(1) Let (X, d) be a metric space. A set $A \subseteq X$ is said to be closed if $X \setminus A$ is open. Then prove the following property. If $A \subseteq X$ is closed, then A contains all its limit points.

(1) Pf. If $A \subset X$ is closed. Let $N_r(x) := \{y \mid d(x,y) < r\}$

by definition $X \setminus A$ is open.

$\forall x \in X \setminus A.\ \exists r > 0 \quad N_r(x) \subset X \setminus A$

Let A' be all the limit points of A. by definition

$\forall x \in A'.\ \forall r > 0.\ \exists y \in A.\ N_r(x) \cap A \setminus \{x\} \neq \phi$

If $A' \setminus A \neq \phi$. suppose $x_0 \in A' \setminus A \subset X \setminus A$

then $\exists r_0 > 0$ s.t. $N_{r_0}(x_0) \subset X \setminus A$

i.e. let $r_1 = \frac{1}{2} r_0 \quad N_{r_1}(x_0) \cap A \setminus \{x_0\} \subset N_{r_1}(x_0) \cap A = \phi$

this contradicts with the fact that x_0 is a limit point of A.

So our assumption is false.

$A' \setminus A = \phi \quad$ i.e. $A' \subset A$.

$\square$

(2) Prove that $\sqrt{2}$ is irrational.

(2). Pf. If $\sqrt{2} \in \mathbb{Q}$.

$\exists p.q. \in \mathbb{Z}$ s.t. $\sqrt{2} = \frac{p}{q}$ and $\gcd(p,q) = 1$

$\Rightarrow 2q^2 = p^2 \Rightarrow 2 \mid p^2$

so p must be even. let $p = 2k$. $k \in \mathbb{Z}$

$$2q^2 = 4k^2 \Rightarrow 2 \mid q^2$$

so q must be even

so $\gcd(p, q) \geq 2$. contradiction.

so $\sqrt{2} \notin \mathbb{Q}$. $\square$

(3) Prove that $\mathbb{Q}$ is not closed in $\mathbb{R}$. (Hint: Problem (5) in Assignment 4 may be used here)

Pf. Let $\mathbb{Q}'$ be the set of all the limit points of $\mathbb{Q}$

If $\mathbb{Q}$ is closed in $\mathbb{R}$

we have $\mathbb{Q}' \subset \mathbb{Q}$ (by conclusion of (1))

Claim. $\sqrt{2} \in \mathbb{Q}' \setminus \mathbb{Q}$

In HW4 P5 (b) I showed that

$$\forall x \in \mathbb{R}.\ \forall \varepsilon > 0.\ \exists q \in \mathbb{Q} \text{ s.t. } q < x < q + \varepsilon$$

i.e. $x - \varepsilon < q < x$

so $\forall x \in \mathbb{R}.\ \forall r > 0.\ N_r(x) \cap \mathbb{Q} \setminus \{x\} \neq \emptyset$

so $\forall x \in \mathbb{R}.\ x \in \mathbb{Q}'$

so $\sqrt{2} \in \mathbb{Q}'$

but $\sqrt{2} \notin \mathbb{Q}$ (conclusion of (2))

then $\mathbb{Q}' \not\subset \mathbb{Q}$. $\square$