Midterm 1 (100(+20) points) Name: ______________________

Feb. 2024, 11:00-12:15 Student ID:______________________

Please Read!!! No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work.
Question 1 and Question 2 are standard. Question 3 to 6 give you bonus points. **If your total points exceed 100, then it will be 100**.

1. (50 points) Count the following quantities. (Show the reasoning.)

    1. The number of positive divisors of $2^5 \times 3^{10} \times 7^9$.
    2. 10 boys and 15 girls stand in a row. No two boys are adjacent. How many different ways are there?
    3. The number of 10-permutations of the multiset $\{1 \cdot a, 2 \cdot b, 3 \cdot c, 4 \cdot d\}$.
    4. Find the number of solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 10$, where $x_1 \geq 0, x_2 \geq 2, x_3 \geq -1, x_4 \geq 2, x_5 \geq 0$ and they are integers.
    5. Randomly choose a subset with cardinality 5 from the multiset $\{4{\cdot}1, 4{\cdot}2, \ldots, 4{\cdot}9\}$. What's the probability that you get four same number in the subset.

Proof. 1. 660

2. $\binom{16}{10}15!$ First arrange the positions for girls (15!), then arrange boys into the places between girls.
3. $\frac{10!}{2!3!4!}$ from the theorem in the book
4. $\binom{11}{4}$
5. $\frac{8\times 9}{\binom{36}{5}}$

$\square$

2. (50 points) Answer the following questions.

   1. Simplify $\sum_{k=1}^{99} 2^k 3^{90-k}\binom{99}{k}$.
   2. Let $n \le m_1 \le m_2$. Prove that
   $$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$
   3. Use the combinatorial reasoning to show that
   $$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}.$$
   4. If we expand $(x_1+x_2+x_3)^{100}$, what's the coefficient of $x_1^{10}x_2^{20}x_3^{70}$?
   5. Prove that for any positive integers m and n, we have
   $$m! \mid (n+1)(n+2)\dots(n+m).$$

Proof.

1. $3^{-9}(5^{99} - 3^{99})$
2. Consider two ways to count choosing n objects from $m_1 + m_2$ objects.
3. From n people. elect a committee with a chair and a vice chair (they could be the same person).
4. $\frac{100!}{10!20!70!}$
5. Just note that $\binom{m+n}{m}$ is an integer.

$\square$

Questions 3,4,5,6 give you bonus points.

3. (5 points) Team A and Team B fight in a tournament for 20 rounds. Their final scores are 13 : 7. If the score of Team A is always bigger or equal than the score of Team B throughout the process, how many possible cases are there?

Proof. $\binom{20}{7} - \binom{20}{5}$. □

4. (5 points) A dice has six faces. You color each face using either red or blue. We say two colored dices are the same if one can be obtained from the other by rotation. How many different dices?

   Proof. All red or all blue: 2
   One red or one blue: 2
   Two red or two blue: 4
   Three red: 2
   Total: 10 □

5. (5 points) Suppose there are 10 points $p_1, \ldots, p_{10}$ in $[0,1]^2$ whose coordinates are $p_i = (x_i, y_i)$, where $0 \le x_i, y_i \le 1$. Show that there exists two different points p_i, p_j so that

$$\max\{|x_i - x_j|, |y_i - y_j|\} \le \frac{1}{3}.$$

Proof. Divide $[0,1]^2$ into nine $1/3 \times 1/3$ squares. By pigeonhole principle, two points p_i, p_j lie in the same square. That is what we want. $\square$

6. (5 points) Suppose p is a prime number and $1 \leq k < p$. Prove that $p \mid \binom{p}{k}$.

Proof. Note that $\binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!}$ is an integer, and the denominator does not have factor p (since they are all $< p$). So, p divides $\binom{p}{k}$. $\square$