A Brief Intro of Catalan Numbers

2024 Spring MATH 473

Chentian Wu

Wenhan Zhang

Austin Luo

Kaicheng Xue

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

May 5, 2024

1. Introduction

Catalan Numbers were first discovered by Swiss mathematician *Leonhard Euler* and Hungarian mathematician *Johann Andreas von Segner* by studying the problem of triangulation of the convex polygon[1]. Although recursive relations of these numbers are introduced from Segner, many of the properties and identities of these numbers are discovered by the side of French-Belgian mathematician Eugene Charles Catalan in 1838 through the study of well-formed sequences of parentheses.

We choose this topic, because it is originally covered in our textbook, but removed in the revised version. We believe that this topic is important to be covered in the course, because it is a good example of how to solve a problem using dynamic programming. In addition, it is a good example of how to solve a problem using combinatorial mathematics.

2. Recurrence Relation

The definition of Catalan Numbers is given by the following formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \quad n \geq 0.$$

or is more commonly defined by the following recurrence relation:

$$\begin{aligned} C_0 &= 1, \\ C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-i}, \quad n \geq 0. \end{aligned}$$

These two definitions are proved equivalent by using the method of generating functions. Suppose there is a function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} C_n x^n.$$

where C_n is the n-th Catalan number. Then we have [1]

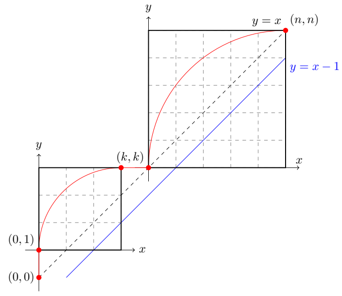
$$\begin{aligned} g^2(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n. \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \\ &= \frac{g(x)-1}{x}. \end{aligned} \quad \overset{g(0)=1}{\Longrightarrow} \quad \begin{aligned} g(x) &= \frac{1-\sqrt{1-4x}}{2x} \\ &= -\frac{1}{2x} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n. \end{aligned}$$

By applying the Taylor Expansion, we can derive that these two definitions are equivalent as the coefficient of x^n in the series expansion of $g(x)$ is C_n.

3. Application

A typical application of Catalan Numbers is the number of ways from $(0, 0)$ to (n, n) in a $n \times n$ grid without crossing the line $y = x - 1$ (1).

[1]Note that in Combinatorics, we don't really care about whether this series converges, we always assume that it will converge in a certain radius.



1

Figure 1: Cambly / SWOT

4. Conclusion

From medicine to applied mathematics, to analytics, and then to logic, Curry adjusted his research direction three times in his life. As an double-major student, I'm deeply inspired by Curry's perseverance and decisiveness. I now major in theoretical computer science and mathematics (especially logic), and hope that I could make a difference in these fields, like Dr. Curry did.

References

[1] John Stillwell. *Mathematics and its History*. Springer, 2020. ISBN: 978-3-030-55193-3. URL: https://www.springer.com/gp/book/9781441960528.

[2] David Acheson. *Mathematics for All*. 1st. Oxford: Oxford University Press, 2012. ISBN: 978-0-19-967796-7. URL: https://global.oup.com/academic/product/mathematics-for-all-9780199677967?cc=gb&lang=en& (visited on 06/01/2021).