WHAT IS HOMOTOPICAL ABOUT HOMOTOPY TYPE THEORY?

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Abstract. The introduction of homotopy theory into type theory was firsyly mentioned in the 1994 paper [1] of Martin Hofmann and Thomas Streicher. In this paper, I'll give a brief answer to the question: "What's homotopical about Homotopy Type Theory". I'll firstly give a introduction of Martin-Löf's dependent type theory, which is the mathematical background (or object) that we care about in the context of HoTT. After that, I'll give a correspondence of common concepts in homotopy theory and HoTT.

Contents

1. Introduction to Type Theory

2. Martin-Löf's Dependent Type Theory

Definition 2.1. Consider a type family B over A. The *dependent pair type* (or Σ-type) is defined to be the inductive type $\Sigma_{(x:A)}B(x)$ equipped with a pairing function

$$(-,-) : \prod_{(x:A)} \Big(B(x) \to \sum_{(y:A)} B(y)\Big)$$

3. Homotopical Explanation

Type Theory	*Homotopy Theory*
Types	Topological Spaces
Dependent Types	Fibrations
Terms	Points
Σ Type	Total Space
Identity Type	Path Fibration
Contractible Type	Contractible Space

Table 1. The homotopy interpretation [2]

Definition 3.1. Let $f, g : \prod_{(x:A)} P(x)$ be two dependent functions. The type of *homotopies* from f to g is defined as

$$f \sim g :\equiv \prod_{(x:A)} f(x) = g(x)$$

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "h"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi_1"] \\
& U
\end{tikzcd}$$

Definition 3.2. A *fiber bundle* structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the following diagram commute

The fiber bundle structure is determined by the projection map $p : E \to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F \to E \to B$, a 'short exact sequence of spaces'. The space B is called the *base space* of the bundle, and E is the *total space*.

Definition 3.3. A *fibration* is a map $p : E \to B$ having the homotopy lifting property with respect to all spaces X.

Definition 3.4. A *contractible type* is a type which has, up to identification, only one term.

4. Conclusion

References

1. Martin Hofmann and Thomas Streicher, *The groupoid model refutes uniqueness of identity proofs*, Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science, IEEE, 1994, pp. 208–212.
2. Egbert Rijke, *Introduction to homotopy type theory*, 2022.