Entropy on quadratic Hubbard trees

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Hubbard trees and core entropy

- Consider quadratic map $z \mapsto z^2 + c$. The critical point is $z = 0$. When the forward orbit of 0 is finite, it is called *post critically finite*. When it is periodic, it is called *critically periodic* or *has super attracting fixed points*, and they are *center* of hyperbolic components, otherwise it is called *Misiurewicz*.
- When the map is PCF, there is a unique way to connect the points in critical orbit within the filled Julia set, such that the intersection with any Fatou component is via segments of internal rays. The quadratic map send this finite tree to itself as a simplicial map, and its entropy is called core entropy.
- In the special case of $c \in \mathbb{R}$, the Hubbard tree is a line segment and study of dynamics on Hubbard tree is the same as the study of unimodal interval maps.

Real parameters

K. Lindsey, D. Davis, H. Bray and I previously found an algorithm that characterizes the "Thurston's teapot", i.e. the closure of the set

$$T = \{(z, \lambda)\}$$

Where $\lambda = e^h$, h is the entropy of a critically periodic interval map, and z a Galois conjugate of λ. Furthermore, we showed that

Theorem [BDLW]

- If $|z| = 1$, then $(z, y) \in T$ iff $1 \leq y \leq 2$.
- If $(z, \lambda) \in T$, $\lambda \leq y \leq 2$, then $(z, y) \in T$.
- The set $\Omega_\lambda = \{z : |z| \leq 1 \text{ or } (z, \lambda) \in T\}$ is continuous with respect to λ.

Key idea of the proof: Milnor-Thurston kneading theory.

The goal of the current project is to do something similar to exponent of core entropy in general.

Thurston's algorithm, Continuity of core entropy

- External rays on Mandelbrot set are image of rays $\{te^{2\pi i\theta} : t > 1\}$ under uniformization map to the complement of the Mandelbrot set.
- Rays with angle in $\mathbb{Q}\pi$ lands on Misiurewicz points or on bases of a hyperbolic component, hence correspond to PCF maps.

The core entropy of such PCF maps can be calculated using Thurston's algorithm as follows:

- Let $x_1 = \theta mod 1$, $x_i = 2x_{i-1} mod 1$. Let S be the finite set consisting of all x_i.
- Divide $\mathbb{R}/\mathbb{Z}$ into two segments at $\theta/2$ and $\theta/2 + 1/2$.
- Let V_θ be set of subsets of S with 2 points, M_θ be a map from $\mathbb{R}^{V_\theta}$ to itself, sending (x_i, x_j) to
    - (x_{i+1}, x_{j+1}), if x_i and x_j lie in the closure of the same segment.
    - $(x_1, x_{i+1}) + (x_1, x_{j+1})$, if otherwise. (x_1, x_1) is set as 0.

Example: eigenvalues of M_θ of centers, as a function of θ.
Theorem (Tiozzo) The eigenvalue of M_θ, hence the core entropy, is continuous with respect to θ.

New results

"Theorem (Lindsey-Tiozzo-W)" The eigenvalues of M_θ outside the unit circle changes continuously with θ. Note that all Galois conjugates of e^h must be eigenvalues of M_θ.

Strategy of proofs:

- Consider the directed graph represented by M_θ. Find an infinite weak cover, whose spectral determinant $P(t)$ is well defined. Here

$$P(t) = \sum_{\gamma \text{ multi cycle}} (-1)^{C(\gamma)} t^{l(\gamma)}$$

  Here C is the number of components, l the length.
- Approximate the infinite weak cover by finite weak covers

- Show that the reciprocal of the spectral determinant of the finite weak covers is the characteristic polynomial multiplies with z^k and some cyclotomics.
- Bound the coefficients of the spectral determinants of the finite weak covers. This implies that them, hence their roots, converges to the roots of the spectral determinant of infinite graphs.
- The spectral determinant of these infinite graphs change continuously.
- Extra treatment for periodic θ.

Further questions

- For roots inside the unit circle, there are ways to relate kneading theory on intervals with kneading theories on some trees. However we don't yet know how general the statement may be.
- Irreducibility of the polynomials involved?
- Higher degree?