Pullback Property: Quotient Topologies to Categories

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1 Introduction

The most elegant theorem I've learnt in this course is the pullback property of quotient topologies. While I've been learning Abstract Algebra this semester, I found some generalized facts about this property in category theory. The four problems and sections corresponding answers of this homework are as follows:

1. Prove the theorem. (Section 2.1)
2. In your opinion, what is the most crucial assumption of the theorem? Does the theorem still hold if you remove this assumption? If not, can you construct an example? (Section 2.2)
3. Describe the importance of the theorem. (Section 2.3, Section 3)
4. Provide a typical example to exhibit the application of theorem. (Section 4)

2 Pullback Property

2.1 Pullbacks in Quotient Topologies

Definition. Quotient map Let X and Y be topological spaces. A map $f : X \to Y$ is said to be a *quotient map* if for any set $U \subset Y$, U is open in Y iff $f^{-1}(U)$ is open in X.

By the way a quotient map is defined, it's obviously continuous. It's sometimes called *strongly continous* for this reason.

Theorem (Pullback Property). Let $\pi : X \to Y$ be a quotient map. Let Z be an arbitrary topological space. Then there is a bijection between

- $\mathscr{H} := \{h \in \mathtt{Map}(X, Z) \mid h \text{ is constant on every } \pi^{-1}(\{y\}), y \in Y\}$, and;
- $\mathscr{F} := \mathtt{Map}(Y, Z)$,

which can be realized by the *pullback*

$$\begin{aligned} \pi^* : \mathscr{F} &\to \mathscr{H} \\ f &\mapsto f \circ \pi \end{aligned}$$

Moreover, the following holds:
1. $f \in \mathscr{F}$ is continuous if and only if $h = \pi^*(f)$ in $\mathscr{H}$ is continuous.
2. $f \in \mathscr{F}$ is a quotient map if and only if $h = \pi^*(f)$ in $\mathscr{H}$ is a quotient map.

Proof. For each $y \in Y$, $h(\pi^{-1}(\{y\}))$ is a singleton in Z, denoted the element in the singleton by $f(y)$, we have acturally defined a map implicitly by $f \circ \pi = h$. Since the choice of h is arbitrary, π^* is surjective. Suppose $\pi^*(f_1) = f_1 \circ \pi = h_1 = h_2 = f_2 \circ \pi = \pi^*(f_2)$, let $x_0 \in \pi^{-1}(y_0)$ for any $y \in Y$, then $f_1(y) = f_1 \circ \pi(x_0) = h_1(x_0) = h_2(x_0) = f_2(y)$, so $f_1 = f_2$, i.e. π^* is injective.

Let U be any open set in Z, $h^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1} \circ f^{-1}(U)$. Since π is a quotient map, $h^{-1}(U)$ is open in X if and only if $f^{-1}(U)$ is open in Y . Therefore $f \circ \pi$ is continuous if and only if f is continuous.

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In fact, the Pullback Property could be described using the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f \circ \pi"] \arrow[d, "\pi"] & Z \\
Y \arrow[ur, "f"'] &
\end{tikzcd}$$

2.2 Analysis of the Crucial Assumption

The most crucial assumption in the Pullback Property theorem is that $\pi : X \to Y$ must be a **quotient map**, not just any continuous surjection. This assumption is essential because:

1. It ensures that the topology on Y is the finest topology making π continuous (the quotient topology)
2. It provides the necessary and sufficient conditions for lifting continuity through the pullback

Proposition. The theorem fails if we only assume π is a continuous surjection.

Proof. We construct a counterexample:

Let $X = \mathbb{R}$ with the usual topology, and let $Y = \mathbb{R}$ with the indiscrete topology. Define $\pi : X \to Y$ as the identity function. Then:

1. π is continuous (since every set in the indiscrete topology is open)
2. π is surjective (as the identity map)
3. However, π is not a quotient map because the preimage of any non-empty proper subset of Y is open in X, but no such subset is open in Y

Now let $Z = \mathbb{R}$ with the usual topology, and consider:
- $h : X \to Z$ defined by $h(x) = x$ (the identity function)
- $f : Y \to Z$ defined by $f(y) = y$ (also the identity function)

Then:
1. $h = f \circ \pi$ (so h is constant on the fibers of π)
2. h is continuous (identity map between spaces with usual topology)
3. But f is not continuous (identity map from indiscrete to usual topology)

This violates the conclusion of the theorem that h is continuous if and only if f is continuous. Therefore, the quotient map assumption cannot be weakened to just continuous surjection.

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Remark. This counterexample illustrates the universal property of quotient maps: they are precisely the maps that allow us to “lift” continuity through the pullback construction. Without this property, we lose the equivalence of continuity between the original and pulled-back functions.

2.3 Pullbacks in Categories

Definition. (Category) A category $\mathbf{C}$ consists of:

1. A collection of objects
2. A collection of morphisms (or arrows) between objects
3. A composition operation for morphisms that is associative
4. An identity morphism for each object

The category $\mathbf{Top}$ of topological spaces consists of:
1. Objects: Topological spaces
2. Morphisms: Continuous functions
3. Composition: Usual function composition
4. Identity: Identity function on each space

Let’s understand how pullbacks generalize in categorical settings:

Definition. (Universal Property) In any category $\mathbf{C}$, a pullback of arrows $f : A \to C$ and $g : B \to C$ consists of:

1. Arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that $fp_1 = gp_2$
2. For any object Z with maps $z_1 : Z \to A$ and $z_2 : Z \to B$ satisfying $fz_1 = gz_2$, there exists a unique $u : Z \to P$ making all diagrams commute.

3 Importance of the Theorem

The importance of the pullback property lies in its fundamental role in connecting different mathematical structures. Let’s explore this in detail:

3.1 Universal Nature of Quotient Maps

Theorem. The pullback property provides a universal characterization of quotient maps in the category $\mathbf{Top}$.

This universality manifests in several ways:

1. **Factor Space Construction**: Given any continuous map $f : X \to Z$ that is constant on the fibers of a quotient map $\pi : X \to Y$, there exists a unique continuous map $\tilde{f} : Y \to Z$ making the diagram commute.

2. **Preservation of Properties**: The pullback property shows that certain topological properties are preserved under quotient maps, making it a powerful tool for studying topological spaces through their quotients.

3.2 Categorical Perspective

Definition. (Pullback) In any category $\mathbf{C}$, a pullback of arrows $f : A \to C$ and $g : B \to C$ consists of arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that $fp_1 = gp_2$, and universal with this property. i.e. Given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $fz_1 = gz_2$, there exists a unique $u : Z \to P$ such that $z_1 = p_1 u$ and $z_2 = p_2 u$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

The pullback construction provides several key insights:

1. **Universality**: The pullback property is universal in the sense that it characterizes the quotient map up to unique isomorphism.
2. **Functoriality**: The pullback operation defines a functor between appropriate categories, preserving the structural relationships between spaces.
3. **Naturality**: The construction is natural in the sense of category theory, meaning it commutes with the relevant morphisms in a functorial way.

4 Applications of the Theorem

The pullback property finds applications across various areas of mathematics:

4.1 Topological Applications

Example 1. (Quotient Spaces) Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ with the quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}$ being the projection onto the first coordinate. For any continuous function $f : \mathbb{R} \to \mathbb{R}$, the pullback $\pi^*(f)$ gives us the function $h(x, y) = f(x)$, which is constant on vertical lines (the fibers of π).

Example 2. (Identification Spaces) Consider the torus T^2 as a quotient of the square $[0,1] \times [0,1]$. The pullback property allows us to characterize continuous functions on the torus in terms of periodic functions on the square.

4.2 Group Theory Applications

Example 3. (Normal Subgroups) Let G be a group and $N \trianglelefteq G$ a normal subgroup. The quotient map $\pi : G \to G/N$ and any homomorphism $f : H \to G/N$ give rise to a pullback diagram:

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"] & H \arrow[d, "f"] \\
G \arrow[r, "\pi"] & G/N
\end{tikzcd}
$$

The pullback P is isomorphic to the fiber product $H \times_{\frac{G}{N}} G$.

Bibliography

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