1. Let A be a set.

(a) Let S be a non empty set of equivalence relations on A. Show that $\bigcap S$ is an equivalence relation on A.

(b) Let R be a relation between A and A. Show that there is a unique equivalence relation on A, called $\sim_R$, such that any equivalence relation $\sim$ on A which contains R has $\sim_R$ as a subset ($R \subseteq \sim \implies \sim_R \subseteq \sim$).

1. (a). Pf. let $R = \bigcap S := \{ r \in \bigcup S \mid \forall s \in S . r \in s \}$

given $x . y \in A$

$(x.y) \in R$ iff $\forall r \in S . (x.y) \in r$

but by definition $\forall s \in S$. s is an equivalence relation on A.

1. symmetry. $(x.y) \in \bigcap S \Leftrightarrow \forall r \in S . (x.y) \in r$

$\Rightarrow \forall r \in S . (y.x) \in r$

$\Leftrightarrow (y.x) \in \bigcap S$

2. reflexity. $\forall r \in S , \forall x \in A . (x.x) \in r$

$\Rightarrow \forall x \in A . (x.x) \in \bigcap S$

3. transitivity. $(x.y) \in \bigcap S . (y.z) \in \bigcap S$

$\Leftrightarrow \forall r \in S . ((x.y) \in r \wedge (y.z) \in r)$

$\Rightarrow \forall r \in S . (x.z) \in r$

$\Leftrightarrow (x.z) \in \bigcap S$

$\square$

1. (b).

Pf. let $S_R = \{ \sim \subset A \times A \mid R \subset \sim$ and $\sim$ is an equivalence relation $\}$

claim. $\sim_R = \bigcap S_R$

consider the relation $A\times A$. It can be easily checked that this is an equivalence relation. since every pair of elements are equivalent.

by definition $A\times A \in S_R$.

so $S_R \neq \phi$.

Using the conclusion from 1(a). $\bigcap S_R$ is an equivalence relation on A

by definition $\bigcap S_R$ is unique.

since $\forall r \in S_R$. $R \subset r$, by definition $R \subset \bigcap S_R$.

we only need to show that for any equivalence relation $\sim$ containing R.

$\bigcap S_R \subset \sim$

this can be proved by definition of $\bigcap S_R$. that is:

$(x,y) \in \bigcap S_R \Leftrightarrow$ for any equivalence relation $\sim$ containing R. $(x,y) \in \sim$

so $\bigcap S_R \subset \sim$.

So our claim is true. that is, we can let $\sim_R = \bigcap S_R$ $\square$

2. Let A and B be two sets, $f : A \to B$ a function. Define function F : $P(B) \to P(A)$ as $F(C) = f^{-1}(C)$. Show that F is an injection iff f is a surjection, F is a surjection iff f is an injection.

2.

Pf. If f is a surjection, I'll show that F is an injection.

$\forall y \in B$. $\exists x \in A$. $f(x) = y$.

i.e. $\forall y \in B$. $f^{-1}(\{y\}) = F(\{y\}) \neq \phi$

Observe that by definition $\forall y_1, y_2$ with $y_1 \neq y_2$.

$$f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \{x \in A \mid f(x) = y_1\} \cap \{x \in A \mid f(x) = y_2\}$$
$$= \{x \in A \mid f(x) = y_1 \text{ and } f(x) = y_2\}$$
$$= \phi$$

If $C_1, C_2 \subset B$ with $C_1 \neq C_2$.

W.L.O.G. suppose $C_1 \setminus C_2 \neq \phi$

$$F(C_1) \setminus F(C_2) = \{x \in A \mid f(x) \in C_1\} \setminus \{x \in A \mid f(x) \in C_2\}$$
$$= \{x \in A \mid f(x) \in C_1 \setminus C_2\}$$
$$= F(C_1 \setminus C_2) \neq \phi$$

so $F(C_1) \neq F(C_2)$.

If. F is an injection, I'll show that f is a surjection.

suppose $\exists y \in B.\ \forall x \in A.\ f(x) \neq y$

then F is not even a function, let alone an injection.

contradiction

so f is a surjection $\Leftrightarrow$ F is an injection.

If. f is an injection. I'll show that F is a surjection.

$\forall D \subset A$. $f(D) \in P(B)$ satisfies that

$$F(f(D)) := \{x \in A \mid f(x) \in f(D)\} \supset D$$

since f is injective, $f^{-1}(f(D)) = D$. so $\forall D \in P(A). \exists f(D) \in P(B)$ s.t. $F(f(D)) = D$

so F is a surjection.

If F is a surjection, I'll show that f is an injection.

If f is not an injection. i.e. $\exists x_1, x_2 \in A$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

$$F^{-1}(\{\{x_1\}\}) := \{C \subset B \mid F(C) = \{x_1\}\} = \{C \subset B \mid f^{-1}(C) = \{x_1\}\} = \phi$$

because whatever $C \subset B$ is. as long as $f(x_1) = f(x_2)$

$f^{-1}(C)$ either contains x_1 and x_2. or contains neither.

Since $\nexists C \in P(B)$ s.t $F(C) = \{x_1\} \in P(A)$. F is not surjective

So F is a surjection iff f is an injection. $\square$

3. Show that $\sim = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Q}\}$ is an equivalence relation on $\mathbb{R}$.

3. Pf. reflexity: $\forall x \in \mathbb{R}$. $x - x = 0 \in \mathbb{Q}$

$\Rightarrow (x, x) \in \sim$

Symmetry: Given $x, y \in \mathbb{R}$. $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$

$\Leftrightarrow y - x \in \mathbb{Q} \Leftrightarrow y \sim x$

transitivity: Given $x, y, z \in \mathbb{R}$

$$x \sim y \wedge y \sim z \Leftrightarrow x-y \in \mathbb{Q} \wedge y-z \in \mathbb{Q}$$

$$\Rightarrow x-z = (x-y)+(y-z) \in \mathbb{Q}$$

$$\Leftrightarrow x \sim z$$

$\square$

4. Let A and B be two sets. $C = \{(x,i) \in (A \cup B) \times \{0,1\} : x \in A \text{ if } i = 0, x \in B \text{ if } i = 1\}$. Show that there are injections $k : A \to C$, $j : B \to C$, such that $C = k(A) \cup j(B)$ and $k(A) \cap j(B) = \emptyset$.

4. Pf. let $k : A \to C$ $\quad$ $j : B \to C$

$x \mapsto (x,0)$ $\quad$ $x \mapsto (x,1)$

$k(A) \cap j(B) = \phi$ is obvious.

$$C = \{(x,i) \in (A \cup B) \times \{0,1\} \mid x \in A \text{ if } i=0,\ x \in B \text{ if } i=1\}$$

$$= \{(x,0) \mid x \in A\} \cup \{(x,1) \mid x \in B\}$$

$$= k(A) \cup j(B).$$

$\square$

5. Let $f : A \to B$ be a function. Show that there is a set C, an injection $g : A \to C$, and a surjection $h : C \to B$, such that $f = h \circ g$. (Hint: You may want to use the solution for the previous problem).

5. Pf. let $\Delta = B \setminus f(A)$.

$$C := \left\{ (x,i) \in (A \cup \Delta) \times \{0,1\} \;\middle|\; \begin{matrix} x \in A \text{ if } i=0 \\ x \in \Delta \text{ if } i=1 \end{matrix} \right\}$$

define $g: A \to C$ $\qquad$ $h: C \to B$

$$x \mapsto (x, 0) \qquad (x, 0) \mapsto f(x)$$

$$(y, 1) \mapsto y$$

since $\forall y \in B$. either $\exists x \in A$ or not s.t. $f(x) = y$

so h is a surjection.

g is obviously injection.

$\square$