HW4

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Let $\langle 12 \rangle$ be the subgroup of $(\mathbb{Z}, +)$ consisting of all integers divisible by 12. Let G be the quotient group $\mathbb{Z}/\langle 12 \rangle$. Find all the normal subgroups of G and count the number of elements of the corresponding quotient groups.

2. Let G be a group, N a normal subgroup, $p : G \to G/N$ a quotient map. Let S be the set of subgroups of G that contains N, S' be the set of subgroups of G/N.

   (a) Show that the map $F : S \to S'$ defined by $F(H) = p(H)$ is a bijection. (Hint: show that the map $H' \mapsto p^{-1}(H')$ is its inverse)

   (b) Show that $H \in S$ is a normal subgroup of G iff $p(H)$ is a normal subgroup of G/N.

   (c) If $H \in S$ and $H \trianglelefteq G$, show that there is an isomorphism from G/H to $(G/N)/p(H)$ defined as $aH \mapsto (aN)p(H)$ (Need to first show that it is well defined.)

   (This is usually called the Third Isomorphism Theorem.)

3. Let G be a group, $H \leq G$ a subgroup. Show that H is a normal subgroup of G if and only if the sets $\{gh : h \in H\}$ and $\{hg : h \in H\}$ are equal for all $g \in G$. (The set $\{hg : h \in H\}$ is often denoted as Hg and called a "right coset".)

4. Let G be a group, N a normal subgroup, H a subgroup.

   (a) Show that the set $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G.

   (b) Show that N is a normal subgroup of NH.

   (c) Show that $N \cap H$ is a normal subgroup of H.

   (d) Show that the map $f : (NH)/N \to H/(N \cap H)$ defined as $(nh)N \mapsto h(N \cap H)$ is a group isomorphism. (Need to first show that it is well defined.)

   (This is called the Second Isomorphism Theorem.)

Answer:

1. Because the group itself is abelian all subgroups are normal subgroups. To find the possible normal subgroups, start with $\{0 + \langle 12 \rangle$, add elements to it till it becomes G itself, and list all the groups one obtained via this process. Below are all the possibilities and the corresponding cardinality of the quotient groups:

$\{0 + \langle 12 \rangle\}$	12
G	1
$\langle 2 + \langle 12 \rangle \rangle$	2
$\langle 3 + \langle 12 \rangle \rangle$	3
$\langle 4 + \langle 12 \rangle \rangle$	4
$\langle 6 + \langle 12 \rangle \rangle$	6

2. (a) Let $F' : S' \to S$ be $F'(H') = p^{-1}(H')$. We will now show that F and F' are inverses of one another.

   i. For every $H \in S$, $F'(F(H)) = \{g \in G : p(g) = p(g') \text{ for some } g' \in H\} = \{g \in G : g = g'n \text{ for some } n \in H, g' \in H\}$. Because $N \leq H$, this set is just H.

   ii. For every $H' \in S'$, $F(F'(H')) = \{p(g) : p(g) \in H'\} = H' \cap p(G) = H'$.

   (b) If $H \in S$ is normal, then for every $g \in G$, $H = gHg^{-1}$. Apply p to elements on both sides, we get $p(H) = p(g)H(p(g))^{-1}$. Because p is a surjection, $p(H)$ is normal. On the other hand, if $H \in S$ is not normal, there is some $g \in G$ such that $gHg^{-1} \neq H$. However $N = gNg^{-1} \leq gHg^{-1}$, hence $gHg^{-1} \in S$. Now apply p to both the elements of H and of gHg^{-1}, from (a) we know that the resulting subgroups under p are also different, hence $p(H)$ is not normal in G/N.

   (c) Denote this map as ϕ.

   i. To show ϕ is well defined, if $aH = a'H$, then $a' = ah$ for some $h \in H$, hence $a'N = (aN)(hN)$. Because $hN \in p(H)$, we have $(a'N)p(H) = (aN)p(H)$.

   ii. To show that it is a group homomorphism, let $aH, bH \in G/H$, $\phi(aH)\phi(bH) = ((aN)p(H))((bN)p(H)) = ((aN)(bN))p(H) = ((ab)N)p(H) = \phi((ab)H) = \phi((aH)(bH))$.

   iii. ϕ is a surjection because every element of $(G/N)/p(H)$ can be written as $(aN)p(H)$ for some $a \in G$, hence equals $\phi(aH)$.

   iv. Now we show ϕ is an injection, which we can do by showing that the kernel is trivial. If $\phi(aH) = (eN)p(H)$, $aN \in (eN)p(H) = p(H)$, in other words $p(a) \in p(H)$. However by (a) above, $p^{-1}(p(H)) = H$, hence $a \in H$, $aH = eH$.

3. If H is normal, then any element of the form gh where $h \in H$ can be written as $(ghg^{-1})g$, so $gH \subseteq Hg$; any element of the form hg where $h \in H$ can be written as $g(g^{-1})h(g^{-1})^{-1}$ which is in gH, so $Hg \subseteq gH$.

If H is not normal, by HW3 Problem 3 there is some $g \in G$, $h \in H$ such that $ghg^{-1} \notin H$, hence $gh \in gH$ but $gh \notin Hg$.

4. (a) $e = ee \in NH$. For any $nh, n'h' \in NH$, $(nh)(n'h') = (n(hn'h^{-1}))(hh') \in NH$, $(nh)^{-1} = (h^{-1}n^{-1}h)h^{-1}$.

   (b) It is easy to see that N is a subgroup of NH. N being normal in G implies that $gNg^{-1} = N$ for all $g \in G$. Because $NH \leq G$, $gNg^{-1} = N$ for all $g \in NH$, which implies that N is a normal subgroup of NH.

   (c) For any $g \in H$, $n \in H \cap N$, $gng^{-1} \in H$ because $H \leq G$, $gng^{-1} \in N$ because N is normal, hence $gng^{-1} \in H \cap N$. Now apply HW3 Problem 3.

   (d) i. First we show that it is well defined. If $(nh)N = (n'h')N$, there is some $n'' \in N$ such that $n'h' = nhn''$, hence $h' = h(h^{-1}(n'^{-1}n)h)n''$. Here $(h^{-1}(n'^{-1}n)h)n''$ is in N because N is normal, and it is in H because it equals $h^{-1}h'$.

      ii. Next we show that it is a group homomorphism. For $(nh)N, (n'h')N \in NH/N$, $f(((nh)N)((n'h')N)) = f((nhn'h')N) = f(((n(hn'h^{-1}))(hh'))N) = (hh')(N \cap H) = (h(N \cap H))(h'(N \cap H)) = f((nh)N)f((n'h')N)$

      iii. Surjectivity is because $h(N \cap H) = f(hN)$.

      iv. To show injectivity, we only need to show the kernel is trivial. If $nh \in NH$, $n \in N$, $h \in H$ is sent to identity by f, then $h \in H \cap N$, hence $nh \in N$, $(nh)N = eN$.