§1. Intro to Mathematical Logic

之所以叫 Meta-Math：是因为用数学的方法研究数学本身.（妙啊）

形式化地证明定理工作量巨大

Definition: Proof. 这个定义依然是相对模糊的.

A proof is an argument that uses logical steps to show that a mathematical statement follows from certain assumptions

i.e Rules

Hilbert's Program

纯形式化地证明一切数学定理（仅通过符号的推导, 哪怕不知道符号的意思）

in a purely mechanical way

Definition. Formal System

A Formal System is a tuple of 3 things

$$(\mathcal{L}, R, A)$$

① L : Language

$$\mathcal{L} = (\underset{\uparrow}{\Sigma}, \underset{\uparrow}{G})$$

G — Grammar：语法规则

Σ — Symbols：

包括变量、括号、运算符、量词等一切能写下来的东西

② Rules.

指的是 rules of logic that we use when we write proofs.

例子：如果能证明 $\neg(\phi)$，也能证明 $\phi \vee \psi$，那么就能证明 ψ

Denoted by $\dfrac{\neg(\phi), \phi \vee \psi}{\psi}$ $\Longleftarrow$ 这个符号表示 deduce

③ Axioms.

指的是 Statements that describe the very basic behavior of numbers, sets, or any math objects.

We use Axioms as assumptions in our proofs

i.e. Axioms 不需要被证明

在一个 Formal System (L, R, A) 中.

① 可以证明 Language L is complete: 所有的定理可以用语言表述

② Gödel's Completeness Theorem: Rules 是完备的

i.e. 如果某个 statement 能被证明.

就可以只用这些 Rules 证明.

③ Gödel's Incompleteness Theorem: Axioms 是不完备的.

有的 statement 不能通过公理证明.

{ Mathematical Logic 主要关于 Language 和 Rules 部分.
Set Theory 主要关于 Axioms 部分

妙啊! 讲得超级清晰

§2. Strings

Definition n-tuple.

let A be <u>any set</u>. $A^n := \{\langle a_1, a_2 \cdots, a_n\rangle \mid a_i \in A\}$.

特别地. $A^0 = \{\langle\rangle\}$.

Definition: the set of strings of A.

$$A^{<\omega} := A^0 \cup A^1 \cup A^2 \cup \cdots$$
$$= \bigcup_{k=0}^{\infty} A^0$$

Notation: 我们可以省略“<”,“>”和“,”.

$$\langle a_1, a_2 \cdots, a_n\rangle =: a_1 a_2 \cdots a_n. \quad a_i \in A$$

§3. Logic. The Language of Sentential Logic

Sentential Logic = Propositional Logic

都是"命题逻辑".

之所以叫 Sentential Logic，是因为构成该种逻辑的基本 block 为 Sentence.

Language of Sentential Logic.

① Symbols: $($, $)$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ — 这五个叫 **Logic Connectives 逻辑连接词**

Σ $A_1, A_2 \cdots$ (variables)

② Grammar (G):

- An expression is a string of Σ.
- A well-formed formula (wff) is an expression that makes sense.

这里实际上比较 trivial. 所以只记录一部分.

Definition: The wff are strings of Σ that are built in the following way

(1). Every variable symbol is a wff.

(2) If ϕ and ψ are wffs, then

$\Rightarrow \neg\phi$. (ϕ). $\phi \wedge \psi$. $\phi \vee \psi$. $\phi \to \psi$. $\phi \leftrightarrow \psi$ are all wffs.

§. 4. Truth Assignments

We want to assign a truth value to each formula

let S be a set of sentential variables.

$$S=\{A_1, A_2, \cdots, A_k\}.$$

A truth assignment is a function

$$v: S \to \{T, F\}.$$

e.g. $v(A_1)=T$. $v(A_2)=F$. etc.

△ Whenever we have $v: S \to \{T, F\}$ we have the extended truth assignment $\bar{v}: \bar{S} \to \{T, F\}$.

where $\bar{S} := \{$all the wffs constructed from elements in $S\}$.

We are going to do the recursion: (具体实现).

(1) let ϕ be a sentential variable. then $\bar{v}(\phi)=\phi$

(2) let ϕ be $\neg\psi$. then $\bar{v}(\phi) \in \{T, F\} \setminus \bar{v}(\psi)$.

(3) $\bar{v}(\phi \wedge \psi) = \begin{cases} T & \text{if } \bar{v}(\phi)=T \text{ and } \bar{v}(\psi)=T \\ F & \text{else} \end{cases}$

. . .

都是一些熟知的定义

可以用表格的形式呈现 truth value，称为真值表.

ϕ	ψ	$\neg\phi$	$\phi\wedge\psi$	$\phi\vee\psi$	$\phi\rightarrow\psi$	$\phi\leftrightarrow\psi$
T	T					
T	F					
F	T					
F	F					

略

一些不显然的地方是，"$\rightarrow$"和"$\leftrightarrow$"的也被赋予了真值

§5. Tautological Implication. 重言蕴含.

e.g. $\dfrac{(A\vee B)\to C,\quad B\leftrightarrow\neg C}{A\vee\neg B}$

Definition of "Tautological Implication",

Let Σ be a set of wff, τ be a single wff.
(注意不同于前文)

we say that $\Sigma \vDash \tau$ iff any truth assignment v
"tautologically implies"
that satisfies all formulars in Σ, also satisfies τ

e.g. $\{(A\vee B)\to C,\ B\leftrightarrow\neg C\} \vDash A\vee\neg B$

要使得 Σ 中的 wff 均为真, 必须有

$$\begin{cases} v(A)=T \\ v(B)=F \\ v(C)=T \end{cases} \quad 或 \quad \begin{cases} v(A)=F \\ v(B)=F \\ v(C)=T \end{cases}$$

而此时 $v(A\vee\neg B)=T$

根据定义. Check.

Definition. We say two wffs ϕ and ψ are tautologically equiv.

$$\{\phi\} \models \psi \text{ and } \{\psi\} \models \phi.$$

记作 $\phi \models\!\dashv \psi$.

重要的 tautologically equivalence.

Associativity 结合律

$$(A \wedge B) \wedge C \models\!\dashv A \wedge (B \wedge C).$$

$$(A \vee B) \vee C \models\!\dashv A \vee (B \vee C).$$

$$(A \leftrightarrow B) \leftrightarrow C \models\!\dashv A \leftrightarrow (B \leftrightarrow C)$$

Distributivity 分配律.

$$A \wedge (B \vee C) \models\!\dashv (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) \models\!\dashv (A \vee B) \wedge (A \vee C).$$

Negation

$$\neg\neg A \models\!\dashv A.$$

$$\neg(A \to B) \models\!\dashv \neg B \to \neg A$$

$$\neg(A \wedge B) \models\!\dashv (\neg A) \vee (\neg B).$$

$$\neg(A \vee B) \models\!\dashv (\neg A) \wedge (\neg B)$$

直观感受是, 原本写 "=" 的地方, 可以写成 t-equivalence, 显得很高级.

§6. Completeness of Sentential Logic

Question. Sentential Logic 中的 5个 connectives 够了吗.

$\{\neg, \wedge, \vee, \to, \leftrightarrow\}$.

假如有一个新的连接符 #. 是三元连接符.

$\bar{v}(\#\phi\psi\theta) = \begin{cases} T, & \text{if } \{\phi, \psi, \theta\} \text{ 中至少有2个的} \bar{v} \text{为} T. \\ F, & \text{else.} \end{cases}$

可以证明. $\#\phi\psi\theta \models\!\!\!\!= (\phi\wedge\psi)\vee(\psi\wedge\theta)\vee(\phi\wedge\theta)$

Definition k-place boolean function.

function $f: \{T, F\}^k \to \{T, F\}$ is a k-place boolean function

(※ 这里有点 Synthesis 的味道).

显然. 如果一个 wff α 用到了 k个变量, 那它就和一个 k-place boolean function B_α^k 相互对应.

△ 我们称 B_α^k is realized by α.

△ Observation

(1). α, β are two wffs with n variables $A_1, A_2, \cdots, A_n$.

then $\alpha \models\!\dashv \beta$ iff $\underbrace{B_\alpha^n = B_\beta^n}$

(只关注I/O而不注意内部实现)

i.e. $\forall x \in \{T, F\}^n \quad B_\alpha^n(x) = B_\beta^n(x)$

(2) α is a tautology (永真式) iff $\forall x \in \{T, F\}^n;\ B_\alpha(x) = T$

这里"$\forall$"符号还没介绍，但我们假定读者都熟知. lmao.

△ Question:

Can every k-place boolean function be realized by a wff?

答案是可以

这里太像 synthesis 了.

Thm: (Emil Leon, 1921, Introduction to General Theory of Elementary Propositions)

Every n-place boolean function f can be realized by a wff with n variables. $A_1, A_2, \cdots, A_n$.

Pf of thm:

① 如果只有一个 $v \in \{T, F\}^n$ s.t. $f(v) = T$.

假设 $v = (a_1, a_2, \cdots, a_n)$ $\quad a_i \in \{T, F\}$.

则 $\alpha = \bigwedge_{i=1}^{n} B_i$ 其中 $B_i = \begin{cases} A_i, & \text{if } a_i = T \\ \neg A_i, & \text{if } a_i = F \end{cases}$

$\Uparrow$

非常自然的想法.

② 如果 $f(v_i) = T$ for $i = 1, 2, \cdots, l$. $v_i \in \{T, F\}^n$.

假设 $v_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ $\quad a_{ij} \in \{T, F\}$.

令 $\alpha = \bigvee_{i=1}^{l} \left(\bigwedge_{j=1}^{n} B_{ij} \right)$

注意下标

其中 $B_{ij} = \begin{cases} A_j, & \text{if } a_{ij} = T \\ \neg A_j, & \text{if } a_{ij} = F \end{cases}$

△ 以上的证明说明，给定任何一个 finite n-place boolean function. 都可以用包含 n 个变量的命题逻辑 wff 和 $\{\vee, \wedge, \neg\}$ 来 realize. 因此称在命题逻辑中, $\{\vee, \wedge, \neg\}$ 是完备的，也可以说. 命题逻辑的语言 (Language)

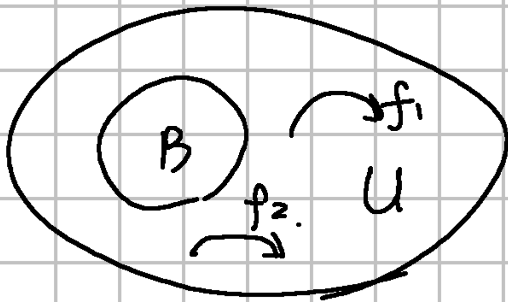
是完备的 (Complete)

或者说 Propositional Logic is <u>Sematically</u> complete

(but not syntactically).

§7. Generating Sets out of Functions.

Let U be a set, $B \subset U$. $\underline{\underline{F}}$ be a set of operations on U.



Definition. $\boxed{\text{Closed}}$ $\Longleftarrow$ 一个 Top-down 定义.

A subset $S \subset U$ is closed under F iff

$$\forall f \in F, \forall s \in S^{<\omega}: f(s) \in S$$

在此基础上，称S还是 inductive的

如果有: $B \subset S$ （B是我们自己定义的 Base 集）

其中 $S^{<\omega} := \bigcup_{n=0}^{\infty} S^n$

令 $C^* = \bigcap_{\substack{S \subset U \\ S \text{ is inductive}}} S$

直观上（从拓扑的角度讲. Closed 且 $\forall S_\alpha \cap S_\beta \neq \phi \Rightarrow C^* \neq \phi$）

可以注意到:

① C^* 仍然是 inductive 的.

根据定义仍然有 $B \subset C^*$. $C^* \subset U$

下面证明 C^* is closed take $\bar{c} \in C^*$, $f \in F$.

take an <u>inductive</u> set S. then

$$\bar{c} \in C^* \subset S.$$

$\Rightarrow f(\bar{c}) \in S.$ (因为 S is closed).

$\Rightarrow f(\bar{c}) \in \bigcap_{S \text{ is inductive}} S$ (由 S 的任意性可得)

$= C^*$ (由任意性有 C^* is closed)

(说明 the intersection of all closed sets (with...) is closed)

② C^* 是最小的 inductive set.

这是显然的.

此时称 C^* 是 B 在 $\mathcal{F}$ 下的 <u>Closure</u>

Definition. Closed. ⟸ bottom up 的定义.

令 $C_0 = B$, $C_1 := \{ f(\bar{c}) \mid f \in \mathcal{F}, \bar{c} \in C_0^{<\omega} \} \cup C_0$ 好啊!

...

$$C_n := \{ f(\bar{c}) \mid f \in \mathcal{F}, \bar{c} \in C_{n-1}^{<\omega} \} \cup \left(\bigcup_{k=0}^{n-1} C_k \right)$$

...

令 $C_* = \bigcup_{n=0}^{\infty} C_n$ (通过无穷并避免了使用 lim 符号)

此时同样称 C_* 是 B 在 $\mathcal{F}$ 下的 <u>Closure</u>

※ 我们要证明 $C^* = C_*$

在此之前我们先举个例子. $U=\mathbb{R}$. $B=\{0\}$. $\mathcal{F}=\{f\}$

$f(x)=x+1$. $x\in\mathbb{R}$

所以 $C_* = \{0\}\cup\{1\}\cup\cdots = \mathbb{Z}_{\geq 0}$.

$C^* = \mathbb{Z}_{\geq 0}$ (这个例是不怎么好说明)

$U=\Sigma$ (symbols in Propositional Logic). $B=\{A_i \mid i\in\mathbb{N}\}$.

$\mathcal{F}=\{f_\wedge, f_\vee, f_\neg, f_\rightarrow, f_\leftrightarrow\}$.

$f_\wedge: U\times U\to U$ ...

$(x,y)\to(x\wedge y)$.

then the <u>Closure of B under $\mathcal{F}$</u> would be all the wffs in propositional logic.

(Meet in the middle!)

① $C_* \subset C^*$.

只要证明 $\forall n\in\mathbb{Z}_{\geq 0}$, $C_n\subset C^*$.

Proof by induction on n.

$C_0 = B \subset C^*$ by definition. (C^* is inductive)

if $C_n\subset C^*$ for some $n\in\mathbb{N}$. since C^* is closed under $\mathcal{F}$, 根据定义 $C_{n+1}\subset C^*$. □

② $C^* \subset C_*$

只要证明 C_* is inductive. (因为 C^* 是最小的 inductive set).

首先 $B \subset C_n \ (\forall n) \Rightarrow B \subset C_*$.

如果 $\bar{x} \in C_*$. $f \in \mathcal{F}$. (只要证明 $f(\bar{x}) \in C_*$)

$$\bar{x} \in \bigcup_{n=0}^{\infty} C_n \Rightarrow \exists n \in \mathbb{Z}_{\geq 0}.\ \bar{x} \in C_n$$
$$\Rightarrow f(\bar{x}) \in C_{n+1}$$
$$\Rightarrow f(\bar{x}) \in C_* = \bigcup_{n=0}^{\infty} C_n.$$

$$\frac{C_* \subset C^*,\ C^* \subset C_*}{C^* = C_*} \quad !!!$$

此时我们记 $C = C^*$ $(= C_*)$. 称 C is the closure of B under $\mathcal{F}$.

§8. Structional Induction and Recursion.

Theorem (Induction Principle).

Assume C is generated by $\mathcal{F}$ starting from B.
Consider a property P and suppose that

① all elements in B satisfy P.

② for all $f \in \mathcal{F}$, if $\bar{c} \in U$ satisfies P, then $f(\bar{c})$ satisfies P.

then all the elements in C satisfies property P.

Pf. Claim: $P = \{u \in U \mid u \text{ satisfies } P\}$ is inductive

(这样的话. $C \subset P$ ✓)

首先根据定义的要求①: $B \subset P$

根据要求②: P is closed

所以 P is inductive. 故 $C \subset P$.

习题2. (Recursion Principle)

Setting $\mathcal{F} = \{f, g\}$, where $f: U \times U \to U$, $g: U \to U$.

Assume that C is generated by $\mathcal{F}$ starting from B.

Definition. C is generated by $\mathcal{F}$ starting from B **freely** if B, $f(C,C)$, $g(C)$ are disjoint.

这里指的是 $x \in C^2$

i.e. $\forall \bar{c} \in C$,

$$\boxed{C = B \sqcup f(C^2) \sqcup g(C)}$$

(1) either $\bar{c} \in B$

(2) or $\bar{c} = f(\bar{s})$ for some unique $\bar{s} \in C^2$ (或 $C^{<\omega}$)

(3) or $\bar{c} = g(s)$ for some unique $s \in C$ (或 $C^{<\omega}$)

e.g. $B = \{A_1, A_2, A_3\}$. $\mathcal{F} = \{f_\to, f_\vee\}$.

如果 $C = \{(A_1 \vee A_2) \to A_3\}$. 则 C is freely ....

如果 C 包含了 $A_1 \vee A_2 \to A_3$. 则 C 不是 freely ...

Theorem (Recursion Principle). 仍然使用上面的 setting (B, C, f, g).

Assume C is freely generated. Consider a new set V and functions h, F, G s.t.

$h: B \to V$

$F: V^2 \to V$ $\quad f: C^2 \to C$

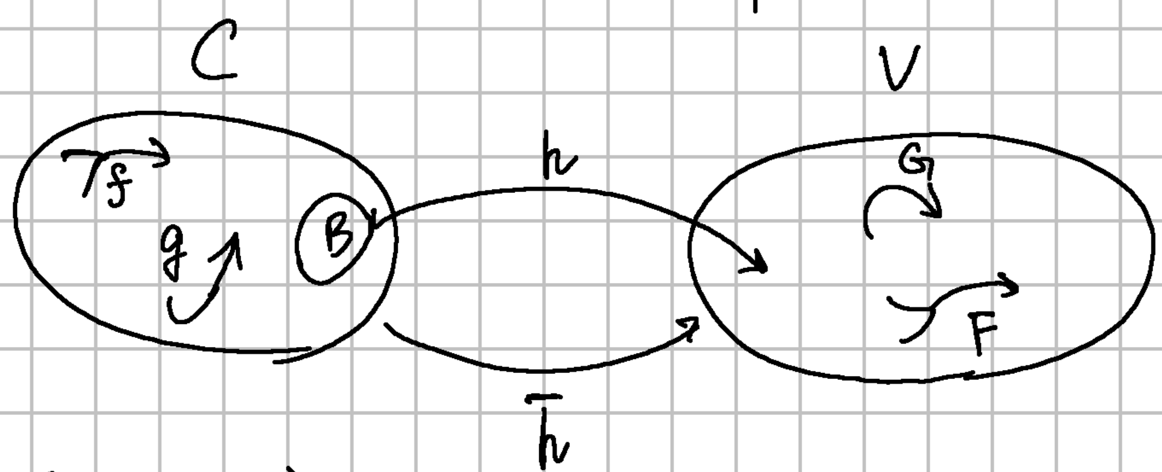
$G: V \to V$ $\quad g: C \to C$

then there exists a unique function $\bar{h}: C \to V$ s.t.

$$\begin{cases} \bar{h}(x) = h(x) & \forall x \in B \\ \bar{h}(f(x,y)) = F(\bar{h}(x), \bar{h}(y)) & x, y \in C \\ \bar{h}(g(x)) = G(\bar{h}(x)) & x \in C \end{cases}$$

根据定义 B, $f(C^2)$, $g(C)$ 互不相交.

$\Rightarrow$ 直观上，最后总是可以从 C 中缩回到 B 中.



原本的 h 只定义在 B 上，现在我们 Claim 有一个 $\bar{h}$ 在 C 上处处有定义（$\bar{h}$ 视为 h 的 extension）

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e.g. v: Variables $\to \{T, F\}$

$\bar{v}$: wff $\to \{T, F\}$.

$\bar{v}(\phi \wedge \psi) = B^2_{A_1 \wedge A_2}(\bar{v}(\phi), \bar{v}(\psi))$. 好啊

Proof of Recursion Principle/Theorem.

这里我们考虑更为一般的情况. $\mathcal{F} = \{f_n: U^n \to U \mid n \in \mathbb{Z}_{\geq 0}\}$.

称 B generated C freely under operations of $\mathcal{F}$, iff.

$$C = \coprod_{n=0}^{\infty} f_n(C^n).$$

Note 1: $f_0(C^0)$ 即为 B, $f_1(C) \Leftrightarrow h: B \to V$

Note 2: 认为 f_n 自动取有意义的定义.

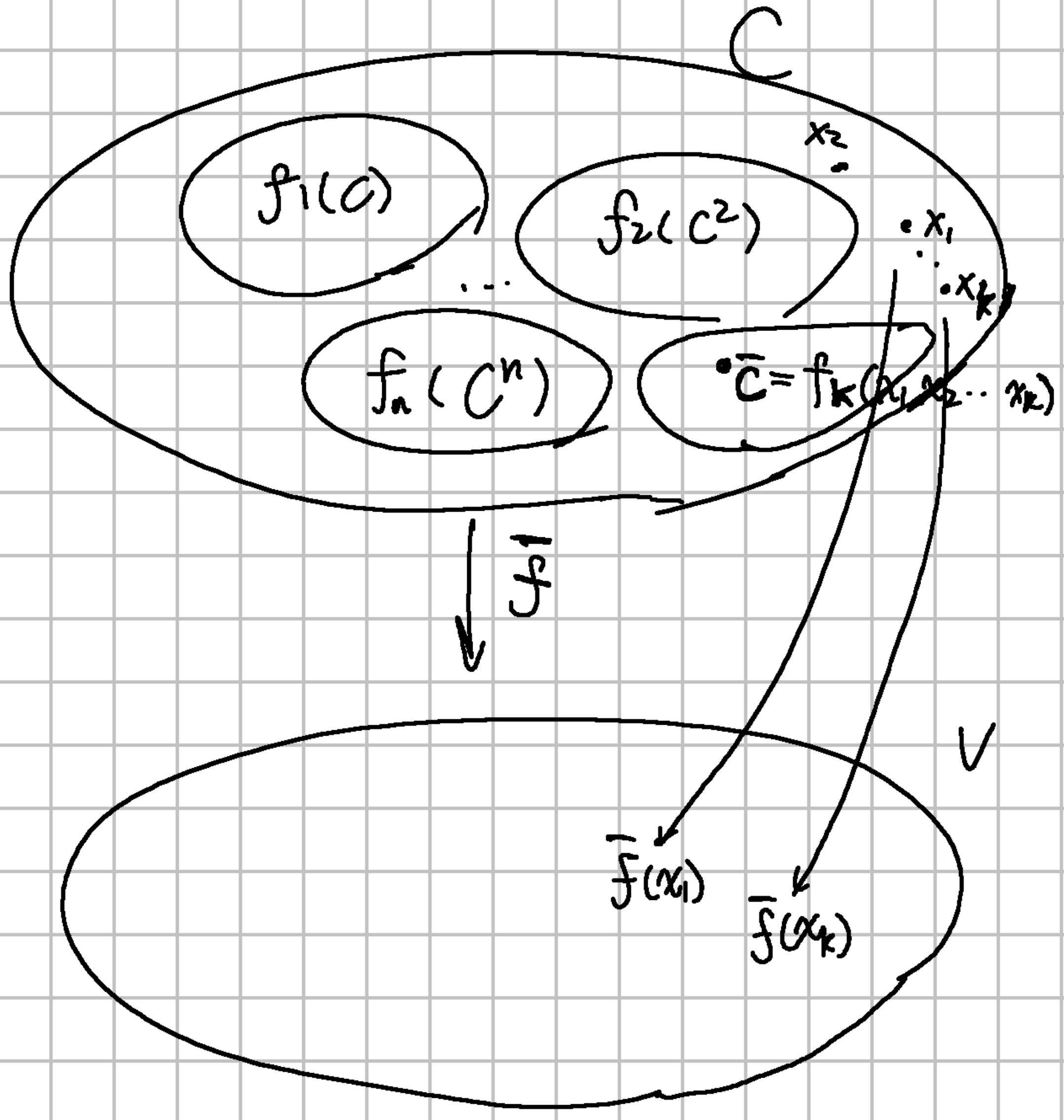
$\forall\ F_n : V^n \to V.\quad n \in \mathbb{Z}_{\geq 0}$

$\exists!\quad \bar{f} : C \to V$

满足 $\bar{f}(f_n(x_1, x_2 \cdots x_n)) = F_n(\bar{f}(x_1), \bar{f}(x_2) \cdots \bar{f}(x_n))$.

$n \in \mathbb{Z}_{\geq 0}$ (分段函数)

$x_1, x_2, \cdots, x_n \in C$



这里目前还是不会证明.

§.9. The Language of First Order Logic.

Symbols:

logic.

① equality: $=$

② connectives: $\wedge$. $\vee$. $\rightarrow$. $\leftrightarrow$. $\neg$. $\forall$. $\exists$.   谓多了这两个.

③ variables: v_1. v_2 ...

④ punctuations: (.)

Vocabulary

- Constant Symbols: c_0. c_1 ...
- Function Symbols: f_0, f_1 ...
- Relation Symbols: R_0, R_1 ...

Definition of Terms (项, 短语)

Terms are strings of symbols that represent elements.

e.g. $1+1$.

Definition of Formulas. (陈述句).

Formulas are strings of symbols that represent statements.

e.g. $1+1=2$. (formula 是有真值的)

更标准地

- constants: c_0. c_1 ...
- variables: v_0, v_1 ...
- functions on terms: $f_i(t_1 . t_2 \cdots t_n)$, t_j is a term

Definition of Atomic Formulas

1. $t_1 = t_2$. where t_1, t_2 are terms
2. $R(t_1, t_2 \cdots t_n)$ where $t_1, t_2 \cdots t_n$ are terms.

※这么多集终于分清关系R和函数f的关系了. lmao

Formal Definition of Formulas.

好啊!

1. All atomic formulas
2. $\neg\varphi$ (where φ is a formula)
3. $(\varphi \wedge \psi)$ (where φ and ψ are formulas)
   $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$
4. $\forall v_i \varphi$ 和 $\exists v_i \varphi$ (where v_i is a variable and φ is a formula.)

$$\text{formula} ::= \text{"}\forall\text{"} + \$\text{var} + \$\text{formula}$$
$$\mid \text{"}\exists\text{"} + \$\text{var} + \$\text{formula}$$
$$\mid \cdots$$

§10. Structures

> Structure (结构)可以理解为论境. 同样的 formula, 在不同的 structure 下可能有不同的真值.

> 考虑 vocabulary $\tau = \{e, *\}$.

> $\forall x \exists y (y * x = e)$ is true in $(\mathbb{Z}; \overset{e}{0}, \overset{*}{+})$

false in $(\mathbb{N}; 0, +)$

ex. 是否存在一个 wff 在 $(\mathbb{R}^+; 1, *)$ 中正确, 但在 $(\mathbb{Q}^+; 1, *)$ 中错误

$\forall x \exists y \quad x * y * y = e$

Definition. Structure

Consider a vocabulary $\tau = \{\underbrace{c_0, c_1 \cdots}_{\text{constant}}, \underbrace{f_0, f_1 \cdots}_{\text{func}}, \underbrace{R_0, R_1 \cdots}_{\text{relations}}\}$

A τ-structure is a tuple

$$\mathcal{M} := (M : a_0, a_1 \cdots g_0, g_1 \cdots H_0, H_1 \cdots)$$

Where.

(1) M is a set we call universe/domain of $\mathcal{M}$.

(2) $a_i \in M$

(3) $g_i : M^\alpha \to M$ (of the same arity as f_i resp.)

(4) $H_i \subset M^\alpha$ are relations of same arity as R_i.

§ 11. Logic. Free Variables.

Definition. Free Variable.

A free variable is the variable that is not bounded by quantifier ($\forall, \exists$).

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e.g. $\exists x (x * x = y)$, structure is $(\mathbb{Q} : +)$

in this case. x is bounded, y is free.

(你甚至可以对 y 分类讨论

Definition. $\boxed{\text{Sentence.}}$

A wff without free variables is called a sentence

指称更明确的

Formal Definition of Free Variables

定义一个从terms集合T 的幂集 映射到变量集合V的函数

$var : T \to \mathcal{P}(V)$ (找到T中所有的变量)

满足:

① $var(v) = \{v\}$ if $v \in V$

② $var(c) = \phi$ if c is a constant

③ $var(f(v_1, v_2, \cdots, v_n)) = \bigcup_{k=1}^{n} var(v_k)$

(where f is an n-ary function)

这里由于函数只会调用已经声明过的变量/函数/值等. 所以我们总是可以假设 $var(v_k)$ 可以得到

在 $var: T \to V$ 的定义的基础上, 我们定义一个新的函数 FV.

$$FV: F \to P(V)$$

where F is the set of formulas, V is the set of variables

(Recall the recursive definition of formulas.

$$\text{formula} = \begin{cases} \text{atomic formula (两种)} \\ \forall x \varphi.\ \exists x \varphi \\ \text{命题逻辑中所有的 wff.} \end{cases}$$
)

($t_1 = t_2$ 也类似)

① $FV(R(t_1, t_2 \cdots t_n)) = \bigcup_{k=1}^{n} var(t_k)$

② $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$

$\phi \to \psi.\ \neg\phi.\ \psi \vee \phi.\ \phi \Leftrightarrow \psi$ (也类似)

③ $FV(\forall x \varphi) = FV(\varphi) \setminus \{x\}$.

$FV(\exists x \varphi) = FV(\varphi) \setminus \{x\}$

整个定义还是比较 trivial 的.

8. Formal Definition of sentence.

A wff φ is called sentence in <u>First Order Logic</u>

iff $FV(\varphi) = \phi$.

§12. Interpretation of Terms.

Definition of $\boxed{\text{Variable Assignment}}$ (通常为 S)

A variable assignment is a function $S: V \to M$

where $\begin{cases} V \text{ is the set of variables} \\ M \text{ is the } \underline{\text{domain}} \text{ of our structure} \end{cases}$ 论域.

We define the extended assignment function

$\bar{S}: T \to M$ where T is the $\underline{\text{set of terms}}$

(依然是 Recursive 定义).

$$\begin{array}{ccccccc} & c_0 & \cdots & f_0 & \cdots & R_0 & \cdots \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

在 Structure $\mathcal{M} = (M: a_0, \cdots, g_0, \cdots, H_0, \cdots)$ 中

$\bar{S}(v) = S(v)$ 如果 $v \in V$.

$\bar{S}(c_i) = a_i$ 如果 c_i is constant

$\bar{S}(f_i(t_1, \cdots t_k)) = g_i(\bar{S}(t_1), \bar{S}(t_2), \cdots \bar{S}(t_k))$

where $\bar{S}(t_\square) \in M$. $g_i: M^k \to M$

---

e.g. $\mathfrak{M} = (\mathbb{Z}: 0, +)$ $\tau = (e, *)$. $S(x) = 1$. $S(y) = 3$.

问 $\bar{S}((x*e)*y) = ?$

Sol. $\bar{S}((x*e)*y) = \bar{S}((x*e)) + \bar{S}(y)$

$= \bar{S}(x) + \bar{S}(e) + \bar{S}(y)$

$= 1 + 0 + 3 = 4$

§13. Interpretation of Formulas

(truth assignment).

> Formula 自带返回值 $r \in \{T, F\}$.

$\models_{\mathcal{M}} \varphi[s]$: Formula φ is true in structure $\mathcal{M}$ according to variable assignment $\underline{s: V \to M}$

(where V is the set of variables and M is the domain of $\mathcal{M}$).

Formal Definition of "$\models_{\mathcal{M}} \varphi[s]$".

①和②是 atomic formula

① $\models_{\mathcal{M}} (t_1 = t_2)[s]$ iff $\underbrace{\bar{s}(t_1)}_{\in M} = \bar{s}(t_2)$.

② $\models_{\mathcal{M}} R_i(t_1, t_2, \cdots, t_k)[s]$ iff $(\bar{s}(t_1), \bar{s}(t_2), \cdots, \bar{s}(t_k)) \in H_i$

其中 H_i 是模型 $\mathcal{M}$ 中解释 R_i 的关系实例.

③ $\models_{\mathcal{M}} \neg\varphi[s]$ iff $\models_{\mathcal{M}} \varphi[s]$ <u>doesn't hold</u>

④ $\models_{\mathcal{M}} \varphi \wedge \psi[s]$ iff $\models_{\mathcal{M}} \varphi[s]$ and $\models_{\mathcal{M}} \psi[s]$.

⑤ $\models_{\mathcal{M}} \varphi \vee \psi[s]$ iff $\models_{\mathcal{M}} \varphi[s]$ or $\models_{\mathcal{M}} \psi[s]$

⑥ $\models_{\mathcal{M}} \varphi \to \psi[s]$ iff $\models_{\mathcal{M}} \neg\varphi[s]$ or $\models_{\mathcal{M}} \psi[s]$

⑦ $\models_{\mathcal{M}} \varphi \leftrightarrow \psi$ ...

既然我们知道"→"和"↔"可以用与或非表示，后面就不赘述了

⑧ $\models_{\mathcal{M}} \forall x \varphi[s]$ iff $\forall a \in M$. $\models_{\mathcal{M}} \varphi[s(x \mapsto a)]$

（关键是把量词移到外面来了，现在 $\models_{\mathcal{M}} \varphi[\ ]$ 形式的我们还没给出定义了.）

⑨ $\models_{\mathcal{M}} \exists x \varphi[s]$ iff $\exists a \in M$ $\models_{\mathcal{M}} \varphi[s(x \mapsto a)]$

（这里 $s(x \mapsto a)(y) := \begin{cases} s(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$

除了将 x 赋成 a 之外，别的保持原有的赋值方式）

Theorem. 非自由变量的 assignment 没有意义（意义不大）.

Let s_1 and s_2 be variable assignments. φ is a wff. Suppose s_1 and s_2 coincide on the free variables of φ. Then,

$$\models_{\mathcal{M}} \varphi[s_1] \text{ iff } \models_{\mathcal{M}} \varphi[s_2]$$

Pf.（目前什么都不会，只能使用最基本的手段证明）

§14. Elementary Classes

Recall that $\models_{\mathcal{M}} \varphi[s]$ 读作.

- ① $\mathcal{M}$ satisfies φ with s
- ② φ is true in $\mathcal{M}$ according to s.
- ③ $\mathcal{M}, s$ model φ

根据 §13. 的最后一个 thm. $\models_{\mathcal{M}} \varphi[s]$ depends only on the values of $s(x)$ for $x \in FV(\varphi)$.

△ 特殊地. 如果 φ is a sentence (i.e. $FV(\varphi) = \phi$). 那么. $\models_{\mathcal{M}} \varphi[s]$ doesn't depend on s.

我们记. $\models_{\mathcal{M}} \varphi$ 为

- ① $\mathcal{M}$ satisfies sentence φ.
- ② sentence φ is true in $\mathcal{M}$
- ③ $\mathcal{M}$ model φ

Definition. $\models_{\mathcal{M}} \Gamma[s]$.

如果 Γ is a set of formulas,

$$\models_{\mathcal{M}} \Gamma[s] := (\forall \varphi \in \Gamma.\ \models_{\mathcal{M}} \varphi[s])$$

Definition. Group

(或者说 Example)

A group is a structure $\mathcal{K} = (K; e, \otimes)$ in the vocab $\tau = \{e, \otimes\}$. where

① e is a constant symbol

② $\otimes$ is a binary function symbol

satisfying

① $\forall x, y, z\, (x \otimes (y \otimes z) = (x \otimes y) \otimes z)$. 结合

② $\forall x\, (x \otimes e = x)$ 有零

③ $\forall x\, (\exists y\, (x \otimes y = e))$ 有逆

这里还给出了线性序 (linear order) 和序域 (ordered field) 的例子

下面又给出了 $\mathbb{N}$ 的 Peano Axioms 和集合的 ZFC 公理

$\tau = \{\in\}$

Definition. Mod.

Given a set of well-formed sentences T, define

$$\operatorname{Mod} T := \{\mathcal{M} \mid \models_{\mathcal{M}} T\}$$

(the class of all structures which model T).

e.g. T is the set of group axioms. then

$\text{Mod}\, T$ is the class of all groups.

结构类 模型论

Definition. Elementary Class (初等类)

A class of structures $\mathcal{K}$ is an elementary class iff there exists a wff φ s.t. $\mathcal{K} = \text{Mod}\, \varphi$.

e.g. the class of all groups is an elementary class.

Q: 不是有三条群公理吗?

A: $g_1 \wedge g_2 \wedge g_3$. lmao.

结构类

Definition. Weakly Elementary Class (弱初等类)

A class of structures $\mathcal{K}$ is an weakly elementary class iff there exists a set of wff s.t.

$$\mathcal{K} = \text{Mod}\, T$$

★ Elementary Class 一定是 weakly elementary class

e.g $T = \{\text{Group-Axiom-1}, \text{Group-Axiom-2}, \text{Group-Axiom-3}\}$

$\cup \{\varphi_n : n \in \mathbb{N}\}$.

其中 φ_n 表示 $\exists x_1, x_2, \cdots x_n$. s.t. $(x_i \neq x_j)$

我们发现Γ中的wff不能合成为一个wff.

△ 有限个 $\rightarrow$ 1个

无限个 $\not\rightarrow$ 1个.

所以无穷群 (infinite group) (也就是Mod Γ) 是弱初等类 (weakly elementary class).

§15. Elementary Equivalence 初等等价

① Definition. (结构的) Elementary Equivalence.

两个 structures M and N are elementary equivalent (写作) $M \equiv N$ iff. for all sentences φ.

$$\models_M \varphi \iff \models_N \varphi.$$

e.g.1. $(\mathbb{Q}; 0, +) \not\equiv (\mathbb{Z}; 0, +)$

因为考虑句子 φ: $\forall x \exists y (y+y=x)$ 作为反例.

e.g.2. $(\mathbb{N}; 0, +) \not\equiv (\mathbb{Z}; 0, +)$

考虑句子 φ: $\forall x \exists y (x+y=0)$ 作为反例.

e.g.3. $(\mathbb{R}; 0, +) \equiv (\mathbb{Q}; 0, +)$ 后面会证明.

(太强了! 这是本课至今的第一个 Non-trivial 结论. $\mathbb{Q}$ 上的洞在 $\tau = \{0, +\}$ 下无法观察到).

e.g.4 $(\mathbb{Z}; \leq) \not\equiv (\mathbb{Q}; \leq)$

只要注意到 $<(x,y) := (\leq(x,y) \wedge \neg(x=y))$.

e.g.5. $(\mathbb{Z}; \leq) \equiv (\mathbb{Z}+\mathbb{Z}; \leq)$. 后面会证明

e.g.6. $(\mathbb{Q}; \leq) \equiv (\mathbb{R}; \leq)$ 后面会证明

Definition. 结构的 Theory.

$$Th(\mathcal{M}) := \{ \varphi \text{ (sentence)} \mid \models_{\mathcal{M}} \varphi \}.$$

Observation. Theory 和结构的关系

$\mathcal{M} \equiv \mathcal{N}$ iff $\boxed{Th(\mathcal{M}) = Th(\mathcal{N})}$

注意. 是"所有的" sentence!

△ Note. $Th(\mathbb{N}; 0, 1, +, \times)$ is not computable.

§16. Logical Implication 邏輯蘊含

Definition. Logical Implication in 一阶邏輯.

Γ is a set of wffs. φ is a wff.

Γ logically implies φ iff.

for all structure $\mathcal{M}$ and every variable assignment $s: V \to M$

$$\models_{\mathcal{M}} \Gamma[s] \Rightarrow \models_{\mathcal{M}} \varphi[s].$$

怎么解释都要强.

记作 $\Gamma \models \varphi$.

特殊地. 如果 Γ is a set of sentences. φ is a sentence

则 $\Gamma \models \varphi := \left(\text{for all structure } \mathcal{M}.\ \models_{\mathcal{M}} \Gamma \Rightarrow \models_{\mathcal{M}} \varphi\right)$.

Definition. Logical equivalence (邏輯等价).

(定义和 Sentential Logic 中一样)

称 wff. ϕ and ψ are logically equivalent. iff.

$$\{\phi\} \models \psi \text{ and } \{\psi\} \models \phi.$$

记作 $\phi \models\!\!\dashv \psi$.

Definition. Satisfiability.

A set Γ of wffs. is called satisfiable iff. there exists a structure $\mathcal{M}$ and $s: V \to M$ s.t.

$$\forall \varphi \in \Gamma: \models_{\mathcal{M}} \varphi[s]$$

Lemma. $\Gamma \models \varphi \Leftrightarrow (\Gamma \cup \{\neg\varphi\}$ is not satisfiable$)$

表示在任意模型中对于任意赋值函数 s.

Pf. if $\Gamma \cup \{\neg\varphi\}$ is satisfiable.

then there is a structure $\mathcal{M}$ and variable assignment $s: V \to M$.

$$\models_{\mathcal{M}} \neg\varphi[s]$$

由于 $\Gamma \models \varphi$. 根据定义 $\models_{\mathcal{M}} \varphi[s]$

根据定义 $\models_{\mathcal{M}} \varphi \wedge \neg\varphi[s]$

反过来也易证.

§17. Definable Sets.

Notation. If $FV(\varphi) \subset \{x_1, x_2, \cdots, x_n\}$ then we write

$$\varphi = \varphi(x_1, x_2, \cdots, x_n).$$

Notation. If $a_1, a_2, \cdots, a_n \in M$, we write

$$\models_{\mathcal{M}} \varphi(a_1, a_2, \cdots, a_n)$$

to mean:

$$\models_{\mathcal{M}} \varphi[s] \text{ for all } s: V \to M \text{ with } s(v_i) = a_i \quad (1 \le i \le n)$$

Definition. Definable ← 最抽象的一集.

A set $A \subset M^n$ is definable in structure $\mathcal{M}$ if there is a wff. $\varphi(x_1, x_2, \cdots, x_n)$ s.t.

$$A = \{(a_1, a_2, \cdots, a_n) \in M^n \mid \models_{\mathcal{M}} \varphi(a_1, a_2, \cdots, a_n)\}.$$

(大概就是，一个集合在给定的结构下可定义，当且仅当它能够用该结构的语言给描述出来)

e.g. The set of even numbers is definable in

$\mathcal{M} = (\mathbb{Z}; +)$

考虑 $\varphi(y) = \exists x\,(x * x = y)$ → y是自由变量

则集合 $A = \{a \in \mathbb{Z} \mid \models_{\mathcal{M}} \varphi(a)\}$ 表示所求的集合.

Note: 原来自由变量的作用在于<u>定义</u>!

e.g. $A = \{(a,b) \in \mathbb{R}^2 \mid a \leq b\}$ is definable in $(\mathbb{R}; \times, +)$

只要考虑 $\varphi(x,y) = \exists z \left(y = x \underset{\substack{\uparrow \\ +}}{\oplus} (z \underset{\substack{\uparrow \\ \times}}{\otimes} z) \right)$

因此<u>序</u>$\leq$ 也被定义出来了.

$\leq(x,y) \iff (x,y) \in A$.

§. 18. Homomorphisms 同态.

这里先给出<u>群同态</u>. <u>环同态</u>. <u>序环同态</u>的例子.

Definition. (结构的) 同态映射 (Homomorphism).

A homomorphism from structure $\mathcal{M}$ to $\mathcal{N}$ is a function

$h: M \to N$, s.t.

$\boxed{\text{The whole structure is preserved}}$

(注意这里是 domain → domain)

① for all constant symbols c in τ

$$h(c^{\mathcal{M}}) = c^{\mathcal{N}}$$

② for all function symbols f in τ, $\forall m_i \in M$

$$h(f^{\mathcal{M}}(m_1, m_2, \cdots m_k)) = f^{\mathcal{N}}(h(m_1), h(m_2), \cdots, h(m_k)).$$

③ for all relation symbols R in τ. $\forall m_i \in M$.

$$(m_1, m_2, \cdots, m_k) \in R^{\mathcal{M}} \Leftrightarrow (h(m_1), h(m_2) \cdots h(m_k)) \in R^{\mathcal{N}}$$

其中. $\mathcal{M}$, $\mathcal{N}$ are structures over the same vocab τ

$$\mathcal{M} = (M; c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, \cdots f_1^{\mathcal{M}}, f_2^{\mathcal{M}}, \cdots, R_1^{\mathcal{M}}, R_2^{\mathcal{M}} \cdots)$$

$$\mathcal{N} = (N; c_1^{\mathcal{N}}, c_2^{\mathcal{N}}, \cdots f_1^{\mathcal{N}}, f_2^{\mathcal{N}}, \cdots R_1^{\mathcal{N}}, R_2^{\mathcal{N}} \cdots).$$

△ 再次强调 Structure 的定义.

$f_i \in \mathcal{L}$ 是一个 function symbol. 但是这里的 f_i^M 不是 symbol. 而是一个 function $f_i^M: M^k \to M$

△ 注意这里和之前的递归定义不一样. 这里的 $h: M \to N$ 是很明确的定义.

Definition. Embedding 嵌入映射

An embedding (or isomorphism embedding) is a injective homomorphism.

同构

单射

Definition. Isomorphic 同构性

我们称结构 $\mathcal{M}$ 和 $\mathcal{N}$ 为 isomorphic, iff there is an embedding from $\mathcal{M}$ onto $\mathcal{N}$.

surjection: 满射

Definition Automorphism 自同构映射

An embedding from a structure onto itself is called an automorphism.

△ 注意. homomorphism, embedding, automorphism 都是映射, 而 isomorphic 是性质

§ 19. Preservation Results

Lemma 1. 项(term)在 homomorphism 下的作用

Suppose $h: M \to N$ is a <u>homomorphism</u>, then for all terms $t(x_1, x_2, \cdots, x_k)$ and $m_1, m_2, \cdots, m_k \in M$,

$$h(t^M(m_1, m_2, \cdots, m_k)) = t^N(h(m_1), h(m_2) \cdots h(m_k))$$

$$\left(\begin{tikzcd} x \in M^k \arrow[r, "h"] \arrow[d, "t^M"'] & \cdot \arrow[d, "t^N"] \\ \cdot \arrow[r, "h"'] & \square \end{tikzcd}\right)$$ ⇐ 大概是这么个意思

Pf. h 是 homomorphism.

(注意到根据 homomorphism 的定义.

$h(f^M(m_1, m_2, \cdots m_k)) = f^N(h(m_1). h(m_2). \cdots h(m_k))$

对于任意的对应的 f^M 和 f^N)

对 term 归纳.

对于常数 term. $h(c^M) = c^N = t^N(\emptyset)$. ✓.

对于变量 term v^M 可以视为单变量函数 $f^M(v)$

取遍 $v \in M$. 可以一一得到 $h(v) \in N$

所以 $h(t^M(m)) = t^N(h(m))$

对于函数 term. 根据定义 (homomorphism). $\square$

(单射)

Lemma 2. Suppose $h: M \to N$ is an embedding. Then for all quantifier-free wff. $\varphi(x_1, x_2, \cdots x_k)$ and $m_1, m_2, \cdots, m_k \in M$ we have.

$$\models_M \varphi(m_1, m_2, \cdots m_k) \Leftrightarrow \models_N \varphi(h(m_1), h(m_2), \cdots h(m_k)).$$

(不含非自由变量的公式在同构结构下保真).

Pf. ① 如果 φ is atomic. $\varphi = R(t_1, t_2, \cdots, t_\ell)$.

$$\models_M \varphi(\underbrace{m_1, m_2, \cdots, m_k}_{=:\overline{m}}) \Leftrightarrow (t_1^M(\overline{m}), t_2^M(\overline{m}), \cdots t_\ell^M(\overline{m})) \in R^M$$

$$\Leftrightarrow (h(t_1^M(\overline{m})), h(t_2^M(\overline{m})) \ldots h(t_\ell^M(\overline{m})) \in R^N$$

$$\overset{\text{lemma 1}}{\Longleftrightarrow} (t_1^N(h(\overline{m})), t_2^N(h(\overline{m})) \ldots t_\ell^N(h(\overline{m}))) \in R^N$$

$$\Leftrightarrow \models_N \varphi(h(m_1), h(m_2) \cdots h(m_k)).$$

② 如果 $\varphi = \neg\phi$. $\phi \wedge \psi$. $\phi \vee \psi$

我们用 $\overline{m}$ 表示 $(m_1, m_2, \cdots m_k)$. 以 $\varphi = \phi \wedge \psi$ 为例.

$$\models_M \varphi(\overline{m}) \Leftrightarrow \models_M \phi(\overline{m}) \text{ and } \models_M \psi(\overline{m})$$

$$\Leftrightarrow \models_N \phi(h(\overline{m})) \text{ and } \models_N \psi(h(\overline{m})) \Leftrightarrow \models_N \varphi(h(\overline{m}))$$

(Inductive Hypothesis) $\square$

上面的条件还是不够强。

(双射，同构)

Lemma 3. Suppose $h: M \to N$ is an onto isomorphism, then for all wffs $\varphi(x_1, x_2, \cdots, x_k)$ and $\overline{m} = (m_1, m_2, \cdots, m_k) \in M^k$.

$$\models_M \varphi(\overline{m}) \Leftrightarrow \models_N \varphi(h(\overline{m})),\quad 记作 M \cong N$$

(其中 $h(\overline{m}) := (h(m_1), h(m_2), \cdots, h(m_k))$.)

Pf. of lemma 3. 假设 $\varphi(\overline{x}) = \forall y\, \phi(y, \overline{x})$

其中 y 是 ϕ 中唯一的非自由变量。

$$\models_M \varphi(\overline{m}) \Leftrightarrow \forall a \in M \models_M \phi(a, \overline{m})$$

$$\underset{\text{(Inductive Hypothesis)}}{\Longleftrightarrow} \forall a \in M \models_N \phi(h(a), h(\overline{m}))$$

$$\Leftrightarrow \forall b \in N \models_N \phi(b, h(\overline{m}))$$

$$\Leftrightarrow \models_N \forall y\, \phi(y, h(\overline{m}))$$

$$=: \models_N \varphi(h(\overline{m}))$$

(实际上是对 formula 中非自由变量的个数 n 进行归纳)

$\square$

Note. 如果两个结构 M 和 N 是 isomorphic (同构) 的，则这两个结构是基本等价的 (elementary equivalent)

$\mathcal{M} \equiv \mathcal{N} := \models_{\mathcal{M}} \varphi \Leftrightarrow \models_{\mathcal{N}} \varphi$ for all sentence φ.

所以在 lemma 3 中的 $\varphi(\cdots)$ 就可以替换成 φ.

即证.

(自同构. 双射)

Corollary 4. 令 $h: M \to M$ be an automorphism. Then for all definable set $A \subset M^n$. $h[\![A]\!] = A$.

(Recall that a set A is definable in structure $\mathcal{M}$ iff there exists a wff φ s.t.

$$A = \{(a_1, a_2, \cdots, a_n) \in M^n \mid \models_{\mathcal{M}} \varphi(a_1, a_2, \cdots, a_n)\}.$$
)

$$h[\![A]\!] := \{(h(a_1), h(a_2), \cdots, h(a_n)) \mid (a_1, a_2, \cdots, a_n) \in A\}$$

$h[\![A]\!] = A$ 说明 A 对于运算 h 是封闭的 (甚至是不动点)

Pf. of Col 4. 如果 A is defined by $\varphi(x_1, x_2, \cdots, x_n)$

$$\bar{x}_0 \in A \Leftrightarrow \models_{\mathcal{M}} \varphi(\bar{x}_0) \overset{\text{lemma 3}}{\Longleftrightarrow} \models_{\mathcal{M}} \varphi(h(\bar{x}_0))$$

$$\Leftrightarrow h(\bar{x}_0) \in A$$

即. 自同构结构上的任意可定义集都是自同构映射的不动点.

见书 (P96). 描述有出入.

§20. Non-definability via automorphisms.

→ 指的是没有一阶逻辑公式可以定义它.

回顾§19 的最后一个 Corollary: Definable sets are preserved under automorphisms.

e.g. 证明 $\mathbb{N}$ is not definable in $(\mathbb{Z}; 0, +)$.

Pf. 如果 $\mathbb{N}$ is definable in $(\mathbb{Z}; 0, +)$. 则对于 $(\mathbb{Z}; 0, +)$ 的一个 automorphism $h: \mathbb{Z} \to \mathbb{Z}$, $x \to -x$

$h[\![\mathbb{N}]\!] \neq \mathbb{N}$ 矛盾.

e.g. 证明 Multiplication is not definable in $(\mathbb{Z}; 0, +)$

Pf. $M = \{(a, b, c) \in \mathbb{Z}^3 \mid a \times b = c\}$. 我们通过证明此集合 M 是不可定义的. 来说明乘法是不可定义的

注意到 M 实际上是 graph of $X(x, y)$.

let $h: \mathbb{Z} \to \mathbb{Z}$, $x \to -x$. h 是一个自同构映射 of $(\mathbb{Z}; 0, +)$

注意到 $(1, 1, 1) \in M$. 但 $(h(1), h(1), h(1)) \notin M$.

所以 M 没有被 automorphism preserve 住.

说明 M 在 $(\mathbb{Z}; 0, +)$ 下是不可定义的 $\square$

妙啊!!!!

e.g. 证明 $\{2\}$ is not definable in $(\mathbb{N}; \times)$.

(This one is tricky)

Pf. 考虑这样的 $h: \mathbb{N} \to \mathbb{N}$

Given $n \in \mathbb{N}$, decompose it into prime factors.

$$n = p_1^{k_1} \times p_2^{k_2} \cdots\cdots p_l^{k_l}$$

$$h(n) = \left(\frac{2}{3}\right)^{k_2 - k_1} \times n = \left(p_1^{k_2} \times p_2^{k_1}\right) \times \cdots \times p_l^{k_l}$$

Claim. h is automorphism of $\mathbb{N}$

但注意到 $h(2) = \left(\frac{2}{3}\right)^{-1} \times 2 = 3 \notin \{2\}$

§.21. Substructures. 子结构

子结构

Definition. Given τ-structure $\mathcal{M}$ and $\mathcal{N}$. we say that

$\mathcal{M}$ is a substructure of $\mathcal{N}$ iff:

① $M \subseteq N$ (再次强调是 domain)

② $c^{\mathcal{M}} = c^{\mathcal{N}}$ for all constant symbols in τ

③ $f^{\mathcal{M}}$ is the restriction of $f^{\mathcal{N}}$ to M^k. for all k-ary function symbol in τ

④ $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^k$ for all k-ary relation symbols in τ.

可以用符号 $f^{\mathcal{M}} = f^{\mathcal{N}} \upharpoonright M^k$ 表示

e.g. $(\mathbb{Z}; 0, 1, +, \times, <)$ is a substructure of $(\mathbb{R}; 0, 1, +, \times, <)$.

Observe. $\mathcal{M}$ is a substructure of $\mathcal{N}$ iff identity map

$$\begin{aligned} id: M &\to N \\ x &\mapsto x \end{aligned}$$

is an embedding.

单射.

Definition. Elementary Substructure. 初等子结构.

Given τ-structure $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ (记作 $\mathcal{M} \prec \mathcal{N}$) iff.

$\begin{cases} ① \ M \subseteq N \\ ② \text{ for every first-order wff } \varphi(x_1, x_2, \cdots, x_k) \text{ and every} \end{cases}$

$m_1, m_2, \cdots, m_k \in M$.

$$\models_M \varphi(m_1, m_2, \cdots, m_k) \Leftrightarrow \models_N \varphi(m_1, m_2, \cdots, m_k)$$

(即. $M \equiv N$ 对于 M 的部分总成立)

注意到根据§19. lemma 2.

如果 M 是 N 的 substructure. 则 $id: M \to N$ 是一个 embedding.

则对所有的 quantifier-free wff φ $\models_M \varphi \Leftrightarrow \models_N \varphi$ 成立

但是如果 M 是 N 的 elementary substructure.

对所有的 first-order wff φ 都有 $\models_M \varphi \Leftrightarrow \models_N \varphi$ 成立.

所以初等子结构是比子结构更强的条件.

e.g. $(\mathbb{Z}; +) \not\preceq (\mathbb{Q}; +)$

Antonio: "They are not even elementary equivalent".

e.g. $(2\mathbb{Z}; +) \not\preceq (\mathbb{Z}; +)$

考虑双射 $h: 2\mathbb{Z} \to \mathbb{Z}$. 所以 $(2\mathbb{Z}; +) \cong (\mathbb{Z}; +)$

$x \to \frac{1}{2}x$ (isomorphism).

根据§19. lemma 3 的推论. 对所有的 sentence φ.

$$\models_{(2\mathbb{Z};+)} \varphi \Leftrightarrow \models_{(\mathbb{Z};+)} \varphi$$

故 $(2\mathbb{Z};+)$ 和 $(\mathbb{Z};+)$ 是 elementary equivalent 的.

但是 $(2\mathbb{Z};+)$ 不是 $(\mathbb{Z};+)$ 的 elementary substructure.

考虑 formula $\varphi(y) = \neg(\exists x.\ x+x=y)$

$$\models_{(2\mathbb{Z};+)} \varphi(2), \quad \not\models_{(\mathbb{Z};+)} \varphi(2)$$

△ 注意一些定义的区别.(关键部分. 非正式)

elementary equivalent: 任意的 sentence φ. $\models_{\mathcal{M}} \varphi \Leftrightarrow \models_{\mathcal{N}} \varphi$

isomorphic: 任意的 wff $\varphi(x_1, x_2, \dots, x_n)$. $\models_{\mathcal{M}} \varphi(\vec{x}) \Leftrightarrow \models_{\mathcal{N}} \varphi \circ h(\vec{x})$

elementary substructure 任意的 wff. φ. $\models_{\mathcal{M}} \varphi \Leftrightarrow \models_{\mathcal{N}} \varphi$

e.g. $(\mathbb{Z};+) \prec (\mathbb{Z}+\mathbb{Z};+)$

e.g. $(\mathbb{Q};+) \prec (\mathbb{R};+) \prec (\mathbb{C};+)$

e.g. $(\mathbb{Q};\leqslant) \prec (\mathbb{R};\leqslant)$

§22 Compactness.

Definition. Model. 模型.

A model of a set of well-formed sentences is an elem of $\{\mathcal{M} \mid \models_{\mathcal{M}} \Gamma\} =: \operatorname{Mod}\Gamma$

Recall that a set Γ of well-formed sentences is called <u>satisfiable</u> if $\operatorname{Mod}\Gamma \neq \phi$.

Theorem. Compactness Theorem. 紧致性定理.

If every finite subset of Γ is satisfiable (by different models), then Γ is satisfiable.

e.g. $\tau = \{0, 1, +, \times, <, c\}$

let $T = \operatorname{Th}(\mathbb{N}; 0, 1, +, \times, <) \cup \{\varphi_p : p \text{ is prime}\}$ ordered ring.

where $\varphi_p := \exists x\,(x \times (\underbrace{1+1+\cdots+1}_{p\text{ times}}) = c)$. 也即 $(p \mid c)$

(Recall that $\operatorname{Th}\mathcal{M} := \{\text{sentences true in } \mathcal{M}\}$.)

问: Is T satisfiable? (是否有 τ-structure $\mathcal{M}$ s.t. $\models_{\mathcal{M}} T$)

Pf. Let T_0 be a finite subset of T

令 $\mathcal{N} = (\mathbb{N}; 0, 1, +, \times, <)$.

则 $\exists P \in \mathbb{N}$ $T_0 \subset Th(\mathcal{N}) \cup \{\varphi_p \mid p < P\}$

考虑 $\mathcal{M}_0 = (\mathbb{N}; 0, 1, +, \times, <, c^{\mathcal{M}})$

其中 $c^{\mathcal{M}} := \prod_{\substack{p<P \\ p \text{ is prime}}} p$

则 $\models_{\mathcal{M}_0} Th(\mathcal{N}) \cup \{\varphi_p \mid p < P\}$

根据 compactness theorem (假设我们已经知道), T is satisfiable. $\square$

Compactness Theorem 的证明要在 Completeness 的证明之后才可以得到.

§23. An application of Compactness.

Compactness Theorem 的应用之一.

证明 classes are not elementary.

(Recall that a class of structures is elementary iff there is a sentence φ and all the elements of the class are exactly models of φ.)

Definition. A undirected graph is a structure $G = (V; E)$, where

E is a binary relation satisfying

$$\forall x \forall y.\ xEy \rightarrow yEx$$

two points $x, y \in V$ are connected iff xEy

e.g. The class of unconnected graph is not (even) weakly elementary.

(Recall that a class of structures is weakly elementary iff there exists a set Γ of well-formed sentences s.t. ⋯)

(可以是有穷的或无穷的)

i.e. there is no set Γ of wfs s.t. $\models_G \Gamma \Leftrightarrow G$ is a connected graph.

Pf. 如果存在这样的 Γ. let $\tau = \{E, \underbrace{c, d}_{\text{constant}}\}$

Let $T = \Gamma \cup \{\varphi_n \mid n \in \mathbb{N}\}$ where $\underline{\varphi_n \text{ is the wfs}}$

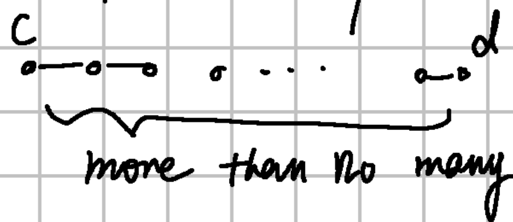
$$\neg\left(\exists x_1, x_2 \cdots x_n \left((c, x_1), (x_1, x_2) \cdots (x_n, d) \in E\right)\right).$$

(不存在有从 c 到 d 的长度为 n 的 walk).

Let $T_0 \subset T$ be a finite subset. there is $n_0 \in \mathbb{N}$ s.t.

$$T_0 \subset \Gamma \cup \{\varphi_n \mid n < n_0 \text{ and } n \in \mathbb{N}\}.$$

Observe that T_0 is satisfiable. as long as the model

$G =$ 

则根据 compactness thm. T is satisfiable.

根据定义. G is <u>disconnected</u>.

这与 G is connected graph 矛盾 □

△ 注意. 是在 "First Order System" 中无法 axiomitize.

Class of <u>connected graph</u>.

§ 24 Proving non-definability via Compactness

Theorem. The connectness relation is not <u>definable</u> in <u>graphs</u> (使用一阶逻辑).

i.e. There is no FO wff $\varphi(x,y)$ on all graphs $(V;E)$ and all $v,w \in V$, v and w are connected $\Leftrightarrow$

$$\models_{(V;E)} \varphi(v,w)$$

我们假设存在这样的 $\varphi(x,y)$　Let $\tau = \{E, \underline{c,d}\}$ constant.

(Axioms of Graph)

let $T = \underline{AG} \cup \{\varphi(c,d)\} \cup \{\varphi_n \mid n \in \mathbb{N}\}$.

$$\varphi_n := \neg\left(\exists x_1 \exists x_2 \cdots \exists x_n \left((c,x_1),(x_1,x_2),\cdots,(x_n,d) \in E\right)\right)$$

通过和上次类似的构造可以证明. T is not satisfiable

(什么鬼…)

§.25. Compactness via Implication

Theorem. If $\Gamma \models \varphi$, then there is a finite $\Gamma_0 \subset \Gamma$ s.t. $\Gamma_0 \models \varphi$

(一阶逻辑的有限蕴含定理. 是紧致性定理的推论).

(Recall that $T \models \varphi$ means that every model of T is a model of φ.)

Pf. $\Gamma \models \varphi \iff \Gamma \cup \{\neg\varphi\}$ is not satisfiable

$\iff$ there is finite set $\Gamma_0 \subset \Gamma$ s.t. $\Gamma_0 \cup \{\neg\varphi\}$ is not satisfiable

$\iff \Gamma_0 \models \varphi$

□

Application of 有限蕴含定理

Thm. If a sentence φ is true in all infinite groups, it must be true in some finite group too.

假设三条 Axiom of Group 称为 ψ_1, ψ_2, ψ_3.

let $\tau = \{e, *\}$. $T = \{\psi_1, \psi_2, \psi_3\} \cup \{\varphi_n \mid n \in \mathbb{N}\}$

where $\varphi_n := \exists x_1 x_2 \cdots x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge \cdots)$

观察到 the models of T are exactly the

infinite groups.

Pf. 根据定义. $T \models \varphi$

所以 there exists finite $T_0 \subset T$: $T_0 \models \varphi$.

so $\exists n \in \mathbb{N}$: $T_0 \subset \{\psi_1, \psi_2, \psi_3\} \cup \{\varphi_m \mid m \leq n\}$

so $\{\psi_1, \psi_2, \psi_3\} \cup \{\varphi_m \mid m \leq n\} \models \varphi$. $\square$

§26. Compactness in Sentential Logic

Thm. Compactness Theorem in Sentential Logic

(Recall that Γ is satisfiable iff there is a truth assignment $v: \{A_1, A_2, \cdots, A_n\} \to \{T, F\}$ that makes every wff true.)

$\boxed{\text{如果 a set of wffs } \Gamma \text{ 中每一个 finite subset 都是 satisfiable 的，那么 } \Gamma \text{ 也是 satisfiable 的}}$

i.e. 如果 $\Gamma \vDash \varphi$，则存在 finite $\Gamma_0 \subset \Gamma$ s.t. $\Gamma_0 \vDash \varphi$

(△ 这个直观上和 SAT 问题得有点关系)

pf. Claim 必然存在一个 assignment $S: V \to \{T, F\}$

对于每一个有限的 $\Gamma_0 \subset \Gamma$，$\vDash \Gamma_0[S]$ 且满足下列两种情况之一：

① $S(A_1) \equiv T$ （在所有 Γ_0 中）

② $S(A_1) \equiv F$ （在所有 Γ_0 中）

否则对于任意一个 assignment function S，

总存在有限的 $\Gamma_1 \subset \Gamma$，$\Gamma_2 \subset \Gamma$，$\Gamma_1 \neq \Gamma_2$

（两个集）

s.t. $\begin{cases} \vDash \Gamma_1[S] \text{ 时 } S(A_1) = T \\ \vDash \Gamma_2[S] \text{ 时 } S(A_2) = F \end{cases}$

所以对任意的赋值函数 S，$\nvDash \Gamma_1 \cup \Gamma_2[S]$，矛盾

我们定义一系列的赋值函数 $\{v_n \mid n \in \mathbb{N}\}$.

如果是上述情况① . $v_1(A_1) = T$

② $v_1(A_1) = F$

Having defined $v_1, v_2, \dots, v_n$,

$$v_n : \{A_1, A_2, \dots, A_n\} \to \{T, F\}$$

s.t. every finite $T_0 \subset T$ is compatible with v_n.

如果 for all finite $T_0 \subset T$ $v_n(A_{n+1}) = T$

则定义 $v_{n+1}(A_{n+1}) = T$

如果 for all finite $T_0 \subset T$. $v_n(A_{n+1}) = F$

则定义 $v_{n+1}(A_{n+1}) = F$

Claim (类似地). 对于每个可满足的 finite $T_0 \subset T$. v_n 要么一直将 A_{n+1} 赋成 T. 要么一直将 A_{n+1} 赋成 F.

所以我们可以用上面的方式逐步迭代赋值函数 $\{v_n\}$.

根据 induction. T is satisfiable. □

§27. An application of Compactness for Sentential Logic

Definition. Partial Ordering 偏序

A partial ordering is a structure $P=(V;\preceq)$ where $\preceq$ is a binary relation satisfying transitivity and reflexity (自反性) and antisymmetry (反对称性).

woc!

传递性

$$\begin{cases} ① & x \preceq x \\ ② & x \preceq y \wedge y \preceq z \rightarrow x \preceq z \\ ③ & x \preceq y \wedge y \preceq x \rightarrow x = y \end{cases}$$

Definition. Linear Ordering 线性序.

A linear ordering is a partial ordering that also satisfies:

$\forall x, y:\ x \preceq y \vee y \preceq x$ (任意两个可比)

e.g. 字典序也是 linear ordering.

Definition. Linearization 线性化.

A Linearization of a partial ordering $P=(V;\preceq)$

is a linear ordering $\trianglelefteq$ on V compatible with $\preceq$.

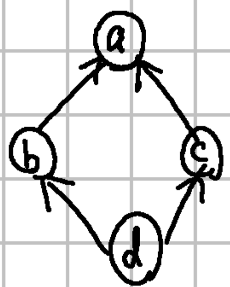
$\trianglelefteq$- 可以想象为把非线性序（格式）拆扁.

i.e. $\forall p,q\in V: p\preceq q \Rightarrow p\trianglelefteq q$

Theorem. Every Partial Ordering has a linearization.

偏序可以线性化.

Pf. 首先证明 Every finite partial ordering 都有 linearization. 这是易证的.



类似于图的拓扑排序.

接着用 Sentential Logic 的 Compactness Theorem 来证明无穷的情况

let $P=(V;\preceq)$ be a partial ordering.

For each $p,q\in V$. Let A_{pq} be a sentential variable symbol $\longrightarrow$ $A_{pq} := p\trianglelefteq q$.

我们想找一个赋值函数 $v:\{A_{pq} \mid p,q\in V\}\to\{T,F\}$

我们考虑这样一个刻画了线性序性质的集合 Γ, where.

$$\Gamma := \{(A_{pq} \wedge A_{qr}) \to A_{pr} \mid p,q,r \in V\} \quad // \text{ 传递性}$$

$$\cup \{A_{pp} \mid p \in V\} \quad // \text{ 自反性（被第5条包括，可删去）}$$

$$\cup \{(A_{pq} \wedge A_{qp}) \to q = p \mid p,q \in V\} \quad // \text{ 反对称性}$$

$$\cup \{A_{pq} \vee A_{qp} \vee p = q \mid p,q \in V\} \quad // \text{ 三歧性}$$

$$\cup \{A_{pq} \mid p,q \in V\} \quad // \text{ 线性}$$

只要说明 Γ is satisfiable

let $\Gamma_0 \subset \Gamma$ be finite. 考虑 $V_0 := \{p \in V \mid$ some variable in Γ_0 uses $p\}$

则 $(V_0; \leq)$ 是一个 finite partial ordering，根据归纳假设它存在一个 linearization $\trianglelefteq_0$.

接下来要跳过一大部分和 Compactness 的应用相关的

先不看这些偏难的部分 看懂论文要紧

§ 34. Syntactical Implication

Definition. Syntactical Implication. 句法蕴含.

Given a set of wffs Γ and a wff φ. we say that Γ syntactically implies φ (write $\Gamma \vdash \varphi$) if there is a $\boxed{\text{formal proof}}$ of φ from Γ.

△ $\Gamma \vDash \varphi$ 称为 Semantical Implication 语义
(或 Logic Implication).

Inference Rule

Modus Ponens (肯定前件. 拉丁语):

指 From α and $\alpha \to \beta$ we may infer β.

Definition. Deduction

a set of wffs

A deduction of φ from Γ is a finite sequence $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ of wffs s.t. $\alpha_n = \varphi$ and $\forall k \leq n$, either

① $\alpha_k \in \Gamma \cup \Lambda$, or

② α_k is obtained by modus ponens from earlier formulas

其中. Λ 是 a set of axioms.

可以理解为 deduction 是证明的一系列步骤.

每一步 { 要么是之前的步骤
要么可以由之前的步骤直接推导得到.
(Modus Ponens)

When such a deduction exists, we say that φ is deducible from T, or that φ is a theorem of T, and write $T \vdash \varphi$.

definition of theorem.

(\$\vdash\$)

(这才是 syntactically imply 的正式定义)

Definition. Closed under Modus Ponens

我们称集合 S (of wffs) 在 modus ponens 下为 closed, if

$$(\alpha \in S \wedge (\alpha \to \beta) \in S) \Rightarrow \beta \in S$$

Principle of Induction. 归纳原理

T, Λ and S are sets of wffs. If $T \cup \Lambda \subset S$ and S is closed under modus ponens, then S contains every theorem of T

其中 T 是 assumptions, Λ 是 axioms 集

§35. Logical Axioms 逻辑公理

我们对上次的 Logical Axioms 展开讨论.

Definition. Generalization 泛化.

A wff φ is a generalization of a wff ψ if

$$\varphi = \forall x_1 \forall x_2 \cdots \forall x_n \psi$$

其中 $x_1 . x_2 \cdots x_n$ 是 ψ 中原本出现的 free variables.

Definition. Logical Axioms

The logical axioms are generalization of the following wffs: 对于任意的 wff $\alpha . \beta$

① Tautologies.

(如果 t 可替代的话, 后面介绍)

② $(\forall x \alpha) \to \alpha^x_t$ 其中 t 是 x 在 α 中的 substitution

③ $(\forall x(\alpha \to \beta)) \to (\forall x \alpha \to \forall x \beta)$

④ $\alpha \to \forall x \alpha$. 其中 x 在 α 中不是自由变量.

(也就是 x 在 α 中不出现. 或就算出现, $\forall x$ 也指代不到里面去. Antonio: "Dummy Quantifier")

⑤ $x = x$

⑥ $x = y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing x in some places by y.

(其中 ⑤ 和 ⑥ 是比较trivial的)

Recall that $\forall x = \neg \exists x \neg$，所以上面的 axiom 只考虑 $\forall$.

Definition. Substitution

对于wff. α，α^x_t is obtained from α by replacing x by t whenever x occurs free.

e.g. $(Q(x) \to \forall x P(x))^x_y$，P. Q 为一元谓词

↳ $= (Q(y) \to \forall x P(x))$

e.g. $\alpha = \neg \forall y (x = y)$，$(\forall x \alpha) \to \alpha^x_y$ (陷阱例子)

$= (\forall x \alpha) \to \neg \forall y (y = y)$

$= \forall x \exists y (x \neq y) \to \exists y (y \neq y)$

但 $(\forall x \alpha) \to \alpha^x_y$ 是公理，不应出现 $\exists y (y \neq y)$ 的情况

※ 注意 Axiom 2 要求 t 是 substitutable 的

Definition. Substitutable

称 t is not substitutable for x in α iff there is a variable y in t captured by some quantifier $\forall y$ or $\exists y$ in α_t^x.

e.g. $x+3$ is <u>not</u> substitutable for y in

$$\alpha = \exists x (y = x-1)$$

△在下一部分我们介绍公理的第一条：重言式

§3.6. Tautologies in First Order Logic

Recall that a tautology ~~is a wff~~ that is True on any assignments in Propositional Logic

Definition. Tautology in First Order Logic.

A tautology in First-order logic is obtained from a sentential tautology, replacing each sentential variable by a first-order wff.

(妙啊！串起来了！)

e.g. $A \vee \neg A$

$$\left(\forall x (P(x) \to Q(x)) \vee \neg (\forall x P(x) \to Q(x))\right).$$

上面是 FO 中的 tautology.

$$\forall z \, (\cdots)$$

再外面加若干个 $\forall x_i$ 可以形成 axiom 的形式（注意不能和非自由变量冲突）.

Observation. (并不显然. 课本 P130 ex3)

Tautological implication $\Longrightarrow$ Logical Implication.

(大概就是 $\models \forall x P(x) \Rightarrow \models P(c)$).

Theorem. $\Gamma \vdash \varphi$ iff $\Gamma \cup \Lambda$ tautologically implies φ.

(Antonio: Interesting to prove and is an interesting technique)

注意. "Λ" 默认是 set of axioms

Pf. ($\Rightarrow$) Let $S = \{\varphi \mid \Gamma \cup \Lambda$ tautologically implies $\varphi\}$.

首先 $\Gamma \cup \Lambda \subseteq S$

如果 $\Gamma \cup \Lambda$ tautologically implies $\alpha \to \beta$
and $\Gamma \cup \Lambda$ tautologically implies α.

注意一阶逻辑的 tautology 这边开始有两个赋值函数.

一个是 sentential logic 部分的

一个是 first-order logic 部分的

则 $\Gamma \cap \Lambda$ tautologically implies β

i.e. $(\alpha \to \beta \in S \wedge \alpha \in S) \Rightarrow \beta \in S$

i.e. S is closed under MP.

根据 Induction Principle. S contains every theorem of Γ.

i.e. $\Gamma \cup \Lambda$ tautologically implies φ.

$(\Leftarrow)$ T-Imply $\Rightarrow$ ⊢-imply，根据 Compactness Theorem.

there are $\{\gamma_0, \gamma_1 \cdots \gamma_k\} \subset \Gamma$ (finite)

$\{\lambda_0, \lambda_1 \cdots \lambda_l\} \subset \Lambda$. (finite)

s.t. $\{\gamma_0, \gamma_1, \cdots, \gamma_k, \lambda_0, \lambda_1, \cdots, \lambda_l\} \vDash \varphi$.

注意到 $\underline{\psi = \gamma_0 \to (\gamma_1 \to \cdots (\gamma_k \to (\lambda_0 \to (\lambda_1 \to \cdots \to (\lambda_l \to \varphi))))}$.

is a tautology. (因为 $p \to q \Leftrightarrow \neg p \vee q$)

则对 ψ 使用 MP $(k+l)$ 次，可以得到 φ.

根据"⊢"的定义，存在这样的 deduction from $\Gamma \cup \Lambda$ to φ.

故 $\Gamma \vdash \varphi$ （注意根据定义不是 $\Gamma \cup \Lambda \vdash \varphi$）.

§ 36. The Generalization Metatheorem

Theorem. In any first order τ-structure, P is a unary relation symbol. then

$$\vdash \big(P(x) \to \exists y P(y)\big)$$

Pf. 注意到 $\forall y \neg P(y) \to \neg P(x)$ (Axiom 2)

$(\forall y \neg P(y) \to \neg P(x)) \to (P(x) \to \exists y P(y))$ (Axiom 1)

一个 tautology.

则 $P(x) \to \exists y P(y)$ (MP)

Note. $\vdash \varphi$ 实际上是 $\phi \vdash \varphi$. 即只需要公理就可以证明. 不需要任何的 assumptions (集合 $\Gamma = \phi$)

这里验证了 Thm: $\underline{\Gamma \vdash \varphi}$ iff $\underline{\Gamma \cup \Lambda}$ tautologically implies $\underline{\varphi}$.

沿用上面的 Notations.

Theorem. $\vdash \forall x\,(P(x) \to \exists y P(y))$

Pf. 在上面我们证明了. $P(x) \to \exists y P(y)$.

根据 Axioms 的定义.

Axiom 2: $\forall y \neg P(y) \to \neg P(x)$

Axiom 0 (加 $\forall$): $\forall x (\forall y \neg P(y) \to \neg P(x))$ (I)

Axiom 1: $(\forall y \neg P(y) \to \neg P(x)) \to (P(x) \to \neg \forall y \neg P(y))$

Axiom 0: $\forall x(\cdots)$

Axiom 3: $(\forall x(\ ) \to \forall x(\ ))$ (II)

由(I), (II). MP. 有 $\forall x(P(x) \to \exists y P(y))$

说明 $\vdash \forall x(P(x) \to \exists y P(y))$. □

Theorem

Generalization Theorem. 泛化定理

If $\Gamma \vdash \varphi$ and $x \notin FV(\varphi)$, then $\Gamma \vdash \forall x \varphi$

Pf. 考虑使用 Induction Principle.

Let $S = \{\varphi \mid \Gamma \vdash \forall x \varphi\}$

首先注意到 如果 $\varphi \in \Gamma$. 根据 Axiom 4. $\varphi \to \forall x \varphi$. $(x \notin FV(\varphi))$

则 $\Gamma \subset S$

再次注意力惊人地注意到 $\Lambda \subset S$. 因为

如果 $\varphi \in \Lambda$ 则根据 Axiom 0. $\forall x \varphi \in \Lambda$

说明 $\Gamma \vdash \forall x \varphi$, $\varphi \in \Gamma$

即. $\Lambda \subset S$

再次注意力惊人地注意到 S is closed under MP.

因为假设 α. $\alpha \to \beta \in S$.

i.e. $\Gamma \vdash \forall x \alpha$, $\Gamma \vdash \forall x(\alpha \to \beta)$.

根据 Axiom 3. $\Gamma \vdash (\forall x \alpha \to \forall x \beta)$

根据 MP. $\Gamma \vdash \forall x \beta$ i.e. $\beta \in S$

说明 S is closed under MP.

根据 Induction Principle. S contains every theorem of Γ.

说明. $\{\varphi \mid \Gamma \vdash \varphi\} \subset \{\varphi \mid \Gamma \vdash \forall x \varphi\}$

i.e. $\Gamma \vdash \varphi \Rightarrow \Gamma \vdash \forall x \varphi$ 好啊! $\square$

§37. More Metatheorems

Lemma (Rule T)

如果 $\Gamma \vdash \alpha_1$, $\Gamma \vdash \alpha_2$ $\cdots$ $\Gamma \vdash \alpha_n$.

and $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ tautologically implies β.

then $\Gamma \vdash \beta$.

Pf of Rule T.

> 这里前几章没有对 tautologically imply 讲清楚.
> 我们假设 Γ tautologically implies φ iff.
> 不管所有的 $\psi \in \Gamma$ 是真还是假, φ 都为真
> 可以写成 "$\psi_1 \to (\psi_2 \to \cdots \to (\psi_n \to \varphi)))$ 为 tautology".

由于 $\{\alpha_i\}$ tautologically implies β.

则 $\underbrace{\alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_n \to \beta))\cdots)}_{=\psi_1}$ is a tautology

我们令 $[\alpha_1]$ 表示 the deduction of $\Gamma \vdash \alpha_1$ (ends with α_1)

则 $[\alpha_1], \psi \Rightarrow \psi_2 = \alpha_2 \to (\alpha_3 \to \cdots \to (\alpha_n \to \beta)\cdots)$.

进行n次相同的操作之后, 有 $\Gamma \vdash \beta$ $\square$

Deduction Theorem.

If $\Gamma;\gamma\vdash\varphi$. 则 $\Gamma\vdash\gamma\to\varphi$.

Pf. 注意 $\Gamma;\gamma\vdash\varphi := \Gamma\cup\{\gamma\}\vdash\varphi$.

根据定义. $\Gamma;\gamma\vdash\varphi := \Gamma\cup\{\gamma\}\cup\Lambda$ tautologically implies φ.

所以 $\Gamma\cup\Lambda$ tautologically implies. $\gamma\to\varphi$

(还是可以和 Rule T 一样展开证明).

所以 $\Gamma\vdash\gamma\to\varphi$ (根据定义.) $\square$

(△ 反过来也显然. 所以可以作为充要条件)

Contraposition

If $\Gamma;\varphi\vdash\neg\psi$. 则 $\Gamma;\psi\vdash\neg\varphi$.

Pf. 根据 deduction theorem. $\Gamma;\varphi\vdash\neg\psi$

$\Rightarrow \Gamma\vdash\varphi\to\neg\psi$

$\Rightarrow \Gamma\vdash\psi\to\neg\varphi$

(由于根据 Rule T. 令 $\alpha_1=\varphi\to\neg\psi$ $\beta=\psi\to\neg\varphi$.

$\Gamma\vdash\alpha_1$. α_1 tau.. implies β.

则 $\Gamma\vdash\beta$)

所以 $\Gamma;\psi\vdash\neg\varphi$

Reduction and Absurdum：归谬法

Definition. Inconsistency of wff set.

称 a set of wffs. Γ is inconsistent iff there exists a wff β. $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$.

If $\Gamma ; \varphi$ is inconsistent, then $\Gamma \vdash \neg\varphi$.

Pf. $\Gamma ; \varphi \vdash \beta$, $\Gamma ; \varphi \vdash \neg\beta$

根据 deduction theorem. $\Gamma \vdash \varphi \to \beta$

$\Gamma \vdash \varphi \to \neg\beta$.

根据 Rule T. 由于 $\{\varphi \to \beta, \varphi \to \neg\beta\}$ tautologically implies $\neg\varphi$.

则 $\Gamma \vdash \neg\varphi$. $\square$

§38. Generalization of Constants.

Change of variables and equality.

Theorem. Generalization of Constants

Let c be a constant symbol <u>not</u> in Γ and y a variable <u>not</u> in φ. (y is not free in Γ) then,

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vdash \forall y\, \varphi^{c}_{y}.$$

pf. Let $\langle \alpha_0, \alpha_1 \dots \alpha_k \rangle$ be a formal deduction of φ from Γ ($\varphi = \alpha_k$)

Claim. $\langle \alpha_0{}^{c}_{y}, \alpha_1{}^{c}_{y}, \dots \alpha_k{}^{c}_{y} \rangle$ is a formal deduction,

我们考虑分类处理(递归): For each $i \leq k$

① 如果 α_i is a logical axiom,

我们 check axiom 的 6 种情况 发现都会成立.

② 如果 $\alpha_i \in \Gamma$. 则.

$\alpha_i{}^{c}_{y} = \underset{\in \Gamma}{\alpha_i}$ (由于 y 在 Γ 中不是自由变量).

③ 如果 $\exists k, l < i$ and $\alpha_l = \alpha_k \rightarrow \alpha_i$

则 $\alpha_l{}^{c}_{y} = (\alpha_k \rightarrow \alpha_i)^{c}_{y} = \alpha_k{}^{c}_{y} \rightarrow \alpha_i{}^{c}_{y}$

由 ①~③ $\Gamma \vdash \varphi^{c}_{y}$. 由 Generalization thm, $\Gamma \vdash \forall y\, \varphi^{c}_{y}$.

Corollary 1. Let c be a constant not in Γ or θ.

then $\Gamma \vdash \theta^x_c \Rightarrow \Gamma \vdash \forall x \theta$

Corollary 2. Let c be a constant not in Γ, θ, ψ.

then $\Gamma ; \theta^x_c \vdash \psi \Rightarrow \Gamma ; \exists x \theta \vdash \psi$

Alphabetic Variants.

Thm 如果φ和φ'只在量词上有区别，则 $\varphi \models\!\dashv \varphi'$

§. Soundness Theorem

Thm. Soundness Theorem

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi.$$

(能证明的就是对的)

只是MP推导. 不考虑含义

(Note: formal implication 又叫 syntactical implication)

Pf. 使用 Induction Principle.

Let $S = \{\varphi \mid \Gamma \vDash \varphi\}$

Claim. S is closed under MP.

如果 $\alpha \in S$, $\alpha \to \beta \in S$, 则 $\Gamma \vDash \alpha$ and $\Gamma \vDash \alpha \to \beta$

则 every model $\mathcal{M}$ of Γ satisfies α and $\alpha \to \beta$,

we have $\vDash_{\mathcal{M}} \beta$

i.e. $\Gamma \vDash \beta$. $\beta \in S$

Claim. $\Gamma \subset S$ 显然

Claim $\Lambda \subset S$ 注意 logical axioms are true in every structure

(Set of axioms)

所以 for all $\mathcal{M}$, $\vDash_{\mathcal{M}} \lambda$ $(\lambda \in \Lambda)$

所以. $\Gamma \cup \Lambda \subset S$ and S is closed under MP.

根据 Induction principle. S contains all the theorems of Γ.

i.e. $\{\varphi \mid \Gamma \vdash \varphi\} \subset \{\varphi \mid \Gamma \models \varphi\}$

$\Rightarrow \quad \Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$ □

Corollary. Any satisfiable set of wffs is consistent

Pf. 考虑任何一个模型 $\mathcal{M} \in \{\mathcal{M} \mid \models_{\mathcal{M}} \Gamma\}$.

则对于 wff β. $\not\models_{\mathcal{M}} \beta$ 和 $\not\models_{\mathcal{M}} \neg\beta$ 至少有1成立.

说明 $\Gamma \not\models \beta$ 和 $\Gamma \not\models \neg\beta$ 至少有1成立

根据 Soundness thm. $\Gamma \nvdash \beta$ 和 $\Gamma \nvdash \neg\beta$ 至少有1成立

i.e. Γ is <u>not inconsistent</u>

i.e. Γ is consistent. □

§. 40. Maximal consistent of sets of axioms.

Definition. Maximal

A set Γ of sentences is called <u>maximal</u> if for any sentence φ. either $\varphi \in \Gamma$. or $\neg\varphi \in \Gamma$

再要大一点就会爆炸 (inconsistent)

e.g. $Th(\mathcal{M})$ is maximal and consistent for every structure $\mathcal{M}$.

Recall that $Th(\mathcal{M}) := \{ \text{sentence } \varphi \mid \models_{\mathcal{M}} \varphi \}$.

Observation 1. 可满足 $\Rightarrow$ 最大一致存在

If a set of wfss. Σ is satisfiable. then ~~there~~ exists Γ s.t.

$$\begin{cases} \Sigma \subset \Gamma \\ \Gamma \text{ is maximal and cons··tent.} \end{cases}$$

$\mathrm{Mod}\,\Sigma \neq \emptyset$ 考虑 $\Gamma = Th(\mathcal{M}_0)$ 对某个 $\mathcal{M}_0 \in \underline{\mathrm{Mod}\,\Sigma}$.

Lemma 2 一致性 ⇒ 最大一致性.

如果 Σ is consistent, there is $\Gamma \supset \Sigma$ that is maximal and consistent.

Pf. Claim. If Σ is consistent, φ is a sentence. then one of $\Sigma;\varphi$ and $\Sigma;\neg\varphi$ is also consistent.

pf of claim. 如果 $\Sigma;\varphi$ is consistent, we are done.

否则 如果 $\Sigma;\varphi$ is inconsistent (1/2的写法)

$\Rightarrow \Sigma \vdash \neg\varphi$ (I)

Claim $\Sigma;\neg\varphi$ is consistent.

否则 $\Sigma;\neg\varphi \vdash \beta$. $\Sigma;\neg\varphi \vdash \neg\beta$

根据 deduction theorem

$$\Sigma \vdash \neg\varphi \to \beta \quad \Sigma \vdash \neg\varphi \to \neg\beta$$

根据 (I). MP. 有 $\Sigma \vdash \beta$. $\Sigma \vdash \neg\beta$

说明 Σ is inconsistent. 和假设矛盾.

说明 Claim 为真

假设 $\{\varphi_n\}$ be an enumeration of all sentences.

Let $\Sigma_0 = \Sigma$ is consistent.

根据 Claim. 我们找到一个算法 使 $\Sigma_{n+1} = \Sigma_n \cup \{\varphi_n\}$ 或

$$\Sigma_{n+1} = \Sigma_n \cup \{\neg\varphi_n\}$$

根据数学归纳法. $\forall n \in \mathbb{N}$: Σ_n is consistent.

令 $T = \bigcup_{n\in\mathbb{N}} \Sigma_n \quad (\mathbb{N}^2 \sim \mathbb{N})$

根据定义. T is maximal.

$\square$.

妙啊!

§41. Gödel's Completeness Theorem.

Thm. Completeness Theorem (Gödel, 1930)

① If $\Gamma \models \varphi$. then $\Gamma \vdash \varphi$

② any consistent set of formulas is satisfiable.

Restatement of ①: If a formula is logically valid, then there is a finite deduction (formal proof) of the formula.

① 和 ② 是等价的表述

Pf. 首先我们证明 ① ⇒ ②

如果 Γ is not satisfiable. then there is no model $\mathcal{M}$ can make $\models_{\mathcal{M}} \Gamma$ stands. so trivially forall sentence φ. we have $\Gamma \models \varphi$. (also $\Gamma \models \neg\varphi$)

by ① $\Gamma \vdash \varphi$, $\Gamma \vdash \neg\varphi$

by definition Γ is not consistent.

by contrapositive "Γ is consistent ⇒ Γ is satisfiable".

接着我们证明 ② ⇒ ① 根据归谬原理.

$\Gamma; \varphi$ is inconsistent ⇒ $\Gamma \vdash \neg\varphi$

考虑它的 contrapositive. $\Gamma \nvdash \varphi \Rightarrow \Gamma; \neg\varphi$ is consistent

说明 $\Gamma; \neg\varphi$ is satisfiable

说明 $\Gamma \not\models \varphi$

by contrapositive. ① stands.

在证明了①与②等价之后，我们要证明②为真
在之后的几节中

Recall the Compactness Theorem

(我们一直没有证明 Compactness Thm，它实际上是 Completeness 的一个 Corollary.)

① 如果 $\Gamma \models \varphi$. $\exists$ finite $\Gamma_0 \subset \Gamma$: $\Gamma_0 \models \varphi$.

根据 completeness thm. $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$

则存在一个有限的 formal proof from Γ to φ

则存在有限的 $\Gamma_0 \subset \Gamma$. 使得 formal proof 中每一项对于 $\Gamma_0 \cup \Lambda$ 均为 MP-closed.

则 $\Gamma_0 \vdash \varphi$ 根据 soundness thm. $\Gamma_0 \models \varphi$.

② If every finite $\Gamma_0 \subset \Gamma$ is satisfiable, then Γ is satisfiable (之前证明过和①等价)

§42. The set of Closed Terms

Consider a vocabulary τ, and a set of consistent sentences Γ. let:

① $\mathfrak{A}$ be the set of τ closed terms (terms without variables)

② For $s, t \in \mathfrak{A}$, let $E = \{(s,t) \in \mathfrak{A}^2 \mid \Gamma \vdash s = t\}$

E is a binary relation. $xEy := (x, y) \in E$

③ let $M^\Gamma = \mathfrak{A}/E$

(也就是 $\mathfrak{A}$ 中关于关系 E 的等价类集合)

e.g. $\tau = \{e, *, c\}$, Γ be 群公理.

$\mathfrak{A} = \{e, c, e*c, e*c*c, e*e*e*c, \cdots\}$

注意到 $c*e \; E \; e$ because $\Gamma \vdash \forall x (x = x*e)$

by quantifier ~~axiom~~ 2. $\Gamma \vdash c = c*e$

$M^\Gamma = \{[e], [c], [c*c], [c*c*c] \cdots\}$ (好啊!)

可以注意到 $[c]$ has no inverse in M^Γ (实际上是 monoid, 幺半群)

e.g. $\tau = \{e, *, (\cdot)^{-1}, a, b\}$ and Γ be group axioms

可以证明 M^Γ looks like $\mathbb{Q}$.

Lemma. E is a equivalent relation on $\mathcal{A}$.

Pf of lemma. reflexity. 根据 logical axiom 5.

$$\forall x. x E x$$

transitivity. $\Gamma \vdash s=t$, $\Gamma \vdash t=r$

根据 Group Axiom $\Gamma \vdash \forall x \forall y \forall z (x=y \wedge y=z \to x=z)$

根据 logical Axiom 2. $\Gamma \vdash s=r$.

symmetry 类似于传递性的证明. $\square$

~~这一~~ lemma 证明了要求③实际上 make sense.

所以 $M^{\Gamma} = \mathcal{A}/E := \{[s] \in \mathcal{A} \mid [s] := \{t \in \mathcal{A} \mid sEt\}\}$

$[s] = [t]$ iff sEt

§43. The Term Model

Definition. Suppose $\Gamma = \{c_0, c_1, \cdots, f_0, f_1, \cdots, R_0, R_1 \cdots\}$

let $[t]$ be the E-equivalent class of t. let

$$\mathcal{M}^\Gamma = (M^\Gamma; c_0^\Gamma, c_1^\Gamma, \cdots, f_0^\Gamma, f_1^\Gamma, \cdots, R_0^\Gamma, R_1^\Gamma \cdots).$$

where $M^\Gamma = A/E$ and for all i.

① $c_i^\Gamma := [c_i]$

② $f_i^\Gamma([t_1], [t_2], \cdots, [t_k]) = [f_i(t_1, t_2, \cdots, t_k)]$

③ $\langle [t_1], [t_2], \cdots, [t_k] \rangle \in R_i^\Gamma \Leftrightarrow \Gamma \vdash R_i(t_0, t_1, \cdots, t_k)$.

其中 Γ is a consistent set of sentences.

Note. △ $f_i^\Gamma : (A/E)^k \to A/E$

△ $R_i^\Gamma \subset (A/E)^k$

Antonio: When we define a function or relation like that, we should always be careful that it's well-defined.

在这里，我们要确保 如果 $[\overline{s_j}] E [\overline{t_j}]$ resp.

则 $f_i^\Gamma([\overline{s_j}]) E f_i^\Gamma([\overline{t_j}])$ resp.

可以使用 "The E relation is a congruence relation (同余关系) 来证明".

§ 44. Term Models of Maximal Theories

Theorem. Let Γ be maximal, consistent and φ is a quantifier-free <u>sentence</u>.

Then $\models_{\mathcal{M}^\Gamma} \varphi$ iff $\Gamma \vdash \varphi$.

Pf. (Γ is maximal iff for every sentence ψ, either $\Gamma \vdash \psi$ or $\Gamma \vdash \neg\psi$)

1. If φ is atomic:

1.1. $\Gamma \vdash s=t \Leftrightarrow s E t \Leftrightarrow \models_{\mathcal{M}^\Gamma} s=t$

(根据 $\mathcal{M}^\Gamma$ 的定义可证).

1.2. $\Gamma \vdash R(t_1, t_2, \cdots, t_k)$

$\Leftrightarrow ([t_1], [t_2], \cdots [t_k]) \in R^\Gamma$

$\Leftrightarrow \models_{\mathcal{M}^\Gamma} R(t_1, t_2, \cdots, t_k)$

2. If $\varphi = \psi \wedge \theta,\ \psi \vee \theta,\ \neg\psi$.

以 $\psi \wedge \theta$ 为例

$\Gamma \vdash \psi \wedge \theta \Leftrightarrow \Gamma \vdash \psi \wedge \Gamma \vdash \theta$

(同时根据 Rule T. $\psi \wedge \theta \to \psi$, $\psi \wedge \theta \to \theta$)

△突然意识到 $\psi_1 \to \psi_2$ 不是 ψ_1 tautologically implies ψ_2 !!)

故 $\models_{\mathcal{M}^\Gamma} \psi$ and $\models_{\mathcal{M}^\Gamma} \theta$ (Induction Hypothesis).

故 $\models_{\mathcal{M}^\Gamma} \psi \wedge \theta$ (根据定义)

注意到 φ 是无量词的 sentence. 所以结束证明. $\square$

§45. Sets (Theories) that Contain Term Witness.

Definition. Term Witness

Γ contains term witness if whenever $\Gamma \vdash \exists x \varphi$, there is some closed term t s.t. $\Gamma \vdash \varphi^x_t$

(强有的的定义)

Lemma.

If Γ is maximal, consistent and contains term witnesses, then for every wff φ

$$\models_{\mathfrak{A}^\Gamma} \varphi \Leftrightarrow \Gamma \vdash \varphi.$$

Pf of lemma.

Recall that if φ is a quantifier-free sentence.
则结论正确

如果 $\varphi = \exists x \psi$, 则 ...

$\Gamma \vdash \exists x \psi \Leftrightarrow$ there is a closed term t s.t. $\Gamma \vdash \psi^x_t$

($\Rightarrow$: 由于 Γ contains witness

$\Leftarrow$: 根据 logical axioms in Contrapositive)

根据 Induction Hypothesis. $\Leftrightarrow \models_{\mathfrak{A}^\Gamma} \psi^x_t$

$$\Leftrightarrow \models_{\mathcal{M}^\Gamma} \psi(x \mapsto [t])$$

$$\underset{\text{定义}}{\Leftrightarrow} \models_{\mathcal{M}^\Gamma} \exists x \psi \Leftrightarrow \models_{\mathcal{M}^\Gamma} \varphi.$$

如果 $\varphi = \forall x \psi$. 只需注意到 $\forall x \psi = \neg \exists x \neg \psi$

§ 46. Finishing the Proof of Completeness

Theorem.

If Σ is consistent, there is a set $\Gamma \supseteq \Sigma$ that is maximal, consistent and contains term witness

为了证明定理. 先证明 Lemma 1~2

Lemma 1

Let c be a constant not in $\Sigma; \varphi$.
If Σ is consistent, then so is $\Sigma; \exists x\varphi \to \varphi^x_c$

Pf of lemma. 如果 $\Sigma; \exists x\varphi \to \varphi^x_c$.
is inconsistent.

$\Leftrightarrow \Sigma \vdash \neg(\exists x\varphi \to \varphi^x_c)$

i.e. $\Sigma \vdash (\exists x\varphi \wedge \neg\varphi^x_c)$

根据 Rule T $\Leftrightarrow \Sigma \vdash \exists x\varphi$. 且 $\Sigma \vdash \neg\varphi^x_c$

根据 Generalization of Constant Thm (Corollary)

(Let c be a constant symbol not in Γ or θ.
If $\Gamma \vdash \theta^x_c$. then $\Gamma \vdash \forall x\theta$)

$$\begin{cases} \Sigma \vdash \exists x\varphi \Leftrightarrow \Sigma \vdash \neg\forall x\neg\varphi \\ \Sigma \vdash \neg\varphi^x_c \Leftrightarrow \Sigma \vdash \forall x\neg\varphi \end{cases} \Rightarrow \Sigma \text{ is inconsistent.}$$

Lemma 2.

For each wff φ consider a new constant c_φ. Let

$\Theta = \{\exists x \varphi \to \varphi^x_{c_\varphi} \mid \varphi \text{ is a wff}\}$

If Σ is consistent, so is $\Sigma \cup \Theta$

Pf, Let $\{\varphi_n \mid n \in \mathbb{N}\}$ be an enumeration of all wffs

By induction, show that for every $n \in \mathbb{N}$,

$\Sigma \cup \{\exists x \varphi_i \to (\varphi_i)^x_{c_{\varphi_i}} \mid i < n\}$ is consistent.

Induction 的过程就是我们 in Lemma 1.

(向 consistent 集合中加入 $\exists x \varphi \to \varphi^x_c$ 不改变 consistency 性质)

所以 $\forall n \in \mathbb{N}$: $\Sigma_n = \Sigma \cup \{\exists x \varphi_i \to (\varphi_i)^x_{c_{\varphi_i}} \mid i < n\}$ is consistent.

如果 $\Sigma \cup \Theta$ is inconsistent.

则 there exists wff β s.t. $\begin{cases} \Sigma \cup \Theta \vdash \beta \\ \Sigma \cup \Theta \vdash \neg\beta \end{cases}$

Since the deduction is finite,

there exists $N \in \mathbb{N}$ s.t. Σ_N is inconsistent

Contradiction!

Lemma 2 说明. Θ 中每一项都有 witness.

(Θ 的定义是合理的, 因为 φ 显然是 countable 的)

Pf of Gödel's Completeness Theorem.

根据 §40. lemma 2. 一致性 ⇒ 最大一致存在.

let Γ be a maximal consistent set of wffs containing $\Sigma \cup \Theta$

根据 §45. lemma. for every wff φ.

$$\models_{M^{\Gamma}} \varphi \Leftrightarrow \Gamma \vdash \varphi$$

这证明了 Gödel's Completeness Theorem 的第 II 表述 $\square$

§47. Summing Up.

- We have a deduction system for formal proofs that is complete.

- The language for first order logic is very good, but it has limitations: Compactness.

  (e.g. 图的 connectness 无法用 FO-language 表示)

- In the right setting, FO language is complete.

- 在集合论中，有一系列 Axioms 称为 ZFC-Axioms over $\tau = \{\in\}$, where essentially all of mathematics can be developed.

- Gödel's Incompleteness Theorem.

  If T is consistent, and contains enough of arithmetic, and is computably enumerable, then it cannot be maximal.