# Reconciling two models of public debt and interest rates

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Abstract: This paper provides a careful replication and evaluation of the differences between two models of the relationship between public debt and interest rates. The American Economic Association presidential address of Olivier Blanchard (2019) provided evidence that increasing government debt in times of low interest rates could be welfare improving. Evans (2020) recreates the Blanchard approach and finds an opposite result. Evans mentions some possibilities of what might be different in his study from Blanchard's, but he provides no direct mapping between the two studies. This paper seeks to identify the differences with the goal of providing further evidence for the relationship between public debt and interest rates.

# 1. Introduction

In recent years, taking on more and more public debts has become the norm for many developed countries, and Debt- to-GDP ratios across developed economies have reached a historical high level. Currently, the U.S. government's public debt is more than $22 trillion. With the continuously growing debt level, the concerns for the cost of high public debt is growing as well. As the amount of government debt is not only very important in terms of central government economic policy, but also is at the center for making household decisions and business decisions, whether government should continue increasing debt level has become one of the most important economic issue.

The President of the American Economic Association, Olivier Blanchard, has addressed this issue during the January 2019 annual meeting AEA Presidential Address. Blanchard (2019) purposed that given that the interest rate is lower than the growth rate, the fiscal and welfare costs of the public debt could be small. He concluded that the safe interest rates in U.S. were lower than the growth rates for a long time in the past, so they will continue stay below the growth rates and in this low interest rate environment, the fiscal and welfare costs of the public debt can be much lower than expected. He argued from four aspects. First, he suggested that it is normal to have the interest rate lower than the growth rate and the situation will continue in the future, so the public debt will have no fiscal cost. Second, although public debt reduces capital accumulation, a safe rate lower than the growth rate indicates that the risk-adjusted rate of return on capital is also low, so the welfare cost could be smaller than expected. Third, while the measure rate of earnings has been quite high, the marginal product of capital is lower and in turn will lead to a lower welfare cost of debt. Fourth, although investors might require risk premium to balance the high public debt and the fiscal burden will increase, he claims that this argument has no straightforward implications for the appropriate level of debt.

Evans (2020) recreates the modeling and calibration approaches of Blanchard and finds contrasting results. The attempted replication of Blanchard's stated approach could not find any long-run average welfare gains from increased government debt. Moreover, Evans's research has found that with some relax on the strong risk-reducing assumptions, those welfare losses are exacerbated. As there is no direct mapping between the two studies, the purpose of my research is to explore the difference between Blanchard's model and Evans's model. By trying to identify the reasons that give rise to the exact opposite results, this paper seeks to further explore the relationship between the increased public debt and the low interest rates.

# 2. Methodology and Model

## 2.1 Economic Model

Blanchard (2019) first looks at the effects of an intergenerational transfer in an overlapping generation model with uncertainty. As in the Diamond model (1965), the effect of such an intergenerational transfer depends on both the average safe rate and the average risky rate. A transfer has two effects on welfare that has opposite signs: an effect through reduced capital accumulation and an indirect effect through the induced change in the returns to labor and capital. If the riskless rate is lower than the growth rate, the welfare effect through lower capital accumulation will be positive. On the other hand, when the average risky rate is higher than the growth rate, the welfare effect through the induced change in returns to labor and capital will be negative.

Although in current settings, there is a safe rate lower than the growth rate, the average risky rate exceeds the growth rate, which brings the two effects into opposite direction. In turn, the overall effect on welfare costs becomes unclear. Therefore, in the following simulations, Blanchard (2019) has chosen a set of average riskless rate with different average safe rate to explore how the different rates influence this transfer.

The basic setup of the overlapping generations model for the two papers is the same: the economy is consisted of people who only live for two periods. In the first period, the young will work and consume, and then in the second period, the old will consume all savings. The utility maximization function is an Epstein-Zin utility function as below.

$$U_t = \max_{k_{2,t+1}} (1-\beta)\ln\left(c_{1,t}\right) + \beta \frac{1}{1-\gamma} \ln\left(E_t\left[\left(c_{2,t+1}\right)^{1-\gamma}\right]\right) \ \ \forall t$$

$$c_{1,t} + k_{2,t+1} = w_t + x_1 - H_t$$

$$c_{2,t+1} = R_{t+1} k_{2,t+1} + H_{t+1}$$

where $c_{1,t}$ and $c_{2,t+1}$ are consumption in each period of life respectively. $w_t$ denotes the wage on the unit of inelastically supplied labor by the young and $R_t$ represents the return on risky savings. $H_t$ represents the intergenerational transfer from the young to the old, and $x_1$ represents an endowment to the young so that the government will always be guaranteed with a promised amount of transfer $H_t$ from young to the old.

The endowment amount $x_1$ is treated differently in the two papers, and I will address this difference in modification 5. In Blanchard model, the endowment is simply set as a non-stochastic value that always equal to the average steady state wage.

$k_{2,t+1}$ denotes the optimal risky savings and needs to be obtained through a dynamic optimization. Using the first order conditions from the utility function, we get a Euler equation that could later be used in simulations to find the optimal value for risky savings.

As the strength of the second effect on welfare is related to the elasticity of substitution between capital and labor, Blanchard (2019) assumes a constant elasticity of substitution production function with multiplicative uncertainty.

$$Y_t = A_t(\alpha K_{t-1}^{\rho} + (1-\alpha)N^{\rho})^{1/\rho} = A_t(\alpha K_{t-1}^{\rho} + (1-\alpha))^{1/\rho}$$

where $\rho = (\eta\text{-}1)/\eta$, $\eta$ is the elasticity of substitution.

Blanchard Model has discussed two special cases for the production model: when $\eta = \infty$ and $\rho = 1$, the production function is a linear model. When $\eta = 1$ and $\rho = \infty$, the production function is a Cobb-Douglas function which provides a good description for a medium-run production function.

$A_t$ is defined differently between two papers. In Blanchard (2019), $A_t$ is simply defined as a white noise that distributes log normally with $lnA_t \sim N(\mu, \sigma^2)$ . While in Evans (2020), $A_t$ is not only defined as a log normal distribution but also, $A_t = e^{z_t}$ where $z_t$ follows a normally distributed AR(1) process.

$$z_t = \rho z_{t-1} + (1-\rho)\mu + \varepsilon_t \text{ where } \rho \in [0,1) \text{ and } \varepsilon_t \sim N(0, \sigma^2)$$

To match the two sets of codes, I set $\rho = 0$ to make the Evans model consistent with the Blanchard model.

The other parameters are all set completely identical between two papers. $\alpha$ is set to 1/3 and $\sigma$ is set as an annual value of 4%, which implies a value of 0.2% for a 25-years period. The average risky rate is chosen between 0% and 4% which implies a 25-year gross risky rate between 1% and 2.66%. The average riskless rate is chosen between -2% and 1%, which implies a 25-year gross safe rate between 0.6% and 1.28%. The choice of $\gamma$ is also identical between two papers. There is some discrepancy in the choices of parameters $\beta$ and $\mu$ and I will address both of them in the next section that discusses specific modifications.

## 2.2 Methodology

This paper provides a detailed comparison between the codes used for running simulations for the intergenerational transfer model and seeks to identify the differences of Blanchard (2019) and Evans (2020). Although Blanchard (2019)'s code is written in MATLAB, Evans (2020) code is written is Python. To eliminate any possible discrepancy caused by programming languages, I used Julien Acalin's Replication of Blanchard (2019) code that is written in Python. It fully replicates the analysis and derivation of the stochastic overlapping generations(OLG) model developed in Blanchard(2019), and yields exactly the same results, so it is sufficient to become the substitute in my comparison between Blanchard(2019) and Evans (2020). Therefore, for the comparisons of both theoretical formula and programming code, I use mostly Acalin (2019) as major reference.

Blanchard (2019) also includes a discussion of debt and shows how a debt rollover differs from the transfer scheme. As Evans (2020) focuses mostly on the replication of Figure 7 in Blanchard (2019), the welfare effect of a transfer of 5% of saving using linear production function, and Figure 9, the welfare effect of a transfer of 5% of saving using Cobb-Douglas production function, I will not consider cases with debt rollovers. For Evans (2020) code, as the major structure for linear production function and Cobb-Douglas function is the almost identical, the only differences are some equation within defined functions, I focus my comparison on the code for linear production model.

# 3. Modifications and Results

In total, I have made 9 modifications to the Evans' code to match the Blanchard Python code. After all modifications, the code will fully replicate the results from Blanchard. For each modification I made, I recorded the results and values for each variable in Appendix. Each modification is built on the previous modification code, so I will only compare results between each modified version of code and their previous modified version.

Additionally, I used the MATLAB code from Blanchard model to produce figures in the same fashion as Figure 7 in Blanchard (2019). The effect of each modification can be directly shown from each figure. All figures are summarized in Appendix 1.9.

### 3.1.1 Modification 1 (Appendix 1.1.1)

$$\max_{k_{2,t+1}} (1-\beta)\ln\left(c_{1,t}\right) + \beta\frac{1}{1-\gamma}\ln\left(E_t\left[\left(c_{2,t+1}\right)^{1-\gamma}\right]\right) \quad \forall t \tag{1}$$

$$c_{1,t} + k_{2,t+1} = w_t + x_1 - H_t \tag{2}$$

$$c_{2,t+1} = R_{t+1}k_{2,t+1} + H_{t+1} \tag{3}$$

For all tables in Evans (2020) (Table 2, 3, 4) which shows the percentage change in average lifetime utility from increased transfer, despite the different parameters of $\mu$ or $\beta$ used, we can notice that for Linear model predictions, when the average risky rate $r_t$ doesn't change and only the average riskless rate changes, the three percentage changes are the same. While in the Blanchard model, the result has a different pattern. In Figure 7 of Blanchard (2019), we can see that when the average riskless rate is the same, the percentage change in utility is very similar across all different values of average risky rate.

Although it could be a result of rounding or there could be theoretical explanation for the constant result, I suspect that there is something going on in the calculation of the utility function, so I reviewed the code for "sim_timepath" and "get_k2tp1". Comparing to the Blanchard Python code, in the Blanchard code, the calculation was an exact match like in equation (1). While in Evans' Python code, instead of the expectation of the consumption in period two, $E_t[(c_{2,t+1})^{1-\gamma}]$, the $(c_{2,t+1})^{1-\gamma}$ was used directly in the computation of utility function. This discrepancy contributes a large change into the completely opposite final result in the end.

Evidently, after changing the consumption in period two to the expectation of consumption, we can see from Appendix 1.1.1, the result changed significantly. When the risky rate is 0% and the safe rate is -2% or -0.5%, we can see that the percentage change has flipped positive to 1.00% and 0.19%, while all the other percentage changes remain negative. Another significant change is that, when average risky rate is constant, the persistence in the percentage change is gone. Now when the average riskless rate changes, the utility percentage change will be different as well. Figure 1 is generated using Table 4 in Evans (2020), and we can see that it is greatly different from Figure 3.1.1 generated using the modified code.

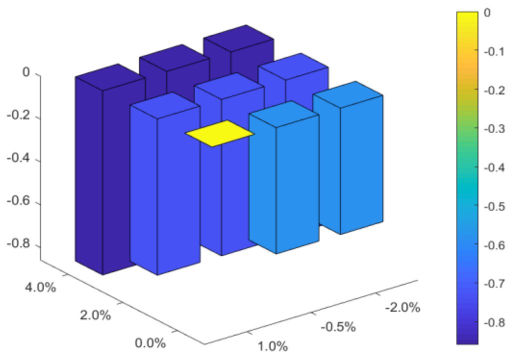


Figure 1. Table 4 in Evans (2020)

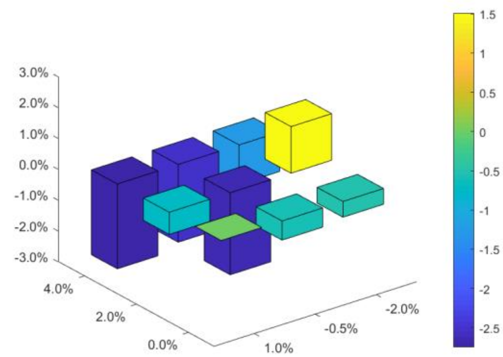


Figure 3.1.1. Modification 1

### 3.1.2 Modification 1 (Appendix 1.1.2)

$$\mu = ln(E_t[R_{t+1}]) - \ln(\alpha) - \frac{\sigma^2}{2} \tag{4}$$

The other modification I made is a minor change that addresses the problem of the choice of parameters $\mu$. In Evans (2020), there is one table that attempts to replicate Blanchard (2019, Figure 7) where all the $\mu$ is kept constant at 1.0786. However, in a closer examination of the Blanchard Python code, I find that Figure 7 is not generated with a constant $\mu$. It is still generated using $\mu$ as a function in equation (4), but on Figure 7, $\mu$ is written as $\mu = 1.0786$ and is not defined clearly.

The influence of the change of $\mu$ is significant. In Appendix 1.1.1 and Appendix 1.1.2, all other part of code is identical. Appendix 1.1.2 shows the result of keeping $\mu$ a constant at 1.0786, and appendix 1.1.1 uses $\mu$ as a function shown in equation (4). When $\mu$ is constant, we can see that when risky rate is 0%, the percentage change in utility is back to negative. Comparing the specific utility in the two cases of different transfer amount, we can see that by keeping $\mu$ constant, the utility remains always below 1. Once $\mu$ can change with the risky rate, the utility increases with the value of $\mu$. The same pattern can be found in the value of capital $k_{2,t+1}$, and the consumption in period one $c_{1,t}$. The expectation of consumption in period two, $E_t[(c_{2,t+1})^{1-\gamma}]$ is greatly influenced by the value of $\mu$. With a constant $\mu$, the value has gone to the level of $10^{13}$, while after setting $\mu$ to a function, the value will be close to 0.

Therefore, for all other modifications, I set $\mu$ as a function as shown in equation 4.

### 3.2. Modification 2 (Appendix 1.2)

$$\frac{1-\beta}{c_{1,t}} = \beta \frac{E_t[R_{t+1}(c_{2,t+1})^{-\gamma}]}{E_t[(c_{2,t+1})^{1-\gamma}]} \quad \forall t \tag{5}$$

The second modification is the most influential one that makes all the difference. To maximize the lifetime utility function with optimal risky savings, we get a resulting Euler equation from the first order conditions from equation (1). The dynamics of capital accumulation is given above in equation (5). This Euler function is identical for both Blanchard (2019, Appendix C.3.4, equation (9)) and Evans (2020, Appendix, equation (5)).

However, in the actual code, the value of $R_{t+1}$ is defined differently. The calculation for $R_{t+1}$ is identical for both papers. Therefore, in the Python code, before the optimization starts, $R_{t+1}$ has the same initial values. In Evans (2020), the Python code used $(1 + R_{t+1})$, while Blanchard directly used $R_{t+1}$. Because of this discrepancy, the Euler function and also the computation for $c_{2,t+1}$ are all different between two papers. In Evans (2020) Python code, equation (3) and (5) are computed as below in equation (6) and (7), which are different from the formulas written in the paper.

$$c_{2,t+1} = (1 + R_{t+1})k_{2,t+1} + H_{t+1} \tag{6}$$

$$\frac{1-\beta}{c_{1,t}} = \beta \frac{E_t[(1+R_{t+1})(c_{2,t+1})^{-\gamma}]}{E_t[(c_{2,t+1})^{1-\gamma}]} \quad \forall t \tag{7}$$

Because the expression of $R_{t+1}$ is different, the optimization value of the Euler function is different as well. Therefore, this discrepancy changes everything for the final result. After I changed $(1 + R_{t+1})$ to $R_{t+1}$ in Evans (2020) code, half of the results for percentage change in utility has flipped to positive values.

As we can see from Appendix 1.2, when average riskless rate is at -2%, for all values of average risky rate, the percentage change is positive. Additionally, when average risky rate is 0% and average riskless rate is -0.5%, the percentage change becomes 0.2%. From the second table that shows the value of utility function in both cases, we can see that although the values between Appendix 1.1.1 and Appendix 1.2 are similar before and after the modification, the dynamics has changed. The values of optimal savings $k_{2,t+1}$ has a bigger change as the Euler equation used for optimization is different. The consumption $c_{1,t}$ in period one is similar, while the value of $E_t[(c_{2,t+1})^{1-\gamma}]$ has a very large change because of the inconsistency in the calculation of $c_{2,t+1}$.

### 3.3. Modification 3 (Appendix 1.3)

Then I made another minor modification, the choice of variable $\beta$. Although the other choices of parameters are identical in both papers, there are slight differences in the choice of parameters for $\beta$. In Blanchard's paper, for the linear case, he derived most formulas without the presence of $\beta$, so the parameter $\beta$ is set randomly at 0.325 for all simulations. While in the Evans (2020), $\beta$ is calibrated using Cobb-Douglas expected value expression, and this calibrated value is used throughout the linear model.

For the Blanchard model, I tested other values of $\beta$ ranging from 0.2 to 0.65, and after a series of simulations, I find that the value of $\beta$ do not change the final result. As $E[R_{t+1}]$ does not depend on capital $k_{2,t+1}$ or $\beta$, and is only calibrated with $\mu$, the value of $\beta$ in the linear model seems to be irrelevant. Therefore, to better match compare results between Blanchard (2019) and Evans (2020), I set the $\beta$ value as the Blanchard model with a constant of 0.325.

Although a change of value in $\beta$ does not cause results to change in Blanchard model, in Evans model, as shown in Appendix 1.3, we observe that an additional percentage change has flipped positive. Comparing to Appendix 1.2, when the average risky rate is 2% and the average riskless rate is -0.5%, the percentage change has gone from -0.18% to 0.15%. Moreover, when the average risky rate is 4% and average safe rate is -0.5%, the percentage change in utility has increased from -0.33% to -0.01%, which almost approaches a positive value.

In the second table, we can see that the value of utility function when the average risky rate is 4% is on average larger than the values from Appendix 1.2. The change in $k_{2,t+1}$ is more significant as it depends on the value of $\beta$. When the average risky rate is 4%, $k_{2,t+1}$ on average is around 5 while in previous modified simulations, the value is always around 1.

Due to the change in $\beta$, as shown in the last table, $E_t[(c_{2,t+1})^{1-\gamma}]$ has gone through a noticeable change. In previous simulations, the expectation is relatively large and when the average risky rate is 0% and average riskless rate is -2%, the expectation is around 34, which really stands out from all other values. After the change in $\beta$, the expectation shrinks significantly to smaller than 1, and when the average risky rate is 4%, the number has reduced to the level of $10^{-28}$.

### 3.4. Modification 4 (Appendix 1.4)

The other modification I made is relevant to $H_t$, the lump sum government transfer taken from the young and given to the old in each period. In both papers, the change of utility is compared between no intergenerational transfer and a transfer amount of 5% of savings. Therefore, in the Blanchard Python code, the value of $H_t$ is set as a non-stochastic value as 5% of the steady state savings K, which is denoted with steady state average wage and $\beta$ as below in equation (8). While in Evans (2020), $H_t$ is computed as a function shown in equation (9), so the value will change with the simulation.

$$H_t = \ 2\beta x_1 = 2\beta w_t \tag{8}$$

$$H_t = \min(\overline{H}, w_t + x_1 - c_{min} - K_{min}\ ) \tag{9}$$

In the Blanchard model, as $\beta$ is set as a constant, and steady state wage is also set at the beginning, the value of the transfer is constant through all simulations, which is different from Evans model. After I changed the expression for $H_t$, I got the results shown in Appendix 1.4. Comparing to the set of tables in Appendix 1.3, the change in all variables is really small, but the percentage change in utility for average risky rate at 4% and average riskless rate at -0.5% has gone from -0.1% to 0.1%. Again, even though the sign is flipped, the change is really small.

### 3.5. Modification 5 (Appendix 1.5)

In both papers, the endowment $x_1$ is defined in the same way as the equal value of the wage. However, in the code, there is a slight difference. In Blanchard's paper, the endowment $x_1$ is set to always equal to the average steady state wage corresponding to the pair of average risky rate and average riskless rate without transfer or debt. Like the value of $H_t$, $x_1$ is computed as a non-stochastic value that remains as a constant throughout the simulations.

$$x_1 = (1 - \alpha)e^{\mu + \frac{\sigma^2}{2}} \tag{10}$$

$$x_1 = (1 - \alpha)e^{\mu + \frac{\sigma^2}{2}}(2\beta^{\alpha})^{\frac{1}{1-\alpha}} \tag{11}$$

As shown in equation (10), the Blanchard model $x_1$ does not change with $\beta$, while in equation (11), the Evans model $x_1$ depends on $\beta$. The differences in the computation of $x_1$ could partially explain why the change of value in $\beta$ causes no change in Blanchard model but has a big influence in the Evans model in Modification 3.

After I modified the value of $x_1$ to resemble the Blanchard model, the results are shown in Appendix 1.5. The percentage changes are similar to the previous simulations, but the value of the optimal level of capital $k_{2,t+1}$ is different from Appendix 1.4. When the average risky rate is

4%, the optimal risky savings change from around 5 to around 3, which are approaching the optimal savings from the Blanchard model (around 3 as well).

### 3.6. Modification 6 (Appendix 1.6)

This modification is just a simple calibration difference that does not change the result much, but in order to compare the codes more accurately, I change the Evans code to resemble to Blanchard Python code. In the final computation of lifetime utility function, the Evans code omits the last value of $c_{1,t}$ and the first value of $E_t[(c_{2,t+1})^{1-\gamma}]$, so there are in total 24 periods of simulation for the utility function. In the Blanchard Python code, last value of $c_{1,t}$ is also dropped, but the first value for $c_{1,t}$, the consumption for "the first generation" of old when they were young is fixed at 1. After changing the value for the first period, I got the result in Appendix 1.6.

When the average safe rate is -2% and -0.5%, all values of average risky rate could produce a positive percentage change in utility. From the tables we can see that all other variables are very similar from Appendix 1.5. After the above six modifications, we get a similar result that closely resembles the Blanchard model.

### 3.7. Modification 7 (Appendix 1.7)

To get the optimal risky savings $k_{2,t+1}$, both sets of codes used the built-in optimizer function in Python to find the solution for the Euler Equation. To eliminate any differences caused by the optimization algorithm in Python, I change the optimizer used in the Evans code, "scipy.optimize.root", to "scipy.optimize.least_squares" to match the Blanchard Python code. Although this modification does not change the economic model, it creates slightly different results from Appendix 1.6. As shown in Appendix 1.7, we can see that when the risky rate is 2% and the safe rate is -0.5%, the percentage change increases to 0.16% from 0.01%.

### 3.8. Modification 8 (Appendix 1.8)

$$\frac{1-\beta}{c_{1,t}} = \beta \frac{E_t[R_{t+1}(c_{2,t+1})^{-\gamma}]}{E_t[(c_{2,t+1})^{1-\gamma}]} \quad \forall t \tag{5}$$

$$k_{init} = \frac{1}{2}(w_t + x_1 - c_{min} - H_t) \tag{12}$$

$$k_{init} = 2\beta w_t \tag{13}$$

Again, this modification is about the optimizer used to find the optimal risky savings $k_{2,t+1}$. The optimizer in Python requires an input of “initial value” as the start value of the algorithm, and the start value is set differently between the Blanchard model (equation 12) and the Evans model (equation 13).

After changing the initial value to resemble the Blanchard model, I got the result shown in Appendix 1.8. Comparing to results in Appendix 1.7, we can see that there is not much change except when the average risky rate is 4% and the average riskless rate is -0.5%, the percentage change increases from 0.05% to 0.12%.

### 3.9. Modification 9

The last discrepancy between the Blanchard model and the Evans model is also a minor change. While in the Evans paper, the effect of a transfer is shown by the percentage change in average lifetime utility, Blanchard paper did not use percentage change in Figure 7. The values in Figure 7 is the difference in average lifetime utility multiply by 100. Therefore, after changing the percentage change to the difference between two cases, I got the final results in Table 1.9 and generated Figure 1.9.2.

As the Evans model only simulated 8 values, I used the Blanchard model results to produce Figure 1.9.1. We can see that Figure 1.9.2 is an exact match with Figure 1.9.1. Therefore, after all 9 changes, the Evans model can completely replicate the Blanchard model.

| | Table 1.9. Difference in Mean Utility | | | | | |
|---|---|---|---|---|---|---|
| | Modified | | | Blanchard | | |
| | -2.00% | -0.50% | 1.00% | -2.00% | -0.50% | 1.00% |
| 0.00% | 1.0804 | 0.2755 | NaN | 1.0180 | 0.2120 | NaN |
| 2.00% | 1.0270 | 0.2456 | -0.3051 | 0.9680 | 0.1870 | -0.3690 |
| 4.00% | 0.9685 | 0.2235 | -0.3203 | 0.9220 | 0.1620 | -0.3810 |

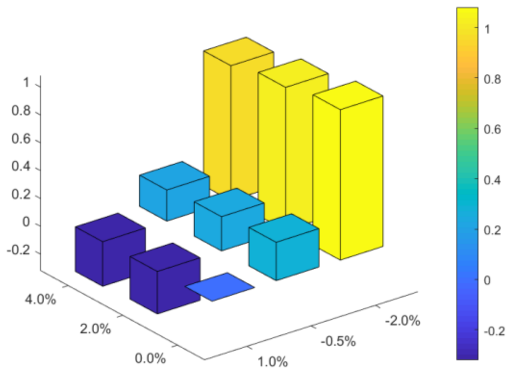


Figure 3.9.1. Modification 9

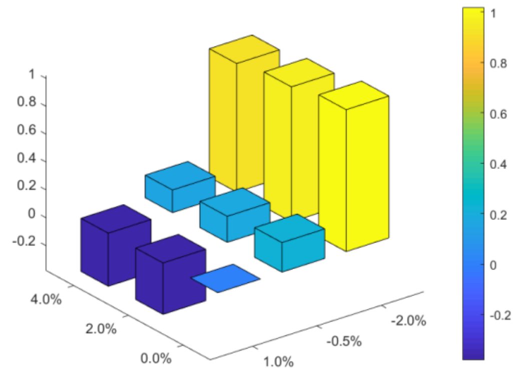


Figure 3.9.2

Blanchard Model Figure 7 (2019)

# 4. Conclusion

Through a complete examination and direct mapping between the Blanchard Python code and Evans code, this paper has identified 9 differences which lead to the final opposite results. While most modifications are just differences in the definition of variables or in the choice of parameter values, after accumulating all modifications, the Evans model can fully replicate the Blanchard model.

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