Problem 1 (Ternary expansion). To formulate this problem, we first assume some basic knowledge in mathematical analysis:

- (Convergence of geometric series) If $q \in (0,1)$, then geometric series $\sum_{n=0}^{\infty} q^n$ converges to $\frac{1}{1-q}$.
- (Comparison principle) Let $a_n \geqslant 0$ and $b_n \geqslant 0$ satisfy $b_n \geqslant a_n$ for any $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges to $B \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n$ converges to some real number A. Moreover, $A \leqslant B$.

Let us formulate the ternary expansions for all real numbers in $[0,1]$ as follows: for any $x \in [0,1]$, we write

$$[x]_3 = 0.d_1d_2d_3\ldots \equiv \sum_{n=1}^{\infty} \frac{d_n}{3^n}, \quad d_n \in \{0,1,2\}.$$

For example, $\frac{1}{3}$ has two different ternary expansions:

$$\left[\frac{1}{3}\right]_3 = 0.100000\ldots = 0.0222222\ldots.$$

Now prove the following property. If $x \in [0,1]$ has two *distinct* ternary expansions

$$[x]_3 = 0.d_1d_2\ldots d_n\ldots = 0.e_1e_2\ldots e_n\ldots,$$

then the following holds. Let $n \equiv \min\{k \in \mathbb{Z}_+ : d_k \neq e_k\}$. Then $e_n = d_n + 1$ and

$$d_k = 2, \quad e_k = 0, \quad \forall\, k \geqslant n+1.$$

Pf. W.L.O.G $e_n - d_n \in \{1, 2\}$ where $n = \min\{k \in \mathbb{Z}_+ \mid d_k \neq e_k\}$

if $e_n - d_n = 2$

$$\sum_{k=1}^{\infty} \frac{d_k}{3^k} - \sum_{k=1}^{\infty} \frac{e_k}{3^k} = \frac{2}{3^n} + \sum_{k=n+1}^{\infty} \frac{d_k}{3^k} - \sum_{k=n+1}^{\infty} \frac{e_k}{3^k} \qquad (\mathrm{I})$$

since $0 \leqslant \frac{d_k}{3^k}, \frac{e_k}{3^k} \leqslant \frac{2}{3^k}$. $\sum_{k=n+1}^{\infty} \frac{2}{3^k} = \frac{1}{3^n} < \infty$ for a given $n \in \mathbb{Z}$.

By comparison principle, we have $\sum_{k=n+1}^{\infty} \frac{d_k}{3^k} < \infty$, $\sum_{k=n+1}^{\infty} \frac{e_k}{3^k} < \infty$

so $(I) = \frac{2}{3^n} + \sum_{k=n+1}^{\infty} \frac{d_k - e_k}{3^k} \geqslant \frac{2}{3^n} - \sum_{k=n+1}^{\infty} \frac{2}{3^k} = \frac{1}{3^n} > 0$

contradiction

So $e_n - d_n = 1$.

$$\sum_{k=1}^{\infty} \frac{e_k}{3^k} - \sum_{k=1}^{\infty} \frac{d_k}{3^k} = \left(\sum_{k=n+1}^{\infty} \frac{e_k - d_k}{3^k}\right) + \frac{1}{3^n} = \sum_{k=n+1}^{\infty} \frac{e_k - d_k + 2}{3^k} = 0$$

But $e_k - d_k + 2 \geqslant 0$

the only possibility is that $d_k - e_k = 2$. $(\forall k \geqslant n+1)$

so $e_n = d_n + 1$. $\forall k \geqslant n+1$. $e_k = 0$, $d_k = 2$. $\square$

Problem 2. Let us construct an infinite subset $C \subset [0,1]$ in the following inductive process.

$$F_0 = [0,1],$$
$$F_1 = \underbrace{\left[0, \frac{1}{3}\right]}_{I_1^{(1)}} \cup \underbrace{\left[\frac{2}{3}, 1\right]}_{J_1^{(1)}},$$
$$F_2 = \underbrace{\left[0, \frac{1}{9}\right]}_{I_1^{(2)}} \cup \underbrace{\left[\frac{2}{9}, \frac{1}{3}\right]}_{J_1^{(2)}} \cup \underbrace{\left[\frac{2}{3}, \frac{7}{9}\right]}_{I_2^{(2)}} \cup \underbrace{\left[\frac{8}{9}, 1\right]}_{J_2^{(2)}},$$
$$\vdots$$
$$F = \bigcap_{n=0}^{\infty} F_n.$$

$= 0.011$

$\frac{4}{27} = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{9} + 1 \cdot \frac{1}{27}$

$d_3 = g_3(\frac{4}{27}) = 2$

$d_2 = g_2(\frac{4}{27} - \frac{1}{27}) = g_2(\frac{1}{9}) = 0$

$d_1 = g_1(\frac{1}{9} - 9$

$[0, \frac{1}{27}]$ $[\frac{2}{27}, \frac{3}{27}]$

That is, in the nth step, there are 2^n-intervals $I_k^{(n)}$ and $J_k^{(n)}$ in the set F_n. The union $I_k \cup J_k$ comes from deleting an open interval that contributes *the central* 1/3 in its predecessor. We will prove that F is uncountable by achieve the following steps.

(1) For any $n \in \mathbb{Z}_+$, F_n is identical to the following set F_n' of ternary decimals

$$\{0.d_1d_2d_3d_4\ldots | d_j \in \{0,2\} \ \forall 1 \leqslant j \leqslant n\}.$$

Also prove that any element $x \in F$ has a unique ternary expansion in $\bigcap_{n=1}^{\infty} F_n'$.

(2) We take a countable subset $G = \{x^1, x^2, x^3 \ldots\} \subseteq F$ and write them in the ternary expansion as described above,

$$x^1 = 0.d_1^1 d_2^1 d_3^1 d_4^1 \ldots$$
$$x^2 = 0.d_1^2 d_2^2 d_3^2 d_4^2 \ldots$$
$$x^3 = 0.d_1^3 d_2^3 d_3^3 d_4^3 \ldots$$
$$x^4 = 0.d_1^4 d_2^4 d_3^4 d_4^4 \ldots$$
$$\vdots,$$

where $d_i^j \in \{0,2\}$ for any $i, j \in \mathbb{Z}_+$. We define an element $p \in F$ with a ternary expansion $[p]_3 = 0.p_1p_2p_3p_4\dots$ such that

$$p_j = \begin{cases} 0 & \text{if } d_j^j = 2, \\ 2 & \text{if } d_j^j = 0. \end{cases}$$

Prove that $p \notin G$.

(3) Based on the previous steps, prove that the set F is uncountable.

(1) Pf. The correspondence between F_n and F_n' can be clarified as follows.

For convenience, define function

$$t : [0,1] \to \mathbb{Z} \qquad t(m) = \begin{cases} 0 & 0 \leq m \leq \frac{1}{3} \\ 1 & \frac{1}{3} < m < \frac{2}{3} \\ 2 & \frac{2}{3} \leq m \leq 1 \end{cases}$$

denote $f : F_n \to F_n'$

$$m \mapsto 0.d_1d_2\cdots d_n$$

where $d_1 = t(m)$

$$d_{i+1} = t\left(3\cdot\left(m - \frac{d_i}{3^i}\right)\right) \qquad 1 \leq i \leq n-1. \qquad (\mathrm{I})$$

Inversely, given $0.d_1d_2\cdots d_n$, for $1 \leq k \leq n$

pick I_k at step k if $d_k = 0$

pick J_k at step k if $d_k = 2$.

So F_n and F_n' are corresponding.

Using function f. $\forall x \in F$. $f(x)$ is the unique ternary expansion of x.

$f(x)$ is unique because by the recurrence relation if d_i is unique. then d_{i+1} is unique. Inductively $f(x)$ is unique

$\square$.

(2). Pf. If $p \in G$

$\exists j \in \mathbb{Z}_+$ s.t. $0.d_1^j d_2^j \cdots d_j^j \cdots = p$

but by definition $p_j \neq d_j^j$.

contradiction.

So $p \notin G$.

$\square$

(3). Suppose F is countable

then F is a countable subset of itself.

by (2) $\exists p \in F$ s.t. $p \notin G = F$ contradiction

obviously F is infinite.

so F is uncountable.

$\square$

Problem 3. This problem is to prove **Cantor's Theorem**: *Given any set A, denote by $\mathscr{P}(A)$ the power set of A. Then there does not exist a surjective function $f : A \to \mathscr{P}(A)$.*

(1) First, consider a simpler case of Cantor's Theorem. Let $D = \{1, 2, 3, 4\}$. Then construct an injective function $f : D \to \mathscr{P}(D)$. For the function f you just constructed, write down all the elements of the set

$$B \equiv \{x \in D | x \notin f(x)\}.$$

(2) Show that there exists no surjective function $f : D \to \mathscr{P}(D)$ for any finite set.

(3) Using the constructive strategy in (1), prove Cantor's Theorem in full generality.

(1). Pf. let $f : D \to P(D)$ f is obviously injective.

$$x \mapsto \{x\}$$

$$B := \{x \in D \mid x \notin \{x\}\} = \phi.$$

(2). Pf. if f is surjective.

$\forall y_0 \in P(D).\ \exists x_0 \in D.\ f(x_0) = y_0.$

so $|D| \geq |P(D)|$ by definition. when $|D| < \infty$

but this implies $|D| \geq 2^{|D|}$

which is false for all $|D| \in \mathbb{Z}_{\geq 0}$.

contradiction.

$\square$

(3) If there's a surjection $f : D \to P(D)$.

let $B := \{x \in D \mid x \notin f(x)\} \in P(D)$

since f is a surjection. $\exists x_0 \in D : f(x_0) = B$

if $x_0 \in B$ then $x_0 \notin f(x_0) = B$

if $x_0 \notin B$ then $x_0 \in f(x_0) = B$

contradiction.

So there is no such surjection.

$\square$