Fibered cone, sphere complexes and geometry of metric graph

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Free splitting complex of F_n

Let F_n be a free group of n generators. Two ways of defining $FS(F_n)$:

- A simplex is an action of F on a non-trivial tree T with no F-invariant subtree, all edge stablizers trivial. Boundary map means collapsing edges equivariantly.
- Consider all simplicial graphs G of n loops, no valence 1 vertices, minimal and has a marking a.k.a. isomorphism from F_n to its fundamental group. Set total edge length 1, and allow some edge lengths to be 0, then we get a simplicial complex.

We can give a metric on $FS(F_n)$ by setting all edges to have length 1.

Let ψ be an outer automorphism of F_n, then ψ acts on $FS(F_n)$ by isometry, and the stable translation length

$$l(\psi) := \lim\inf_{n\to\infty} \frac{d(v, \psi^n(v))}{n}$$

We found upper bounds for the $l(\psi)$ for a large family of ψ, in particular, we showed that:
Theorem: The smallest non-zero $l(\psi)$ decays at least as fast as n^{-2}.

Thickening and sphere complex

A **doubled handlebody** M_n is the connected sum of n $S^2 \times S^1$. Any $\psi \in Out(F_n)$ induces a homeomorphism from M_n to itself, and the $FF(F_n)$ is the **complex of spheres** in M_n, where simplices are isotopy classes of disjoint spheres, and boundary map is by removing a sphere. The dual graph of the spheres is the graph in the definition of $FF(F_n)$.

Mapping torus and generalized mapping cone

Let M be any compact manifold, f a map from M to itself, $N = M \times [0,1]/(f(x),0) \sim (x,1)$ the mapping cone. Let the elements in N be denoted by (x,t), then dt is a closed 1-form whose Poincare dual is the fiber M, which is an integer point in $H^1(M,\mathbb{R})$. Integer points in nearby directions will also be non-singular like dt hence their dual will be other closed manifolds and they will correspond to other fiberations of N over the circle.
Question: Is there a description of these integer points and their corresponding fibering over the circle?

Definition: Consider the maximal free abelian cover of N, then M is lifted to a free abelian cover $\tilde{M}$ with deck group H, ψ lifted to $\tilde{\psi}$. Let D be a fundamental domain of $\tilde{M}$. Then $\tilde{\psi}^k(D)$ hits finitely many fundamental domains of the form hD, all such h form a subset of H called Ω_k. Consider $\Omega = \cup_k - \Omega_k \times \{k\}$, then the generalized fibered cone is

$$C = \{x \in ((H \oplus \mathbb{Z}) \otimes \mathbb{R})^* = H^1(N, \mathbb{R}) :$$

$$\exists M > 0, (h, t) \in \Omega \implies sign(t)x(h, t) > 0\}$$

Lemma: If $\beta = (h, t)$ is an integer point in C, then the fiber over the circle of N corresponding to C is as follows:

$$M_\beta = \tilde{M}/\beta^\perp, \tilde{\psi_\beta}^t = \tilde{\psi}$$

Main Theorem: If V is a proper subcone of a rational subcone of C, L a rational subspace of $H^1(N)$ intersecting with V at V', then for integer elements β on V', the stable translation length on the sphere complex on M_β is bounded from above by

$$l(\psi_\beta) \leq C/\|\beta\|^{1+1/(\dim(L)-1)}$$

Examples

- Let M be closed surface, then the sphere complex is the curve complex, earlier work by Baik-Shin-W shows that the generalized fibered cone is Thurston's fibered cone, i.e. it is a rational cone. The Main Theorem is a previous result by Baik-Shin-W. This upper bound was known in special cases previously by Gadre-Tsai and Kin-Shin, and is shown to be optimal in some cases by Baik-Kin-Shin-W.
- Let M be doubled handlebody, the generalized fibered cone is the intersection of two dual "McMullen cones" defined by Dowdall-Kapovich-Leininger, hence is also a rational cone. The sphere complex is the free factor complex.
- Let M be handlebody and replace sphere complexes with disk complexes, the Main Theorem still holds, and it gives a new bound on the minimal stable translation length for handlebody groups.

Further questions

- Lower bound?
- Optimal lifting from $Out(F_n)$ to handlebody group.
- Relationship with L^2 torsion.