Midterm 2 (100(+10) points) Name: ______________________________

April 2024, 11:00-12:30 Student ID: ______________________________

Please Read!!! No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work.

If your total points exceed 100, then it will be 100.

1. (15 points) Count the number of integers n in $[1, 50000]$ such that in the factorization of n, the power of 2 is exactly 3, the power of 3 is exactly 2, then power of 5 is exactly 1.

   Proof. Such n is of form $n = m \times 2^3 \times 3^2 \times 5$, where $2, 3, 5$ doesn't divide m. Note $1 \leq m \leq [\frac{50000}{360}] = 138$ The problem is equivalent to find the number of m in $[1, 138]$ such that $2, 3, 5$ doesn't divide m. By inclusion-exclusion lemma,

$$138 - [138/2] - [138/3] - [138/5] + [138/6] + [138/10] + [138/15] - [138/30] = 37.$$

$\square$

2. (15 points) What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

that satisfy

$$1 \leq x_1 \leq 5, -1 \leq x_2 \leq 4, 0 \leq x_3 \leq 3, 0 \leq x_4.$$

Proof. By change of variables, we just need to count the integral solutions for

$$y_1 + y_2 + y_3 + y_4 = 10,$$

with

$$0 \leq y_1 \leq 4, 0 \leq y_2 \leq 5, 0 \leq y_3 \leq 3, 0 \leq y_4.$$

Denote A_i be the set of nonnegative solutions where

$$y_i \geq \begin{cases} 5 & when\ i = 1 \\ 6 & when\ i = 2 \\ 4 & when\ i = 3 \end{cases}$$

Total nonnegative solutions$=\binom{13}{3} = 286$

$|A_1| = \binom{8}{3} = 56$

$|A_2| = \binom{7}{3} = 35$

$|A_3| = \binom{9}{3} = 84$

$|A_1 \cap A_2| = 0$

$|A_1 \cap A_3| = 4$

$|A_2 \cap A_3| = 1$

$|A_1 \cap A_2 \cap A_3| = 0$

By inclusion-exclusion,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 286 - 56 - 35 - 84 + 4 + 1 = 116.$$

$\square$

3. (30 points) Find the expression for $\{f_n\}$ which satisfy

$$f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3} + 2^n + 2n, \quad f_0 = 0, \ f_1 = 12, \ f_2 = 38.$$

1. (5 points) Find the solution for homogeneous part.
2. (20 points) Find the particular solution.
3. (5 points) Find the expression for f_n.

Proof. (1) For the homogeneous part $f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3}$, the characteristic polynomial is $x^3 - 5x + 8x - 4 = (x-1)(x-2)^2$. (To factorize, the idea is to first guess a trivial root which is $x = 1$.) Therefore, the solution for homogeneous part is $f_n = a + b \cdot 2^n + c \cdot n \cdot 2^n$.

(2.1) Consider $f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3} + 2^n$. We guess the solution is of form $f_n = d \cdot n^2 \cdot 2^n$. (As discussed in the class, we first try $f_n = d \cdot 2^n$ which doesn't work. Then try $d \cdot n \cdot 2^n$, and then try $d \cdot n^2 \cdot 2^n$ until it works.) Plugging into the equation, we get

$$dn^2 2^n = 5d(n-1)^2 2^{n-1} - 8d(n-2)^2 2^{n-2} + 4(n-3)^2 2^{n-3} + 2^n.$$

It's equivalent to

$$d(2n^2 - 5(n-1)^2 + 4(n-2)^2 - (n-3)^2) = 2.$$

We get $d = 1$.

So, the particular solution is $n^2 \cdot 2^n$.

(2.2) Consider $f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3} + 2n$. We guess the solution is of form $f_n = d \cdot n^2 + e \cdot n$. Plugging into the equation, we get

$$d(n^2 - 5(n-1)^2 + 8(n-2)^2 - 4(n-3)^2) + e(n - 5(n-1) + 8(n-2) - 4(n-3)) = 2n,$$

Equivalently,

$$e + d(2n - 9) = 2n$$

which gives $d = 1, e = 9$. So, the particular solution is $f_n = n^2 + 9n$.

(3) By the linearity, the solution is the sum of (1),(2.1),(2.2), which is

$$f_n = a + b \cdot 2^n + c \cdot n \cdot 2^n + n^2 2^n + n^2 + 9n.$$

Checking for $f_0 = 0, f_1 = 12, f_2 = 38$, we get $a = b = c = 0$. Therefore, the solution is

$$f_n = n^2 2^n + n^2 + 9n.$$

$\square$

4. (15 points) Suppose $f_n = 2 \cdot (\frac{1+\sqrt{3}i}{2})^n + 2 \cdot (\frac{1-\sqrt{3}i}{2})^n + 2^n + 1$. Determine all the integers $n \geq 0$ such that 3 divides f_n.

Proof. We can look at $g_n = 2 \cdot (\frac{1+\sqrt{3}i}{2})^n + 2 \cdot (\frac{1-\sqrt{3}i}{2})^n + 2^n$.

The three roots are $\omega_{1,2} = \frac{1\pm\sqrt{3}i}{2}, \omega_3 = 2$.

The characteristic polynomial is $(x^2 - x + 1)(x - 2) = x^3 - 3x^2 + 3x - 2$.

The recurrence relation is $g_n = 3f_{n-1} - 3g_{n-2} + 2g_{n-3}$. $g_0 = 5, g_1 = 4, g_2 = 2$.

We see that $g_n \equiv -g_{n-3} (mod\ 3)$, so under mod 3, the sequence $\{g_n\}$ is

$$-1, 1, -1, 1, -1, 1, ...$$

with period 6. Note $f_n = g_n + 1$, so 3 divides f_n if and only if $n \equiv 0, 2, 4 (mod\ 6)$. $\square$

5. (25 points) For $n \in \mathbb{N}$, let α_n denote the number of ways to write n as a sum of 1's and 2's, and let β_n denote the number of ways to write n as a sum of integers ≥ 2. Here, different orders of the summands are counted as different ways. For example: $\alpha_3 = 3$, since $3 = 1{+}1{+}1 = 1{+}2 = 2{+}1$; $\beta_5 = 3$, since $5 = 2+3 = 3+3 = 5$.

   1. (10 points) Find the generating function of $\{\alpha_n\}$.
   2. (10 points) Find the generating function of $\{\beta_n\}$.
   3. (5 points) Show that $\alpha_n = \beta_{n+2}$.

   Proof. (1) $\sum_{n=0}^{\infty} \alpha_n x^n = 1 + (x + x^2) + (x + x^2)^2 + \cdots = \frac{1}{1-x-x^2}$.

   (2) $\sum_{n=0}^{\infty} \beta_n x^n = 1 + (x^2 + x^3 + \dots) + (x^2 + x^3 + \dots)^2 + \cdots = \frac{1}{1-(x^2+x^3+\dots)} = \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2} = 1 + \frac{x^2}{1-x-x^2}$.

   (3) By (1), (2), we have $\sum_{n=0}^{\infty} \beta_n x^n = 1 + \sum_{n=0}^{\infty} \alpha_n x^{n+2}$. Therefore, $\alpha_n = \beta_{n+2}$. $\square$

6. (5 points) Suppose $a_1, a_2, \ldots, a_{2n}$ is a sequence that satisfy: each $a_i \in \{1, -1\}$; $a_1 + a_2 + \cdots + a_k \geq 0$ for $k = 1, \ldots, 2n$. We define another sequence $b_1, b_2, \ldots, b_{2n}$ recursively in the following way: Let $b_{2n} = a_{2n}$. Suppose we have defined $b_{k+1}, \ldots, b_{2n}$. Let $b_k = a_k$, if $b_{k+1} + \cdots + b_{2n} \leq 0$; let $b_k = -a_k$ if $b_{k+1} + \cdots + b_{2n} > 0$.

   Show that $b_1 + b_2 + \cdots + b_{2n} = 0$.

   Proof. We prove by induction on n. If there exists m such that

$$b_m + b_{m+1} + \cdots + b_{2n} = 0, \tag{1}$$

   then we work with $a_1, \ldots, a_{m-1}$. Note that $b_{m-1} = a_{m-1}$. Because of (1), for $k < m - 1$, $b_k = a_k$ if $b_{k+1} + \cdots + b_{m-1} \leq 0$; $b_k = -a_k$ if $b_{k+1} + \cdots + b_{m-1} > 0$. We can treat $m - 1$ as new n and use induction to get $b_1 + \cdots + b_{m-1} = 0$. Together with (1), we finish the proof.

   Suppose (1) fails. There are two cases.

   (1) $b_m + b_{m+1} + \cdots + b_{2n} \leq -1$ for all m. Then by definition, $b_k = a_k$ for all k. However,

$$a_1 + \cdots + a_{2n} = b_1 + \cdots + b_{2n} \leq -1$$

   which contradicts the condition that $a_1 + \cdots + a_{2n} \geq 0$.

   (2) $b_m + b_{m+1} + \cdots + b_{2n} \geq 1$ for all m. Then $b_{2n} = a_{2n}$, $b_k = -a_k$ for $k \leq 2n - 1$. Then

$$1 \leq b_1 + \cdots + b_{2n} = -(a_1 + \cdots + a_{2n-1}) + a_{2n}.$$

   Since $a_1 + \cdots + a_{2n-1} \geq 0$ and is an odd number, so $a_1 + \cdots + a_{2n-1} \geq 1$. We get

$$1 \leq b_1 + \cdots + b_{2n} = -(a_1 + \cdots + a_{2n-1}) + a_{2n} \leq -1 + 1 = 0,$$

   a contradiction. $\square$

7. (5 points) There is a 3×3 board with eight 1×1 tiles in it. These tiles occupy eight spots, leaving one spot empty. These tiles are labeled $1, 2, \ldots, 8$. You are allowed to do this operation: If a tile is adjacent to the empty spot, you can slide the tile to the empty spot.

1	2	3
4	5	6
7	8	

1	2	3
4	5	6
8	7	

Can you do several operations to get from the first pattern to the second pattern? (Answering Yes or No doesn't count any point.)

Proof. We give the order to the nine positions. The order is $(1,1) < (1,2) < (1,3) < (2,1) < \cdots < (3,3)$. Here (i,j) denotes the position at i-th row j-th column. For a pattern, write the labels of tiles according to the order, so that we obtain an arrangement of $\{1, 2, \ldots, 8\}$.

The first pattern corresponds to 12345678; the second pattern corresponds to 12345687.

Next, we note if we slide the tile horizontally, the sequence doesn't change. If we slide label i up, then in the corresponding sequence, the number i is moved two positions left. If we slide label i down, then in the corresponding sequence, the number i is moved two positions right. No matter which operations we do, the number of inversion pairs of the sequence has the same parity!

Note 12345678 has 0 inversion pair, while 12345687 has 1 inversion pair. So first pattern cannot be changed to second pattern. $\square$