第三次作业

1 1. 投硬币

1.1 1.1 问题重述

考虑掷硬币试验。分别使用参数为(a,b)=(1,1)和(a,b)=(10,5)的贝塔分布作为先验，用程序分别画出出现下列正面向上的计数结果时，硬币向上的概率参数的后验分布：

1. 投掷0次，0次正面向上
2. 投掷1次，1次正面向上
3. 投掷2次，2次正面向上
4. 投掷3次，2次正面向上
5. 投掷8次，4次正面向上
6. 投掷15次，6次正面向上
7. 投掷50次，24次正面向上
8. 投掷500次，263次正面向上

2 1.2 解答

抛硬币场景，θ为硬币正面向上的概率，x是n次实验观测到正面向上的次数

$$P(\theta \mid x)=\frac{P(x \mid \theta)P(\theta)}{P(x)}=\frac{P(x \mid \theta)P(\theta)}{\int P(x \mid \theta)P(\theta)d\theta} \tag{26}$$

由于原问题为多次抛硬币，分布为二项分布

$$P(x \mid \theta)=C_n^x\theta^x(1-\theta)^{n-x} \tag{27}$$

由于题目假设参数θ的先验分布为Beta分布:

$$P(\theta)=\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1} \tag{28}$$

所以:

$$\begin{aligned}
P(x)&=\int P(x \mid \theta)P(\theta)d\theta \\
&=\int_0^1 C_n^x\theta^x(1-\theta)^{n-x}\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}d\theta \\
&=\frac{C_n^x}{B(a,b)}\int_0^1\theta^{a+x-1}(1-\theta)^{b+n-x-1}d\theta \\
&=C_n^x\frac{B(a+x,b+n-x)}{B(a,b)}\int_0^1\frac{1}{B(a+x,b+n-x)}\theta^{a+x-1}(1-\theta)^{b+n-x-1}d\theta \\
&=C_n^x\frac{B(a+x,b+n-x)}{B(a,b)}\int_0^1 Beta(a+x,b+n-x)d\theta \\
&=C_n^x\frac{B(a+x,b+n-x)}{B(a,b)}
\end{aligned} \tag{29}$$

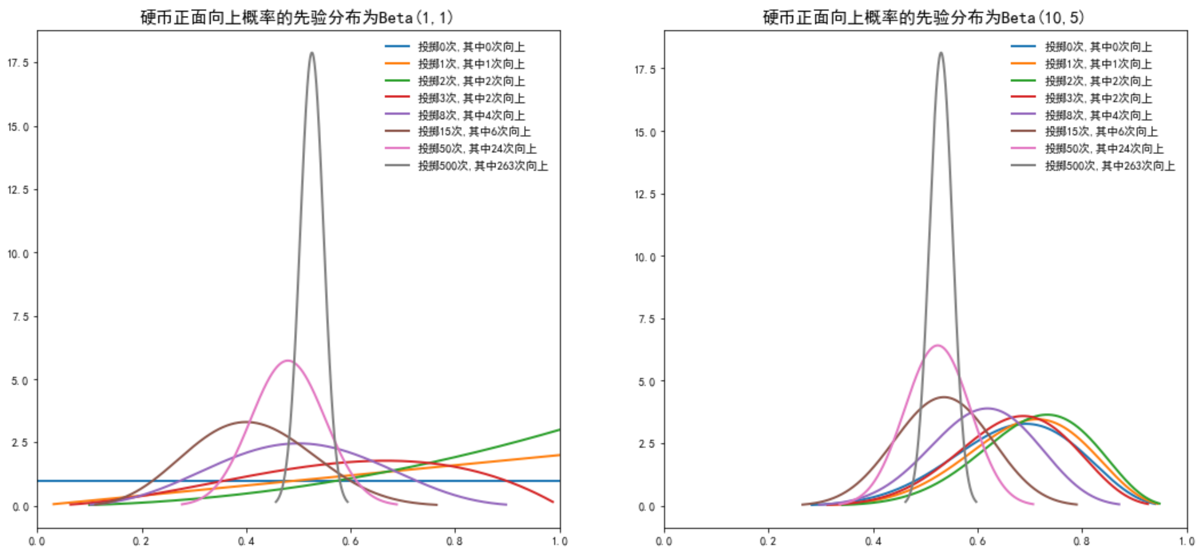
所以后验分布:

$$\begin{aligned}
P(\theta \mid x)&=\frac{C_n^x\theta^x(1-\theta)^{n-x}\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}}{C_n^x\frac{B(a+x,b+n-x)}{B(a,b)}} \\
&=\frac{1}{B(a+x,b+n-x)}\theta^{a+x-1}(1-\theta)^{n+b-x-1} \\
&=Beta(\theta \mid a+x,b+n-x)
\end{aligned} \tag{30}$$

```python
import numpy as np
# https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.beta.html
from scipy.stats import beta
import matplotlib.pyplot as plt
plt.rcParams['font.sans-serif'] = ["SimHei"]  # 用来正常显示中文标签
plt.rcParams['axes.unicode_minus'] = False  # 用来正常显示负号


def draw_beta_plot(ax, a, b, n, x, style='r-'):  # 其中a,b为先验分布的参数，n为总投掷次数，x为正面向上次数
    result_a = a+x
    result_b = b+n-x
    x_line = np.linspace(beta.ppf(0.001, result_a, result_b),
                         beta.ppf(0.999, result_a, result_b), 1000)
    ax.plot(x_line, beta.pdf(x_line, result_a, result_b),
            lw=2, label="投掷{n}次,其中{x}次向上".format(n=n, x=x))
    ax.legend(loc='best', frameon=False)
    plt.xlim(0,1)


# 先验为(a,b)=(1,1)和(a,b)=(10,5)
# 定义一组alpha 跟 beta值
n_list = [0,1,2,3,8,15,50,500]
x_list = [0,1,2,2,4,6,24,263]
plt.figure(figsize=(18, 8))
ax = plt.subplot(1,2,1)
ax.set_title("硬币正面向上概率的先验分布为Beta(1,1)", fontsize=15)
for n,x in zip(n_list,x_list):
    draw_beta_plot(ax,a=1,b=1,n=n,x=x)
ax = plt.subplot(1,2,2)
ax.set_title("硬币正面向上概率的先验分布为Beta(10,5)", fontsize=15)
for n,x in zip(n_list,x_list):
    draw_beta_plot(ax,a=10,b=5,n=n,x=x)
plt.savefig("抛硬币.png",dpi=300)
```



3 2.分布证明

3.1 2.1 多项分布的共轭先验是狄利克雷分布

似然函数是多项分布,其中θ_i表示第i类出现的概率，n_i表示第i类出现的数量。通过伽马函数$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$对阶乘进行近似,有$\Gamma(x+1)=x!$：

$$P(x\mid\theta)=\frac{n!}{n_1!n_2!\ldots n_k!}\prod_{i=1}^k\theta_i^{n_i}=\frac{\Gamma(n+1)}{\prod_{i=1}^k\Gamma(n_i+1)}\prod_{i=1}^k\theta_i^{n_i} \tag{31}$$

其中，$\sum_{i=1}^k\theta_i=1$.

假设概率$\theta=(\theta_1,\theta_2,\ldots,\theta_k)$的先验分布为参数是$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)$的狄利克雷分布$Dir(\alpha)$:

$$P(\theta)=\frac{\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(\alpha_i)}\prod_{i=1}^k\theta_i^{\alpha_i-1} \tag{32}$$

则可以算出归一化因子:

$$\begin{aligned}P(x)&=\int P(x\mid\theta)P(\theta)d\theta\\&=\int_0^1\frac{\Gamma(n+1)}{\prod_{i=1}^k\Gamma(n_i+1)}\prod_{i=1}^k\theta_i^{n_i}\frac{\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(\alpha_i)}\prod_{i=1}^k\theta_i^{\alpha_i-1}d\theta\\&=\int_0^1\frac{\Gamma(n+1)\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(n_i+1)\Gamma(\alpha_i)}\prod_{i=1}^k\theta_i^{n_i+\alpha_i-1}d\theta\\&=\frac{\Gamma(n+1)\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(n_i+1)\Gamma(\alpha_i)}\frac{\prod_{i=1}^k\Gamma(n_i+\alpha_i)}{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}\int_0^1\frac{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}{\prod_{i=1}^k\Gamma(n_i+\alpha_i)}\prod_{i=1}^k\theta_i^{n_i+\alpha_i-1}d\theta\\&=\frac{\Gamma(n+1)\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(n_i+1)\Gamma(\alpha_i)}\frac{\prod_{i=1}^k\Gamma(n_i+\alpha_i)}{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}\int_0^1 Dir(n+\alpha)d\theta\\&=\frac{\Gamma(n+1)\Gamma(\sum_{i=1}^k\alpha_i)}{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}\prod_{i=1}^k\frac{\Gamma(n_i+\alpha_i)}{\Gamma(n_i+1)\Gamma(\alpha_i)}\end{aligned} \tag{33}$$

所以θ的后验分布:

$$\begin{aligned}P(\theta\mid x)&=\frac{P(x\mid\theta)P(\theta)}{P(x)}\\&=\frac{\frac{\Gamma(n+1)}{\prod_{i=1}^k\Gamma(n_i+1)}\prod_{i=1}^k\theta_i^{n_i}\frac{\Gamma(\sum_{i=1}^k\alpha_i)}{\prod_{i=1}^k\Gamma(\alpha_i)}\prod_{i=1}^k\theta_i^{\alpha_i-1}}{\frac{\Gamma(n+1)\Gamma(\sum_{i=1}^k\alpha_i)}{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}\prod_{i=1}^k\frac{\Gamma(n_i+\alpha_i)}{\Gamma(n_i+1)\Gamma(\alpha_i)}}\\&=\frac{\Gamma(\sum_{i=1}^k n_i+\alpha_i)}{\prod_{i=1}^k\Gamma(n_i+\alpha_i)}\prod_{i=1}^k\theta_i^{n_i+\alpha_i-1}\\&=Dir(n+\alpha)\end{aligned} \tag{34}$$

θ的先验和后验分布都是狄利克雷分布，似然函数是多项分布，所以多项分布的共轭先验是狄利克雷分布

3.2 2.2 泊松分布的共轭先验是伽马分布

似然函数是泊松分布,n 表示事件发生的次数，λ表示单位时间内随机事件的平均发生次数，通过伽马函数$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$对阶乘进行近似,有$\Gamma(x+1)=x!$：

$$P(x=n\mid\lambda)=\frac{\lambda^n}{n!}e^{-\lambda}=\frac{\lambda^n}{\Gamma(n+1)}e^{-\lambda} \tag{35}$$

假设λ服从参数为(a,b)的伽马分布Ga(a,b):

$$P(\lambda)=\frac{\lambda^{a-1}e^{-b\lambda}b^a}{\Gamma(a)} \tag{36}$$

则可以算出归一化因子:

$$\begin{aligned}
P(x) &= \int P(x \mid \lambda) P(\lambda) d\lambda \\
&= \int_0^1 \frac{\lambda^n}{\Gamma(n+1)} e^{-\lambda} \frac{\lambda^{a-1} e^{-b\lambda} b^a}{\Gamma(a)} d\lambda \\
&= \frac{b^a}{\Gamma(n+1)\Gamma(a)} \frac{\Gamma(n+a)}{(b+1)^{n+a}} \int_0^1 \lambda^{n+a-1} e^{-(b+1)\lambda} \frac{(b+1)^{n+a}}{\Gamma(n+a)} d\lambda \\
&= \frac{b^a}{\Gamma(n+1)\Gamma(a)} \frac{\Gamma(n+a)}{(b+1)^{n+a}} \int_0^1 Ga(n+a, b+1) d\lambda \\
&= \frac{b^a}{\Gamma(n+1)\Gamma(a)} \frac{\Gamma(n+a)}{(b+1)^{n+a}}
\end{aligned} \tag{37}$$

所以λ的后验分布:

$$\begin{aligned}
P(\lambda \mid x) &= \frac{P(x \mid \lambda) P(\lambda)}{P(x)} \\
&= \frac{\frac{\lambda^n}{\Gamma(n+1)} e^{-\lambda} \frac{\lambda^{a-1} e^{-b\lambda} b^a}{\Gamma(a)}}{\frac{b^a}{\Gamma(n+1)\Gamma(a)} \frac{\Gamma(n+a)}{(b+1)^{n+a}}} \\
&= \frac{(b+1)^{n+a}}{\Gamma(n+a)} \lambda^{n+a-1} e^{-(b+1)\lambda} \\
&= Ga(n+a, b+1)
\end{aligned} \tag{38}$$

λ的先验和后验分布都是伽马分布，似然函数是泊松分布，所以泊松分布的共轭先验是伽马分布.

3.3 2.3 指数分布的共轭先验是伽马分布

似然函数是指数分布,n 表示事件发生的次数，λ表示单位时间内随机事件的平均发生次数:

$$P(x = n \mid \lambda) = \lambda e^{-\lambda x} \tag{39}$$

假设λ服从参数为(a,b)的伽马分布Ga(a,b):

$$P(\lambda) = \frac{\lambda^{a-1} e^{-b\lambda} b^a}{\Gamma(a)} \tag{40}$$

则可以算出归一化因子:

$$\begin{aligned}
P(x) &= \int P(x \mid \lambda) P(\lambda) d\lambda \\
&= \int_0^1 \lambda e^{-\lambda x} \frac{\lambda^{a-1} e^{-b\lambda} b^a}{\Gamma(a)} d\lambda \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+1)}{(b+1)^{a+1}} \int_0^1 \lambda^a e^{-(b+1)\lambda} \frac{(b+1)^{a+1}}{\Gamma(a+1)} d\lambda \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+1)}{(b+1)^{a+1}} \int_0^1 Ga(a+1, b+1) d\lambda \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+1)}{(b+1)^{a+1}}
\end{aligned} \tag{41}$$

所以λ的后验分布:

$$\begin{aligned}
P(\lambda \mid x) &= \frac{P(x \mid \lambda) P(\lambda)}{P(x)} \\
&= \frac{\lambda e^{-\lambda x} \frac{\lambda^{a-1} e^{-b\lambda} b^a}{\Gamma(a)}}{\frac{b^a}{\Gamma(a)} \frac{\Gamma(a+1)}{(b+1)^{a+1}}} \\
& (b+1)^{a+1}
\end{aligned} \tag{42}$$

$$
\begin{aligned}
&= \frac{(b+1)^{a+1}}{\Gamma(a+1)} \lambda^a e^{-(b+1)\lambda} \\
&= Ga(a+1, b+1)
\end{aligned}
$$

λ的先验和后验分布都是伽马分布，似然函数是指数分布，所以指数分布的共轭先验是伽马分布.

3.4 2.4 方差已知的正态分布的共轭先验是正态分布

似然函数是方差已知的正态分布分布, σ^2表示分布的方差为已知固定值不是随机变量，μ表示分布的均值，x是随机变量:

$$
P(x \mid \mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{43}
$$

假设参数μ服从参数为(a, b^2)的正态分布$\mu \sim N(a, b^2)$:

$$
P(\mu) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-a}{b}\right)^2} \tag{44}
$$

则可以算出归一化因子:

$$
\begin{aligned}
P(x) &= \int P(x \mid \mu) P(\mu) d\mu \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-a}{b}\right)^2} d\mu \\
&= \frac{1}{2\sigma b\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-\mu)^2 b^2 + (\mu-a)^2\sigma^2}{\sigma^2 b^2}} d\mu \\
&= \frac{1}{2\sigma b\pi} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(\frac{\left(\mu - \frac{xb^2+a\sigma^2}{\sigma^2+b^2}\right)^2}{\frac{\sigma^2 b^2}{\sigma^2+b^2}} + \frac{(x-a)^2}{\sigma^2+b^2}\right)\right) d\mu \\
&= \frac{e^{-\frac{1}{2}\frac{(x-a)^2}{\sigma^2+b^2}}}{2\sigma b\pi} \frac{\sigma b}{\sqrt{\sigma^2+b^2}} \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{1}{\frac{\sigma b}{\sqrt{\sigma^2+b^2}}\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{\left(\mu - \frac{xb^2+a\sigma^2}{\sigma^2+b^2}\right)^2}{\frac{\sigma^2 b^2}{\sigma^2+b^2}}\right)\right) d\mu \\
&= \frac{1}{\sqrt{\sigma^2+b^2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-a)^2}{\sigma^2+b^2}} \int_{-\infty}^{\infty} \mu \sim N\left(\frac{xb^2+a\sigma^2}{\sigma^2+b^2}, \frac{\sigma b}{\sqrt{\sigma^2+b^2}}\right) d\mu \\
&= \frac{1}{\sqrt{\sigma^2+b^2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-a)^2}{\sigma^2+b^2}}
\end{aligned} \tag{45}
$$

所以μ的后验分布:

$$
\begin{aligned}
P(\mu \mid x) &= \frac{P(x \mid \mu) P(\mu)}{P(x)} \\
&= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-a}{b}\right)^2}}{\frac{1}{\sqrt{\sigma^2+b^2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-a)^2}{\sigma^2+b^2}}} \\
&= \frac{\sqrt{\sigma^2+b^2}}{\sigma b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{\mu-a}{b}\right)^2 - \frac{(x-a)^2}{\sigma^2+b^2}\right)} \\
&= \frac{1}{\frac{\sigma b}{\sqrt{\sigma^2+b^2}}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\mu - \frac{xb^2+a\sigma^2}{\sigma^2+b^2}\right)^2}{\frac{\sigma^2 b^2}{\sigma^2+b^2}}} \\
&= N\left(\frac{xb^2+a\sigma^2}{\sigma^2+b^2}, \frac{\sigma b}{\sqrt{\sigma^2+b^2}}\right)
\end{aligned} \tag{46}
$$

λ的先验和后验分布都是正态分布，似然函数是正态分布分布，所以方差已知的正态分布的共轭先验是正态分布。

3.5 2.5 均值已知的正态分布的共轭先验是逆伽马分布

似然函数是均值已知的正态分布分布, σ^2表示分布的方差，μ表示分布的均值为已知固定值不是随机变量，x是随机变量:

$$P(x \mid \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{47}$$

假设参数σ^2服从参数为(a,b)的逆伽马分布$\sigma^2 \sim IGa(a,b)$:

$$p(\sigma^2) = \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{b}{\sigma^2}} \tag{48}$$

则可以算出归一化因子:

$$\begin{aligned}
P(x) &= \int P(x \mid \sigma^2) P(\sigma^2) d\sigma^2 \\
&= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{b}{\sigma^2}} d\sigma^2 \\
&= \frac{1}{\sqrt{2\pi}} \frac{b^a}{\Gamma(a)} \int_0^\infty \left(\frac{1}{\sigma^2}\right)^{a+\frac{3}{2}} e^{-\frac{b}{\sigma^2}-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d\sigma^2 \\
&= \frac{1}{\sqrt{2\pi}} \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+\frac{1}{2})}{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}} \int_0^\infty \frac{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}}{\Gamma(a+\frac{1}{2})} \left(\frac{1}{\sigma^2}\right)^{a+1+\frac{1}{2}} e^{-\frac{b+\frac{1}{2}(x-\mu)^2}{\sigma^2}} d\sigma^2 \\
&= \frac{1}{\sqrt{2\pi}} \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+\frac{1}{2})}{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}} \int_0^\infty \sigma^2 \sim IGa\left(a+\frac{1}{2}, b+\frac{1}{2}(x-\mu)^2\right) d\sigma^2 \\
&= \frac{1}{\sqrt{2\pi}} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \frac{b^a}{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}}
\end{aligned} \tag{49}$$

所以μ的后验分布:

$$\begin{aligned}
P(\sigma^2 \mid x) &= \frac{P(x \mid \sigma^2) P(\sigma^2)}{P(x)} \\
&= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{b}{\sigma^2}}}{\frac{1}{\sqrt{2\pi}} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \frac{b^a}{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}}} \\
&= \frac{\left(b+\frac{1}{2}(x-\mu)^2\right)^{a+\frac{1}{2}}}{\Gamma(a+\frac{1}{2})} \left(\frac{1}{\sigma^2}\right)^{a+1+\frac{1}{2}} e^{-\frac{b+\frac{1}{2}(x-\mu)^2}{\sigma^2}} \\
&= IGa\left(a+\frac{1}{2}, b+\frac{1}{2}(x-\mu)^2\right)
\end{aligned} \tag{50}$$

λ的先验和后验分布都是逆伽马函数，似然函数是正态分布分布，所以均值已知的正态分布的共轭先验是逆伽马分布。