EXTRA ASSIGNMENT 1

Due: 30 September, 11:59pm

Problem 1 (Ternary expansion). To formulate this problem, we first assume some basic knowledge in mathematical analysis:

- (Convergence of geometric series) If $q \in (0,1)$, then geometric series $\sum_{n=0}^{\infty} q^n$ converges to $\frac{1}{1-q}$.
- (Comparison principle) Let $a_n \geqslant 0$ and $b_n \geqslant 0$ satisfy $b_n \geqslant a_n$ for any $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges to $B \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n$ converges to some real number A. Moreover, $A \leqslant B$.

Let us formulate the ternary expansions for all real numbers in $[0,1]$ as follows: for any $x \in [0,1]$, we write

$$[x]_3 = 0.d_1d_2d_3\ldots \equiv \sum_{n=1}^{\infty} \frac{d_n}{3^n}, \quad d_n \in \{0,1,2\}.$$

For example, $\frac{1}{3}$ has two different ternary expansions:

$$\left[\frac{1}{3}\right]_3 = 0.100000\ldots = 0.0222222\ldots.$$

Now prove the following property. If $x \in [0,1]$ has two *distinct* ternary expansions

$$[x]_3 = 0.d_1d_2\ldots d_n\ldots = 0.e_1e_2\ldots e_n\ldots,$$

then the following holds. Let $n \equiv \min\{k \in \mathbb{Z}_+ : d_k \neq e_k\}$. Then $e_n = d_n + 1$ and

$$d_k = 2, \quad e_k = 0, \quad \forall\, k \geqslant n+1.$$

Problem 2. Let us construct an infinite subset $C \subset [0,1]$ in the following inductive process.

$$\begin{aligned}
F_0 &= [0,1],\\
F_1 &= \underbrace{\left[0,\frac{1}{3}\right]}_{I_1^{(1)}} \cup \underbrace{\left[\frac{2}{3},1\right]}_{J_1^{(1)}},\\
F_2 &= \underbrace{\left[0,\frac{1}{9}\right]}_{I_1^{(2)}} \cup \underbrace{\left[\frac{2}{9},\frac{1}{3}\right]}_{J_1^{(2)}} \cup \underbrace{\left[\frac{2}{3},\frac{7}{9}\right]}_{I_2^{(2)}} \cup \underbrace{\left[\frac{8}{9},1\right]}_{J_2^{(2)}},\\
&\vdots\\
F &= \bigcap_{n=0}^{\infty} F_n.
\end{aligned}$$

That is, in the nth step, there are 2^n-intervals $I_k^{(n)}$ and $J_k^{(n)}$ in the set F_n. The union $I_k^- \cup I_k^+$ comes from deleting an open interval that contributes *the central* $1/3$ in its predecessor. We will prove that F is uncountable by achieve the following steps.

(1) For any $n \in \mathbb{Z}_+$, F_n is identical to the following set F_n' of ternary decimals

$$\{0.d_1d_2d_3d_4\ldots | d_j \in \{0,2\}\ \forall 1 \leqslant j \leqslant n\}.$$

Also prove that any element $x \in F_n$ has a unique ternary expansion in F_n'.

(2) We take a countable subset $G = \{x^1, x^2, x^3 \ldots\} \subseteq F$ and write them in the ternary expansion as described above,

$$\begin{aligned}
x^1 &= 0.d_1^1d_2^1d_3^1d_4^1\ldots\\
x^2 &= 0.d_1^2d_2^2d_3^2d_4^2\ldots\\
x^3 &= 0.d_1^3d_2^3d_3^3d_4^3\ldots\\
x^4 &= 0.d_1^4d_2^4d_3^4d_4^4\ldots\\
&\vdots,
\end{aligned}$$

where $d_i^j \in \{0,2\}$ for any $i,j \in \mathbb{Z}_+$. We define an element $p \in F$ with a ternary expansion $[p]_3 = 0.p_1p_2p_3p_4\ldots$ such that

$$p_j = \begin{cases} 0 & \text{if } d_j^j = 2,\\ 2 & \text{if } d_j^j = 0.\end{cases}$$

Prove that $p \notin G$.

(3) Based on the previous steps, prove that the set F is uncountable.

Problem 3. This problem is to prove **Cantor's Theorem**: *Given any set A, denote by $\mathscr{P}(A)$ the power set of A. Then there does not exist a surjective function $f : A \to \mathscr{P}(A)$.*

(1) First, consider a simpler case of Cantor's Theorem. Let $D = \{1, 2, 3, 4\}$. Then construct an injective function $f : D \to \mathscr{P}(D)$. For the function f you just constructed, write down all the elements of the set

$$B \equiv \{x \in D | x \notin f(x)\}.$$

(2) Show that there exists no surjective function $f : D \to \mathscr{P}(D)$ for any finite set.

(3) Using the constructive strategy in (1), prove Cantor's Theorem in full generality.