Midterm 1 Practice 100 points) Name: ______________________

Feb. 2024, 11:00-12:15 Student ID:______________________

Please Read!!! No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work.

1. (50 points) Count the following quantities. (Show the reasoning.)

   1. The number of positive divisors of 100^{100}.
   2. Let $n_1 + n_2 = n$. Count the number of n-permutations of the multiset $\{n_1 \cdot a, n_2 \cdot b\}$.
   3. Find the number of solutions for $x_1 + x_2 + x_3 = 5$, where $x_1 \geq 1, x_2 \geq 2, x_3 \geq 0$ and they are integers.
   4. Randomly choose a subset with cardinality 5 from the multiset $\{4\cdot 1, 4\cdot 2, \ldots, 4\cdot 9\}$. What's the probability that the subset is of form $\{a, a, a, b, b\}$.
   5. Count the number of 5-digit passwords, which consists of numbers $\{1, 2, \ldots, 9\}$ and letters $\{a, b, c\}$ so that the letters are not adjacent.

   A sketch of the solution. You should provide all the details in the exam.

   1. $100^{100} = 2^{200}5^{200}$. The number of positive divisors is 201^2.
   2. Use the formula discussed in class. We get $\binom{n_1+n_2}{n_1}$.
   3. It's equivalent to finding the number of solutions to
   $$y_1 + y_2 + y_3 = 5,$$
   where each $y_i \geq 1$. This will be $\binom{4}{2} = 6$.
   4. Since we are counting the probability, it's good to view all the numbers in the set as different elements. (Think about the poker.) The number of subsets with cardinality 5 is $\binom{36}{5}$. The number of subsets with form $\{a, a, a, b, b\}$ is counted as follows. 9 ways to choose a; 8 ways to choose b; 4 ways to choose three a; 6 ways to choose two b. In total, $9 \times 8 \times 4 \times 6$. Answer: $\frac{9\times 8\times 4\times 6}{\binom{36}{5}}$.
   5. Case 1: No letter. 9^5.
   Case 2: One letter. The letter can lie in 5 positions with 3 choices of the letters. 15×9^4.
   Case 3: Two letters. Note that the two positions that the letters lie can be
   $$(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5).$$
   There are $6 \times 3^2 \times 9^3$ cases.
   Case 4: Three letters. The three letters must lie at positions $(1, 3, 5)$. $3^3 \times 9^2$.
   Answer: $9^5 + 15 \times 9^4 + 6 \times 3^2 \times 9^3 + 3^3 \times 9^2$.

$\square$

2. (50 points) Answer the following questions.

   1. Simplify $\sum_{k=0}^{99} (\frac{7}{5})^k \binom{100}{k}$.
   2. Simplify
   $$\sum_{t_1+t_2+t_3=n, t_1, t_2, t_3 \geq 0} \binom{m_1}{t_1}\binom{m_2}{t_2}\binom{m_3}{t_3}.$$
   3. Use the combinatorial reasoning to count
   $$\sum_{k=0}^{n} \binom{n}{k}^2$$
   4. If we expand $(x_1 + x_2 + x_3 + x_4)^{20}$, what's the coefficient of $x_1^2 x_2^3 x_3^5 x_4^{10}$?
   5. Prove that for integers r, k, m, we have
   $$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

A sketch of proof. You should provide all the details in the exam.

1. $(\frac{12}{5})^{100} - (\frac{7}{5})^{100}$.
2. $\binom{m_1+m_2+m_3}{n}$
3. Count how many ways to select n people from $2n$ people. The number of ways is $\binom{2n}{n}$. On the other hand, it's equivalent to choose k people from the first n people and choose $n-k$ people from the remaining n people. It gives $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$.
4. $\frac{20!}{2!3!5!10!}$
5. Use the definition $\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$.

$\square$