Applied Bayesian Method HW2

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1 Assignment Description

Given that our target distribution $\pi(j) = Cb_j$, where $b_j = \dfrac{1}{j^2}$ and C is a normalization constant which is not required in the algorithm. Do Metropolis-Hastings with proposal $q(y) = \dfrac{1}{2^y}$.

2 Methodology

In this assignment, we will use Metropolis-Hastings algorithm, which is a part of Markov Chain Monte Carlo (MCMC) methods. This algorithm aims to generate a sequence of samples from a target distribution ($\pi(j)$) by constructing a Markov chain that has $\pi(j)$ as its equilibrium distribution. Following are steps of the Metropolis-Hastings Algorithm:

1. Initialization: Start with an initial value x_0, which can be randomly chosen from the support of the target distribution.

2. Proposal distribution: Choosen a proposal distribution $q(y|x)$ to generate a candidate state y given the current state x. In this assignment, the proposal distribution is

$$q(y) = \frac{1}{2^y}$$

3. Acceptance Probability: Calculate the acceptance probability α for moving from the current state x to the proposed state y:

$$\alpha = \min\left(\frac{\pi(y)q(x|y)}{\pi(x)q(y|x)}, 1\right)$$

In this assignment, we can easily calculate the acceptance probability α as

$$\alpha = \min\left(\frac{i^2 2^j}{j^2 2^i}, 1\right)$$

where i is the current value and j is the new value sapled from the proposal distribution.

4. Accept or Reject: Generate a uniform random number u from the interval $[0, 1]$. If $u < \alpha$, we will accept the new value. Otherwise, we reject and retain the current value.
5. Iteration: We repeat steps 2-4 for a large number of iterations to create a Markov chain and these samples can be considered as coming from the target distribution.

3 Result

Before we start our simulation, we briefly talk about how to do the sampling from proposal distribution $q(y) = 1/2^y$. First, we compute the cumulative distribution function (CDF) of the distribution and obtain

$$CDF(y) = \sum_{k=1}^{y} \frac{1}{2^k} = 1 - \frac{1}{2^y}$$

Next, we generate a random number u from a uniform distribution between 0 and 1, and we can find the value y such that $CDF(y) = u$ by rerranging the CDF equation

$$1 - \frac{1}{2^y} = u \Rightarrow y = -\log_2(1-u)$$

Thus, we can sample y from our proposal distribution using

$$y = -\log_2(1-u)$$

Fig (1) shows our simulation result. Moreover, we calculate the probabilities of the occurrence of value 1 to 10 (table 1) and find that the ratio of probability for each value approximately follow the rule below:

$$P(i=1) : P(i=2) : \cdots : P(i=10) = 1 : \frac{1}{2^2} : \cdots : \frac{1}{10^2}$$

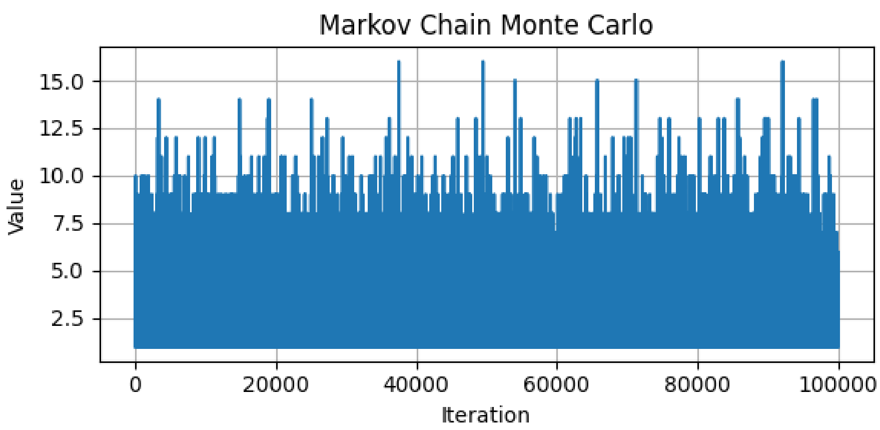

Figure 1: MCMC simulation for 100000 times

Value	Occurrence Times	Probability
1	63592	0.6359
2	15603	0.1560
3	6918	0.0692
4	3923	0.0392
5	2501	0.0250
6	1732	0.0173
7	1229	0.0123
8	1065	0.0106
9	737	0.0074
10	556	0.0056

Table 1: Probabilities for the occurrence of value 1 to 10