HW4

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Let $\langle 12 \rangle$ be the subgroup of $(\mathbb{Z}, +)$ consisting of all integers divisible by 12. Let G be the quotient group $\mathbb{Z}/\langle 12 \rangle$. Find all the normal subgroups of G and count the number of elements of the corresponding quotient groups.

2. Let G be a group, N a normal subgroup, $p : G \to G/N$ a quotient map. Let S be the set of subgroups of G that contains N, S' be the set of subgroups of G/N.

   (a) Show that the map $F : S \to S'$ defined by $F(H) = p(H)$ is a bijection. (Hint: show that the map $H' \mapsto p^{-1}(H')$ is its inverse)

   (b) Show that $H \in S$ is a normal subgroup of G iff $p(H)$ is a normal subgroup of G/N.

   (c) If $H \in S$ and $H \trianglelefteq G$, show that there is an isomorphism from G/H to $(G/N)/p(H)$ defined as $aH \mapsto (aN)p(H)$ (Need to first show that it is well defined.)

   (This is usually called the Third Isomorphism Theorem.)

3. Let G be a group, $H \leq G$ a subgroup. Show that H is a normal subgroup of G if and only if the sets $\{gh : h \in H\}$ and $\{hg : h \in H\}$ are equal for all $g \in G$. (The set $\{hg : h \in H\}$ is often denoted as Hg and called a "right coset".)

4. Let G be a group, N a normal subgroup, H a subgroup.

   (a) Show that the set $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G.

   (b) Show that N is a normal subgroup of NH.

   (c) Show that $N \cap H$ is a normal subgroup of H.

   (d) Show that the map $f : (NH)/N \to H/(N \cap H)$ defined as $(nh)N \mapsto h(N \cap H)$ is a group isomorphism. (Need to first show that it is well defined.)

   (This is called the Second Isomorphism Theorem.)