1. Let $\langle 12\rangle$ be the subgroup of $(\mathbb{Z},+)$ consisting of all integers divisible by 12. Let G be the quotient group $\mathbb{Z}/\langle 12\rangle$. Find all the normal subgroups of G and count the number of elements of the corresponding quotient groups.

Pf. since $(\mathbb{Z},+)$ is abelian, $\langle 12\rangle \leq \mathbb{Z}$.

so $\langle 12\rangle \trianglelefteq \mathbb{Z}$.

by definition $G=\{[0],[1],\cdots,[11]\}$ with the group operation defined as $[a]\cdot[b] \mapsto [ab]$

since $[b]\cdot[a]=[ba]=[ab]=[a]\cdot[b]$, G is abelian.

so we only need to find all the subgroups of G.

$\{[0]\}\leq G$, $G\leq G$ are vacuously true.

If $H\leq G$ and $[h]\in H$, $1\leq h\leq 11$

then $[h]^{-1}=[h^{-1}]=[12+(-h)]\in H$

① If $h=1$. then $\underbrace{[h][h]\cdots[h]}_{k \text{ times}}=[kh]\in H$ for all $k\in\mathbb{N}$. then $H=G$

More generally if $h\mid 12$. let $kh=12$. $1\leq k\leq 12$.

then $H=\{[0],[h],[2h],\cdots[12-h]\}$ is a subgroup of G because it's closed under inverse and group operation

② If $h\nmid 12$. since $1\leq h\leq 11$. we have $\gcd(h,12)=1$.

By Bezout Theorem. $\exists m,n\in\mathbb{Z}$ s.t. $m\cdot h+12\cdot n=1$

then $[mh+12\cdot n]=[mh]\cdot[12n]=[0]^n\cdot[h]^m=[h]^m=[1]$

if $H \leq G$. then $[1] \in H$. then $H = G$.

So the number of normal subgroup of G is $\#\{k \in \mathbb{N} \mid k \mid 12\} = 6$.

that is.

$H_1 = \{[0]\}$. $|G/H_1| = 12$

$H_2 = \{[0].[2].[4].[6].[8].[10]\}$. $|G/H_2| = 2$.

$H_3 = \{[0].[3].[6].[9].\}$ $|G/H_3| = 3$

$H_4 = \{[0].[4].[8]\}$ $|G/H_4| = 4$

$H_5 = \{[0].[6]\}$ $|G/H_5| = 6$

$H_6 = G$. $|G/H_6| = 1$.

2. Let G be a group, N a normal subgroup, $p : G \to G/N$ a quotient map. Let S be the set of subgroups of G that contains N, S' be the set of subgroups of G/N.

   (a) Show that the map $F : S \to S'$ defined by $F(H) = p(H)$ is a bijection. (Hint: show that the map $H' \mapsto p^{-1}(H')$ is its inverse)

   (b) Show that $H \in S$ is a normal subgroup of G iff $p(H)$ is a normal subgroup of G/N.

   (c) If $H \in S$ and $H \trianglelefteq G$, show that there is an isomorphism from G/H to $(G/N)/p(H)$ defined as $aH \mapsto (aN)p(H)$ (Need to first show that it is well defined.)

   (This is usually called the Third Isomorphism Theorem.)

(a). Pf. Consider the map $\tilde{F} : S' \to S$. $H' \mapsto p^{-1}(H')$

If $H' \leq G/N$. let $H' = \{g_0 N, g_1 N \cdots\}$ let $g_0 = e$.

By definition $\tilde{F}(H') = p^{-1}(H') = \{g \in G \mid gN \in H'\}$

$\tilde{F}$ is well-defined because p^{-1} is a map defined on $G/N \to \mathcal{P}(G)$ and $\tilde{F}(H') \in S$ because. ①~④.

① $\forall n \in N \quad nN = N \in H'$ so $N \subset \tilde{F}(H')$

② Identity. $e \in \tilde{F}(H')$ because $N \in H'$

③ Closed under inversion. if $g \in \tilde{F}(H')$. $g^{-1}N = (gN)^{-1}$ so $g^{-1} \in \tilde{F}(H')$

④ Closed under group operation. if $g_1 . g_2 \in \tilde{F}(H')$.

$g_1 N \cdot g_2 N := (g_1 g_2) N \in H'$. so $g_1 g_2 \in \tilde{F}(H')$

$$F \circ \tilde{F}(H') = F(\{g \in G \mid gN \in H'\})$$
$$= \{p(g) \in G/N \mid gN \in H'\}$$
$$= H'$$

Claim. If $H \in S$. then $N \trianglelefteq H$.

Hw 2.2(b) showed that. $\left.\begin{matrix} H \leq G \\ N \leq G \\ N \subset H \end{matrix}\right\} \Rightarrow N \leq H$

If $\exists h \in H. n \in N$ s.t. $hnh^{-1} \in H \setminus N$ since $h \in G$. we have $N \ntrianglelefteq G$. contradiction.

so $N \trianglelefteq H$. then $\tilde{F} \circ F(H) = \tilde{F}(\{gN \mid g \in H\})$

$$= \{g \in G \mid gN \in \{gN \mid g \in H\}\} = H.$$

so $F \circ \tilde{F} = \hat{F} \circ F = id$. $\hat{F} = F^{-1}$ and F is a bijection. $\square$

(b). Pf. If $H \in S$. $p(H) = \{hN \mid h \in H\}$

If. $p(H) \trianglelefteq G/N$. then $\forall g \in G$.

$$(gN)\,p(H)\,(gN)^{-1} = (gN)\,p(H)\,(g^{-1}N) = \{(ghg^{-1})N \mid h \in H\}$$
$$= p(H) = \{hN \mid h \in H\}$$

If $\exists g_0 \in G,\ h_0 \in H$ s.t. $g_0 h_0 g_0^{-1} \notin H$. but $\exists h_1 \in H$ s.t. $h_1^{-1}(g_0 h_0 g_0^{-1}) \in N$

then $\underbrace{h_1^{-1}(g_0 h_0 g_0^{-1}) \notin H}_{(*)}$ because otherwise $g_0 h_0 g_0^{-1} = h_1 (h_1^{-1} g_0 h_0 g_0^{-1}) \in H$

according to $(*)$ $h_1^{-1}(g_0 h_0 g_0^{-1}) \notin N$. i.e. $(g_0 h_0 g_0^{-1})N \notin p(H)$. contradiction

so $gHg^{-1} \subset H$ i.e. $H \trianglelefteq G$.

If $H \trianglelefteq G$. $\forall g \in G$. $\forall h \in H$ $ghg^{-1} \in H$

since the quotient map p is surjection. and $N \trianglelefteq G$

$$\forall g \in G.\ (gN)\,p(H)\,(gN)^{-1} = (gN)\,p(H)\,(g^{-1}N)$$
$$= \{(ghg^{-1})N \mid h \in H\} \subset p(H)$$

so $p(H) \trianglelefteq G/N$ $\square$

(c). Pf. If $H \in S$. $H \trianglelefteq G$. I showed in (a) that $N \trianglelefteq H$.

I'll show that the map $f: aH \mapsto (aN)\,p(H)$ is well-defined because ① - ③

① $p(H) \trianglelefteq G/N$. as I've showed in (b)

② $f(G/H) \subset (G/N)/p(H)$

given $g \in G$. $f(gH) = (gN)\,p(H) \in (G/N)/p(H)$

③ $y_1 = f(x_1)$. $y_2 = f(x_2)$. $x_1 = x_2 \Rightarrow y_1 = y_2$

given $g_1, g_2 \in G$ with $g_1H = g_2H$, then $g_1^{-1}g_2 \in H$

then $(g_1N)^{-1}(g_2N) = (g_1^{-1}g_2)N \in p(H)$, i.e. $f(g_1H) = f(g_2H)$.

so the map $f: aH \mapsto (aN)\,p(H)$ is well defined.

Next I'll show that f is a homomorphism.

$f(H) = (eN)\,p(H)$ is the identity in $(G/N)/p(H)$ because N is the identity in G/N.

$$f(aH \cdot bH) = f(abH) = (abN)\,p(H) = ((aN)(bN)\,p(H))$$
$$= ((aN)\,p(H))((bN)\,p(H)) = f(aH)\,f(bH)$$

so $f \in \mathrm{Hom}(G/H,\ (G/N)/p(H))$.

Next I'll show that f is a bijection.

① f is a surjection

pick $(gN)\,p(H) \in f(G/H)$. apparently $gH \in G/H$ satisfies that $f(gH) = (gN)\,p(H)$.

② f is injection

$g_1^{-1}g_2 \notin H \Rightarrow g_1H \neq g_2H \Rightarrow (g_1N)^{-1}(g_2N) = (g_1^{-1}g_2)N \notin p(H)$

because if $\exists h_0 \in H$ s.t. $g_1^{-1} g_2 h_0^{-1} \in N$

then $g_1^{-1} g_2 = (g_1^{-1} g_2 h_0^{-1})(h_0) \in N \subset H$ contradiction

so $f(g_1H) \neq f(g_2H)$.

so f is a isomorphism from G/H to $(G/N)/p(H)$. $\square$

3. Let G be a group, $H \leq G$ a subgroup. Show that H is a normal subgroup of G if and only if the sets $\{gh : h \in H\}$ and $\{hg : h \in H\}$ are equal for all $g \in G$. (The set $\{hg : h \in H\}$ is often denoted as Hg and called a "right coset".)

3. pf. If $H \trianglelefteq G$. Let $L_g := \{gh \mid h \in H\}$. $R_g := \{hg \mid h \in H\}$

If $\exists g_0 \in G$. s.t. $L_{g_0} \neq R_{g_0}$

① If $\exists h_0 \in H$ s.t $g_0 h_0 \notin R_{g_0}$ then $\forall h \in H$. $g_0 h_0 \neq h g_0$

then $\forall h \in H$. $g_0 h_0 g_0^{-1} \neq h$. i.e. $g_0 h_0 g_0^{-1} \notin H$. this contradicts with $H \trianglelefteq G$

② If $\exists h_0 \in H$ s.t. $h_0 g_0 \notin L_{g_0}$. then $\forall h \in H$. $h_0 g_0 \neq g_0 h$

then $\forall h \in H$ $g_0^{-1} h_0 g_0 \neq h$. then $\forall h \in H$. $g_0 h_0^{-1} g_0^{-1} \neq h$

i.e. $g_0 h_0^{-1} g_0^{-1} \notin H$. this contradicts with $H \trianglelefteq G$.

so $H \trianglelefteq G \Rightarrow \forall g \in G$. $gH = Hg$.

If $\forall g \in G$. $gH = Hg$.

then $gHg^{-1} := \{ghg^{-1} \mid h \in H\} = Hgg^{-1} = \{hgg^{-1} \mid h \in H\}$

$= H$. for all $g \in G$.

By definition $H \trianglelefteq G$. $\square$

4. Let G be a group, N a normal subgroup, H a subgroup.

(a) Show that the set $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G.

(b) Show that N is a normal subgroup of NH.

(c) Show that $N \cap H$ is a normal subgroup of H.

(d) Show that the map $f : (NH)/N \to H/(N \cap H)$ defined as $(nh)N \mapsto h(N \cap H)$ is a group isomorphism. (Need to first show that it is well defined.)

(This is called the Second Isomorphism Theorem.)

4. (a). Pf. $N \trianglelefteq G$. $H \leq G$, so

① $e \in N$. $e \in H$. so $e \in NH$

② $\forall n_1, n_2 \in N$. $\forall h_1, h_2 \in H$. $n_1h_1 \cdot n_2h_2 \in NH$.

$$n_1 h_1 n_2 h_2 = h_1\left(\left(h_1^{-1} n_1 h_1\right) n_2\right) h_1^{-1} h_1 h_2$$

since $N \trianglelefteq G$. $h_1^{-1} n_1 h_1 \in N$. $h_1 h_2 \in H$

$\Rightarrow (h_1^{-1} n_1 h_1) n_2 \in N$

$\Rightarrow h_1^{-1} n_1 h_1 n_2 h^{-1} \in N$

$\Rightarrow n_1 h_1 n_2 h_2 \in NH$

③ $\forall n \in N$. $h \in H$. $nh \in NH$.

$$(nh)^{-1} = h^{-1} n^{-1} = h^{-1} n^{-1} h \cdot h^{-1} \in NH$$

so $NH \trianglelefteq G$. $\square$

(b). Pf. $\forall n_0 \in N$. $h_0 \in H$. $n_0 h_0 \in NH$

$$(n_0 h_0) N (n_0 h_0)^{-1} = n_0 \left(h_0 N h_0^{-1}\right) n_0^{-1}$$

$$= n_0 N n_0^{-1}$$

$$= N$$

so $N \trianglelefteq NH$. $\square$

(c). Pf. Firstly. $N \le G$. $H \le G$. Hw2.2.(a) showed that $N \cap H \le G$

Hw2.2(b) showed that $N \cap H \le N$. $N \cap H \le H$

$\forall h \in H \quad h(N \cap H) h^{-1} \subset h N h^{-1} = N$

$h(N \cap H) h^{-1} \subset h H h^{-1} = H$

so $h(N \cap H) h^{-1} \subset N \cap H$

so $N \cap H \trianglelefteq H$.

(d). Pf. f is well-defined because ① and ②

① $N \trianglelefteq NH$. $N \cap H \trianglelefteq H$. so NH/N and $H/(N \cap H)$ makes sense.

② given $n_1, n_2 \in N$. $h_1 h_2 \in H$ $\quad n_1 h_1 (n_2 h_2)^{-1} \in N$. i.e. $n_1 h_1 N = n_2 h_2 N$.

then $h_1 h_2^{-1} = n_1^{-1} \left(n_1 h_1 (n_2 h_2)^{-1} \right) n_2 \in N$. $h_1 h_2^{-1} \in H$

$\Rightarrow h_1 h_2^{-1} \in N \cap H$

$\Rightarrow f(n_1 h_1) = f(n_2 h_2)$

Then I'll show that $f \in \mathrm{Hom}\left((NH)/N, H/(N \cap H) \right)$. because ③④

③ $f(nhN) = h(H \cap N)$ when $nh \in N$. $h \in N \Rightarrow h \in H \cap N$

$\Rightarrow h(H \cap N)$ is the identity in $H/(N \cap H)$.

④ $f(n_1 h_1 N \cdot n_2 h_2 N) = f(n_1 h_1 n_2 h_2 N) = f\left(h_1 \left((h_1^{-1} n_1 h_1) n_2 \right) h_1^{-1} h_1 h_2 \right)$

$= h_1 h_2 (N \cap H) = \left(h_1 (N \cap H) \right) \left(h_2 (N \cap H) \right)$

$= f(n_1h_1N) \cdot f(n_2h_2N)$

so $f \in \mathrm{Hom}(NH/N,\ H/(N\cap H))$

Next I'll show that f is a bijection because ⑤. ⑥

⑤ f is a surjection

given $h_0(N\cap H) \in H/(N\cap H)$. $eh_0N \in NH/N$

and $f(eh_0N) = h_0(N\cap H)$

⑥ f is an injection

given. $n_1 . n_2 \in N$. $h_1 . h_2 \in H$ with $n_1h_1(n_2h_2)^{-1} \notin N$

then $h_1h_2^{-1} = n_1^{-1}\left(n_1h_1(n_2h_2)^{-1}\right) n_2 \notin N$

then $h_1h_2^{-1} \notin N\cap H$

i.e. $f(n_1h_1N) \neq f(n_2h_2N)$

so f is an isomorphism from NH/N to $H/(N\cap H)$. $\square$