HW5

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Show that the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $(t, x) \mapsto xe^t$ is a left $(\mathbb{R}, +)$ action on $\mathbb{R}$. Write down the corresponding permutation representation. Is the action effective? Is the action transitive? Is the action free?

2. Let G be a group, X and Y be two left G sets.
   (a) Show that $\cdot : G \times (X \times Y) \to X \times Y$ defined as $(g, (x, y)) = (gx, gy)$ is a left G action on $X \times Y$.
   (b) If $f : X \to Y$ is a map, show that f is G equivariant if and only if $\{(x, f(x)) : x \in X\}$ is a G invariant subset of the left G set $(X \times Y, \cdot)$.

3. Let G be a group with more than one elements, X be a non empty left G set. Show that $\cdot : G \times P(X) \to P(X)$ defined as $(g, A) \mapsto \{ga : a \in A\}$ is a left G action on $P(A)$. Can this action be free? Can this action be transitive?

4. Any non empty set X is a left S_X set via the action $\cdot : S_X \times X \to X$, $(\sigma, x) \mapsto \sigma(x)$. Show that if X has 3 or more elements then the only S_X equivariant map from X to X is the identity.

5. Write down a group G, two left G sets X and Y, such that there are no G equivariant map from X to Y.

Answer:

1. $t \mapsto (x \mapsto xe^t)$. The kernel is $\{0\}$ so it is effective. The stablizer of 0 is $\mathbb{R}$ so it is not free. There are no real number t such that $e^t \times 1 = 0$ so it is not transitive.

2. (a) $e \cdot (x, y) = (ex, ey) = (x, y)$, $a \cdot (b \cdot (x, y)) = a \cdot (bx, by) = (abx, aby) = (ab) \cdot (x, y)$.
   (b) If f is G-equivariant, for any $(x, f(x))$, $g \cdot (x, f(x)) = (gx, gf(x)) = (gx, f(gx))$, so the set $\{(x, f(x) : x \in X\}$ is G-invariant. If the set $\{(x, f(x)) : x \in X\}$ is G invariant, for any $g \in G$, $x \in X$, $g \cdot (x, f(x)) = (gx, gf(x)) \in \{(x, f(x)) : x \in X\}$, hence $gf(x) = f(gx)$, which implies that f is G equivariant.

3. $e \cdot A = \{ea : a \in A\} = A$. $g \cdot (h \cdot A) = g \cdot \{ha : a \in A\} = \{gha : a \in A\} = (gh) \cdot A$. The action is neither free nor transitive, because the stablizer of $X \in P(X)$ is G, and no $g \in G$ has $g \cdot \emptyset = X$.

4. If there is such a S_X equivariant map f, let $x \in X$, we must have $(S_X)_x \subseteq (S_X)_{f(x)}$. We will now show that $f(x) = x$. Suppose not, let $a \in X$ be an element which is neither x nor $f(x)$, let $g \in S_X$ be the element that switches $f(x)$ and a while fixing every other element, then $g \in (S_X)_x \backslash (S_X)_{f(x)}$, a contradiction.

5. Let G be a group with more than one elements, X be a set with a single element x and the group action is trivial, Y be G with left action. Then if f is such an equivariant map, let $g \neq e$, then $f(x) = f(gx) = gf(x) \neq f(x)$, a contradiction.