1. (§ 1.6. Ex 4 (a))

Pf. Consider induction on the cardinality of finite set A

If $|A| = 1$, the largest element would be the only element.

Suppose $|A| = k$ and A has a largest element for some $k \in \mathbb{N}$.

when $|A| = k+1$,

randomly remove an element a_0 from A

$|A \setminus \{a_0\}| = k$. by hypothesis $A \setminus \{a_0\}$ has a largest element. denoted by a_1

if $a_0 < a_1$ the largest element of A would be a_1

else the largest element of A would be a_0.

By induction. A nonempty finite simply ordered set has a largest element.

$\square$

2. (§ 1.6. Ex 6).

(a). Pf: Let $f: \mathcal{P}(A) \to X^n$

$$\{i_1, i_2, \cdots, i_k\} \mapsto (0, \cdots \underset{\substack{\uparrow \\ i_1\text{-th}}}{1}, \cdots \underset{\substack{\uparrow \\ i_2\text{-th}}}{1}, \cdots, \underset{\substack{\uparrow \\ i_k\text{-th}}}{1} \cdots 0)$$

i.e. $X(i) = 1$ iff $i \in \{i_1, i_2, \cdots, i_k\}$. where $1 \le i_1 < i_2 < \cdots < i_k \le n$

$k \in \mathbb{N}$.

f is obviously a bijection.

given $x=(x_1, x_2, \cdots, x_n) \in X^n$. $f^{-1}(x) = \{ i \in A \mid x_i = 1 \} \in \mathcal{P}(A)$.

$\square$

(b). Pf. In (a). Since there's a bijection from $\mathcal{P}(A)$ to X^n.

When A is finite. n is finite. Also $X=\{0,1\}$ is finite

By Corollary 6.8. X^n is finite.

so $\mathcal{P}(A)$ is finite.

$\square$

3. (§1.6. Ex7).

Pf. Denote the set of all functions $f: A \to B$ by B^A

By definition $\forall f \in B^A$. $f \subset A \times B \Rightarrow B^A \subset \mathcal{P}(A\times B)$

Since A and B are finite.

By Corollary 6.8 $A\times B$ is finite

By Ex 6. (b). $\mathcal{P}(A\times B)$ is finite

so $B^A \subset \mathcal{P}(A\times B)$ should be finite.

$\square$

4. (§1.7. Ex 1)

Pf. In Corollary 7.4. It was proved that $\mathbb{Q}^+$ is countable.

let $\mathbb{Q}^-$ be the set of all negative rational numbers.

there's a bijection $f: \mathbb{Q}^+ \to \mathbb{Q}^-$

$$q \mapsto -q$$

so $\mathbb{Q}^-$ is countably infinite.

By Theorem 7.5 since $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$

so $\mathbb{Q}$ is countably infinite.

$\square$

5. (§1.7. Ex 4).

(a). Pf. let $\mathbb{P}_n(x)$ be the set of polynomial of degree n with rational coefficients. there's a bijection

$$f: \mathbb{Q}^n \to \mathbb{P}_n(x)$$

$$(a_0, a_1, \cdots, a_{n-1}) \mapsto x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

In §1.7. Ex 1. I showed that $\mathbb{Q}$ is countable

so by Theorem 7.6. $\forall n \in \mathbb{N}$. $\mathbb{Q}^n$ is countable

by Theorem 7.5. $\bigcup_{n\in\mathbb{N}} \mathbb{Q}^n$ is countable

So $\mathbb{P}(x) := \bigcup_{n\in\mathbb{N}} \mathbb{P}_n(x)$ is countable

since $\forall p \in \mathbb{P}(x)$. $p(x)=0$ has only finitely many roots.

$\{x \mid p(x)=0.\ p(x)\in\mathbb{P}(x)\}$ by Theorem 7.5 is countable

i.e. the set of algebraic numbers is countable. $\square$

(b). Pf. since $\mathbb{R}$ is uncountable.

if the set of transcendental numbers is countable.

by definition $\mathbb{R} = \{a \mid a \text{ is algebraic}\} \cup \{t \mid t \text{ is transcendental}\}$

by Theorem 7.5. $\mathbb{R}$ should be countable.

contradiction.

so the set of transcendental numbers is uncountable. $\square$.

6. (§ 1.7. Ex 5)

(a). Pf. Yes.

Denote the set of all functions from $\{0.1\}$ to $\mathbb{Z}_+$ by $\mathbb{Z}_+^{\{0.1\}}$

by definition $\mathbb{Z}_+^{\{0,1\}} = \{(0,m) \mid m\in\mathbb{Z}_+\} \times \{(1.n) \mid n\in\mathbb{Z}_+\}$

by Theorem 7.6. $\mathbb{Z}_+^{\{0.1\}}$ is countable. $\square$

(ii). Pf. let $f: I \to A$

$$\{a,b\} \mapsto \begin{array}{l} 0 \mapsto \min\{a,b\} \\ 1 \mapsto \max\{a,b\} \end{array}$$

by definition f is injective.

as proved in (a) that A is countable

so I is countable.

I is obviously infinite because it contains $\{(0,z) \mid z \in \mathbb{Z}_+\}$

$\square$