Solutions Manual to Einstein Gravity in a Nutshell

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0 Setting the Stage

Prologue: Three Stories

1 Solution in back of book.

2 Denote the position of the ant by A and the point on the edge halfway between the ant and the drop of honey by M. The shortest path between A and M is the great circle joining the two points. By symmetry, the second half of the trek from M to the honey is also determined.

3 Use the same "cut the cylinder and unroll it" trick as in the text, and follow the discussion in the text allowing for the difference in speed. Evidently, the ant now spends more time on the outside of the glass than on the inside.

I From Newton to Riemann: Coordinates to Curvature

I.1 Newton's Laws

1 Solution in back of book.

2 Solution in back of book.

3 Solution in back of book.

4, 5 Solution in back of book.

6 Let R denote the radius of the earth, d the distance from the midpoint of the tunnel to the center of the earth, r the distance from an arbitrary point along the tunnel to the center of the earth, and w the linear coordinate measuring the distance along the tunnel joining city A and city B. (You need to draw a figure to follow this discussion.) One check to see if you drew the figure correctly: r ranges between d and R. Let the w coordinate be set up so that it ranges from $-\sqrt{R^2-d^2}$ to $+\sqrt{R^2-d^2}$, vanishing at the midpoint of the tunnel. According to Newton's two superb theorems, the force acting on the railroad car (of mass m) when it is at the point specified by r is given by $F_c = -G(Mr^3/R^3)m/r^2 = -GMmr/R^3$. But this force is pointed toward the center of the earth. What we want is the force F along the tunnel, and so we have to multiply F_c by the cosine of some angle; you can see from your figure that the cosine equals w/r. Thus, we have $F = F_c w/r = -GMmw/R^3$. Now Newton's $ma = F$ tells us that $\ddot{w} = -GMw/R^3 = -(g/R)w$ with g the acceleration due to gravity at the surface of the earth (which you learned in school is given by $\sim$32 feet/sec^2 or $\sim$9.8 meters/sec^2). This equation, supplemented by the boundary condition $w(t=-T/2) = -\sqrt{R^2-d^2}$ and $\dot{w}(t=-T/2)=0$ (in light of the symmetry of the situation, we set up coordinates so that $w(t=0)=0$), has the solution $w(t) = -\sqrt{R^2-d^2}\cos[\sqrt{g/R}(t+(T/2))]$. The condition $w(t=0)=0$ then fixes the period $T = \pi\sqrt{R/g}$. The remarkable result is that the transit time, which works out to be about 42 minutes, does not depend on d (and hence the location of the two cities.)

I.2 Conservation Is Good

1 We have $\frac{d}{dt}\sum_a m_a \frac{dx_a^i}{dt} = -\sum_a \frac{\partial V(x)}{\partial x_a^i} = 0$. If you are confused about the last equality, consider the $N = 2$ case in which $(\frac{\partial}{\partial x_1^i} + \frac{\partial}{\partial x_2^i})V(x_1 - x_2) = 0$.

I.3 Rotation: Invariance and Infinitesimal Transformation

1 Consider an infinitesimal rotation by an angle $\delta\theta$ in the (1-2) or (x-y) plane. Then since $\vec{p}$ is a vector, its components transform like (using (14))

$$\begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \to \begin{pmatrix} p^1 + \delta\theta\, p^2 \\ p^2 - \delta\theta\, p^1 \\ p^3 \end{pmatrix}$$

and likewise for $\vec{q}$. The array of three numbers then transforms like

$$\begin{pmatrix} p^2 q^3 \\ p^3 q^1 \\ p^1 q^2 \end{pmatrix} \to \begin{pmatrix} \left(p^2 - \delta\theta\, p^1\right) q^3 \\ p^3 \left(q^1 + \delta\theta\, q^2\right) \\ \left(p^1 + \delta\theta\, p^2\right)\left(q^2 - \delta\theta\, q^1\right) \end{pmatrix}$$

But if this array were a vector, it should have transformed like

$$\begin{pmatrix} p^2 q^3 \\ p^3 q^1 \\ p^1 q^2 \end{pmatrix} \to \begin{pmatrix} p^2 q^3 + \delta\theta\, p^3 q^1 \\ p^3 q^1 - \delta\theta\, p^2 q^3 \\ p^1 q^2 \end{pmatrix}$$

Since the array does not transform as it should under this rotation, it does not transform like a vector under rotations and is therefore not a vector.

For the other array, let us define the coefficients a^i as

$$\begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} p^2 q^3 - p^3 q^2 \\ p^3 q^1 - p^1 q^3 \\ p^1 q^2 - p^2 q^1 \end{pmatrix}$$

Then we note that in index notation, we can write

$$a^i = \varepsilon^{ijk} p^j q^k$$

Under an infinitesimal rotation, $p^j \to (\delta^{jk} + A^{jk})p^k$, where by (14) the matrix A is antisymmetric and traceless. Then the a^i transform like

$$\begin{aligned} a^i &\to \varepsilon^{ijk}\left(\delta^{j\ell} + A^{j\ell}\right)\left(\delta^{km} + A^{km}\right) p^\ell q^m \\ &= \varepsilon^{ijk} p^j q^k + \left(\varepsilon^{i\ell k} A^{km} + \varepsilon^{ijm} A^{j\ell}\right) p^\ell q^m \\ &= \varepsilon^{ijk} p^j q^k + \left(\varepsilon^{inj} A^{jp} \delta^{\ell n} \delta^{mp} - \varepsilon^{inj} A^{jp} \delta^{\ell p} \delta^{mn}\right) p^\ell q^m \\ &= \varepsilon^{ijk} p^j q^k + \left(\delta^{\ell n} \delta^{mp} - \delta^{\ell p} \delta^{mn}\right) \varepsilon^{inj} A^{jp} p^\ell q^m \end{aligned}$$

To get to the second line, we only kept terms linear in A (since this is supposed to be an infinitesimal rotation), and to get to the third line, we used the antisymmetry of ε^{ijk}, relabeled indices, and explicitly included the Kronecker deltas. Now, we can use the identity $\varepsilon^{ijk}\varepsilon^{k\ell m} = \delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}$ to write

$$\begin{aligned}
a^i &\to \varepsilon^{ijk}p^j q^k + \varepsilon^{\ell mk}\varepsilon^{knp}\varepsilon^{inj}A^{jp}p^\ell q^m \\
&= \varepsilon^{ijk}p^j q^k + \varepsilon^{\ell mk}\left(\delta^{pj}\delta^{ik} - \delta^{pi}\delta^{jk}\right)A^{jp}p^\ell q^m \\
&= \varepsilon^{ijk}p^j q^k + \varepsilon^{\ell mk}\left(A^{jj}\delta^{ik} - A^{ki}\right)p^\ell q^m \\
&= \varepsilon^{ijk}p^j q^k + A^{ik}\varepsilon^{k\ell m}p^\ell q^m \\
&= \left(\delta^{ik} + A^{ik}\right)\varepsilon^{k\ell m}p^\ell q^m = \left(\delta^{ik} + A^{ik}\right)a^k
\end{aligned}$$

To get to the second line, we used our Levi-Civita identity again. To get to the fourth line, we used that fact that $A^{jj} = 0$, since A is traceless. Finally, to get to the fifth line, we used the antisymmetry of A and ε and relabeled some indices. Thus the components a^k transform correctly under infinitesimal rotations, and so they do form a vector.

2 Solution in back of book.

3 The infinitesimal generator of rotations about the x-axis is $\mathcal{J}_x$ given in (14); the generator of finite rotations is $R_x(\theta_x) = e^{\theta_x \mathcal{J}_x}$. Note that for integer n, we have

$$\mathcal{J}^{2n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathcal{J}^{2n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then in the power series expansion for $R_x(\theta_x)$, we can separate even and odd powers:

$$\begin{aligned}
R_x(\theta_x) &= \sum_{n=0}^{\infty} \frac{1}{n!}\theta_x^n \mathcal{J}_x^n \\
&= I + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\theta_x^{2n}\mathcal{J}_x^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta_x^{2n+1}\mathcal{J}_x^{2n+1} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sum_{n=0}^{\infty} \frac{1}{(2n)!}\theta_x^{2n} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta_x^{2n+1} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos\theta_x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sin\theta_x \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}
\end{aligned}$$

For $R_y(\theta_y)$, note that we can simply permute the axes $x \to y$, $y \to z$, and $z \to x$, which corresponds to just reshuffling the rows and columns of the matrix $R_x(\theta_x)$, so we can immediately write

$$R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

By explicit computation, note that

$$\begin{aligned} R_x(\theta_x)R_y(\theta_y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ -\sin\theta_x \sin\theta_y & \cos\theta_x & \sin\theta_x \cos\theta_y \\ -\sin\theta_y \cos\theta_x & -\sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix} \\ &\neq \begin{pmatrix} \cos\theta_y & -\sin\theta_x \sin\theta_y & \cos\theta_x \sin\theta_y \\ 0 & \cos\theta_x & \sin\theta_x \\ -\sin\theta_y & -\sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y \end{pmatrix} = R_y(\theta_y)R_x(\theta_x) \end{aligned}$$

4 Let's start with (19):

$$J^{ij}_{(mn)} = -i\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$$

Then we have

$$\begin{aligned} \left[J_{(mn)}, J_{(pq)}\right]^{ij} &= J^{ik}_{(mn)} J^{kj}_{(pq)} - (m \leftrightarrow p,\, n \leftrightarrow q) \\ &= -\left(\delta^{mi}\delta^{nk} - \delta^{mk}\delta^{ni}\right)\left(\delta^{pk}\delta^{qj} - \delta^{pj}\delta^{qk}\right) - (m \leftrightarrow p,\, n \leftrightarrow q) \\ &= -\delta^{np}\delta^{mi}\delta^{qj} + \delta^{nq}\delta^{mi}\delta^{pj} + \delta^{mp}\delta^{ni}\delta^{qj} - \delta^{mq}\delta^{ni}\delta^{pj} - (m \leftrightarrow p,\, n \leftrightarrow q) \\ &= \delta^{np}\left(\delta^{mj}\delta^{qi} - \delta^{mi}\delta^{qj}\right) + \delta^{nq}\left(\delta^{mi}\delta^{pj} - \delta^{pi}\delta^{mj}\right) \\ &\quad + \delta^{mp}\left(\delta^{ni}\delta^{qj} - \delta^{qi}\delta^{nj}\right) + \delta^{mq}\left(\delta^{pi}\delta^{nj} - \delta^{ni}\delta^{pj}\right) \\ &= -i\delta^{np}J^{ij}_{(mq)} + i\delta^{nq}J^{ij}_{(mp)} + i\delta^{mp}J^{ij}_{(nq)} - i\delta^{mq}J^{ij}_{(np)} \end{aligned} \tag{1}$$

and thus

$$\left[J_{(mn)}, J_{(pq)}\right] = i\left(\delta^{nq}J^{ij}_{(mp)} + \delta^{mp}J^{ij}_{(nq)} - \delta^{np}J^{ij}_{(mq)} - \delta^{mq}J^{ij}_{(np)}\right)$$

5 Solution in back of book.

I.4 Who Is Afraid of Tensors?

1 This follows from the observation that $\vec{\nabla}$ transforms like a vector, as suggested by the notation.

2 Just change the sign in (5):

$$\begin{aligned} S^{ij} \to S'^{ij} &= T'^{ij} + T'^{ji} = R^{ik}R^{jl}T^{kl} + R^{jk}R^{il}T^{kl} \\ &= R^{ik}R^{jl}T^{kl} + R^{jl}R^{ik}T^{lk} = R^{ik}R^{jl}(T^{kl} + T^{lk}) = R^{ik}R^{jl}S^{kl} \end{aligned} \tag{2}$$

3 $d^3x' = d^3xJ$, where the Jacobian $J = \det \frac{\partial x'^i}{\partial x^j} = \det R = 1$.

4 Solution in back of book.

5 Solution in back of book.

6 The first index i can take on D values. Then for a specific i, j can take on $D-1$ values, and k only $D-2$ values. Since T^{ijk} is totally antisymmetric, we see that T has $\frac{1}{3!}D(D-1)(D-2)$ components. For $D=3$, there is only one component, namely T^{123}.

7 It has only one component, which we can write as $T^{123} = \frac{1}{3!}\varepsilon^{ijk}T^{ijk}$. Under a rotation, $\varepsilon^{ijk}T^{ijk} \to \varepsilon^{ijk}R^{ii'}R^{jj'}R^{kk'}T^{i'j'k'} = (\det R)\varepsilon^{i'j'k'}T^{i'j'k'} = \varepsilon^{i'j'k'}T^{i'j'k'}$, where we used the definition of the determinant.

8 Solution in back of book.

9 Solution in back of book.

I.5 From Change of Coordinates to Curved Spaces

1 Solution in back of book.

2 We have $x = \theta\cos\varphi$ and $y = \theta\sin\varphi$, so that

$$\begin{aligned} dx &= \cos\varphi\, d\theta - \theta\sin\varphi\, d\varphi & dy &= \sin\varphi\, d\theta + \theta\cos\varphi\, d\varphi \\ d\theta &= \cos\varphi\, dx + \sin\varphi\, dy & \theta\, d\varphi &= -\sin\varphi\, dx + \cos\varphi\, dy \end{aligned}$$

Small x and y implies small θ, so in the metric $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, we expand $\sin^2\theta = \theta^2 - \theta^4/3 + \mathcal{O}(\theta^6)$. The metric then becomes

$$\begin{aligned} ds^2 &= d\theta^2 + \left(1 - \tfrac{1}{3}\theta^2 + \cdots\right)(\theta\, d\varphi)^2 \\ &= (\cos\varphi\, dx + \sin\varphi\, dy)^2 + \left(1 - \tfrac{1}{3}\theta^2 + \cdots\right)(-\sin\varphi\, dx + \cos\varphi\, dy)^2 \\ &= dx^2 + dy^2 - \tfrac{1}{3}\left(-\theta\sin\varphi\, dx + \theta\cos\varphi\, dy\right)^2 + \cdots \\ &= dx^2 + dy^2 - \tfrac{1}{3}\left(-y\, dx + x\, dy\right)^2 + \cdots \\ &= \left(1 - \frac{y^2}{3}\right)dx^2 + \left(1 - \frac{x^2}{3}\right)dy^2 + \tfrac{2}{3}\, xy\, dx\, dy + \cdots \end{aligned}$$

3 Solution in back of book.

4 Solution in back of book.

5 Solution in back of book.

6 Recall that the curvature of a space can be evaluated by considering the ratio of the circumference to the ratio of infinitesimally small circles. If the radius of the circles is ε, the curvature is

$$R = \lim_{\varepsilon \to 0} \frac{6}{\varepsilon^2} \left(1 - \frac{C}{2\pi\varepsilon}\right)$$

where C is the circumference of the circles. We are told that $\varepsilon = \theta$ and $C = 2\pi \sin\theta$, so we have

$$\begin{aligned} R &= \lim_{\varepsilon \to 0} \frac{6}{\varepsilon^2} \left(1 - \frac{C}{2\pi\varepsilon}\right) \\ &= \lim_{\theta \to 0} \frac{6}{\theta^2} \left(1 - \frac{2\pi \sin\theta}{2\pi\theta}\right) \\ &= \lim_{\theta \to 0} \frac{6}{\theta^2} \left(1 - \left(1 - \tfrac{1}{6}\theta^2 + \mathcal{O}(\theta^4)\right)\right) \\ &= \lim_{\theta \to 0} \frac{6}{\theta^2} \left(\tfrac{1}{6}\theta^2 + \mathcal{O}(\theta^4)\right) \\ &= 1 \end{aligned}$$

7 Define the circle by $x^2 + y^2 = 1$. First, obtain the metric: $ds^2 = dx^2 + dy^2 = dx^2/(1 - x^2)$. In polar coordinates, $\theta = \arctan y/x = \arctan \sqrt{1 - x^2}/x$. By direct differentiation, we obtain $d\theta = -dx/\sqrt{1 - x^2}$. We recognize $1/\sqrt{1 - x^2}$ as precisely $\sqrt{g}$.

8 The actual length width and height of the volume element are given by adx, bdy, and cdz, and so the volume is indeed given by $d^3x\sqrt{g} = d^3x\sqrt{a^2b^2c^2}$.

9 Our coordinate transformation between the Cartesian coordinates X^i and spherical coordinates θ_i is

$$X^n = \cos\theta_n \prod_{i=1}^{n-1} \sin\theta_i \text{ for } n \leq d, \qquad X^{d+1} = \prod_{i=1}^{d} \sin\theta_i$$

To prove the Pythagorean relation, we proceed by induction. For the base case of the S^1, it is trivial: we have $X^1 = \cos\theta_1$, $X^2 = \sin\theta_2$, so clearly $(X^1)^2 + (X^2)^2 = 1$. For the inductive hypothesis, assume the Pythagorean relation is satisfied for S^{d-1};

then we show that it is satisfied for S^d as well:

$$\begin{aligned}
\sum_{n=1}^{d} (X^n)^2 + \left(X^{d+1}\right)^2 &= 1 \\
\sum_{n=1}^{d} \cos^2\theta_n \prod_{i=1}^{n-1} \sin^2\theta_i + \prod_{i=1}^{d} \sin^2\theta_i &= 1 \\
\sum_{n=1}^{d-1} \cos^2\theta_n \prod_{i=1}^{n-1} \sin^2\theta_i + \cos^2\theta_d \prod_{i=1}^{d-1} \sin^2\theta_i + \sin^2\theta_d \prod_{i=1}^{d-1} \sin^2\theta_i &= 1 \\
\sum_{n=1}^{d-1} \cos^2\theta_n \prod_{i=1}^{n-1} \sin^2\theta_i + \left(\cos^2\theta_d + \sin^2\theta_d\right) \prod_{i=1}^{d-1} \sin^2\theta_i &= 1 \\
\sum_{n=1}^{d-1} \cos^2\theta_n \prod_{i=1}^{n-1} \sin^2\theta_i + \prod_{i=1}^{d-1} \sin^2\theta_i &= 1
\end{aligned}$$

To get to the third line, we explicitly removed the last term from the summation and from the second product. Now we recognize the final expression as the Pythagorean relation for S^{d-1}, which is satisfied by our inductive hypothesis, and so the Pythagorean relationship is satisfied for S^d.

To find the metric on S^d, we proceed recursively. Consider the coordinates X^i, $i = 1, \ldots, d$ in which we embed an S^{d-1}:

$$X^n = \cos\theta_n \prod_{i=1}^{n-1} \sin\theta_i \text{ for } n \leq d-1, \qquad X^d = \prod_{i=1}^{d-1} \sin\theta_i$$

and the coordinates $\widetilde{X}^i$, $i = 1, \ldots, d+1$ in which we embed an S^d:

$$\widetilde{X}^n = \cos\theta_n \prod_{i=1}^{n-1} \sin\theta_i \text{ for } n \leq d, \qquad \widetilde{X}^{d+1} = \prod_{i=1}^{d} \sin\theta_i$$

We can relate these coordinates via

$$\widetilde{X}^i = X^i \text{ for } i = 1, \ldots, d-1, \qquad \widetilde{X}^d = X^d \cos\theta_d, \qquad \widetilde{X}^{d+1} = X^d \sin\theta_d$$

Then the metric on S^d will be

$$
\begin{aligned}
d\Omega_d^2 &= \sum_{i=1}^{d+1} \left(d\widetilde{X}^i\right)^2 \\
&= \sum_{i=1}^{d-1} \left(d\widetilde{X}^i\right)^2 + \left(d\widetilde{X}^d\right)^2 + \left(d\widetilde{X}^{d+1}\right)^2 \\
&= \sum_{i=1}^{d-1} \left(dX^i\right)^2 + \left(\cos\theta_d\, dX^d - X^d \sin\theta_d\, d\theta_d\right)^2 + \left(\sin\theta_d\, dX^d + X^d\cos\theta_d\, d\theta_d\right)^2 \\
&= \sum_{i=1}^{d-1} \left(dX^i\right)^2 + \left(dX^d\right)^2 + \left(X^d\right)^2 d\theta_d^2 \\
&= \sum_{i=1}^{d} \left(dX^i\right)^2 + \left(X^d\right)^2 d\theta_d^2
\end{aligned}
$$

The first term is just the metric on an S^{d-1}, so we have

$$d\Omega_d^2 = d\Omega_{d-1}^2 + \left(X^d\right)^2 d\theta_d^2$$

The metric on an S^1 is just $d\Omega_1^2 = d\theta_1^2$, so with our expression for X^d, we can iteratively construct the metric on an arbitrary S^d:

$$d\Omega_d^2 = d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-1}\, d\theta_d^2$$

10 Solution in back of book.

11 We have $\sqrt{g} = \sqrt{g_{11}g_{22}\cdots g_{dd}} = (\sin\theta_1)\cdots(\sin\theta_1\cdots\sin\theta_{d-1})$, where in this product, evidently $\sin\theta_1$ appears $(d-1)$ times, $\sin\theta_2$ appears $(d-3)$ times, and so on. It is easier for you to grasp what is going on than for me to say it! The area element is given by this times $d\theta_1 d\theta_2\cdots d\theta_d$ and thus is equal to $d\theta$ for $d=2$, $\sin\theta_1 d\theta_1 d\theta_2$ for $d=3$, and $\sin^2\theta_1\sin\theta_2 d\theta_1 d\theta_2 d\theta_3$ for $d=4$. (You can easily keep going.) Integrating over $0 \le \theta_i < \pi$ for $1 \le i < d$ and over $0 \le \theta_d < 2\pi$, and using $\int_0^\pi d\theta \sin\theta = 2$ and $\int_0^\pi d\theta\sin^2\theta = \frac{\pi}{2}$, we obtain that the generalized area of (2π), $2(2\pi) = 4\pi$, and $(\frac{\pi}{2})2(2\pi) = 2\pi^2$ for S^1, S^2, and S^3, respectively.

12 Distance around the equator: $2\pi(b^2+a^2)/b$. Distance around a circle of fixed longitude through the poles: $2\int_0^\pi d\theta\sqrt{b^2+a^2\cos^2\theta}$, which can be evaluated in terms of elliptic functions, but for $a=b$, this gives $\simeq 7.64a$. The area is $4\pi(b^2+a^2)$.

13 Comparing two similar triangles in figure 3 gives us $(L - \sqrt{L^2-r^2})/r = 2L/\rho$. Solving, we obtain the stated result.

14 The demonstration of the result, that angles are the same as calculated with the two different metrics, is given in the solution I.6.10.

15 To compare the expression for the divergence given in appendix 3, $\partial_r W^r + \partial_\theta W^\theta + \partial_\varphi W^\varphi + \frac{2}{r} W^r + \frac{\cos\theta}{\sin\theta} W^\theta = \frac{1}{r^2}\partial_r(r^2 W^r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta W^\theta) + \partial_\varphi W^\varphi$ (notice that we have merely gone back to the first form given in (26)) with the expression $\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\partial_r(r^2 E_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\partial_\varphi E_\varphi$ given in elementary texts, we see that we have to lower indices and to make a trivial change in normalization. We have $W^\mu = g^{\mu\nu} W_\nu$ with $g^{rr} = 1,\ g^{\theta\theta} = 1/r^2$, and $g^{\varphi\varphi} = 1/(r^2\sin^2\theta)$. Furthermore, from the combination $W^\mu\partial_\mu$ in the expression we started the discussion with in appendix 3, we see that W^θ and W^φ have an extra dimension of inverse length compared to W^r, while $E_r,\ E_\theta$, and E_φ in the elementary discussion all have the same dimension. Thus, we define $W_r = E_r,\ W_\theta = rE_\theta,\ W_\varphi = r\sin\theta E_\varphi$. We recover the usual expression.

16 Solution in back of book.

17 Solution in back of book.

18 Solution in back of book.

I.6 Curved Spaces: Gauss and Riemann

1 The coordinates (x, y) that the Eskimo mites use correspond to distance traveled from the north pole along the sphere. The coordinates $(\tilde{x}, \tilde{y})$ used in (2), however, correspond to distance traveled from the north pole along a tangent plane. If θ is the polar angle of the sphere, we should therefore expect $x^2 + y^2 = \theta^2$, while $\tilde{x}^2 + \tilde{y}^2 = \sin^2\theta$. We can then write the two coordinates as follows:

$$\begin{aligned} x &= \theta\cos\varphi \qquad & y &= \theta\sin\varphi \\ \tilde{x} &= \sin\theta\cos\varphi \qquad & \tilde{y} &= \sin\theta\sin\varphi \end{aligned}$$

We can rearrange these into the explicit coordinate transformation

$$\begin{aligned} x &= \frac{\arcsin\sqrt{\tilde{x}^2+\tilde{y}^2}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}\,\tilde{x} = \tilde{x} + \tfrac{1}{6}\,\tilde{x}\tilde{y}^2 + \tfrac{1}{6}\,\tilde{x}^3 + \cdots \\ y &= \frac{\arcsin\sqrt{\tilde{x}^2+\tilde{y}^2}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}\,\tilde{y} = \tilde{y} + \tfrac{1}{6}\,\tilde{x}^2\tilde{y} + \tfrac{1}{6}\,\tilde{y}^3 + \cdots \end{aligned}$$

Plugging this into the Eskimo mites' metric from exercise I.5.2, we get

$$\begin{aligned} ds^2 &= \left(1 - \frac{y^2}{3}\right)dx^2 + \left(1 - \frac{x^2}{3}\right)dy^2 + \tfrac{2}{3}\,xy\,dx\,dy + \cdots \\ &= \left(1 + \tilde{x}^2\right)d\tilde{x}^2 + \left(1 + \tilde{y}^2\right)d\tilde{y}^2 + 2\,\tilde{x}\tilde{y}\,d\tilde{x}\,d\tilde{y} + \cdots \end{aligned}$$

This is clearly of the form given in (2), with $a^2 = b^2 = 1$ and $c = 0$.

2 Let us take the torus to be a tube of radius a coiled into a circle of radius L. Embedding the torus in usual 3-dimensional Euclidean space, we take its axis to be the z-axis, so if we introduce the usual azimuthal angle φ in the (x-y) plane and an additional angle θ that parametrizes the circles of constant ϕ, we can parametrize the torus as

$$\begin{aligned}
X &= (L + a\sin\theta)\cos\varphi \\
Y &= (L + a\sin\theta)\sin\varphi \\
Z &= a\cos\theta
\end{aligned}$$

Now, consider some point on the torus at coordinates (φ, θ). The tangent plane at this point is defined by the unit normal to the torus, which is just

$$\vec{N} = (\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta)$$

This plane is also spanned by the two tangent vectors

$$\begin{aligned}
\vec{v}_\varphi &= \partial_\varphi\,(X, Y, Z) = (-(L + a\sin\theta)\sin\varphi, (L + a\sin\theta)\cos\varphi, 0) \\
\vec{v}_\theta &= \partial_\theta\,(X, Y, Z) = (a\cos\theta\,\cos\varphi, a\cos\theta\,\sin\varphi, -a\sin\theta)
\end{aligned}$$

To use the tangent plane method, we want to express the derivatives of the normal vector $\vec{N}$ in the basis $\{\vec{v}_\varphi, \vec{v}_\psi\}$:

$$\begin{aligned}
\partial_\varphi\vec{N} &= (-\sin\theta\,\sin\varphi, \sin\theta\,\cos\varphi, 0) = \frac{\sin\theta}{L + a\sin\theta}\,\vec{v}_\varphi \\
\partial_\theta\vec{N} &= (\cos\theta\,\cos\varphi, \cos\theta\,\sin\varphi, -\sin\theta) = \frac{1}{a}\,\vec{v}_\theta
\end{aligned}$$

The shape operator (that is, the matrix M) is then

$$M = \begin{pmatrix} \frac{\sin\theta}{L+a\sin\theta} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

so the intrinsic curvature of the torus is

$$\det M = \frac{\sin\theta}{a(L + a\sin\theta)}$$

3 (a) The metric in the new coordinates becomes

$$\begin{aligned}
ds^2 &= d\kappa^2 + (1 + 2\kappa)\,d\zeta^2 + \cdots \\
&= (d\omega + \phi d\phi)^2 + \left(1 + 2\omega + \phi^2\right)((1 - \omega)d\phi - \phi d\omega)^2 + \cdots \\
&= d\omega^2 + d\phi^2 + \cdots
\end{aligned}$$

All linear terms have now canceled.

(b) If we take ζ to be the angular coordinate of the polar coordinates scaled by r^* and κ to be the normalized deviation of the radial coordinate from r^*, we can write a coordinate transformation as

$$\kappa = r - r^* \qquad \zeta = r^*\theta$$

Then starting from the metric in polar coordinates, the metric becomes

$$\begin{aligned} ds^2 &= dr^2 + r^2\, d\theta^2 \\ &= d\kappa^2 + \left(1 + \frac{\kappa}{r^*}\right)^2 d\zeta^2 \\ &= d\kappa^2 + \left(1 + \frac{2\kappa}{r^*}\right) d\zeta^2 + \cdots \end{aligned}$$

which for $r^* = 1$ is exactly the metric we started with. The civilization has thus been living in a neighborhood of the point $(r, \theta) = (1, 0)$.

4 Solution in back of book.

5 Solution in back of book.

6 Let's express the metric in exercise 3 in the locally flat coordinates $(\omega,\, \phi)$. We know that the space is really flat, and that we can obtain the metric in the coordinates $(\omega,\, \phi)$ from the metric in polar coordinates via

$$\begin{aligned} r &= 1 + \omega + \tfrac{1}{2}\, \phi^2 \\ \theta &= \phi - \phi\omega \end{aligned}$$

Thus we have

$$\begin{aligned} ds^2 &= dr^2 + r^2\, d\theta^2 \\ &= (d\omega + \phi\, d\phi)^2 + \left(1 + \omega + \tfrac{1}{2}\, \phi^2\right)^2 (-\phi\, d\omega + (1-\omega)\, d\phi)^2 \\ &= \left(1 + \phi^2\right) d\omega^2 - 2\phi\omega\, d\phi\, d\omega + \left(1 + 2\phi^2 - 2\omega^2\right) d\phi^2 + \cdots \end{aligned} \tag{3}$$

where the dots represent terms cubic or higher in ω and ϕ. We can now read off the coefficients $B_{\mu\nu,\lambda\sigma}$:

$$\begin{aligned} B_{\omega\omega,\phi\phi} &= 1 \\ B_{\phi\phi,\omega\omega} &= -2 \\ B_{\omega\phi,\omega\phi} &= -\tfrac{1}{2} \end{aligned} \tag{4}$$

The intrinsic curvature is then

$$2B_{\omega\phi,\omega\phi} - B_{\omega\omega,\phi\phi} - B_{\phi\phi,\omega\omega} = 2\left(-\tfrac{1}{2}\right) + 2 - 1 = 0$$

as we should expect, since we know the space is just flat space.

7 Any metric in $D = 1$ will take the form $ds^2 = g_{11}(x)\, dx^2$. But if we change coordinates to $y = \int \sqrt{g_{11}(x)}\, dx$ (which will always be well defined at least in some region of the space), the metric becomes $ds^2 = dy^2$. In this new coordinate, all derivatives of the metric are zero, so curves have no intrinsic curvature.

8 Solution in back of book.

9 As is often the case, it is simpler to work out things in general than this specific example. Suppose we have arrived at $g_{\mu\nu}(x) = \delta_{\mu\nu} + 2A_{\mu\nu,\lambda}x^{\lambda} + \cdots$. (We define $A_{\mu\nu,\lambda}$ here with an extra 2 for convenience.) Now let $x^{\mu} = x'^{\mu} + L^{\mu}{}_{\nu\lambda}x'^{\nu}x'^{\lambda} + \cdots$. Then

$$\begin{aligned} g'_{\rho\sigma} &= (\delta_{\mu\nu} + 2A_{\mu\nu,\lambda}x^{\lambda})(\delta^{\mu}_{\rho} + 2L^{\mu}{}_{\rho\psi}x'^{\psi})(\delta^{\nu}_{\sigma} + 2L^{\nu}{}_{\sigma\phi}x'^{\phi}) + \cdots \\ &= \delta_{\rho\sigma} + 2(A_{\rho\sigma,\lambda} + L_{\sigma,\rho\lambda} + L_{\rho,\sigma\lambda})x^{\lambda} + \cdots \end{aligned}$$

where the first index in $L_{\sigma,\rho\lambda}$ is lowered with the Euclidean metric $\delta_{\mu\nu}$ and I have introduced a comma to emphasize its symmetry properties. Thus, we could remove the linear terms in $g_{\mu\nu}$ by $L_{\sigma,\rho\lambda} + L_{\rho,\sigma\lambda} = -A_{\rho\sigma,\lambda}$ for L. Using an identity similar to that in the appendix to chapter I.4, or simply by inspection, we find $L_{\rho,\sigma\lambda} = -\frac{1}{2}(A_{\rho\sigma,\lambda} + A_{\rho\lambda,\sigma} - A_{\lambda\sigma,\rho})$. Note that, not surprisingly, this is closely related to the formula for the Christoffel symbol in terms of the derivatives of the metric.

10 Solution in back of book.

11 Solution in back of book.

12 We immediately see with no extra work that the Poincaré half plane is conformally flat:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \Omega^2(y)\left(dx^2 + dy^2\right)$$

where $\Omega(y) = 1/y$. For the sphere, we need to do a little more work. Start with the metric on S^2 in standard coordinates:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = \sin^2\theta\left(\frac{d\theta^2}{\sin^2\theta} + d\phi^2\right)$$

Define a new coordinate $\psi = \int d\theta/\sin\theta = \ln\tan(\theta/2)$; then $\sin^2\theta = \operatorname{sech}^2\psi$, so that we have

$$ds^2 = \Omega^2(\psi)\left(d\psi^2 + d\phi^2\right)$$

with $\Omega(\psi) = \operatorname{sech}\psi$. Thus the sphere is also conformally flat.

13 Solution in back of book.

14 According to appendix 3, the n-dimensional hyperbolic space is described by $ds^2 = \frac{dr^2}{1+r^2} + r^2 d\Omega^2_{n-1}$. Set this equal to $ds^2 = \Omega^2(d\rho^2 + \rho^2 d\Omega^2_{n-1})$, and we have the two conditions $r = \Omega\rho$ and $dr/\sqrt{1+r^2} = \Omega d\rho = r d\rho/\rho$. Integrating this equation, we find $r = 2\rho/(1-\rho^2)$ and $\Omega = 2/(1-\rho^2)$.

I.7 Differential Geometry Made Easy, but Not Any Easier!

1 We easily calculate $\vec{t} = \frac{d\vec{X}}{dl} = \frac{d\vec{X}}{d\varphi}\frac{d\varphi}{dl}$. From $\vec{t}^2 = 1$, we obtain $\frac{d\varphi}{dl} = e^{-a\varphi}/\sqrt{a^2+1}$. We then calculate $\frac{d\vec{t}}{dl} = \frac{d\vec{t}}{d\varphi}\frac{d\varphi}{dl}$, the magnitude of which gives $\kappa = e^{-a\varphi}/\sqrt{a^2+1}$. Notice that $\kappa \to 0$ as $\varphi \to \infty$, but $\kappa \to \infty$ as $\varphi \to -\infty$; that is, as the spiral spirals in toward the origin, it winds itself tighter and tighter. We next find $\vec{b}$, which turns out to be just $(0, 0, 1)$. As might be expected, $\tau = 0$.

2 Solution in back of book.

3 Solution in back of book.

4 A simple calculation shows that $\kappa = a$ and $\tau = b$, in complete accordance with our geometric intuition. For $b = 0$, the curve does not twist.

5 Evaluating Gauss's equation, we find that $K_{\mu\nu} = -g_{\mu\nu}$ and thus $\mathcal{G} = 1$.

6 Solution in back of book.

II Action, Symmetry, and Conservation

II.1 The Hanging String and Variational Calculus

1 We have $F'(u) = u/\sqrt{1+u^2}$. Thus, instead of the first term in (3), we would obtain $T\frac{d}{dx}\left(\frac{1}{\sqrt{1+(\frac{d\phi}{dx})^2}}\frac{d\phi}{dx}\right)$.

2 Solution in back of book.

3 Solution in back of book.

4 The metric is given by $ds^2 = dx^2 + dy^2 + d\phi^2 = dx^2 + dy^2 + (\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy)^2 = (1+(\frac{\partial\phi}{\partial x})^2)dx^2 + (1+(\frac{\partial\phi}{\partial y})^2)dy^2 + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}dxdy$. The area element $dxdy\sqrt{g}$ is then up to quadratic order in ϕ given by $dxdy\sqrt{1+(\frac{\partial\phi}{\partial x})^2+(\frac{\partial\phi}{\partial y})^2}$. Thus, the amount by which the area is stretched is given by $\int dxdy\ \frac{1}{2}[(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2]$.

5 Solution in back of book.

II.2 The Shortest Distance between Two Points

1 From (11) we obtain $\sin^2\theta\frac{d\varphi}{dl} = K \le 1$, which when plugged into (12) gives us $(\frac{d\theta}{dl})^2 + \frac{K^2}{\sin^2\theta} = 1$. The potential has a minimum value of $K^2 \le 1$ at $\theta = \pi/2$. The particle oscillates between two values of θ determined by $\sin^2\theta = K^2$.

2 We need to consider only three cases (up to permutations of the indices) in which the Christoffel symbol does not vanish: (1) $\Gamma^1_{11} = -\frac{1}{2}g^{11}\partial_1 g_{11}$, (2) $\Gamma^2_{11} = \frac{1}{2}g^{22}\partial_2 g_{11}$, and (3) $\Gamma^1_{12} = \frac{1}{2}g^{11}\partial_2 g_{11}$. Note that $\Gamma^1_{23} = 0$.

3 The solution is actually given in the text.

4 The definition of length l gives us $(\frac{dx}{dl})^2 + (\frac{dy}{dl})^2 = y^2$. The geodesic equation $\frac{d}{dl}(\frac{dx}{dl}/y^2) = 0$ gives $\frac{dx}{dl} = y^2/b$. Plugging this in, we obtain $(\frac{dy}{dl})^2 = y^2 - y^4/b^2$. With the change of variable $y = b\sin\theta$, we find $x = \mp b\cos\theta + x_*$, in agreement with the result in appendix 4.

5 The Christoffel symbol $\Gamma^{\eta}_{\omega\sigma}$ vanishes in Cartesian coordinates. Thus, the Christoffel symbol in polar coordinates is given by $\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\eta}} \frac{\partial^2 x^{\eta}}{\partial x'^{\mu} \partial x'^{\nu}}$. We obtain $\Gamma^{r}_{\theta\theta} = \frac{\partial r}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial r}{\partial y}\frac{\partial^2 y}{\partial \theta^2} = -r$. Similarly, $\Gamma^{\theta}_{r\theta} = \frac{\partial \theta}{\partial x}\frac{\partial^2 x}{\partial r \partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial^2 y}{\partial r \partial \theta} = \frac{1}{r}$.

6 Solution in back of book.

7 Plugging into (30), we obtain $\Gamma'^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}$, and so set $L^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\mu\nu}$.

8 Solution in back of book.

II.3 Physics Is Where the Action Is

1 Set $m = 2$, so that $S = \int_0^T (\dot{q}^2 - \omega^2 q^2)$. Write the solution of the equation of motion $\ddot{q} = -\omega^2 q$ as $q_0(t) = a\cos\omega t + b \sin\omega t$ with a, b fixed by some boundary conditions (with $q(0) \neq q(T)$.) Let $q = q_0 + \delta q$. Then the second order variation of the action is $\delta S^{(2)} = \int_0^T ((\delta\dot{q})^2 - \omega^2 \delta q^2)$ with the boundary conditions $\delta q(0) = \delta q(T) = 0$. We can already see that we could make $\delta S^{(2)} < 0$ if δq varies slowly enough in time. More explicitly, write $\delta q = \sum_1^\infty c_n \sin(n\pi t/T)$. Then $\delta S^{(2)} = \frac{1}{2}\sum_1^\infty ((n\pi/T)^2 - \omega^2)c_n^2$. For T large enough such that $\pi/T < \omega$, we could, by choosing c_n appropriately, get either sign for $\delta S^{(2)}$.

2 We find $S(q_2) = \frac{2h^2}{3T} - \frac{hT}{3}$, and so $S(q_2) - S_{\min} = \frac{h^2}{6T} + \frac{hT}{6} + \frac{T^3}{24} > 0$.

3 We simply take the answer given in the text, set $h = 0$ and restore g.

4 Solution in back of book.

5 $dx^+ \frac{\partial}{\partial x^+} = dx^+ \frac{\partial x'^+}{\partial x^+}\frac{\partial}{\partial x'^+} = dx'^+ \frac{\partial}{\partial x'^+}$.

6 Use Cartesian coordinates. Let a string of length $2L$ be tied down at $x = \pm a$. Call the area enclosed between the string and the x-axis $2A = \int_{-a}^{a} dx\ y(x)$. The constraint reads $2P \equiv \int_{-a}^{a} dx\ \sqrt{1 + (\frac{dy}{dx})^2} = 2L$. (The various factors of 2 are for convenience.) Thanks to Lagrange, we simply plug and chug. We have the obvious symmetry condition $y(x) = y(-x)$. Extremizing $A + \lambda P = \int_0^a dx\ \left(y + \lambda\sqrt{1 + (\frac{dy}{dx})^2}\right)$ with the boundary condition $y(a) = 0$ and the Lagrange multiplier λ, we obtain the Euler-Lagrange equation $\frac{d}{dx}\frac{\lambda \frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}} = 1$, which implies $(\frac{dy}{dx})^2 = \frac{x^2}{\lambda^2 - x^2}$. We integrate to obtain $y = \sqrt{\lambda^2 - x^2} = \sqrt{a^2 - x^2}$ with the second equality fixed by the boundary condition. We find the circle as expected. Imposing the constraint, we find $2L = 2\int_0^a dx\ \sqrt{1 + (\frac{dy}{dx})^2} = 2a\int_0^a dx\ 1/\sqrt{a^2 - x^2} = \pi a$. By the way, the geometrical proof I learned as a kid is given here: www.math.upenn.edu/˜shiydong/Math501X-5-Isoperimetric.pdf.

II.4 Symmetry and Conservation

1 We now have rotational invariance around the x-, y-, and z-axes. Simply repeat what we did to get to (10). The 3 conserved quantities are $L_i = m\epsilon_{ijk}x^j\frac{dx^k}{dt}$, where ϵ_{ijk} is the totally antisymmetric symbol defined in chapter I.3. For example, $L_1 = m\left(x^2\frac{dx^3}{dt} - x^3\frac{dx^2}{dt}\right)$. This is of course just the familiar formula $\vec{L} = \vec{x}\times\vec{p}$ from elementary physics.

III Space and Time Unified

III.1 Galileo versus Maxwell

1 Solution in back of book.

III.2 Einstein's Clock and Lorentz's Transformation

1 Squaring, we find $(\Delta t')^2 = \frac{4L^2}{c^2-u^2} = \frac{(\Delta t)^2}{1-\frac{u^2}{c^2}}$ and thus the relation between $\Delta t'$ and Δt. The rest follows.

2 You can simply plug in and verify that one matrix is the inverse of the other. The easiest way to show this is to use the form of the transformation in light cone coordinates (5). Then $u \to -u$ corresponds to $\phi \to -\phi$ and e^ϕ going into its inverse.

3 With no loss of generality, let the object be moving in the (x-y) plane. Then the differential form of the Lorentz transformation gives $dt' = \frac{dt-udx}{\sqrt{1-u^2}}$, $dx' = \frac{dx-udt}{\sqrt{1-u^2}}$, $dy' = dy$. Dividing $dx' = \frac{dx-udt}{\sqrt{1-u^2}}$, dy' by dt', we obtain $v'_x = \frac{v_x+u}{1+uv_x}$, $v'_y = \sqrt{1-u^2}\frac{v_y}{1+uv_x}$. If we insist on writing this using vectors, we obtain the rather unwieldy expression

$$\vec{v}' = [\sqrt{1-u^2}u^2\vec{v} + ((1-\sqrt{1-u^2})\vec{u}\cdot\vec{v} + u^2)\vec{u}]/(u^2(1+\vec{u}\cdot\vec{v}))$$

This exercise reinforces the point, to be made in chapter III.6, that it is advisable to deal with objects that transform "naturally."

4 The limiting value v_* is determined by $v_* = \frac{v_*+u}{1+uv_*}$. Solving, we obtain $v_*^2 = 1$, as expected. Note that v_0 does not enter, also as expected.

III.3 Minkowski and the Geometry of Spacetime

1 Solution in back of book.

2 We have $dt = \rho\cosh T\ dT + \sinh T\ d\rho$, $dr = \rho\sinh T\ dT + \cosh T\ d\rho$. Thus, $dt^2 - dr^2 = \rho^2 dT^2 - d\rho^2$.

3 Solution in back of book.

4 Solution in back of book.

5 Let $\omega^{\mu\nu} \equiv a^\mu V^\nu - V^\mu a^\nu$ so that $\frac{dW^\mu}{d\tau} = -\omega^{\mu\nu} W_\nu$. Then

$$\frac{d}{d\tau}(W_\mu W^\mu) = 2W_\mu \frac{dW^\mu}{d\tau} = -2\omega^{\mu\nu} W_\mu W_\nu = 2\omega^{\nu\mu} W_\nu W_\mu = 0$$

where the last step follows from the fact that something equal to minus itself vanishes. This is just an example of the general result we learned in chapter I.4 that the product of an antisymmetric tenor (namely $\omega^{\mu\nu}$) and a symmetric tensor (namely $W_\mu W_\nu$) vanishes. Next, $(a^\mu V^\nu - V^\mu a^\nu)V_\nu = -a^\mu = -\frac{dV^\mu}{d\tau}$. Finally, $-\frac{d}{d\tau}(V_\mu W^\mu) = V_\mu \omega^{\mu\nu} W_\nu + W_\mu \omega^{\mu\nu} V_\nu = 0$.

6 Solution in back of book.

7 In the context of appendix 1, the statement is true by definition. The metric $\eta_{\mu\nu}$ regarded as a matrix is left invariant; in particular, the number of eigenvalues equal to $+1$ or -1 cannot change. This statement has more content for curved spaces or spacetimes. It amounts to the intuitive statement that a space cannot be turned into a spacetime by a coordinate transformation, and vice versa. In the context of chapter I.6, when we go to locally flat coordinates, we could scale the diagonal elements of $g_{\mu\nu}(0)$ to ± 1, but we cannot change their signs.

8 Solution in back of book.

9 In the experimentalist's frame, $u^\mu = (1, \vec{0})$ and hence the energy of the particle is given by $E = -u^\mu p_\mu$. Since (in any frame) $\vec{p}^2 = E^2 + p^2$, plugging in what we have for E, the stated result follows.

10 We have $d(t,x) \equiv d_{\mathrm{AB}} + d_{\mathrm{BC}} = \sqrt{t^2 - x^2} + \sqrt{(2-t)^2 - x^2}$. By symmetry, take $0 < t < 1$. Since

$$\frac{\partial d(t,x)}{\partial x} = -x \left((\sqrt{t^2 - x^2})^{-1} + (\sqrt{(2-t)^2 - x^2}\Big)^{-1} \right) < 0$$

the distance $d(t,x)$ reaches it maximum at $x = 0$, where $d(t,0) = 2$.

III.5 The Worldline Action and the Unification of Material Particles with Light

1 Solution in back of book.

2 Under a Weyl transformation, we have $\gamma \to e^{4\omega(\tau,\sigma)}\gamma$ and $\gamma^{\alpha\beta}(\tau,\sigma) \to e^{-2\omega(\tau,\sigma)}\gamma^{\alpha\beta}(\tau,\sigma)$. Thus $S = \frac{1}{2}T \int d\tau d\sigma \gamma^{\frac{1}{2}} (e^{4\omega(\tau,\sigma)})^{\frac{1}{2}} \gamma^{\alpha\beta} e^{-2\omega(\tau,\sigma)} (\partial_\alpha X^\mu \partial_\beta X_\mu) = S$ is left invariant.

III.6 Completion, Promotion, and the Nature of the Gravitational Field

1 Solution in back of book.

2 Solution in back of book.

3 Solution in back of book.

4 Solution in back of book.

5 Solution in back of book.

6 The string current found in the text takes the form

$$J^{\mu\nu}(x) = \int d\tau\, d\sigma \det \begin{pmatrix} \partial_\tau q^\mu & \partial_\tau q^\nu \\ \partial_\sigma q^\mu & \partial_\sigma q^\nu \end{pmatrix} \delta^{(d)}(x - q(\tau, \sigma))$$

Now, consider making a change of variables to $\tau' = \tau'(\tau, \sigma)$ and $\sigma' = \sigma'(\tau, \sigma)$. Then the measure in the integral changes by a Jacobian:

$$d\tau\, d\sigma = \det \begin{pmatrix} \frac{\partial \tau}{\partial \tau'} & \frac{\partial \sigma}{\partial \tau'} \\ \frac{\partial \tau}{\partial \sigma'} & \frac{\partial \sigma}{\partial \sigma'} \end{pmatrix} d\tau'\, d\sigma' = J\, d\tau'\, d\sigma'$$

Meanwhile, the partial derivatives transform like

$$\begin{aligned} \partial_\tau q^\mu &= \frac{\partial \tau'}{\partial \tau} \partial_{\tau'} q^\mu + \frac{\partial \sigma'}{\partial \tau} \partial_{\sigma'} q^\mu \\ \partial_\sigma q^\mu &= \frac{\partial \tau'}{\partial \sigma} \partial_{\tau'} q^\mu + \frac{\partial \sigma'}{\partial \sigma} \partial_{\sigma'} q^\mu \end{aligned}$$

so that we have

$$\begin{aligned} \det \begin{pmatrix} \partial_\tau q^\mu & \partial_\tau q^\nu \\ \partial_\sigma q^\mu & \partial_\sigma q^\nu \end{pmatrix} &= \det \begin{pmatrix} \frac{\partial \tau'}{\partial \tau} \partial_{\tau'} q^\mu + \frac{\partial \sigma'}{\partial \tau} \partial_{\sigma'} q^\mu & \frac{\partial \tau'}{\partial \tau} \partial_{\tau'} q^\nu + \frac{\partial \sigma'}{\partial \tau} \partial_{\sigma'} q^\nu \\ \frac{\partial \tau'}{\partial \sigma} \partial_{\tau'} q^\mu + \frac{\partial \sigma'}{\partial \sigma} \partial_{\sigma'} q^\mu & \frac{\partial \tau'}{\partial \sigma} \partial_{\tau'} q^\nu + \frac{\partial \sigma'}{\partial \sigma} \partial_{\sigma'} q^\nu \end{pmatrix} \\ &= \det \left[\begin{pmatrix} \frac{\partial \tau'}{\partial \tau} & \frac{\partial \sigma'}{\partial \tau} \\ \frac{\partial \tau'}{\partial \sigma} & \frac{\partial \sigma'}{\partial \sigma} \end{pmatrix} \begin{pmatrix} \partial_{\tau'} q^\mu & \partial_{\tau'} q^\nu \\ \partial_{\sigma'} q^\mu & \partial_{\sigma'} q^\nu \end{pmatrix} \right] \\ &= \det \begin{pmatrix} \frac{\partial \tau'}{\partial \tau} & \frac{\partial \sigma'}{\partial \tau} \\ \frac{\partial \tau'}{\partial \sigma} & \frac{\partial \sigma'}{\partial \sigma} \end{pmatrix} \det \begin{pmatrix} \partial_{\tau'} q^\mu & \partial_{\tau'} q^\nu \\ \partial_{\sigma'} q^\mu & \partial_{\sigma'} q^\nu \end{pmatrix} \\ &= J^{-1} \det \begin{pmatrix} \partial_{\tau'} q^\mu & \partial_{\tau'} q^\nu \\ \partial_{\sigma'} q^\mu & \partial_{\sigma'} q^\nu \end{pmatrix} \end{aligned}$$

Putting these back together, we see that the factors of the Jacobian J cancel precisely, and we get

$$J^{\mu\nu}(x) = \int d\tau'\, d\sigma' \det \begin{pmatrix} \partial_{\tau'} q^\mu & \partial_{\tau'} q^\nu \\ \partial_{\sigma'} q^\mu & \partial_{\sigma'} q^\nu \end{pmatrix} \delta^{(d)}(x - q(\tau', \sigma'))$$

so the string current is indeed invariant under transformations of string coordinates.

7 Recall that angular momentum can be written as $\vec{J} = \vec{r} \times \vec{p}$, so we have, for instance, $J_z = xp_y - yp_x$, and so forth. Of course, we really should think of rotations in general dimensions, in which case we want to label them with two indices, not one, so we instead consider $\mathcal{J}^{ij} = x^i p^j - x^j p^i$. Now, the components of momentum are related to the components of the stress tensor as $p^i = T^{0i}$, so we might be tempted to say that the angular momentum is $\mathcal{J}^{ij} = x^i T^{0j} - x^j T^{0i}$. However, this doesn't transform like a tensor; we really need to integrate it over some 3-volume, so we write

$$\mathcal{M}^{ij} = \int d^3x \left(x^i T^{0j} - x^j T^{0i}\right)$$

We conclude that the $\mathcal{M}^{ij}$ describe the angular momentum of a system. If we upgrade from only spatial indices to Lorentz indices, and define

$$\mathcal{M}^{\mu\nu} = \int d^3x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right)$$

we get an object that describes both angular momentum ($\mathcal{M}^{ij}$) and conserved quantities associated with Lorentz boosts ($\mathcal{M}^{0i}$).

Note that the 12-component $\mathcal{M}^{12} = \int d^3x \left(x^1 T^{02} - x^2 T^{01}\right)$ is precisely the elementary definition of angular momentum around the 3rd axis for an extended body. It seemingly could depend on the time t when the integral is evaluated. Let us check that in fact it does not (namely that angular momentum is conserved):

$$\frac{d}{dt}\mathcal{M}^{ij} = \int d^3x \left(x^i \partial_0 T^{0j} - x^j \partial_0 T^{0i}\right) = \int d^3x \left(T^{ij} - T^{ji}\right) = 0$$

In the second equality we used $\partial_0 T^{0\nu} = -\partial_i T^{i\nu}$, and in the third equality, the fact that $T^{\mu\nu}$ is symmetric.

IV Electromagnetism and Gravity

IV.1 You Discover Electromagnetism and Gravity!

1 We have $\frac{dp^2}{d\tau} = 2p_\mu \frac{dp^\mu}{d\tau} = m^{-1} F^{\mu\nu} p_\mu p_\nu = 0$. The last equality is due to the antisymmetry of $F^{\mu\nu}$.

IV.2 Electromagnetism Goes Live

1 $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(-2F_{0i}^2 + F_{ij}^2) = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$.

2 (a) $\partial_\mu T^{\mu\nu} = (\partial_\mu F^\mu{}_\lambda) F^{\nu\lambda} + F^\mu{}_\lambda \partial_\mu F^{\nu\lambda} - \frac{1}{2} F_{\sigma\rho} \partial^\nu F^{\sigma\rho}$. In the absence of charged particles, the first term vanishes by Maxwell's equation (13). Next, (16) tells us that the sum of $\partial^\nu F^{\sigma\rho}$ and the two terms obtained from it by cyclically permuting the indices $\nu\sigma\rho$ vanish. Hence the third term in the expression above for $\partial_\mu T^{\mu\nu}$ can be rewritten as $+\frac{1}{2} F_{\sigma\rho}(\partial^\sigma F^{\rho\nu} + \partial^\rho F^{\nu\sigma}) = F_{\sigma\rho} \partial^\sigma F^{\rho\nu}$. In contrast, the second term in $\partial_\mu T^{\mu\nu}$, namely $F^{\mu}{}_{\lambda}\partial_\mu F^{\nu\lambda}$, can be written as $-F_{\mu\lambda}\partial^\mu F^{\lambda\nu}$. We thus obtained the desired result. Throughout, we raise and lower indices with the Minkowski metric and use the antisymmetry of $F_{\mu\nu}$.

(b) In the presence of charged particles, the first term in the expression in part a for $\partial_\mu T^{\mu\nu}$ does not vanish; instead, Maxwell's equation (13) gives $\partial_\mu T^{\mu\nu}_{\text{electromagnetic}} = -J_\lambda F^{\nu\lambda}$. We have to add to this $\partial_\mu T^{\mu\nu}_{\text{particles}}$. We simply repeat the calculation in exercise III.6.3 except that in the last step, we have to use the equation of motion in the presence of the electromagnetic field, namely (IV.1.23) from the preceding chapter: $\frac{dp^\mu_a}{d\tau_a} = e_a F^\mu{}_\nu(X_a(\tau_a))\frac{dX^\nu_a}{d\tau_a}$. We obtain $\partial_\mu T^{\mu\nu}_{\text{particles}} = F^{\nu\lambda}J_\lambda = -\partial_\mu T^{\mu\nu}_{\text{electromagnetic}}$. Hence the desired result.

3 We have $T^{00} = F^{0i}F^{0i} + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}$. Plugging in (IV.1.17), we obtain the stated result.

4 Solution in back of book.

5 Solution in back of book.

6 Use the definition $\tilde{F}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$ and plug the identity $\eta_{\lambda\sigma}\epsilon^{\lambda\mu\rho\omega}\epsilon^{\sigma\nu\zeta\psi} = -(\eta^{\mu\nu}\eta^{\rho\zeta}\eta^{\omega\psi} + \ldots)$ into $\eta_{\lambda\sigma}\tilde{F}^{\mu\lambda}\tilde{F}^{\nu\sigma}\ldots$.

7 Solution in back of book.

8 Solution in back of book.

9 This follows almost instantly: $T \equiv \eta_{\mu\nu}T^{\mu\nu} = F_{\mu\lambda}F^{\mu\lambda} - (\frac{1}{4})4F_{\sigma\rho}F^{\sigma\rho} = 0$.

10 Solution in back of book.

Prologue to Book Two: The Happiest Thought

1 Solution in back of book.

V Equivalence Principle and Curved Spacetime

V.1 Spacetime Becomes Curved

1 Solution in back of book.

V.2 The Power of the Equivalence Principle

1 Solution in back of book.

V.3 The Universe as a Curved Spacetime

1 We find $\frac{d}{d\tau}\left((\frac{dt}{d\tau})^2 - a^2(\frac{d\vec{x}}{d\tau})^2\right) = 2(\frac{dt}{d\tau})\frac{d^2t}{d\tau^2} - 2a\dot{a}(\frac{dt}{d\tau})(\frac{d\vec{x}}{d\tau})^2 - 2a^2(\frac{d\vec{x}}{d\tau})\frac{d^2\vec{x}}{d\tau^2} = 0$ upon using (4) and (5).

2 With $g_{00} = -1,\ g_{01} = -\sin x,\ g_{11} = \cos^2 x$, we evaluate (9) to find $dl^2 = dx^2$. To understand this simple result, note that if we replace t by $\tilde{t} = t - \cos x$, the spacetime is in fact flat.

3 For $a(t) = \beta t^\alpha$, we obtain $2\beta(1-\alpha)R = (t_R^{1-\alpha} - t_S^{1-\alpha})$ and thus $D(R; t_S) = \frac{1}{2}[(2\beta(1-\alpha)R + t_S^{1-\alpha})^{\frac{1}{1-\alpha}} - t_S]$. For $\alpha < 1$, $(1-\alpha) > 0$, so that as R tends infinity, so does t_R. In the limit $R \to \infty$, we obtain $D(R; t_S) \propto R^{\frac{1}{1-\alpha}}$, namely $\propto R^3$ for a matter dominated universe and $\propto R^2$ for a radiation dominated universe. In the opposite limit, for $2\beta(1-\alpha)R \ll t_S^{1-\alpha}$, we find that $D(R; t_S) \simeq \beta t_S^\alpha R = a(t_S)R$, as expected.

In general, for R small, we have $t_R \simeq t_S$, and thus $2R = \int_{t_S}^{t_R} \frac{dt}{a(t)} \simeq (t_R - t_S)/a(t_S)$ and so $D(R; t_S) = \frac{1}{2}(t_R - t_S) \simeq a(t_S)R$.

4 It is merely a matter of evaluating $1 + z \equiv \frac{\omega_e}{\omega_r} = \frac{a(T)}{a(t_S)}$ with $T = t(R; t_S)$ determined by $R = \int_0^R dr = \int_{t_S}^{t(R;t_S)} \frac{dt}{a(t)}$.

For $a(t) = e^t$ we obtain $(1+z) = 1/(1 - Re^{t_S})$. In the $R \to 0$ limit, $z \simeq Re^{t_S}$.

For $a(t) = \beta t^\alpha$ we obtain $(1+z) = t_S^{-\alpha}(\beta(1-\alpha)R + t_S^{1-\alpha})^{\frac{1}{1-\alpha}}$. In the $R \to 0$ limit, we have $z \simeq \alpha\beta R/{t_S}^{1-\alpha}$.

5 Solution in back of book.

6 Solution in back of book.

7 Solution in back of book.

8 Consider the discussion in the text about sending a message carried by electromagnetic wave at time t_S to our friend located at $r = R$ and getting a response back at time t_R. We now have $\frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - k\frac{r^2}{L^2}}}$. Integrating this along the outbound trip and the return trip, we obtain

$$\int_{t_S}^{t_R} \frac{dt}{a(t)} = 2\int \frac{dr}{\sqrt{1 - k\frac{r^2}{L^2}}} = 2L(\sin^{-1}(R/L) \quad \text{or} \quad \sinh^{-1}(R/L))$$

for a closed and open universe, respectively. For a given $a(t)$, this determines $t_R = t_R(R, t_S)$, which in turn fixes the distance between us and our friend $D(R; t_S) = \frac{1}{2}(t_R - t_S)$.

V.4 Motion in Curved Spacetime

1 Simply plug (23) into (17).

2 Solution in back of book.

3 Simply vary to obtain the equations of motion with A and B dependent on t as well as r.

4 Solution in back of book.

5 Solution in back of book.

6 Solution in back of book.

V.6 Covariant Differentiation

1 You can show this either by verifying that the divergence as stated transforms correctly or by pretending that $T^{\mu\nu}$ is given by $W^{\mu}U^{\nu}$.

2 Simply plug in and do an ordinary integration by parts.

3 For the last part of the exercise: $D_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^{\mu}_{\lambda\rho} g_{\rho\nu} + \Gamma^{\nu}_{\lambda\rho} g_{\mu\rho} = -g^{\mu\sigma}(\partial_\lambda g_{\sigma\omega} - \Gamma^{\eta}_{\lambda\sigma} g_{\eta\omega} - \Gamma^{\eta}_{\lambda\omega} g_{\sigma\eta}) g^{\omega\nu} = -g^{\mu\sigma}(D_\lambda g_{\sigma\omega}) g^{\omega\nu} = 0$, where we used (7) in the second equality.

4 Solution in back of book.

5 Solution in back of book.

6 For a scalar S define $\frac{DS(\zeta)}{D\zeta} = \frac{dS(\zeta)}{d\zeta}$. This transforms like a scalar.

7 In exercise 6, let $S = U_\mu W^\mu$. Then as we would expect, the covariant derivative of a vector with a lower index along a curve is given by $\frac{DU_\mu(\zeta)}{D\zeta} = \frac{dU_\mu(\zeta)}{d\zeta} - \Gamma^{\nu}_{\mu\lambda}(X(\zeta))V^{\lambda}(\zeta)W_\nu(\zeta)$. Given how tensors transform as if they were made of products of vectors with an upper index and of vectors with a lower index, we can now immediately write down the covariant derivative of tensors along a curve. For example,

$$\frac{DT^{\mu}{}_{\nu}}{D\zeta} = \frac{dT^{\mu}{}_{\nu}}{d\zeta} + \Gamma^{\mu}_{\lambda\rho} V^{\lambda} T^{\rho}{}_{\nu} - \Gamma^{\rho}_{\lambda\nu} V^{\lambda} T^{\mu}{}_{\rho}$$

VI Einstein's Field Equation Derived and Put to Work

VI.1 To Einstein's Field Equation as Quickly as Possible

1 Note: A solution was already provided in the back of the book, but here is a more detailed solution.

This is just plug-and-chug, using the Christoffel symbols we already know:

$$\begin{aligned}
D_\theta W_\theta &= \partial_\theta W_\theta - \Gamma^\theta_{\theta\theta} W_\theta - \Gamma^\varphi_{\theta\theta} W_\varphi \\
&= \partial_\theta W_\theta \\
D_\theta W_\varphi &= \partial_\theta W_\varphi - \Gamma^\theta_{\theta\varphi} W_\theta - \Gamma^\varphi_{\theta\varphi} W_\varphi \\
&= \partial_\theta W_\varphi - \cot\theta\, W_\varphi \\
D_\varphi W_\theta &= \partial_\varphi W_\theta - \Gamma^\theta_{\varphi\theta} W_\theta - \Gamma^\varphi_{\varphi\theta} W_\varphi \\
&= \partial_\varphi W_\theta - \cot\theta\, W_\varphi \\
D_\varphi W_\varphi &= \partial_\varphi W_\varphi - \Gamma^\theta_{\varphi\varphi} W_\theta - \Gamma^\varphi_{\varphi\varphi} W_\varphi \\
&= \partial_\varphi W_\varphi + \sin\theta\cos\theta\, W_\theta
\end{aligned}$$

$$\begin{aligned}
D_\theta U^\theta &= \partial_\theta U^\theta + \Gamma^\theta_{\theta\theta} U^\theta + \Gamma^\theta_{\theta\varphi} U^\varphi \\
&= \partial_\theta U^\theta \\
D_\theta U^\varphi &= \partial_\theta U^\varphi + \Gamma^\varphi_{\theta\theta} U^\theta + \Gamma^\varphi_{\theta\varphi} U^\varphi \\
&= \partial_\theta U^\varphi + \cot\theta\, U^\varphi \\
D_\varphi U^\theta &= \partial_\varphi U^\theta + \Gamma^\theta_{\varphi\theta} U^\theta + \Gamma^\theta_{\varphi\varphi} U^\varphi \\
&= \partial_\varphi U^\theta - \sin\theta\cos\theta\, U^\varphi \\
D_\varphi U^\varphi &= \partial_\varphi U^\varphi + \Gamma^\varphi_{\varphi\theta} U^\theta + \Gamma^\varphi_{\varphi\varphi} U^\varphi \\
&= \partial_\varphi U^\varphi + \cot\theta\, U^\theta
\end{aligned}$$

Next, note that

$$\begin{aligned}
(D_\theta W_\rho)\, U^\rho + W_\rho\,(D_\theta U^\rho) &= (D_\theta W_\theta)\, U^\theta + (D_\theta W_\varphi)\, U^\varphi + W_\theta \left(D_\theta U^\theta\right) + W_\varphi\,(D_\theta U^\varphi) \\
&= (\partial_\theta W_\theta)\, U^\theta + (\partial_\theta W_\varphi - \cot\theta\, W_\varphi)\, U^\varphi \\
&+ W_\theta \left(\partial_\theta U^\theta\right) + W_\varphi\,(\partial_\theta U^\varphi + \cot\theta\, U^\varphi) \\
&= (\partial_\theta W_\theta)\, U^\theta + W_\theta \left(\partial_\theta U^\theta\right) + (\partial_\theta W_\varphi)\, U^\varphi + W_\varphi\,(\partial_\theta U^\varphi) \\
&= \partial_\theta \left(W_\theta U^\theta + W_\varphi U^\varphi\right) \\
&= D_\theta\,(W_\rho U^\rho)
\end{aligned}$$

$$\begin{aligned}
(D_\varphi W_\rho)\, U^\rho + W_\rho\,(D_\varphi U^\rho) &= (D_\varphi W_\theta)\, U^\theta + (D_\varphi W_\varphi)\, U^\varphi + W_\theta \left(D_\varphi U^\theta\right) + W_\varphi\,(D_\varphi U^\varphi) \\
&= (\partial_\varphi W_\theta - \cot\theta\, W_\varphi)\, U^\theta + (\partial_\varphi W_\varphi + \sin\theta\cos\theta\, W_\theta)\, U^\varphi \\
&+ W_\theta \left(\partial_\varphi U^\theta - \sin\theta\cos\theta\, U^\varphi\right) + W_\varphi \left(\partial_\varphi U^\varphi + \cot\theta\, U^\theta\right) \\
&= (\partial_\varphi W_\theta)\, U^\theta + W_\theta \left(\partial_\varphi U^\theta\right) + (\partial_\varphi W_\varphi)\, U^\varphi + W_\varphi\,(\partial_\varphi U^\varphi) \\
&= \partial_\varphi \left(W_\theta U^\theta + W_\varphi U^\varphi\right) \\
&= D_\varphi\,(W_\rho U^\rho)
\end{aligned}$$

and thus we have shown explicitly that $(D_\mu W_\nu)\, U^\nu + W_\nu\,(D_\mu U^\nu) = D_\mu\,(W_\nu U^\nu)$.
Solution in back of book.

3 Simply plug in and verify.

4 Again, simply plug in and verify.

5 In proving relations like this, we can pretend, once again, that $T_{\mu\rho}$ is equal to the product of two vectors $W_\mu Y_\rho$.

6–7 Solution in back of book.

8 Solution in back of book.

9 The only nonzero Christoffel symbols are $\Gamma^y_{y\mu} = (\partial_\mu\phi)/\phi$ and $\Gamma^\mu_{yy} = -\eta^{\mu\nu}\partial_\nu\phi$. Then the nonzero components of the Ricci tensor are

$$\begin{aligned} R_{\mu\nu} &= R^\sigma{}_{\mu\sigma\nu} + R^y{}_{\mu y\nu} \\ &= -\partial_\nu\Gamma^y_{y\mu} - \Gamma^y_{\nu y}\Gamma^y_{y\mu} \\ &= -\frac{\partial_\mu\partial_\nu\phi}{\phi} \\ R_{yy} &= R^\mu{}_{y\mu y} + R^y{}_{yyy} \\ &= \partial_\mu\Gamma^\mu_{yy} - \Gamma^y_{y\mu}\Gamma^\mu_{yy} \\ &= -\phi\Box\phi \end{aligned}$$

where $\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_\mu(\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\nu\phi)/\sqrt{-\eta}$. The Ricci scalar is then

$$R = \eta^{\mu\nu}R_{\mu\nu} + g^{yy}R_{yy} = -\frac{2\Box\phi}{\phi}$$

10 Solution in back of book.

11 Solution in back of book.

12 Go to a locally flat system of coordinates to verify the stated relation.

13 Solution in back of book.

14 We are given $ds^2 = \left(1 + \frac{\rho^2}{4L^2}\right)^{-2}(dx^2 + dy^2)$ with $\rho^2 = x^2 + y^2$, that is $g_{ij} = \Omega^2\delta_{ij}$. To calculate the scalar curvature using (32), we merely have to set $d = 2$ and to calculate $\partial_i\Omega$ and $\partial_i\partial_j\Omega$ with $\Omega = \left(1 + \frac{\rho^2}{4L^2}\right)^{-1}$. We find that $R = 2$.

15 (a) Since the Weyl tensor has all the same symmetries as the Riemann tensor, the only trace we need to evaluate is $g^{\mu\rho}C_{\mu\nu\rho\sigma} = R_{\nu\sigma} + (d-2)^{-1}(R_{\nu\sigma}(2-d) - g_{\nu\sigma}R) + ((d-1)(d-2))^{-1}(d-1)g_{\sigma\nu}R = 0$.

(b) Simply evaluate $\tilde{C}_{\mu\nu\rho\sigma}$ by plugging the result of exercise 12 into (33).

16 $D_U D_V W^\lambda = D_U V^\nu W^\lambda{}_{;\nu} = U^\mu(V^\nu W^\lambda{}_{;\nu})_{;\mu} = U^\mu V^\nu{}_{;\mu} W^\lambda{}_{;\nu} + U^\mu V^\nu W^\lambda{}_{;\nu;\mu}$. Interchange U and V and subtract. The stated result follows.

17 (a) By direct calculation: the metric $g_{xx} = y^2,\ g_{yy} = x^2,\ g_{xy} = 0$ leads to $\Gamma^x_{xy} = 1/y,\ \Gamma^x_{yy} = -x/y^2$, with the other components either vanishing or given by $x \leftrightarrow y$. Then $R^x{}_{yxy} = \partial_x \Gamma^x_{yy} - \partial_y \Gamma^x_{xy} - \Gamma^x_{xy}\Gamma^x_{xy} - \Gamma^x_{yy}\Gamma^y_{xy} = 0$. (2) Let $x = e^u,\ y = e^v$. Then $ds^2 = e^{2(u+v)}(du^2 + dv^2)$. So $\Omega = e^{(u+v)} = xy$.

(b) From exercise 13: we have $\tilde{R} = 0 - 2\delta^{\rho\omega}\frac{\partial_\rho\partial_\omega\Omega}{\Omega^3} - (-2)\delta^{\rho\omega}\frac{(\partial_\rho\Omega)(\partial_\omega\Omega)}{\Omega^4} = 0$.

VI.2 To Cosmology as Quickly as Possible

1 Solution in back of book.

2 Again, Einstein's equation is solved if $p + q + r + s = p^2 + q^2 + r^2 + s^2 = 1$. This has the interesting solution $p = q = r = \frac{1}{2},\ s = -\frac{1}{2}$ so that the 5th dimension (for the meaning of 5, see chapter X.1) contracts.

3 Calculate the Christoffel symbol, then the Ricci tensor. Then solve.

VI.3 The Schwarzschild-Droste Metric and Solar System Tests of Einstein Gravity

1 Just continue the computation started in the text.

2 Solution in back of book.

3 Solution in back of book.

4 Solution in back of book.

5 Solution in back of book.

6 The solution is given in the appendix to chapter VII.4.

7 This involves a straightforward computation best performed with the help of a computer.

VI.4 Energy Momentum Distribution Tells Spacetime How to Curve

1 The electromagnetic stress tensor was given in (8); since we're working in flat spacetime here, we'll go ahead and replace $g_{\mu\nu} \to \eta_{\mu\nu}$:

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}\,\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}$$

Getting T_{ij} is now simply a matter of breaking up the index summations into sums over time and space components (using the fact that $F_{\mu\nu}$ is antisymmetric in its indices), making the substitutions $F_{0i} = E_i$ and $F_{ij} = \varepsilon_{ijk} B_k$, and using some identities involving the Levi-Civita symbol:

$$\begin{aligned}
T_{ij} &= F_{i\lambda} F_j{}^\lambda - \tfrac{1}{4}\,\eta_{ij} F_{\sigma\rho} F^{\sigma\rho} \\
&= F_{i0} F_j{}^0 + F_{ik} F_j{}^k - \tfrac{1}{4}\,\delta_{ij}\left(2F_{0k}F^{0k} + F_{k\ell}F^{k\ell}\right) \\
&= -E_i E_j + \varepsilon_{ik\ell}\varepsilon_{jkm} B_\ell B_m - \delta_{ij}\left(-2E_k E_k + \tfrac{1}{4}\,\varepsilon_{k\ell m}\varepsilon_{k\ell n} B_m B_n\right) \\
&= -E_i E_j + \left(\delta_{ij}\delta_{\ell m} - \delta_{im}\delta_{\ell j}\right) B_\ell B_m - \tfrac{1}{4}\,\delta_{ij}\left(-2\vec{E}^2 + 2\delta_{\mu\nu} B_m B_n\right) \\
&= -E_i E_j + \delta_{ij} B_\ell B_\ell - B_j B_i - \tfrac{1}{4}\,\delta_{ij}\left(-2\vec{E}^2 + 2B_m B_m\right) \\
&= -E_i E_j - B_i B_j + \tfrac{1}{2}\,\delta_{ij}\left(\vec{E}^2 + \vec{B}^2\right)
\end{aligned}$$

The trace of the stress tensor is thus

$$\begin{aligned}
T &= \eta^{\mu\nu} T_{\mu\nu} \\
&= -T_{00} + \delta^{ij} T_{ij} \\
&= -\tfrac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) - \vec{E}^2 - \vec{B}^2 + \tfrac{3}{2}\left(\vec{E}^2 + \vec{B}^2\right) \\
&= 0
\end{aligned}$$

as it should.

2 The divergence of the stress energy tensor of a gas of particles is

$$\begin{aligned}
D_\mu T^{\mu\nu}_{\text{particles}} &= \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{dX_a^\nu}{d\tau_a} \frac{dX_a^\mu}{d\tau_a} D_\mu \delta^4(x - X_a) \\
&= \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{dX_a^\nu}{d\tau_a} \frac{d}{d\tau_a} \delta^4(x - X_a) \\
&= -\frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{d^2 X_a^\nu}{d\tau_a^2} \delta^4(x - X_a)
\end{aligned}$$

To get to the second line, we used the chain rule to write $(dX_a^\mu/d\tau_a) D_\mu = d/d\tau_a$, and to get to the last line, we integrated by parts. But since no external forces act on the particles, we have by the geodesic equation that

$$m_a \frac{dX_a^\mu}{d\tau_a} D_\mu \frac{dX_a^\nu}{d\tau_a} = m_a \frac{d^2 X_a^\nu}{d\tau_a^2} = 0$$

and thus $D_\mu T^{\mu\nu}_{\text{particles}} = 0$.

3 The stress energy tensor is

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{dX_a^\mu}{d\tau_a} \frac{dX_a^\nu}{d\tau_a} \delta^4(x - X_a) - g_{\sigma\lambda} F^{\mu\sigma} F^{\nu\lambda} + \tfrac{1}{4} g^{\mu\nu} F_{\sigma\lambda} F^{\sigma\lambda}$$

Let's take the divergence term-by-term, beginning with the last two terms (that is, the stess energy tensor of the electromagnetic field):

$$\begin{aligned} D_\mu T^{\mu\nu}_{\mathrm{EM}} &= D_\mu \left(-g_{\sigma\lambda} F^{\mu\sigma} F^{\nu\lambda} + \tfrac{1}{4}\, g^{\mu\nu} F_{\sigma\lambda} F^{\sigma\lambda} \right) \\ &= -g_{\sigma\lambda} \left(F^{\mu\sigma} D_\mu F^{\nu\lambda} + F^{\nu\lambda} D_\mu F^{\mu\sigma} \right) + \tfrac{1}{4}\, g^{\mu\nu} D_\mu \left(F_{\sigma\lambda} F^{\sigma\lambda} \right) \end{aligned}$$

Maxwell's equations with a source give us that $D_\mu F^{\mu\sigma} = -J^\sigma$, and noting that $D_\mu \left(F_{\sigma\lambda} F^{\sigma\lambda} \right) = 2F^{\sigma\lambda} D_\mu F_{\sigma\lambda}$, we are left with

$$\begin{aligned} D_\mu T^{\mu\nu}_{\mathrm{EM}} &= -g_{\sigma\lambda} \left(F^{\mu\sigma} D_\mu F^{\nu\lambda} - F^{\nu\lambda} J^\sigma \right) + \tfrac{1}{2}\, g^{\mu\nu} F^{\sigma\lambda} D_\mu F_{\sigma\lambda} \\ &= F^\nu{}_\sigma J^\sigma - g^{\sigma\lambda} F^\mu{}_\sigma D_\mu F^\nu{}_\lambda + \tfrac{1}{2}\, g^{\mu\nu} F^{\sigma\lambda} D_\mu F_{\sigma\lambda} \\ &= F^\nu{}_\sigma J^\sigma - g^{\sigma\nu} F^{\mu\lambda} D_\mu F_{\sigma\lambda} + \tfrac{1}{2}\, g^{\mu\nu} F^{\sigma\lambda} D_\mu F_{\sigma\lambda} \end{aligned}$$

Relabeling indices in the last two terms allows us to rewrite them as

$$\begin{aligned} D_\mu T^{\mu\nu}_{\mathrm{EM}} &= F^\nu{}_\sigma J^\sigma - \tfrac{1}{2} g^{\mu\nu} F^{\sigma\lambda} \left(2D_\sigma F_{\mu\lambda} - D_\mu F_{\sigma\lambda} \right) \\ &= F^\nu{}_\sigma J^\sigma - \tfrac{1}{2} g^{\mu\nu} F^{\sigma\lambda} \left(2D_\sigma F_{\mu\lambda} + D_\mu F_{\lambda\sigma} \right) \end{aligned}$$

The first term in the parentheses can be rewritten as

$$\begin{aligned} 2F^{\sigma\lambda} D_\sigma F_{\mu\lambda} &= F^{\sigma\lambda} D_\sigma F_{\mu\lambda} + F^{\sigma\lambda} D_\sigma F_{\mu\lambda} \\ &= F^{\sigma\lambda} D_\sigma F_{\mu\lambda} + F^{\lambda\sigma} D_\sigma F_{\lambda\mu} \\ &= F^{\sigma\lambda} D_\sigma F_{\mu\lambda} + F^{\sigma\lambda} D_\lambda F_{\sigma\mu} \\ &= F^{\sigma\lambda} \left(D_\sigma F_{\mu\lambda} + D_\lambda F_{\sigma\mu} \right) \end{aligned}$$

This leaves us with

$$\begin{aligned} D_\mu T^{\mu\nu}_{\mathrm{EM}} &= F^\nu{}_\sigma J^\sigma - \tfrac{1}{2} g^{\mu\nu} F^{\sigma\lambda} \left(D_\sigma F_{\mu\lambda} + D_\lambda F_{\sigma\mu} + D_\mu F_{\lambda\sigma} \right) \\ &= F^\nu{}_\sigma J^\sigma \end{aligned}$$

where we used the Bianchi identity to kill the quantity in parentheses. The current of a collection of charged particles is

$$J^\sigma = \frac{1}{\sqrt{-g}} \sum_a e_a \int d\tau_a\, \frac{dX^\sigma_a}{d\tau_a} \delta^4(x - X_a)$$

Therefore we have

$$\begin{aligned} D_\mu T^{\mu\nu}_{\mathrm{EM}} &= \frac{1}{\sqrt{-g}} \sum_a e_a \int d\tau_a\, F^\nu{}_\sigma \frac{dX^\sigma_a}{d\tau_a} \delta^4(x - X_a) \\ &= \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a\, \frac{d^2 X^\nu_a}{d\tau_a^2} \delta^4(x - X_a) \end{aligned}$$

where we used the Lorentz force law

$$m_a \frac{dX^\mu_a}{d\tau_a} D_\mu \frac{dX^\nu_a}{d\tau_a} = m_a \frac{d^2 X^\nu_a}{d\tau_a^2} = e_a F^\nu{}_\sigma \frac{dX^\sigma_a}{d\tau_a}$$

Next, we need to find the divergence of the stress energy tensor of the particles. Since all the x dependence is in the delta function, we have

$$\begin{aligned} D_\mu T^{\mu\nu}_{\text{particles}} &= \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{dX^\nu_a}{d\tau_a} \frac{dX^\mu_a}{d\tau_a} D_\mu \delta^4(x - X_a) \\ &= \frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{dX^\nu_a}{d\tau_a} \frac{d}{d\tau_a} \delta^4(x - X_a) \\ &= -\frac{1}{\sqrt{-g}} \sum_a m_a \int d\tau_a \, \frac{d^2 X^\nu_a}{d\tau_a^2} \delta^4(x - X_a) \end{aligned}$$

To get to the second line, we used the chain rule to write $(dX^\mu_a/d\tau_a)D_\mu = d/d\tau_a$, and to get to the last line, we integrated by parts. This result is precisely the opposite of the divergence of the stress energy tensor of the electromagnetic field, so we have found that

$$D_\mu T^{\mu\nu} = D_\mu T^{\mu\nu}_{\text{particles}} + D_\mu T^{\mu\nu}_{\text{EM}} = 0$$

4 Solution in back of book.

5 Solution in back of book.

6 Just follow the same steps leading to (18).

7 Solution in back of book.

8 Solution in back of book.

VI.5 Gravity Goes Live

1 The Palatini action is

$$S = \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, g^{\mu\nu} R_{\mu\nu}(\Gamma) + S_{\text{matter}}$$

The variation with respect to $g^{\mu\nu}$ is easy:

$$\begin{aligned} \delta S &= \frac{1}{16\pi G} \int d^4x \, \left[\left(\sqrt{-g} \, \delta g^{\mu\nu} + \delta\sqrt{-g} \, g^{\mu\nu} \right) R_{\mu\nu}(\Gamma) + \sqrt{-g} \, g^{\mu\nu} \delta R_{\mu\nu}(\Gamma) \right] + \delta S_{\text{matter}} \\ &= \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \Big[\Big(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \Big) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}(\Gamma) \Big] + \delta S_{\text{matter}} \end{aligned}$$

The quantity in brackets being contracted with $\delta g^{\mu\nu}$ gives us the Einstein field equations, as usual:

$$R_{\mu\nu} - \tfrac{1}{2} \, g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Next, we need to vary the piece containing $\delta R_{\mu\nu}(\Gamma)$:

$$\begin{aligned} \sqrt{-g} \, g^{\mu\nu} \delta R_{\mu\nu}(\Gamma) &= \sqrt{-g} \, g^{\mu\nu} \delta \left[\partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\mu \Gamma^\sigma_{\sigma\nu} + \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\lambda\mu} \right] \\ &= \sqrt{-g} \, g^{\mu\nu} \left[\partial_\sigma \delta\Gamma^\sigma_{\mu\nu} - \partial_\mu \delta\Gamma^\sigma_{\sigma\nu} + \delta\Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} + \Gamma^\lambda_{\mu\nu} \delta\Gamma^\sigma_{\lambda\sigma} - 2\delta\Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\lambda\mu} \right] \end{aligned}$$

Now, let's work on integrating the first two terms by parts. Using our handy identities

$$\begin{aligned}
\partial_\sigma g^{\mu\nu} &= -g^{\mu\lambda} g^{\nu\rho} \partial_\sigma g_{\lambda\rho} \\
\partial_\sigma \sqrt{-g} &= \frac{1}{2} \sqrt{-g}\, g^{\lambda\rho} \partial_\sigma g_{\lambda\rho}
\end{aligned}$$

we can write (dropping total derivatives, since they'll drop out when we integrate by parts)

$$\begin{aligned}
\sqrt{-g}\, g^{\mu\nu} \partial_\sigma \delta\Gamma^\sigma_{\mu\nu} &= -\partial_\sigma \left(\sqrt{-g}\, g^{\mu\nu}\right) \delta\Gamma^\sigma_{\mu\nu} \\
&= \sqrt{-g} \left[g^{\mu\lambda} g^{\rho\nu} \partial_\sigma g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\sigma g_{\lambda\rho} \right] \delta\Gamma^\sigma_{\mu\nu} \\
-\sqrt{-g}\, g^{\mu\nu} \partial_\mu \delta\Gamma^\sigma_{\sigma\nu} &= -\sqrt{-g} \left[g^{\mu\lambda} g^{\rho\nu} \partial_\mu g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\mu g_{\lambda\rho} \right] \delta\Gamma^\sigma_{\sigma\nu}
\end{aligned}$$

As a result, we find

$$\begin{aligned}
&\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}(\Gamma) = \\
&\sqrt{-g} \Big[\Big(g^{\mu\lambda} g^{\rho\nu} \partial_\sigma g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\sigma g_{\lambda\rho} \Big) \delta\Gamma^\sigma_{\mu\nu} - \Big(g^{\mu\lambda} g^{\rho\nu} \partial_\mu g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\mu g_{\lambda\rho} \Big) \delta\Gamma^\sigma_{\sigma\nu} \\
&\qquad\qquad + g^{\mu\nu} \Big(\delta\Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} + \Gamma^\lambda_{\mu\nu} \delta\Gamma^\sigma_{\lambda\sigma} - 2\delta\Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\lambda\mu} \Big) \Big]
\end{aligned}$$

$$\begin{aligned}
&\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}(\Gamma) = \sqrt{-g} \Big[g^{\mu\lambda} g^{\rho\nu} \partial_\sigma g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\sigma g_{\lambda\rho} \\
&\quad - \Big(g^{\delta\lambda} g^{\rho\nu} \partial_\delta g_{\lambda\rho} - \tfrac{1}{2} g^{\delta\nu} g^{\lambda\rho} \partial_\delta g_{\lambda\rho} \Big) \delta^\mu_\sigma + g^{\mu\nu} \Gamma^\lambda_{\sigma\lambda} + g^{\lambda\rho} \Gamma^\mu_{\lambda\rho} \delta^\nu_\sigma - 2 g^{\lambda\nu} \Gamma^\mu_{\sigma\lambda} \Big] \delta\Gamma^\sigma_{\mu\nu}
\end{aligned}$$

Now, we must assume that the connection Γ is symmetric in its two lower indices, so we really need to symmetrize the expression in brackets above in the indices μ and ν. Then assuming the connection Γ doesn't appear in S_{matter}, the equations of motion for the connection are

$$\begin{aligned}
&g^{\lambda(\mu} g^{\nu)\rho} \partial_\sigma g_{\lambda\rho} - \tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\sigma g_{\lambda\rho} \\
&\quad - \Big(g^{\delta\lambda} g^{\rho(\nu} \partial_\delta g_{\lambda\rho} - \tfrac{1}{2} g^{\lambda\rho} g^{\delta(\nu} \partial_\delta g_{\lambda\rho} \Big) \delta^{\mu)}_\sigma + g^{\mu\nu} \Gamma^\lambda_{\sigma\lambda} + g^{\lambda\rho} \Gamma^{(\mu}_{\lambda\rho} \delta^{\nu)}_\sigma - 2 g^{\lambda(\nu} \Gamma^{\mu)}_{\sigma\lambda} = 0
\end{aligned}$$

The parenthesis notation used here means symmetrization: for example, $g^{\lambda(\nu}\Gamma^{\mu)}_{\sigma\lambda} = \frac{1}{2}(g^{\lambda\nu}\Gamma^\mu_{\sigma\lambda} + g^{\lambda\mu}\Gamma^\nu_{\sigma\lambda})$.

By contracting μ and ν, we get a formula for $\Gamma^\lambda_{\sigma\lambda}$ in terms of $g^{\lambda\rho}\Gamma^\mu_{\lambda\rho}$. Plugging back in and contracting σ and ν then gives a formula for $g^{\lambda\rho}\Gamma^\mu_{\lambda\rho}$. Plugging that back in gives a formula for $g^{\lambda(\nu}\Gamma^{\mu)}_{\sigma\lambda}$. The intermediate results are:

$$\begin{aligned}
\Gamma^\mu_{\sigma\mu} &= \tfrac{1}{4} g^{\mu\nu} \partial_\sigma g_{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} \partial_\mu g_{\nu\sigma} - \tfrac{1}{2} g_{\mu\sigma} g^{\nu\lambda} \Gamma^\mu_{\nu\lambda} \\
g^{\lambda\rho} \Gamma^\mu_{\lambda\rho} &= g^{\lambda\nu} g^{\mu\rho} \partial_\nu g_{\lambda\rho} - \tfrac{1}{2} g^{\lambda\rho} g^{\sigma\mu} \partial_\sigma g_{\lambda\rho} \\
g^{\lambda(\nu} \Gamma^{\mu)}_{\sigma\lambda} &= \frac{1}{2} g^{\lambda(\mu} g^{\nu)\rho} \partial_\sigma g_{\lambda\rho}
\end{aligned}$$

We can use this last result to reconstruct $\Gamma^{\mu}_{\nu\sigma}$ by themselves by using the symmetry of the connections. Indeed, first let us remove the symmetrization parentheses by writing

$$g^{\lambda\nu}\Gamma^{\mu}_{\sigma\lambda} = \tfrac{1}{2}g^{\lambda\nu}g^{\mu\rho}\partial_\sigma g_{\lambda\rho} + C^{\mu\nu}{}_{\sigma}$$

where $C^{\mu\nu}{}_{\sigma}$ are constants (to be determined) antisymmetric in μ and ν. Lowering the index ν gives

$$\Gamma^{\mu}_{\sigma\nu} = \tfrac{1}{2}g^{\mu\rho}\partial_\sigma g_{\nu\rho} + C^{\mu}{}_{\nu\sigma}$$

Since the right-hand side must be symmetric in σ and ν, we conclude that $C^{\mu}{}_{\nu\sigma}$ must provide a contribution of $(1/2)g^{\mu\rho}\partial_\nu g_{\sigma\rho}$. Then the antisymmetry of $C^{\mu}{}_{\nu\sigma}$ in μ and ν requires another term $-(1/2)g^{\mu\rho}\partial_\rho g_{\sigma\nu}$, so that we have

$$C^{\mu}{}_{\nu\sigma} = \tfrac{1}{2}g^{\mu\rho}\partial_\nu g_{\sigma\rho} - \tfrac{1}{2}g^{\mu\rho}\partial_\rho g_{\sigma\nu}$$

These are all the terms that the coefficients C may contain, subject to the symmetry requirements and the fact that the C contains only first derivatives of the metric. Thus we have specified the C uniquely, and we are left with

$$\Gamma^{\mu}_{\sigma\nu} = \tfrac{1}{2}g^{\mu\rho}\left(\partial_\sigma g_{\nu\rho} + \partial_\nu g_{\sigma\rho} - \partial_\rho g_{\sigma\nu}\right)$$

as expected.

2 Solution in back of book.

VI.6 Initial Value Problems and Numerical Relativity

1 Solution in back of book.

2 Solution in back of book.

VII Black Holes

VII.1 Particles and Light around a Black Hole

1 Plugging in the values of various quantities given in the text, we obtain $g_{tt}(V^t)^2 + g_{\varphi\varphi}(V^\varphi)^2 = -\epsilon^2/(1-\frac{r_S}{r}) + l^2/r^2 = -(1-\frac{r_S}{r})(1-\frac{3r_S}{2r})^{-1} + \frac{r_S}{2r}(1-\frac{3r_S}{2r})^{-1} = -1$.

2 The desired function $r(\varphi)$ is readily obtained by integrating (10).

3 Solution in back of book.

4 Solution in back of book.

VII.2 Black Holes and the Causal Structure of Spacetime

1 The flat space metric in spherical coordinates is

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2$$

We wish to find a curve $z = f(r)$ so that the induced metric on the curve reproduces the metric of a constant-time slice of Schwarzschild, (12). Thus $dz = f'(r)\,dr$, and the induced metric becomes

$$ds^2 = (1 + f'(r)^2)dr^2 + r^2 d\phi^2$$

For this to agree with a constant-time slice of Schwarzschild, we require $1 + f'(r)^2 = 1/(1 - r_{\mathrm{S}}/r)$, so we have

$$f'(r) = \sqrt{\frac{r_{\mathrm{S}}}{r - r_{\mathrm{S}}}} \Rightarrow f(r) = 2\sqrt{r_{\mathrm{S}}(r - r_{\mathrm{S}})}$$

VII.3 Hawking Radiation

1 Solution in back of book.

VII.4 Relativistic Stellar Interiors

1 Plugging $\mathcal{M}(r) = (4\pi/3)r^3\rho = (r/R)^3 M$ into (13), we see that the expression in the second parentheses on the right hand side becomes $\left(1+\frac{4\pi R^3 P(r)}{M}\right)$, so that the equation simplifies to

$$\frac{dP}{\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi R^3 P}{M}\right)} = -\frac{GM_\rho}{R^3}\frac{rdr}{\left(1-\frac{2GMr^2}{R^3}\right)}$$

which can now be integrated to give the stated result for the pressure profile.

2 Look at Tolman's paper.

VII.5 Rotating Black Holes

1 Solution in back of book.

2 Solution in back of book.

3 This is totally straightforward. Since $\rho^2 = r^2 + O(a^2)$ and $\Delta = r^2 + O(a^2)$, we have $ds^2_{\mathrm{Kerr}} = ds^2_{\mathrm{Schwarzschild}} - (2r_{\mathrm{S}} a \sin^2\theta/r)dtd\varphi + O(a^2)$.

4 This is also straightforward. We read off from the Kerr metric that $\Sigma^2 = \rho^2(r^2+a^2) + r_{\mathrm{S}}a^2 r\sin^2\theta = (r^2 + a^2 - a^2\sin^2\theta)(r^2+a^2) + r_{\mathrm{S}}a^2 r\sin^2\theta = (r^2+a^2)^2 - \Delta a^2\sin^2\theta$. Substituting, we obtained the stated form.

5 Expand out (23) and compare with the result of the previous exercise.

6 We find that

$$\omega(r,\theta) = \frac{r_{\rm S} a r}{(r^2 + a^2\cos^2\theta)(r^2+a^2) + r_{\rm S} a^2 r \sin^2\theta}$$

For large r, $\omega \to r_{\rm S} a/r^3$ independent of θ, while for small r, $\omega \to r_{\rm S} r/(a^3\cos^2\theta)$, which $\to 0$ except for $\theta = \pi/2$, that is, in the equatorial plane. Thus, for $\theta \neq \pi/2$, $\omega(r,\theta)$ attains a maximum at a value of r determined by solving a quadratic equation. You could verify that $\omega(r,\theta)$ does not depend on θ for r satisfying $r^2 + a^2 - r_{\rm S} r = 0$, that is, on the outer and inner horizons. When plotted as a function of r, the curves $\omega(r,\theta)$ for various values of θ intersect for r on the outer and inner horizons. We plot $\omega(r,\pi/2)$, $\omega(r,\pi/4)$, $\omega(r,0)$ in two figures, VII.5.6.a for $r_{\rm S} = 3,\ a = 1$, and VII.5.6.b for $r_{\rm S} = 3,\ a = 3/2$ (an extremal case). I intentionally left the curves unlabeled.

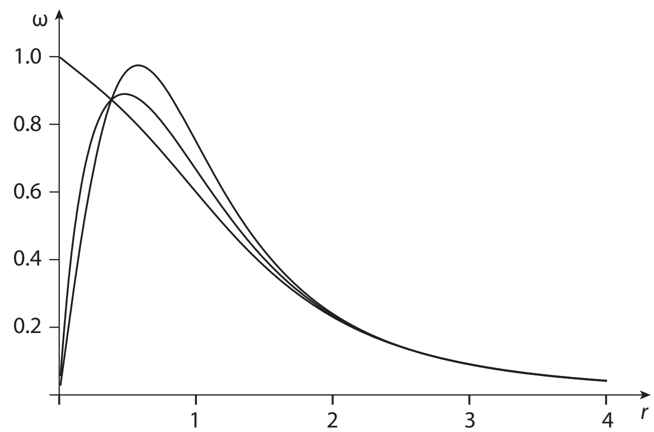

Figure VII.5.6.a

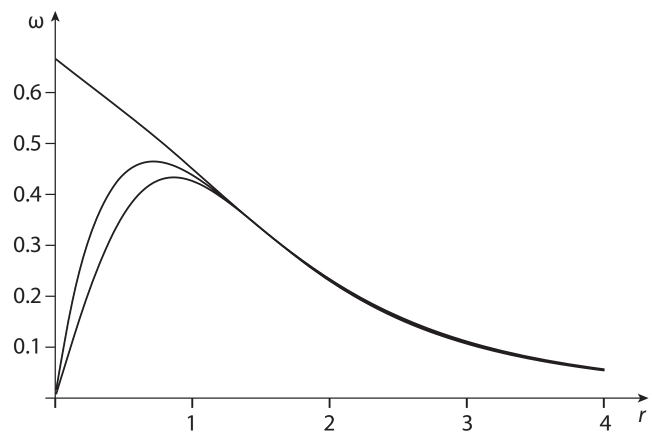

Figure VII.5.6.b

7 We have from the text that

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 + \left|\frac{g_{tt}}{g_{\varphi\varphi}}\right|}$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi}$. At $\theta = \pi/2$, we have

$$\begin{aligned}
-g_{tt} &= \frac{\Delta - a^2}{r^2} = 1 - \frac{r_\mathrm{S}}{r} = 1 - \frac{1}{x} \\
-g_{t\varphi} &= \frac{a r_\mathrm{S}}{r} = \frac{r_\mathrm{S} g}{x} \\
g_{\varphi\varphi} &= \frac{(r^2+a^2)^2 - \Delta a^2}{r^2} = r^2 + a^2\left(1 + \frac{r_\mathrm{S}}{r}\right) = r_\mathrm{S}^2\left[x^2 + g^2\left(1 + \frac{1}{x}\right)\right]
\end{aligned}$$

Then it follows that

$$\begin{aligned}
r_\mathrm{S}\Omega_\pm &= r_s \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 + g_{\varphi\varphi}|g_{tt}|}}{g_{\varphi\varphi}} \\
&= \frac{g/x \pm \sqrt{g^2/x^2 + (x^2 + g^2(1+1/x))(1-1/x)}}{x^2 + g^2(1+1/x)} \\
&= \frac{g \pm \sqrt{g^2 + (x^3 + g^2(x+1))(x-1)}}{x^3 + g^2(x+1)} \\
&= \frac{g \pm x\sqrt{g^2 + x(x-1)}}{x^3 + g^2(x+1)}
\end{aligned}$$

8 By now, you should find this straightforward to work out, following the general treatment first given in chapter V.4. Allow me to point out though, that you could save considerable labor by realizing that you can simply make some minor adjustments to the same problem worked out for a massive particle in appendix 3.

So, start from (46). We have the same equation except that we should now interpret Newton's dot notation as $\dot{t} = \frac{dt}{d\zeta}$ and $\dot{\zeta} = \frac{d\varphi}{d\tau}$, with ζ the affine parameter. Now, after setting $\dot\theta = 0$, we have $g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{\varphi\varphi}\dot{\varphi}^2 + g_{rr}\dot{r}^2 = -x$, with the parameter x equal to 0 rather than 1. Thus, with the proper understanding of what Newton's dot means, we can now treat the massive and the massless cases at the same time. After a few arithmetic steps, we arrive at an equation of the form $\dot{r}^2 + \cdots = -x/g_{rr} = -x((r^2 + a^2 - r r_\mathrm{S})/r^2) = -x(1 + \frac{a^2}{r^2} - \frac{r_\mathrm{S}}{r})$.

But we already did the arithmetic in appendix 3! All we have to do now is to replace the 0 on the right hand side of (47) by the round parentheses in the preceding equation. Thus, we have

$$\dot{r}^2 - \frac{r_\mathrm{S}}{r} + \frac{l^2 + a^2\left(1-\epsilon^2\right)}{r^2} - \frac{r_\mathrm{S}(l - a\epsilon)^2}{r^3} - \epsilon^2 + 1 = 1 + \frac{a^2}{r^2} - \frac{r_\mathrm{S}}{r}$$

We see that several terms cancel between the left and the right hand sides, and we end up with $\dot{r}^2 + \frac{1}{r^2}\{l^2 - a^2\epsilon^2 - (l - a\epsilon)^2\frac{r_\mathrm{S}}{r}\} = \epsilon^2$. As was explained in chapter VI.3, the freedom to scale ζ means that the physics can only depend on the impact parameter $b \equiv |l|/\epsilon$, and not on l and ϵ separately. Dividing by l^2 (which we can absorb into $\dot{r}^2$), we obtain the stated result.

Note that b is defined to be positive. Taking $r \to \infty$ in (4) and (5) in the text (with τ replaced by ζ), we see that $\epsilon > 0$, while l can take on either sign. This explains the appearance of $\mathrm{sign}(l)$ in the equation of motion.

9 To transform Kerr I to Boyer-Lindquist, set $du = dT + dr + r_S r dr/(r^2 + a^2 - r_S r)$ and $d\varphi = -d\Phi - a dr/(r^2 + a^2 - r_S r)$. After some arithmetic, we recover the BL form (15) of the Kerr metric (with t renamed T, and φ renamed Φ to avoid obvious confusion.)

To transform Kerr II to Kerr I, set $t = u - r$, $x = \sin\theta(a \sin\varphi + r\cos\varphi)$, $y = \sin\theta(r\sin\varphi - a\cos\varphi)$, $z = r\cos\theta$. Again, after some arithmetic, we obtain Kerr I.

10 In contrast to the Boyer-Lindquist form used in the text, the Kerr I form of the metric mentioned in exercise VII.5.9 depends on r_S in only one place, and in a very simple way. Indeed, we have

$$\begin{aligned} ds^2 &= -(du + a\sin^2\theta d\varphi)^2 + 2(du + a\sin^2\theta d\varphi)(dr + a\sin^2\theta d\varphi) \\ &\quad +\rho^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{r r_S}{\rho^2}(du + a\sin^2\theta d\varphi)^2 \end{aligned}$$

For a given a, the M dependence is isolated in the fourth term. Write $du + a\sin^2\theta d\varphi$ as $l_\mu dx^\mu$, which defines the vector l_μ. As we will soon see, it is advantageous to depart from tradition and order the coordinates as $x^\mu = (u, r, \varphi, \theta)$. With this convention, we have $l_\mu = (1, 0, a\sin^2\theta, 0)$. The first three terms define

$$g^0_{\mu\nu} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & a\sin^2\theta & 0 \\ 0 & a\sin^2\theta & (r^2 + a^2)\sin^2\theta & 0 \\ 0 & 0 & 0 & r^2 + a^2\cos^2\theta \end{pmatrix}$$

With our unusual ordering of coordinates, the 4×4 block that is the metric breaks up into a 3×3 block and a 1×1 block.

Next, we have to invert $g_{\mu\nu} = g^0_{\mu\nu} + (r r_S/\rho^2) l_\mu l_\nu$ to obtain $l^\mu = g^{\mu\nu} l_\nu$. But we can save ourselves some arithmetical labor by noticing that $l_\mu = (1, 0, a\sin^2\theta, 0)$ is just the second row of $g^0_{\mu\nu}$, and thus we see that $l^\mu = (0, 1, 0, 0)$. This solves $l_\mu = g_{\mu\nu} l^\nu = g^0_{\mu\nu} l^\nu$ since manifestly $l_\nu l^\nu = 0$. (Now that we have found l^μ by eyeball so to speak, we can easily argue that the solution is unique, since $g_{\mu\nu}$ is not singular at a generic point in spacetime.)

Now we obtain immediately $g_{\mu\nu} l^\mu l^\nu = 0$ and $g^0_{\mu\nu} l^\mu l^\nu = 0$, as desired.

11 The calculation is straightforward but tedious and is perhaps best done by a computer. Satisfyingly, the Riemann curvature tensor turns out to vanish. For the reader who wishes to do a few terms by hand, here are a couple of Christoffel symbols to provide a check: $\Gamma^u_{\theta\theta} = -2r\left(r^2 + a^2\right)/(2r^2 + a^2 + a^2\cos 2\theta)$ and $\Gamma^r_{r\varphi} = -2ar\sin^2\theta/(2r^2 + a^2 + a^2\cos 2\theta)$.

VII.6 Charged Black Holes

1 The Reissner-Nordström metric is

$$ds^2 = -A(r)\,dt^2 + \frac{1}{A(r)}\,dr^2 + r^2 d\Omega^2$$

where $A(r) = (r-r_+)(r-r_-)/r^2$. A photon moving in a radial direction has 4-velocity $u^\mu = (u^t, u^r, 0, 0)$, and normalization requires

$$u_\mu u^\mu = -A(r)\left(u^t\right)^2 + \frac{1}{A(r)}\left(u^r\right)^2 = 0$$

Thus, since $dt/dr = u^t/u^r$, we obtain

$$\left(\frac{dt}{dr}\right)^2 = \frac{1}{A(r)^2} \Rightarrow \frac{dt}{dr} = \pm\frac{1}{A(r)}$$

Integrating, we get

$$\begin{aligned}\pm(t-t_0) &= \int \frac{dr}{A(r)} \\ &= r - r_0 + \frac{1}{r_+ - r_-}\left[r_+^2 \ln\left(\frac{r-r_+}{r_0-r_+}\right) - r_-^2 \ln\left(\frac{r-r_-}{r_0-r_-}\right)\right]\end{aligned}$$

where we introduced constants of integration so that the photon is at $r = r_0$ at time $t = t_0$.

2 Setting $dt = d\bar{t} - (A^{-1} - 1)dr$, the metric becomes

$$\begin{aligned}ds^2 &= -A\left(d\bar{t} - \left(\frac{1}{A} - 1\right)dr\right)^2 + \frac{1}{A}\,dr^2 + r^2 d\Omega^2 \\ &= -A\,d\bar{t}^2 + 2(1-A)\,d\bar{t}\,dr + (2-A)\,dr^2 + r^2\,d\Omega^2\end{aligned}$$

3 Maxwell's equations are invariant under the duality transformation $\vec{E} \to -\vec{B}$, $\vec{B} \to \vec{E}$, $J_e^\mu \to -J_m^\mu$, $J_m^\mu \to J_e^\mu$, where J_e^μ and J_m^μ are the electric and magnetic 4-currents, respectively. We can therefore take the Reissner-Norström solution and replace the electric charge Q_e with a magnetic charge $-Q_m$ to obtain the solutions for a black hole with magnetic charge. We get the same metric again:

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}\,dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}\,dr^2 + r^2 d\Omega^2$$

except that now, $r_\pm$ is given by

$$r_\pm = M \pm \sqrt{M^2 - Q_m^2}$$

In general, for a solution containing both electric and magnetic charge, we would replace $Q^2 \to Q_e^2 + Q_m^2$.

VIII Introduction to Our Universe

VIII.1 The Dynamic Universe

1 Solution in back of book.

2 The corresponding equation is now $\dot{R}^2 = \frac{T^2}{4R^2} + 1$, with the solution $R(t) = \sqrt{t(T+t)}$. The small t behavior is the same whether the universe is closed or open.

3 Solution in back of book.

4 Set up the problem as in the preceding exercise. In this case, the cosmological equation reads $\dot{R}^2 = \frac{2T}{R} + 1$. The solution is given parametrically by $R(\eta) = T(\cosh\eta - 1)$ and $t(\eta) = T(\sinh\eta - \eta)$. Again, check the small t behavior of $R(t)$.

5 Simply solve (26) and (27). It is instructive to plot $R(t)$ for the closed, flat, and open universes and compare. For large t, the curvature term does not matter.

VIII.2 Cosmic Struggle between Dark Matter and Dark Energy

1 Solution in back of book.

2 Solution in back of book.

3 Solution in back of book.

4 Solution in back of book.

5 Solution in back of book.

IX Aspects of Gravity

IX.1 Parallel Transport

1 Solution in back of book.

2 Let the circle be described by $(x, y) = (L + \varepsilon\cos\eta, \varepsilon\sin\eta)$. Then $r = \sqrt{x^2 + y^2} \simeq L + \varepsilon\cos\eta$ and $\theta = \arctan(y/x) \simeq \varepsilon\sin\eta/L$. We find

$$\int dr\ \theta = 2\int_{L-\varepsilon}^{L+\varepsilon} dr\theta = (2\varepsilon^2/L)\int_0^\pi d\eta\sin^2\eta = \pi\varepsilon^2/L$$

As a check, $\int d\theta\ r = \pi\varepsilon^2/L$. This example shows that unless the coordinates all have dimensions of length, we cannot expect $a^{\sigma\lambda}$ to have dimensions of length squared, of course.

3 Solution in back of book.

IX.3 Geodesic Deviation

1 On the sphere, we know that geodesics follow great circles. Consider two nearby great circles of constant longitude; we can parametrize the geodesics following these circles as $x^\mu(\tau) = (\tau, 0)$, so that the velocity of each geodesic is $u^\mu = dx^\mu/d\tau = (1, 0)$, which is properly normalized. We take $\tau = 0$ at the pole, so $\tau = \pi/2$ at the equator. If the geodesics are separated by a latitude of φ_0 at the equator, we can then write the separation vector as $\varepsilon^\mu = (0, \varphi_0 \sin\tau)$. Finally, the only nonzero Christoffel symbols are $\Gamma^\theta_{\varphi\varphi} = -\cos\theta\,\sin\theta$, $\Gamma^\varphi_{\theta\varphi} = \cot\theta$, and the only independent component of the Riemann tensor is $R_{\theta\varphi\theta\varphi} = \sin^2\theta$. We are now ready to verify (6). First, note that

$$R^\theta{}_{\sigma\rho\lambda} u^\sigma u^\rho \varepsilon^\lambda = 0, \qquad R^\varphi{}_{\sigma\rho\lambda} u^\sigma u^\rho \varepsilon^\lambda = -\varphi_0 \sin\tau$$

Next, note that

$$\frac{D^2\varepsilon^\theta}{D\tau^2} = 0, \qquad \frac{D^2\varepsilon^\varphi}{D\tau^2} = -\varphi_0 \sin\tau$$

thus verifying (6) for the sphere.

2 Solution in back of book.

3 It's just a matter of repackaging (6) $\frac{D^2\varepsilon^\mu}{D\tau^2} = R^\mu{}_{\sigma\rho\lambda} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} \varepsilon^\lambda$ and evaluating the left hand side as in (34) and (35).

4 Solution in back of book.

IX.4 Linearized Gravity, Gravitational Waves, and the Angular Momentum of Rotating Bodies

1 Solution in back of book.

2 As indicated in the hint, $\int d^3y\ T^{ij}(t, \vec{y}) = -\int d^3y\ y^j \partial_k T^{ik} = \int d^3y\ y^j \partial_0 T^{i0} = \frac{d}{dt} \int d^3y\ y^j T^{i0} = (\frac{1}{2} \frac{d}{dt} \int d^3y\ (y^j T^{i0} + y^j T^{i0}))$. Repeat the same trick: $\int d^3y\ (y^j T^{i0} + y^j T^{i0}) = -\int d^3y\ y^i y^j \partial_k T^{0k} = (\frac{d}{dt} \int d^3y\ y^i y^j T^{00})$. We obtain the stated result.

3 Solution in back of book.

IX.6 Isometry, Killing Vector Fields, and Maximally Symmetric Spaces

1 The first equation in (5) gives $\xi^\theta = F'(\varphi)$, the second then gives $\xi^\varphi = -\cot\theta F(\varphi) + g(\theta)$, so that the third gives $F''(\varphi) + F(\varphi) + g'(\theta) = 0$. This forces $g'(\theta)$ to be a constant, and so $g(\theta) = c + c_1\theta$, but periodicity implies that $c_1 = 0$. The rest is elementary.

2 Our three Killing vector fields are

$$\xi_{(1)} = \sin\varphi\,\partial_\theta + \cot\theta\,\cos\varphi\,\partial_\varphi, \qquad \xi_{(2)} = \cos\varphi\,\partial_\theta - \cot\theta\,\sin\varphi\,\partial_\varphi, \qquad \xi_{(3)} = \partial_\varphi$$

On the equator, $\theta = \pi/2$, we get

$$\xi_{(1)} = \sin\varphi\,\partial_\theta, \qquad \xi_{(2)} = \cos\varphi\,\partial_\theta$$

so $\xi_{(1)}$ and $\xi_{(2)}$ point "up" and "down" the sphere, depending on the value of φ. Near the north pole of the sphere, where the coordinates fail to be well defined, $\cot\theta = 1/\theta + \cdots$, so we have

$$\xi_{(1)} = \sin\varphi\,\partial_\theta + \frac{1}{\theta}\cos\varphi\,\partial_\varphi + \cdots, \qquad \xi_{(2)} = \cos\varphi\,\partial_\theta - \frac{1}{\theta}\sin\varphi\,\partial_\varphi + \cdots$$

and thus $\xi_{(1)}$ and $\xi_{(2)}$ are dominated by a component in the φ direction, unless $\cos\varphi = 0$ or $\sin\varphi = 0$.

3 Solution in back of book.

IX.7 Differential Forms and Vielbein

1 Solution in back of book.

2 Solution in back of book.

3 The transformation from spherical to Cartesian coordinates is

$$\tan\varphi = \frac{y}{x} \qquad \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

It then follows that

$$d\varphi = \frac{1}{x^2+y^2}(x\,dy - y\,dx)$$

and

$$\begin{aligned} d\cos\theta &= \frac{dz}{r} - (\frac{z}{r^2})\frac{1}{r}(x\ dx + y\,dy + z\ dz) \\ &= (x^2+y^2)\frac{dz}{r^3} - \frac{z}{r^3}(x\ dx + y\,dy) \end{aligned}$$

Note that $(x\ dx + y\,dy)(x\,dy - y\,dx) = (x^2+y^2)dxdy$ and hence

$$\begin{aligned} d\cos\theta\,d\varphi &= (xdzdy - ydzdx)/r^3 - zdxdy/r^3 \\ &= -(xdydz + ydzdx + zdxdy)/r^3 = -\epsilon_{ijk}x^i dx^j dx^k/r^3 \end{aligned}$$

Thus, $F = \frac{g}{4\pi} d\cos\theta\,d\varphi$ clearly represents a radial magnetic field. We can see this by writing F in terms of components:

$$F_{\mu\nu} \propto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & -y \\ 0 & -z & 0 & x \\ 0 & y & -x & 0 \end{pmatrix}$$

But the components of the field strength tensor are related to the electric and magnetic fields by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Thus for our field strength tensor, the electric field is zero, and components of the magnetic field go like $B_x \propto x$, $B_y \propto y$, $B_z \propto z$ with the same proportionality constant, indicating that the magnetic field is radial.

4 Solution in back of book.

5 Solution in back of book.

6 Solution in back of book.

IX.8 Differential Forms Applied

1 Solution in back of book.

2 Solution in back of book.

3 Solution in back of book.

4 Solution in back of book.

IX.9 Conformal Algebra

1 Solution in back of book.

2 A straightforward calculation gives

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (f^8/(y^2)^5)(y^2\eta_{\mu\nu} + 8y_\lambda y_\rho) dy^\mu dy^\nu,$$

which is assuredly not conformally flat.

3 Solution in back of book.

IX.10 De Sitter Spacetime

1 This is just a matter of straightforward calculation.

2 After working through a few metrics, you will realize that it would be easiest to write a computer program to do the actual calculation

3 We begin with

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2\, d\Omega^2$$

Defining $dp = dt + dr/(1-r^2)$ and $dq = -dt + dr/(1-r^2)$, we find that $dt = (dp-dq)/2$ and $dr = (1-r^2)(dp+dq)/2$, so that the metric becomes

$$\begin{aligned} ds^2 &= -\tfrac{1}{4}(1-r^2)\,(dp-dq)^2 + \frac{(1-r^2)^2}{4}\frac{(dp+dq)^2}{1-r^2} + r^2\,d\Omega^2 \\ &= \tfrac{1}{4}(1-r^2)\left(-(dp-dq)^2 + (dp+dq)^2\right) + r^2\,d\Omega^2 \\ &= (1-r^2)\,dp\,dq + r^2\,d\Omega^2 \end{aligned}$$

4 This is also just a matter of straightforward calculation.

5 Solution in back of book.

6 Again, this is just a matter of straightforward calculation.

7 de Sitter's original metric is

$$ds^2 = -\cos^2\chi\,dt^2 + \left(d\chi^2 + \sin^2\chi\,d\Omega^2\right)$$

The 4-velocity of a line of constant χ, θ, φ is $u^\mu = (u^t, 0, 0, 0)$, so the geodesic equation becomes

$$\begin{aligned} 0 &= u^\mu D_\mu u^\nu \\ &= \frac{\partial u^\nu}{\partial \tau} + \Gamma^\nu_{\mu\lambda} u^\mu u^\lambda \\ &= \frac{\partial u^\nu}{\partial \tau} + \Gamma^\nu_{tt}\left(u^t\right)^2 \end{aligned}$$

The $\nu = t$ component is satisfied automatically; to satisfy the others, we need $\Gamma^\nu_{tt} = 0$ for $\nu = \chi$, θ, φ. We have

$$\begin{aligned} \Gamma^\chi_{tt} &= -\tfrac{1}{2}\,g^{\chi\chi}\partial_\chi g_{tt} = -\sin\chi\,\cos\chi \\ \Gamma^\theta_{tt} &= 0 = \Gamma^\varphi_{tt} \end{aligned}$$

So indeed, the geodesic equation can only be satisfied if $\chi = 0$, so that we have $\Gamma^\chi_{tt} = 0$.

X Gravity Past, Present, and Future

X.3 Effective Field Theory Approach to Gravity

1 Solution in back of book.

X.4 Finite Sized Objects and Tidal Forces in Einstein Gravity

1 The gravitational field we feel every day is given, to first approximation, by the Schwarzschild metric produced by the earth. For an everyday slowly moving object, with 3-velocity essentially zero, so its 4-velocity is $\dot{X}^\mu = (1/\sqrt{1 - r_S/r}, 0, 0, 0)$, where r_S is the Schwarzschild radius of the earth. (Note that $g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = -1$.) As per the definition of the gravitational electric field given in the text, $E_{\mu\nu} = R_{\mu\rho\nu\sigma}\dot{X}^\rho\dot{X}^\sigma$, we need

$$E_{rr} = R_{rtrt}(V^t)^2 = g_{rr}R^r{}_{trt}/(1 - r_S/r) = -\frac{r_S}{r^3}(\frac{1}{1-\frac{r_S}{r}})$$

Similarly, we obtain $E_{\theta\theta} = E_{\varphi\varphi} = -\frac{1}{2}E_{rr}$. For the gravitational magnetic field $B_{\mu\nu}$, we need all components of the dual of the curvature tensor with two t indices, which in turn requires all components of the Riemann tensor with only one t index. The gravitational magnetic field is zero (this isn't so surprising: we expect magnetic fields to be produced by moving charges, but relative to us standing on the surface of the earth, the earth isn't moving).

X.5 Topological Field Theory

1 We have $E = 20 \times 3/2 = 30$, and so, using Euler's theorem, $V = 2 + E - F = 2 + 30 - 20 = 12$. How many triangles meet at each vertex? The number is $3 \times 20/V = 60/12 = 5$. The angular deficit at each vertex is $2\pi - 5(\pi/3) = \pi/3$. The total angular deficit is $12 \times \pi/3 = 4\pi$. Descartes is some kind of genius!

2 We have, from exercise I.5.16, $\sqrt{g} = a(L + a\sin\theta)$, and hence $\int d^2x\sqrt{g}R = 2\int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta = 0$.

3 Solution in back of book.

X.6 A Brief Introduction to Twistors

1 We have

$$p \cdot q = (\varepsilon^{\alpha\beta}\lambda_\alpha\mu_\beta)\left(\varepsilon\dot{\alpha}\dot{\beta}\tilde{\lambda}_{\dot{\alpha}}\tilde{\mu}_{\dot{\beta}}\right) \equiv \langle\lambda,\mu\rangle[\lambda,\mu]$$

Note the appearance of the two Lorentz invariants $\langle\lambda,\mu\rangle$ and $[\lambda,\mu]$ defined in the text.

2 Solution in back of book.

3 Solution in back of book.