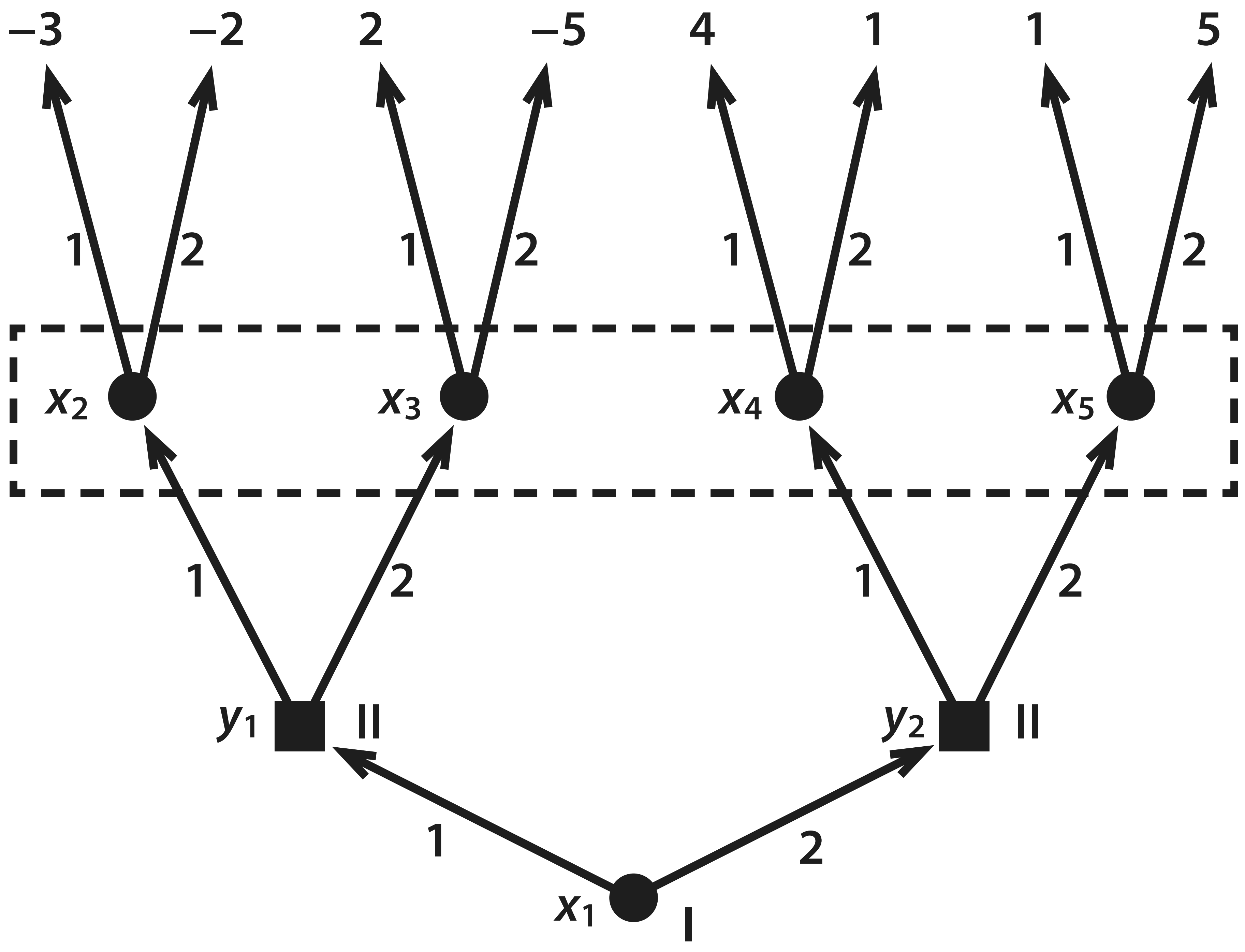



Suppose $u_1 = (2, 1)$, $u_2 = (1, 1)$:

- $u_1(x_1) = 2$, $u_1(x_2, x_3, x_4, x_5) = 1$,
- $u_2(y_1) = 1$, $u_2(y_2) = 1$.

Payoffs in strategy profile $(u_1, u_2)$:
$(K_1(u_1, u_2), K_2(u_1, u_2)) = (4, -4)$.

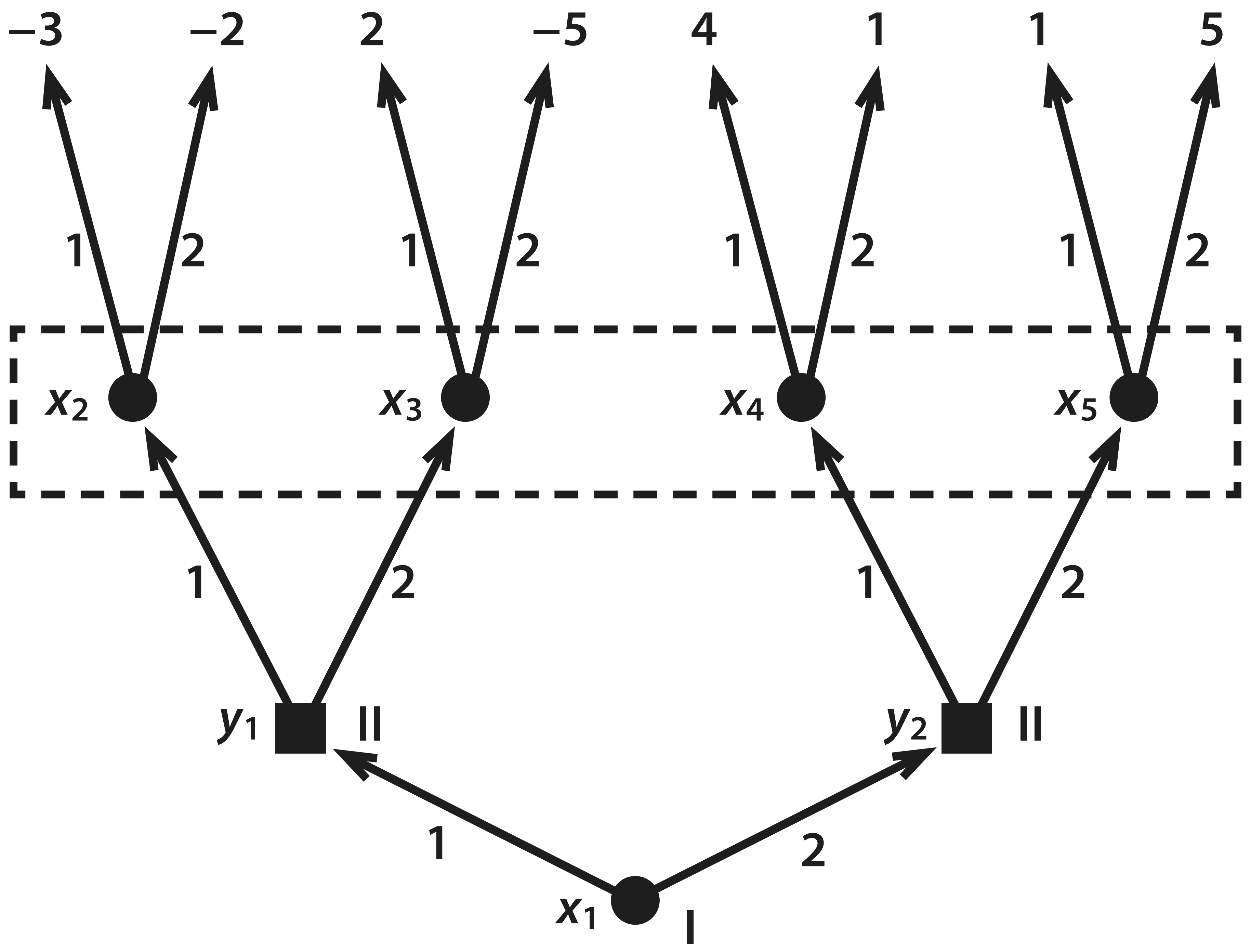
# Multistage Games with Imperfect Information




$$\longrightarrow$$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} (1.1) & (1.2) & (2.1) & (2.2) \end{array} \\
\begin{array}{c} (1.1) \\ (1.2) \\ (2.1) \\ (2.2) \end{array} & \begin{bmatrix} -3 & -3 & 2 & 2 \\ -2 & -2 & -5 & -5 \\ 4 & 1 & 4 & 1 \\ 1 & 5 & 1 & 5 \end{bmatrix}
\end{array}
$$

# Multistage Games with Imperfect Information



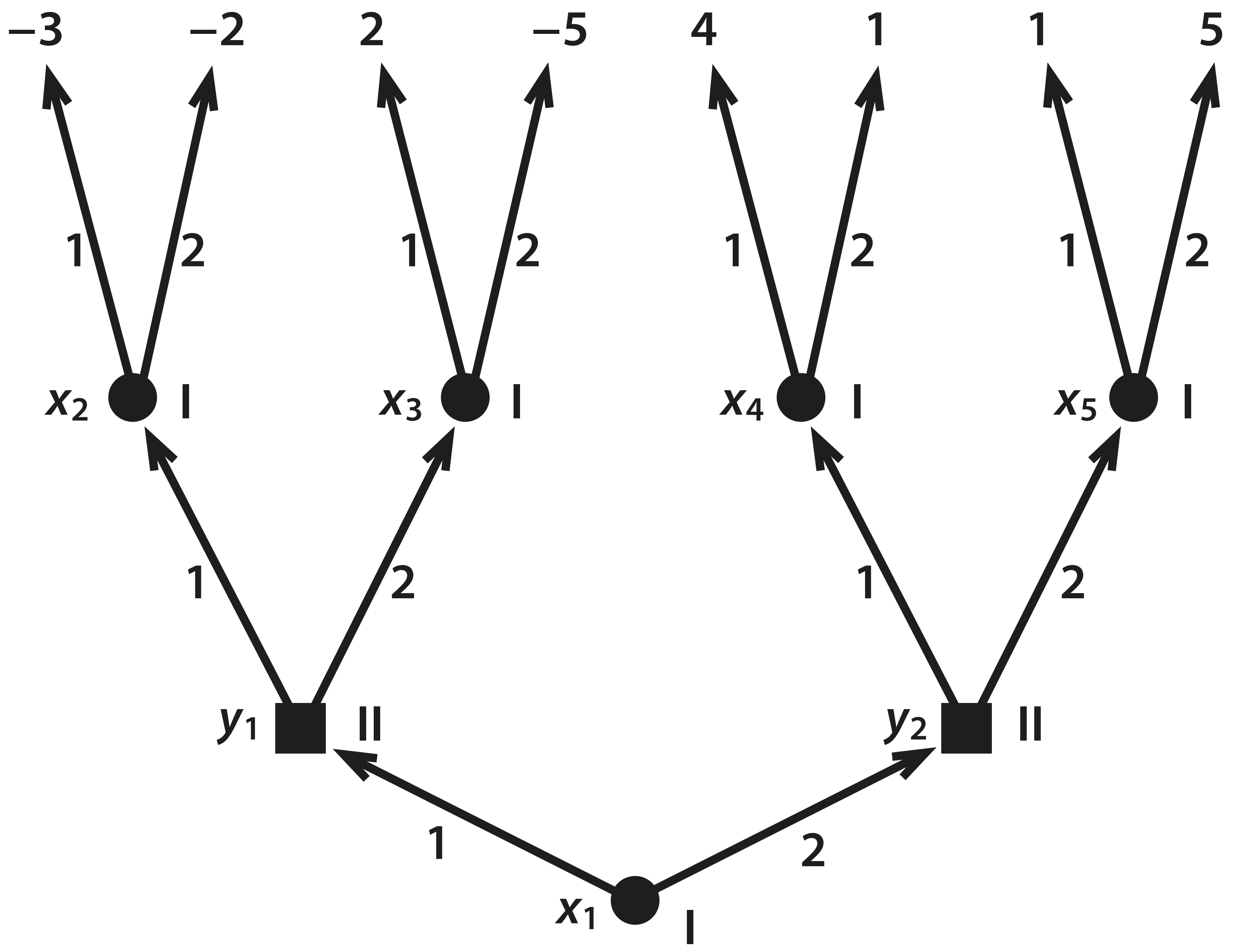

Saddle point $(u_1^*, u_2^*)$:

- $u_1^* = (0, 0, \frac{4}{7}, \frac{3}{7})$,
- $u_2^* = (\frac{4}{7}, \frac{3}{7}, 0, 0)$.

Payoffs in saddle point $(u_1^*, u_2^*)$:
$(K_1(u_1^*, u_2^*), K_2(u_1^*, u_2^*)) = (\frac{19}{7}, -\frac{19}{7})$.

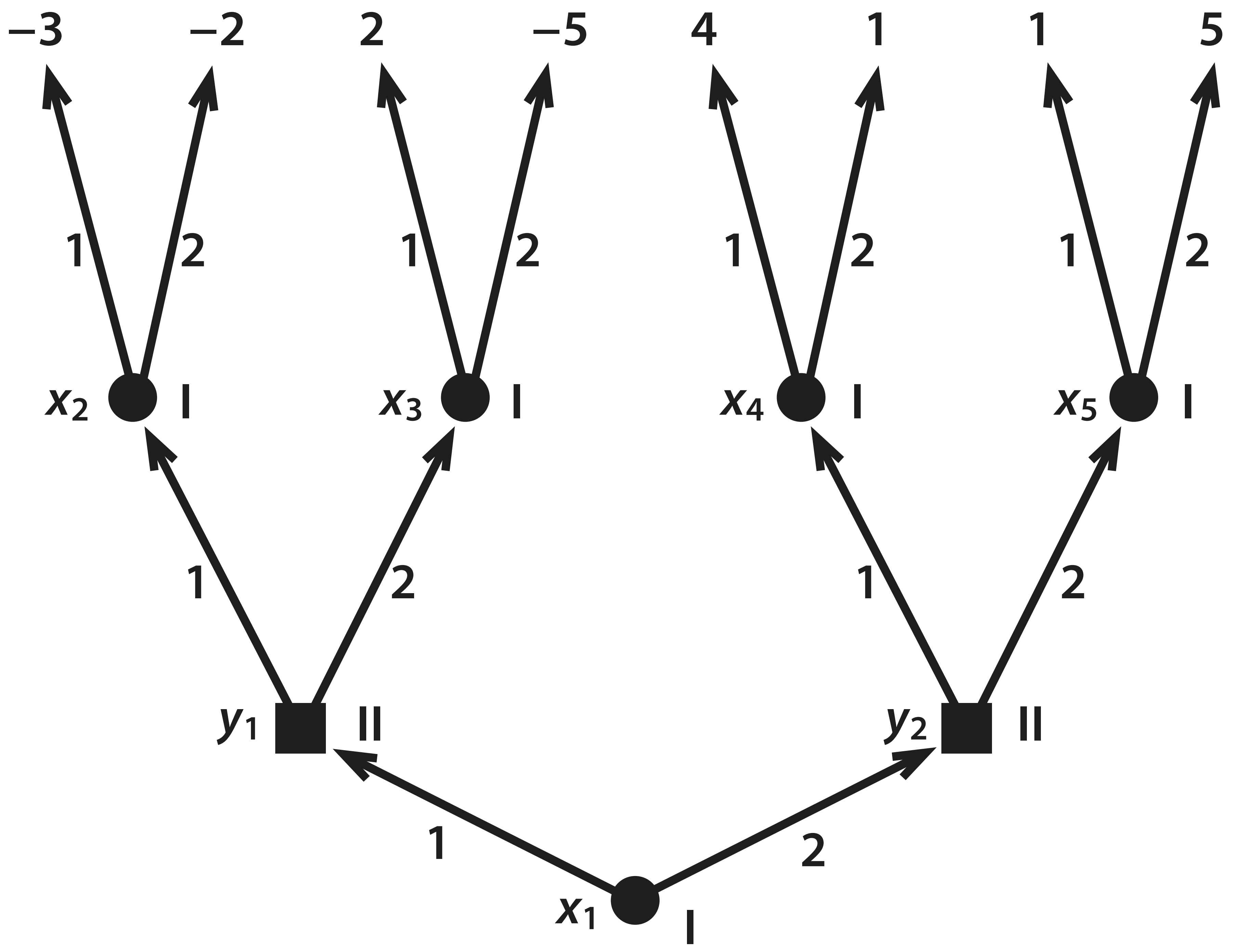
Here the saddle point is in mixed strategies!

# Multistage Games with Imperfect Information




Matrix game: $[32 \times 4]$.

# Multistage Games with Imperfect Information




Saddle point $(u_1^*, u_2^*)$:

- $u_1^* = (2, 1, 1, 1, 2), (2, 1, 2, 1, 2), (2, 2, 1, 1, 2), (2, 2, 2, 1, 2).$
- $u_1^* = (1, 1), (2, 1).$

Payoffs in saddle point $(u_1^*, u_2^*)$:
$(K_1(u_1^*, u_2^*), K_2(u_1^*, u_2^*)) = (4, -4).$

Here the saddle point is in pure strategies!

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2. Fudenberg, D. & Tirole, J. (2000). Game Theory. Cambridge: MIT-press.
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# Games with Imperfect Information

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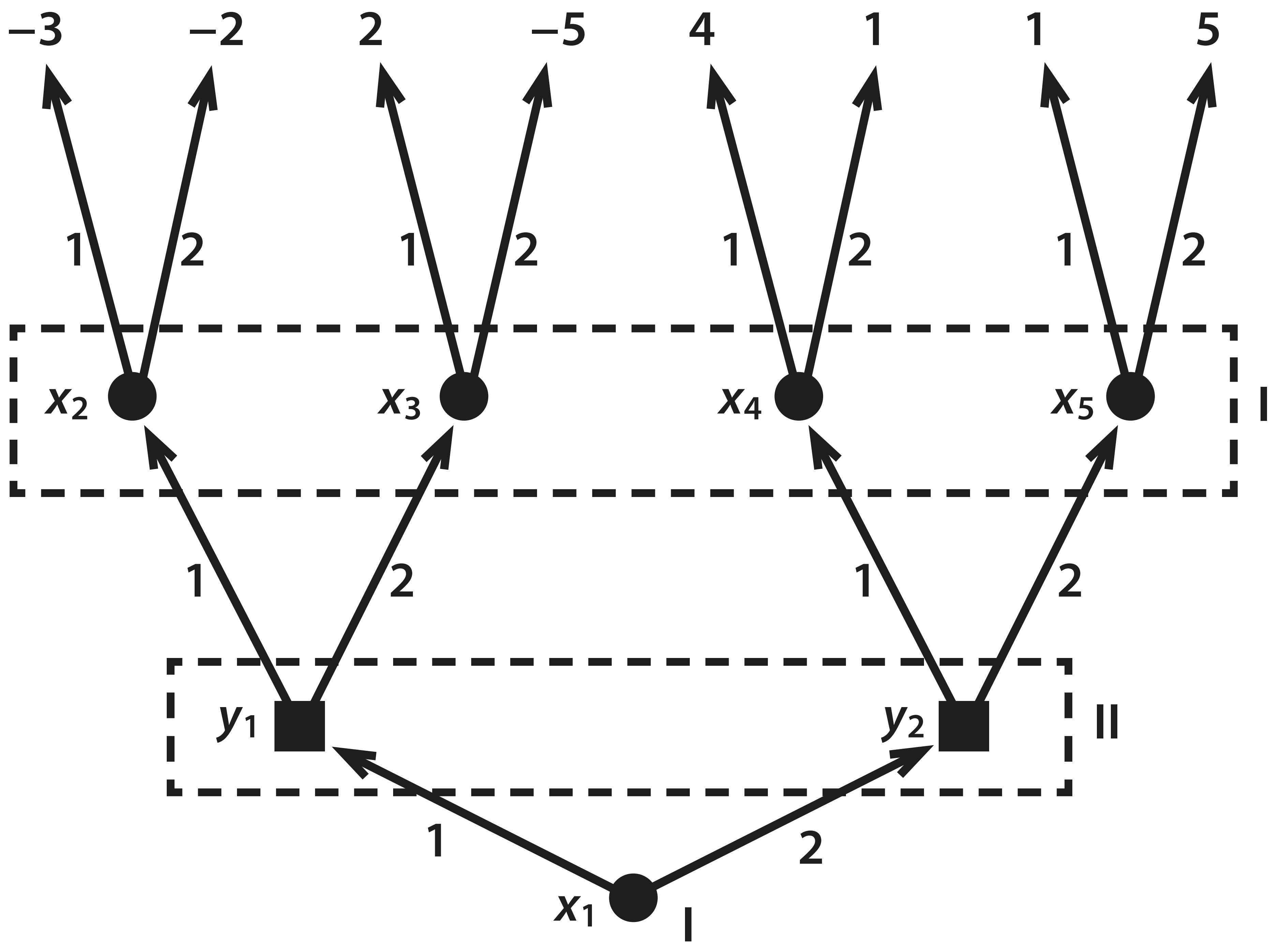
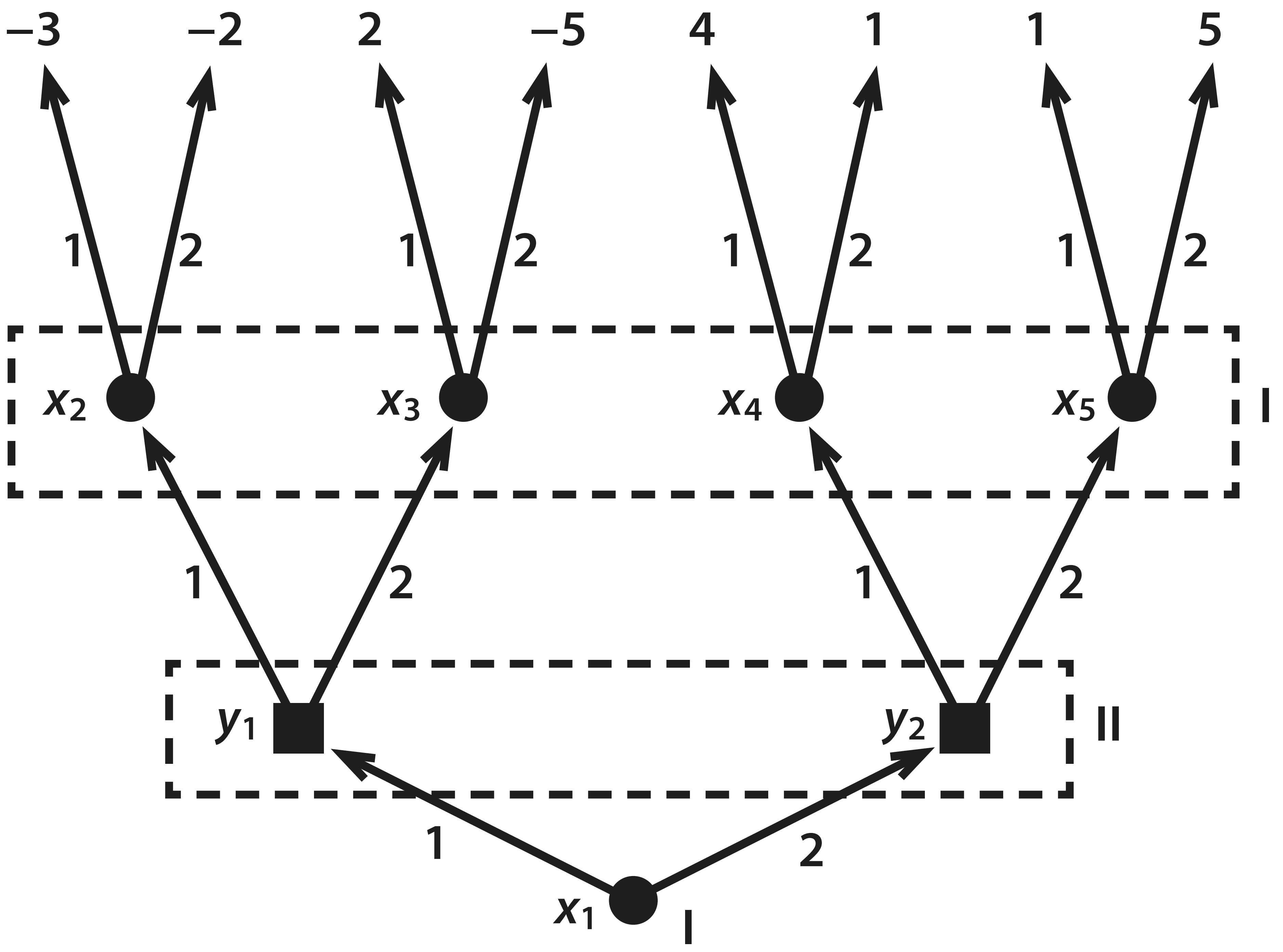
# Multistage Games with Imperfect Information




$$
\begin{array}{c c}
 & \begin{array}{cc} 1 & 2 \end{array} \\
\begin{array}{c} (1.1) \\ (1.2) \\ (2.1) \\ (2.2) \end{array} &
\begin{bmatrix} -3 & 2 \\ -2 & -5 \\ 4 & 1 \\ 1 & 5 \end{bmatrix}
\end{array}
$$

# Multistage Games with Imperfect Information

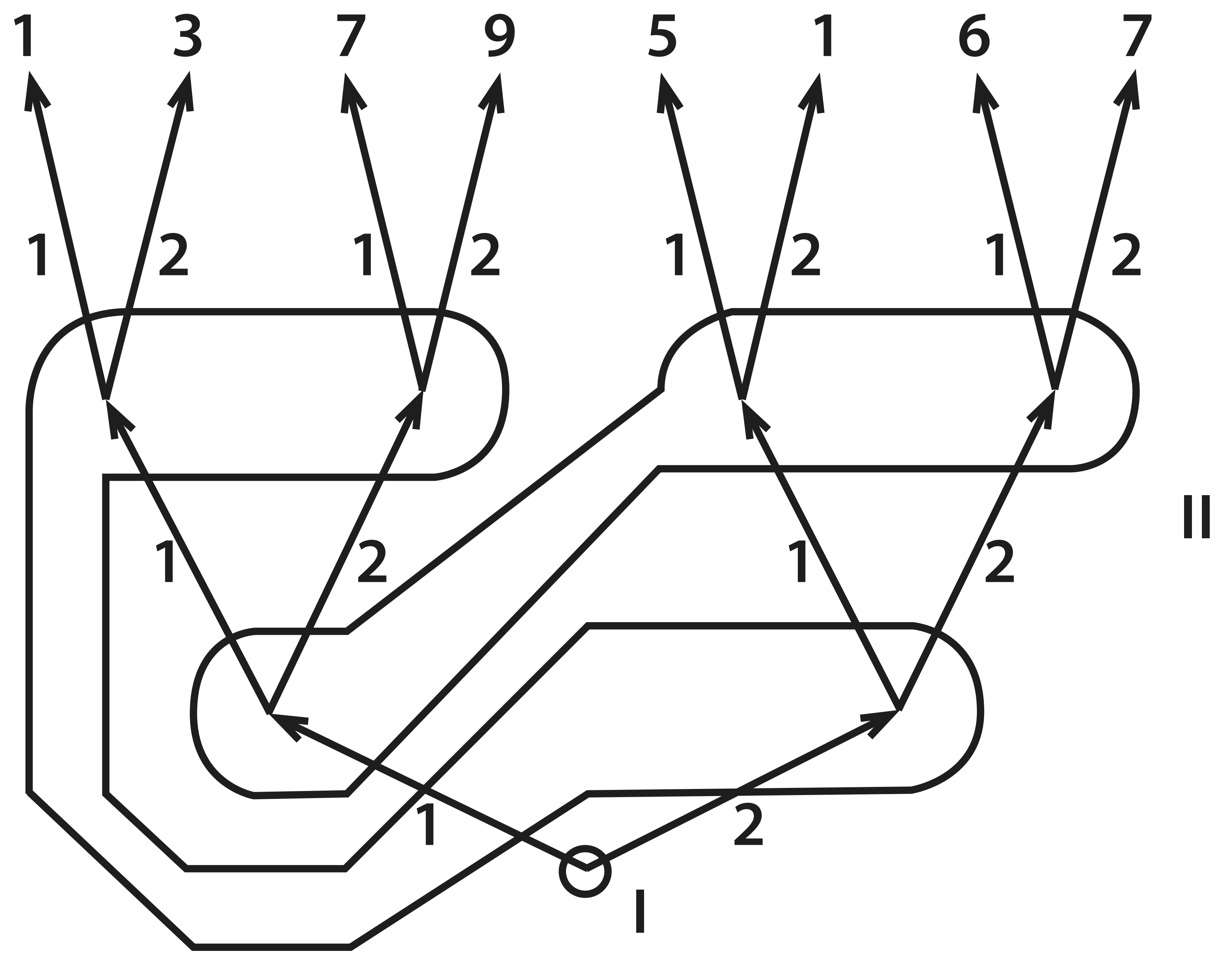



Saddle point $(u_1^*, u_2^*)$:

- $u_1^* = (0, 0, \frac{4}{7}, \frac{3}{7})$,
- $u_2^* = (\frac{4}{7}, \frac{3}{7})$.

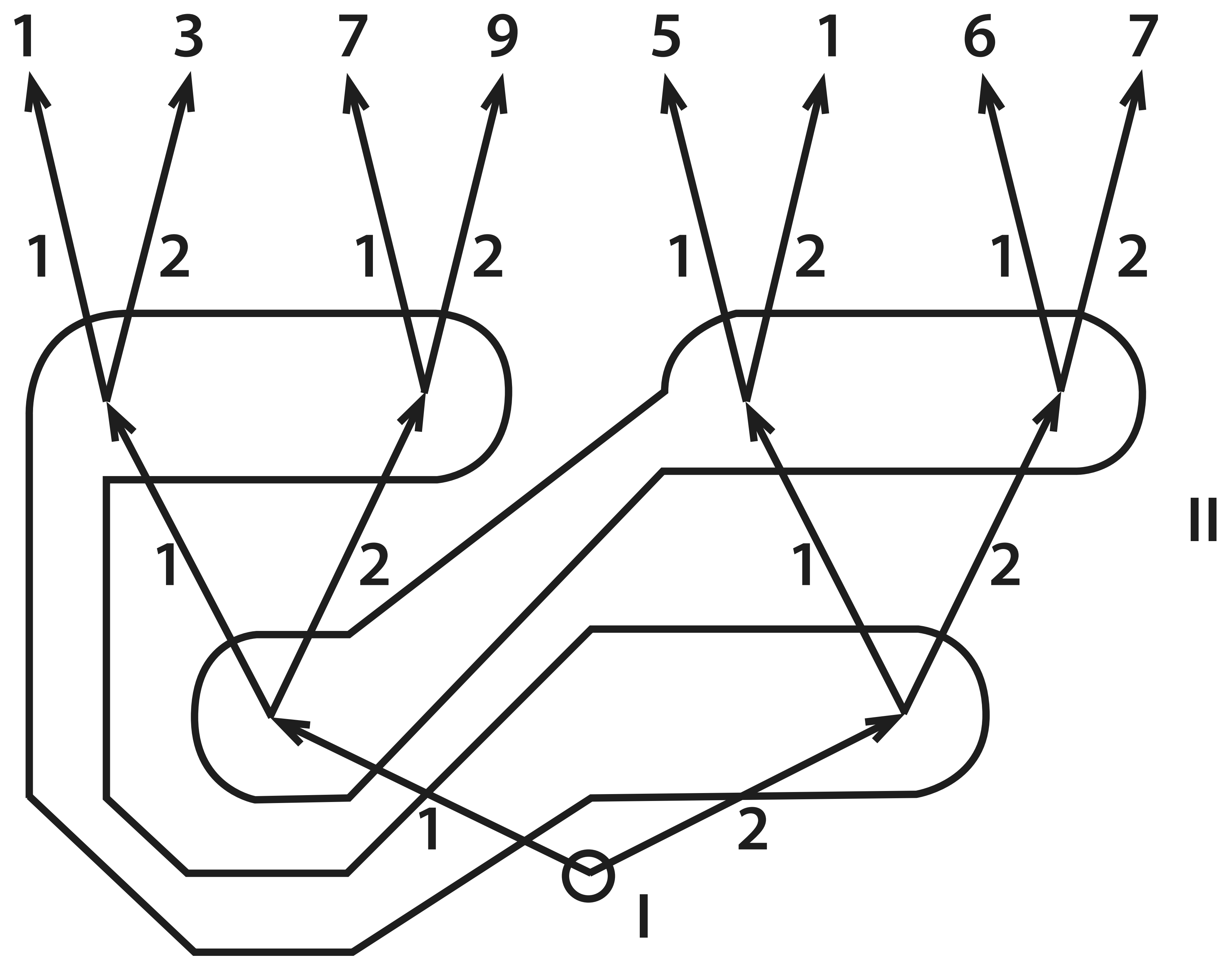
Payoffs in saddle point $(u_1^*, u_2^*)$:
$(K_1(u_1^*, u_2^*), K_2(u_1^*, u_2^*)) = (\frac{19}{7}, -\frac{19}{7})$.

# Multistage Games with Imperfect Information




$$\begin{array}{cc} & \begin{array}{cccc} (1.1) & (1.2) & (2.1) & (2.2) \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 3 & 7 & 9 \\ 5 & 6 & 1 & 7 \end{bmatrix} \end{array}$$

# Multistage Games with Imperfect Information




Saddle point $(u_1^*, u_2^*)$:

- $u_1 = (\frac{2}{5}, \frac{3}{5})$,
- $u_2 = (\frac{3}{5}, 0, \frac{2}{5}, 0)$.

Payoffs in saddle point $(u_1, u_2)$:
$(K_1(u_1, u_2), K_2(u_1, u_2)) = (\frac{17}{5}, -\frac{17}{5})$.

# References


1. Basar, T. & Olsder, G. J. (1998). Dynamic Noncooperative Game Theory. (2nd ed.). New York: Academic Press.
2. Fudenberg, D. & Tirole, J. (2000). Game Theory. Cambridge: MIT-press.
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# Repeated Games

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# Prisoner’s Dilemma

**“Illustrations to the works M. Lermontov”,**
V. A. Polyakov, 1900

Two criminals are interrogated independently of each other. Everyone can betray the other or remain silent:

- both remain silent — both of them will only serve six months in prison,
- each betray the other — each of them serves 2 years in prison,
- one betrays, another remains silent — first will be set free and second will serve 10 years in prison.

# Prisoner's Dilemma

**Noncooperative game in normal form $\Gamma$** $= (N, \{X_i\}_{i \in N}, \{K_i\}_{i \in N})$:

- $N = \{1, 2\}$.
- $X_1 = (x_1, x_2)$, where $x_1$ — remain silent, $x_2$ — betray,
  $X_2 = (y_1, y_2)$, where $y_1$ — remain silent, $y_2$ — betray,
- $K_1(x_1, y_1) = K_2(x_1, y_1) = -0.5$ — both remain silent,
  $K_1(x_2, y_2) = K_2(x_2, y_2) = -2$ — each betray the other,
  $K_1(x_2, y_1) = K_1(x_1, y_2) = 0$ — one betrays, another remains silent,
  $K_2(x_2, y_1) = K_1(x_1, y_2) = -10$.

# Prisoner's Dilemma

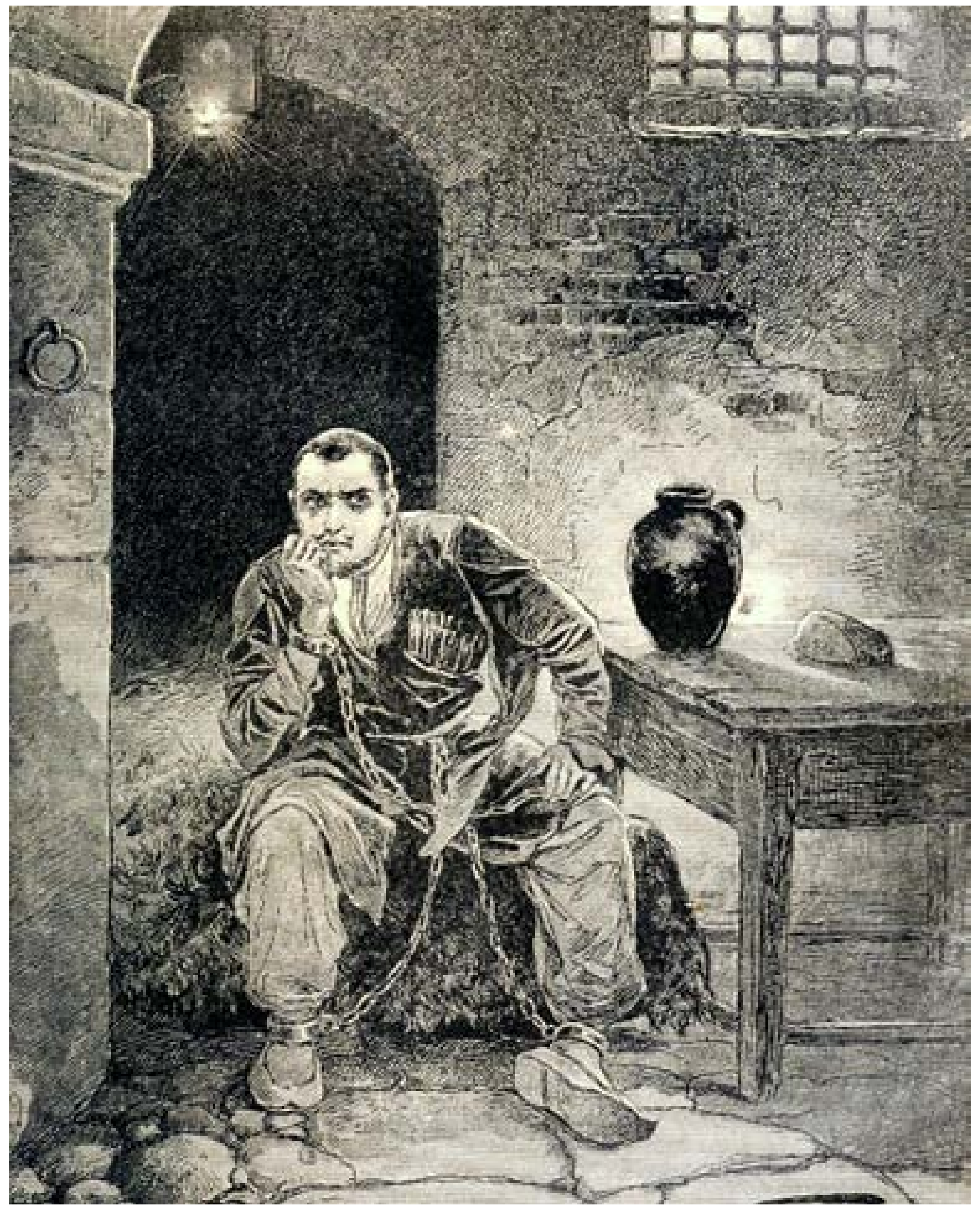
$$(A, B) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} (-0.5, -0.5) & (-10, 0) \\ (0, -10) & (-2, -2) \end{pmatrix} \end{array}$$

Strategy profiles:

- $(x_1, y_1)$ — both remain silent.
- $(x_2, y_2)$ — each betray the other.
- $(x_2, y_1)$, $(x_1, y_2)$ — one betrays, another remains silent.

Nash equilibrium: $(x_2, y_2)$

# Repeated Games



“Illustrations to the works M. Lermontov”,
V. A. Polyakov, 1900

Assume that the prisoner dilemma game is repeated infinite number of times, at instants $t = 0, 1, 2, \ldots$ .

Which strategies will form the Nash equilibrium?

# Repeated Games

**Prisoner's dilemma repeated game $\Gamma_\delta^\infty$:**

- $N = \{1, 2\}$.
- $x = (x_{i_0}, x_{i_1}, \dots, x_{i_k}, \dots) \in X_1$ is a strategy of prisoner 1,
  $y = (y_{j_0}, y_{j_1}, \dots, y_{j_k}, \dots) \in X_2$ is a strategy of prisoner 2,
  where $x_1$, $y_1$ — remain silent, $x_2$, $y_2$ — betray,
- $K_1(x, y) = \sum_{t=0}^{\infty} \delta^t K_1^t(x_{i_t}, y_{j_t})$, $K_2(x, y) = \sum_{t=0}^{\infty} \delta^t K_2^t(x_{i_t}, y_{j_t})$,

  where $K_1^t(x_{i_t}, y_{j_t})$, $K_2^t(x_{i_t}, y_{j_t})$ are the payoffs of prisoners in singleshot games in instant $t$, $0 < \delta < 1$ is a discount factor.

# Repeated Games

Denote $x_2^\infty = (x_2, x_2, \dots, x_2, \dots)$.

Consider strategy profile $(x, y) = (x_2^\infty, y_2^\infty)$:

- $K_1(x_2^\infty, y_2^\infty) = \sum_{t=0}^{\infty} \delta K_1^t(x_2, y_2) = \sum_{t=0}^{\infty} -2\delta^t = -\frac{2}{1-\delta}$,
- $K_2(x_2^\infty, y_2^\infty) = \sum_{t=0}^{\infty} \delta K_2^t(x_2, y_2) = \sum_{t=0}^{\infty} -2\delta^t = -\frac{2}{1-\delta}$.

Strategy profile $(x_2^\infty, y_2^\infty)$ is subgame-perfect Nash equilibrium.

# Repeated Games

**Definition.**
Trigger (penalty) strategy $Tr_i$ of player $i$ is the strategy of form:

$$Tr_1 = \begin{cases} x^t = x_1, \text{ if } (x^{t-1}, y^{t-1}) = (x_1, y_1), \\ x^t = x_2, \text{ if } \exists \tau < t : y^\tau \neq y_1. \end{cases}$$

$$Tr_2 = \begin{cases} y^t = y_1, \text{ if } (x^{t-1}, y^{t-1}) = (x_1, y_1), \\ y^t = y_2, \text{ if } \exists \tau < t : x^\tau \neq x_1. \end{cases}$$

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# Repeated Games

Payoffs $(K_1(Tr_1, Tr_2), K_2(Tr_1, Tr_2))$ in strategy profile $(Tr_1, Tr_2)$ are

- $K_1(Tr_1, Tr_2) = \sum_{t=0}^{\infty} \delta K_1^t(x_1, y_1) = \sum_{t=0}^{\infty} -0.5\delta^t = -\frac{1}{2(1-\delta)}$,
- $K_2(Tr_1, Tr_2) = \sum_{t=0}^{\infty} \delta K_2^t(x_1, y_1) = \sum_{t=0}^{\infty} -0.5\delta^t = -\frac{1}{2(1-\delta)}$.

# Repeated Games

Strategy profile ($Tr_1$, $Tr_2$) in the game $\Gamma_\delta^\infty$ is subgame-perfect Nash equilibrium, if $\delta \geq 0.25$:

- if $x^t = x_2$, then according to $Tr$ player 1 in the subgame starting on the step $t$ receives payoff:

  $$\delta^t K_1^t(x_2, y_1) + \sum_{l=t+1}^{\infty} \delta^l K_1^l(x_2, y_2) = 0 + \sum_{l=t+1}^{\infty} -2\delta^t = \frac{-2\delta^{t+1}}{1-\delta}.$$

- $-\dfrac{\delta^t}{2(1-\delta)} \geq -\dfrac{2\delta^{t+1}}{1-\delta} \longrightarrow \delta \geq 0.25.$

Similar for player 2.

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# Cooperative Multistage Games

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PhD

# Signing Documents

**“The head of Secret Chancellery, A. S. Ushakov interrogates princess Yusupova”,**
N. Nevrev, 1886

Consider the bureaucratic process of approving documents:

- Company needs to sign the package of documents successively in three departments.
- If in one department the documents are not approved, then the process (game) terminates.
- If the documents are signed by all departments, then the process (game) also terminates.

# Cooperative Multistage Games with Perfect Information

**Definition.**

Cooperative multistage game with perfect information $\Gamma$:

- $N = \{1, \dots, n\}$ is the set of players.

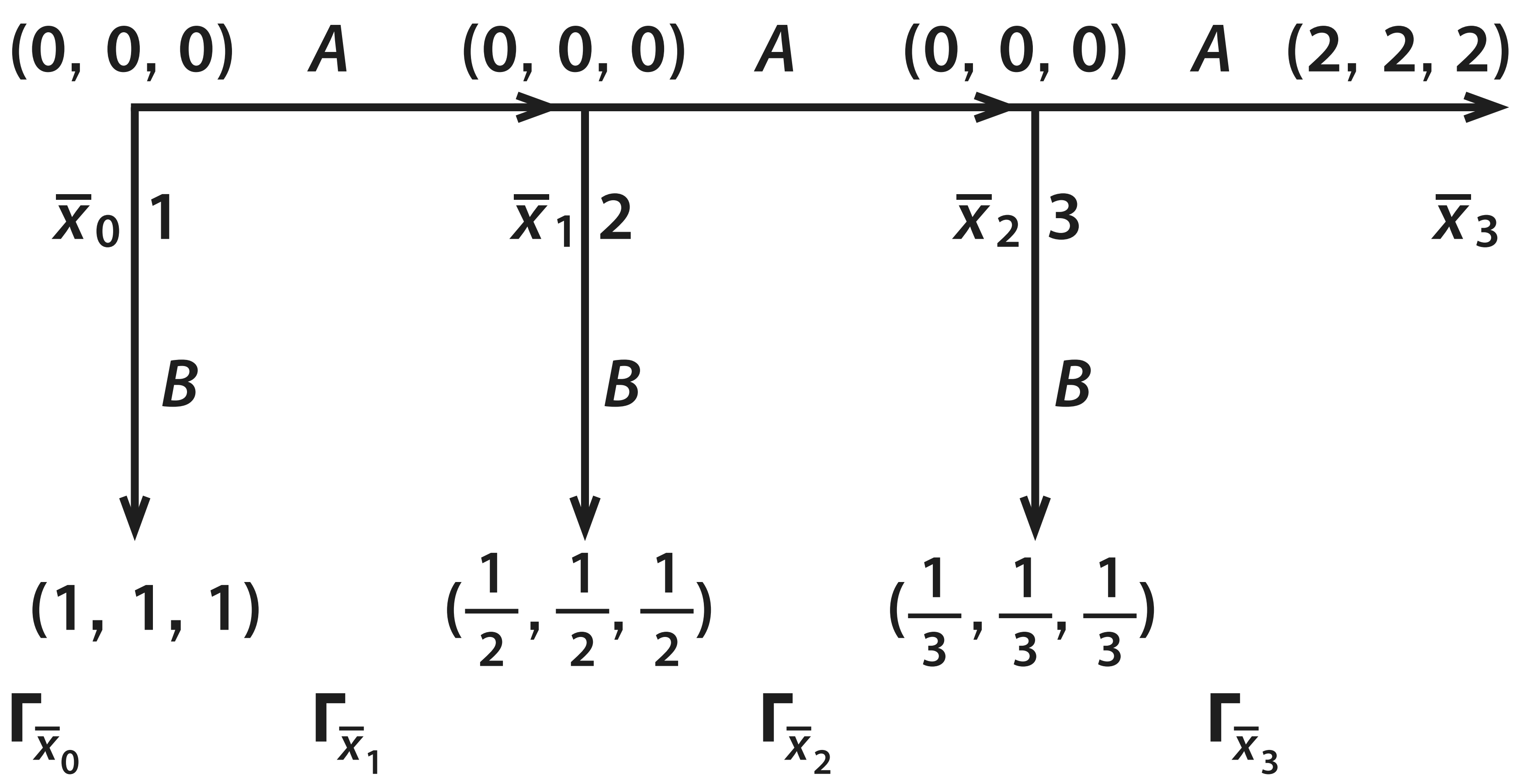
- $X = \bigcup\limits_{i=1}^{n+1} X_i$ is the set of personal positions,

  $X_i : X_k \cap X_e = \varnothing,\ k \neq e,\ i = 1, \dots, n$ is the set of personal positions of player $i$,

  $X_{n+1} = \{x : Fx = \varnothing\}$ is the set of final or terminal positions.

- $K_i(u_1, \dots, u_n) = H_i(x) = \sum_{k=0}^{l} h_i(x_k)$ is the payoff function of player $i$, where each strategy profile $(u_1, \dots, u_n)$ determines a path $x = (x_0, \dots, x_l)$, $x_l \in X_{n+1}$, $h_i(x)$, $i = 1, \dots, n$ are payoffs determined in each position $x \in X$.

# Cooperative Multistage Games with Perfect Information




**Example.**

- $N = \{1, 2, 3\}$.
- Strategies: $\{A, B\}$.
- Payoff function: $K_i(A, A, A) = 2$,

$$K_i(B, u_2, u_3) = 1,\ \forall u_2, u_3 \in \{A, B\},$$

$$K_i(A, B, u_3) = \frac{1}{2},\ \forall\ u_3 \in \{A, B\},$$

$$K_i(A, A, B) = \frac{1}{3},\ i = 1, 2, 3.$$

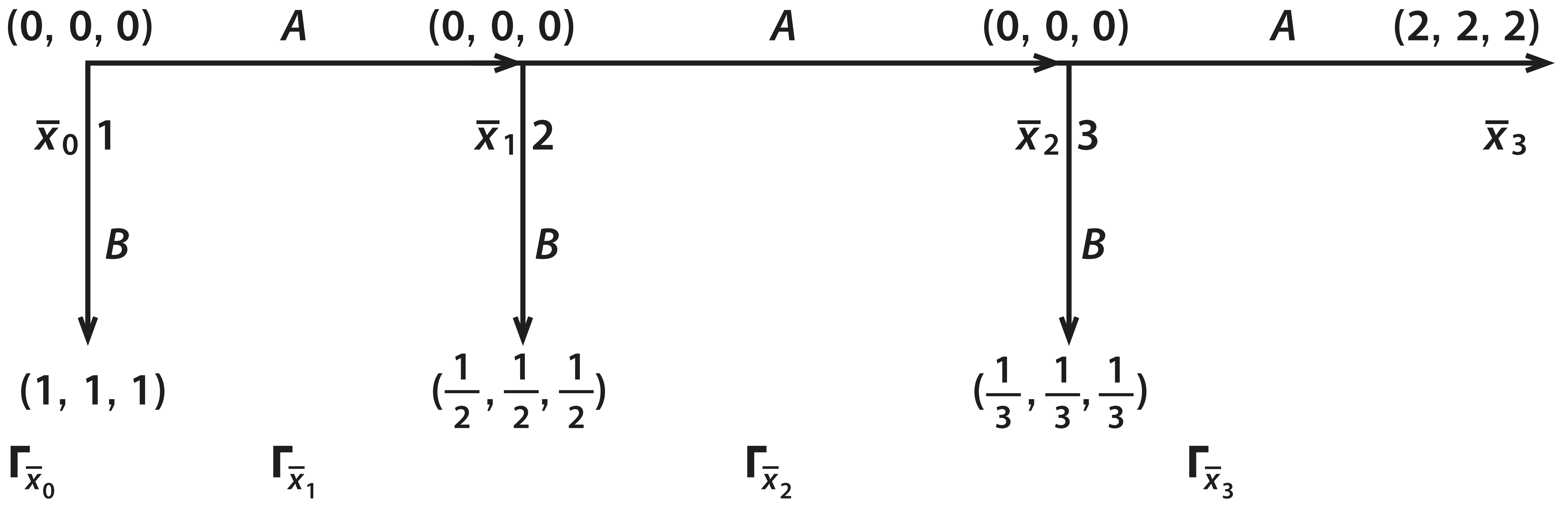
# Cooperative Multistage Games with Perfect Information

At the beginning of game, players agree to choose strategies $\bar{u}(\cdot) = (\bar{u}_1, \dots, \bar{u}_i, \dots, \bar{u}_n)$ and path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_k, \dots, \bar{x}_l)$, $x_l \in X_{n+1}$ according to the following optimization problem:

$$\sum_{i=1}^{n} \sum_{k=0}^{l} h_i(x_k) \underset{x_0, \dots, x_k, \dots, x_l}{\longrightarrow} \max.$$

**Definition.**
The path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_k, \dots, \bar{x}_l)$ is called cooperative trajectory and strategy profile $\bar{u}(\cdot) = (\bar{u}_1, \dots, \bar{u}_i, \dots, \bar{u}_n)$ is called cooperative strategies.

# Cooperative Multistage Games with Perfect Information




$$\max_{x_0, \dots, x_k, \dots, x_l} \sum_{i=1}^{n} \sum_{k=0}^{l} h_i(x_k) = \max\{3, 1.5, 1, 6\}.$$

Cooperative trajectory: $\overline{x} = (x_0, \overline{x}_1, \overline{x}_2, \overline{x}_3)$.

Cooperative strategies: $\overline{u} = (A, A, A)$.

# Cooperative Multistage Games with Perfect Information

**Definition.**
Define characteristic function as a value of zero-sum game between coalitions $S \subseteq N$ and $N \setminus S$, it satisfies the conditions:

- $V(N) = \sum_{i=1}^{n} \sum_{k=0}^{l} h_i(\overline{x}_k)$, $N = \{1, \dots, n\}$,
- $V(S_1 \cup S_2) \geq V(S_1) + V(S_2)$, $V(\varnothing) = 0$ for $S_1 \subseteq N$, $S_2 \subseteq N$, $S_1 \cap S_2 = \varnothing$.

**Definition.**
Define imputation set in the game $\Gamma$:

$$C = \left\{ \xi = (\xi_1, \dots, \xi_n) : \sum_{i=1}^{n} \xi_i = V(N),\ \xi_i \geq V(\{i\}),\ i = 1, \dots, n \right\}$$

# Cooperative Multistage Games with Perfect Information

**Definition.**

Cooperative solution $M \subseteq C$ is defined as some fixed subset of imputation set.

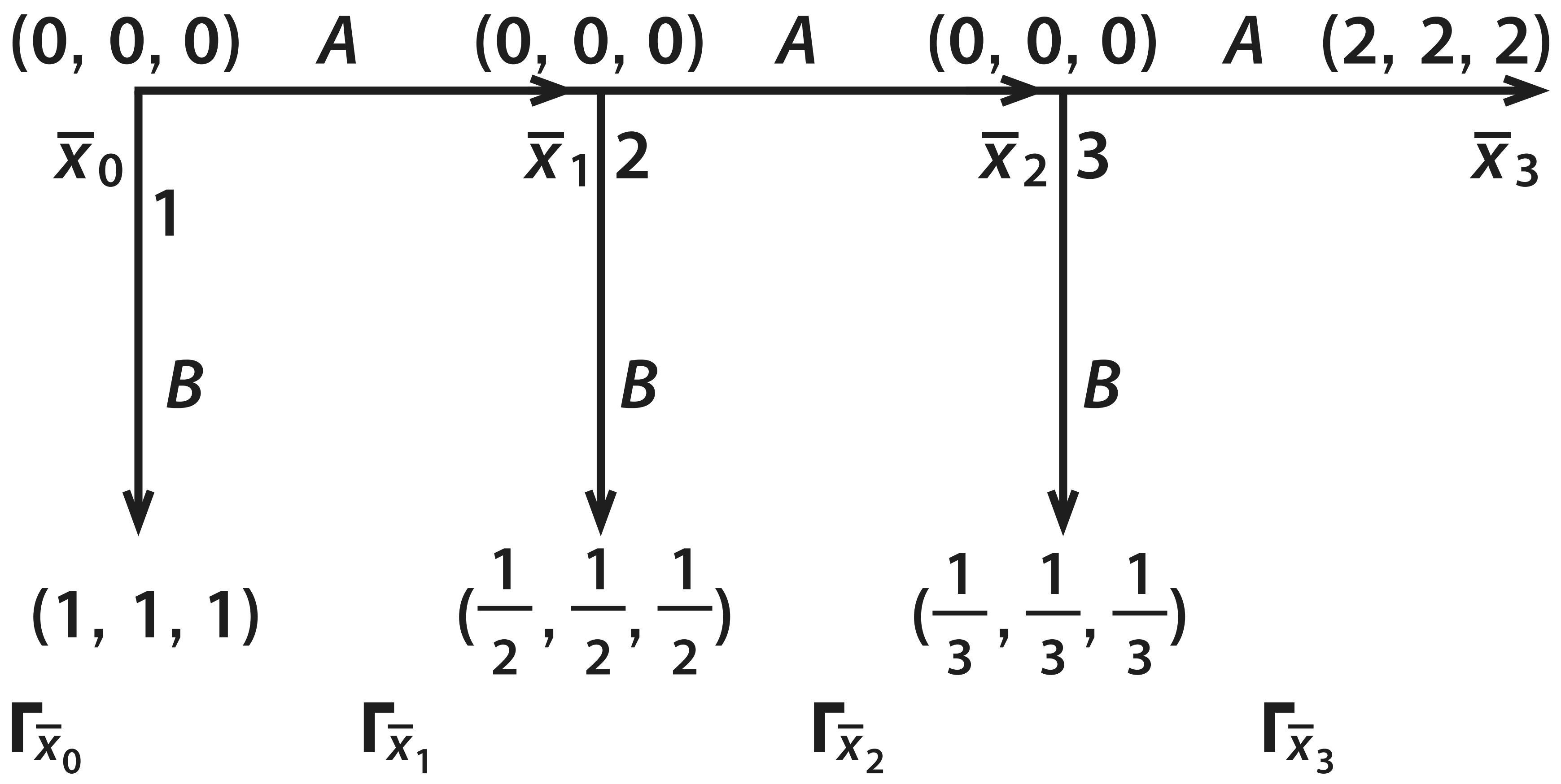
**Definition.**

Shapley Value is defined as

$$\varphi_i\,[v] = \sum_{S \subseteq N | i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!}\,[V(S) - V(S \setminus \{i\})],\ i \in N,$$

where $|S|$ is the number of players in coalition $S \subseteq N$.

# Cooperative Multistage Games with Perfect Information




Characteristic function:

$V(\{1, 2, 3\}) = 6,$

$V(\{1, 2\}) = 2,$

$V(\{1, 3\}) = 2,$

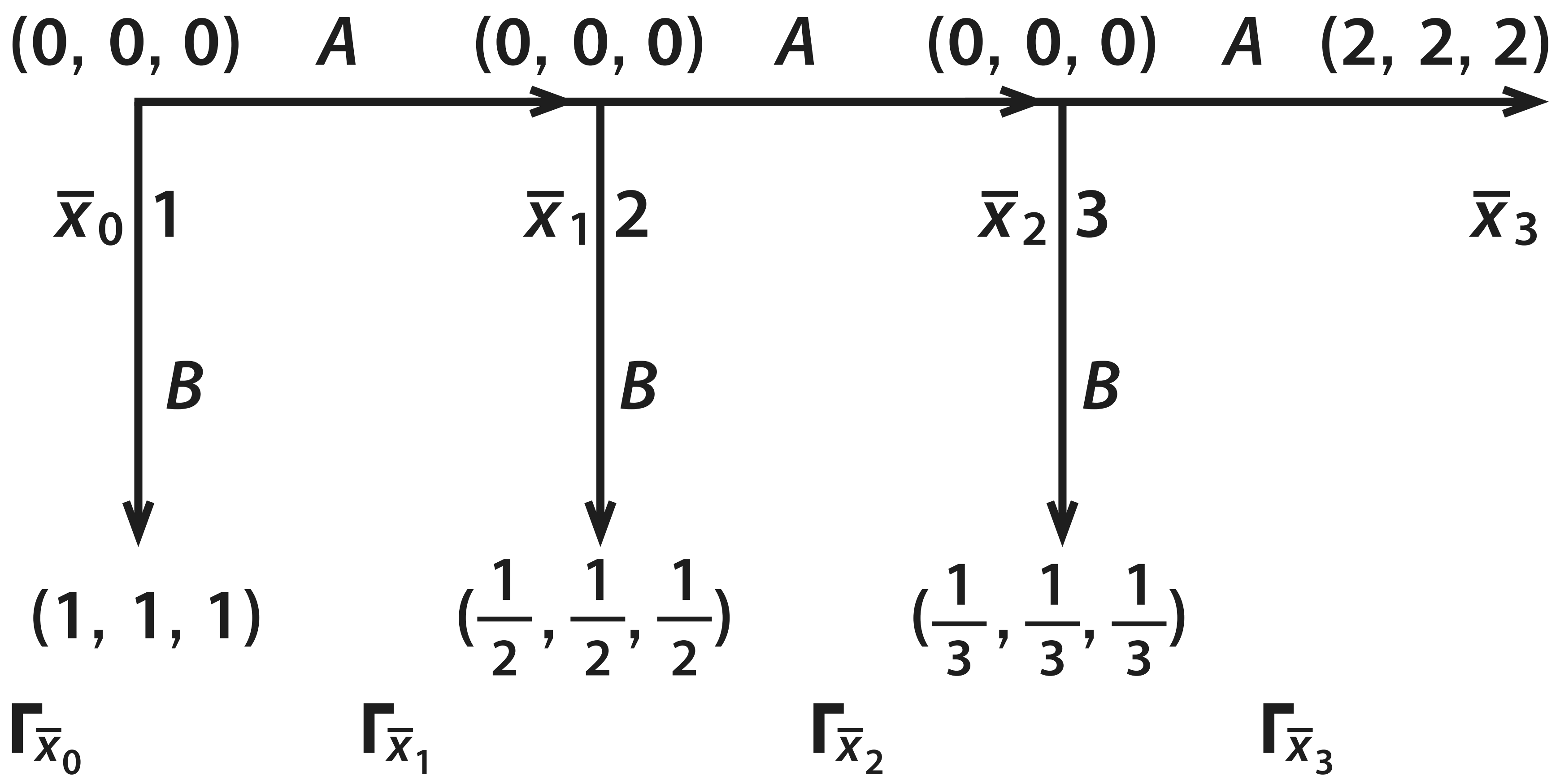
$V(\{2, 3\}) = 2,$

$V(\{1\}) = 1,$

$V(\{2\}) = \frac{1}{2},$

$V(\{3\}) = \frac{1}{2}.$

# Cooperative Multistage Games with Perfect Information




Imputation set:

$$C = \left\{ \xi = (\xi_1, \xi_2, \xi_3) : \xi_1 \geq 1, \xi_2 \geq \frac{1}{2}, \xi_3 \geq \frac{1}{2}, \xi_1 + \xi_2 + \xi_3 = 6 \right\}$$

Shapley Value:

$$Sh = \left(\frac{26}{12}, \frac{23}{12}, \frac{23}{12}\right).$$

# References

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# Cooperative subgame

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# Cooperative Multistage Games with Perfect Information

Consider subgame $\Gamma_{\overline{x}_k}$ starting in the position $\overline{x}_k$, $k = 0, \dots, l$, then solution of optimization problem

$$\sum_{i=1}^{n} \sum_{j=k}^{l} h_i(x_j) \longrightarrow \max_{x_k, \dots, x_j, \dots, x_l}$$

is the truncation of cooperative trajectory $\overline{x} = (\overline{x}_0, \dots, \overline{x}_k, \dots, \overline{x}_l)$ of game $\mathbf{\Gamma}$.

How can we determine imputations in the subgame?

# Cooperative Multistage Games with Perfect Information

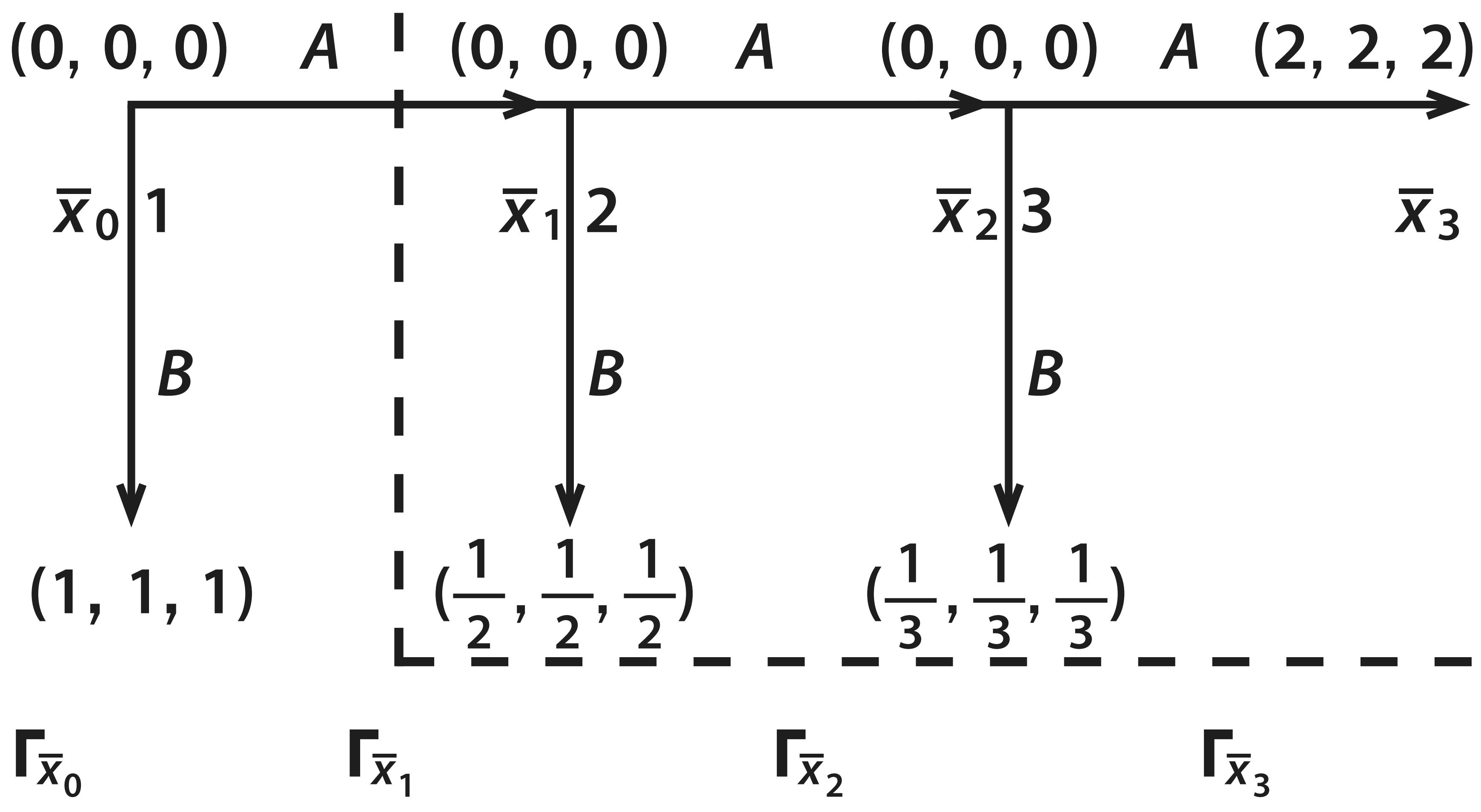
**Definition.**
Characteristic function $V(S; \overline{x}_k)$, $S \subseteq N$ in the subgame $\Gamma_{\overline{x}_k}$, $k = 1, \dots, l$ is defined as a value of zero-sum game between coalitions $S \subseteq N$ and $N \setminus S$.

**Definition.**
Imputation set in the subgame $\Gamma_{\overline{x}_k}$ , $k = 0, \dots, l$ is defined as

$$C(\overline{x}_k) = \left\{ \xi = (\xi_1, \dots, \xi_n) : \sum_{i=1}^{n} \xi_i = V(N; \overline{x}_k), \, \xi_i \geq V(\{i\}; \overline{x}_k), \, i = 1, \dots, n \right\}.$$

**Definition.**
$M(\overline{x}_k) \subseteq C(\overline{x}_k)$ is the cooperative solution in the subgame $\Gamma_{\overline{x}_k}$.

# Cooperative Multistage Games with Perfect Information




Characteristic function in the subgame $\Gamma_{\bar{x}_1}$:

$V(\{1, 2, 3\}; \bar{x}_1) = 6,$

$V(\{1, 2\}; \bar{x}_1) = 1,$

$V(\{1, 3\}; \bar{x}_1) = 1,$

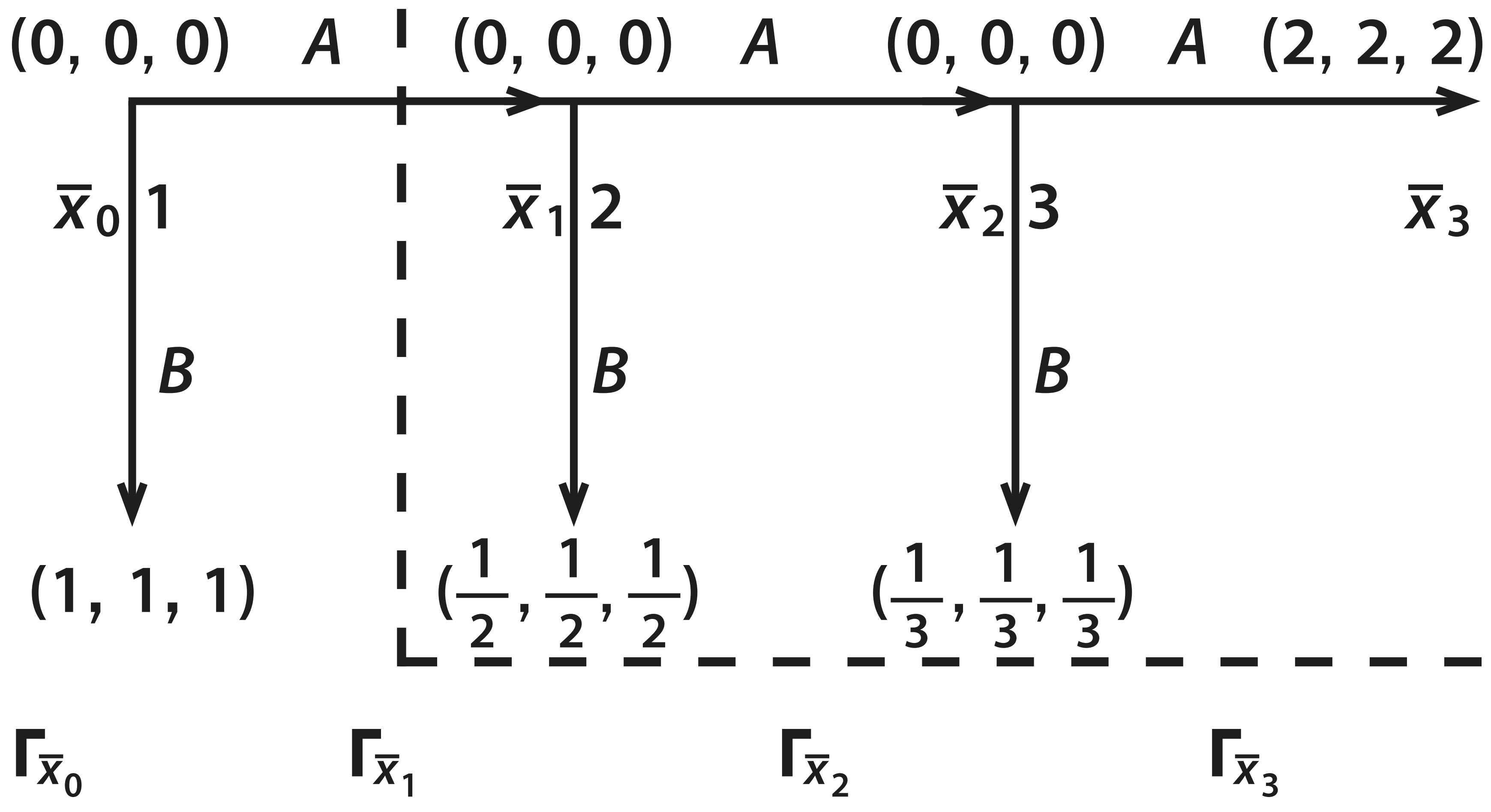
$V(\{2, 3\}; \bar{x}_1) = 4,$

$V(\{1\}; \bar{x}_1) = \frac{1}{3},$

$V(\{2\}; \bar{x}_1) = \frac{1}{2},$

$V(\{3\}; \bar{x}_1) = \frac{1}{2}.$

# Cooperative Multistage Games with Perfect Information




Imputation set in the subgame $\Gamma_{\bar{x}_1}$:

$$C(\bar{x}_1) = \left\{ \xi = (\xi_1, \xi_2, \xi_3): \right.$$

$$\xi_1 \geq \frac{1}{3},\ \xi_2 \geq \frac{1}{2},\ \xi_3 \geq \frac{1}{2},$$

$$\left. \xi_1 + \xi_2 + \xi_3 = 6 \right\}.$$

Shapley Value in the subgame $\Gamma_{\bar{x}_1}$:

$$Sh\,(\bar{x}_1) = \left(\frac{34}{36}, \frac{91}{36}, \frac{91}{36}\right).$$

# References

1. Basar, T. & Zaccour, G. (2018). Handbook of Dynamic Game Theory. New York: Springer-Verlag.
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# Imputation Distribution Procedure and Time-consistency

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# Imputation Distribution Procedure

**Definition.**
Suppose that $\xi = \{\xi_1, \dots, \xi_i, \dots, \xi_n\} \in M(x_0)$, any matrix $\{\beta_i^k\}$ such that

$$\xi_1 = \sum_{k=0}^{l-1} \beta_1^k,$$

$$\dots$$

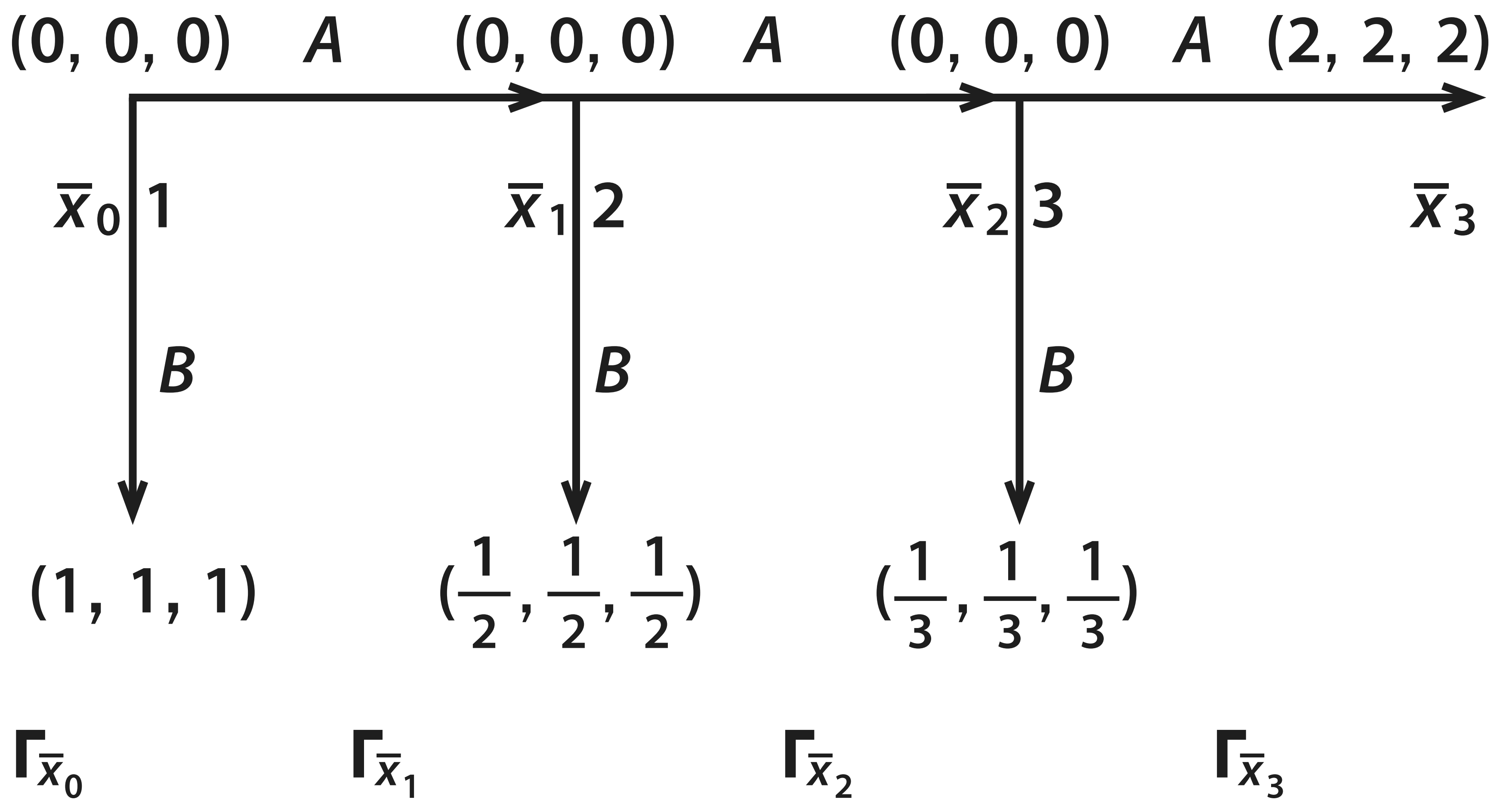
$$\xi_i = \sum_{k=0}^{l-1} \beta_i^k,$$

$$\dots$$

$$\xi_n = \sum_{k=0}^{l-1} \beta_n^k,$$

is called the imputation distribution procedure (IDP).

# Imputation Distribution Procedure




Shapley Value:

$Sh(x_0) = \left(\frac{26}{12}, \frac{23}{12}, \frac{23}{12}\right).$

Imputation distribution procedure:

$$\beta^0 = (\beta_1^0; \beta_2^0; \beta_3^0) = \left(\frac{5}{12}, \frac{8}{12}, \frac{3}{12}\right),$$

$$\beta^1 = (\beta_1^1; \beta_2^1; \beta_3^1) = \left(\frac{13}{12}, \frac{5}{12}, \frac{11}{12}\right),$$

$$\beta^2 = (\beta_1^2; \beta_2^2; \beta_3^2) = \left(\frac{8}{12}, \frac{10}{12}, \frac{9}{12}\right).$$

# Time-consistency

**“General George Washington Resigning his Commission”,**
John Trumbull, 1817

Time-consistency property of cooperative solution guarantees that the reconsideration of this solution at any given instant will result in the same solution.

# Time-consistency

**Definition.**
Cooperative solution $M(x_0)$ is called time-consistent, if for each $\xi \in M(x_0)$ corresponding IDP $\beta$ satisfies the conditions:

$$\xi(\bar{x}_1) = \xi - \beta^0 \in M(\bar{x}_1),$$

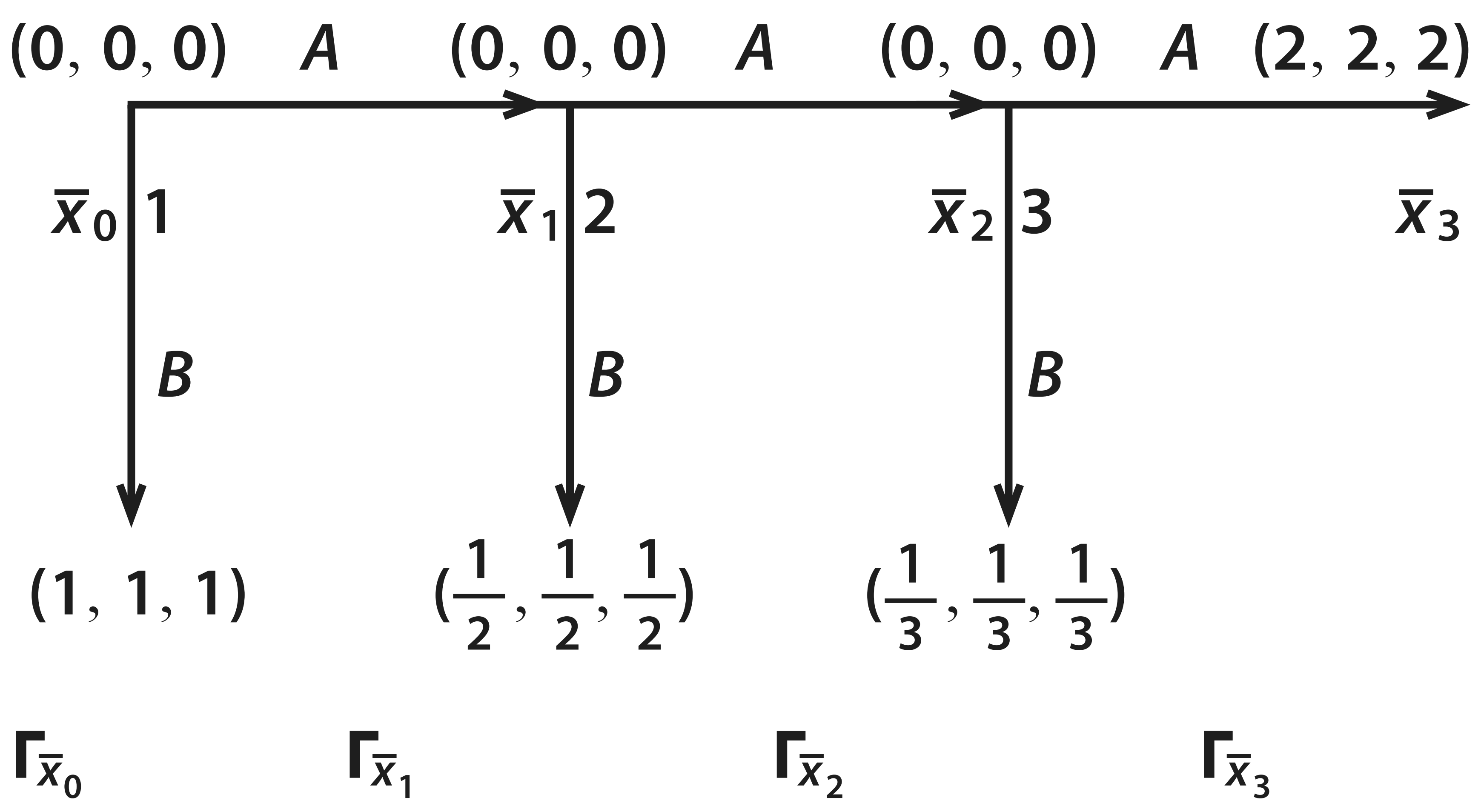
$$\dots$$

$$\xi(\bar{x}_k) = \xi - \sum_{j=0}^{k-1} \beta^j \in M(\bar{x}_k),$$

$$\dots$$

$$\xi(\bar{x}_l) = \xi - \sum_{j=0}^{l-1} \beta^j \in M(\bar{x}_l).$$

# Time-consistency




Time-consistent solution $Sh(x_0)$ and its IDP:

$$Sh(x_0) - Sh(\bar{x}_1) = \beta^0 = \left(\frac{44}{36}, -\frac{22}{36}, -\frac{22}{36}\right),$$

$$Sh(\bar{x}_1) - Sh(\bar{x}_2) = \beta^1 = \left(-\frac{8}{36}, \frac{49}{36}, -\frac{41}{36}\right),$$

$$Sh(\bar{x}_2) - Sh(\bar{x}_3) = \beta^2 = \left(-\frac{15}{18}, -\frac{15}{18}, \frac{30}{18}\right).$$

# Strong Time-consistency

**“The Congress of Paris”,**
Edouard Louis Dubufe, 1856

**Definition.**
Cooperative solution $M(x_0)$ is called strong time-consistent, if for every $\xi \in M(x_0)$ there exists IDP $\beta$ such that:

$$\sum_{j=0}^{k-1} \beta^j \oplus M(\bar{x}_k) \subseteq M(x_0),\ k = 1, \dots, l,$$

where $a \oplus A = \{a + a' : a' \in A,\ \ a \in R^1, A \subset R^1\}$.

# References


1. Basar, T. & Zaccour, G. (2018). Handbook of Dynamic Game Theory. New York: Springer-Verlag.
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# Multistage Games with Non-transferable Payoffs

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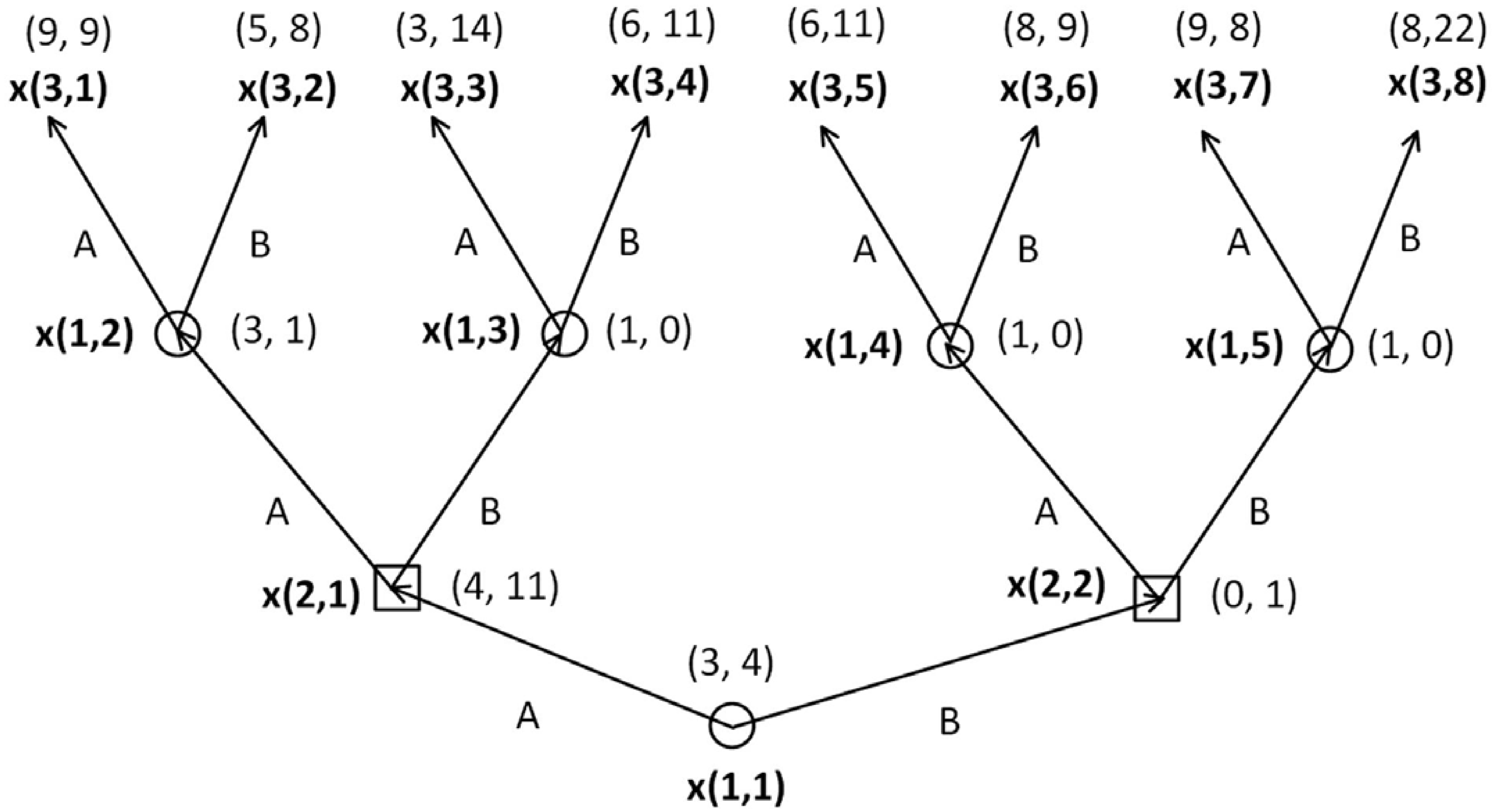
PhD

# Non-transferable Payoffs

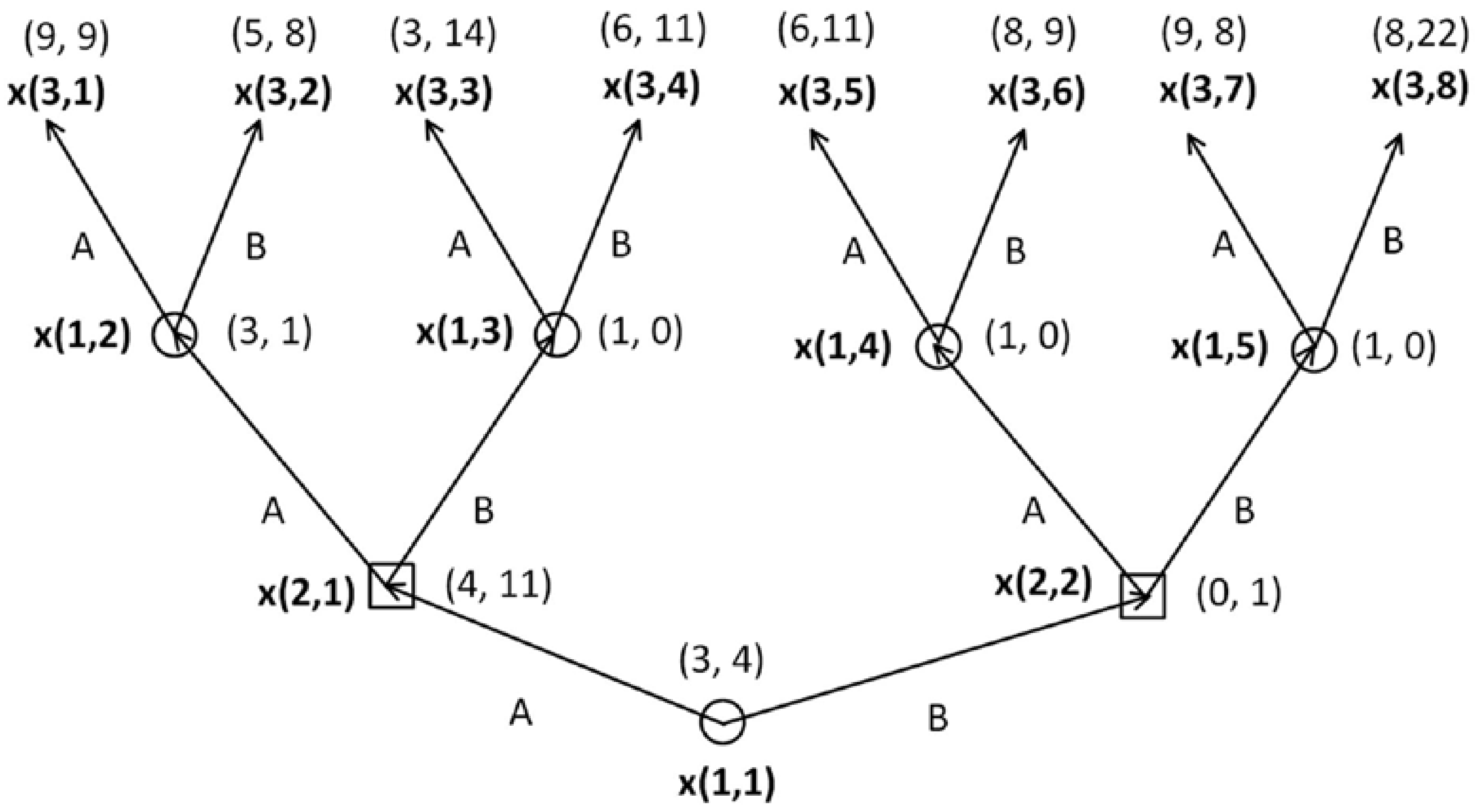
**“The Conspiracy of Claudius Civilis”,**
Rembrandt Harmenszoon van Rijn, 1659

In this game model players make an agreement on strategies that they will use in the game.

# Non-transferable Payoffs



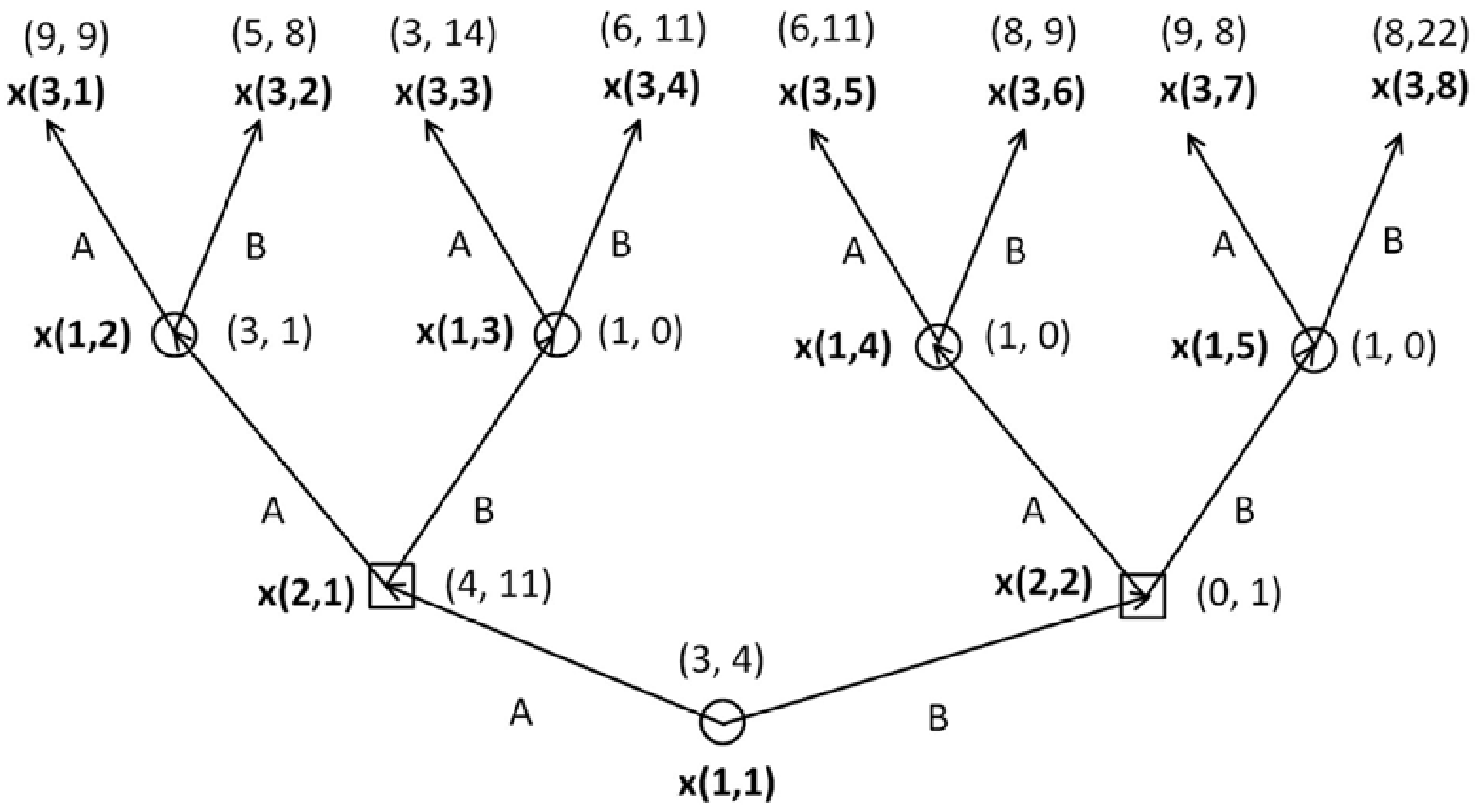
# Non-transferable Payoffs

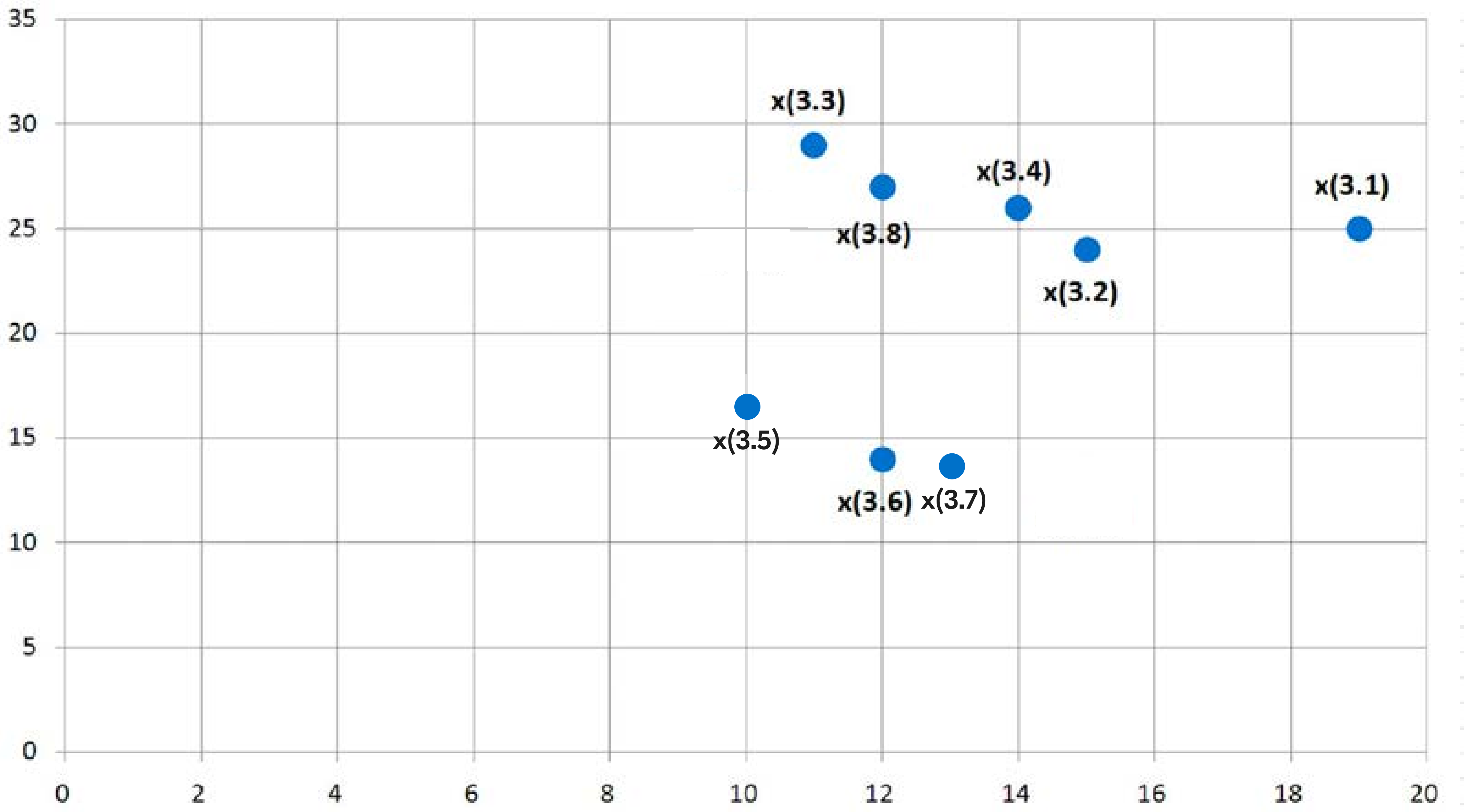



Payoffs on paths:

- $(x_{(1.1)}, x_{(2.1)}, x_{(1.2)}, x_{(3.1)}) \longrightarrow (H_1, H_2) = (19, 25)$,
- $(x_{(1.1)}, x_{(2.1)}, x_{(1.2)}, x_{(3.2)}) \longrightarrow (H_1, H_2) = (15, 24)$,
- $(x_{(1.1)}, x_{(2.1)}, x_{(1.3)}, x_{(3.3)}) \longrightarrow (H_1, H_2) = (11, 29)$,
- $(x_{(1.1)}, x_{(2.1)}, x_{(1.3)}, x_{(3.4)}) \longrightarrow (H_1, H_2) = (14, 26)$,
- $(x_{(1.1)}, x_{(2.2)}, x_{(1.4)}, x_{(3.5)}) \longrightarrow (H_1, H_2) = (10, 16)$,
- $(x_{(1.1)}, x_{(2.2)}, x_{(1.4)}, x_{(3.6)}) \longrightarrow (H_1, H_2) = (12, 14)$,
- $(x_{(1.1)}, x_{(2.2)}, x_{(1.5)}, x_{(3.7)}) \longrightarrow (H_1, H_2) = (13, 13)$,
- $(x_{(1.1)}, x_{(2.2)}, x_{(1.5)}, x_{(3.8)}) \longrightarrow (H_1, H_2) = (12, 27)$.

# Non-transferable Payoffs



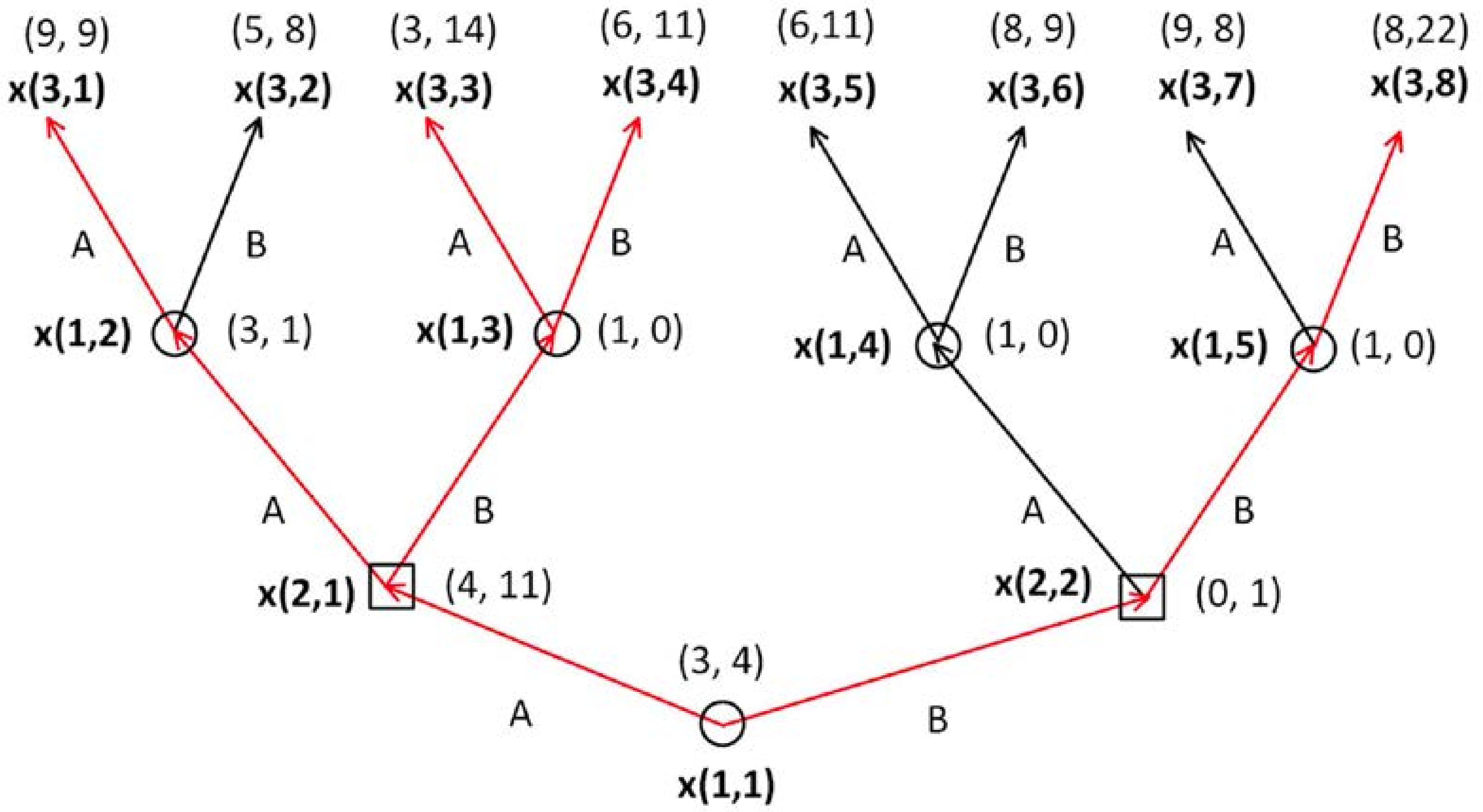

Set of all possible payoffs

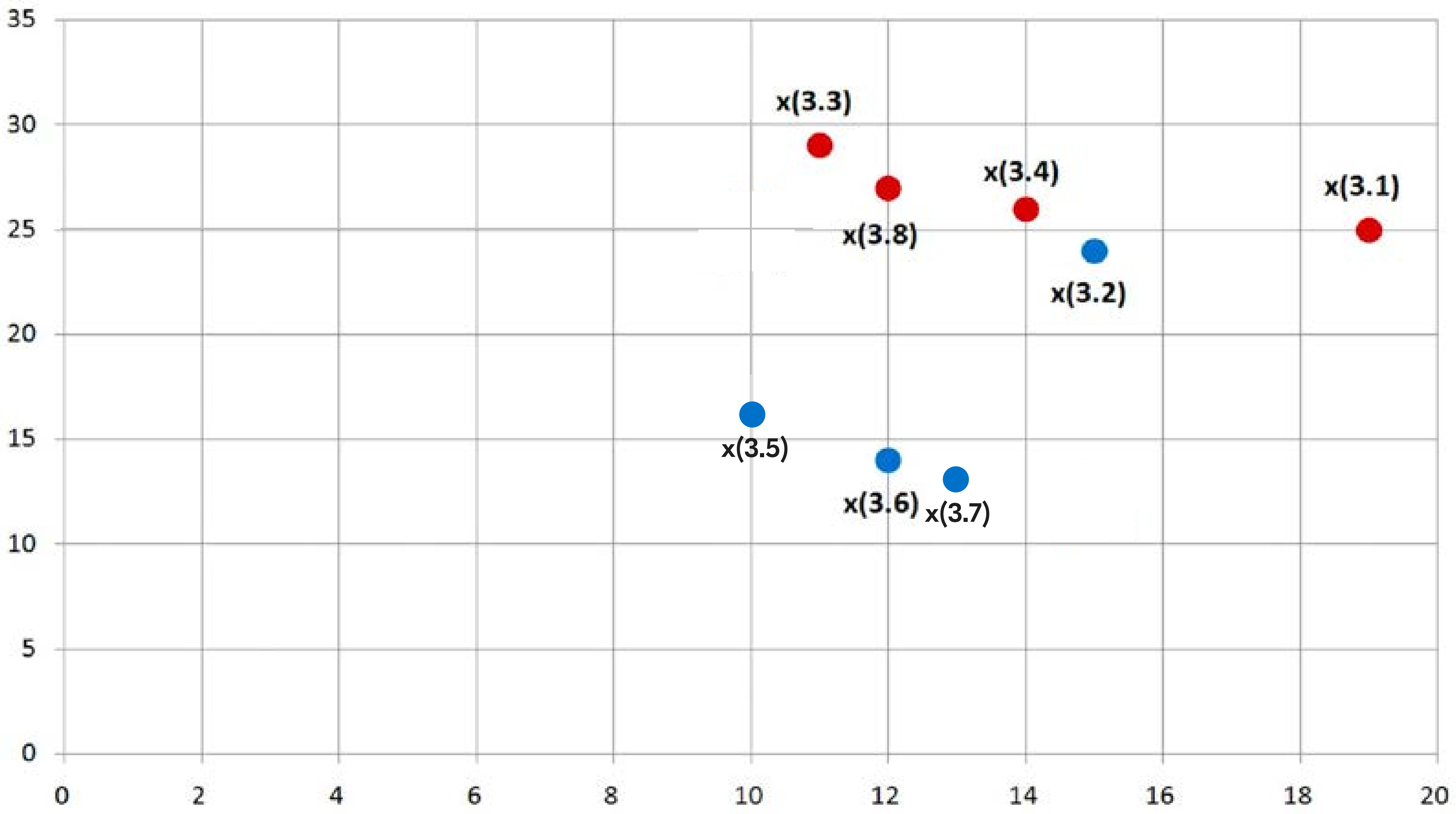
## Pareto-optimal Solutions

**Definition.**
Strategy profile $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ is called Pareto-optimal, if there exists no strategy profile $u$, for which the following inequalities hold:

- $K_i(u) \geq K_i(\overline{u})$ for all $i \in N$,
- $K_{i_0}(u) > K_{i_0}(\overline{u})$ for at least one $i_0 \in N$.

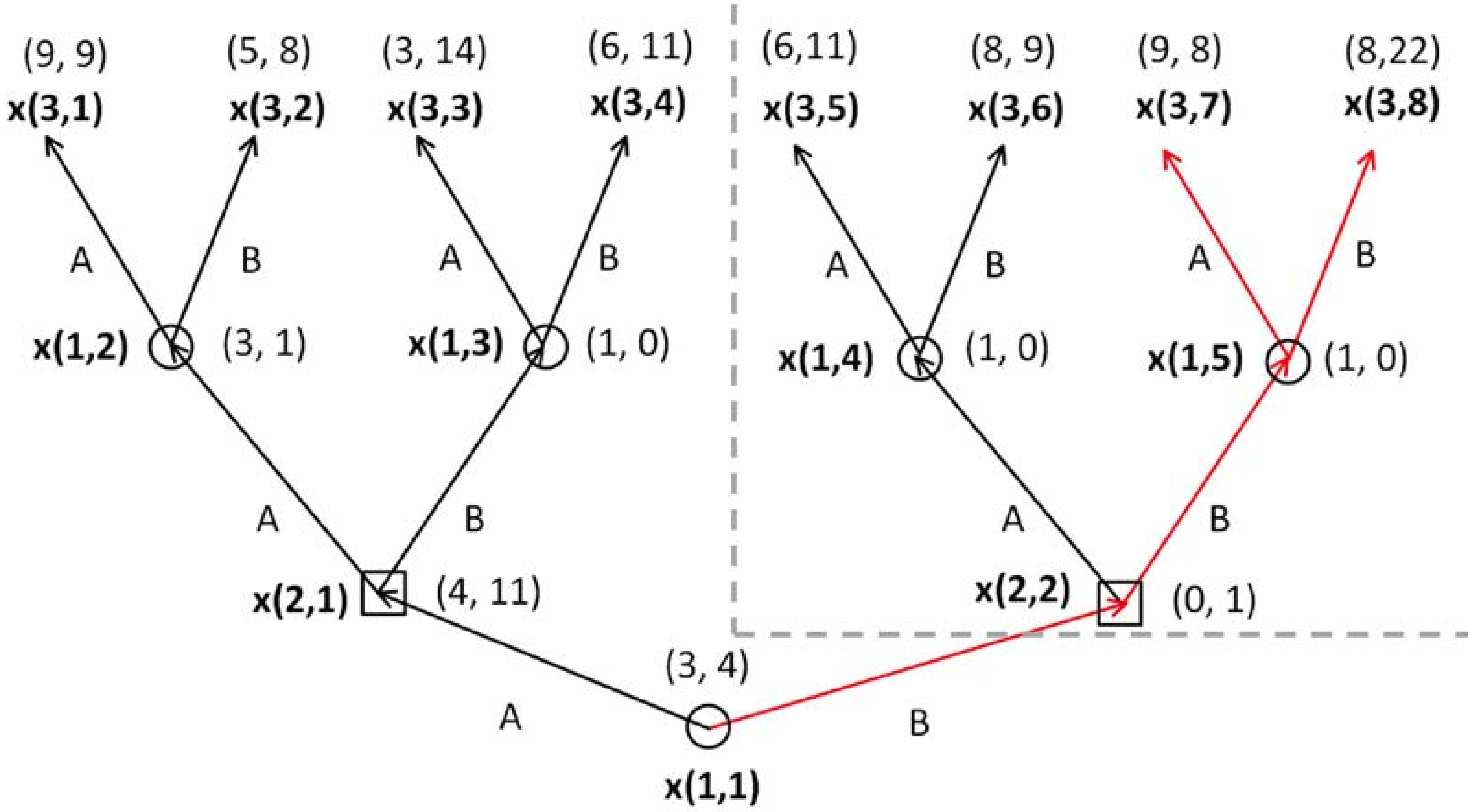
# Pareto-optimal Solutions




Set of Pareto-optimal solutions

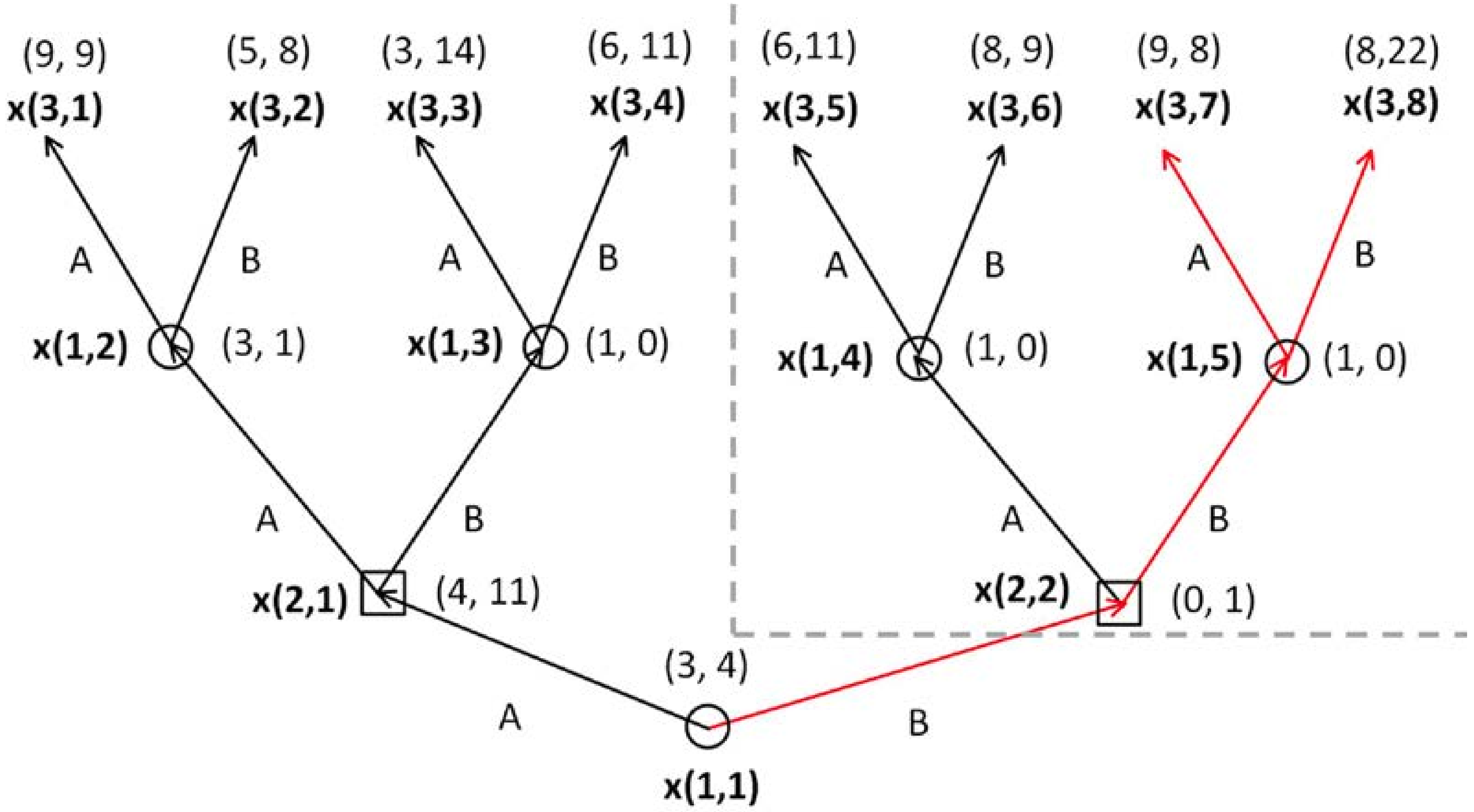
# Pareto-optimal Solutions

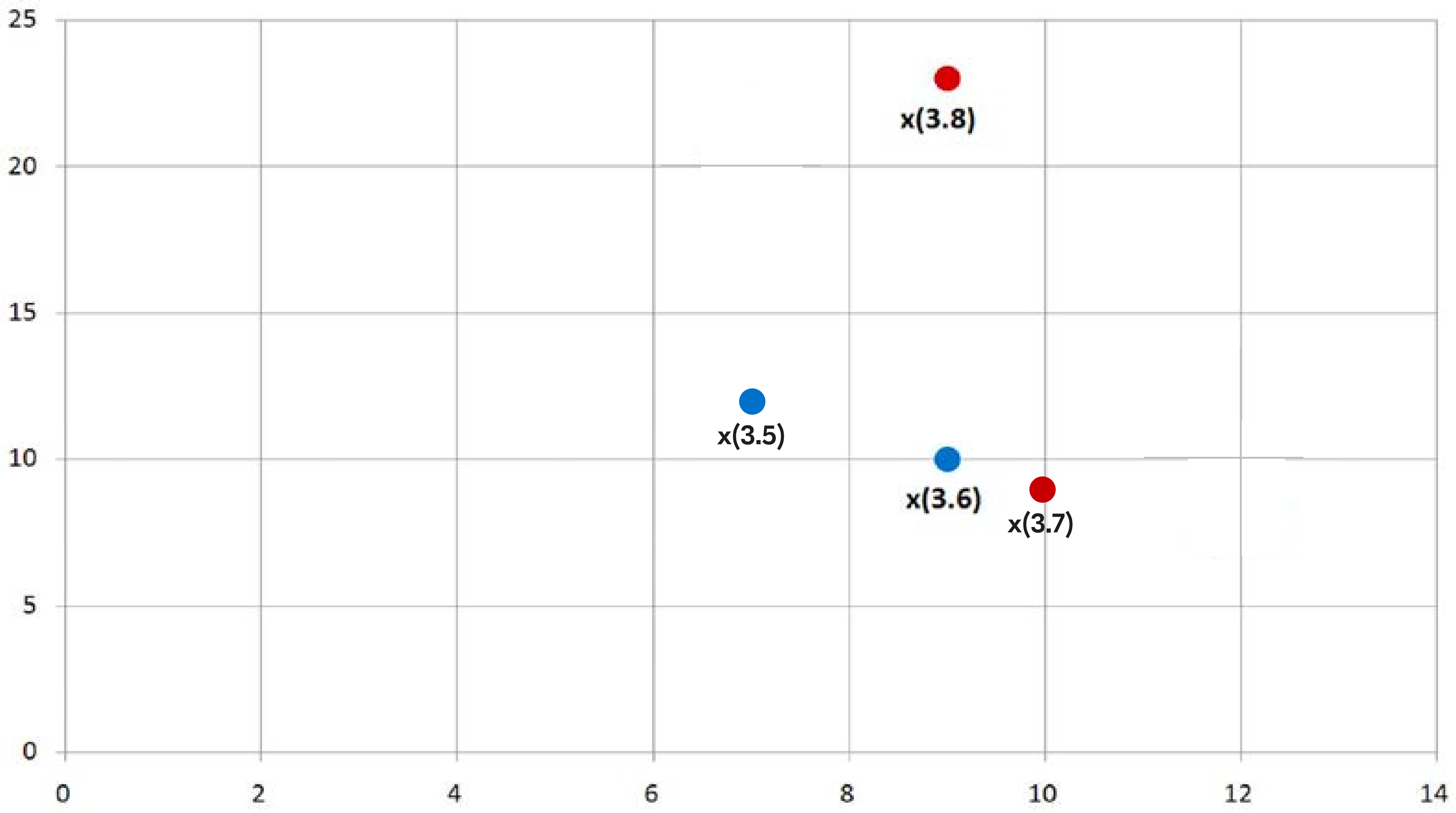



Consider subgame $\Gamma_{x_{(2.2)}}$ along the selected Pareto-optimal path $(x_{(1.1)}, x_{(2.2)}, x_{(1.5)}, x_{(3.8)})$, where the payoffs are

- $(x_{(2.2)}, x_{(1.4)}, x_{(3.5)}) \longrightarrow (H_1, H_2) = (7, 12)$,
- $(x_{(2.2)}, x_{(1.4)}, x_{(3.6)}) \longrightarrow (H_1, H_2) = (9, 10)$,
- $(x_{(2.2)}, x_{(1.5)}, x_{(3.7)}) \longrightarrow (H_1, H_2) = (10, 9)$,
- $(x_{(2.2)}, x_{(1.5)}, x_{(3.8)}) \longrightarrow (H_1, H_2) = (9, 23)$.

# Pareto-optimal Solutions




In the subgame $\Gamma_{x(2.1)}$ set of Pareto-optimal solutions contains elements not belonging to the corresponding solution in the initial game, it shows that the Pareto-optimal set is not strong time-consistent!

SPbSU

# Nash Bargaining Solution

**Definition.**

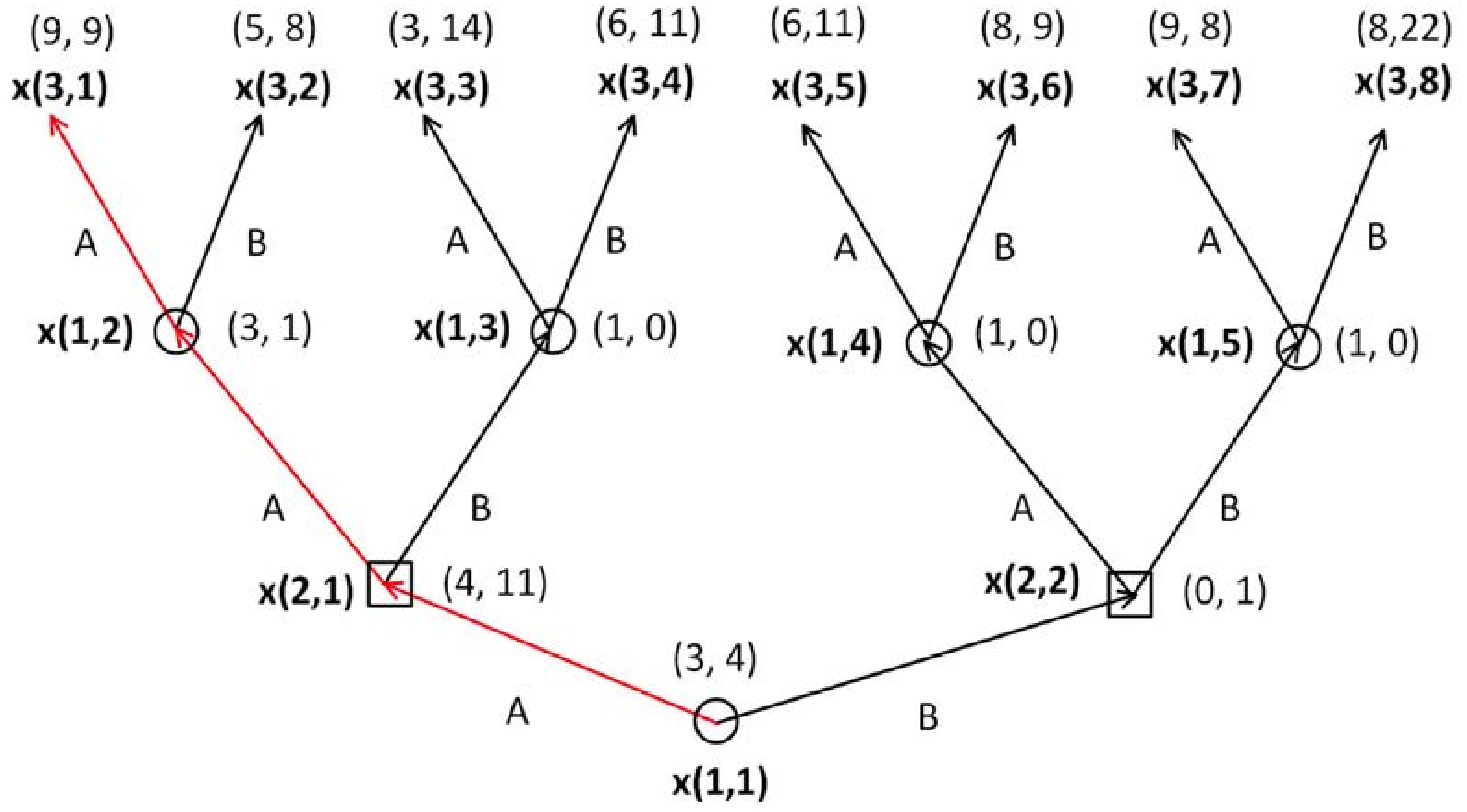
Nash bargaining solution we define as a strategy profile for which maximum of the following functional is achieved:

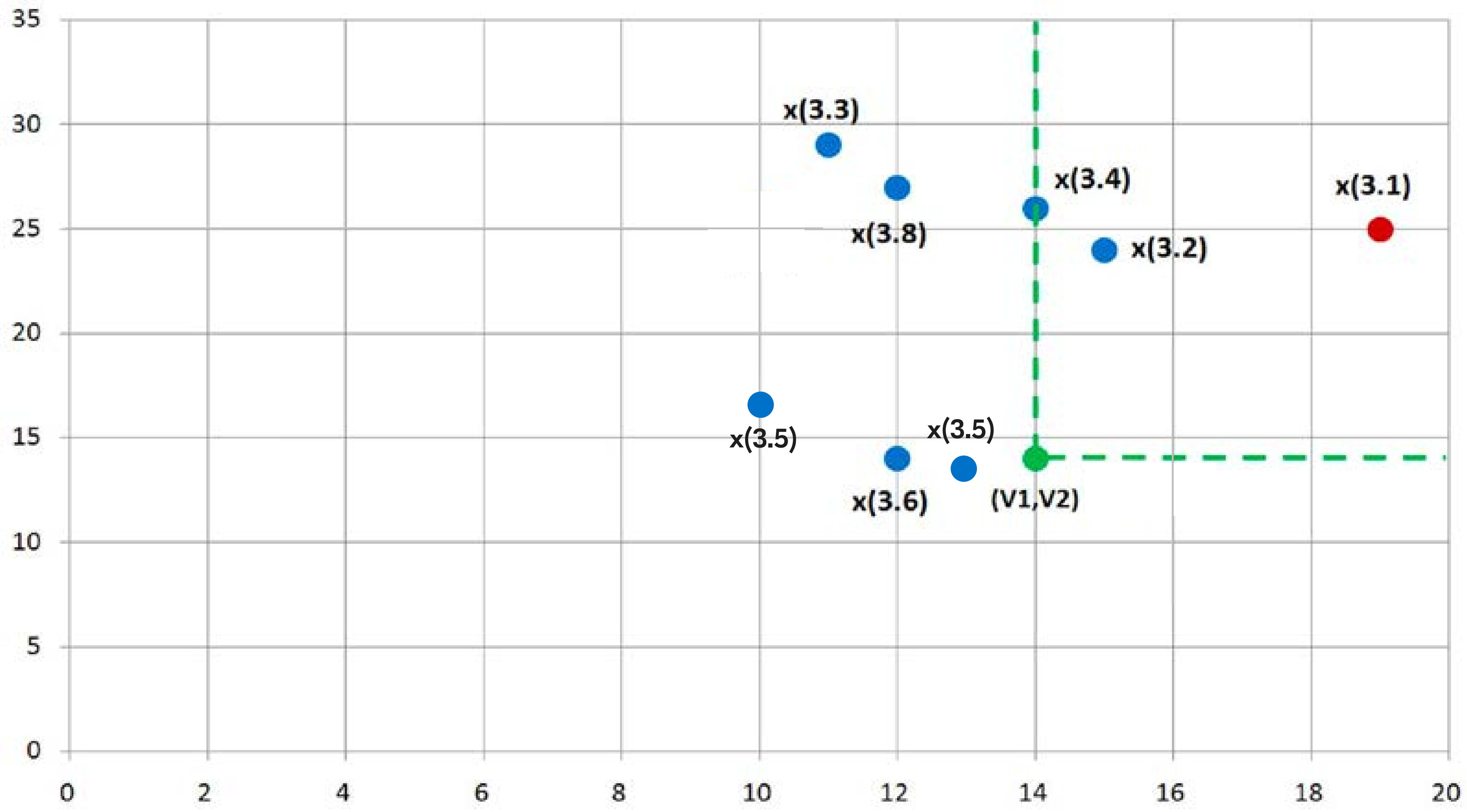
$$\max_{\substack{u_1,\dots,u_n \\ K_i(u_1,\dots,u_n)\geq V^i(x_0)}} \prod_{i=1}^{n}\left[K_i(u_1,\dots,u_n)-V^i(x_0)\right]=\prod_{i=1}^{n}\left[K_i(\overline{u}_1,\dots,\overline{u}_n)-V^i(x_0)\right]$$

**Definition.**

Path $\overline{x}=(x_0,\overline{x}_1,\dots,\overline{x}_l)$, $\overline{x}_l\in X_{n+1}$ corresponding to the strategies $(\overline{u}_1,\dots,\overline{u}_n)$ we will call optimal trajectory.

# Nash Bargaining Solution



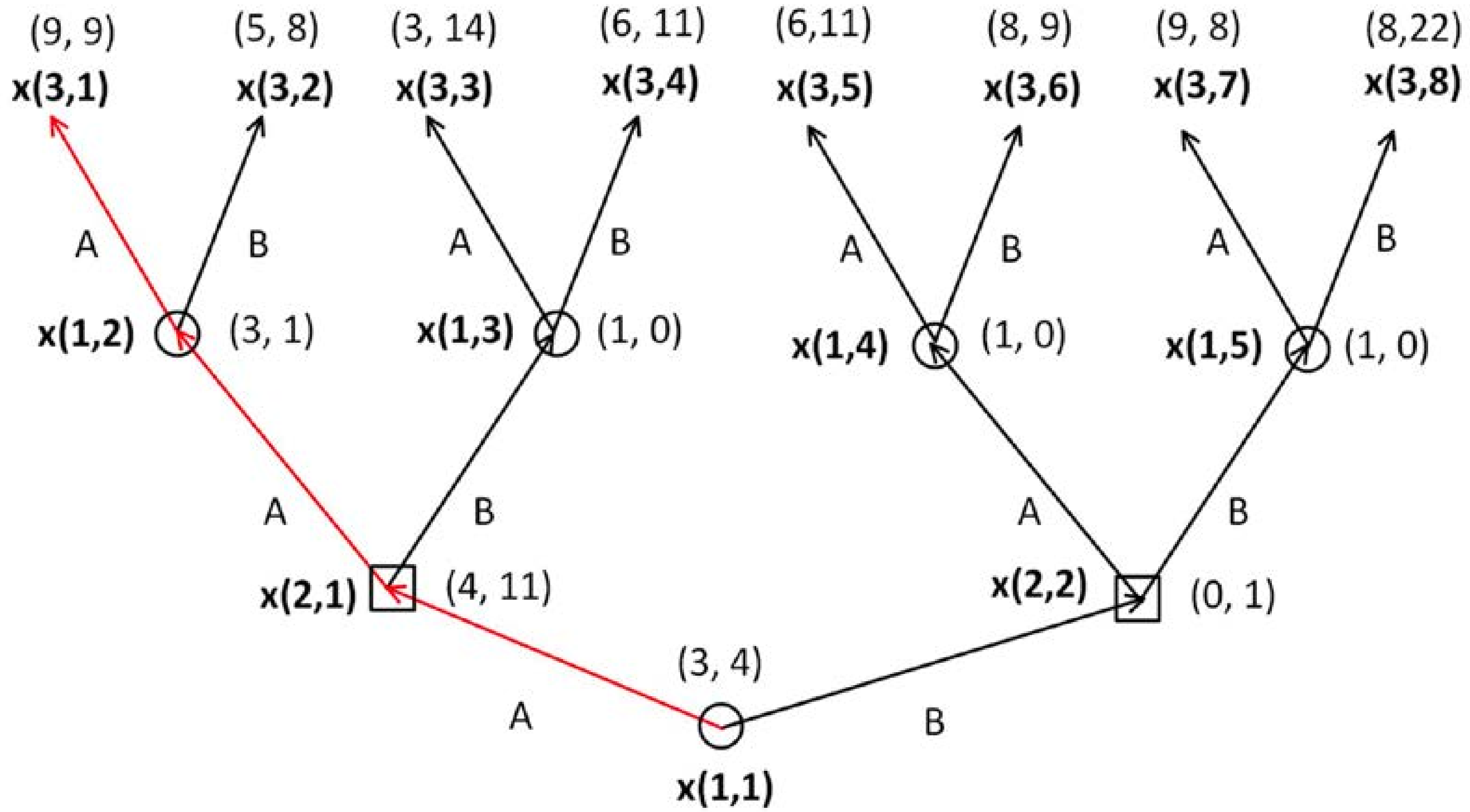




Status quo: $V^1(x_{(1.1)}) = 14$, $V^2(x_{(1.1)}) = 14$.

Nash bargaining solution in the game **Γ**:

Path $(x_{(1.1)}, x_{(2.1)}, x_{(1.2)}, x_{(3.1)}) \longrightarrow (H_1(x_{(3.1)}), H_2(x_{(3.1)})) = (19, 25)$.

# Nash Bargaining Solution

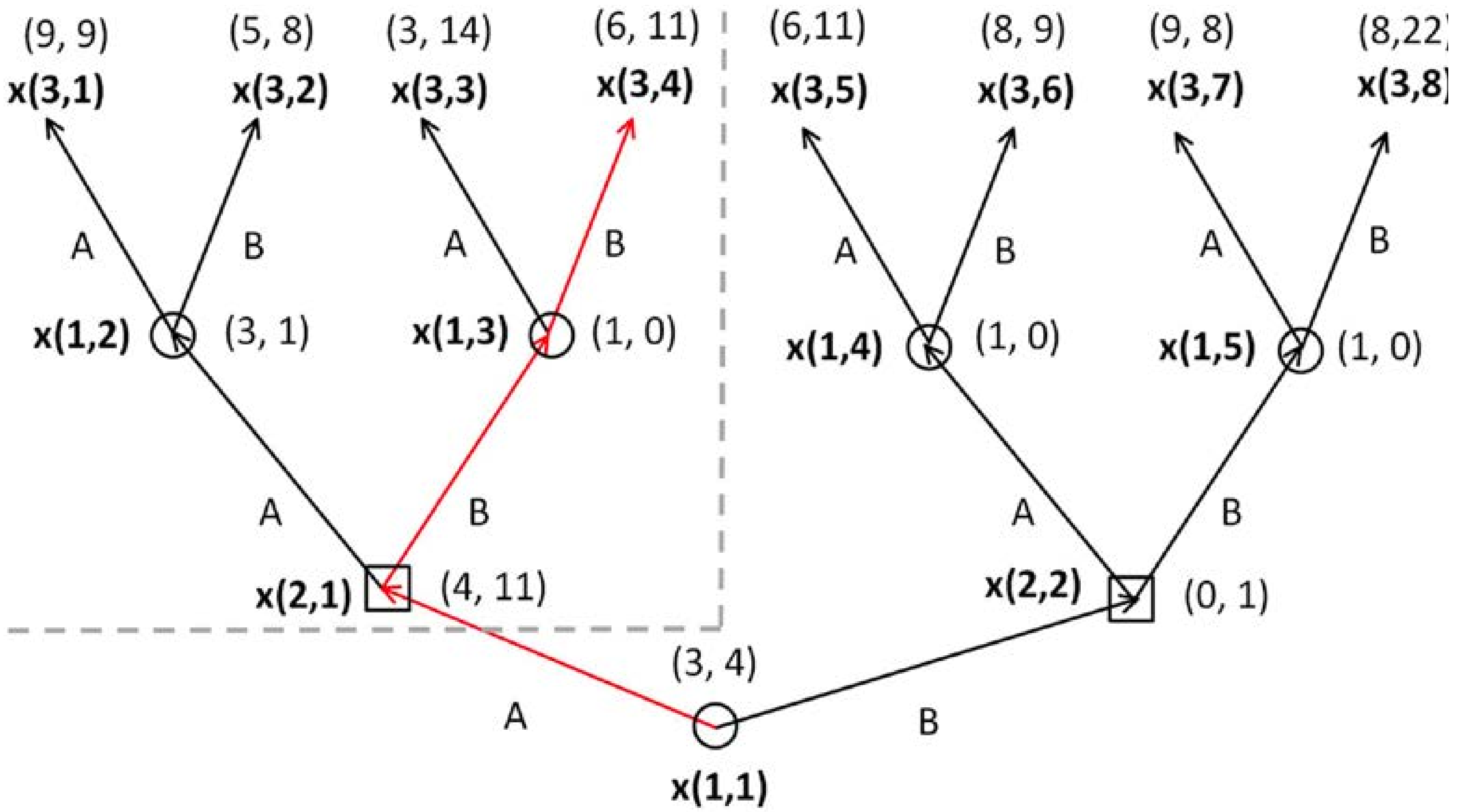


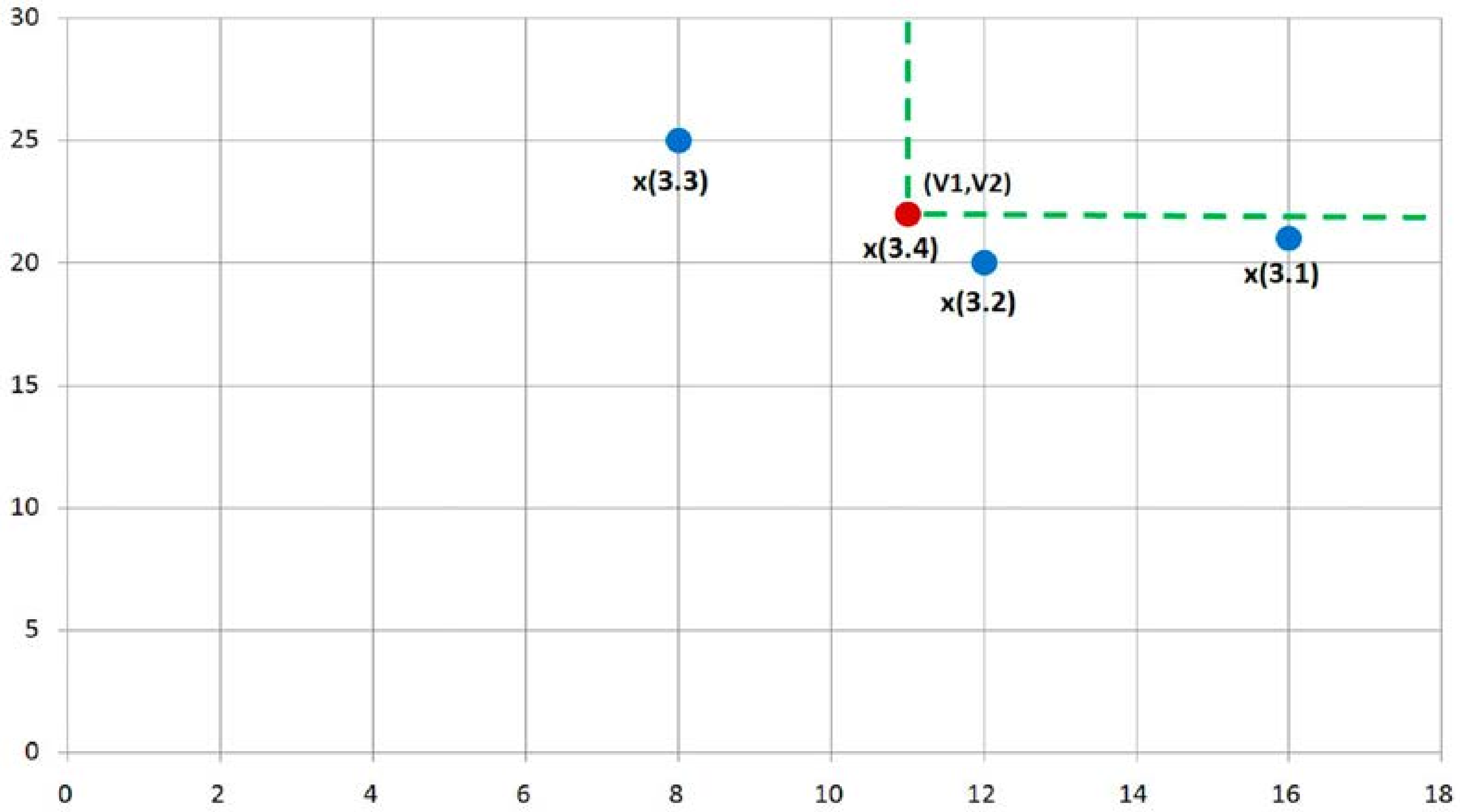

Consider subgame $\Gamma_{x_{(2.1)}}$, set of possible outcomes:

- $(x_{(2.1)}, x_{(1.2)}, x_{(3.1)}) \longrightarrow (16, 21)$,
- $(x_{(2.1)}, x_{(1.2)}, x_{(3.2)}) \longrightarrow (12, 20)$,
- $(x_{(2.1)}, x_{(1.3)}, x_{(3.3)}) \longrightarrow (8, 25)$,
- $(x_{(2.1)}, x_{(1.3)}, x_{(3.4)}) \longrightarrow (11, 22)$.

Status quo: $V^1(x_{(2.1)}) = 11$,
$V^2(x_{(2.1)}) = 22$.

# Nash Bargaining Solution

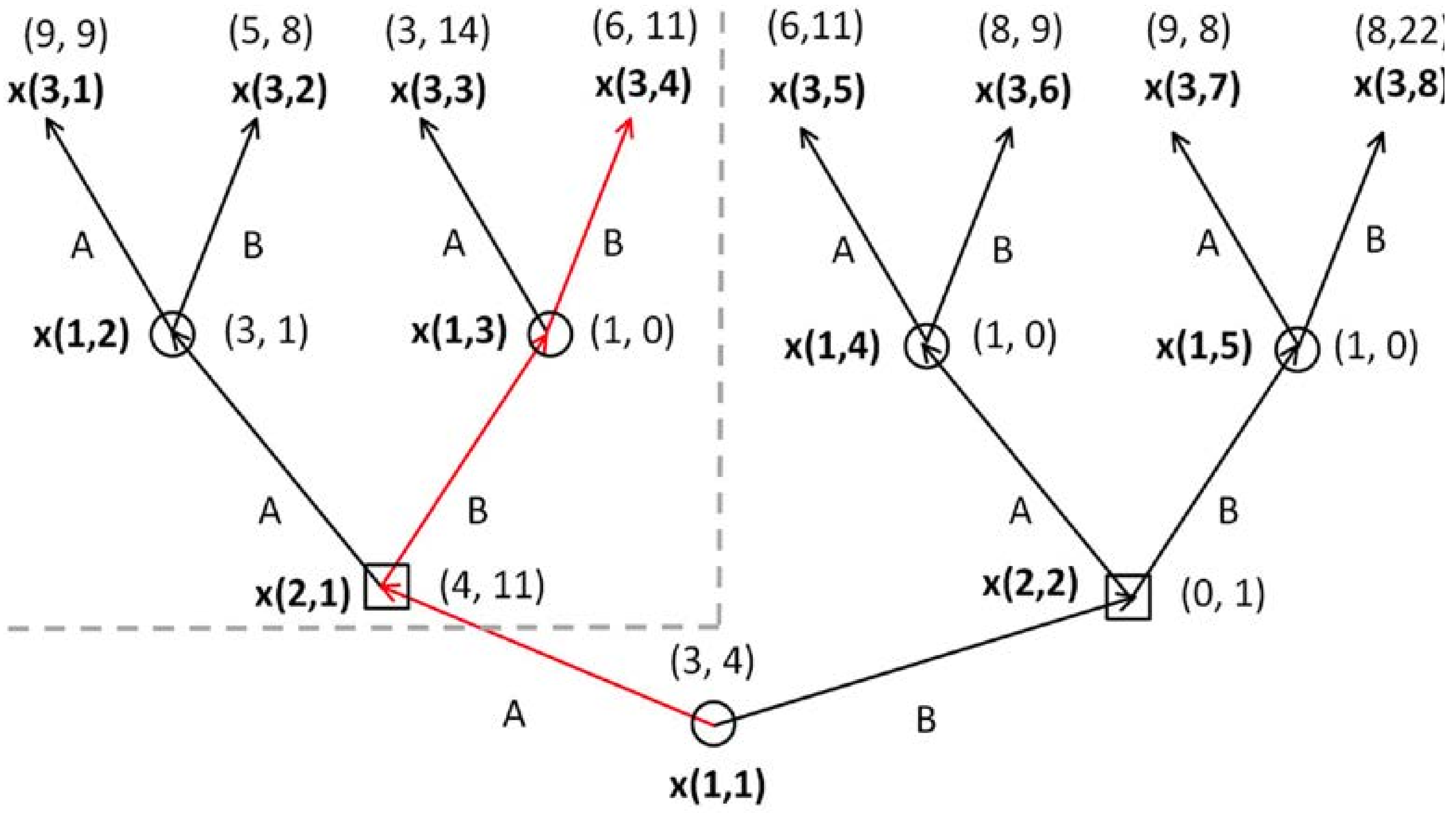


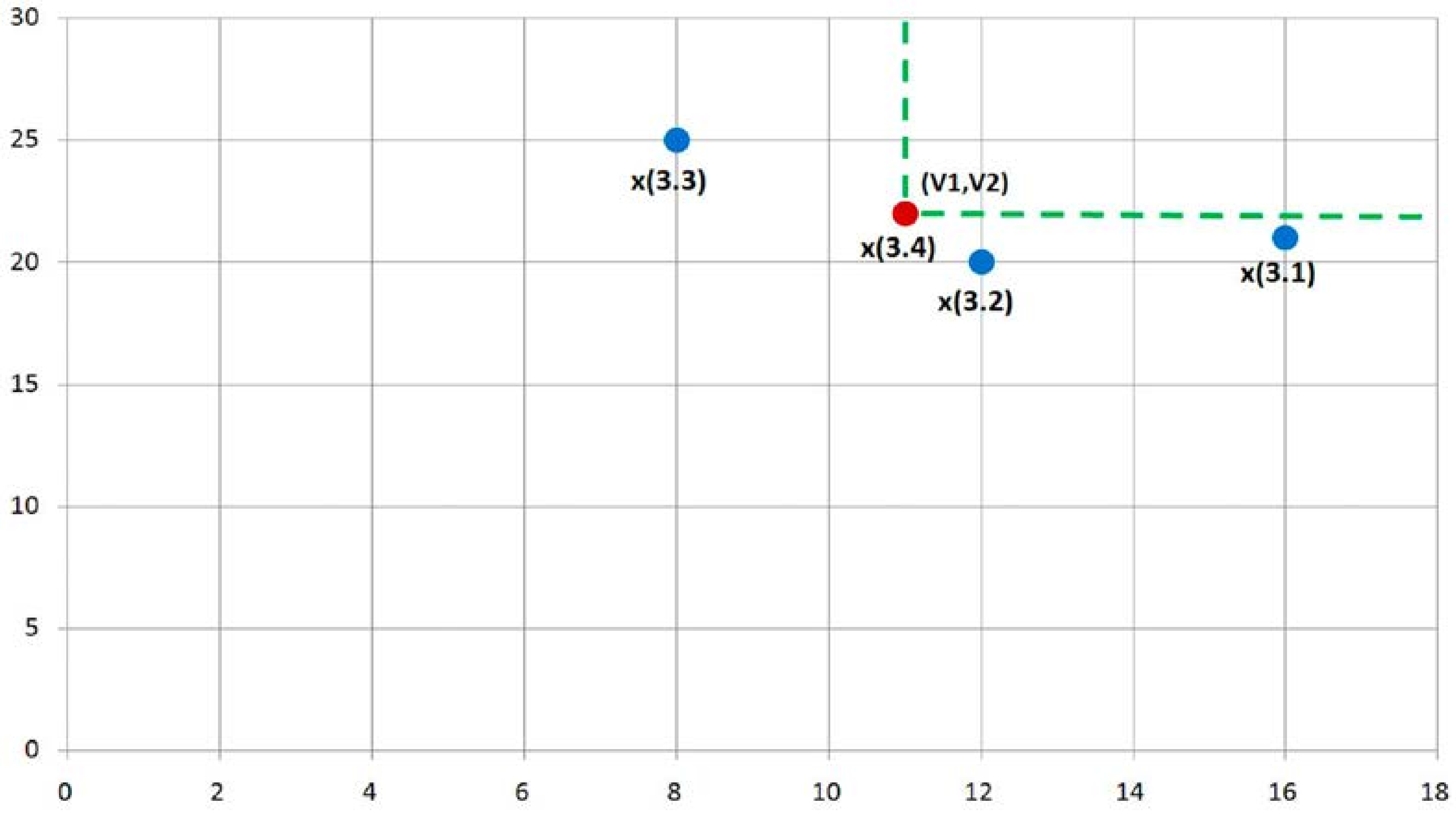




Nash bargaining solution in the subgame $\Gamma_{x_{(2.1)}}$:

Path $(x_{(2.1)}, x_{(1.3)}, x_{(3.4)}) \longrightarrow (H_1(x_{(3.4)})), H_2(x_{(3.4)})) = (11, 22)$.

# Nash Bargaining Solution




Nash bargaining solution in the subgame $\Gamma_{x(2.1)}$ and in the initial game do not match, it shows that the Nash bargaining solution is not time-consistent!

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1. Basar, T. & Zaccour, G. (2018). Handbook of Dynamic Game Theory. New York: Springer-Verlag.
2. Yeung, D. W. K. & Petrosyan, L. A. (2016). Subgame Consistent Cooperation. A Comprehensive Treatse. Singapore: Springer-Verlag.
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# Payoff Distribution Procedure and Subgame Consistency

**O. Petrosian**

PhD

# Subgame Consistency

**“Signing the agreement”,**
Fritz Wagner, 1924

Usually cooperative agreements lose relevance over time. However, it is possible to find subgame consistent cooperative agreement, where the terms for participants remain beneficial throughout the duration of contract.

# Subgame Consistency

**Definition.**

Cooperative solution (Pareto-optimal solution) is called subgame consistent, if it is

- Pareto-optimal in every subgame along the selected path.
- Individually rational along the selected path:

$$\sum_{k=0}^{l} h_i(\bar{x}_k) \geq V^i(\bar{x}_0),\ i \in N,$$

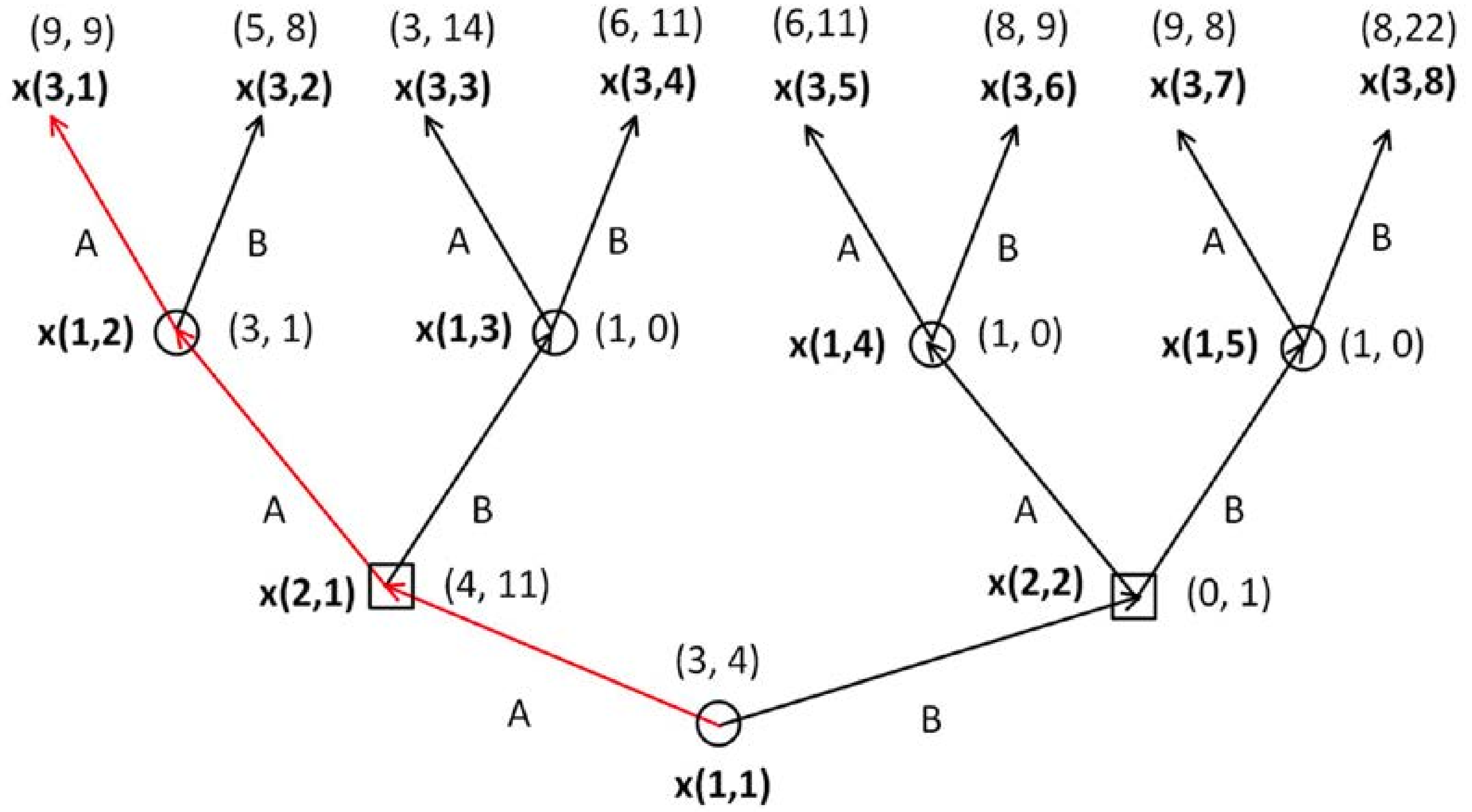
$$\dots$$

$$\sum_{k=m}^{l} h_i(\bar{x}_k) \geq V^i(\bar{x}_m), \text{ in the subgame } \Gamma_{\mathrm{m}}^{\bar{x}}, \text{ on the path } \bar{x}\ i \in N,$$

$$\dots$$

$$h_i(\bar{x}_l) = V^i(\bar{x}_l),\ i \in N.$$

# Subgame Consistency




Individual rationality along Nash bargaining path in the subgame $\Gamma_{x_{(2.1)}}$:

$$H_1(\bar{x}_1) = \sum_{k=1}^{3} h_1(\bar{x}_k) = 4 + 3 + 9 = 16 \geq V^1(\bar{x}_1) = 11,$$

$$H_2(\bar{x}_1) = \sum_{k=1}^{3} h_2(\bar{x}_k) = 11 + 1 + 9 = 21 < V^2(\bar{x}_1) = 22.$$

Subgame consistency for Nash bargaining solution in the subgame $\mathbf{\Gamma}_{x_{(2.1)}}$ is not satisfied!

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# Payoff Distribution Procedure

**Definition.**
Payoff distribution procedure (PDP) $H_i(\overline{x})$ along the selected path $\overline{x} = (\overline{x}_0, \dots, \overline{x}_l)$ is a matrix $\{\gamma_i^k\}$ such that:

$$H_1(\overline{x}) = \sum_{k=0}^{l} \gamma_1^k,$$

$$\dots$$

$$H_i(\overline{x}) = \sum_{k=0}^{l} \gamma_i^k,$$

$$\dots$$

$$H_n(\overline{x}) = \sum_{k=0}^{l} \gamma_n^k.$$

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# Payoff Distribution Procedure

**Definition.**

Payoff distribution procedure (PDP) $H_i(\overline{x})$ along the selected path $\overline{x} = (\overline{x}_0, \dots, \overline{x}_l)$ is a matrix $\{\gamma_i^k\}$ such that:

$$H_i(\overline{x}) = \sum_{k=0}^{l} h_i(\overline{x}_k) = \sum_{k=0}^{l} \gamma_i^k, \ i \in N.$$

**Theorem.**

For the following payoff distribution procedure the property of individual rationality along the path $\overline{x} = (x_0, \overline{x}_1, \dots, \overline{x}_l)$ is fulfilled, if

$$\gamma_i^k = \frac{H_i(\overline{x}) - V^i(x_0)}{l+1} - [V^i(\overline{x}_{k+1}) - V^i(\overline{x}_k)], \ i \in N.$$

# Payoff Distribution Procedure

**Construction of PDP for the game Γ:**

PDP $\gamma_1^k$, $k \in 0, 1, 2, 3$ of the player 1

$$\gamma_1^0 = \frac{H_1(\bar{x}) - V^1(x_0)}{4} - [V^1(\bar{x}_1) - V^1(x_0)] = \frac{19-14}{4} - [11 - 14] = 4.25,$$

$$\gamma_1^1 = \frac{19-14}{4} - [12 - 11] = 0.25,$$

$$\gamma_1^2 = \frac{19-14}{4} - [9 - 12] = 4.25,$$

$$\gamma_1^3 = \frac{19-14}{4} - [0 - 9] = 10.25.$$

Consistency requirement:

$$\sum_{k=0}^{3} h_1(\bar{x}_k) = 3 + 4 + 3 + 9 = 19 = 4.25 + 0.25 + 4.25 + 10.25 = \sum_{k=0}^{3} \gamma_1^k.$$

# Payoff Distribution Procedure

**Construction of PDP for the game Γ:**

PDP $\gamma_2^k$, $k \in 0, 1, 2, 3$ of the player 2

$$\gamma_2^0 = \frac{H_2(\overline{x}) - V^2(x_0)}{4} - [V^2(\overline{x}_1) - V^2(x_0)] = \frac{25 - 14}{4} - [22 - 14] = -5.25,$$

$$\gamma_2^1 = \frac{25 - 14}{4} - [9 - 22] = 15.75,$$

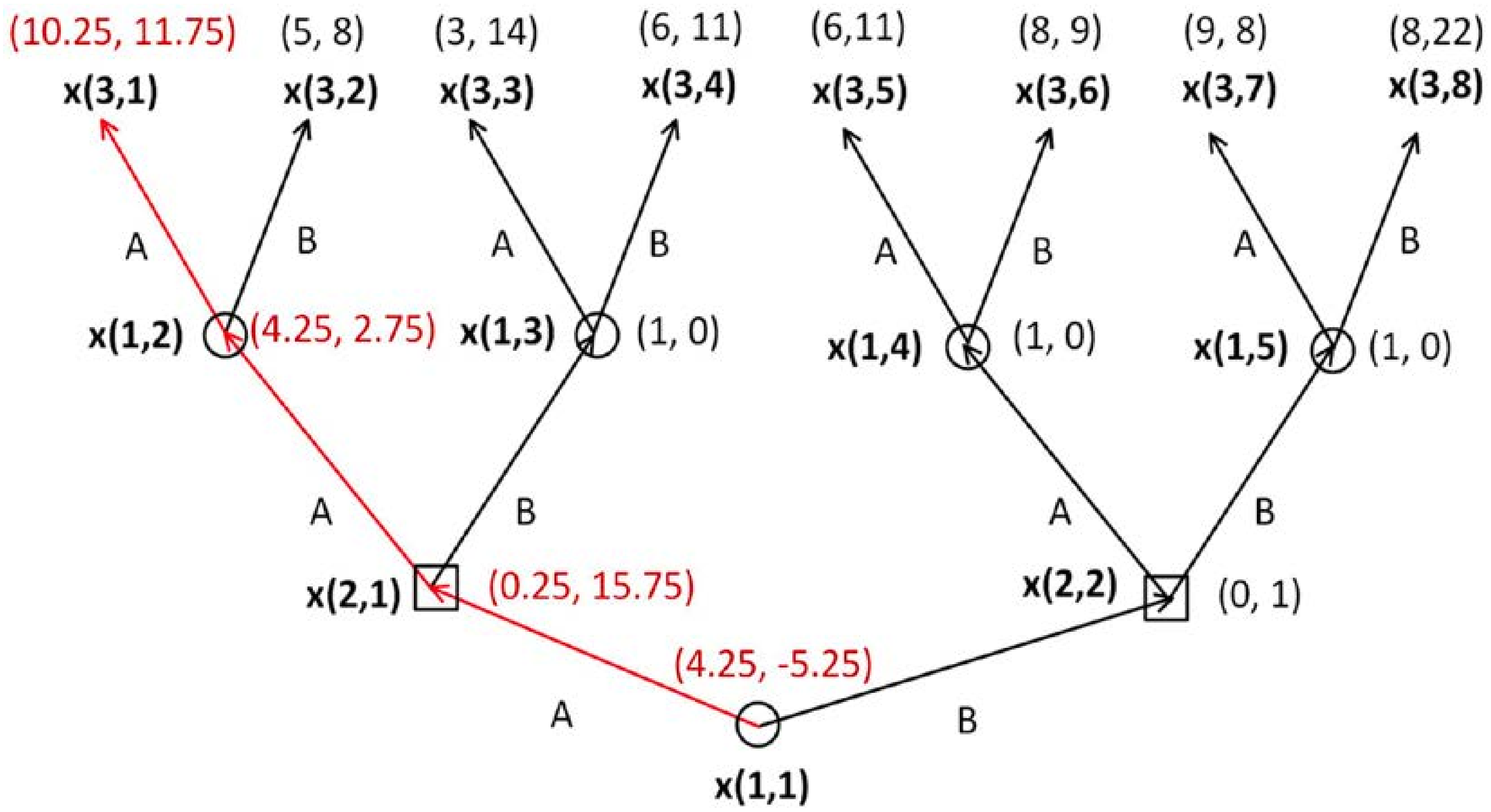
$$\gamma_2^2 = \frac{25 - 14}{4} - [9 - 9] = 2.75,$$

$$\gamma_2^3 = \frac{25 - 14}{4} - [0 - 9] = 11.75.$$

Consistency requirement:

$$\sum_{k=0}^{3} h_2(\overline{x}_k) = 4 + 11 + 1 + 9 = 25 = -5.25 + 15.75 + 2.75 + 11.75 = \sum_{k=0}^{3} \gamma_2^k.$$

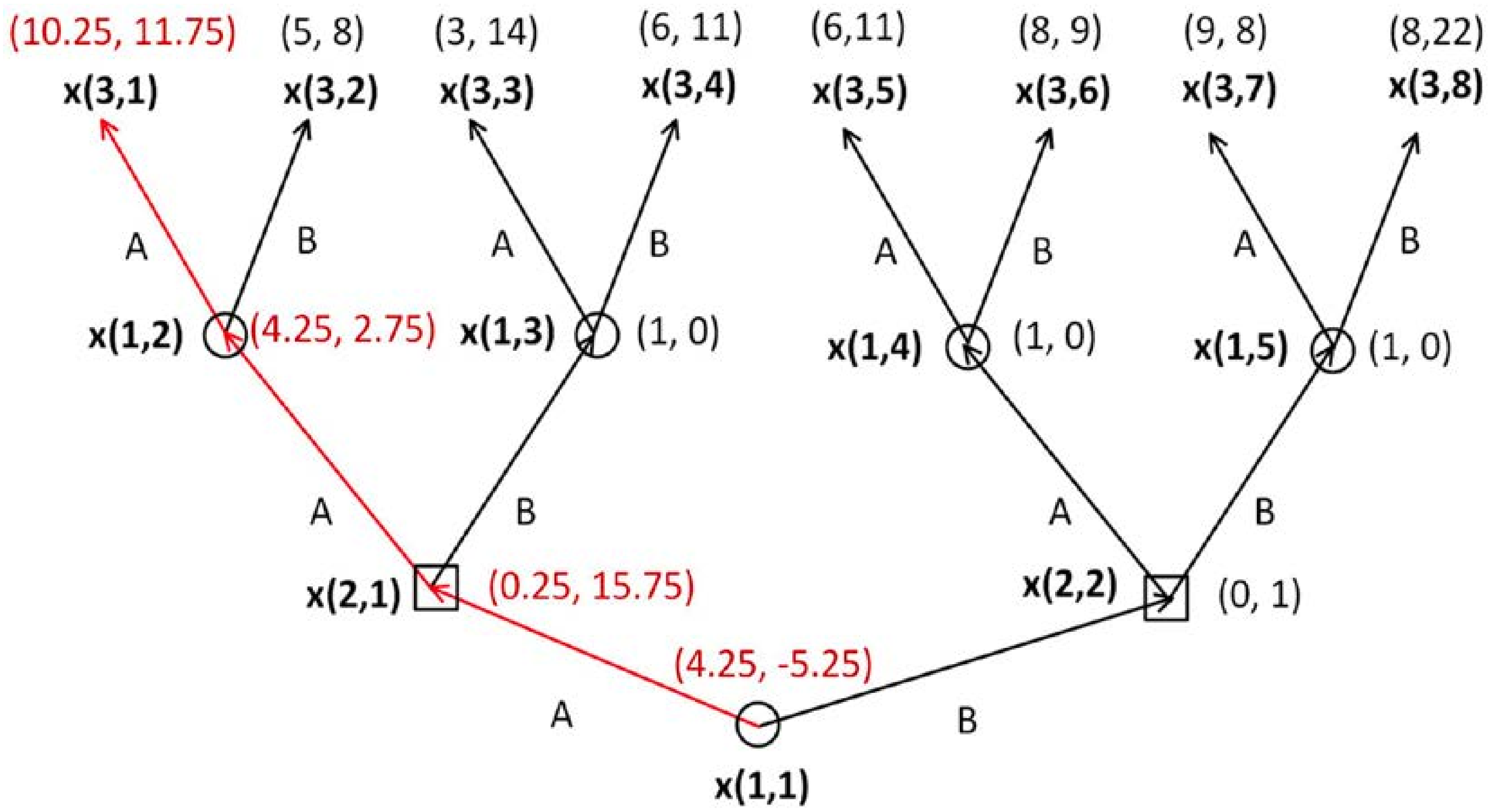
# Payoff Distribution Procedure




Tree graph of game Γ with recalculated payoffs along the optimal trajectory $(x_{(1.1)}, x_{(2.1)}, x_{(1.2)}, x_{(3.1)})$:

$h_i(\bar{x}_k) = \gamma_i^k$, $i = 1, 2$, $k = 0, \dots, 3$.

# Payoff Distribution Procedure

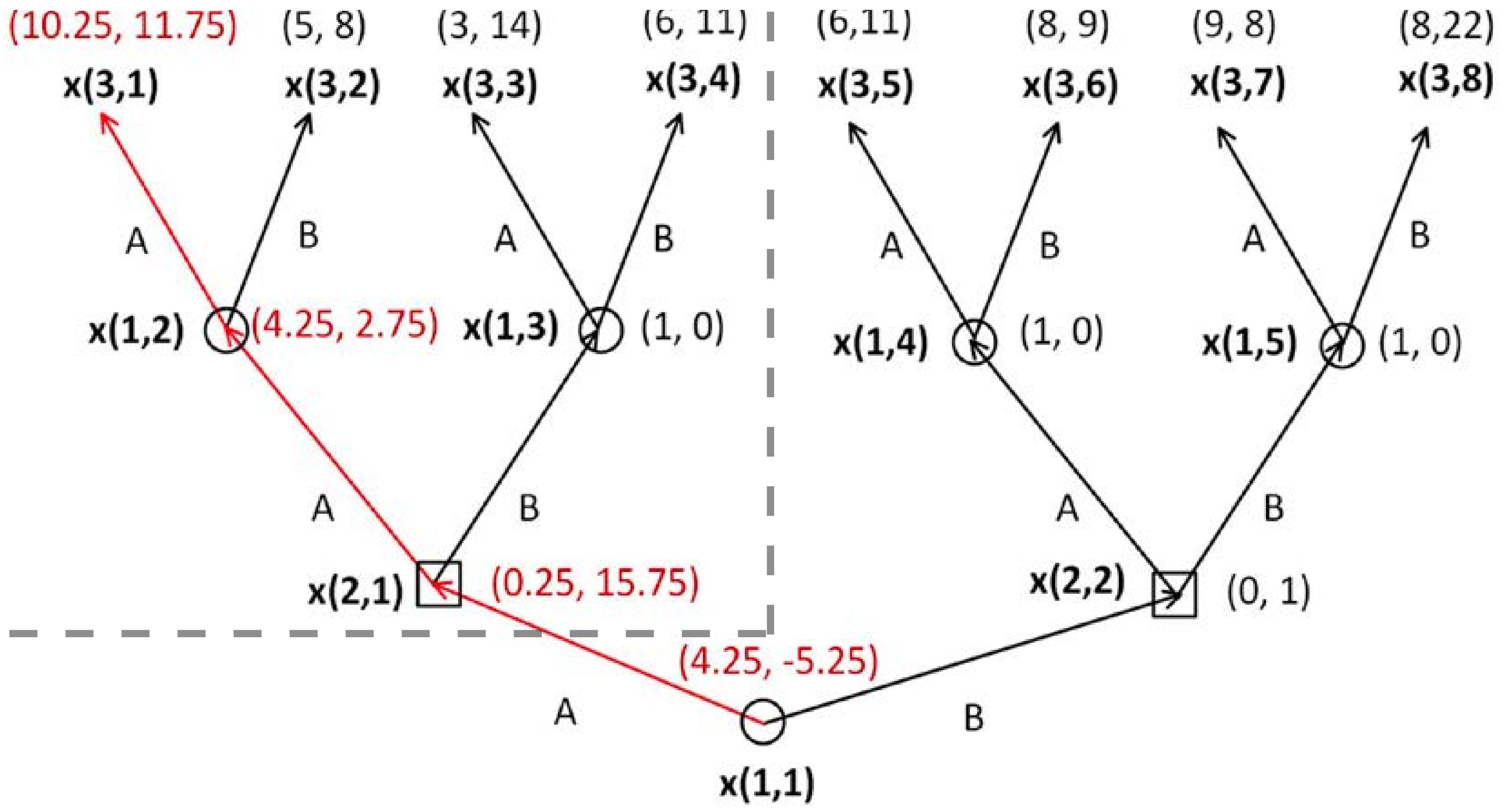



Individual rationality in the game **Γ**:

$$H_1(\bar{x}_0) = \gamma_1^0 + \gamma_1^1 + \gamma_1^2 + \gamma_1^3 = 5.25 + 0.25 + 4.25 + 10.25 = 19 \geq V^1(x_{(1.1)}) = 14,$$

$$H_2(\bar{x}_0) = \gamma_2^0 + \gamma_2^1 + \gamma_2^2 + \gamma_2^3 = -5.25 + 15.75 + 2.75 + 11.75 = 25 \geq V^2(x_{(1.1)}) = 14.$$

# Payoff Distribution Procedure

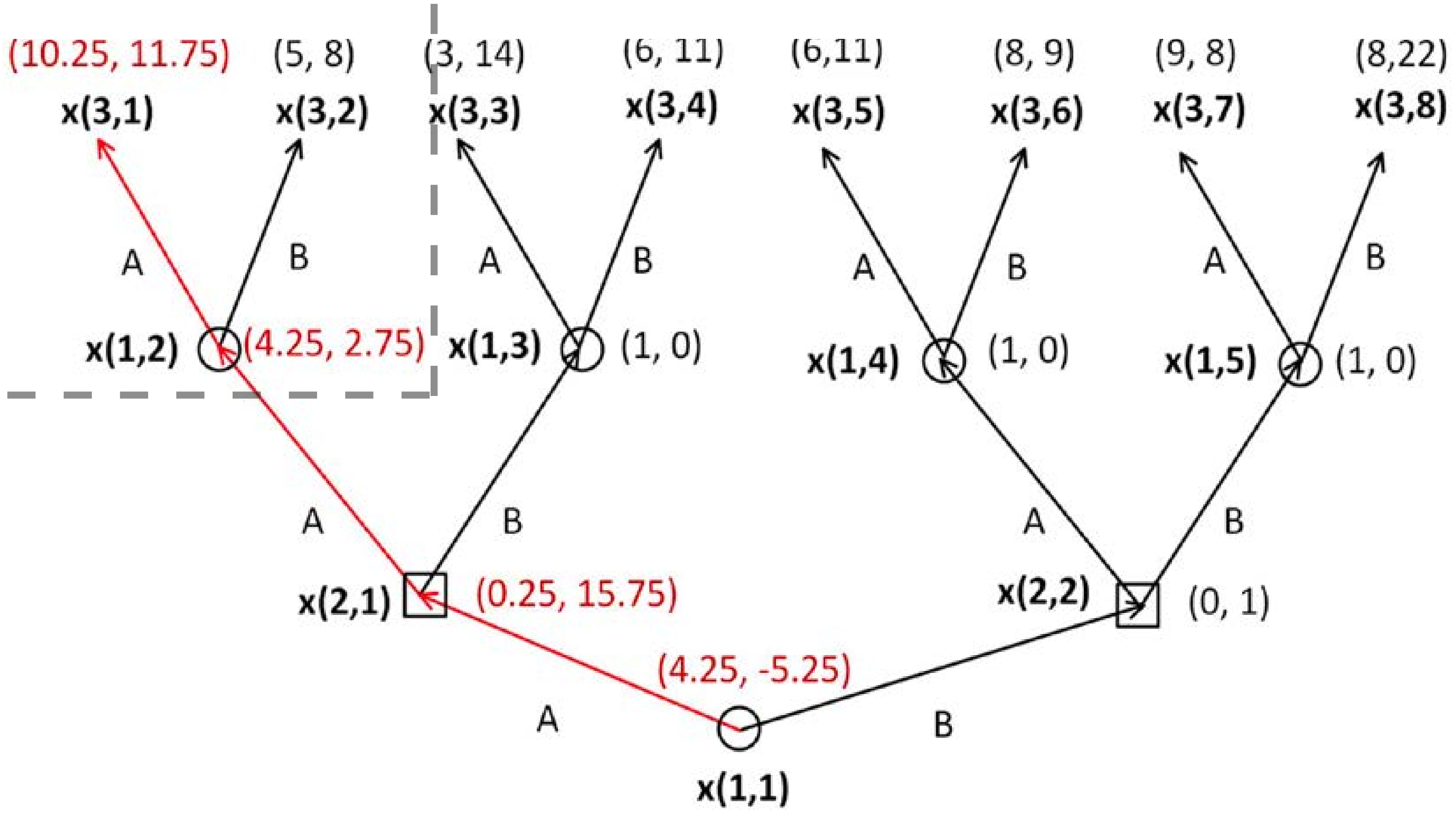



Individual rationality in the subgame $\Gamma_{x_{(2.1)}}$:

$H_1(\overline{x}_0) = \gamma_1^1 + \gamma_1^2 + \gamma_1^3 = 0.25 + 4.25 + 10.25 =$
$= 14.75 \geq V^1(x_{(2.1)}) = 11,$

$H_2(\overline{x}_0) = \gamma_2^1 + \gamma_2^2 + \gamma_2^3 = 15.75 + 2.74 + 11.75 =$
$= 30.25 \geq V^2(x_{(2.1)}) = 22.$

# Payoff Distribution Procedure




Individual rationality in the subgame $\Gamma_{x_{(1.2)}}$:

$$H_1(\bar{x}_0) = \gamma_1^2 + \gamma_1^3 = 4.25 + 10.25 = 14.5 \geq V^1(x_{(1.2)}) = 12,$$

$$H_2(\bar{x}_0) = \gamma_2^2 + \gamma_2^3 = 2.75 + 11.75 = 14.5 \geq V^2(x_{(1.2)}) = 9.$$

# References


1. Basar, T. & Zaccour, G. (2018). Handbook of Dynamic Game Theory. New York: Springer-Verlag.
2. Yeung, D. W. K. & Petrosyan, L. A. (2016). Subgame Consistent Cooperation. A Comprehensive Treatse. Singapore: Springer-Verlag.
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# Optimal Control Problem and Dynamic Programming

**O. Petrosian**

PhD

# Optimization of Advertising Costs

**“Knight at the Crossroads”,**
V. Vasnetsov, 1877

How should firm choose the strategy?

- Consider a firm whose revenue depends on its market share.
- The only market tool that the firm uses to increase share is the advertising.

# Optimal Control Problem

Consider optimal control problem:

$$\int_{t_0}^{T} g[t, x(t), u(t)]\, dt + q(x(T)) \longrightarrow \max_{u \in U} \qquad (1)$$

subject to

$$\begin{aligned} \dot{x}(t) &= f[t, x(t), u(t)], \\ x(t_0) &= x_0, \end{aligned} \qquad (2)$$

where $x(t) \in R^1$ is a state and $u(t) \in U^1 \in \mathrm{CompR}^1$ is a feedback control (we consider one dimensional space). Functions $f[t, x, u]$, $g[t, x, u]$ and $q(x)$ are assumed to be integrable.

**Definition.**
Feedback control $u^*(t)$ that maximizes functional (1) subject to (2) is called optimal and corresponding solution of (2) $x^*(t)$ optimal trajectory.

# Optimal Control Problem

Optimal control problem for advertising costs:

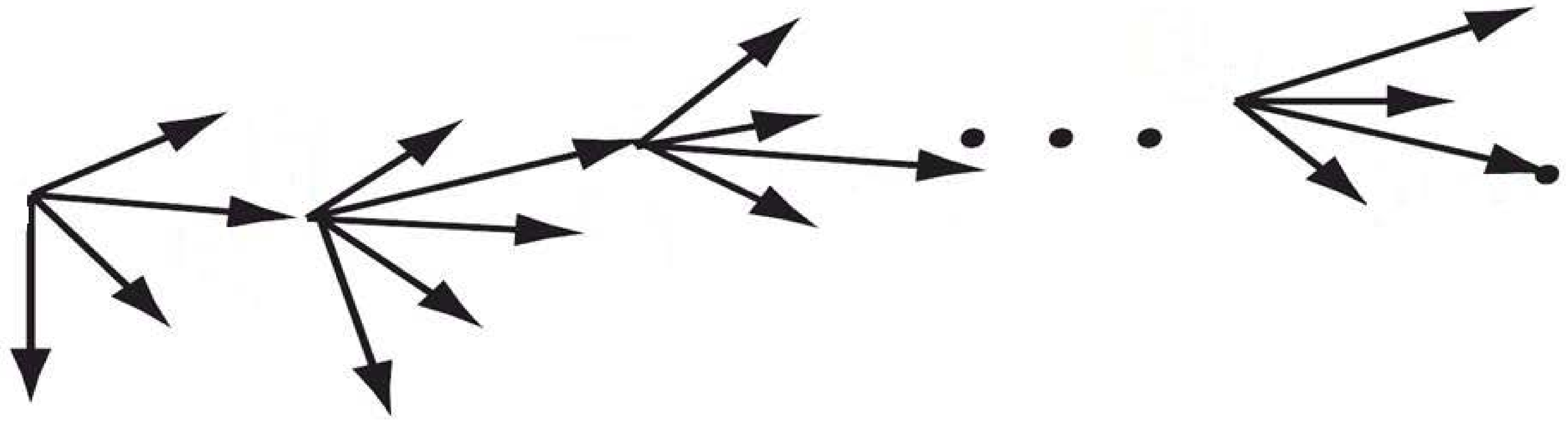
$$\int_0^T \exp[-rt]\Big[-x(t) - cu(\mathrm{t})^2\Big]dt + \exp[-rT]\; qx(T) \longrightarrow \max_{u\in U} \qquad (3)$$

subject to

$$\begin{aligned} \dot{x}(t) &= a - u(t)\,(x(t))^{1/2}, \\ x(0) &= x_0, \\ u(t) &\geq 0, \end{aligned} \qquad (4)$$

where $a$, $c$, $x_0$, $r$, $q$ are positive real numbers.

# Dynamic Programming Principle



Principle of dynamic programming is used to determine the optimal control in the problem (1), (2):

Suppose $\boldsymbol{x}^*(\boldsymbol{t})$ is optimal trajectory and $\boldsymbol{u}^*(\boldsymbol{t})$ corresponding optimal control in the problem (1), (2) defined on the interval $[\boldsymbol{t_0}, \boldsymbol{T}]$, then $\boldsymbol{x}^*(\boldsymbol{t})$, $\boldsymbol{u}^*(\boldsymbol{t})$ will also be optimal in the subproblem on the interval $[\boldsymbol{t'}, \boldsymbol{T}]$, $\boldsymbol{t_0} < \boldsymbol{t'} < \boldsymbol{T}$ from initial state $\boldsymbol{x}^*(\boldsymbol{t'})$.

# Dynamic Programming Principle

Define Bellman function as follows:

$$V(t, x) = \int_t^T g[s, x^*(s), u^*(s)]ds + q(x^*(T)),$$

where $t$, $x$ are the initial time and state of subproblem (1), (2). From the dynamic programming principle follows:

**Theorem (Hamilton-Jacobi-Bellman equation).**
Control $u^*(t) = \varphi^*(t, x^*(t))$ is optimal in the problem (1), (2), if there exists a continuously differentiable function $V(t, x) : [t_0, T] \times R^1 \longrightarrow R^1$, satisfying the following equation:

$$-V_t(t, x) = \max_{u \in U}\{g[t, x, u] + V_x(t, x)\, f\,[t, x, u]\} = g[t, x, \varphi^*(t, x)] + V_x(t, x)\, f\,[t, x, \varphi^*(t, x)], \tag{5}$$
$$V(T, x) = q(x).$$

# Dynamic Programming Principle

Bellman equation for optimal control problem of advertising costs:

$$-V_t(t, x) = \max_{u \in U} \left\{ [-x - cu^2] \exp[-rt] + V_x(t,x)[a - ux^{1/2}] \right\},$$

$$V(T, x) = \exp[-rT]\, qx(T).$$

Bellman function is defined in the form:

$$V(t, x) = \exp[-rt]\, [A(t)x + B(t)],$$

then optimal control will be

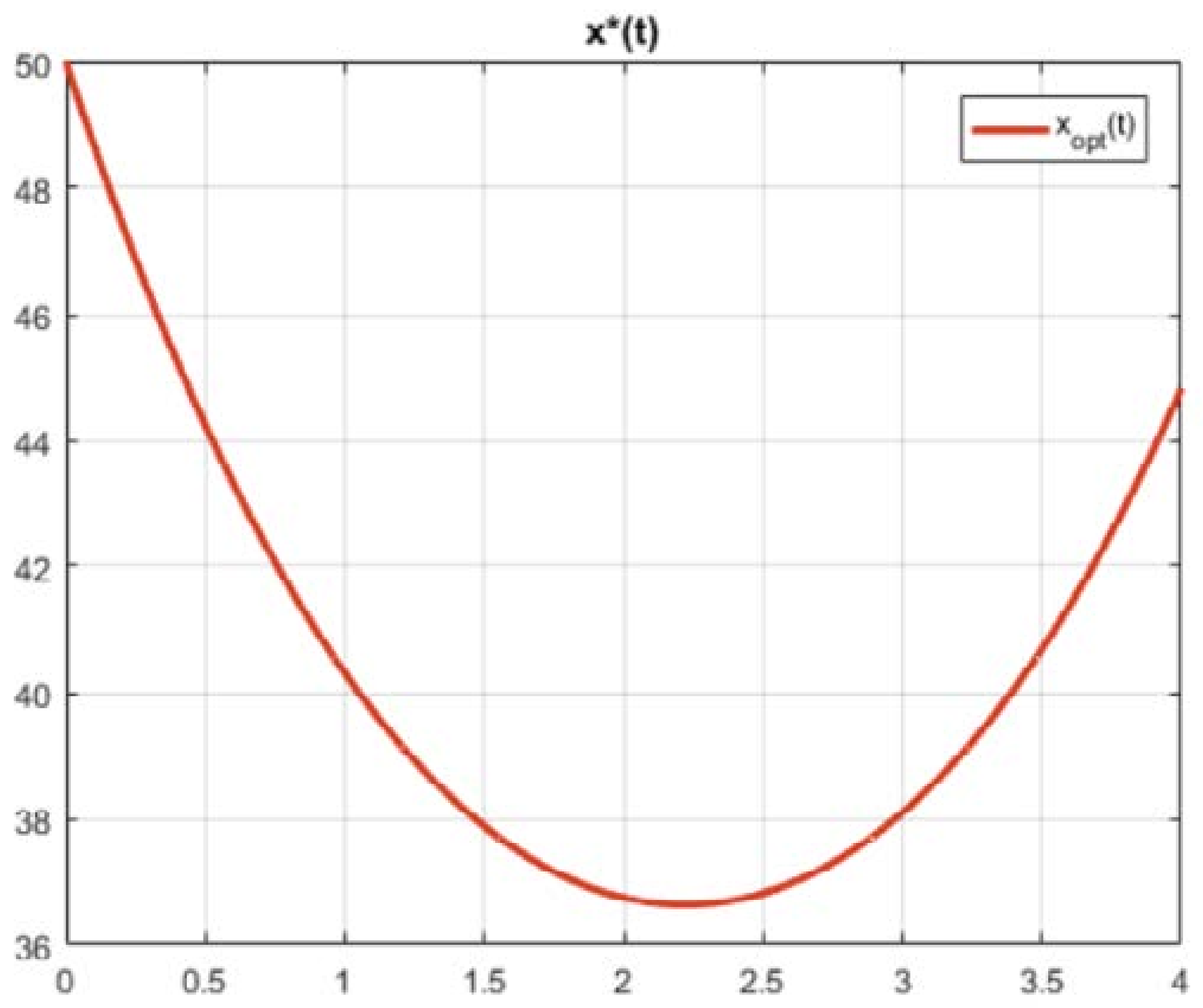
$$\varphi^*(t, x) = \frac{-A(t)x^{1/2}}{2c},$$

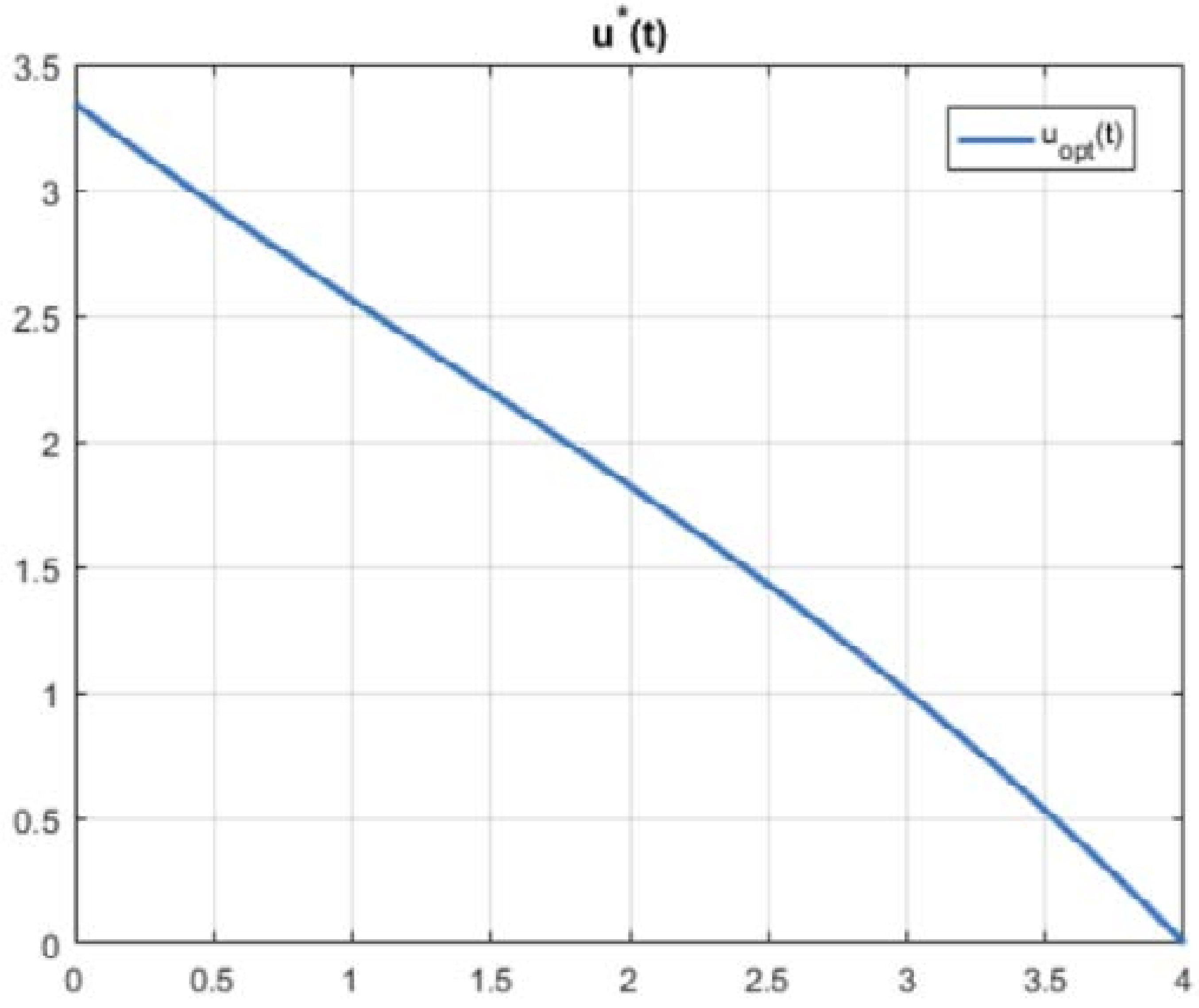
where

$$\dot{A}(t) = rA(t) - \frac{A(t)^2 e^{rt}}{4c} + 1, \quad \dot{B}(t) = rB(t) - aA(t)e^{rt},$$

$$A(T) = q, \quad B(T) = 0.$$

# Dynamic Programming Principle




Optimal trajectory $x^*(t)$

Optimal control $u^*(t) = \varphi^*(t, x^*(t))$

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3. Pontryagin L. S., Boltiansky, V.G., Gamkrelidze R.V., and Mishchenko E.F. (1962). Mathematical theory of optimal processes. New York: Wiley.

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# Noncooperative Differential Games and Nash Equilibrium

**O. Petrosian**

PhD

# Optimization of Advertising Costs

**“Novgorod bargaining”,**
A. Vasnetsov, 1912

- Consider two firms whose revenues depend on their market shares.
- Each firm wants to increase its revenue by increasing the market share (capacity of the market is limited).
- Firms can change market share by changing investments in advertising.

How should firms behave in such conflict situation?

# Noncooperative Differential Game

- Set of players $N = \{1, 2, \dots, n\}$.
- Payoff function of player $i \in N$:

$$K_i(x_0, T - t_0; u) = \int_{t_0}^{T} g^i[t, x(t), u_1(t, x), \dots, u_n(t, x)]dt + q^i(x(T)). \qquad (6)$$

- State of game $x(t)$ is defined by motion equations:

$$\begin{aligned} \dot{x}(t) &= f[t, x(t), u_1(t, x), \dots, u_n(t, x)], \\ x(t_0) &= x_0, \end{aligned} \qquad (7)$$

where $x(t) \in R^1$ is state at instant $s$, $u_i(t, x) \in U_i \subset R^1$ is strategy of player $i \in N$.

# Noncooperative Differential Game

Game model of advertising costs optimization:

- Set of players $N = \{1, 2\}$.
- Payoff function of player $i \in \{1, 2\}$:

$$K_i(x_0, T - t_0; u) = \int_0^T [q_i x_i(t) - \frac{c_i}{2} u_i(t, x)^2] \exp(-rt)\, dt + \exp(-rT)\, S_i x_i(T),$$

where $r$, $q_i$, $c_i$, $S_i$ for $i \in \{1, 2\}$ are positive constants,
$x_1(t) = x(t)$ is a market share of firm 1 at instant $t$,
$x_2(t) = [1 - x(t)]$ is a market share of firm 2,
$u_i$ is investment of firm $i \in \{1, 2\}$ in advertising.

- First firm's market share is defined by

$$\dot{x}(t) = u_1(t, x)\, [1 - x(t)]^{1/2} - u_2(t, x) x(t)^{1/2},$$
$$x(0) = x_0. \qquad (8)$$

Second player's market share is $1 - x(t)$.

# Nash Equilibrium

**Definition.**
Strategy profile $u^*(t,x) = (\varphi_1^*(t,x), \dots, \varphi_n^*(t,x))$ is a feedback Nash equilibrium in the game (5), (6), if for any player $i \in N$ and $t \in [t_0, T]$ conditions are met:

$$\int_{t_0}^{T} g^i[t, x^*(t), \varphi_1^*(t,x^*(t)), \dots, \varphi_i^*(t,x^*(t)), \dots, \varphi_n^*(t,x^*(t))]\, dt + q^i(x^*(T)) \geq$$
$$\int_{t_0}^{T} g^i[t, x^{[i]}(t), \varphi_1^*(t,x^*(t)), \dots, \varphi_i^*(t,x^*(t)), \dots, \varphi_n^*(t,x^*(t))]\, dt + q^i(x^{[i]}(T)),$$

where $t \in [t_0, T]$:

$$\dot{x}^*(t) = f[t, x^*(s), \varphi_1^*(t,x^*(t)), \dots, \varphi_i^*(t,x^*(t)), \dots, \varphi_n^*(t,x^*(t))],\ x^*(t_0) = x_0,$$
$$\dot{x}^{[i]}(t) = f[t, x^{[i]}(s), \varphi_1^*(t,x^*(t)), \dots, \varphi_i(t,x^*(t)), \dots, \varphi_n^*(t,x^*(t))],\ x^{[i]}(t_0) = x_0.$$

# Nash Equilibrium

By $V_i(t, x)$ we denote payoff of player $i \in N$ in feedback Nash equilibrium in subgame defined on the interval $[t, T]$ starting from the initial state x. In what follows this function will be called Bellman function.

**Theorem (system of Hamilton-Jacobi-Isaacs-Bellman equations).**
Set of strategies $\{u_i^*(t, x) = \varphi_i^*(t, x) \in U_i,\ i \in N\}$ is feedback Nash equilibrium in the game (6), (7), if there exist continuously-differentiable functions $V_i(t, x) : [t_0, T] \times R^1 \longrightarrow R^1$, $i \in N$, satisfying the following system of equations:

$$-V_t^i(t, x) =$$

$$\max_{u_i \in U_i}\{g^i[t, x, \varphi_1^*(t, x), \ldots, \varphi_{i-1}^*(t, x), u^i(t, x), \varphi_{i+1}^*(t, x), \ldots, \varphi_n^*(t, x)]$$
$$+ V_x^i(t, x)\, f[t, x, \varphi_1^*(t, x), \ldots, \varphi_{i-1}^*(t, x), u^i(t, x), \varphi_{i+1}^*(t, x), \ldots, \varphi_n^*(t, x)]\} =$$

$$g^i[t, x, \varphi_1^*(t, x), \varphi_2^*(t, x), \ldots, \varphi_n^*(t, x)] + V_x^i(t, x)\, f[t, x, \varphi_1^*(t, x), \ldots, \varphi_n^*(t, x)]\},$$

$$V^i(T, x) = q^i(x),\ i \in N.$$

# Nash Equilibrium

System of Hamilton-Jacobi-Isaacs-Bellman equations for feedback Nash equilibrium:

$$-V^1_t(t, x) = \max_{u_1 \in U_1}\left\{[q_1 x - \frac{c_1}{2}\, u_1^2] \exp(-rt) + V^1_x(t, x)(u_1[1 - x]^{1/2} - \varphi^*_2(t, x)\, x^{1/2})\right\},$$

$$-V^2_t(t, x) = \max_{u_2 \in U_2}\left\{[q_2(1 - x) - \frac{c_2}{2}\, u_2^2] \exp(-rt) + V^2_x(t, x)\, (\varphi^*_1(t, x)\, [1 - x]^{1/2} - u_2 x^{1/2})\right\},$$

$$V^1(T, x) = \exp(-rT)\, S_1 x,$$

$$V^2(T, x) = \exp(-rT)\, S_2(1 - x). \qquad (9)$$

# Nash Equilibrium

We will define Bellman functions in the form:

$$V^1(t, x) = \exp\,[-r(t)]\,[A_1(t)x + B_1(t)],$$
$$V^2(t, x) = \exp\,[-r(t)]\,[A_2(t)(1 - x) + B_2(t)], \qquad (10)$$

then equilibrium strategies are

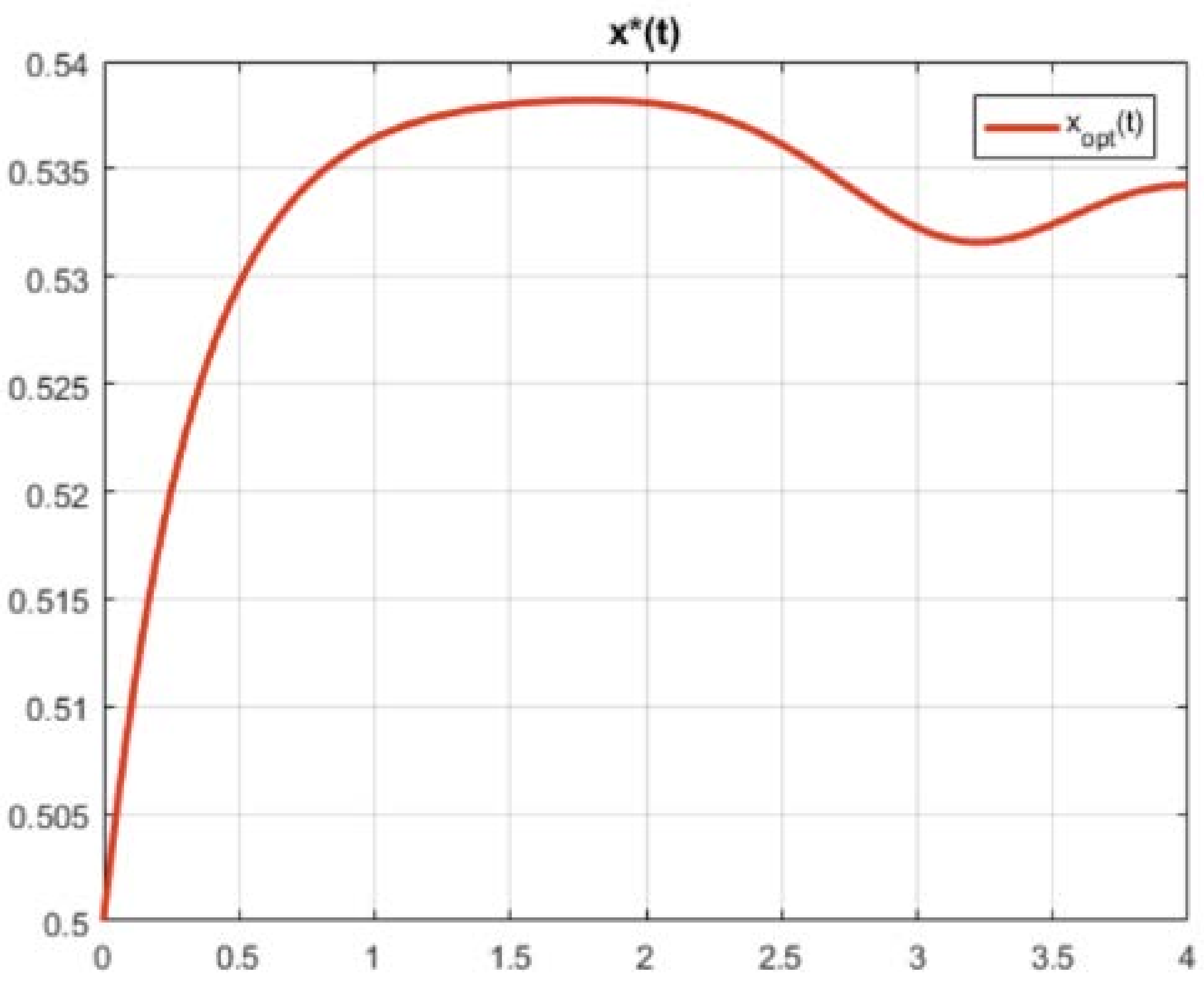
$$\varphi_1^*(t, x) = \frac{A_1(t)}{c_1}\,[1 - x]^{1/2},\ \varphi_2^*(t, x) = \frac{A_2(t)}{c_2}\,[x]^{1/2}, \qquad (11)$$

where $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ satisfy the equations:

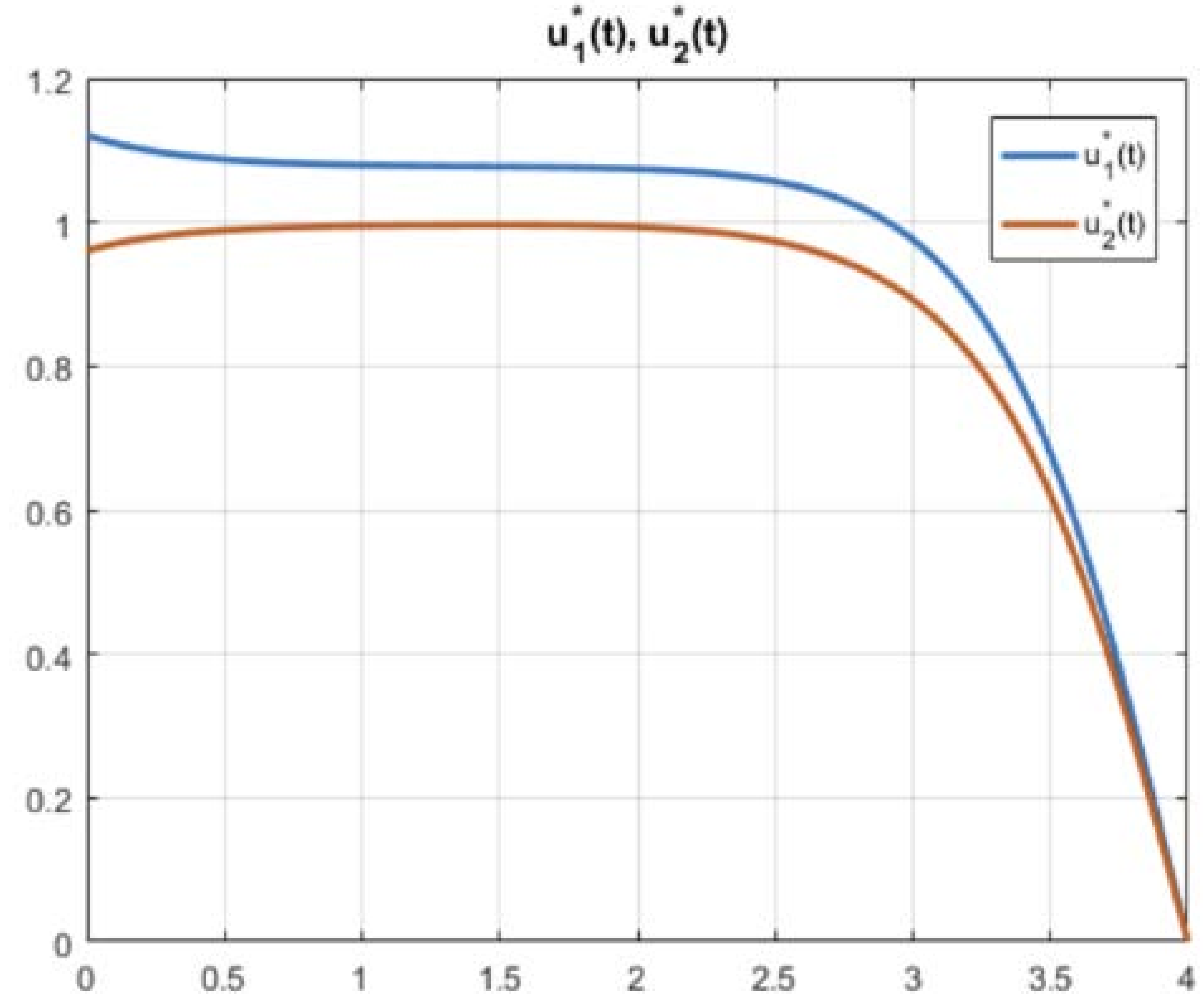
$$\dot{A}_1(t) = rA_1(t) - q_1 + \frac{A_1(t)^2}{2c_1} + \frac{A_1(t)\,A_2(t)}{2c_2},$$
$$\dot{A}_2(t) = rA_2(t) - q_2 + \frac{A_2(t)^2}{2c_2} + \frac{A_1(t)\,A_2(t)}{2c_1},$$

with boundary conditions $A_1(T) = S_1$, $B_1(T) = 0$, $A_2(T) = S_2$, $B_2(T) = 0$.

# Nash Equilibrium




Optimal trajectory $x^*(t)$




"Optimal" control $\varphi_i^*(t, x^*(t)), i \in N$

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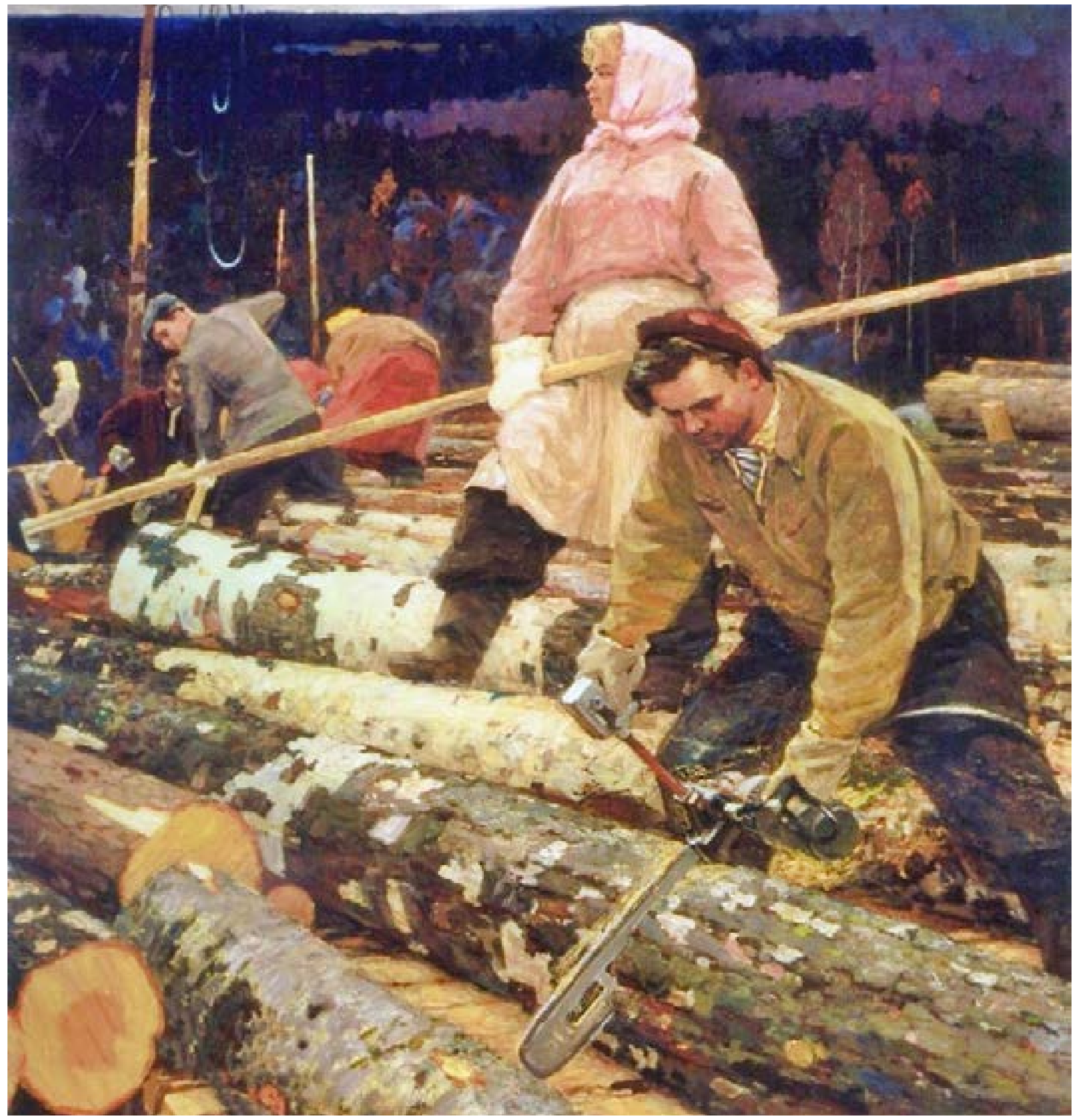
# Differential Games in Charasteristic Function Form

**O. Petrosian**

PhD

# Limited Resource Extraction



**“Woodcutters”,**
Alexey Belyh, 1923

Three firms are extracting limited renewable resources. Each firm wants to maximize the amount of extracted resource.

How can firms make an agreement on profit allocation?

# Differential Games in Characteristic Function Form

Optimality principle in the cooperative differential game:

1. Agreement on cooperative strategies $u_i^*(t,x)$, $i \in N$ and corresponding cooperative trajectory $x^*(t)$:

$$\sum_{i=1}^{n} \int_{t}^{T} g^i[t, x(t), u_1(t,x), \dots, u_n(t,x)]dt \longrightarrow \max_{u_1, \dots, u_n}$$

subject to

$$\begin{aligned} \dot{x}(t) &= f[t, x(t), u_1(t,x), \dots, u_n(t,x)], \\ x(t_0) &= x_0. \end{aligned} \qquad (12)$$

2. Allocation mechanism for maximum total payoff between the players.

# Differential Games in Characteristic Function Form

Optimal control problem for limited resource extraction game (agreement on cooperative strategies):

$$\sum_{i=1}^{3} K_i(x_0,\, T-t_0;\, u_1,\, u_2,\, u_3) = \sum_{i=1}^{3} \int_{t_0}^{T} \sqrt{u_i(t,x)} - \frac{c_i}{\sqrt{x(t)}}\, u_i(t,x)dt \longrightarrow \max_{u_1,\,\dots\,,\,u_n} \qquad (13)$$

subject to

$$\begin{aligned} &\dot{x} = a\sqrt{x(t)} - bx(t) - \sum_{i=1}^{3} u_i(t,x), \\ &x(t_0) = x_0, \end{aligned} \qquad (14)$$

where

- $a\sqrt{x(t)} - bx(t)$ is the growth rate of renewable resources,
- $u_i(t,x) \in [0,\, \mathrm{d}]$, $d > 0$ is the amount of resource extracted by firm $i$ at instant $t$,
- $c_i = \text{const}$, $c_i \neq c_k$, $\forall i \neq k = \overline{1,3}$,
- $x_0$, $T$, $a$, $b$, $d$, $c_i$, $i = \overline{1,3}$ are nonnegative values.

# Differential Games in Characteristic Function Form

**Theorem.**
Suppose that there exists contnuously differentiable function $W(t, x) : [t_0, T] \times R^1 \longrightarrow R^1$, satisfying the Bellman equation:

$$W_t(t, x) + \max_{u \in U}\left\{\sum_{i \in N} g_i(t, x, u) + W_x(t, x) f(t, x, u)\right\} = 0,$$

where $W(T, x) = 0$. If maximum in the right hand side of the formula above is achieved on $u^*(t, x)$, then $u^*(t, x)$ is optimal in the control problem (13), (14).

Here $W(t, x)$ is the maximum joint payoff of players in subgame defined on the interval $[t, T]$ starting from the initial state $x$.

# Differential Games in Characteristic Function Form

Define Bellman function as follows:

$$W(t, x) = \sum_{i=1}^{3} \int_{t}^{T} \sqrt{u_i^*(s, x^*(s))} - \frac{c_i}{\sqrt{x^*(s)}} u_i^*(s, x^*(s)) ds,$$

where $t$, $x$ is the initial instant and state of subproblem (8), (9), then Bellman equation will take the form:

$$-W_t(t, x) = \max_{u \in U} \left\{ \sum_{i=1}^{3} \left[ \sqrt{u_i} - \frac{c_i}{\sqrt{x(t)}} u_i \right] + W_x(t, x) \left[ a\sqrt{x(t)} - bx(t) - \sum_{i=1}^{3} u_i \right] \right\},$$

$$W(T, x) = 0.$$

# Differential Games in Characteristic Function Form

Bellman function are defined in the form:

$$W(t, x) = A(t)\sqrt{x} + C(t), \qquad (15)$$

then optimal controls are

$$u_i\,(t, x) = \frac{x}{4[c_i + A(t)/2]^2}\;,\; i = \overline{1,3}, \qquad (16)$$

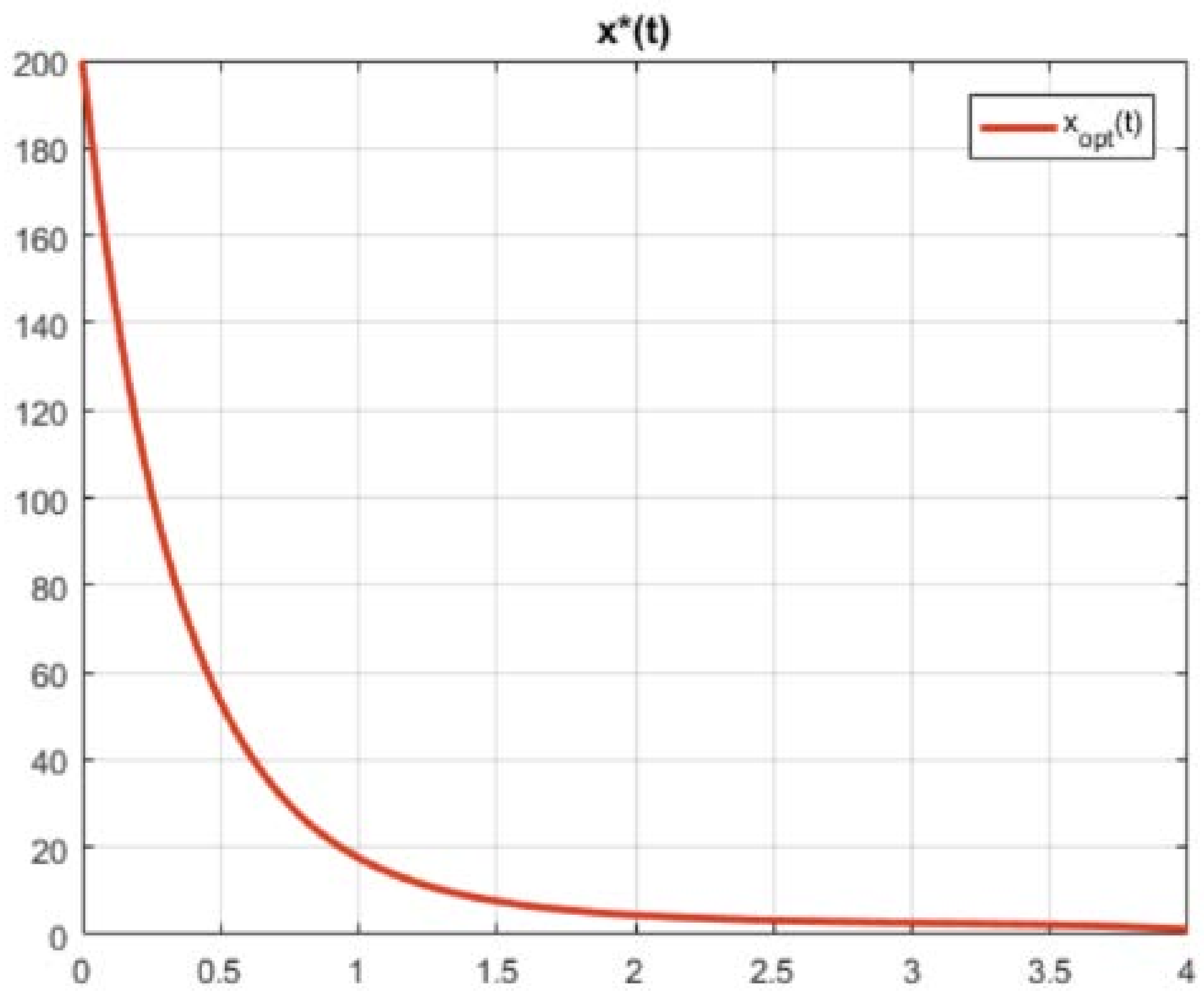
where

$$\dot{A}(t) = \frac{b}{2}\,\mathrm{A}(t) - \sum_{i=1}^{3}\left[\frac{x}{4\left[c_i + \frac{A^j(t)}{2}\right]}\right]\;,$$

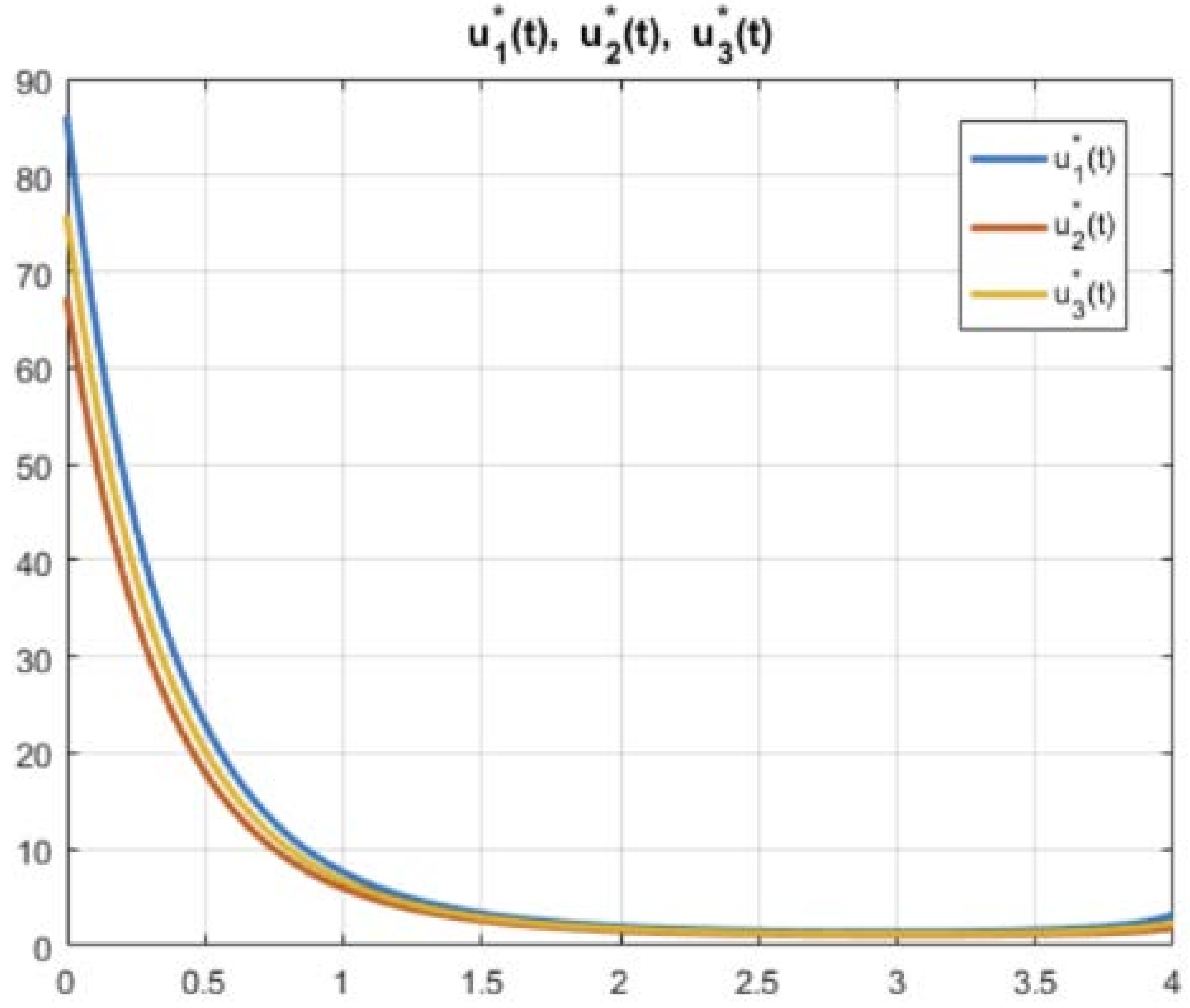
$$\dot{C}(t) = C(t) - \frac{a}{2}\,A^j(t). \qquad (17)$$

with boundary conditions $A(T) = 0$, $C(T) = 0$.

# Differential Games in Characteristic Function Form




Optimal (cooperative) trajectory $x^*(t)$




Optimal controls
$u_i^*(t) = \varphi_i^*(t, x^*(t)), i = \overline{1,3}$

# Characteristic Function

Characteristic function:

$$V(S; x^*(t), T-t) = \begin{cases} \sum_{i\in N} K_i(x^*(t), \mathrm{T}-t; u_1^*(t,x),\ldots,u_n^*(t,x)), & S = N, \\ \tilde{V}(S, x^*(t), T-t), & S \subset N, \\ 0, & S = \varnothing, \end{cases} \tag{18}$$

where $\tilde{V}(S, x^*(t), T-t)$ is the total payoff of players from coalition S in Nash equilibrium $u^* = (u_1^*, \ldots, u_n^*)$ in the game with $|N \setminus S| + 1$ players:

- coalition $S$,
- players from the set $N \setminus S$.

# Characteristic Function

Using the system of Hamilton-Jacobi-Isaacs-Bellman equations for characteristic function we obtain, $V(S; x, T - t) = V_S(t, x)$:

$V_S(t, x) = A_S(t)\sqrt{x} + C_S(t)$,
$V_i(t, x) = A_i(t)\sqrt{x} + C_i(t)$, $i \in N \setminus S$,
where

$$\dot{A}_S(t) = A_S(t)\left[\frac{b}{2} + \sum_{N\setminus S} \frac{1}{8(c_k + A_k(t)/2)^2}\right] - \sum_{k\in S} \frac{1}{4(c_k + A_s(t)/2)},$$

$$\dot{A}_i(t) = A_i(t)\left[\frac{b}{2} + \sum_{k\in S} \frac{1}{8(c_k + A_s(t)/2)^2}\right] - \sum_{l\in N\setminus\{S\cup i\}} \frac{1}{4(c_l + A_l(t)/2)}, \quad i \in N \setminus S,$$

$$\dot{C}_S(t) = -\frac{a}{2}A_S(t),$$

$$\dot{C}_i(t) = -\frac{a}{2}A_i(t)$$

with boundary conditions $A_S(T) = A_i(T) = 0$, $C_S(T) = C_i(T) = 0$.

# Imputations in Dynamics

Set of imputations in the subgame along the cooperative trajectory $x^*(t)$:

$$E(x^*(t), T-t) = \Big\{ \xi(x^*(t), T-t) \in R^n \mid$$
$$\xi_i(x^*(t), T-t) \geq V(\{i\}; x^*(t), T-t), \; i = \overline{1,n};$$
$$\sum_{i \in N} \xi_i (x^*(t), T-t) = V(N; x^*(t), T-t) \Big\}, \qquad (19)$$

where

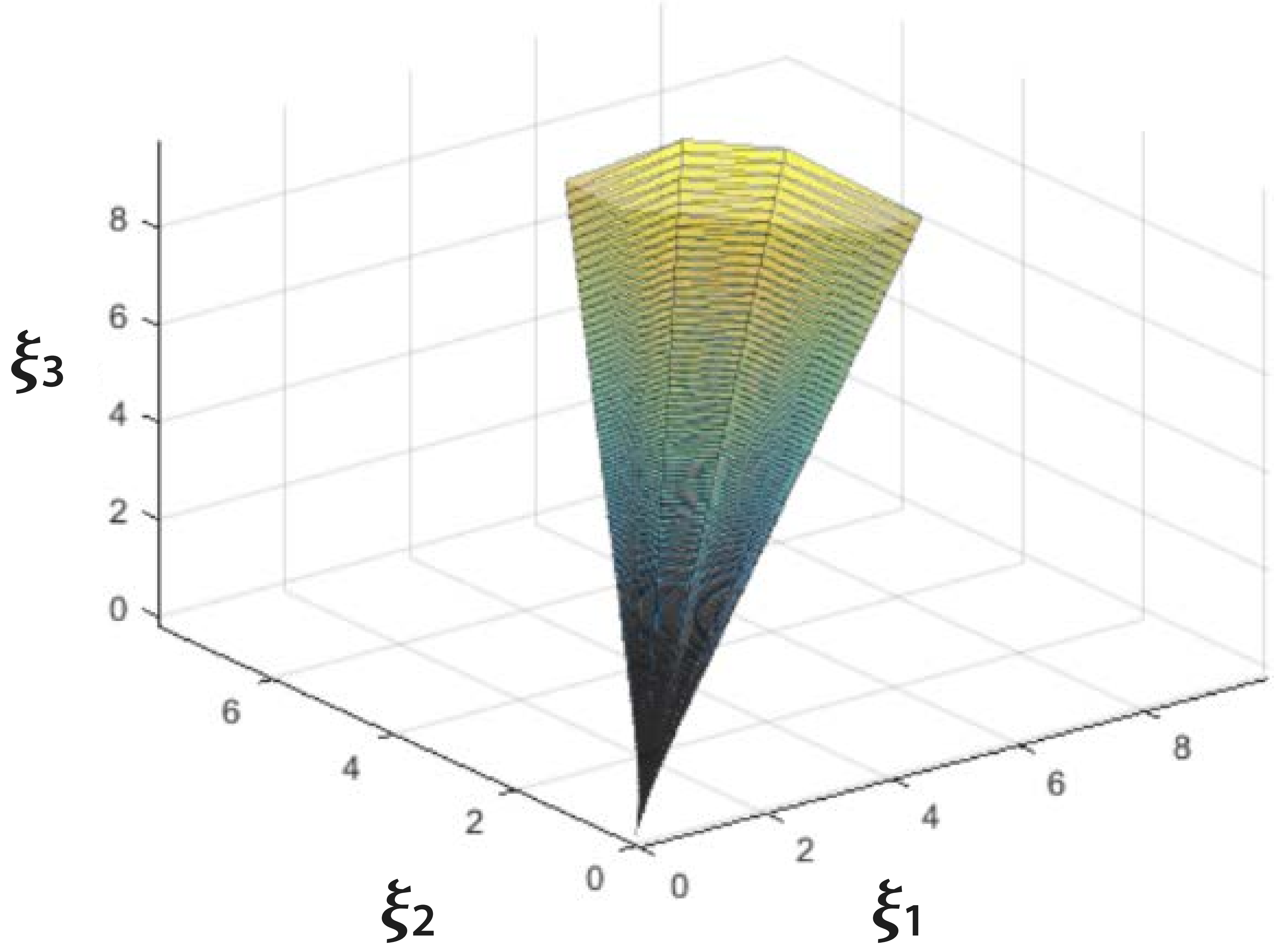
$$V(N; x^*(t), T-t) = V(N; x_0, T-t_0) - \sum_{i=1}^{n} \left\{ \int_{t_0}^{t} g_i[t, x^*(t), u^*(t)]dt \right\}.$$

# Imputations in Dynamics

Imputation set for resource extraction game:

$$E(x^*(t), T-t) = \Big\{ \xi(x^*(t), T-t) \in R^3 \,\Big|$$
$$\xi_1(x^*(t), T-t) \geq A_1(t)\sqrt{x^*(t)} + C_1(t),$$
$$\xi_2(x^*(t), T-t) \geq A_2(t)\sqrt{x^*(t)} + C_2(t),$$
$$\xi_3(x^*(t), T-t) \geq A_3(t)\sqrt{x^*(t)} + C_3(t),$$
$$\xi_1(x^*(t), T-t) + \xi_2(x^*(t), T-t) + \xi_3(x^*(t), T-t) = A(t)\sqrt{x^*(t)} + C(t) \Big\}.$$

# Imputations in Dynamics




Imputation set $E(x^*(t), T - t)$

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# Time-consistent Optimality Principles

**O. Petrosian**

PhD

# Shapley Value

Vector $\varphi(x_0, T - t_0) = \{\varphi_i(x_0, T - t_0), i = \overline{1,n}\}$ is called the Shapley value, if for $i \in N$:

$$\varphi_i(x_0, T - t_0) = \sum_{S \subseteq N \,|\, S \in i} \frac{(n - s)!(s - 1)!}{n!}[v(S; x_0, T - t_0) - v(S \setminus i; x_0, T - t_0)],$$

where $|S|$ is the number of players in coalition $S \subseteq N$.

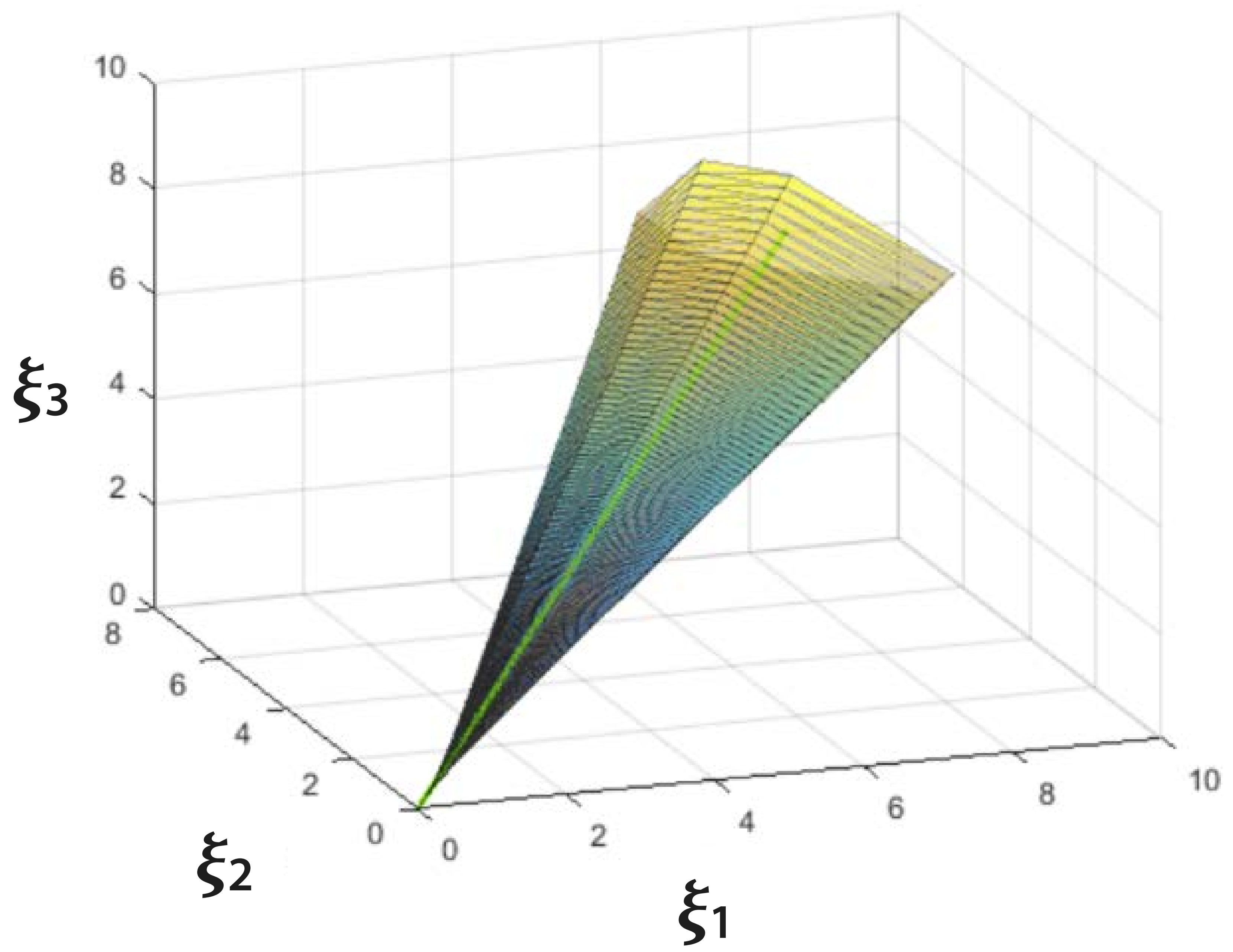
# Shapley Value

$$\Phi_i^v(x_0,\ T - t_0) = \frac{[v(i);\, x_0,\ T - t_0) - v(\emptyset;\, x_0,\ T - t_0)]}{3} + \frac{[v(ij;\, x_0,\ T - t_0) - v(j;\, x_0,\ T - t_0) + v(ik;\, x_0,\ T - t_0) - v(k;\, x_0,\ T - t_0)}{6} +$$

$$+ \frac{[v(N;\, x_0,\ T - t_0) - v(jk;\, x_0,\ T - t_0)]}{3} = \frac{[A_{\{i\}}(T - t_0)\sqrt{x_0} + C_{\{i\}}(T - t_0)]}{3} + \frac{[A(T - t_0)\sqrt{x_0} + C(T - t_0) - A_{\{j,k\}}(T - t_0)\sqrt{x_0} + C_{\{j,k\}}(T - t_0)]}{3} +$$

$$+ \frac{[A_{\{i,j\}}(T - t_0)\sqrt{x_0} + C_{\{i,j\}}(T - t_0) - A_{\{j\}}(T - t_0)\sqrt{x_0} + C_{\{j\}}(T - t_0)]}{6} + \frac{[A_{\{i,k\}}(T - t_0)\sqrt{x_0} + C_{\{i,k\}}(T - t_0) - A_{\{k\}}(T - t_0)\sqrt{x_0} + C_{\{k\}}(T - t_0)]}{6}$$

# Shapley Value




Shapley Value in the imputation set,
$\varphi_i(x^*(t), T-t), i \in N$

# Time-consistency

**“George Washington resigning his commission”,**
John Trumbull, 1817

Time-consistency property of cooperative solution guarantees that the reconsideration of this solution at any given instant will result in the same solution.

# Time-consistency

**Definition.**
Function $\beta(t, x^*)$, $t \in [t_0, T]$ is called the Imputation Distribution Procedure (IDP) of imputation $\xi(x_0, T - t_0) \in E(x_0, T - t_0)$, if

$$\xi(x_0, T - t_0) = \int_{t_0}^{T} \beta(t, x^*(\mathrm{t}))dt. \tag{20}$$

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# Time-consistency

**Definition.**

Cooperative solution $W(x_0, T - t_0)$ ($\xi(x_0, T - t_0)$) is called time-consistent, if for any imputation $\xi(x_0, T - t_0) \in W(x_0, T - t_0)$ there exist IDP $\beta(t, x^*)$, $\forall t \in [t_0, T]$ such that:

$$\left\{ \int_t^T \beta(\tau, x^*(\tau)) d\tau \right\} \in W(x^*(t), T - t) \tag{21}$$

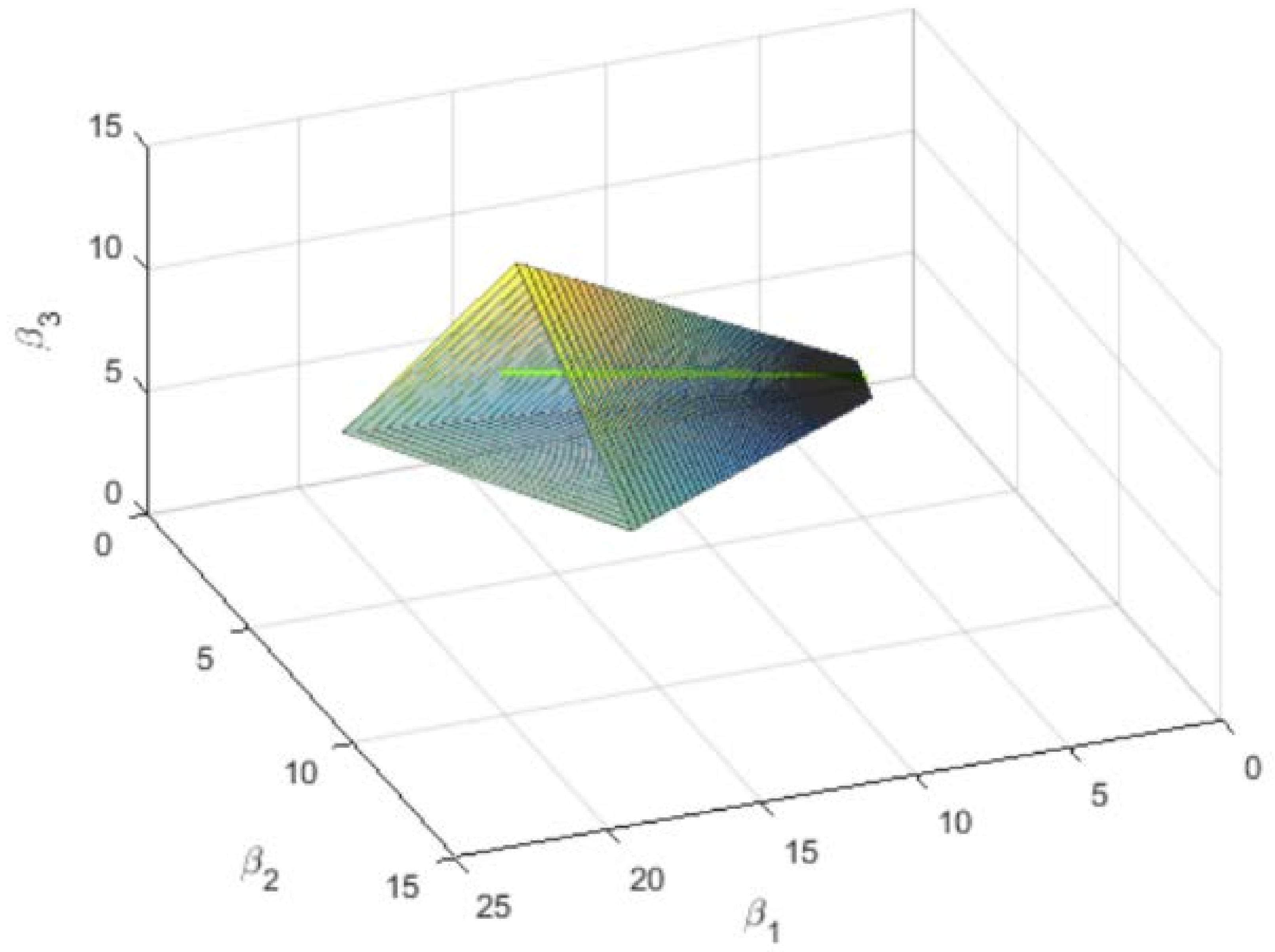
$$\left( \int_t^T \beta(\tau, x^*(\tau)) d\tau \ = \xi(x^*(t), T - t) \right).$$

**Theorem.**

Imputation Distribution Procedure $\beta(t, x^*)$ that provides time-consistent imputation $\xi(x_0, T - t_0)$ has the form:

$$\beta(t, x^*(t)) = -\frac{d}{dt} \xi(x^*(t), T - t). \tag{22}$$

# Shapley Value




IDP of Shapley value, $\beta_i(x^*(t), T - t), i \in N$

# Strong Time-consistency

**Definition.**
Cooperative solution $W(x_0, T - t_0)$ is called strong time-consistent, if

- $W(x^*(t), T - t) \neq \varnothing, \forall t \in [t_0, T]$,
- for any $\xi(x^*(t), T - t) \in W(x^*(t), T - t)$ exist IDP $\beta(t, x^*) = (\beta_1(t, x^*), \dots, \beta_n(t, x^*))$, $t \in [t_0, T]$, such that

$$\int_{t_0}^{t} \beta(\tau, x^*(\tau))d\tau \oplus W(x^*(t), T - t) \subseteq W(x_0, T - t_0)$$

for any $t \in [t_0, T]$, where
$\oplus$: $a \oplus B = \{a + b : b \in B\}$, $a \in R^n$, $B \subset R^n$.

# IDP-core

**Definition.**
Set $B(t_0, x_0)$ of integrable vector functions $\beta(t, x^*) = (\beta_1(t, x^*), \dots, \beta_n(t, x^*))$ is defined as

$$\sum_{i\in S} \beta_i(t, x^*(t)) \geq -\frac{d}{dt} V(S; x^*(t), T-t), \ \forall S \subseteq N,$$
$$\sum_{i\in N} \beta_i(t, x^*(t)) = -\frac{d}{dt} V(N; x^*(t), T-t), \ t \in [t_0, T]. \tag{23}$$

**Definition.**
IDP-core $\bar{C}(x^*(t), T-t)$ is the set of vectors $\xi(x^*(t), T-t)$ of the form:

$$\xi(x^*(t), t) = \int_t^T \beta(\tau, x^*)d\tau, \ \beta(t, x^*) \in B(t, x^*), \ t \in [t_0, T]$$

for all possible vector functions $\beta(t, x^*) \in B(t_0, x_0)$.

# IDP-core

Set $B(t_0, x_0)$ for limited resource extraction game:

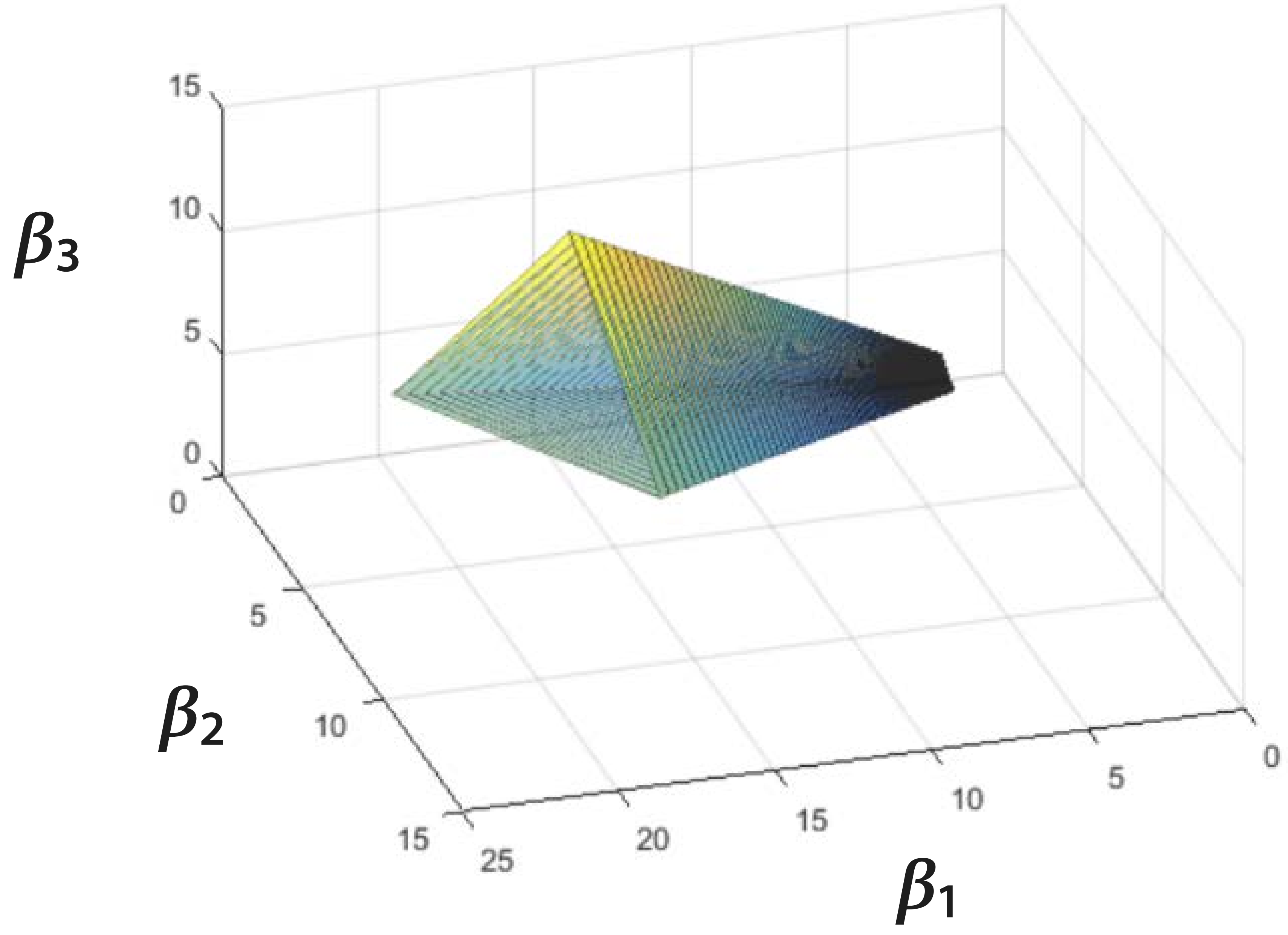
$$\begin{aligned}
&\sum \beta_i(t, x^*) = -\frac{d}{dt} V(\{1, 2, 3\}; x^*(t), T - t),\\
&\beta_1(t, x^*) + \beta_2(t, x^*) \geq -\frac{d}{dt} V(\{1, 2\}; x^*(t), T - t),\\
&\beta_1(t, x^*) + \beta_3(t, x^*) \geq -\frac{d}{dt} V(\{1, 3\}; x^*(t), T - t),\\
&\beta_2(t, x^*) + \beta_3(t, x^*) \geq -\frac{d}{dt} V(\{2, 3\}; x^*(t), T - t),\\
&\beta_1(t, x^*) \geq -\frac{d}{dt} V(\{1\}; x^*(t), T - t),\\
&\beta_2(t, x^*) \geq -\frac{d}{dt} V(\{2\}; x^*(t), T - t),\\
&\beta_3(t, x^*) \geq -\frac{d}{dt} V(\{3\}; x^*(t), T - t).
\end{aligned} \tag{24}$$

# IDP-core

**Theorem.**
Set $\bar{C}(x^*(t), T - t)$ is a subset of Core $C(x^*(t), T - t)$ in cooperative differential game, $t \in [t_0, T]$.

**Theorem.**
IDP-core $\bar{C}(x_0, T - t_0)$ is strong time-consistent in cooperative differential game.

# IDP-core




IDP-core $\bar{C}(x^*(t), T - t)$

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2. Yeung, D. W. K. & Petrosyan, L. A. (2016). Subgame Consistent Cooperation. A Comprehensive Treatse. Singapore: Springer-Verlag.
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