HW1

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Let A be a set.
   (a) Let S be a non empty set of equivalence relations on A. Show that $\bigcap S$ is an equivalence relation on A.
   (b) Let R be a relation between A and A. Show that there is a unique equivalence relation on A, called $\sim_R$, such that any equivalence relation $\sim$ on A which contains R has $\sim_R$ as a subset ($R \subseteq \sim \implies \sim_R \subseteq \sim$).
2. Let A and B be two sets, $f : A \to B$ a function. Define function $F : P(B) \to P(A)$ as $F(C) = f^{-1}(C)$. Show that F is an injection iff f is a surjection, F is a surjection iff f is an injection.
3. Show that $\sim = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Q}\}$ is an equivalence relation on $\mathbb{R}$.
4. Let A and B be two sets. $C = \{(x, i) \in (A \cup B) \times \{0, 1\} : x \in A \text{ if } i = 0, x \in B \text{ if } i = 1\}$. Show that there are injections $k : A \to C$, $j : B \to C$, such that $C = k(A) \cup j(B)$ and $k(A) \cap j(B) = \emptyset$.
5. Let $f : A \to B$ be a function. Show that there is a set C, an injection $g : A \to C$, and a surjection $h : C \to B$, such that $f = h \circ g$. (Hint: You may want to use the solution for the previous problem).

Answer:

1. (a) For every $C \in S$, because C is an equivalence relation, we have $id_A \subseteq C$, hence $id_A \subseteq \bigcap S$. For any $a, b \in A$, if $(a, b) \in \bigcap S$ then $(a, b) \in C$ for all $C \in S$, hence $(b, a) \in C$ for all $C \in S$, which implies that $(b, a) \in \bigcap S$. Lastly, for any $a, b, c \in A$, $(a, b) \in \bigcap S, (b, c) \in \bigcap S$ implies that for every $C \in S$, $(a, b) \in C$ and $(b, c) \in C$, hence $(a, c) \in C$, which implies that $(a, c) \in \bigcap S$.
   (b) Let S_R be the set of equivalence relations on A which has S as a subset. $R \subseteq A \times A$ so $A \times A \in S_R$, S_R is non-empty. Let $\sim_R$ be $\bigcap S_R$. Then due to (a), $\sim_R$ is an equivalence relation. By construction, $R \subseteq \sim_R$, and any equivalence relation $\sim$ that satisfies $R \subseteq \sim$ must

be in S_R, hence $\sim_R \subseteq \sim$. Lastly, suppose there is some equivalence relation $\sim'$ such that $R \subseteq \sim'$ and $\sim' \subseteq \sim$ for every equivalence relation $\sim$ such that $R \subseteq \sim$, then $\sim' \subseteq \sim_R$ and $\sim_R \subseteq \sim'$, hence they must be equal.

2. (a) i. Suppose f is not a surjection, there must be some $b \in B$ which is not in $f(A)$, then $F(\emptyset) = \emptyset = F(\{b\})$, hence F is not an injection.

   ii. Suppose F is not an injection, then there must be some $C, D \in P(B)$ such that $F(C) = F(D)$ and $C \neq D$. Suppose $C \backslash D \neq \emptyset$ (swap C and D if $D \backslash C \neq \emptyset$), let $b \in C \backslash D$. If there is some $a \in A$ such that $f(a) = b$, then $a \in F(C)$ and $a \notin F(D)$, a contradiction, hence $b \notin f(A)$, f is not a surjection.

   (b) i. Suppose f is an injection, then for every $E \in P(A)$, $F(f(E)) = f^{-1}(f(E)) = \{a \in A : \text{ there exists } a' \in E, f(a) = f(a')\} = \{a \in A : \text{ there exists } a' \in E, a = a'\} = E$, hence F is a surjection.

   ii. Suppose f is not an injection, there must be $a, b \in A$, $f(a) = f(b)$ and $a \neq b$. Consider $\{a\} \in P(A)$. If $\{a\} = F(C)$, then $f(a) \in C$, hence $f(b) \in C$, $b \in F(C)$, a contradiction. Hence F can not be a surjection.

3. For any $x, y, z \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q}$, hence $x \sim x$. If $x \sim y$, then $x - y \in \mathbb{Q}$, hence $y - x = -(x - y) \in \mathbb{Q}$, $y \sim x$. If $x \sim y, y \sim z$, then $x - y \in \mathbb{Q}$, $y - z \in \mathbb{Q}$, hence $x - z = (x - y) + (y - z) \in \mathbb{Q}$, $x \sim z$.

4. Let k ad j be defined as $k : a \mapsto (a, 0)$, $j : b \mapsto (b, 1)$. By the construction of C both are well defined, and every element in C is of the form $(a, 0)$ where $a \in A$ or $(b, 1)$ where $b \in B$, in the former case it would be in $k(A)$ and in the latter case it would be in $j(B)$, hence $C = k(A) \cup j(B)$. If $(x, i) \in k(A) \cap j(B)$, $(x, i) \in k(A)$ implies $i = 0$, $(x, i) \in j(B)$ implies $i = 1$, contradiction. Hence $k(A) \cap j(B) = \emptyset$.

5. Let $C = \{(x, i) \in (A \cup B) \times \{0, 1\} : x \in A, i = 0 \text{ or } x \in B \backslash f(A), i = 1\}$, g be $a \mapsto (a, 0)$, and h be $h((x, i)) = \begin{cases} f(x) & i = 0 \\ x & i = 1 \end{cases}$. Then one can easily verify that g is an injection, h is a surjection, and $h \circ g = f$.