Q1 Mathematical Proof

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(1) Assume the radius is r

$D = 2r$.

Since it is a black circle with white background. We want to find the σ s.t. obtain the maximum of $\sigma^2 \iint_{R^2} \nabla^2 G(x,y,\sigma) \cdot I(x,y)\, dxdy$.

Since the circle is black. Then if $x^2+y^2 \le r^2$ $I(x,y)=0$. Then we only need to consider $x^2+y^2 \ge r^2$.

i.e. $\sigma^2 \iint_D \nabla^2 G\, dx\, dy$ where $D = [(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \ge r^2]$.

Let $x = \rho\cos\theta$, $y = \rho\sin\theta$, $\rho \in [r, +\infty)$ $\theta \in [0, 2\pi]$

$$\sigma^2 \iint_D \nabla^2 G\, dxdy = \sigma^2 \cdot \int_0^{2\pi} d\theta \int_r^{+\infty} \frac{1}{\pi\sigma^4}\left(\frac{\rho^2}{2\sigma^2} - 1\right) e^{-\frac{\rho^2}{2\sigma^2}} \cdot \rho\, d\rho$$

$$= 2\pi\sigma^2 \cdot \frac{1}{\pi\sigma^4}\Big[\underbrace{\int_r^{+\infty} \frac{\rho^2}{2\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} \cdot \rho\, d\rho}_{A} - \underbrace{\int_r^{\infty} e^{-\frac{\rho^2}{2\sigma^2}} \rho\, d\rho}_{B}\Big]$$

$$= \frac{2}{\sigma^2}[A-B]$$

$$A = \frac{1}{2\sigma^2}\int_r^{+\infty} \rho^2 \cdot e^{-\frac{\rho^2}{2\sigma^2}} \rho\, d\rho$$

$$= \frac{1}{2\sigma^2} \cdot \frac{1}{2}\int_r^{+\infty} \rho^2 e^{-\frac{\rho^2}{2\sigma^2}} d\rho^2$$

$$= -\frac{1}{2}\int_r^{+\infty} \rho^2\, d e^{-\frac{\rho^2}{2\sigma^2}}$$

$$= -\frac{1}{2}\left[\rho^2 e^{-\frac{\rho^2}{2\sigma^2}}\Big|_r^{\infty} - \int_r^{+\infty} e^{-\frac{\rho^2}{2\sigma^2}} d\rho^2\right]$$

$$= -\frac{1}{2}\rho^2 e^{-\frac{\rho^2}{2\sigma^2}}\Big|_r^{\infty} + \frac{1}{2}\int_r^{\infty} e^{-\frac{\rho^2}{2\sigma^2}} d\rho^2$$

$$= -\frac{1}{2}\rho^2 e^{-\frac{\rho^2}{2\sigma^2}}\Big|_r^{\infty} + \underbrace{\int_r^{\infty} e^{-\frac{\rho^2}{2\sigma^2}} d\rho}$$

equal

Then $\frac{2}{\sigma^2}[A-B] = -\frac{1}{\sigma^2}\rho^2 e^{-\frac{\rho^2}{2\sigma^2}}\Big|_r^{\infty}$

$$= -\frac{1}{\sigma^2}\left[\lim_{\rho\to\infty} \frac{\rho^2}{e^{\frac{\rho^2}{2\sigma^2}}} - r^2 e^{-\frac{r^2}{2\sigma^2}}\right]$$

$$\lim_{\rho\to\infty} \frac{\rho^2}{e^{\frac{\rho^2}{2\sigma^2}}} = \lim_{\rho\to\infty} \frac{2\rho}{\frac{2\rho}{2\sigma^2} e^{\frac{\rho^2}{2\sigma^2}}} \quad \text{(L'Hôpital's Rule)}$$

$$= \lim_{\rho\to\infty} \frac{2\sigma^2}{e^{\frac{\rho^2}{2\sigma^2}}} = 0$$

Then $\frac{2}{\sigma^2}[A-B] = \frac{1}{\sigma^2} r^2 e^{-\frac{r^2}{2\sigma^2}}$

Since we want to find the maximum value,

Take deratives

$$f(\sigma) = \frac{1}{\sigma^2} r^2 e^{-\frac{r^2}{2\sigma^2}}$$

$$f' = (-2)\sigma^{-3} r^2 e^{-\frac{r^2}{2\sigma^2}} + \frac{1}{\sigma^2} r^2 \cdot \left(-\frac{r^2}{2}\cdot(-2)\sigma^{-3}\right) e^{-\frac{r^2}{2\sigma^2}}$$

$$= -2 \cdot \frac{1}{\sigma^3} r^2 e^{-\frac{r^2}{2\sigma^2}} + \frac{1}{\sigma^2} r^2 \cdot \frac{r^2}{\sigma^3} e^{-\frac{r^2}{2\sigma^2}}$$

$$= \underbrace{\frac{r^2}{\sigma^3} e^{-\frac{r^2}{2\sigma^2}}}_{>0\ (\text{positive})}\left(-2 + \frac{r^2}{\sigma^2}\right)$$

Then if
$$\begin{cases} \sigma = \frac{r}{\sqrt{2}}, & f' = 0 \\ \sigma < \frac{r}{\sqrt{2}}, & f' > 0 \\ \sigma > \frac{r}{\sqrt{2}}, & f' < 0 \end{cases}$$

Then when $\sigma = \frac{r}{\sqrt{2}}$, f attains maximum.

Then Since $2r = D$

Then $\sigma = \frac{D}{2\sqrt{2}}$

(2) The proof step is quite similar to the first question. The differece is we want to find the minimum of the result and change the integral domain since it is a white circle.

In this way, we want to find the mimimum value of $\sigma^2 \iint_D \frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2} - 1\right) e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$

Applying the same trick (chang of variables in last part) We can get

$$-\frac{1}{\sigma^2}\rho^2 e^{-\frac{\rho^2}{2\sigma^2}}\Big|_0^r = -\frac{1}{\sigma^2} r^2 e^{-\frac{r^2}{2\sigma^2}}$$

Compared to last part

$$g(\sigma) = -\frac{1}{\sigma^2} r^2 e^{-\frac{\rho^2}{2\sigma^2}} = -f(\sigma)$$

Then when $\sigma = \frac{r}{\sqrt{2}}$, $g(\sigma)$ attains its minimum which is the maximum magnitude response.