Minimax Strategy for Planning Accelerated Life Tests

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Abstract

Due to time constraint and experimental cost, how to plan an efficient accelerated life test (ALT) to obtain more accurate lifetime information of products is an important research issue. Many strategies were proposed to design a locally optimal planning of an ALT under the pre-specified planning values of parameters. However, the optimal design for an ALT also depends on model parameters are usually unknown before the experiment. Alternatively, the Bayesian technique is used by pre-specifying the prior information for the unknown parameters. Instead of specifying prior distributions, this study adopts a minimax criterion to obtain a more robust design for conducting an ALT. Particularly, the minimax design is determined once we specify the range of sample failure probability under a specific failure model. To find the minimax design efficiently, this study adopts the particle swarm optimization (PSO) technique. Finally, compared to the locally optimal design via simulation study, the minimax design is more robust and more practical.

1 Introduction

For highly reliable products, it is critical to obtain their reliability information at the use condition, including the mean time to failure or the quantile of lifetime. For collecting failure information efficiently, an accelerated life test (ALT) is often adopted to obtain more failures at higher testing stress levels, usually higher than the use stress level. For obtaining the failure information at the use condition, one can use the data from the ALT to construct an accelerated function related to the lifetime information and testing stress levels. Using the fitted accelerated function, the failure information at the use stress level could be easily extrapolated by the fitted values of parameters. More specifically, the purpose of an ALT is to make inference or make predictions of lifetime information at the use condition. Due to the limited experimental period, the censoring scheme is often adopted in the ALT. For example, if the product's failure time lasts too long, then we record the unit as a censored unit at the pre-given censoring time C. For highly reliable products, there are more censored units at the stress levels close to the use condition in an ALT. However, too many censored units can not provide much information to make inference of reliability information at the use condition. Hence, to design an efficient ALT so that make the prediction of reliability information at the use condition more precise is an critical and practical research. Several studies proposed their planning strategies for ALTs, including the locally optimum design (Meeker and Escobar, 1998), the Bayesian technique (Zhang and Meeker, 2006), the sequential strategies (Lee et al., 2018). In practice, the locally optimum design needs the planning value, the Bayesian technique needs the prior information of model parameters, and the sequential strategies needs historical data and the prior information of model parameters.

Although we can consult with experts, a more robust design over parameters information for an ALT is still worth exploring. This study proposes the minimax or maximin (Chen et al., 2017) strategy to develop an ALT planning. In practice, the plausible region is usually formed by specifying an interval of plausible values for each model parameter. This is likely more feasible in practice since it only requires the user to provide a range of possible values for each model parameter of interest. Some work for finding minimax (or maximin) optimal designs are Müller and Pázman (1998); Huang and Lin (2006); Chen et al. (2008, 2015) and Lukemire et al. (2008). In the area of reliability, the strategies in Ginebra and Sen (1998) and Pascual and Montepiedra (2002) are most related to the concept of a minimax design. Ginebra and Sen (1998) restricted their candidate designs in the minimax criterion since they indicated finding the global minimax design is too difficult. Pascual and Montepiedra (2002) and Pascual and Montepiedra (2003) constructed the minimax criterion for model robustness by specifying planning values. Due to the complexity of the minimax criterion, searching a global minimax design is not easy. In the literature, the particle swarm optimization (PSO), or a modified version of it, is used to tackle more challenging optimal design problem in Chen et al. (2017, 2020) and Chen et al. (2022). In this study, we adopt the PSO-based algorithm to search a global minimax design, and we compare the proposed design to the existing designs for an ALT via the simulation study.

2 Lifetime Model and Minimax optimal designs for ALT

2.1 Lifetime model

In the reliability literature, using a log-location-scale distribution to model the cycles-to-failure, T. The cumulative distribution function (cdf) and the probability density function (pdf) are given as

$$F(t;\mu,\sigma) = \Phi\left[\frac{\log(t)-\mu}{\sigma}\right] \quad \text{and} \quad f(t;\mu,\sigma) = \frac{1}{\sigma t}\phi\left[\frac{\log(t)-\mu}{\sigma}\right],$$

where μ and σ are the location and the scale parameters, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard cdf and pdf, respectively. The lognormal and Weibull distributions are the common distributions for analyzing lifetime data. In the ALT modeling, we assume a constant scale parameter σ and the location parameter is a function of the stress level of the accelerating variable. Under the stress level S, the location parameter μ is denoted by $\mu(S) = a + bX(S)$, where $X(S)$ is the standardized function depending on different accelerating variable. The function $X(S)$ is defined as follows:

$$X(S) = \begin{cases} \dfrac{S - S_U}{S_H - S_U} & \text{linear relationship} \\ \dfrac{1/S - 1/S_U}{1/S_H - 1/S_U} & \text{Arrhenius model} \\ \dfrac{\ln S - \ln S_U}{\ln S_H - \ln S_U} & \text{inverse-power law,} \end{cases}$$

where S_U is the use stress level, S_H is a given highest stress level, and $S_U \leq S \leq S_H$. For simplicity, we denote $x = X(S)$ as the standardized stress level hereafter, where $0 \leq x \leq 1$, and $x_U = 0$ and $x_H = 1$ are the standardized use and highest stress levels, respectively. Thus, the relationship between the location parameter and the standardized stress level is re-expressed as $\mu_{\boldsymbol{\beta}}(x) = \beta_0 + \beta_1 x$, where $\beta_0 = a + bX(S_0)$, $\beta_1 = b[X(S_U) - X(S_0)]$, and

$\boldsymbol{\beta} = (\beta_0, \beta_1)'$ is the unknown parameter of the linkage function. In the ALT modeling, the model parameters are $\boldsymbol{\theta} = (\boldsymbol{\beta}', \sigma)'$.

To provide a suitable and meaningful planning values, one way is to re-parameterize the model parameters. It is much easier for experts or engineers to specify the planning values based on reasonable re-parameterization. Similar concept shown in Zhang and Meeker (2006) and Weaver et al. (2016), they used $(\xi_{p,U}, \sigma, -\beta_1)'$ to be the re-parameterization for planning ALTs. This study defines $\boldsymbol{\gamma} = (p_U, p_H, \sigma)'$ as the proportions of failing at the censoring time t_c under the use level and the highest stress level and the scale parameter (Meeker, 1984; Pascual and Montepiedra, 2002), where

$$p_U = \Phi\left[\frac{\log t_c - \beta_0}{\sigma}\right] \text{ and } p_H = \Phi\left[\frac{\log t_c - \beta_0 - \beta_1}{\sigma}\right]. \tag{1}$$

In general, $0 \leq p_U \leq p_H \leq 1$, which is much easier and to specify the planning values for a locally optimum design. Taking the dataset of device A (Meeker and Escobar, 1998) as the example, the proportion of failing at the censoring time (5000 hours) under the use level 10°C is 0/30, and the proportion of failing at the censoring time under 80°C is 14/15. For the planning stage, it is more reasonable to give planning vales of $\boldsymbol{\gamma}$.

2.2 Design criterion

For the purpose of the ALT, a lower quantile of the lifetime distribution is of interest. Under the log-location-scale family, the logarithm of the q quantile at the the standardized stress level $x_U = 0$, denoted by $\xi_{q,U}$, is

$$\log(\xi_{q,U}) = \mu_{\boldsymbol{\gamma}}(x_U) + z_q\sigma, \tag{2}$$

where

$$\mu_{\boldsymbol{\gamma}}(x_U) = \log(t_c) - \Phi^{-1}(p_U)\sigma + [\Phi^{-1}(p_U) - \Phi^{-1}(p_H)]\sigma x_U,$$

$z_q = \Phi^{-1}(q)$ is the q quantile of the standard distribution, and $\Phi^{-1}(q)$ is the inverse function of the cdf of the standard distribution. For example, $\Phi^{-1}(q) = \log(-\log(1-q))$ if the distribution is a Weibull distribution, and $\Phi^{-1}(q) = \Phi^{-1}_{norm}(q)$ if the distribution is a log-normal distribution and $\Phi_{norm}(z)$ is the cdf of the normal distribution.

By maximizing the log-likelihood function, the ML estimates of $\boldsymbol{\gamma}$ are obtained, denoted by $\widehat{\boldsymbol{\gamma}} = (\widehat{p}_U, \widehat{p}_H, \widehat{\sigma})'$. In addition, the estimated logarithm of the q quantile at stress level x_U is also obtained by substituting the values $\widehat{\boldsymbol{\gamma}}$ into Equation (2). That is, $\log(\widehat{\xi}_{q,U}) = \mu_{\hat{\boldsymbol{\gamma}}}(x_U) + z_q\hat{\sigma}$. By the delta method and the derivative of inverse function, the asymptotic variance of $\log(\widehat{\xi}_{p,U})$ under the stress level x_U is

$$\text{AVar}\left[\log\left(\widehat{\xi}_{p,U}\right)\right] = \boldsymbol{c}'\mathbf{I}^{-1}\left(\boldsymbol{\gamma}|\boldsymbol{x}_n\right)\boldsymbol{c}, \tag{3}$$

where $\mathbf{I}\left(\boldsymbol{\gamma}|\boldsymbol{x}_n\right)$ is the Fisher information matrix based on the given stress level $\boldsymbol{x}_n$, and

$$\boldsymbol{c} = \left[\frac{\partial\log(\widehat{\xi}_{p,U})}{\partial p_U}, \frac{\partial\log(\widehat{\xi}_{p,U})}{\partial p_H}, \frac{\partial\log(\widehat{\xi}_{p,U})}{\partial\sigma}\right]' = \left[\frac{-\sigma}{\phi\left[\Phi^{-1}(p_U)\right]}, 0, z_p - \Phi^{-1}(p_U)\right]'.$$

Let (x_i, t_i, δ_i) denote the ith observation with its standardized stress level x_i and the corresponding failure (or censoring) time and status, t_i and δ_i, and the Fisher information is expressed as $\mathbf{I}\left(\boldsymbol{\gamma}|\boldsymbol{x}_n\right) = \sum_{i=1}^{n}\mathbf{I}_i\left(\boldsymbol{\gamma}|x_i\right)$, where

$$\mathbf{I}_i\left(\boldsymbol{\gamma}|x_i\right) = E\left[-\frac{\partial^2 l\left(\boldsymbol{\gamma}|x_i, t_i, \delta_i\right)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'}\right],$$

and $l\left(\boldsymbol{\gamma}|x_i, t_i, \delta_i\right)$ is the log-likelihood function of the ith observation.

From the definition of locally optimum design, it is always needed to specify the values of model parameters as the planning values $\boldsymbol{\theta}_0$. Based on the planning values, the optimum design is determined by the optimization of a design criterion. Define

$$\boldsymbol{\eta} = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_k \\ \pi_1 & \pi_2 & \cdots & \pi_k \end{bmatrix}, \tag{4}$$

be an approximate design with k stress levels, where ξ_l and π_l are the lth stress level and its corresponding proportion of units, $0 \le \xi_l \le 1$, $0 \le \pi_l \le 1$, $\forall l = 1, \ldots, k$, and $\sum_{l=1}^{k}\pi_l = 1$. For determining a good design to obtain a precise estimate, lots of studies proposed their strategies based on the locally c-optimal design, which is determined by minimizing the asymptotic variance in Equation (3) under possible planning values of parameters $\boldsymbol{\gamma}_0$. Then, the objective function of the locally c-optimal design given $\boldsymbol{\gamma}_0$ by Equation (3) is $V_{\boldsymbol{\gamma}_0}(\boldsymbol{\eta}) = \boldsymbol{c}'\mathbf{I}^{-1}\left(\boldsymbol{\gamma}_0|\boldsymbol{\eta}\right)\boldsymbol{c}$, where the approximate design $\boldsymbol{\eta}$ is substituted into the Fisher information matrix as follows:

$$\mathbf{I}\left(\boldsymbol{\gamma}|\boldsymbol{\eta}\right) = n\sum_{l=1}^{k}\pi_l\mathbf{I}_l\left(\boldsymbol{\gamma}|\xi_l\right).$$

Among all possible design space $\boldsymbol{\Xi}$, the locally c-optimal design, $\boldsymbol{\eta}_{lc}^*$, is determined by minimizing the asymptotic variance, which is

$$\boldsymbol{\eta}_{lc}^* = \arg\min_{\boldsymbol{\eta}\in\boldsymbol{\Xi}} V_{\boldsymbol{\gamma}_0}(\boldsymbol{\eta}) = \arg\min_{\boldsymbol{\eta}\in\boldsymbol{\Xi}} \boldsymbol{c}'\mathbf{I}^{-1}\left(\boldsymbol{\gamma}_0|\boldsymbol{\eta}\right)\boldsymbol{c}.$$

In practice, the true values of parameters or the planning values are not exactly known at the early stage of experiments. Instead, it is much easier to provide a suitable ranges of unknown parameters based on experience. Hence, this study aims to determine a robust design over parameter space $\boldsymbol{\Gamma}$ by the framework of minimax design for the early stage of experiments. Given a design, one way consider the inefficiency of the design over the plausible region and then find a standardized minimax optimal design that minimizes the maximal inefficiency. For our problem, we define the objective function of the standardized c-optimal design, which is the c-efficiency or the asymptotic sample ratio (ASR) in Pascual and Montepiedra (2002). The ASR of a design $\boldsymbol{\eta}$ under given parameter $\boldsymbol{\gamma}$ is defined as

$$\mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta}) = \frac{V_{\boldsymbol{\gamma}}(\boldsymbol{\eta})}{V_{\boldsymbol{\gamma}}(\boldsymbol{\eta}^*_{lc})}. \tag{5}$$

Then, the standardized minimax design (smMD) in this study is determined by

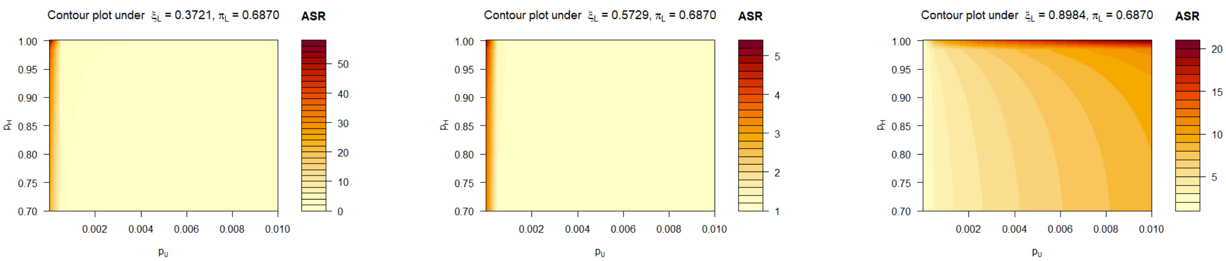
$$\boldsymbol{\eta}^*_{mM} = \arg\min_{\boldsymbol{\eta}\in\boldsymbol{\Xi}} \max_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta}) = \arg\min_{\boldsymbol{\eta}\in\boldsymbol{\Xi}} \max_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \frac{V_{\boldsymbol{\gamma}}(\boldsymbol{\eta})}{V_{\boldsymbol{\gamma}}(\boldsymbol{\eta}^*_{lc})}. \tag{6}$$

In Equation (6), we need to pre-specify the settings of parameter space Γ and design space $\boldsymbol{\Xi}$. From Pascual and Montepiedra (2003), the values of $\mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta})$ given a specific design $\boldsymbol{\eta}$ doesn't depend on parameter σ, which depends on the parameters $\boldsymbol{\rho} = (p_U, p_H)'$. Hence, at the design stage, we set the value of σ is given. For the parameter space of $\boldsymbol{\rho}$, we set $(p_U, p_H) \in [p_U^L, p_U^U] \times [p_H^L, p_H^U]$, where p^L and p^U are the lower and upper bounds of p. Thus, the parameter space of the re-parameterization (p_U, p_H) is easier to be pre-specified in practice. For the design space, the set contains all possible values satisfy the condition in Equation (4). It is noted that the complexity of the optimization arises because there are two layers of optimization, including the parameter space and the design space. In the following, we introduce the particle swarm optimization (PSO) to handle the difficulty.

2.3 The PSO-based algorithm for minimax design

Particle swarm optimization (PSO) proposed by Kennedy and Eberhart (1995) is a commonly used metaheuristic optimization method for complex and high dimension optimization problems. Basically PSO allows to search better solutions among all possible candidate solutions (particles), which are iteratively updated the j-the particle's position (x_j^t) and velocity (v_j^t) at the tth iteration by simple formula (7-8), i.e.,

$$v_j^{t+1} = \omega^t v_j^t + c_1 R_1 \otimes \left(x_{j,p}^t - x_j^t\right) + c_2 R_2 \otimes \left(x_G^t - x_j^t\right), \tag{7}$$



(a) Design 1 (b) Design 2 (c) Design 3

Figure 1: The heatmap of ASR over parameter space for the different designs

$$x_j^{t+1} = x_j^t + v_j^{t+1}, j = 1, \ldots, P,\ t = 1, \ldots, t_{max}, \tag{8}$$

where $x_{j,p}^t$ and x_G^t are the best solutions for the jth particle and all particle, respectively. In addition, the notation P is the number of particles, t_{max} is the maximum number of iteration, c_1 and c_2 are acceleration constants, and R_1 and R_2 are random selected from $[0, 1]$. Here c_1 and c_2 are two tuning parameters and usually are fixed as 2. But the parameters P and t_{max} would be problem dependent. To deal with higher dimension optimization problems, we would suggest choosing the larger values of P and t_{max}.

Chen et al. (2017) modified this nested PSO for the standardized maximin designs. Instead of the nested PSO methods, Chen et al. (2020) introduced an efficient hybrid method for minimax problems by combining PSO with gradient-based methods. Following Chen et al. (2020), the PSO is used to search the best design over the design space, and for each particle, a gradient based method is adopted to work on the inner-loop optimization with respect to parameters. Figure 1 illustrates the trends of $\mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta}_0)$ over the parameter space with $10^{-6} \le p_U \le 10^{-2}$ and $0.7 \le p_H \le 1$ for three given designs, $\boldsymbol{\eta}_0$. It is clear that no matter what design is, the maximum value of $\mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta}_0)$ is located at one of the corners of the parameter region, and $\mathrm{ASR}_{\boldsymbol{\gamma}}(\boldsymbol{\eta}_0)$ is quite smooth over the parameter region. In this study, a gradient-based method implemented by the function "optim()" in R, is used to solve the inner-loop optimization problem for the parameter vector $\boldsymbol{\gamma}$.

3 Numerical Minimax Designs

In this study, the design we consider is the probability measure with three levels (support points),

$$\boldsymbol{\eta}_3 = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix},$$

where $\pi_i \geq 0$ and $\sum_i \pi_i = 1$. Thus, there are 5 decision variables, including ξ_1, ξ_2, ξ_3, π_1 and π_2, for the outer loop of the minimax problem shown in Equation (6). Here the hybrid algorithm, combined the PSO with the gradient based method, is used as the design generator to search the best minimax design. We further use the equivalence theorem (Huang and Lin, 2006) to verify if the design generated by the hybrid algorithm is the standardized minimax c-optimal design, $\boldsymbol{\eta}^*_{mM}$, or with a higher efficiency compared to the best design. For the minimax design problem, we need to pre-specify the parameter spaces as follows.

(a) The ranges of p_U are set to be $[10^{-6}, 10^{-2}]$, and $[10^{-6}, 10^{-3}]$, and $[10^{-6}, 10^{-4}]$.

(b) The ranges of p_H are set to be $[0.7, 1]$.

The other setups are the same as these shown in Example 20.4 of Meeker and Escobar (1998). For example, the censoring time is 183 days, the use and highest stress levels are 50°C and 120°C, the total sample size is 300, and the value of σ is 0.6. These settings could be found in Pascual and Montepiedra (2003) and Zhang and Meeker (2006). The settings of PSO are that $c_1 = c_2 = 2$, and the numbers of particle and iteration to be $P = 32$ and $t_{max} = 128$, respectively. In the following, the Weibull distribution with different ranges of (p_U, p_H) are used to demonstrate the performance of the proposed strategy. The results of the best designs, smMD, found by the hybrid algorithm over given $p_H \in [0.7, 1]$ with different ranges of p_U are shown in Table 1. For example, supposed $p_U \in [10^{-6}, 10^{-2}]$, the corresponding smMD allocates the sample units to the stress levels $(0.5476, 0.8168, 1)$ with proportions of units $(0.4007, 0.3166, 0.2827)$, and when $p_U \in [10^{-6}, 10^{-3}]$, the corresponding smMD allocates the sample units to the stress levels $(0.6627, 0.8277, 1)$ with proportions of units $(0.4445, 0.2397, 0.3158)$. These are three-level designs, and the lowest stress level (ξ_1) shifts from 0.55 to 0.66 when the upper bound of p_U is from 10^{-2} to 10^{-3}. It is reasonable because the smaller failing proportion under the use condition at the censoring time leads an higher lowest stress level, and is straightforward to conduct an experiment under higher stress levels to obtain more failure information if the failing proportion at the use condition is smaller. When $p_U \in [10^{-6}, 10^{-4}]$, the smMD has the stress levels $(0.7545, 1)$ and its allocation $(0.6379, 0.3621)$, which is a two-level design. It indicates a two-level design is good enough for a shorter range of parameters which represents the stronger failure information considered at the design stage.

In the following, we demonstrate the equivalence theorem to show the results in Table 1 is an standardized minimax design. The verification of smMDs is shown in Figure 2. From Figure 2, the stress levels of each smMD with the corresponding allocation proportion are verified under the three ranges of p_U. It is found that the directional derivative of the equivalence theorem has the maximum value of 1 achieved at the corresponding stress levels

Table 1: The demonstrated standardized minimax design based on Weibull distribution

p_U	p_H	(ξ_1, ξ_2, ξ_3)	(π_1, π_2, π_3)
$[10^{-6}, 10^{-2}]$	$[0.7, 1]$	$(0.5476, 0.8168, 1)$	$(0.4007, 0.3166, 0.2827)$
$[10^{-6}, 10^{-3}]$	$[0.7, 1]$	$(0.6627, 0.8277, 1)$	$(0.4445, 0.2397, 0.3158)$
$[10^{-6}, 10^{-4}]$	$[0.7, 1]$	$(0.7545, \quad - \quad, 1)$	$(0.6379, \quad - \quad, 0.3621)$

in Table 1. It shows the designs in Table 1 determined by the PSO is the standardized minimax c-optimal design. It also shows the proposed procedure works well.

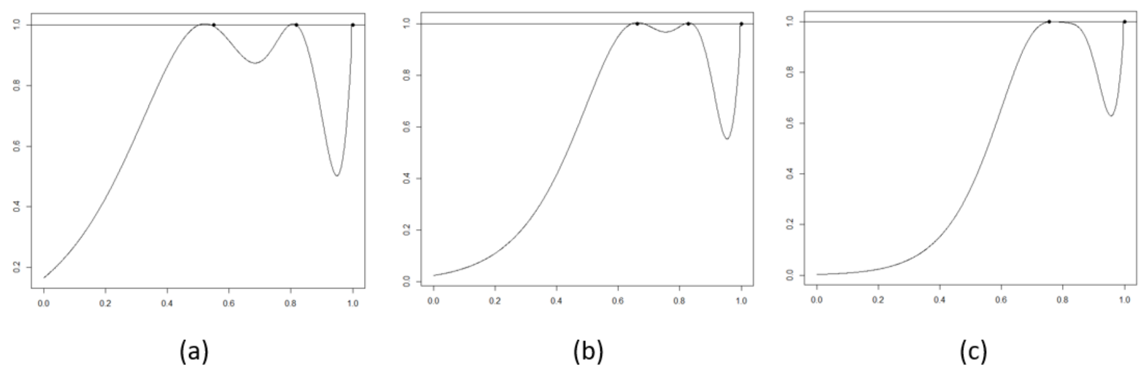
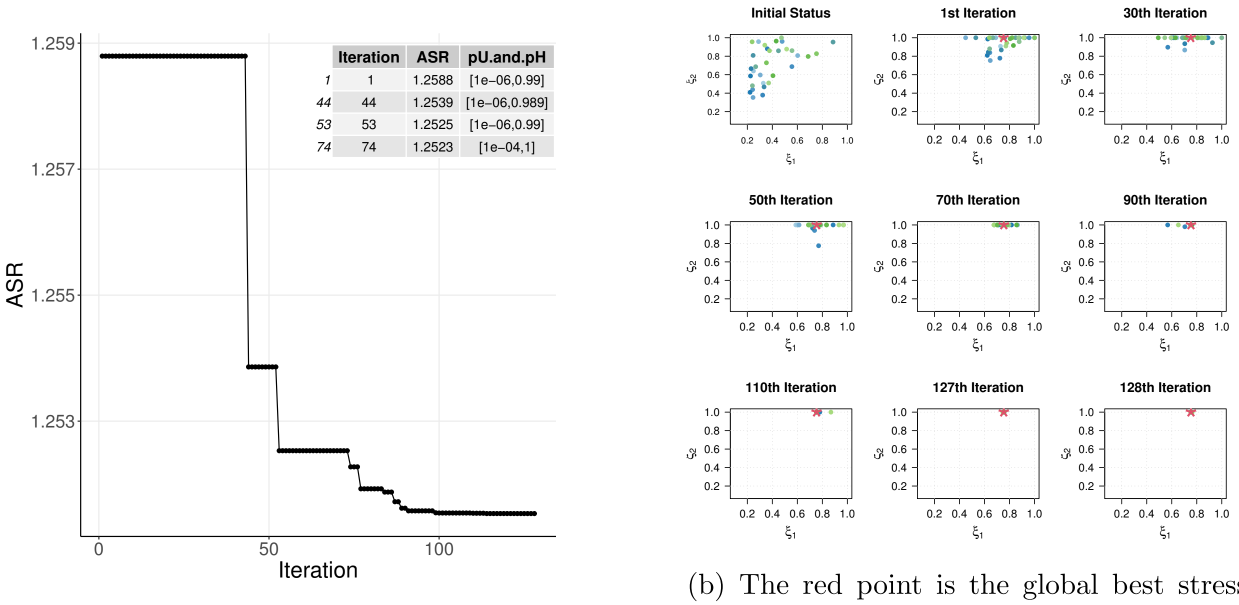


Figure 2: Use the equivalence theorem to verified smMDs under (a) $p_U \in [10^{-6}, 10^{-2}]$, (b) $p_U \in [10^{-6}, 10^{-3}]$, and (c) $p_U \in [10^{-6}, 10^{-4}]$.

Moreover, we also provide the detailed information of PSO for searching a smMD at each iteration. For the two-level design (under $p_U \in [10^{-6}, 10^{-4}]$), Figure 3 (a) shows the objective function ASR in Equation (5) at each iteration. It shows the rate of convergence to the best value is quick if the range of parameter is small. Figure 3 (b) shows the positions of 32 particles of stress levels move at 9 iterations. It is found that the particles are gathering to the global best stress levels (red point in the figure) as the iteration is getting larger.

3.1 Comparison with the existing methods

In this subsection, we compare the proposed procedure to the existing strategies, including the locally optimum design ($\boldsymbol{\eta}^*_{lc}$), Bayesian planning ($\boldsymbol{\eta}^*_B$) (Zhang and Meeker, 2006), and model-robust planning ($\boldsymbol{\eta}^*_{mr}$) (Pascual and Montepiedra, 2003). Since the criteria of these methods are different, the fair comparison for the precise prediction is demonstrated based on the simulation. Typically, we generate experimental data of an ALT according to each test plan for 4000 times given a specific true model with its corresponding parameters ($\boldsymbol{\gamma}_0$).



(a) The values of ASR with iterations.

(b) The red point is the global best stress levels, and the other colors are positions of each particle of stress levels in the PSO.

Figure 3: The details of PSO at each iteration for the smMD based on Weibull distribution with $p_U \in [10^{-6}, 10^{-4}]$ and $p_H \in [0.7, 1]$.

For the ith simulated experimental dataset based on a specific $\boldsymbol{\eta}$, we can obtain the prediction of $\log t_{0.1}$ by the ML estimates $\hat{\boldsymbol{\gamma}}_{\boldsymbol{\eta},i}$, and compare to the true value of $\log t_{0.1}$. Let $\log t_{0.1,\hat{\boldsymbol{\gamma}}_{\boldsymbol{\eta},i}}$ and $\log t_{0.1,\boldsymbol{\gamma}_0}$ be the prediction and the true value of $\log t_{0.1}$, respectively. Then, the square root of the mean square error (RMSE) of the prediction is the measure of design performance, defined as

$$\mathrm{RMSE}(\boldsymbol{\eta}) = \sqrt{\frac{\sum_{i=1}^{4000} \left(\log t_{0.1,\hat{\boldsymbol{\gamma}}_{\boldsymbol{\eta},i}} - \log t_{0.1,\boldsymbol{\gamma}_0}\right)^2}{4000}}.$$

Let the failure time distribution be the Weibull distribution, and the true values of parameters are $\boldsymbol{\gamma}_0 = (0.001, 0.9, 0.6)'$. For the candidate planning, we use the Bayesian planning and the locally optimum design shown in the table 2 and table 8 of Zhang and Meeker (2006), respectively. For the model-robust planning, we use the planning at $(p_U, p_H) = (0.001, 0.9)$ in the table 2 of Pascual and Montepiedra (2003). For the smMD with $p_H \in [0.7, 1]$, we consider the designs at $p_U \in [10^{-6}, 10^{-2}]$ and $p_U \in [10^{-6}, 10^{-3}]$ in Table 1, denoted by $\boldsymbol{\eta}^{*a}_{mM}$ and $\boldsymbol{\eta}^{*b}_{mM}$. Then, the candidate planning strategies are shown in Table 2. Under the settings $\boldsymbol{\gamma}_0 = (0.001, 0.9, 0.6)'$, only the proposed planning is a three-level design, the other strategies are two-level designs. The lowest stress levels for $\boldsymbol{\eta}^*_{mM}$ and $\boldsymbol{\eta}^*_{mr}$ are close, and the lowest stress levels for $\boldsymbol{\eta}^*_{lc}$ and $\boldsymbol{\eta}^*_{B}$ are close, but with different proportions of units.

For each simulation trial based on the specific planning , we simulate the failure observations and the censoring time 183 days with the total sample size $n = 50$, 150, and 300. We summarize the values of RMSE under different total sample size in Table 2. For the values of RMSE, the proposed smMDs are robust compared to the locally optimum design $\boldsymbol{\eta}^*_{lc}$, and performs better than $\boldsymbol{\eta}^*_B$ and $\boldsymbol{\eta}^*_{mr}$. Under the cases of $n = 50$, the planning $\boldsymbol{\eta}^{*b}_{mM}$ has the smallest value of RMSE, the planning $\boldsymbol{\eta}^{*a}_{mM}$ and the planning $\boldsymbol{\eta}^*_{lc}$ are similar, and the planning $\boldsymbol{\eta}^*_{mr}$ has the largest value of RMSE. Under the case of $n = 150$, the planning $\boldsymbol{\eta}^*_{lc}$ has the smallest value of RMSE because the planning is suggested according to the true values $\boldsymbol{\gamma}_0$. However, the planning $\boldsymbol{\eta}^{*b}_{mM}$ still works well with the efficiency $0.5519/0.5820 \times 100\% = 94.8\%$. Under the case of $n = 300$, the planning $\boldsymbol{\eta}^*_{lc}$ is still the best, and $\boldsymbol{\eta}^{*b}_{mM}$ and $\boldsymbol{\eta}^*_B$ are with the efficiency 90% with respect to the planning $\boldsymbol{\eta}^*_{lc}$. The planning $\boldsymbol{\eta}^*_{mr}$ doesn't perform well because the strategy considers the robustness between the Weibull and log-normal distributions. From the results, it is found that the locally optimum design at $\boldsymbol{\gamma}_0$ has more accurate prediction compared to other designs when the sample size is getting larger. However, the true values of parameters are unknown in real application, and the planning $\boldsymbol{\eta}^{mM}$ and the planning $\boldsymbol{\eta}^B$ are constructed by consideration of uncertainty of model parameters. If the ranges of parameters are wider or far from the true parameters, then the performance on RMSE of the prediction is worse. For example, the planning $\boldsymbol{\eta}^{*a}_{mM}$ is worse than the planning $\boldsymbol{\eta}^{*b}_{mM}$ since $p_U \in [10^{-6}, 10^{-2}]$ is wider than $p_U \in [10^{-6}, 10^{-3}]$.

Overall, the proposed smMD performs well especially for the smaller sample size. Compared to the Bayesian design, we don't need to specify the prior distributions with hyperparameters when adopting the procedure of the smMD. Hence, the proposed smMD is much easier to be determined by specifying the failure distribution and the corresponding ranges of parameters.

4 Concluding Remarks and Future Work

This study adopts a minimax criterion to obtain a more robust design for conducting an ALT. Particularly, the minimax design is determined once we specify the range of sample failure probability under a specific failure model. To find the minimax design efficiently, this study adopts the particle swarm optimization (PSO) technique. Compared to the locally optimal design via simulation study, the minimax design is more robust and more practical. The log-normal distribution and other distributions could be extended as a future work.

Parts of this report is extracted from 李馨茹 (2020) and 尹思懿 (2022), master's thesis of the Institute of Statistics, National Cheng Kung University.

Table 2: The values of RMSE of $\log \hat{t}_{0.1}$ based on different strategies under the Weibull distribution with sample sizes $n = 50$, $n = 150$, and $n = 300$ based on experimental settings of Zhang and Meeker (2006).

Planning ($\boldsymbol{\eta}$)	Stress levels	Allocation	RMSE($\boldsymbol{\eta}$) $n = 50$	$n = 150$	$n = 300$
$\boldsymbol{\eta}^{*a}_{mM}$	$(0.5476, 0.8168, 1)$	$(0.4007, 0.3166, 0.2827)$	1.1651	0.6183	0.4329
$\boldsymbol{\eta}^{*b}_{mM}$	$(0.6627, 0.8277, 1)$	$(0.4445, 0.2397, 0.3158)$	1.0832	0.5820	0.4119
$\boldsymbol{\eta}^{*}_{lc}$	$(0.682, 1)$	$(0.706, 0.294)$	1.1527	0.5519	0.3741
$\boldsymbol{\eta}^{*}_{B}$	$(0.671, 1)$	$(0.501, 0.499)$	1.5606	0.6065	0.4162
$\boldsymbol{\eta}^{*}_{mr}$	$(0.5371, 1)$	$(0.7134, 0.2866)$	2.8188	0.7045	0.4386

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