Honor's Assignment 2

December 7, 2020

1. (Exercise 7.13) Show that the composite Trapezium rule always give accurate answer to $\int_0^{2\pi} \sin(x)dx$.

Answer: The composite trapezium rule with n subintervals is

$$I_n = \frac{1}{n}\sum_{k=1}^{n-1} \sin(\frac{2\pi k}{n}) = \frac{1}{n}\sum_{k=1}^{n-1}(\sin(\frac{2\pi k}{n}) + \sin(\frac{2\pi(n-k)}{n}))/2$$

$$= 0 = \int_0^{2\pi} \sin(x)dx$$

2. (Exercise 10.7) Let $[a, b] = [-1, 1]$, let p_{n-1} be the degree $n-1$ orthogonal polynomial of weight $1 - x^2$, and let I_n be the quadrature rule where the quadrature points are roots of $(x^2 - 1)p_{n-1}(x)$.

- Show that if q is a polynomial of degree no more than $2n - 1$, then $\int_{-1}^{1} q dx = I_n(q)$.
- Show that all quadrature weights are positive.
- Suppose f is smooth, find a constant C such that

$$|\int_{-1}^{1} f dx - I_n(f)| \leq C \max_{x\in[-1,1]} |f^{(2n)}(x)|$$

Answer:

- I_n has $n+1$ quadrature points hence gives accurate answer to any polynomial of degree up to n. If q is of degree no more than $2n-1$, by long division of polynomials we have $q = (x^2-1)p_{n-1}q_1 + r$, where r is of degree at most n, and q_1 is a polynomial of degree no more than $n-2$. Hence $I_n(q) = I_n(r) = \int_{-1}^{1} r dx = \int_{-1}^{1} q dx$.

- The j-th quadrature weight is

$$w_j = \int_{-1}^{1} \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)} dx$$

  If $j = 0$ or $j = n$, the function being integrated is non negative, hence $w_j > 0$. Now suppose $0 < j < n$, then by calculation,

$$w_j = \int_{-1}^{1} \frac{x^2-1}{x_j^2-1} \cdot \frac{\prod_{i \neq j, 1<i<n}(x - x_i)^2}{\prod_{i \neq j, 1<i<n}(x_j - x_i)^2} dx$$

  Which are all positive.

- Let p be the polynomial of degree no more than $2n-1$ such that $p(x_i) = f(x_i)$, and for all $1 < i < n$, $p'(x_i) = f'(x_i)$. Then by a similar argument to the error bound of Hermite interpolation polynomials we have

$$|f(x) - p(x)| \leq \max |f^{(2n)}| \frac{(1-x^2)\prod_{1<i<n}(x - x_i)^2}{(2n)!}$$

  So

$$C = \int_{-1}^{1} \frac{(1-x^2)\prod_{1<i<n}(x - x_i)^2}{(2n)!} dx$$

3. Consider the initial value problem $y' = \sin(y)$, $y(0) = 1$.

- Write down the formula for two step Adams-Bashforth.

- Show that the two step Adams-Bashforth has order of accuracy 2 for this problem.
- Suppose we use starting points $z(0) = 1$, $z(h) = 1 + h\sin(1)$ to carry out Adams-Bashforth till time $t = nh = 1$. Find number C such that

$$|z(1) - y(1)| \leq Ch^2$$

Answer:

- The quadrature weights for $\int_{t+h}^{t+2h}$, using $x_0 = t$, $x_1 = t + h$, are

$$w_0 = \int_{t+h}^{t+2h} \frac{(s-t-h)}{-h} ds = -h/2$$

$$w_1 = \int_{t+h}^{t+2h} \frac{(s-t)}{h} ds = 3h/2$$

  So the 2nd order Adams-Bashforth is

$$z(t+2h) - z(t+h) = h(\frac{3}{2}f(t+h, z(t+h)) - \frac{1}{2}f(t, z(t)))$$

- This can be done by doing power series expansion on both sides, or via the error formula for quadrature rules.
- Suppose $z_k(nh)$ satisfies

$$z_k(nh) = \begin{cases} y(nh) & n \leq k \\ z_k((n-1)h) + \frac{3h}{2}\sin(z_k((n-1)h)) - \frac{h}{2}\sin(z_k((n-2)h)) & n > k, n \geq 2 \\ 1 + h & n = 1, k = 0 \end{cases}$$

  Then by analyzing the truncated error for Euler's and Adams-Bashforth methods, we get

$$|z_k((k+1)h) - z_{k+1}((k+1)h)| = |z_k((k+1)h) - y((k+1)h)| \leq \begin{cases} \frac{h^2}{2} & k = 0 \\ \frac{5h^3}{6} & k > 0 \end{cases}$$

You may be able to find better bounds.

Now we prove by induction on m that $|z_k((k+1+m)h) - z_{k+1}((k+1+m)h)| \leq (1+2h)^m |z_k((k+1)h) - z_{k+1}((k+1)h)|$: when $m=0$ or $m=1$ one can verify it directly. If $m>1$, the left hand side is bounded by

$$|z_k((k+m)h)-z_{k+1}((k+m)h)|+\frac{3h}{2}|z_k((k+m)h)-z_{k+1}((k+m)h)|+\frac{h}{2}|z_k((k+m-1)h)-z_{k+1}((k+m-1)h)|$$

$$\begin{aligned}
&\leq ((1+2h)^{m-1} + \frac{3h}{2}(1+2h)^{m-1} + \frac{h}{2}(1+2h)^{m-2})|z_k((k+1)h) - z_{k+1}((k+1)h)| \\
&\leq (1+2h)^m |z_k((k+1)h) - z_{k+1}((k+1)h)|
\end{aligned}$$

So

$$\begin{aligned}
|z(nh) - y(nh)| &\leq \sum_k |z_k(nh) - z_{k+1}(nh)| \\
&\leq \sum_{k=0}^{n-1} (1+2h)^{n-k-1} h^2 \\
&\leq e^2 \frac{h^2}{2} + \frac{e^2-1}{2h}\frac{5h^3}{6} \\
&= \frac{11e^2-5}{12} h^2
\end{aligned}$$

Sorry for the typo in the previous version of problem 2 and part 3 of problem 3. They won't be counted.