1. Show that the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $(t, x) \mapsto xe^t$ is a left $(\mathbb{R}, +)$ action on $\mathbb{R}$. Write down the corresponding permutation representation. Is the action effective? Is the action transitive? Is the action free?

1. Pf. $\begin{cases} \forall x \in \mathbb{R} \quad f(0,x) = xe^0 = x \\ \forall a, b \in \mathbb{R} \quad (a,(b,x)) \mapsto xe^b e^a = e^{a+b}x = f(a+b, x) \end{cases}$

so this is a left action on $\mathbb{R}$.

the corresponding permutation representation is defined as

$$p : t \mapsto (x \mapsto xe^t)$$

By definition. If X is a left G-set, if the kernel of the left G-action equals $\{e_G\}$. then we say the action is effective

$$\ker(p) = \{t \in \mathbb{R} \mid e^t = 1\} = \{0\} = e_{(\mathbb{R},+)}$$

so the action is effective.

By definition, we say a left G action on an nonempty set X is transitive if $\forall x, y \in X$. $\exists g \in G$ s.t. $y = gx = xe^g$

here $X = \mathbb{R}$ $G = (\mathbb{R}, +)$ let $x = -1$. $y = 1$. $\forall g \in G$. $xe^g < 0 < y$.

So this action is not transitive.

By definition. X is a left G-set. $f : G \times X \to X$ is free on X if

$$\forall x \in X. \; G_x = \{g \in G \mid gx = x\} = \{e_G\}$$

Here $X = \mathbb{R}$. $G = (\mathbb{R}, +)$. Given $x \in X$

$$G_x := \{g \in \mathbb{R} \mid xe^g = x\} = \begin{cases} \{0\} & x \neq 0 \\ \mathbb{R} & x = 0 \end{cases}$$

So this action f is not free. $\square$

2. Let G be a group, X and Y be two left G sets.

(a) Show that $\cdot : G \times (X \times Y) \to X \times Y$ defined as $(g, (x, y)) = (gx, gy)$ is a left G action on $X \times Y$.

(b) If $f : X \to Y$ is a map, show that f is G equivariant if and only if $\{(x, f(x)) : x \in X\}$ is a G invariant subset of the left G set $(X \times Y, \cdot)$.

(a). Pf. Let $u: G \times X \to X$. $v: G \times Y \to Y$ be the G-action on X and Y correspondingly

then $\begin{cases} \forall x \in X & u(e.x) = x \\ \forall x \in X.\ a.b \in G. & u(a.\ u(b.x)) = u(ab.x) \end{cases}$

$\begin{cases} \forall y \in Y. & v(e.y) = y \\ \forall y \in Y.\ a.b \in G. & v(a.\ v(b.y)) = v(ab.y) \end{cases}$

then "$\cdot$" is a left G-action on $X \times Y$ because ① ②

① $\forall x \in X.\ \forall y \in Y.\ \cdot(e.(x,y)) = (ex.ey) = (u(e.x).\ v(e.y)) = (x.y)$

② $\forall x \in X.\ \forall y \in Y.\ \forall a.b \in G.$

$$\begin{aligned} \cdot(a.\cdot(b.(x.y))) &= \cdot(a, (u(b.x).\ v(b.y))) \\ &= (u(a.u(b.x)),\ v(a.v(b.y))) \\ &= (u(ab.x)\ v(ab.y)) \\ &= \cdot(ab, (x.y)) \end{aligned}$$

$\square$

(b). Pf. If $\Delta = \{(x, f(x)) \mid x \in X\}$ is a G-invariant subset of $(X \times Y, \cdot)$

$\forall x \in X.\ \forall g \in G\ \ \cdot(g, (x, f(x))) = (gx, g(f(x))) \in \Delta$

Claim. if $(x_1, f(x_1)), (x_2, f(x_2))$ are different in Δ, then $x_1 \neq x_2$.

Otherwise if $x_1 = x_2$, then $f(x_1) = f(x_2)$. contradiction.

Since $(gx, f(gx)) \in \Delta$. we have $f(gx) = g(f(x))$. i.e.

$\forall x \in X.\ \forall g \in G.\ f(gx) = g(f(x))$.

so f is G equivariant.

If f is G equivariant. $\forall g \in G.\ \forall x \in X\ \ f(gx) = g(f(x))$

then $\forall x \in X.\ \forall g \in G.\ \ \cdot(g, (x, f(x))) = (gx, g f(x))$

$= (gx, f(gx)) \in \Delta$

because $gx \in X$.

$\square$

3. Let G be a group with more than one elements, X be a non empty left G set. Show that $\cdot : G \times P(X) \to P(X)$ defined as $(g, A) \mapsto \{ga : a \in A\}$ is a left G action on $P(X)$. Can this action be free? Can this action be transitive?

Pf. the map $\cdot : (g, A) \mapsto \{ga \mid a \in A\}$ is a left G-action because: ① ②

① $\forall A \in g(X)$. $e \cdot A = \{ea \mid a \in A\}$

$= \{a \mid a \in A\}$ because X is left G-set

$= A$

② $\forall A \in g(X)$. $\forall a.b \in G$.

$$a \cdot (b \cdot A) = (a \cdot \{bx \mid x \in A\})$$
$$= \{a(bx) \mid x \in A\}$$
$$= \{(ab)x \mid x \in A\} \quad \text{because } X \text{ is a left } G\text{-set.}$$
$$= ab \cdot A$$

So "$\cdot$" is a left G action on $P(A)$.

Claim. This action cannot be free.

This action is free iff. $\forall A \in P(X)$, $G_A := \{g \in G \mid \{ga \mid a \in A\} = A\} = \{e_G\}$

Consider $A = \phi \in P(X)$. $G_\phi = \{g \in G \mid \{ga \mid a \in A\} = A\} = G \neq \{e_G\}$

Since $A = \{ga \mid a \in A\}$ will be vacuously true. there's no constraint over g.

So this action cannot be free.

Claim. This action cannot be transitive

The action is transitive iff $\forall A.B \in P(X)$. $\exists g \in G$. s.t.

$$B = gA = \{ga \mid a \in A\}.$$

Let $A = \phi$. $B = X \neq \phi = gA$

So this action is not transitive.

$\square$

4. Any non empty set X is a left S_X set via the action $\cdot : S_X \times X \to X$, $(\sigma, x) \mapsto \sigma(x)$. Show that the only S_X equivariant map from X to X is the identity.

If $|X| \geq 3$

Pf. Let $f: X \to X$ be a S_X equivariant map. By definition.

$$\forall \sigma \in S_X.\ \forall x \in X.\quad f(\sigma(x)) = \sigma(f(x))$$

It can be checked that id_X is indeed an S_X-equivariant map.

pick $x_0 \in X$. $\quad \sigma_y(x) := \begin{cases} x_0 . & x = y \\ y . & x = x_0 \\ x . & \text{otherwise} \end{cases}$

Specially if $x_0 = y$. $\sigma_y = id_X$.

$$x_0 = \sigma_{f(x_0)}(f(x_0)) = f(\sigma_{f(x_0)}(x_0)) = f^2(x_0)$$

since x_0 is arbitrary. $\forall x \in X.\ f^2(x) = x$

Pick $x_1 . x_2 . x_3 \in X$. $x_1 . x_2 . x_3$ are different. (since $|X| \geq 3$)

let $\sigma_3(x) = \begin{cases} x_2 & x = x_1 \\ x_3 & x = x_2 \\ x_1 & x = x_3 \\ x & \text{otherwise} \end{cases}$ $\qquad$ Observe that $\sigma_3^3 = id_X$.

then $f(\sigma_3(x_1)) = f(x_2) = \sigma_3(f(x_1))$

$\qquad f(\sigma_3(x_2)) = f(x_3) = \sigma_3(f(x_2))$

$$f(\sigma_3(x_3)) = f(x_1) = \sigma_3(f(x_3))$$

$$f(\sigma_3(x)) = f(x) = \sigma_3(f(x)) \quad x \neq x_1. x_2. x_3.$$

So $x \in \{x_1 . x_2 . x_3\} \Leftrightarrow f(x) \in \{x_1 . x_2 . x_3\}$

If $f(x_1) = x_2$. $f(x_2) = f^2(x_1) = x_1 \neq x_3$ contradiction

If $f(x_1) = x_3$. $f(\sigma_3(x_3)) = f(x_1) = \sigma_3(f(x_3)) = x_3$

$\Rightarrow f(x_3) = x_2$. $f(x_2) = x_1$

$\Rightarrow f^2(x_3) = f(x_2) = x_1 \neq x_3$ contradiction

so $f(x_1) = x_1$. $f(\sigma_3(x_1)) = f(x_2) = \sigma_3(x_1) = x_2$

$f(\sigma_3(x_2)) = f(x_3) = \sigma_3(f(x_2)) = x_3$

so $f(x) = x$. $x \in X$.

$\square$

5. Write down a group G, two left G sets X and Y, such that there are no G equivariant map from X to Y.

pf. Let $G = S_2$ $X = \{1\}$ $Y = \{1.2\}$

$f_X : G \times X \to X$ $\quad$ $f_Y = G \times Y \to Y$

$\sigma\ 1 \mapsto 1$ $\quad$ $\sigma \cdot y \mapsto \sigma(y)$

The only two possible map f are ① and ②

①. $f: 1 \mapsto 1$

$f((1\ 2)\ 1) = f(1) = 1 \neq (1\ 2) f(1) = 2$

② $f: 1 \mapsto 2$

$$f((1\ 1)\ 1) = f(1) = 1 \neq (1\ 1)\ f(1) = 2.$$

so no such G-equivariant exists. $\square$