Midterm Review Problems

1. Write down all the elements of the subgroup of S_4 generated by $(1,2)(3,4)$ and $(1,3)(2,4)$, and write down all its subgroups. There is no need for justification. Is this group abelian?

2. Let $G = S_3$. Consider the map $\cdot : G \times Map(G,G) \to Map(G,G)$ defined as $(g,f) \mapsto (x \mapsto gf(g^{-1}x))$.

   (a) Show that $\cdot$ is a left G action.

   (b) In the left G-set $(Map(G,G),\cdot)$, is there a G-orbit with 3 elements? Write down one or prove that such a G-orbit does not exist.

3. Let G be a group, H a normal subgroup. Recall that for a group G, a conjugacy class is an orbit of the action $(g,x) \mapsto gxg^{-1}$.

   (a) Show that each conjugacy class of H is contained in a unique conjugacy class of G.

   (b) If G is a finite group, show that all the conjugacy classes of H that lies in the same conjugacy class of G have the same number of elements.

4. Let G be a group, consider the map $G \times Map(G,\mathbb{R}) \to Map(G,\mathbb{R})$ defined as $(g,f) \mapsto (x \mapsto f(gx))$.

   (a) Show that this map is a left G-action if and only if G is abelian.

   (b) When G is abelian, show that any subgroup H of G is the stablizer of some $f_H \in Map(G,\mathbb{R})$.

Answer:

1. The subgroup is $\{e,(1,3)(2,4),(1,2)(3,4),(1,4)(2,3)\}$. It is abelian. Its subgroups are $\{e\}$, itself, $\{e,(1,3)(2,4)\}$, $\{e,(1,2)(3,4)\}$ and $\{e,(1,4)(2,3)\}$.

2. (a) $(e \cdot f)(x)ef(e^{-1}x) = f(x)$, $(a \cdot (b \cdot f))(x) = a(bf(b^{-1}(a^{-1}x))) = (ab)f((ab)^{-1}x)$.

   (b) We only need find an element f whose stablizer is a subgroup of S_3 with 3 elements. Pick such a subgroup $H = \langle (1,2) \rangle$, then we can pick f as, for example,

$$f(x) = \begin{cases} x & x \notin H \\ xa & x \in H \end{cases}$$

   where $a \in G \backslash H$.

3. (a) Two elements x and y are in the same conjugate class of H iff $y = hxh^{-1}$ for some $h \in H$, which implies that they are in the same conjugacy class of G.

   (b) Suppose $x, y \in H$ are in the same conjugacy class of G but not the same conjugacy class of H, then there is some $g \in G \backslash H$ such that $y = gxg^{-1}$. By orbit stablizer theorem we only need to show that the conjugacy action by H has the isomorphic stablizer. The isomorphism and its inverse are just conjugation by g and g^{-1}.

4. (a) The "if" part is similar to Problem 2(a) above. To show the "only if" part, consider f_1 as the function that sends e to 1 and every other element to 0. Let $a, b \in G$, $(ab)f$ sends $(ab)^{-1}$ to 1 and all else to 0, while $a(bf)$ sends $(ba)^{-1}$ to 1 and all else to 0. Hence $ab = ba$ and the group is abelian.

   (b) We can pick the function as $f_H(x) = 1$ if $x \in H$ and 0 if otherwise. Then, the stablizer equals

$$\{g \in G : gx \in H \text{ iff } x \in H\} = H$$