HW 4

November 17, 2020

1. Let $f(x) = x^3$, p be the Lagrange interpolation polynomial of f using interpolation points $x = 0$, $x = 1$. On the interval $[0, 1]$, find the point c that maximizes the interpolation error $|f(c) - p(c)|$, and find another point $s \in [0, 1]$ such that

$$f(c) - p(c) = f''(s)c(c-1)/2$$

Answer:

$$p(x) = 0 \cdot \frac{x-1}{0-1} + 1 \cdot \frac{x-0}{1-0} = x$$
$$|f - p| = |x^3 - x|$$

So this is maximalized at point $c = \frac{\sqrt{3}}{3}$.

$$f(c) - p(c) = c^3 - c = 3sc(c-1)$$

So

$$s = \frac{c+1}{3} = \frac{\sqrt{3}+3}{9}$$

2. Let $f(x) = e^x$, p be the Lagrange interpolation polynomial of f on interval $[0, 2]$ using interpolation points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, find an upper bound for the L^∞ norm of $f(x) - p(x)$ on $[0, 2]$, using the error bound of Lagrange polynomial we covered in the lecture (Theorem 6.2 in textbook, Theorem 1.5 in lecture notes).

Answer:

The error bound of Lagrange polynomial is

$$|f(x) - p(x)| = \frac{|f'''(c)||x(x-1)(x-2)|}{3!}$$

When $c \in [0, 2]$, $|f'''(c)| \le e^2$.
When $x \in [0, 2]$, $|x(x-1)(x-2)| \le \frac{2\sqrt{3}}{9}$.
Hence an upper bound for this error is $\frac{e^2\sqrt{3}}{27}$.
It's ok if you get a slightly larger error bound, for example $4e^2/3$.

3. Suppose f is continuous and with continuous derivatives of order up to and including 5 on $[a, b]$, and there are three distinct points x_0, x_1, x_2 in $[a, b]$. Let $y_i = f(x_i)$, $i = 0, 1, 2$; $z_j = f'(x_j)$, $j = 0, 2$.

(i) Find a polynomial p of degree at most 4, such that $p(x_i) = y_i$, $i = 0, 1, 2$; $p'(x_j) = z_j$, $j = 0, 2$.

(ii) Use an argument similar to the error estimate of Hermite interpolation polynomial to show that for any $x \in [a, b]$, there is some number $s \in [a, b]$ such that

$$f(x) - p(x) = f^{(5)}(s)(x - x_0)^2(x - x_1)(x - x_2)^2/5!$$

Answer:

(i)
- Approach I: We can find five polynomials p_0, p_1, p_2, q_0, q_2, such that

$$p_0(x_0) = p_1(x_1) = p_2(x_2) = q_0'(x_0) = q_2'(x_2) = 1$$

$$p_i(x_j) = 0 \text{ when } i \neq j$$

$$p_i'(x_j) = 0 \text{ when } j = 0, 2$$

$$q_0'(x_2) = q_2'(x_0) = 0$$

$$q_i(x_j) = 0$$

Then the answer can be written as

$$p = \sum_i y_i p_i + z_0 q_0 + z_2 q_2$$

To get p_0, from $p_0(x_1) = p_0(x_2) = p_0'(x_2) = 0$ we get $p_0 = (x - x_1)(x - x_2)^2(Ax + B)$, now use the remaining two conditions, $p_0(x_0) = 1$, $p_0'(x_0) = 0$, to solve for A and B, we get

$$p_0 = \frac{(x - x_1)(x - x_2)^2}{(x_0 - x_1)(x_0 - x_2)^2}(1 - (x - x_0)(\frac{1}{x_0 - x_1} + \frac{2}{x_0 - x_2}))$$

Similarly,

$$p_1 = \frac{(x-x_0)^2(x-x_2)^2}{(x_1-x_0)^2(x_1-x_2)^2}$$

$$p_2 = \frac{(x-x_0)^2(x-x_1)}{(x_2-x_0)^2(x_2-x_1)}(1-(x-x_2)(\frac{2}{x_2-x_0}+\frac{1}{x_2-x_1}))$$

$$q_0 = \frac{(x-x_0)(x-x_1)(x-x_2)^2}{(x_0-x_1)(x_0-x_2)^2}$$

$$q_2 = \frac{(x-x_0)^2(x-x_1)(x-x_2)}{(x_2-x_0)^2(x_2-x_1)}$$

So

$$\begin{aligned} p = {} & y_0\frac{(x-x_1)(x-x_2)^2}{(x_0-x_1)(x_0-x_2)^2}(1-(x-x_0)(\frac{1}{x_0-x_1}+\frac{2}{x_0-x_2})) \\ & +y_1\frac{(x-x_0)^2(x-x_2)^2}{(x_1-x_0)^2(x_1-x_2)^2} \\ & +y_2\frac{(x-x_0)^2(x-x_1)}{(x_2-x_0)^2(x_2-x_1)}(1-(x-x_2)(\frac{2}{x_2-x_0}+\frac{1}{x_2-x_1})) \\ & +z_0\frac{(x-x_0)(x-x_1)(x-x_2)^2}{(x_0-x_1)(x_0-x_2)^2} \\ & +z_2\frac{(x-x_0)^2(x-x_1)(x-x_2)}{(x_2-x_0)^2(x_2-x_1)} \end{aligned}$$

- We can also use Hermite interpolation polynomial: suppose $p'(x_1) = a$, use all information for $p(x_i)$, $i = 0, 1, 2$ and $p'(x_i)$, $i = 0, 1, 2$, we can write down the Hermite interpolation polynomial which is a polynomial of degree at most 5. The coefficient for x^5 is

$$-\frac{y_0}{(x_0-x_1)^2(x_0-x_2)^2}(\frac{2}{x_0-x_1}+\frac{2}{x_0-x_2})$$

$$-\frac{y_1}{(x_1-x_0)^2(x_1-x_2)^2}(\frac{2}{x_1-x_0}+\frac{2}{x_1-x_2})$$

$$-\frac{y_2}{(x_2-x_1)^2(x_2-x_0)^2}\left(\frac{2}{x_2-x_1}+\frac{2}{x_2-x_0}\right)$$
$$+\frac{z_0}{(x_0-x_1)^2(x_0-x_2)^2}+\frac{a}{(x_1-x_0)^2(x_1-x_2)^2}$$
$$+\frac{z_2}{(x_2-x_0)^2(x_2-x_1)^2}$$

Since we want p to be of degree no more than 4, we must set this coefficient to be 0. Hence

$$a=\frac{y_0(x_1-x_2)^2}{(x_0-x_2)^2}\left(\frac{2}{x_0-x_1}+\frac{2}{x_0-x_2}\right)$$
$$+y_1\left(\frac{2}{x_1-x_0}+\frac{2}{x_1-x_2}\right)$$
$$+\frac{y_2(x_1-x_0)^2}{(x_2-x_0)^2}\left(\frac{2}{x_2-x_1}+\frac{2}{x_2-x_0}\right)$$
$$-\frac{z_0(x_1-x_2)^2}{(x_0-x_2)^2}-\frac{z_2(x_1-x_0)^2}{(x_2-x_0)^2}$$

Now put this in the formula for Hermite interpolation polynomials, you'll get the exact same answer as above.

(ii) If $x=x_i$ it's trivially true. Now suppose $x\neq x_i$ for any i, consider

$$G(t)=f(t)-p(t)-\frac{(f(x)-p(x))(t-x_0)^2(t-x_1)(t-x_2)^2}{(x-x_0)^2(x-x_1)(x-x_2)^2}$$

$G(x)=G(x_i)=G'(x_0)=G'(x_2)=0$, so G' is zero at at least 5 points, $G^{(5)}$ is zero at at least one point. Let that point be s, then $G^{(5)}(s)=0$ implies the equation we need to prove.

4. Let $q_j = (1-x^2)^j$, $\varphi_j = q_j^{(j)}$, show that φ_j are orthogonal to each other in $L^2([-1,1])$. In other words, if $j \neq j'$, $\int_{-1}^{1} \varphi_j \varphi_{j'} dx = 0$.

Answer:

Firstly we show that if $i < j$, then $q_j^{(i)}$ has a factor $(1-x^2)^{j-i}$. Do induction on i. It is trivially true for $i = 0$. Now, suppose $q_j^{(i)} = (1-x^2)^{j-i} h(x)$ where h is a polynomial, then, by product rule,

$$
\begin{aligned}
q_j^{(i+1)} &= ((1-x^2)^{j-i} h(x))' = -2(j-i)x(1-x^2)^{j-i-1} h(x) + (1-x^2)^{j-i} h'(x) \\
&= (1-x^2)^{j-i-1}(-2(j-i)xh(x) + (1-x^2)h'(x))
\end{aligned}
$$

Hence by induction this statement is proved.

Now, because φ_i are all non-zero, they all have non-zero L^2 norms on $[-1,1]$. We only need to show that when $i \neq j$, $\int_{-1}^{1} \varphi_i \varphi_j dx = 0$. Without loss of generality assume $i < j$, then by integration by parts and the conclusion in the previous step,

$$
\begin{gathered}
\int_{-1}^{1} \varphi_i \varphi_j dx = \int_{-1}^{1} q_i^{(i)} q_j^{(j)} dx \\
= -\int_{-1}^{1} q_i^{(i+1)} q_j^{(j-1)} dx \\
= \int_{-1}^{1} q_i^{(i+2)} q_j^{(j-2)} dx \\
= \cdots = (-1)^j \int_{-1}^{1} q_i^{(i+j)} q_j dx
\end{gathered}
$$

However the degree of q_i is $2i < i + j$, hence $q_i^{(i+j)} = 0$, which implies that the integration is zero.

5. Find three distinct points x_0, x_1 and x_2 in $(-1, 1)$, such that for any polynomial function f of degree 3, the best approximation of f under L^2 norm on $[-1, 1]$ of degree at most 2 coincides with the Lagrange interpolation polynomial of f using interpolation points x_0, x_1 and x_2.

Answer:

Suppose f is the degree 3 Legendre polynomial $f_3 = x^3 - \frac{3}{5}x$, then, because it is orthogonal to the degree 0, 1, and 2 Legendre polynomials under $L^2([-1, 1])$, and these three Legendre polynomials form an orthogonal basis of the space V_2 of polynomials of degree no more than 2, the best approximation formula in inner product space implies that the best approximation of f on V_2 under $L^2([-1, 1])$ norm must be 0. By assumption, the Legendre interpolation of f_3 at x_0, x_1 and x_2 must also be zero, so these three points can only be the three roots of $x^3 - \frac{3}{5}x$, which are $0, \pm\sqrt{\frac{3}{5}}$.

Now suppose $f = \sum_{i=0}^{3} a_i x^i$ is any degree 3 polynomial. Then, because $f - a_3 f_3$ is of degree at most 2 and is identical to f at $0, \pm\sqrt{\frac{3}{5}}$, the Lagrange interpolation of f at x_i is $f - a_3 f_3$. On the other hand, let e_0, e_1, e_2 be any orthogonal basis of V_2, then the best approximation of f on V_2 under $L^2([-1, 1])$ norm is $\sum_i (f, e_i) e_i$. However, because f_3 is orthogonal to V_2, $(f, e_i) = (f - a_3 f_3, e_i)$, so the best approximation of f is the same as the best approximation of $f - a_3 f_3$, which must be $f - a_3 f_3$ itself as $f - a_3 f_3 \in V_2$. This proves that x_i being $0, \pm\sqrt{\frac{3}{5}}$ satisfies the requirement in the problem.

6. Let f be a continuous function on $[0,1]$, p_n be the polynomial of best approximation of degree no more than n under the L^2 norm. Then, after studying Theorem 9.5 in the textbook, which proved that $f - p_n$ is zero at at least $n+1$ distinct points in $(0,1)$, find a function f such that $f - p_2$ is zero at 4 distinct points in $(0,1)$.

Answer:

Let V_2 be the space of polynomials of degree no more than 2. If we pick f to be anything orthogonal to V_2 under the $L^2([0,1])$ norm, then the best approximation of f on V_2 must be zero, so we just need to pick such a f with 4 or more zeros. So, for example, we can pick the degree 4 orthogonal polynomials with weight 1 on $[0,1]$, which is $70x^4 - 140x^3 + 90x^2 - 20x + 1$.