Project Summary

The research project will be mostly about following two related topics:

- The study of canonical metrics and harmonic forms on metric graphs, simplicial complexes and Riemannian or complex manifolds.
- The study of asymptotic translation length of pseudo-Anosov maps on curve graphs, of train track maps on free factor complexes, as well as related problems.

Some specific questions the PI is trying to answer includes the estimate of minimal translation lengths of pseudo-Anosov maps on curve graphs, estimate of the τ-invariant (which is the capacity of the canonical measure with a kernel defined using Green's function) on compact metric graphs, bounds on the spectral radius of the train track map action on the space of L^2 integrable harmonic 1-forms of infinite graphs, and the convergence of canonical (or Bergman) metrics on coverings of complex manifolds.

Both topics are about the study of group actions on metric spaces, and are related to or motivated by the study of mapping class groups and the geometry of Riemann surfaces. A main motivation for asking these problems is the analogy between surfaces and graphs which can be seen both in the theory of Berkovich spaces and in the theory of Culler-Vogtmann outer spaces.

Intellectual Merit

The groups involved in this project includes the mapping class group, three manifold groups, and the outer automorphism groups of the free group, which are all of interest in geometric group theory. Furthermore, a main motivation for the study of canonical metric of finite metric graphs is the analogy between metric graphs and Riemann surfaces, which is provided concretely by the concept of Berkovich spaces in tropical geometry, and can also be seen in the analogy between outer space and Teichmuller spaces. Our project will result in more examples, and deeper understanding, of these interesting connections.

Broader Impacts

The study on metric graphs and closed surfaces can provide many opportunities to engage undergraduate students in both classroom learning and undergraduate research. As an example, the PI use some of the results related to this project as examples in an introductory short course on L^2-techniques in the KAIX summer school in 2019 with an audience of undergraduate and lower level graduate students. Furthermore, concepts like canonical metric, L^2 integrable harmonic forms, and the relationship between topological and geometric quantities and group action, have potential applications on topological and geometric methods in data analysis and may provide new insights or techniques for dimension reduction, feature selection, and hypothesis testing.

Project Description

1 Introduction

1.1 Motivation and background

This research project is built upon the PI's prior works [4, 38, 3], and can be roughly divided into the following two types of questions:

- Questions on finite metric graphs that are inspired by known properties of Riemann surfaces.
- Questions related to the asymptotic translation length of pseudo-Anosov maps on curve complexes.

The project are primarily motivated by the following three ideas: the analogy between Riemann surfaces and finite metric graphs; the Masur-Minsky theory on curve graphs and mapping class group action on them; the question of discrete uniformization in the case of graphs and other simplicial complexes, which we will explain in more details below:

1.1.1 Analogy between surfaces and graphs

There is a well studied anology between compact Riemann surfaces and metric graphs which has results in many interesting and important results in topology, dynamics and number theory. For example, because graphs and surfaces with hyperbolic structure are both Eilenberg–MacLane spaces, the outer automorphism groups of free groups are the group of homotopy equivalences from a graph to itself, while the outer automorphism groups of surface groups are the group of homotopy equivalences from a surface to itself. Furthermore, Culler and Vogtmann [16] proposed the concept of outer space which serves the same purpose of Teichmüller space for the study of $Out(F_n)$. More specifically, the outer space for $Out(F_n)$ consists of all finite metric graphs with n loops and no degree 1 vertices, together with a marking map which is a homotopy equivalence between this metric graph and the wedge of n circles, just as the Teichmuller surface for closed surfaces of genus g consists of Riemann surfaces of genus g together with a marking map, which is a homotopy equivalence from this surface to a fixed "model surface" of genus g. Both spaces are contractable, and just as the mapping class group acts on Teichümuller space via precomposition with the marking map, $Out(F_n)$ acts on the Culler-Vogtmann outer space via precomposition with the marking map. This analogy made it possible to generalize many concepts and properties on Riemann surfaces and Teichmuller spaces, e.g. Nielsen–Thurston classification, train tracks, measured laminations, Teichmuller metrics, and curve complexes. Also, by reducing the problem to the topology and geometry of outer space, this analogy provides a new and fruitful approach to study the algebraic and topological properties of the outer automorphism group which is a cell complex consisting of simplicial cells and are assigned an asymmetric metric, resulting in many important results by Bestvina, Feign, Handel, Dowdall, Kapovich, Leininger (cf. [11, 10, 18]) etc.

In algebraic geometry and number theory, there is also a useful analogy between Riemann surfaces and metric graphs which happens e.g. in the setting of tropical geometry and Berkovich spaces (cf. [8]). Just as Riemann surfaces can be seen as analytic spaces of complex algebraic curves, the Berkovich spaces of dimension one serve as an analytic space for curves on non-Archimedean algebraic field. They have a natural geometric structure and by collapsing $\mathbb{R}$-trees can be turned into finite metric graphs. Motivated this analogy, it is possible to define the analogy of harmonic functions and maps, Jacobians (which are real tori, unlike the Jacobian of Riemann surfaces), and line bundles, and show that they satisfies many of the same properties as in the case of Riemann surfaces, cf. the works of Baker, Chinberg, Rumely, de Jong, Shokrieh etc. (cf. [28, 6]) The problems in the first half of the project, as well as some of the questions in the second half, are motivated by these two ways to view finite metric graphs as an analogy for compact Riemann surfaces.

1.1.2 Mazur-Minsky theory on curve graphs

As mentioned above, the topology of mapping class group can be studied via its action on the Teichmüller spaces. However, this action has the drawback that it is not an action on hyperbolic spaces. To get hyperbolicity, a possible strategy is to consider the action on the curve graphs. Let S be a closed surface of finite genus. The curve graph $\mathcal{C}(S)$ is a graph where each vertex represent an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. The curve complex is the flag complex obtained from a curve graph. It is easy to see that the mapping class group of S acts on curve graph and curve complex by isometry. Masur-Minsky [31] show that $\mathcal{C}(S)$ is δ-hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserves a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. Further results on the conjugacy problems of mapping class group as well as many other properties can be obtained by showing that the subsurface projection (the projection to the curve graph of a subsurface) gives a hirerachical hyperbolic structure on the mapping class group [32, 7].

Since the pseudo-Anosov elements acts on the curve graph loxodromically, one can define the asymptotic translation lengths of such elements which are always positive real numbers. Furthermore, there is a relationship between this length and the length of the Teichmüller geodesic represented by this pseudo-Anosov element shown in [21]. The behavior of the length of the Teichmüller geodesics corresponding to pseudo-Anosov maps have been extensively studied: for example, asymptotics of the function $l(k, g)$ which is the smallest length of such a geodesic corresponding to a pseudo-Anosov map on a surface of genus g that preserves a dimension-k subspace of the homology is shown in [25], and the Teichmüller geodesic length corresponding to pseudo-Anosovs with the same mapping tori can be encoded by finitely many polynomials as shown in [33]. A main problem in the second half of the project is to show analogous results for the asymptotic translation lengths on curve complexes. Furthermore, if a pseudo-Anosov map corresponds to a short geodesic in the moduli space, it can not belong to any proper normal subgroup of the mapping class groups. Many of the problems in the second half of the project is aiming at showing something similar but for the asymptotic translation length on curve complexes.

The analogy between finite metric graphs and Riemann surface, reviewed in the subsection above also indicates that there might be an analogy for the Masur-Minsky theory for the outer automorphism groups of free groups. In particular, the subsurface projection now becomes subfactor projections, while the analogy of the curve complex could be the free factor complex, free splitting complex, cyclic splitting complex etc. Just as Penner's example provides the optimal asymptotic upper bound on the minimal translation lengths for pseudo-Anosov maps on closed surface of genus g, it is also conjectured that it has an analogy which provides an optimal asymptotic upper bound on the translation lengths on free factor complexes for $Out(F_n)$. Another main goal of the second half of the project is to investigate these analogies and try and generalize the results on curve complex translation lengths to the case of free factor complex or other complexes related to $Out(F_n)$.

1.1.3 Discrete uniformization on metric graphs and simplicial complices

One of the most important classical results in one variable complex analysis is the Riemann uniformization theorem, which states that all Riemann surfaces admits a conformal metric that turns it into a quotient of the sphere S^2, the plane $\mathbb{C}$ or the hyperbolic disc $\mathbb{H}^2$ under a discrete group of isometries. It has long been a question how one may be able to generalize this uniformization to the case of discrete geometrical objects, as such uniformization would both be interesting in mathematics and be useful in many tasks in statistics and machine learning e.g. dimension reduction and manifold reconstruction. In the case of piecewise Eucidean surfaces, there are approaches based on circle packing [36], circle patterns [12, 39], scaling of the metric on triangulations [23], conjugate harmonic functions [37] and many others. A main motivation for the PI and his collaborator's work on limits of canonical metrics on graphs [38] is to devise a similar uniformization for metric graphs. The idea is that since the canonical (Arakelov) metric of finite normal covers of closed Riemann surface converges to a multiple of the hyperbolic metric when the covers get close to the universal cover, one may use the limit of the canonical metrics of normal covers of a finite metric graph, as the covers get close to the universal cover, as the analogy for hyperbolic metric in the case of graphs. However, this analogy has the issue that if one starts with the new, limiting metric and go through the same process, one generally get another metric. One of the goal of the project is to see if repeating the process of taking limit of canonical metrics on covers would result in convergence to a fixed point, and what kind of geometric properties the new metric would satisfy. The approach of [38, 4] can also be generalized to other settings, including piecewise Euclidean surfaces, so it would also be an interesting problem to compare the result from [4] with results from other approaches of discrete uniformization in that setting.

1.2 Summary of main questions and prior results

The questions the PI and his collaborators will work on regarding finite metric graphs are motivated by the following theorems by Kazhdan and McMullen:

Theorem 1. [26] Let $S \leftarrow S_1 \leftarrow S_2 \ldots$ be a tower of finite regular covers of a compact Riemann surface S, $\cap_i \pi_1(S_i) = \{1\}$, then the canonical metric on S_i are pullbacks of metric d_i defined on S, and d_i converges uniformly to a multiple of the hyperbolic metric. Here, the canonical metric on a compact Riemann surface S is defined as:

$$\|v\| = \sup_{\|w\|=1, w \text{ holomorphic 1-form on } S} w(v)$$

Where v is any tangent vector.

Theorem 7. [34] Let ϕ be a pseudo-Anosov map on Riemann surface. Let λ be the stretch factor of ϕ on the invariant measured foliations, ρ be the size of the leading eigenvalue of the action of ϕ on the homology of the surface. Then, either $\lambda = \phi$ when passing to a double cover, or there is some $\epsilon > 0$ so that when passing to any finite cover of the surface, $\lambda - \rho > \epsilon$.

Together with his collaborators, the PI found an analogy of Theorem 1 to the case of the metric graphs:

Theorem 2. [38] Let $G \leftarrow G_1 \leftarrow G_2 \ldots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'. Here the canonical metric of metric graphs are as defined in [43, 6] and depends on Foster's coefficients.

And they also found generalizations of Theorem 1 to other settings. A first goal of this part of the project is to generalize this result to the case of Bergman metric on complex manifolds, and the second goal is to use the ideas and methods used in the proof of Theorem 2 to prove analogous results of Theorem 7 in the case of metric graphs, and a third, related goal is to established the conjectured lower bound of the τ-constant of metric graphs which was raised and partially proved in [15].

The second set of questions the PI will work on in the project concerns with the asymptotic translation lengths of pseudo-Anosov maps on curve complexes. In [31], it is shown that the curve graph is hyperbolic and that the asymptotic translation length of the action of a pseudo-Anosov map on it is always non zero, this asymptotic translation length is further shown to be rational in [13] an algorithm for its calculation is obtained in [42], and in [22] the techniques in [31] is used to obtained a good asymptotic lower bound as genus increases. Other important works on this question has been done by [41, 19, 1, 2] as well as many others.

Furthermore, in [22], an example of a sequence of pseudo-Anosov maps is provided that reaches this asymptotic lower bound. A key feature of this sequence is that they have the same mapping torus and lie on the same fibered cone. This technique is further generalized in [27], and by the PI and his collaborators at [3]. In an upcoming paper, the PI and his collaborators would show that the asymptotic upper bound in [3] is sharp when the fiber cone is of dimension at most 3, and that the techniques in [3] can be used to show that in any 2-dimensional slices, all but finitely many resulting pseudo-Anosov maps are normal generators of the corresponding mapping class group. A project the PI and his collaborators will work on is to strengthen these results, and also to potentially generalize them to the fibered face of mapping tori of graph maps [18, 20] and the induced action on the free factor, free splitting or cyclic splitting complexes.

In Section 2 below, we will review in greater details the various properties on Riemann surfaces the PI and his collaborators will look into, as well as their analogies in the setting of finite metric graphs, some of which have been proven by the PI and his collaborators in their prior works and some are questions they are currently working on. Next, in Section 3, we will outline the prior works the PI and his collaborators has done on the estimation of asymptotic translation length and normal generation on curve complex in various settings, and also outline the improvements and generalizations of these results the PI and his collaborator are currently working on.

2 Questions about metric graphs

2.1 Limit behavior of canonical metric

By a metric graph, we mean an undirected simplicial graph with a positive edge length function l defined on the edges, and where each edge e is made into an interval of length $l(e)$. When the graph has finitely many edges we call it a finite metric graph. The following definitions are standard:

Definition 1. Let G be metric graph.

1. By a 1-form on G we mean a map from the directed edges of G to $\mathbb{R}$, where an edge and its inverse takes on opposite values.

2. Let G be a metric graph with edge set $E(G)$, w a 1-form, we define the norm of w as $\|w\| = (\sum_{e\in E(G)} w(e)^2/l(e))^{1/2}$. We call a 1-form L^2 if it has a finite norm.

3. We call a 1-form harmonic, if for every vertex v, for all the outgoing edges e_i from v, $\sum_i w(e_i)/l(e_i) = 0$.

It is easy to see that L^2 integrable 1-forms form a Hilbert space under the norm, and that L^2 integrable harmonic 1-forms is a closed subspace of it.

The following definition has previously appear in [43, 6] and is stated in a way analogous to the Bergman metric in [34, Alppendix]:

Definition 2. The canonical, or Bergman, metric on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

As a comparison, the canonical metric on Riemann surface is defined in [34, Alppendix] as follows:

Definition 3. Let S be a closed Riemann surface, the canonical metric is a Riemannian metric where the length of any tangent vector v is defined as

$$\|v\| = \sup_{\|w\|=1, w \text{ holomorphic } 1-form} \|w(e)\| \ .$$

From the definition, one see immediately that in both the case of Riemann surfaces and metric graphs, if there is a regular cover $X \leftarrow X'$ then the canonical metric on X' is the pull back of some other metric on X.

A classical result on the canonical metric on Riemann surfaces by Kazhdan [26], which is also described in [34, Alppendix], is the following:

Theorem 1. [26] Let $S \leftarrow S_1 \leftarrow S_2 \ldots$ be a tower of finite regular covers of a compact Riemann surface S, $\cap_i \pi_1(S_i) = \{1\}$, then the canonical metric on S_i are pullbacks of metric d_i defined on S, and d_i converges uniformly to a multiple of the hyperbolic metric.

In [38], the PI and his collaborators show the analogy of the previous theorem on the case of metric graphs as follows:

Theorem 2. [38] Let $G \leftarrow G_1 \leftarrow G_2 \ldots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.

In the case when $\cap_i \pi_1(G_i)$ is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of G.

The key ingredient in the proof of the theorem is Lück's approximation theorem:

Theorem 3. [29] Let X be a cell complex, G a group acting on it geometrically and cellularly, let G_i be a sequence of normal subgroups of G of finite index such that $G_{i+1} \subseteq G_i$ and $\cap_i G_i = \{1\}$. Then, the L^2 betti numbers of X seen as a G-space $b_{d,L^2}(X)$ satisfies:

$$b_{d,L^2}(X) = \lim_{i \to \infty} \frac{b_d(X/G_i)}{[G : G_i]}$$

The approach to the proof of Theorem 2 allow for easy generalization to other settings. For instance, in [4], the PI and his collaborators showed that:

Theorem 4. [4] Let $S \leftarrow S_1 \leftarrow S_2 \ldots$ be a tower of finite regular covers of a compact Riemann surface S, then the canonical metric on S_i are pullbacks of metric d_i defined on S, and d_i converges uniformly.

One can further define the concept of canonical metric to more general manifolds or simplicial complexes:

Definition 4. Let M be a Riemannian manifold.

1. We define the canonical metric $\|\cdot\|_{can}$ on M as follows: for any tangent vector v, let

$$\|v\|_{can} = \sup_{w \text{ harmonic } 1-form, \|w\|=1} |w(v)|$$

2. [4] We define the d-th canonical measure on M as:

$$\mu_{can}(A) = \int_A \sum_j \|w_j\|^2 dA$$

Where $\{w_j\}$ is an orthonormal basis of the space of L^2 integrable harmonic d-forms, and dA is the area form from the Riemannian metric.

Definition 5. Let X be a metrized simplicial complex (a simplicial complex where each face has the geometry of a simplex in Euclidean space and the boundary maps are isometries).

1. A k-form is a map sending oriented k-faces to $\mathbb{R}$, so that when one reverse the orientation the image is the opposite. In other words, it is an element in the d-th simplicial cochain $C^k(X)$.

2. The norm on the space of d form w on X is defined as $\sum_F w(F)^2/Vol(F)$, where $Vol(\cdot)$ is the volume under Euclidean metric, and the summation goes through all faces of dimension k.

3. Let d be the coboundary map, δ be its dual under the norm defined above, then a k-form w is called harmonic iff $\Delta w = (\delta d + d\delta)w = 0$.

4. The k-th canonical measure m_k of X is a function on k-faces of X defined as

$$m_k(F) = \sum_i w_i(F)^2/Vol(F)$$

   Where $\{w_i\}$ is a orthnormal basis of the space of L^2 integrable harmonic k forms.

We indicated in [4], though not stated explicitly, that:

Theorem 5. Let M be a compact Riemannian manifold or finite metrized simplicial complex, $M \leftarrow M_1 \leftarrow M_2 \ldots$ a tower of finite regular covers of M, then the k-canonical measure on M_i are lifts of measures on M and these measures converges to a measure whose pullback on M' is the canonical measure there. Here, $M \leftarrow M'$ is a regular cover and $\pi_1(M') = \cap_i \pi_1(M_i)$.

With some analysis one can strengthen the above result in the case $k = 1$, and M being a Riemannian manifold, into the following:

Theorem 6. Let M be a compact Riemannian manifold, $M \leftarrow M_1 \leftarrow M_2 \ldots$ a tower of finite regular covers of M, then the k-canonical measure on M_i are lifts of Riemannian metrics on M and these metrics converges uniformly to a metric whose pullback on M' is the canonical metric there. Here, $M \leftarrow M'$ is a regular cover and $\pi_1(M') = \cap_i \pi_1(M_i)$.

A main motivation for these convergence results is that the limit of canonical metric can be seen as a generalization of the hyperbolic metric on the unit disc. However, this analogy is not perfect as if one start with the limiting metric, lift it to a sequence of regular covers that converges to the universal cover, calculate the corresponding canonical metrics and take limit, the result is not always the same as the limiting metric one starts with. However, Huiping Pan and the PI observed that if one repeat this process the metric usually converges to a fixed point.

Together with his collaborators, the PI is trying to study the following questions:

Question 6. Can one give explicit calculations for the limiting metric in Theorem 2 and 4 when $\cap_i \pi_1 \neq \{1\}$? More specifically, what if the covers involved are all abelian? What if the surface covers converges to the Schottky cover?

Question 7. What's the limiting measure, or metric, for Theorem 5 and Theorem 6? If one lift the limiting measure or metric to the sequence of covers and carry out the process again and again, does the limit exist? Is it unique?

Question 8. Can the Theorems be generalized to the case of certain classes of branched cover where the deck group acts transitively?

Definition 9. Let M be a complex manifold of complex dimension n. The Bergman volume form on M is defined as:

$$\sum_i w_i \wedge \overline{w_i}$$

Where $\{w_i\}$ is an orthornormal basis of the space of L^2 integrable n-forms on M.

Question 10. Is there convergence result for the Bergman volume form defined above?

2.2 Spectral radius of maps on L^2 harmonic forms

The concept of train-track maps are between graphs are first proposed in [11] and they play an important role in the study of the geometric of outer spaces and the properties of outer automorphism groups of free groups.

Definition 11. Let $\phi : G \to G$ be a map from finite graph G to itself which is a homotopy equivalent and where vertices are sent to vertices. Let $E(G)$ be the set of edges and $V(G)$ the set of vertices.

1. The incidence matrix M_ϕ is a $|E(G)| \times |E(G)|$ square matrix, where the i, j-th entry is the number of times for $\phi(e_i)$ to ge through e_j, regardless of the direction.
2. [11] We call ϕ a traintrack map if the image of any edge under ϕ^n for any n is a path without back-tracking, i.e. an immersion of an interval to the graph G.

In this section it is further assumed that ϕ has an irreducible incidence matrix, i.e. there is some n so that the image of every edge under ϕ^n covers the whole graph G.

A related concept is the invariant laminations under the train track map:

Definition 12. [10] Let $\phi : G \to G$ be a train track map. An invariant lamination of ϕ is a immersion $i : \mathbb{R} \to G$ so that there is a linear map f on $\mathbb{R}$ with slope larger than 1, so that it is conjugate to ϕ by i.

When ϕ is a traintrack with irreducible incidence matrix it is easy to see, for example by Perron-Frobenuous, that the invariant lamination exists and is unique, and also when one consider increasingly large intervals I_n on $\mathbb{R}$ the number of times they passes through each edge converges in the projectivization of $\mathbb{R}^{E(G)}$. We call this limit the weight of the lamination on each edge. Let λ be the slope of the linear map f in the definition of the invariant lamination, it is evident that it is the leading eigenvalue, and also the spectral radius, of the incidence matrix, which we call the stretch factor of ϕ.

There is a conjecture on train track maps motivated by the following theorem in [34]:

Theorem 7. [34] Let ϕ be a pseudo-Anosov map on Riemann surface. Let λ be the stretch factor of ϕ on the invariant measured foliations, ρ be the size of the leading eigenvalue of the action of ϕ on the homology of the surface. Then, either $\lambda = \phi$ when passing to a double cover, or there is some $\epsilon > 0$ so that when passing to any finite cover of the surface, $\lambda - \rho > \epsilon$.

In particular, the first case happens when the singularities of measured foliations are all of even order, and the latter happens when otherwise.

Motivated by this observation, the PI and his collaborators proposed the following local conditions for invariant laminations of a train track map:

Definition 13. An invariant lamination i of a train track map ψ is called "locally orientable" if for every vertex $v \in V(G)$, there is a partition of edges incident to v into two sets where the sum of weights of the lamination on the edges in these two sets are the same.

The following conjecture is proposed by Koberda:

Conjecture 14. Let ψ be a train track map on a finite graph, with irreducible incidence matrix. Then, either after passing to some finite cover the stretch factor is identical to the spectral radius of the induced map on homology, or there is some $\epsilon > 0$ such that when passing to arbitrary finite cover, the stretch factor is at least ϵ above the spectral radius of the induced map on homology.

Due to the analogy between outer space and Teichmüller space this is saying that along a geodesic on the outer space, the Abel-Jacobi map is either uniformly contracting however one pass to a finite cover, or will become an isometry when one passes to a finite cover. Here, the Jacobian of a metric graph, as well as the Torelli maps, are defined as follows [28, 35]:

Definition 15. Let G be a finite metric graph. Let $\mathcal{H}$ be the space of harmonic 1-forms on G with the inner product structure defined above, then the Jacobian of G is a flat, real tours:

$$J(G) = \mathcal{H}^*/H_1(G; \mathbb{Z})$$

Where each loop γ in G is seen as a functional on $\mathcal{H}$ whose value on each 1-form w is the sum of w evaluated on consecutive edges of γ. The map $G \mapsto J(G)$, from the outer space $CV_{g(G)}$ to the moduli space of real tori with metric, is the graph Torelli map.

The PI and his collaborators expect that non-local orientability of the invariant lamination would be an obstacle for the first case. They are currently trying to answer the following question:

Question 16. Prove Conjecture 14 when the invariant lamination of ψ is not locally orientable.

As a possible intermediate step, motivated by the proof of Theorem 7 in [34], the PI and his collaborators proposed the following related question:

Question 17. Let ψ be a train track map on a finite graph G, with irreducible incidence matrix, and whose invariant lamination is non locally orientable. Give G a metric such that ψ is uniformly expanding by λ. Let $\widetilde{G}$ be the universal cover of G, and $\widetilde{\psi}$ a lift of ψ on this universal cover. Let $\mathcal{H}$ be the Hilbert space of L^2-integrable hamonic 1-forms on G. Is it true that the spectral radius of the self map on $\mathcal{H}$ induced by ψ is less than λ? Can this spectral radius be expressed or estimated by the L^2 torsion of the mapping torus of ψ?

2.3 Capacity of the canonical metric

Using the concept of canonical metric in the previous section, one can define an important invariant of the metric graph τ as follows, cf. [14, 43] and also [17]:

Definition 18. Let G be a finite metric graph.

1. If f is a piecewise linear function, subdivide the edges of G when needed so that f is linear on each edge, then the 1-form *df* is defined as one sending each oriented edge e to $f(e^+) - f(e^-)$ where $e^\pm$ are the head/tail of e.

2. Let p, q be two points on G. The effective resistence $r(p,q)$ between p and q is defined as follows: let f be the unique piecewise linear function on f such that df is harmonic on $G\backslash\{p,q\}$, and for all outgoing edges e_i at p, $\sum_i df(e_i)/l(e_i) = 1$. Then $r(p,q) = |r(p) - r(q)|$.

3. The τ constant of G is defined as the capacity of canonical measure normalized into volume 1 with the kernel $\frac{1}{2}r(\cdot,\cdot)$. Or, alternatively,

$$r(G) = \int_G (f'(x))^2 dx$$

Where dx is the original Lebesgue measure and $f(x) = \frac{1}{2}r(x,q)$ for some $q \in G$.

A problem proposed by Cinkir [15], and the PI and his collaborators are currently working on, is the following:

Question 19. Let G be any finite metric graph, then $\tau(G) \geq \epsilon \sum_e l(e)$ for some universal constant $\epsilon > 0$.

It is conjectured that $\epsilon = 1/108$, which is proved by Cinkir in many cases and also tested and confirmed by the PI and his collaborators in numerical experiments.

An alternative characterization of the τ invariant is the following (cf. [5]): let Δ be the Laplacian operator on the set of functions that are smooth on edges, π be the orthogonal projection to the subspace of functions which averages to 0 under the canonical measure, λ_i be the eigenvalues of $\pi\Delta$, then

$$\tau(G)\sum_e l(e) = \sum_i \frac{1}{\lambda_i}$$

In particular, τ has an lower bound if the minimum of λ_i has an upper bound. Motivated by this, the PI and his collaborators are also investigating the upper bound on the smallest eigenvalue of the analogous operator in the case of hyperbolic surfaces.

3 Questions about pseudo-Anosov maps and curve complexes

3.1 Asymptotic translation length on curve graphs

Let S be a surface of finite type. The curve graph of S, denoted as $\mathcal{C}(S)$, is a graph where the vertices are isotopy classes of simple closed curves on S and there is an edge between two vertices if they have disjoint representations. A metric can be assigned to the curve graph by setting all edge lengths to be 1. The mapping class group of S acts isometrically on S, and:

Definition 20. The asymptotic translation length of a mapping class g on $\mathcal{C} = \mathcal{C}(S)$ is

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$.

It is shown in [31] that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov. Furthermore, the technique in [31] in showing the positivity of $l_{\mathcal{C}}$, which is based on studying the incidence matrix on the induced map on invariant traintracks, have been used by [22, 41, 2] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases.

In [27], a sequence of pseudo-Anosov maps in different genus are constructed that realized the asymptotic lower bound. The PI and his collaborators generalized the argument in [27] and proved the following:

Definition 21. [40]

- Let M be a hyperbolic 3-manifold, for every integer cohomology class $\alpha \in H^1(M;\mathbb{Z})$, the *Thurston norm* of α is defined as

$$\|\alpha\| = \min_S \max\{0, -\chi(S_i)\}$$

  Where $S = \bigcup_i S_i$ is an embedded surface that represents α. The Thurston norm can be extended to $H^1(M;\mathbb{R})$ as a piecewise linear function with rational coefficients.

- If M is homeomorphic to a mapping torus of a surface map ϕ (seen as a surface bundle over the circle), let the cohomology class associated with ϕ, denoted as α_ϕ, be the pullback of the generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called *fibered cones*, and the map associated with a primitive integer class α in the fibered cone are denoted as ϕ_α.

Theorem 8. [3] Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

In an upcoming paper, the PI and his collaborators will show that this asymptotic upper bound is sharp when $r \leq 3$.

There is also the question on how constraints on the dimension of invariant cohomology may affect the minimal asymptotic translation length. For example, in [2], it is shown that

Theorem 9. [2] The minimal asymptotic translation length on the curve graph of a pseudo-Anosov map in the Torelli group for closed surfaces of genus g grows at $O(1/g)$ as genus g increases.

The PI and his collaborators are able to get the following:

Theorem 10. Let $L_c(k, g)$ be the minimal asymptotic translation length on curve graph of a pseudo-Anosov map on closed surface of genus at most g that preserved a subset of cohomology of dimension at least k. Then

$$\frac{1}{g(2g-k+1)} \lesssim L_c(k,g) \lesssim \frac{k+1}{g^2}$$

The PI and his collaborators will work on the following further questions:

Question 22. Is the bound in Theorem 8 sharp for larger r? Can this be generalized to 3-manifolds with torus boundaries? Can Theorem 10 be improved to make the asymptotic upper and lower bounds identical?

3.2 Normal generation

In an upcoming paper, the PI and his collaborators uses techniques similar to [3] to showed the following:

Theorem 11. Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.

The PI and his collaborators are working on the following question:

Question 23. Can the condition that L has dimension 2 be removed from Theorem 11?

3.3 Generalization to outer space

There is an analogy between mapping class groups which are the groups of homotopy equivalences from closed surfaces to themselves, and outer automorphism groups of the free groups, which are the groups of homotopy equivalences from finite graphs to themselves, c.f. [11, 10]. Under this analogy between closed surfaces and finite graphs, we have a concept of the Thurston norms for mapping tori of graph maps which is described in [20], and there is also a concept of fibered cones on the first cohomology of these mapping tori described in [18]:

Definition 24. [18] Suppose $\phi \in Out(F_n)$ has an irreducible, expanding train track representation, let γ_ϕ be a folding path of ϕ in the Culler-Vogtmann outer space. The *folded mapping torus* is a 2-d cell complex built from γ_ϕ, which has a surjection over the circle and the fibers are the graphs in the folding path. The orientation on the circle gives an orientation of all 1-cells in this cell complex. An analogy for the fibered face containing ϕ is $\mathcal{A}$ which consists of first cohomology classes of this cell complex which are positive on all positive cells.

Similarly, there are many analogies for the concept of curve complex for finite graphs, e.g. the free splitting complex [24], free factor complex [9], and the cyclic splitting complex [30]:

Definition 25. [24, 9, 30]

1. The free factor complex of a free group F_n is a simplicial complex whose vertices are proper free factors of F_n, and k vertices associated with free factors A_i form a simplex iff for any pair i, j, $A_i \subset A_j$ or $A_j \subset A_i$.
2. A free splitting over F_n is a minimal, simplicial action of F_n on a simplicial tree with trivial edge stablizer. The free splitting complex of F_n is a simplicial complex whose vertices are free splittings of F_n and k vertices form a simplex if for any two free splittings among them one can be obtained from another via equivariant edge collapsing.
3. The cyclic splitting complex has the same vertices as the free splitting complex, but k vertices form a simplex if for any two free splittings among them, either one can be obtained from another with edge collapsing, or after folding an element in the vertex groups of them over a trivial edge they have the same $\mathbb{Z}$-splitting.

A problem the PI and his collaborators are working on is:

Question 26. Is it possible to prove the analogy for the results in Theorems 8, 11 above to the case of finite graphs?

Broader Impact

Much of the results and techniques in this project are accessible to undergraduate students and can hence benefit both classroom teaching and learning and are good materials for undergraduate research. As an example, the PI discussed some of his prior results related to this project in a summer course in KAIX summer school in Daejeon, Korea with an audience consisting of undergraduate and lower level graduate students from Korea, Thailand and Vietnam, and received active participation and feedback from the students. Furthermore, many of the concepts and techniques involved, e.g. harmonic analysis on discrete spaces and the Alexander polynomial, has applications in other areas of science and engineering and the study of them may provide new insights to real life problems like shape reconstruction and topological data analysis.

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Biographical Sketch: Chenxi Wu

Professional Preperaton

- Cornell University, Ithaca NY, Mathematics, PhD 2016
- Peking University, Beijing, China, Mathematics BS 2010

Appointments

- Rutgers University, Hill visiting assistant professor, 2017-
- Max-Planck Institute of Mathematics, Postdoc, 2016-2017

Publications

Publications related to the project

- Hyungryul Baik, Farbod Shokrieh, Chenxi Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019. doi: 10.1515/crelle-2019-0007.
- Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019. doi: 10.1007/s00222-018-0838-5
- Hyungryul Baik, Hyunshik Shin and Chenxi Wu. An upper bound on the asymptotic translation length on the curve graph and fibered faces *arXiv:* 1801.06638

Other significant publications

- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805
- Chenxi Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics*, 214(2), 733-740, 2016. doi: 10.1007/s11856-016-1357-y
- Chenxi Wu. The relative cohomology of abelian covers of the flat pillowcase. *Journal of Modern Dynamics*, 9, 123-140, 2015. doi:10.3934/jmd.2015.9.123
- Hyungryul Baik, Chenxi Wu, KyeongRo Kim, and TaeHyouk Jo. An algorithm to compute Teichmüller polynomial from matrices *Geometriae Dedicata* 2019 doi: 10.1007/s10711-019-00450-4
- Hyungryul Baik, Ahmad Rafiqi and Chenxi Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019. doi: 10.1017/etds.2017.109

Synergistic Activities

- Gave lectures in KAIX Summer School in August 2019
- Volunteered for F.E.M.M.E.S. event at University of Michigan in November 2017
- Gave lectures in Math Explorer's Club at Cornell in March 2016