A proof that McMullen's $\Omega_1 E_D$ are closed invariant subsets under $SL(2,R)$ action

Denote $\Gamma = SP(4,\mathbb{Z})$, here we let the sympletic form be $diag(J,J)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\Gamma \subset GL(4,\mathbb{C})$ induces a right action of Γ on $\mathbb{C}^4$.

Proposition 1: for all $v = (a_1, b_1, a_2, b_2) \in \mathbb{C}^4$, the followings are equivalent:

a) For any element $g \in \Gamma$, let $(a_1', b_1', a_2', b_2') = vg$, there exists a integer matrix $P \in M(2,\mathbb{Z})$ with positive determinant, positive real number λ, such that $\lambda(a_1', b_1') = (a_2', b_2')P$.

b) For two different elements $g_1, g_2 \in \Gamma$ the assumption in a) is satisfied.

c) There exists integers d, e, positive real number λ and 2-by-2 integer matrix P, such that $\lambda^2 = e\lambda + d$, $d = det(P) > 0$ and $\lambda(a_1, b_1) = (a_2, b_2)P$. If P is also primitive, then we call $D = 4d + e^2$ the discriminant.

d) For any element $g \in \Gamma$, $(a_1', b_1', a_2', b_2') = vg$ satisfies all conditions in c) with a fixed discriminant D not depend on g.

Sketch of Proof: $d) \Rightarrow a) \Rightarrow b)$ are trivial.

$c) \Rightarrow d)$: Let $B \in M(2,\mathbb{Z})$ such that $JB^t = PJ$, denote $T = \begin{pmatrix} 0 & B \\ P & eI \end{pmatrix} \in M(4,\mathbb{Z})$ which is (1)primitive, (2)self-adjoint with respect to the bilinear form $diag(J,J)$, (3)$vT = \lambda v$ and (4)$T^2 = eT + d$. For any $g \in \Lambda$, $g^{-1}Tg$ satisfies (1)-(4). By the proof of [M] Theorem 8.3 $g^{-1}Tg = \begin{pmatrix} f'I & B' \\ P' & e'I \end{pmatrix}$. Denote $T' = g^{-1}Tg - f'I$, then T' satisfies (1), (2) and (3')$vgT' = \lambda' vg$ and (4')$T'^2 = e'T + d'$. By the construction of T' $e'^2 + 4d' = e^2 + d$, and by (3') and the fact that T' is of the form $\begin{pmatrix} 0 & B' \\ P' & (e' - f')I \end{pmatrix}$ we know vg, e', d', λ', P' also satisfy all the conditions in c), i.e. d) holds.

$b) \Rightarrow c)$: If one of the λ is rational, then c) is evident.

If otherwise, without losing generality suppose $\lambda(a_1, b_1) = (a_2, b_2)P$, $\lambda'(a_1', b_1') = (a_2', b_2')P'$, $(a_1', b_1', a_2', b_2') = vg \neq v$, and P, P' are both primitive.

We define the action of $SL(2,\mathbb{R})$ on $\mathbb{C}^4$ as acting diagonally on all for components as the standard action on $\mathbb{R}^2 = \mathbb{C}$. Consider the homology affine group $\mathrm{Aff} = \{\Psi \in SP(4,\mathbb{Z}) | \exists \psi \in SL(2,\mathbb{R}), v\Psi = \psi v\}$. Because a_1, b_1, a_2, b_2 are $\mathbb{Q}-$linear independent it is natually isomorphic to the group $G = \{\psi\} \subset SL(2,\mathbb{R})$, because $g^{-1}\mathrm{Aff}g \subset \mathrm{Aff}$, G is larger than $SL(2,\mathbb{Z})$, hence there is $\Psi_0 \in \mathrm{Aff}$ such that its corresponded element $\psi_0 \in G$ has non-rational trace. By [M] Theorem 5.3 its trace must be in a integer in a quadratic field, hence $\Psi_0 + \Psi_0^{-1}$ divided by some integer if needed satisfies (1)-(3) in the previous proof. Because $\psi_0 + \psi_0^{-1}$ satisfies (4) and Aff is bijectively identified with G it also satisfies (4), hence c) holds by the proof above.$\square$

Proposition 2: Let the set L_D consisting of complex vector (a_1, b_1, a_2, b_2) such that (a_1, b_1, a_2, b_2) satisfies a)-d) in Proposition 1 with discriminant D, and $\frac{\sqrt{-1}}{2}(a_1\bar{b_1} - b_1\bar{a}_1 +$

$a_2\bar{b_2}-b_2\bar{a_2})=1$, then they are closed in manifold $U=\{(a_1,b_1,a_2,b_2)\in\mathbb{C}^4|\frac{\sqrt{-1}}{2}(a_1\bar{b_1}-b_1\bar{a_1}+a_2\bar{b_2}-b_2\bar{a_2})=1\}$.

Proof: for any $M>0$, let $U_M=\{(a_1,b_1,a_2,b_2)\in\mathbb{C}^4||a_1|\leq M,|b_1|\leq M,|a_2|\leq M,|b_2|\leq M,\frac{\sqrt{-1}}{2}(a_1\bar{b_1}-b_1\bar{a_1}+a_2\bar{b_2}-b_2\bar{a_2})=1\}$, then we only need to show $L_D\cap U_M$ is closed for any M.

L_D is the union of countably many connected components indexed by (λ,P), each of which is a embedded submanifold of U, so we only need to show that only finitely many of them have non-empty intersection with U_M. Firstly, fixing D there are only finitely many choices of λ,d. Fixing a pair λ,d, we have $\frac{\sqrt{-1}}{2}(a_2\bar{b_2}-b_2\bar{a_2})=\frac{\lambda^2}{d+\lambda^2}$, i.e. the parallelogram formed by a_2,b_2 has fixed area, similarly so does the parallelogram spanned by a_1,b_1. Therefore, if a point in a component indexed by (λ,P) also lies in U_M, because $|a_2|,|b_2$ are bounded by M and the parallelogram formed by $a_2,b_2|$ has fixed area, the sin of the angle between a_2,b_2 must satisfy a lower bound depended on M, and $|a_2|,|b_2|$ are also bounded below. On the other hand, $|a_1|,|b_1|$ are bounded above by M, hence from $\lambda(a_1,b_1)=(a_2,b_2)P$ we have an upper bound on $||P||$ by M,λ,d. Hence, there can only be finitely many P for each λ.$\square$

For any genus-2 translation surface M, let $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ be a bases of $H_1(M;\mathbb{Z})$ with intersection form $diag(J,J)$.

Let $\Omega_1 E_D$ be the set of translation surface whose absolute periods with regard to $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ lies in L_D. By proposition 1 $\Omega_1 E_D$ are well-defined and $SL(2,\mathbb{R})$-invariant, and by proposition 2 they are closed in the moduli space of all translation surfaces with area normalized to 1.

In general, for genus $g>1$, let $U=\{(a_1,\ldots,a_g,b_1,\ldots,b_g)\in\mathbb{C}^{2g}|\prod a_i\prod b_i\neq 0,\frac{\sqrt{-1}}{2}\sum(a_i\bar{b_i}-b_i\bar{a_i})=1\}$, $GL(2,\mathbb{R})$ and $SP(2g,\mathbb{Z})$ both acts on it as described above, and if we can find a closed subset $V\subset U$ invariant under both group actions then the set of genus-g translation surface whose absolute period under a sympletic bases of H_1 lies in V would be a well-defined, closed $SL(2,\mathbb{R})$-invariant subset in the moduli space of all translation surfaces of genus g.

[M]McMullen, Curtis. Dynamics of $SL^2(\mathbb{R})$ over moduli space in genus two