Notes on ODE

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This is a review on some basics of the theory of ordinary differential equations.

An **ordinary differential equation** is an equation relating a function on $\mathbb{R}$ and its derivatives. For example, the followings are ordinary differential equations:

$$y' = t\sin(y)$$
$$y' = sin(t)$$
$$y' = y\cos(t) + e^t$$

We can also have systems of equations like the following

$$y_1' = y_2, y_2' = -y_1$$

The **initial value problem** of an ordinary differential equation means finding a solution after specifying the value of the solution at some time t_0, which, for convenience, we can choose to be 0. For example

$$y' = y\cos(t) + e^t, y(0) = 0$$

In general the solution of an ODE can not be written down explicitly, however, in some situations we can get explicit solutions. For example, if the equation is of the form $y' = f(t)g(y)$, then the general solution is of the form

$$\int_0^y \frac{ds}{g(s)} = \int_0^t f(s)ds + C$$

This is called **separation of variables**.

The most important result in the theory of ODE is Picard's theorem:

Theorem 0.1. *If $f(t, y)$ is continuous and Lipschitz in the second parameter with Lipschitz constant L, then $y' = f(t, y)$, $y(0) = a$ always has a unique solution.*

Proof. Firstly let's show uniqueness: if y_1 and y_2 are two solutions, then $|y_1(t) - y_2(t)|e^{-L|t|}$ is non increasing when $t > 0$ and non decreasing when $t < 0$, hence must always be 0.

Now let's show existence: consider a sequence of functions defined as below:

$$y_0(t) = a$$

$$y_i(t) = a + \int_0^t f(s, y_{i-1}(s))ds$$

Then f being Lipschitz implies that

$$|y_i(t) - y_{i-1}(t)| \leq \int_0^t L|y_{i-1}(s) - y_{i-2}(s)|ds$$

$$\leq \int_0^t L^2(t-s)|y_{i-2}(s) - y_{i-3}(s)|ds$$

$$\leq \int_0^t L^3 \frac{(t-s)^2}{2}|y_{i-3}(s) - y_{i-4}(s)|ds$$

$$\leq \cdots \leq \int_0^t L^{i-1} \frac{(t-s)^{i-2}}{(i-2)!}|y_1(s) - y_0(s)|ds$$

$$\leq \frac{\max(|y_1 - y_0|)L^{i-1}t^{i-1}}{(i-1)!}$$

Hence the sequence converges uniformly on any finite interval, and fundamental theorem of calculus implies that the limiting function y is the solution. □

The argument above can be used to show that if f is real analytic (i.e. has Taylor series convergent to itself), so is y.