ASSIGNMENT 3

Due: 3 October, 11:59pm

(1) Let $A, B \subset \mathbb{R}^1$ be two bounded sets such that $x \leqslant y$ for any $x \in A$ and $y \in B$. Prove that $\sup A \leqslant \inf B$.

(2) Prove the following inequality (called Triangle Inequality) by induction: for any positive integer $n \in \mathbb{Z}_+$ and any real number s $a_1, a_2, \ldots, a_n$,

$$\left| \sum_{i=1}^{m} a_i \right| \leqslant \sum_{i=1}^{m} |a_i| .$$

(3) Let $f : [0,1] \to \mathbb{R}$ be a real-valued function. Assume that there exists some number $Q > 0$ such that for any choice of a finite number of points $p_1, \ldots, p_n$ in $[0,1]$,

$$\left| \sum_{i=1}^{n} f(x_i) \right| \leqslant Q.$$

Prove the following:

(a) We define $\Sigma \equiv \{x \in [0,1] | f(x) \neq 0\}$. Show that

$$\Sigma = \bigcup_{m=1}^{\infty} \Sigma_m, \quad \text{where } \Sigma_m \equiv \left\{ x \in [0,1] \middle| |f(x)| \geqslant \frac{1}{m} \right\}.$$

(b) Σ is countable.

(4) Determine all the limit points of the following sets and decide whether the sets are open or closed or neither (justify your answers):

(a) $A_1 = (a, b]$;

(b) $A_2 = \{1/n^2 | n \in \mathbb{Z}_+\}$;

(c) $A_3 = \{2^{-n} + 3^{-m} | m, n \in \mathbb{Z}_+\}$.

(5) Let $A, B \subseteq \mathbb{R}^n$ be two subsets. Show that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$