HW3

Please type or photograph your solution and turn it into a pdf before submitting it to Canvas. The first two problems will be graded for correctness.

1. Recall that by S_n we mean the permutation group of $\{1, 2, \dots, n\}$.

   (a) Find all the automorphisms of S_2.

   (b) Find all the automorphisms of S_3.

   Hint: If $f : G \to G$ is a group isomorphism, $g \in G$, then $g^n = e$ iff $f(g)^n = e$, because $f(g)^n = f(g^n)$ and f is a bijection that sends the identity e to itself.

2. Let G be a group, $f : G \to G$ a function, and $\sim$ an equivalence relation on G. Let $G \times G$ be the direct product of G with itself, i. e. with group operation defined as $((a, b), (c, d)) \mapsto (ac, bd)$

   (a) Show that $G_f = \{(g, f(g)) : g \in G\}$ is a subgroup of $G \times G$ iff f is a group homomorphism.

   (b) Show that $G_\sim = \{(a, b) \in G \times G : a \sim b\}$ is a subgroup of $G \times G$ iff there is a normal subgroup H of G, such that $\sim = \{(a, b) \in G \times G : b^{-1}a \in H\}$.

3. Let G be a group, S a subset of G. For every $g \in G$, define S^g as $S^g = \{gsg^{-1} : s \in S\}$. Suppose for every $g \in G$, $S^g \subseteq S$, show that for every $g \in G$, $S^g = S$.

4. Let G be a group, S a subset of G. Let H_S be a subset of G consisting of identity e together with all elements of the form $s_1 s_2 \dots s_n$, where each s_j is either in S or its inverse is in S. Show that H_S is a subgroup of G, and any subgroup of G containing all elements in S must have H_S as a subgroup, i. e. $H_S = \langle S \rangle$

5. Recall that if group G satisfies $G = \langle S \rangle$, we say S is a generating set of G. Let $n > 2$ be an integer.

   (a) Let S be a finite subset of $(\mathbb{Q}, +)$, show that $\langle S \rangle \neq \mathbb{Q}$.

   (b) Show that S_n, which is the group of bijections from $\{1, \dots, n\}$ to itself, with group operation being the composition, has a generating set with no more than $n - 1$ elements.

   (c) Write down a generating set of S_n with only two elements.