Pullback Property: Quotient Topologies to Categories

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1 Introduction

The most elegant theorem I've learnt in this course is the pullback property of quotient topologies. While I've been learning Abstract Algebra this semester, I found some generalized facts about this property in category theory. The four problems and sections corresponding answers of this homework are as follows:

- 1. Prove the theorem. (Section 2.1)
- 2. In your opinion, what is the most crucial assumption of the theorem? Does the theorem still hold if you remove this assumption? If not, can you construct an example? (Section 2.2)
- 3. Describe the importance of the theorem. (Section 2.3, Section 3)
- 4. Provide a typical example to exhibit the application of theorem. (Section 4)

2 Pullback Property

2.1 Pullbacks in Quotient Topologies

Definition. Quotient map Let X and Y be topological spaces. A map $f:X\to Y$ is said to be a *quotient map* if for any set $U\subset Y$, U is open in Y iff $f^{-1}(U)$ is open in X.

By the way a quotient map is defined, it's obviously continuous. It's sometimes called *strongly continuous* for this reason.

Theorem (Pullback Property). Let $\pi:X\to Y$ be a quotient map. Let Z be an arbitrary topological space. Then there is a bijection between

- $\mathcal{H} := \{ h \in \mathsf{Map}(X, Z) \mid h \text{ is constant on every } \pi^{-1}(\{y\}), y \in Y \}$, and;
- ullet $\mathscr{F}:=\operatorname{Map}(Y,Z)$,

which can be realized by the pullback

$$\pi^*: \mathcal{F} \to \mathcal{H}$$

$$f \mapsto f \circ \pi$$

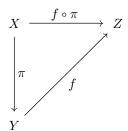
Moreover, the following holds:

- 1. $f \in \mathcal{F}$ is continuous if and only if $h = \pi^*(f)$ in \mathscr{H} is continuous.
- 2. $f \in \mathcal{F}$ is a quotient map if and only if $h = \pi^*(f)$ in \mathcal{H} is a quotient map.

Proof. For each $y\in Y$, $h(\pi^{-1}(\{y\}))$ is a singleton in Z, denoted the element in the singleton by f(y), we have acturally defined a map implicitly by $f\circ\pi=h$. Since the choice of h is arbitrary, π^* is surjective. Suppose $\pi^*(f_1)=f_1\circ\pi=h_1=h_2=f_2\circ\pi=\pi^*(f_2)$, let $x_0\in\pi^{-1}(y_0)$ for any $y\in Y$, then $f_1(y)=f_1\circ\pi(x_0)=h_1(x_0)=h_2(x_0)=f_2(y)$, so $f_1=f_2$, i.e. π^* is injective.

Let U be any open set in Z, $h^{-1}(U)=(f\circ\pi)^{-1}(U)=\pi^{-1}\circ f^{-1}(U)$. Since π is a quotient map, $h^{-1}(U)$ is open in X if and only if $f^{-1}(U)$ is open in Y. Therefore $f\circ\pi$ is continuous if and only if f is continuous.

In fact, the Pullback Property could be described using the following commutative diagram.



2.2 Analysis of the Crucial Assumption

The most crucial assumption in the Pullback Property theorem is that $\pi:X\to Y$ must be a **quotient map**, not just any continuous surjection. This assumption is essential because:

- 1. It ensures that the topology on Y is the finest topology making π continuous (the quotient topology)
- 2. It provides the necessary and sufficient conditions for lifting continuity through the pullback

Proposition. The theorem fails if we only assume π is a continuous surjection.

Proof. We construct a counterexample:

Let $X=\mathbb{R}$ with the usual topology, and let $Y=\mathbb{R}$ with the indiscrete topology. Define $\pi:X\to Y$ as the identity function. Then:

- 1. π is continuous (since every set in the indiscrete topology is open)
- 2. π is surjective (as the identity map)
- 3. However, π is not a quotient map because the preimage of any non-empty proper subset of Y is open in X, but no such subset is open in Y

Now let $Z=\mathbb{R}$ with the usual topology, and consider:

- $h: X \to Z$ defined by h(x) = x (the identity function)
- $f: Y \to Z$ defined by f(y) = y (also the identity function)

Then:

- 1. $h = f \circ \pi$ (so h is constant on the fibers of π)
- 2. h is continuous (identity map between spaces with usual topology)
- 3. But f is not continuous (identity map from indiscrete to usual topology)

This violates the conclusion of the theorem that h is continuous if and only if f is continuous. Therefore, the quotient map assumption cannot be weakened to just continuous surjection.

Remark. This counterexample illustrates the universal property of quotient maps: they are precisely the maps that allow us to "lift" continuity through the pullback construction. Without this property, we lose the equivalence of continuity between the original and pulled-back functions.

2.3 Pullbacks in Categories

Definition. (Category) A category C consists of:

- 1. A collection of objects
- 2. A collection of morphisms (or arrows) between objects
- 3. A composition operation for morphisms that is associative
- 4. An identity morphism for each object

The category Top of topological spaces consists of:

- 1. Objects: Topological spaces
- 2. Morphisms: Continuous functions
- 3. Composition: Usual function composition
- 4. Identity: Identity function on each space

Let's understand how pullbacks generalize in categorical settings:

Definition. (Universal Property) In any category C, a pullback of arrows $f:A\to C$ and $g:B\to C$ consists of:

- 1. Arrows $p_1:P\to A$ and $p_2:P\to B$ such that $fp_1=gp_2$
- 2. For any object Z with maps $z_1:Z\to A$ and $z_2:Z\to B$ satisfying $fz_1=gz_2$, there exists a unique $u:Z\to P$ making all diagrams commute.

3 Importance of the Theorem

The importance of the pullback property lies in its fundamental role in connecting different mathematical structures. Let's explore this in detail:

3.1 Universal Nature of Quotient Maps

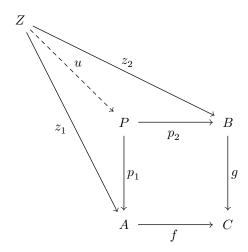
Theorem. The pullback property provides a universal characterization of quotient maps in the category Top.

This universality manifests in several ways:

- 1. Factor Space Construction: Given any continuous map $f:X\to Z$ that is constant on the fibers of a quotient map $\pi:X\to Y$, there exists a unique continuous map $\tilde{f}:Y\to Z$ making the diagram commute.
- 2. **Preservation of Properties**: The pullback property shows that certain topological properties are preserved under quotient maps, making it a powerful tool for studying topological spaces through their quotients.

3.2 Categorical Perspective

Definition. (Pullback) In any category C, a pullback of arrows $f:A\to C$ and $g:B\to C$ consists of arrows $p_1:P\to A$ and $p_2:P\to B$ such that $fp_1=gp_2$, and universal with this property. i.e. Given any $z_1:Z\to A$ and $z_2:Z\to B$ with $fz_1=gz_2$, there exists a unique $u:Z\to P$ such that $z_1=p_1u$ and $z_2=p_2u$.



The pullback construction provides several key insights:

- 1. **Universality**: The pullback property is universal in the sense that it characterizes the quotient map up to unique isomorphism.
- 2. **Functoriality**: The pullback operation defines a functor between appropriate categories, preserving the structural relationships between spaces.
- 3. **Naturality**: The construction is natural in the sense of category theory, meaning it commutes with the relevant morphisms in a functorial way.

4 Applications of the Theorem

The pullback property finds applications across various areas of mathematics:

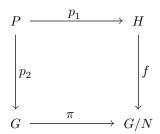
4.1 Topological Applications

Example 1. (Quotient Spaces) Let $X=\mathbb{R}^2$ and $Y=\mathbb{R}$ with the quotient map $\pi:\mathbb{R}^2\to\mathbb{R}$ being the projection onto the first coordinate. For any continuous function $f:\mathbb{R}\to\mathbb{R}$, the pullback $\pi^*(f)$ gives us the function h(x,y)=f(x), which is constant on vertical lines (the fibers of π).

Example 2. (Identification Spaces) Consider the torus T^2 as a quotient of the square $[0,1]\times[0,1]$. The pullback property allows us to characterize continuous functions on the torus in terms of periodic functions on the square.

4.2 Group Theory Applications

Example 3. (Normal Subgroups) Let G be a group and $N \unlhd G$ a normal subgroup. The quotient map $\pi: G \to G/N$ and any homomorphism $f: H \to G/N$ give rise to a pullback diagram:



The pullback P is isomorphic to the fiber product $H \times_{\frac{G}{N}} G$.

Bibliography

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