Name: ______________________________

Final (100 points)
May 9 2024, 19:25-21:25

Student ID: ______________________________

No notes, no calculators!

1. (10 points) Determine the number of 12-combinations of the multiset

$$S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.$$

Proof. Note that the number of k-combinations of $\{\infty \cdot a_1, \ldots, \infty \cdot a_n\}$ is $\binom{n+k-1}{k-1}$.

Denote A_a to be the set of 12-combinations such that a appears ≥ 5 times. Similarly define A_b, A_c, A_d.

We have $|A_a| = \binom{10}{3} = 120$, $|A_b| = \binom{11}{3} = 165$, $|A_c| = \binom{10}{3} = 120$, $|A_d| = \binom{9}{3} = 84$.

$|A_a \cap A_b| = \binom{6}{3} = 20$, $|A_a \cap A_c| = \binom{5}{3} = 10$, $|A_a \cap A_d| = \binom{4}{3} = 4$, $|A_b \cap A_c| = 20$, $|A_b \cap A_d| = 10$, $|A_c \cap A_d| = 4$.

$|A_a \cap A_b \cap A_c| = 0$.

By inclusion-exclusion theorem, what we want is

$$\binom{15}{3} - 120 - 165 - 120 - 84 + 20 + 10 + 4 + 20 + 10 + 4 = 34.$$

$\square$

2. (10 points) How many sets of three integers between 1 and 20 are possible if no two consecutive integers are to be in a set?

Proof. Choosing three separated numbers from [1,20] is equivalent to choosing three numbers from [1,18] (by deleting a number from each gap). Hence, what we want is $\binom{18}{3} = 816$. $\square$

3. (10 points) What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

that satisfy

$$1 \leq x_1 \leq 5, -1 \leq x_2 \leq 4, 0 \leq x_3 \leq 3, 0 \leq x_4.$$

Proof. See midterm 2. $\square$

4. (10 points) Let n be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

sketch of proof. You need to compute the coefficient of x^n in $(1-x^2)^n = (1-x)^n(1+x)^n$. Compute for each of them then compare. $\square$

5. (10 points) Prove that the number of $2 \times n$ arrays

$$\begin{bmatrix} x_{11} & x_{12} \ldots x_{1n} \\ x_{21} & x_{22} \ldots x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2 \ldots, 2n$ such that

$$x_{11} < x_{12} < \cdots < x_{1n}$$

$$x_{21} < x_{22} < \cdots < x_{2n}$$

$$x_{11} < x_{21}, x_{12} < x_{22}, \ldots, x_{1n} < x_{2n},$$

equals the nth Catalan number C_n.

sketch of proof. $C_n = \frac{1}{n+1}\binom{2n}{n}$.

You need to find a correspondence between this array to a path from $(0, 0)$ to (n, n). $\square$

6. (20 points) Find the expression for $\{f_n\}$ which satisfy

$$f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3} + 2^n + 2n, \quad f_0 = 0, \ f_1 = 12, \ f_2 = 38.$$

   1. (5 points) Find the solution for homogeneous part.
   2. (10 points) Find the particular solution.
   3. (5 points) Find the expression for f_n.

Proof. See midterm 2. □

7. (10 points) What is the chromatic number of the graph obtained from K_n by removing one edge?

sketch of proof. $n - 1$.

Since this graph contains a subgraph K_{n-1}, we must need at least $n - 1$ colors. On the other hand, $n - 1$ colors are enough, since one vertex has degree $n - 2$. $\square$

8. (10 points) Prove that the chromatic polynomial of a cycle graph C_n equals

$$p(k) = (k-1)^n + (-1)^n (k-1).$$

sketch of proof. You use induction on n to prove the result. Deleting one edge from C_n, we obtain a tree T_n. Contracting one edge, we obtain C_n. Then,

$$p_{C_n}(k) = p_{T_n}(k) - p_{C_n}(k) = k(k-1)^n - p_{C_{n-1}}(k).$$

Just need to prove

$$p_{C_3}(k) = (k-1)^3 - (k-1).$$

and

$$(k-1)^n + (-1)^n(k-1) = k(k-1)^n - [(k-1)^{n-1} + (-1)^{n-1}(k-1)].$$

$\square$

9. (10 points) Let $n \geq 3$. Let K_n be an order n complete graph whose vertices are labeled $1, \ldots, n$. We define a map ϕ from the set of spanning trees of K_n to the set of sequences $a_1 a_2 \cdots a_{n-2}$ (where each $a_i \in \{1, 2 \ldots, n\}$). Suppose T is a spanning tree of K_n, we define the sequence $\phi(T) = a_1 \cdots a_{n-2}$ as follows.

Suppose b_1 is the smallest vertex among all degree-one vertices of T. Set a_1 to be the neighbor of the vertex b_1. Delete vertex b_1 and denote the remaining tree to be T_1.

For $k = 1, \ldots, n-3$, we do the following recursively. Suppose we have defined $a_1 \cdots a_k$ and obtain a tree T_k. Let b_{k+1} be the smallest vertex among all degree-one vertices of T_k. Set a_{k+1} to be the neighbor of the vertex b_{k+1}. Delete vertex b_{k+1} and denote the remaining tree to be T_{k+1}.

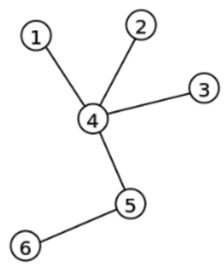

Figure 1:

For example, in Figure 1, it is a spanning tree T of K_6. By running the process above, you get $b_1 = 1, a_1 = 4; b_2 = 2, a_2 = 4; b_3 = 3, a_3 = 4; b_4 = 4, a_4 = 5$. Hence, $\phi(T) = 4445$.

1. (5 points) Find a tree T such that $\phi(T) = 11111$?
2. (5 points) Find a tree T such that $\phi(T) = 12345$?

Proof. 1. T is a tree with a degree-6 vertex labeled as 1, and other vertices are all adjacent to 1.

2. T is of form: $6 - 1 - 2 - 3 - 4 - 5 - 7$. $\square$