1.25

Chapter 2.

§2.1 4 Counting Principle

① Addition
② Multiplication
③ Subtraction
④ Division

§2.2 Permutation of Sets 排列

$P(n,r) = \frac{n!}{(n-r)!}$

Ex1.  rotation (same) diff 1-9 → 6 faces

$\frac{P(9,6)}{6\times 4}$
(底面) (方向)

9个数字选6个.

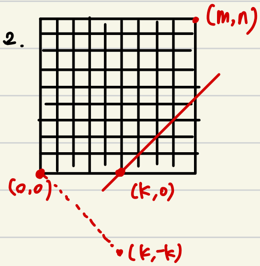
§2.3 Combinations (Subsets) of Sets. 组合

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Ex1. $x_1 + x_2 + \cdots + x_r = n$,

1 1 1 1
n个

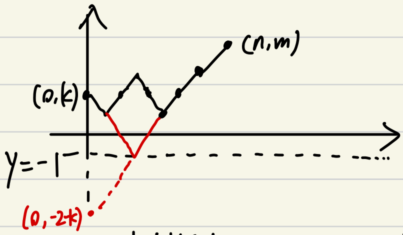
$+: \binom{n-1}{r-1}$ 非负: $\binom{n+r-1}{r-1}$

Ex2. 

$\binom{m+n}{m} - \binom{m+n}{m-k}$

1.30

Ex3. Count # of sequences s.t. $X_0 = k$, $X_n = m$ $X_{i+1} = X_i \pm 1$, $X_i \geq 0$



Draw (i, x_i) $0 \leq i \leq n$

piecewise linear

X intersect $y=-1$

$$\begin{cases} u+v=n \\ u-v=m-k \end{cases} \Rightarrow \begin{cases} u=\frac{n+m-k}{2} \\ v=\frac{n+k-m}{2} \end{cases} \qquad \binom{n}{\frac{m+n-k}{2}} \quad ①$$

$$\begin{cases} u+v=n \\ u-v=m+k+2 \end{cases} \Rightarrow \begin{cases} u=\frac{m+n+k+2}{2} \\ v=\frac{n-m-k-2}{2} \end{cases} \qquad \binom{n}{\frac{m+n+k+2}{2}} \quad ②$$

① − ② ✓

注意是 set！无顺序

§ 2.4 Permutations of Multisets

$S = \{a,a,a,b,c,c\}$ Denote it by $\{3\cdot a, b, 2\cdot c\}$. (repetition number)

r-permutation

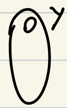
Ex 1. $S = \{\infty\cdot a_1, \infty\cdot a_2, \cdots, \infty\cdot a_k\}$ $a_1,\cdots,a_k$ 可用无限次

$\#\{r\text{-permutation of } S\} = k^r$

Thm 2 $S = \{n_1\cdot a_1, n_2\cdot a_2, \cdots, n_k\cdot a_k\}$, $n = n_1+n_2+\cdots+n_k$

$$\#\{n\text{-permutation of } S\} = \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

$\phi: X \to Y$

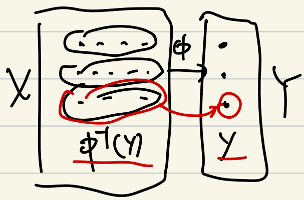


"mapping trick" $\phi: X \to Y$

For $y \in Y$, define preimage of y: $\phi^{-1}(y) = \{x \in X : \phi(x) = y\}$

$y_1 \neq y_2$, $\phi^{-1}(y_1) \cap \phi^{-1}(y_2) = \phi$

Suppose we know $\#X$, $\#\phi^{-1}(y) = m$ for all $y \in Y$, then $\#Y = \frac{\#X}{m}$, $\left(X = \bigsqcup_{y\in Y} \phi^{-1}(y)\right)$



Pf of Thm 2. Define S^*: label the same element in S

$X = \{n\text{-permutation of } S^*\}$ $\#X = n!$

Define ϕ: $a_2, b_1, a_1 \mapsto a, b, a$ $\#\phi^{-1}(y) = n_1!\cdots n_k!$ $\forall y$.

$Y = \{n\text{-permutation of } S\}$, then $\#Y = \frac{\#X}{\#\phi^{-1}(y)}$. //

Ex 3. Suppose we have k Boxes, 共放 $n = n_1+\cdots+n_k$ 个球, Box i : n_i 个

第 m 个球放入 Box l $\Leftrightarrow$ Thm 2 中 排列第 m 个位置为 a_l

§ 2.5 Combinations of multisets.

Alice wants to buy 10 fruits from 5 kinds of fruits, How many choices?

$S = \{\infty \cdot a_1, \infty \cdot a_2, \cdots, \infty \cdot a_5\}$ # 10-Combinations of S = ? $\binom{14}{4}$

Thm 1. $S = \{\infty \cdot a_1, \infty \cdot a_2, \cdots, \infty \cdot a_k\}$, then

of n-Combinations of S is $\binom{n+k-1}{k-1}$

Pf. Suppose a_i choose x_i times, $x_i \geq 0$, $\sum_{i=1}^{k} x_i = n$,

Ex 2. What is number of non-decreasing sequences of length r, whose terms are from $\{1, 2, \cdots, k\}$?

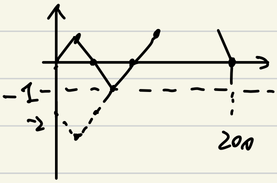
$\binom{r+k-1}{r-1}$

§ 2.6 Finite Probability

Ex1. $\boxed{\begin{matrix} 100\ W \\ 100\ B \end{matrix}}$ Bob will put balls out of the box one by one.
What's the probability that in this process, # B balls outside ≥ # W balls outside?

(i) $S = \{200\text{-arrangements of } \{100 \cdot W, 100 \cdot B\}\}$ $\#S = \frac{200!}{100!\,100!} = \binom{200}{100}$

(ii) Count the arrangements s.t. $\forall i$, in first i balls, $\#B \geq \#W$.



$X_i = \begin{cases} 1 & B \\ -1 & W \end{cases}$ $\quad \sum X_i \geq 0$

$\begin{cases} a+b = 200 \\ a-b = 2 \end{cases} \Rightarrow \begin{cases} a = 101 \\ b = 99 \end{cases}$ $\qquad \binom{200}{100} - \binom{200}{101}$

The answer is $1 - \frac{\binom{200}{99}}{\binom{200}{100}} = 1 - \frac{100}{101} = \frac{1}{101}$.

Ex2. A Poker hand is a set of 5 cards.

(a) Full house (b) Flush (c) Straight

2.6

Pigeonhole principle

1. Ex $a_1, \cdots, a_m$, $\exists\, 1 \le i \le j \le m$, s.t. $m \mid a_i + a_{i+1} + \cdots + a_j$

Pf. Let $S_k = a_1 + a_2 + \cdots + a_k$, $k = 1, \cdots, m$ $\quad S_0 \triangleq 0$.

$\exists\, 0 \le i \le j \le m$, $S_i \equiv S_j \pmod m$ $\Rightarrow$ $m \mid S_j - S_i$.

2. Ex . (Chinese remainder theorem)

Suppose n, m are coprime ($(n,m)=1$) . For any a, b , $\exists\, x$ s.t. $\begin{cases} x \equiv a \pmod m \\ x \equiv b \pmod n \end{cases}$.

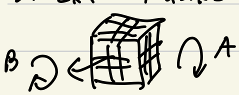
Pf. $kn + b \equiv ln + b \pmod m$ $\quad k, l \in \{0, 1, \cdots, m-1\}$

$(k-l)n \equiv 0 \pmod m$ $\Rightarrow$ $m \mid k-l$ $\quad 1 \le |k-l| < m$ $\Rightarrow$ 矛盾

故 $a, n+b, 2n+b, \cdots, (m-1)n+b$ 模m余数不同 .

故一定有 $kn + b \equiv a \pmod m$.

3. Ex. Rubik's cube



ABAB$\cdots$ $\Rightarrow$ origin state .

4. Thm. (Recurrence Theorem) $S = \{P_1, \cdots, P_n\}$ set of states

the trancition function in S : $f: S \to S$ 双射 bijection

then, $\forall\, a \in S$, $f^{(k)}(a) = f \circ \cdots \circ f(a) = a$. $\exists k$

Pf. 由抽屉原理, $\exists\, i, j$,

$f^{(i)}(a) = f^{(j)}(a) \Leftrightarrow a = f^{(j-i)}(a)$ $\quad 1 \le i < j \le n+1$.

5. Dirichlet's approximation

① Given $\alpha \in (0,1)$, $N \ge 1$, $\exists\, j \in [N]$, s.t. $|\alpha - \frac{j}{N}| \le \frac{1}{N}$.

② Given $\alpha \in (0,1)$, $N \geq 1$, $\exists\, q \in [N], p$, s.t. $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qN}$. $\Leftrightarrow |q\alpha - p| \leq \frac{1}{N}$.

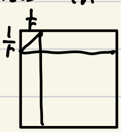
Pf. For $q\alpha = m_q + r_q$, $r_q \in [0,1)$, $q = 1, 2, \cdots, N$

$[0,1) = [0,\frac{1}{N}) \cup [\frac{1}{N}, \frac{2}{N}) \cup \cdots \cup [\frac{N-1}{N}, 1)$, $\exists\, l, k \in [N]$. s.t.

$\frac{1}{N} \geq |r_l - r_k| = |(l\alpha - m_l) - (k\alpha - m_k)| = |(l-k)\alpha - (m_l - m_k)|$

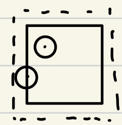
let $q = l-k$, $p = m_l - m_k$ ✓

6. Ex. We have N points in $[0,1]^2$. Show 2 of them have distance $\leq \frac{1}{4}$.

Idea 1. $\underbrace{r^2+1}_{N}$ points, $\frac{1}{r}$  r^2 squares dist $\leq \frac{\sqrt{2}}{r} \leq \frac{1}{4}$ $\Big(\underbrace{r^2+1}_{N} \geq 33\Big)$

Idea 2. draw a disc of radius $\frac{1}{8}$, centred at each point,

$N \cdot \pi \cdot \frac{1}{64} > (1+\frac{1}{4})^2$ $\quad N \geq 32$.



2.8

Permutation group 置换群

$S=[n]$, $G_n=\{(i_1,i_2,\cdots,i_n)\mid\{i_1,i_2,\cdots,i_n\}=[n]\}$

Elements in G_n are maps $g: S\to S$ (bijection)

Composition of functions : Multiplication in G_n

Ex. $(1\ 4\ 2\ 3)*(2\ 4\ 1\ 3)=(4\ 3\ 1\ 2)$

(Identity) $e=(1,2,3,\cdots)$

(Inverse) $h*g=g*h=e$ $\quad g^{-1}:=h$

Ex. $(2\ 4\ 1\ 3)^{-1}=(3\ 1\ 4\ 2)$

Q. Is g^{-1} unique ?

$$\left.\begin{array}{l} e=h_1*g=(h_1\circ g(1),h_1\circ g(2),\cdots) \\ e=h_2*g=(h_2\circ g(1),h_2\circ g(2),\cdots)\end{array}\right\}\Rightarrow h_1=h_2$$

1. Def. 群 G : ⓪二元运算封闭，①结合律，②单位元，③逆元.
   Associatity Identity Inverse

2. Ex. Group :
   (i) $(G_n,*)$
   (ii) The set of invertible $n\times n$ matrices with the matrix multiplication $(GL_n,\times)$
   (iii) $(\mathbb{R}\setminus\{0\},\times)$

3. Galois Theory. Cannot "solve" the roots for polynomial with degree ≥ 5.

Sign of a permutation

1. Def. Simple element in G_n. If $g \in G_n$ that only switches two numbers
We call g simple : $g(i)=j,\ g(j)=i,\ g(k)=k,\ k \neq i,j$ Denote $g=(i,j)$

2. Thm. $\forall g \in G_n$ can be written as multiplications of simple elements. 例：(冒泡排序)

3. Ex. $(2\,4\,1\,3) \xrightarrow{*(1,3)} (1\,4\,2\,3) \xrightarrow{*(2,3)} (1\,2\,4\,3) \xrightarrow{*(3,4)} (1\,2\,3\,4) = e$

$$(2\,4\,1\,3)*(1,3)*(2,3)*(3,4) = e$$

$$(2\,4\,1\,3) = [(1,3)*(2,3)*(3,4)]^{-1} = (3,4)*(2,3)*(1,3)$$

$$((hg)^{-1} = g^{-1}h^{-1},\ (i,j)^{-1} = (i,j),\ (i,j) \in G_n)$$

逆序数 $\quad g_1 \underset{m\text{ pairs}}{\overset{k\text{ pairs}}{\rightrightarrows}} g_2 \quad 2 \mid k-m$, 行列式：$(-1)^{2n} = 1$.

4. Def. $g=(i_1, i_2, \cdots, i_n) = (j_1,k_1)*\cdots*(j_m,k_m) \in G_n$

$$\varepsilon(g) := \begin{cases} 1, & \#\text{ inversion pairs of } g \text{ is even} \\ -1, & \text{— — — odd} \end{cases} = \begin{cases} 1, & 2 \mid m \\ -1, & 2 \nmid m \end{cases}$$

5. Lemma. #IP changed by an odd number

6. $A = [a_{ij}]_{1 \le i,j \le n}$, $\det A = \sum_{g \in G_n} \varepsilon(g)\, a_{1g(1)} \cdots a_{ng(n)}$ $A = [v_1, \cdots, v_n]$

$$= \varepsilon(g) \det [\vec{v}_{g(1)} \cdots \vec{v}_{g(n)}]$$

2.13

1. $\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$ 0th, 1th, 2th

$\cdots$

ith row jth position (i,j)

$a(i,j) = a(i-1,j-1) + a(i-1,j)$

① $a(i,j) = \#$ paths from $(0,0)$ to (i,j) (length i, turn right j times)

$= \binom{i}{j}$

② (induction) Verifying (NP) is much easer than proving (P).

If a problem can be verified in polynomial time, the proof can be …?

2. $\sum\limits_{\substack{k+l\le n \\ k,l\ge 1}} \binom{n}{k}\binom{n-k}{l} kl = n(n-1)3^{n-2}$

n people, committee $\underset{k}{A}$, $\underset{l}{B}$, 选 chair (主席)

2.15

1. Thm. If n is odd, then $\binom{n}{0}<\binom{n}{1}<\cdots<\binom{n}{\frac{n-1}{2}}=\binom{n}{\frac{n+1}{2}}>\binom{n}{\frac{n+3}{2}}>\cdots>\binom{n}{n}$

If n is even, then $\binom{n}{0}<\binom{n}{1}<\cdots<\binom{n}{\frac{n}{2}}>\binom{n}{\frac{n+2}{2}}>\cdots>\binom{n}{n}$

2. Def. S ① $\mathcal{A}$: antichain of S if $\mathcal{A}$ is a collection of subsets of S

s.t. no subset in $\mathcal{A}$ is contained in another subset in $\mathcal{A}$ （不互相包含）

② $\mathcal{A}$: chain of S if $\mathcal{A}=\{X_1,\cdots,X_k\}$ $X_i\subset S$

$X_1\subsetneqq X_2\subsetneqq\cdots\subsetneqq X_k$. （真包含链）

3. Thm. (Sperner) $\#S=n$, $\mathcal{A}$ is antichain of S, then $\#\mathcal{A}\le\binom{n}{\lfloor\frac{n}{2}\rfloor}$.

$\sup\#\mathcal{A}$

Remark. The upper bound is sharp (tight, optional)

i.e. $\exists\mathcal{A}$, s.t. $\#\mathcal{A}=\binom{n}{\lfloor\frac{n}{2}\rfloor}$. $\mathcal{A}_k=\{T\subset S\mid |T|=k\}$ ✓

Trick. "Double counting"

Pf. Consider all the pairs $(A,\mathcal{C})$, $A\in\mathcal{A}$, $\mathcal{C}$ is a maximal chain of S.

Also, $A\in\mathcal{C}$.

1° Fix $A\in\mathcal{A}$, $\#A=k$. $\phi\subsetneqq A_1\subsetneqq\cdots\subsetneqq A\subsetneqq\cdots\subsetneqq A_k\subsetneqq S$

then # of $\mathcal{C}$ $=k!(n-k)!\ge\lfloor\frac{n}{2}\rfloor!(n-\lfloor\frac{n}{2}\rfloor)!$ $\left(\binom{n}{k}\le\binom{n}{\lfloor\frac{n}{2}\rfloor}\right)$

then $\#(A,\mathcal{C})\ge\#\mathcal{A}\cdot\lfloor\frac{n}{2}\rfloor!(n-\lfloor\frac{n}{2}\rfloor)!$

2° Fix $\mathcal{C}$, $\#(A,\mathcal{C})\le\#$ of $\mathcal{C}=n!$

Lemma. Intersection of a maximal chain and an antichain contains at most 1 element. i.e. $\#(\mathcal{A}\cap\mathcal{C})\le 1$

Q : Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^2$.

For integer $r \geq 2$, we define the set of r-rich points

$$P_r = \{x \in \mathbb{R}^2, \geq r \text{ lines from } \mathcal{L} \text{ pass through } x\}$$

then $\# P_r \leq \frac{(\#\mathcal{L})^2}{r^2}$.

Pf. 1° Fix $x \in P_r$. $\#$ pass $(l_1, l_2) \geq r^2$

2° Fix (l_1, l_2). $\#(l_1 \cap l_2) \leq 1$

$$\{(x, l_1, l_2) : x \in P_r, l_1, l_2 \in \mathcal{L}, x \in l_1 \cap l_2\} \begin{cases} \leq \#\mathcal{L} \cdot \#\mathcal{L} \\ \geq P_r \cdot r^2 \end{cases}$$

2.20

多项式

1. Multinomial Theorem

$$(x_1+x_2+\cdots+x_t)^n = \sum_{n_1+\cdots+n_t=n} \frac{n!}{n_1!\cdots n_t!} x_1^{n_1}\cdots x_t^{n_t}$$

2. Def. $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ $\quad \alpha\in\mathbb{R},\ k\in\mathbb{N}$

3. Thm. $\alpha\in\mathbb{R}$, $0<|x|<|y|$, then

$$(x+y)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k y^{\alpha-k}.$$

$$\Updownarrow$$

$$(1+z)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}z^k \quad |z|<1 \quad (\text{Taylor})$$

$k\geq 1$:
$$\left|\binom{\alpha}{k}\right| = \left|\frac{\alpha+1-k}{k}\cdot\frac{\alpha+1-(k-1)}{k-1}\cdot\cdots\cdot\frac{\alpha+1-1}{1}\right|$$
$$= \left|\left(\frac{\alpha+1}{k}-1\right)\left(\frac{\alpha+1}{k-1}-1\right)\cdots\left(\frac{\alpha+1}{1}-1\right)\right|$$
$$\leq \prod_{i=1}^{\lfloor|\alpha|\rfloor}\left|\frac{\alpha+1}{i}-1\right| \leq |\alpha|^{|\alpha|} \quad \left(i>|\alpha|,\ \left|\frac{\alpha+1}{i}-1\right|\leq 1\right)$$

so $\left|\binom{\alpha}{k}\right| \leq C_\alpha$

$$\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$$

$$(1+z)^{-n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}z^k$$
$$(1-z)^{-n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}z^k.$$

Tricks ① Calculus ② induction ③ recursion
④ generating function

① $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}$

Let $x=1$, $\sum_{k=1}^{n} \binom{n}{k} k = n 2^{n-1}$

$x=-1$, $\sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} = \begin{cases} 1 & n=1 \\ 0 & n \geq 2 \end{cases}$

$\int_0^t (1+x)^n dx = \int_0^t \sum_{k=0}^{n} \binom{n}{k} x^k dx$

$\Rightarrow \frac{(1+t)^{n+1}-1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^{k+1}}{k+1}$ Let $t=1$, $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = \frac{2^{n+1}-1}{n+1}$.

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} \frac{1}{k} &= \int_0^1 \frac{(1+x)^n - 1}{(1+x)-1} dx \\
&\overset{y=1+x}{=} \int_1^2 \frac{y^n - 1}{y-1} dy \\
&= \int_1^2 y^{n-1} + y^{n-2} + \cdots + 1 \, dy \\
&= \frac{y^n}{n} + \frac{y^{n-1}}{n-1} + \cdots + y \Big|_1^2 \\
&= \sum_{k=1}^{n} \frac{2^k - 1}{k}
\end{aligned}$$

证一. $\sum_{k=1}^{n} \frac{k}{k+1} \binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k} - \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k} = 2^n - \frac{2^{n+1}-1}{n+1} = \frac{(n-1)2^n + 1}{n+1}$

证二. $\int_0^t (1+x)^n dx = \int_0^t \sum_{k=0}^{n} \binom{n}{k} x^k dx$

$\Rightarrow \frac{(1+t)^{n+1}-1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^{k+1}}{k+1} \Rightarrow \frac{(1+t)^{n+1}-1}{(n+1)t} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{k+1}$ 再求导 令 $t=1$ ✓

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{m}{m+k} = \frac{1}{\binom{m+n}{n}}$$

② Pf.
$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\frac{m}{m+k} = \sum_{k=1}^{n}(-1)^k\binom{n+1}{k}\frac{m}{m+k} + 1 + (-1)^{n+1}\frac{m}{m+n+1}$$
$$= \left[\sum_{k=1}^n (-1)^k\binom{n}{k}\frac{m}{m+k}+1\right] + \sum_{k=1}^n(-1)^k\binom{n}{k-1}\frac{m}{m+k} + (-1)^{n+1}\frac{m}{m+n+1}$$
$$= \sum_{k=0}^n (-1)^k\binom{n}{k}\frac{m}{m+k} + \sum_{k=0}^n (-1)^{k+1}\binom{n}{k}\frac{m+1}{m+k+1}\cdot\frac{m}{m+1}$$
$$= \frac{1}{\binom{m+n}{n}} - \frac{m}{m+1}\sum_{k=0}^n(-1)^k\binom{n}{k}\frac{m+1}{m+1+k}$$
$$= \frac{1}{\binom{m+n}{n}} - \frac{m}{m+1}\cdot\frac{1}{\binom{m+n+1}{n}}$$
$$= \frac{1}{\binom{m+n+1}{n+1}\frac{n+1}{m+n+1}} - \frac{\frac{m}{m+1}}{\binom{m+n+1}{n+1}\frac{n+1}{m+1}}$$
$$= \frac{1}{\binom{m+n+1}{n+1}}$$

③ Pf.
$$b_n = \text{原式} = 1 + \sum_{k=1}^{n-1}(-1)^k\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]\frac{m}{m+k} + (-1)^n\frac{m}{m+k}$$
$$= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{m}{m+k} + \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{m}{m+k}$$
$$= b_{n-1} + \sum_{k=1}^n (-1)^k\binom{n}{k}\frac{k}{n}\frac{m}{m+k}$$
$$= b_{n-1} + \frac{m}{n}\sum_{k=0}^n(-1)^k\binom{n}{k}\left(1-\frac{m}{m+k}\right)$$
$$= b_{n-1} - \frac{m}{n}b_n$$
$$\Rightarrow b_n = \frac{n}{m+n}b_{n-1} = \frac{n!\,m!}{(m+n)!}b_0 = \frac{1}{\binom{m+n}{n}}$$

④ 2 polynomials $p(x)=q(x)$, then their corresponding coefficients are equal.

Q. $\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n} = n+1$

Pf. Consider $(1+x)^{n+1}$ the coefficient of x^n is $n+1$

$2k \quad n-2k \quad k=0,1,\cdots,\lfloor \frac{n}{2}\rfloor$.

$$(1+x)^{n+1} = \frac{(1-x^2)^{n+1}}{(1-x)^{n+1}} = \left(\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k x^{2k}\right)\left(\sum_{k=0}^{\infty}\binom{n+k}{k}x^k\right)$$
$$x^n: \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{k}(-1)^k\binom{2n-2k}{n-2k}$$

$$(1+x+x^2)^n = \frac{(1-x^3)^n}{(1-x)^n}$$

compare coef of x^N $\qquad N \in \mathbb{N}$

coef on LHS = # solutions of $\begin{cases} i_1+i_2+\cdots+i_n = N \\ \text{each } i_j \in \{0,1,2\},\ (1\le j\le n) \end{cases}$

$$= \sum_{\substack{i_1+\cdots+i_n=N \\ 1\le i_1,\cdots,i_n\le N}} 1$$

$$(1-x^3)^n(1-x)^{-n} = \Big(\sum_{k=0}^{n}\binom{n}{k}(-1)^k \underbrace{x^{3k}}_{x^{3k}}\Big)\Big(\sum_{k=0}^{\infty}\binom{n+k-1}{k}\underbrace{x^k}_{x^{N-3k}}\Big) \qquad k=0,1,\cdots\lfloor\tfrac{N}{3}\rfloor$$

coef on RHS = $\displaystyle\sum_{k=0}^{\lfloor\frac{N}{3}\rfloor}\binom{n}{k}(-1)^k\binom{n+N-3k-1}{N-3k}$

$$(x^4+x^2+1)^n = (x^2+x+1)^n(x^2-x+1)^n = \frac{(1-x^3)^n}{(1-x)^n}\,\frac{(1+x^3)^n}{(1+x)^n}$$

coef of x^{4n}

coef of LHS = 1

$$\text{RHS} = \Big(\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{3k}\Big)\Big(\sum_{k=0}^{n}\binom{n}{k}x^{3k}\Big)\Big(\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k\Big)\Big(\sum_{k=0}^{\infty}\binom{n+k-1}{k}(-1)^k x^k\Big)$$

$$x^{3k} \qquad x^{3l} \qquad x^{j} \qquad x^{4n-3k-3l-j}$$

coef of RHS = $\displaystyle\sum_{\substack{k,l,j \\ 3k+3l+j\le 4n}}\binom{n}{k}\binom{n}{l}\binom{n+j-1}{j}\binom{n+4n-3k-3l-j-1}{4n-3k-3l-j}(-1)^{-3l-j}$

2.29

Inclusion-Exclusion Principle

1. Thm. $A_1, \cdots, A_n \subset S$

(1) $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$

(2) $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| + (-1)^n |A_1 \cap \cdots \cap A_n|$

Pf of (2). "Characteristic function method"

$$1_X(x) := \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases}$$

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= \sum_{x \in S} 1_{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}}(x) \\ &= \sum_{x \in S} 1_{\overline{A_1}}(x) 1_{\overline{A_2}}(x) \cdots 1_{\overline{A_n}}(x) \\ &= \sum_{x \in S} (1 - 1_{A_1}(x))(1 - 1_{A_2}(x)) \cdots (1 - 1_{A_n}(x)) \\ &= \sum_{x \in S} \left[1 - \sum_i 1_{A_i}(x) + \cdots \right] \end{aligned}$$

2. Ex. ① $\#\{1 \le n \le 1000,\ 2|n \text{ or } 3|n \text{ or } 5|n\}$

② # of 8-combinations of $\{2 \cdot a, 3 \cdot b, 4 \cdot c\}$

③ $x_1 + x_2 + x_3 = 10 \quad 0 \le x_1 \le 8,\ -2 \le x_2 \le 5,\ 1 \le x_3 \le 3$

④ Arrangement of $[n]$, s.t. number i is not in the position i.

① Let $A_k = \{1 \le n \le 1000 : k|n\}$

$$\begin{aligned} |A_2 \cup A_3 \cup A_5| &= |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_3 \cap A_5| - |A_2 \cap A_5| + |A_2 \cap A_3 \cap A_5| \\ &= 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734. \end{aligned}$$

② Let $A_a = \{\text{8-combinations of } \{\infty \cdot a, \infty \cdot b, \infty \cdot c\},\ a \text{ appears} \ge 3\}$

$A_b = \{\text{8-combinations of } \{\infty \cdot a, \infty \cdot b, \infty \cdot c\},\ b \text{ appears} \ge 4\}$

$A_c = \{\text{8-combinations of } \{\infty \cdot a, \infty \cdot b, \infty \cdot c\},\ c \text{ appears} \ge 5\}$

$$\begin{aligned} |\overline{A_a} \cap \overline{A_b} \cap \overline{A_c}| &= |S| - |A_a| - |A_b| - |A_c| + |A_a \cap A_b| + |A_b \cap A_c| + |A_c \cap A_a| - |A_a \cap A_b \cap A_c| \\ &= \binom{8+2}{2} - \binom{5+2}{2} - \binom{4+2}{2} - \binom{3+2}{2} + \binom{3}{2} + 0 + 1 - 0 \end{aligned}$$

④ A_i = {arrangements where i is in the i position}

$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$

$= |S| - \sum |A_i| + \sum |A_i A_j| - \cdots$

$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n$

$= \left(\sum_{k=2}^{n} (-1)^k \frac{1}{k!}\right) n!$

Möbius inversion

1. Def. (Convolution "$*$") For $F, G : \mathbb{N} \to \mathbb{C}$,

$$F * G(n) = \sum_{k \mid n} F(k) G\left(\frac{n}{k}\right)$$

Properties. 1) $F * G = G * F$

2) $(F * G) * H(n) = F * (G * H)(n) = \sum_{m_1 \cdot m_2 \cdot m_3 = n} F(m_1) G(m_2) H(m_3)$

2. Def $1(n) = 1 \quad \forall n \in \mathbb{N}$.

Prop. $\sum_{k \mid n} F(k) = 1 * F(n)$

3. Def $\delta(n) = \begin{cases} 1, & n = 1 \\ 0, & \text{else} \end{cases}$

Prop. $\delta * F(n) = \sum_{k \mid n} \delta(k) F\left(\frac{n}{k}\right) = F(n)$

4. How to find G s.t. $F * G = \delta$? $\quad (F^{-1} := G)$

Ex. If $F \equiv 0 \quad \nexists G$

If $F(1) \neq 0 \quad \exists G$

$n = 1, \; F(1) G(1) = 1 \Leftrightarrow G(1) = F(1)^{-1}$

$n = 2, \; F(1) G(2) + F(2) G(1) = 0 \Leftrightarrow G(2) = \cdots$

$\cdots$

5. Remark. $(\mathbb{N} \to \mathbb{R}$ where at 1 is non-zero, $*)$ is a group.

6. Find $\mu = 1^{-1} \; (1 * \mu = \delta)$:

$\begin{cases} \mu(1) = 1 \\ \sum_{k \mid n} \mu(k) = 0, & n \geq 2 \end{cases}$.

$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n \text{ is the product of } k \text{ different primes} \\ 0 & \text{else} \end{cases}$

(i) $n = p$ is a prime,

$\mu(1) + \mu(p) = 0 \Rightarrow \mu(p) = -1$

(ii) $n = p^2 \quad \mu(1) + \mu(p) + \mu(p^2) = 0 \Rightarrow \mu(p^2) = 0 \quad \Rightarrow \mu(p^k) = 0$

7. Def. (Euler ϕ function)

$\phi(n) := \#\{1 \le k \le n,\ (k,n)=1\}$

$$\underline{G(n)} \triangleq \sum_{k|n} \phi(k) = \sum_{k|n} \phi\left(\frac{n}{k}\right) = \sum_{k|n} \#\left\{1 \le d \le \frac{n}{k} : \left(d, \frac{n}{k}\right)=1\right\}$$

$$= \sum_{k|n} \#\{k \le kd \le n,\ (kd, n) = k\}$$

$$= \sum_{k|n} \#\{1 \le r \le n : (r,n)=k\}$$

$$= \underline{n}.$$

(Ex)

$n = 12 = 2^2 \times 3$

$$k = \overset{2^0\times3^0}{1},\ \overset{2^1\times3^0}{2},\ \overset{2^0\times3^1}{3},\ \overset{2^2\times3^0}{4},\ \overset{2^1\times3^1}{6},\ \overset{2^2\times3^1}{12}$$

$1 \le r \le 12$

按照含2和3的因数情况分类

8. By Möbius inversion, $1 * \phi = \underline{G} \Rightarrow \phi = \mu * G \Rightarrow \phi(n) = \sum_{k|n} \mu(k) \frac{n}{k}$

$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, $\phi(n) = \sum_{i_1,\cdots,i_m \in \{0,1\}} \mu(p_1^{i_1} \cdots p_m^{i_m}) \frac{n}{p_1^{i_1} \cdots p_m^{i_m}}$

$$= \sum_{i_1,\cdots,i_m \in \{0,1\}} (-1)^{i_1 + \cdots + i_m} \frac{n}{p_1^{i_1} \cdots p_m^{i_m}}$$

$$= \sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \left(\frac{-1}{p_1}\right)^{i_1} \cdots \left(\frac{-1}{p_m}\right)^{i_m} n$$

$$= \left(1-\frac{1}{p_1}\right) \cdots \left(1-\frac{1}{p_m}\right) n.$$

9. Def. $X_n = [n]$, $P(X_n)$ is the collection of all subsets of X_n.

$|P(X_n)| = 2^n$.

10. Def. $1(L) = 1 \quad \forall L \subset X_n$

11. Def. $F, G : P(X_n) \to \mathbb{R}$

$$F * G(L) = \sum_{K \subset L} F(K) G(L \setminus K) = \sum_{L = K_1 \sqcup K_2} F(K_1) G(K_2)$$

$$F_1 * F_2 * F_3(L) = \sum_{\substack{L = K_1 \sqcup K_2 \sqcup K_3 \\ (\text{有序})}} F_1(K_1) F_2(K_2) F_3(K_3)$$

12. Def. $\delta(L) = \begin{cases} 1, & L = \emptyset \\ 0, & L \neq \emptyset \end{cases}$

$$\delta * F(L) = \sum_{K \subset L} \delta(K) F(L \setminus K) = F(L) \qquad \delta * F = F.$$

13. Find $\mu = 1^{-1}$ $\quad \mu * 1 = \delta$

$$\begin{cases}\mu(\emptyset)=1\\ \sum_{K\subset L}\mu(K)=0, \quad L\neq\emptyset\end{cases}$$

$|L|=1$, $\mu(\emptyset)+\mu(L)=0 \Rightarrow \mu(L)=-\mu(\emptyset) \Rightarrow \mu(L)=(-1)^{|L|}$.

14. Def. $A_1,\cdots,A_n$ CS $L\subset[n]$.

$F(L):=\#[\bigcap_{i\in L}A_i\setminus\bigcap_{i\notin L}A_i]$

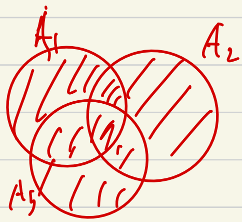
$G(L):=F*1(L)=\sum_{K\subset L}F(K)\underset{\Delta}{=}\#(\bigcup_{i\in L}A_i)$

Note: For different $K\subset L$, $\bigcap_{i\in K}A_i\setminus\bigcap_{i\notin K}A_i$ are disjoint.

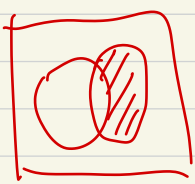
$\bigcap_{i\in L}A_i=\bigsqcup_{K\subset L}[\bigcap_{i\in K}A_i\setminus\bigcap_{i\notin K}A_i]$

$\Rightarrow F(L)=\sum_{K\subset L}\mu(L\setminus K)G(K)=\sum_{K\subset L}(-1)^{|L|-|K|}|\bigcap_{i\in K}A_i|$

$[3]$



$=\bigcup_{i\in L}A_1$



Recurrence relation

1. Def. (Linear recurrence relation)

$m \in \mathbb{N}$. $a_m, a_{m-1}, \cdots, a_0, b \in \mathbb{R}$, $a_m \neq 0$, $\underline{a_m f_n + a_{m-1} f_{n-1} + \cdots + a_0 f_{n-m} + b = 0}$, $(*)$

① $b = 0$, homogeneous.

② $b \neq 0$, homogeneous.

2. Def. The characteristic polynomial of $(*)$: $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$.

Ob 1. $P(w) = 0$, $f_n = w^n$ satisfies $(*)$

Ob 2. $w_1, \cdots, w_m$ are solutions of $P(x) = 0$, then $\forall\, d_1, \cdots, d_m \in \mathbb{R}$, $f_n = d_1 w_1^n + \cdots + d_m w_m^n$ satisfies $(*)$

① $w_1, \cdots, w_m$ are different, we can get $d_1, \cdots, d_m$

② otherwise, we can't get $d_1, \cdots, d_m$, so we need other approaches

3. When $p(x)$ has multiple roots: $m_1 \cdot w_1, m_2 \cdot w_2, \cdots, m_j \cdot w_j$

$$f_n = (d_{1,0} + d_{1,1} n + \cdots + d_{1,m_1-1} n^{m_1-1}) w_1^n + \cdots + (d_{j,0} + d_{j,1} n + \cdots + d_{j,m_j-1} n^{m_j-1}) w_j^n$$

4. Thm. (Generating function)

$\{h_n\}_{n=0}^{\infty}$, $g(x) = h_0 + h_1 x + h_2 x^2 + \cdots$

5. Ex. ① Let h_n be the # of nonnegative integral solution for $e_1 + e_2 + \cdots + e_r = n$.

Then $g(x) = \underbrace{(1 + x + x^2 + \cdots)}_{e_1} \cdots \underbrace{(1 + x + x^2 + \cdots)}_{e_r} = \frac{1}{(1-x)^r}$.

② $e_1 + e_2 + e_3 + e_4 = n$ $\quad$ e_1 is even, e_2 is odd, $0 \le e_3 \le 4$, $e_4 \ge 1$

$$(1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots)$$

$$= \frac{1}{1-x^2} \cdot \frac{x}{1-x^2} \cdot \frac{1-x^5}{1-x} \cdot \frac{x}{1-x}$$

③ $e_1 + 2e_2 + 3e_3 + 4e_4 = n$

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)$$
$$=\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}$$

6. $h_n = \#$ of ways to write $n = x_1+\cdots+x_m$ where $1\le x_1\le x_2\le\cdots\le x_m$,

3.12

目的是算 x^n 的系数.

Exponential generating function (EGF)

1. Def. EGF of $\{h_n\}$: $g^{(e)}(x)=\sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$

2. Thm. $S=\{n_1\cdot a_1, n_2\cdot a_2, \cdots, n_k\cdot a_k\}$, $h_n=\#$ of n-permutations of S.

Then the EGF of $\{h_n\}$ is $g^{(e)}(x)=\left(1+\frac{x}{1}+\frac{x^2}{2!}+\cdots+\frac{x^{n_1}}{n_1!}\right)\cdots\left(1+\cdots+\frac{x^{n_k}}{n_k!}\right)$.

Pf. $h_n=\sum_{\substack{m_i\le n_i\\ \sum n_i=n}} \frac{n!}{m_1!\cdots m_k!}$, $g^{(e)}(x)=\sum_{n=0}^{\infty} h_n\frac{x^n}{n!}=\sum_{n=0}^{\infty}\sum \frac{x^{n_1}\cdots x^{n_k}}{m_1!\cdots m_k!}=\text{RHS}$.

3. Ex. $S=\{\infty\cdot a, \infty\cdot b, \infty\cdot c\}$

① $h_n=\#\{$ n-permutation of S where a appears even of number of times $\}$

$$g^{(e)}(x)=\sum_{n=0}^{\infty} h_n\frac{x^n}{n!}=\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2$$
$$=\frac{1}{2}(e^x+e^{-x})e^{2x}=\frac{1}{2}e^{3x}+\frac{1}{2}e^x=\sum_{n=0}^{\infty}\left(\frac{1}{2}\cdot\frac{3^n}{n!}+\frac{1}{2}+\frac{1}{n!}\right)x^n.\quad \frac{3^n+1}{2}$$

② $h_n=\#\{$ n-permutation of S where a appears $3k$ of number of times, $\exists k\in\mathbb{N}\}$

$$g^{(e)}(x)=\sum_{n=0}^{\infty} h_n\frac{x^n}{n!}=\left(1+\frac{x^3}{3!}+\frac{x^6}{6!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2$$
$$=\frac{1}{3}(e^x+e^{\omega x}+e^{\omega^2 x})e^{2x}=\sum_{n=0}^{\infty}\frac{1}{3}\frac{1}{n!}\left[3^n+\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)^n+\left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)^n\right]x^n$$

4. Ex. If w is a root of $P(x)$ with multiplicity k, (w是k重根)

$f(n)=f_n=w^n, nw^n, n^2w^n, \cdots, n^{k-1}w^n$

$k=2$, $p(x)=(x-w)^2q(x)$

perturb. $p^{(t)}(x)=(x-w)(x-(w+t))q(x)=a_n^{(t)}x^n+\cdots+a_0^{(t)}$

$h_n^{(t)}=w^n, (w+t)^n, \frac{(w+t)^n-w^n}{t}$

$h_n=\lim_{t\to 0}h_n^{(t)}=\lim_{t\to 0}\frac{(w+t)^n-w^n}{t}=\frac{d}{dt}t^n\Big|_{t=w}=nw^{n-1}$

5. Ex. Use GF to solve recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \ f_1 = 1.$$

$$g(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

$$x g(x) = f_0 x + f_1 x^2 + \cdots$$

$$x^2 g(x) = f_0 x^2 + \cdots$$

$$(1 - x - x^2) g(x) = f_0 + f_1 x - f_0 x = x$$

$$\Rightarrow g(x) = \frac{-x}{x^2 + x - 1} = -x \frac{1}{(x - x_1)(x - x_2)} = x \left(\frac{1}{x - x_1} - \frac{1}{x - x_2} \right) \cdot \frac{1}{x_2 - x_1}.$$

$$= \frac{-x}{\sqrt{5}} \left(\frac{1}{x - x_1} - \frac{1}{x - x_2} \right) = \frac{-x}{\sqrt{5}} \left(-\frac{1}{x_1} \cdot \frac{1}{1 - \frac{x}{x_1}} + \frac{1}{x_2} \cdot \frac{1}{1 - \frac{x}{x_2}} \right)$$

$$= \frac{-x}{\sqrt{5}} \left(-\frac{1}{x_1} \sum_{n=0}^{\infty} \left(\frac{x}{x_1} \right)^n + \frac{1}{x_2} \sum_{n=0}^{\infty} \left(\frac{x}{x_2} \right)^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\sum_{n=1}^{\infty} \left(\frac{1}{x_1} \right)^n x^n - \left(\frac{1}{x_2} \right)^n x^n \right) \Rightarrow f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1}{x_1} \right)^n - \left(\frac{1}{x_2} \right)^n \right)$$

6. Ex. $f_n + f_{n-1} - f_{n-2} - f_{n-3} = 0 \quad f_0 = f_1 = 0, \ f_2 = 1$

$$g(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

$$x g(x) = f_0 x + f_1 x^2 + \cdots$$

$$x^2 g(x) = f_0 x + \cdots$$

$$x^3 g(x) = \cdots$$

$$(1 + x - x^2 - x^3) g(x) = x^2$$

$$\Rightarrow g(x) = \frac{x^2}{1 + x - x^2 - x^3} = -\frac{x^2}{(x-1)(x+1)^2} = -x^2 \left(\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2} \right)$$

$$= \cdots$$

3.14

1. $h_n + 3h_{n-1} - 4h_{n-3} = 0$, $h_0 = 0, h_1 = 1, h_2 = 2$,

Method 1. $p(x) = x^3 + 3x^2 - 4 = (x-1)(x+2)^2$

So $h_n = a \cdot 1^n + b(-2)^n + c \cdot n(-2)^n$

Method 2. $g(x) := h_0 + h_1 x + \cdots$

$(1+3x-4x^3)g(x) = x + 5x^2$

$\Rightarrow g(x) = (x+5x^2)\left(\frac{a}{1-x} + \frac{b}{1+2x} + \frac{c}{(1+2x)^2}\right)$

$= (x+5x^2)\sum_{n=0}^{\infty} x^n \left(\frac{1}{9} + \frac{2}{9}(-2)^n + \frac{2}{3}(n+1)(-2)^n\right)$

$= 0 + x + \sum_{n=2}^{\infty} x^n \left(\frac{1}{9} + \frac{2}{9}(-2)^{n-1} + \frac{2}{3}n(-2)^{n-1} + 5\left(\frac{1}{9} + \frac{2}{9}(-2)^{n-2} + \frac{2}{3}(n-1)(-2)^{n-2}\right)\right)$

$= \cdots$

2. Non homogeneous

Steps 1. Find the solution for homogeneous part.

2. Find a particular solution.

3. plug in initial values.

Ex. $h_n = 2h_{n-1} + 3^n$. $h_0 = 2$

1. $p(x) = x - 2$, $f_n = a \cdot 2^n$

2. Guess $g_n = b \cdot 3^n = 3^{n-1}$

3. $h_n = f_n + g_n = a \cdot 2^n + 3^{n-1} = -2^n + 3^{n-1}$.

有时 b 无解 $\Rightarrow g_n = b \cdot n \cdot 3^n \Rightarrow g_n = b \cdot n^2 \cdot 3^n$

n^2 : $g_n = an^2 + bn + c$

n : $g_n = an + b$, $an^2 + bn (+c)$

某次数

如果有次项请掉

后面不需要考虑该次数

3.19

1. Suppose $f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Show that $3 \mid f_n \Leftrightarrow 4 \mid n$

2. $f_n = \frac{1}{\sqrt{3}}\left[(\underbrace{5+\sqrt{3}}_{w_1})^n - (\underbrace{5-\sqrt{3}}_{w_2})^n\right] + \underbrace{2}_{w_3}^n$

$p(x) = (x^2 - 10x + 22)(x-2)$

$= x^3 - 12x^2 + 42x - 44$

$f_n = 12f_{n-1} - 42f_{n-2} + 44f_{n-3}$

$f_0 = 1$ $\qquad$ $f_n \equiv -f_{n-3} \pmod 3$

$f_1 = 4$ $\qquad$ $\Leftrightarrow n \equiv 0, 1 \pmod 6$

$f_2 = 24$

Chapter 8

1. Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \#\{\text{paths } (0,0)\to(n,n) \text{ avoiding } y=x-1\}$$
$$= \binom{2n}{n} - \binom{2n}{n-1}$$

2. Recurren relation of C_n

(1) $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$ for $n \geq 2$, $C_0 = 1$. $C_1 = 1$

Pf. At some step, the path first interset a point (k,k) on the diagonal.

View the path as 2 parts: ① $(0,1)\to(k-1,k)$ avoids $y=x$

$\Leftrightarrow (0,0)\to(k-1,k-1)$ avoids $y=x-1$ $\quad C_{k-1}$

② $(k,k)\to(n,n)$ avoids $y=x-1$ $\quad C_{n-k}$

(2) $C_n = \sum_{n_1+\cdots+n_\ell=n} (C_{n_1-1}\times C_{n_2-1}\times\cdots\times C_{n_\ell-1})$

Difference operation

n	0	1	2
h_n	1	6	15
Δh_n	5	9	
$\Delta^2 h_n$	4		

0-th diag

1. $\Delta h_n = h_{n+1} - h_n$

   $\Delta^p h_n = \Delta^{p-1} h_{n+1} - \Delta^{p-1} h_n$

2. Lemma. $\{h_n\}$ is determined by 0-th diag

3. Lemma. h_n is a polynomial in n

   $\Leftrightarrow$ there are finitely many nonzero numbers on 0-th diag of $\{h_n\}$.

Pf. "$\Rightarrow$" $h_n = a_p n^p + \cdots + a_0$ 每做一次Δ, degree -1

$\cdots$ $\Delta^p h_n = p!\, a_p$, $\Delta^{p+1} h_n = 0$.

"$\Leftarrow$" Given $\{c_0, \cdots, c_p, 0, 0, \cdots, 0\}$

Construct the corresponding seq $\{h_n\}$

$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_p\binom{n}{p}$.

4. Remark. Under the condition in Lemma 3, $h_n = \sum_{k=0}^{\infty} \Delta^k h_0 \binom{n}{k}$.

5. Ex. $I h_n := \sum_{k=0}^{n} h_k$ $I^p h_n = \sum_{k=0}^{n} I^{(p-1)} h_k$

   If $h_n = \binom{n}{r}$, $I^p h_n = \binom{n+p}{r+p}$

6. Def. poly in n with deg$=k$: $[n]_k := P(n,k) = k!\binom{n}{k} = \#\{k\text{-permutations of } [n]\}$

7. Given n^p, we want to express it as

$$n^p = C(p,0)\binom{n}{0} + C(p,1)\binom{n}{1} + \cdots + C(p,p)\binom{n}{p}$$
$$= \sum_{k=0}^{p} S(p,k)[n]_k$$

$S(p,k)$: the Stirling numbers of second kind

$[n]_0 = 0!\binom{n}{0} = 1$, $S(p,p) = 1$, $S(p,0) = 0$.

8. Thm. $1 \le k \le p-1$, $S(p,k) = S(p-1,k-1) + kS(p-1,k)$

Pf. $\sum_{k=0}^{p} S(p,k)[n]_k = n \cdot n^{p-1} = \sum_{k=0}^{p-1} n \cdot S(p-1,k)[n]_k$

$$= \sum_{k=0}^{p-1}\left((n-k)S(p-1,k)[n]_k + kS(p-1,k)[n]_k\right)$$

$$= \sum_{k=0}^{p-1}\left(S(p-1,k)[n]_{k+1} + kS(p-1,k)[n]_k\right) \qquad (n-k)[n]_k = [n]_{k+1}$$

$$= \sum_{k=1}^{p-1} S(p-1,k-1)[n]_k + \sum_{k=0}^{p-1} kS(p-1,k)[n]_k$$

9. Def. $S^{\#}(p,k) :=$ # of partitions of $[p]$ into k nonempty subsets.

$S'(p,k) :=$ # of partitions of $[p]$ into k different nonempty subsets.

$S'(p,k) = k!\, S^{\#}(p,k)$

10. Thm. $S^{\#}(p,k) = S(p,k)$

Pf. We only need to prove $S^{\#}(p,k) = S^{\#}(p-1,k-1) + kS^{\#}(p-1,k)$

1° one subset contains only p

2° put $[p-1]$ in k. then put p in any of them.

11. Compute $S'(p,k)$. Idea: $k \cdot k \cdot \cdots \cdot k = k^p$.

$A_i = \{$ways to put $[p]$ into k different boxes, s.t. box i is empty$\}$.

Then $S'(p,k) = |\bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_k|$

$$= |S| - \sum|A_i| + \sum|A_i \cap A_j| - \sum|A_i \cap A_j \cap A_k| + \cdots + (-1)^p|A_1 \cap \cdots \cap A_k|$$

$$= k^p - \binom{k}{1}(k-1)^p + \binom{k}{2}(k-2)^p - \binom{k}{3}(k-3)^p + \cdots + (-1)^p \cdot 0$$

$$= \sum_{t=0}^{k} (-1)^t (k-t)^p$$

$[n]_0 = 1 = (-1)^0 s(0,0)$ $\qquad$ $s(p,p) = 1 \quad p \geq 0$

$s(p,0) = 0 \quad p \geq 1$

12. Def. Stirling number of first kind

$$n^p = \sum_{k=0}^{p} S(p,k)[n]_k, \quad [n]_p = \sum_{k=0}^{p} (-1)^{p-k} s(p,k)\, n^k .$$

13. Thm. $s(p,k) = s(p-1,k-1) + (p-1)s(p-1,k)$

Pf. $[n]^p = [n]^{p-1}(n-p+1) = (n-p+1) \sum_{k=0}^{p-1} (-1)^{p-1-k} s(p-1,k)\, n^k$

$$= \sum_{k=1}^{p} (-1)^{p-k} s(p-1,k-1)\, n^k + \sum_{k=0}^{p-1} (-1)^{p-k} (p-1)\, s(p-1,k)\, n^k$$

14. Def. $s^\#(p,k)$ = # of ways to partition [n] into k circular arrangement.

15. Thm. $s^\#(p,k) = s(p,k)$.

Pf. We only need to prove $s^\#(p,k) = s^\#(p-1,k-1) + (p-1)s^\#(p-1,k)$.

1° ○ 1 of p

2° put [p-1] in k. 有 p-1 种插法.

4.2

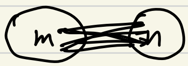
Chapter 11 Graph theory

Cycle : 只有首尾共点

1. Def. A (general) graph : $G = (V, E)$, $E \subset V \times V$
   (V: vertices, E: edges)

2. Def. Simple graph : at most one edge connecting any pair of parts, and there is no loop

3. Def. Complete graph K_n : n vertices, $\binom{n}{2}$ edges.

4. Def. Bipartite graph $K_{m,n}$: 

   mn edges connecting vertices from different group.

5. Def. Ramsey number $R(s,t)$: smallest n s.t. $\forall$ graph with n vertices has either s vertices (构成完全图), or t vertices (no edge connect)

6. Ex. $R(3,3) = 6$

7. Def. $G = (V, E)$, $x \in V$, $\deg(x) :=$ # of edges incident to x.

8. Thm. $\sum_{x \in V} \deg(x) = 2|E|$.

   Pf. $\#\{(x,e) \in V \times E : x \text{ is incident to } e\} = \begin{cases} \sum_{x \in V} \deg(x) \\ \sum_{e \in E} 2 = 2|E| \end{cases}$

9. Def. Walk. (closed walk)

10. Def. Trail (无重复 edge walk)
    Path (无重复 vertices walk except for $x_0 = x_m$)

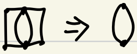
11. Def. Eulerian trail : a trail which contains all the edges.

12. Def. A graph is connected : $\forall x, y \in V$, there is a walk from x to y.

13. Thm. G is a connected general graph.

There is an closed Eulerian trail $\Leftrightarrow$ $\deg(x)$ is even for any $x \in V$.

There is an open Eulerian trail $\Leftrightarrow$ $\deg(x)$ is even for any $x \in V$ except 2.

Pf.  , $\exists$ a closed trail in G.

($G=(V,E)$ $\deg(x)$ is even $\Rightarrow$ E is a disjoint union of closed trails.)

14. Def. $x_0 - x_1 - \cdots - x_n$ $\quad x_0, \cdots, x_n$ are distinct.

is a Hamilton path : if $V = \{x_0, x_1, \cdots, x_n\}$ (不重复地走过所有点)

Hamilton cycle

15. Thm. If a graph has a bridge, it doesn't have a Hamilton cycle.

16. Ex. The complete graph K_n has Hamilton cycle.

17. Ore property. In a graph, for any 2 vertices x, y that are not adjacent, we have $\deg(x) + \deg(y) \geq n = |V|$

18. Thm. If G has order $n \geq 3$ and satisfy Ore property, then G has a Hamilton cycle.

19. Cor. In G, if $\deg(x) \geq \frac{n}{2}$, $\forall x \in V$, then G has a Hamilton cycle.

20. Def. (Bipartite graph) $G=(V,E)$, $V = V_1 \sqcup V_2$. For any edge $x_1 - x_2$, it has from $\{x_1, x_2\}$ with $x_1 \in V_1$, $x_2 \in V_2$

21. Ex. $K_{m,n}$ is the complete bipartite graph.

22. Thm. A graph is bipartite iff each of its cycle has even length..

P_{26}

23. Thm.(Def) G : connected graph with order n. The followings are equivalent:

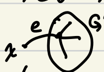
1) G has $n-1$ edge
2) Upon the removal of any edge, G becomes disconnected
3) G doesn't have any cycle
4) Any pair of vertices can be joined by a unique path.

Pf. 1) $\Rightarrow$ 2) Suppose 2) fails, G' has $n-2$ edges and G' is connected.

(Lemma.) If $G=(V,E)$ is connected, then $\#E \geq \#V - 1$.

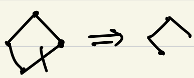
Pf. Induction on n.

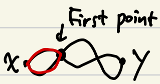
(i) $\deg(x) \geq 2$, $\forall x \in V$, $2\#V \leq \sum_{x\in V} \deg(x) = 2\#E \Rightarrow \#E \geq \#V$

(ii) $\deg(x) = 1$, 

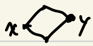
$G' := \{V\setminus\{x\}, E\setminus\{e\}\}$, G' is connected.

$\#E = 1 + \#E' \geq 1 + \#V' - 1 = \#V - 1$

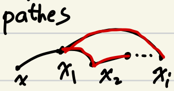
2) $\Rightarrow$ 3) If G has a cycle,  is still connected.

3) $\Rightarrow$ 4) If 

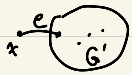
2 pathes, there exists a cycle.

4) $\Rightarrow$ 3) cycle 

$\Rightarrow$ 2 pathes

3) $\Rightarrow$ 1) 1° $\deg(x) \geq 2$, $\forall x \in V$. 

$\times$

2° $\exists x \in V$, $\deg(x) = 1$ 

$|E| = 1$:  $|V| = 2$ ✓

By induction, $G' = (E', V')$ $|E'| = |V'| - 1 \Rightarrow |E| = |V| - 1$. ✓

24. Def. If G has no cycles, we call G is a forest.

25. Prop. If G is a forest with m trees, n vertices, e edges.

then $m+e=n$.

26. Def In a graph, if x has degree 1, then we call it a pendent.

27. Thm. G : a tree with order $n \geq 2$, then G has ≥ 2 pendents.

Pf. $V_1 := \{x \in V : deg(V) = 1\}$, $V_2 := \{x \in V : deg(V) \geq 2\}$

$$\left.\begin{array}{l} \sum_{x \in V} deg(x) \geq |V_1| + 2|V_2| = |V_1| + 2(n - |V_1|) = 2n - |V_1| \\ \sum_{x \in V} deg(x) = 2|E| = 2n-2 \end{array}\right\} \Rightarrow |V_1| \geq 2.$$

28. Def. G is a graph. We call T a spanning tree of G if T is a subgraph of G & T is a tree & $V(T) = V(G)$.

29. Thm. Any connected graph G has a spanning tree.

Pf. We keep deleting edges in the cycles of G, until there is no cycle.

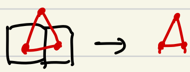
30. Q. How many spanning trees are there in K_n? n^{n-2}.

Prüfer Sequence

Chapter 12. Chromatic number

1. Def. Chromatic number of $G=(V,E)$ is the smallest integer K s.t. $\exists$ a $\underline{K\text{-coloring of } V}$, denoted by $\chi(G)$.
   (color V using K different colors and adjacent vertices have different colors.)

2. Def. Planar graph.

3. Def. Dual graph. 

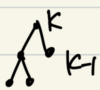
4. Thm. $\chi(G) \leq \Delta + 1$, $\Delta := \max_{x \in V} \deg(x)$

   Pf. (Greedy alg.) $\quad x_1 \quad x_2 \quad \cdots \quad x_i \quad x_{i+1}$
   $\quad 1$

   we need $\deg x_{i+1} \leq \chi(G) - 1$ ✓

5. Def. $P_G(K) = \#$ of K-colorings of G, s.t. adjocent certices have different colors.

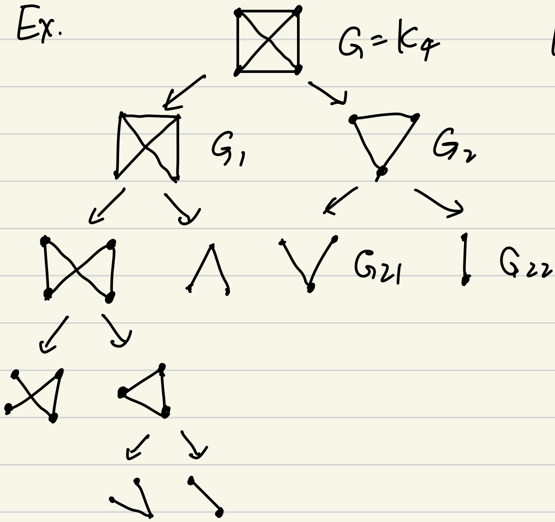
6. Thm. If T is a tree of order n, then $P_T(K) = K(K-1)^{n-1}$.

   Pf. 

7. Thm. $P_G(K)$ is a polymonial in K with degree n (n is the order of G)

   Pf. By induction, G_1 is obtained from G by removing e
   G_2 is obtained from G by merging x, y
   Suppose we have a K-coloring of G
   $P_G(K) = P_{G_1}(K) - P_{G_2}(K)$ By Thm 6 ✓.

8. Ex. $k(k-1)^3-3k(k-1)^2+2k(k-1)$



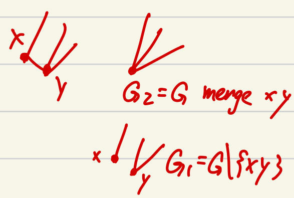
(Tree)

9. Def. Chromatic index
$\chi'(G)$ = smallest k s.t. $\exists$ k-coloring of edges of G

10. Thm. $\Delta \le \chi'(G) \le \Delta + 1$

11. Thm. G is a planar graph. Then $\chi(G) \le 5$

$P_G(k) = P_{G_1}(k) - P_{G_2}(k)$



Pf. Claim 1. n = # of vertices
e = # of edges $\quad n-e+m=2$
m = # of regions

When G is a tree $e=n-1$, $m=1$, $n-e+m=2$

Claim 2. $\exists x \in V$, $\deg(x) \le 5$

$N = \#\{(e,u): e \in E,\ u$ is a region, e is on the boundary of $u\}$

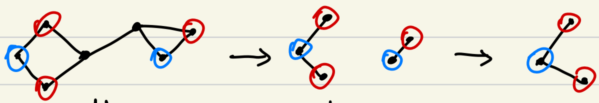
$N \le 2e$ (一条边分割两个面)

$N \ge 3m$ (一个 region 至少由3条 edges 围成) (G是 simple graph ×)

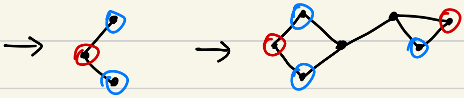
By Claim 1. $2 = n-e+m \le n-e+\frac{2}{3}e = n-\frac{1}{3}e = n-\frac{1}{6}\sum_{x\in V}\deg(x)$

$\Rightarrow \sum_{x \in V} \deg(x) \le 6n-12 < 6n$ //

Claim 3.



H　　Hr,b　　Cr,b (connected component)



✓

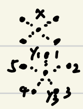
Induction on n.

By Claim 2. $\exists x$ s.t. $\deg(x) \le 5$,

Denote H from G by removing x. H has $n-1$ vertices

By Induction, we have 5-coloring of H.

i) $\deg(x) \le 4$, ✓



ii) $\deg(x) = 5$,

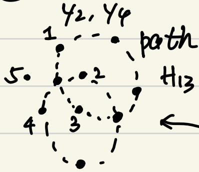
For example. $H_{1,3}$

① $\exists$ y_1, y_3 lie in different connected component.

$y_1: 1 \to 3$　　$x: 1$ ✓

② $\forall$ y_1, y_3 lie in the same ... .

y_2, y_4 path H_{13}

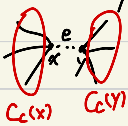


← planar graph 一定会相交

所以只用了3个颜色. (1. 2, 3. 4. 5)

12. Thm. $G=(V,E)$, $\Delta := \max_{x\in V} \deg(x)$. Then $\exists$ a $(\Delta+1)$-coloring of the edges of G.

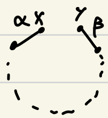
Pf. Prove by induction on $|E|$. By induction. $G-e$ has a $(\Delta+1)$-coloring



c: one $(\Delta+1)$-coloring

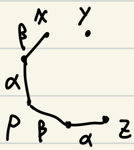
If $C_c(x) \cup C_c(y) = [\Delta+1]$ (W.L.O.G.)

Since $\Delta+1 > \max_{x\in V} \deg x$, $\exists\ \alpha, \beta \in [\Delta+1]$. s.t.



$\beta \notin C_c(x)$, $\alpha \notin C_c(y)$ ($\alpha \in C_c(x)$, $\beta \in C_c(y)$).

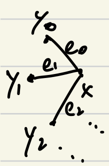
P: maximal (α,β)-path starting at x, if it ends at $z \neq y$.



By $\underbrace{\text{swapping } \alpha, \beta \text{ on } P}_{c \to c'}$, $\alpha \notin C_{c'}(x) \cup C_{c'}(y)$

Then we find a $(\Delta+1)$-coloring of G.

Now we suppose P must ends at y

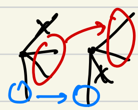


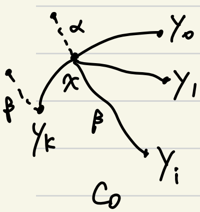
Def. $c_i : G - e_i$, $\exists \{y_i\}_{i=0}^{k}$ sequence 互不相同.

1) $c_0(xy_i) \notin C_{c_0}(y_{i-1})$, $1 \le i \le k$, 2) maximal

$\begin{cases} c_1(e_0) = c_0(e_1), \\ c_1(e_1) \times \\ c_1(e) = c_0(e), \text{ otherwise} \end{cases}$

$\begin{cases} c_i(e_j) = c_0(e_{j+1}), & 0 \le j < i \\ c_i(e_i) \times \\ c_i(e) = c_0(e), & \text{otherwise} \end{cases}$





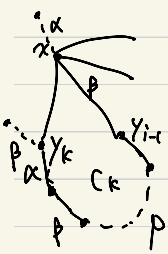
$\exists \beta \notin C_{c_0}(y_k) \Rightarrow \beta \in C_{c_0}(x)$

$\exists i, c_0(xy_i) = \beta$

$c_{i-1}(xy_i) = \beta$

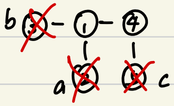
$c_k(xy_{i-1}) = \beta$

Let $\alpha \notin C_{c_0}(x)$, $\exists$ maximal (α,β)-path under c_k.



13. Thm. The number of spanning trees of $K_n = n^{n-2}$.

Pf. $\phi : \{\text{spanning trees of } K_n\} \xrightarrow{\cong} \{a_1 a_2 \cdots a_{n-2} : a_i \in [n]\}$

T 

a_i : 删掉叶子的悬挂处.（按从小到大顺序）

$\phi(T) = 114$

How to find $T = \phi^{-1}(a_1 \cdots a_{n-2})$

$\deg(x) = \#\{1 \le i \le n-2 ; a_i = x\} + 1$

Chapter 14 Pólya counting

1. Def. $(G,\cdot)$ is a group: $\forall f,g,h\in G$,

   1) $e\in G$, $e\cdot g = g\cdot e = g$.

   2) $f\cdot(g\cdot h) = (f\cdot g)\cdot h$

   3) $\forall g\in G$, $\exists\, h\in G$, s.t. $g\cdot h = h\cdot g = e$. Denote $h = g^{-1}$.

2. Ex. S_n permutation group.

   $f\in S_n$ is of form $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$

   $\# S_n = n!$, $f: [n]\to[n]$, $f(k) = i_k$

   S_n has the multiplication structure.

3. Def. (Group action) G is a group. X is a set,

   G acts on X: $\exists$ map $G\times X\to X$

   $$(g,x)\mapsto g(x)$$

   1) $e(x) = x$, $\forall x\in X$

   2) $f(g(x)) = (f\cdot g)(x)$

4. Ex. ① S_n acts on $[n]$

   ② Gl_n acts on $\mathbb{R}^n$ $(A,x)\mapsto Ax$

5. Def. G acts on X, $x\in X$,

   $\mathrm{orb}(x) = \{g(x) : g\in G\}\subset X$

6. Lemma. If $y\in\mathrm{orb}(x)$, then $\mathrm{orb}(y) = \mathrm{orb}(x)$. (orbit 不交)

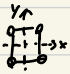
   Pf. $y\in\mathrm{orb}(x)\Rightarrow\exists\, g\in G$, $y = g(x)\Rightarrow x = g^{-1}(y)$

   $\mathrm{orb}(x) = \{f(x) : f\in G\} = \{f(g^{-1}(y)) : f\in G\} = \{(fg^{-1})(y) : f\in G\} = \{f(y) : f\in G\} = \mathrm{orb}(y)$.

7. Lemma. G acts on X, then $|X| = \sum_{j\in J} |\mathrm{orb}_j|$

If all orbits have same cardinality, then # of orbits $= \dfrac{|X|}{|\mathrm{orb}_j|}$.

(use 4 different colors)

8. Ex.  10 colors color corners (rigid motion is the same)

X = set of patterns, ${}_k^i\square_\ell^j$ $i, j, k, \ell \in [10]$.

$G = \{$rotation by $\underset{e}{0^\circ}$, $\underset{\rho}{90^\circ}$, $\underset{\rho^2}{180^\circ}$, $\underset{\rho^3}{270^\circ}$, (5)

reflection w.r.t. $\underset{\tau}{\{x=0\}}$, $\underset{\rho\circ\tau}{\{x=y\}}$, $\underset{\rho^3\circ\tau}{\{x=-y\}}$, $\underset{\rho^3\circ\tau}{\{y=0\}}$. $\}$

$= \{e, \rho, \rho^2, \rho^3, \tau, \rho\circ\tau, \rho^2\circ\tau, \rho^3\circ\tau\}$

$\rho^4 = e$, $\tau^2 = e$, $\rho\circ\tau = \tau\circ\rho^3$

All the orbits have the same cardinality. $|\mathrm{orb}_j| = |G|$

? = # of orbits in X under the action by G

$= P(10,4)/8 = 630$.

9. Ex. S_m, $X = \{a_1 a_2 \cdots a_m :$ permutation of $[n]\}$

S_m acts on X.

m-combination = # of orbits $= \dfrac{|X|}{|S_m|} = \dfrac{n!}{(n-m)!\,m!}$.

10. Def. G acts on X for $x \in X$, define the stabilizer

$\mathrm{stab}_G(x) = \{g \in G : g(x) = x\}$.

11. Ex. $G = (\mathbb{R}^n, +)$ $X = M_{n\times n} \to$ $n\times n$ matrices.

Action of G on X : $v(A) = Av$. $v \in G$, $A \in X$.

Action of X on G

4.30

1. Thm. (Mantel) Any trangle-free graph of order n has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.
Sharp : $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor}$.

Pf.

5.2

1.