WHAT IS HOMOTOPICAL ABOUT HOMOTOPY TYPE THEORY?

CHENTIAN WU

ABSTRACT. The introduction of homotopy theory into type theory was firsyly mentioned in the 1994 paper [1] of Martin Hofmann and Thomas Streicher. In this paper, I'll give a brief answer to the question: "What's homotopical about Homotopy Type Theory". I'll firstly give a introduction of Martin-Löf's dependent type theory, which is the mathematical background (or object) that we care about in the context of HoTT. After that, I'll give a correspondence of common concepts in homotopy theory and HoTT.

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1. Introduction to Type Theory

2. Martin-Löf's Dependent Type Theory

Definition 2.1. Consider a type family B over A. The dependent pair type (or Σ -type) is defined to be the inductive type $\Sigma_{(x:A)}B(x)$ equipped with a pairing function

$$(-,-):\prod_{(x:A)}\left(B(x)\to\sum_{(y:A)}B(y)\right)$$

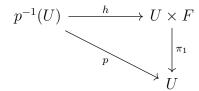
3. Homotopical Explanation

Type Theory	Homotopy Theory
Types	Topological Spaces
Dependent Types	Fibrations
Terms	Points
Σ Type	Total Space
Identity Type	Path Fibration
Contractible Type	Contractible Space

Table 1. The homotopy interpretation [2]

Definition 3.1. Let $f, g: \prod_{(x:A)} P(x)$ be two dependent functions. The type of *homotopies* from f to g is defined as

$$f \sim g :\equiv \prod_{\substack{(x:A) \\ 1}} f(x) = g(x)$$



Definition 3.2. A fiber bundle structure on a space E, with fiber F, consists of a projection map $p: E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h: p^{-1}(U) \to U \times F$ making the following diagram commute

The fiber bundle structure is determined by the projection map $p: E \to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F \to E \to B$, a 'short exact sequence of spaces'. The space B is called the *base space* of the bundle, and E is the *total space*.

Definition 3.3. A *fibration* is a map $p: E \to B$ having the homotopy lifting property with respect to all spaces X.

Definition 3.4. A contractible type is a type which has, up to identification, only one term.

4. Conclusion

References

- 1. Martin Hofmann and Thomas Streicher, *The groupoid model refutes uniqueness of identity proofs*, Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science, IEEE, 1994, pp. 208–212.
- 2. Egbert Rijke, Introduction to homotopy type theory, 2022.