Essential concepts covered so far

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The following topics are the most important topics and will be covered in final exam. I try to present them in a way with minimal technical details and I hope this presentation would be more accessible.

1 Polynomial interpolation

$x_0, \ldots, x_n$ distinct points on the domain of some function f.

Two kinds of polynomial interpolations:

(i) Lagrange interpolation: find a polynomial p of degree at most n, such that $p(x_i) = f(x_i)$.

(ii) Hermite interpolation: find a polynomial p of degree at most $2n+1$, such that $p(x_i) = f(x_i)$, $p'(x_i) = f'(x_i)$.

How to find interpolation polynomials:

(i) Lagrange interpolation:

$$p(x) = \sum_i \left(f(x_i) \cdot \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} \right)$$

(ii) Hermite interpolation:

$$p(x) = \sum_i \left(f'(x_i) \cdot \frac{(x - x_i)\prod_{j \neq i}(x - x_j)^2}{\prod_{j \neq i}(x_i - x_j)^2} \right.$$

$$\left. + f(x_i) \cdot \left(1 - (x - x_i) \sum_{j \neq i} \frac{2}{x_i - x_j} \right) \cdot \frac{\prod_{j \neq i}(x - x_j)^2}{\prod_{j \neq i}(x_i - x_j)^2} \right)$$

Why are these formulas valid?

(i) Lagrange interpolation: $p = \sum_i f(x_i) p_i$, where

$$p_i = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

By calculation, we can verify that

$$p_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

So $p(x_i) = f(x_i)p_i(x_i) + \sum_{j \neq i} f(x_j)p_j(x_i) = p(x_i)$.

(ii) Hermite interpolation: $p = \sum_i f(x_i)p_i + \sum_j f'(x_i)q_i$.

$$p_i = \left(1 - (x - x_i)\sum_{j \neq i} \frac{2}{x_i - x_j}\right) \cdot \frac{\prod_{j \neq i}(x - x_j)^2}{\prod_{j \neq i}(x_i - x_j)^2}$$

$$q_i = \frac{(x - x_i)\prod_{j \neq i}(x - x_j)^2}{\prod_{j \neq i}(x_i - x_j)^2}$$

By calculation, we can verify that

$$p_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$p_i'(x_j) = 0$$

$$q_i(x_j) = 0$$

$$q_i'(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 1.1.

Both Lagrange and Hermite interpolations are unique given f and interpolation points.
If p is the Lagrange interpolation of $f \in C^{n+1}$, $x, x_i \in [a, b]$, then there is $s \in [a, b]$:

$$f(x) - p(x) = \frac{f^{(n+1)}(s) \prod_i (x - x_i)}{(n+1)!}$$

If p is the Hermite interpolation of $f \in C^{2n+2}$, $x, x_i \in [a, b]$, then there is $s \in [a, b]$:

$$f(x) - p(x) = \frac{f^{(2n+2)}(s) \prod_i (x - x_i)^2}{(2n+2)!}$$

Intuition: If higher order derivative is zero, it's polynomial, hence error must be zero. So error should depend on higher order derivative. It takes time for higher order derivative to have an effect, so the more points there are the function being integrated should be smaller.

2 Approximation theory

- $L^\infty([a,b])$ norm: $\|f-g\|_\infty = \sup_{x\in[a,b]} |f(x)-g(x)|$.
- $L^2([a,b])$ norm: $\|f-g\|_2 = (\int_a^b (f-g)^2 dx$.
- Weighted $L^2([a,b])$ norm: $\|f-g\|_2 = (\int_a^b w(f-g)^2 dx$.

Orthogonal polynomials:

Definition 2.1. *We call ϕ_j, $j = 0, 1, 2, \ldots$ a system of* **orthogonal polynomials** *with weight w, if*

(i) ϕ_j is of degree j.

(ii) $\int_a^b w\phi_j\phi_i dx$ is non-zero iff $i = j$.

Orthogonal polynomials can be found via Gram-Schmidt applied to $\{1, x, x^2, x^3, \ldots\}$.

$$\phi_0 = 1$$

$$\phi_j = x^j - \sum_{i=0}^{j-1} \frac{\int_a^b wt^j \phi_i(t)dt}{\int_a^b w\phi_i^2(t)dt}\phi_i(x)$$

L^2 best approximation (used in two problems in HW4 but less important in final exam):

Let $f, \psi_1, \ldots, \psi_n \in L^2([a, b])$. Find a linear combination of ψ_i, $\sum_i a_i \psi_i$, such that $\|f - \sum_i a_i \psi_i$ is minimized.

Key idea: see them as if they are vectors in $\mathbb{R}^m$, $f, g \mapsto \int_a^b fg dx$ as if it is the inner product in $\mathbb{R}^m$.

The best approximation $f^* = \sum_i a_i \psi_i$ is the **orthogonal projection**, i.e. $\int_a^b \psi_i (f - f^*) dx = 0$ for all i.

3 Numerical Integration (Chapter 7, 10)

3.1 Quadrature rule

Question: Estimate $\int_a^b f(x)dx$.

Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$, then we can use the Lagrange interpolation polynomial to estimate f, and hence

$$\int_a^b f(x)dx \approx \sum_k w_k f(x_k)$$

Where

$$w_k = \int_a^b \frac{\prod_{j \neq k}(x - x_j)}{\prod_{j \neq k}(x_k - x_j)} dx$$

To estimate the integration.

The points x_i are called **quadrature points**, and w_i called **quadrature weights**.

- Newton-Cotes quadrature: quadrature points evenly spaces on $[a, b]$.
- Gauss quadrature: quadrature points are roots of orthogonal polynomials.
- Error bound via interpolation error bound.
- Composite method: divide the interval, then use quadrature rule on subintervals.