# HW 4

November 2, 2020

1. Let $f(x) = x^3$, $p$ be the Lagrange interpolation polynomial of $f$ using interpolation points $x = 0$, $x = 1$. On the interval $[0, 1]$, find the point $c$ that maximizes the interpolation error $|f(c) - p(c)|$, and find another point $s \in [0, 1]$ such that

$$f(c) - p(c) = f''(s)c(c-1)/2$$

2. Let $f(x) = e^x$, $p$ be the Lagrange interpolation polynomial of $f$ on interval $[0, 2]$ using interpolation points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, find an upper bound for the $L^\infty$ norm of $f(x) - p(x)$ on $[0, 2]$, using the error bound of Lagrange polynomial we covered in the lecture (Theorem 6.2 in textbook, Theorem 1.5 in lecture notes).

3. Suppose $f$ is continuous and with continuous derivatives of order up to and including 5 on $[a, b]$, and there are three distinct

points $x_0$, $x_1$, $x_2$ in $[a,b]$. Let $y_i = f(x_i)$, $i = 0,1,2$; $z_j = f'(x_j)$, $j = 0,2$.

(i) Find a polynomial $p$ of degree at most 4, such that $p(x_i) = y_i$, $i = 0,1,2$; $p'(x_j) = z_j$, $j = 0,2$.

(ii) Use an argument similar to the error estimate of Hermite interpolation polynomial to show that for any $x \in [a,b]$, there is some number $s \in [a,b]$ such that

$$f(x) - p(x) = f^{(5)}(s)(x-x_0)^2(x-x_1)(x-x_2)^2/5!$$

4. Let $q_j = (1-x^2)^j$, $\varphi_j = q_j^{(j)}$, show that $\varphi_j$ are orthogonal to each other in $L^2([-1,1])$. In other words, if $j \neq j'$, $\int_{-1}^{1} \varphi_j \varphi_{j'}\, dx = 0$.

5. Find three distinct points $x_0$, $x_1$ and $x_2$ in $(-1,1)$, such that for any polynomial function $f$ of degree 3, the best approximation of $f$ under $L^2$ norm on $[-1,1]$ of degree at most 2 coincides with the Lagrange interpolation polynomial of $f$ using interpolation points $x_0$, $x_1$ and $x_2$.

6. Let $f$ be a continuous function on $[0,1]$, $p_n$ be the polynomial of best approximation of degree no more than $n$ under the $L^2$ norm. Then, after studying Theorem 9.5 in the textbook, which proved that $f - p_n$ is zero at at least $n+1$ distinct points in $(0,1)$, find a function $f$ such that $f - p_2$ is zero at 4 distinct points in $(0,1)$.