A Brief Intro of Catalan Numbers

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1. Introduction

Catalan Numbers is a kind of counting sequence which was first discovered by Swiss mathematician *Leonhard Euler* and Hungarian mathematician *Johann Andreas von Segner*[1] by studying the problem of triangulation of the convex polygon[2]. Although recursive relations of these numbers are introduced from Segner, many of the properties and identities of these numbers are discovered by the side of French-Belgian mathematician Eugene Charles Catalan in 1838 through the study of well-formed sequences of parentheses.

We choose this topic, because it is originally covered in our textbook, but removed in the revised version. We believe this topic is still interesting and important in Computational Mathematics.

2. Recurrence Relation

The definition of Catalan Numbers is given by the following formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \quad n \geq 0.$$

or is more commonly defined by the following recurrence relation:

$$\begin{aligned} C_0 &= 1, \\ C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-i}, \quad n \geq 0. \end{aligned}$$

These two definitions are proved equivalent by using the method of generating functions. Suppose there is a function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} C_n x^n.$$

where C_n is the n-th Catalan number. Then we have [1]

$$\begin{aligned} g^2(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n. \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \\ &= \frac{g(x) - 1}{x}. \end{aligned} \quad \xRightarrow{g(0)=1} \quad \begin{aligned} g(x) &= \frac{1 - \sqrt{1-4x}}{2x} \\ &= -\frac{1}{2x} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n. \end{aligned}$$

By applying the Taylor Expansion, we can derive that these two definitions are equivalent as the coefficient of x^n in the series expansion of $g(x)$ is C_n.

3. Application

A typical application of Catalan Numbers is the number of ways from $(0, 0)$ to (n, n) in a $n \times n$ grid without crossing the line $y = x - 1$ (**??**).

we can explain both the general formula and the recurrence formula in the context of this application. The general formula is used to calculate the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$ grid without crossing the line $y = x - 1$. The recurrence formula is used to calculate the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$ grid without crossing the line $y = x - 1$ by using the number of ways to reach $(n - 1, n - 1)$ from $(0, 0)$ in a $(n - 1) \times (n - 1)$ grid without crossing the line $y = x - 1$.

It's clear that the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$

[1] Note that in Combinatorics, we don't really care about whether this series converges, we always assume that it will converge in a certain radius.

grid without crossing the line $y = x - 1$ is equal to the number of ways to reach (n, n) from $(0, 0)$ without constraints minus the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$ grid without crossing the line $y = x - 1$. But we can always do a symmetry transformation with respect to the line $y = x - 1$ as Figure 1 shows. Then

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

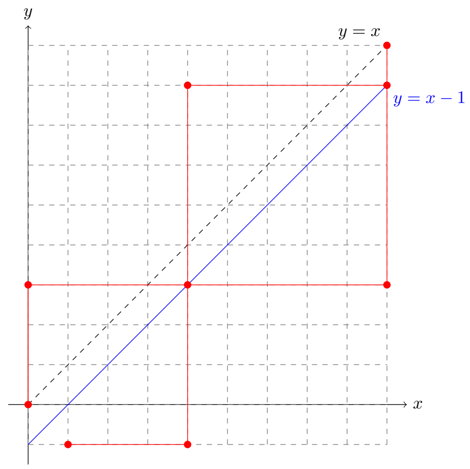

Figure 1: general Formula

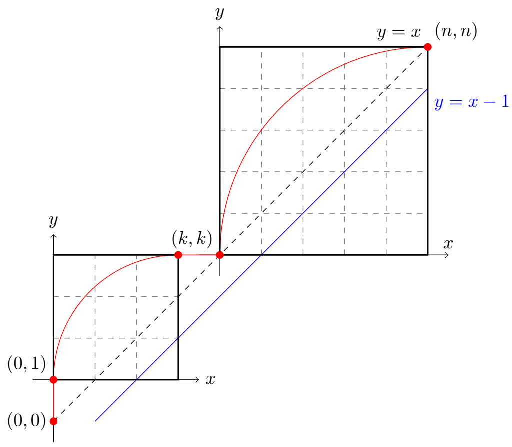

Figure 2: recurrence Formula

On the other hand, suppose we firstly touch $y = x$ at point (k, k), then we can divide the path into two parts: the first part is from $(0, 0)$ to (k, k) (i.e. $(0, 1) \to (k - 1, k)$), and the second part is from (k, k) to (n, n). By principle of multiplication, we can traverse the first part in C_k ways and the second part in C_{n-1-k} ways. Therefore,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

References

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[3] Aybeyan Selimi and Muzafer Saračević. "Catalan Numbers and Applications - Vision Journal". In: *Vision Journal* (2019). URL: http://visionjournal.edu.mk/wp-content/uploads/2019/08/aybeyan-pdf.pdf.

[4] J J O'Connor and E F Robertson. *Leonhard Euler - Biography*. Maths History. 1998. URL: http://mathshistory.st-andrews.ac.uk/Biographies/Euler/.

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[6] Robert M Dickau and Heinz Klaus Strick. *Catalan Numbers*. URL: http://mathshistory.st-andrews.ac.uk/Extras/Catalan/ (visited on 04/30/2024).