HW2

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Let A be a set. $+_A : P(A) \times P(A) \to P(A)$ defined as $(B, C) \mapsto (B \cup C) \backslash (B \cap C)$. Then:
   (a) Show that $(P(A), +_A)$ is an abelian group.
   (b) Let $A' \subseteq A$, show that $B \mapsto B \cap A'$ is a homomorphism from $(P(A), +_A)$ to $(P(A'), +_{A'})$.
   (c) Let $F = \{B \in P(A) : B \text{ is finite or } A \backslash B \text{ is finite}\}$. Show that B is a subgroup of $(P(A), +_A)$.
2. Let G be a group, H_1, H_2 be two subgroups.
   (a) Show that $H_1 \cap H_2 \leq G$.
   (b) Show that $H_1 \cup H_2 \leq G$ iff $H_1 \leq H_2$ or $H_2 \leq H_1$.
   (c) Let G be the group of integers and the group operation is addition. Write down two subgroups whose union is no longer a subgroup.
3. Show that the set of $n \times n$ matrices with integer entries and determinant 1 form a group under matrix multiplication. (These groups are denoted as $SL(n, \mathbb{Z})$.
4. Let G be a group, show that G has only the identity element iff for any group H, $Hom(H, G)$ has exactly one element.
5. Show that for any group G, any $g \in G$, there is a unique group homomorphism from $(\mathbb{Z}, +)$ to G, sending 1 to g.
6. Let M be a set, $* : M \times M \to M$ be a function, such that for any $a, b, c \in M$, $*(a, *(b, c)) = *(*(a, b), c)$, $*(a, b) = *(b, a)$, and there is an element $e \in M$ such that for any $a \in M$, $*(e, a) = *(a, e) = a$. Let $\cdot : (M \times M) \times (M \times M) \to M \times M$ be $((a, b), (c, d)) \mapsto (*(a, c), *(b, d))$, $\sim$ a relation on $M \times M$ defined as $\sim = \{((a, b), (c, d)) \in (M \times M) \times (M \times M) :$ there exists $k \in M, *(*(a, d), k) = *(*(b, c), k)\}$
   (a) Show that $\sim$ is an equivalence relation.
   (b) Let $G = (M \times M)/\sim$. Show that $([a], [b]) \mapsto [\cdot(a, b)]$ is a function from $G \times G$ to G. Denote it as $\cdot'$.

(c) Show that $(G, \cdot')$ is an abelian group. This is called the Grothendieck group of $(M, *)$.

(d) Show that there is a bijective homomorphism from the Grothendieck group of $(\mathbb{Z}\backslash\{0\}, \times)$ to the group $(\mathbb{Q}, \times)$.