ASSIGNMENT 4

Due: 10 October, 11:59pm

(1) We define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{3x^2+y^2}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Use the definition of continuous function (in terms of "$\epsilon - \delta$") to show that f is discontinuous at $(0,0)$.

(2) Let $\{x_j\}_{j=1}^{\infty}$ be an infinite sequence in $\mathbb{R}^n$. If the sequence is bounded, show that it has a converging subsequence. (Hint: applying Bolzano-Weierstraß Theorem)

(3) Consider two sequences $L = \{a_j\}_{j=1}^{\infty}$ and $R = \{b_j\}_{j=1}^{\infty}$ in $\mathbb{R}^1$. If

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{j-1} \leqslant a_j \leqslant \ldots \leqslant b_j \leqslant b_{j-1} \leqslant \ldots \leqslant b_1,$$

and

$$\lim_{j \to \infty} |b_j - a_j| = 0,$$

then show that

$$\sup L = \inf R.$$

(4) **Textbook Chapter 1.4.** Exercise 9(b): if $x \in \mathbb{R} \setminus \mathbb{Z}$, then prove that there exists a **unique** integer $n_0 \in \mathbb{Z}$ such that

$$n_0 < x < n_0 + 1.$$

(5) Use (4) to prove that for any real number $x \in \mathbb{R}$ and for any $\epsilon > 0$, there exists a rational number q such that

$$q < x < q + \epsilon.$$

(6) Let us recall the definition of "subspace topology" for subsets of $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be a nonempty subset. We call a subset $W \subseteq U$ **an open set in** U (or an open set with respect to U) if there exists an open set $O \subset \mathbb{R}^n$ such that $W = O \cap U$.

(a) Consider the set $S \equiv \{\frac{1}{n} : n \in \mathbb{Z}_+\}$. Let us regard S as a subspace of $\mathbb{R}$. Show that every *singleton* (a set with exactly one element) of S is an open set in S.

(b) Consider the set $\mathbb{Q}$ of all rational numbers. If $\mathbb{Q}$ is regarded as a subspace of $\mathbb{R}$, show that no *singleton* of $\mathbb{Q}$ is an open set in $\mathbb{Q}$.