Final Review Problems

November 27, 2020

1 Basic Problems

1. Write down the Hermite interpolation polynomial $p(x)$ of $f(x) = \sin(x)$ at $x_0 = 0$, $x_1 = \pi$, and find an upper bound of $|f(x) - p(x)|$ using the error bound of Hermite interpolation.

Answer: The Hermite interpolation polynomial is

$$p(x) = \frac{1}{\pi^2}(x(x-\pi)^2 + x^2(\pi - x))$$

And

$$|f(x) - p(x)| = \frac{|-\sin(s)|x^2(x-\pi)^2}{4!} \leq \frac{x^2(x-\pi)^2}{24}$$

2. Find two points x_0 and x_1, such that for any polynomial f of degree no more than 3,

$$\int_0^\pi \sin(x) f(x) dx = \frac{\pi}{2}(f(x_0) + f(x_1))$$

And find constant c such that if $g \in C^6([0,\pi])$,

$$|\int_0^\pi \sin(x) g(x) dx - \frac{\pi}{2}(g(x_0) + g(x_1))| \leq C \max |g^{(6)}|$$

Answer: These two points are the Gauss quadrature points on interval $[0,\pi]$ with weight function $\sin(x)$, hence must be the root of the degree-2 orthogonal polynomial on $[0,\pi]$ with weight $\sin(x)$. Suppose this polynomial is $p_2 = x^2 + ax + b$, then

$$0 = \int_0^\pi \sin(x)p_2(x)dx = \pi^2 - 4 + a\pi + 2b$$

$$0 = \int_0^\pi \sin(x)xp_2(x)dx = \pi^3 - 6\pi + a(\pi^2 - 4) + b\pi$$

So $a = -\pi$, $b = 2$, $x_0 = \frac{\pi - \sqrt{\pi^2-8}}{2}$, $x_1 = \frac{\pi+\sqrt{\pi^2-8}}{2}$. And by Theorem 10.1 from the textbook or 3.19(iv) in the Lecture notes,

$$C = \frac{\int_0^\pi \sin(x)(x^2 - \pi x + 2)^2 dx}{6!} = \frac{10 - \pi^2}{180}$$

3. Estimate the solution of $y' = \sin(y)$, $y(0) = 1$ at time 0.1 using Euler's method, improved Euler's method, and rk4, using time step $h = 0.1$.

Answer:

- Euler's method gets $z(0.1) = 1 + 0.1 \times \sin(1) = 1 + \sin(1)/10 \approx 1.0841471$.
- Improved Euler's method gets $z(0.1) = 1 + \frac{1}{20}(\sin(1) + \sin(1 + \sin(1)/10)) \approx 1.0862688$
- Runge-Kutta 4-th order gets $k_1 = \sin(1)$, $k_2 = \sin(1 + k_1/20)$, $k_3 = \sin(1 + k_2/20)$, $k_4 = \sin(1 + k_3/10)$

$$\begin{aligned} z(0.1) = 1 + \frac{\sin(1)}{60} &+ \frac{\sin(1+\sin(1)/20)}{30} \\ &+ \frac{\sin(1+\sin(1+\sin(1)/20)/20)}{30} \\ &+ \frac{\sin(1+\sin(1+\sin(1+\sin(1)/20)/20)/10)}{60} \approx 1.0863557 \end{aligned}$$

The accurate answer is 1.0863558.

4. Consider explicit 2-step method for $y' = f(t, y)$:

$$z((n+2)h) = az((n+1)h) + bz(nh) + chf((n+1)h, z((n+1)h)) + dhf(nh, z(nh))$$

Where h is step size and $z(t)$ is the estimate for $y(t)$. Find all real numbers a, b, c, d such that the method is zero stable and has order of accuracy at least 2.

Answer: The first characteristic polynomial is

$$\rho(z) = z^2 - az - b$$

To make it consistent, $\rho(1) = 0$, $c + d = 2 - a$, so $1 - a - b = 0$, $b = 1 - a$, and the other root must be $a - 1$, so $0 \leq a < 2$ and $b = 1 - a$.

Now let's calculate the order of accuracy. Firstly, let $t = nh$, suppose y is the solution of the IVP, then

$$y'(t) = f(t, y(t))$$

$$y''(t) = \partial_t f(t, y(t)) + \partial_y f(t, y(t)) y'(t)$$

Now let's do power series expansion, with respect to h, for

$$y(t + 2h) - ay(t + h) - (1 - a)y(t) - chf(t + h, y(t + h)) - dhf(t, y(t))$$

And after cancelling some terms, we get

$$2y''(t)h^2 - ay''(t)h^2/2 - c\partial_t f(t, y(t))h^2 - c\partial_y f(t, y(t))y'(t)h^2 + O(h^3)$$

So $c = 2 - a/2$, $d = -a/2$. Note that when $a = 1$ this is 2-step Adams-Bashforth.

2 More advanced problems

Problems like the ones below will account for no more than 10% of the final exam, so don't worry about them unless you have a lot of time during final review.

5. Suppose f is smooth and periodic with period 1, $|f^{(4)}| \leq 1$. Let I_n be the result of composite trapezium rule for $\int_0^1 f dx$ using n subintervals. Find a number C, such that

$$|\int_0^1 f dx - I_n(f)| \leq \frac{C}{n^4}$$

Answer: Consider the function $f_n(x) = \sum_{i=0}^{n-1}(x + i/n)$. Then f_n is periodic with period $1/n$, and it is easy to see that the composite trapezium rule for $\int_0^1 f dx$ using n subintervals is the same as the trapezium rule for $\int_0^{1/n} f_n dx$.

Now let p_n be the Hermite interpolation of f_n at 0 and $1/n$. Then because $f_n(0) = f_n(1/n)$, $f_n'(0) = f_n'(1/n)$, we have

$$p_n(x) = f_n(0) + 2f_n'(0)n^2 x(x - \frac{1}{2n})(x - \frac{1}{n})$$

$$\int_0^{1/n} p_n(x)dx = f_n(0)/n = I_n(f)$$

So

$$|\int_0^1 f dx - I_n(f)| \leq \int_0^{1/n} |f_n(x) - p_n(x)| dx \leq \int_0^{1/n} \frac{\max|f_n^{(4)}| x^2(x-1/n)^2}{24} dx \leq \frac{1}{720n^4}$$

6. Consider the 3-step Adams-Bashforth method for $y' = \cos(y)$:

$$z(t+3h) = z(t+2h) + \frac{23h}{12} f(z(t+2h)) - \frac{4h}{3} f(z(t+h)) + \frac{5h}{12} f(z(t))$$

Suppose $z(t) = y(t)$, $z(t+h) = y(t+h)$, $z(t+2h) = y(t+2h)$, find C such that

$$|z(t+3h) - y(t+3h)| \leq Ch^4$$

Answer: Let $g(t) = y'(t) = \cos(y(t))$, p_3 be the Lagrange interpolation of g at t, $t+h$, $t+2h$, then the 3-step Adams-Bashforth can be written as

$$z(t+3h) = y(t+2h) + \int_{t+2h}^{t+3h} p_3(s) ds$$

So

$$|z(t+3h) - y(t+3h)| \leq \int_{t+2h}^{t+3h} |g(s) - p_3(s)| ds$$

Now by error estimate of Lagrange interpolation,

$$|g(s) - p_3(s)| \leq \frac{\max|g^{(3)}|(s-t)(s-t-h)(s-t-2h)}{6}$$

So after integration we get

$$|z(t+3h) - y(t+3h)| \leq \max|g^{(3)}| \cdot \frac{3h^4}{8}$$

$$g' = -y' \sin(y) = -\cos(y)\sin(y) = -\frac{\sin(2y)}{2}$$
$$g'' = -\cos(y)\cos(2y) = -\frac{\cos(3y)+\cos(y)}{2}$$
$$g''' = \frac{3\sin(3y)\cos(y) + \sin(y)\cos(y)}{2}$$

So $|g'''| \leq 7/4$, or you can use a better bound, and $C = 21/32$.