NOTES ON ELEMENTARY TOPOLOGY

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1. Elementary set theory and logic

1.1. **Fundamental concepts.**

Naive set theory v.s. Zermelo-Frænkel's formalized set theory (ZFC)

1.1.1. *Elementary operations in set theory.* A set is usually written as a capital letter A, and it can be represented as follows

$$A = \{x|P(x)\}, \tag{1.1}$$

where x is an element (or member) in A and $P(x)$ is the property that x satisfies. We start with basic notion in set theory.

- $a \in A$: a belongs to A.
- $a \notin A$: a does not belong to A.
- $A = B$ if and only if they contain the same elements.
- $A \subseteq B$: A is a subset of B.
- $A \subsetneq B$: A is a proper subset of B.

Example 1.1. *There are some sets of numbers:*
$\mathbb{Z}$*: the set of all integers;*
$\mathbb{Z}_+$ *or* $\mathbb{N}$*: the set of all positive integers;*
$\mathbb{N}_0$*: the set of nonnegative integers;*
$\mathbb{Q}$*: the set of all rational numbers;*
$\mathbb{R}$*: the set of all real numbers;*

Next, we introduce some elementary operations of sets.

Definition 1.1 (Union and intersection)**.** *Let A and B be two sets. Then we define*

$$A \cup B \equiv \{x|x \in A \text{ or } x \in B\}, \tag{1.2}$$

$$A \cap B \equiv \{x|x \in A \text{ and } x \in B\}. \tag{1.3}$$

In the context of set theory, "x is in A or B" means that x belongs to at least one of the two sets A and B. In particular, there are three possibilities:
(1) $x \in A$ and $x \notin B$,
(2) $x \in B$ and $x \notin A$,
(3) $x \in A$ and $x \in B$.

Definition 1.2 (Difference and complement)**.** *Let A and B be two sets. Then we define*

$$A - B = A \setminus B \equiv \{x|x \in A \text{ and } x \notin B\}. \tag{1.4}$$

When $B \subseteq A$, we denote by $B^c \equiv A - B$ the complement of B in A.

In the following proposition, we collect some formulae of operations in set theory.

Lemma 1.1. *Let A, B, C be sets. The the following holds:*

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(2) The following formulae are called DeMorgan's laws:

$$A - (B \cup C) = (A - B) \cap (A - C), \tag{1.5}$$

$$A - (B \cap C) = (A - B) \cup (A - C). \tag{1.6}$$

(3) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(4) $\emptyset \subseteq A$ for any set A, and $\emptyset \in \{\emptyset\}$.

Next, we defined the cartesian product for two sets.

Definition 1.3 (Cartesian product)**.** *Let A and B be two sets. Their cartesian product $A \times B$ is defined as the following set of ordered pairs,*

$$A \times B \equiv \{(x, y) | x \in A, y \in B\}. \tag{1.7}$$

For two ordered pairs (x, y) and (x', y'),

$$(x, y) = (x', y') \text{ if and only if } x = x', y = y'. \tag{1.8}$$

1.1.2. *Logic calculus and set theory.* The operations introduced above can be translated to logic terms. In mathematical logic, a **predicate** is a symbol that represents a property or a relation. The notation $P(x)$ stands for a logic formula which means the symbol (or predicate) P applies to the element x.

Let $A = \{x|P(x)\}$ and $B = \{x|Q(x)\}$. Then we have the following correspondence:

$$x \in A \cup B \iff x \text{ satisfies } \underline{P \text{ or } Q} \ (P \vee Q), \tag{1.9}$$

$$x \in A \cap B \iff x \text{ satisfies } \underline{P \text{ and } Q} \ (P \wedge Q), \tag{1.10}$$

respectively. In a logic flow of mathematical reasoning, the sentence "if …, then …" is usually written as

$$P \implies Q, \tag{1.11}$$

which means that if property P holds, then property Q holds. In terms of set theory, one can write if $x \in A$ then $x \in B$, where $A = \{x|P(x)\}$ and $B = \{x|Q(x)\}$. In other words, $A \subseteq B$.

Example 1.2. *Let P be a property that can never be true. Then for any property Q, the statement $P \implies Q$ always holds. In set theory, this means that $A \equiv \{x|P(x)\} = \emptyset$. Since $\emptyset \subseteq B$ for any set B, we have that $P \implies Q$ always holds for any property Q. Here is a concrete example: if a real number x satisfies $x^2 < -1$, then $x = 5$.*

We denote by $\neg P$ the **negation** of P. For any predicate P, either P or $\neg P$ holds. For example, the negation of "$x^2-1>0$" is "$x^2-1\leqslant 0$". For a set $A=\{x\in X|P(x)\}$, we have

$$x \in X - A = \{x \in X|\neg P(x)\}. \tag{1.12}$$

That is, $x \notin A$ if and only if the property P does not hold for x. We have introduced three types of logic relations $P\vee Q$, $P\wedge Q$, and $P \implies Q$. Their negations are given by

- $\neg(P\vee Q)=(\neg P)\wedge(\neg Q)$,
- $\neg(P\wedge Q)=(\neg P)\vee(\neg Q)$,
- $\neg(P \implies Q)=P\wedge(\neg Q)$.

Example 1.3. *Consider the two predicates:*
P: a real number x satisfies $x^2>0$;
Q: the same number x satisfies $x^3\geqslant 0$.
Then $\neg(P \implies Q)$ is $P\wedge(\neg Q)$, which reads

$$\exists x \text{ such that } x^2>0 \text{ and } x^3<0. \tag{1.13}$$

In terms of sets, let $A\equiv\{x|P(x)\}$ and $B\equiv\{x|Q(x)\}$. $(P \implies Q)$ is equivalent to $A\subseteq B$. Then the negation $\neg(P \implies Q)$ is equivalent to $A-B\neq\emptyset$. That is, (1.13) *holds.*

We are now in a position to introduce how to negate a predicate when logical quantifiers are involved. The most commonly used quantifiers are the **universal** quantifier and the **existential** quantifier.

- (Universal quantifier) $\forall$: for every (for all); everything in the domain satisfies a given property.
- (Existential quantifier) $\exists$: there exists a (there is at least one) member in the domain satisfies a given property.

Now let $A\subseteq X$. The negation $\neg R$ of the statement

$$R: \forall\ x\in A,\ P \text{ holds for } x \tag{1.14}$$

is

$$\exists\ x\in A \text{ such that } P \text{ does not hold for } x. \tag{1.15}$$

To show this, let us take $B\equiv\{x|P(x)\}$. Then statement R is equivalent to $A\subseteq B$. So the negation of R is that A has an element that does not belong to B. Equivalently, there exists some element $x\in A$ such that P does not hold for x.

Similarly, let

$$R: \exists\ x\in A,\ P \text{ holds for } x. \tag{1.16}$$

Then its negation $\neg R$ is given by

$$\neg R: \forall\ x\in A,\ P \text{ does not hold for } x. \tag{1.17}$$

There are more examples of negations.

Example 1.4. *Let A, B be two sets, and let R be the statement*

$$R : M \implies N, \tag{1.18}$$

where

$$\begin{aligned} &M : \forall\ x \in A, P \text{ holds for } x, && (1.19)\\ &N : \exists\ x \in B \text{ such that } Q \text{ holds for } x. && (1.20) \end{aligned}$$

Then the negation $\neg R$ is

$$R : M \text{ and } \neg N. \tag{1.21}$$

That is, P holds for $x \in A$, and property Q fails for every $x \in B$.

Example 1.5. *Let us compute the negation of $P \wedge (Q \implies R)$ in this example.*

$$\begin{aligned} \neg\,(P \wedge (Q \implies R)) &= (\neg P) \vee (\neg(Q \implies R)) \\ &= (\neg P) \vee (Q \wedge \neg R). \end{aligned} \tag{1.22}$$

For a statement $P \implies Q$, its **converse** is $Q \implies P$. If both $P \implies Q$ and its converse hold, then we write $P \iff Q$, which is read P holds if and only if Q holds.

Definition 1.4 (Contrapositive)**.** *The contrapositive of $P \implies Q$ is $(\neg Q) \implies (\neg P)$.*

Lemma 1.2. *$P \implies Q$ is equivalent to $(\neg Q) \implies (\neg P)$.*

Proof. It suffices to show that $\neg(P \implies Q)$ is equivalent to $\neg((\neg Q) \implies (\neg P))$. Indeed,

$$\begin{aligned} \neg(P \implies Q) &= P \wedge (\neg Q), \\ \neg((\neg Q) \implies (\neg P)) &= (\neg Q) \wedge \neg(\neg P) = (\neg Q) \wedge P, \end{aligned} \tag{1.23}$$

which completes the proof. □

In practical, contrapositive or proof by contradiction is commonly used in arguments.

1.1.3. *Collections of sets.* This paragraph is devoted to formulating collections of sets. We will also discuss arbitrary unions and intersections.

Definition 1.5 (Power set)**.** *Let A be a set. Its power set $\mathscr{P}(A)$ is the set defined as*

$$\mathscr{P}(A) \equiv \{B | B \subset A\}. \tag{1.24}$$

From the definition, the power set is a set A whose elements are all subsets of A. In general, a **collection of sets** is a set whose elements are sets, which is denoted by a script letter such as $\mathscr{A}$, $\mathscr{B}$, $\mathscr{T}$, etc. We also denote by $\{A_j\}_{j\in J}$ a collection of sets, where J is called the set of indices.

Definition 1.6 (Arbitrary union and arbitrary intersection)**.** *Given a collection of sets $\mathscr{A}$, we define*

$$\begin{aligned}\bigcup_{A\in\mathscr{A}} A &\equiv \{x|\exists A\in\mathscr{A} \text{ s.t. } x\in A\},\\ \bigcup_{A\in\mathscr{A}} A &\equiv \{x|\forall A\in\mathscr{A},\quad x\in A\}.\end{aligned} \tag{1.25}$$

When the index set, one can similarly define $\bigcup_{j\in J} A_j$ *and* $\bigcap_{j\in J} A_j$.

1.1.4. *A remark on axiomatic set theory.* In the history of mathematics, there is a famous paradox, called **Russell's Paradox**. It can be stated as follows:

Let $A\equiv\{x|x\notin x\}$. The question is whether $A\in A$.

If $A\in A$, then A, as an element of A, satisfies $A\notin A$. Otherwise, if $A\notin A$, then by definition, $A\in A$. Contradiction arises anyway.

The problem essentially comes from an implicit assumption in the native set theory "any description gives a (possibly empty) set". To resolve this issue, one needs to systematically formulate a series of axioms to guarantee that the set in Russell's Paradox does not exist.

Axiom 1.1 (Axiom schema of Separation)**.** *Given a set A, let $P(z)$ be a predicate which is defined for every element in A. Then there exists a subset $B\subseteq A$ such that $x\in B$ if and only if x, as an element of A, satisfies P.*

Axiom 1.2 (Axiom of Regularity)**.** *For any nonempty set A, one can find an element $x\in A$ such that $x\cap A=\emptyset$.*

In axiomatic set theory, one can prove the following theorems.

Theorem 1.3. *There exists no set that contains all sets as its elements.*

Theorem 1.4. *For any set A, we have $A\notin A$.*

Corollary 1.5. *There exists no set $\mathcal{P}$ which is defined by*

$$\mathcal{P}\equiv\{A|A\notin A\}. \tag{1.26}$$

1.2. **Functions, relations and cartesian products.**

Definition 1.7 (Rule of assignment)**.** *Let A and B be two sets. A rule of assignment is a subset $\rho\subseteq A\times B$, which satisfies the property that for any $a\in A$, there is at most one element $(x,y)\in\rho$ with $x=a$. In other words,*

$$[(x,y)\in\rho,\ (x,z)\in\rho] \implies [y=z]. \tag{1.27}$$

If $(x, y) \in \rho$, one also writes $x \mapsto y$, which is read "x is assigned to y". We also define the domain and image set of ρ,

$$\mathrm{Dom}(\rho) \equiv \{a \in A | \exists b \in B \text{ s.t. } (a, b) \in \rho\}, \tag{1.28}$$

$$\mathrm{Image}(\rho) \equiv \{b \in B | \exists a \in A \text{ s.t. } (a, b) \in \rho\}. \tag{1.29}$$

Definition 1.8 (Function). *Let A and B be two sets. We say f is a function with domain D (or a function from D to B) if f is a rule of assignment ρ in $A \times B$ with $\mathrm{Dom}(\rho) = D$. One also denotes by $f : D \to B$ a function from D to B.*

Definition 1.9 (Restriction). *Let $f : A \to B$ be a function and let $A_0 \subseteq A$. The restriction of f to A_0 (or f restricted to A_0), denoted by $f|_{A_0}$, is defined as the function mapping from A_0 to B whose rule of assignment is*

$$\{(a, f(a)) | a \in A_0\}. \tag{1.30}$$

Definition 1.10 (Composite). *Given functions $f : A \to B$ and $g : B \to C$, their composite $g \circ f$ is defined as*

$$g \circ f : A \to C : a \mapsto g(f(a)). \tag{1.31}$$

In other words, the rule of $g \circ f$ is

$$\{(a, c) \in A \times C | \exists b \in B \text{ s.t. } b = f(a),\ g(b) = c\}. \tag{1.32}$$

The composite $g \circ f$ can be defined only when $\mathrm{Image}(f) \subseteq \mathrm{Dom}(g)$.

Definition 1.11. *Let $f : A \to B$ be a function. Then we define the following terms.*

- *f is called* **injective** *(or one-to-one) if $f(a) = f(b) \implies a = b$.*
- *f is called* **surjective** *(or onto) if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. In other words, $B = \mathrm{Image}(f)$.*
- *f is called* **bijective** *(or one-to-one correspondence) if f is both injective and surjective.*

Let $f : A \to B$ be a bijection. Then its **inverse** f^{-1} is defined as for any $b \in B$, $f^{-1}(b) = a$, where $a \in A$ is the only element such that $b = f(a)$

Definition 1.12 (Image and preimage). *Let $f : X \to Y$ be a function. For any $A \subseteq X$ and $B \subseteq Y$, we define*

$$f(A) \equiv \{y \in Y | \exists a \in A \text{ s.t. } y = f(a)\}, \tag{1.33}$$

$$f^{-1}(B) \equiv \{x \in X | \exists b \in B \text{ s.t. } b = f(x)\}. \tag{1.34}$$

Definition 1.13 (Relation on a set). *A* **relation** *on a set A is a subset $C \subseteq A \times A$. xCy: x is in the relation C to y.*

Definition 1.14 (Equivalence relations)**.** *Given a set A, an equivalence relation on A is a relation $C \subseteq A \times A$ that satisfies the following properties.*

- *(Reflexivity) $\forall x \in A$, xCx.*
- *(Symmetry) $xCy \implies yCx$.*
- *(Transitivity) $[xCy$ and $yCz] \implies xCz$.*

The case xCy is called x is equivalent to y, which is also denoted as $x \sim y$. Given $x \in A$, the set

$$[x] = E_x \equiv \{y \in A | y \sim x\} \tag{1.35}$$

is called the **equivalence class** *determined by x.*

Lemma 1.6. *Let C be an equivalence relation on A. For any two equivalence classes E_1 and E_2, either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.*

Proof. It suffices to show that $E_1 = E_2$ if $E_1 \cap E_2 \neq \emptyset$. Let $w \in E_1 \cap E_2$ be any element. We denote $E_1 \equiv [x]_C$ and $E_2 = [y]_C$, and we will prove that $x \sim y$.

Indeed, by assumption, $w \in E_1 \cap E_2 \subseteq E_1 = [x]_{E_1}$, we have $w \sim x$. On the other hand, we also have $w \in E_1 \cap E_2 \subseteq E_2 = [y]_{E_2}$, which implies that $w \sim y$. Applying the symmetry property of equivalence relation, we conclude that $x \sim y$. Therefore, $E_1 = E_2$. □

Definition 1.15 (Partition)**.** *A partition of a set A is a collection of sets $\mathscr{E} \subseteq \mathscr{P}(A)$ such that the following holds:*

- *$\bigcup\limits_{E \in \mathscr{E}} E = A$.*
- *$\forall\ X, Y \in \mathscr{E}$, either $X = Y$ or $X \cap Y = \emptyset$ holds.*

Clearly, given an equivalence relation C on A, one can obtain a partition $\mathscr{E}$ of A so that each member of $\mathscr{E}$ is an equivalence class. In this case, $\mathscr{E}$ is called the **partition induced by the equivalence relation**.

Lemma 1.7. *Let A be a set. Given any partition $\mathscr{D}$ of A, there exists a unique equivalence relation C on A that induces $\mathscr{D}$.*

Proof. First, let us start with a partition $\mathscr{D}$ of A and construct an equivalence relation C that induces $\mathscr{D}$.

We write $A = \bigcup\limits_{E \in \mathscr{D}} E$. Given any $x \in A$, y is said to be equivalent to x if y belongs to the same set $E \in \mathscr{D}$ as x. One can easily check that this relation satisfies the reflexivity, symmetry and transitivity.

Next, we will show the uniqueness property in the lemma. Let C_1 and C_2 be two equivalence relations which both induce $\mathscr{D}$. We will prove that xC_1y iff xC_2y. Assuming xC_1y, we define $E_1 \equiv [x]_{C_1}$ and $E_2 \equiv [x]_{C_2}$. By definition, E_1 is a member of the partition $\mathscr{D}$, and

E_2 is also a member of $\mathscr{D}$. Since $x \in E_1 \cap E_2$, the intersection of E_1 and E_2 is not empty. By the definition of partition, $E_1 = E_2$. Therefore, as two sets, we have $[x]_{C_1} = [x]_{C_2}$, which implies that yC_1x iff yC_2x. The proof is done. $\square$

Example 1.6. *Let $P[x]$ be the set of all polynomials with real coefficients. We define an equivalence relation D on $P[x]$ as follows: for two polynomials $f, h \in P[x]$, we say fDh if*

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} \in \mathbb{R} \setminus \{0\}. \tag{1.36}$$

One can show that fDg iff $\deg(f) = \deg(h)$.

Definition 1.16 (Order relation)**.** *A relation C on a set A is called an* **order relation** *(or a simple order, or a linear order) if it satisfies the following.*

- *(Comparability) $\forall x, y \in A$ with $x \neq y$, either xCy or yCx.*
- *(Nonreflexivity) There does not exist $x \in A$ that satisfies xCx.*
- *(Transitivity) xCy and $yCz \implies xCz$.*

The "less than" symbol, $<$, or $<_A$, is commonly used to denote an order relation. In terms of $<$, the order relation C can be stated as follows:

- *(Comparability) $\forall x, y \in A$ with $x \neq y$, either $x < y$ or $y < x$.*
- *(Nonreflexivity) $x < y \implies x \neq y$.*
- *(Transitivity) $x < y$ and $y < z \implies x < z$.*

Definition 1.17 (Open interval)**.** *Let $(X, <)$ be a set with an order relation. If $a < b$, then (a, b) denotes the set*

$$\{x \in X | a < x < b\} \tag{1.37}$$

is called an open interval in X. If $(a, b) = \emptyset$, then a is called an immediate predecessor of b, and b is called an immediate successor of a.

Definition 1.18 (Order type)**.** *Let $(A, <_A)$ and $(B, <_B)$ be two ordered sets. We say A and B have the same order type if there exists a bijection $f : A \to B$ such that*

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2). \tag{1.38}$$

Definition 1.19 (Dictionary order relation)**.** *Let $(A, <_A)$ and $(B, <_B)$ be two ordered sets. An order relation $<$ on $A \times B$ is called the dictionary order relation on $A \times B$ if $<$ is defined by*

$$a_1 \times b_1 < a_2 \times b_2 \Longleftrightarrow a_1 <_A a_2, \text{ or } a_1 = a_2,\ b_1 <_B b_2. \tag{1.39}$$

Definition 1.20. *Let $(X, <)$ be a set with an order relation. Let $A \subseteq X$. We define the following.*

- *An element $b \in A$ is called the* **largest element** *of A if $x < b$ for each $x \in A$. Similarly, b is called the* **smallest element** *of A if $b < x$ for each $x \in A$.*
- *An element $b \in X$ is called an* **upper bound** *of A if $x < b$ for any $x \in A$. In this case, A is called bounded above. If the set*

$$U \equiv \{b \in X | b \text{ is an upper bound of } A\} \tag{1.40}$$

has the smallest element b_0, then b_0 is called the **least upper bound**, *or the* **supremum**, *of A. Similarly, one can define a lower bound and the greatest lower bound (or infimum) of A.*

Definition 1.21. *Let $(X, <)$ be a set with an order relation. X is said to have the least upper bound property if for any subset $A \subseteq X$ that is bounded above then A has the least upper bound in X.*

Definition 1.22 (J-tuple)**.** *Let J be an index set. Given a set X, a J-tuple of elements of X is defined as a function*

$$\boldsymbol{x} : J \to X. \tag{1.41}$$

For any $\alpha \in J$, $x_\alpha = x(\alpha)$ is called the αth coordinate of $\boldsymbol{x}$. Sometimes we also denote the function $\boldsymbol{x}$ by $(x_\alpha)_{\alpha \in J}$.

Definition 1.23 (Cartesian product)**.** *Let $\{A_j\}_{j \in J}$ be a collection of sets. Let $X = \bigcup\limits_{j \in J} A_j$. The cartesian product $\prod\limits_{j \in J} A_j$ is defined as the set of all J-tuples $\boldsymbol{x} = (x_j)_{j \in J}$ of X such $x_j \in A_j$ for all $j \in J$. In other words, the cartesian product $\prod\limits_{j \in J} A_j$ is the set of all functions*

$$\boldsymbol{x} : J \to X \text{ such that } x(j) \in A_j \text{ for every } j \in J. \tag{1.42}$$

Example 1.7. *Given $m \in \mathbb{Z}_+$, the cartesian product $\prod\limits_{j=1}^{m} A_i$ or $A_1 \times \ldots \times A_m$ is the set of all m-tuples $(x_1, \ldots, x_m)$ such that $x_j \in A_j$ for each $1 \leqslant j \leqslant m$.*

Definition 1.24 (ω-tuple)**.** *Given a set X, a ω-tuple is a function $\boldsymbol{x} : \mathbb{Z}_+ \to X$. Such a function is also called a sequence, or an infinite sequence of elements of X. The cartesian product $\prod\limits_{j \in \mathbb{Z}_+} A_j$ is the set of all ω-tuples $(x_1, x_2, \ldots)$ such that $x_j \in A_j$ for every $j \in \mathbb{Z}_+$.*

Example 1.8. *Given $m \in \mathbb{Z}_+$, the cartesian product*

$$\mathbb{R}^m \equiv \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{m} \tag{1.43}$$

is called the m-dimensional Euclidean space. The infinite dimensional Euclidean space $\mathbb{R}^\omega$ is defined as the set of all ω-tuples with real entries, or the set of all real infinite sequences.

1.3. Integers and real numbers.

Definition 1.25 (Binary operation). *A binary operation on a set A is a function $f : A \times A \to A$.*

Definition 1.26 (Group). *A set G together with a binary operation $\cdot$ is called a group if the following holds.*

- *(Associativity) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ $\forall x, y, z \in G$.*
- *(Existence of identity) $\exists e \in G$, called the identity elementy, such that $e \cdot x = x \cdot e$ for all $x \in G$*
- *(Existence of inverse) $\forall x \in G$, $\exists y \in G$, called the inverse of x, such that $x \cdot y = y \cdot x = e$.*

In addition, if $x \cdot y = y \cdot x$ for all $x, y \in G$, then G is called a commutative group (or an abelian group).

Lemma 1.8. *Let G be a group. Then the following holds.*

1. *There exists a unique identity element.*
2. *Every element has a unique.*

Definition 1.27. *The set of real numbers $\mathbb{R}$, together with two binary operations $+$, $\cdot$ and the natural linear order $<$, is an ordered field and a linear continuum. That is $\mathbb{R}$ satisfies axioms (F1)-(F4) of ordered field and axiom (C1) of linear continuum:*

(F1) $(\mathbb{R}, +)$ is an abelian group with the additive identity 0.
(F2) $(\mathbb{R} \setminus \{0\}, \cdot)$ is an abelian group with the multiplicative identity 1.
(F3) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in \mathbb{R}$.
(F4) If $x > y$ and $z \in \mathbb{R}$, then $x + z > y + z$. If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.
(C1) The order relation $<$ has the least upper bound property.

Lemma 1.9. *Let $x, y \in \mathbb{R}$. If $x < y$, then there exists an element $z \in \mathbb{R}$ such that $x < z < y$.*

Proof. We take $z \equiv 2^{-1} \cdot (x + y)$. Immediately,

$$z - x = 2^{-1} \cdot (x + y) - 2^{-1} \cdot 2 \cdot x = 2^{-1} \cdot (y - x) > 0. \tag{1.44}$$

Therefore, $z > x$. Similarly, one can show that $z < y$. □

In ZF set theory, the natural numbers are defined as follow

Definition 1.28 (Set theoretic definition of $\mathbb{N}_0$). *The set of natural numbers $\mathbb{N}_0$ is defined in the following way*

- *The first few elements are $0 \equiv \emptyset$, $1 \equiv \{0\} = \{\emptyset\}$, $2 \equiv \{0, 1\} = \{0, \{0\}\}$.*
- *Given any n, $n + 1 \equiv n \cup \{n\}$.*

Proposition 1.10. *The following holds.*

(1) $-1 < 0 < 1$.
(2) *For any* $n \in \mathbb{Z}_+$, $0 < n$.

Hint: To prove (1), one needs to establish the following steps.
(a) $x > 0 \Longleftrightarrow -x < 0$.
(b) $x^2 > 0$ for any $x \neq 0$.

Definition 1.29 (Alternative definition of $\mathbb{Z}_+$)**.** *A subset* $A \subseteq \mathbb{R}$ *is called inductive if*

- $1 \in A$,
- $x \in A \implies x + 1 \in A$.

Let $\mathscr{A}$ *be the collection of all inductive subsets of* $\mathbb{R}$. *Then*

$$\mathbb{Z}_+ \equiv \bigcap_{A \in \mathscr{A}} A. \tag{1.45}$$

Immediately, by definition, if $A \subseteq \mathbb{Z}_+$ is inductive, then $A = \mathbb{Z}_+$.

Now we introduce the sections of positive integers of given lengths. For any $n \in \mathbb{Z}_+$, let us denote by S_n a section of positive integers, which is defined as

$$S_n \equiv \begin{cases} \emptyset, & n = 1 \\ \{k \in \mathbb{Z}_+ | k < n\}, & n \geqslant 2. \end{cases} \tag{1.46}$$

Lemma 1.11. *If* $n \geqslant 2$, *then for any* $\ell \in S_n$ *and* $k \in \mathbb{Z}_+ \setminus S_n$, *we have*

$$k > \ell. \tag{1.47}$$

Theorem 1.12 (Well-ordering property)**.** *Let* $A \subseteq \mathbb{Z}_+$ *be nonempty. Then* A *has a smallest element.*

Proof. In the first step, we will prove the following claim.

Claim: For any $n \geqslant 2$, the well-ordering property holds for all non-empty subsets of S_n.

First, the well-ordering property obviously holds for $n = 2$. We now assume that the well-ordering property holds for $n = k$. Then we will show that any subset $W \subseteq S_{k+1}$ has a smallest element. Indeed, we can show this by analyzing the two cases:

(1) If $k \notin W$, then $W \subseteq S_k$. The induction hypothesis implies that W has a smallest element.
(2) If $k \in W$, then let us consider $W \cap S_k$. Again, by the induction hypothesis, $W \cap S_k$ has a smallest element. It follows from the definition of S_n that k is larger than any element in $W \cap S_k$. By Lemma 1.11, we conclude that the smallest element of $W \cap S_k$ is automatically the smallest element of W.

Then the proof of the case $n = k + 1$ is done, which completes the induction argument.

We are now in a position to complete the proof of theorem. Let $A \subseteq \mathbb{Z}_+$ be non-empty. We choose any positive integer $n \geqslant 2$ and consider the intersection $A \cap S_n$. The claim implies that $A \cap S_n$ has a smallest element k, and k is the smallest element of A. $\square$

Theorem 1.13 (Strong induction principle)**.** *Given a set $A \subseteq \mathbb{Z}_+$, if $S_n \subseteq A$ implies $n \in A$, then $A = \mathbb{Z}_+$.*

Proof. Suppose $A \neq \mathbb{Z}_+$ and we define

$$n_0 \equiv \min\{n \in \mathbb{Z}_+ | n \notin A\}. \tag{1.48}$$

Then we have $n \in A$ for any $n \leqslant n_0 - 1$, which implies $S_{n_0} \subset A$. By assumption, $n_0 \in A$. So the desired contradiction arises. $\square$

1.4. Finite sets.

Definition 1.30 (Finite set). *A set X is said to be finite if there exists a bijection $f : X \to S_n$ to a section of positive integers. If $X = \emptyset$, we say X has cardinality 0; otherwise we say X has cardinality $n - 1$.*

Lemma 1.14 (Reduction Lemma). *Let $n \in \mathbb{Z}_+$ and $A \neq \emptyset$ with $a_0 \in A$. Then*

$$\exists a \text{ bijection } h : A \to S_{n+2} \Longleftrightarrow \exists a \text{ bijection } f : A \setminus \{a_0\} \to S_{n+1}. \tag{1.49}$$

Proof. First, we will prove "$\Longleftarrow$". Let $f : A \setminus \{a_0\} \to S_{n+1}$ be a bijection. We define

$$h(x) \equiv \begin{cases} f(x), & x \neq a_0, \\ n+1, & x = a_0. \end{cases} \tag{1.50}$$

Then h is a bijection from A to S_{n+2}.

Next, we will prove the other direction. Assume that $h : A \to S_{n+2}$ is a bijection. We will analyze the following two cases:

(1) If $h(a_0) = n + 1$, then one can simply define $f(x) \equiv h(x)$ for any $x \in A \setminus \{a_0\}$. Clearly, f is a bijection from A to S_{n+1}.
(2) Consider the case $h(a_0) = m \neq n + 1$. Since $h : A \to S_{n+2}$ is a bijection, there exists a unique element $w \in A$ that satisfies $h(w) = n + 1$. We consider the map $\iota : S_{n+2} \to S_{n+2}$:

$$\begin{aligned} &\iota(x) \equiv x, \quad \text{for any } \ x \neq m \text{ and } k \neq n+1, \\ &\iota(m) \equiv n+1, \\ &\iota(n+1) \equiv m. \end{aligned} \tag{1.51}$$

Obviously, ι is a bijection. Next, we define

$$\bar{h} \equiv \iota \circ h : A \to S_{n+2} \tag{1.52}$$

and $\bar{h}$ is also a bijection. Notice that $\bar{h}(a_0) = \iota(m) = n + 1$, and the situation is reduced to case (1).

The proof is now complete. □

Theorem 1.15. *Let $A \neq \emptyset$. Assume that there exists a bijection $f : A \to S_{n+1}$ for some $n \in \mathbb{Z}_+$. Let $B \subsetneq A$ be nonempty. Then there exists no bijection $g : B \to S_{n+1}$, but there exists a bijection $h : B \to S_{m+1}$ for some $m < n$.*

Proof. We prove the theorem by induction. Let $\mathfrak{D} \subseteq \mathbb{Z}_+$ be the set of integers n for which the theorem holds. We will show that $\mathfrak{D} = \mathbb{Z}_+$.

First, $1 \in \mathfrak{D}$. In this case, $B = \emptyset$ and there is no bijection between $\emptyset$ and any non-emptyset.

Assume that $k \in \mathfrak{D}$. We will show that $k+1 \in \mathfrak{D}$ which will imply that $\mathfrak{D}$ is inductive. Let $f : A \to S_{k+2}$ be a bijection and let $B \neq \emptyset$ be a proper subset of A. Let $a_0 \in B$. By Lemma 1.14, there exists a bijection

$$g : A \setminus \{a_0\} \longrightarrow S_{k+1}. \tag{1.53}$$

Notice that $B \setminus \{a_0\} \subsetneq A \setminus \{a_0\}$. Applying the induction hypothesis, we can conclude the following:

(1) There exists no bijection $h : B \setminus \{a_0\} \to S_{k+1}$.
(2) $B = \{a_0\}$, or there exists a bijection

$$\varphi : B \setminus \{a_0\} \to S_m \quad \text{for some } m < k+1. \tag{1.54}$$

Applying Lemma 1.14 to item (1), there exists no bijection $\psi : B \to S_{k+2}$, which completes the proof of the first part. Now we prove the second part. Based on item (2), there are two cases:

(1) If $B - \{a_0\} = \emptyset$, then there exists a bijection from B to $\{1\}$.
(2) If $B - \{a_0\} \neq \emptyset$, then Lemma 1.14 gives a bijection

$$\bar{\varphi} : B \to S_{m+1}. \tag{1.55}$$

Here $m+1 < k+2$.

In summary, $k+1 \in \mathfrak{D}$, and hence $\mathfrak{D} = \mathbb{Z}_+$, which completes the proof of the theorem. □

Corollary 1.16. *If A is finite, then there exists no bijection from A to a proper subset of A.*

Proof. We only need to consider the case $A \neq \emptyset$. Suppose the statement of the corollary does not hold. Namely, one can find a proper subset $B \subset A$ together with a bijection $\psi : B \to A$.

Since A is finite, by definition, there exists a bijection $\varphi : A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Then the composed mapping $\varphi \circ \psi : B \to \{1, \ldots, n\}$ is a bijection, which contradicts to Theorem 1.15. □

Corollary 1.17. *$\mathbb{Z}_+$ is not finite.*

Proof. We define the mapping $f : \mathbb{Z}_+ \to \mathbb{Z}_+ \setminus \{0\}$ as

$$f(n) \equiv n+1. \tag{1.56}$$

Clearly, f is a bijection. Applying the previous corollary, we complete the proof. □

Corollary 1.18. *The cardinality of a finite set A is uniquely determined by A.*

Proof. Suppose not. Namely assume that there exist a set A together and two bijections:

$$\begin{aligned} f &: A \longrightarrow \{1, \ldots, n\}, \\ g &: A \longrightarrow \{1, \ldots, m\}, \end{aligned} \tag{1.57}$$

where $n < m$. Then $g \circ f^{-1} : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is a bijection, which contradicts to Corollary (1.16). ☐

Corollary 1.19. *Let A, B be two sets. If $A \subseteq B$, then the following holds.*

(1) If B is finite, then A is finite.
(2) If $A \subsetneq B$, then $|A| < |B|$.

Proof. Homework. ☐

Corollary 1.20. *Let $A \neq \emptyset$. Then the following statements are equivalent.*

(1) A is finite.
(2) $\exists$ a surjection $f : S_n \to A$ for some integer $n \in \mathbb{Z}_+$.
(3) $\exists$ an injection $g : A \to S_n$ for some integer $n \in \mathbb{Z}_+$.

Proof. We will prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2). It follows from the definition of finite set.

(2) $\implies$ (3). Let $f : S_n \to A$ be a surjection. When $n = 1$, both S_n and A are the empty set. The proof immediately follows.

Let $n \geqslant 2$. We define a mapping $g : A \to S_n$ as follows:

$$g(a) \equiv \text{the smallest element in the pre-image } f^{-1}(a). \tag{1.58}$$

Since f is surjective, every set $f^{-1}(\{a\}) \subset S_n$ is non-empty. By Theorem 1.12, g is well-defined for each $a \in A$. Next, we check the injectivity of g. Given two distinct elements $a, \hat{a} \in A$, by definition,

$$f^{-1}(\{a\}) \cap f^{-1}(\{\hat{a}\}) = \emptyset. \tag{1.59}$$

Therefore, $g(a) \neq g(\hat{a})$, which implies the injectivity of g.

(3) $\implies$ (1). If g is surjective, then the conclusion immediately follows. If g is not surjective, then $g(A)$ is a proper subset of S_n. By Theorem 1.15, there exists a bijection $h : g(A) \to S_m$ for some $m < n$. Then the composed map $h \circ g : A \to S_m$ is a bijection, which implies the finiteness of A. ☐

Corollary 1.21. *Let $A_1, \ldots A_n$ be finite sets. Then both $\bigcup_{j=1}^{n} A_j$ and $\prod_{j=1}^{n} A_j$ are finite.*

1.5. **Countable and uncountable sets.**

Definition 1.31 (Countable set)**.** *Let A be set.*

- *A is said to be infinite if it is not finite.*
- *A is said to be countably infinite if there exists a bijection $f : A \to \mathbb{Z}_+$.*
- *A is said to be countable if it is either finite or countably infinite.*
- *A is said to be uncountable if it is not countable.*

Theorem 1.22. *Let $A \neq \emptyset$. Then the following are equivalent.*

(1) A is countable.
(2) $\exists$ a surjection $f : \mathbb{Z}_+ \to A$.
(3) $\exists$ an injection $g : A \to \mathbb{Z}_+$.

Proof. We will prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2). It follows from the definition of countable set.

(2) $\implies$ (3). It is a simple corollary of the well-ordering property of $\mathbb{Z}_+$. Indeed, for any $a \in A$, we define

$$g(a) \equiv \text{the smallest element in } f^{-1}(\{a\}). \tag{1.60}$$

The well-definedness of g is due to Theorem 1.12, and the injectivity of g is obvious.

(3) $\implies$ (1). We prove this part modulo the proof of Lemma 1.23. It suffices to show that $g(A)$ is countable. If $g(A)$ is finite, then by definition $g(A)$ is countable. If $g(A)$ is infinite, as a subset of $\mathbb{Z}_+$, it is countably infinite. Then the image set $g(A)$ is countable, so is A. $\square$

Lemma 1.23. *Let $A \subseteq \mathbb{Z}_+$ be an infinite subset. Then A is countably infinite.*

Proof. Let us define a bijection

$$f : \mathbb{Z}_+ \longrightarrow A \tag{1.61}$$

as follows:

$$f(n) \equiv \begin{cases} \text{the smallest element of } A, & n = 1, \\ \text{the smallest element of } A \setminus \{f(1), \dots, f(n-1)\}, & n \geqslant 2. \end{cases} \tag{1.62}$$

Immediately, f is injective since

$$(A \setminus \{f(1), \dots, f(n-1)\}) \cap (A \setminus \{f(1), \dots, f(m-1)\}) = \emptyset \tag{1.63}$$

whenever $m \neq n$. To show the surjectivity of f, let us take any element $\alpha \in A$. We will prove that there exists a positive integer $m \in \mathbb{Z}_+$ that satisfies $f(m) = \alpha$. For this purpose, we define

$$n_0 \equiv \text{the smallest integer } n \text{ that satisfies } f(n) \geqslant \alpha. \tag{1.64}$$

Immediately, $f(n_0) \geqslant \alpha$ and $f(j) < \alpha$ for each $1 \leqslant j \leqslant n_0 - 1$. Therefore,

$$\alpha \in A \setminus \{f(1), \dots, f(n_0 - 1)\}. \tag{1.65}$$

By the definition of f, we have $f(n_0) \leqslant \alpha$. Therefore, $f(n_0) = \alpha$, which proves the surjectivity of f. $\square$

Corollary 1.24. *Any subset of a countable set is countable.*

Corollary 1.25. *A countable union of countable sets is countable.*

Corollary 1.26. *A finite product of countable sets is countable.*

Theorem 1.27 (Construction of uncountable set)**.** *Let* $X = \{0, 1\}$*. Then the set* X^ω *is uncountable.*

Proof. We will prove that any map

$$g : \mathbb{Z}_+ \longrightarrow X^\omega \tag{1.66}$$

cannot be surjective. For this purpose, let us denote by x_{nm} the m-th coordinate of $g(n) \in X^\omega$, i.e.,

$$g(n) = (x_{n1}, x_{n2}, x_{n3}, \dots, x_{nm}, \ ), \tag{1.67}$$

where $x_{ij} \in \{0, 1\}$. Now let us take an element $\boldsymbol{y} = (y_1, y_2, \dots, y_n, \dots) \in X^\omega$ which is defined by

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1, \\ 1 & \text{if } x_{nn} = 0. \end{cases} \tag{1.68}$$

Then it follows that $\boldsymbol{y} \neq g(n)$ for any $n \in \mathbb{Z}_+$. □

Theorem 1.28 (Cantor)**.** *Let A be a set. Then there exists no injective map* $f : \mathscr{P}(A) \to A$*, and there exists no surjective map* $g : A \to \mathscr{P}(A)$*.*

Proof. We assume $A \neq \emptyset$. Otherwise the proof is immediate.

We also notice that the existence of an injective map $f : \mathscr{P}(A) \to A$ implies the existence of a surjective map $g : A \to \mathscr{P}(A)$. Therefore, it suffices to prove the non-existence of a surjective map $g : A \to \mathscr{P}(A)$.

Suppose that any subset $B \subseteq \mathscr{P}(A)$ has a pre-image in A. In particular, for any $a \in A$, we construct

$$B_0 \equiv \{a \in A | a \in A \setminus g(a)\}. \tag{1.69}$$

By assumption, there exists some $a_0 \in A$ such that $g(a_0) = B_0$. We notice the following equivalence

$$a_0 \in B_0 \Longleftrightarrow a_0 \in A \setminus g(a_0) \Longleftrightarrow a_0 \in A \setminus B_0 \Longleftrightarrow a_0 \notin B_0. \tag{1.70}$$

The desired contradiction arises. □

Corollary 1.29. $\mathscr{P}(\mathbb{Z}_+)$ *is uncountable.*

Continuum Hypothesis: There exists no set whose cardinality is strictly between that of the integers and the real numbers.

We denote by $\aleph_0$ the cardinality of $\mathbb{Z}_+$ and by $\aleph_1$ the cardinality of $\mathbb{R}$.

In Zermelo-Frænkel set theory with the axiom of choice (ZFC), the Continuum Hypothesis is equivalent to the equation

$$2^{\aleph_0} = \aleph_1. \tag{1.71}$$

1.6. **Axiom of choice and well-ordered sets.**

Theorem 1.30. *Let A be a set. The following statements are equivalent.*

(1) A is infinite;
(2) $\exists$ an injection $f : \mathbb{Z}_+ \to A$;
(3) $\exists$ a bijection between A with a proper subset of itself.

Proof. We prove that (2) $\implies$ (3) $\implies$ (1), and we prove (1) $\implies$ (2) in the last step.

(2) $\implies$ (3). Let $f : \mathbb{Z}_+ \to A$ be an injection. Let us denote $B \equiv f(\mathbb{Z}_+)$ and $a_n \equiv f(n)$. We can construct a bijection

$$g : A \longrightarrow A \setminus \{a_1\} \tag{1.72}$$

as follows:

$$g(x) \equiv \begin{cases} a_{n+1}, & x = a_n \in B, \\ x, & x \in A \setminus B. \end{cases} \tag{1.73}$$

It is easy to verify that g is a bijection.

(3) $\implies$ (1). Suppose A is finite. By Theorem 1.15, there exists no bijection between A and any of its proper subset.

Heuristic proof of (1) $\implies$ (2). For any fixed $a_1 \in A$, let us run the following "recursive process" by defining a function

$$\begin{cases} f(1) \equiv a_1, \\ f(j) \equiv \text{an arbitrary element in} \left\{A \setminus f(\{1, \ldots, j-1\})\right\}, \quad j \geqslant 2. \end{cases} \tag{1.74}$$

$\square$

Principle of recursive definition. Let A be a set. Given a formula that defines $h(1)$ as a unique element of A, and for any $i \geqslant 2$, defines $h(i)$ uniquely as an element of A in terms of the values $h(k)$ with $k \leqslant i-1$, this formula determines a unique function $h : \mathbb{Z}_+ \to A$.

Axiom 1.3 (Axiom of Choice)**.** *Given a collection $\mathscr{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathscr{A}$. That is, there exists a set C such that $C \subset \bigcup_{A \in \mathscr{A}} A$ and for each $A \in \mathscr{A}$ the set $C \cap A$ contains a single element.*

Corollary 1.31 (Choice function)**.** *Given any collection $\mathscr{B}$ of nonempty sets (unnecessarily disjoint), there exists a function*

$$\varphi : \mathscr{B} \longrightarrow \bigcup_{B\in\mathscr{B}} B \tag{1.75}$$

such that $\varphi(B)$ is an element of B for each $B \in \mathscr{B}$.

2. Point-set topology in metric spaces

2.1. **Basic concepts.** Here are different descriptions (or characterizations) of the Euclidean space $\mathbb{R}^n$:

- linear algebra: vector space
- topology:
- geometry: holonomy,
- metric geometry: a metric cone whose cross section is the unit sphere
- analysis: curvature $\equiv 0$

The **Euclidean norm** is defined as

$$\|\boldsymbol{x}\| \equiv (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}, \quad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{2.1}$$

Then the Euclidean norm naturally induces the Euclidean distance function

$$d(\boldsymbol{x},\boldsymbol{y}) \equiv \|\boldsymbol{x}-\boldsymbol{y}\|, \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n. \tag{2.2}$$

Lemma 2.1. *For any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$, the following holds.*

(1) $\|\boldsymbol{x}\| \geqslant 0$, and equality holds iff $\boldsymbol{x} = 0$.
(2) For any $\lambda \in \mathbb{R}$, $\|\lambda\cdot\boldsymbol{x}\| = |\lambda|\cdot\|\boldsymbol{x}\|$.
(3) $\|\boldsymbol{x}-\boldsymbol{y}\| = \|\boldsymbol{y}-\boldsymbol{x}\|$.
(4) Cauchy-Schwarz Inequality: $|\boldsymbol{x}\cdot\boldsymbol{y}| \leqslant \|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|$. Moreover, equality holds iff $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent.
(5) Triangle inequality: $\|\boldsymbol{x}-\boldsymbol{z}\| \leqslant \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}-\boldsymbol{z}\|$.

Proof of (4). First, consider the special case: let $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ satisfy $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$. Then

$$0 \leqslant (\boldsymbol{x}-\boldsymbol{y})^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} = 2(1-\boldsymbol{x}\cdot\boldsymbol{y}), \tag{2.3}$$

which implies that $\boldsymbol{x}\cdot\boldsymbol{y} \leqslant 1$. Similarly, using the fact $(\boldsymbol{x}+\boldsymbol{y})^2 \geqslant 0$, one can prove that $\boldsymbol{x}\cdot\boldsymbol{y} \geqslant -1$. Then we complete the proof of inequality in the special case. Moreover, equality holds if and only if $\boldsymbol{x} = \boldsymbol{y}$ or $\boldsymbol{x} = -\boldsymbol{y}$.

Next, we prove the inequality in the general case. Without loss of generality, we assume that $\boldsymbol{x} \neq 0$ and $\boldsymbol{y} \neq 0$. Otherwise, the inequality would be trivial. We now define $\boldsymbol{u} \equiv \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$

and $\boldsymbol{v} \equiv \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$. Obviously, $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$. Applying Cauchy-Schwarz inequality in the special case,

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leqslant 1. \tag{2.4}$$

Therefore,

$$\left| \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|} \right| \leqslant 1, \tag{2.5}$$

which implies that $|\boldsymbol{x} \cdot \boldsymbol{y}| \leqslant \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$. Equality holds iff

$$\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \quad \text{or} \quad \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = -\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}. \tag{2.6}$$

□

Let $\boldsymbol{x} \in \mathbb{R}^n$ and let $r > 0$ be a given number. We denote by $B_r(\boldsymbol{x}) \subset \mathbb{R}^n$ the open ball centered at $\boldsymbol{x}$ of radius r:

$$B_r(\boldsymbol{x}) \equiv \{\boldsymbol{y} \in \mathbb{R}^n | d(\boldsymbol{y}, \boldsymbol{x}) < r\}. \tag{2.7}$$

The concept of open ball can be generalized as follows.

Definition 2.1 (Open set). *Let $U \subseteq \mathbb{R}^n$ be a subset.*

- *A point $p \in U$ is said to be an interior point if there exists $r > 0$ that satisfies $B_r(p) \subset U$.*
- *The set of interior points of U is called the interior of U.*
- *A subset $U \subseteq \mathbb{R}^n$ is said to be open if every point of U is an interior point. In particular, the interior of an open set is itself.*
- *If $p \in U$ and U is open, then U is also called a neighborhood of p.*

Definition 2.2. *A subset $U \subseteq \mathbb{R}^n$ is said to be closed if $\mathbb{R}^n \setminus U$ is open.*

Lemma 2.2 (Basic properties of open sets). *The following properties hold.*

(1) $\emptyset$ is open and X is open.
(2) For any collection $\mathscr{A}$ of open sets, $\bigcup_{U \in \mathscr{A}} U$ is open.
(3) If $U_1, \ldots, U_m$ are open, then $\bigcap_{j=1}^{m} U_j$ is open.

Proof. We only prove (3). Let us take any $p \in \bigcap_{j=1}^{m} U_j$ and we will show that p is an interior point. Since $p \in U_j$ and U_j is open for any $1 \leqslant j \leqslant m$, there exists $\delta_j > 0$ such that

$$B_{\delta_j}(p) \subseteq U_j. \tag{2.8}$$

Let us define $\delta_0 \equiv \min\{\delta_1, \ldots, \delta_m\}$. Then one can easily check that $B_{\delta_0}(p) \subset U_j$ for any $1 \leqslant j \leqslant m$. Therefore, $B_{\delta_0}(p) \subseteq \bigcap_{j=1}^{m} U_j$. □

Corollary 2.3 (Basic properties of closed sets)**.** *The following properties hold.*

(1) $\emptyset$ is closed and X is closed.

(2) For any collection $\mathscr{A}$ of closed sets, $\bigcap\limits_{U\in\mathscr{A}} U$ is closed.

(3) If $U_1, \ldots, U_m$ are closed, then $\bigcup\limits_{j=1}^{m} U_j$ is closed.

Definition 2.3 (Limit point and derived set)**.** *Let $U \subseteq \mathbb{R}^n$ be a subset. A point $p \in U$ is called a limit point if*

$$B_r(p) \cap (S \setminus \{p\}) \neq \emptyset, \quad \forall\ r > 0. \tag{2.9}$$

The set of all limit points of U is called the derived set of U, which is denoted as U'.

It is clear that U' is closed. To see this, it suffices to show that $\mathbb{R}^n \setminus U'$ is open. In fact, any point $p \in \mathbb{R}^n \setminus U'$ is not a limit point of U', that is, there exists $r_p > 0$ such that $B_{r_p}(p) \cap U = \emptyset$. This implies that any point p is an interior point of $\mathbb{R}^n \setminus U'$. Therefore, $\mathbb{R}^n \setminus U'$ is open and U' is closed.

Definition 2.4 (closure)**.** *Let $U \subseteq \mathbb{R}^n$ be a subset. The closure of U, denoted by $\overline{U}$, is defined as the intersection of all closed sets that contain U.*

Combining the above two definitions, we have

$$\overline{U} = U \cup U'. \tag{2.10}$$

We will show that for any closed set $W \supseteq U$, we have $W \supseteq U \cup U'$. It suffices to show that $\mathbb{R}^n \setminus W \subseteq \mathbb{R}^n \setminus U'$. In fact, taking any $p \in \mathbb{R}^n \setminus W$, the openness of $\mathbb{R}^n \setminus W$ implies that there exists $r_p > 0$ such that $B_{r_p}(p) \subseteq \mathbb{R}^n \setminus W$. Then p is not a limit point of U, namely $p \in \mathbb{R}^n \setminus U'$. This completes the proof.

There is a useful lemma about limit point.

Lemma 2.4. *Let p be a limit point of U. Then for any $r > 0$, the open ball $B_r(p)$ contains infinitely many points of $U \setminus \{p\}$.*

Proof. We will prove it by contradiction. Suppose that there exists some $r_0 > 0$ such that the intersection $B_{r_0}(p) \cap (U \setminus \{p\})$ has finitely many points $\{q_1, \ldots, q_m\}$. Let us define

$$s_0 \equiv \min\{\|p - q_j\| : 1 \leqslant j \leqslant m\}. \tag{2.11}$$

Then the open ball $B_{\frac{s_0}{2}}(p)$ does not contain any point in $U \setminus \{p\}$, which contradicts the definition of limit point. $\square$

Now we have the following characterization for closed sets.

Theorem 2.5. *Let $U \subseteq \mathbb{R}^n$ be a subset. Then the following statements are equivalent.*

(1) U is closed.

(2) $\overline{U} = U$.
(3) U *contains all its limit points.*

Proof. We will prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2). Assume that U is closed. Then we will show that $\overline{U} \subseteq U$. It suffices to take any limit point $p \in U'$, we will show that $p \in U$. If not, we have $p \in \mathbb{R}^n \setminus U$. By assumption, $\mathbb{R}^n \setminus U$ is open, which implies that there exists some $\delta > 0$ such that $B_\delta(p) \subseteq \mathbb{R}^n \setminus U$. This tells us that p cannot be a limit point of U.

(2) $\implies$ (3). It follows from the definition.

(3) $\implies$ (1). Assuming (3), we will show that $\mathbb{R}^n \setminus U$ is open. If not, there exists a point $p \in \mathbb{R}^n \setminus U$ such that for any $r > 0$, $B_r(p) \cap U \neq \emptyset$. Since $p \notin U$, we have $B_r(p) \cap (U \setminus \{p\}) \neq \emptyset$ for any $r > 0$. Therefore, p is a limit point of U, which contradicts the assumption that U contains all its limit points. The proof is complete. $\square$

The next topic is the structure of open sets in $\mathbb{R}^1$. Let us start with the definition.

Definition 2.5 (Component interval)**.** *Let* $U \subset \mathbb{R}^1$ *be open. An open interval* I *is called a component interval of* U *if* $I \subseteq U$ *and if there exists no open interval* $J \neq I$ *such that* $I \subseteq J \subseteq U$.

Lemma 2.6. *Every point* x *of a nonempty open set* $U \subset \mathbb{R}^1$ *is contained in precisely one component interval* I_x *of* S.

Proof. The largest interval containing x can be constructed as follows. For any $x \in U$, we define

$$\alpha(x) \equiv \inf\{\alpha \in \mathbb{R}^1 : (\alpha, x) \subseteq U\}, \quad \beta(x) \equiv \inf\{\alpha \in \mathbb{R}^1 : (x, \beta) \subseteq U\}. \tag{2.12}$$

Then one can check that the interval $I_x \equiv (\alpha(x), \beta(x))$ is the desired component interval. $\square$

Theorem 2.7. *Every non-empty open set* $U \subseteq \mathbb{R}^1$ *is the union of a countable collection of disjoint component intervals of* U.

Proof. Let us consider the countable set $W \equiv \mathbb{Q} \cap U = \{p_1, p_2, \ldots, p_k, \ldots\}$. By Lemma 2.1, for any $x \in U$, there exists precisely component interval $I_x \subseteq U$ containing x. We define the following function

$$\pi : \text{ collection of component intervals of } U \longrightarrow \mathbb{Z}_+ \tag{2.13}$$

by

$$I_x \mapsto m, \tag{2.14}$$

where p_m has the smallest index in $W \cap I_x$. By the well-ordering property of $\mathbb{Z}_+$, the function π is well-defined.

Next, we will prove that π is injective. In fact, if $\pi(I_x) = \pi(I_y) = m$, then $p_m \in I_x \cap I_y$. It follows from the definition of component interval, $I_x = I_y$, which completes the proof. $\square$

2.2. **Limits, continuity and homeomorphism.** In this subsection, we will discuss the notion of continuity in $\mathbb{R}^n$.

Definition 2.6 (Sequential limit)**.** *A sequence $\{x_j\}_{j=1}^{\infty} \subseteq \mathbb{R}^n$ is said to converge to a limit $x_0 \in \mathbb{R}^n$ if for any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for any $n \geqslant N$,*

$$\|x_j - x_0\| < \epsilon. \tag{2.15}$$

Example 2.1. *If $\{x_j\}_{j=1}^{\infty}$ converges to x_0, then x_0 is a limit point of $\{x_j\}_{j=1}^{\infty}$. In general, any bounded sequence $\{x_j\}_{j=1}^{\infty}$ has a limit point, but it does not have to converge. For example, $x_j = (-1)^j + \frac{1}{j}$ has two distinct limit points 1 and -1, but x_j does not converge.*

Lemma 2.8 (Uniqueness of sequential limit)**.** *If a sequence $\{x_j\}_{j=1}^{\infty} \subseteq \mathbb{R}^n$ converges in $\mathbb{R}^n$, then it has a unique limit.*

Proof. Let x_0 and $\bar{x}_0$ be two limits of the sequence. We will prove that $x_0 = \bar{x}_0$. We particularly need to show that $\|x_0 - \bar{x}_0\| < \epsilon$ for any $\epsilon > 0$. This follows from triangle inequality. $\square$

Lemma 2.9. *Let $A \subseteq \mathbb{R}^n$ be a subset. A point $x_0 \in \mathbb{R}^n$ is a limit point of A if and only if there exists a sequence $\{x_j\}_{j=1}^{\infty} \subseteq A$ such that $\lim\limits_{j\to\infty} x_j = x_0$.*

Proof. " $\Longrightarrow$ ": Let x_0 be a limit of A and let us take a sequence $r_j = 2^{-j} \to 0$. By definition, for any $j \in \mathbb{Z}_+$,

$$(B_{r_j}(x_0) \setminus \{x_0\}) \cap A \neq \emptyset. \tag{2.16}$$

We pick an arbitrary point $x_j \in (B_{r_j}(x_0) \setminus \{x_0\}) \cap A$ for each $j \in \mathbb{Z}_+$. Then we will show that $x_j \to x_0$ as $j \to \infty$. In fact, for any $\epsilon > 0$, one can choose sufficiently large number

$$j_0 \equiv \left[\frac{\log(\frac{1}{\epsilon})}{\log 2} \right] + 1, \tag{2.17}$$

so that $r_j < \epsilon$ for any $j \geqslant j_0$. It follows that $\|x_j - x_0\| < \epsilon$ for any $j \geqslant j_0$, which completes the proof.

" $\Longleftarrow$ ": Let $\{x_j\}_{j=1}^{\infty} \subseteq A$ be a sequence converging to $x_0 \in \mathbb{R}^n$. We will show that x_0 is a limit point of A. By the definition of sequential limit, for any $\epsilon > 0$, there exists $j_0 > 0$ such that $x_j \in B_\epsilon(x_0)$ for any $j \geqslant j_0$. Since $x_j \neq x_0$, we have

$$(B_\epsilon(x_0) \setminus \{x_0\}) \cap A \neq \emptyset, \tag{2.18}$$

which implies that x_0 is a limit point of A. $\square$

Definition 2.7 (Limit at a point). *Let $U \subseteq \mathbb{R}^n$ be an open set and let $L \in \mathbb{R}^1$. A function $f : U \to \mathbb{R}$ is said to have a limit L at $p \in U$, denoted by $\lim_{x\to p} f(x) = L$, if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x \in B_\delta(p)$, then $|f(x) - L| < \epsilon$.*

Lemma 2.10. *Pointwise limit is always unique.*

Proof. Assume that L and L' are both limits of f at p. Then for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(x) - L'| < \frac{\epsilon}{2} \quad \text{and} \quad |f(x) - L| < \frac{\epsilon}{2}. \tag{2.19}$$

Applying triangle inequality,

$$|L - L'| \leqslant |f(x) - L'| + |f(x) - L| < \epsilon. \tag{2.20}$$

Therefore, $L = L'$. □

Definition 2.8 (Continuity). *Let $U \subseteq \mathbb{R}^n$ be a subset. A function $f : U \to \mathbb{R}$ is said to be continuous at $p \in U$ if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x \in B_\delta(p)$, then $|f(x) - f(p)| < \epsilon$. A function f is continuous on an open set U if f is continuous at every point $p \in U$.*

By definition, the statement that a function $f : U \to \mathbb{R}$ fails to be continuous at $p \in U$ can be converted to the following: there exists some $\epsilon_0 > 0$ such that for any $\delta > 0$ one can find some point $x \in B_\delta(p)$ such that

$$|f(x) - f(p)| \geqslant \epsilon_0. \tag{2.21}$$

Theorem 2.11. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a real valued function. Then the following properties are equivalent.*

(1) f is continuous on U.
(2) For any open set $V \subset \mathbb{R}$, $f^{-1}(V) \cap U$ is open.
(3) For any $p \in U$ and any neighborhood V of $f(p)$, then there exists a neighborhood $W \subseteq U$ of p such that $f(W) \subseteq V$.

Proof. (1) $\implies$ (2). Assume $f^{-1}(V) \cap U \neq \emptyset$ and let us take any $p \in f^{-1}(V) \cap U$. The openness of V implies that there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(f(p)) \subseteq V$. Since f is continuous on U, there exists $\delta_0 > 0$ such that $B_{\delta_0}(p) \subseteq U$ for any $x \in B_{\delta_0}(p)$, we have

$$|f(x) - f(p)| < \epsilon_0. \tag{2.22}$$

That is, $f(B_{\delta_0}(p)) \subseteq B_{\epsilon_0}(f(p)) \subseteq V$. Therefore, $B_{\delta_0}(p) \subseteq U \cap f^{-1}(V)$, which implies that p is an interior point of $U \cap f^{-1}(V)$. Since p is arbitrary in $f^{-1}(V) \cap U$, by definition, $f^{-1}(V) \cap U$ is open.

(2) $\implies$ (3). The proof is immediate.

(3) $\implies$ (1). Let us take any $\epsilon > 0$ and consider the open ball $B_\epsilon(f(p)) \subseteq \mathbb{R}$. By assumption, there exists some neighborhood $W \subseteq U$ of p that satisfies $f(W) \subseteq B_\epsilon(f(p))$. Since W is open, there exists $\delta > 0$ that satisfies $B_\delta(p) \subseteq W$. That is, for any $x \in B_\delta(p)$, it holds that

$$|f(x) - f(p)| < \epsilon, \tag{2.23}$$

which proves the continuity at p. □

One can define subspace topology in $\mathbb{R}^n$.

Definition 2.9. *Let $U \subseteq \mathbb{R}^n$ be a subset. A set $V \subseteq U$ is said to be open in U (or open with respect to U) if there exists an open set $W \subseteq \mathbb{R}^n$ such that $V = U \cap W$.*

One can define continuous function with respect to subspace topology.

Definition 2.10 (Continuous function with respect to subspace topology)**.** *Let $U, V \subseteq \mathbb{R}^n$ be two subsets and let $f : U \to V$ be a function.*

(1) f is said to be continuous on U if for any open set W in V, the preimage $f^{-1}(W)$ is open in U.
(2) f is said to be a homeomorphism if
- *f is bijective,*
- *both f and f^{-1} are continuous.*

Example 2.2. *A bijective continuous function does not have to be a homeomorphism. Here we have an intuitive example. Let $U = [0, 1) \subset \mathbb{R}$ and $V = S^1 \subset \mathbb{R}^2$. A continuous function $f : U \to V$ can be chosen as*

$$f(t) \equiv (\cos 2\pi t, \sin 2\pi t). \tag{2.24}$$

It is clear that $f : U \to V$ is bijective and continuous. However, f^{-1} is not continuous at $(-1, 0) \in S^1$. Indeed, $f^{-1}((-1, 0)) = 0$. One can find a sequence $j \to \infty$ and a sequence of points $x_j \in B_{j^{-1}}((-1, 0)) \cap S^1$ such that $|f^{-1}(x_j) - 1| < 10^{-2}$ for any j.

2.3. **Fundamental theorems and their relations.**

Axiom 2.1 (Axiom of Supremum or Axiom of Linear Continuum)**.** *If $A \subset \mathbb{R}^1$ has an upper bound, then A has a supremum.*

Lemma 2.12. *Let x be a nonnegative real number. If $x < \epsilon$ for any $\epsilon > 0$, then $x = 0$.*

Proof. Suppose there exists a positive number $x_0 > 0$ that satisfies $x_0 < \epsilon$ for any $\epsilon > 0$. Let us consider $\frac{1}{x_0}$. It follows that there exists $N_0 \in \mathbb{Z}_+$ such that $\frac{1}{x_0} < N_0$, which implies that $x_0 > \frac{1}{N_0}$. Contradiction. □

Theorem 2.13 (Bolzano-Weierstraß)**.** *If $A \subseteq \mathbb{R}^n$ is bounded and contains infinitely many points, then A has a limit point.*

Remark 2.1. *The proof shows "Axiom 2.1 $\Longrightarrow$ Bolzano-Weierstraß."*

Proof. We first prove the theorem for $n = 1$. Let $A \subseteq [-a_0, a_0]$ be a set that contains infinitely many points, and we define

$$I_0 \equiv [-a_0, 0], \quad J_0 \equiv [0, a_0]. \tag{2.25}$$

Then at least one of I_0 and J_0 contains infinitely many points of A. Without loss of generality, assume that I_0 contains infinitely many points of A. Next, I_0 can be written as the union of two intervals I_1 and J_1 such that $|I_1| = |J_1| = \frac{a_0}{2}$. Let $I_1 \equiv [a_1, b_1]$ be the interval that contains infinitely many points of A. In the jth-step, we will obtain a closed interval $I_j = [a_j, b_j] \subset I_{j-1}$ that satisfies $b_j - a_j = \frac{a_0}{2^j}$ and contains infinitely many points of A.

We now collect all the end points of I_j for all $j \in \mathbb{Z}_+$, and we obtain two sets of end points

$$L \equiv \{a_1, a_2, \ldots, a_j, \ldots\}, \quad R \equiv \{b_1, b_2, \ldots, b_j, \ldots\}. \tag{2.26}$$

Clearly, $L \cup R \subseteq [-a_0, a_0]$. By Axiom 2.1, both $\sup L$ and $\inf R$ exist. Moreover,

$$\sup L = \inf R, \tag{2.27}$$

which is denoted as w (Question: how to prove this?). One can show that w is a limit point. In fact, for any $r > 0$, let us take an interval $B_r(w)$. Then $I_j \subset B_r(w)$ for any $j \in \mathbb{Z}_+$ that satisfies

$$r > 2(b_j - a_j). \tag{2.28}$$

Then $B_r(w)$ contains infinitely many points of A, which implies that w is a limit point of A.

Let us prove the general case when $n \geqslant 2$. Since $A \subset \mathbb{R}^n$ is bounded, A must contained contained in some n-cube

$$I_0^n \equiv [-a_0, a_0]^n = \underbrace{[-a_0, a_0] \times \ldots \times [-a_0, a_0]}_{n}. \tag{2.29}$$

Similar to the previous arguments, one can divide the I_0^n into 2^n sub-cubes, each of which has side length equal to $\frac{a_0}{2}$. Note that A contains infinitely many points, which implies that at least one of those sub-cubes, denoted by

$$I_1^n = \left[a_1^{(1)}, b_1^{(1)}\right] \times \ldots \times \left[a_1^{(n)}, b_1^{(n)}\right]. \tag{2.30}$$

We can further divide I_1^n into 2^n copies of sub-cubes such that each of them has side length equal to $\frac{a_0}{4}$, and then we select a copy, called I_2^n, that contains infinitely many points of A. Repeating the above process, in the jth step, we will obtain a sub-cube

$$I_j^n = \left[a_j^{(1)}, b_j^{(1)}\right] \times \ldots \times \left[a_j^{(n)}, b_j^{(n)}\right], \tag{2.31}$$

where

$$b_j^{(k)} - a_j^{(k)} = \frac{a_0}{2^j} \quad \forall\ 1 \leqslant k \leqslant n. \tag{2.32}$$

It is the same as the case $n = 1$ that for any $j \in \mathbb{Z}_+$ and $1 \leqslant k \leqslant n$, we will the following two subsets of the cube I_0^n:

$$L^{(k)} \equiv \{a_j^{(k)} : j \in \mathbb{Z}_+\}, \quad R^{(k)} \equiv \{a_j^{(k)} : j \in \mathbb{Z}_+\}. \tag{2.33}$$

By Axiom 2.1, one can show that for every $1 \leqslant k \leqslant n$, both $\sup L^{(k)}$ and $\inf R^{(k)}$ exist and they are equal, denoted as $w^{(k)}$. This process produces a point

$$w = (w^{(1)}, \ldots, w^{(n)}) \in [-a_0, a_0]. \tag{2.34}$$

Using the same arguments as in the first step, we can show that w is a limit point of A. Indeed, for any $r > 0$, if we take a large positive integer j that satisfies

$$r > 2(b_j^{(k)} - a_j^{(k)}) = 2^{1-j} a_0, \quad 1 \leqslant k \leqslant n, \tag{2.35}$$

we have $I_j^n \subset B_r(w)$. Therefore, $B_r(w)$ contains infinitely many points of A, which implies that w is a limit point of A. The proof is complete. □

Theorem 2.14 (Cantor's Intersection Theorem)**.** *Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of non-empty closed subsets of $\mathbb{R}^n$ such that A_1 is bounded and*

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_j \supseteq A_{j+1} \supseteq \ldots \tag{2.36}$$

Then

$$\bigcap_{j=1}^{\infty} A_j \neq \emptyset. \tag{2.37}$$

Example 2.3. *The assumption of the boundedness of A_1 is necessary. For example, if we define $A_j \equiv [j, \infty)$, then clearly $\bigcap\limits_{j=1}^{\infty} A_j = \emptyset$.*

Remark 2.2. *This proof shows the fact that "Bolzano-Weierstraß $\Longrightarrow$ Cantor's Intersection Theorem."*

Proof. One can assume that, for any $j \in \mathbb{Z}_+$, the closed set A_j is an infinite set. Otherwise, if A_{j_0} is a finite set for some $j_0 \in \mathbb{Z}_+$, then any A_j with $j \geqslant j_0$ is finite, which implies that there exists some $j_1 \in \mathbb{Z}_+$ such that $A_j = A_{j_1}$ for any $j \geqslant j_1$. Immediately, $\bigcap_{j=1}^{\infty} = A_{j_1} \neq \emptyset$.

Next, let us take $A \equiv \bigcap_{j=1}^{\infty} A_j$. Then by Corollary 2.23, A is closed. We can take a set $W = \{x_1, x_2, \ldots\}$ of distinct points such that $x_j \in A_j$ for each $j \in \mathbb{Z}_+$. Such a selection always exists since each A_j is infinite set. Then W is a bounded infinite set. Applying Bolzano-Weierstraß Theorem, W has a limit point $w_0 \in \mathbb{R}^n$.

We will prove that w_0 is contained in A_j for each $j \in \mathbb{Z}_+$. Since A_j's are all closed, it suffices to show that w_0 is a limit point for each A_j. For any $\epsilon > 0$, the open ball $B_\epsilon(w_0)$ contains infinitely many points of W. On the other hand, by the definition of W, for any $j \in \mathbb{Z}_+$, all except for a finite number of the points of W belong to A_j. Therefore, for any $\epsilon > 0$ and for any $j \in \mathbb{Z}_+$, the open ball $B_\epsilon(w_0)$ contains infinitely many points of A_j, which implies that w_0 is a limit point of A_j for any $j \in \mathbb{Z}_+$. The proof is complete. □

Definition 2.11 (Diameter). *Let $A \subseteq \mathbb{R}^n$ be a subset. The diameter of A, denoted by* $\operatorname{diam}(A)$*, is defined by*

$$\operatorname{diam}(A) \equiv \sup\{d(p,q) | p, q \in A\}. \tag{2.38}$$

Corollary 2.15. *Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of non-empty closed subsets of $\mathbb{R}^n$ such that*

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_j \supseteq A_{j+1} \supseteq \ldots \tag{2.39}$$

and $\operatorname{diam}(A_j) \to 0$ *as $j \to \infty$. Then there exists a unique point $x_0 \in \mathbb{R}^n$ such that*

$$\bigcap_{j=1}^{\infty} A_j = \{x_0\}. \tag{2.40}$$

Proof. By Theorem 2.14, $\bigcap_{j=1}^{\infty} A_j \neq \emptyset$. By assumption, $\operatorname{diam}\left(\bigcap_{j=1}^{\infty} A_j\right) = 0$. We are now ready to show that this intersection is a set of single point. Suppose not and $\bigcap_{j=1}^{\infty} A_j \neq \emptyset$ contains at least two points. Then its diameter must be positive. So the desired contradiction arises. □

Corollary 2.16. *If Cantor's Intersection Theorem is true, then Bolzano-Weierstraß Theorem is true.*

Proof. The main argument follows from Corollary 2.15. □

Definition 2.12 (Cauchy sequence). *A sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists a number $N_0 \in \mathbb{Z}_+$ such that*

$$\|x_j - x_k\| < \epsilon \quad \forall\ j, k \geqslant N_0. \tag{2.41}$$

Lemma 2.17. *Let $\{x_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. Then it is bounded.*

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a Cauchy sequence. For $\epsilon = 10^{-3}$, there exists $N_0 \in \mathbb{Z}_+$ such that for any $j \geqslant N_0 + 1$ we have $\|x_j - x_{N_0}\| < 10^{-3}$. Now for any $n \in \mathbb{Z}_+$,

$$\|x_j\| \leqslant \max\{\|x_1\|, \ldots, \|x_{N_0-1}\|, \|x_{N_0}\| + 10^{-3}\}, \tag{2.42}$$

which proves the boundedness of $\{x_j\}_{j=1}^{\infty}$.

□

Lemma 2.18. *If a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ has a converging subsequence, then $\{x_j\}_{j=1}^{\infty}$ itself also converges.*

Proof. Let $\{x_{j_k}\}_{k=1}^{\infty}$ be a converging subsequence with a limit x_0, where $j_{k_1} \leqslant j_{k_2}$ for any $k_1 \leqslant k_2$. Then for any $\epsilon > 0$, there exists $K_0 \in \mathbb{Z}_+$ such that

$$\|x_{j_k} - x_0\| < \frac{\epsilon}{2}, \quad \forall\ k \geqslant K_0. \tag{2.43}$$

On the other hand, for any $\epsilon > 0$, there exists $N_0 \in \mathbb{Z}_+$ such that for any $j, k \geqslant N_0$,

$$\|x_j - x_k\| < \frac{\epsilon}{2}. \tag{2.44}$$

Letting $Q_0 \equiv \max\{N_0, j_{K_0}\}$, for any $j \geqslant Q_0$, we have

$$\|x_j - x_0\| \leqslant \|x_j - x_{j_k}\| + \|x_{j_k} - x_0\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{2.45}$$

which completes the proof. □

Theorem 2.19 (Cauchy Convergence Theorem)**.** *$\mathbb{R}^n$ is complete. That is, any Cauchy sequence in $\mathbb{R}^n$ converges.*

Remark 2.3. *The proof follows "Bolzano-Weierstraß $\Longrightarrow$ Cauchy Convergence Theorem".*

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. By Lemma 2.17, it is bounded. Applying Bolzano-Weierstraß Theorem, $\{x_j\}_{j=1}^{\infty}$ has a limit point $x_0 \in \mathbb{R}^n$. Then there exists a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ that limits to x_0. Finally, Lemma 2.18, implies that $\{x_j\}_{j=1}^{\infty}$ converges to x_0 as well. □

Theorem 2.20. *If the Cauchy's Convergence Theorem is true, then the Axiom of Supremum is true.*

Proof. Let $W \subset \mathbb{R}$ be a non-empty set that is bounded from above. We will apply a constructive argument to produce a supremum of W.

Without loss of generality, we assume that W has at least two elements. Then one can find two numbers $A_1, B_1 \in \mathbb{R}$ such that B_1 is an upper bound for W and A_1 is not an upper bound for W. Now we define two sequences of numbers

$$\begin{aligned} &A_1,\ A_2, \ldots, A_n, \ldots \\ &B_1,\ B_2, \ldots, B_n, \ldots \end{aligned} \tag{2.46}$$

inductively:

(1) If $\frac{A_n+B_n}{2}$ is an upper bound for W, then we define

$$A_{n+1} \equiv A_n \quad \text{and} \quad B_{n+1} \equiv \frac{A_n + B_n}{2}. \tag{2.47}$$

(2) If $\frac{A_n+B_n}{2}$ is not an upper bound for W, then there exists some number $\beta \in W$ that satisfies $\beta > \frac{A_n+B_n}{2}$ and that is not an upper bound for W. Then we define

$$A_{n+1} \equiv \beta \quad \text{and} \quad B_{n+1} \equiv B_n. \tag{2.48}$$

One can prove by induction that the two sequences satisfy the following comparisons

(a) $A_{n+1} \geqslant A_n$ for any $n \in \mathbb{Z}_+$,
(b) $B_{n+1} \leqslant B_n$ for any $n \in \mathbb{Z}_+$,
(c) $A_n \leqslant B_n$ for any $n \in \mathbb{Z}_+$.

Therefore, we have

$$A_1 \leqslant A_2 \leqslant A_3 \leqslant \ldots \leqslant B_3 \leqslant B_2 \leqslant B_1. \tag{2.49}$$

Moreover, one can show that $|B_n - A_n| \to 0$ as $n \to \infty$. Applying (2.49) to the above squeezing, one can conclude that both $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are Cauchy sequences. Then they converge to the same limit x_0.

We are ready to show that x_0 is the supremum of W. First, x_0 must be an upper bound of W. In fact, it suffices to show that $x_0 + \epsilon > x$ for any $x \in W$. Since $B_n \to x_0$, by definition, for any $\epsilon > 0$, there exists some $N_0 \in \mathbb{Z}_+$ such that for all $n \geqslant N_0$

$$x_0 + \epsilon > B_n \geqslant x, \quad \forall x \in W. \tag{2.50}$$

The last inequality holds since every B_n is an upper bound for W. Next, we will show that for any $\epsilon > 0$, $x_0 - \epsilon$ is not an upper bound for W. Suppose not, then for some $\epsilon_0 > 0$, the number $x_0 - \epsilon_0$ would be an upper bound for W. However, by the definition of sequential limit, there exists $n_0 \in \mathbb{Z}_+$ such that $A_{n_0} > x_0 - \epsilon_0$, which contradicts the fact that no A_n is an upper bound for W. This completes the proof. □

So far, we have established the following equivalence.

Theorem 2.21. *The following items are equivalent in* $\mathbb{R}^n$.

(1) Bolzano-Weierstraß Theorem
(2) Cantor's Intersection Theorem
(3) Cauchy Criterion for Convergence

If $n = 1$, *then the Axiom of Supremum is equivalent to any one of the above.*

2.4. **Metric space.**

Definition 2.13 (Metric space). *A set X together with a function $d : X \times X \to \mathbb{R}$ is called a metric space if the following holds:*

(1) (Positivity) $d(p, q) \geqslant 0$ for all $p, q \in X$. Moreover, equality holds iff $p = q$.
(2) (Symmetry) $d(p, q) = d(q, p)$ for all $p, q \in X$.
(3) (Triangle inequality) $d(p, q) \leqslant d(p, x) + d(x, q)$ for all $p, q, x \in X$.

On a metric space (X, d), one can define notion of metric ball. Given $p \in X$ and $r > 0$, we define open metric ball, closed metric ball and metric sphere as follows:

$$\begin{aligned} B_r(p) \equiv& \{x \in X | d(x, p) < r\}, \\ \overline{B_r(p)} \equiv& \{x \in X | d(x, p) \leqslant r\}, \\ S_r(p) \equiv& \overline{B_r(p)} \setminus B_r(p) = \{x \in X | d(x, p) = r\}. \end{aligned} \tag{2.51}$$

Based on the definition of metric ball, one can define open set and closed set in a metric space.

Definition 2.14 (Open set and closed set). *Let (X, d) be a metric space and let $U \subseteq X$ be a subset.*

- *A point $p \in U$ is said to be an* **interior point** *if there exists $r > 0$ that satisfies $B_r(p) \subseteq U$.*
- *The set of interior points of U is called the interior of U.*
- *U is said to be* **open** *if every point of U is an interior point. In particular, the interior of an open set is itself.*
- *If $p \in U$ and U is open, then U is also called a neighborhood of p.*
- *U is said to be* **closed** *if $X \setminus U$ is open.*
- *A point $p \in U$ is called a* **limit point** *if*

$$B_r(p) \cap (S \setminus \{p\}) \neq \emptyset, \quad \forall\ r > 0. \tag{2.52}$$

  The set of all limit points of U is called the derived set of U, which is denoted as U'.
- *The* **closure** *of U, denoted by $\overline{U}$, is defined by $\overline{U} = U \cup U'$.*

Lemma 2.22 (Basic properties of open sets). *Let (X, d) be a metric space. Then the following properties hold.*

(1) $\emptyset$ is open and X is open.
(2) For any collection $\mathscr{A}$ of open sets, $\bigcup_{U \in \mathscr{A}} U$ is open.
(3) If $U_1, \ldots, U_m$ are open, then $\bigcap_{j=1}^{m} U_j$ is open.

Corollary 2.23 (Basic properties of closed sets). *Let (X, d) be a metric space. Then the following properties hold.*

(1) $\emptyset$ is closed and X is closed.
(2) For any collection $\mathscr{A}$ of closed sets, $\bigcap_{U\in\mathscr{A}} U$ is closed.
(3) If $U_1, \ldots, U_m$ are closed, then $\bigcup_{j=1}^{m} U_j$ is closed.

Theorem 2.24. *Let (X,d) be a metric space and let $U \subseteq X$ be a subset. Then the following statements are equivalent.*

(1) U is closed.
(2) $\overline{U} = U$.
(3) U contains all its limit points.

Definition 2.15 (Subspace)**.** *A metric space $(U,\hat{d})$ is called a subspace of a metric space (X,d) if $U \subseteq X$ and $\hat{d} = d|_U$.*

The next theorem relates the topology of (X,d) and that of a subspace.

Theorem 2.25. *Let (S,d) be a subspace of (X,d) and let $W \subseteq S$ be a subset. Then the following holds.*

(1) W is open in S iff $W = U \cap S$ for some open set $U \subseteq X$.
(2) W is closed in S iff $U = \Sigma \cap S$ for some open set $\Sigma \subseteq X$.

Proof. We only prove item (1).

" $\Longrightarrow$ ": Assume that W is open in S. By definition, for any $x \in W$, there exists some $r_x > 0$ such that $B^S_{r_x}(x) \subseteq W$. Also, it follows from the definition of the restricted metric ball

$$B^S_{r_x}(x) = \{z \in S : d(z,x) < r_x\} = B^X_{r_x}(x) \cap S. \tag{2.53}$$

Therefore,

$$W = \bigcup_{x\in W} B^S_{r_x}(x) = \bigcup_{x\in W} (B^X_{r_x}(x) \cap S) = \left(\bigcup_{x\in W} B^X_{r_x}(x)\right) \cap S. \tag{2.54}$$

The open set $U \subseteq X$ can be just taken as

$$U = \bigcup_{x\in W} B^X_{r_x}(x). \tag{2.55}$$

" $\Longleftarrow$ ": Assume that $W = U \cap S$ for some open set $U \subseteq X$. Since U is open in X and $W \subseteq U$, for any $x \in W$, there exists $r_x > 0$ that satisfies $B^X_{r_x}(x) \subseteq U$. Therefore,

$$B^S_{r_x}(x) = B^X_{r_x}(x) \cap S \subseteq U \cap S = W, \tag{2.56}$$

which implies that W is open. $\square$

Next, we define continuity of a function on a metric space.

Definition 2.16 (Continuity and homeomorphism). *Let (X, d_X) and (Y, d_Y) be metric spaces, and let us take a function $f : X \to Y$*

(1) f is said to be continuous at $p \in X$ if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $f(B^X_\delta(p)) \subseteq B^Y_\epsilon(f(p))$. In other words, for any $w \in B^X_\delta(p)$,

$$d_Y(f(w), f(p)) < \epsilon. \tag{2.57}$$

(2) f is said to be continuous on X if f is continuous at any point $p \in X$.

(3) $f : X \to Y$ is called a homeomorphism if f is bijective and f, f^{-1} are both continuous.

Theorem 2.26. *Let (X, d_X) and (Y, d_Y) be metric spaces. Given a function $f : X \to Y$, then the following statements are equivalent.*

(1) f is continuous on X.
(2) For any open subset $O \subset Y$, $f^{-1}(O)$ is open in X.
(3) For any closed subset $W \subset Y$, $f^{-1}(W)$ is closed in X.
(4) For any $p \in X$ and for any neighborhood V of $f(p)$, there exists a neighborhood U of p such that $f(U) \subseteq V$.

2.5. **Examples of metric spaces.**

2.5.1. *Simple examples.*

Example 2.4. *For any set X, one can equip it with a discrete metric*

$$d(x, y) \equiv \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases} \tag{2.58}$$

Example 2.5. *The Euclidean space $\mathbb{R}^n$ can be equipped with a bounded distance*

$$d(\boldsymbol{x}, \boldsymbol{y}) \equiv \frac{\|\boldsymbol{x} - \boldsymbol{y}\|}{1 + \|\boldsymbol{x} - \boldsymbol{y}\|}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{2.59}$$

Example 2.6. *Let $1 \leqslant p \leqslant \infty$. The L^p metric on $\mathbb{R}^n$ is defined as follows: for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,*

$$\begin{aligned} d_p(\boldsymbol{x}, \boldsymbol{y}) &\equiv \left(\sum_{j=1}^n |x_j - y_j|^p \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \\ d_\infty(\boldsymbol{x}, \boldsymbol{y}) &\equiv (\sum_{j=1}^n |x_j - y_j|^p)^{\frac{1}{p}}, \quad p = \infty. \end{aligned} \tag{2.60}$$

2.5.2. *Spaces of matrices.* Let $\mathcal{M}_\mathbb{R}(m, n)$ be the set of all $(m \times n)$ matrices with real entries. One can define different metrics on $\mathcal{M}_\mathbb{R}(m, n)$.

Let $A = (a_{ij}) \in \mathcal{M}_\mathbb{R}(m, n)$ and $B = (b_{ij}) \in \mathcal{M}_\mathbb{R}(m, n)$

- (L^1-metric)
$$d_1(A,B) \equiv \max_{1\leqslant j\leqslant n} \sum_{i=1}^{m} |a_{ij} - b_{ij}|. \tag{2.61}$$
It is the maximal absolute column sum of $A - B$.
- (L^∞-metric)
$$d_1(A,B) \equiv \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij} - b_{ij}|. \tag{2.62}$$
It is the maximal absolute row sum.
- (L^p-metric with $1 \leqslant p \leqslant \infty$)
$$d_p(A,B) \equiv \sup_{v\neq 0} \frac{\|Av\|_p}{\|v\|_p}. \tag{2.63}$$

2.5.3. *Spaces of sequences.* We have three examples:
$(\mathbb{R}^\omega, d)$,

$$d(\boldsymbol{x}, \boldsymbol{y}) \equiv \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|x_j - y_j|}{1 + |x_j - y_j|} \tag{2.64}$$

$\ell^1 \equiv \{\boldsymbol{x} \in \mathbb{R}^\omega : \sum\limits_{j=1}^{\infty} |x_j| < \infty\}$,

$$d_{\ell_1}(\boldsymbol{x}, \boldsymbol{y}) \equiv \sum_{j=1}^{\infty} |x_j - y_j| \tag{2.65}$$

$\ell^2 \equiv \{\boldsymbol{x} \in \mathbb{R}^\omega : \sum\limits_{j=1}^{\infty} |x_j|^2 < \infty\}$,

$$d_{\ell_2}(\boldsymbol{x}, \boldsymbol{y}) \equiv \left(\sum_{j=1}^{\infty} |x_j - y_j|^2\right)^{\frac{1}{2}} \tag{2.66}$$

2.5.4. *Spaces of functions.* Let $C^0([a,b])$ be the set of all real valued continuous functions on $[a,b]$. One can equip it with a C^0-metric
$$d_0(f,g) \equiv \|f - g\|_{C^0([a,b])} = \max_{x\in[a,b]} |f(x) - g(x)|. \tag{2.67}$$
Then $(C^0([a,b]), d_0)$ is a complete metric space.

Let $C^1([a,b])$ be the set of all real valued C^1-functions on $[a,b]$. One can equip it with a C^1-metric
$$d_1(f,g) \equiv \|f - g\|_{C^0([a,b])} + \|f' - g'\|_{C^0([a,b])}. \tag{2.68}$$

Then $(C^1([a,b]), d_1)$ is also a complete metric space.

2.5.5. *Intrinsic metric.* For any geodesic space (X,d), one can define a metric $\hat{d}$ that is induced from the original metric d and the length structure of X.

2.5.6. *Word metric on groups.* Let G be a finitely generated group with a finite set $S = \{s_1,\ldots,s_m\}$ of generators. By definition, for any $g \in G$, one can find a sequence of generators $\{s_{k_j}\}_{j=1}^n \subseteq S \cup S^{-1}$ such that

$$g = \prod_{j=1}^{n} s_{k_j}. \tag{2.69}$$

2.6. **Constructions of metric spaces.** We will mainly discuss product metric spaces and quotient metric spaces.

Definition 2.17 (Product metric)**.**

Definition 2.18 (Quotient metric)**.** *Let (X,d) be a complete metric space and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ the set of equivalence classes. If for every $p \in X$, its equivalence class $[x]$ is closed in X, then the quotient metric $\bar{d}$ on $X/\sim$ is defined as*

$$\bar{d}([x],[y]) \equiv \inf\{d(p_1,q_1) + \ldots + d(p_n,q_n)\}. \tag{2.70}$$

Here the infimum is taken among all finite sequences $(p_1,\ldots,p_n)$, $(q_1,\ldots,q_n)$, and

$$[p_1] = [x],\ \ [q_n] = [y],\ \ [q_j] = [p_{j+1}], \quad 1 \leqslant j \leqslant n-1. \tag{2.71}$$

Here we give an example of quotient metric space.

Example 2.7. *Let $\mathbb{R}$ be the real line equipped with the standard Euclidean distance d_0. We define an equivalence relation as follows. For any $x \in \mathbb{R}$, its equivalence class is defined as*

$$[x] \equiv \{x + n : n \in \mathbb{Z}\}. \tag{2.72}$$

Then we define a metric on $\mathbb{R}/\sim$,

$$\bar{d}([x],[y]) \equiv \inf\{d(p,q) : p \in [x], q \in [y]\}. \tag{2.73}$$

One can check that the quotient space $(\mathbb{R}/\sim, \bar{d})$ is a sequentially compact space.

Example 2.8. *Let $[a,b]$ be a closed interval with $b > a$. We denote by $\sim$ the following equivalence relation on $[a,b]$: $x \sim y$ iff $x = y$ or $y = b, x = a$. Then we define a metric $\bar{d}$ on $[a,b]/\sim$ in terms of Definition 2.18. Then one can check that the quotient space $([a,b]/\sim, \bar{d})$ is a sequentially compact space. Moreover, the diameter of the quotient space is equal to $\frac{b-a}{2}$.*

Example 2.9 (Flat cone)**.** *Given $\beta \in (0,1]$,*

(S_β, d_S),

$C(S_{2\pi\beta})$

2.7. **Sequential limit and completeness.**

Definition 2.19 (Sequential limit)**.** *Let (X, d) be a metric space. A sequence $\{x_j\}_{j=1}^{\infty} \subseteq X$ is said to converge to a limit $x_0 \in X$ if for any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for any $n \geqslant N$,*

$$d(x_j, x_0) < \epsilon. \tag{2.74}$$

Lemma 2.27. *Let (X, d) be a metric space and let $S \subseteq X$. A point p is a limit point of S if and only if there exists a sequence $\{x_j\}_{j=1}^{\infty} \subseteq S$ that converges to p.*

Definition 2.20 (Cauchy sequence)**.** *Let (X, d) be a metric space. A sequence $\{x_j\}_{j=1}^{\infty} \subseteq X$ is called a Cauchy sequence if for any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for any $m, n \geqslant N$,*

$$d(x_m, x_n) < \epsilon. \tag{2.75}$$

Lemma 2.28. *Let (X, d) be a metric space. Then any converging sequence must be a Cauchy sequence.*

Proof. The proof follows immediately from triangle inequality. □

Lemma 2.29. *Let (X, d) be a metric space. If a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ has a converging subsequence, then $\{x_j\}_{j=1}^{\infty}$ itself also converges.*

Definition 2.21 (Complete metric space)**.** *A metric space (X, d) is said to be complete if any Cauchy sequence in X converges.*

Theorem 2.30. *Let (X, d) be a complete metric space. A subspace (S, d) is complete if and only if S is closed in X.*

Proof. " $\Longrightarrow$ ": Assume that (S, d) is complete. We will prove that S is closed. It suffices to show that $S' \subseteq S$. Let $x_0 \in X$ be any limit point of S. We will show that $x_0 \in S$. By Lemma 2.27, there exists a sequence $\{x_j\}_{j=1}^{\infty} \subseteq S$ such that $\lim\limits_{j \to \infty} x_j = x_0$. Note that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in S. The completeness of S implies that $\{x_j\}_{j=1}^{\infty}$ converges in S. Then the uniqueness of limit implies that $x_0 \in S$, which proves that S is closed.

" $\Longleftarrow$ ": Assume that S is closed in X. To show the completeness of the subspace (S, d), it suffices to show that for any Cauchy sequence $\{x_j\}_{j=1}^{\infty} \subseteq S$, its limit x_0 is contained in S. In fact, the completeness of (X, d) guarantees that $\{x_j\}_{j=1}^{\infty}$ converges to some point $x_0 \in X$. Note that x_0 is a limit point of S. Then the closedness of S implies that $x_0 \in S$. □

Example 2.10. $\mathbb{Q}$ *is not complete in* $\mathbb{R}$.

2.8. **Compact metric spaces.** In a metric space, the notion of compact set is a generalization of finite set. Roughly speaking, a compact set can be effectively approximated by a sequence of finite sets. We give two examples of applications of this idea.

Example 2.11. *The idea of finite point approximation enables one to implement "finite dimensional reduction". For example, given a metric space (X, d) and a subset $K \subseteq X$, we assume that for any $\delta > 0$, K is "δ-close" to a finite subset $N_\delta = \{x_1, \dots, x_m\} \subseteq K$. Then the distance information of K can be δ-approximated by the finite set N_δ together with an $m \times m$ matrix $(d_{ij})_{1 \leqslant i,j \leqslant m}$, where $d_{ij} \equiv d(x_i, x_j)$.*

Example 2.12. *Let (X, d) be a metric space and let $f : A \to \mathbb{R}$ be a real valued function defined on a finite set $A \subseteq X$. Obviously, $|f(A)| \leqslant |A|$ so that f achieves both maximum and minimum on A. This simple fact can be generalized to a powerful theorem for continuous functions on compact sets.*

For a metric space (X, d), the topology is completely determined by all its converging sequences. More precisely, we have the following theorem.

Theorem 2.31. *Let d_1 and d_2 be two metrics on X. Then the following properties are equivalent.*

(1) $U \subseteq X$ is d_1-open if and only if it is d_2-open.
(2) $A \subseteq X$ is d_1-closed if and only if it is d_2-closed.
(3) A sequence $\{x_j\}_{j=1}^\infty \subseteq X$ converges w.r.t. d_1 if and only if it is converging w.r.t. d_2.

We are now ready to introduce the definition of compact set.

Definition 2.22 (Open covering)**.** *Let (X, d) be a metric space and let $U \subseteq X$ be a subset. A collection of open sets $\mathcal{O} \equiv \{O_\beta\}_{\beta \in B}$ is called an open covering of U if*

$$U \subseteq \bigcup_{\beta \in B} O_\beta. \tag{2.76}$$

Definition 2.23 (Compact set)**.** *Let (X, d) be a metric space. A subset $K \subseteq X$ is said to be compact if any open covering of K is reduced to a finite sub-covering.*

Definition 2.24 (Sequential compactness)**.** *Given a metric space (X, d), a subset $K \subseteq X$ is called sequentially compact if any sequence has a converging subsequence.*

Lemma 2.32. *If (X, d) is a compact metric space, then it is sequentially compact.*

Proof. Suppose the conclusion does not hold. That is, one can find a sequence $\{x_n\}_{n=1}^\infty$ in X which does not have any converging subsequence.

We observe that for any $x \in X$, there exists some number $r_x > 0$ such that

$$(B_{r_x}(x) \setminus \{x\}) \cap \{x_n\}_{n=1}^\infty = \emptyset. \tag{2.77}$$

Otherwise, for some $x_0 \in X$, there exists a sequence $r_j \to 0$ and a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$d(x_{n_j}, x_0) < r_j. \tag{2.78}$$

Then $x_{n_j} \to x_0$, so the desired contradiction arises.

For any $p \in X$, let $r_p > 0$ be the number in the above observation. Then

$$\mathcal{O} \equiv \{B_{r_p}(p) : p \in X\} \tag{2.79}$$

is an open covering of X. The compactness of X implies that $\mathcal{O}$ is reduced to a finite covering

$$\mathcal{O}^* \equiv \{B_{r_k}(p_k) : 1 \leqslant k \leqslant m\}. \tag{2.80}$$

However, $\mathcal{O}^*$ contains at most finitely many points of $\{x_n\}_{n=1}^{\infty}$. So it cannot cover the whole sequence, which implies that $\mathcal{O}^*$ does not cover X. Contradiction. $\square$

The next theorem is an alternative version of Bolzano-Weierstraß Theorem.

Theorem 2.33 (Bolzano-Weierstraß). *Let $K \subset \mathbb{R}^n$ be bounded and closed, then K is sequentially compact.*

Example 2.13. *Any closed interval $[a, b] \subset \mathbb{R}^1$ is sequentially compact, and any closed metric ball $\overline{B_R(p)} \subset \mathbb{R}^n$ is sequentially compact. Any finite union of those objects is sequentially compact.*

Example 2.14. *The Cantor set $\mathscr{C} \subset [0, 1]$ is sequentially compact, uncountable, and nowhere dense.*

Here we have a brief remark regarding the nowhere density of $\mathscr{C}$. It follows from the fact that the Lebesgue measure of $\mathscr{C}$ is zero so that it cannot contain any interval.

Example 2.15. *Consider a sequence $\{\boldsymbol{e}_j\}_{j=1}^{\infty} \subset (\ell^2, d_{\ell^2})$, where $\boldsymbol{e}_j = \{x_j^{(k)}\}_{k=1}^{\infty} \in \ell^2$ is defined as*

$$x_j^{(k)} = \begin{cases} 1, & k = j, \\ 0, & k = 0. \end{cases} \tag{2.81}$$

Then for any $j \in \mathbb{Z}_+$, $d_{\ell^2}(\mathbf{0}, \boldsymbol{e}_j) = 1$, but for any $i, j \in \mathbb{Z}_+$, $d_{\ell^2}(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sqrt{2}$ so that no subsequence of $\{\boldsymbol{e}_j\}_{j=1}^{\infty}$ is converging. Therefore, closed bounded subsets in (ℓ^2, d_{ℓ^2}) may not be sequentially compact.

One can apply the same idea to construct a bounded diverging sequence in $L^2([-\pi, \pi])$. In fact, we can take an L^2-orthonormal basis $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$, where

$$\boldsymbol{e}_n = \frac{1}{\pi} \cos(n \cdot x). \tag{2.82}$$

Example 2.16. *Let $\mathbb{R}$ be the real line equipped with the bounded metric*

$$d_b(x,y) = \frac{|x-y|}{1+|x-y|}. \tag{2.83}$$

Then $(\mathbb{R}, d_b)$ is closed and bounded, but is not sequentially compact. In fact, $\mathbb{Z}_+$ has no converging subsequence since for any $j \neq k$,

$$d_b(j,k) = 1 - \frac{1}{1+|j-k|} \geqslant \frac{1}{2}. \tag{2.84}$$

Proposition 2.34. *If (X,d) is a sequentially compact metric space, then it is complete.*

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a Cauchy sequence in X. We will show that $\{x_j\}_{j=1}^{\infty}$ converges in X. By the sequential compactness of (X,d), $\{x_j\}_{j=1}^{\infty}$ has a converging subsequence $\{x_{j_m}\}_{m=1}^{\infty}$ that converges to $x_0 \in X$. Applying Lemma 2.29, $\{x_j\}_{j=1}^{\infty}$ also converges to x_0, which completes the proof. □

Corollary 2.35. *If (X,d) is a compact metric space, then it is complete.*

Proof. Since (X,d) is compact, by Lemma 2.32, (X,d) is sequentially compact. Then the conclusion follows from Proposition 2.34. □

Definition 2.25 (ϵ-net)**.** *Let (X,d) be a metric space. Given $\epsilon > 0$, a subset $X(\epsilon)$ is called an ϵ-net of X if $X(\epsilon)$ is ϵ-dense in X. That is, for any $x \in X$, there exists $p \in X(\epsilon)$ such that $d(p,x) < \epsilon$.*

To measure the boundedness of a metric space (X,d), we introduce the notation of diameter:

$$\mathrm{diam}(X) \equiv \sup\{d(p,q) | p,q \in X\}. \tag{2.85}$$

Definition 2.26. *A metric space (X,d) is called bounded if* $\mathrm{diam}(X) < \infty$.

Definition 2.27. *A metric space (X,d) is said to be totally bounded if for any $\epsilon > 0$, there exists an ϵ-net $X(\epsilon)$ that is a finite set.*

Lemma 2.36. *If (X,d) is a totally bounded metric space, then (X,d) is bounded.*

Proof. Let us fix $\epsilon = 10^{-2}$. Then there exists a finite ϵ-net $X(\epsilon) = \{p_1, \ldots, p_m\}$ of X. Let us denote

$$D_0 \equiv \max\{d(p_i,p_j) : 1 \leqslant i,j \leqslant m\}. \tag{2.86}$$

Taking any points $q, w \in X$, there exist $p_i, p_j \in X(\epsilon)$ such that

$$d(q,p_i) \leqslant 10^{-2} \quad \text{and} \quad d(w,p_j) \leqslant 10^{-2}. \tag{2.87}$$

Applying triangle inequality, we obtain a uniform upper bound of distance,

$$d(q,w) \leqslant d(q,p_i) + d(p_i,p_j) + d(p_j,w) \leqslant 2 \cdot 10^{-2} + D_0, \tag{2.88}$$

which completes the proof. □

Example 2.17. *Here we exhibit two examples of metric spaces which are bounded but not totally bounded:*

(1) The subspace $\{\boldsymbol{e}_j\}_{j=1}^{\infty} \subset (\ell^2, d_{\ell^2})$,
(2) The set $\{\overline{OQ_j}\}_{j=1}^{\infty}$ *equipped with intrinsic distance, where* $\{Q_j\}_{j=1}^{\infty} = S^1 \cap \mathbb{Q}^2$.

Lemma 2.37. *If* (X,d) *is sequentially compact, then it is totally bounded.*

Proof. We will prove the lemma by contradiction. Suppose (X,d) is not totally bounded but it is sequentially compact. Then there exists some $\epsilon > 0$ such that every ϵ-net $X(\epsilon)$ of X is an infinite set.

We are now constructing a sequence by induction. First, pick any point $x_1 \in X$. The non-existence of finite ϵ-net of X implies that $B_\epsilon(x_1)$ is a proper subset of X. Then one picks any point $x_2 \in X \setminus B_\epsilon(x_1)$. Again, the non-existence of finite ϵ-net of X implies that one can find $x_3 \in X \setminus (B_\epsilon(x_1) \cup B_\epsilon(x_2))$. In m-steps, one selects a set $\{x_1, \ldots, x_m\}$ such that

$$d(x_i, x_j) \geqslant \epsilon, \ \forall 1 \leqslant i,j \leqslant m. \tag{2.89}$$

Notice that $\bigcup\limits_{i=1}^{m} B_\epsilon(x_i)$ is a proper subset of X. Then one can pick $x_{m+1} \in X$ such that

$$d(x_i, x_j) \geqslant \epsilon, \ \forall 1 \leqslant i,j \leqslant m+1. \tag{2.90}$$

Inductively, one obtains an infinite sequence $\{x_i\}_{i=1}^{\infty}$ such that

$$d(x_i, x_j) \geqslant \epsilon, \ \forall 1 \leqslant i,j < \infty. \tag{2.91}$$

Obviously, $\{x_i\}_{i=1}^{\infty}$ does not have any Cauchy subsequence, which implies that $\{x_i\}_{i=1}^{\infty}$ has no converging subsequence. The desired contradiction arises. □

Theorem 2.38 (The Lebesgue number lemma)**.** *Let* (X,d) *be a metric space and let* $\mathcal{O} = \{O_\alpha\}_{\alpha\in\Lambda}$ *be an open covering of* X. *If* X *is sequentially compact, then there exists some* $\delta > 0$, *called a Lebesgue number for the covering* $\mathcal{O}$, *such that for any subset* $U \subseteq X$ *with* $\operatorname{diam}(U) < \delta$, *there exists some* $O_\alpha \in \mathcal{O}$ *that contains* U.

Proof. We will prove it by contradiction. Suppose for an open covering $\mathcal{O} = \{O_\alpha\}_{\alpha\in\Lambda}$ of X, no Lebesgue number can be found. That is, there exists a sequence of points $x_j \in X$ such that for any $j \in \mathbb{Z}_+$,

$$B_{j^{-1}}(x_j) \nsubseteq O_\alpha \tag{2.92}$$

for any $O_\alpha \in \mathcal{O}$. By assumption, the sequence $\{x_j\}_{j=1}^\infty$ has a converging subsequence $\{x_{j_k}\}_{k=1}^\infty$ such that

$$x_{j_k} \to x_0 \quad \text{as } k \to \infty. \tag{2.93}$$

Since $\mathcal{O}$ is a covering of X, there exists some open set $\mathcal{O}_{\alpha_0} \in \mathcal{O}$ that contains x_0. Thus, there exists some number $\epsilon > 0$ such that $B_\epsilon(x_0) \subseteq \mathcal{O}_{\alpha_0}$. By (2.93), there exists a number $K_0 = K_0(\epsilon) > 0$ such that for all $k \geqslant K_0$, $d(x_{j_k}, x_0) < \frac{\epsilon}{3}$. Applying triangle inequality, for any $y \in B_{\frac{\epsilon}{3}}(x_{j_k})$,

$$d(y, x_0) \leqslant d(y, x_{j_k}) + d(x_{j_k}, x_0) < \frac{2\epsilon}{3} < \epsilon, \tag{2.94}$$

which implies that $B_{\frac{\epsilon}{3}}(x_{j_k}) \subset B_\epsilon(x_0) \subseteq \mathcal{O}_{\alpha_0}$. The desired contradiction arises when k is sufficiently large. □

Theorem 2.39 (Heine-Borel Theorem for metric space)**.** *Let (X, d) be a metric space. Then the following statements are equivalent.*

(1) (X, d) is compact.
(2) (X, d) is sequentially compact.
(3) (X, d) is complete and totally bounded.

Proof. We will first prove (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2): Lemma 2.32.

(2) $\Longrightarrow$ (1): To prove the compactness of (X, d), we will show that any open covering $\mathcal{O}$ can be reduced to a finite covering. Since (X, d) is sequentially compact, by Theorem 2.38, let $\delta > 0$ be a Lebesgue number for $\mathcal{O}$. Applying Lemma 2.37, there exists a δ-net

$$X(\delta) = \{x_1, x_2, \ldots, x_N\}. \tag{2.95}$$

By Theorem 2.38, for any $1 \leqslant j \leqslant N$, there exists $O_{\alpha_j} \in \mathcal{O}$ that contains $B_\delta(x_j)$. So it follows that

$$X = \bigcup_{j=1}^N B_\delta(x_j) = \bigcup_{j=1}^N O_{\alpha_j}, \tag{2.96}$$

and we manage to reduce $\mathcal{O}$ to a finite covering.

Next, we prove (2) $\Longleftrightarrow$ (3).

(2) $\Longrightarrow$ (3):

Sequentially compact implies complete: Proposition 2.34

Sequentially compact implies totally bounded: Lemma 2.37.

(3) $\Longrightarrow$ (2): Let $\{x_j\}_{j=1}^\infty \subseteq X$ be any infinite sequence, and we will construct a converging subsequence. For this purpose, let $X(2^{-1})$ be a finite 2^{-1}-net of X. Then there exists some $p_1 \in X(1)$ such that $B_{2^{-1}}(p_1)$ contains infinitely many points of the sequence $\{x_j\}_{j=1}^\infty$. We pick any point x_{n_1} in the sequence that belongs to $\in B_{2^{-1}}(p_1)$. Next, we pick a finite 2^{-2}-net

$X(2^{-2})$ of $B_{2^{-1}}(p_1)$. Similarly, there exists a point $p_2 \in X(2^{-2})$ such that $B_{2^{-2}}(p_2)$ contains infinitely many points of the sequence, and we pick a point x_{n_2} in the sequence that satisfies

$$x_{n_2} \neq x_{n_1} \quad \text{and} \quad x_{n_2} \in B_{2^{-2}}(p_2). \tag{2.97}$$

Up to the jth step, we have picked j distinct points $\{x_{n_1}, \ldots, x_{n_j}\}$ from the original sequence with $n_1 < n_2 < \ldots < n_j$ such that x_{n_j} from the intersection $B_{2^{-j}}(p_j)$ and p_j is a point of a finite 2^{-j}-net of $B_{2^{-(j-1)}}(p_{j-1})$. Therefore, for any $k \leqslant j$, both x_{n_k} and x_{n_j} are contained in $B_{2^{-k}}(p_k)$. Applying triangle inequality,

$$d(x_{n_k}, x_{n_j}) \leqslant 2^{1-k}. \tag{2.98}$$

Repeating this inductive construction, we obtain a Cauchy sequence $\{x_{n_1}, \ldots, x_{n_j}, \ldots\}$. In fact, for any $\epsilon > 0$, we can choose $k_0 \in \mathbb{Z}_+$ such that $2^{1-k_0} < \epsilon$. Then for any $k_0 \leqslant i \leqslant j$,

$$d(x_{n_i}, x_{n_j}) \leqslant 2^{1-i} \leqslant 2^{1-k_0} < \epsilon. \tag{2.99}$$

Since X is complete, $\{x_{n_1}, \ldots, x_{n_j}, \ldots\}$ converges, which completes the proof. □

Corollary 2.40. *Let (X, d) be a compact metric space. Then any closed subset $K \subset X$ is compact.*

The next theorem is the Euclidean version of the Heine-Borel Theorem.

Theorem 2.41 (Heine-Borel Theorem for Euclidean space)**.** *Let $K \subseteq \mathbb{R}^n$ be a non-empty subset. Then the following are equivalent.*

(1) K is compact.
(2) K is closed and bounded.
(3) Any infinite subset $S \subseteq K$ has a limit point in K.

Proof. We will prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

To prove the step (1) $\implies$ (2). Let us first observe the following property.

Claim. Let $S \subseteq \mathbb{R}^n$ be a subset and let $p \in \mathbb{R}^n$ be a limit point of S. Let $\mathcal{O} = \{U_1, \ldots, U_m\}$ be a collection of finitely many open sets. If for any $1 \leqslant j \leqslant m$, there exists some open neighborhood W_j of p such that $U_j \cap W_j = \emptyset$, then $\mathcal{O}$ cannot be an open cover of S.

Proof of the Claim. Since the family $\mathcal{O}$ is finite, $W_0 \equiv \bigcap_{j=1}^{m} W_j$ is a non-empty open set of p. Then there exists some $r_0 > 0$ such that $B_{r_0}(p) \subset W_0$. Since p is a limit point of S, $B_{r_0}(p) \setminus \{p\}$ must contain a point of S. However, by assumption, $U_j \cap W_0 = \emptyset$ for any $1 \leqslant j \leqslant m$, that is, $\mathcal{O}$ cannot cover S. □

We are now in a position to complete the proof of (1) $\implies$ (2). We will argue by contradiction. Suppose K is compact but not closed. Let p be a limit point of K and $p \notin K$.

Then we define an open cover

$$\mathcal{O} \equiv \left\{ B_{r_x}(x) \middle| p \notin B_{r_x}(x) \cap K,\ x \in K \right\}. \tag{2.100}$$

Then any finite subcollection of $\mathcal{O}$ satisfies the assumption in the Claim, and it cannot be a finite open cover of K. But this contradicts the compactness of K.

(3) $\implies$ (1)

sequentially compact implies totally bounded

Lebesgue's Number Lemma

Corollary 2.42. *If K is a compact subset in $\mathbb{R}$, then K has a minimum and maximum.*

2.9. **Continuous functions on compact metric spaces.**

Theorem 2.43. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a continuous function. Then for any compact subset $K \subseteq X$, the image $f(K)$ is compact in Y.*

Proof. To show the compactness of Y, let us take any open covering $\mathcal{O} = \{O_\alpha\}_{\alpha\in\Lambda}$ of $f(K)$. By continuity, for any open set $O_\alpha \in \mathcal{O}$, the pre-image $f^{-1}(O_\alpha)$ is open in X. Notice that the collection of open sets $\mathcal{O}^* = \{f^{-1}(O_\alpha)\}_{\alpha\in\Lambda}$ covers K since $\mathcal{O}$ covers $f(K)$. By assumption, X is compact, which implies that there exists a finite open sub-covering

$$\mathcal{O}^*_m = \{f^{-1}(O_1), \dots, f^{-1}(O_m)\} \subseteq \mathcal{O}^* \tag{2.101}$$

of K. Then the open sets $\{O_1, \dots, O_m\}$ covers $f(K)$, which completes the proof. □

Corollary 2.44. *Let (X, d) be a compact metric space and let $f : X \to \mathbb{R}$ be a real valued continuous function on X. Then f achieves both maximum and minimum in its range.*

Proof. By Theorem 2.43, $f(X)$ is compact in $\mathbb{R}$. The Euclidean Heine-Borel Theorem implies that $f(X)$ is bounded and closed. Therefore, both $\sup f(X)$ and $\inf f(X)$ can be achieved. The proof is done. □

Theorem 2.45. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a bijective continuous function. If X is compact, then f is a homeomorphism.*

Proof. We will prove that $f^{-1} : Y \to X$ is continuous. By Theorem 2.26, it suffices to show that for any closed subset $A \subseteq X$, its image $f(A)$ under f is closed. Since X is compact, by Corollary 2.40, any closed subset $A \subseteq X$ is compact. We just proved in Theorem 2.43 that $f(A)$ is compact since f is continuous. Therefore, $f(A)$ is closed. □

3. Topological spaces and continuous functions

It is a fundamental and natural question that given a topological space $(X, \mathfrak{T})$ whether one can find a metric d on X such that the topology induced by d is the same as $\mathfrak{T}$. That is, whether every topological space is metrizable. The answer is no. Indeed, there are a large variety of non-metrizable topological spaces from various subjects in mathematics. This motivates us to investigate general topological spaces.

3.1. Basic concepts in topological spaces.

Definition 3.1 (Topology). *Let X be a set. A collection $\mathfrak{T}$ of subsets of X is called a topology on X if the following holds:*

(1) $\emptyset \in \mathfrak{T}$ and $X \in \mathfrak{T}$;
(2) if $\{U_\alpha\}_{\alpha\in\Lambda}$ is any family of members in $\mathfrak{T}$, then $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \mathfrak{T}$;
(3) if $\{U_i\}_{i=1}^m$ is a finite subset of $\mathfrak{T}$, then $\bigcap\limits_{i=1}^{m} U_i \in \mathfrak{T}$.

If $\mathfrak{T}$ is a topology on X, then every member $U \in \mathfrak{T}$ is called an open set.

Example 3.1. *Let $X = \{a, b, c\}$ be a finite set. There are different topologies:*

(1) $\mathfrak{T}_1 = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$;
(2) $\mathfrak{T}_2 = \{X, \emptyset\}$;
(3) $\mathfrak{T}_3 = \mathscr{P}(X)$.

Example 3.2. *Let X be a non-empty set. Then one can define two simple topologies:*

- *(Trivial topology) $\mathfrak{T} = \{\emptyset, X\}$.*
- *(Discrete topology) $\mathfrak{S} = \mathscr{P}(X)$.*

Definition 3.2 (Basis). *Let X be a set. A basis $\mathfrak{B}$ for a topology on X is a collection of subsets of X that satisfy*

(1) for any $p \in X$, there exists some $B \in \mathfrak{B}$ that contains p, that is, $X = \bigcup\limits_{B\in\mathfrak{B}} B$;
(2) if $p \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathfrak{B}$, there exists $B_3 \subseteq B_1 \cap B_2$ that contains p.

In this situation, we define the collection $\mathfrak{T}$ **generated by** $\mathfrak{B}$ *as follows: A subset $U \subseteq X$ is an element of $\mathfrak{T}$ if for any $p \in U$, there exists some $B \in \mathfrak{B}$ such that*

$$p \in B \subseteq U. \tag{3.1}$$

In particular, $\mathfrak{B} \subseteq \mathfrak{T}$.

Example 3.3. *Different bases of the standard Euclidean topology on $\mathbb{R}^n$:*

- *Collection of all open balls*

$$\{B_r(x) : r \geqslant 0,\ x \in \mathbb{R}^n\}. \tag{3.2}$$

- *Collection of all open cubes*

$$\left\{\prod_{i=1}^{n}(a_i, b_i) : a_i \leqslant b_i\right\}. \tag{3.3}$$

Example 3.4. *Let (X, d) be a metric space. The collection $\mathfrak{T}$ of all open subsets of X is a topology on X. The collection $\mathfrak{B}$ of all open metric balls in X is a basis for the topology $\mathfrak{T}$ on X.*

Lemma 3.1. *Let X be a set with a basis $\mathfrak{B}$. Let $\mathfrak{T}$ be the collection of sets that is generated by $\mathfrak{B}$. Then $\mathfrak{T}$ is a topology on X.*

Proof. Since $X = \bigcup\limits_{B\in\mathfrak{B}} B$, we have $X \in \mathfrak{T}$. Also, it is automatically true that $\emptyset \in \mathfrak{T}$. Next, given a family of sets $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda} \subseteq \mathfrak{T}$, by definition, for any $\alpha \in \Lambda$ and for any $x \in U_\alpha$, there exists $B_{x,\alpha} \in \mathfrak{B}$ such that $x \in B_{x,\alpha} \subseteq U_\alpha$. Now for every $x \in \bigcup\limits_{\alpha\in\Lambda} U_\alpha$, there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$, which implies that there exists some $B_{x,\alpha_0} \in \mathfrak{B}$ such that $x \in B_{x,\alpha_0} \subseteq U_{\alpha_0}$. This immediately implies that $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \mathfrak{T}$.

We are now ready to show that $\mathfrak{T}$ is closed under the operation of finite intersection. That is, $U_1, \ldots, U_m \in \mathfrak{T}$ implies $\bigcap\limits_{i=1}^{m} U_i \in \mathfrak{T}$. We first prove the case when $m = 2$. Let $U_1, U_2 \in \mathfrak{T}$. Taking any $x \in U_1 \cap U_2$, by the definition of $\mathfrak{T}$, there exist $B_1, B_2 \subseteq \mathfrak{B}$ which both contain x. Since $\mathfrak{B}$ is a basis, there exists $B_3 \subseteq B_1 \cap B_2$ that contains x as well. Therefore, $U_1 \cap U_2 \in \mathfrak{T}$. For general $m \geqslant 2$, we will prove by induction. Assume that the property holds for any $k \leqslant m - 1$. Then

$$\bigcap_{i=1}^{m} U_i = \left(\bigcap_{i=1}^{m-1} U_i\right) \cap U_m. \tag{3.4}$$

By the induction hypothesis, $\bigcap\limits_{i=1}^{m-1} U_i \in \mathfrak{T}$. Then applying the base step, we conclude that $\bigcap\limits_{i=1}^{m} U_i \in \mathfrak{T}$, which completes the proof. □

Given a basis $\mathfrak{B}$, one can represent any open set in the topology generated by $\mathfrak{B}$ in terms of elements in $\mathfrak{B}$. Conversely, given a topology $\mathfrak{T}$, one can find a basis $\mathfrak{B}$ that generated $\mathfrak{T}$.

Lemma 3.2. *Let X be a set. Then the following holds:*

(1) Let $\mathfrak{B}$ be a basis for a topology $\mathfrak{T}$ on X. Then

$$\mathfrak{T} = \left\{U \subseteq X : U = \bigcup_{\alpha\in\Lambda} B_\alpha, B_\alpha \in \mathfrak{B}\right\}. \tag{3.5}$$

(2) Let $\mathfrak{D}$ be a collection of open sets of X with respect to a topology $\mathfrak{T}$ such that for any $U \in \mathfrak{T}$ and for any $x \in U$, there exists some $D \in \mathfrak{D}$ such that $x \in D \in \mathfrak{D}$. Then $\mathfrak{D}$ is a basis for the topology $\mathfrak{T}$ on X.

Proof. The proof of (1) is tautological. Given any set $U \in \mathfrak{T}$, by definition, for any $x \in U$, there exists some $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} B_x. \tag{3.6}$$

Now let us prove (2).

First, we will show that $\mathfrak{D}$ is a basis. We notice that $D_1 \cap D_2 \in \mathfrak{T}$ for any $D_1, D_2 \in \mathfrak{D}$. Then by the definition of $\mathfrak{D}$, for any $x \in D_1 \cap D_2$, there exists some $D \in \mathfrak{D}$ that contains $x \in D \subseteq D_1 \cap D_2$. It is easy to show that arbitrary union is closed in $\mathfrak{D}$.

Next, we will show that $\mathfrak{T}$ is generated by $\mathfrak{D}$. Let $\mathfrak{T}'$ be the topology generated by the basis $\mathfrak{D}$. By the definition of $\mathfrak{D}$, any element $U \in \mathfrak{T}$ is naturally contained in $\mathfrak{T}'$. Now let us take any $W \in \mathfrak{T}'$ and item (1) implies that W can be represented as $W = \bigcup_{\alpha \in \Lambda} D_\alpha$ with $D_\alpha \in \mathfrak{D}$. Since each basis element D_α is open for $\mathfrak{T}$, we have $W \in \mathfrak{T}$. □

Given two topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on a given set X, if $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$, then $\mathfrak{T}_2$ is said to be **finer** than $\mathfrak{T}_1$ ($\mathfrak{T}_1$ is called **coarser** than $\mathfrak{T}_2$).

Lemma 3.3. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathfrak{T}$ and $\mathfrak{T}'$ on X, respectively. Then the following are equivalent:*

(1) $\mathfrak{T}'$ is finer than $\mathfrak{T}$.
(2) For each $x \in X$ and $B \in \mathfrak{B}$ with $x \in B$, there exists some element $B' \in \mathfrak{B}'$ such that

$$x \in B' \subseteq B. \tag{3.7}$$

Proof. We first prove (2) $\implies$ (1). Taking any $U \in \mathfrak{T}$ and $x \in U$, there exists some $B \in \mathfrak{B}$ such that $x \in B \subseteq U$. By (2), one can find some $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$. Since $\mathfrak{T}'$ is the topology generated by $\mathfrak{B}'$, by definition, U is also an open set in $\mathfrak{T}'$.

Next, we prove (1) $\implies$ (2). Let us take any $x \in X$ and $B \in \mathfrak{B}$ such that $x \in B \in \mathfrak{B}$. Since $\mathfrak{B} \subset \mathfrak{T}$, we have $B \in \mathfrak{T}$. By assumption, $B \in \mathfrak{T}'$, which implies that there exists $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. The proof is done. □

Definition 3.3 (Subspace topology)**.** *Let $(X, \mathfrak{T})$ be a topological space. Given a subset $A \subseteq X$, the collection*

$$\mathfrak{T}_A \equiv \{A \cap U | U \in \mathfrak{T}\} \tag{3.8}$$

is a topology on A, which is said to be the subspace topology. $(Y, \mathfrak{T}_A)$ is called a subspace of $(X, \mathfrak{T})$.

Lemma 3.4. *Let $\mathfrak{B}$ be a basis for $(X, \mathfrak{T})$ and let $Y \subseteq X$ be a subspace. Then the collection*

$$\mathfrak{B}_Y \equiv \{B \cap Y | B \in \mathfrak{B}\} \tag{3.9}$$

is a basis for the subspace topology of Y.

3.2. **Closed sets and limit points.**

Definition 3.4 (Closed set)**.** *Let X be a topological space. A set A is said to be closed if $X \setminus A$ is open.*

Lemma 3.5. *Let X be a topological space. Then the following holds.*

(1) $\emptyset$ and X are closed.
(2) Arbitrary intersections of closed sets are closed.
(3) Finite unions of closed sets are closed.

Lemma 3.6 (Subspace property)**.** *Let Y be a subspace of a topological space X. Then a set A is closed in Y if and only if there exists a closed set $E \subseteq X$ such that $A = E \cap Y$.*

Proof. A is closed in $Y \iff Y \setminus A$ is open in $Y \iff \exists$ an open set $U \in X$ such that $Y \setminus A = U \cap Y$.

Notice that $A = (X \setminus U) \cap Y$ and $X \setminus U$ is a closed subset in X, which completes the proof. □

Corollary 3.7. *Let Y be a closed subspace of X. A subset $A \subseteq Y$ is closed in Y if and only if it is closed in X.*

Next, we will introduce several related technical notions in the context of topological space.

Definition 3.5. *Let X be a topological space and let $A \subseteq X$. Then we define the interior and closure of A as follows:*

$$\begin{aligned} \operatorname{Int}(A) &\equiv \bigcup \{U \subseteq A | U \text{ is open in } X\}, \\ \overline{A} &\equiv \bigcap \{A \subseteq E | E \text{ is closed in } X\}. \end{aligned} \tag{3.10}$$

Points in $\operatorname{Int}(A)$ *are called interior points of A.*

Lemma 3.8. *Let Y be a subspace of X and let $A \subseteq Y$. Then*

$$\overline{A}^Y = \overline{A}^X \cap Y. \tag{3.11}$$

Proof. By definition,

$$\overline{A}^Y = \bigcap \{A \subseteq E | E \text{ is closed in } Y\}. \tag{3.12}$$

Let E be a closed subset of Y. Applying Lemma 3.6, there exists a closed subset $G \subseteq X$ such that $E = G \cap Y$. Therefore,

$$\begin{aligned}\overline{A}^Y &= \bigcap\{A \subseteq G \cap Y | G \text{ is closed in } X\} \\ &= \Big(\bigcap\{A \subseteq G | G \text{ is closed in } X\}\Big) \bigcap Y \\ &= \overline{A}^X \cap Y, \end{aligned} \tag{3.13}$$

which completes the proof. □

Theorem 3.9. *Let X be a topological space and let $A \subseteq X$.*

(1) Then $p \in \overline{A}$ if and only if every open set U that contains p satisfies $U \cap A \neq \emptyset$.
(2) Let $\mathfrak{B}$ be a basis of the topology of X. Then $p \in \overline{A}$ if and only if for any $B \in \mathfrak{B}$ that contains p satisfies $B \cap A \neq \emptyset$.

Proof. Proof of (1): we will show that $p \notin \overline{A}$ is equivalent to the property that one can find an open set U such that $p \in U$ and $U \cap A = \emptyset$. By definition, $p \notin \overline{A}$ means one can find a closed subset $E_0 \subseteq X$ such that $A \subseteq E_0$ and $p \notin E_0$. Let $U_0 = X \setminus E_0$, which is obviously open. Then $p \notin \overline{A}$ is equivalent to the property that one can find an open set U_0 such that $p \in U_0$ and $U_0 \cap A = \emptyset$.

To finish the proof of item (2), we observe that any open set U with $x \in U$, there exists an element $B \in \mathfrak{B}$ such that $x \in B \in \mathfrak{B}$. Now let us take any open set U that contains p. One can find some $B_p \in \mathfrak{B}$ that contains p such that $B_p \cap A \neq \emptyset$. Then $U \cap A \supset B_p \cap A \neq \emptyset$. □

Definition 3.6 (Limit point)**.** *Let $A \subseteq X$ be a subset. A point p is said to be a limit point of A if for any neighborhood U of p,*

$$(U \setminus \{p\}) \cap A \neq \emptyset. \tag{3.14}$$

Theorem 3.10. *Let X be a topological space and let $A \subseteq X$. Then $\overline{A} = A \cup A'$.*

Proof. Let $p \in \overline{A} \setminus A$. Taking any neighborhood U of p, we have $U \cap A = (U \setminus \{p\}) \cap A$. By Theorem 3.9 (1),

$$(U \setminus \{p\}) \cap A \neq \emptyset, \tag{3.15}$$

which implies $p \in A'$. On the other hand, let $p \in A' \setminus A$. Then for any neighborhood U of p,

$$(U \setminus \{p\}) \cap A \neq \emptyset. \tag{3.16}$$

Since $p \notin A$, we have $U \cap A = (U \setminus \{p\}) \cap A$. Therefore, $U \cap A \neq \emptyset$, which shows $p \in \overline{A}$. □

Immediately, we have the following theorem.

Theorem 3.11. *Let X be a topological space and let $A \subseteq X$. Then the following are equivalent.*

(1) A is closed.
(2) $\overline{A} = A$.
(3) A contains all its limit points.

Example 3.5. *Let X be a non-emptyset equipped with the discrete topology. Then every subset is closed in X. That is, if $A \subset X$ be a proper subset, then a point $p \in A'$ if and only if $p \in A$.*

3.3. **Sequential limits and Hausdorff space.**

Definition 3.7. *Let X be a topological space. A sequence of points $\{x_j\}_{j=1}^{\infty} \subseteq X$ is said to converge to $x_0 \in X$ if for any open neighborhood U of x_0, there exists $N_0 \in \mathbb{Z}_+$ such that $x_j \in U$ for any $j \geqslant N_0$.*

Definition 3.8 (Hausdorff space)**.** *A topological space X is said to be a Hausdorff space if for any two distinct points $p_1, p_2 \in X$, there exist neighborhoods U_1 of p_1 and U_2 of p_2 that are disjoint.*

Theorem 3.12. *Let X be a Hausdorff space. Then the following properties holds.*

(1) Every finite set is closed.
(2) If a sequence converges, then its limit is unique.
(3) Let $A \subseteq X$ be a subset. A point p is a limit point of A if and only if any neighborhood of p contains infinitely many points of A.

Proof. (1) It suffices to show that any singleton is closed. Suppose not and there exists a singleton $A = \{p\}$ that is not closed. This implies that $\overline{A}$ contains a point $q \neq p$. Since X is Hausdorff, there exists two disjoint neighborhoods U_q and U_p of q and p, respectively. This contradicts to the definition of closure.

(2) is obvious.

(3) We only need to prove " $\Longrightarrow$ ". We will argue by contradiction. Let $p \in A$ be a limit point and let U be a neighborhood of p that contains only finitely many points $S = \{p_1, \ldots, p_m\} \subset A$. For each $1 \leqslant i \leqslant m$, there exists a neighborhood U_i of p such that $p_i \notin U_i$ (since the space is Hausdorff). Then

$$S \cap \left(\bigcap_{i=1}^{m} U_i \right) = \emptyset. \tag{3.17}$$

Contradiction. ∎

3.4. **Continuous functions.**

Definition 3.9 (Continuous function)**.** *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous on X if for any open subset $V \subseteq Y$, the pre-image $f^{-1}(V)$ is open in X.*

Theorem 3.13. *Let X and Y be topological spaces, and let $f : X \to Y$ be a function defined on X. Then the following properties are equivalent.*

(1) *f is continuous on X.*
(2) *For any $A \subseteq X$, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
(3) *For any closed set $W \subseteq Y$, the pre-image $f^{-1}(W)$ is closed in X.*
(4) *For any $p \in X$ and any neighborhood V of $f(p)$, there exists a neighborhood U of p such that $f(U) \subseteq V$.*

In particular, if (4) holds for some $p \in X$, we say f is continuous at p.

Definition 3.10 (Homeomorphism)**.** *Let X, Y be topological spaces. A bijective function $f : X \to Y$ is said to be a homeomorphism if both f and f^{-1} are continuous.*

Definition 3.11. *Let X, Y be topological spaces, and let $f : X \to Y$ be an injective function. We denote by $Z \equiv f(X)$. The function f is said to be a topological embedding if $f : X \to Z$ is homeomorphism between X and Z, where Z is regarded as a subspace of Y.*

Theorem 3.14. *Let X, Y, Z be topological spaces. Then the following properties hold.*

(1) *(Constant function) Let $f : X \to Y$ be a function that satisfies $f(X) = \{y_0\} \subset Y$. Then f is continuous.*
(2) *(Inclusion) If $A \subseteq X$ is a subspace, then the inclusion function $\iota : A \hookrightarrow X$, $a \mapsto a$ is continuous.*
(3) *(Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composite $g \circ f : X \to Z$ is continuous.*
(4) *(Restriction) If $f : X \to Y$ is continuous and $A \subseteq X$ is a subspace, then $f|_A : A \to Y$ is continuous.*
(5) *(Restricting and expanding the range) Let $f : X \to Y$ be continuous:*
    - *if $Z \subseteq Y$ satisfies $f(X) \subset Z$, then the function $\bar{f} : X \to Z$ determined by $\bar{f} = f$ on X is continuous;*
    - *if $Y \subseteq Z$, then the function $\hat{f} : X \to Z$ determined by $\bar{f} = f$ on X is continuous.*
(6) *(Localization) $f : X \to Y$ is continuous if $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ for a family of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ and $f|_{U_\alpha}$ is continuous for each $\alpha \in \Lambda$.*

There is a natural question about continuous functions.

Question: given a function $f : X \to Y$, from a set X to a topological space $(Y, \mathfrak{T}_Y)$, what is the smallest topology $\mathfrak{T}_X$ of X so that f is continuous w.r.t. $\mathfrak{T}_X$?

It is natural to define a topology

$$\mathfrak{T}_X \equiv \{f^{-1}(U) : U \text{ is open in } Y\}. \tag{3.18}$$

Clearly, $\mathfrak{T}_X$ is a topology on X since we notice that

$$\bigcap_{j=1}^{m} f^{-1}(U_j) = f^{-1}\left(\bigcap_{j=1}^{m} U_j\right) \quad \text{and} \quad \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right). \tag{3.19}$$

Moreover, if $\mathfrak{T}$ is a topology on X such that f is continuous with respect to $\mathfrak{T}$, by the definition of continuity, $\mathfrak{T}_X \subseteq \mathfrak{T}$.

This discussion also leads to a more general question.

Question: Given a set X, a family of topological spaces $\{(Y_\alpha, \mathfrak{T}_\alpha)\}_{\alpha\in\Lambda}$ and a family of functions $f_\alpha : X \to Y_\alpha$. What is the smallest topology $\mathfrak{T}_X$ of X so that for each $\alpha \in \Lambda$, f_α is continuous with respect to $\mathfrak{T}_X$?

Notice that necessary conditions for $\mathfrak{T}_X$ to be such a topology are:

(1) members are closed under finite intersections;
(2) members are closed under arbitrary unions.

By the definition of continuity, any pre-image $f_\alpha^{-1}(U_\alpha)$ with $U_\alpha \in \mathfrak{T}_\alpha$ must be an element of $\mathfrak{T}_X$. The collection $\mathfrak{S}_X$ of all "open" pre-images $f_\alpha^{-1}(U_\alpha)$ is not a topology yet since it doest not satisfy (1) or (2).

Condition (1) requires $\mathfrak{T}$ to satisfy this property: for any finite subset $\{\alpha_1, \ldots, \alpha_m\} \subset \Lambda$,

$$U_{\alpha_i} \in \mathfrak{T}_{\alpha_i} \implies \bigcap_{i=1}^{m} f_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathfrak{T}_X. \tag{3.20}$$

So an effective way of defining $\mathfrak{T}_X$ is to construct a basis $\mathfrak{B}_X$ that contains all finite intersections as in (3.20). Finally, $\mathfrak{T}_X$ is defined as the topology generated by the basis $\mathfrak{B}_X$.

This motivates us to introduce the notion of subbasis.

Definition 3.12 (Subbasis)**.** *Let X be a set. A subbasis $\mathfrak{S}$ for a topology of X is a collection of subsets of X such that $\bigcup_{S\in\mathfrak{S}} S = X$. The topology $\mathfrak{T}$ generated by the subbasis $\mathfrak{S}$ is defined as follows:*

$$U \in \mathfrak{T} \Longleftrightarrow U = \bigcup_{B\in\mathfrak{B}} B, \tag{3.21}$$

where

$$\mathfrak{B} \equiv \left\{ B \subseteq X : B = \bigcap_{i=1}^{m} S_i,\ S_i \in \mathfrak{S} \right\}. \tag{3.22}$$

Example 3.6. $\mathfrak{S} \equiv \{(a,\infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$ *is a subbasis of* $\mathbb{R}$.

Lemma 3.15. *Let X and Y be topological spaces, and let $f : X \to Y$ be a function defined on X. Then the following properties hold.*

(1) If $\mathfrak{B}$ is a basis for the topology of Y, then f is continuous on X if and only if $f^{-1}(B)$ is open for any $B \in \mathfrak{B}$.
(2) If $\mathfrak{S}$ is a subbasis for the topology of Y, then f is continuous on X if and only if $f^{-1}(S)$ is open for any $S \in \mathfrak{S}$.

Proof. (1) Given any open set $U \subseteq Y$, one can write it as $U = \bigcup_{\alpha\in\Lambda} B_\alpha$ with $B_\alpha \in \mathfrak{B}$. Then

$$f^{-1}(U) = \bigcup_{\alpha\in\Lambda} f^{-1}(B_\alpha), \tag{3.23}$$

which is open in X.

(2) Let $B \in \mathfrak{B}$ be any element. Then one can write B as $B = \bigcap_{i=1}^{m} S_i$. Therefore,

$$f^{-1}(B) = \bigcap_{i=1}^{m} f^{-1}(S_i), \tag{3.24}$$

which is open. By (1), we conclude that f is continuous. □

Corollary 3.16. *Let $f : U \to \mathbb{R}$ be a real valued function defined on an open set $U \subset \mathbb{R}^n$. Then the following statements are equivalent:*
(1) f is continuous on U.
(2) For any $t \in \mathbb{R}$, both the sub-level set $f^{-1}((-\infty, t))$ and super-level set $f^{-1}((t, \infty))$ are open.

Example 3.7 (Lower semicontinuous and upper semicontinuous functions)**.** *Let U be an open subset in $\mathbb{R}$. A function $f : U \to \mathbb{R}$ is said to be*

- *upper semicontinuous if for every $t \in \mathbb{R}$, the pre-image $f^{-1}((-\infty, t))$ is open;*
- *lower semicontinuous if for every $t \in \mathbb{R}$, the pre-image $f^{-1}((t, \infty))$ is open.*

Therefore, a function is continuous if and only if it is both lower semicontinuous and upper semicontinuous. There are two simple graphs.

3.5. **The (finite) product topology.** This subsection is to define the product topology on a product space with finite factors.

Definition 3.13 (Product topology)**.** *Let $(X_1, \mathfrak{T}_1), \dots, (X_m, \mathfrak{T}_m)$ be topological spaces. The product topology $\mathfrak{T}$ on $\prod_{i=1}^m X_i$ is the topology generated by a basis*

$$\mathfrak{B} \equiv \left\{ \prod_{i=1}^m U_i : U_i \in \mathfrak{T}_i \right\}. \tag{3.25}$$

Lemma 3.17. *In the above definition, $\mathfrak{B}$ is a basis on $\prod_{i=1}^m X_i$.*

Proof. Given two sequences of open sets $U_i \in \mathfrak{T}_i$ and $V_i \in \mathfrak{T}_i$, we have

$$\left(\prod_{i=1}^m U_i \right) \cap \left(\prod_{i=1}^m V_i \right) = \prod_{i=1}^m (U_i \cap V_i). \tag{3.26}$$

For each $1 \leqslant i \leqslant m$, $U_i \cap V_i \in \mathfrak{T}_i$. Then

$$\left(\prod_{i=1}^m U_i \right) \cap \left(\prod_{i=1}^m V_i \right) \in \mathfrak{B}. \tag{3.27}$$

which completes the proof. □

Lemma 3.18. *Let X_i, $1 \leqslant i \leqslant m$, be a sequence of topological spaces with bases $\mathfrak{B}_i$'s for the topologies of X_i's. Then the collection*

$$\widehat{\mathfrak{B}} \equiv \left\{ \prod_{i=1}^m B_i : B_i \in \mathfrak{B}_i \right\} \tag{3.28}$$

is a basis for the product topology of $\prod_{i=1}^m X_i$.

Proof. Let U be any open set in $\prod_{i=1}^m X_i$. We denote by $\mathfrak{B}$ be a basis for the product topology. Then for any $p = (p_1, \dots, p_m) \in U$, there exists $\prod_{i=1}^m U_i$ such that, for each $1 \leqslant i \leqslant m$, U_i is open in X_i and

$$p \in \prod_{i=1}^m U_i \subseteq U. \tag{3.29}$$

Now for any $1 \leqslant i \leqslant m$, there exists $B_i \in \mathfrak{B}_i$ such that $p_i \in B_i \subseteq U_i$ since $\mathfrak{B}_i$ is a basis for the topology of X_i, which implies that

$$p \in \prod_{i=1}^m B_i \subseteq \prod_{i=1}^m U_i \subseteq U. \tag{3.30}$$

This completes the proof. □

Theorem 3.19. *Let X_i, $1 \leqslant i \leqslant m$, be topological spaces. If for each $1 \leqslant i \leqslant m$, $A_i \subseteq X_i$ is a subspace. Then $\prod_{i=1}^{m} A_i$ is a subspace of $\prod_{i=1}^{m} X_i$.*

Definition 3.14 (Projection)**.** *Let $X \equiv \prod_{i=1}^{m} X_i$ be a product space. For each $1 \leqslant i \leqslant m$, the projection $\pi_i : X \equiv X_i$ onto the i-th factor is defined as*

$$\pi_i(p_1, \ldots, p_m) = p_i. \tag{3.31}$$

Lemma 3.20. *Let $X \equiv \prod_{i=1}^{m} X_i$ be a product space. Then for each $1 \leqslant i \leq m$,*

(1) $\pi_i : X \to X_i$ is surjective;
(2) $\pi_i^{-1}(U_i) = X_1 \times \ldots \times X_{i-1} \times U_i \times X_{i+1} \times \ldots \times X_m$ for any $U_i \subseteq X_i$;
(3) $\bigcap_{i=1}^{m} \pi_i^{-1}(U_i) = \prod_{i=1}^{m} U_i$ for any $U_i \subseteq X_i$.

In particular, each projection map π_i is continuous.

Theorem 3.21. *Let $X \equiv \prod_{i=1}^{m} X_i$ be a product space of a sequence of topological spaces $(X_i, \mathfrak{T}_i)$. Then the collection*

$$\mathfrak{S} \equiv \bigcup_{i=1}^{m} \{\pi_i^{-1}(U_i) : U_i \text{ is open in } X_i\} \tag{3.32}$$

is a subbasis for the product topology of X.

Proof. Let $\prod_{i=1}^{m} U_i \in \mathfrak{B}$ with $U_i \in \mathfrak{T}_i$ be any base element. Then

$$\prod_{i=1}^{m} U_i = \bigcap_{i=1}^{m} \pi_i^{-1}(U_i), \tag{3.33}$$

which implies that $\mathfrak{S}$ is a basis for the product topology of X. □

Theorem 3.22. *Let A be a topological space and Let $X \equiv \prod_{i=1}^{m} X_i$ be a finite Cartesian product. Then the product topology on X is the smallest topology that makes each projection $\pi_i : X \to X_i$ is continuous.*

Theorem 3.23 (Maps into products)**.** *Let A be a topological space and let $X \equiv \prod_{i=1}^{m} X_i$ be a finite product space. Then a function $f : A \to X$ is continuous if and only if for any $1 \leqslant i \leqslant m$, $f_i \equiv \pi_i \circ f : A \to X_i$ is continuous.*

Proof. Proof of " $\Longrightarrow$ " is trivial.

Now we prove the other direction " $\Longleftarrow$ ". Let $\mathfrak{S}$ be a subbasis for the product topology of X. It suffices to show that for any member S in the subbasis $\mathfrak{S}$, its pre-image $f^{-1}(S)$ is open in A. In fact, by definition, any subbasis element S can be written as $S = \pi_i^{-1}(U_i)$ for some open set $U_i \in X_i$. Then

$$f^{-1}(S) = f^{-1} \circ \pi_i^{-1}(U_i) = f_i^{-1}(U_i). \tag{3.34}$$

The continuity of f_i implies that $f_i^{-1}(U_i)$ is open in A, which completes the proof. $\square$

Example 3.8. *Given continuous functions f_1, f_2, f_3 on $\mathbb{R}^3$, the map*

$$V(p) = (f_1(p), f_2(p), f_3(p)),\ \ p \in \mathbb{R}^3, \tag{3.35}$$

is a continuous vector field on $\mathbb{R}^3$.

Theorem 3.24. *Let Z be any topological space and let $\{X_i\}_{i=1}^m$ be a sequence of topological spaces. For each $1 \leqslant i \leqslant m$, let $f_i : Z \to X_i$ be a continuous function. Denote by $X \equiv \prod_{i=1}^m X_i$ the product space. Then there exists a unique continuous function $f : Z \to X$ such that $f_i = \pi_i \circ f$ for each $1 \leqslant i \leqslant m$. In other words, the following diagram commutes:*

$$\begin{tikzcd} Z \arrow[r, dashed, "f"] \arrow[dr, "f_i"'] & X \arrow[d, "\pi_i"] \\ & X_i. \end{tikzcd} \tag{3.36}$$

The digram in the case of $m = 2$ is

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, dashed, "f"] & \\ X_1 & & X_2 \\ & X_1 \times X_2 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & . \end{tikzcd} \tag{3.37}$$

3.6. **Box topology and product topology.** In this subsection, we will define the box topology and product topology on a general (possibly infinite) product set.

Let $X \equiv \prod_{\alpha\in\Lambda}$ be the product set of a family of topological spaces $\{X_\alpha\}_{\alpha\in\Lambda} X_\alpha$. We define the box topology of X as follows.

Definition 3.15 (Box topology)**.** *We define the collection*

$$\mathfrak{B}_\square \equiv \left\{ \prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \text{ is open in } X_\alpha \right\} \tag{3.38}$$

as the basis for the box topology of the product set X.

Definition 3.16 (Product topology). *On the product set $X = \prod_{\alpha\in\Lambda} X_\alpha$, we denote by*

$$\mathfrak{S} \equiv \bigcup_{\alpha\in\Lambda} \left\{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \text{ is open in } X_\alpha\right\}. \tag{3.39}$$

The topology generated by the subbasis $\mathfrak{S}$ is called the product topology of X. In the product topology, the product set X is called a product space.

Lemma 3.25. *In the product topology, let $\mathfrak{B}$ be the basis generated by $\mathfrak{S}$. Then any $B \in \mathfrak{B}$ can be written as*

$$B = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}). \tag{3.40}$$

Theorem 3.26 (Comparison of the box and product topologies). *The box topology on $X = \prod_{\alpha\in\Lambda} U_\alpha$ has basis*

$$\mathfrak{B}_\Box \equiv \left\{\prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \text{ is open in } X_\alpha\right\}. \tag{3.41}$$

The product topology on X has basis

$$\mathfrak{B} \equiv \left\{\prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \text{ is open in } X_\alpha, \text{ and } U_\alpha = X_\alpha \text{ except for finitely many } \alpha' s\right\}. \tag{3.42}$$

Box topology is in general finer than product topology.

Theorem 3.27. *For each $\alpha \in \Lambda$, let $\mathfrak{B}_\alpha$ be a basis for the topology of X_α. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the product set. Then*

(1) the collection of all sets of the form $\prod_\alpha B_\alpha$, where $B_\alpha \in \mathfrak{B}_\alpha$ for each $\alpha \in \Lambda$, is a basis for the box topology of X;

(2) The collection of all sets of the form $\prod_\alpha B_\alpha$, where $B_\alpha \in \mathfrak{B}_\alpha$ for finitely many α's and $B_\alpha = X_\alpha$ for all the remaining indices, is a basis for the product topology of X.

Theorem 3.28. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the product set of a family of topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$. Then for each $\alpha \in \Lambda$, the projection function π_α is continuous with respect to either the box topology or the product topology of X.*

Theorem 3.29 (Subspace property and closure property). *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of topological spaces. For each $\alpha \in \Lambda$, let A_α be a subspace of X_α. Then the following properties hold.*

(1) $\prod_{\alpha\in\Lambda} A_\alpha$ is a subspace of $X = \prod_{\alpha\in\Lambda} X_\alpha$ with respect to either the box topology or the product topology of X.

(2) The equality

$$\prod_{\alpha\in\Lambda}\overline{A}_\alpha = \overline{\prod_{\alpha\in\Lambda}A_\alpha} \tag{3.43}$$

holds for either the box topology or the product topology of X.

Theorem 3.30. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be a product topological space and let A be a topological space. A function $f : A \to X$ is continuous if and only if $f_\alpha = \pi_\alpha \circ f$ is continuous on A for each $\alpha \in \Lambda$.*

Proof. The proof of the direction " $\Longrightarrow$ " is easy since f_α is the composite of two continuous functions.

Now we will prove " $\Longleftarrow$ ". It suffices to show that for any element $\pi_\alpha^{-1}(U_\alpha)$, in the subbasis for the product topology, where $\alpha \in \Lambda$ and $U_\alpha \in \mathfrak{T}_\alpha$, the pre-image $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open. In fact,

$$f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha). \tag{3.44}$$

Since f_α is continuous for any $\alpha \in \Lambda$, the pre-image $f_\alpha^{-1}(U_\alpha)$ is open. □

Example 3.9. *Let $\mathbb{R}^\omega$ be the space equipped with the box topology. Consider the function $f : \mathbb{R} \to \mathbb{R}^\omega$ defined by $f_i(t) = t$ for any $i \in \mathbb{Z}_+$ and $t \in \mathbb{R}$, where $f_i = \pi_i \circ f$. We consider the open set $U \equiv \prod_{j=1}^{\infty}(-j^{-1}, j^{-1})$ in the box topology of $\mathbb{R}^\omega$. Then*

$$f^{-1}(U) = \bigcap_{j\in\mathbb{Z}_+}(-j^{-1}, j^{-1}) = \{0\}. \tag{3.45}$$

But $\{0\}$ is not open in $\mathbb{R}$.

Example 3.10 (General circular cylinder and solid torus)**.** *Let X be a topological space. One can construct a general circular cylinder $X \times S^1$ as a subspace of $X \times \mathbb{C}$ (using a quotient map that is similar to the previous one). The product space $X \times \mathbb{R}$ is called an infinite cylinder, and $X \times [a, b]$ is called a finite cylinder. The product space $\mathbb{D}^2 \times S^1$ is also known as the solid torus.*

3.7. **Equivalence relation and quotient set.** Let X be a set and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ the set of all equivalence classes.

Definition 3.17 (Equivalence relations)**.** *Given a set A, an equivalence relation on A is a relation $C \subseteq A \times A$ that satisfies the following properties.*

- *(Reflexivity) $\forall x \in A$, xCx.*
- *(Symmetry) $xCy \implies yCx$.*
- *(Transitivity) $[xCy$ and $yCz] \implies xCz$.*

The case xCy is called x is equivalent to y, which is also denoted as $x \sim y$. Given $x \in A$, the set

$$[x] = E_x \equiv \{y \in A | y \sim x\} \tag{3.46}$$

is called the **equivalence class** *determined by x.*

Lemma 3.31. *Let C be an equivalence relation on A. For any two equivalence classes E_1 and E_2, either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.*

Definition 3.18 (Partition)**.** *A partition of a set A is a collection of sets $\mathscr{E} \subseteq \mathscr{P}(A)$ such that the following holds:*

- $\bigcup_{E \in \mathscr{E}} E = A$.
- $\forall\ X, Y \in \mathscr{E}$, *either $X = Y$ or $X \cap Y = \emptyset$ holds.*

Clearly, given an equivalence relation C on A, one can obtain a partition $\mathscr{E}$ of A so that each member of $\mathscr{E}$ is an equivalence class. In this case, $\mathscr{E}$ is called the **partition induced by the equivalence relation**.

Example 3.11 ("circle")**.** *Let $\mathbb{R}$ be the real line. We define an equivalence relation $\sim$ on $\mathbb{R}$ by*

$$x \sim y \Longleftrightarrow \exists n \in \mathbb{Z} \text{ s.t. } x = y + n. \tag{3.47}$$

Then for any $x \in \mathbb{R}$, the equivalence class $[x]$ that contains x is given by

$$[x] = \{x + n | n \in \mathbb{Z}\}. \tag{3.48}$$

One can quickly check that

$$[x] = [y] \Longleftrightarrow x - y \in \mathbb{Z}. \tag{3.49}$$

Also, it is easy to check that $\mathbb{R}$ can be represented as the union of distinct equivalence classes:

$$\mathbb{R} = \bigcup_{x \in [0,1)} [x]. \tag{3.50}$$

Namely, the equivalence relation $\sim$ gives a partition of $\mathbb{R}$.

We denote by S^1 the quotient set $\mathbb{R}/\sim$.

Recall that any partition of X also provides an equivalence relation.

Lemma 3.32. *Let X be a set. Given any partition $\mathscr{D}$ of X, there exists a unique equivalence relation $\sim$ on X that induces $\mathscr{D}$.*

The following example is important in applications.

Example 3.12 (Partition associated to a surjective map)**.** *Let $\pi : X \to Y$ be a surjective map between two sets. Then there is a natural partition of X corresponding to π.*

We define a collection $\mathscr{P}$ of subsets of X by

$$\mathscr{P} \equiv \{\pi^{-1}(y) : y \in Y\}. \tag{3.51}$$

It is clear that for distinct $y_1, y_2 \in Y$,

$$\pi^{-1}(y_1) \cap \pi^{-1}(y_2) = \emptyset \tag{3.52}$$

and

$$X = \bigcup_{y \in Y} \pi^{-1}(y). \tag{3.53}$$

Therefore, $\mathscr{P}$ is a partition of X.

A concrete example is the natural projection

$$\pi_1 : X_1 \times X_2 \to X_1, (p_1, p_2) \mapsto p_1. \tag{3.54}$$

Obviously, π_1 is surjective. Then π defines a partition of the product set $X \equiv X_1 \times X_2$:

$$\mathscr{P} \equiv \{\pi_1^{-1}(p_1) : p_1 \in X_1\}, \tag{3.55}$$

where for each $p_1 \in X_1$, the pre-image $\pi_1^{-1}(p_1) = \{p_1\} \times X_2$. The product set $X \equiv X_1 \times X_2$ can be also represented as the union

$$X = X_1 \times X_2 = \bigcup_{p_1 \in X_1} \{p_1\} \times X_2. \tag{3.56}$$

Example 3.13 (Equivalence class from a partition)**.** *Let $\pi : X \to Y$ be a surjection. We consider the partition defined $\mathscr{P}$ in Example 3.12. Then we have an equivalence relation $\sim$: letting $p, q \in X$,*

$$p \sim q \Longleftrightarrow \pi(p) = \pi(q). \tag{3.57}$$

Then for any $p \in X$, the equivalence class $[p]$ contains all points $w \in X$ that satisfies $\pi(w) = \pi(p)$. We denote by X^* the quotient set $X/\sim$.

Now we give a concrete example. Consider a real valued function

$$f : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (\cos 2\pi t, \sin 2\pi t). \tag{3.58}$$

Let $X = \mathbb{R}$ and $Y = f(X) \subset \mathbb{R}^2$. Obviously $f : X \to Y$ is a surjection. We will describe the partition $\mathscr{P}$ of X consisting of the pre-images of the surjective map f.

For any fixed $\boldsymbol{y} \in f(X)$, let us analyze it pre-image $f^{-1}(\boldsymbol{y})$. Observe that any point $\boldsymbol{y} \in \mathbb{R}^2$ that satisfies $\|\boldsymbol{y}\| = 1$ is a point contained in Y. It follows from basic properties of trigonometric functions that $f(t) = f(s)$ if and only if there exists an integer $n \in \mathbb{Z}$ such that

$t = s + n$. On the other hand, the equivalence relation $\sim_{\mathscr{P}}$ induced by $\mathscr{P}$ can be described as: for any $t \in \mathbb{R}$,

$$[t] = \{t + n | n \in \mathbb{Z}\} = f^{-1}(f(t)) \tag{3.59}$$

The partition $\mathscr{P}$ associated to f can be represented as

$$\mathbb{R} = \bigcup_{t\in[0,1)} [t], \tag{3.60}$$

which is the same as the partition in Example 3.11. Then the quotient set $\mathbb{R}/\sim_{\mathscr{P}}$ is the same as the the quotient set in Example 3.11. Therefore, $\mathbb{R}/\sim_{\mathscr{P}} = S^1$.

We also notice that the function f satisfies an important property. For any $\boldsymbol{y} \in f(X)$, the function f is constant on the pre-image $f^{-1}(\boldsymbol{y})$. Indeed, for any $t, s \in f^{-1}(\boldsymbol{y})$, we have $t = s + n$ for some integer $n \in \mathbb{Z}$. Then

$$f(t) = f(s). \tag{3.61}$$

Let $\pi : X \to Y$ be a surjection. The next example relates the functions on Y and the functions on X.

Example 3.14. *Let Z be any set and let $f : Y \to Z$ be a function. Then the composite $h \equiv f \circ \pi$ is a function from X to Z. We observe that the function $h : X \to Z$ is a "periodic" function in the sense that for any $y \in Y$ and for any $p_1, p_2 \in \pi^{-1}(y)$, we have $h(p_1) = h(p_2)$. Indeed, $h(p_1) = f \circ \pi(p_1) = f(y) = f \circ \pi(p_2) = h(p_2)$.*

In the previous example, one can construct a large variety of "periodic" functions on $\mathbb{R}$ that are constant on each pre-image of f. For example, $h_1(t) \equiv \cos(4\pi t)$, $h_2(t) = \exp(\sin^4(10\pi t))$, and $h_3(t) = \sin^6(8\pi t)$ are all constant on each pre-image of f since for every $i \in \{1, 2, 3\}$,

$$h_i(t) = h_i(t + n), \forall\ t \in \mathbb{R},\ n \in \mathbb{Z}. \tag{3.62}$$

The following lemma is very useful.

Lemma 3.33. *Let $\pi : X \to Y$ be a surjection. Let Z be an arbitrary set. If $h : X \to Z$ is a function that is constant on each pre-image of π, then one can define a function $f : Y \to Z$ that satisfies $h = f \circ \pi$. In other words, the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd} \tag{3.63}$$

Proof. Let $y \in Y$ be any element and let us take any arbitrary representative $x \in \pi^{-1}(y)$, we define

$$f(y) \equiv h(x). \tag{3.64}$$

First, we must prove that the definition of f is independent of the choice of the representative $x \in \pi^{-1}(y)$. In other words, f is well-defined. In fact, if $x_1, x_2 \in \pi^{-1}(y)$, then $\pi(x_1) = \pi(x_2) = y$. By the assumption that h is constant on each pre-image of π,

$$h(x_1) = h(x_2) \tag{3.65}$$

since x_1 and x_2 belong to the same pre-image of π. This implies that f is well-defined.

Next, by definition, for any $x \in X$ with $y = \pi(x)$, we have that $h(x) = f(y) = f \circ \pi(x)$. That is, the diagram (3.63) commutes. □

Example 3.15. *Each of the functions h_1, h_2, h_3 in Example 3.14 is constant on every pre-image of $f : \mathbb{R} \to f(\mathbb{R})$, where f is defined as*

$$f(t) = (\cos(2\pi t), \sin(2\pi t)). \tag{3.66}$$

By Lemma 3.33, each $h_i : \mathbb{R} \to \mathbb{R}$ induces a function $g_i : f(X) \to \mathbb{R}$ that satisfies $h_i = g_i \circ f$.

In summary, we have the following theorem.

Theorem 3.34 (Pull-back property)**.** *Let $\pi : X \to Y$ be a surjection. Let Z be an arbitrary set. Then there is a one-to-one correspondence between*

$$\mathscr{H} \equiv \{h : X \to Z | h \text{ is constant on every } \pi^{-1}(y),\ y \in Y\} \text{ and } \mathscr{F} \equiv \{f : Y \to Z\}, \tag{3.67}$$

which can be realized by the pull-back $\pi^* : \mathscr{F} \to \mathscr{H}$,

$$\pi^*(f) \equiv f \circ \pi, \tag{3.68}$$

as in the diagram (3.69)*.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd} \tag{3.69}$$

Proof. To start with, it is clear that π^* sends any element of $\mathscr{F}$ into $\mathscr{H}$. That is, for any map $f : Y \to Z$, the composite map $h \equiv f \circ \pi : X \to Z$ is constant on each pre-image of π. This has been shown in Example 3.14.

Now we check the injectivity of the map $\pi^* : \mathscr{F} \to \mathscr{H}$. Let $h_1, h_2 \in \mathscr{H}$ induce two elements $f_1, f_2 \in \mathscr{F}$ such that $f_1 \circ \pi = h_1$ and $f_2 \circ \pi = h_2$. We will prove that $f_1 = f_2$ provided $h_1 = h_2$. In fact, given for any $y \in Y$, let x be an arbitrary element in $\pi^{-1}(y)$. Note that x can be always found since π is a surjection. Then we have that

$$f_1(y) = f_1 \circ \pi(x) = h_1(x) = h_2(x) = f_2 \circ \pi(x) = f_2(y). \tag{3.70}$$

Next, we will only prove that π^* is surjective. That is, any map $h \in \mathscr{H}$ induces a map $f \in \mathscr{F}$. Indeed, by setting $f(y) \equiv h(x)$ for any $x \in \pi^{-1}(y)$. We have shown in Lemma 3.33 that f is well defined. It follows that for any $x \in X$ with $y = \pi(x)$, $f(\pi(x)) = f(y) = h(x)$. That is, $f \circ \pi = h$. □

3.8. Quotient topology.

Definition 3.19 (Quotient map and quotient topology). *Let X be a topological space, Y be a set, $p : X \to Y$ be a surjection.*

- *p is said to be a* **quotient map** *if Y is a topological space and*

$$U \text{ is open in } Y \text{ if and only if } p^{-1}(U) \text{ is open in } X. \tag{3.71}$$

- *There exists a unique topology $\mathfrak{T}$ on Y that makes $p : X \to Y$ a quotient map: $\mathfrak{T}$ is called the* **quotient topology** *on Y.*

Lemma 3.35. *Let X be a topological space, Y be a set, $p : X \to Y$ be a surjection. Then the quotient topology $\mathfrak{T}$ on Y is the largest (finest) possible topology on Y such that $p : X \to Y$ is a continuous map. That is, if $\mathfrak{T}'$ is a topology on Y such that p is continuous, then $\mathfrak{T}' \subseteq \mathfrak{T}$.*

Example 3.16. *Consider a subspace $H = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ of $\mathbb{R}^n$. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the projection $(x, y) \mapsto x$. The restricted map*

$$p : H \cup \{(0, 0)\} \longrightarrow \mathbb{R} \tag{3.72}$$

is not a quotient map. Note that $p^{-1}(\{0\}) = \{(0, 0)\}$, $\{(0, 0)\}$ is open in $H \cup \{(0, 0)\}$, but $\{0\}$ is not open in $\mathbb{R}$.

Definition 3.20. *Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be topological spaces. A map $f : X \to Y$ is said to be*

- *an open map if $U \in \mathfrak{T}_X \implies f(U) \in \mathfrak{T}_Y$;*
- *an open map if $f(E)$ is closed in Y for any closed set $E \subseteq X$.*

Example 3.17 (Open map and continuous map). *The map $f : [0, 1) \to S^1$ defined by*

$$t \mapsto (\cos 2\pi t, \sin 2\pi t) \tag{3.73}$$

is a continuous bijection, but f is not an open map and f is not a closed map either.

The inverse map $f^{-1} : S^1 \to [0, 1)$ is an open map, but it is not continuous.

Lemma 3.36. *Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a continuous surjective map. Then f is a quotient map provided that the map f is either open or closed.*

Proof. By assumption, f is surjective, which implies that for any subset $U \subseteq Y$,

$$f(f^{-1}(U)) = U. \tag{3.74}$$

Taking a pre-image $f^{-1}(U)$ that is open in X. There are two cases:

If f is an open map, then the set U, as the image of an open set $f^{-1}(U)$, is open in Y.

If f is a closed map, then by computations

$$Y \setminus U = f(f^{-1}(Y \setminus U)) = f(X \setminus f^{-1}(U)), \tag{3.75}$$

it follows that $Y \setminus U$ is closed in Y. In other words, U is an open subset of Y. □

Corollary 3.37. *Let $f : X \to Y$ be a continuous surjective map between two metric spaces. If X is compact, then f is a quotient map.*

Proof. Let K be any closed subset of X. Since X is compact, then K is compact. The continuity of f implies that $f(K)$ is compact. Therefore, K is closed. By Lemma 3.36, f is a quotient map. □

Definition 3.21 (Quotient or identification space)**.** *Let X be a topological space and let $\mathscr{P}$ be an equivalence relation on X that also gives a partition of X. Denote by $Y \equiv X/\mathscr{P}$ the set of equivalence classes. Let $\pi : X \to Y$, $x \mapsto [x]$ be the natural projection. The topology $\mathfrak{T}$ on Y is called the* **quotient topology** *or* **identification topology** *of Y if $\mathfrak{T}$ is the largest for which π is continuous. The space $(Y, \mathfrak{T})$ is called the quotient space (or identification space).*

Lemma 3.38. *Let $Y \equiv X/\sim$ be a quotient space from X. Then the natural projection $\pi : X \to Y$, $x \mapsto [x]$ is a quotient map.*

Proof. Let $\mathfrak{T}$ be the quotient topology of Y. Now let us collect all subsets $U \subseteq Y$ such that $\pi^{-1}(U)$ is open in X, namely we define

$$\mathfrak{T}' \equiv \{U \subseteq Y | \pi^{-1}(U) \in \mathfrak{T}_X\}. \tag{3.76}$$

Immediately, $\mathfrak{T} \subseteq \mathfrak{T}'$. Also it is easy to check that $\mathfrak{T}'$ is a topology on Y, and $\pi : X \to Y$ is continuous for $\mathfrak{T}'$. Thus, it follows from the definition of quotient space that $\mathfrak{T} = \mathfrak{T}'$. In other words, $U \in \mathfrak{T}$ if and only if $\pi^{-1}(U) \in \mathfrak{T}_X$. Therefore, π is a quotient map. □

Example 3.18 (Quotient map is in general not open)**.** *Let $I = [0, 1]$ be the unit interval and we denote by $S^1 \equiv I/\sim$, where $\sim$ identifies 0 and 1. Then the natural projection $\pi : I \to S^1$ is a quotient map provided that S^1 is equipped with the quotient topology.*

Notice that $[0, \frac{1}{2})$ is open in I, but $V = \pi([0, \frac{1}{2}))$ is not open in S^1. In fact, $\pi^{-1}(V) = [0, \frac{1}{2}) \cap \{1\}$, which is not open in I.

Theorem 3.39 (Pull-back property)**.** *Let $\pi : X \to Y$ be a quotient map. Let Z be an arbitrary topological space. Then there is a one-to-one correspondence between*

$$\mathscr{H} \equiv \{h : X \to Z | h \text{ is constant on every } \pi^{-1}(y),\ y \in Y\} \text{ and } \mathscr{F} \equiv \{f : Y \to Z\}, \tag{3.77}$$

which can be realized by the pull-back $\pi^* : \mathscr{F} \to \mathscr{H}$,

$$\pi^*(f) \equiv f \circ \pi, \tag{3.78}$$

see (3.79). Moreover, the following holds:

(1) $f \in \mathscr{F}$ is continuous if and only if $h = \pi^*(f) \in \mathscr{H}$ is continuous.
(2) $f \in \mathscr{F}$ is a quotient map if and only if $h = \pi^*(f) \in \mathscr{H}$ is a quotient map.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd} \tag{3.79}$$

Proof. To start with, it is clear that π^* send any element of $\mathscr{F}$ into $\mathscr{H}$.

Now we check the injectivity of π^*. Let $h_1, h_2 \in \mathscr{H}$ induce two elements $f_1, f_2 \in \mathscr{F}$ such that $f_1 \circ \pi = h_1 = h_2 = f_2 \circ \pi$. Given for any $y \in Y$, let x be an arbitrary element in $\pi^{-1}(y)$. Then

$$f_1(y) = f_1 \circ \pi(x) = h_1(x) = h_2(x) = f_2 \circ \pi(x) = f_2(y). \tag{3.80}$$

Here we used the fact that both h_1 and h_2 are constant on any pre-image of π.

Next, we will only prove that π^* is surjective. That is, any map $h \in \mathscr{H}$ induces a map $f \in \mathscr{F}$. Indeed, by setting $f(y) \equiv h(x)$ for any $x \in \pi^{-1}(y)$. The above f is well defined. In fact, f is constant on each pre-image $\pi^{-1}(y)$ so that the assignment is independent of the choice of the elements in the pre-image. Obviously, for any $x \in X$, $f(\pi(x)) = h(x)$. That is, $f \circ \pi = h$.

The last step is to prove that the continuity of f is equivalent to that of h. Let U be any open set in Z. Then

$$h^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1} \circ f^{-1}(U). \tag{3.81}$$

Since π is a quotient map, $h^{-1}(U)$ is open in X if and only if $f^{-1}(U)$ is open in Y. Therefore, $f \circ \pi$ is continuous if and only if f is continuous. $\square$

Corollary 3.40. *If $f : X \to Y$ is a quotient map, $g : Y \to Z$ is a quotient map, then $g \circ f : X \to Z$ is a quotient map.*

Theorem 3.41 (Universal property of quotient topology)**.** *Let $Y \equiv X/\sim$ be a quotient space from a topological space X and let $\pi : X \to Y$ be the natural projection. Let Z be an arbitrary topological space. A function $f : Y \to Z$ is continuous if and only if its pull-back $\pi^*(f) = f \circ \pi : X \to Z$ is continuous; see* (3.82)*.*

$$\begin{tikzcd} X \arrow[r, "f\circ\pi"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "f"'] & \end{tikzcd} \tag{3.82}$$

Proof. By Lemma 3.38, π becomes a quotient map. Then the conclusion follows from Theorem 3.39. $\square$

The next theorem gives an alternative characterization of the quotient space.

Theorem 3.42. *Let $h : X \to Z$ be a surjective continuous map. We denote by*

$$X^* \equiv \{h^{-1}(z) : z \in Z\} \tag{3.83}$$

and equip X^* with the quotient topology induced from X. Then the following holds.

(1) The map $h : X \to Z$ induces a continuous bijection $f : X^* \to Z$ such that (3.84) commutes. Moreover, f is a homeomorphism if and only if $h : X \to Z$ is a quotient map.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\pi"'] & Z \\ X^* \arrow[ur, "f"'] & \end{tikzcd} \tag{3.84}$$

(2) If Z is Hausdorff, so is X^*.

Proof. (1) Denote by $[x] \equiv \{q \in X : h(q) = h(x)\}$. We define a function $f : X^* \to Z$ by

$$f([x]) \equiv h(x). \tag{3.85}$$

Obviously, h is constant on the equivalence class $[x]$ for any $x \in X$, which implies that f is well defined. Notice that f is both surjective and injective. It is also easy to check that $h = f \circ \pi$, namely (3.84) commutes.

Next, we check that f is continuous. Let $W \subset Z$ be any open subset.

$$\pi^{-1} \circ f^{-1}(W) = h^{-1}(W). \tag{3.86}$$

Since h is continuous, $h^{-1}(W)$ is open in X. By Lemma 3.38, $\pi : X \to X^*$ is a quotient map. Therefore, by the definition of quotient map, $f^{-1}(W)$ is open in X^*, which implies that f is continuous.

The last part is to verify the criterion for f to be a homeomorphism. Since f is a continuous bijection, f is a homeomorphism if and only if f is an open map. First, assuming f is a homeomorphism, we will show that h is a quotient map. For any subset $W \subset Z$,

$$h^{-1}(W) = \pi^{-1} \circ f^{-1}(W). \tag{3.87}$$

Assume that W is open. Since π is a quotient map, $f^{-1}(W)$ is open. It follows that W is open in Z since f is a homeomorphism.

Now we assume that h is a quotient map. Let $f^{-1}(W)$ be any open set in X. Using $h^{-1} = \pi^{-1} \circ f^{-1}$, we have $h^{-1}(W) = \pi^{-1} \circ f^{-1}(W)$. Since π is continuous and $f^{-1}(W)$ is open, $h^{-1}(W)$ is open in X. By assumption, W is open in Z. Therefore, f is a quotient map. Since f is a continuous bijection, f must be a homeomorphism.

(2) Let $[p] \neq [q]$ be two distinct points in X^*. Since f constructed in item (1) is bijective, $f([p]) \neq f([q])$. Since Z is Hausdorff, one can find two disjoint neighborhoods W_1 and W_2 of $z_1 \equiv f([p])$ and $z_2 \equiv f([q])$, respectively. Then

$$f^{-1}(W_1) \cap f^{-1}(W_2) = f^{-1}(W_1 \cap W_2) = \emptyset. \tag{3.88}$$

Therefore, X^* is Hausdorff. $\square$

3.9. **Examples of quotient spaces.**

Example 3.19 (Circle). *Let $I = [0, 1]$ and consider the natural map*

$$\pi_c : I \to \mathbb{R}^2, \quad t \mapsto (\cos 2\pi t, \sin 2\pi t). \tag{3.89}$$

We denote $S^1 \equiv \pi_c(I)$. Then π_c is a continuous surjection onto S^1. By Corollary 3.37, π_c is a quotient map.

On the other hand, the identification space I^* corresponding π_c is given by gluing the two points $\{t = 0\}$ and $\{t = 1\}$. If we equip I^* with the quotient topology, by Theorem 3.42, then I^* is homeomorphic to S^1.

Example 3.20 (Cylinder and torus). *One can realize cylinders and tori as quotient spaces of squares. Let $I \equiv [0, 1]$ and let $Q \equiv [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be the unit square.*

Consider the continuous surjection

$$\pi_c : Q \longrightarrow \mathbb{C}^2, \ (\tau, \theta) \mapsto (\tau, e^{\sqrt{-1}2\pi\theta}). \tag{3.90}$$

We denote by $I \times S^1 \equiv \pi_c(Q)$ the image of Q, called a cylinder. By Corollary 3.37, π_c is a quotient map. The identification space $Q^* = \{\pi_c^{-1}(z) \subset Q : z \in I \times S^1\}$ is obtained from gluing the two lateral sides $\{t = 0\}$ of $\{t = 1\}$ of Q. By Theorem 3.42, Q^* is homeomorphic to $I \times S^1$.

A torus can be constructed in a similar way. Define a continuous surjection

$$\pi_{\mathbb{T}^2} : Q \longrightarrow \mathbb{C}^2, \ (\theta_1, \theta_2) \mapsto \left(e^{\sqrt{-1}\cdot 2\pi\theta_1}, e^{\sqrt{-1}\cdot 2\pi\theta_2}\right). \tag{3.91}$$

The compactness of Q implies that $\pi_{\mathbb{T}^2}$ is a quotient map. We denote by

$$\mathbb{T}^2 = S^1 \times S^1 = \pi_{\mathbb{T}^2}(Q), \tag{3.92}$$

as a subspace of $\mathbb{C} \times \mathbb{C}$. Applying Theorem 3.42, $\mathbb{T}^2$ is homeomorphic to

$$Q^* = \{\pi_{\mathbb{T}^2}^{-1}(z) \subset Q : z \in \mathbb{T}^2\}. \tag{3.93}$$

Intuitively, Q^* is obtained from gluing two opposite sides of Q (in the same orientation).

Example 3.21 (Möbius strip). *Let $I_1 \times I_2 \equiv [0, 1] \times [-1, 1]$. The equivalence relation $\sim$ on $I_1 \times I_2$ is as follows: any point (θ, t) with $0 < \theta < 1$ is identified to itself;*

$$(0, t) \sim (1, -t), \tag{3.94}$$

for all $t \in [0, 1]$. The quotient set $\mathbb{M} \equiv (I_1 \times I_2)/\sim$ equipped with the quotient topology is called the Möbius band. Notice that $\mathbb{M}$ can be realized as a surface inside a solid torus $S^1 \times \mathbb{D}^2 \subset \mathbb{C} \times \mathbb{C}$. We define a map $\Phi : I_1 \times I_2 \to \mathbb{C} \times \mathbb{C}$ by

$$(\theta, t) \mapsto (e^{\sqrt{-1}\cdot 2\pi\theta}, e^{\sqrt{-1}\cdot \pi\theta} \cdot t), \quad (\theta, t) \in I_1 \times I_2. \tag{3.95}$$

By Theorem 3.42, $\mathbb{M}$ is homeomorphic to $\Phi(I_1 \times I_2) \subset S^1 \times \mathbb{D}^2$.

Similarly, one can define an infinite Möbius band as a quotient space from $\mathbb{R}\times[0,1]$ (twisted $\mathbb{R}$ bundle over S^1).

Example 3.22 (Topological cone and suspension)**.** *Let $X \times T$ be a product space.*

- *Let $T = [0, \infty)$. The quotient space*
$$C(X) \equiv (X \times [0, \infty))/ \sim_c \tag{3.96}$$
is called a topological cone over the cross section X, where the equivalence relation $\sim_c$ identifies the bottom slice $X \times \{0\}$ with a single point.
- *Let $T = [0, \pi]$. The quotient space*
$$S(X) \equiv (X \times [0, \infty))/ \sim_s \tag{3.97}$$
is called the suspension over cross section X, where the equivalence relation $\sim_s$ identifies the bottom slice $(X \times \{0\})$ to a point and the top slice $(X \times \{\pi\})$ to another point. For example, $\mathbb{S}^3$ can be viewed as the suspension over $\mathbb{S}^2$.

Example 3.23 (Klein bottle)**.** *The Klein bottle $\mathbb{K}$ can be constructed as follows. Let $I_1\times I_2 \equiv [0, 1]\times[0, 1]$. The equivalence relation $\sim$ on $I_1\times I_2$ is as follows: any point (θ, t) with $0 < \theta < 1$ and $0 < t < 1$ is identified to itself;*
$$(0, t) \sim (1, -t) \text{ and } (\theta, 0) \sim (\theta, 1) \tag{3.98}$$
for all $x \in [0, 1]$ and $y \in [0, 1]$.

The Klein bottle can be also regarded as a surface (a nontrivial circle bundle over S^1) in the solid torus $S^1 \times \mathbb{D}^2 \subset \mathbb{C} \times \mathbb{C}$. We define a map
$$\Psi(\theta, t) = (e^{\sqrt{-1}\cdot 2\pi\theta}, e^{\sqrt{-1}\cdot(\pi\theta+2\pi t)}). \tag{3.99}$$
By Theorem 3.42, $\mathbb{K}$ is homeomorphic to $\Psi(I_1 \times I_2)$.

The next goal is to prove that a closed ball $\overline{\mathbb{B}^n}$ quotient by its boundary is homeomorphic to a sphere. Before the proof, we provide lemma which identifies the topology of $\mathbb{R}^n$ using two distinct geometric models.

Lemma 3.43. *Let us denote $\mathbb{B}^n \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < 1\}$, $\mathbb{S}^n \equiv \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \|\boldsymbol{x}\| = 1\}$, and $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Then $\mathbb{B}^n$ and $\mathbb{S}^n \setminus \{N\}$ are both homeomorphic to $\mathbb{R}^n$.*

Proof. First, one can prove that
$$\mathscr{R} : \mathbb{B}^n \to \mathbb{R}^n, \quad \boldsymbol{x} \mapsto \frac{\boldsymbol{x}}{1 - \|\boldsymbol{x}\|}, \tag{3.100}$$
gives a homeomorphism between $\mathbb{B}^n$ and $\mathbb{R}^n$ (Homework problem: check the continuity of $\mathscr{R}^{-1}$).

Next, we define the *stereographic projection* which gives a homeomorphism between $\mathbb{S}^n \setminus N$ and $\mathbb{R}^n$.

For simplicity, consider the special $n = 2$. Let $P = (x, y, z)$ be any point in $\mathbb{S}^2$. Then $x^2 + y^2 + z^2 = 1$. We now define a map $\Phi : \mathbb{S}^2 \setminus N \to \mathbb{R}^2$ Let $(u_1, u_2, 0)$ be the unique intersection of segment $\overline{NP}$ and the Euclidean plane $\mathbb{R}^2_{x,y} \equiv \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Then we define $\Phi(P) = (u_1, u_2)$. One can compute that

$$u_1 = \frac{x}{1-z} \quad \text{and} \quad u_2 = \frac{y}{1-z}. \tag{3.101}$$

One can show that both Φ and Φ^{-1} are continuous (Homework problem: check this, and generalize it in higher dimensions). □

Theorem 3.44. *$\overline{\mathbb{B}^n}/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$, where $\mathbb{B}^n \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < 1\}$ is the unit ball and $\mathbb{S}^{n-1}$ is the unit sphere which is the boundary of $\mathbb{B}^n$.*

Proof. We will construct a $F : \overline{\mathbb{B}^n} \to \mathbb{S}^n$ that satisfies the following properties:

(1) F is a continuous surjection.
(2) $F|_{\mathbb{B}^n}$ is a continuous bijection onto $\mathbb{S}^n \setminus \{N\}$.
(3) $F|_{\mathbb{S}^{n-1}}$ is a constant map with $F(\mathbb{S}^{n-1}) = \{N\}$.

Let $\mathscr{R} : \mathbb{B}^n \to \mathbb{R}^n$ be the homeomorphism constructed in the previous lemma, and let $\Phi : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ be the stereographic projection. Then $\Phi \circ \mathscr{R}^{-1} : \mathbb{B}^n \to \mathbb{S}^n \setminus \{N\}$ is a homeomorphism. Then we define

$$F(x) \equiv \begin{cases} \Phi \circ \mathscr{R}^{-1}(x), & x \in \mathbb{B}^n, \\ \{N\}, & x \in \mathbb{S}^{n-1}. \end{cases} \tag{3.102}$$

Then F is a continuous surjection on $\overline{\mathbb{B}^n}$ (why?).

Since $\overline{\mathbb{B}^n}$ is compact, by Corollary 3.37, $F : \overline{\mathbb{B}^n} \to \mathbb{S}^n$ is a quotient map. One can realize $\overline{\mathbb{B}^n}/\mathbb{S}^{n-1}$ as the set of pre-images

$$\{F^{-1}(p) : p \in \mathbb{S}^n\}, \tag{3.103}$$

and equip it with the quotient topology. Applying Theorem 3.42, $\overline{\mathbb{B}^n}/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$. □

Example 3.24 (Projective space)**.** *One can describe the projective space $\mathbb{R}P^n$ in three equivalent ways.*

(1) Taking the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we define an equivalence relation on $\mathbb{S}^n$: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n$

$$\boldsymbol{x} \sim_s \boldsymbol{y} \iff \boldsymbol{x} = -\boldsymbol{y}. \tag{3.104}$$

The quotient set $\mathbb{S}^n/\sim_s$ equipped with the quotient topology is the projective space $\mathbb{R}P^n$.

(2) We define an equivalence relation on $\mathbb{R}^{n+1}$: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+1}$

$$\boldsymbol{x} \sim_r \boldsymbol{y} \iff \exists \text{ a line } L \subset \mathbb{R} \text{ that connects } \boldsymbol{x}, \boldsymbol{y}. \tag{3.105}$$

The quotient set $\mathbb{R}^{n+1}/\sim_r$ equipped with the quotient topology is the projective space $\mathbb{R}P^n$.

(3) We define an equivalence relation on $\overline{\mathbb{B}^n}$: any point $\boldsymbol{x} \in \overline{\mathbb{B}^n}$ is only identified with itself; for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n$

$$\boldsymbol{x} \sim_b \boldsymbol{y} \iff \boldsymbol{x} = -\boldsymbol{y}. \tag{3.106}$$

The quotient set $\overline{\mathbb{B}^n}/\sim_b$ equipped with the quotient topology is the projective space $\mathbb{R}P^n$.

For example, let us check (1) $\iff$ (2). Denote by $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ the natural embedding. Let $\pi_s : \mathbb{S}^n \to \mathbb{S}^n/\sim_s$ be the natural projection. Then π_s is the natural quotient map with respect to the quotient topology of $\mathbb{S}^n/\sim_s$. Similarly, let $\pi_r : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/\sim_r$ be the natural quotient map with respect to the quotient topology of $\mathbb{R}^{n+1}/\sim_r$. Then the composite $h \equiv \pi_r \circ \iota : \mathbb{S}^n \to \mathbb{R}^{n+1}/\sim_r$ is a continuous surjection. By Corollary 3.37, h is a quotient map. Now we have the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "\pi_s"'] \arrow[d, hook, "\iota"] \arrow[dr, "h"'] & \mathbb{S}^n/\sim_s \arrow[d, dashed, "\mathscr{H}"] \\
\mathbb{R}^{n+1} \arrow[r, "f"'] & \mathbb{R}^{n+1}/\sim_r
\end{tikzcd} \tag{3.107}$$

By Theorem 3.42, $\mathbb{S}^n/\sim_s$ is homeomorphic to $\mathbb{R}^{n+1}/\sim_r$.

Example 3.25. *Möbius strip, Klein bottle, projective space*

4. Connectedness and compactness

4.1. **Compactness.** If we review the definition of compact metric space, one can see that the definition only needs to the notion of openness. That is, the definition of compactness does not rely on the distance structure, and hence one can generalize the notion of compactness to general topological spaces.

Definition 4.1 (Compact space)**.** *A topological space X is said to be compact if any open covering of X admits a finite sub-covering.*

Lemma 4.1. *Let X be a compact space and let K be a closed subspace. Then K is compact in X.*

Proof. Let $\mathcal{O}$ be any open covering of K. Since K is closed in X, we have an extended open covering

$$\widehat{\mathcal{O}} \equiv \mathcal{O} \cup \{X \setminus K\}. \tag{4.1}$$

By assumption, X is compact, which implies that $\widehat{\mathcal{O}}$ can be reduced to a finite covering $\mathcal{O}^*$ of X. Then $\mathcal{O}^* \setminus \{X \setminus K\}$ becomes a finite covering of K. □

Theorem 4.2. *Let X be a Hausdorff space and let K be a compact subspace. Then K is closed in X.*

Proof. We will show that $X \setminus K$ is open. It suffices to show that for any $x \in X \setminus K$, there exists a neighborhood W_x of x that satisfies $W_x \subseteq K$. Since X is a Hausdorff space, for any fixed $x \in X \setminus K$ and for any $p \in K$, there exist neighborhoods $W_p(x)$ and U_p of x and p, respectively, such that $W_p(x) \cap U_p = \emptyset$. Obviously, $\mathcal{U} \equiv \{U_p : p \in K\}$ is an open covering of K. Due to the compactness of K, $\mathcal{U}$ can be reduced to a finite sub-covering $\mathcal{U}^* \equiv \{U_{p_1}, \ldots, U_{p_m}\}$.

Notice that for any $1 \leqslant i \leqslant m$, $U_{p_i} \cap W_{p_i}(x) = \emptyset$, which implies that

$$W_{p_i}(x) \cap \left(\bigcup_{i=1}^{m} U_{p_i} \right) = \emptyset. \tag{4.2}$$

Since $\mathcal{U}^*$ covers K, for any $1 \leqslant i \leqslant m$, $W_{p_i}(x) \cap K = \emptyset$, and hence $\subseteq X \setminus K$. We define an open set

$$W_x \equiv \bigcap_{j=1}^{m} W_{\alpha_j}. \tag{4.3}$$

(Indeed, W_x is an open set since it is the intersection of finitely many open sets) Therefore, we conclude that $W_x \subseteq X \setminus K$, which completes the proof. □

Theorem 4.3. *Let $f : X \to Y$ be continuous. If K is a compact subspace, then $f(K)$ is compact in Y.*

Proof. To show the compactness of Y, let us take any open covering $\mathcal{O} = \{O_\alpha\}_{\alpha\in\Lambda}$ of $f(K)$. By continuity, for any open set $O_\alpha \in \mathcal{O}$, the pre-image $f^{-1}(O_\alpha)$ is open in X. Notice that the collection of open sets $\mathcal{O}^* = \{f^{-1}(O_\alpha)\}_{\alpha\in\Lambda}$ covers K since $\mathcal{O}$ covers $f(K)$. By assumption, X is compact, which implies that there exists a finite open sub-covering

$$\mathcal{O}^*_m = \{f^{-1}(O_1), \ldots, f^{-1}(O_m)\} \subseteq \mathcal{O}^* \tag{4.4}$$

of K. Then the open sets $\{O_1, \ldots, O_m\}$ covers $f(K)$, which completes the proof. $\square$

Corollary 4.4. *Let X be a compact space and let Y be a Hasudorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. It suffices to show that $f : X \to Y$ is a closed map. In fact, given any closed subset $K \subseteq X$, by Lemma 4.1, K is compact in X. Corollary 4.4 implies that $f(K)$ is compact in Y. By Theorem 4.2, $f(K)$ is a closed subset. Therefore, f is a closed map, which completes the proof. $\square$

Corollary 4.5. *Let X be a compact space and let Y be a Hasudorff space. If $f : X \to Y$ is a continuous surjection, then f is a quotient map.*

Theorem 4.6 (Tychonoff's Theorem (baby version))**.** *If X, Y are compact spaces, then $X \times Y$ is also compact.*

Lemma 4.7 (The tube lemma)**.** *Consider the product space $X \times Y$, where Y is compact. If O is an open set of $X \times Y$ that contains the slice $\{x_0\} \times Y$ of $X \times Y$, then O contains some tube $U_{x_0} \times Y$, where U_{x_0} is a neighborhood of x_0.*

Proof of Theorem 4.6 modulo the proof of Lemma 4.7. Let $\mathcal{O}$ be any open covering of the product space $X \times Y$. Observe that for any $x_0 \in X$, the slice $\{x_0\} \times Y$ is a compact subspace in $X \times Y$. Therefore, $\mathcal{O}$ contains a sub-covering $\mathcal{O}_{x_0} = \{O_j\}_{j=1}^{m_{x_0}}$ of $\{x_0\} \times Y$. In other words,

$$W_{x_0} \equiv \bigcup_{j=1}^{m_{x_0}} O_j \tag{4.5}$$

contains $\{x_0\} \times Y$. By Lemma 4.7, W_{x_0} contains an open tube $U_{x_0} \times Y$. Since $x_0 \in X$ is arbitrary, the collection $\mathcal{U} \equiv \{U_{x_0} : x_0 \in X\}$ gives an open covering of X. The compactness of X implies that $\mathcal{U}$ can be reduced to a finite covering $\mathcal{U}^* \equiv \{U_k : 1 \leqslant k \leqslant N\}$ of X. Since for any $1 \leqslant k \leqslant N$, $U_k \times Y$ is covered by finitely many elements in $\mathcal{O}$, the finite union $\bigcup_{k=1}^m (U_k \times K)$ is also covered by finitely many elements in $\mathcal{O}$. Therefore, $\mathcal{O}$ can be reduced to a finite covering of the product $X \times Y$. $\square$

Proof of Lemma 4.7. One can write the open set O as the union of basis elements

$$\{U_\alpha \times E_\alpha \in \mathfrak{B}_{X\times Y} : (U_\alpha \times E_\alpha) \cap (\{x_0\} \times Y) \neq \emptyset,\ \alpha \in \Lambda\}\,. \tag{4.6}$$

Then $\mathcal{E} \equiv \{E_\alpha\}_{\alpha\in\Lambda}$ becomes an open covering of Y. Since Y is compact, $\mathcal{E}$ reduces to a finite sub-covering $\mathcal{E}^* \equiv \{E_{\alpha_j}\}_{j=1}^m$ of Y. Then we take

$$U_{x_0} \equiv \bigcap_{j=1}^{m} U_{\alpha_j}. \tag{4.7}$$

Obviously, $U_{x_0} \times Y$ is an open subset of O containing the slice $\{x_0\} \times Y$. □

Corollary 4.8. *If $\{X_j\}_{j=1}^m$ is a sequence of compact spaces, then $\prod_{j=1}^{m} X_j$ is compact.*

Proof. The corollary can be proved by induction. The base step is given by Theorem 4.6. □

The following theorem is named after A. Tikhonov.

Theorem 4.9 (Tychonoff's Theorem 1935)**.** *An arbitrary product of compact spaces is a compact with respect to the product topology.*

4.2. **Connectedness.**

Definition 4.2 (Connected space)**.** *A separation of a topological space X is a pair of disjoint nonempty open subsets U, V such that $X = U \cup V$. A topological space X is said to be connected if it does not admit a separation.*

Lemma 4.10. *A topological space X is connected if and only if the only subsets that are both open and closed are $\emptyset$ and X.*

Theorem 4.11. *Let $f : X \to Y$ be a continuous map. If X is connected, so is $f(X)$.*

Proof. Notice that f is a surjection onto $Z \equiv f(X)$. Let $U \subset Z$ be a non-empty subset that is open and closed. Then $f^{-1}(U)$ is both open and closed in X. Since f is surjective, $f^{-1}(U) \neq \emptyset$. The connectedness of X implies that $f^{-1}(U) = X$. Therefore,

$$f(X) \supseteq U \supseteq f(f^{-1}(U)) = f(X), \tag{4.8}$$

which completes the proof. □

Lemma 4.12. *Let X admit a separation given by C and D. If $Y \subset X$ is a connected subspace, then either $Y \subseteq C$ or $Y \subseteq D$.*

Proof. Let $U \equiv Y \cap C \neq \emptyset$ and $V \equiv Y \cap D \neq \emptyset$. Then U and V are two open sets in Y. Moreover, $U \cup V = Y \cap (C \cup D) = Y \cap X = Y$. Since Y is connected, it does not admit any separation. Then $U = \emptyset$ or $V = \emptyset$. Equivalently, $Y \subseteq D$ or $Y \subseteq C$. □

Theorem 4.13. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{Z_\alpha\}_{\alpha\in\Lambda}$ be connected subspaces with

$$\bigcap_{\alpha\in\Lambda} Z_\alpha \neq \emptyset. \tag{4.9}$$

Suppose $Z \equiv \bigcup_{\alpha\in\Lambda} Z_\alpha$ is not connected.

Let C and D be a separation of Z. Without loss of generality, assume that

$$p \in \left(\bigcap_{\alpha\in\Lambda} Z_\alpha\right) \cap C. \tag{4.10}$$

By Lemma 4.12, for any $\alpha \in \Lambda$, $Z_\alpha \subseteq C$, which implies that $Z \subseteq C$. This contradicts the contradiction assumption that C and D give a separation of Z, which completes the proof. □

Theorem 4.14. *A finite cartesian product of connected spaces is connected.*

Proof. We will prove that for two connected spaces X and Y, the product space $X \times Y$ is connected. The general case can be proved by induction.

First, we will show that for any $x \in X$ and $y \in Y$, the union T_{xy} of two slices $H_y \equiv X \times \{y\}$ and $V_x \equiv \{x\} \times Y$ is connected. In fact, $H_y \cap V_x = (x, y)$. By Theorem 4.13, T_{xy} is connected.

Next, for any fixed $x \in X$,

$$\bigcap_{y\in Y} T_{xy} = \{x\} \times Y. \tag{4.11}$$

Also observe that

$$X \times Y = \bigcup_{y\in Y} T_{xy}. \tag{4.12}$$

Applying Theorem 4.13, the conclusion follows. □

Definition 4.3 (Connected component)**.** *Let X be a topological space. For any $p \in X$, the connected component containing p is the largest connected subspace $C_p \subseteq X$ that contains p. In other words, if U is a connected subspace that contains p, then $U \subseteq C_p$.*

Remark 4.1. *Connected components give a partition of a topological space. One can define a natural equivalence relation for this partition.*

4.3. **Applications and examples of connectedness.**

Theorem 4.15. *All intervals and rays in $\mathbb{R}$ are connected.*

Proof. We will show that any convex subset $Y \subset \mathbb{R}$ is connected. Suppose not. Let two open subsets $A, B \subseteq Y$ give a separation. Taking $a \in A$ and $b \in B$, we will analyze the closed interval $[a, b] \subseteq Y$. We denote

$$A^* \equiv A \cap [a, b], \quad B^* \equiv B \cap [a, b], \tag{4.13}$$

and define $s \equiv \sup A^*$. We will show that $s \notin A^*$ and $s \notin B^*$. There are two cases to analyze.

(1) Case 1: $s \in B^*$. Since B^* is open in $[a, b]$, there exists $\epsilon > 0$ such that $(s-\epsilon, s] \subseteq B^*$. By the construction, $A^* \cap B^* = \emptyset$, which implies that $A^* \cap (s-\epsilon, s) = \emptyset$. Then any number $t \in (s-\epsilon, s)$ is an upper bound of A^*, which contradicts the assumption $s = \sup A^*$.
(2) Case 2: $s \in A^*$. The proof in this case is similar. The openness of A^* in $[a, b]$ implies that there exists some number $\epsilon > 0$ such that $[s, s+\epsilon) \subseteq A^*$, which contradicts $s = \sup A^*$.

The proof is complete. □

Lemma 4.16. *Let $U \subset \mathbb{R}$ be a connected subset. If $a < b$ are two numbers in U, then $[a, b] \subseteq U$. That is, any connected subset in $\mathbb{R}$ is convex.*

Proof. We will prove it by contradiction. Suppose that there exists some $t \in (a, b)$ such that $t \notin U$. Then $A \equiv (-\infty, t) \cap U$ and $B \equiv (t, +\infty) \cap U$ are two open sets in U such that $a \in A$ and $b \in B$. Notice that $U = A \cup B$, which implies that A and B give a separation of U. Contradiction. □

Theorem 4.17 (Intermediate Value Theorem)**.** *Let X be a connected space and let $f : X \to \mathbb{R}$ be a real valued continuous function. If $a < b$ are two numbers in $f(X)$, then for any $r \in (a, b)$, there exists some point $p \in X$ such that $f(p) = r$.*

Proof. By Lemma 4.16, $[a, b] \subset f(X)$. In particular, any point $r \in (a, b)$ is an element of $f(X)$, so the conclusion immediately follows. □

Example 4.1. *$\mathbb{R}^\omega$ is connected in the product topology, and it is disconnected in the box topology.*

Definition 4.4 (Continuous path)**.** *A continuous map $\gamma : [0, 1] \to X$ is called a path in X.*

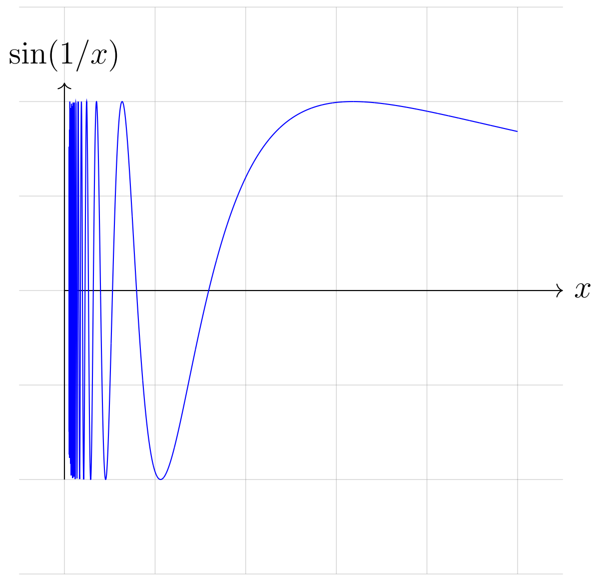
Definition 4.5 (Path connectedness)**.** *A topological space X is said to be path connected if for any distinct points p and q in X, there exists a path $\gamma : [0, 1] \to X$ in X that connects p and q.*

Lemma 4.18. *If X is path connected, then X is connected.*

Proof. We will prove by contradiction. Suppose X is not connected. Let C and D give a separation of X. Since X is path connected, one can take a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = p \in C$ and $\gamma(1) = q \in D$. The connectedness of the image $\Gamma \equiv \gamma([0, 1])$ implies that either $\Gamma \subseteq C$ or $\Gamma \subseteq D$. Contradiction. □

Example 4.2 (Topologist's sine curve)**.** *This is a standard model:*

$$\Sigma \equiv \left\{(x, \sin(1/x)) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1\right\}, \tag{4.14}$$



the closure $\overline{\Sigma}$ *is called topologist's sine curve.*

5. Selected topics

Surface theory.