HW 6

December 9, 2020

1. (Problem 12.7 in textbook) Consider solving $y' = f(t, y), y(0) = y_0$ using the trapezium method

$$z((n+1)h) = z(nh) + \frac{h}{2}(f((n+1)h, z((n+1)h)) + f(nh, z(nh)))$$

Suppose further that $|y'''|$ is uniformly bounded by M.

(i) Prove that

$$|\frac{y((n+1)h) - y(nh)}{h} - \frac{1}{2}(f((n+1)h, y((n+1)h)) + f(nh, y(nh)))| \leq \frac{Mh^2}{12}$$

Hint: You can prove it by applying integration by parts to $\int_{nh}^{(n+1)h} (x - hn)(x - nh - h)y'''(x)dx$.

(ii) Let $e_n = z(nh) - y(nh)$, and assume that f is L-Lipschitz with respect to the second parmeter, then

$$|e_{n+1}| \leq |e_n| + \frac{1}{2}hL(|e_{n+1}| + |e_n|) + \frac{h^3 M}{12}$$

Answer:

(i)
- Approach I: Left hand side is the error for applying trapezium rule to calculate $\int_{nh}^{(n+1)h} f(s, y(s))ds$, and right hand side is the error bound we learned in class.

- Approach II:

$$\frac{h^3M}{6} \geq |\int_{nh}^{(n+1)h} (x-hn)(x-nh-h)y'''(x)dx|$$

$$= |(x-hn)(x-nh-h)y''|_{nh}^{(n+1)h} - \int_{nh}^{(n+1)h} (2x-2nh-h)y''(x)dx|$$

$$= |\int_{nh}^{(n+1)h} (2x-2nh-h)y''(x)dx|$$

$$= |(2x-2nh-h)y'|_{nh}^{(n+1)h} - \int_{nh}^{(n+1)h} 2y'dx|$$

$$= |h(f((n+1)h, y((n+1)h)) + f(nh, y(nh))) - 2(y((n+1)h) - y(nh))|$$

Divide h on both sides we get the required inequality.

(ii) From the inequality proved above, we have

$$y((n+1)h) = y(nh) + \frac{h}{2}(f((n+1)h, y((n+1)h)) + f(nh, y(nh))) + E$$

Where $|E| \leq \frac{Mh^3}{12}$. Hence,

$$|e_{n+1}| = |y((n+1)h)-z((n+1)h)| \leq |y(nh)-z(nh)|+\frac{hL}{2}(|y((n+1)h)-z((n+1)h)|+|y(nh)-z(nh)|)+|E|$$

$$\leq |e_n| + \frac{hL}{2}(|e_{n+1}| + |e_n|) + \frac{h^3M}{12}$$

2. (Problem 12.12 in textbook) Consider solving the initial value problem $y' = f(t,y), y(0) = y_0$ via linear multistep method:

$$z((n+3)h) + bz((n+1)h) + az(nh) = hf((n+2)h, z((n+2)h))$$

(i) Find a, b such that the method is consistent.

(ii) Show that for such a, b, the method is not zero stable.

Answer:

(i) $1 + b + a = 0$, $3 + b = 1$, so $b = -2$, $a = 1$.

(ii) The first characteristic polynomial is now $z^3 - 2z + 1$, which has a root $\frac{-1-\sqrt{5}}{2}$ hence is not zero-stable.