(1) We define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{3x^2+y^2}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Use the definition of continuous function (in terms of "$\epsilon - \delta$") to show that f is discontinuous at $(0,0)$.

Pf. f is continuous if $\forall \varepsilon > 0.\ \exists \delta > 0.$ s.t.

$$\|(x.y) - (0.0)\| < \delta \Rightarrow |f(x.y) - f(0.0)| < \varepsilon$$

let $\varepsilon_0 = \frac{1}{2}$ $\quad |f(x.y) - f(0.0)| = \frac{1}{2} + \frac{\frac{5}{2}x^2}{x^2+2y^2} > \varepsilon_0 \quad \forall x.y.$

let $x_i = 0 \quad y_i = \left(\frac{1}{2}\right)^i$

$(x_i.y_i) \to (0.0)$ as $i \to +\infty$. but $|f(x_i.y_i) - f(0.0)| > \varepsilon_0$

so f is discontinuous at (0.0) by definition

$\square$

(2) Let $\{x_j\}_{j=1}^{\infty}$ be an infinite sequence in $\mathbb{R}^n$. If the sequence is bounded, show that it has a converging subsequence. (Hint: applying Bolzano-Weierstraß Theorem)

Pf. By Bolzano-Weierstrass Theorem. $\{x_j\}_{j=1}^{\infty}$ has a limit point.

denote the limit point by x_0

by definition $\forall r > 0.\ \exists j \in \mathbb{N}.\ 0 < |x_j - x_0| < r$

let $i_1 = 1$. $\quad r_1 = |x_0 - x_{i_1}|$

with r_k picked. $i_{k+1} = j$ where $x_j \in B_{r_k}(x_{i_k}) \cap \{x_j\}_{j=1}^{\infty} \setminus \{x_{i_k}\} \neq \phi$

$r_{k+1} = |x_0 - x_{i_k}|$.

then such an inductively picked sequence $\{x_{i_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{x_j\}_{j=1}^{\infty}$. with $\lim_{k\to\infty} x_{i_k} = x_0$. $\square$

(3) Consider two sequences $L = \{a_j\}_{j=1}^{\infty}$ and $R = \{b_j\}_{j=1}^{\infty}$ in $\mathbb{R}^1$. If

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{j-1} \leqslant a_j \leqslant \ldots \leqslant b_j \leqslant b_{j-1} \leqslant \ldots \leqslant b_1,$$

and

$$\lim_{j\to\infty} |b_j - a_j| = 0,$$

then show that

$$\sup L = \inf R.$$

Pf. By definition $\forall i \in \mathbb{N}$. b_i is an upper bound of L.

so $\sup L \leq b_i \quad (\forall i \in \mathbb{N})$

so $\sup L$ is a lower bound of R

so $\sup L \leq \inf R$.

if $\sup L < \inf R$ let $\Delta = \inf R - \sup L > 0$

since $\forall j \in \mathbb{N}$. $a_j \leq \sup L < \inf R \leq b_j$.

$\Rightarrow |b_j - a_j| \geqslant \Delta \quad (\forall j \in \mathbb{N})$

$\Rightarrow \lim_{j\to\infty} |b_j - a_j| \geqslant \Delta > 0$

this contradicts with $\lim_{j\to\infty} |b_j - a_j| = 0$.

so $\sup L = \inf R$. $\square$

(4) **Textbook Chapter 1.4.** Exercise 9(b): if $x \in \mathbb{R} \setminus \mathbb{Z}$, then prove that there exists a **unique** integer $n_0 \in \mathbb{Z}$ such that

$$n_0 < x < n_0 + 1.$$

Pf. If $x > 0$. Then $A = \{n \in \mathbb{Z}_+ \mid n < x\}$ is finite.

Any nonempty subset of $\mathbb{Z}$ has a maximum element.

let $n_0 = \max_{n \in A} n$. $n_0 < x$ by definition.

Claim. $x < n_0 + 1$.

if not. $n_0 + 1 < x \Rightarrow n_0 + 1 \in A \Rightarrow n_0 \neq \max_{n \in A} n$ contradiction.

so $n_0 < x < n_0 + 1$ when $x > 0$.

if $x < 0$. $\exists n_1 \in \mathbb{Z}_+$ s.t. $n_1 < -x_0 < n_1 + 1$

$\Rightarrow -n_1 - 1 < x_0 < -n_1$

so the conclusion holds for $x < 0$.

so $\forall x \in \mathbb{R} \setminus \mathbb{Z}$. $\exists!\, n_0 \in \mathbb{Z}$ s.t. $n_0 < x < n_0 + 1$ $\square$

(5) Use (4) to prove that for any real number $x \in \mathbb{R}$ and for any $\epsilon > 0$, there exists a rational number q such that

$$q < x < q + \epsilon.$$

Pf. If $x \in \mathbb{R} \setminus \mathbb{Z}$.

$\exists n_0 \in \mathbb{Z}$. s.t. $n_0 < x < n_0 + 1$

since $\varepsilon > 0$. $\exists k \in \mathbb{Z}_+$ s.t. $0 < \frac{1}{k} < \varepsilon$.

consider the sequence $x_i = n_0 + \frac{i}{k}$ $\quad 0 \leq i \leq k$

$\exists !\, i \in \{1, \cdots, k\}$ s.t. $x_{i-1} \leq x < x_i$

$\Rightarrow n_0 + \frac{i-1}{k} \leq x < n_0 + \frac{i}{k}$

let $q = n_0 + \frac{i-1}{k}$ $\quad \frac{1}{k} < \varepsilon \Rightarrow q + \frac{1}{k} < q + \varepsilon$

$\Rightarrow q \leq x < q + \frac{1}{k} < q + \varepsilon.$

If $x \in \mathbb{Z}$. use the k above. let $q = x - \frac{1}{k} \in \mathbb{Q}$

$x - \frac{1}{k} < x < x + (\varepsilon - \frac{1}{k}) \Rightarrow q < x < q + \varepsilon.$

So $\forall x \in \mathbb{R}.\ \varepsilon > 0.\ \exists q \in \mathbb{Q}.$ s.t. $q < x < q + \varepsilon.$ $\quad \square$.

(6) Let us recall the definition of "subspace topology" for subsets of $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be a nonempty subset. We call a subset $W \subseteq U$ **an open set in** U (or an open set with respect to U) if there exists an open set $O \subset \mathbb{R}^n$ such that $W = O \cap U$.

(a) Consider the set $S \equiv \{\frac{1}{n} : n \in \mathbb{Z}_+\}$. Let us regard S as a subspace of $\mathbb{R}$. Show that every *singleton* (a set with exactly one element) of S is an open set in S.

(b) Consider the set $\mathbb{Q}$ of all rational numbers. If $\mathbb{Q}$ is regarded as a subspace of $\mathbb{R}$, show that no *singleton* of $\mathbb{Q}$ is an open set in $\mathbb{Q}$.

(a). Pf. Let $O_n = B_{r_n}(\frac{1}{n})$ where $r_n = \frac{1}{n} - \frac{1}{n+1}$

By definition $\forall n \in \mathbb{Z}_+$. O_n is open.

$S \cap O_n = \{\frac{1}{n}\}$. $\forall n \in \mathbb{Z}_+$

$\Rightarrow \{\frac{1}{n}\}$ is open in S. $\forall n \in \mathbb{Z}_+$.

$\square$

(b). Pf. let $q \in \mathbb{Q}$.

$\{q\}$ is a singleton in $\mathbb{Q}$.

If there is such $O \subset \mathbb{R}$ open in $\mathbb{R}$ s.t. $O \cap \mathbb{Q} = \{q\}$

since every point of O should be an interior point.

then $\exists r_0 > 0$ s.t. $B_{r_0}(q) \subset O$ (I)

the conclusion from (5) shows that:

$$\forall x \in \mathbb{R}.\ \forall \varepsilon > 0.\ \exists q \in \mathbb{Q}:\ q < x < q + \varepsilon$$

$$\Rightarrow\ x - \varepsilon < q < x$$

this indicates that given any two different real numbers, there always exists a rational number between them.

In (I). $q \in \mathbb{R}$. $q + r_0 \in \mathbb{R}$. $\Rightarrow \exists q_1 \in \mathbb{Q}.\ q < q_1 < q + r_0$

$\Rightarrow$ $q_1 \in B_{r_0}(q)$ and $q \neq q_1$

$\Rightarrow$ $\{q. q_1\} \subset O \cap \mathbb{Q}$.

So no such open set O exists in $\mathbb{R}$

$\square$