Real Analysis Theorem Sheet

Chapter 1

Theorem 1.1

There is no number whose square is 2

Assume $\left(\frac{p}{q}\right)^2=2$ and that $p,q\in Z$ and that they have no common factors.

$$\frac{p^2}{q^2}=2\rightarrow p^2=2q^2$$

Since any number multiplied by 2 is even, $p=2n$, so:

$$(2n)^2=2q^2\rightarrow 4n^2=2q^2\rightarrow 2n^2=q^2$$

Which implies q is even. Both p and q are even. But we have already established that p and q are to have no common factors. This is a contradiction. Thus, $\sqrt{2}$ is irrational.

Triangle Inequality

$$|x|=\begin{cases} x\,if\;x\geq 0 \\ -x\,if\;x\leq 0\end{cases}$$

(i) $|ab|=|a|\vee b\vee ¿$;

(ii) $|a+b|\leq|a|+¿b\vee ¿$.

$$|a-b|=|(a-c)+(c-b)|$$

$$|(a-c)+(c-b)|\leq|a-c|+|c-b|$$

$$|a-b|\leq|a-c|+¿c-b\vee ¿$$

Theorem 1.2.6

$a,b\in R, a=b\leftrightarrow\epsilon>0, |a-b|<\epsilon$

Proof. $(\rightarrow)$

If

$$a=b$$

then,

$$a-b=0$$

Proof by Contradiction. $(\leftarrow)$

If for every real number $\epsilon>0$, it follows that $|a-b|<\epsilon$, then $a=b$.

Let $a\neq b$. That is, assume $a=b$ is false. Then:

$$|a-b|>0$$

$$\epsilon_0=|a-b|>0$$

Since $|a-b|<\epsilon$, ϵ_0 must be larger, i.e. $|a-b|<\epsilon_0$. However, $|a-b|=\epsilon_0$ and $|a-b|<\epsilon_0$. This is a contradiction. Thus, for $\epsilon>0$, it follows that:

$$|a-b|<\epsilon \rightarrow a=b$$

<u>**Axiom of Completeness**</u>

Every non-empty set of real numbers that is bounded above has a least upper bound.

Definition 1.3.1. Least Upper Bound

A set $A\subseteq R$ is bounded above if there exists a number $b\in R$ such that $a\leq b$ for all $a\in A$. The number b is called an upper bound.

Definition 1.3.2. Greatest Lower Bound

A real number s is the least upper bound for set $A\in R$ if it meets the following two criteria.

(i) s is an upper bound for A;
(ii) If b is an upper bound for A, then $s\leq b$.

Definition 1.3.4.

A real number is the maximum of a set if a_0 is an element of the set A and:

$$a_0\geq a, \forall_a \in A$$

Similarly, a number a_1 is a minimum of A if $a_1\in A$:

$$a_1\leq a, \forall_a \in A$$

<u>Lemma 1.3.8</u>

Assume $s\in R$ is an upper bound for a set $A\subseteq R$. Then $s=supA$ if and only if for every choice of $\epsilon>0$, there exists an element $a\in A$ satisfying $s-\epsilon<a$.

Theorem 1.4.1: Nestled Interval Property

For every $n \in N$, assume we are given a closed interval:

$$I_n = [a_n, b_n] = \{x \in R \vee a_n \leq x \leq b\}$$

Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals:

$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq I_5 \subseteq \ldots$

Has a nonempty intersection that is ¿1¿$\infty I_n \notin \emptyset$

Proof.

Goal: $x \in I_n \, for\, every\, n \in N$.

Since intervals are nested every b_n, this serves as an upper bound for A. Using the Axiom of Completeness, $x = ^{(A)}$¿

Consider a particular $I_n = [a_n, b_n]$. Because x is a supremum of A, then $a_n \leq x$. Each b_n is an upper bound for A and $x = (A)$. So by 1.3.2 (ii), we know that $x \leq b_n$. Therefore, $a_n \leq x \leq b_n$, which means $x \in I_n$ for any $n \in N$. Hence, $x \in$ ¿1¿∞I_n and the intersection is nonempty.

Theorem 1.4.2: Archimedean Property

(i) Given any number $x \in R$, there exists an $n \in N$ satisfying $n > x$;

(ii) Given any real number $y > 0$, there exists an $n \in N$ satisfying $\frac{1}{n} < y$.

Theorem 1.4.3: Density of Q in R

For every two $a, b \in R$ with $a < b$, there exists a $r \in Q$ satisfying $a < r < b$. “Q is dense in R”.

Goal: $a < \frac{m}{n} < b$, $m \in Z$ and $n \in N$. We want to have big n, so length of interval is small (smaller than length [a,b])

$$\frac{1}{n} < b - a$$

$$\frac{m-1}{n} \leq a < \frac{m}{n}$$

$$an < m \qquad \text{by 1.4.2}$$

So,

$$a < \frac{m}{n}$$

$$\frac{m-1}{n} \leq a \rightarrow m-1 \leq an$$

Recall:

$$\frac{1}{n} < b-a$$

Then,

$$\frac{1}{n} - b \leftarrow a \rightarrow a < b - \frac{1}{n}$$

$$\frac{m-1}{n} < a < b - \frac{1}{n} \rightarrow \frac{m-1}{n} < b - \frac{1}{n} \rightarrow m-1 < \left(b - \frac{1}{n}\right) n$$

$$m-1 < bn-1 \rightarrow m < bn \rightarrow \frac{m}{n} < b$$

$$a < \frac{m}{n} < b$$

Q.E.D

Corollary 1.4.4.

Given any two R numbers $a<b, \exists_{\epsilon}$ s.t. $\epsilon \in Q$ and $a<\epsilon<b$.

Theorem 1.4.5

$\exists_{\alpha}$ s.t. $\alpha \in R$ satisfies $\alpha^2 = 2$

Chapter 2

Definition 2.2.1.

A sequence is a function whose domain is N.

- Given a function $f: N \rightarrow R, f(n)$ is just the n^{th} term.

a) $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

b) $\left(\frac{n+1}{n}\right)_{n=1}^{\infty} = \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\right)$
c) (A_n) where $x_1 = 2, x_{n+1} = 2x_n + 3$

Note: first term doesn't need to be a since it's an infinite list of real numbers.

Definition of Convergence of Sequence

A sequence A_n converges to a R number a if $\forall_{\epsilon^{+¿}, \exists_N, N \in N¿}$ s.t. whenever $n \geq N$, it follows that $|a_n - a| < \varepsilon$.

Definition 2.2.4

Given a real number $a \in R$ and a positive number $\epsilon > 0$, the set

$$\forall_{\varepsilon}(a) = \{x \in R : |x - a| < \epsilon\}$$

is called the $\epsilon - neighborhood\ of\ a$.

Definition of the Topological version of Convergence of a Sequence

A sequence (a_n) converges to a if given any ϵ-neighborhood, $\forall_{\epsilon}(a)$ of a, there exists a point in the sequence after which all the terms are in $\forall_{\epsilon}(a)$.

- The value of N depends on choice of ϵ.
- Not convergent since its not in the $\forall_{\varepsilon}$ for every ϵ.

Template for $(x_n) \rightarrow x$:

- "Let $\epsilon > 0$be arbitrary"
- Demonstrate a choice for $N \in N$. This step usually requires the most work, almost all of which is done prior to writing the formal proof.
- Now, show that N works.
- "Assume $n \geq N$"
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Theorem 2.2.7: Uniqueness of Limits

The limit of a sequence, when it exists, must be unique.

Definition 2.2.9. A sequence that does not converge is said to *diverge*.

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M>0$ such that $|x_n| \leq M$ for all $n \in N$.

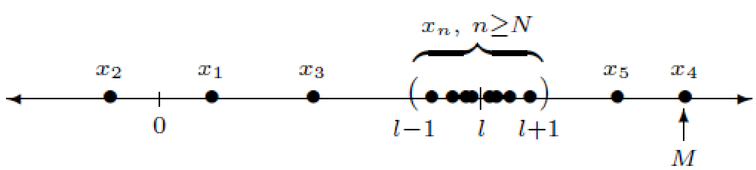
Geometrically, this means we can find an $[-M, M]$ that contains every term in the sequence (x_n).

Definition 2.3.2. Every convergent sequence is bounded.

Proof. Assume (x_n) converges to a limit l. This means that given a particular value of ϵ, say $\epsilon = 1$, we know there must exist an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-1, l+1)$. Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



We still need to worry (slightly) about the terms in the sequence that come before the Nth term. Because there are only a finite number of these, we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l| + 1\}.$$

It follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$, as desired. $\square$

Theorem 2.3.3 (Algebraic Limit Theorem). Let $\lim a_n = a \wedge \lim b_n = b$. Then,

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Proof. (i) Consider the case where $c \neq 0$. We want to show that the sequence (ca_n) converges to ca, so the structure of the proof follows the template we described in Section 2.2. First, we let ϵ be some arbitrary positive number. Our goal is to find some point in the sequence (ca_n) after which we have

$$|ca_n - ca| < \epsilon.$$

Now,

$$|ca_n - ca| = |c||a_n - a|.$$

We are given that $(a_n) \to a$, so we know we can make $|a_n - a|$ as small as we like. In particular, we can choose an N such that

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. To see that this N indeed works, observe that, for all $n \geq N$,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

The case $c = 0$ reduces to showing that the constant sequence $(0, 0, 0, \ldots)$ converges to 0, which is easily verified.

Before continuing with parts (ii), (iii), and (iv), we should point out that the proof of (i), while somewhat short, is extremely typical for a convergence proof. Before embarking on a formal argument, it is a good idea to take an inventory of what we *want* to make less than ϵ, and what we are *given* can be made small for suitable choices of n. For the previous proof, we wanted to make $|ca_n - ca| < \epsilon$, and we were given $|a_n - a| <$ *anything we like* (for large values of n). Notice that in (i), and all of the ensuing arguments, the strategy each time is to bound the quantity we want to be less than ϵ, which in each case is

$$|(\text{terms of sequence}) - (\text{proposed limit})|,$$

with some algebraic combination of quantities over which we have control.

(ii) To prove this statement, we need to argue that the quantity

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ using the assumptions that $|a_n - a|$ and $|b_n - b|$ can be made as small as we like for large n. The first step is to use the triangle inequality (Example 1.2.5) to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Again, we let $\epsilon > 0$ be arbitrary. The technique this time is to divide the ϵ between the two expressions on the right-hand side in the preceding inequality. Using the hypothesis that $(a_n) \to a$, we know there exists an N_1 such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1.$$

Likewise, the assumption that $(b_n) \to b$ means that we can choose an N_2 so that

$$|b_n - b| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

The question now arises as to which of N_1 or N_2 we should take to be our choice of N. By choosing $N = \max\{N_1, N_2\}$, we ensure that if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. This allows us to conclude that

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.

(iii) To show that $(a_n b_n) \to ab$, we begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

In the initial step, we subtracted and then added ab_n, which created an opportunity to use the triangle inequality. Essentially, we have broken up the distance from $a_n b_n$ to ab with a midway point and are using the sum of the two distances to overestimate the original distance. This clever trick will become a familiar technique in arguments to come.

Letting $\epsilon > 0$ be arbitrary, we again proceed with the strategy of making each piece in the preceding inequality less than $\epsilon/2$. For the piece on the right-hand side ($|a||b_n - b|$), if $a \neq 0$ we can choose N_1 so that

(The case when $a = 0$ is handled in Exercise 2.3.9.) Getting the term on the left-hand side ($|b_n||a_n - a|$) to be less than $\epsilon/2$ is complicated by the fact that we have a variable quantity $|b_n|$ to contend with as opposed to the constant $|a|$ we encountered in the right-hand term. The idea is to replace $|b_n|$ with a worst-case estimate. Using the fact that convergent sequences are bounded (Theorem 2.3.2), we know there exists a bound $M > 0$ satisfying $|b_n| \leq M$ for all $n \in \mathbf{N}$. Now, we can choose N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

To finish the argument, pick $N = \max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M\left(\frac{\epsilon}{M2}\right) + |a|\left(\frac{\epsilon}{|a|2}\right) = \epsilon. \end{aligned}$$

(iv) This final statement will follow from (iii) if we can prove that

$$(b_n) \to b \quad \text{implies} \quad \left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$. We begin by observing that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|}.$$

Because $(b_n) \to b$, we can make the preceding numerator as small as we like by choosing n large. The problem comes in that we need a worst-case estimate on the size of $1/(|b||b_n|)$. Because the b_n terms are in the denominator, we are no longer interested in an upper bound on $|b_n|$ but rather in an inequality of the form $|b_n| \geq \delta > 0$. This will then lead to a bound on the size of $1/(|b||b_n|)$.

The trick is to look far enough out into the sequence (b_n) so that the terms are closer to b than they are to 0. Consider the particular value $\epsilon_0 = |b|/2$. Because $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2$.

Next, choose N_2 so that $n \geq N_2$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}.$$

Finally, if we let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

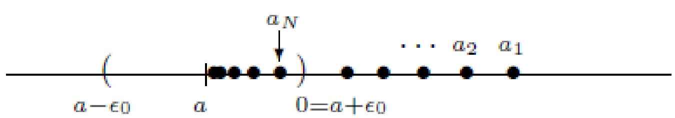
Theorem 2.3.4 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) *If* $a_n \geq 0$ *for all* $n \in \mathbf{N}$, *then* $a \geq 0$.

(ii) *If* $a_n \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $a \leq b$.

(iii) *If there exists* $c \in \mathbf{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbf{N}$, *then* $a \leq c$.

Proof. (i) We will prove this by contradiction; thus, let's assume $a < 0$. The idea is to produce a term in the sequence (a_n) that is also less than zero. To do this, we consider the particular value $\epsilon = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. In particular, this would mean that $|a_N - a| < |a|$, which implies $a_N < 0$. This contradicts our hypothesis that $a_N \geq 0$. We therefore conclude that $a \geq 0$.



(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b - a \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). $\square$

Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2 (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be monotone and bounded. To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}.$$

It seems reasonable to claim that $\lim a_n = s$.

To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired. $\square$

Definition 2.4.3 (Convergence of a Series). Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots .$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem 2.4.6 (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Theorem 2.3.2 guarantees that the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

are bounded; that is, there exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. We want to prove that $\sum_{n=1}^{\infty} b_n$ converges. Because $b_n \geq 0$, we know that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k. \end{aligned}$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

The proof that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges implies $\sum_{n=1}^{\infty} b_n$ diverges is similar to Example 2.4.5. The details are requested in Exercise 2.4.9. □

Corollary 2.4.7. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

A rigorous argument for this corollary requires a few basic facts about geometric series. The proof is requested in Exercise 2.7.5 at the end of Section 2.7 where geometric series are discussed.

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbf{N}$ indexes the subsequence.

Theorem 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Assume $(a_n) \to a$, and let (a_{n_k}) be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is, $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. $\square$

This not too surprising result has several somewhat surprising applications. It is the key ingredient for understanding when infinite sums are associative (Exercise 2.5.3). We can also use it in the following clever way to compute values of some familiar limits.

Example 2.5.4 (Divergence Criterion). Theorem 2.5.2 is also useful for providing economical proofs for divergence. In Example 2.2.8, we were quite sure that

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right)$$

did not converge to any proposed limit. Notice that

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right)$$

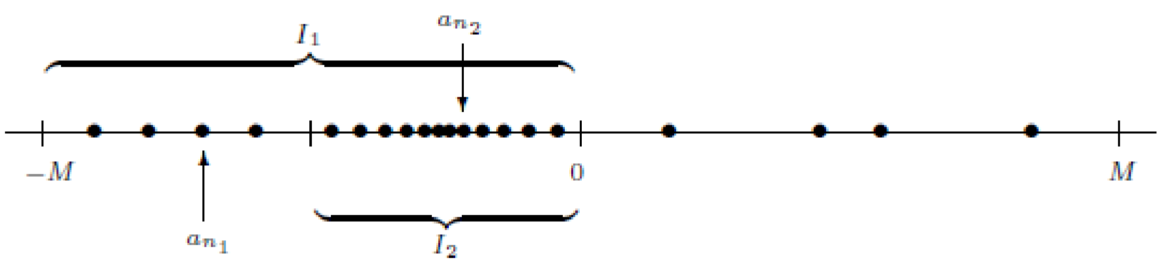
is a subsequence that converges to $1/5$. Also,

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots\right)$$

is a different subsequence of the original sequence that converges to $-1/5$. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges.

Theorem 2.5.5 (Bolzano–Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) Now, it must be that at least one of these closed intervals contains an infinite number of the terms in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some term in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of (a_n) and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k} \in I_k$.

We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $(a_{n_k}) \to x$.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (This follows from Example 2.5.3 and the Algebraic Limit Theorem.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. $\square$

Definition 2.6.1. A sequence (a_n) is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

To make the comparison easier, let's restate the definition of convergence.

Definition 2.2.3. A sequence (a_n) *converges* to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

As we have discussed, the definition of convergence asserts that, given an arbitrary positive ϵ, it is possible to find a point in the sequence after which the terms of the sequence are all closer to the limit a than the given ϵ. On the

Theorem 2.6.2. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x. To prove that (x_n) is Cauchy, we must find a point in the sequence after which we have $|x_n - x_m| < \epsilon$. This can be done using an application of the triangle inequality. The details are requested in Exercise 2.6.1. $\square$

The converse is a bit more difficult to prove, mainly because, in order to prove that a sequence converges, we must have a proposed limit for the sequence to approach. We have been in this situation before in the proofs of the Monotone Convergence Theorem and the Bolzano–Weierstrass Theorem. Our strategy here will be to use the Bolzano–Weierstrass Theorem. This is the reason for the next lemma. (Compare this with Theorem 2.3.2.)

Lemma 2.6.3. *Cauchy sequences are bounded.*

Proof. Given $\epsilon = 1$, there exists an N such that $|x_m - x_n| < 1$ for all $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence (x_n). $\square$

Theorem 2.6.4 (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

Proof. $(\Rightarrow)$ This direction is Theorem 2.6.2.

$(\Leftarrow)$ For this direction, we start with a Cauchy sequence (x_n). Lemma 2.6.3 guarantees that (x_n) is bounded, so we may use the Bolzano–Weierstrass Theorem to produce a convergent subsequence (x_{n_k}). Set

$$x = \lim x_{n_k}.$$

The idea is to show that the original sequence (x_n) converges to this same limit. Once again, we will use a triangle inequality argument. We know the terms in the subsequence are getting close to the limit x, and the assumption that (x_n) is Cauchy implies the terms in the "tail" of the sequence are close to each other. Thus, we want to make each of these distances less than half of the prescribed ϵ.

Let $\epsilon > 0$. Because (x_n) is Cauchy, there exists N such that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, we also know that $(x_{n_k}) \to x$, so choose a term in this subsequence, call it x_{n_K}, with $n_K \geq N$ and

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

To see that N has the desired property (for the original sequence (x_n)), observe that if $n \geq N$, then

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Theorem 2.7.1 (Algebraic Limit Theorem for Series). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbf{R}$ *and*

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Proof. (i) In order to show that $\sum_{k=1}^{\infty} ca_k = cA$, we must argue that the sequence of partial sums

$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m$$

converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning that the partial sums

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$

converge to A. Because $t_m = cs_m$, applying the Algebraic Limit Theorem for sequences (Theorem 2.3.3) yields $(t_m) \to cA$, as desired.

The proof of part (ii) is analogous and is left as an unofficial exercise. □

One way to summarize Theorem 2.7.1 (i) is to say that infinite addition still satisfies the distributive property. Part (ii) verifies that series can be added in the usual way. Missing from this theorem is any statement about the *product* of two infinite series. At the heart of this question is the issue of commutativity, which requires a more delicate analysis and so is postponed until Section 2.8.

Theorem 2.7.2 (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

and apply the Cauchy Criterion for sequences. □

The Cauchy Criterion leads to economical proofs of several basic facts about series.

Theorem 2.7.3. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Proof. Consider the special case $n = m + 1$ in the Cauchy Criterion for Series. □

Every statement of this result should be accompanied with a reminder to look at the harmonic series (Example 2.4.5) to erase any misconception that the converse statement is true. Knowing (a_k) tends to 0 does not imply that the series converges.

Theorem 2.7.4 (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.*

(i) *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*

Proof. Both statements follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|.$$

Alternate proofs using the Monotone Convergence Theorem are requested in the exercises. $\square$

Theorem 2.7.6 (Absolute Convergence Test). *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges as well.*

Proof. This proof makes use of both the necessity (the "if" direction) and the sufficiency (the "only if" direction) of the Cauchy Criterion for Series. Because $\sum_{n=1}^{\infty} |a_n|$ converges, we know that, given an $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{n=1}^{\infty} a_n$ also converges. □

The converse of this theorem is false. In the opening discussion of this chapter, we considered the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Taking absolute values of the terms gives us the harmonic series $\sum_{n=1}^{\infty} 1/n$, which we have seen diverges. However, it is not too difficult to prove that with the alternating negative signs the series indeed converges. This is a special case of the Alternating Series Test.

Theorem 2.7.7 (Alternating Series Test). *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. A consequence of conditions (i) and (ii) is that $a_n \geq 0$. Several proofs of this theorem are outlined in Exercise 2.7.1. □

Definition 2.7.8. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

Definition 2.7.9. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbf{N} \to \mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbf{N}$.

We now have all the tools and notation in place to resolve an issue raised at the beginning of the chapter. In Section 2.1, we constructed a particular rearrangement of the alternating harmonic series that converges to a limit different from that of the original series. This happens because the convergence is *conditional.*

Theorem 2.7.10. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

for the partial sums of the original series and use

$$t_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + \cdots + b_m$$

for the partial sums of the rearranged series. Thus we want to show that $(t_m) \to A$.

Let $\epsilon > 0$. By hypothesis, $(s_n) \to A$, so choose N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, a_3, \ldots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out in the series $\sum_{n=1}^{\infty} b_n$ so that we have included all of these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

It should now be evident that if $m \geq M$, then $(t_m - s_N)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=N+1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned}
|t_m - A| &= |t_m - s_N + s_N - A| \\
&\leq |t_m - s_N| + |s_N - A| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

whenever $m \geq M$. □

Theorem 2.8.1. *Let $\{a_{ij} : i, j \in \mathbf{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover,

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1), define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|.$$

Definition 3.2.1. A set $O \subseteq \mathbf{R}$ is *open* if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.

Theorem 3.2.3. (i) *The union of an arbitrary collection of open sets is open.*

(ii) *The intersection of a finite collection of open sets is open.*

Proof. To prove (i), we let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets and let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Let a be an arbitrary element of O. In order to show that O is open, Definition 3.2.1 insists that we produce an ϵ-neighborhood of a completely contained in O. But $a \in O$ implies that a is an element of at least one particular $O_{\lambda'}$. Because we are assuming $O_{\lambda'}$ is open, we can use Definition 3.2.1 to assert that there exists $V_\epsilon(a) \subseteq O_{\lambda'}$. The fact that $O_{\lambda'} \subseteq O$ allows us to conclude that $V_\epsilon(a) \subseteq O$. This completes the proof of (i).

For (ii), let $\{O_1, O_2, \ldots, O_N\}$ be a finite collection of open sets. Now, if $a \in \bigcap_{k=1}^N O_k$, then a is an element of each of the open sets. By the definition of an open set, we know that, for each $1 \leq k \leq N$, there exists $V_{\epsilon_k}(a) \subseteq O_k$. We are in search of a *single* ϵ-neighborhood of a that is contained in every O_k, so the trick is to take the smallest one. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, it follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a)$ for all k, and hence $V_\epsilon(a) \subseteq \bigcap_{k=1}^N O_k$, as desired. □

Closed Sets

Definition 3.2.4. A point x is a *limit point* of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Limit points are also often referred to as "cluster points" or "accumulation points," but the phrase "x is a limit point of A" has the advantage of explicitly reminding us that x is quite literally the limit of a sequence in A.

Theorem 3.2.5. *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.*

Proof. ($\Rightarrow$) Assume x is a limit point of A. In order to produce a sequence (a_n) converging to x, we are going to consider the particular ϵ-neighborhoods obtained using $\epsilon = 1/n$. By Definition 3.2.4, every neighborhood of x intersects A in some point other than x. This means that, for each $n \in \mathbf{N}$, we are justified in picking a point

$$a_n \in V_{1/n}(x) \cap A$$

with the stipulation that $a_n \neq x$. It should not be too difficult to see why $(a_n) \to x$. Given an arbitrary $\epsilon > 0$, choose N such that $1/N < \epsilon$. It follows that $|a_n - x| < \epsilon$ for all $n \geq N$.

($\Leftarrow$) For the reverse implication we assume $\lim a_n = x$ where $a_n \in A$ but $a_n \neq x$, and let $V_\epsilon(x)$ be an arbitrary ϵ-neighborhood. The definition of convergence assures us that there exists a term a_N in the sequence satisfying $a_N \in V_\epsilon(x)$, and the proof is complete. $\square$

The restriction that $a_n \neq x$ in Theorem 3.2.5 deserves a comment. Given a point $a \in A$, it is always the case that a is the limit of a sequence in A if we are allowed to consider the constant sequence $(a, a, a, \dots)$. There will be occasions where we will want to avoid this somewhat uninteresting situation, so it is important to have a vocabulary that can distinguish limit points of a set from *isolated points*.

Definition 3.2.6. A point $a \in A$ is an *isolated point of* A if it is not a limit point of A.

As a word of caution, we need to be a little careful about how we understand the relationship between these concepts. Whereas an isolated point is always an element of the relevant set A, it is quite possible for a limit point of A not to belong to A. As an example, consider the endpoint of an open interval. This situation is the subject of the next important definition.

Definition 3.2.7. A set $F \subseteq \mathbf{R}$ is *closed* if it contains its limit points.

The adjective "closed" appears in several other mathematical contexts and is usually employed to mean that an operation on the elements of a given set does not take us out of the set. In linear algebra, for example, a vector space is a set that is "closed" under addition and scalar multiplication. In analysis, the operation we are concerned with is the limiting operation. Topologically speaking, a closed set is one where convergent sequences within the set have limits that are also in the set.

Theorem 3.2.8. *A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Proof. Exercise 3.2.5. $\square$

Theorem 3.2.10 (Density of Q in R). *For every $y \in \mathbf{R}$, there exists a sequence of rational numbers that converges to y.*

Proof. Combine the preceding discussion with Theorem 3.2.5. $\square$

Closure

Definition 3.2.11. Given a set $A \subseteq \mathbf{R}$, let L be the set of all limit points of A. The *closure* of A is defined to be $\overline{A} = A \cup L$.

Theorem 3.2.12. *For any $A \subseteq \mathbf{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Proof. If L is the set of limit points of A, then it is immediately clear that $\overline{A}$ contains the limit points of A. There is still something more to prove, however, because taking the union of L with A could potentially produce some new limit points of $\overline{A}$. In Exercise 3.2.7, we outline the argument that this does not happen.

Now, *any* closed set containing A must contain L as well. This shows that $\overline{A} = A \cup L$ is the smallest closed set containing A. $\square$

Theorem 3.2.13. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Proof. Given an open set $O \subseteq \mathbf{R}$, let's first prove that O^c is a closed set. To prove O^c is closed, we need to show that it contains all of its limit points. If x is a limit point of O^c, then *every* neighborhood of x contains some point of O^c. But that is enough to conclude that x cannot be in the open set O because $x \in O$ would imply that there exists a neighborhood $V_\epsilon(x) \subseteq O$. Thus, $x \in O^c$, as desired.

For the converse statement, we assume O^c is closed and argue that O is open. Thus, given an arbitrary point $x \in O$, we must produce an ϵ-neighborhood $V_\epsilon(x) \subseteq O$. Because O^c is closed, we can be sure that x is *not* a limit point of O^c. Looking at the definition of limit point, we see that this implies that there must be some neighborhood $V_\epsilon(x)$ of x that does not intersect the set O^c. But this means $V_\epsilon(x) \subseteq O$, which is precisely what we needed to show.

The second statement in Theorem 3.2.13 follows quickly from the first using the observation that $(E^c)^c = E$ for any set $E \subseteq \mathbf{R}$. $\square$

The last theorem of this section should be compared to Theorem 3.2.3.

Theorem 3.2.14. (i) *The union of a finite collection of closed sets is closed.*

(ii) *The intersection of an arbitrary collection of closed sets is closed.*

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ it is true that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

The result follows directly from these statements and Theorem 3.2.3. The details are requested in Exercise 3.2.9. ∎

Definition 3.3.1 (Compactness). A set $K \subseteq \mathbf{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K.

Definition 3.3.3. A set $A \subseteq \mathbf{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 3.3.4 (Characterization of Compactness in R). *A set $K \subseteq \mathbf{R}$ is compact if and only if it is closed and bounded.*

Proof. Let K be compact. We will first prove that K must be bounded, so assume, for contradiction, that K is not a bounded set. The idea is to produce a sequence in K that marches off to infinity in such a way that it cannot have a convergent subsequence as the definition of compact requires. To do this, notice that because K is not bounded there must exist an element $x_1 \in K$ satisfying $|x_1| > 1$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given any $n \in \mathbf{N}$, we can produce $x_n \in K$ such that $|x_n| > n$.

Now, because K is assumed to be compact, (x_n) should have a convergent subsequence (x_{n_k}). But the elements of the subsequence must satisfy $|x_{n_k}| > n_k$, and consequently (x_{n_k}) is unbounded. Because convergent sequences are bounded (Theorem 2.3.2), we have a contradiction. Thus, K must at least be a bounded set.

Next, we will show that K is also closed. To see that K contains its limit points, we let $x = \lim x_n$, where (x_n) is contained in K and argue that x must be in K as well. By Definition 3.3.1, the sequence (x_n) has a convergent subsequence (x_{n_k}), and by Theorem 2.5.2, we know (x_{n_k}) converges to the same limit x. Finally, Definition 3.3.1 requires that $x \in K$. This proves that K is closed.

The proof of the converse statement is requested in Exercise 3.3.3. $\square$

There may be a temptation to consider closed intervals as being a kind of standard archetype for compact sets, but this is misleading. The structure of compact sets can be much more intricate and interesting. For instance, one implication of Theorem 3.3.4 is that the Cantor set is compact. It is more useful to think of compact sets as generalizations of closed intervals. Whenever a fact involving closed intervals is true, it is often the case that the same result holds when we replace "closed interval" with "compact set." As an example, let's experiment with the Nested Interval Property proved in the first chapter.

Theorem 3.3.5 (Nested Compact Set Property). *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Proof. In order to take advantage of the compactness of each K_n, we are going to produce a sequence that is eventually in each of these sets. Thus, for each $n \in \mathbf{N}$, pick a point $x_n \in K_n$. Because the compact sets are nested, it follows that the sequence (x_n) is contained in K_1. By Definition 3.3.1, (x_n) has a convergent subsequence (x_{n_k}) whose limit $x = \lim x_{n_k}$ is an element of K_1.

In fact, x is an element of *every* K_n for essentially the same reason. Given a particular $n_0 \in \mathbf{N}$, the terms in the sequence (x_n) are contained in K_{n_0} as long as $n \geq n_0$. Ignoring the finite number of terms for which $n_k < n_0$, the same subsequence (x_{n_k}) is then also contained in K_{n_0}. The conclusion is that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$ and the proof is complete. $\square$

Definition 3.3.6. Let $A \subseteq \mathbf{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Theorem 3.3.8 (Heine–Borel Theorem). *Let K be a subset of $\mathbf{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others:*

(i) *K is compact.*

(ii) *K is closed and bounded.*

(iii) *Every open cover for K has a finite subcover.*

Proof. The equivalence of (i) and (ii) is the content of Theorem 3.3.4. What remains is to show that (iii) is equivalent to (i) and (ii). Let's first assume (iii), and prove that it implies (ii) (and thus (i) as well).

To show that K is bounded, we construct an open cover for K by defining O_x to be an open interval of radius 1 around each point $x \in K$. In the language of neighborhoods, $O_x = V_1(x)$. The open cover $\{O_x : x \in K\}$ then must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Because K is contained in a finite union of bounded sets, K must itself be bounded.

The proof that K is closed is more delicate, and we argue it by contradiction. Let (y_n) be a Cauchy sequence contained in K with $\lim y_n = y$. To show that K is closed, we must demonstrate that $y \in K$, so assume for contradiction that this is not the case. If $y \notin K$, then every $x \in K$ is some positive distance away from y. We now construct an open cover by taking O_x to be an interval of radius $|x-y|/2$ around each point x in K. Because we are assuming (iii), the resulting open cover $\{O_x : x \in K\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. The contradiction arises when we realize that, in the spirit of Example 3.3.7, this finite subcover cannot contain all of the elements of the sequence (y_n). To make this explicit, set

$$\epsilon_0 = \min\left\{\frac{|x_i - y|}{2} : 1 \leq i \leq n\right\}.$$

Because $(y_n) \to y$, we can certainly find a term y_N satisfying $|y_N - y| < \epsilon_0$. But such a y_N must necessarily be excluded from each O_{x_i}, meaning that

$$y_N \notin \bigcup_{i=1}^{n} O_{x_i}.$$

Thus our supposed subcover does not actually cover all of K. This contradiction implies that $y \in K$, and hence K is closed and bounded.

The proof that (ii) implies (iii) is outlined in Exercise 3.3.9. To be historically accurate, it is this particular implication that is most appropriately referred to as the Heine–Borel Theorem. $\square$