Applied Bayesian Method HW3

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1 Assignment Description

Let $\theta = (\theta_0, \theta_1)$, consider the logit model $p(y_i|\theta) = \dfrac{\exp(\theta_0 + \theta_1 x_i)}{1 + \exp(\theta_0 + \theta_1 x_i)}$. Do the following analysis:

- Obtain $p(\theta|y)$ by Metropolis-Hastings algorithm using Gaussian proposal $q(y|x) = \phi(y; x, \sigma)$, with $\sigma = 1$ and 0.1.
- Obtain the acceptance rates for $\sigma = 1$ and 0.1.
- Compute the posterior mean and variance.

Please use dataset ”programming” on e-learning website.

2 Methodology

In this assignment, we will use Metropolis-Hastings algorithm, which is a part of Markov Chain Monte Carlo (MCMC) methods. This algorithm aims to generate a sequence of samples from a target distribution ($\pi(j)$) by constructing a Markov chain that has $\pi(j)$ as its equilibrium distribution. Following are steps of the Metropolis-Hastings Algorithm:

1. Initialization: Start with an initial value x_0, which can be randomly chosen from the support of the target distribution.
2. Proposal distribution: Choosen a proposal distribution $q(\theta|\theta')$ to generate a candidate state y given the current state x, where θ is the current value generate by proposal distribution and θ' is the previous value. In this assignment, the proposal distribution is

$$q(\theta|\theta') = \phi(\theta, \theta', \sigma)$$

3. Acceptance Probability: Calculate the acceptance probability α for moving from the current

state x to the proposed state y:

$$\alpha = \min\left(\frac{\pi(\theta)q(\theta'|\theta)}{\pi(\theta')q(\theta|\theta')}, 1\right)$$

In this assignment, since we are given the data, we can calculate the target distribution $p(\theta|y)$ of the distribution of θ by the likelihood and the prior distribution. That is,

$$\alpha = \min\left(\frac{\prod_i^n \left(\frac{\exp(\theta_0+\theta_1 x_i)}{1+\exp(\theta_0+\theta_1 x_i)}\right)^{y_i}\left(\frac{1}{1+\exp(\theta_0+\theta_1 x_i)}\right)^{1-y_i}\prod_j N(\theta_j, 0.25)}{\prod_i^n \left(\frac{\exp(\theta_0'+\theta_1' x_i)}{1+\exp(\theta_0'+\theta_1' x_i)}\right)^{y_i}\left(\frac{1}{1+\exp(\theta_0'+\theta_1' x_i)}\right)^{1-y_i}\prod_j N(\theta_j', 0.25)}, 1\right)$$

where i is the observation of data and $j = 0, 1$. Note that the proposal distribution is not in the formula because in our example the proposal distribution is symmetric so it's ratio will cancel each other out in the calculation, and thus will not appear in the formula.

4. Accept or Reject: Generate a uniform random number u from the interval $[0, 1]$. If $u < \alpha$, we will accept the new value. Otherwise, we reject and retain the current value.

5. Iteration: We repeat steps 2-4 for a large number of iterations to create a Markov chain and these samples can be considered as coming from the target distribution.

3 Result

Before we show the results of the simulation, I want to mention that I take logarithm on both likelihood and prior distribution in my Python code. In fig (1) and fig (2), we show the results of the sampling with σ set to be 1 and 0.1 respectively. For the acceptance rate, posterior mean, and posterior variance, the results are shown in table (1), note that the posterior mean and variance are simply calculated from the sampling results in fig (1) and fig (2).

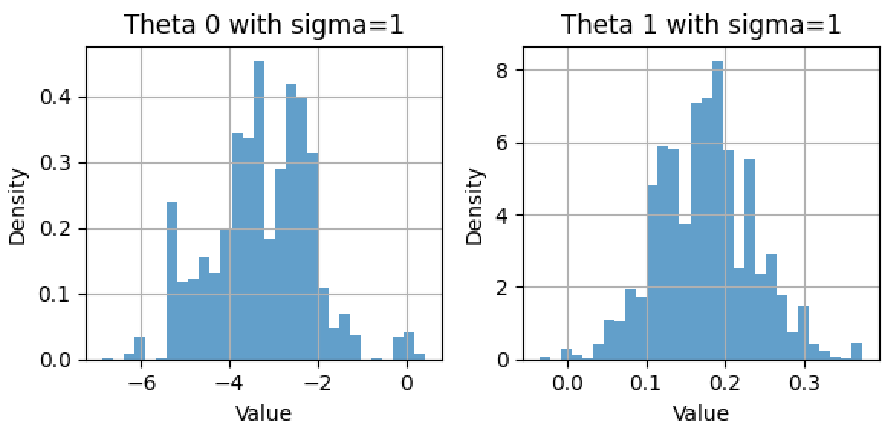

Figure 1: MCMC simulation for 10000 times ($\sigma = 1$)

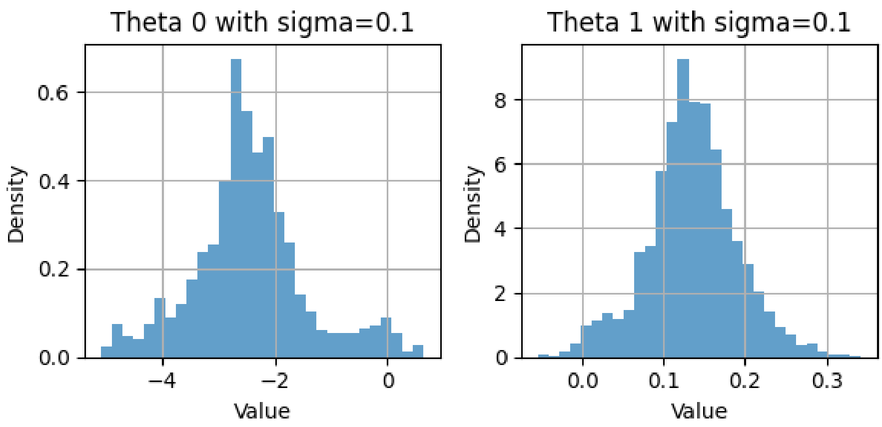

Figure 2: MCMC simulation for 10000 times ($\sigma = 0.1$)

	$\sigma = 1$	$\sigma = 0.1$
Acceptance Rate	0.0309	0.2972
Posterior Mean for θ_0	-3.2728	-2.4513
Posterior Variance θ_0	1.2933	1.0218
Posterior Mean for θ_1	0.1747	0.1348
Posterior Variance for θ_1	0.0037	0.0031

Table 1: Acceptance rate, posterior mean, and posterior variance under different level of σ