Hints for HW 4

November 9, 2020

For problem 3, note that we don't have $f'(x_1)$ so we can't use Hermite interpolation directly. What we need to do is to follow the same procedure we used to derive the Lagrange and Hermite interpolation.

I will show you the answer of a simpler version of problem 3:

Suppose f is continuous and with continuous derivatives of order up to and including 3 on $[a, b]$, and there are three distinct points x_0, x_1 in $[a, b]$. Let $y_i = f(x_i)$, $i = 0, 1$; $z_j = f'(x_j)$, $j = 0$.

(i) Find a polynomial p of degree at most 2, such that $p(x_i) = y_i$, $i = 0, 1$; $p'(x_j) = z_j$, $j = 0$.

(ii) Use an argument similar to the error estimate of Hermite interpolation polynomial to show that for any $x \in [a, b]$, there is some number $s \in [a, b]$ such that

$$f(x) - p(x) = f^{(3)}(s)(x - x_0)^2(x - x_1)/3!$$

Answer: If we can find p_0, p_1 and q_0 of degree at most 2, such that $p_0'(x_0) = p_0(x_1) = p_1(x_0) = p_1'(x_0) = q_0(x_0) = q_0(x_1) = 0$, $p_0(x_0) = p_1(x_1) = q_0'(x_0) = 1$, then we can let $p = y_0p_0 + y_1p_1 + z_0q_0$ and it's easy to check that it satisfies all three conditions.

Now we try and find the three polynomials. By some calculation, we get $p_0 = \frac{x-x_1}{x_0-x_1}(1 - \frac{x-x_0}{x_0-x_1})$, $q_0 = \frac{(x-x_0)(x-x_1)}{x_0-x_1}$, $p_1 = \frac{(x-x_0)^2}{(x_1-x_0)^2}$.

To do the second half of the problem, suppose $x \neq x_0$, $x \neq x_1$, consider function

$$G(t) = f(t) - p(t) - \frac{(f(x) - p(x))(t - x_0)^2(t - x_1)}{(x - x_0)^2(x - x_1)}$$

Then $G(x) = G(x_0) = G(x_1) = G'(x_0) = 0$, hence G' is zero at 3 points, G''' is zero at one point, which is s.

4. Use integration by parts formula: $\int_a^b uv'dx = uv|_a^b - \int_a^b u'vdx$.

5, 6: If p is the L^2 best approximation of f on $L = span\{1, x, x^2\}$, then $f - p \perp L$ under the L^2 inner product. So you may consider using the concept of orthogonal polynomials.