Midterm 2 (100 points) Name: ____________________

April 2024, 11:00-12:15 Student ID: ____________________

Please Read!!! No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work.

Unlike Midterm 1, there is no bonus question.

1. (15 points) Count the number of integers n in $[1, 1000]$ that are not divisible by $5, 6, 8$.

Proof. Define A_1, A_2, A_3 to be the set of integers in $[1, 1000]$ that are divisible by $5, 6, 8$ respectively. Our goal is to count $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$ which is equal to

$$1000 - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$= 1000 - 200 - 166 - 125 + 33 + 25 + 41 - 8 = 600.$$

$\square$

2. What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

that satisfy

$$1 \le x_1 \le 5,\ -2 \le x_2 \le 4,\ 0 \le x_3 \le 5,\ 3 \le x_4 \le 9\ ?$$

Proof. After change of variable, we just need to count the number of solutions for equation

$$y_1 + y_2 + y_3 + y_4 = 16 \tag{1}$$

that satisfy

$$0 \le y_1 \le 4,\ 0 \le y_2 \le 6,\ 0 \le y_3 \le 5,\ 0 \le y_4 \le 6.$$

Let A_1, A_2, A_3, A_4 be the number of nonnegative solutions to (1) that satisfy $y_1 \ge 5,\ y_2 \ge 7,\ y_3 \ge 6,\ y_4 \ge 7$ respectively.

The number of nonnegative solutions to (1) is $\binom{19}{3} = 969$. $|A_1| = \binom{14}{3} = 364$. $|A_2| = |A_4| = \binom{12}{3} = 220$. $|A_3| = \binom{13}{3} = 286$.

$|A_1 \cap A_2| = |A_1 \cap A_4| = \binom{7}{3} = 35$. $|A_1 \cap A_3| = \binom{8}{3} = 56$. $|A_2 \cap A_3| = \binom{6}{3} = 20$. $|A_2 \cap A_4| = \binom{5}{3} = 10$. $|A_3 \cap A_4| = \binom{6}{3} = 20$. The intersection of three sets is empty.

We have

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = 969 - (364 + 220 + 286 + 220) + (35 + 56 + 35 + 20 + 10 + 20) = 55.$$

$\square$

3. Find the expression for $\{f_n\}$ which satisfy

$$f_n = 4f_{n-1} - 4f_{n-2} + 2^n + n, \quad f_0 = 1, \ f_1 = 2.$$

Proof. (1) For the homogeneous part $f_n = 4f_{n-1} - 4f_{n-2}$, the characteristic polynomial is $(x-2)^2$. Therefore, the solution for homogeneous part is $f_n = a \cdot 2^n + b \cdot n \cdot 2^n$.

(2) Consider $f_n = 4f_{n-1} - 4f_{n-2} + 2^n$. We guess the solution is of form $f_n = c \cdot n^2 \cdot 2^n$. Plugging into the equation, we get

$$c\big(n^2 - 2(n-1)^2 + (n-2)^2\big)2^n = 2^n.$$

We get $c = \frac{1}{2}$. So, the particular solution is $\frac{1}{2}n^2 \cdot 2^n$.

(3) Consider $f_n = 4f_{n-1} - 4f_{n-2} + n$. We guess the solution is of form $f_n = c \cdot n + d$. Plugging into the equation, we get

$$c(n - 4(n-1) + 4(n-2)) + d(1 - 4 + 4) = n,$$

which gives $c = 1, d = 4$. So, the particular solution is $f_n = n + 4$.

(4) By the linearity, the solution is the sum of (1),(2),(3), which is

$$f_n = a \cdot 2^n + b \cdot n \cdot 2^n + \frac{1}{2}n^2 \cdot 2^n + n + 4.$$

Checking for $f_0 = 1, f_1 = 2$, we get $a = -3, b = 1$. Therefore, the solution is

$$f_n = -3 \cdot 2^n + n \cdot 2^n + \frac{1}{2}n^2 \cdot 2^n + n + 4$$

□

4. Suppose $f_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$. Determine all the integers $n \geq 0$ such that 3 divides f_n.

Proof. We need to find the recurrence relation of $\{f_n\}$. Note that $f_n = a\omega_1^n + b\omega_2^n$ where $\omega_{1,2} = \frac{1\pm\sqrt{5}}{2}$. The characteristic polynomial is $(x-\omega_1)(x-\omega_2) = x - (\omega_1+\omega_2)x + \omega_1\omega_2 = x^2 - x - 1$. Therefore, the recurrence relation is

$$f_n = f_{n-1} + f_{n-2}.$$

We also compute the initial values: $f_0 = 1, f_1 = 1$. Therefore, under $(mod\ 3)$, the sequence $f_0, f_1, \ldots$ is equal to $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \ldots$. We see the period is 8. So, 3 divides f_n if and only if $n \equiv 0, 4 (mod\ 8)$.

$\square$

5. Fix k. Determine the generating function for the number h_n of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

in nonnegative odd integers $x_1, \ldots, x_k$.

Proof. We have $g(x) = (x + x^3 + x^5 + \ldots)^k = x^k(1 + x^2 + x^4 + \ldots) = \frac{x^k}{(1-x^2)^k}$. $\square$

6. Let h_n denote the number of nonnegative integral solutions of the equation

$$3x_1 + 4x_2 + 2x_3 + 5x_4 = n.$$

Find the generating function for $\{h_n\}$.

Proof. $g(x) = (1 + x^3 + x^6 + \ldots)(1 + x^4 + x^8 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^5 + x^{10} + \ldots) = \frac{1}{1-x^3} \frac{1}{1-x^4} \frac{1}{1-x^2} \frac{1}{1-x^5}$. $\square$