可靠度資料分析
Reliability Data Analysis

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Lecture 5 – System Reliability

System Reliability

- A product or service is usually made out of smaller components or subsystems.

- Understanding the **total system reliability** for the product or service is critical for large and complex products or processes.

- The system reliability is a combination of all the individual reliabilities associated with the components or subsystems within the product or service.

Serial Configuration

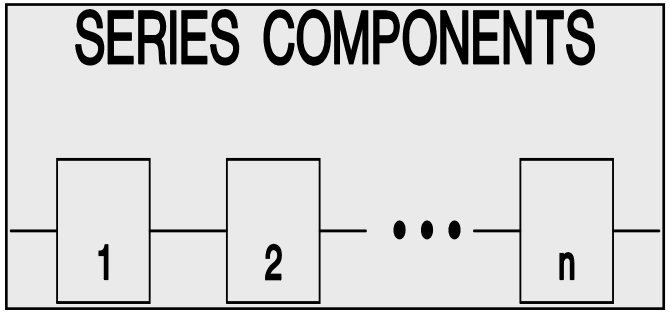
E_1 = the event, component 1 does not fail, and
E_2 = the event, component 2 does not fail

Then $P(E_1) = R_1$ and $P(E_2) = R_2$
where
R_1 = the reliability of component 1, and
R_2 = the reliability of component 2.

Therefore assuming independence:
$R_s = P(E_1 \cap E_2) = P(E_1)\, P(E_2) = R_1\, R_2$

And

Reliability
Block Diagram



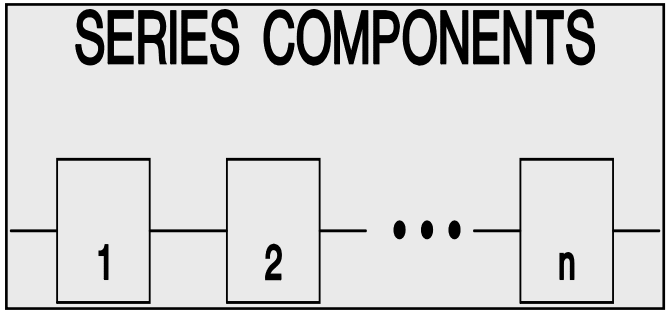
*Independence: Failure or nonfailure of one component does not change the reliability of the other component

In order for the system to function, all components must function!

Multiple Components in Series

Generalizing to n mutually independent components in series;

$$R_s(t) = R_1(t) \times R_2(t) \times \ldots \times R_n(t)$$

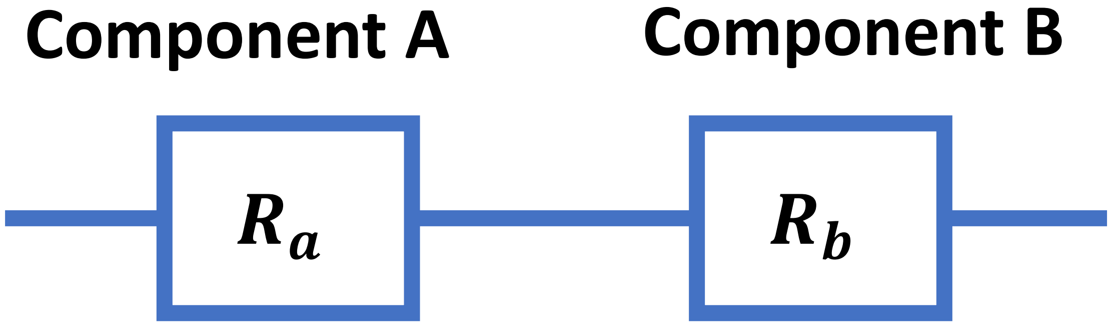


and $$R_s(t) \leq \min\{R_1(t), R_2(t), \ldots, R_n(t)\}$$

The system reliability can be no greater than the <u>smallest</u> component reliability (limiting factor)!

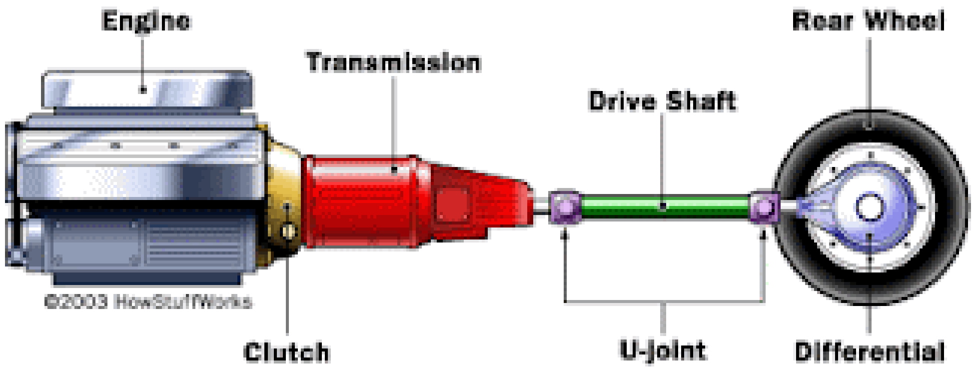
Serial Configuration Activity

- What are all the possible cases of the system below?



Example

Engine → Clutch → Drive Shaft/ U-joint → Rear Wheel/ Differential



Transmission System

transmits power from engine to wheels

After 150,000 miles, each component has a reliability of 95%. What is the current transmission system reliability?

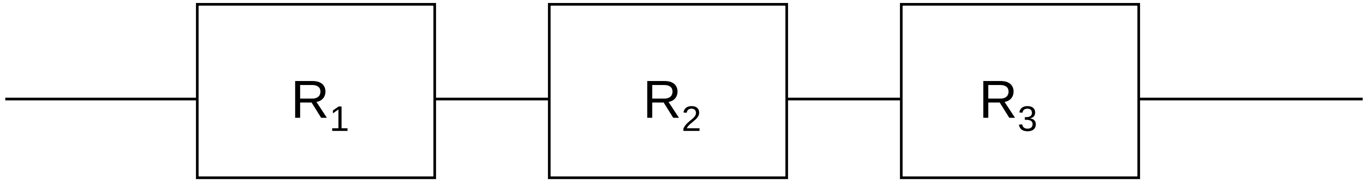
http://differenttypesofengineering.blogspot.com/2016/06/introduction-to-transmission-sstem.html

Exponential (Constant Failure Rate) Series Components

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} \exp\left(-\lambda_i t\right) = \exp\left(-\sum_{i=1}^{n} \lambda_i t\right) = \exp\left(-\lambda_s t\right)$$

$$\text{where } \lambda_s = \sum\nolimits_{i=1}^{n} \lambda_i$$

Example:



$$\boxed{R_{system} = R_1 R_2 R_3}$$

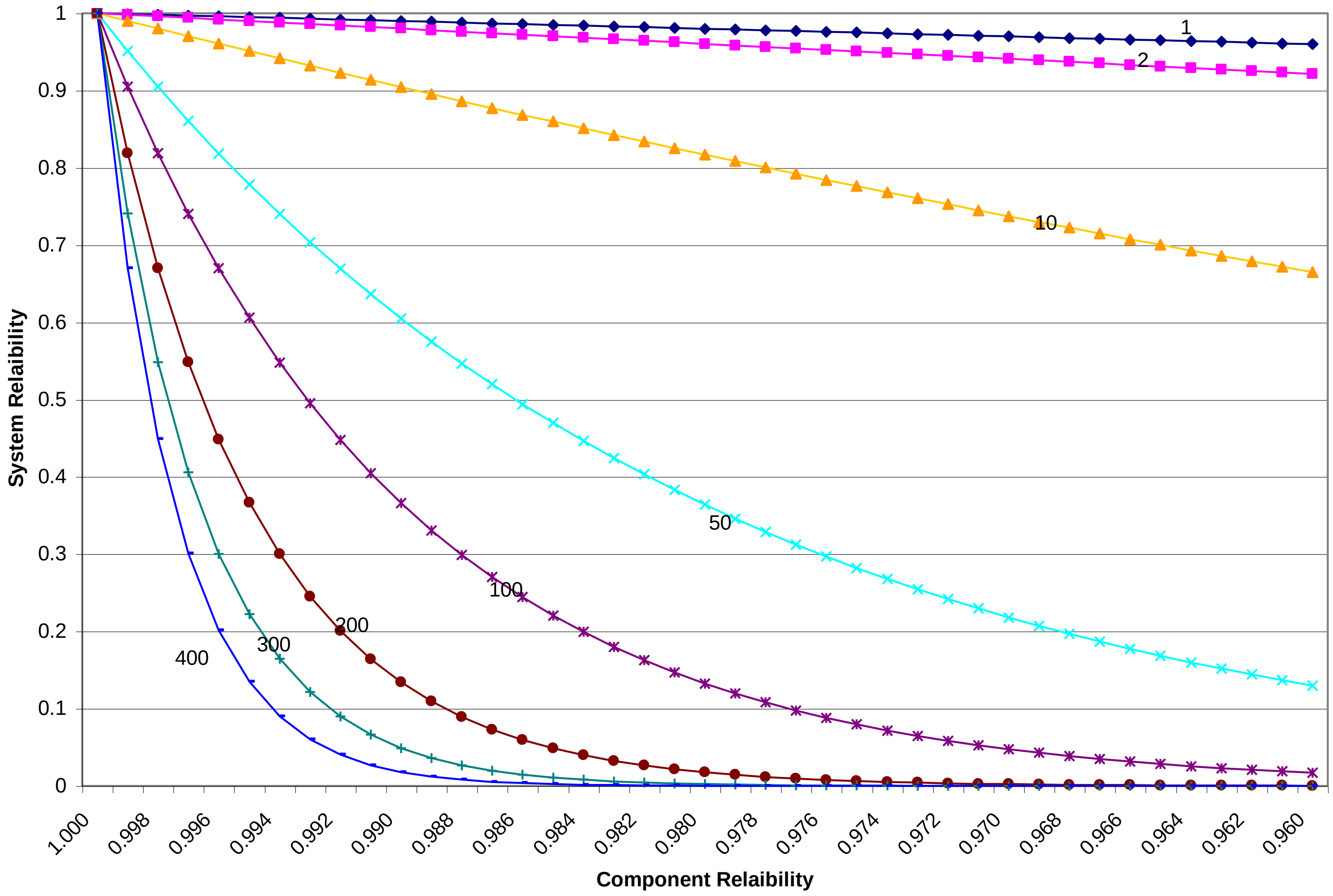
$$R_{system} = e^{-\lambda_1 t} \times e^{-\lambda_2 t} \times e^{-\lambda_3 t}$$

$$R_{system} = e^{-(\lambda_1 + \lambda_2 + \lambda_3)t}$$

$$e^{-\lambda_{system} t} = e^{-(\lambda_1 + \lambda_2 + \lambda_3)t}$$

$$\boxed{\lambda_{system} = \lambda_1 + \lambda_2 + \lambda_3}$$

As complexity of a series system increase, reliability decreases

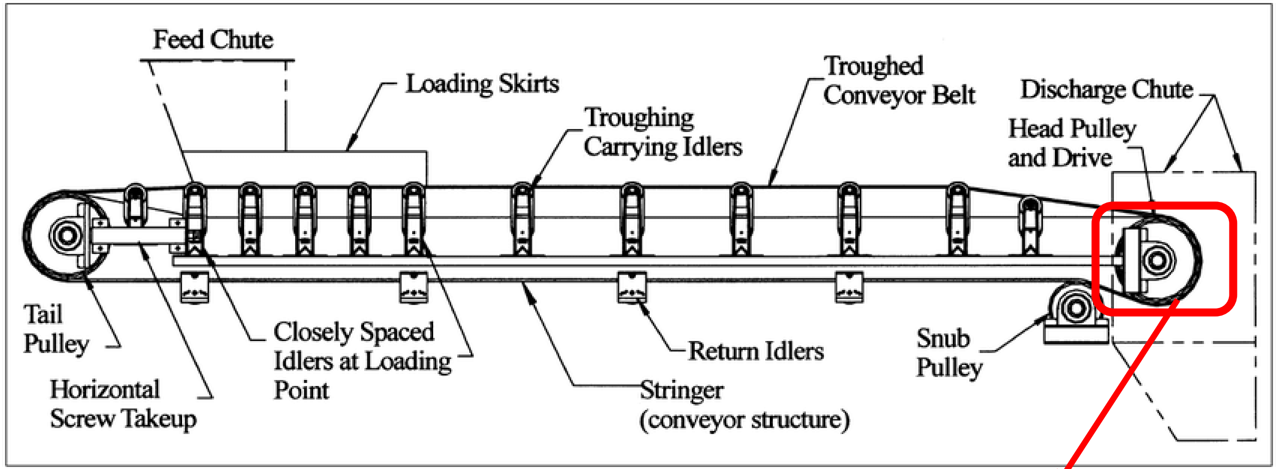


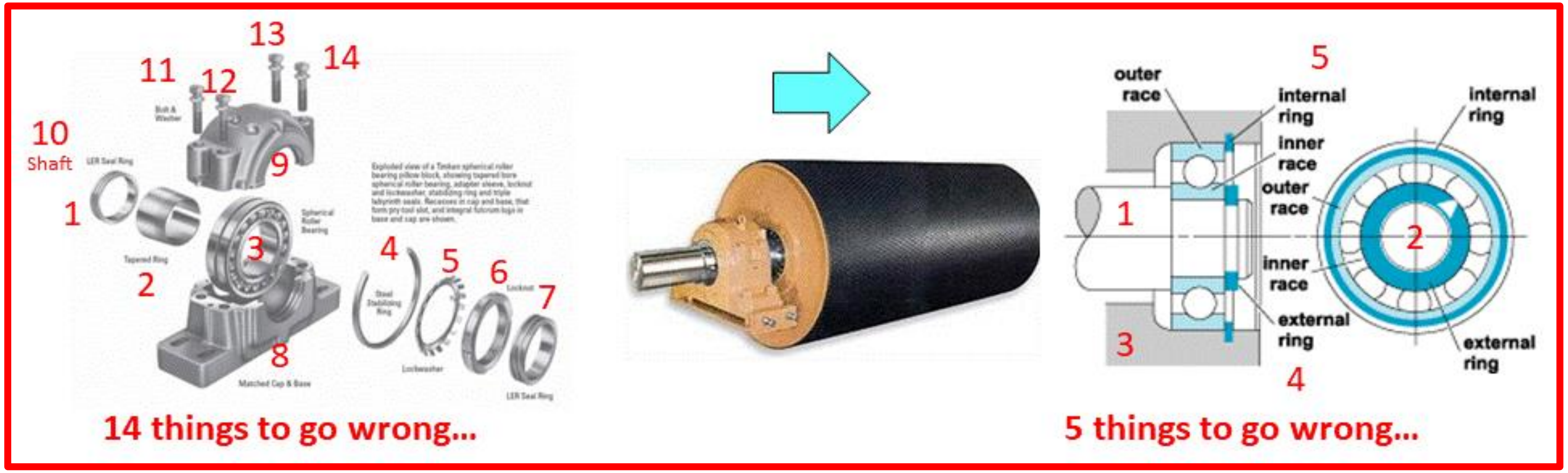
<u>Example:</u>

As number of components is increased from 1 to 400, where each component has a reliability of 0.996, the system reliability is decreased from ~1 to ~0.26.

Series Reliability: Simplify, Simplify & Simplify!

Designing a pulley on a conveyor belt system





https://lifetime-reliability.com/tutorials/series-system-reliability-property-2-examples/
https://oreflow.com.au/a-brief-overview-of-troughed-conveyors/

Weibull Series Components

$$R_s(t) = \prod_{i=1}^{n} \exp\left[-\left(\frac{t}{\theta_i}\right)^{\beta_i}\right] = \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$

Note: θ is the scale parameter here

$$\lambda(t) = \exp\left\{-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right\}\left[\sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i - 1}\right] \Big/ \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$

$$= \sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i - 1}$$

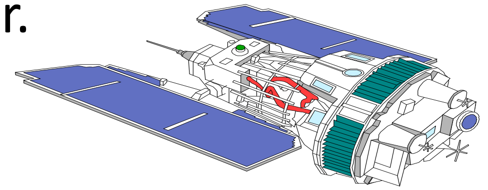
<u>Reminder:</u>
$\lambda(t)$ is the hazard function (instantaneous failure rate), where $\lambda(t) = \frac{f(t)}{R(t)}$

Components in Series - Example

A communications satellite consists of the following components:

Component	Probability Distribution	Shape Parameter	Characteristic life
Power unit	Weibull	2.7	43,800 hr.
Receiver	Weibull	1.4	75,000 hr.
Transmitter	Weibull	1.8	68,000 hr.
Antennae	Exponential	MTTF = 100,000 hr.	

Calculate the system hazard rate and reliability, and the reliability at 17, 520 hr.



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at slido.com with **#1853453**

Parallel Configuration

Or

Probability that all components fail

$R_s = \boxed{P(E_1 \cup E_2)} = 1 - \boxed{P((E_1 \cup E_2)')}$

Probability that at least one component doesn't fail

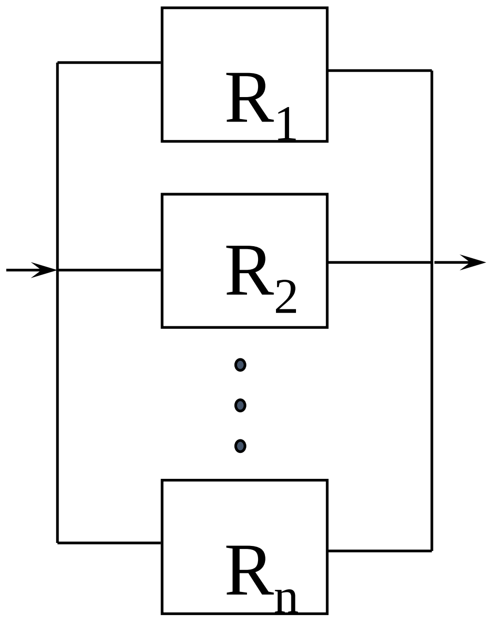
$= 1 - P(E_1' \cap E_2')$

$= 1 - P(E_1')\, P(E_2') = 1 - (1-R_1)\,(1-R_2)$

where E_1 = event that component 1 does not fail

E_2 = event that component 2 does not fail, and

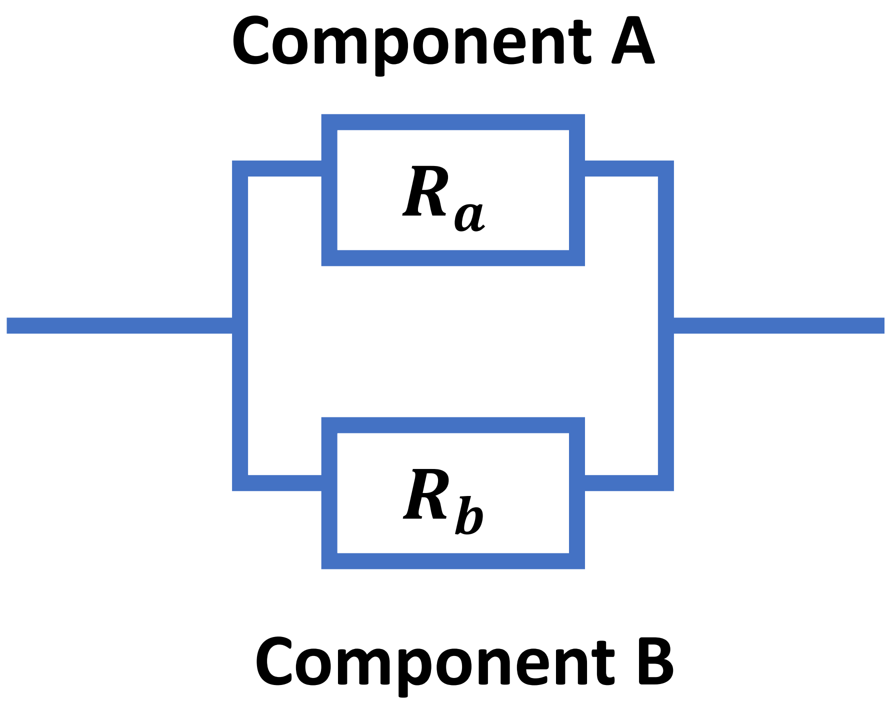
$P(E_1) = R_1$ and $P(E_2) = R_2$



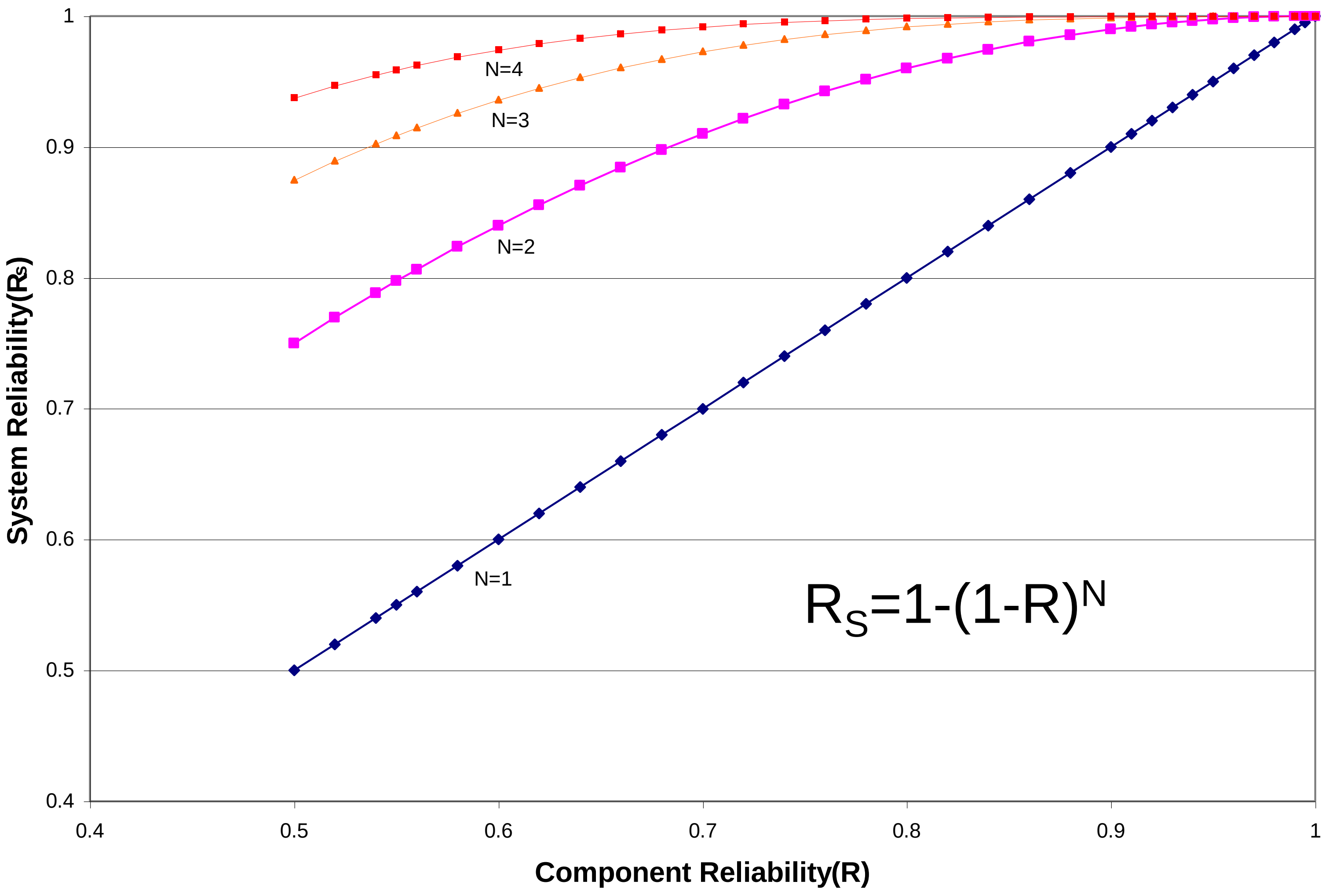
The probability at least one component does not fail!

Parallel Configuration Activity

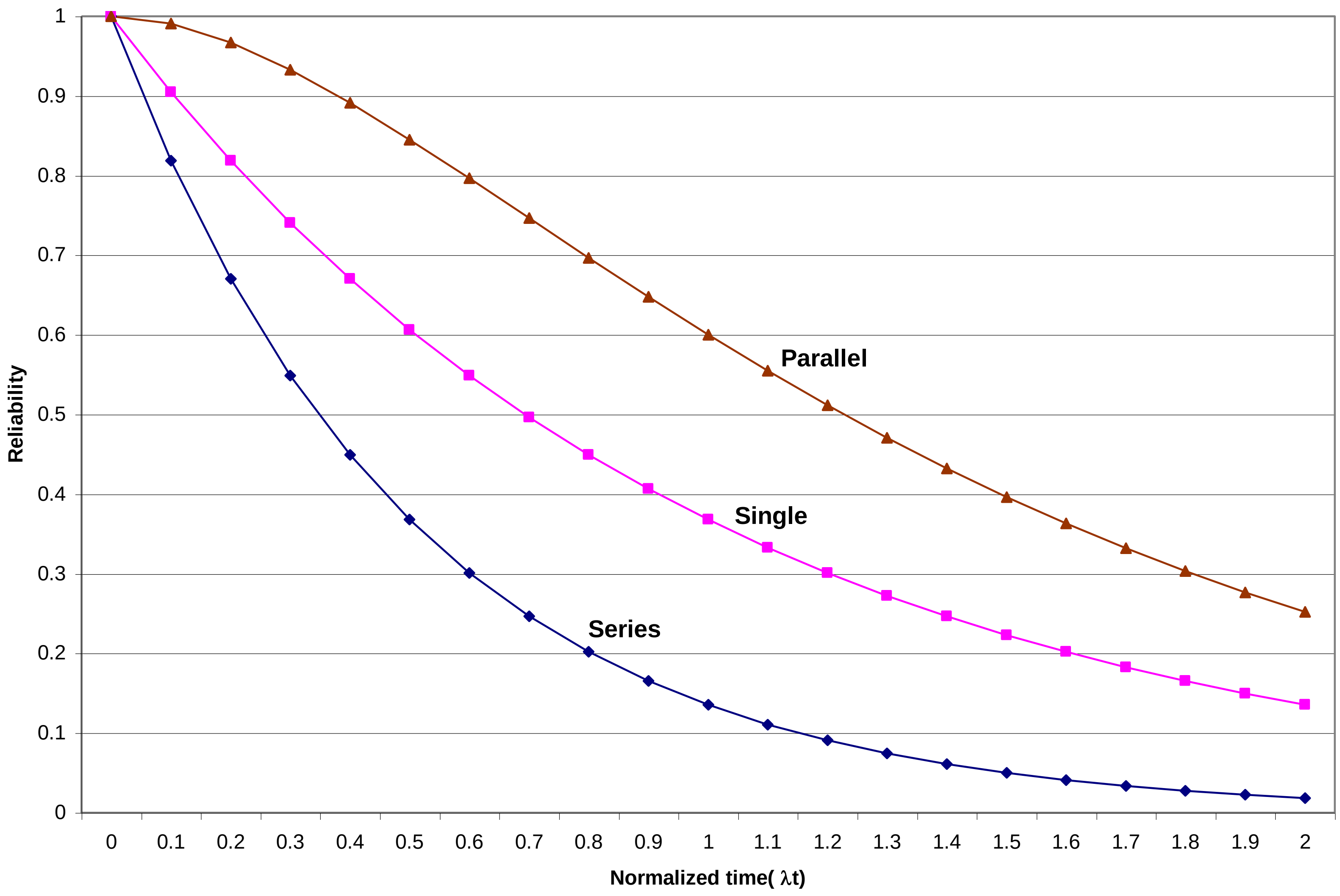
- What are all the possible cases of the system below?



Putting two(or more) subsystems in parallel is often the easiest way to increase reliability



Comparison of Single unit, 2 units in series, 2 units in parallel



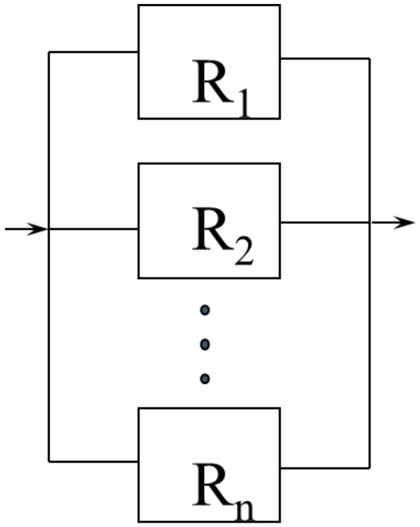
Parallel Configuration - Generalization

$$R_s(t) = 1 - \prod_{i=1}^{n} \left[1 - R_i(t)\right]$$

$$R_s\ (t) \geq \max\ \{R_1(t), R_2(t), \ldots, R_n(t)\}$$

The system reliability can not be less than the largest component reliability!

Note:



For the entire system to fail, all the components have to fail. Therefore, the entire system reliability must be larger the largest component reliability, because if the largest component reliability fails, there are other backup components in place.

Parallel Configuration – 2-component Exponential Model

$$R_s(t) = 1 - \prod_{i=1}^{n} \left[1 - e^{-\lambda_i t} \right]$$

For n = 2:

$$R_s(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$$

$$\text{MTTF} = \int_0^\infty R_s(t)\, dt = \int_0^\infty e^{-\lambda_1 t} dt + \int_0^\infty e^{-\lambda_2 t} dt - \int_0^\infty e^{-(\lambda_1 + \lambda_2)t} dt$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

Parallel Configuration - Weibull Model

$$R_s(t) = 1 - \prod_{i=1}^{n} \left[1 - e^{-\left(\frac{t}{\theta_i}\right)^{B_i}} \right]$$

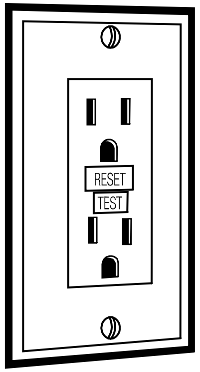
Note: θ is the scale parameter here

If n identical Weibull components are in parallel:

$$R_s(t) = 1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^{B}} \right]^n$$

Example

A circuit breaker has a Weibull failure distribution (against a power surge) with beta equal to .75 and a characteristic life of 12 years.

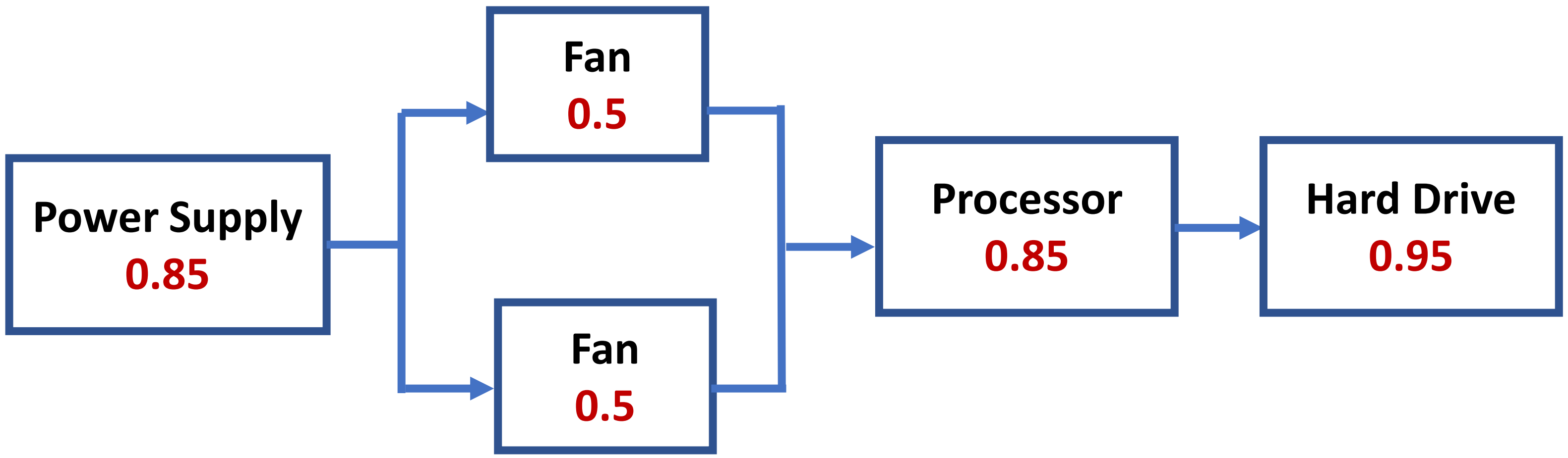


$$R(1) = e^{-\left(\frac{1}{12}\right)^{.75}} = 0.856$$

Note: Weibull Reliability=$e^{-\left(\frac{t}{\eta}\right)^{\beta}}$

Determine the one year reliability if two identical circuit breakers are redundant.

Combined System Reliability

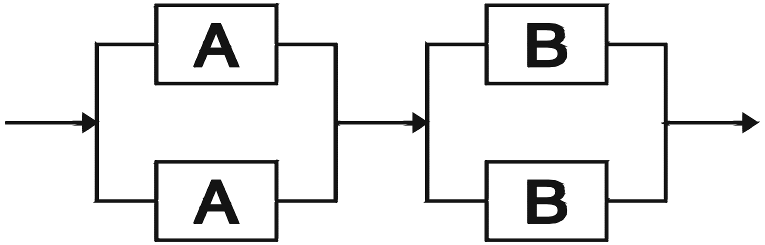


What is the reliability of this simplified computer system, given the component reliabilities shown in diagram?

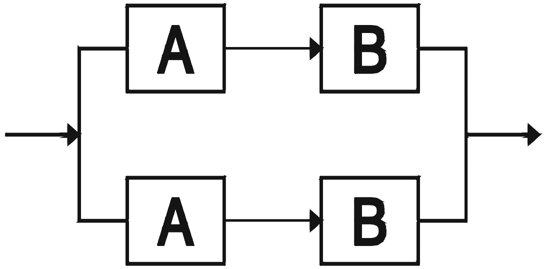
System Redundancy

Redundancy can increase reliability, but how is redundancy designed?

- Low-level redundancy
  - Duplication takes place at the subsystem or component level

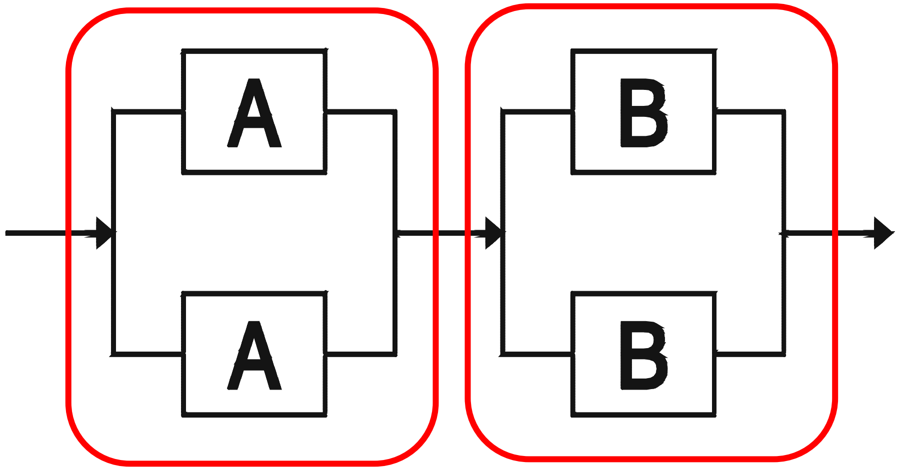


- High-level redundancy
  - Entire system is duplicated



Low Level Redundancy

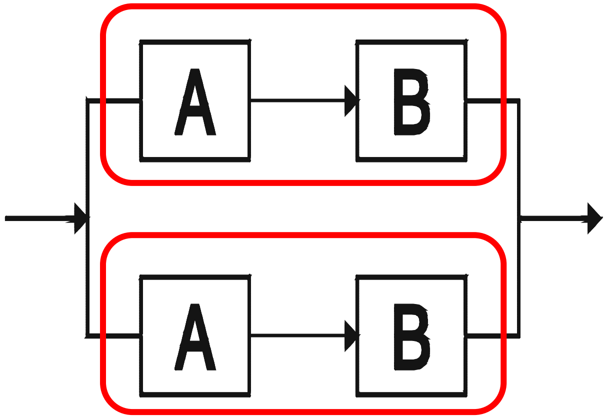
- Each component comprising the system may have one or more parallel components



- **R_{LOW}** = $[1-(1-R)^2]^2 = [1-(1-2R+R^2)]^2 = \mathbf{(2R-R^2)^2}$

High Level Redundancy

- Entire system placed in parallel with one or more identical systems



- $\mathbf{R_{high}} = 1 - (1-R^2)^2 = 1 - [1-2R^2 + R^4] = \mathbf{2R^2 - R^4}$

High vs Low Level Redundancy

$R_{low} - R_{high} = (2R - R^2)^2 - (2R^2 - R^4)$

*Note: Assume components reliabilities are mutually independent and independent of configuration in which they are placed

$= R^2 (2-R)^2 - R^2 (2 - R^2)$

$= R^2 [4 - 4R + R^2 - 2 + R^2]$

$= 2 R^2 [R^2 - 2R + 1] = 2 R^2 (R - 1)^2 \geq 0$ ⇒

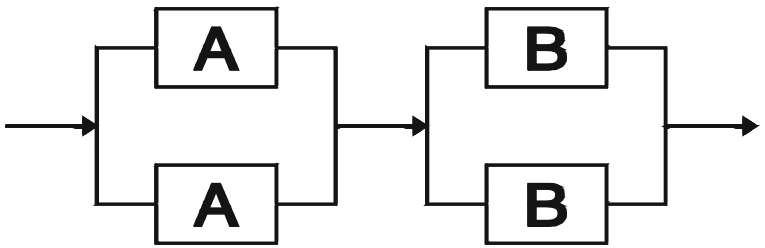
Reliability of low-level redundancy is <u>greater</u> than or equal to high-level redundancy!

*Note: equality is obtained when R=1

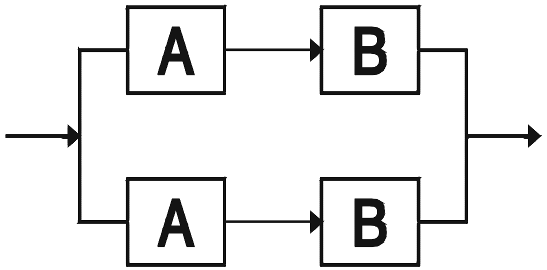
High vs Low Level Redundancy

Reliability of low-level redundancy is <u>greater</u> than or equal to high-level redundancy!

Low-level redundancy



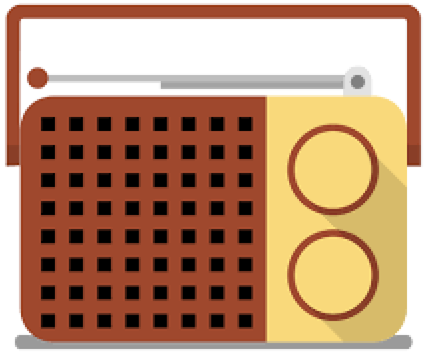
High-level redundancy



- Both systems will fail if either both components A fail or both components B fail
- High-level has one extra failure path → Easier to fail compared to low-level
  - Failure path: One A fails, and one B fails on separate paths

Example

- A radio set consists of three major components: a power supply, a receiver, and an amplifier, having reliabilities of 0.8, 0.9, and 0.85, respectively. Compute system reliabilities for both high-level and low-level redundancy for systems with two parallel components.



Quick Review: Permutations & Combinations

- When the number of possible outcomes is large, fully enumerating the outcomes, or fully tracing the sequence of possible outcomes becomes untenable

- We need to be able to count the number of ways outcomes might happen without explicitly listing them all

Multiplication Rule

- If one event can occur in **m** ways, a second event in **n** ways and a third event in **r**, then the three events can occur in m × **n** × **r** ways.

- **Example:** Erin has 5 tops, 6 skirts and 4 caps from which to choose an outfit. In how many ways can she select one top, one skirt and one cap?

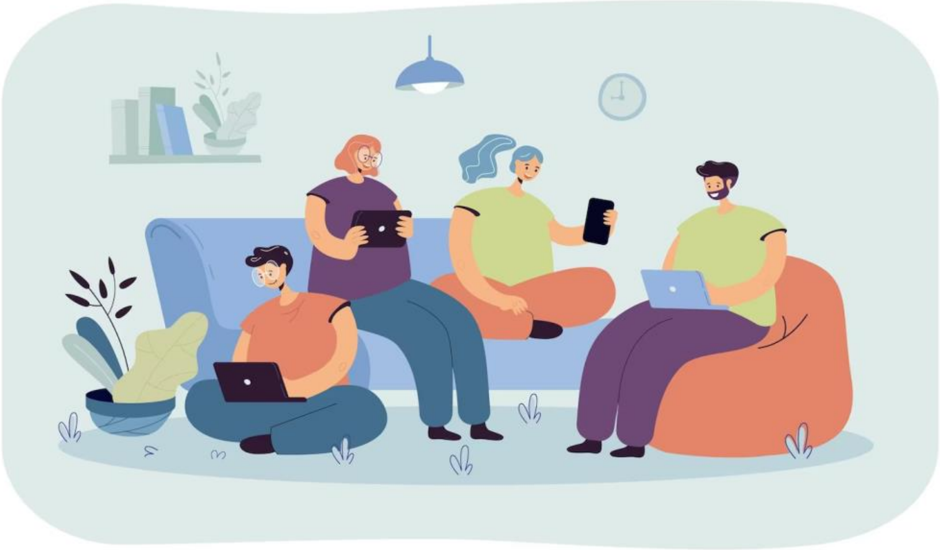
- **Solution:** Ways = 5 × 6 × 4 = 120

Repetition of an Event

- If one event with **n** outcomes occurs **r** times with repetition allowed, then the number of ordered arrangements is n^r

- **Example:** What is the number of arrangements if a dice is rolled
  - (a) 2 times ? $6 \times 6 = 6^2$
  - (b) 3 times ? $6 \times 6 \times 6 = 6^3$
  - (b) r times ? $6 \times 6 \times 6 \times \ldots\ldots = 6^r$

Factorial Representation



- $n! = n(n-1)(n-2)..........3 \times 2 \times 1$
- For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$
  - **Note $0! = 1$**

- **Example:**
- a) In how many ways can 6 people be arranged in a row?

Solution : $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

- b) How many arrangements are possible if only 3 of them sit in a row?

Solution: $6 \times 5 \times 4 = 120$

Arrangements or Permutations

- More generally, we might want to count the number of ways we can pick an ordered subset of size **k** from a larger set of size **n**

$$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

- **Example:** A debating team consists of 4 speakers.
- a) In how many ways can all 4 speakers be arranged in a row for a photo?
- **Solution:** **4 × 3 × 2 × 1 = 4!**
- b) How many ways can the captain and vice-captain be chosen?
- **Solution:** **4 × 3 = 12**

Combinations

- Given a set of size ***n***, how many subsets of size ***k*** are there?

- **Ordering does not matter!**

- 2-permutations of A,B,C: AB, BA, AC, CA, BC, CB
- 2-letter combinations: AB, AC, BC

- The number of *k*-combinations is simply the number of *k*-permutations divided by the number of "duplicates"

$$\frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \binom{n}{k}$$

Example

- There are 12 people on a basketball team, but only 5 people play at a time. How many different 5 person lineups are there?

- **Solution:**
- $\binom{12}{5} = \frac{12!}{(12-5)!} \cdot \frac{1}{5!} = \frac{12!}{7!} \cdot \frac{1}{5!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} = 11 \times 9 \times 8 = 792$

Bernoulli trials

- A Bernoulli trial is a random experiment with two outcomes: "success" and "failure"

$$P(success) = p \qquad P(failure) = 1 - p$$

- Imagine a sequence of independent Bernoulli trials. How can we calculate the probability of (exactly) k successes?
- **Example:** A fair coin is being tossed. What is the probability that exactly 2 out of the next 3 flips will be "heads"?
- **Solution:**

HHH	THT
HHT	TTH
HTH	HTT
THH	TTT

These 3 choices have exactly 2 out of 3 heads → HHT, HTH, THH

Method 1: $\mathrm{P}\left(\{\mathrm{HHT,THH,HTH}\}\right) = \frac{3}{8}$

Method 2: Suppose P(head)=p $\qquad$ P(tail)=1−p

$$\mathrm{P}\left(\{\mathrm{HHT,THH,HTH}\}\right) = \mathrm{P}\left(\mathrm{HHT}\right) + \mathrm{P}\left(\mathrm{THH}\right) + \mathrm{P}\left(\mathrm{HTH}\right)$$

$$P(HHT) = P(H) \cdot P(H) \cdot P(T) = p^2(1-p) = \left(\tfrac{1}{2}\right)^2 \cdot \tfrac{1}{2} = \tfrac{1}{8}$$

$$P(HHT) = P(THH) = P(HTH)$$

$$P\{HHT, THH, HTH\} = \tfrac{3}{8}$$

Bernoulli trials

- In general, for any specific sequence with successes in trials, the probability of that sequence is $p^k(1-p)^{n-k}$

- There are $\binom{n}{k}$ possible sequences with k successes in n trials

$$\mathrm{P}\,(\text{exactly } k \text{ successes out of } n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathrm{P}\,(\text{at least } k \text{ successes out of } n) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Bernoulli trials

- In general, for any specific sequence with successes in trials, the probability of that sequence is $p^k(1-p)^{n-k}$

- There are $\binom{n}{k}$ possible sequences with k successes in n trials

$$\mathrm{P}\left(\text{exactly } k \text{ successes out of } n\right) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial Distribution: $$\mathrm{P}\left(\text{at least } k \text{ successes out of } n\right) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Note: Bernoulli trials are a special case of Binomial distribution for a single trial

Example

- A basketball player is a 70% free throw shooter. What is the probability that the person makes exactly 6 out of my next 10 free throws?

- What is the probability that the person make at least 6 out of 10?

Back to System Reliability….

- k-out-of-n Redundancy
  - A generalization of n parallel components occurs when a requirement exists for k-out-of-n identical and independent components to function for the system to function.
    - $k \leq n$.
    - If $k = 1$, complete redundancy occurs
    - if $k = n$, the n components are, in effect, in series

k-out-of-n Redundancy

Let n = the number of redundant, identical and independent components each having a reliability of R.

Let X = a random variable, the number of components (out of n components) operating. Then

$$\boxed{P(x) = \binom{n}{x} R^x (1-R)^{n-x}}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!}$

If k ≤ n components must operate for the system to operate:

$$\boxed{R_s = \sum_{x=k}^{n} P(x)}$$

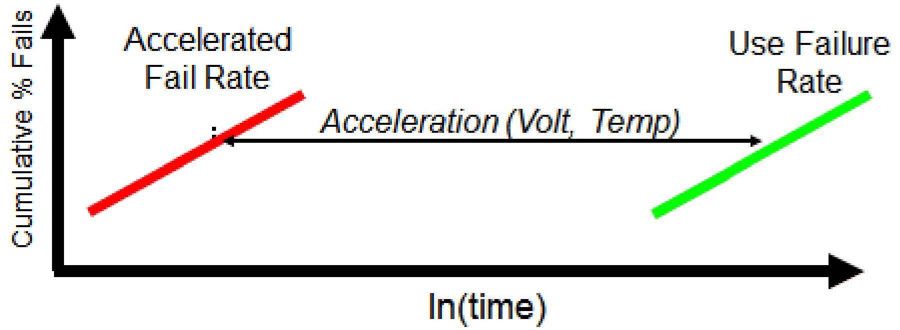
which is the probability of k or more successes from among the n components

$$\mathrm{P}\,(\text{at least } k \text{ successes out of } n) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Example

- A pressure vessel is equipped with six relief valves. Pressure transients can be controlled successfully by any three of these valves. If the probability that any one of these valves will fail to operate on demand is 0.04, what is the probability that the relief valve system will fail to control a pressure transient (on demand)? Assume the failures are independent.

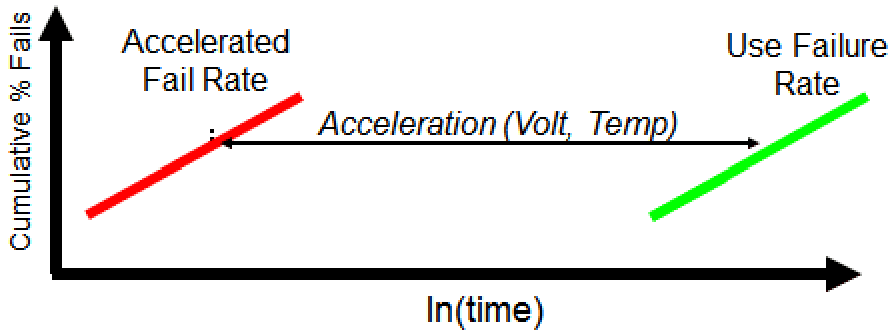
Accelerated Testing



- **Stressing at higher than normal** conditions to make failures occur earlier.
- True acceleration occurs when levels of increasing stress cause “things to happen faster”.
- The failure mechanisms are exactly the same as seen under normal stress, only the time scale has been changed. It’s like watching a movie in fast forward mode.
- True acceleration is, therefore, just a **transformation of the time scale**.

Reference: David C. Trindade, Exploring Reliability with JMP: Accelerated Testing
https://www.ti.com/support-quality/reliability/reliability-testing.html

Physical Acceleration Models



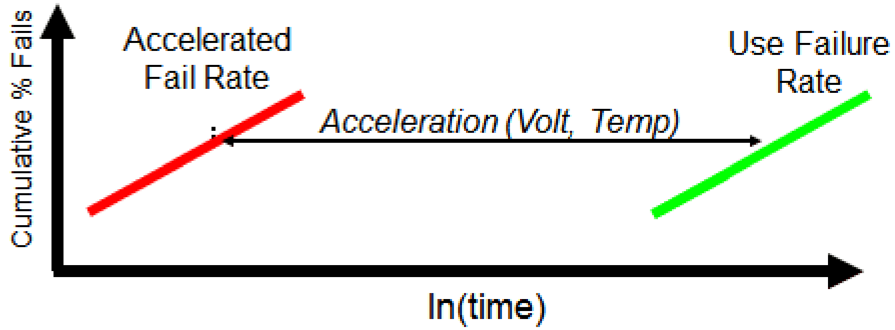
- Linear Acceleration
  - If we know the life distribution of units operating at a high stress, and we apply an appropriate time scale transformation to a lower stress condition, we can derive the life distribution at that lower stress.
  - When every time of failure at high stress is multiplied by the same constant value to obtain the projected results at a low stress, we have linear acceleration.
  - In other words,

failure time low stress = AF x failure time high stress,

where **AF** is called an “**acceleration factor**.”

Reference: David C. Trindade, Exploring Reliability with JMP: Accelerated Testing
https://www.ti.com/support-quality/reliability/reliability-testing.html

Acceleration Factor AF



- The acceleration factor is the ratio of the time to a given CDF value under low stress conditions to the time to that same CDF value under high stress conditions, that is:

$$AF = \left(\frac{t_{S_{low}}}{t_{S_{high}}} \right)$$

- For true, linear acceleration, the acceleration factor, once determined, is the same for any percentile value. Thus,

$$AF = \frac{t_{50_{S_{low}}}}{t_{50_{S_{high}}}} = \frac{t_{10_{S_{low}}}}{t_{10_{S_{high}}}} = \frac{t_{1_{S_{low}}}}{t_{1_{S_{high}}}}$$

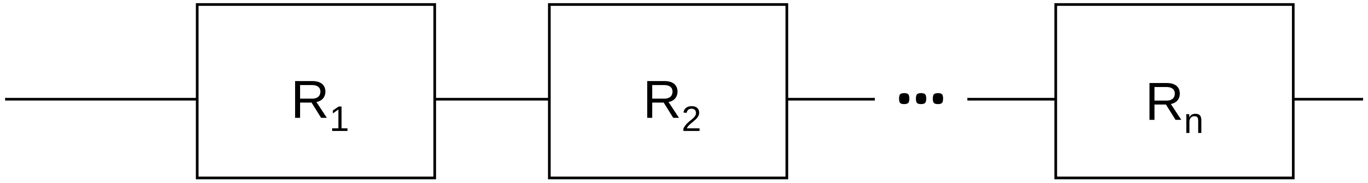
- For example, it takes 1,000 hours to reach 10% failures under high stress. If the AF is 25 to field use, it should take 25,000 hours to achieve 10% failures in the field.

Reference: David C. Trindade, Exploring Reliability with JMP: Accelerated Testing
https://www.ti.com/support-quality/reliability/reliability-testing.html

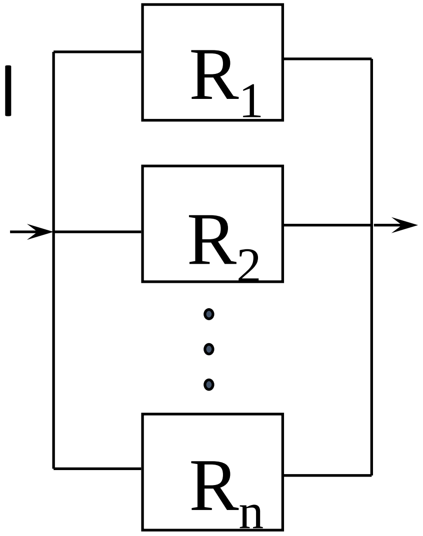
Summary

- Series system reliability: System fails if any components fail

$$R_s(t) \; = \; R_1(t) \; x \; R_2(t) \; x \; \ldots \; x \; Rn(t)$$



- Parallel system reliability: System fails if all components fail



$$R_s(t) = 1 - \prod_{i=1}^{n} \left[1 - R_i(t) \right]$$

Summary

- k-out-of-n Redundancy
  - If k ≤ n components must operate for the system to operate:

$$R_s = \sum_{x=k}^{n} P(x)$$

where $P(x) = \binom{n}{x} R^x (1-R)^{n-x}$

Summary

- Accelerated testing
  - failure time low stress = AF x failure time high stress

$$AF = \left(\frac{t_{S_{low}}}{t_{S_{high}}} \right)$$

where AF is called an “acceleration factor”