CS208 · Algorithm Design and Analysis

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Assignment 4

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Exercise 4.2, 4.8, 4.22

1 Exercise 4.2

(a) True.

Consider the Kruskal's algorithm. We sort the edges by the cost. Then select the edges one by one. When the cost c_e is replaced by c_e^2, the order will not change. Therefore the edges selected will not change, and T is still a minimum spanning tree.

(b) False.

There is an obvious counter-example. Consider the graph G has 4 nodes with 4 directed edges. $(1,2)$ with cost 1, $(2,4)$ with cost 99, $(1,3)$ with cost 50, and $(3,4)$ with cost 51. Then we can easily find the shortest path from 1 to 4 is 100 via 2. However, when c_e is replaced by c_e^2, the shortest path should be 5101 via 3, not 9802 via 2.

2 Exercise 4.8

Suppose $|V| = n$ and $|E| = m$. Firstly we sort the edges by the cost in ascending order, and number them as $e_1, e_2, \cdots, e_m$. In other word, $c(e_1) < c(e_2) < \cdots < c(e_m)$ since the costs are all distinct.

Assume T and T' are two different minimum spanning trees of G. Let the edges of T: $E_1 = \{e_{a_1}, e_{a_2}, \cdots, e_{a_{n-1}}\}$ and let the edge of T': $E_2 = \{e_{b_1}, e_{b_2}, \cdots, e_{b_{n-1}}\}$, where $a_1 < a_2 < \cdots < a_{n-1}$ and so does E_2. Obviously $E_1 \neq E_2$ and let k be the least positive integer such that $a_k \neq b_k$. Without loss of generality, let $a_k < b_k$ then $c(e_{a_k}) < c(e_{b_k})$.

Then add e_{a_k} to T', there must be a cycle in T'. Also we can say that the cost of any element in the cycle except e_{a_k} itself, is greater than $c(e_{a_k})$, otherwise it will make T contain a cycle, since $a_i = b_i, \forall 1 \leq i < k$. Therefore we can delete the edge with the largest cost in the cycle, denoted by e_x, and let the new spanning tree be T''. $c(e_x)$ is larger than $c(e_{a_k})$, hence $Cost(T'') = Cost(T') + c(e_{a_k}) - c(e_x) < Cost(T')$, which is contradicted to the fact that T' is the minimum spanning tree.

Therefore G has a unique minimum spanning tree.

3 Exercise 4.22

False. We can easily construct a counter-example as the figure shown. It is easy to check that there are 4 minimum spanning trees with the cost 6, and $\forall e \in E$, e belongs to some minimum-cost spanning tree. However, the spanning tree T' consists of $(A,B), (A,D), (C,D)$ with cost 8 is not the minimum-cost spanning tree.

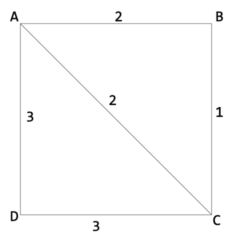

Figure 1: the counter-example