Honors HW1

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas.

1. Let S be a set. $S' = S \times \{-1, 1\}$. Let S^* be the set of finite sequences of elements of S' of the form $s_1 s_2 \dots s_n$. Here we allow $n = 0$, which corresponds to an empty sequence. Define $\sim$ as the smallest equivalence relation on S^* such that if $s_i = (s, d_i)$, $s_{i+1} = (s, -d_i)$ then $s_1 \dots s_i s_{i+1} \dots s_n \sim s_1 \dots s_{i-1} s_{i+2} \dots s_n$ (see HW1(b)). Let $F(S) = S^*/\sim$, define $* : F(S) \times F(S) \to F(S)$ as $([a], [b]) \mapsto [ab]$ where ab is the concatenation of a and b.

   (a) Show that $*$ is well defined, and $(F(S), *)$ is a group. (We call this **the free group generated by** S.)

   (b) Let G be a group, show that for every $f \in Map(S, G)$, there is a unique group homomorphism $F \in Hom(F(S), G)$, such that $F([s]) = f(s)$ for every $s \in S$.

2. Let $\{G_i : i \in I\}$ be a family of groups, $G = \prod_{i \in I} G_i$ their direct product. For every $i \in I$, let $p_i : G \to G_i$ be $\alpha \mapsto \alpha(i)$.

   (a) Show that p_i are all group homomorphisms.

   (b) Let H be a group, for every $i \in I$, pick $f_i \in Hom(H, G_i)$. Show that there is a unique $f \in Hom(H, G)$ such that $f_i = p_i \circ f$ for all $i \in I$.

   (c) Can you find a group G', such that there are injective homomorphisms $j_i : G_i \to G'$ for all $i \in I$, and if for any group H, for every $i \in I$, one pick some arbitrary $g_i \in Hom(G_i, H)$, then there is a unique $g \in Hom(G', H)$ such that $g_i = g \circ j_i$ for all $i \in I$? (Hint: use a construction similar to Problem 1 above. This is called the **free product**.)