Ramsey Theory Meets Infinity

A short introduction to Erdős-Rado Theorem

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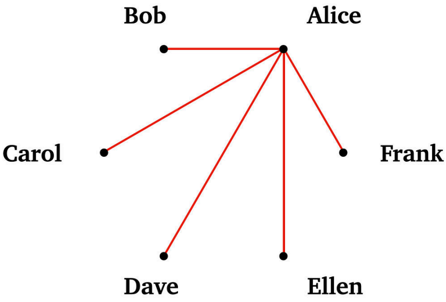
Outline

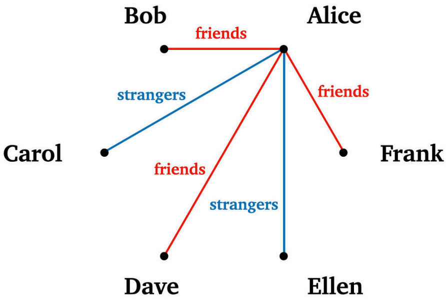
Ramsey’s Theorem

Problem of friends and strangers

Problem: The Friendship Riddle.

Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.



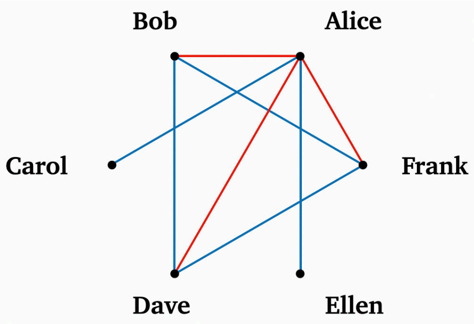


Problem of friends and strangers (ii)

Problem: The Friendship Riddle.

Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.

$$6 \rightarrow (3)_2^2$$



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Notations

Definition 1: Complete hypergraph.

A *complete hypergraph* on a set S, denoted as $[S]^n$, is the set of n-membered subset of S for some $n \geq 1$.

$$[S]^n = \{T \subset S : |T| = n\}$$

Definition 2: Coloring function.

A κ-*coloring* of a complete hypergraph $[S]^n$ is a function $c : [S]^n \to \kappa$ where κ is a cardinal.

Definition 3: Homogeneous set.

A subset $T \subset S$ is said to be *homogeneous* for a κ-coloring $c : [S]^n \to \kappa$ if $c([T]^n)$ is a singleton.

Notations (ii)

Notation: $\lambda \to (\mu)^n_\kappa$.

Let $n < \omega$ and suppose λ, κ and μ are cardinal numbers (not necessarily infinite). We denote by $\lambda \to (\mu)^n_\kappa$ the following statement:

For every κ-coloring of the complete hypergraph $[S]^n$, there exists a homogeneous set $T \subset S$ such that $|T| = \lambda$ and $|c([T]^n)| = \mu$.

Example: Pigeonhole Principle.

Let λ be a finite cardinal number. Then, the statement $\lambda^+ \to (2)^1_\lambda$ is equivalent to the Pigeonhole Principle.

Example: Arrow notation for the Friends and Strangers' Problem.

Let $n = 2, \lambda = 6, \mu = 3$ and $\kappa = 2$. Another notation would be $K_6 \to K_3, K_3$

Ramsey Theorem

Theorem 1: Finite Ramsey Theorem.

For every positive integers n, λ, κ, there exists positive integer γ such that

$$\gamma \to (\lambda)^n_\kappa$$

Theorem 2: Infinite Ramsey Theorem.

$$\aleph_0 \to (\aleph_0)^2_2$$

Theorem 3: Generalized Ramsey Theorem.

For any positive integers n, k,

$$\aleph_0 \to (\aleph_0)^n_k$$

Other Infinite Cardinals?

A natural question is whether the infinite Ramsey Theorem holds for other infinite cardinals. The answer is **NO**.

Theorem 4: Sierpiński 1933.

$$\aleph_1 \nrightarrow (\aleph_1)^2_2$$

What's worse, whether a uncountable cardinal χ satisfies $\chi \rightarrow (\chi)^2_2$ is independent of ZFC.

Theorem 5: Erdős and Tarski 1943.

Let χ be an uncountable cardinal. If $\chi \rightarrow (\chi)^2_2$, then the first order theory ZFC has a model (namely ZFC is consistent).

Erdős-Rado Theorem

Erdős-Rado Theorem

Theorem 6: (Corollary of) Erdős-Rado Theorem.

$$\left(2^{\aleph_0}\right)^+ \to (\aleph_1)^2_2$$

Theorem 7: Erdős-Rado Theorem.

For every natural number n and every infinite cardinal κ

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

where

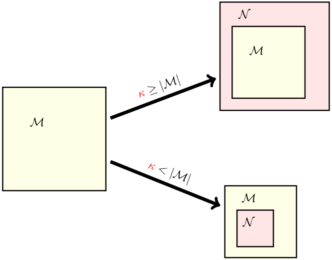
- $\beth_1(\kappa) = 2^\kappa$
- $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$
- $\beth_\lambda(\kappa) = \bigcup_{\alpha<\lambda} \beth_\alpha(\kappa)$ for limit ordinal λ

Downward Löwenheim-Skolem Theorem

Theorem 8: (Downward) Löwenheim-Skolem Theorem.

Let L be a first order language. For every structure N in L, and every subset $A \subset |N|$, there exists a structure M in L such that

- $A \subset |M|$
- $M \preceq N$
- $\|M\| \leq |A| + |L| + \aleph_0$



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Strengthened DLST

Definition 4: Type.

If M is a structure in L and $A \subset |M|$. Let L_A be the language obtained by adding to L constant symbols for each $a \in A$, and $\bar{a} = (a_1, a_2, ..., a_n) \in M^n$, then the type of $\bar{a}$ over A in M, denoted by $\text{tp}^M(\bar{a}/A)$, is defined by

$$\text{tp}^M(\bar{a}/A) = \{\varphi(v_1, ...v_n) \in L_A \mid M \vDash \varphi(a_1, ..., a_n)\}$$

Theorem 9: Strengthened DLST.

For every structure N in L, and every subset $A \subset |N|$, and every cadinal κ where $\kappa \geq 2^{|A|+|L(N)|+\aleph_0}$, there exists a structure M in L such that

- $A \subset |M|$
- $M \preceq N$
- $\|M\| < \kappa$
- For every $\bar{a} \in |N|$, $\text{tp}^N(\bar{a}/A)$ is realized in M

Erdős-Rado Theorem

Theorem 10: Erdős-Rado Theorem.

For every natural number n and every infinite cardinal κ, we have that

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

Pf. We proceed by indction on $n < \omega$ to show that for very infinite κ, the partition relation

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

holds.

1. When $n = 0$.

$\kappa^+ \to (\kappa^+)^1_\kappa$ is true because κ is regular (same as the Pigeonhole Principle).

Erdős-Rado Theorem (ii)

2. Suppose the statement holds for n, we want to show that it holds for $n+1$. *i.e.*

$$\beth_{n+1}(\kappa)^{+} \to (\kappa^{+})^{n+2}_{\kappa}$$

i.e., let κ be given, and denote by μ the carinality $\beth_n(\kappa)$. We want to show that for every coloring function $F : \left[(2^{\mu})^{+}\right]^{n+2} \to \kappa$, there is an F-monochromatic subset $B \subset (2^{\mu})^{+}$ of cardinality κ^{+}.

Erdős-Rado Theorem (iii)

Let $N = \langle (2^\mu)^+, <, F, i \rangle_{i<\kappa}$, where the set of constant symbols $\{i\}_{i<\kappa}$ denotes the colors.

By the Strengthened DLST, we can define an increasing *continous* chain of structures $\{N_i \preceq N \mid i < \mu^+\}$ such that

- $i < j \Rightarrow N_i \preceq N_j$
- if i is a limit ordinal, then $N_i = \bigcup_{j<i} N_j$
- for all i, $\|N_i\| \leq 2^\mu$
- for every $B \subset N_i$ with cardinality $\leq \mu$ and $\bar{a} \in |N|$, $\text{tp}^N(\bar{a}/B)$ is realized in N_{i+1}

Let $M := \bigcup\limits_{i<\mu^+} N_i$, then the construction implies $\|M\| < 2^\mu$. Since $\|N\| = (2^\mu)^+$ is regular, we may fix $\alpha^* \in |N| \setminus \sup(|M|)$.

Erdős-Rado Theorem (iv)

Define $\{a_i \mid i < \mu^+\} \subset |M|$ inductively with $a_i \in N_i$ such that the following holds,

$$\mathsf{tp}^N\left(a_i/\{a_j\}_{j<i}\right) = \mathsf{tp}^N\left(\alpha^*/\{a_j\}_{j<i}\right)$$

By the last requirement on the construction of N_i, this construction is possible.

It's easy to see that for all $i < j < \mu^+$, $a_i < a_j$. Moreover, for every $i_1 < \ldots < i_{n+2} < \mu^+$, we have that

$$F\left(a_{i_1}, \ldots, a_{i_{n+2}}\right) = F\left(a_{i_1}, \ldots, a_{i_{n+1}}, \alpha^*\right)$$

Erdős-Rado Theorem (v)

Now define the new coloring $G : [\{a_i\}]^{n+1} \to \kappa$ by

$$G\left(a_{i_1}, ..., a_{i_{n+1}}\right) = F\left(a_{i_1}, ..., a_{i_{n+1}}, \alpha^*\right)$$

Since $\mu^+ \to (\kappa^+)^{n+1}_{\kappa}$, there is a monochromatic set $B \subset \{a_i\}_{i<\mu^+}$ of cardinality κ^+ and $i_0 \in \kappa$ such that

for every $a_1 < ... < a_{n+1} \in B, G(a_1, ..., a_{n+1}) = i_0$. Then for every $a_1 < ... < a_{n+2} \in B$,

$$F(a_1, ..., a_{n+2}) = F(a_1, ..., a_{n+1}, \alpha^*) = G(a_1, ..., a_{n+1}) = i_0$$

References

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2. Chang, C., Keisler, H.: Model Theory: Third Edition. Dover Publications (2013)
3. Marker, D.: Model Theory: An Introduction. Springer, New York (2002)