可靠度資料分析
Reliability Data Analysis

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Lecture 3 –Reliability Basics

Class Presentation (After Midterm)

- Start thinking about a topic to pick
  - Submit topic title by 3/11/2024 on Moodle
- Apply what you learn to actual applications
- Each person presents 20min presentation on topic of choice
  - Can be application of reliability your research
  - Or, pick topic of interest related to reliability
- Research current state of reliability
- Identify future research / potential solutions to improve reliability
- Presentation format:
  - 15 min talk
  - 5 min Q & A
  - Either English or Chinese

Example Topics

- Software reliability
- Reliability in artificial intelligence
  - Ex. Explainable AI
- Reliability in electronic design automation/circuit testing
- Predictive maintenance
- Control charts
- Data sampling in clinical trials
- Testing in your research area
- Etc…..

Presentation Example Outline

Story Arc
Problem statement
Why important
Background
Issues with current practice
Potential solutions
Recommendations
Future work

Example Part 1 of 3

From researcher in device radiation reliability:

Story Arc	Example of material discussed in presentation
Problem statement	Evaluating impact of new device geometries on sensitivity to ionizing radiation
Why important	Traditional scaling techniques no longer practical - forcing designers to develop new device geometries (example FinFET, 3D, etc).
Background	Characterization of ionizing radiation on traditional device geometry has been well documented, but sensitivity of new geometries presents a risk

Note: Topics related to reliability may not necessarily have the word “reliability” in the subject

Example Part 2 of 3

Story Arc	Example of material discussed in presentation
Issues with current practice	Traditional approach was to run radiation tests after devices have been fabricated and document sensitivity. Additional system level features could be implemented to detect and correct single bit errors. If new geometries are too sensitive, then system level mitigations may not be enough to mitigate device level failure rates.
Potential solutions	In order to rapidly evaluate potential device geometries for radiation, these options were evaluated: [1] development of special test structures [2] creation of tool to model radiation performance based on geometry

Example Part 3 of 3

Story Arc	Example of material discussed in presentation
Recommendations	Based on cost and schedule impacts, it was determined that development of test structures offered the lowest risk of implementation and provides the most realistic results to provide rapid feedback to the design team
Future work	Development of model tool should be investigated, and test structure results could provide validation of model accuracy

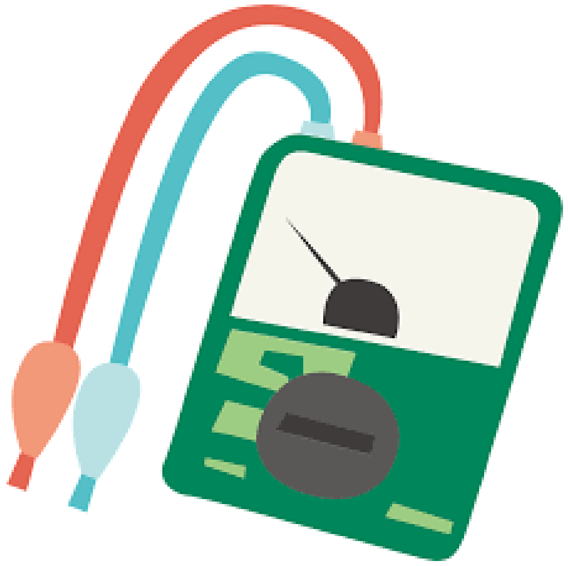
Grading Rubric

Presenter Name:

Element	**Value**	**score**	**Comments**
Clarity of presentation	1 to 10		
Technical content	1 to 10		
Delivery	1 to 10		
Duration (15 min + 5 min Q&A)	1 to 5		
Fielding questions	1 to 5		
Total	40		

Where to find literature…

- Example: Semiconductor Reliability
  - Conferences:
    - IEEE International Reliability Physics Symposium (IRPS)
    - European Symposium on Reliability of Electron Devices Failure Physics and Analysis (ESREF)
    - IEEE International Symposium on the Physical and Failure Analysis of Integrated Circuits (IPFA)
  - Journals:
    - IEEE Transactions on Very Large Scale Integration (VLSI) Systems
    - Can also be found in other IEEE Transactions related journals in semiconductors, devices, and circuits
    - Microelectronics Reliability



Where to find literature…

- Example: EDA/Circuit Reliability Testing
  - Conferences:
    - International Test Conference (ITC)
    - IEEE VLSI Test Symposium (VTS)
    - ACM/IEEE Design Automation Conference (DAC)
    - Asia and South Pacific Design Automation Conference (ASP-DAC)
    - Design, Automation and Test in Europe Conference (DATE)
    - IEEE European Test Symposium (ETS)
    - IEEE Symposium on VLSI Technology and Circuits (VLSI Technology and Circuits)

Where to find literature...

- Example: Reliability (More Statistics-oriented)
  - Journals:
    - IEEE Transactions on Reliability

Last week

- Reviewed probability concepts for single random variables
- What if there are more than one random variable?

Joint Random Variables

- Many problems in statistics and probability lead to models involving more than one random variable simultaneously. In these cases, we want to jointly model the behavior of two or more distributions.

- In this review, we will consider two cases:
  - Joint Discrete Random Variables
  - Joint Continuous Random Variables

Joint Discrete Random Variables

- For two discrete rv's X : $S_X \rightarrow RX$ and Y: $S_Y \rightarrow RY$, we define the joint probability mass function to be a function p : $R_X \times RY \rightarrow [0, 1]$ such that:

$$p(x, y) = P(X = x \text{ and } Y = y).$$

- For any set A consisting of pairs (x, y) define:

$$P\big((X, Y) \in A\big) = \sum_{(x,y)\in A} p(x, y).$$

Joint Discrete Random Variables

- We define the joint cumulative distribution function F : $R_X \times RY \rightarrow$ [0, 1] such that:

joint probability mass function

$$F(x, y) = P(X \leq x \, and \, Y \leq y) = \sum_{s \leq x, t \leq y} p(s, t)$$

- Also we define the marginal probability functions of X and Y and denote p_X, and, respectively, p_Y:

$$p_x(x) = \sum_{y \in R_y} p(x, y)$$

$$p_y(y) = \sum_{x \in R_x} p(x, y)$$

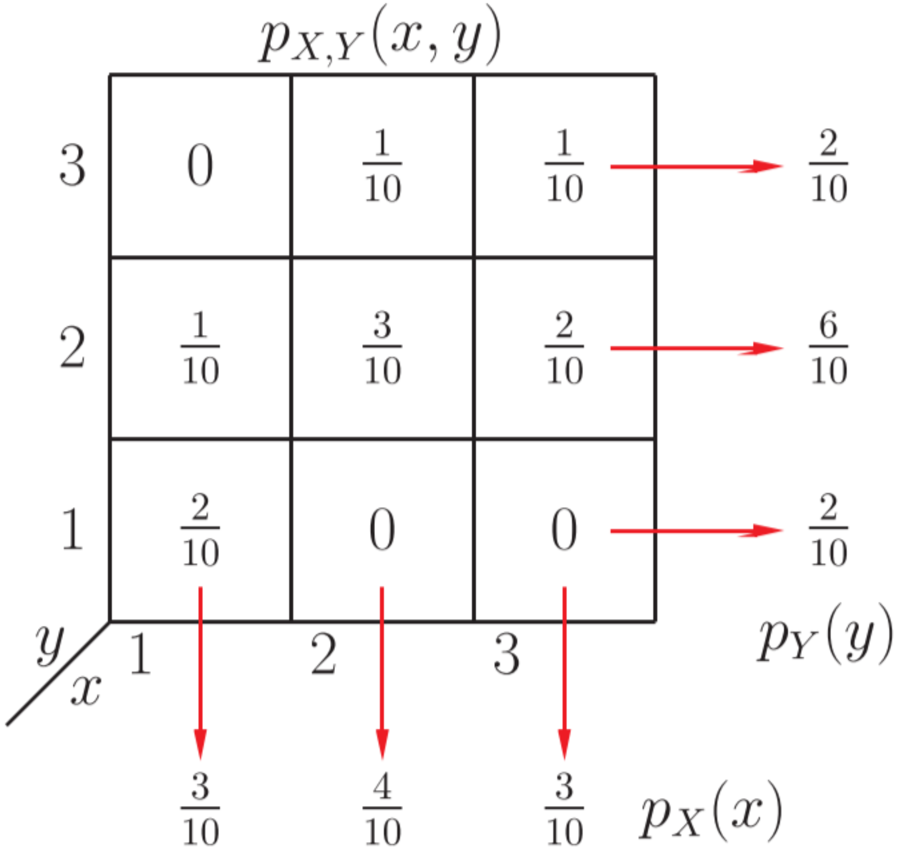
Example

$$p_{X,Y}(x,y) = \mathrm{P}\left(\{X = x\} \cap \{Y = y\}\right) = \mathrm{P}\left(X = x, Y = y\right)$$

Sums to 1 $\rightarrow \sum_{x,y} p_{x,y}(x,y) = 1$ joint probability mass function

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$



Joint Discrete Random Variables

- Similar to conditional probabilities, given the distribution of the rv X we can update/revise its distribution if partial information is available from other distributions. Therefore, given two rv's X and Y and their joint pmf we can define the conditional distribution of X given Y and denote X | Y as follows:

$$p(x|y) = P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}$$

joint probability mass function

marginal probability function

Joint Continuous Random Variables

- For two continuous rv's $X : S_X \rightarrow RX$ and Y: $S_Y \rightarrow RY$, we define the joint probability density function a function $f : RX \times RY \rightarrow \mathbb{R}$ such that:

$$f(x, y) \geq 0$$

$$\int_{R_x R_Y} f(x, y) dx dy = 1$$

- For any set A consisting of pairs (x, y) define:

$$P\big((X, Y) \in A\big) = \iint_A f(x, y) dx dy$$

Joint Continuous Random Variables

- We define the joint cumulative distribution function $F : RX \times RY \rightarrow$ [0, 1] such that:

$$F(x, y) = P(X \leq x \; and \; Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) dx dy$$

- Also we define the marginal probability functions of X and Y and denote f_X, and, respectively, f_Y:

$$f_{\mathrm{X}}(x) = \int_{R_Y} f(x, y) dy$$

$$f_Y(x) = \int_{R_X} f(x, y) dx$$

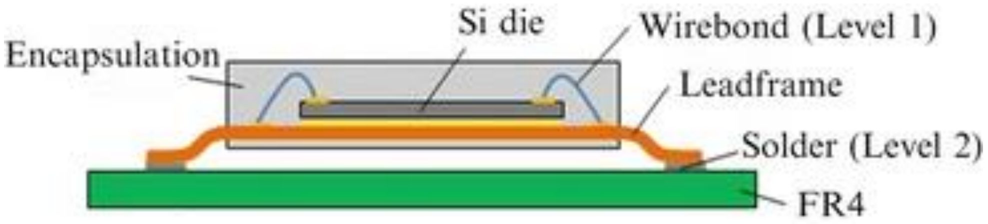
Example

- You are estimating the failure time for the interconnect and solder joint to fail in product testing. Let X be the time it takes the interconnect to fail (in minutes) and Y be the time it takes the solder joint to fail, and suppose X and Y have the joint pdf

$$f_{X,Y}(x.y) = \begin{cases} 2e^{-(x+2y)} & x, y \geq 0 \\ 0 & otherwise \end{cases}$$

- What is the probability that both the interconnect and solder joint fail in the next minute?

Example of interconnect (wire) and solder joint in an electronic product



Nai, S.M.L., Tan, L.B., Selvanayagam, C. (2014). Solder Joint Technology. In: Nee, A. (eds) Handbook of Manufacturing Engineering and Technology. Springer, London. https://doi.org/10.1007/978-1-4471-4976-7_59-1

Example

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$$f_{X,Y}(x.y) = \begin{cases} 2e^{-(x+2y)} & x, y \geq 0 \\ 0 & otherwise \end{cases}$$

- What is the probability that both the interconnect and solder joint fail in the next minute?
- **<u>Solution:</u>**
- The event "both the interconnect and solder joint fail in the next minute" is

$$A = \{(X \leq 1)\ and\ (Y \leq 1)\}$$

- Thus

$$P(A) = \int_0^1 \int_0^1 f_{X,Y}(x.y)dxdy$$
$$= 2\int_0^1 \int_0^1 e^{-(x+2y)}\ dxdy$$
$$= 2\left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{-2y}dy\right)$$
$$= (1 - e^{-1})\ (1 - e^{-2}) \approx 0.5466$$

Joint Continuous Random Variables

- For two continuous rv's X and Y and their joint pdf's we can define the conditional distribution of X given Y and denote X | Y as follows:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

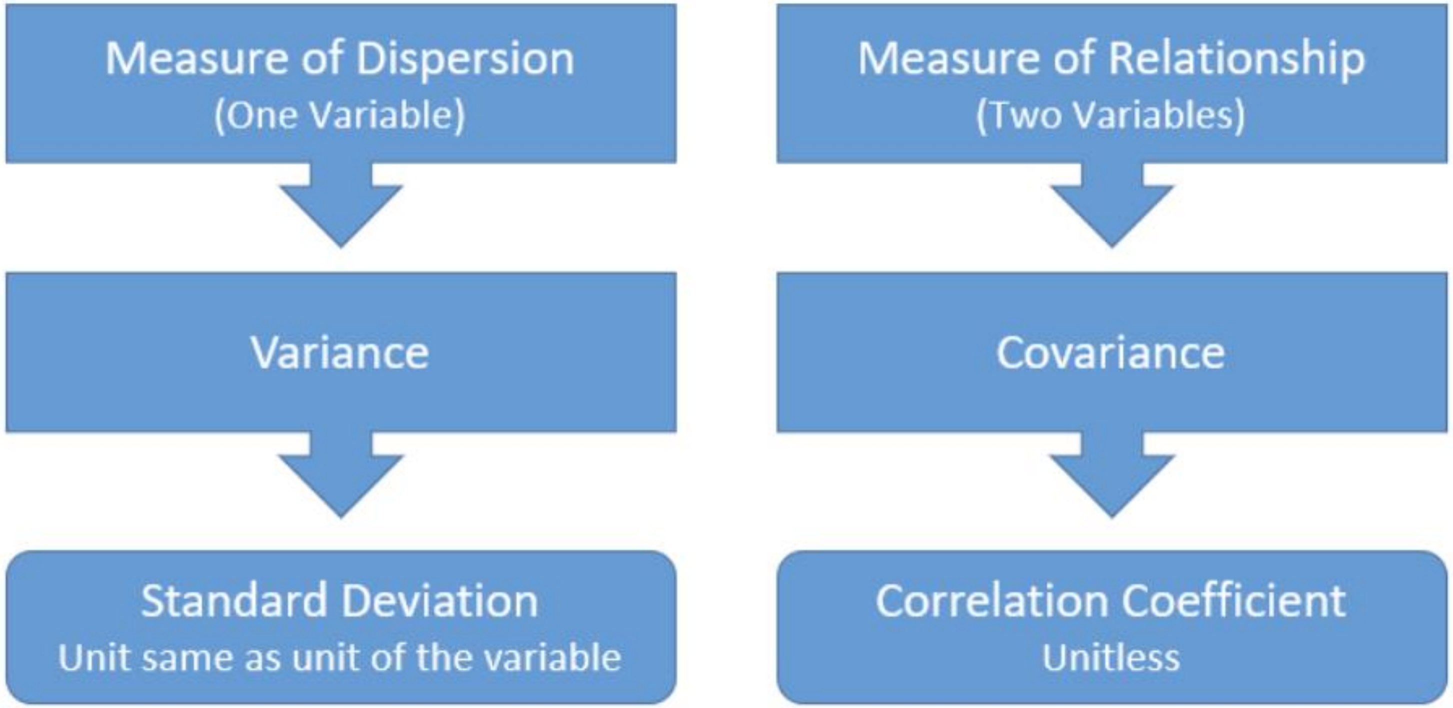
Covariance, Correlation, and Independence

- Given the joint distributions of X and Y, we define their covariance: $cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E[XY] - E[X]E[Y]$
- and we define their correlation:

$$cor(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}}$$

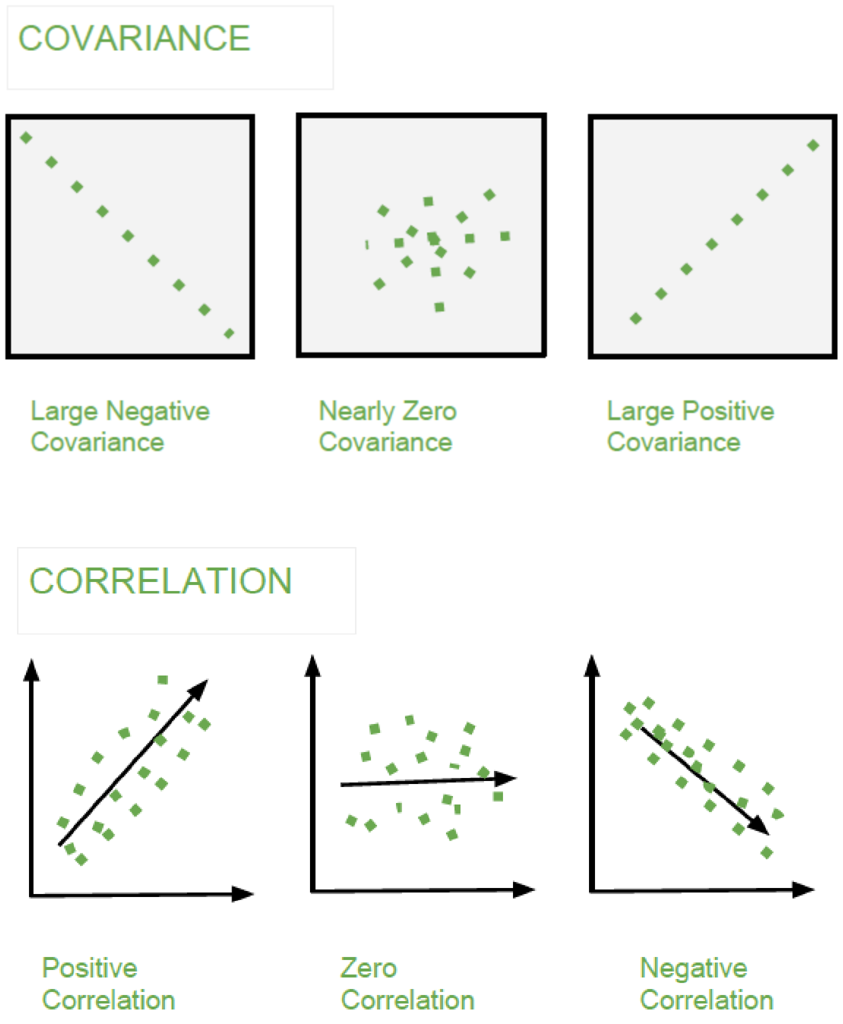
- Two rv's are independent if $cov(X,Y) = 0$ or $cor(X,Y) = 0$. Also we can define independence in terms of the marginal and joint distributions (see next slide).

Variance vs. Covariance



https://www.k2analytics.co.in/covariance-and-correlation/

Covariance vs. Correlation



Covariance	Correlation
Covariance is a measure of how much two random variables vary together	Correlation is a statistical measure that indicates how strongly two variables are related.
involve the relationship between two variables or data sets	involve the relationship between multiple variables as well
Lie between -infinity and +infinity	Lie between -1 and +1
Measure of correlation	Scaled version of covariance
provide direction of relationship	provide direction and strength of relationship
dependent on scale of variable	independent on scale of variable
have dimensions	dimensionless

https://www.geeksforgeeks.org/mathematics-covariance-and-correlation/

Covariance, Correlation, and Independence

- Two discrete rv's X and Y with joint pmf p(x, y) and marginal pmf's $p_X(x)$ and $p_Y(y)$ are independent if:
$$p(x, y) = p_X(x)p_Y(y)$$
- Two continuous rv's X and Y with joint pdf f(x, y) and marginal pdf's $f_X(x)$ and $f_Y(y)$ are independent if:
$$f(x, y) = f_X(x)f_Y(y)$$
- Based on the definition above, two rv's with pmf's or pdf's f_X and f_Y are independent if $f(x \mid y) = f(x)$ (that is, the conditional pmf or pdf of X | Y does not depend on y).

Moving to more reliability specific concepts….

Reliability Function

- Suppose n_o identical components are tested under their designed operating conditions. Furthermore. let us assume that during the interval $(t \ - \Delta t, t)$, we observed $n_f(t)$ failed components, and $n_s(t)$ surviving components such that $[nf(t) \ + \ ns(t) \ = \ no]$
- Reliability at time t, R(t), is defined as the cumulative probability function of success, then;

$$R(t) = \frac{n_s(t)}{n_s(t) + n_f(t)}$$

- If T is a random variable denoting the time to failure, then the reliability function at time t can be expressed as;

$$R(t) = P(T > t)$$

Often called the SURVIVAL FUNCTION!

Reliability, CDF, & PDF

- The cumulative distribution function CDF of failure, F(t) is the complement of R(t), that is;

$$R(t) + F(t) = 1$$

- If T has a probability density function (p.d.f), f(t), then,

$$R(t) = 1 - F(t) = 1 - \int_0^t f(v)dv$$

- Taking the derivative of the above equation with respect to t, we get,

$$\frac{dR(t)}{dt} = -f(t)$$

Mean Time to Failure

- One of the key measures of a system's reliability is the MTTF.
  - Note that MTTF is usually used when the system is nonrepairable. For repairable systems, the failure time between two successive failures is usually referred to as MTBF (Between)

- Consider n identical nonrepairable systems and their time to failure are given by $t_1, t_2, \ldots, t_n$. Then the mean time to failure is given as,

$$\widehat{MTTF} = \frac{1}{n}\sum_{i=1}^{n} t_i$$

Mean Time to Failure

- If t is a random variable representing time to failure, then the Mean-Time-To-Failure, MTTF can be defined as follows:

$$MTTF = E[T] = \int_0^\infty tf(t)dt$$

$$\mu = \int_0^\infty xf(x)dx$$
$$\mu = E[X]$$

- Another measure that is often used to describe the distribution of the time to failure is its variance σ^2

$$\sigma^2 = \int_0^\infty (MTTF)^2 f(t)dt$$

$$\sigma^2 = \int_0^\infty (x-\mu)^2 f(x)dx$$

$$\sigma^2 = \int_0^\infty t^2 f(t)dt - (MTTF)^2$$

Mean Time to Failure

- If t is a random variable representing time to failure, then

$$MTTF = \int_0^{\infty} tf(t)dt$$

Integration by parts

$$\int udv = uv - \int vdu$$

- Recall that $f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$, thus

$$MTTF = \int_0^{\infty} t(-\frac{dR(t)}{dt})dt = -\int_0^{\infty} tdR(t) = -tR(t)|_0^{\infty} + \int_0^{\infty} R(t)dt$$

- Since $R(\infty) = 0$ and $R(0) = 1$, we have,

$$MTTF = \int_0^{\infty} R(t)dt$$

Failure Rate

- The probability of failure of a component in a given interval of time $[t_1, t_2]$ in terms of its reliability function is given as

$$\int_{t_1}^{t_2} f(t)dt = R(t_1) - R(t_2)$$

- Failure Rate in the interval $[t_1, t_2]$ is defined as the probability that a failure per unit time occurs in the interval given that no failure occurred prior to t_1, in other words;

$$\frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)}$$

This is conditional probability given $R(t_1)$

- By replacing t_1 and t_2 with t and $t + \Delta t$, we can rewrite the above equation as follows,

$$\frac{R(t) - R(t + \Delta t)}{\Delta t R(t)}$$

Failure Rate

- Note: There might be other definitions of failure rate in literature. Please check definition before usage.
  - Ex. In industry, it is common to use the definition:
    - Failure rate is defined in failures per hour

Hazard Rate

- The Hazard Function is defined as the limit of the failure rate as Δt approaches zero, i.e., it is the **instantaneous Failure Rate**.

$$\lambda(t) = h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)}$$

$$= \lim_{\Delta t \to 0} \frac{-[R(t+\Delta t)+R(t)]}{\Delta t} \frac{1}{R(t)} = \frac{-dR(t)}{dt} \frac{1}{R(t)} = \frac{f(t)}{R(t)}$$

- The cumulative hazard function H(t) is the condition probability of failure over a period t. It is also the total number of failures during the time interval 0 to t

$$L(t) = H(t) = \int_0^t h(v)dv$$

Hazard Rate

- **A particular hazard rate function will uniquely determine a reliability function.**
  - A.k.a. Can use the hazard rate function to figure out the reliability function
  - To see this, let:

$$\lambda(t) = \frac{-dR(t)}{dt}\frac{1}{R(t)} \text{ or } \lambda(t)dt = \frac{-dR(t)}{R(t)}$$

Integrating, $\int_0^t \lambda(t')dt' = \int_1^{R(t)} \frac{-dR(t')}{R(t')}$,

where $R(0) = 1$ establishes the lower limit in the integral

Then

$$-\int_0^t \lambda(t')dt' = \ln R(t)$$

Or

$$R(t) = \exp[-\int_0^t \lambda(t')dt']$$

Can be used to derive the reliability function from a known hazard rate function

Hazard Rate

- Commonly used form of hazard rate:

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)}$$

- Also called the instantaneous (conditional) failure rate
  - The failure rate is sometimes called a "conditional failure rate" since the denominator $1 - F(t)$ (i.e., the population survivors) converts the expression into a conditional rate, given survival past time t.

Average Hazard Rate

- Another useful function is the Average Hazard Rate, denoted as $AHR(t_1, t_2)$, and defined as,

$$AHR(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \lambda(t') dt' = \frac{lnR(t_1) - \ln R(t_2)}{t_2 - t_1}$$

- If $t_1 = 0$ and $t_2 = t$, then AHR can be written as follows:

$$AHR(t) = \frac{-lnR(t)}{t}$$

Example

- An engineer approximates the reliability of a software system as:

$$R(t) = \begin{cases} (1 - \frac{t}{t_0})^2, & 0 \leq t < t_0 \\ 0, & t \geq t_0 \end{cases}$$

  - a) Determine the hazard rate
  - b) Does the hazard rate increase or decrease with time?
  - c) Determine the MTTF.

Example

- An engineer approximates the reliability of a software system as:

$$R(t) = \begin{cases} (1 - \frac{t}{t_0})^2, & 0 \le t < t_0 \\ 0, & t \ge t_0 \end{cases}$$

  - a) Determine the hazard rate
  - b) Does the hazard rate increase or decrease with time?
  - c) Determine the MTTF.
- Answer:
  - a) $f(t) = -\frac{d}{dt}\left(1 - \frac{t}{t_0}\right)^2 = \frac{2}{t_0}\left(1 - \frac{t}{t_0}\right),\ 0 \le t < t_0$

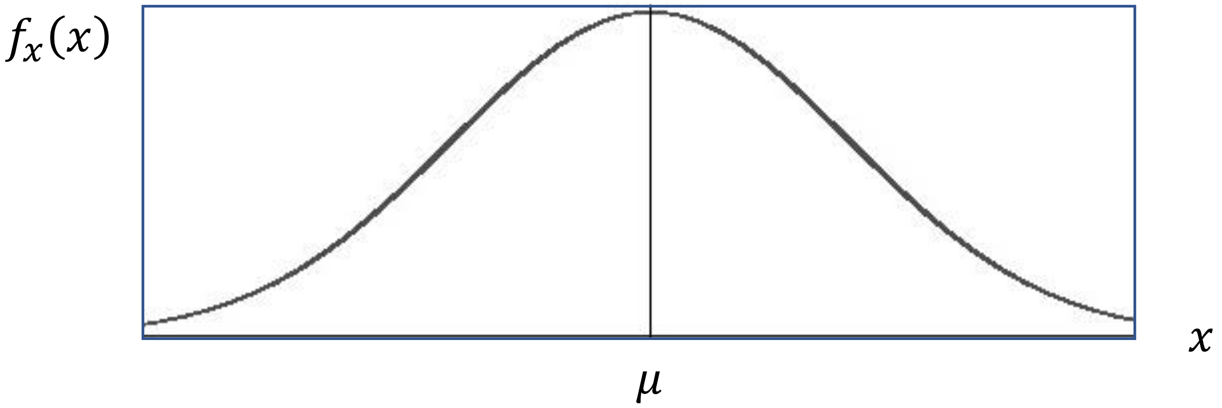
    So, $\lambda(t) = \frac{f(t)}{R(t)} = \frac{2}{t_0(1-\frac{t}{t_0})}, 0 \le t < t_0$
  - b) The failure rate increases from $\frac{2}{t_0}$ at $t = 0$ to infinity at $t = t_0$.
  - c) $MTTF = \int_0^{t_0} \left(1 - \frac{t}{t_0}\right)^2 dt = \frac{t_0}{3}$

Normal distribution

- We call a continuous random variable normal or Gaussian if it has pdf:

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

  where μ and σ are parameters
- (Note: we need $\sigma > 0$ for this to be a valid PMF)



$$E[X] = \mu$$

Normal distribution

- The CDF of a normal random variable X is:

$$P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\, du$$

- There are no closed form expressions for this function. To calculate probabilities for normal random variables, must use tables, software functions, etc.
- For convenience, these are typically stated in terms of a **standard normal random variable** $Y \sim Normal\ (0,1)$ which has CDF

$$\Phi(y) = P(Y \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{\frac{-\mu^2}{2}} du$$

Linear transformations

- A linear transformation of a normal random variable is still a normal random variable.

Theorem:

If $X \sim N(\mu_X, \sigma_X^2)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$ where

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

Proof:

We can write

$$X = \sigma_X Z + \mu_X \qquad \text{where } Z \sim N(0,1).$$

Thus,

$$\begin{aligned} Y &= aX + b \\ &= a(\sigma_X Z + \mu_X) + b \\ &= (a\sigma_X)Z + (a\mu_X + b). \end{aligned}$$

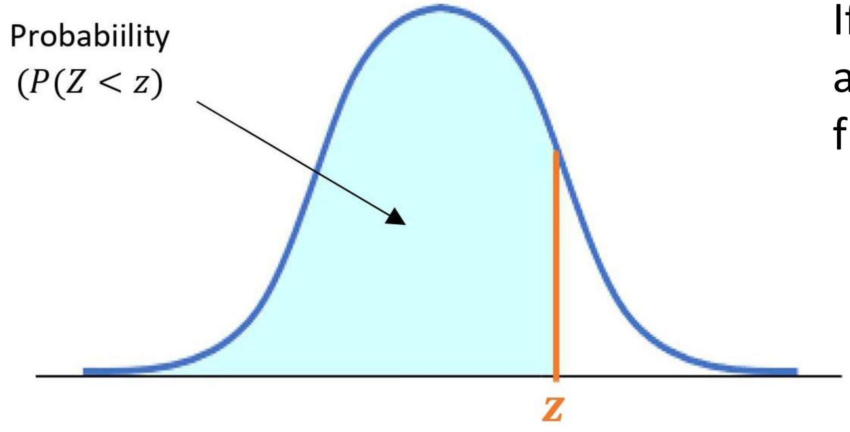
Therefore,

$$Y \sim N(a\mu_X + b, a^2\sigma_X^2).$$

So, what does this all mean?

Normal distribution

- Any normal distribution can be converted to a standard normal distribution to find the CDF



If mean and standard deviation are known, use the Z-Table to find the CDF

Z-Table

$$z = \frac{x - \mu}{\sigma}$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5754
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

⋮

https://www.wizeprep.com/online-courses/16841/chapter/10/core/5/1

Note: Z-Score Tables come in different formats!

- **First type (don’t recommend):**
  - Yields probability or area starting at the mean and going to the right of the mean up to the needed z-score.
  - Usually labeled "**cumulative from mean**“
  - This table basically works with half of the area under the normal curve, and the user must take this into consideration and make adjustments when using this table. This type of table lists positive z-scores only.

https://mathbitsnotebook.com/Algebra2/Statistics/STzScores.html

Note: Z-Score Tables come in different formats!

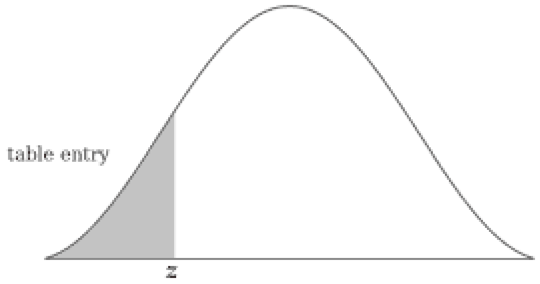
- **<u>Second type (use this):</u>**
  - Yields probability or area starting from negative infinity (the farthest left) and going to the right up to the needed z-score
  - Usually labeled "**cumulative from the left**“
  - This table works with the entire area under the normal curve, and requires less adjustments than the first option. This table lists both positive and negative z-scores.

https://mathbitsnotebook.com/Algebra2/Statistics/STzScores.html

Note: Z-Score Tables come in different formats!

- **<u>Second type (use this):</u>**
  - A.k.a. Need two tables, one positive and one negative.

<u>Negative Z score table</u>



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414
-0.1	.46017	.45620	.45224	.44828	.44433	.44034	.43640	.43251	.42858	.42465
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476

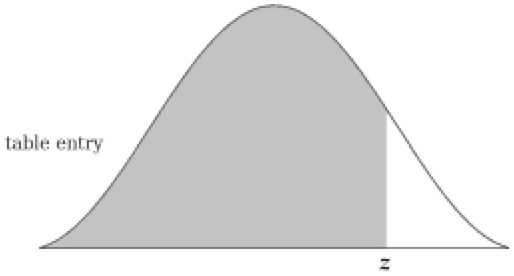
- Use the negative Z score table to find values on the left of the mean
- Corresponding values which are less than the mean are marked with a negative score in the z-table and represent the area under the bell curve to the left of z.

https://mathbitsnotebook.com/Algebra2/Statistics/STzScores.html
https://www.ztable.net/

Note: Z-Score Tables come in different formats!

- **<u>Second type (use this):</u>**
  - A.k.a. Need two tables, one positive and one negative.

<u>Positive Z score table</u>



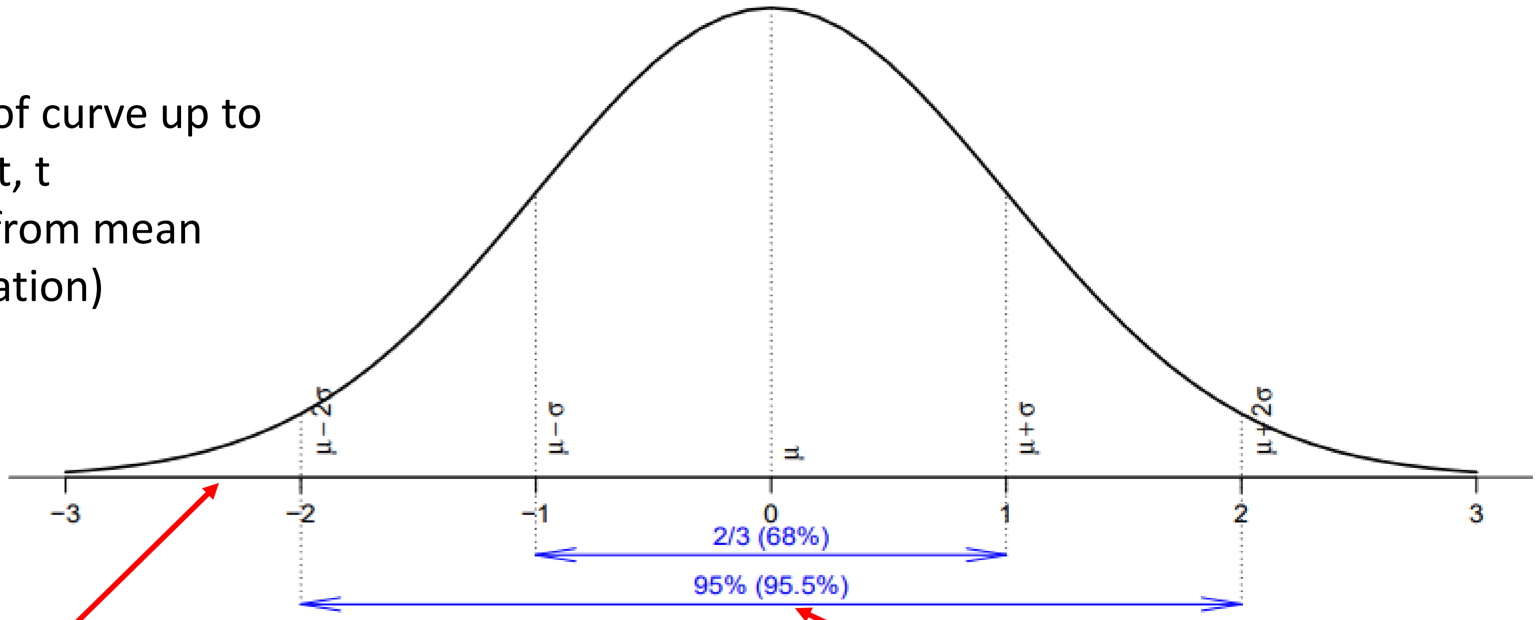
z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
+0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
+0.1	.53983	.54380	.54776	.55172	.55567	.55966	.56360	.56749	.57142	.57535
+0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
+0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
+0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
+0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
+0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
+0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
+0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
+0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
+1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214

https://mathbitsnotebook.com/Algebra2/Statistics/STzScores.html
https://www.ztable.net/

- Use the positive Z score table to find values on the right of the mean.
- Corresponding values which are greater than the mean are marked with a positive score in the z-table and represent the area under the bell curve to the left of z.

Two Ways of Looking at Normal Probability Density Function

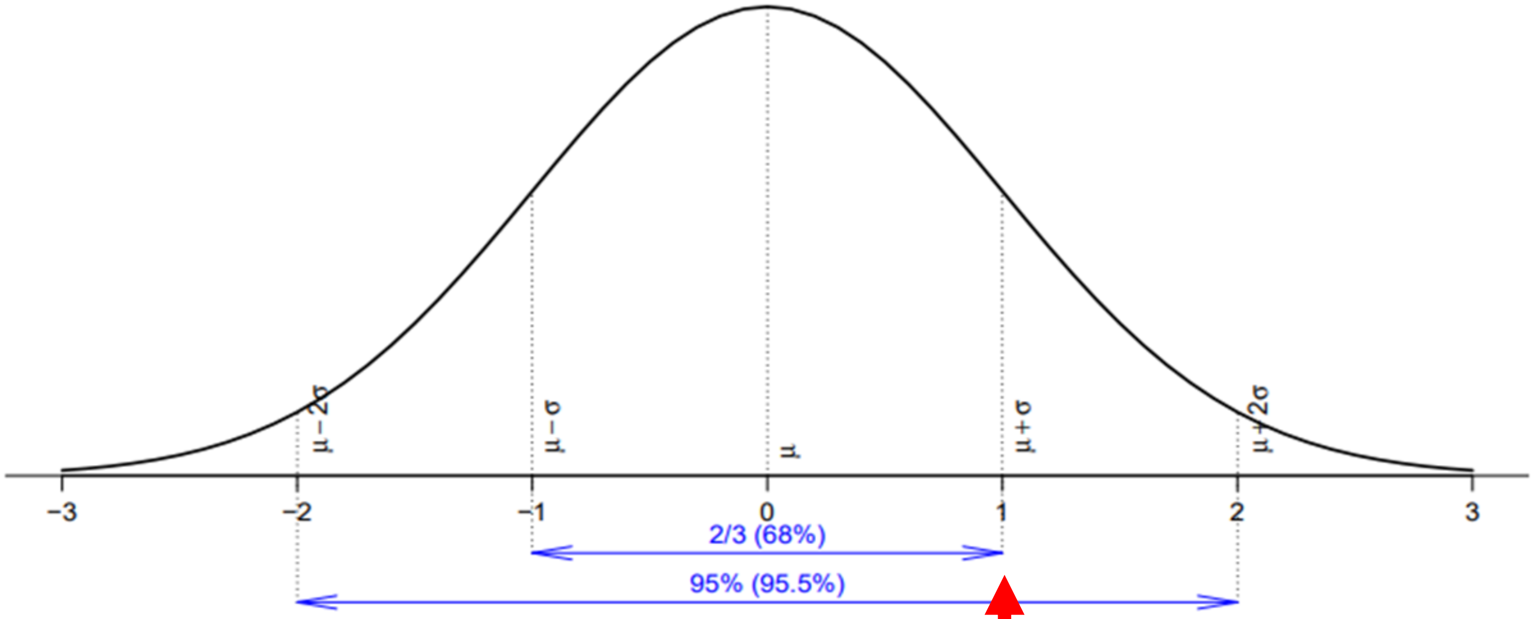
1. From left side of curve up to time of interest, t
2. How far away from mean (standard deviation)



X axis can be anything.
For this discussion it is time,
specifically time to failure

Some applications such as the dimension of an object, one calculates area centered around the mean

Z: Number of Standard Deviations



$$Z = \frac{x - \mu}{\sigma}$$

total number of standard deviations of a data point that lies below or above the mean.

$$Z = \frac{(\mu + \sigma) - \mu}{\sigma} = 1$$

0.833 Area to the left of Z = 1

Z = 1 is one standard deviation above the mean

(Note: Z = -1 is one standard deviation below the mean)

Z-score vs Standard Deviation

- Both describe how far away from the mean the data is.
  - Z-score: $z = \frac{x-\mu}{\sigma}$
  - Standard deviation: $\sigma = \sqrt{\frac{\sum(x_i-\mu)^2}{N}}$
    - x_i = every point in the dataset
    - *N* = number of datapoints
- Difference is that standard deviation has units but Z score doesn’t.
- Z score is helpful when comparing the values of two different samples that carry similar type of values, but are in different units.
- Example:
  - To compare the heights of 14 year old boys from two different schools, but school 1 reports the heights in feet & school 2 reports them in cm. Getting a Z score can help compare the heights without giving an erroneous result.

Example

- You are running a test on 1,000 components. It is known that the product failure times follow a normal distribution with a mean time to failure of 10,000 hours.
- After 15,000 hours you have logged 695 failures.
- What must the standard deviation be of this population of components?

Example

- You are running a test on 1,000 components. It is known that the product failure times follow a normal distribution with a mean time to failure of 10,000 hours.
- After 15,000 hours you have logged 695 failures.
- What must the standard deviation be of this population of components?
- **Solution:**
- Total accumulated percentage of failures = 695/ 1,000=69.5%
- $\Phi(z) = \Phi\left(\frac{x-\mu}{\sigma}\right) = 0.695$
- Need to find the corresponding z score for a CDF of 0.695

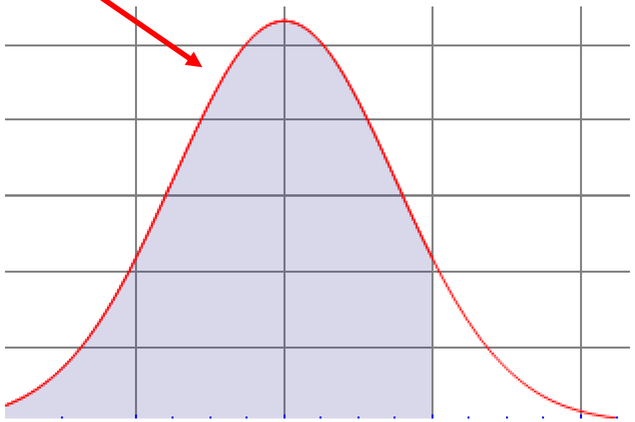
Area Under the Curve

$$Z = \frac{x - \mu}{\sigma}$$

$$\Phi\ (Z)$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051

$$\Phi\left(\frac{x-\mu}{\sigma}\right) = 0.695$$



After looking up on table:

$$Z = \frac{x - \mu}{\sigma} = 0.51$$

$$\frac{15{,}000 - 10{,}000}{\sigma} = 0.51$$

$$\sigma = 9{,}803$$

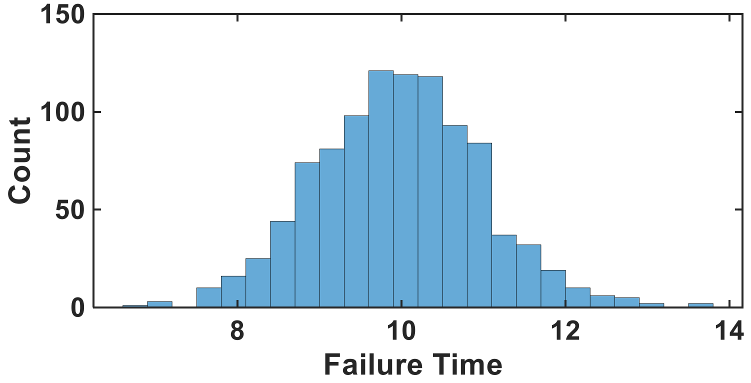
Putting it all together…

Probability / Statistics

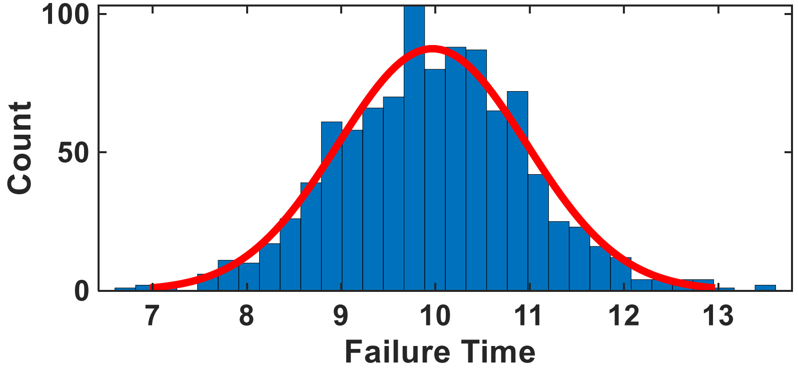
- Start with Histogram of Failures
  - Failure times dictate the functions
  - Create mathematical representations to fit reality

- Mathematical representations
  - Probability Density Function (PDF)
  - Cumulative Distribution Function (CDF)
  - Reliability Function (R)

Histogram

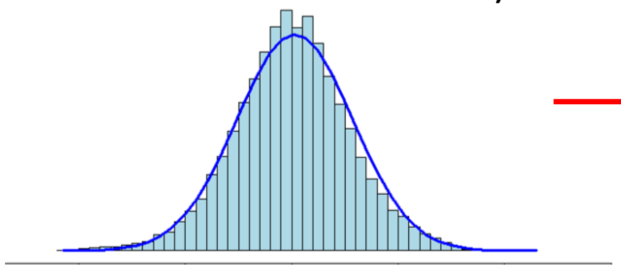


Fit Distribution

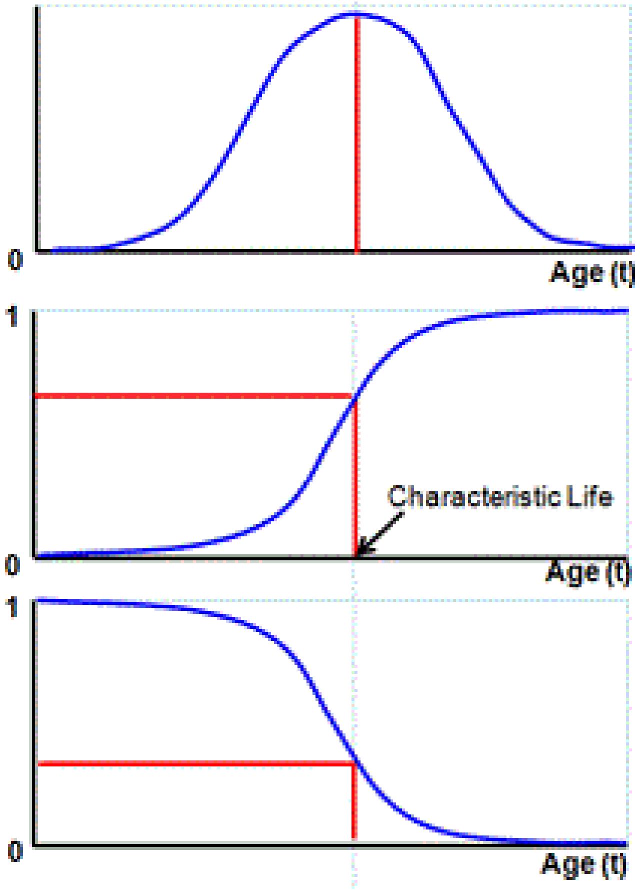


Histogram -> PDF -> CDF -> Rel

(Height in histogram is the number of failures in each time interval)



- CDF is lookup table for area under PDF
- When moving time further to right for CDF, more area under PDF is accumulated, increasing CDF value

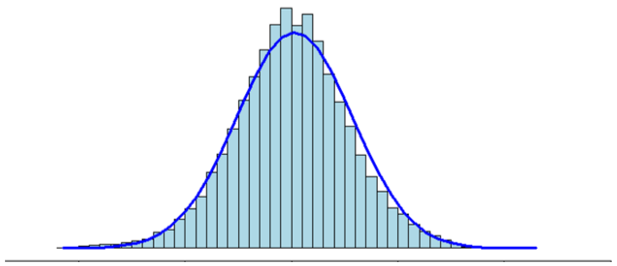


Probability Density Function

Cumulative Failure Distribution F(t)

Reliability Function R(t)

Raw Data to Normal PDF



Mean $$\hat{\mu} = \overline{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$$

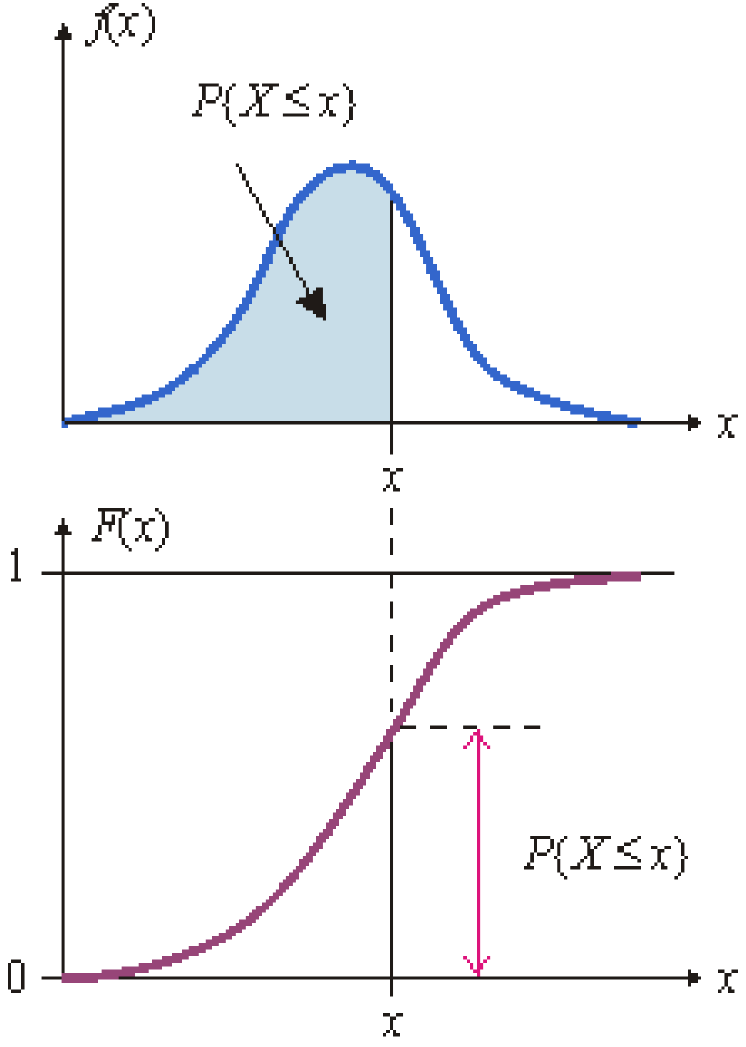
Variance $$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$$\boldsymbol{Standard\ Deviation} = \sqrt{\boldsymbol{variance}} = \sigma$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < \mu < \infty, \quad \sigma^2 > 0.$$

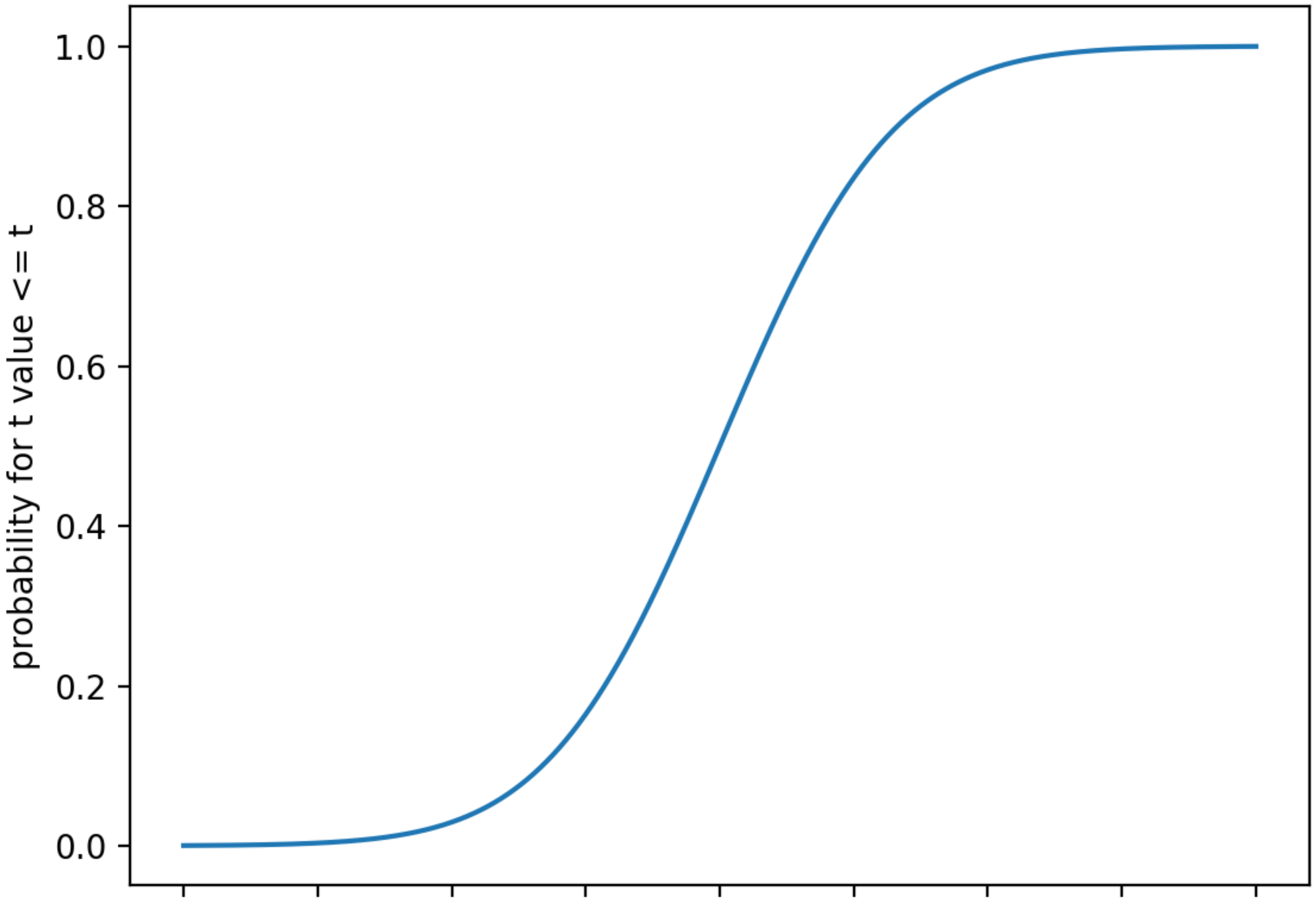
$\boldsymbol{f(x)\ is\ generic}$
$\boldsymbol{f(t) is\ applicable\ to\ time\ to\ failure}$

PDF and CDF Relationship



The CDF acts as a Look up table for area of the PDF for each value of x

Cumulative Distribution Function



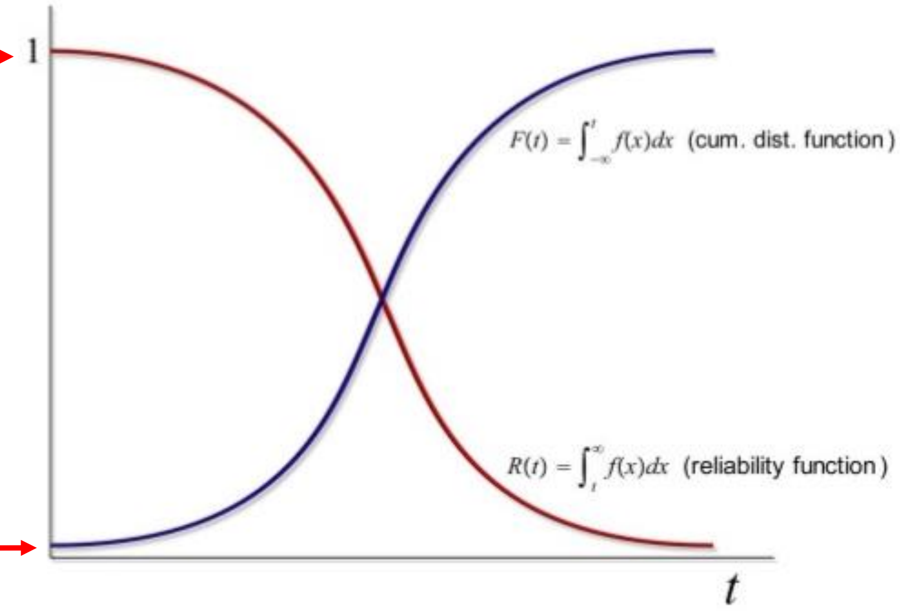
At time 0: probability of failure = 0
As time approaches infinity: probability of failure = 1

CDF and Reliability

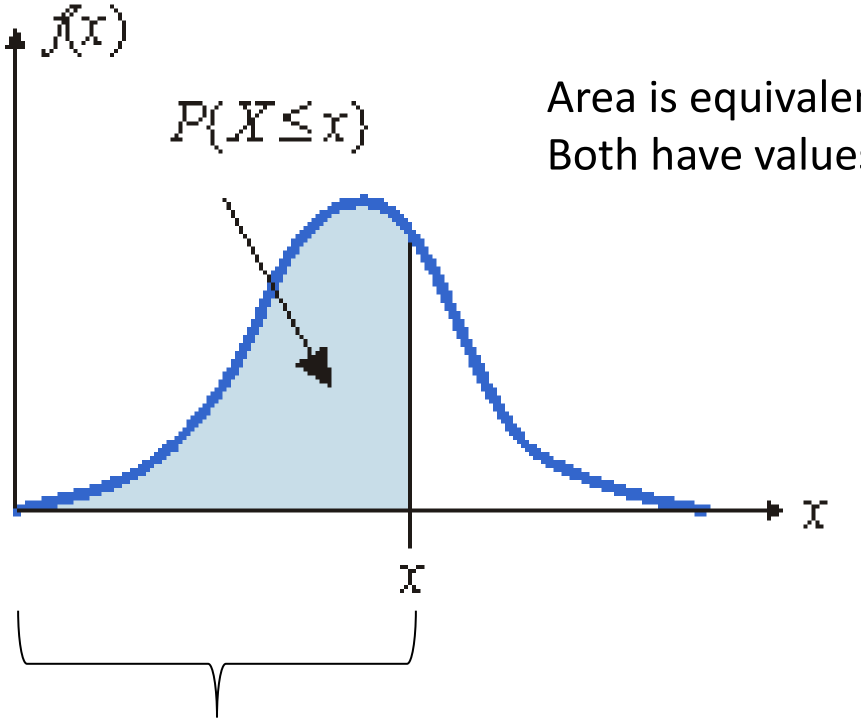
$$R(t) = 1 - F(t)$$

R(t) = 1

F(t) = 0



Area and Probability

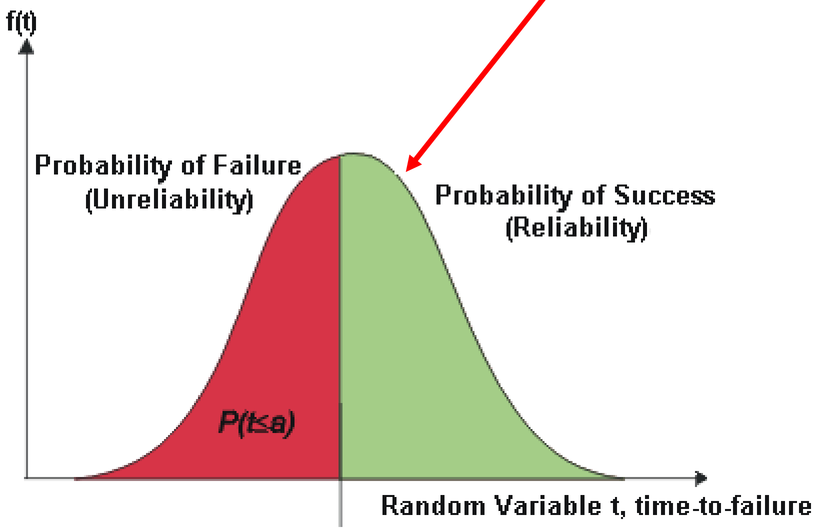


Area is equivalent to Probability
Both have values from 0 -> 1

For this class,
We are interested
In time from 0 to x

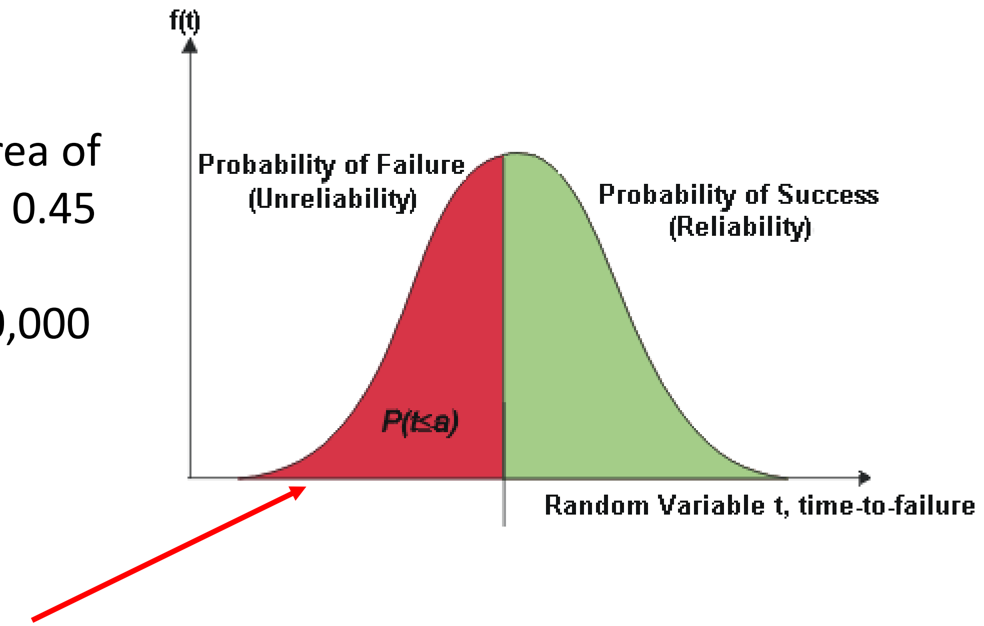
Reliability Function

$$\boldsymbol{R(t) = 1 - F(t) = \int_t^{\infty} f(x)dx}$$



PDF and Reliability

Assume area of
left side is 0.45
and
time = 10,000

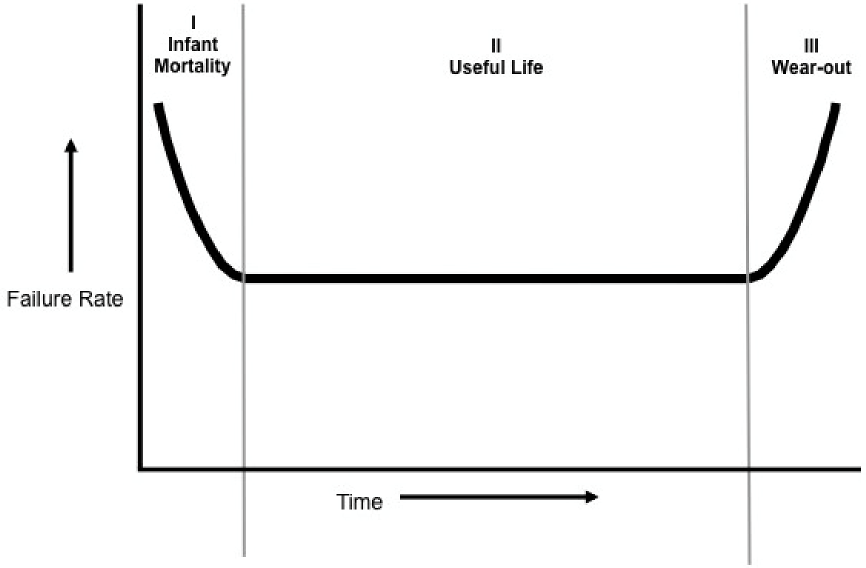


There are two
ways to look at this
area

[A] Probability of
failure of 1 device
is 0.45

[B] Test starts with 100
devices, after 10,000 hours –
45 will fail

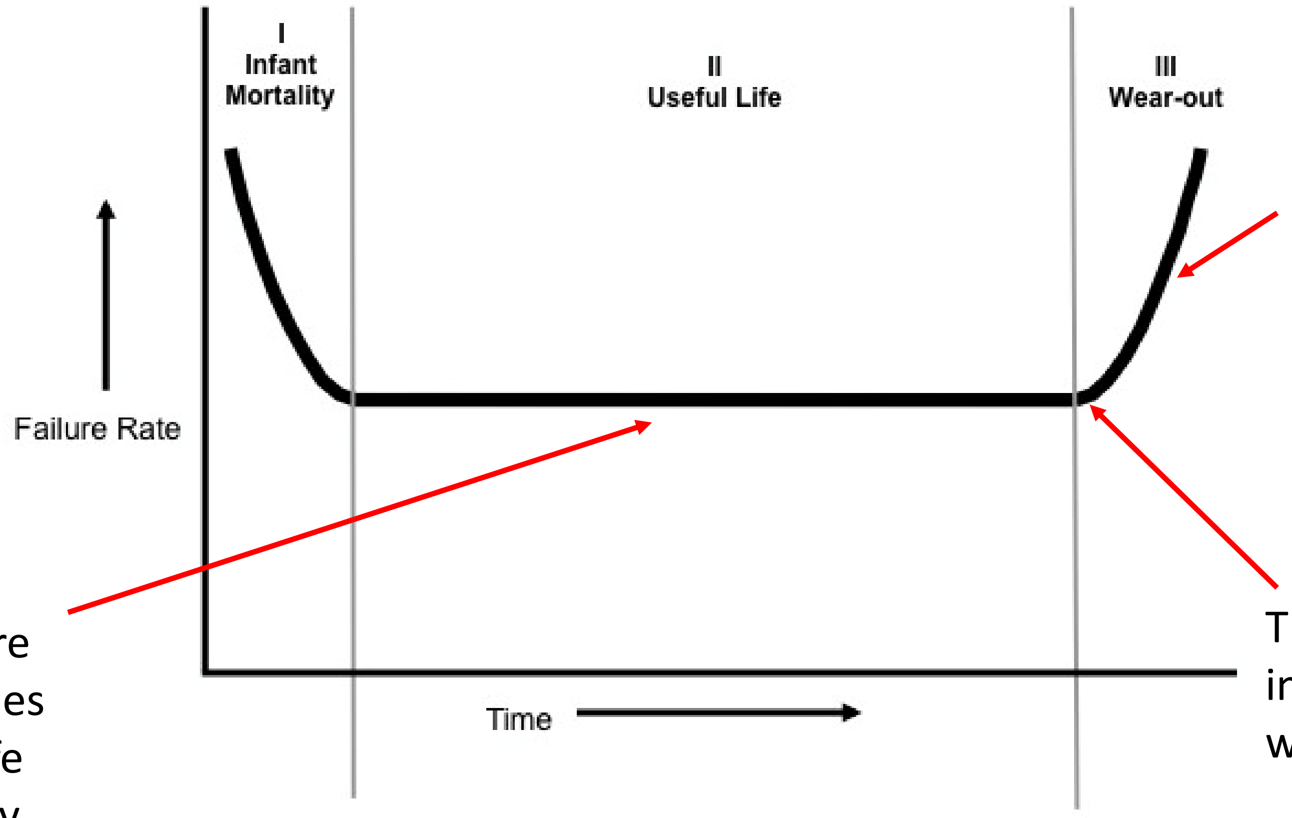
Bathtub Curve



Bathtub

Bathtub Curve

Consider a population of identical components and take a large sample N.

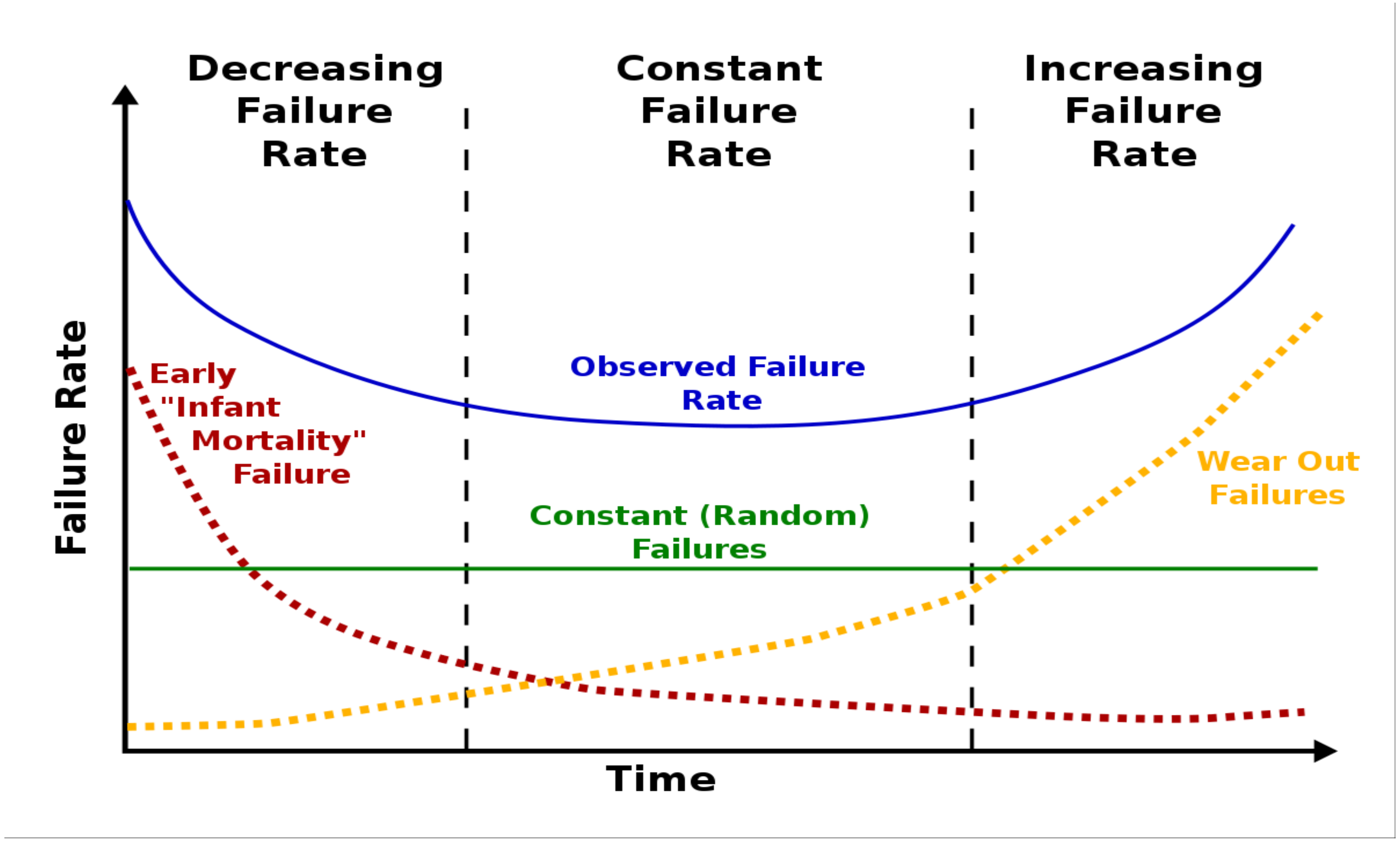


This phase has a different reliability

This point influences warranty

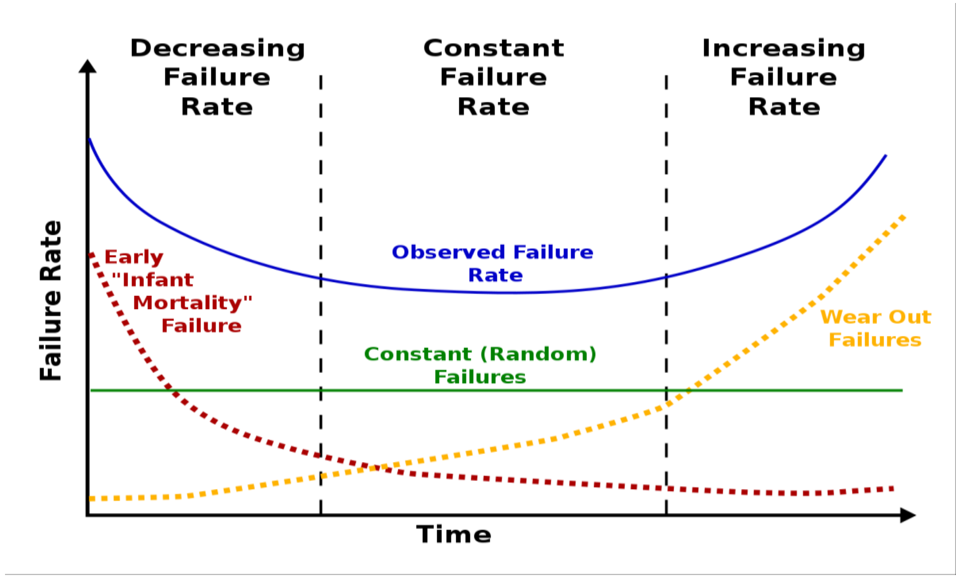
This failure rate defines useful life reliability

Bathtub Curve



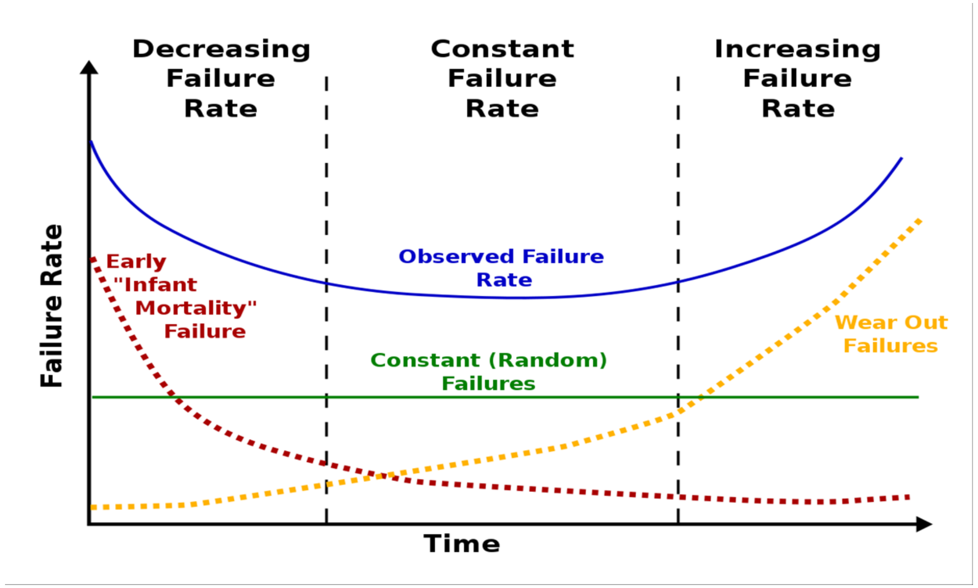
Bathtub Curve–Decreasing Failure Rate

- Units experience a high failure rate at the beginning of the operation time due to manufacturing defects or poor installations and begins to decrease over time (decreasing failure rate, DFR) , i.e., the time between failures begin to increase. This phase is known as infant mortality.
  - This phase is typically caused by manufacturing defects, cracks, poor workmanship, quality control, defective parts, contamination.
  - This phase can be reduced through burn-in testing where the unit is subjected to slightly more severe conditions than those encountered under normal operation.



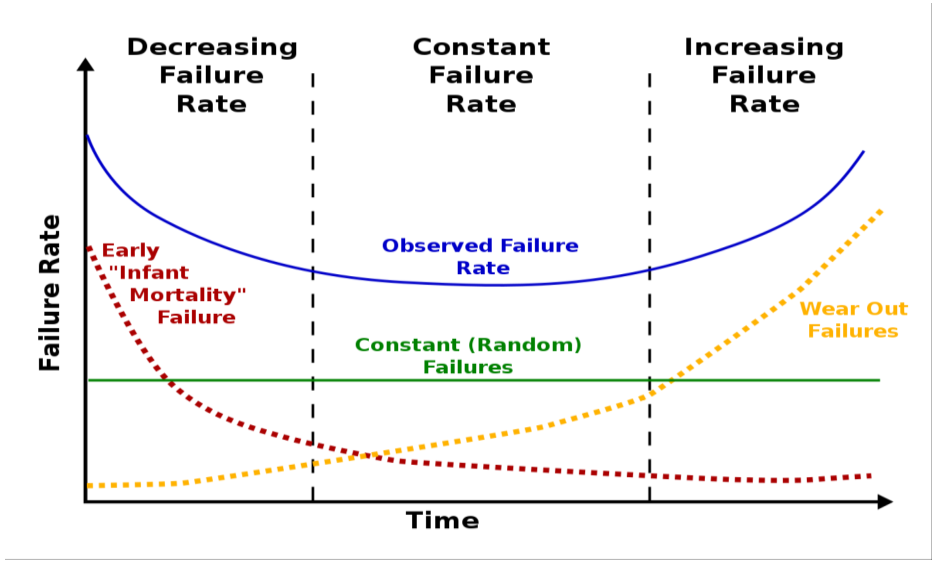
Bathtub Curve–Constant Failure Rate

- The failure rate begins to level for a period of time which is characterized by a constant failure rate (CFR). In this region, failures are random and do not follow a predictable pattern.
  - This phase is typically referred to as the "useful life".
  - Failure may be caused by random loads, human error, chance events, "Act of God"-type events
  - This phase can be reduced by redundancy or excess strength.



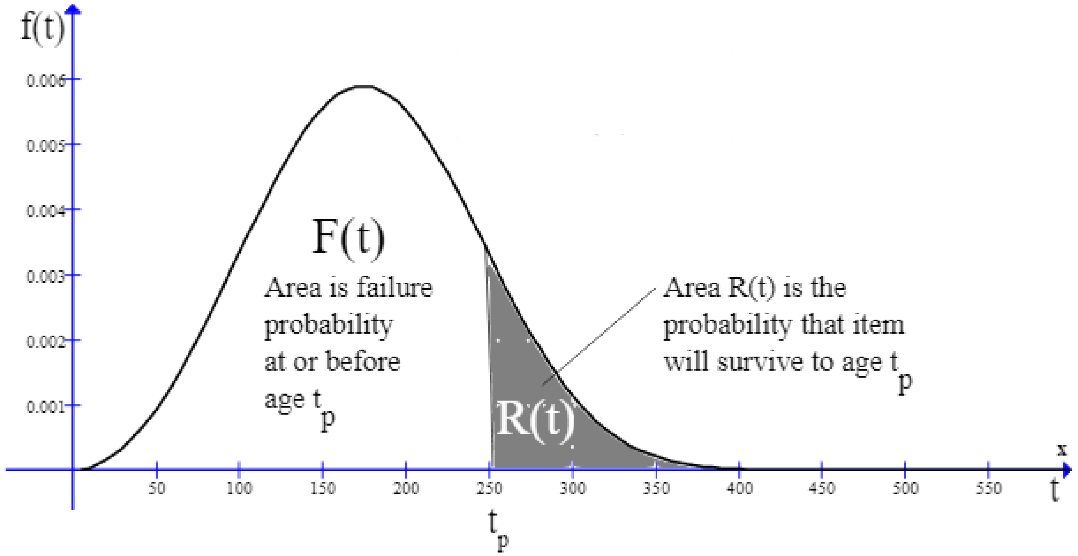
Bathtub Curve–Increasing Failure Rate

- The third region also known as the wear-out phase is characterized by an increasing failure rate (IFR). Failures in this phase are no longer characterized by being random and are mostly due to aging and wear
  - Typical causes of failure in this phase are fatigue due to cyclic loading, wear, corrosion.
  - This phase can be reduced through derating, preventive maintenance, parts replacement, condition monitoring using sensor technology
    - Note: Derating is the practice of limiting thermal, electrical, and mechanical “stresses” to levels below the manufacturer's specified ratings, to improve reliability.



Summary

- Reliability function $R(t)$: 1-CDF
  - Also called survival function
  - $R(t) + F(t) = 1$
  - $\frac{dR(t)}{dt} = -f(t)$
- Mean time to failure:
  - $MTTF = E[T] = \int_0^\infty tf(t)dt$
  - average amount of time a non-repairable asset operates before it fails
  - part's average lifespan
- Mean time between failure:
  - For repairable systems, the failure time between two successive failures is usually referred to as MTBF (Between)



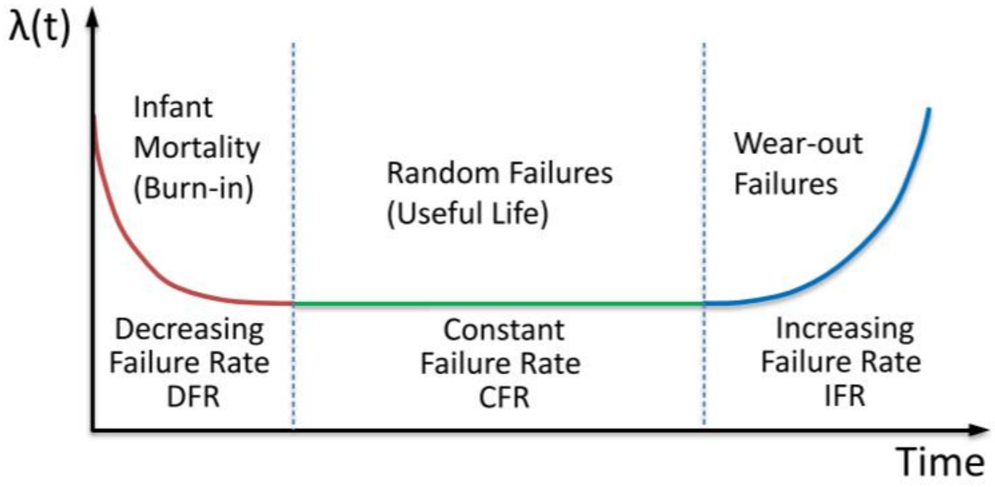
https://www.livingreliability.com/en/posts/real-meaning-of-the-six-rcm-curves/

Summary

- Failure rate: $f(t) = -\frac{dR(t)}{dt}$
  - May have other definitions, ex. number of failures per hour
- Hazard rate: $h(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)}$
  - instantaneous(conditional) failure rate, may also be written as $\lambda(t)$
  - More mathematical definition: $\lambda(t) = \frac{-dR(t)}{dt} \frac{1}{R(t)}$
- Can determine reliability function from hazard rate:
  - $R(t) = \exp[-\int_0^t \lambda(t')dt']$

Summary

Bathtub Curve



	Characterized By	Caused By	Reduced By
Burn-in	DFR	Manufacturing defects: Welding flaws, defective parts, poor quality/workmanship, contamination.	Burn-in testing, screening, acceptance testing, quality control
Useful Life	CFR	Environment, random loads, human error, chance events	Redundancy, excess strength
Wear-out	IFR	Fatigue, corrosion, aging, cyclic loading	Derating, part replacement, preventive maintenance