MATH CAMP – SESSION 1

INTRODUCTION TO PROOF WRITING

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Notation

Notation and Common Symbols I

Sets and Set Operations:

- $A, B, C, \ldots$: Sets (usually uppercase letters)
- $x, y, z, \ldots$: Elements of sets (usually lowercase letters)
- $\emptyset$: Empty set
- $\subset$: Subset
- $\subseteq$: Subset or equal to
- $\cup$: Union
- $\cap$: Intersection
- $\setminus$: Set difference
- $\in$: Belongs to
- $\notin$: Does not belong to

Notation and Common Symbols II

Logic and Quantifiers:

- $\wedge$: Logical AND
- $\vee$: Logical OR
- $\neg$: Logical NOT
- $\forall$: For all (universal quantifier)
- $\exists$: There exists (existential quantifier)

Functions:

- $f : A \to B$: Function from set A to set B
- $f(x)$: Value of function f at point x
- $\mathrm{Dom}(f)$: Domain of function f
- $\mathrm{Im}(f)$: Image of function f
- $\mathrm{Ker}(f)$: Kernel of function f

Notation and Common Symbols III

Proof-Based Notation:

- $\implies$: Implies (logical implication)
- $\iff$: If and only if (biconditional)
- $\square$: End of proof symbol
- $\because$: Because
- $\therefore$: Therefore
- $\forall \varepsilon > 0 \; \exists \, \delta > 0$: Delta-epsilon proofs

Notation and Common Symbols IV

- $\mathbb{N}$ = the set of natural numbers = $\{1, 2, 3, \ldots\}$.
- $\mathbb{Z}$ = the set of integer = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbb{Q}$ = the set of rational numbers (fractions) = $\left\{\frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\right\}$.
- $\mathbb{R}$ = the set of real numbers, represented on the real line.
  - $\mathbb{R}^d$ = the set of real numbers in the d-dimensional Euclidean space.
  - For example, $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ is the set of ordered pairs in the usual Cartesian plane.
  - In words, the set-builder notation for $\mathbb{R}^2$ is "the ordered pairs (x, y) such that x and y are both in the set of real numbers."
- $\mathbb{C}$ = the set of complex numbers = $\{a + bi \mid a, b \in \mathbb{R}$ and $i^2 = -1\}$.

Logic and Set Theory

Necessary and Sufficient Conditions

Let A and B be some arbitrary statements.

- We say 'A is necessary for B' when
  - A must hold for B to hold, or
  - A is true if B is true, or
  - A is implied by B, i.e., $A \impliedby B$.

  Thus, a necessary condition is of the form 'A if B.'
- We say 'A is sufficient for B' when
  - A holds, B must hold, or
  - A is true only if B is true, or
  - A implies B, i.e., $A \implies B$.

  Thus, a sufficient condition is of the form 'A only if B.'
- Note that 'A is necessary for B' is equivalent to 'B is sufficient for A' and vice-versa.

Examples

- *A* if *B* form
  - The demand for a good falls *if* its price increases.
  - Consumer spending may increase *if* the unemployment rate falls.
- *A* only if *B* form
  - A nation achieves a trade surplus *only if* its exports exceed its imports.
  - Workers receive higher wages *only if* their productivity increases.

We may also rephrase the examples above as:

- If *B* then *A* form
  - *If* the price of a good increases *then* its demand falls.
  - *If* the unemployment rate falls, *then* consumer spending may increase.
- If *A* then *B* form
  - *If* a nation achieves a trade surplus *then* its exports exceed its imports.
  - *If* workers receive higher wages *then* their productivity increases.

Equivalence

Consider again the same arbitrary statements A and B, where A is the **premise** and B is the **conclusion**.

- We say that A and B are logically equivalent if
  - A implies B and B implies A, or
  - A if and only if B, i.e., $A \iff B$.
- **Example:** "The rate of inflation decreases if and only if there is a decrease in the money supply." This statement can be broken into two components:
  - The rate of inflation decreases if there is a decrease in the money supply, which is of the form $A \impliedby B$, and
  - The rate of inflation decreases only if there is a decrease in the money supply, which is of the form $A \implies B$.
- **Contrapositive:** If $A \implies B$ is true, then so is $\sim B \implies \sim A$.
  - For example, if the statement "If it is raining, then the streets are wet" ($A \implies B$) is true, then its contrapositive "If the streets are not wet, then it is not raining" ($\sim B \implies \sim A$) is also true.

Primary Techniques of Proof

There are four main proof techniques:

1. Direct or Constructive Proof
2. Contrapositive Proof
3. Proof by Contradiction
4. Proof by Induction
5. Counterexample[1]

[1] To *disprove* a statement.

Direct Proof

Suppose we are to prove a statement of the type $A \implies B$.

- We begin by assuming that A is true and use any previous related results or factual statements to show that B must also be true.
- **Problem:** Prove that if n is even then so is n^2.
- **Proof:** We begin by assuming that n is even. By definition of an even number, it means $n = 2k$ for some $k \in \mathbb{Z}$. Then, $n^2 = (2k)^2 = 4k^2$, which gives $n^2 = 2(2k^2)$. Let $l = 2k^2 \in \mathbb{Z}$ such that $n^2 = 2l \in \mathbb{Z}$, which is an even number by definition.

  $\square$
- **A fun problem:** Sherlock Holmes is investigating a new case. Irene Adler was found murdered. The list of suspects has been narrowed down to his archenemy Professor Moriarty and his own brother Mycroft Holmes. Sherlock has deduced the following: *Either Mycroft is lying or Moriarty is lying. If Mycroft is lying then Irene Adler was murdered using a sword. If Moriarty is lying then Mycroft did not kill Irene Adler.* Watson has just informed Sherlock that Irene Adler was murdered using a gun. Who killed Irene Adler?

Contraposive Proof

Recall the statement we just proved: "if n is even then so is n^2." How would we, instead, go about proving the converse? That is,

- **Prove:** "If n^2 is even then so is n."

  Realization: Not as easy using the direct method.

  Solution? Use the contrapositive method.

  The contrapositive method relies on exactly as the name suggests: proving true the contrapositive of the original statement, which then implies the original statement itself must be true. (Recall the equivalence.)

- In our example, the contrapositive of the original statement is: "If n is not even, then n^2 is not even" or "if n is odd, then n^2 is odd."

- Try this proof by yourself.

  Hint: Use the definition of an odd number – a number k is odd if it can be written in the form $k = 2l + 1$, for $l \in \mathbb{Z}$.

Proof by Contradiction

- A proof by contradiction relies on assuming that A is true, but B is not true and showing that this supposition leads to a logical contradiction. That is, we try to show that $A \implies \sim B$ is not true, so $A \implies B$ must be true.
- **Example:** Prove that $\sqrt{2}$ is irrational.
- **Proof:** We begin by assuming to the contrary that $\sqrt{2}$ is rational. Then $\exists\, a \in \mathbb{Z} \wedge b \in \mathbb{Z} \setminus \{0\}$ such that $\sqrt{2} = a/b$. Assume, without loss of generality, that a and b have no common factors. We write

$$2b^2 = a^2,$$

where a^2 is even. Thus, a is even and so $a = 2k$ for some $k \in \mathbb{Z}$. Further,

$$2b^2 = a^2 = (2k)^2 = 4k^2$$

which gives $b^2 = 2k^2$, so b^2 is even. This, again implies, b is even, i.e., $b = 2l,\ l \in \mathbb{Z}$.

Establishing the Contradiction

- **Contradiction:** However, that means a and b are both multiples of 2, which contradicts our assumption that a and b have no factors in common. Therefore, it must be that $\sqrt{2}$ is irrational. $\square$
- Contradictions are often denoted by ($\Longrightarrow\!\!\!\times\!\!\!\Longleftarrow$).
- From Jehle and Reny 2011: "Sometimes, proofs by contradiction can get the job done very efficiently, yet because they involve no constructive chain of reasoning between A and B as the other two do, they seldom illuminate the relationship between the premise and the conclusion."
  - Yet, this method of proof will be the most helpful for microeconomics.
  - Imperative to practice writing contradiction-based proofs.
- We will focus on proof using contradiction during the second part of the session – the group-based assignment.

Proof by Induction

Mathematical induction is a common method of proof when showing that a statement $S(n)$, which depends on n, is true for all $n \in \mathbb{N}$ (University of Michigan 2023).

- General Setup:
  - The base case: Prove that the statement holds for $n = 0$, or 1.
  - The inductive step: Assume that the statement holds for some k and then prove that it also holds for $k + 1$, i.e., $S(k)$ true $\implies S(k+1)$ true.
  - Conclude that $S(n)$ is true for all $n \in \mathbb{N}$ using *The Principle of Mathematical Induction.*
- The Principle of Mathematical Induction is an Axiom – a statement accepted as true as the basis for argument or inference[2].
  - Axioms along with Theorems, Definitions, Corollaries, Lemmas, Propositions, and Claims establish formal mathematical exposition in the form of results and supplemented by proofs (the topic of discussion here).

[2] Merriam-Webster n.d.

The Principle of Mathematical Induction

The following version presents the axiom in mathematical notation:

Axiom (*The Principle of Mathematical Induction*)

$[S(1) \wedge (\forall k \in \mathbb{N}, S(k) \implies S(k+1))] \implies (\forall m \in \mathbb{N}, S(m)).$[a]

[a] University of Michigan 2023.

The following version presents the axiom in the usual manner:

Axiom (The Axiom of Strong Induction)

Let $S \subset \mathbb{N}$ be any subset of $\mathbb{N}$ with the following properties:

- $0 \in S$,
- *for any n, $\forall m \in \mathbb{N}(m \leq n \implies m \in S) \implies n+1 \in S$, i.e. if, all natural numbers less than or equal to n are in S, then $n+1$ is also in S,*

then, $S = \mathbb{N}$.[a]

[a] University n.d.

Example

Let us try an prove that the sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$. That is,

- **Prove:** $S(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N}.$
- **We prove this by induction.**

  Begin by showing that the base case holds:

$$S(1) : 1 = \frac{1(1+1)}{2} = 1.$$

  Clearly, $S(1)$ is true.

  For the inductive step, assume that for some $k \in \mathbb{N}$, or for $n = k$,

$$S(k) : 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

  is true. We now need to show that $S(k+1)$ also holds. Note that since $S(k)$ is assumed to be true,

$$S(k+1) : (1 + 2 + \cdots + k) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Conclusion

- Recognize that the right-hand side of the previous equation is nothing but $S(n)$ for $n = (k+1)$ and so, $S(k)$ true $\implies S(k+1)$ true, as required. Thus, we have proved that the statement $S(n)$ holds for all $n \in \mathbb{N}$ by induction. $\square$
- Useful tips (University of Michigan 2023):
  - Start your proof by stating that you are doing a proof by induction.
  - Label the two steps clearly.
  - Clearly state where you use the inductive hypothesis.
  - Write some variation of "therefore the statement holds for all n by induction" at the end of your proof.
  - If words like "show for all $n \in \mathbb{N}$" occur in the statement of a problem, then a correct solution will likely involve a proof by induction.
- **Exercise:** Show that the sum of the first n^2 numbers is given by:

$$S(n^2) : 1 + 4 + 9 \cdots + n^2 = \frac{[n(n+1)(2n+1)]}{6}.$$

Counterexamples

Counterexamples are helpful when trying to *disprove* a universal statement, i.e., a statement of the "for all" type, since one only needs to state a single instance to show that the statement cannot hold always.

- **Example:** Suppose your friends claims that *"the demand for all goods is either weakly or strictly decreasing in price."* To disprove their claim, you need only state the existence of Giffen goods (however rare they may be), and you have successfully refuted their claim.
- **Exercise:** All idempotent matrices are the identity matrix. Prove or disprove. (Recall that a matrix $\mathbf{A}$ is idempotent if $\mathbf{A}^2 = \mathbf{A}$.)

  Good candidate to check for a counterexample. Take a simple 2×2 matrix. For example, consider

  $$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

  $\mathbf{A}$ is, indeed, idempotent (verify yourself), but clearly $\mathbf{A} \neq \mathbf{I}$. Thus, the statement is false and we may conclude that not all idempotent matrices are the identity matrix.

Essentials of Set Theory I

Set theory forms the building blocks of formal mathematical writing. Basic competence is necessary to do well in all core economics courses.

We start by restating the very basics (Jehle and Reny 2011):

- A set is any collection of elements.
  - Sets can be defined by enumeration of their elements, e.g., $S = \{2, 4, 6, 8\}$, or by description of their elements. For example:

$$S = \{x \mid x \text{ is a positive even integer greater than zero and less than } 10\}.$$

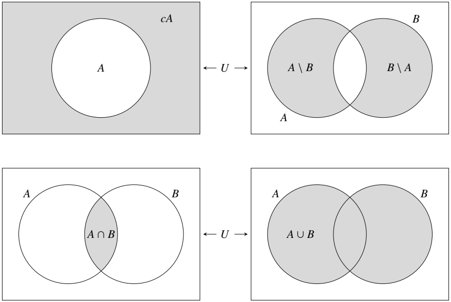
  - When we wish to denote membership or inclusion in a set, we use the symbol $\in$. For example, if $S = \{2, 5, 7\}$, we say that $5 \in S$.
- A set S is a subset of another set T if every element of S is also an element of T.
  - We write $S \subset T$ (S is contained in T) or $T \supset S$ (T contains S).
  - If $S \subset T$, then $x \in S \implies x \in T$.

Essentials of Set Theory II

- Two sets are equal sets if they each contain exactly the same elements. We write $S = T$ whenever $x \in S \implies x \in T$ and $x \in T \implies x \in S$. Thus, S and T are equal sets *if and only* if $S \subset T$ and $T \subset S$.
  - For example, if $S = \{\text{integers}, x \mid x^2 = 1\}$ and $T = \{-1, 1\}$, then $S = T$.
- A set S is empty or is an empty set if it contains no elements at all. For example, if $A = \{x \mid x^2 = 0 \text{ and } x > 1\}$, then A is empty. We denote the empty set by the symbol $\emptyset$ and write $A = \emptyset$.
- The complement of a set S in a universal set U is the set of all elements in U that are not in S and is denoted S^c.
  - If $U = \{2, 4, 6, 8\}$ and $S = \{4, 6\}$, then $S^c = \{2, 8\}$.
- More generally, for any two sets S and T in a universal set U, we define the set difference denoted $S \setminus T$, or $S - T$, as all elements in the set S that are not elements of T.
  - Thus, we can think of the complement of the set S in U as the set difference $S^c = U \setminus S$.

Essentials of Set Theory III

- The basic operations on sets are union and intersection. They correspond to the logical notions of ‘or’ and ‘and’, respectively. For two sets S and T,
  - we define the union of S and T as the set $S \cup T \equiv \{x \mid x \in S \text{ or } x \in T\}$.
  - We define the intersection of S and T as the set $S \cap T \equiv \{x \mid x \in S \text{ and } x \in T\}$.



Note: The top left figure shows the set A and its complement. The top right figure shows the set differences, $A \setminus B$ and $B \setminus A$. The bottom left and right show the intersection and union of A and B, respectively.

Essentials of Set Theory IV

- Sometimes we want to examine sets constructed from an arbitrary number of other sets. We do by collecting the necessary (possibly infinite) number of integers starting with 1 into a set, $I \equiv \{1, 2, 3, \ldots\}$, called an index set, and denote the collection of sets more simply as $\{S_i\}_{i \in I}$.
  - We would denote the union of all sets in the collection by $\cup_{i \in I} S_i$, and
  - the intersection of all sets in the collection as $\cap_{i \in I} S_i$.
- The product of two sets S and T is the set of 'ordered pairs' in the form (s, t), where the first element in the pair is a member of S and the second is a member of T. The product of S and T is denoted

$$S \times T \equiv \{(s, t) \mid s \in S, t \in T\}$$

  and is known as the Cartesian product.
- One familiar set product is the 'Cartesian plane'. This is the plane in which you commonly graph things. It is the visual representation of a set product constructed from the set of real numbers.

Essentials of Set Theory V

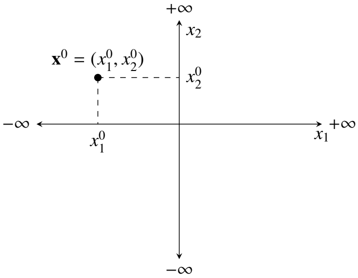
- The set of real numbers is denoted by the symbol $\mathbb{R}$ and is defined as

$$\mathbb{R} \equiv \{x \mid -\infty < x < \infty\}.$$

- If we form the set product

$$\mathbb{R} \times \mathbb{R} \equiv \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\},$$

then any point in the set (any pair of numbers) can be identified with a point in the Cartesian plane depicted below. The set $\mathbb{R} \times \mathbb{R}$ is sometimes called 'two-dimensional Euclidean space' and is commonly denoted $\mathbb{R}^2$.



Essentials of Set Theory VI

- More generally, any n-tuple, or vector, is just an n-dimensional ordered tuple $(x_1, \ldots, x_n)$ and can be thought of as a 'point' in n-dimensional Euclidean space, or 'n-space'. As before, n-space is defined as the set product

$$\mathbb{R}^n \equiv \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \equiv \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, \ldots, n\}.$$

  We usually denote vectors or points in $\mathbb{R}^n$ with boldface type, so that $\mathbf{x} \equiv (x_1, \ldots, x_n)$.

- Often, we want to restrict our attention to a subset of $\mathbb{R}^n$, called the 'non-negative orthant' and denoted $\mathbb{R}^n_+$, where

$$\mathbb{R}^n_+ \equiv \{(x_1, \ldots, x_n) \mid x_i \geq 0,\ i = 1, \ldots, n\} \subset \mathbb{R}^n.$$

  We use the notation $\mathbf{x} \geq 0$ to indicate vectors in $\mathbb{R}^n_+$, where each component x_i is greater than or equal to zero. We use the notation $\mathbf{x} \gg 0$ to indicate vectors where every component of the vector is strictly positive.

- More generally, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we say that $\mathbf{x} \geq \mathbf{y}$ iff $x_i \geq y_i,\ i = 1, \ldots, n$. We say that $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i,\ i = 1, \ldots, n$.

Convex Sets

Definition (Convex Sets in $\mathbb{R}^n$)

$S \subset \mathbb{R}^n$ is a convex set if for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$, we have

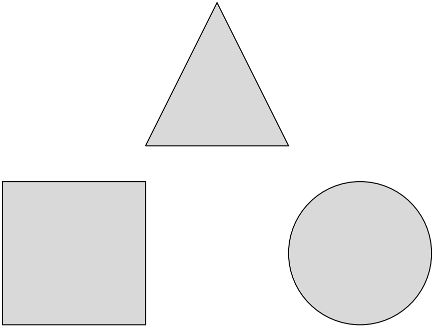
$$t\mathbf{x}^1 + (1 - t)\mathbf{x}^2 \in S$$

for all t in the interval $0 \leq t \leq 1$.

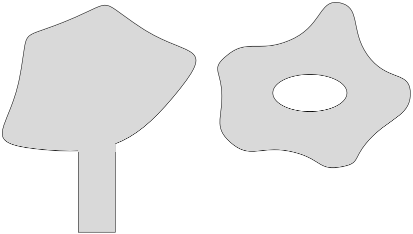
The kind of weighted average used in the definition is called a convex combination.

- We say that $\mathbf{z}$ is a convex combination of $\mathbf{x^1}$ and $\mathbf{x}^2$ if $\mathbf{z} = t\mathbf{x}^1 + (1 - t)\mathbf{x}^2$ for some number t between zero and 1.
- Because t is between zero and 1, so is $(1 - t)$, and the sum of the weights, $t + (1 - t)$, will always equal 1.
- A convex combination $\mathbf{z}$ is thus a point that, in some sense, 'lies between' the two points $\mathbf{x}^1$ and $\mathbf{x}^2$.
- A set is convex iff we can connect any two points in the set by a straight line that lies entirely within the set.

Examples of Convex and Non-Convex Sets



(a) Convex Sets



(b) Non-Convex Sets

Note: To check if the convex sets are, indeed, convex, take any two points and see if the line joining them crosses over the gray region. You should find that that is not possible. For the non-convex sets, however, there are at least two points whose convex combination lies outside the gray region. The takeaway is that convex sets are all 'nicely behaved'. They have no holes, no breaks, and no awkward curvatures on their boundaries. They are nice sets.

Our First Theorem

Let us now state our very first theorem, i.e., *a general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths.*[3]

Theorem (The Intersection of Convex Sets is Convex)

Let S and T be convex sets in $\mathbb{R}^n$. Then $S \cap T$ is a convex set.

- Of the different methods of proofs that we discussed above, which one should we use to prove the theorem?

Proof.

Let S and T be convex sets. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be any two points in $S \cap T$. Because $\mathbf{x}^1 \in S \cap T$, $\mathbf{x}^1 \in S$ and $\mathbf{x}^1 \in T$. Because $\mathbf{x}^2 \in S \cap T$, $\mathbf{x}^2 \in S$ and $\mathbf{x}^2 \in T$. Let $\mathbf{z} = t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, for $t \in [0,1]$, be any convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$. Because S is a convex set, $\mathbf{z} \in S$. Because T is a convex set, $\mathbf{z} \in T$. Because $\mathbf{z} \in S$ and $\mathbf{z} \in T$, $\mathbf{z} \in S \cap T$. Because every convex combination of any two points in $S \cap T$ is also in $S \cap T$, $S \cap T$ is a convex set. $\square$

[3] Oxford English Dictionary 2024.

A Breakdown of the Proof

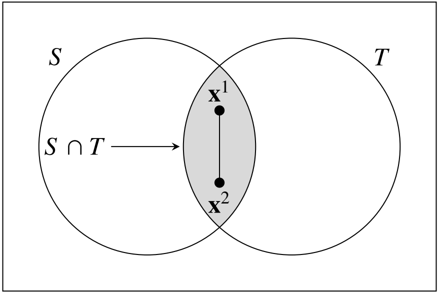
Recall the general guidelines for a direct proof: *begin by assuming that A is true and use any previous related results or factual statements to show that B must also be true.* So, we start by assuming that S and T are convex sets.

- Next, we want to make use of prior results/factual statements – it is helpful to think about, before writing a proof, what we want to show.
  - In this case, we want to show that $S \cap T$ is a convex set. This by itself should give you a hint as to the statements we want to use. Specifically, we call into assistance the definitions of intersection and convex sets.
- We next turn our attention to the statement, "Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be any two points in $S \cap T$."
  - Once we take account of the definitions we need to use, the choice of $\mathbf{x}^1$ and $\mathbf{x}^2$ becomes more obvious. (Keep in mind, however, that even if it does not, it's not a big deal since it takes time and practice to write proficient proofs – precisely what we are all working towards.)
- Following this, we apply the definitions directly. We complete the proof by showing that every convex combination of two points in $S \cap T$ belongs to $S \cap T$, implying that $S \cap T$ must be convex.

A Visual Example

We start by noting that an example does not replace a proof. It only serves to help grasp a deeper understanding for those that are visual learners.

- Start by considering two convex sets S and T in $\mathbb{R}^2$.



- Pick any two points $\mathbf{x}^1$ and $\mathbf{x}^2$ in $S \cap T$ and see that their convex combination always lies in the shaded region.
  - Thus, $S \cap T$ is also ‘nicely behaved.’

Relations

- Any collection of ordered pairs is said to constitute a binary relation between the sets S and T.
- A binary relation is defined by specifying some meaningful relationship that holds between the elements of the pair.
  - For example, let S be the set of cities {Washington, London, Marseilles, Paris}, and T be the set of countries {United States, England, France, Germany}.
  - The statement 'is the capital of' then defines a relation between these two sets that contains the elements {(Washington, United States), (London, England), (Paris, France)}.
  - As this example shows, a binary relation $\mathcal{R}$ on $S \times T$ is always a subset of $S \times T$. When $s \in S$ bears the specified relationship to $t \in T$, we denote membership in the relation $\mathcal{R}$ in one of two ways: either we write $(s, t) \in R$ or, more commonly, we simply write $s\mathcal{R}t$.
- Many familiar binary relations are contained in the product of one set with itself. For example, let S be the closed unit interval, $S = [0, 1]$. Then the binary relation $\geq$ consists of all ordered pairs of numbers in S where the first one in the pair is greater than or equal to the second one.

Properties I

We can build in more structure for a binary relation on some set by requiring that it possess certain properties.

Definition (Completeness)

A relation $\mathcal{R}$ on S is complete if, for all elements x and y in S, $x\mathcal{R}y$ or $y\mathcal{R}x$.

- Suppose that $S = \{1, 2, \ldots, 10\}$, and consider the relation defined by the statement, 'is greater than'. This relation is not complete because one can easily find some $x \in S$ and some $y \in S$ where it is neither true that $x > y$ nor that $y > x$:
  - for example, one could pick $x = y = 1$, or $x = y = 2$, and so on.
  - The definition of completeness does not require the elements x and y to be distinct, so nothing prevents us from choosing them to be the same.
  - Because no integer can be either less than or greater than itself, the relation 'is greater than' is not complete.
- However, the relation on S defined by the statement 'is at least as great as' is complete: for any two integers, whether distinct or not, one will always be at least as great as the other, as completeness requires.

Properties II

Definition (Transitivity)

A relation $\mathcal{R}$ on S is transitive if, for any three elements x, y, and z in S, $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.

- Both the relations just considered are transitive. If x is greater than y and y is greater than z, then x is certainly greater than z. The same is true for the relation defined by the statement 'is at least as great as'.
- More generally, a relation is binary if it satisfies the following three properties:
  - Reflexivity: A relation $\mathcal{R}$ on a set A is reflexive if every element is related to itself, i.e., $(a, a) \in \mathcal{R}$ for all $a \in A$.
  - Symmetry: A relation $\mathcal{R}$ on a set A is symmetric if $(a, b) \in \mathcal{R}$ implies $(b, a) \in \mathcal{R}$.
  - Transitivity: A relation $\mathcal{R}$ on a set A is transitive if $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ imply $(a, c) \in \mathcal{R}$.
- **Exercise:** Prove that the equality relation E on Set $\{1, 2, 3\}$ is a binary relation:

$$E = \{(a, a) \mid a \in \{1, 2, 3\}\} = \{(1, 1), (2, 2), (3, 3)\}.$$

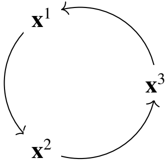
Preference Relation

The binary relation we deal with most often in microeconomics is $\succsim$. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be two consumption bundles, then:

- $\mathbf{x}^1 \succsim \mathbf{x}^2 \implies \mathbf{x}^1$ is at least as good as $\mathbf{x}^2$.
  (weakly preferred to)
- Completeness and Transitivity are the first two *Axioms of Choice*.
  - They imply that $\succsim$ is a preference relation.
  - They imply that the consumer can order all bundles from best to worst, with possible indifferences.

    Ranking: $\mathbf{x}^1 \succsim \mathbf{x}^2 \succsim \mathbf{x}^3 \succsim \dots$

    Rules out cycles:



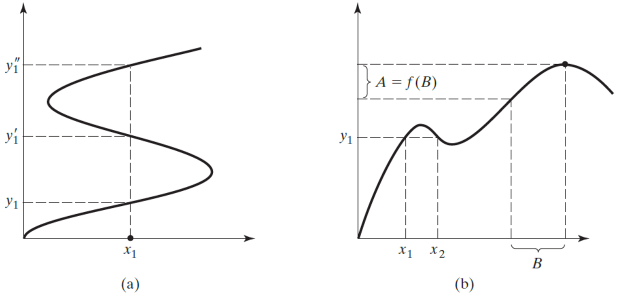
- More on this in ECON 601.

Functions I

We close out our discussion with functions:

- A function is a very common though very special kind of relation.
- Specifically, a function is a relation that associates each element of one set with a single, unique element of another set.
- We say that the function f is a mapping from one set D to another set R and write $f : D \to R$.
  - We call the set D the domain and the set R the range of the mapping.
  - If y is the point in the range mapped into by the point x in the domain, we write $y = f(x)$. To denote the entire set of points A in the range that is mapped into by a set of points B in the domain, we write $A = f(B)$.
- The next slide shows an example of a function and non-function.

Functions II



Note: The graph in panel (a) is not a function, because more than one point in the range is assigned to points in the domain, such as x_1. Panel (b) does depict a function because every point in the domain is assigned some unique point in the range.

Functions III

- The image of f is that set of points in the range into which some point in the domain is mapped, i.e.,

$$I \equiv \{y \mid y = f(x),\ \text{for some } x \in D\} \subset R.$$

- The inverse image of a set of points $S \subset I$ is defined as

$$f^{-1}(S) \equiv \{x \mid x \in D,\ f(x) \in S\}.$$

- The graph of the function f is familiar and is the set of ordered pairs

$$G \equiv \{(x, y) \mid x \in D,\ y = f(x)\}.$$

Injective and Surjective Functions

- There is nothing in the definition of a function that prohibits more than one element in the domain from being mapped into the same element in the range.
  - In panel (b) of the figure above, for example, both x_1 and x_2 are mapped into y_1, yet the mapping satisfies the requirements of a function.
- If, however, every point in the range is assigned to at most a single point in the domain, the function is called one-to-one.
  - That is, for sets D and R a function $f : D \to R$ is one-to-one, also called injective, provided that for all $a, b \in D$, if $f(a) = f(b)$, then $a = b$.
- If the image is equal to the range – if every point in the range is mapped into by some point in the domain – the function is said to be onto.
  - That is, for sets D and R a function $g : D \to R$ is onto, also called surjective, provided that for every $y \in R$ there exists $x \in D$ such that $g(x) = y$.
- If a function is one-to-one and onto, then an inverse function $f^{-1} : R \to D$ exists that is also one-to-one and onto.
  - A function is bijective provided that it is both injective and surjective.

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