(1) Let $A, B \subset \mathbb{R}^1$ be two bounded sets such that $x \leqslant y$ for any $x \in A$ and $y \in B$. Prove that $\sup A \leqslant \inf B$.

Pf. Claim. $\forall y \in B$. $\sup A \leq y$

Suppose not. $\exists y_0 \in B$. $y_0 < \sup A$

since $\forall x \in A$. $x \leqslant y_0$ y_0 is an upper bound of A

so this contradicts with the definition of $\sup A$.

So $\forall y \in B$. $\sup A \leq y$. By definition $\sup A$ is a lower bound of B

By definition $\inf B \geqslant l$ where l is any lower bound of B.

so $\sup A \leq \inf B$. $\square$

(2) Prove the following inequality (called Triangle Inequality) by induction: for any positive integer $n \in \mathbb{Z}_+$ and any real number s $a_1, a_2, \ldots, a_n$,

$$\left|\sum_{i=1}^{m} a_i\right| \leqslant \sum_{i=1}^{m} |a_i| .$$

Pf. I'll prove by induction on m.

When $m = 1$. $LHS = RHS = a_1$.

Suppose the inequality holds when $m = k \in \mathbb{Z}_+$

i.e. $\left|\sum_{i=1}^{k} a_i\right| \leq \sum_{i=1}^{k} |a_i|$

then $\left|\sum_{i=1}^{k+1} a_i\right| = \left|\left(\sum_{i=1}^{k} a_i\right) + a_{k+1}\right| \leqslant \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}|$

$$\leqslant \left(\sum_{i=1}^{k}|a_i|\right)+|a_{k+1}| = \sum_{i=1}^{k+1}|a_i|.$$

So the inequality holds for any $m \in \mathbb{N}$. $\square$

(3) Let $f:[0,1]\to\mathbb{R}$ be a real-valued function. Assume that there exists some number $Q>0$ such that for any choice of a finite number of points $p_1,\ldots,p_n$ in $[0,1]$,

$$\left|\sum_{i=1}^{n} f(p_i)\right| \leqslant Q.$$

Prove the following:

(a) We define $\Sigma \equiv \{x\in[0,1] | f(x)\neq 0\}$. Show that

$$\Sigma = \bigcup_{m=1}^{\infty}\Sigma_m, \quad \text{where } \Sigma_m \equiv \left\{x\in[0,1]\middle| |f(x)| \geqslant \frac{1}{m}\right\}.$$

(b) Σ is countable.

(a). Pf. $f(x)\neq 0 \iff |f(x)|>0$

suppose $|f(x)|=\varepsilon>0$. let $m=\lceil\frac{1}{\varepsilon}\rceil+1. >0$

by definition $0<\frac{1}{m}\leqslant|f(x)|$

i.e. $x\in\Sigma_m$.

So $\forall y\in f([0,1])$ with $y\neq 0$. $\exists m\in\mathbb{N}$. s.t. $x\in\Sigma_m$ and $y=f(x)$

so $\Sigma \subset \bigcup_{m=1}^{\infty}\Sigma_m$. (I)

From the other direction.

$\forall m\in\mathbb{N}$. $\Sigma_m\subset\Sigma$ by definition

so. $\bigcup_{m=1}^{\infty}\Sigma_m\subset\Sigma$. (II)

By (I) and (II). $\Sigma = \bigcup_{m=1}^{\infty} \Sigma_m$. $\square$

(b). Pf. Claim. $\forall m \in \mathbb{N}$. Σ_m is finite.

if $\exists m_0 \in \mathbb{N}$. Σ_{m_0} is infinite.

$\Sigma_{m_0} = \Sigma_{m_0}^{+} \sqcup \Sigma_{m_0}^{-}$ where $\Sigma_{m_0}^{+} := \Sigma_{m_0} \cap f^{-1}(\mathbb{R}_{>0})$

$\Sigma_{m_0}^{-} := \Sigma_{m_0} \cap f^{-1}(\mathbb{R}_{<0})$

so at least one of $\Sigma_{m_0}^{+}$ and $\Sigma_{m_0}^{-}$ should be infinite.

W.L.O.G. suppose $\Sigma_{m_0}^{+}$ is infinite.

pick $N = \lceil m_0 Q \rceil + 1$. elements from $\Sigma_{m_0}^{+}$. denoted as $\{p_i\}_{i=1}^{N}$

$$\left| \sum_{i=1}^{N} f(p_i) \right| = \sum_{i=1}^{N} f(p_i) \geqslant \frac{\lceil m_0 Q \rceil + 1}{m_0} > Q \quad \text{contradiction.}$$

so our claim is true.

Countable union of finite sets is countable.

so $\Sigma := \bigcup_{m=1}^{\infty} \Sigma_m$ is countable. $\square$

(4) Determine all the limit points of the following sets and decide whether the sets are open or closed or neither (justify your answers):

(a) $A_1 = (a, b]$;

(b) $A_2 = \{1/n^2 | n \in \mathbb{Z}_+\}$;

(c) $A_3 = \{2^{-n} + 3^{-m} | m, n \in \mathbb{Z}_+\}$.

(a) $A_1' = [a, b]$. A_1 is not open. not closed.

Pf. a is a limit point because

$$\forall r>0 \quad (B_r(a)\setminus\{a\})\cap A_1 = (a, a+r) \neq \phi$$

b is a limit point because

$$\forall r>0. \quad (B_r(b)\setminus\{b\})\cap A_1 = (b-r, b] \neq \phi$$

$\forall x_0 \in (a, b)$, x_0 is a limit point because.

$\forall r>0$. let $r_0 = \min\{x_0 - a, b - x_0\}$

$$(B_r(x_0)\setminus\{x_0\})\cap A_1 \supset B_{r_0}(x_0)\setminus\{x_0\} \neq \phi$$

If $x_0 < a$. $x_0 \notin A_1'$ because let $r = \frac{1}{2}(a - x_0)$. $(B_r(x_0)\setminus\{x_0\})\cap A_1 = \phi$

so $A_1' = [a, b]$.) similarly if $x_0 > b$, $x_0 \notin A_1'$

$A_1' \neq A_1$ so A_1 is not closed

No ball containing b can be a subset of A_1. so A_1 is not open.

$\square$

(b) $A_2' = \{0\}$ A_2 is not open. not closed.

Pf. 0 is a limit point because

$$\forall r>0 \quad (B_r(0)\setminus\{0\})\cap A_2 = \left\{\frac{1}{n^2} \,\middle|\, n \geq n_0.\ n \in \mathbb{N}\right\} \text{ where } n_0 = \left\lceil \frac{1}{\sqrt{r}} \right\rceil.$$
$$= \phi$$

If $x_0 < 0$ then $x_0 \notin A_2'$. because

$$B_{-\frac{x_0}{2}}(x_0) \subset \mathbb{R}_{<0} \quad \left(B_{-\frac{x_0}{2}}(x_0) \setminus \{x_0\}\right) \cap A_2 = \phi$$

If $x_0 = \frac{1}{n^2}$ for some $n \in \mathbb{N}$ then $x_0 \notin A_2'$. because.

let $r_0 = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$$\left(B_{r_0}(x_0) \setminus \{x_0\}\right) \cap A_2 = \phi$$

If $x_0 > 0$ and $x_0 \notin A_2$. then $x_0 \notin A_2'$ because.

$\exists n_0 \in \mathbb{N}$. $\frac{1}{(n_0+1)^2} < x_0 < \frac{1}{n_0^2}$

let $r_0 = \min\left\{\frac{1}{n_0^2} - x_0,\ x_0 - \frac{1}{(n_0+1)^2}\right\}$

$$\left(B_{r_0}(x_0) \setminus \{x_0\}\right) \cap A_2 = \phi$$

so $A_2' = \{0\}$

Since $A_2' \not\subset A_2$. so A_2 is not closed.

$\forall r > 0$. $B_r(1) \cap A_2^c \supset (1, 1+r) \neq \phi$. so A_2 is not open.

$\square$

(c) $A_3' = \left\{\frac{1}{2^n} \,\middle|\, n \in \mathbb{Z}_+\right\} \cup \left\{\frac{1}{3^n} \,\middle|\, n \in \mathbb{Z}_+\right\} \cup \{0\}$

A_3 is not closed. not open.

pf. 0 is a limit point because. $\forall r>0$

let $m = \lceil \log_2 \frac{2}{r} \rceil + 1$, $n = \lceil \log_3 \frac{2}{r} \rceil + 1$. $0 < 2^{-m} + 3^{-n} < \frac{r}{2} + \frac{r}{2} = r$

If $x_0 \in \{ \frac{1}{2^n} \mid n \in \mathbb{Z}_+ \}$

$\forall r>0$. let $m_0 = \log_3 \lceil \frac{1}{r} \rceil + 1$

$(B_r(x_0) \setminus \{x_0\}) \cap A_3 \supset \{ x_0 + 3^{-n} \mid n \geq m_0 \} \neq \phi$

so any element in $\{ 2^{-n} \mid n \in \mathbb{Z}_+ \}$ would be a limit point.

Similarly any element in $\{ 3^{-n} \mid n \in \mathbb{Z}_+ \}$ is a limit point.

If $x_0 > \frac{5}{6}$. then $\forall x \in A_3 : x < x_0$

take $r_0 = x_0 - \frac{5}{6}$ $(B_{r_0}(x_0) \setminus \{x_0\}) \cap A_3 = \phi$.

Similarly if $x_0 < 0$ x_0 cannot be a limit point.

If $0 < x_0 < \frac{5}{6}$. and $x_0 \notin \{ 2^{-n} \mid n \in \mathbb{Z}_+ \} \cup \{ 3^{-n} \mid n \in \mathbb{Z}_+ \}$.

pick $r>0$ that is sufficiently small s.t. $B_r(x_0) \subset A_3$.

find $m_0, n_0 \in \mathbb{Z}_+$ s.t. $2^{-m_0} < x_0 < 2^{-m_0+1}$

$$3^{-n_0} < x_0 < 3^{-n_0+1}$$

let $\Delta = \frac{1}{2} \cdot \min \{ x_0 - 2^{-m_0}, x_0 - 3^{-n_0}, 3^{-n_0+1} - x_0, 2^{-m_0+1} - x_0 \}$

let $r = \Delta$.

Claim. $\left|B_r(x_0) \cap A_3\right| < \infty$.

if $(m,n) \in \mathbb{Z}_+^2$. $2^{-m} + 3^{-n} \in B_r(x_0)$.

by definition $m \geq m_0$

and $\forall m$. $n \leq \left\lceil \log_3\left(\frac{1}{x_0 - 2^{-m}}\right) \right\rceil$ (I)

If m is large enough s.t. $2^{-m} + 3^{-n_0} < x_0 - r$

i.e. $m > \left\lceil \log_2\left(\frac{1}{x_0 - r - 3^{-n_0}}\right) \right\rceil$ then $\nexists n$ s.t. $2^{-m} + 3^{-n} \in B_r(x_0)$

so m is bounded. and for each m. only finite $n \in \mathbb{Z}_+$ satisfies that $2^{-m} + 3^{-n} \in B_r(x_0)$

so $(B_r(x_0) \setminus \{x_0\}) \cap A_3$ is finite.

$A_3' \not\subset A_3$ so A_3 is not closed

let $r_0 = \frac{1}{2}\min\{|x - x_0|\}$ where $x \in \left(B_{\frac{\Delta}{2}}(x_0) \setminus \{x_0\}\right) \cap A_3$.

then $B_{r_0}(x_0) \cap A_3^c = B_{r_0}(x_0) \setminus \{x_0\} \neq \phi$ so A_3 is not open

$\square$

(5) Let $A, B \subseteq \mathbb{R}^n$ be two subsets. Show that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Pf. $\overline{A \cap B} = (A \cap B) \cup (A \cap B)'$

$\overline{A} \cap \overline{B} = (A \cap B) \cup (A' \cap B')$

we only need to show that $(A \cap B)' \subset A' \cap B'$.

$x \in (A \cap B)'$

$\Leftrightarrow \forall r > 0.\ \exists p \in (A \cap B) \setminus \{x\}.\ \|x - p\| < r$

since $p \in A \cap B$. we have $p \in A$ and $p \in B$

$\Rightarrow \forall r > 0.\ \exists p \in A \setminus \{x\}\ \|x - p\| < r$

and $p \in B \setminus \{x\}\ \|x - p\| < r$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

so $(A \cap B)' \subset A' \cap B'$

so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ $\square$