Notes on Linear Difference and Differential Equations and Taylor Series

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1 Linear Difference Equations

A homogeneous linear difference equation is an iterative relationship:

$$z_{n+k} + a_{k-1} z_{n+k-1} + \cdots + a_0 z_n = 0$$

The general solution of a homogeneous linear difference equation can be obtained as follows:

- Firstly, define the characteristic polynomial $\chi(z) = z^k + a_{k-1} z^{k-1} + \cdots + a_0$.
- Let $\lambda_1, \ldots, \lambda_l$ be its distinct roots, $m_1, \ldots, m_l$ their multiplicities (hence $\sum_i m_i = k$).
- Then, the general solution can be written as

$$z_n = \sum_i p_i(n) \lambda_i^n$$

  Where $p_i(n)$ is any polynomial of degree no more than $m_i - 1$.

2 Linear Differential Equations

Similarly, a homogeneous linear differential equation is

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0$$

The general solution of a homogeneous linear differential equation is as follows:

- Firstly, define the characteristic polynomial $\chi(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$.
- Let $\lambda_1, \ldots, \lambda_l$ be its distinct roots, $m_1, \ldots, m_l$ their multiplicities (hence $\sum_i m_i = k$).
- Then, the general solution can be written as

$$y(t) = \sum_i p_i(t)e^{\lambda_i t}$$

  Where $p_i(n)$ is any polynomial of degree no more than $m_i - 1$.

3 Taylor Series

If $f \in C^{k+1}$, then the Taylor series of f at a, with Lagrange remainder, is

$$f(x) = f(a) + \sum_{j=1}^{k} \frac{f^{(j)}(a)(x-a)^j}{j!} + \frac{f^{(j+1)}(c)(x-a)^{j+1}}{(j+1)!}$$

Where c is in the closed interval between x and a.