A Brief Intro of Catalan Numbers

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1. Introduction

Catalan Numbers is a kind of counting sequence which was first discovered by Swiss mathematician *Leonhard Euler* and Hungarian mathematician *Johann Andreas von Segner*[1], who study the problem of triangulation of the convex polygon[2]. Although recursive relations of these numbers are introduced from Segner, many of the properties and identities of these numbers are discovered by the side of French-Belgian mathematician Eugene Charles Catalan in 1838 in the study of well-formed sequences of parentheses.

This topic is originally covered in the textbook but removed in the revised version. The reason for choosing it is that we still believe this topic is interesting and important in Computational Mathematics. Also, this topic includes what we learned in week 04/01: using power series to expand the generating function (formula 3).

2. Recurrence Relation

The definition of Catalan Numbers is given by the following formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \quad n \geq 0. \tag{1}$$

or common recurrence relation:

$$\begin{aligned} C_0 &= 1, \\ C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-i}, \quad n \geq 0. \end{aligned} \tag{2}$$

These two definitions are proved equivalent by using the method of generating functions. Suppose there is a function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} C_n x^n. \tag{3}$$

where C_n is the n-th Catalan number. Then we have [1]

$$\begin{aligned} g^2(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n. \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \\ &= \frac{g(x)-1}{x}. \end{aligned} \quad \overset{g(0)=1}{\Longrightarrow} \quad \begin{aligned} g(x) &= \frac{1-\sqrt{1-4x}}{2x} \\ &= -\frac{1}{2x} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n. \end{aligned} \tag{4}$$

By applying the Taylor Expansion, we can derive that these two definitions are equivalent as the coefficient of x^n in the series expansion of $g(x)$ is C_n.

3. Application

A typical application of Catalan Numbers is counting the number of ways starting from $(0,0)$ to (n,n) in a $n \times n$ grid without crossing the line $y = x - 1$ (figure 1).

The general formula and the recurrence formula can be interpreted through this application. The general formula is used to directly calculate the number of ways to reach (n,n) from $(0,0)$ in a $n \times n$ grid without crossing the line $y = x - 1$. The recurrence formula is used to calculate the number of ways to reach point (n,n) from $(0,0)$ in a $n \times n$ grid without crossing the line $y = x - 1$ by using previous number derived from point $(n-1,\ n-1)$.

For general formula, it is clear that the number of ways to reach (n,n) from

[1] Note that in Combinatorics, we don't really care about whether this series converges, we always assume that it will converge in a certain radius.

$(0,0)$ in a $n \times n$ grid without crossing the line $y = x - 1$ is equal to the total number of ways to reach (n,n) from $(0,0)$ minus the number of ways to reach (n,n) from $(0,0)$ in a $n \times n$ grid crossing the line $y = x - 1$. The Figure 1 shows to do a symmetry transformation with respect to the line $y = x - 1$. Then

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \tag{5}$$

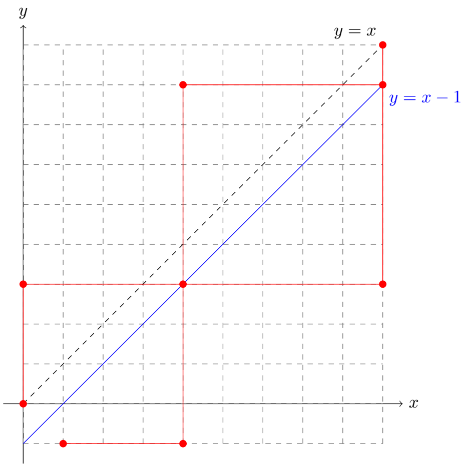

Figure 1: general Formula

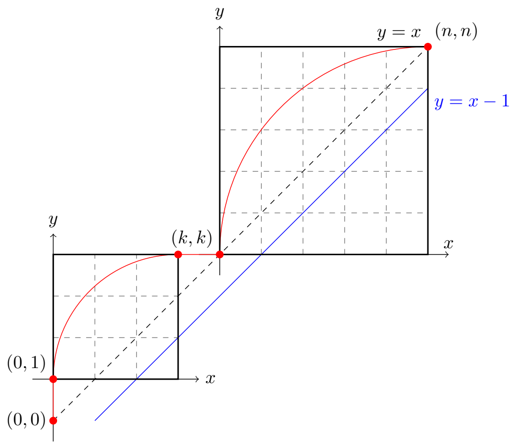

Figure 2: recurrence Formula

For recurrence relation, suppose a point in the moving process touches the line $y = x$ at point (k,k), then the path can be divided into two parts: the first part is from $(0,0)$ to (k,k) (i.e. $(0,1) \to (k-1,k)$), and the second part is from (k,k) to (n,n). By principle of multiplication, the first part will be in C_k ways and the second part will be in C_{n-1-k} ways. Therefore,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}. \tag{6}$$

References

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[4] J J O'Connor and E F Robertson. *Leonhard Euler - Biography*. Maths History. 1998. URL: http://mathshistory.st-andrews.ac.uk/Biographies/Euler/.

[5] J J O'Connor and E F Robertson. *Eugène Catalan - Biography*. Maths History. 2012. URL: http://mathshistory.st-andrews.ac.uk/Biographies/Catalan/.

[6] Robert M Dickau and Heinz Klaus Strick. *Catalan Numbers*. URL: http://mathshistory.st-andrews.ac.uk/Extras/Catalan/ (visited on 04/30/2024).