Math 541 Midterm F24

The exam is open book and open notes. You do not need to justify your answer for problem 1, but for problem 2 and problem 3 please do write down a full proof.

1. Let S_4 be the permutation group of $\{1,2,3,4\}$. $\sigma = (1,2,3,4) \in S_4$ (σ sends 1 to 2, 2 to 3, 3 to 4 and 4 to 1). Let H be the smallest subgroup of S_4 containing σ, $\cdot : H \times \{1,2,3,4\} \to \{1,2,3,4\}$ a left H action defined as $h \cdot a = h(a)$.
   (a) Write down all the elements of H. (20 points)
   (b) Is the action $\cdot$ free? Is it transitive? Is it effective? (15 points)

2. Let G be a group, X a non empty set, $\cdot : G \times X \to X$ a left G action on X, and $\rho : G \to S_X$ defined as $\rho(g) = (x \mapsto g \cdot x)$ is the corresponding permutation representation. Let $\cdot' : G \times X \to X$ be defined as $g \cdot' x = g \cdot (g \cdot x)$, $m : G \to G$ be defined as $m(g) = g^2$.
   (a) Show that if G is abelian, then m is a group homomorphism. (15 point)
   (b) Show that if G is abelian, then $\cdot'$ is a left G action on X. (15 points)
   (c) Show that $\cdot'$ is a left G action if and only if the image of ρ is abelian. (10 points)

3. Let G be a group, N a normal subgroup of G, $p : G \to G/N$ the quotient map to the quotient group G/N. Let $G \times G$ be the direct product of G with itself (group operation on $G \times G$ is defined as $(a,b)(c,d) = (ac, bd)$), $H = \{(a,b) \in G \times G : p(a) = p(b)\}$.
   (a) Show that H is a subgroup of $G \times G$. (20 points)
   (b) Show that the set $(G \times G)/H$ has a bijection to G/N. (5 points)

Answer:

1. (a) $e, (1,2,3,4), (1,3,2,4), (1,4,3,2)$
   (b) It is free, transitive and effective.

2. (a) $m(a)m(b) = a^2b^2 = abab = m(ab)$.

(b) $(ab) \cdot' x = (ab) \cdot ((ab) \cdot x) = (abab) \cdot x = (aabb) \cdot x = (aa) \cdot ((bb) \cdot x) = a \cdot' (b \cdot' x)$, $e \cdot' x = e \cdot (e \cdot x) = x$.

(c) For any $g \in G$, $x \in X$, $g \cdot' x = g \cdot (g \cdot x) = (\rho(g) \circ \rho(g))(x)$.

Suppose $\rho(g)$ is abelian. $e \cdot' x = e \cdot (e \cdot x) = x$, $a \cdot' (b \cdot' x) = (\rho(a) \circ \rho(a) \circ \rho(b) \circ \rho(b))(x) = (\rho(ab) \circ \rho(ab))(x) = (ab) \cdot' x$. So $\cdot'$ is a left G action.

Suppose $\cdot'$ is a left G action, then for any $x \in X$, any $a, b \in G$, $(\rho(a) \circ \rho(a) \circ \rho(b) \circ \rho(b))(x) = a \cdot' (b \cdot' x) = (ab) \cdot' x = (\rho(a) \circ \rho(b) \circ \rho(a) \circ \rho(b))(x)$. Hence $\rho(a) \circ \rho(a) \circ \rho(b) \circ \rho(b) = \rho(a) \circ \rho(b) \circ \rho(a) \circ \rho(b)$. Apply cancellation law in the group S_X, we see that $\rho(a) \circ \rho(b) = \rho(b) \circ \rho(a)$, hence $\rho(G)$ is abelian.

3. (a) $p(e) = p(e)$ hence $(e, e) \in H$. If $(a, b), (a', b') \in H$, then their product in $G \times G$, which is (aa', bb'), is also in H, because $p(aa') = p(a)p(a') = p(b)p(b') = p(bb')$, and the inverse of (a, b) in $G \times G$, which equals (a^{-1}, b^{-1}), is also in H because $p(a^{-1}) = (p(a))^{-1} = (p(b))^{-1} = p(b^{-1})$. These show that H is a subgroup of G.

(b) The bijection can be defined as $F : (g, h)H \mapsto p(g)p(h)^{-1}$. To show that it is well defined, if $(g', h') = (g, h)(a, b)$ for $(a, b) \in H$, then $p(g')p(h')^{-1} = p(g)p(a)p(b)^{-1}p(h)^{-1} = p(g)p(h)^{-1}$. To show that it is an injection, if $p(g)p(h)^{-1} = p(g')p(h')^{-1}$, then $p(g^{-1}g') = p(h^{-1}h')$, so $(g', h')H = (g, h)H$. To show that it is a surjection, for every $q \in G/N$, let $a \in G$ such that $p(a) = q$, then $q = F((a, e)H)$.