Normalization Proof

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$$\nabla^2 G = \underbrace{\frac{1}{\pi \sigma^4}}_{\text{positive}} \underbrace{\left(\frac{x^2+y^2}{2\sigma^2} - 1\right)}_{\text{positive/ negative.}} \underbrace{e^{-\frac{x^2+y^2}{2\sigma^2}}}_{\text{positive}}$$

I will consider the central negative part.

boundary is $x^2+y^2 = 2\sigma^2 \Rightarrow r = \sqrt{2}\sigma$

Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 2\sigma^2\}$

We want find $\iint_D \nabla G^2 \, dx dy$

$\Rightarrow$ Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \sqrt{2}\sigma]$, $\theta \in [0, 2\pi]$

$$\iint_D \nabla G^2 \, dx dy = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}\sigma} \frac{1}{\pi \sigma^4} \left(\frac{r^2}{2\sigma^2} - 1\right) e^{-\frac{r^2}{2\sigma^2}} \cdot r dr$$

$$= 2\pi \cdot \frac{1}{\pi\sigma^4} \cdot \Big[\underbrace{\int_0^{\sqrt{2}\sigma} \frac{r^2}{2\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot r dr}_{A} - \underbrace{\int_0^{\sqrt{2}\sigma} e^{-\frac{r^2}{2\sigma^2}} \cdot r dr}_{B}\Big]$$

$$= \frac{2}{\sigma^4} [A - B]$$

$$A = \int_0^{\sqrt{2}\sigma} \frac{r^2}{2\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot r dr$$

$$= \frac{1}{2\sigma^2} \int_0^{\sqrt{2}\sigma} r^2 e^{-\frac{r^2}{2\sigma^2}} \cdot r dr$$

$$= \frac{1}{2\sigma^2} \cdot \frac{1}{2} \int_0^{\sqrt{2}\sigma} r^2 e^{-\frac{r^2}{2\sigma^2}} \, d\underbrace{r^2}_{t}$$

$$= \frac{1}{4\sigma^2} \int_0^{2\sigma^2} t e^{-\frac{t}{2\sigma^2}} dt$$

$$= -\frac{1}{2} \cdot \int_0^{2\sigma^2} t \, d e^{-\frac{t}{2\sigma^2}}$$

$$= -\frac{1}{2} \cdot \left[t e^{-\frac{t}{2\sigma^2}} \Big|_0^{2\sigma^2} - \int_0^{2\sigma^2} e^{-\frac{t}{2\sigma^2}} dt\right]$$

$$= -\frac{1}{2} \cdot \left[2\sigma^2 e^{-1} - \int_0^{2\sigma^2} e^{-\frac{t}{2\sigma^2}} dt\right]$$

$$= \underbrace{\frac{1}{2} \int_0^{2\sigma^2} e^{-\frac{t}{2\sigma^2}} dt} - \sigma^2 e^{-1}$$

$$B = \int_0^{\sqrt{2}\sigma} e^{-\frac{r^2}{2\sigma^2}} \cdot r dr$$

$$= \frac{1}{2} \int_0^{\sqrt{2}\sigma} e^{-\frac{r^2}{2\sigma^2}} \, d\underbrace{r^2}_{t}$$

$$= \underbrace{\frac{1}{2} \int_0^{2\sigma^2} e^{-\frac{t}{2\sigma^2}} dt}$$

equal.

Then $A - B = -\sigma^2 e^{-1}$

$$\iint_D \nabla G^2 \, dx dy = \frac{2}{\sigma^4} [A - B]$$

$$= \frac{2}{\sigma^4} (-\sigma^2 e^{-1})$$

$$= -\frac{2}{\sigma^2} e^{-1}$$

We can notice, there is a $\frac{2}{\sigma^2}$, which can have impact on the magnitude of the response. Therefore we need to do the normalization.