HW 5

November 22, 2020

1. (Problem 7.6 in textbook) Consider the trapezium and Simpson's rule applied to $\int_0^1 (x^5 - Cx^4)dx$.

- Write down the error for trapezium and Simpson's rule, as functions of C.
- Find C that makes the error under trapezium rule is 0.
- Find the range of C where the trapezium rule is more accurate than Simpson's rule.

2. (Problem 7.11 in textbook) Suppose $f \in C^4([-1,1])$. Let p be the Hermite interpolation polynomial of f at $x_0 = -1$, $x_1 = 1$.

- Calculate $\int_{-1}^1 p dx$ and write it as a linear combination of $f(\pm 1)$, $f'(\pm 1)$.
- Prove that

$$|\int_{-1}^1 f dx - \int_{-1}^1 p dx| \leq \frac{2}{45} \max_{c \in [-1,1]} |f^{(4)}(c)|$$

3. (Problem 10.3 in textbook) Show that if $f \in C^2([0,1])$, then there is some point $c \in (0,1)$ such that

$$\int_0^1 x f dx = \frac{1}{2} f(2/3) + \frac{1}{72} f''(c)$$

Hint: use Gauss quadrature with weight x.

4.

- Suppose f is continuous on $[0,1]$. Let I_n be the estimate of $\int_0^1 f dx$ using composite trapezium rule with n subintervals. Show that

$$\lim_{n\to\infty} |\int_0^1 f dx - I_n| = 0$$

Hint: There are many possible approaches. You can use the fact that any continuous function on a closed interval is absolutely continuous, or use the Weierstrass approximation theorem.

- (Optional) Find a continuous function f, such that there is $C > 0$ such that

$$|\int_0^1 f dx - I_n| \geq \frac{C}{n}$$

Hint: if f has bounded second derivative then the error decays like $O(1/n^2)$, so you need to find some f that doesn't have second order derivative or has unbounded second order derivative.