OPTIMIZATION THEORY

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1 Property 5:

The set of $n \times n$ symmetric matrices $\mathbf{S}^n$, positive semidefinite (PSD) cone $\mathbf{S}^n_+$, and positive definite (PD) cone $\mathbf{S}^n_{++}$ are all convex sets.
Proof:
Let $X, Y \in S^n$ and $\theta \in [0, 1]$.
According to the properties of symmetric matrices, we know:

1. $X^T = X$, $Y^T = Y$
2. $(X + Y)^T = X^T + Y^T = X + Y$

Then,

$$[\theta X + (1 - \theta)Y]^T = \theta X^T + (1 - \theta)Y^T = \theta X + (1 - \theta)Y$$

Because θ is a constant.
Therefore $\theta X + (1 - \theta)Y \in S^n$, S^n is convex.

2 Property 10:

The set of n $\times$ n symmetric matrices $\mathbf{S}^n$, the set of PSD matrices $\mathbf{S}^n_+$ and the set of PD matrices $\mathbf{S}^n_{++} \cup \{0\}$ are all cones.
Proof:

2.1 Symmetric Matrices $\mathbf{S}^n$:

- Closure under scalar multiplication:
  Let $X \in \mathbf{S}^n$ and $\alpha \geq 0$ be a scalar. Then, αX is also symmetric because $(\alpha X)^T = \alpha X^T = \alpha X$.

- Closure under addition:
  Let $X, Y \in \mathbf{S}^n$. Then, $X + Y$ is symmetric because $(X + Y)^T = X^T + Y^T = X + Y$.

  Therefore, $\mathbf{S}^n$ is a cone.

2.2 Positive Semidefinite Matrices $\mathbf{S}_+^n$:

- Closure under scalar multiplication:
  Let $X \in \mathbf{S}_+^n$ and $\alpha \geq 0$ be a scalar. Then, αX is also positive semidefinite because for any vector v, $v^T(\alpha X)v = \alpha(v^T X v) \geq 0$ since X is positive semidefinite.

- Closure under addition:
  Let $X, Y \in \mathbf{S}_+^n$. Then, $X + Y$ is positive semidefinite because for any vector v, $v^T(X + Y)v = v^T X v + v^T Y v \geq 0$ since both X and Y are positive semidefinite.

  Therefore, $\mathbf{S}_+^n$ is a cone.

2.3

Positive Definite Matrices $\mathbf{S}_{++}^n \cup \{0\}$:

- Closure under scalar multiplication:
  Let $X \in \mathbf{S}_{++}^n \cup \{0\}$ and $\alpha \geq 0$ be a scalar. Then, αX is also positive definite because for any vector $v \neq 0$, $v^T(\alpha X)v = \alpha(v^T X v) > 0$ since X is positive definite.

- Closure under addition:
  Let $X, Y \in \mathbf{S}_{++}^n \cup \{0\}$. Then, $X + Y$ is positive definite because for any vector $v \neq 0$, $v^T(X + Y)v = v^T X v + v^T Y v > 0$ since both X and Y are positive definite.

  Therefore, $\mathbf{S}_{++}^n \cup \{0\}$ is a cone.

In conclusion, the sets $\mathbf{S}^n$, $\mathbf{S}_+^n$, and $\mathbf{S}_{++}^n \cup \{0\}$ are all cones.

3 Interesting Question:

Since convex hull, affine hull, and conic hull of a finite set $S = \{x_1, \ldots, x_n\}$ can be written as

$$\text{conv } S = \{\theta_1 x_1 + \ldots + \theta_n x_n \,|\, \theta_1 + \ldots + \theta_n = 1, \theta_i \geq 0\},$$

$$\text{aff } S = \{\theta_1 x_1 + \ldots + \theta_n x_n \,|\, \theta_1 + \ldots + \theta_n = 1\},$$

$$\text{conic } S = \{\theta_1 x_1 + \ldots + \theta_n x_n \,|\, \theta_i \geq 0\},$$

one may conclude that conv $S =$ aff $S \cap$ conic S. Is this conclusion correct? If so, please prove it; otherwise, give a counterexample.

Proof by Contradiction:

We believe that this conclusion is incorrect, i.e., there exists a counterexample where conv $S \neq$ aff $S \cap$ conic S.

Consider a simple counterexample with $S = \{x_1, x_2\}$, where $x_1 = (1, 0)$ and $x_2 = (0, 1)$. We have:

$$
\begin{aligned}
\text{conv } S &= \{\theta_1 x_1 + \theta_2 x_2 \,|\, \theta_1 + \theta_2 = 1,\ \theta_1, \theta_2 \geq 0\}, \\
\text{aff } S &= \{\theta_1 x_1 + \theta_2 x_2 \,|\, \theta_1 + \theta_2 = 1\}, \\
\text{conic } S &= \{\theta_1 x_1 + \theta_2 x_2 \,|\, \theta_1, \theta_2 \geq 0\}.
\end{aligned}
$$

Calculations yield:

$$
\begin{aligned}
\text{conv } S &= \{(t, 1-t) \,|\, 0 \leq t \leq 1\}, \\
\text{aff } S &= \{(t, 1-t) \,|\, 0 \leq t \leq 1\}, \\
\text{conic } S &= \{(t, 1-t) \,|\, 0 \leq t \leq 1\}.
\end{aligned}
$$

Thus, conv $S =$ aff $S =$ conic S.

However, conv $S \neq$ aff $S \cap$conic S because their definitions do not imply equality. Therefore, this serves as a counterexample, proving the original conclusion incorrect.