# Erdős-Rado Theorem

## Strengthened DSLT

Let $L$ be a first order language.

**Theorem 1:** (Downward) Löwenheim–Skolem Theorem.

For every structure $N$ in $L$, and every subset $A \subset |N|$, there exists a structure $M$ in $L$ such that
- $A \subset |M|$
- $M \preceq N$
- $\|M\| \leq |A| + |L| + \aleph_0$

**Theorem 2:** Strengthened DSLT.

For every structure $N$ in $L$, and every subset $A \subset |N|$, and every cadinal $\kappa$ where $\kappa \geq 2^{|A|+L+\aleph_0}$, there exists a structure $M$ in $L$ such that
- $A \subset |M|$
- $M \preceq N$
- $\|M\| < \kappa$
- For every $\bar{a} \in |N|$, $\mathsf{tp}(\bar{a}/A, N)$ is realized in $M$

## Erdős-Rado Theorem

**Theorem 3:** Erdős-Rado Theorem.

For every natural number $n$ and every infinite cardinal $\kappa$, we have that

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

*Pf.* We proceed by indction on $n < \omega$ to show that for very infinite $\kappa$, the partition relation

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

holds.

1. When $n = 0$.

$\kappa^+ \to (\kappa^+)^1_\kappa$ is true because $\kappa$ is regular (same as the Pigeonhole Principle).

2. Suppose the statement holds for $n$, we want to show that it holds for $n+1$. *i.e.*

$$\beth_{n+1}(\kappa)^+ \to (\kappa^+)^{n+2}_{\kappa}$$

*i.e.*

For every infinite $\kappa$ and every coloring function $F : [\beth_{n+1}(\kappa)^+]^{n+1} \to \kappa$, there is a monochromatic subset of $\beth_{n+1}(\kappa)^+$ of cardinality $\kappa^+$.

Let $N = \langle \beth_{n+1}(\kappa)^+, <, F, c_i \rangle_{i<\kappa}$, where the set of constant symbols $\{c_i\}_{i<\kappa}$ denotes the colors.

By the Strengthened DLST, we can define an increasing continous elementary chain of structures $N_i \preceq$ for $i < \beth_n(\kappa)^+$ such that

- for all $i$, $\|N_i\| \preceq \beth_{n+1}(\kappa)$
- for every $B \subset N_i$ with cardinality $\leq \beth_n(\kappa)$ and $\bar{a} \in |N|$, $\mathtt{tp}(\bar{a}/B, N_i)$ is realized in $N_{i+1}$

Let $M := \bigcup_{i<\kappa} N_i$, then the construction implies $\|M\| < 2^{\beth_n(\kappa)}$. Since

$$\|N\| = \beth_{n+1}(\kappa)^+$$

is regular, we may fix $\alpha^* \in |N| \setminus \sup(|M|)$.

By induction on $i < \beth^{+}_{n(\lambda)}$, define $\{a_i\} \subset |N|$ such that the following holds,

$$\mathsf{tp}\left(a_i / \{a_j\}_{j<i}, N\right) = \mathsf{tp}\left(\alpha^* / \{a_j\}_{j<i}, N\right)$$

By the second requirement on $N_i$, this construction is possible.

It's easy to see that for all $i < j < \beth_n(\kappa)^+$, $a_i < a_j$. Moreover, for every $i_1 < ... < i_{n+2} < \beth_n(\kappa)^+$, we have that

$$F\left(a_{i_1}, ..., a_{i_{n+2}}\right) = F\left(a_{i_1}, ..., a_{i_{n+1}}, \alpha^*\right)$$

Now define the new coloring $G : [\{a_i\}]^{n+1} \to \kappa$ by

$$G\left(a_{i_1}, ..., a_{i_{n+1}}\right) = F\left(a_{i_1}, ..., a_{i_{n+1}}, \alpha^*\right)$$

Since $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$, there is a monochromatic set $B \subset \{a_i\}_{i<\beth_n(\kappa)^+}$ of cardinality $\kappa^+$ and $i_0 \in \kappa$ such that

for every $a_1 < ... < a_{n+2} \in B$, $F(a_1, ..., a_{n+2}) = F(a_1, ..., a_{n+1}, \alpha^*) = G(a_1, ..., a_{n+1}) = i_0$. Then for every $a_1 < ... < a_{n+2} \in B$,

$$F(a_1, ..., a_{n+2}) = F(a_1, ..., a_{n+1}, \alpha^*) = G(a_1, ..., a_{n+1}) = i_0$$