Project Summary

Overview

The main goal of the project is to study the topology and dynamics of self maps on graphs, on surfaces and on handlebodies. Self maps of closed Riemann surfaces and self maps on finite metric graphs have important parallels which can be seen from the perspective of Teichmüller spaces and mapping class group action as compared to the Culler-Vogtmann outer spaces and $Out(F_n)$ action, from Berkovich analytic spaces of curves, or from train-track maps induced by a pseudo-Anosov mapping class. Isotopy classes of self homeomorphisms on handlebodies are related to a subgroup of the mapping class group called handlebody groups, and they have connection with both graph maps (because handlebodies are homotopically equivalent to graphs) and surface maps (because the boundary of handlebodies are closed surfaces). Furthermore, the PI will also carry out educational and outreach activities in connection with the research project.

Intellectual Merit

Handlebodies and handlebody groups are important tools in low dimensional topology and geometry. Through this research project the PI hopes to shed new lights on the connection between 3-dimensional topology, symbolic dynamics and Teichmüller theory, and it is hoped that the results will be helpful to the study of certain open problems in these area, e.g. the approximation conjecture for L^2 torsions, determining whether or not an algebraic integer is a pseudo-Anosov stretch factor, lower bounds on the combinatorial complexity of pseudo-Anosov elements (as measured by stable length on curve or arc complexes) in the handlebody group as well as other important subgroups of the mapping class group, and the various conjectures on the shape of Mandelbrot set and its higher degree analogies.

Broader Impacts

The project involves questions at varying levels of abstraction and therefore is well suited for integration into teaching and outreach activities. Furthermore, algorithms obtained in this project have potential real-life applications in areas like machine learning and numerical analysis. The teaching and outreach activities proposed will be helpful for the academic and professional development of math students in both the undergraduate and graduate level and therefore contribute to building a more diverse and competitive STEM workforce, as well as increase mathematical literacy among the general public, as well as improve STEM education, particularly in underserved communities.

Project Description

0. Background and Overview

The study of mapping class groups (the group of homotopy classes of homeomorphisms from a closed oriented surface to itself) as well as the $Out(F_n)$ (the group of outer automorphism groups on free groups) are important topics in geometric group theory, low dimensional topology and dynamics. The mapping class group acts on the *Teichmüller space* (which is the space of marked complex structures on the given surface, and is contractible and admits a complex structure and a number of geometrically important metrics) [37] as well as the *curve complex* of the surface [45] (which is Gromov hyperbolic), and a generic element in the mapping class group is *pseudo-Anosov* [27] (can be represented by a homeomorphism that preserves a pair of transverse singular foliations). Analogously, the group $Out(F_n)$ acts on the *Culler-Vogtmann outer space* [21], as well as the *free factor* [34] and *free splitting* [33] complexes, and one possible analogy of being pseudo-Anosov would be admitting an irreducible train-track representation [14].

One can go from a pseudo-Anosov map to a train track map on graph by simply taking its invariant traintrack [15]. This construction can be (fully or partially) reversed for certain $Out(F_n)$ elements [32, 23]. Another way to establish connections between graph and surface maps is via the concept of handlebody groups [35]. Furthermore, the analogy between mapping class groups and $Out(F_n)$, between closed Riemann surfaces and finite metric graphs, can also be understood via the perspective of Berkovich spaces [11], as the Berkovich space of curves can be represented as $\mathbb{R}$-trees glued on finite metric graphs. One of the main goals of the research program of the PI is to deepen the understanding of these connections and apply them to important questions regarding the geometry and dynamics of surfaces and graphs.

The research project can be roughly divided into three interconnected components:

(i) Topological entropy of graph maps.

(ii) Relationship between mapping class group, handlebody group and Thurston's fibered cones.

(iii) Spectral radius of infinite graph maps.

Below is a more detailed description of the three components of the research plan:

1. Topological entropy of graph maps

1.1 Background and prior works

The first part of the research concerns with simplicial maps on finite graphs. Here a finite graph is a connected finite 1-dimensional simplicial complex, and a simplicial map is a continuous map that sends vertices to vertices and edges to a collection of edges.

The study of simplicial maps on metric graphs has many applications in topology and dynamics: the self maps on intervals are related to conttinuous fractions and β-expansions [38, 51, 30, 58, 61], symbolic dynamics and iterative function systems [9, 8, 59, 20], point processes [52], and the study of similarity structure on surfaces; dynamics of post-critically finite polynomial maps can be related to simplicial maps on trees via the Hubbard tree [24, 41, 53, 49, 62, 65, 66]; pseudo-Anosov maps on surfaces can be related to maps on traintrack graphs [15]. Also, since the fundamental group of graphs is a free group, graph maps are related to outer endormorphisms or automorphisms of the free group.

A first example of simplicial maps on graphs is post-critically finite maps on intervals. Here by *post-critically finite* we mean the forward orbit of all critical points consists of finitely many points, and by *critically periodic* we mean the forward orbit of all critical points are periodic. A key tool in the study of self maps on intervals is the Milnor-Thurston kneading theory [50] which has also been generalized to the case of trees and more general graphs, as well as higher dimensional objects [1, 22].

When the map is critically periodic or postcritically finite, the dynamical system admits a Markov decomposition, the exponent of the topological entropy is the eigenvalue of a Perron-Frobenious matrix hence must be an algebraic integer, hence one can study the set of Galois conjugates of this exponent. In his last paper [63], Willian Thurston proposed the *master teapot*, which is defined below:

Definition 1.

- *An interval map is unimodal iff it has a single critical point c in the interior of the interval. Hence, a unimodal map is critically periodic iff $f^{\circ n}(c) = c$ for some $n > 0$.*
- *Now the* master teapot *is*

$$T := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

  Where h_{top} is the topological entropy, $\mathcal{F}$ is the set of unimocal maps with periodic critical orbit.

- *The projection of the Master teapot on $\mathbb{C}$ is called the* Thurston set.

One application of the Thurston set and Thurston's Master teapot is that they provide necessary conditions on when an algebraic integer can be the exponent of topological entropy of a critically periodic unimodal map, as such an integer λ must have $(z, \lambda) \in T$ for all Galois conjugate z of λ. The properties of these sets have been extensively studied. In particular, [67] gives a characterization of the Thurston set and relate it to the roots of Littlewood polynomials (polynomials with all coefficients ± 1) [43]. Figure 1 and Figure 2 are finite approximations of the Thurston set and Thurston teapot respectively.

In some prior works by the PI and his collaborators [19, 42], an analogous characterization of the Thurston teapot is found, and an algorithm is given that can be used to certify a point not belonging to the Thurston teapot. In particular, given any $\lambda \in (1, 2)$, the PI and his collaborators found a subset M_λ of $\{0, 1\}^{\mathbb{N}}$ invariant under shift, non decreasing as λ increases, such that for any $|z| < 1$, $(z, \lambda) \in T$ iff there is some $w = w_1 w_2 \cdots \in M_\lambda$, such that

$$G(w, z) := \lim_{n \to \infty} f_{w_1, z} \circ \ldots \circ f_{w_n, z}(1) = 1$$

Here $f_{0,z}(x) = zx$ and $f_{1,z}(x) = 2 - zx$.

As a consequence, the PI and his collaborators found this surprising fact:

Theorem 2. *If $(z, t) \in T$, $|z| < 1$, then $(z, y) \in T$ for all $t \leq y \leq 2$.*

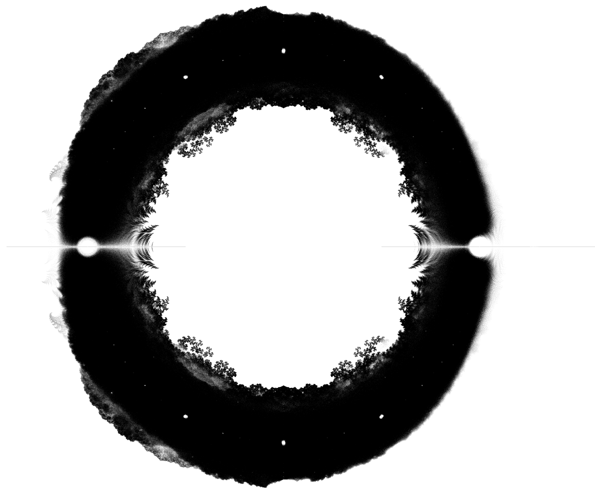

Figure 1: A picture of the Thurston set

The interval can be seen as a special case of the Hubbard tree [24]. Here, the *Hubbard tree* is a finite simplicial tree whose vertices are orbits of the critical point and is contained in the Julia set. The quadratic map on the Julia set induces a simplicial map on the Hubbard tree, and the topological entropy of this simplicial map is called the *core entropy*. Now a critically periodic unimodal map can be seen as a superattracting point ($c \in \mathbb{C}$ such that $f_c : z \mapsto z^2 + c$ satisfies $f_c^{\circ n}(0) = 0$ for some $n > 0$) on the real slice of the *Mandelbrot set*, which consists of all c such that the forward orbit of 0 under f_c does not go to infinity.

In an upcoming paper by the PI with Kathryn Lindsey and Guilio Tiozzo, the characterization of the Thurston teapot in [19, 42] have been generalized to the principal veins of the Mandelbrot sets. Here, the p/q-principal vein of a Mandelbrot set, for p and q coprime, would be an arc connecting the main cardioid and the parameter $c_{p/q}$ in the p/q-limb (which consists of parameters where $f_c = z^2 + c$ has rotation number p/q around the α-fixed point), where 0 is sent to the β fixed point in precisely q steps.

More precisely, for every superattracting parameter c, the Hubbard tree admits a Markov decomposition, so one can write down the incidence matrices and find all the eigenvalues. Now consider the vein with each hyperbolic component collapsed into one point, which becomes an interval I, and consider the product $I \times \mathbb{C}$. Now the analogy of the Thurston teapot T_q is the closure of pairs (z, λ) where z an eigenvalue of the incidence matrix for some superattracting parameter in the p/q vein, and λ is the largest eigenvalue. The PI and his collaborators are able to prove:

Theorem 3.

- *The set* $O_p = \{z : |z| \leq 1$ *or* $(p, z) \in T'\}$ *changes continuously with* p *under Hausdorff topology.*
- *The set* $I_p = \{z : |z| < 1$ *and* $(p, z) \in T'\}$ *is monotone increasing (in the sense of containment) as* p *moves towards the tip of the principal veins.*

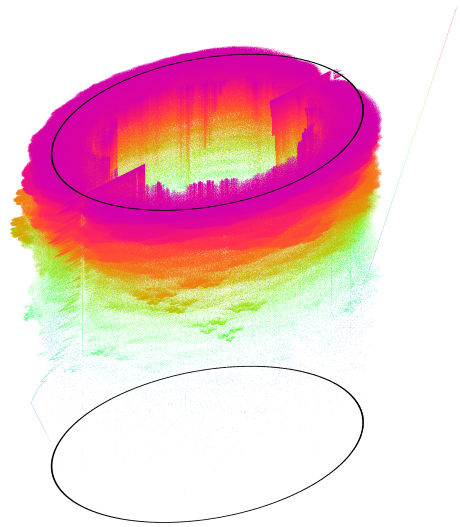

Figure 2: A picture of the Master Teapot

The key idea of the proof is an alternative kneading theory on trees which is different from the one in [1] which allow more results from the Milnor-Thurston kneading theory [50] to be generalized to the case of trees.

Furthermore, in the same upcoming paper, the PI and his collaborators would also prove that the set of eigenvalues of incidence matrix outside unit circle moves continuously with external angle, which generalized the main result in [66]. The key to this result is an extension of Thurston's algorithm for core entropy using infinite graphs, and the concept of weak covering on graphs. The PI and his collaborators are working on generalizing this part of the result to Misiurewicz points.

1.2 Problems and possible strategies

The first main question for this project is:

Research question 1. *Can the characterization of the Thurston set above be generalized to other settings? In particular, the PI will be working on the following cases:*

(i) Superattracting maps on Hubbard trees along non principal veins.

(ii) Post critically finite maps, which would allow the results to be generalized to Misiurewicz points.

(iii) Interval maps with two critical values.

(iv) Self maps on graphs of given topology or genus

The first and second case would provide more information on quadratic core entropy as well as the geometry of Mandelbrot sets. The third case is a natural extension of Thurston's "master teapot", which according to the numerical experiment by the PI and his collaborators, should have similar shape as the teapot and has many of the same properties. It has applications in the core entropy of higher degree polynomial maps as well as the study of generalized β-extensions. The fourth case, when the self map is a homotopy equivalence, is related to the study of geodesics in Culler-Vogtmann Outer space, and, via the train-track construction, the question of what algebraic integers can be pseudo-Anosov stretch factors, which has been extensively studied by Penner, Long, Leininger, Hamenstädt and many others.

The question can also be seen as a refinement of the results in Thurston's last paper [63], where it is shown any Perron number can be the exponent of entropy of a 1-dimensional maps. Here the question is what would happen if more constraints are imposed on the map.

For the first case, the first step would be to generalize the kneading theory on Hubbard trees on principal veins developed by the PI and collaborators to maps on more general Hubbard trees, and the next step would be extending the "combinatorial surgery" by Douady and made more explicit in Tiozzo [65] to this more general setting. Basically the idea behind "combinatorial surgery" is taking an appropriate subinterval on the tree and do first return maps, so some attempts need to be made to pick the appropriate interval.

For the second case, once the steps in the first case are done then the roots of Markov incidence matrix outside the unit circle changing continuously can be established. For roots inside the unit circle, more numerical experiments need to be done to formulate the right conjecture.

For the third case, together with Kathryn Lindsey and Ethan Farber the PI has formulated analogies of the main results in [19, 42], and tested the conjectures via numerical experiments. At the moment the only remaining step is a combinatorial fact regarding the denseness of dominant itineraries (as is defined in [67]). The validity of the statement has been verified via experiment but the PI and his collaborators are still working on a proof of it.

For the fourth case, the first step would be formulating a more general kneading theory in the style of Milnor-Thurston, which the PI is working on, then more experiments need to be done to decide the correct conjecture.

2. Relationship between mapping class group, handlebody group and Thurston's fibered cones

2.1 Motivation and prior works

Surface maps with the same mapping torus can be organized into *fibered cone*, which was introduced in [64]:

Definition 4. *[64]*

- *Let M be a hyperbolic 3-manifold, for every integer cohomology class $\alpha \in H^1(M;\mathbb{Z})$, the* Thurston norm *of α is defined as*

$$\|\alpha\| = \min_S \max\{0, -\chi(S_i)\}$$

  Where $S = \bigcup_i S_i$ is an embedded surface that represents α. The Thurston norm can be extended to $H^1(M;\mathbb{R})$ as a piecewise linear function with rational coefficients.

- *If M is homeomorphic to a mapping torus of a surface map ϕ (seen as a surface bundle over the circle), let the cohomology class associated with ϕ, denoted as α_ϕ, be the pullback of the* *generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called* fibered cones*, and the map associated with a primitive integer class α in the fibered cone are denoted as ϕ_α.*

Recall that a surface map is called *pseudo-Anosov* iff it preserves two mutually transverse singular measured foliations. It is known in [64] that if one surface map in a fibered cone is pseudo-Anosov, so are all the rest. We call the *stretch factor* of a pseudo-Anosov map the exponent of its topological entropy. It has been shown in [28] that the mapping tori of pseudo-Anosov maps with bounded dilatation lies in a finite list of 3-manifolds up to Dehn filling. An important result by McMullen [47] is that pseudo-Anosov maps that have homeomorphic mapping torus and lie in the same *fibered cone* have their stretch factors related to one another via a Laurant polynomial called *Teichmüller polynomial*, which is related to the Alexander polynomial of the mapping torus.

Besides stretch factors, another way of characterizing the topological complexity of a pseudo-Anosov map is through its asymptotic translation length on the curve graph of S. The *curve graph* $\mathcal{C}(S)$ is a graph where each vertex represent an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. It is easy to see that the mapping class group of S acts on curve graph and curve complex by isometry. Masur-Minsky [45] show that $\mathcal{C}(S)$ is δ-hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserves a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. The study of curve graphs are also related to the hirerachical hyperbolic structure on the mapping class group [46, 10]. The asymptotic translation length of a pseudo-Anosov map g on $\mathcal{C}(S)$ can now be defined as

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$.

It is shown in [45] that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov. Furthermore, the technique in [45] in showing the positivity of $l_{\mathcal{C}}$, which is based on studying the incidence matrix on the induced map on invariant traintracks, have been used by [29, 68, 5] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases. Furthermore, in [18] the asymptotic translation length is shown to be a rational number, and in [56, 16] algorithms for its computation are described.

In [39], a sequence of pseudo-Anosov maps in different genus are constructed that realized the asymptotic lower bound. The PI and his collaborators generalized the argument in [39] and proved the following:

Theorem 5. *[6] Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,*

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

Balázs Strenner [60] also proved a stronger result for the asymptotic translation length of arc complexes.

In [4], the PI and his collaborators have shown that this asymptotic upper bound is sharp when $r \leq 3$. Furthermore, in [4], the PI and his collaborators uses techniques similar to [6] to show the following:

Theorem 6. *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.*

Motivated by the relationship between pseudo-Anosov maps and the induced map on the invariant train track as well as the analogy between $Out(F_n)$ and the mapping class group c.f. [14, 13], the PI and his collaborators are working on generalizing Theorem 5 to the case of the metric graph. In an upcoming paper, the PI and his collaborators will prove an analogy of the above theorem in the setting of asymptotic translation lengths on the sphere complex for train track maps.

Let G be a finite simplicial graph. A cellular map $\psi : G \to G$ is called a train track map if the restriction of ψ^n to e for each $n \geq 1$ and each edge e is an immersion (no back-tracking condition). We further assume ψ to be irreducible as an element of $Out(F_n)$. One can make a 3-manifold M_G from G by replacing every edge with $S^2 \times I$ and every vertex with $\mathbb{S}^3$. In the case when ψ is a train track map, ψ induces a homeomorphism ψ_1 on M_G. Let $\mathcal{S}(G)$ be the simplicial graph whose vertices are isotopy classes of embedded spheres in M_G, and there is an edge of length 1 between two vertices if and only if they are disjoint up to isotopy, then it is easy to see that ψ_1 is an isometry of $\mathcal{S}$ and we can define the concept of asymptotic translation length of ψ_1 analogously.

In [26, 25], the argument in [64] and [47] are generalized to the case of maps on finite graphs as follows:

Definition 7. *[26, 25] Suppose ψ is an irreducible train track map, let γ_ψ be a folding path of ψ in the Culler-Vogtmann outer space. The* folded mapping torus *N is a 2-d cell complex built from γ_ψ, which has a surjection over the circle and the fibers are the graphs in the folding path. A flow on N is defined such that any flow line is the orbit of a point on the graph under folding, and an analogy for the fibered face containing ϕ is $\mathcal{S}$ which consists of first cohomology classes whose dual are transverse to all flow lines. This is a rational cone call the "cone of sections" or "McMullen cone" in [25].*

In a recent paper [3], Hyungryul Baik, Dongryul Kim and the PI are able to show the following:

Theorem 8. *[3] Given any finite graph G and any irreducible train track map ψ on G, let $\mathcal{C}$ be any proper subcone of the intersection of the McMullen cone in [47] containing ψ and the negative of the McMullen cone containing ψ^{-1}, then any primitive integer element α in $\mathcal{C}$ must satisfies*

$$l(\psi_\alpha) \lesssim n_\alpha^{-1-1/d}$$

Where d is the dimension of the fibered cone, n_α the genus of the fiber corresponding to α and ψ_α the corresponding monodromy, and $l(\cdot)$ the translation length on the spherical complex obtained by thickening the graph G.

The sphere complex is related to the free splitting [33] and free factor complexes [12], hence the theorem above gives us estimates on these complexes as well.

In the same paper [3], the PI and his collaborators also provided estimates on the curve complex translation length for elements in the handlebody group.

Definition 9. *[35]*

- *A* handlebody *is a closed ball with finitely many "handles" of the form $D_2 \times I$, where D_2 is a 2-disk and I is a closed interval, being glued to its surface. The homotopy type of a handlebody with g handles is a ribbon graph $(S^1)^{\vee g}$, and the surface of the handlebody is a closed oriented surface with genus g.*

- *The* handlebody group *is the group of orientation preserving self homeomorphisms of the handlebody to itself up to isotopy.*

As a consequence, any handlebody group element gives rise to an element in $Out(F_n)$ (due to the homotopy equivalence between handlebodies and graphs), as well as an element in the mapping class group of genus g (by restriction to the boundary). It is known (cf. [35]) that any $Out(F_n)$ element can be associated with infinitely many handlebody group elements. Furthermore, Hensel and Kielak [36] proved that under certain homological conditions, all pseudo-Anosov maps in a mapping class would be in the handlebody group. More precisely,

Theorem 10. *[36, Theorem 1.2] If the mapping torus M of a handlebody homeomorphism corresponding to a handlebody group element has the property that the inclusion map of its boundary induces isomorphism in first homology, then all the monodromies corresponding to primitive integer vectors in the fibered cone of this handlebody group element are also in the handlebody group.*

2.2 Problems and possible strategies

The second main problem for this project is the following:

Research question 2.

(i) Can anything be stated about the minimal stable length on curve complex of pseudo-Anosov handlebody group elements of given genus?

(ii) Given a handlebody group element, is there a way to lift it to a handlebody homeomorphism with "minimal complexity"?

(iii) If the mapping torus of a handlebody homeomorphism has smaller first betti number than its boundary, does it mean that there are elements in the fibered cone of the induced map of its boundary corresponding to non-handlebody group elements?

The main motivation of these problems is that handlebody groups can serve as a way relating $Out(F_n)$ and the mapping class groups, and mapping torus technique is useful for obtaining pseudo-Anosov elements of small complexity (as measured by stretch factor, or by asymptotic translation lengths). Hence, understanding the relationship between the two can potentially help one create many examples as well as non-examples of handlebody group elements, and hence be able to understand how the handlebody group lies inside a mapping class group.

The first question should be doable by modifying the results in the end of the PI's paper [3], possibly by combining it with the techniques in [36]. For the second question, if the "complexity" is measured by stable translation length on disc graphs, then the technique in [3] might be helpful. If the "complexity" is measured by minimal Lipschitz constant, or "stretch factor", one possible approach would be to pick and fix some "canonical" metric on the handlebody and estimate the Lipschitz constant.

For the third question, at the moment the PI and his collaborators conjectured that for a generic primitive integer element of this fibered cone there won't be handlebody extension. Under the assumption of this question, following [36], there are proper slices of the fibered cone where all primitive integer elements would be in the handlebody group. Hence one would need to show that the set of the highest dimensional slices is finite, or at least locally finite. The PI and his collaborators think that the argument by Long [44] on the finiteness of handlebody extension of a mapping class might be adapted to show such a finiteness.

3. Spectral radius of infinite graph maps

3.1 Background and prior work

The analogy between Riemann surfaces and finite metric graphs allow many important concepts and properties on Riemann surfaces to be generalized to the graph-theoretic setting. For example, analogous to the Arakelov metric on Riemann surfaces, there is the "canonical metric" on finite metric graphs [69, 7], which is defined as

Definition 11. *The canonical metric[69, 7] on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:*

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

This is shown by [2, 54] to be related to the distribution of Weistrass points of line bundles on finite metric graphs.

In a prior work of the PI and his collaborator, an analogous result on a property of the Arakelov metric on Riemann surfaces in [48, Appendix] was found, which shows that when passing to larger and larger normal covers the canonical metric on metric graphs converges:

Theorem 12. *[57] Let $G \leftarrow G_1 \leftarrow G_2 \ldots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.*

In the case when $\cap_i \pi_1(G_i)$ is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of G. Because in [48, Appendix] the limit of the Arakelov metrics for a closed Riemann surface of genus ≥ 2 under a tower of normal covers that converges to the universal cover would converge to the hyperbolic metric, one can see the limiting metric in Theorem 12 as a discrete analogy of the hyperbolic metric. However it is unclear what the relationship between this limiting metrics and other approaches of discrete uniformization [17, 31, 55] are. The key idea in the proof of Theorem 12 is Lück's approximation theorem on L^2 betti numbers.

The main theorem in [48] is the following:

Theorem 13. *Let ψ be a pseudo-Anosov map on a closed surface S. Let ρ be the stretch factor (i.e. exponent of the topological entropy) of ψ. Then, either there is a finite cover of S, where ψ lifts and the induced map on homology has spectral radius ρ, or there exists $\epsilon > 0$ such that for any finite cover of S where ψ lifts, the spectral radius on the homology is no more than $\rho - \epsilon$.*

It is conjectured by McMullen and Koberda that the same is true for irreducible train-track maps:

Conjecture 14. *[40] Let ψ be an irreducible traintrack map on a finite graph G. Let ρ be the stretch factor (i.e. exponent of the topological entropy) of ψ. Then, either there is a finite cover of G, where ψ lifts and the induced map on homology has spectral radius ρ, or there exists $\epsilon > 0$ such that for any finite cover of S where ψ lifts, the spectral radius on the homology is no more than $\rho - \epsilon$.*

Furthermore, the PI and his collaborators conjectured the following:

Conjecture 15. *When the lifting of ψ on universal cover $\tilde{G}$ acting on the space of L^2 integrable harminoc 1-forms has spectral radius equals to ρ, there is a finite cover of the graph G preserved by ψ, and the action of the lifting of ψ on its homology has spectral radius ρ. When the spectral radius on the space of L^2 integrable harmonic 1-forms on universal cover is strictly less than ρ, there is a gap between ρ and the spectral radius of the lifting of ψ on the homology of any finite cover of G.*

3.2 Problems and possible strategies

The third major question for this project is:

Research question 3. *(i) What is the criteria for a fully irreducible train track map to have a gap between the stretch factor and the spectral radius of the induced map on L^2 harmonic 1-forms on its universal cover?*

(ii) Prove or disprove Conjectures 15 and 14.

The main motivation for the above questions is to understand the connection between surface maps and graph maps better. Both questions have solutions in the surface case due to [48]. In particular, in the surface case, the answer for the first question is that its quadratic differential has at least one odd order singularity.

For problem 1, at the moment the PI and his collaborator are looking into harmonic analysis techniques at the boundary to study the induced map there. For problem 2, the PI and his collaborator are considering "smaller" coverings of the graph like abelian and or other amenable covers first, in the hope that approximation theorems of Fuglede-Kadison determinants can be used.

Broader Impacts

The PI has prior experience in various outreach activities that has a positive impact to the wider community. While at Cornell the PI volunteered in the math club of local high school, as well as in the Math Explorer's Club which is a series of weekend lectures aiming at middle school and high school students. Currently the PI is working in the Undergraduate Research Scholars program with an undergraduate student from non math background.

The project involves questions at varying levels of abstraction and therefore should be useful for integration into teaching and outreach activities. Furthermore, algorithms obtained in this project has potential real life applications in areas like machine learning and numerical analysis, and has the potential of furthering partnership between academia and industries. The mentoring of undergraduate and graduate students, as well as summer dynamics workshop, would be helpful for the academic and career development of math students and contribute to building a more diverse and competitive STEM workforce. The other teaching and outreach activities proposed will also increase mathematical literacy among the general public, as well as improve STEM education, particularly in underserved communities.

Results from Prior NSF Support

The PI hasn't received any prior NSF support.

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