可靠度資料分析
Reliability Data Analysis

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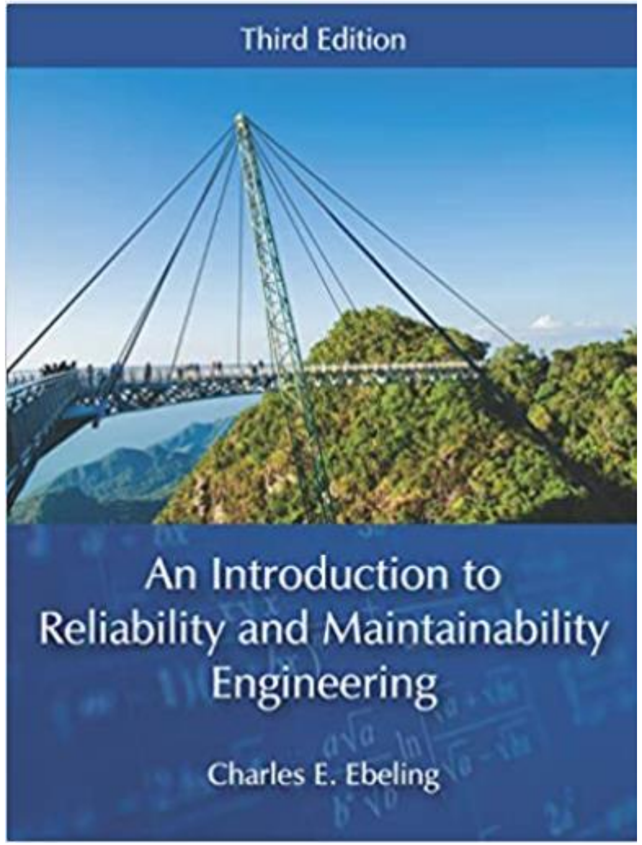
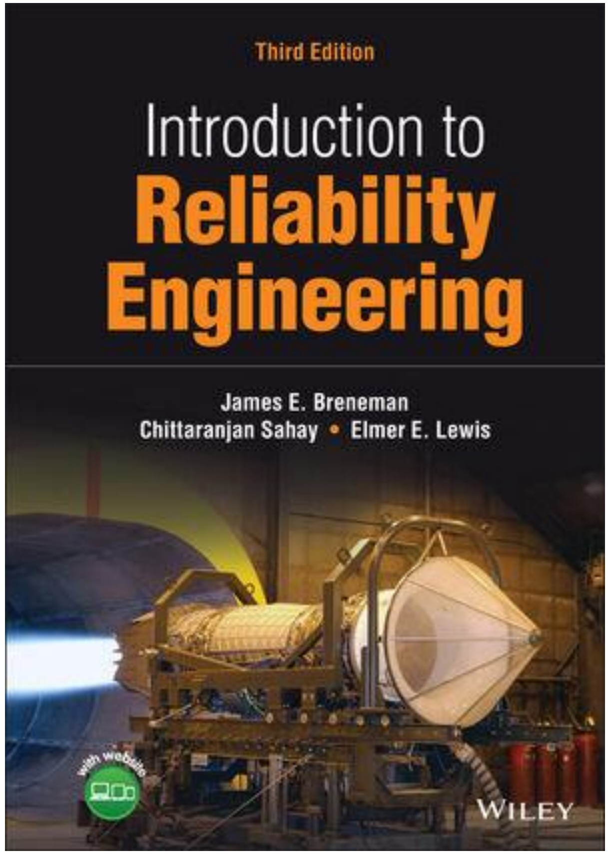
Lecture 2 – Probability & Statistics Concepts

Textbook



Can only buy e-book for $55US (~$1674 NT)



華通書局
$1793 NT for print version

Recap: What is Reliability?

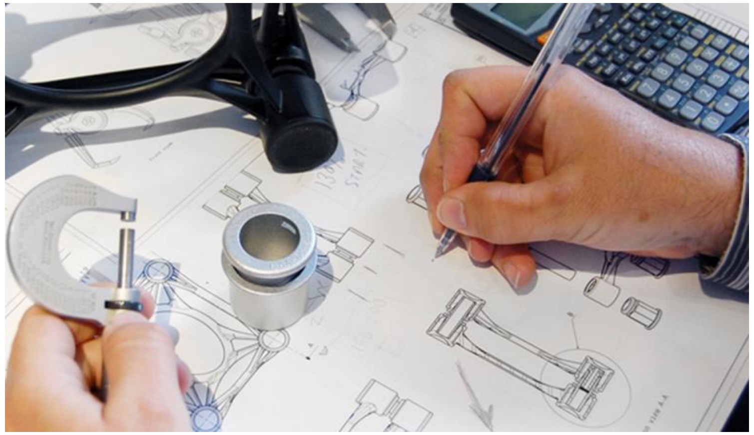
- **Reliability** is the **probability** that a product will operate or a service will be provided properly for a specified period of time (known as the design life) under the intended operating conditions (designated temperature, load, speed, etc.) without failure.

- **Reliability Engineering** attempts to study, characterize, measure, and analyze system failures in order to improve their operational use by increasing their **design life** and reducing the likelihood of unexpected failures, and **downtime**, thereby increasing **availability**.

How reliability can be used

- Designing a device / product / system

- Correcting product reliability/warranty issue



https://www.sparkinnovations.com/product-design/

Designing a Device/Product

- Incremental Update
- Significant Update
- New Technology / Material

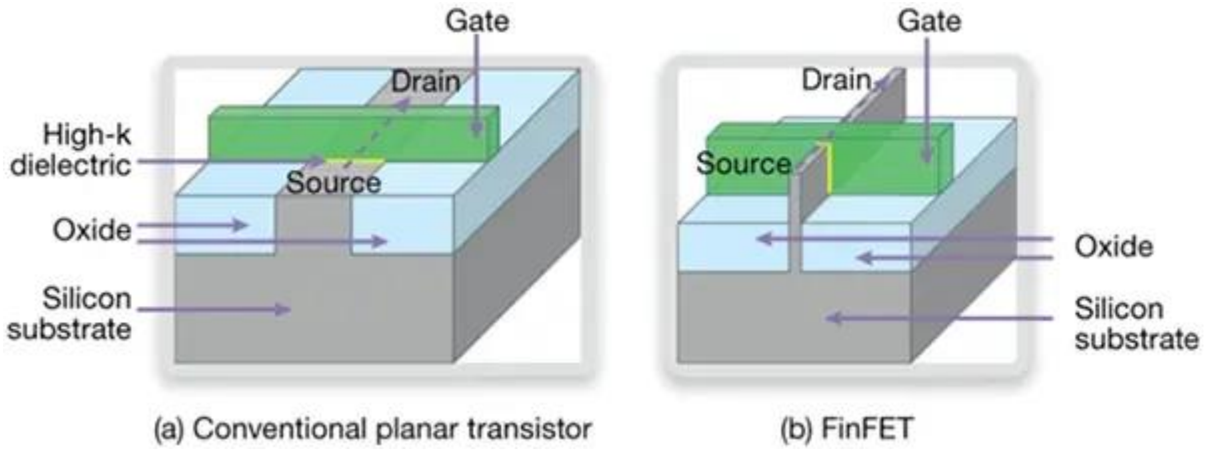
Incremental Update

- Typical Business Rhythm
- Well Defined Schedule / Low Risk
- Design Guidelines
- Routine Reliability Testing
- Examples
  - 45nm -> 32 nm process node
  - Liquid Crystal Display (LCD) 720 ->1080
  - iPhone updates

Significant Product Change

- Business Decision to Disrupt Market
- Multiple Design Changes
- Risks: Cost, Schedule, Performance
- Design Guidelines Adjusted
- Highly Accelerated Life Testing
- Examples
  - FinFET process
  - Introduction of iPhone

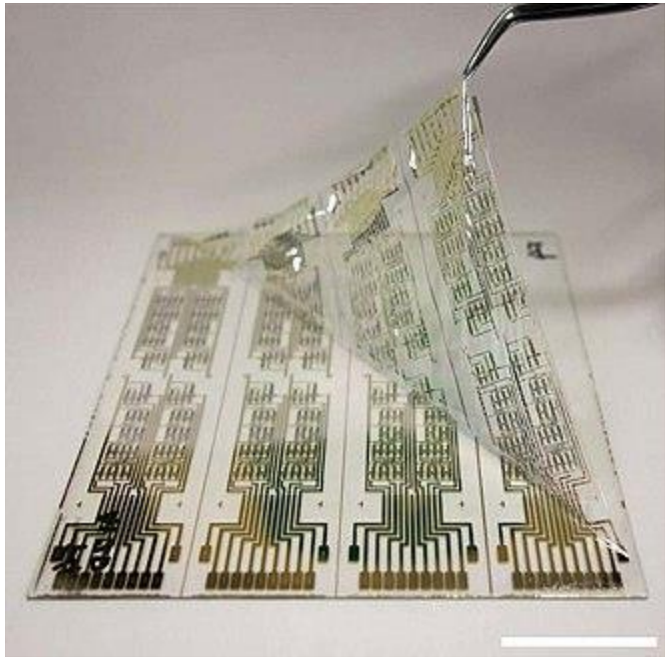
Semiconductor Process Change



https://www.engineersgarage.com/all-about-finfet/

New Technology / Material

- Investment Driven by:
  - Physical limitations of Existing Technology
  - Entry of Company in New Market
- Reliability / Failure Mechanisms Unknown
- Risks
  - Warranty costs
  - Company reputation
- Examples
  - Organic Transistor
  - Organic Light Emitting Diode (OLED Display)



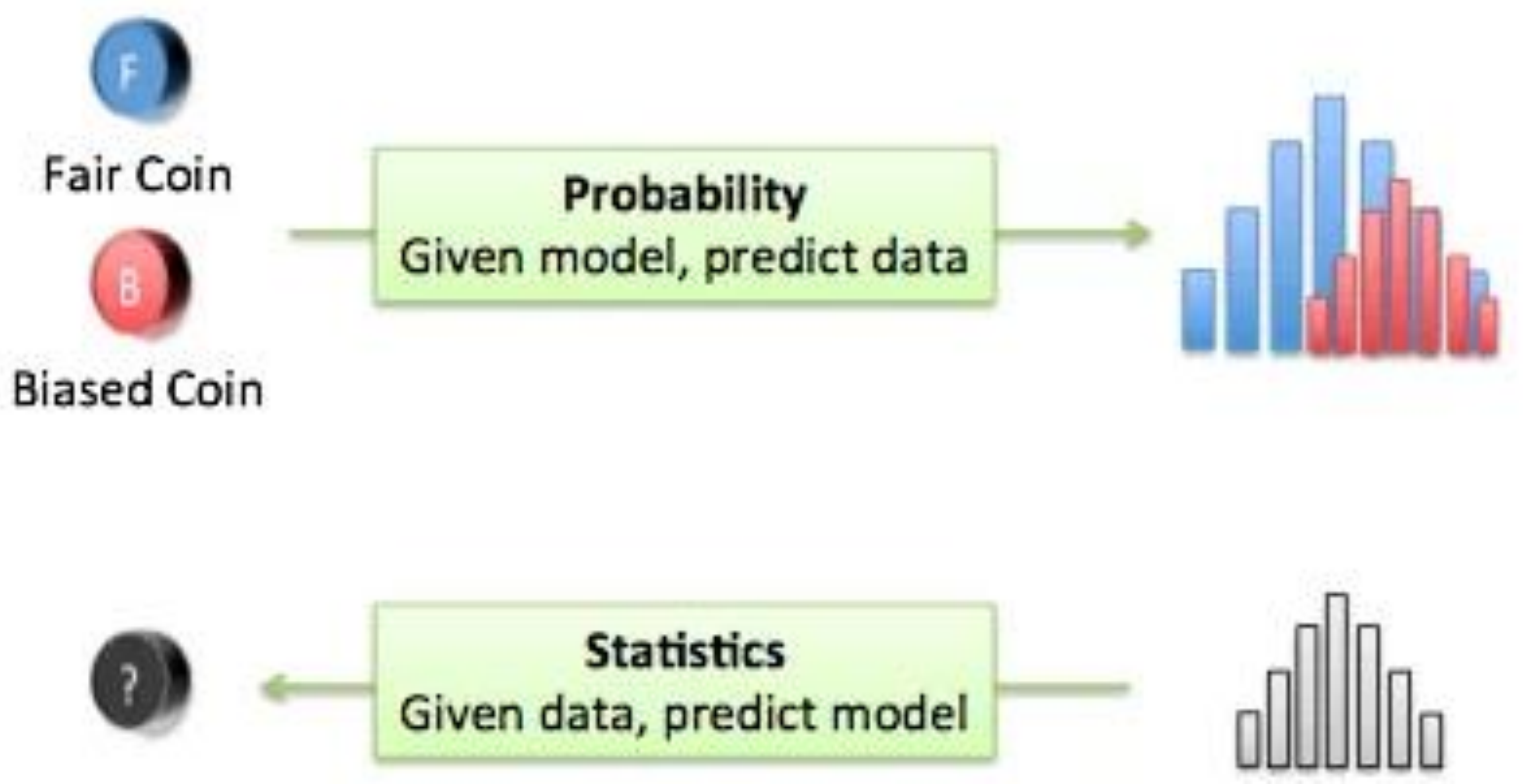
Organic CMOS logic circuit

https://en.wikipedia.org/wiki/Organic_field-effect_transistor

Product Field Reliability Issue

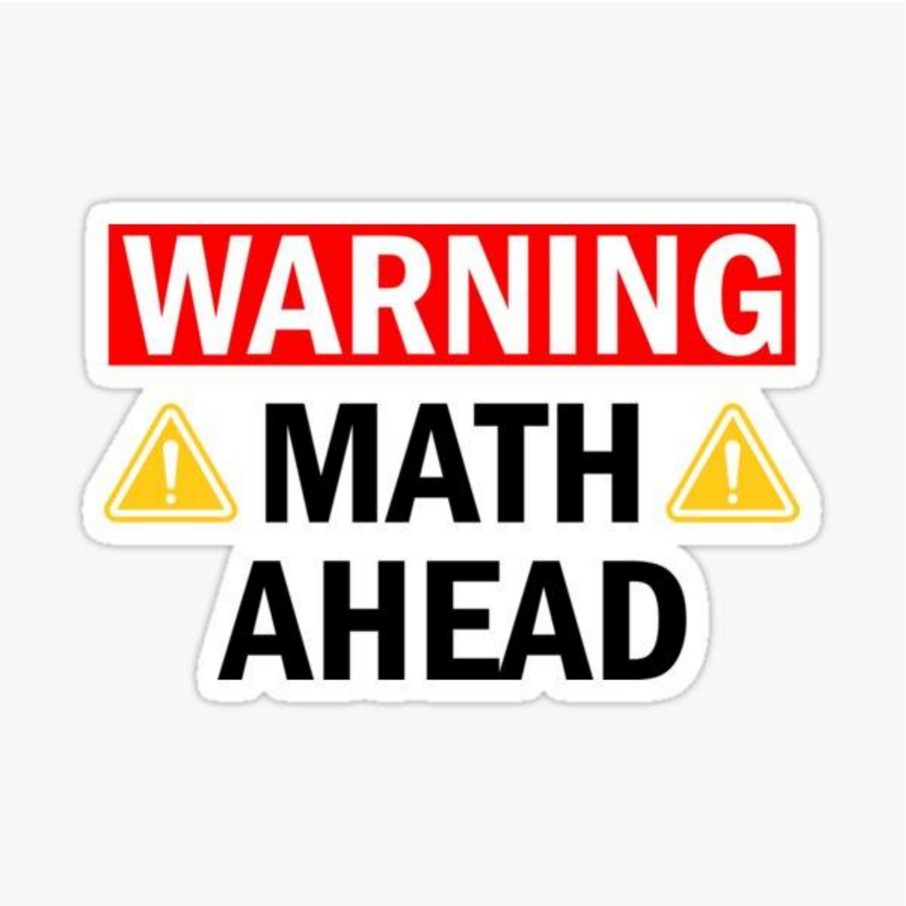
- Determine Root Cause
  - Failure analysis
  - Unbiased perspective
- Identify Corrective Action
  - Test solution thoroughly
  - Balance solution with severity of situation
- Preventative Action
  - Often overlooked
  - Improve design process

Why Probability & Statistics?



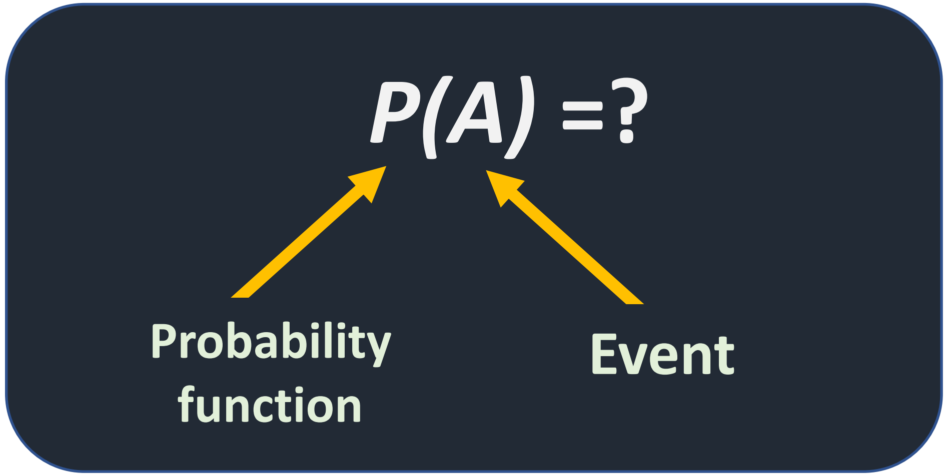
https://betterexplained.com/articles/a-brief-introduction-to-probability-statistics/

Probability Concepts Review



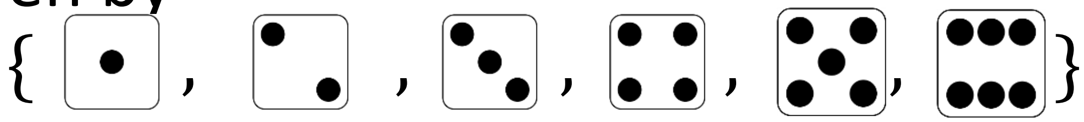
Basic Probability Model

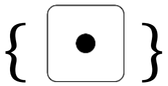
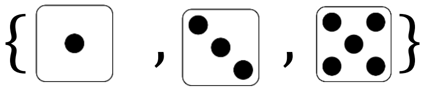
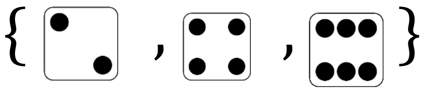
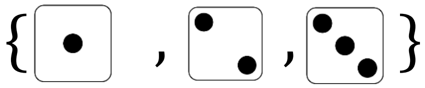
- A probability model consists of an experiment which produces exactly one out of several mutually exclusive (doesn't occur at same time) outcomes
- Essential elements are:
  - Sample space S (some texts use Ω or U)
    - Collection or list (set) of possible outcomes
  - Probability law $P(\cdot)$
    - assigns a "likelihood" to different events
      - Event: a subset of the sample space
    - Ex. Probability of A is given by $P(A)$



Example: Roll of a dice

- Consider a fair six-sided die. The experiment is rolling the dice. The sample space is given by

$\Omega =$ 

- Possible events might include
  - the result is a “1”: 
  - the result is odd: 
  - the result is even: 
  - the result is less than or equal to “3”: 
- There are 2^6 different possible events if we allow Ω and $\emptyset$ to count as events

Example: Roll of a dice

- Since the die is "fair", a natural probability law is to assign each of the six possible outcomes the same value

$$P(\{\text{⚀}\}) = P(\{\text{⚁}\}) = \cdots P(\{\text{⚅}\}) = \frac{1}{6}$$

- It is then straightforward to compute the corresponding probability of different events:
  - $P(\{\text{⚀}, \text{⚂}, \text{⚅}\}) = \frac{1}{2}$
  - $P(\{\text{⚀}, \text{⚁}, \text{⚂}\}) = \frac{1}{2}$
  - $P(\{\text{⚀}, \text{⚁}\}) = \frac{1}{3}$

What conditions must a probability function satisfy?

$P(A)$ =?

Probability function

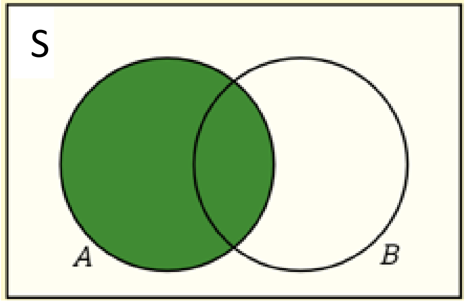
Event

Axioms of Probability

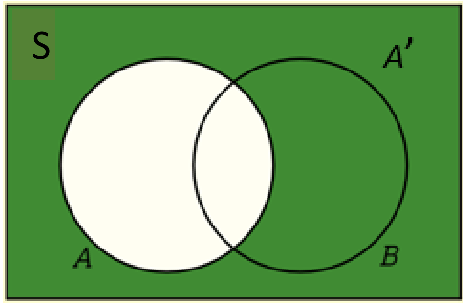
Axiom 1: (non-negativity)	$P(A) \geq 0$
Axiom 2: (normalization)	$P(S) = 1$
Axiom 3: (finite additivity)	empty set If $A \cap B = \emptyset$, Then $P(A \cup B) = P(A) + P(B)$

Probability of an Event

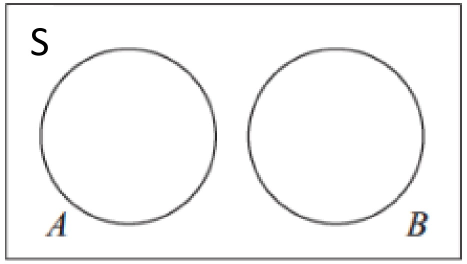
- We can define operations on events based on set theory.
- If A and B are two events:



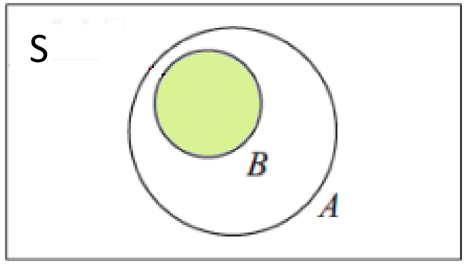
Set *A*



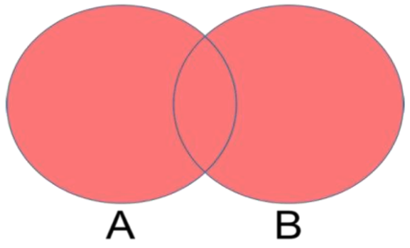
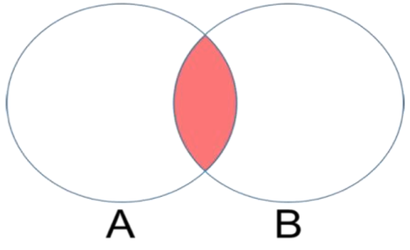
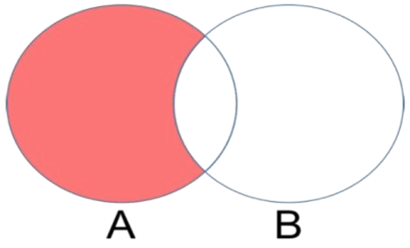
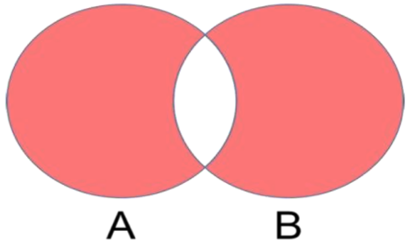
A′ the complement of *A*



A and *B* are disjoint sets (mutually exclusive)



B is proper subset of *A* $B \subset A$

Set Operation	Venn Diagram	Interpretation
Union		$A \cup B$, is the set of all values that are a member of *A*, or *B*, or both.
Intersection		$A \cap B$, is the set of all values that are members of both *A* and *B*.
Difference		$A \setminus B$, is the set of all values of *A* that are not members of *B* (A minus B)
Symmetric Difference (parts of a union)		$A \triangle B$, is the set of all values which are in one of the sets, but not both.

1. https://www.datacamp.com/community/tutorials/sets-in-python
2. https://medium.com/@sukhrobgolibboev/understanding-set-theory-de2532f746ac

Useful Probability Axioms

- $P(A \cup \mathrm{B}) = P(A) + P(B) - P(A \cap B)$
- $P(A') = 1 - P(A)$
- $P(S) = 1$ and $P(\emptyset) = 0$ (something has to happen)
- $P(A \backslash \mathrm{B}) = P(A) - P(A \cap B)$
- $0 \leq P(A) \leq 1$ for any event A

$P(A \cap \mathrm{B}) = 0$

A and B cannot occur at same time

- It follows that when $A_1, \ldots, A_n$ are $\boxed{\text{mutually exclusive}}$ events, then:

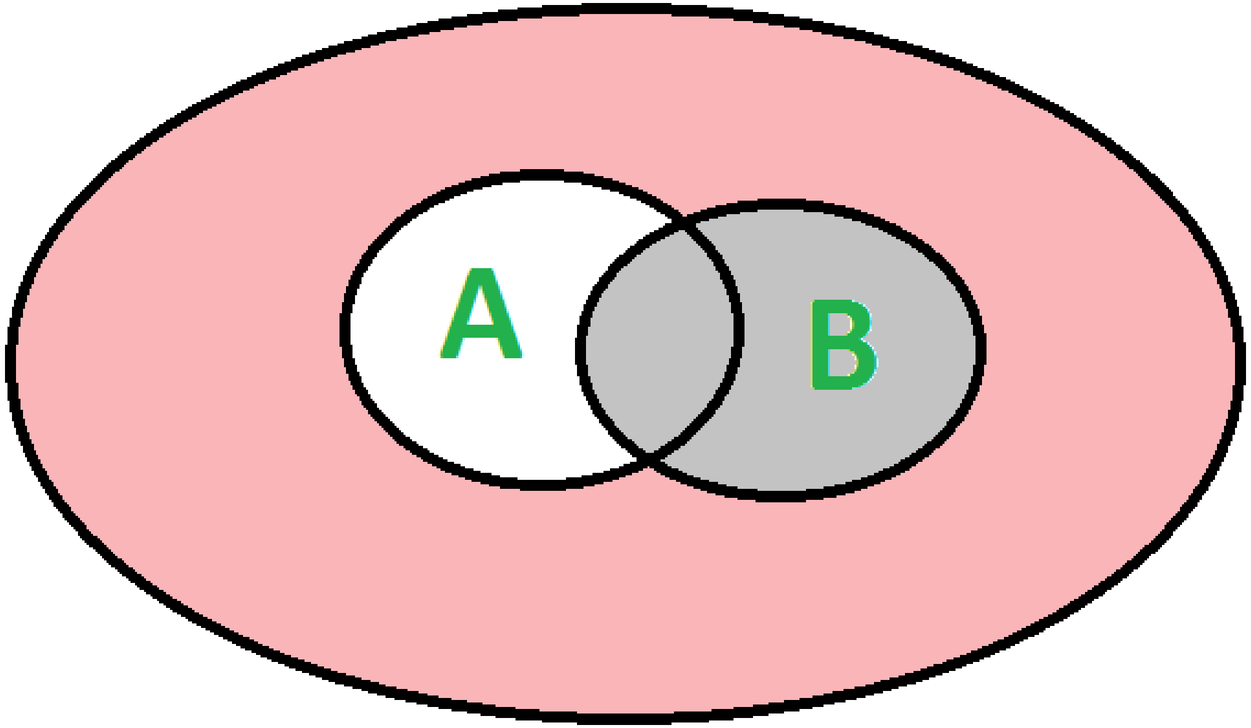
$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

Conditional Probability

- Conditional probability gives us a way to reason about the outcome of an experiment given such partial information.

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Not Possible to Happen as B already happened

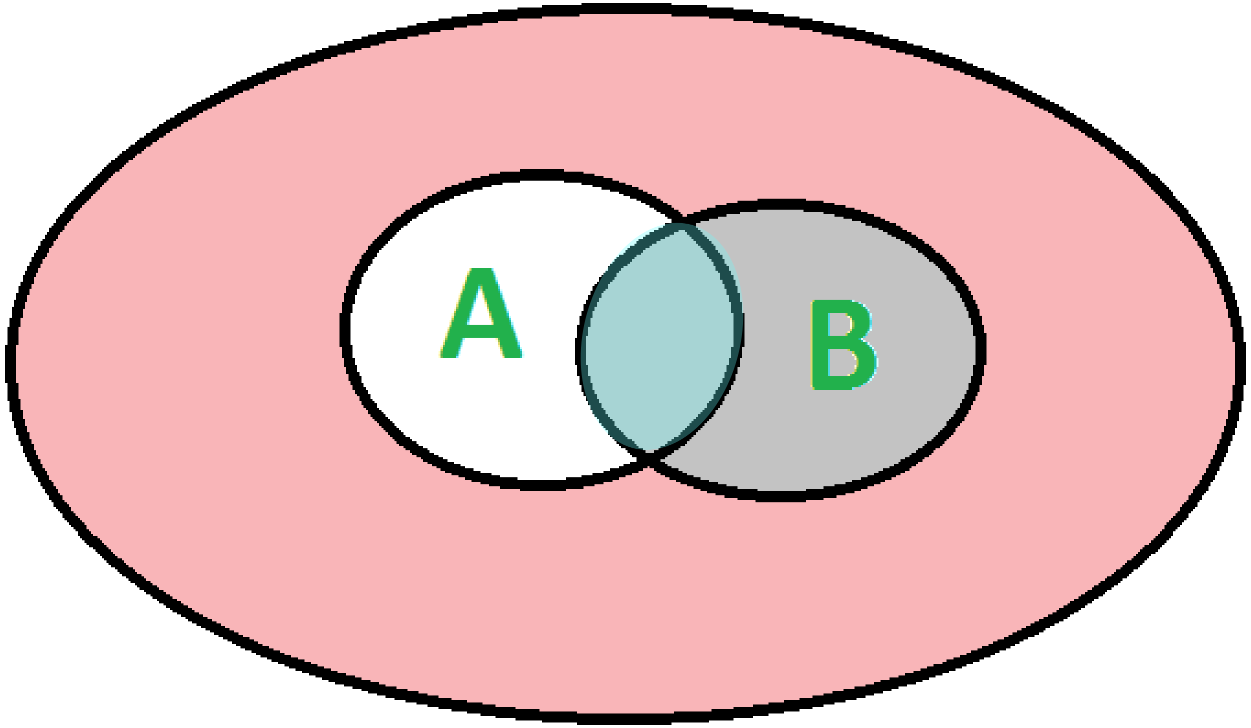
New Sample Space (After B happened)

Since B has already happened, the sample space reduces to B.

https://www.geeksforgeeks.org/conditional-probability/

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Not Possible to Happen as B already happened

New Sample Space (After B happened)

Since B has already happened, the sample space reduces to B.
=> Can think of this as redefining the sample space to be B, and then calculating the relative size of A within this new sample space.

https://www.geeksforgeeks.org/conditional-probability/

Example

- You purchase a certain product. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{\frac{-t}{5}}, \text{ for all } t \geq 0.$$

- For example, the probability that the product lasts more than (or equal to) 2 years is $P(T \geq 2) = e^{\frac{-2}{5}}$ =0.6703. If the product is purchased and used for two years without any problems, what is the probability that it breaks down in the third year?

Independence

- Initial assignment of A doesn't change given that event B occurred
- A and B are independent if

$$P(A|B) = P(A) \text{ and/or } P(B|A) = P(B)$$

Otherwise, they are dependent

- When A and B are independent,

$$P(A \cap B) = P(A)\, P(B)$$

General Multiplicative Law

- From the definition of the conditional probability:

$$P(A \cap B) = P(A|B)P(B)$$
$$P(A \cap B) = P(B|A)P(A)$$

- So if we extend this rule at three events, we obtain:

$$P(A \cap B \cap C) = P(A \cap B)P(C|A \cap B) = P(B|A)P(A)P(C|A \cap B)$$

- and if we extend it to n events, we obtain:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Bayes's theorem allows us to relate $P(A|B)$ to $P(B|A)$

- Provides a way to revise existing predictions or theories (update probabilities), $P(A|B)$, given new or additional evidence, $P(B|A)$

Note: $P(A|B) \neq P(B|A)$ unless $P(A \cap B) = 0$ or $P(A) = P(B)$

The Reverend
Thomas Bayes

Bayes' Theorem Physical Understanding

Likelihood:
How probable is the evidence given hypothesis is true?

Prior:
How probable was the hypothesis before observing the evidence?

$$\boldsymbol{P(H|e) = \frac{P(e|H)P(H)}{P(e)}}$$

Posterior:
How probable is the hypothesis given the observed evidence?

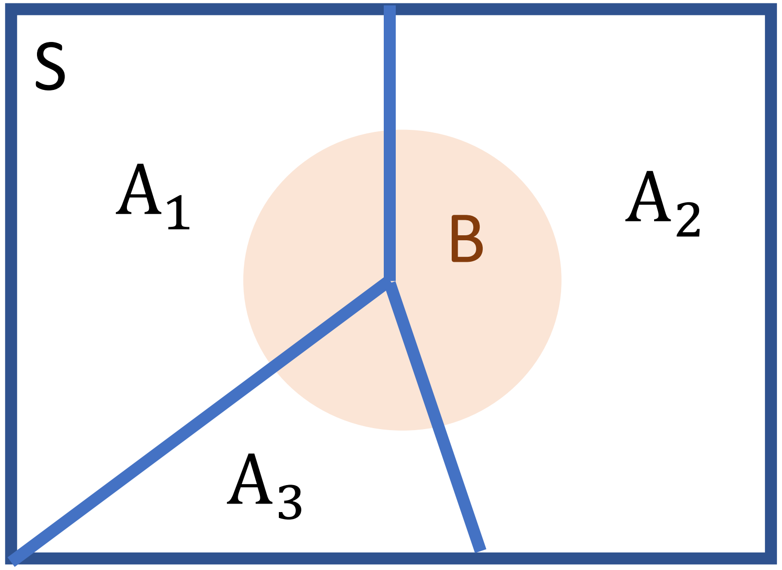
Marginal:
How probable is the new evidence under all possible hypothesis?

Law of Total Probabilities

- $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$

  where $A_1, \ldots, A_n$ is a partition of the sample space

  $$S(A_i \cap A_j = \emptyset \ for\ any\ i \neq j\ and\ A_1 \cup A_2 \cup \cdots A_n = S)$$



Bayes' Theorem

- Applying Law of Total Probabilities to Bayes' Theorem

$$P(H|e) = \frac{P(e|H)P(H)}{P(e)}$$

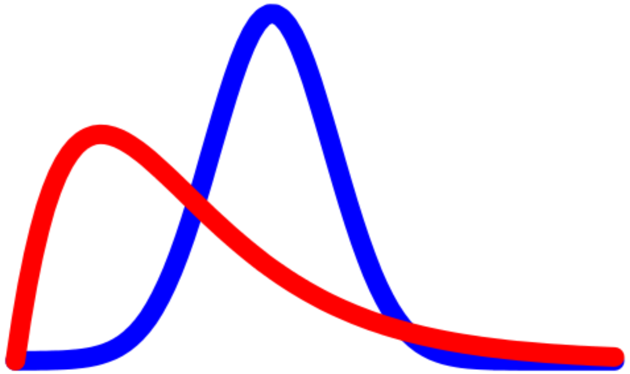
$$= \frac{P(e|H)P(H)}{P(H)P(e|H) + P(\neg H)P(e|\neg H)}$$

$\neg$ *means "not"*

Bayes' Theorem

- Given a priori probabilities (assignments) $P(A_1)$, $P(A_2)$, …, $P(A_n)$, (where A_1; …, A_n is a partition of the sample space S), we can compute (**update**) the posterior probabilities $P(A_i|B)$, for *i* = 1, …, *n* based on the conditional probabilities $P(B|A_i)$, for *i* = 1, …, *n* using:

$$\boxed{P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{k=1}^{n} P(B|A_k)\, P(A_k)}}$$

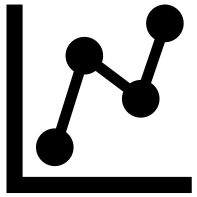


When to use Bayes’ Theorem

You have a hypothesis

?

You’ve observed evidence



You want

***P*(hypothesis given evidence)**

- Quantify and systematize the idea of changing beliefs or updating info
    - Science: analyzing extent to which new data validates or invalidates model
    - AI: explicitly and numerically model a machine's belief
    - Reframes how you think about thought itself

See: 3blue1brown for more intuitive info
https://www.youtube.com/watch?v=HZGCoVF3YvM&t=206s

Spam Mail Example

- It is estimated that 50% of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%. Now if an email is detected as spam, then what is the probability that it is in fact a non-spam email?

Random Variables

- A random variable, *RV*, is a mapping from the sample space S to the real line:

$$RV\colon S \to \mathbb{R}$$

- i.e., assigns a real number to every possible outcome in the sample space, where *S* is the sample space (the space of all possible outcomes) and $\mathbb{R}$ is called state space or range space.

- Application:
  - In a complex system
    - – the number of days until a part failure
    - – the number of parts that have failed today
    - – the number of customers affected by a failure

Notation

- The state space consists of the associated values of the outcomes in the sample space according to the rule *RV*:

$$RV(o_i) = x_i$$

through which we assign a numerical value x_i to each outcome o_i .

- Can also be written as:

$$P(\{X = k\}) \;\; for\; k = 2,3,4$$

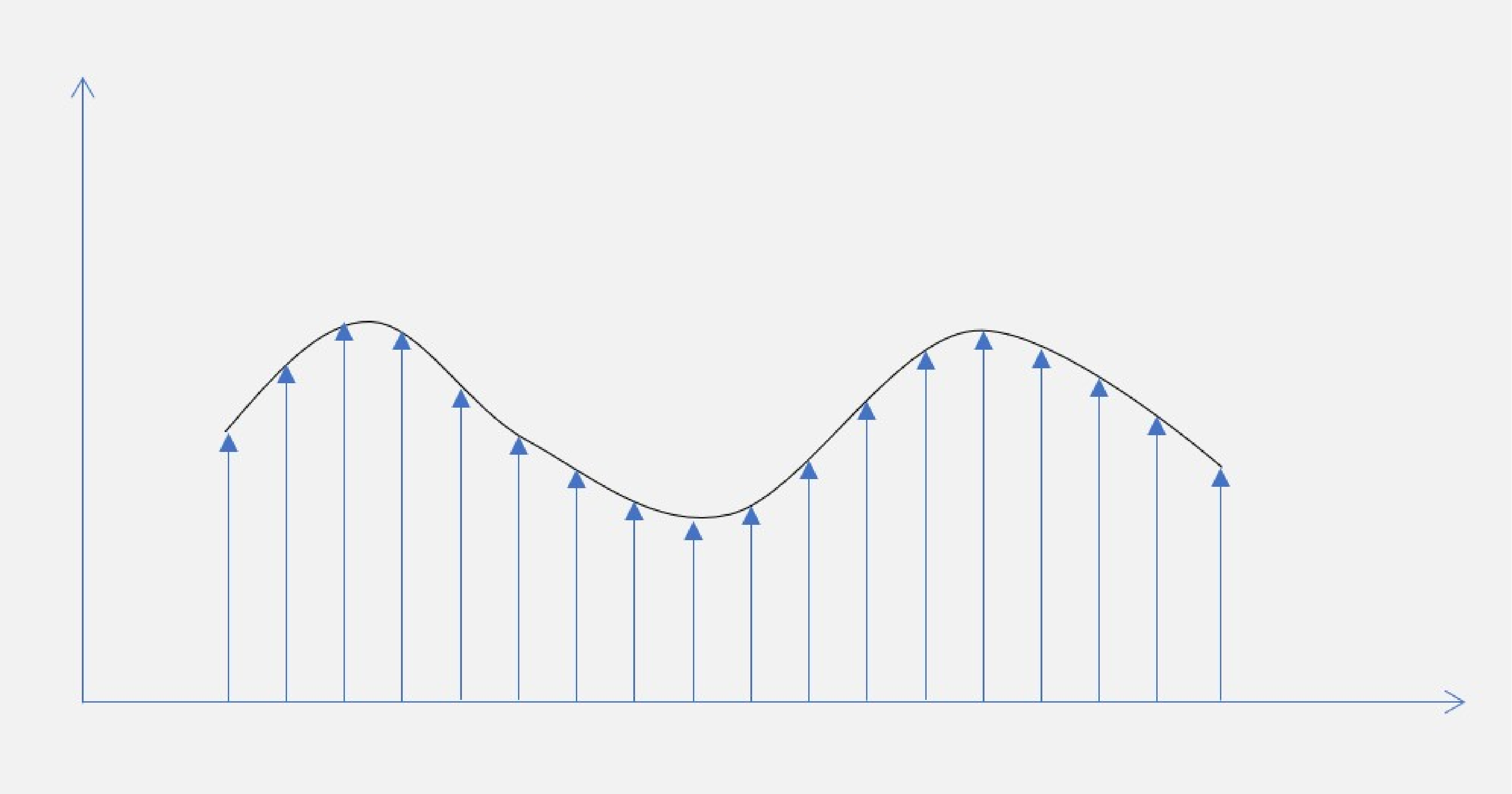
  - Capital letters: denote random variables
  - Lowercase letters: denote particular outcomes a random variable might take

Example

- In any experiment there are various characteristics that can be observed or measured. In most of the studies, an experimenter will focus on some aspect of the experiment.
- A researcher may test a sample of components from a production line and record only the number of components that have failed within 100 hours. In this example, for each component we observe, 0 (failed) or 1 (not failed).
- If observe one outcome to be 100 of 1’s (none of the components failed), call it o_{100}. Another outcome may be observing 90 of 1's (only 10 of the components failed), call it o_{90}, and so on. Here one random variable associates the frequency value 100 to o_{100} and the frequency value 90 to o_{90}:

$$RV(o_{100}) = 100, \;\; RV(o_{90}) = 90$$

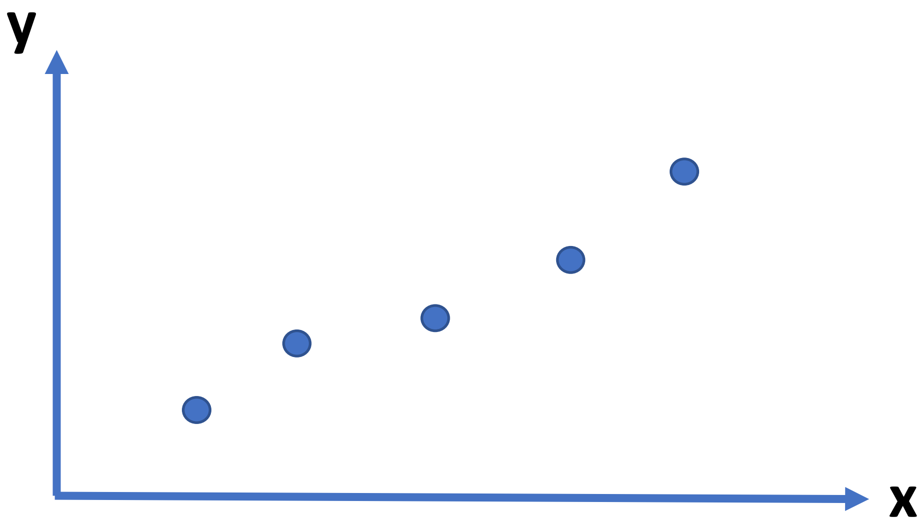
Discrete vs. Continuous Numbers



https://www.sigmamagic.com/blogs/is-my-data-continuous-or-discrete/

Discrete Random Variables

- For a discrete *RV*, the state space $\mathcal{R}$ is discrete or countable.
- Example of discrete space is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and an example of a countable space is the set of all integers $\mathbb{Z}$.



Probability Mass Function (PMF)

- For a discrete rv *X*, we define the probability mass function to be a function:

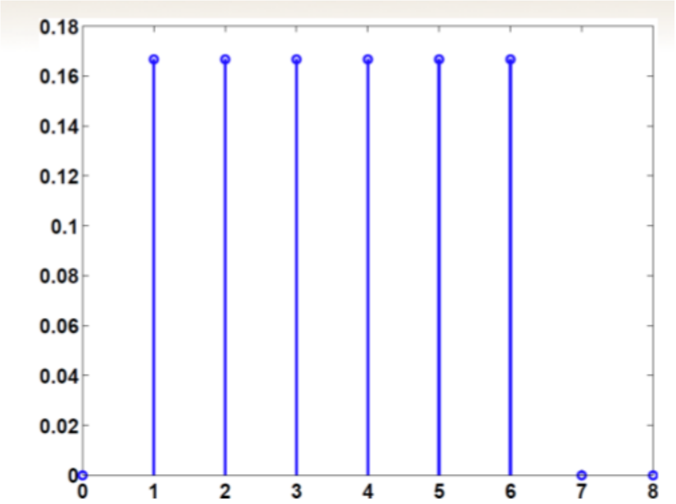
$$P_x\colon \mathcal{R} \to [0,1]$$

  for a state space $\mathcal{R} = \{x_1, x_2, \cdots, x_n\}$ such that

$$P_x(x_i) = p_i, \qquad i = 1, \cdots, n$$

- The values p_i for $i = 1, \ldots, n$ are such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$
- This is a simple function that tells us the probability of each possible outcome
- Used for **<u>discrete</u>** probability distributions

Roll of a fair die



$$p_X(k) = \begin{cases} \frac{1}{6} & k = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$$

Cumulative Distribution (CDF for discrete RV)

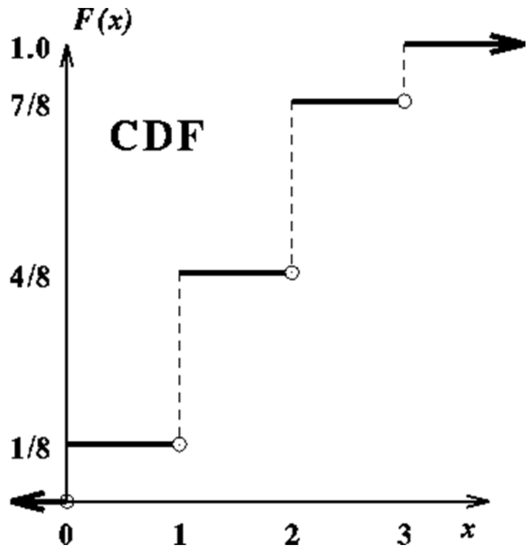
- For a discrete rv X, we define the cumulative distribution function (cdf) a function:

$$F_x\colon \mathcal{R} \to [0,1]$$

for a state space $\mathcal{R} = \{x_1, x_2, \cdots, x_n\}$ such that

$$F_x(x) = P_x(X \le x) = \sum_{x_i \le x} P_x(X = x_i)$$

- In other words, the cdf is the probability that a random variable is less than or equal to a certain real number



https://www.ibiblio.org/links/devmodules/probstat/concepts/html/cointoss.html

Mean

- For a discrete rv X with a state space $\mathcal{R} = \{x_1, x_2, \cdots, x_n\}$ and pmf P_X , we define its expectation (expected value) or mean to be a weighted average of the values in the state space:

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$$

- The expectation is a measure for the average value in the state space
  - a "weighted average" over all the values X can take, where the weights are given by the probabilities of each of those values
- Note: expectation is not random
  - it is a deterministic function of $P(X = x_i)$
- Note: expectation does not need to be one of the possible outcomes

Example

- Apple is interested in hiring you to do some consulting for them. They have two projects they would like your help with, but they will only pay you if they are satisfied with your work.
  - Project 1 pays $1000 and you believe that the probability you will complete the project to Apple's satisfaction is 0.8.
  - Project 2 pays $2000 and you believe that the probability you will complete the project to Apple's satisfaction is 0.5.
- If Apple is happy with your work on whichever project you choose to do first, they will give you the chance to do the second project, but if they don't like your work they will send you on your way. Which project should you take first to maximize your expected earnings?

Variance

- Expectation tells us about the average outcome, but also often need to know how likely rv is close to the average outcome
- For a discrete *X* with a state space $\mathcal{R} = \{x_1, x_2, \cdots, x_n\}$ and pmf P_X, we define its variance to be:

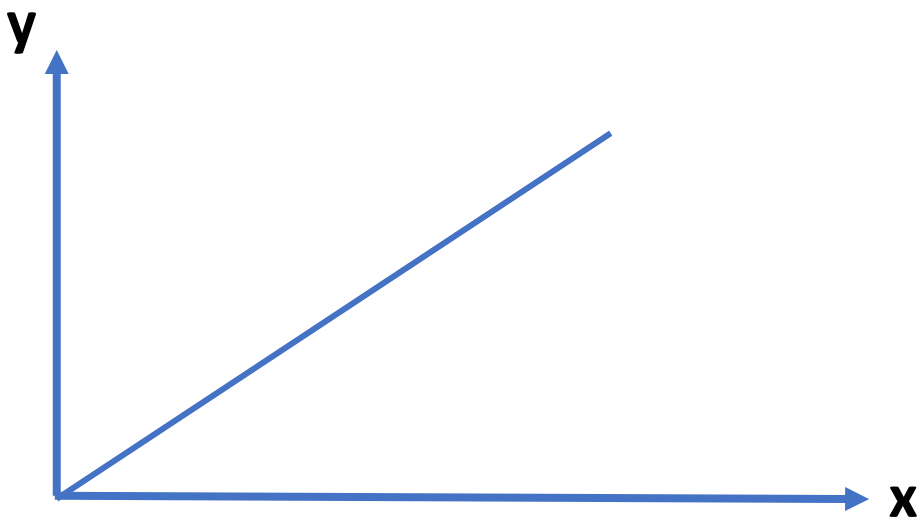
$$V(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^2 P(X = x_i)$$

$$= \sum_{i=1}^{n} {x_i}^2 P(X = x_i) - \left(\sum_{i=1}^{n} x_i P(X = x_i)\right)^2$$

- The variance is a measure for the variability or spread in the state space.
- Describes how much a random variable differs from its expected value.
- The standard deviation of *X* is $\sqrt{V(X)}$
  - Standard deviation is often easier to interpret since it has the same units as *X*

Continuous Random Variables

- For a continuous rv, the state space $\mathcal{R}$ is infinite. Examples of infinite spaces are [0, 1] and the set of all real numbers.



Probability Density Function (PDF)

- For a continuous rv X for a state space R, we define its probability density function (pdf) to be a function:

$$f_x \colon \mathcal{R} \to \mathbb{R}$$

such that

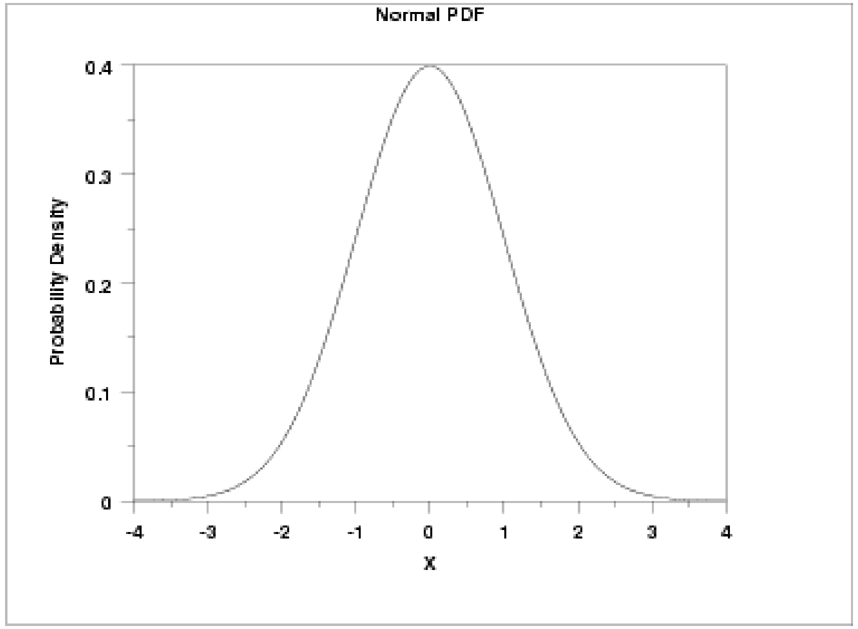
(1) $f(x) \geq 0$

(2) $\int f(x)dx = 1$

- We define the density area above an interval [a, b] for a ≤ b to be:

$$P(a \leq X \leq b) = \int_a^b f_x(x)dx$$

- Used for **continuous** probability distributions



https://www.itl.nist.gov/div898/handbook/eda/section3/eda362.htm

Cumulative Distribution Function (CDF for continuous RV)

- For a continuous rv X with a state space R, we define its cumulative distribution function (cdf) a function:

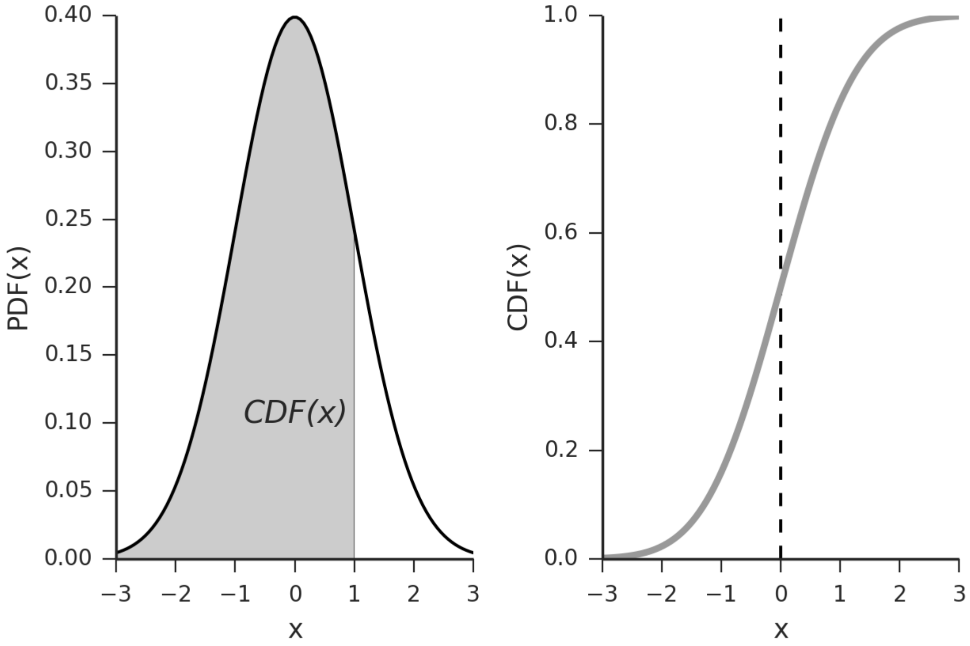
$$F_x\colon \mathcal{R} \rightarrow [0,1]$$

such that

$$F_x(x) = P(X \leq x) = \int_{-\infty}^{x} f_x(x)dx$$

- One important property of the cdf is that its first derivative is:

$$F_x'(x) = f_x(x)$$



http://work.thaslwanter.at/Stats/html/statsDistributions.html

Mean

- For a continuous rv X with a state space R and pdf f_x, we define its expectation or mean to be:

$$E(X) = \int x f_x(x) dx$$

The expectation is a measure for the average value in the state space.

Example

- The time to failure in thousands of hours of an important piece of electronic equipment used in a manufactured radio has the density function

$$f(x)=\begin{cases}3e^{-3x}, & x>0\\ 0, & otherwise\end{cases}$$

- Find the expected life of the piece of equipment.

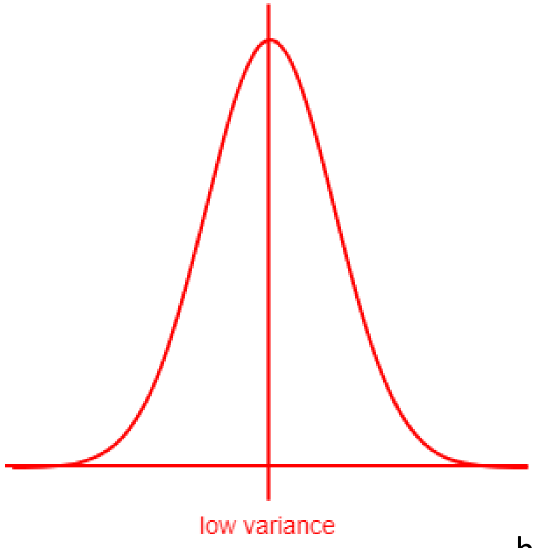
https://www.brainkart.com/article/Mathematical-Expectation--Solved-Example-Problems_38977/

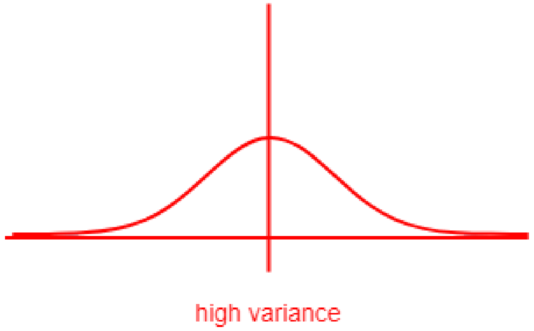
Variance

- For a continuous rv X with a state space R and pdf $f(x)$, we define its variance to be:

$$V(X) = \int \left(x - E(X)\right)^2 f(x)dx = \int x^2 f(x)dx - \left(\int xf(x)dx\right)^2$$

- The variance is a measure for the variability or spread about the mean in the state space. The standard deviation of X is $\sqrt{V(X)}$.





https://nowke.github.io/stats/normal-distributions/

Properties of Mean and Variance

- Useful properties for a random variable X, with constansts $a, b \in \mathbb{R}$:
  - $E[X + b] = E[X] + b$
  - $E[aX] = aE[X]$

- Can combine above two properties into one statement:
  - $E[aX + b] = aE[X] + b$

Properties of Mean and Variance

- One important property for variance for both discrete and continuous distributions is:

$$V(X) = E(X^2) - (E(X))^2$$

- Proof:
- $V(X) = E[(X - E[X])^2]$ (original definition of variance)

$$= E[(X^2 - 2XE(X)) + (E[X])^2]$$

$$= E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$= E(X^2) - (E(X))^2$$

Properties of Mean and Variance

- $Var(X + c) = Var(X)$
- Proof:

$$\begin{aligned}\mathrm{Var}(X+c) &= E\left[(X+c)^2\right] - E(X+c)^2 \\ &= E\left(X^2 + 2cX + c^2\right) - E(X+c)E(X+c)\end{aligned}$$

Expanding the first term,

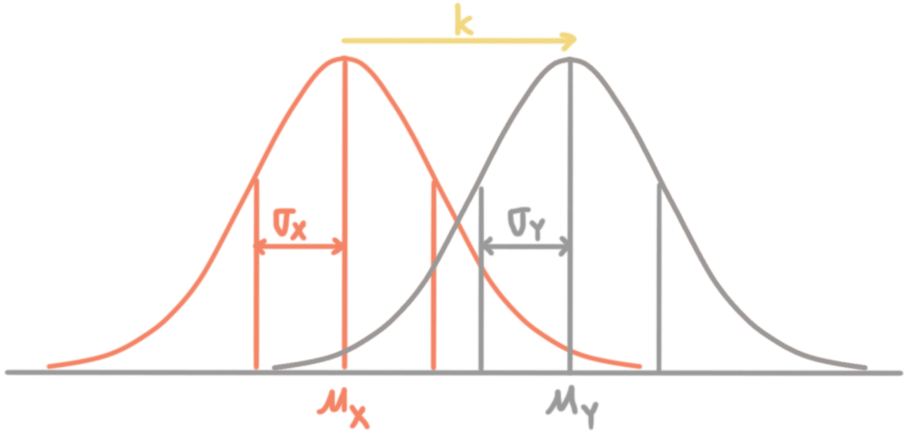
$$E\left(X^2 + 2cX + c^2\right) = E\left(X^2\right) + 2cE(X) + c^2$$

Expanding the second term,

$$\begin{aligned}E(X+c)E(X+c) &= E(X)E(X+c) + E(c)E(X+c) \\ &= E[XE(X) + cE(X)] + cE(X) + cE(c) \\ &= E(X)^2 + cE(X) + cE(X) + c^2 \\ &= E(X)^2 + 2cE(X) + c^2\end{aligned}$$

Putting it all together,

$$\begin{aligned}\mathrm{Var}(X+c) &= E\left(X^2\right) + 2cE(X) + c^2 - E(X)^2 - 2cE(X) - c^2 \\ &= E\left(X^2\right) - E(X)^2 \\ &= \mathrm{Var}(X)\end{aligned}$$



- Since the variance measures the amount of spread of the distribution, shifting the distribution left or right by a constant doesn't affect that spread and therefore shouldn't affect the variance.

https://www.kristakingmath.com/blog/shifting-and-scaling-data-sets
https://mbernste.github.io/files/notes/Variance.pdf

Properties of Mean and Variance

- $Var(cX) \ = c^2\, var(X)$
- Proof:

$$\begin{aligned}\mathrm{Var}(cX) &= E\left[(cX)^2\right] - E(cX)^2 \\ &= c^2E\left(X^2\right) - c^2E(X)^2 \\ &= c^2[E\left(X^2\right) - E(X)^2] \\ &= c^2\mathrm{Var}(X)\end{aligned}$$

https://mbernste.github.io/files/notes/Variance.pdf

Example

- A fisherman is weighing each of 50 fishes. Their mean weight worked out is 50 gm and a standard deviation of 2.5 gm. Later it was found that the measuring scale was misaligned and always under reported every fish weight by 2.5 gm. Find the mean and standard deviation of fishes.

Moments of a Distribution

- We define the k^{th} moment of a distribution X (for both discrete and continuous distributions) to be the expectation of X^k .

- For a discrete distribution, the k^{th} moment is

$$E(X^k) = \sum_{x \in \mathcal{R}} x^k P(X = x)$$

- For a continuous distribution, the kth moment is

$$E(X^k) = \int x^k f_x(x) dx$$

Combination and functions of random variables

- If Y is a random variable such that it can be expressed and the form:

$$Y = aX + b$$

where X is a random variable then we have:

$$E(Y) = aE(X) + b$$
$$V(Y) = a^2 V(X)$$

- For a sequence of rv's $X_1, X_2, \ldots, X_n$:

$$E(a_1 X_1 + \cdots + a_n X_n) = aE(X_1) + \cdots + aE(X_n)$$

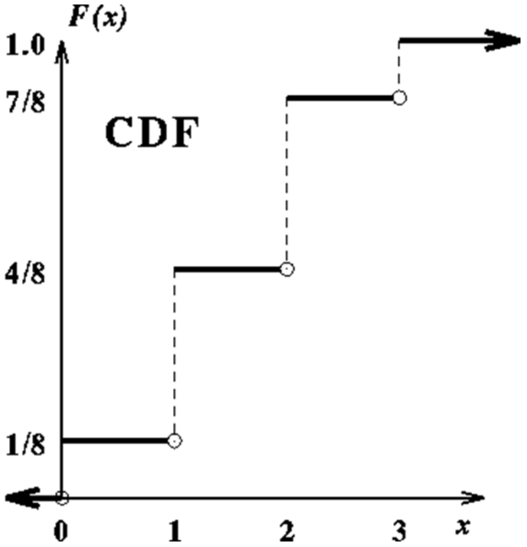
Summary

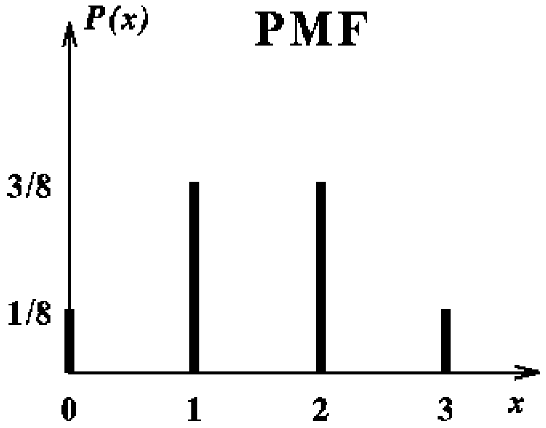
- Independence: $P(A \cap B) = P(A)\ P(B)$
- Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Bayes' Theorem : $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$
  - Used to update pre-existing condition
  - $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Summary

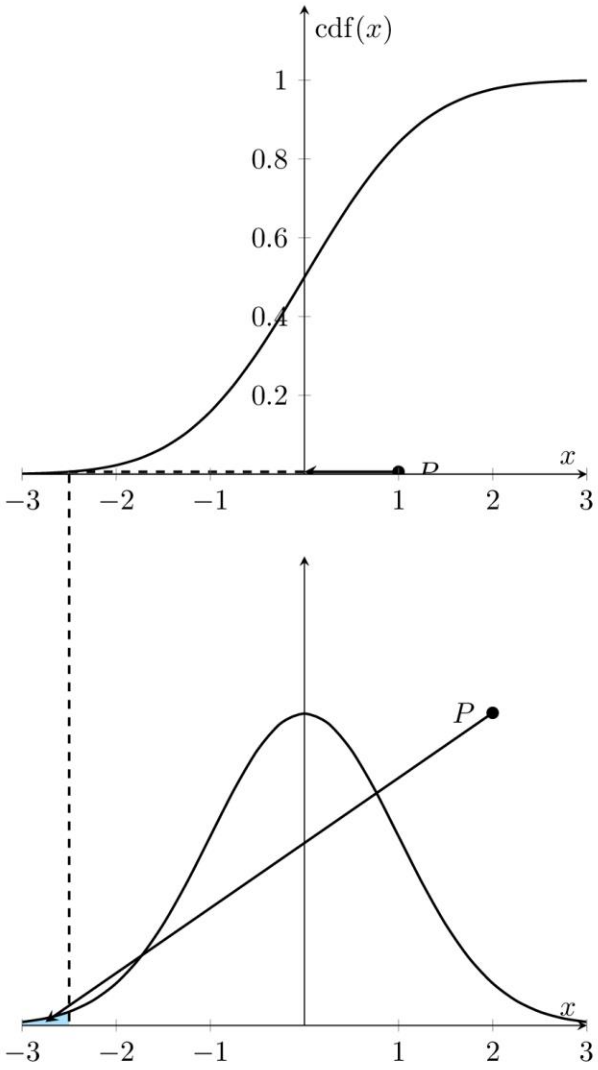
- Cumulative distribution function (CDF): probability that a random variable is less than or equal to a certain real number
- Probability mass function(PMF): probability that a discrete random variable equals a specific value or density for a discrete random variable
- Probability distribution function (PDF): describes the probability distribution of a continuous random variable (relative probability)

Discrete Random Variable





Continuous Random Variable



https://tex.stackexchange.com/questions/515670/probability-density-function-and-cumulative-distribution-function-for-normal-dis
https://www.ibiblio.org/links/devmodules/probstat/concepts/html/cointoss.html

Summary

- Useful equations:
  - $Var(X) = E(X^2) - (E(X))^2$
  - $E[aX + b] = aE[X] + b$
  - $\text{V}ar(X \ + c) = Var(X)$
  - $Var(cX) \ = c^2 \, var(X)$