WHAT IS HOMOTOPICAL ABOUT HOMOTOPY TYPE THEORY?

CHENTIAN WU

Abstract. This paper explores the profound connection between homotopy theory and dependent type theory that gave rise to Homotopy Type Theory (HoTT [1]). We investigate how the homotopical interpretation transforms our understanding of type theory, where types are viewed not merely as sets but as topological spaces up to homotopy equivalence. By examining the unexpected isomorphism between Martin-Löf's identity types and path spaces, we reveal how this correspondence naturally led to the univalence axioma principle with no precedent in classical foundations. We analyze specific constructions in HoTT that directly mirror algebraic topology concepts, including higher inductive types as synthetic representations of cell complexes. This exploration illuminates why homotopyrather than other mathematical structuresoffered the perfect framework for resolving long-standing issues in intensional type theory while simultaneously providing new foundations for mathematics that naturally accommodate higher-dimensional structures.

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1. Introduction

(1) Brief history of type theory and its challenges
(2) The "identity type problem" in intensional type theory
(3) Emergence of homotopical interpretations
(4) Question: What makes homotopy uniquely suitable for type theory?

2. The Homotopical Structure of Type Theory

In this section we give the basic idea of HoTT

(1) Martin-Löf's identity types as path spaces
(2) The Hofmann-Streicher groupoid interpretation
(3) n-types and truncation: from sets to -groupoids
(4) Transport and path induction as homotopical operations

3. Univalence: The Core Homotopical Axiom

In this section we mainly talk about the univalence axiom in HoTT.

(1) Informal meaning: equivalent types are equal
(2) Topological motivation: homotopy equivalence and its role
(3) Consequences of univalence for mathematics
(4) Comparison with traditional foundations

4. Higher Inductive Types: Synthetic Topology

(1) Circle S^1, spheres S^n, and torus T^2 as HITs
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7. Conclusion and Future Directions

(1) The essential homotopical nature of HoTT
(2) Applications in mathematics and computer science
(3) Open questions and research opportunities
(4) The future of homotopical foundations

References

1. The Univalent Foundations Program, *Homotopy type theory: Univalent foundations of mathematics*, `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.