Pullback Property: Quotient Topologies to Categories

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1. Introduction

The most elegant theorem I've learnt in this course is the pullback property of quotient topologies. Since I've been learning Abstract Algebra this semester, I've found some generalized facts about this property in category theory. The four problems and sections corresponding answers of this homework are as follows

1. Prove the theorem. (Section 2.1)
2. In your opinion, what is the most crucial assumption of the theorem? Does the theorem still hold if you remove this assumption? If not, can you construct an example? (Section 2.1)
3. Describe the importance of the theorem. (Section 2.2, Section 3)
4. Provide a typical example to exhibit the application of theorem. (Section 4)

2. Pullback Property

2.1. Pullbacks in Quotient Topologies

Definition. Quotient map Let X and Y be topological spaces. A map $f : X \to Y$ is said to be a *quotient map* if for any set $U \subset Y$, U is open in Y iff $f^{-1}(U)$ is open in X.

By the way a quotient map is defined, it's obviously continuous. It's sometimes called *strongly continous* for this reason.

Theorem (Pullback Property). Let $\pi : X \to Y$ be a quotient map. Let Z be an arbitrary topological space. Then there is a bijection between

- $\mathcal{H} := \{h \in \texttt{Map}(X, Z) \mid h \text{ is constant on every } \pi^{-1}(\{y\}), y \in Y\}$, and;
- $\mathcal{F} := \texttt{Map}(Y, Z)$,

which can be realized by the *pullback*

$$\begin{aligned} \pi^* : \mathcal{F} &\to \mathcal{H} \\ f &\mapsto f \circ \pi \end{aligned}$$

Moreover, the following holds:
1. $f \in \mathcal{F}$ is continuous if and only if $h = \pi^*(f)$ in $\mathcal{H}$ is continuous.
2. $f \in \mathcal{F}$ is a quotient map if and only if $h = \pi^*(f)$ in $\mathcal{H}$ is a quotient map.

Proof. For each $y \in Y$, $h(\pi^{-1}(\{y\}))$ is a singleton in Z, denoted the element in the singleton by $f(y)$, we have acturally defined a map implicitly by $f \circ \pi = h$. Since the choice of h is arbitrary, π^* is surjective. Suppose $\pi^*(f_1) = f_1 \circ \pi = h_1 = h_2 = f_2 \circ \pi = \pi^*(f_2)$, let $x_0 \in \pi^{-1}(y_0)$ for any $y \in Y$, then $f_1(y) = f_1 \circ \pi(x_0) = h_1(x_0) = h_2(x_0) = f_2(y)$, so $f_1 = f_2$, i.e. π^* is injective.

Let U be any open set in Z, $h^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1} \circ f^{-1}(U)$. Since π is a quotient map, $h^{-1}(U)$ is open in X if and only if $f^{-1}(U)$ is open in Y . Therefore $f \circ \pi$ is continuous if and only if f is continuous.

In fact, the Pullback Property could be described using the following commutative diagram. ■

$$
\begin{tikzcd}
X \arrow[r, "f \circ \pi"] \arrow[d, "\pi"] & Z \\
Y \arrow[ur, "f"'] &
\end{tikzcd}
$$

2.2. Pullbacks in Categories

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\
X / \ker(f) \arrow[ur, hook, dashed, "\tilde{f}"'] &
\end{tikzcd}
$$

3. Importance of the Theorem

Definition. (Pullback) In any category $\mathbf{C}$, a pullback of arrows $f : A \to C$ and $g : B \to C$ consists of arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that $fp_1 = gp_2$, and universal with this property. i.e. Given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $fz_1 = gz_2$, there exists a unique $u : Z \to P$ such that $z_1 = p_1 u$ and $z_2 = p_2 u$.

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

4. Applications of the Theorem

Bibliography

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