A Brief Intro of Catalan Numbers

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1. Introduction

Catalan Numbers is a kind of counting sequence which was first discovered by Swiss mathematician *Leonhard Euler* and Hungarian mathematician *Johann Andreas von Segner* (see O'Connor and Robertson, 2004), who study the problem of triangulation of the convex polygon (see Stillwell, 2020). Although recursive relations of these numbers are introduced from Segner, many of the properties and identities of these numbers are discovered by the side of French-Belgian mathematician Eugene Charles Catalan in the study of well-formed sequences of parentheses.

Catalan published his study of Catalan Numbers, Solution d'un problème de Probabilité relatif au jeu de rencontre, in the second volume of Liouville's Journal de Mathématiques Pures et Appliquées in 1837. Later, Catalan published some further study of Catalan Numbers in the same journal in 1838 and 1839.

This topic is originally covered in the Chapter 25 "Combinatorics" of 3rd edition textbook but removed in the revised version. The reason for choosing it is that we still believe this topic is interesting and important in Computational Mathematics. Also, this topic includes what we learned in week 04/01: using power series to expand the generating function (formula 3).

2. Recurrence Relation

The definition of Catalan Numbers is given by the following formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \quad n \geq 0. \tag{1}$$

or common recurrence relation:

$$\begin{aligned} C_0 &= 1, \\ C_n &= \sum_{i=0}^{n-1} C_i C_{n-i-1}, \quad n \geq 1. \end{aligned} \tag{2}$$

These two definitions are proved equivalent by using the method of generating functions.

Suppose there is a function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} C_n x^n. \tag{3}$$

where C_n is the n-th Catalan number. Then we have [1]

$$\begin{aligned} g^2(x) &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} C_i C_{n-i}\right) x^n. \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \\ &= \frac{g(x)-1}{x}. \end{aligned} \quad \overset{g(0)=1}{\Longrightarrow} \quad \begin{aligned} g(x) &= \frac{1-\sqrt{1-4x}}{2x} \\ &= -\frac{1}{2x}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n. \end{aligned} \tag{4}$$

By applying the Taylor Expansion, we can derive that these two definitions are equivalent as the coefficient of x^n in the series expansion of $g(x)$ is C_n.

3. Application

A typical application of Catalan Numbers is counting the number of ways starting from $(0,0)$ to (n,n) in a $n \times n$ grid without crossing the line $y = x - 1$ (figure 1).

The general formula and the recurrence formula can be interpreted through this application. The general formula is used to directly calculate the number of ways to reach (n,n) from $(0,0)$ in a $n \times n$ grid without crossing the line $y = x - 1$. The recurrence formula is used to calculate the number of ways to reach point (n,n) from $(0,0)$ in a $n \times n$ grid without crossing the line $y = x - 1$ by using previous number derived from point $(n-1,\ n-1)$.

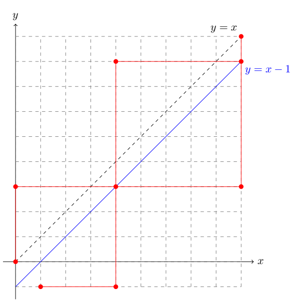

Figure 1: general Formula

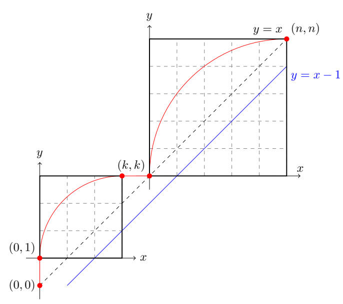

Figure 2: recurrence Formula

[1] Note that in Combinatorics, we don't really care about whether this series converges, we always assume that it will converge in a certain radius.

For general formula, it is clear that the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$ grid without crossing the line $y = x - 1$ is equal to the total number of ways to reach (n, n) from $(0, 0)$ minus the number of ways to reach (n, n) from $(0, 0)$ in a $n \times n$ grid crossing the line $y = x - 1$. The Figure 1 shows to do a symmetry transformation with respect to the line $y = x - 1$. Then

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \tag{5}$$

For recurrence relation, suppose a point in the moving process first touches the line $y = x$ at point (k, k), then the path can be divided into two parts: the first part is from $(0, 0)$ to (k, k), and the second part is from (k, k) to (n, n). By definition, the second part will be C_{n-k} ways. And for first part, because next step of $(0, 0)$ must be $(0, 1)$, and the former step of (k, k) must be $(k - 1, k)$. So, the first step can be changed to $(0, 1)$ to $(k - 1, k)$. And we build new coordinate system with $(0, 1)$ be the new origin, $(k - 1, k)$ becomes $(k - 1, k - 1)$. Then, the first step by definition will be C_{k-1}. By principle of multiplication, it will be in $C_{k-1}\ C_{n-k}$ ways. Therefore,

$$\begin{aligned} C_n &= \sum_{k=1}^{n} C_{k-1} C_{n-k} \\ &= \sum_{k=0}^{n-1} C_{(k+1)-1} C_{n-(k+1)} \\ &= \sum_{i=0}^{n-1} C_i C_{n-i-1} \end{aligned} \tag{6}$$

4. Reflection

Everyone contributed a lot to the group task and was actively involved in discussing and completing the presentation and the paper, not only that, in order to work more efficiently and perfectly, we took out two weekends at the end of April and the beginning of May to meet and communicate offline to finalize our slides presentation and final paper. It's hard to ditinguish each group member's role, but we all put all efforts on this.

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