Assignment. Chentian Wu.

Pt I.

1. (§1.1. Ex 4)

(a) there exists $a \in A$, such that $a^2 \notin B$

(b) forall $a \in A$, ~~there exists~~ $a^2 \notin B$.

(c) there exists $a \in A$, such that $a^2 \in B$

(d) forall a where $a \notin A$. $a \notin B$

2. (§1.2. Ex 1).

(a). Pf. $f(A_0) := \{ f(x) \in B \mid x \in A_0 \}$

$f^{-1}(f(A_0)) := \{ x \in A \mid f(x) \in f(A_0) \}$

then by definition $x \in A_0 \Rightarrow f(x) \in f(A_0) \Rightarrow x \in f^{-1}(f(A_0))$.

since x is arbitrary.

$A_0 \subset f^{-1}(f(A_0))$.

When f is injective, given $x_1, x_2 \in A_0$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

suppose. $x_0 \in \left(f^{-1}(f(A_0)) \setminus A_0\right) \neq \phi$

then $f(x_0) \in f(A_0)$ and $x_0 \notin A_0$

but since $f^{-1}(f(A_0)) \neq \phi$

there must be some $x_0' \in A_0 : f(x_0') = f(x_0)$

since $x_0' \in A_0$, $x_0 \notin A_0$ ~~and~~

so $x_0 \neq x_0'$

but $f(x_0) = f(x_0')$

so f is not an injection.

this is against our assumption

so $f^{-1}(f(A_0)) \setminus A_0 = \phi$

i.e. $f^{-1}(f(A_0)) \subset A_0$

since $A_0 \subset f^{-1}(f(A_0))$

so the equality holds when f is an injection. $\square$

(b) $f^{-1}(B_0) := \{x \in A \mid f(x) \in B_0\}$

$f(f^{-1}(B_0)) = \{y \in B \mid y = f(x),\ x \in f^{-1}(B_0)\}$

by definition $y \in f(f^{-1}(B_0))$

$\Rightarrow y = f(x) \in B_0$

which indicates that $f(f^{-1}(B_0)) \subset B_0$.

When f is surjective.

~~suppose $\Delta = f(f^{-1}(B_0)) \setminus B_0 \neq \phi$~~

~~suppose $y_0 \in \Delta$, $y_0 \notin B_0$~~

~~then by definition $\exists x_0 \in f^{-1}(B_0)$: $y_0 = f(x_0)$~~

suppose $\Delta = B_0 \setminus f(f^{-1}(B_0)) \neq \phi$ $y_0 \in \Delta$

then $y_0 \in B_0$ and $y_0 \notin f(f^{-1}(B_0))$

by definition there exists NO $x_0 \in f^{-1}(B_0)$ s.t. $y_0 = f(x_0)$

but $y_0 \in B_0$

so f is not surjective

this contradicts with our assumption that f is surjective.

so $\Delta = \phi$ i.e. $B_0 \subset f(f^{-1}(B_0))$

also we have $f(f^{-1}(B_0)) \subset B_0$

so the equality holds when f is surjective. $\square$

3. (§1.2. Ex 2).

(a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$

Pf. let $\Delta = B_1 \setminus B_0$ $B_1 = B_0 \sqcup \Delta$

$$
\begin{aligned}
f^{-1}(B_1) &:= \{x \in A \mid f(x) \in B_1\} \\
&= \{x \in A \mid f(x) \in B_0\} \cup \{x \in A \mid f(x) \in \Delta\} \\
&= f^{-1}(B_0) \cup f^{-1}(\Delta) \\
&\supset f^{-1}(B_0)
\end{aligned}
$$

$\square$

(b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

Pf. I have shown this in (a), but I'll prove again by definition

$$f^{-1}(B_0 \cup B_1) := \{x \in A \mid f(x) \in B_0 \cup B_1\}$$
$$= \{x \in A \mid f(x) \in B_0\} \cup \{x \in A \mid f(x) \in B_1\}$$
$$=: f^{-1}(B_0) \cup f^{-1}(B_1)$$
$\square$

(c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

Pf. $$f^{-1}(B_0 \cap B_1) := \{x \in A \mid f(x) \in (B_0 \cap B_1)\}$$
$$= \{x \in A \mid f(x) \in B_0 \text{ and } f(x) \in B_1\}$$
$$= \{x \in A \mid f(x) \in B_0\} \cap \{x \in A \mid f(x) \in B_1\}$$
$$= f^{-1}(B_0) \cap f^{-1}(B_1)$$
$\square$

(d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$

Pf. $$f^{-1}(B_0 - B_1) := \{x \in A \mid f(x) \in B_0 - B_1\}$$
$$= \{x \in A \mid f(x) \in B_0 \text{ and } f(x) \notin B_1\}$$
$$= \{x \in A \mid f(x) \in B_0\} - \{x \in A \mid f(x) \in B_1\}$$
$$= f^{-1}(B_0) - f^{-1}(B_1)$$
$\square$

(e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.

Pf. $f(A_1) := \{f(x) \mid x \in A_1\} = \{f(x) \mid x \in A_0 \cup (A_1 \setminus A_0)\}$

$= \{f(x) \mid x \in A_0\} \cup \{f(x) \mid x \in A_1 \setminus A_0\}$

$= f(A_0) \cup f(A_1 \setminus A_0)$

$\supset f(A_0)$ $\square$

(f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

Pf. $f(A_0 \cup A_1) = \{f(x) \mid x \in A_0 \cup A_1\}$

$= \{f(x) \mid x \in A_0 \text{ or } x \in A_1\}$

$= \{f(x) \mid x \in A_0\} \cup \{f(x) \mid x \in A_1\}$

$=: f(A_0) \cup f(A_1)$ $\square$

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$

Pf. let $y \in f(A_0 \cap A_1) := \{f(x) \mid x \in A_0 \cap A_1\}$

then $\exists x_0 \in A_0 \cap A_1$. s.t. $f(x_0) \in A_0 \cap A_1$

$\Rightarrow f(x_0) \in A_0$ and $f(x_0) \in A_1$

$\Rightarrow y \in f(A_0)$ and $y \in f(A_1)$

$\Rightarrow y \in f(A_0) \cap f(A_1)$

since y is arbitrary, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

when f is injective

let $\Delta = (f(A_0) \cap f(A_1)) \setminus f(A_0 \cap A_1)$, suppose $\Delta \neq \phi$

let $y_0 \in \Delta$. so $y_0 \in f(A_0)$. $y_0 \in f(A_1)$ $y_0 \notin f(A_0 \cap A_1)$

$$\begin{cases} y_0 \in f(A_0) \Rightarrow \exists x_0 \in A_0 \ \text{s.t.}\ f(x_0) = y_0 \\ y_0 \in f(A_1) \Rightarrow \exists x_1 \in A_1 \ \text{s.t.}\ f(x_1) = y_0 \\ y_0 \notin f(A_0 \cap A_1) \Rightarrow \nexists x \in A_0 \cap A_1 \ \text{s.t.}\ f(x) = y_0 \end{cases}$$

$\Rightarrow x_0 \neq x_1$, and $f(x_0) = f(x_1)$

this contradicts with the fact that f is injective

so our assumption is false. i.e. $\Delta = \phi$.

so $f(A_0 \cap A_1) \supseteq f(A_0) \cap f(A_1)$

but also $f(A_0) \cap f(A_1) \supset f(A_0 \cap A_1)$

so when f is injective, the equality holds. $\square$

(h). $f(A_0 - A_1) \supset f(A_0) - f(A_1)$.

if $f(A_0) - f(A_1) = \phi$ then we are done.

pf. let $y_0 \in f(A_0) - f(A_1)$

then $y_0 \in f(A_0)$: $\exists x_0 \in A_0$ $f(x_0) = y_0$

and $y_0 \notin f(A_1)$: $\nexists x_1 \in A_1$ $f(x_1) = y_0$

$\Rightarrow \exists x_0 \in A_0 - A_1$ s.t. $f(x_0) = y_0$

$\Rightarrow y_0 \in f(A_0 - A_1)$

since y_0 is arbitrary. $f(A_0) - f(A_1) \subset f(A_0 - A_1)$

when f is injective,

let $\Delta = f(A_0 - A_1) \setminus (f(A_0) - f(A_1))$ suppose $\Delta \neq \phi$ $y_0 \in \Delta$

$y_0 \in f(A_0 - A_1)$ $y_0 \notin f(A_0) - f(A_1)$

$y_0 \in f(A_0 - A_1) \Rightarrow \exists x_0 \in A_0 - A_1$ s.t. $f(x_0) = y_0$ $(*)$

$\Rightarrow \exists x_0 \in A_0$ s.t. $f(x_0) = y_0$

$\Rightarrow y_0 \in f(A_0)$

but $y_0 \notin f(A_0) - f(A_1)$

so $y_0 \in (f(A_0) \cup f(A_1)) - (f(A_0) - f(A_1)) = f(A_1)$

so $\exists x_1 \in A_1$. s.t. $f(x_1) = y_0$

but according to $(*)$. $\exists x_0 \in A_0 - A_1$. $f(x_0) = y_0$.

and $A_1 \cap (A_0 - A_1) = \phi$

so $x_0 \neq x_1$. but $f(x_0) = f(x_1)$

this contradicts with the fact that f is injective

so our assumption is false. i.e. $\Delta = \phi$.

so $f(A_0 - A_1) \subset f(A_0) - f(A_1)$

but also $f(A_0) - f(A_1) \subset f(A_0 - A_1)$

so the equality holds when f is injective. □

4. (§1.2. Ex 4)

(a) pf. $g^{-1}(C_0) := \{b \in B \mid g(b) \in C_0\}$

$f^{-1}(g^{-1}(C_0)) := \{a \in A \mid f(a) \in g^{-1}(C_0)\}$

$= \{a \in A \mid (g \circ f)(a) \in C_0\}$

~~As we proved in Chapter 1.2. Ex 1 (b)~~

by definition $(g\circ f)^{-1}(C_0) := \{a\in A \mid (g\circ f)(a)\in C_0\}$

so $f^{-1}(g^{-1}(C_0)) = (g\circ f)^{-1}(C_0)$ $\square$

(b) Pf. ~~g is injective: $\forall c_1, c_2 \in C$, $c_1 \neq c_2$~~

f is injective: suppose $a_1, a_2 \in A$. $a_1 \neq a_2$

then $f(a_1) \neq f(a_2)$

g is injective, and $f(a_1), f(a_2) \in B$

then $(g\circ f)(a_1) \neq (g\circ f)(a_2)$

so $g\circ f$ is injective. $\square$

(d) Pf. g is surjective: $\forall c\in C$: $\exists b\in B$ s.t. $g(b)=c$

f is surjective: consider the b above, $\exists a\in A$ s.t. $f(a)=b$

i.e. $\exists a\in A$. $(g\circ f)(a)=c$

so $(g\circ f)$ is surjective $\square$

5 (§1.3. Ex 1)

Solution. reflexity: $\forall p=(x,y)\in\mathbb{R}^2$. $y-x^2=y-x^2$

symmetry: $y_0-x_0^2=y_1-x_1^2 \Rightarrow y_1-x_1^2=y_0-x_0^2$

transitivity: $y_0-x_0^2=y_1-x_1^2$, $y_1-x_1^2=y_2-x_2^2 \Rightarrow y_0-x_0^2=y_2-x_2^2$

so by definition, this is an equivalence relation

equivalent classes are curve $y=x^2+c$

6. (§1.3 Ex 3).

Pf. given statements p and q. $p \Rightarrow q$ is logically equivalent to $\neg p \vee q$

so consider $C=\phi$

by definition C is also symmetric and transitive

but C is not reflexive. $\square$

7. (§1.3. Ex 4)

(a) Pf. symmetry : $a_0 \sim a_1 \Leftrightarrow f(a_0)=f(a_1) \Leftrightarrow f(a_1)=f(a_0) \Leftrightarrow a_1 \sim a_0$

reflexity : $\forall a \in A \cdot f(a)=f(a)$

transivity : $a_0 \sim a_1,\ a_1 \sim a_2 \Leftrightarrow f(a_0)=f(a_1)=f(a_2) \xcancel{\Leftrightarrow} \Rightarrow a_0 \sim a_2$

so "$\sim$" is an equivalence relation $\square$

(b). Pf. $A^* := \{ f^{-1}(\{b\}) \mid b \in B \}$ (by definition)

~~claim~~. since $f: A \to B$ is surjective

$\forall b \in B .\ f^{-1}(\{b\}) \neq \phi$ (*)

consider mapping $g: B \to \cancel{P(A)}\ A^*$

$b \mapsto f^{-1}(\{b\})$

according to (*), g is well-defined.

① g is injective.

consider $b_1, b_2 \in B$. $b_1 \neq b_2$

according to the definition of pre-image

$$f^{-1}(\{b_1\}) =: g(b_1) = \{a \in A \mid f(a) = b_1\}$$
$$\neq g(b_2) = \{a \in A \mid f(a) = b_2\}$$

so g is injective

② g is surjective

this is obvious because $A^* := g(B)$

so g is surjective

from ① and ②. g is bijective $\square$

8. (§1.3. Ex6)

Pf. if $y_0 - x_0^2 < y_1 - x_1^2$ then $(x_0, y_0) < (x_1, y_1)$

if $y_0 - x_0^2 = y_1 - x_1^2$

if $x_0 < x_1$ then $(x_0, y_0) < (x_1, y_1)$

if $x_0 = x_1$ then $(x_0, y_0) = (x_1, y_1)$

if $x_0 > x_1$ then $(x_0, y_0) > (x_1, y_1)$ i.e. $(x_1, y_1) < (x_0, y_0)$

if $y_0 - x_0^2 > y_1 - x_1^2$ then $(x_0, y_0) > (x_1, y_1)$ i.e. $(x_1, y_1) < (x_0, y_0)$

so the relation "<" satisfies: either $(x_0, y_0) < (x_1, y_1)$ or $(x_1, y_1) < (x_0, y_0)$ or $(x_1, y_1) = (x_0, y_0)$

If $(x_0, y_0) < (x_1, y_1)$. then $(x_0, y_0) \neq (x_1, y_1)$ can instantly come from the definition.

If $(x_0, y_0) < (x_1, y_1)$ and $(x_1, y_1) < (x_2, y_2)$

if $x_0 < x_1$ and $x_1 < x_2$. then $x_0 < x_2 \Rightarrow (x_0, y_0) < (x_2, y_2)$

if $x_0 < x_1$ and $x_1 = x_2$ and $y_1 < y_2$. then $x_0 < x_2 \Rightarrow (x_0, y_0) < (x_2, y_2)$

if $x_0 = x_1$ and $y_0 < y_1$ and $x_1 < x_2$. then $x_0 < x_2 \Rightarrow (x_0, y_0) < (x_2, y_2)$

if $x_0 = x_1$ and $y_0 < y_1$ and $x_1 = x_2$ and $y_1 < y_2$.

then $x_0 = x_2$ and $y_0 < y_2$. $\Rightarrow (x_0, y_0) < (x_2, y_2)$

So tranisivity can be attained.

so $< = \mathbb{R}^2$ is an order relation on the plane

Geometrically. $P_1, P_2 \in \mathbb{R}^2$ satisfies $P_1 < P_2$ iff

P_1 and P_2 can be separated by a vertical line with P_1 on the left side, or $\overline{P_1P_2}$ is vertical to x-axis with P_1 down P_2.

9. (§1.3. Ex 12)

Solution.

In (i) $\forall p \in \mathbb{Z}_+ \times (\mathbb{Z}_+ \setminus \{1\})$ has immediate predecessors.

the smallest element is $(1, 1)$.

Pf. consider $P = (x, y) \in \mathbb{Z}_+^2$, $y \neq 1$

then the immediate predecessor should be $(x, y-1)$

where $y-1 \geq 1$. (by definition this is obvious) consider $p=(x,1)$ with $x \geq 1$.

suppose $p'=(x',y')$ is an immediate pred of p

then by definition $x' < x$

W.L.O.G. suppose $x'=x-1$. then $p''=(x', y'+1)$ satisfies that

$$p' < p'' < p$$

this contradicts with our assumption.

there's no such "immediate pred" of p with $y=1$.

The smallest element is $(1,1)$ comes instantly because

$\forall x \in \mathbb{Z}_+. \forall y \in \mathbb{Z}_+. \quad (1,1) \leq (x,y)$

and "$=$" can be attained only when $x=y=1$. □

In (ii) $\forall p \in (\mathbb{Z}_+ \setminus \{1\}) \times (\mathbb{Z}_+ \setminus \{1\})$ has an immediate pred.

the smallest element is $(1,1)$

Pf. if $p \notin (\mathbb{Z} \setminus \{1\}) \times (\mathbb{Z}_+ \setminus \{1\})$

i.e. $p \in \{(x,1) \mid x \in \mathbb{Z}_+\} \cup \{(1,y) \mid y \in \mathbb{Z}_+\}$. $p=(x,y)$

then there is no such $(x',y') \in \mathbb{Z}_+^2$ s.t.

$x'-y' = x-y$ and $y' < y$

so if an ~~immed~~ immediate pred exists, it should be like:

(x',y') where $x'-y' < x-y$

but $(x'+1, y'+1) < x-y$ and $(x',y') < (x'+1, y'+1)$.

so then immediate~~ly~~ pred doesn't exist.

If $p \in (\mathbb{Z}_+ \setminus \{1\}) \times (\mathbb{Z}_+ \setminus \{1\})$

i.e. $p = (x.y)$ with $x \geq 2$. $y \geq 2$.

then pred of p is $(x-1. y-1)$

The smallest element in this case should be $(1,1)$

$\square$ $\forall x \in \mathbb{Z}_+$. $\forall y \in \mathbb{Z}_+$. $(1,1) < (x.y)$ or $(1,1) = (x.y)$

and "=" can only be attained if $x = y = 1$.

$\square$

In (iii) Every element (except $(1,1)$) has immediate pred.

and the smallest element is (1.1).

pf. claim.

$$\text{pred}(x.y) = \begin{cases} (x+1, y-1) & y \neq 1, \\ (1, x-1) & x \neq 1, y = 1 \end{cases}$$

if $y \neq 1$. my claim is obviously true

if $y = 1$. $1 + x - 1 < x + y = x + 1$ so pred$(x.y) < (x.y)$

if $(x_0. y_0)$ satisfies pred$(x.y) < (x_0.y_0) < (x.y)$

then since $\forall n \in \mathbb{Z}_+$: $x < n < x+1$

if $x_0 + y_0 < x + y$. then $x_0 + y_0$ should be $x_{pred} + y_{pred} = x$

since $y_{pred} = x - 1$ is already largest possibly ~~when $y_0 < y_{pred}$~~

so this can't be true.

if $x_0+y_0 = x+y$ then $y_0 < y = 1$. obviously impossible.

The smallest element is $(1,1)$.

because $\forall x \in \mathbb{Z}_+ \ \forall y \in \mathbb{Z}_+$. ~~$x+y \geq$~~ $1+1 < x+y$ or $1+1 = x+y$ and "$=$" can only be attained when $x=1$. $y=1$. also $\forall y \in \mathbb{Z}_+$. $1 \leq y$. □

So the 3 orders are different. □

10. (§1.3. Ex 13)

Pf. since ordered set A has least upper bound property.

~~suppose~~ suppose the order is denoted as "$<$"

consider $E \subset A$ $E \neq \phi$ and E is bounded below.

then let $F = \{x \in A \mid \forall y \in E.\ x < y\} \subset A$

since E is bounded below. $F \neq \phi$

since $E \neq \phi$. then F is bounded above

since $F \subset A$. F has the LUB property. $\sup F$ exists and is unique

claim. $\inf E = \sup F$.

we only need ~~this suffices~~ to show that ① $\forall x \in E.\ \sup F < x$

and ② $\nexists x \in A$: $\sup F < x$ and x is a lower bound of E.

①

if $\exists x_0 \in E$. $x_0 < \sup F$

but by definition of F. $\forall x \in F: x < x_0$

so x_0 is an upper bound of F and $x_0 < \sup F$. contradiction

So $\forall x \in E$. $\sup F \le x$

i.e. $\sup F$ is a lower bound of E.

Pf of ②.

if $\exists x_0 \in A$. $\sup F < x_0$ and x_0 is a lower bound of E

then $x_0 \in F$

then $\exists x \in F$. $x < \sup F$ contradiction.

so $\nexists x_0 \in A$ $\sup F < x_0$ and $\forall y \in E$. $x_0 < y$.

i.e. $\sup F = \inf E$. □

11. (§1.4. Ex 2)

(a) Pf. According to axiom (6)

$x > y \Rightarrow x + w > y + w$

$w > z \Rightarrow w + y > z + y$ (I)

According to axiom (2)

$w + y = y + w > y + z = z + y$ (II)

According to (I) and (II)

$x + w > y + z$. □

(g). Pf. According to axiom (3). $1 \neq 0$.

If $1<0$ $\quad$ (A6) $\Rightarrow 1+(-1) < 0+(-1) \Rightarrow 0<-1$

(A6) $\Rightarrow 1\times(-1) < 0\times(-1) \Rightarrow -1<0$

contradiction.

so $1>0$ (property of order)

Again apply A6. $\quad 1+(-1) > 0+(-1)$

$\Rightarrow 0>-1$

so $-1<0<1$

□

(k) Pf. According to (g) $1>0$

so $2:=1+1>1+0=1>0$

~~(A6) $\Rightarrow$ $x<y \Rightarrow 2\cdot x < 2\cdot y$~~

~~(A5) $\Rightarrow$ $x+x<y+y$~~

(A6) $\quad x<y \Rightarrow x+x<y+x, \ x+y<y+y$

(A5) $\quad 2x<y+x, \ x+y<2y$

(A2) $\quad 2x<x+y, \ x+y<2y$

(A4) $\quad$ Inverse of 2 exists and is unique. $(\neq 0)$

If $2^{-1}<0$ then A6 $\Rightarrow 2\cdot 2^{-1} < 0\cdot 2 \Rightarrow 1<0$

so $2^{-1}>0$

(A6) $\quad 2^{-1}\cdot 2\cdot x < (x+y)/2 < 2^{-1}\cdot 2\cdot y$ □

12. (§1.4. Ex4(a))

Pf. When $n=1$. $\{1\}$ has only one nonempty subset $\{1\}$ and 1 is the largest element.

suppose the conclusion is true for $n=k$ $k\in\mathbb{N}$

Observe $\{1,2,\cdots,k+1\} = \{1,2,\cdots,k\} \cup \{k+1\}$

Let P_k be the set of all non-empty subsets of $\{1,2,\cdots,k\}$

$$P_{k+1} = \{p\cup\{k+1\} \mid p\in P_k\} \cup P_k \cup \{\{k+1\}\}$$

According to induction hypothesis. $\forall p\in P_k$ has the largest element.

$\forall p\in P_{k+1}\setminus P_k$. it's obvious that the largest element is $k+1$.

so $\forall n\in\mathbb{N}$. ~~P_n~~ every element of P_n has its largest element. $\square$

13. (§1.4. Ex.9)

(a) Pf. Let $E\subset\mathbb{Z}$ be any nonempty subset of $\mathbb{Z}$ that is bounded above, u is its upper bound.

~~$\forall n\in E$. $n=u$ or $n<u$~~

can be attained

~~if $n=u$, then we have nothing more to prove~~.

~~if $n<u$ forall $n\in E$~~.

If $E \cap \mathbb{N}_0 = \phi$

let $-E := \{-n \mid n \in E\} \subset \mathbb{Z}_+$

by theorem 4.1 $-E$ has a smallest element

so E has a largest element.

If $E \cap \mathbb{N}_0 \neq \phi$

then the largest element of E should be the largest element of $E \cap \mathbb{N}_0$

since E is bounded above (suppose the upper bound is u)

then $E \cap \mathbb{N}_0$ is the subset of $\{1, 2, \cdots, u\}$

by §1.4 Ex 4 (a). $E \cap \mathbb{N}_0$ has a largest element.

i.e. E has a largest element. $\square$

(c). Pf. If $\nexists n \in \mathbb{Z}$. $y < n < x$

~~by~~ $\exists n \in \mathbb{Z}$. $n \leq y < x \leq n+1$

and the two "="s cannot be attained together

(because otherwise $x - y = 1$)

so $x - y < n+1-n = 1$ $\square$

but $x - y > 1$ contradiction.

14. ($\S$ 1.5. Ex 4.) ~~Ex~~

(d). Pf. let $k: X^n \times X^\omega \to X^\omega$

$$((x_i)_{i=1}^n, (y_n)_{n\in\mathbb{Z}_+}) \mapsto (x_1, x_2, \cdots x_n, y_1, y_2, \cdots)$$

this map is obviously bijective. $\square$

(e). Pf. let $l: X^\omega \times X^\omega \to X^\omega$

$$((x_n)_{n\in\mathbb{Z}_+}, (y_n)_{n\in\mathbb{Z}_+}) \mapsto (x_1, y_1, x_2, y_2, \cdots x_n, y_n, \cdots)$$

given $(x_n)_{n\in\mathbb{Z}_+}$, $l^{-1}((x_n)) = ((x_{2k-1})_{k\in\mathbb{Z}_+}, (x_{2k})_{k\in\mathbb{Z}_+})$

so l is bijective. $\square$

Part II.

15. (II.1).

(1). Pf. $X - \bigcup_{A\in\mathcal{A}} A := \{x \in X \mid x \notin \bigcup_{A\in\mathcal{A}} A\}$

$$\bigcap_{A\in\mathcal{A}} (X-A) := \{x \in \bigcup_{A\in\mathcal{A}} (X-A) \mid \forall A \in \mathcal{A}.\ x \in X-A\}$$

$$= \{x \mid x \in X.\ \forall A \in \mathcal{A}.\ x \notin A\}$$

$$= \{x \mid x \in X.\ x \notin \bigcup_{A\in\mathcal{A}} A\}$$

$$= X - \bigcup_{A\in\mathcal{A}} A$$ $\square$

(2) $X - \bigcap_{A\in\mathcal{A}} A := \{x \in X \mid x \notin \bigcap_{A\in\mathcal{A}} A\}$

$$\bigcup_{A\in\mathcal{A}} (X-A) := \{x \mid \forall A \in \mathcal{A}.\ x \in X - A\}$$

$$= \{x \mid \forall A \in \mathcal{A}.\ x \in X \wedge x \notin A\}$$

$$= \{x \in X \mid \forall A \in \mathcal{A}.\ x \notin A\}$$

$$= \{x \in X \mid x \notin \bigcap_{A\in\mathcal{A}} A\}$$

$$= X - \bigcap_{A\in\mathcal{A}} A.$$

□

16. (II.2).

Pf. given a set A. suppose a_1 and a_2 are both sup of A

W.L.O.G suppose $a_1 < a_2$.

by definition a_2 is not the sup of A. contradiction.

so $a_1 = a_2 = \sup A$.

so sup of a set is unique.

similarly inf of a set is also unique.

□