HW 5

December 6, 2020

1. (Problem 7.6 in textbook) Consider the trapezium and Simpson's rule applied to $\int_0^1 (x^5 - Cx^4)dx$.

- Write down the error for trapezium and Simpson's rule, as functions of C.
- Find C that makes the error under trapezium rule is 0.
- Find the range of C where the trapezium rule is more accurate than Simpson's rule.

Answer:

- The true answer is $1/6 - C/5$. The result of the trapezium rule is $\frac{1-c}{2}$, and the result of Simpson's rule is $\frac{2}{3}(1/32 - C/16) + \frac{1}{6}(1-c) = 3/16 - 5c/24$. So the error under trapezium rule is $|1/3 - 3C/10|$, the error under Simpson's rule is $|1/48 - C/120|$. It's ok if you do not write the absolute value.
- We need to have $|1/3 - 3C/10| = 0$, hence $C = 10/9$.
- This is the range of C such that $|1/3 - 3C/10| < |1/48 - C/120|$, hence $C \in (15/14, 85/74)$.

2. (Problem 7.11 in textbook) Suppose $f \in C^4([-1,1])$. Let p be the Hermite interpolation polynomial of f at $x_0 = -1$, $x_1 = 1$.

- Calculate $\int_{-1}^{1} p dx$ and write it as a linear combination of $f(\pm 1)$, $f'(\pm 1)$.

- Prove that
$$|\int_{-1}^{1} f dx - \int_{-1}^{1} p dx| \leq \frac{2}{45} \max_{c\in[-1,1]} |f^{(4)}(c)|$$

Answer:

- $$p(x) = f'(-1)\frac{(x+1)(x-1)^2}{4} + f'(1)\frac{(x-1)(x+1)^2}{4}$$
$$+f(-1)\frac{(x-1)^2(1+(x+1))}{4} + f(1)\frac{(x+1)^2(1-(x-1))}{4}$$

  So
$$\int_{-1}^{1} p dx = \frac{f'(-1)}{3} - \frac{f'(1)}{3} + f(-1) + f(1)$$

- The error bound for Hermite interpolation tells us
$$|f(x) - p(x)| \leq \frac{\max|f^{(4)}|(x+1)^2(x-1)^2}{4!}$$

  So
$$|\int_{-1}^{1} f dx - \int_{-1}^{1} p dx| \leq \int_{-1}^{1} |f-p| dx$$
$$\leq \max|f^{(4)}| \cdot \int_{-1}^{1} \frac{(x+1)^2(x-1)^2}{4!} dx$$
$$= \frac{2}{45} \max_{c\in[-1,1]} |f^{(4)}(c)|$$

3. (Problem 10.3 in textbook) Show that if $f \in C^2([0,1])$, then there is some point $c \in (0,1)$ such that
$$\int_0^1 x f dx = \frac{1}{2} f(2/3) + \frac{1}{72} f''(c)$$

Hint: use Gauss quadrature with weight x.

Answer: Because for any constant function C, $\int_0^1 xC(x-2/3)dx = 0$, $x-2/3$ is the degree-1 orthogonal polynomial on $[0,1]$ with weight x. Hence the weight x Gauss quadrature for $\int_0^1 xf dx$ should be the $I_0(f) = w_0 f(x_0)$, where x_0 is the root of $x - 2/3$ which is $2/3$, and

$$w_0 = \int_0^1 w(x) \frac{\prod_{i\neq 0}(x - x_i)}{\prod_{i\neq 0}(x_0 - x_i)} dx = \int_0^1 w dx = \frac{1}{2}$$

Now the error formula for Gauss quadrature tells us

$$\int_0^1 xf dx - I_0(f) = f''(c) \int_0^1 \frac{x(x-2/3)^2}{2!} dx = \frac{f''(c)}{72}$$

4.

- Suppose f is continuous on $[0,1]$. Let I_n be the estimate of $\int_0^1 f dx$ using composite trapezium rule with n subintervals. Show that

$$\lim_{n\to\infty} |\int_0^1 f dx - I_n| = 0$$

  Hint: There are many possible approaches. You can use the fact that any continuous function on a closed interval is uniformly continuous (for any $\epsilon > 0$, there is some δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$), or use the Weierstrass approximation theorem.

- (Optional) Find a continuous function f, such that there is $C > 0$ such that

$$|\int_0^1 f dx - I_n| \geq \frac{C}{n}$$

  Hint: if f has bounded second derivative then the error decays like $O(1/n^2)$, so you need to find some f that doesn't have second order derivative or has unbounded second order derivative.

Answer:

- - Approach I: f is continuous on $[0,1]$ hence uniformly continuous. For any $\epsilon > 0$, find δ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Now let N be some integer larger than $1/\epsilon$. For any $n > N$, consider the composite trapezium rule I_n, then f sends each of the n subintervals to an interval of length no more than ϵ, hence the error of the trapezium rule on this subinterval is no more than ϵ/n, and the error of the composite trapezium rule is no more than ϵ, hence $\lim_{n\to\infty} I_n(f) = \int_0^1 f dx$ by definition of limit.
  - Approach II: For any $\epsilon > 0$, find polynomial p such that $|f - p|_\infty < \epsilon/3$, then $|\int_0^1 f dx - \int_0^1 p dx| < \epsilon/3$, and for any n, $|I_n(f) - I_n(p)| < \epsilon/3$. Now let N be large enough such that $\frac{\max|p''|}{12N^2} < \epsilon/3$, then for any $n > N$, $|I_n(p) - \int_0^1 p dx| < \epsilon/3$, hence

$$|I_n(f) - \int_0^1 f dx| \leq |\int_0^1 f dx - \int_0^1 p dx| + |I_n(p) - \int_0^1 p dx| + |I_n(f) - I_n(p)| < \epsilon$$

- Let $f(x) = \sum_{i=1}^\infty \frac{1}{i(i+1)} \cos(2\pi i! x)$. Then $\int_0^1 f dx = 0$, and by trigonometry, $I_n(f) = \frac{1}{n}$.