Optimization Theory HW2

RE6124019 吳明軒

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1 Prove the convexity of the following functions

- **$-\log(x)$ in $(0, \infty)$:**

  The domain (0,∞) is a convex set. For any x,y>0 and $\theta \in$[0,1], we have:

  $f(\theta x + (1-\theta)y) = -\log(\theta x + (1-\theta)y) \quad \leq -\theta\log(x) - (1-\theta)\log(y) = \theta f(x) + (1-\theta)f(y)$

- **$\exp(x)$ in $\mathbb{R}$:**

  The domain R is a convex set. For any x,y∈$\mathbb{R}$ and $\theta \in$[0,1], we have:

  $f(\theta x + (1-\theta)y) = \exp(\theta x + (1-\theta)y) \quad \leq \theta\exp(x) + (1-\theta)\exp(y) = \theta f(x) + (1-\theta)f(y)$

- **x^4 in $\mathbb{R}$:**

  The domain R is a convex set. For any x,y∈$\mathbb{R}$ and $\theta \in$[0,1], we have

  $f(\theta x + (1-\theta)y) = (\theta x + (1-\theta)y)^4 \quad \leq \theta x^4 + (1-\theta)y^4 = \theta f(x) + (1-\theta)f(y)$

- **$\|x\|_2^2$ in $\mathbb{R}^n$:**

  The domain R^n is a convex set. For any x,y∈$\mathbb{R}^n$ and $\theta \in$[0,1], we have:

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \|\theta x + (1-\theta)y\|_2^2 \\ &\leq \theta^2\|x\|_2^2 + (1-\theta)^2\|y\|_2^2 + 2\theta(1-\theta)\langle x, y\rangle \\ &\leq \theta\|x\|_2^2 + (1-\theta)\|y\|_2^2 \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

2 Prove that the set $C = \{X \mid X \succeq A_i, i = 1, \ldots, m\}$ (for symmetric $A_i \in \mathbb{S}^n$) is also a convex set.

Let $C = \{X \mid X \geq A_i, i = 1, \ldots, m\}$ where A_i are symmetric.
We need to show that C is convex.
Let $X, Y \in C$ and $\theta \in [0, 1]$. Then $X \geq A_i$ and $Y \geq A_i$ for all i.
Therefore, $\theta X + (1-\theta)Y \geq \theta A_i + (1-\theta)A_i = A_i$ for all i
(since A_i is symmetric).
Thus, $\theta X + (1-\theta)Y$ is in C. Therefore, C is convex.

3 Prove the equality in (5.7). [Hint: you can assume that $S_2 = \mathbb{R}^m$.]

Let $g(z) = f_2(z) + \frac{c}{2}\|z - (Ax^q - d^q)\|_2^2$.
The optimality condition is:

$$\nabla g(z^*) = \nabla f_2(z^*) + \frac{1}{c}(z^* - (Ax^q - d^q)) = 0$$

Therefore, $z^* = \text{prox}_{\frac{1}{c}} f_2(Ax^q - d^q)$.
Comparing with the ADMM update $z^{q+1} = \arg\min_z g(z)$, we get:

$$z^{q+1} = \text{prox}_{\frac{1}{c}} f_2(Ax^q - d^q)$$

Which is the desired equality.

4 Please implement the above CVX command. Please demo your code based on the FIR filter parameterized by $\{h_i\}_{i=-n}$. [Hint: you can freely specify a suitable size (n, P), frequency samples $\{\omega_1, \ldots, \omega_P\} \in [0, \pi]$, as well as the desired frequency response $H_{\mathbf{des}}(\omega_p)$ with "symmetricity" (i.e., $h_i = h_{-i}$).]

```
% number of FIR coefficients
n = 10;
P = 100;
w = linspace(0,pi,P)'; % omega

% Gaussian filter with linear phase
var1 = 0.1;
Hdes = 1/(sqrt(2*pi*var1))*exp(-(w-pi/2).^2/(2*var1));
Hdes = Hdes.*exp(-j*n/2*w);

% optimal Chebyshev filter formulation
cvx_begin
  variables h(n+1) t;
  minimize(t)
  subject to
    for p = 1:P
        total = 0;
        for i = 2:n+1
            total = total+h(i)*cos(w(p)*(i-1));
```

```
        end
        abs(Hdes(p)-h(1)-2*total) <= t;
    end
cvx_end

disp(['Problem is ' cvx_status])
if ~strfind(cvx_status,'Solved')
  h = [];
end

% plot the FIR impulse reponse
figure(1)
stem([0:n],h)
xlabel('n')
ylabel('h(n)')

% plot the frequency response
H = [exp(-j*kron(w,[0:n]))]*h;
figure(2)
% magnitude
subplot(2,1,1);
plot(w,20*log10(abs(H)),w,20*log10(abs(Hdes)),'--')
xlabel('w')
ylabel('mag H in dB')
axis([0 pi -30 10])
legend('optimized','desired','Location','SouthEast')
% phase
subplot(2,1,2)
plot(w,angle(H))
axis([0,pi,-pi,pi])
xlabel('w'), ylabel('phase H(w)')
```

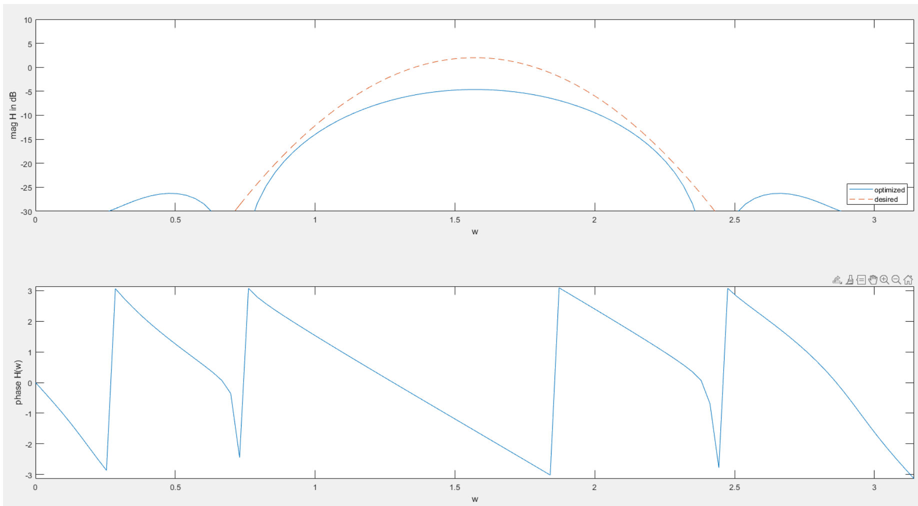

Figure 1: plot the frequency response

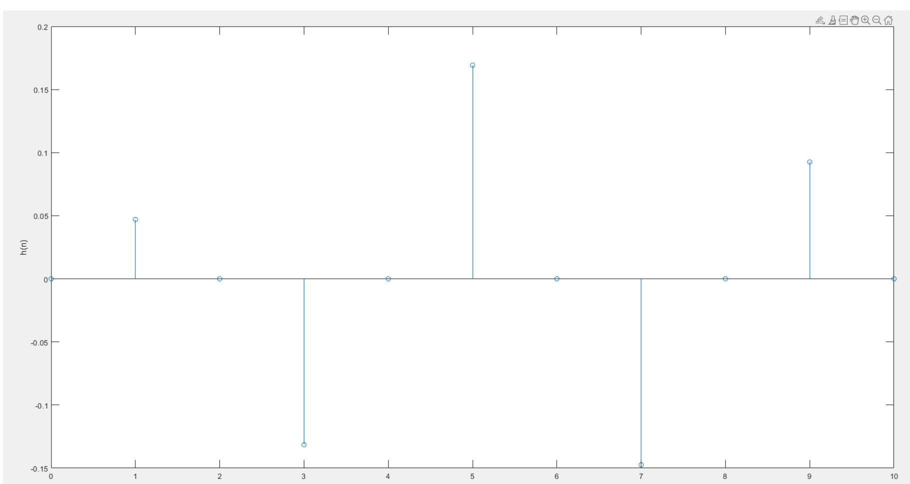

Figure 2: plot the FIR impulse reponse