RESEARCH STATEMENT

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My research is mostly on discrete geometry, geometric group theory and ergodic theory. The followings are some of my current research project:

1. Generalizing various results on Riemann surface to the setting of metric graphs. For example, we have done works on the properties of harmonic forms on metric graphs and graph maps and understanding possible connections to L^2 techniques, and are also looking into possible analysis proofs for the Riemann-Roch on metric graphs.
2. Studying the Galois conjugate of exponent of entropies of post-critically finite interval maps and maps on Hubbard trees induced by complex polynomial maps. When the interval map is unimodal and has periodic critical orbit, we provided a full characterization of the closure of the set $\{(z, \lambda)\}$ where λ is the exponent of a possible entropies and z a Galois conjugate of λ (called "Thurston's teapot"), and found connection with the study of iterated function system. We also managed to generalize this characterization to more general interval maps as well as maps on quadratic Hubbard trees, and in the process we proposed some more general conjectures on generalizing Mandelbrot-Julia correspondence to iterated function systems with parameters.
3. Studying the asymptotic translation length on curve graph induced by surface homeomorphisms, as well as on free factor, or free splitting complexes induced by $Out(F_n)$ action, which can be seen as a way to characterize the "combinatorial complexity" of these maps. The main question we are looking at is trying to understand the relationship of translation lengths of surface or graph maps that have the same mapping torus, and the main tool we use is McMullen's polynomial which gives a description of the entropies of such maps.

1. Generalizing various results on Riemann surfaces to metric graphs

With Farbod Shokrieh, we generalized a classical result of Kazhdan on the convergence of canonical/Arakelov metric under coverings of Riemann surface ([K], also cf. [M, Appendix]) to the setting of finite metric graphs and proves the following:

Theorem 1. [SW] *Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_1(G_i)$, then the limiting metric pulls back to the canonical metric on G'. Here, the canonical metric on a metric graph G is defined as $d_{can}(e) = \frac{1}{l(e)} \sup_{\|w\| \leq 1, w \in \mathcal{H}^1(G)} w(e)$, where $\mathcal{H}^1$ is the space of harmonic 1-forms (1-forms which are locally the differential of graph harmonic functons) on G*

Due to the analogy with the Riemann surface situation, the limiting metric can be seen as some kind of "discrete" uniformization metric. Here by "uniformization metric" of a compact Riemann surface we mean the metric of constant curvature defined on its universal cover.

Furthermore, we proved many generalizations of this statement, for example, to the case of triangulated surfaces with piecewise Euclidean structure on each simplex, which gives interesting insights to the problem of discrete uniformization.

At the moment, Farbod Shokrieh and I are working on a conjecture by Thomas Koberda, which is a generaliztion of the main theorem of [M] to metric graphs, and says that for any irreducible train track map ϕ, the gap between its stretch factor ρ and the spectral radius of the induced map on first homology is either bounded away from 0 under all finite covers, or disappears under some finite cover.

Also, Farbod Shokieh and I are working on finding generalizations of the Riemann-Roch theorem on metric graphs [BN], as well as new proofs of it that are more based on analysis, for example via heat equations.

2. Thurston's "master teapot" and related results

The dynamics of unimodal maps, as well as other continuous maps, on a finite interval is a classical subject that has been extensively studied. A main tool for studying interval maps is the kneading theory of Milnor-Thurston, which says that the dynamic of such a map is semiconjugate to both a real quadratic map and a subshift, and [Ti] developed this idea and found connections between the study of Galois conjugates of entropies of interval maps with the study of iterated function systems.

In [T], to study the number-theoretic properties of the entropies of 1-dimensional dynamical systems, Thurston proposed the "master teapot" which is a closed subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, defined as follows:

$$T = \overline{\{(z, \lambda) : z \text{ is a Galois conjugate of } \lambda = e^h, h \text{ is the entropy of a unimodal map with periodic critical orbit}\}}$$

In [BDLW, LW], Harrison Bray, Diana Davis, Kathryn Lindsey and I gave a full characterization of this set and an algorithm to test if a point does not belong to it. For example, the characterization for $T \cap \{(z, \lambda) : |z| < 1\}$ is as follows: we found shift-invariant subset $S_\lambda \subset \{0, 1\}^{\mathcal{N}}$ such that if $\lambda < \lambda'$ then $S_\lambda \subseteq S_{\lambda'}$, and for any $|z| < 1$, $\lambda \in [1, 2]$, $(z, \lambda) \in T$ iff there is some $w = (w_1, w_2, \dots) \in S_\lambda$ such that $\lim_{n\to\infty} f^z_{w_1}(f^z_{w_2}(\dots(1)\dots)) = 1$ where $f^z_0(x) = xz$ and $f^z_1(x) = 2 - xz$.

As a consequence, we showed many interesting properties of the set T, for example, if $(z, \lambda) \in T$ and $|z| < 1$ then so is (z, y) for all $y \in [\lambda, 1]$. Also as an application, this gives a necessary condition for an algebraic integer to be the exponent of entropy of a unimodal map with periodic critical orbit.

In upcoming papers with Kathryn Lindsey and Giulio Tiozzo, we are able to generalize the result in [BDLW, LW] to interval maps with larger λ, as well as to maps on quadratic Hubbard trees that lie in a principal vein of the Mandelbrot set via the "combinatorial surgery" introduced by Tiozzo.

(i) During numerical experiments with T Kathryn and I found interesting Julia-Mandelbrot like similarities in this shape. Hence, we propose the following analogy of Julia-Mandelbrot correspondance for the teapot:

Conjecture 2.1. Let $M_\lambda = \{z : |z| < 1, (z, \lambda) \in T\}$ which is the "Mandelbrot set". For every point $z \in M_\lambda$, a neighborhood of z is asymptotically similar to a neighborhood of 1 in the "filled Julia set" $J_z = \{\lim_{n\to\infty} f^z_{w_1}(f^z_{w_2}(\dots(1)\dots)) : w \in S_\lambda\}$.

An application is that this would allow us to understand the fractal-like structure of T. Also, we hope that by studying this conjecture we can also have deeper understanding of possible generalization of Julia-Mandelbrot correspondence to more general iterated function system.

(ii) Also motivated by the numerical experiment, Kathryn and I want to see what kind of shapes can be approximated by the "Mandelbrot set" of some iterated function system.

3. Asymptotic translation length on curve complexes and related results

Let S be a surface of finite type. The *curve graph* of S, denoted as $\mathcal{C}(S)$, is the graph where the vertices are isotopy classes of simple closed curves on S and there is an edge between two vertices if they have disjoint representations. A metric can be assigned to the curve graph by setting all edge lengths to be 1. The mapping class group of S acts isometrically on S, and the *asymptotic translation length* of a mapping class g on $\mathcal{C} = \mathcal{C}(S)$ is

$$\ell_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$. $l_{\mathcal{C}}$ can be seen as a possible way to characterize the "combinatorial complexity" of the pseudo-Anosov map.

In [KS], a sequence of pseudo-Anosov maps in different genus is constructed that realizes the asymptotic lower bound. Generalizing the construction in [KS], Hyungryul Baik, Hyunshik Shin and I proved the following:

Theorem 2. [BShiW] *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone* [T2] *in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M; \mathbb{Z})$, let ϕ_α be the monodromy corresponding to the fibering of M over the circle constructed from α, then*

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))} .$$

In a follow-up paper [BKSW], Hyungryul Baik, Eiko Kin, Hyunshik Shin, and I showed that this asymptotic upper bound is sharp when $r \leq 3$.

Furthermore, in [BKSW], we proved the following:

Theorem 3. [BKSW] *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimension 2 that has non empty intersection with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M; \mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.*

Inspired by suggestions by Nathalie Wahl and Karen Vogtmann, Hyungryul Baik, Dongryul Kim and I were able to get an analogy for [BShiW] in the setting of free factor and free splitting complexes as follows:

Theorem 4. [BKW] *Let L be a rational slice of a proper subcone P' of a "McMullen cone" P (defined in* [DKL]*), passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$, l is the asymptotic translation length on the 1-skeleton of the free factor or free splitting complexes, and $\psi_\beta \in Out(F_{\|\beta\|})$.*

My collaborators and I are working on 3 main problems regarding this project:

(i) Applying these ideas to the study of pseudo-Anosov maps which preserves a subspace of homology of given dimension. We hope to give asymptotic lower and upper bound for the minimal asymptotic translation length for a pseudo-Anosov map on a surface of genus g that preserves a k dimensional subspace of homology. At the moment our upper bound is $O(\frac{k+1}{g^2})$ and lower bound is $O(\frac{1}{g(2g+1-k)})$, we would like to make these two closer to each other.
(ii) Study the lower bound for the asymptotic translation length on free factor and free splitting complexes.
(iii) Study the minimal number of iterations of a train track map to send an edge back to itself, and via this hopefully generalize the result in [BKSW] to other three manifolds and show that the bound in [BShiW] is indeed sharp.

4. Other past and current projects

The following are some of my past and current research projects since grad school:

- Non-uniform discreteness of holonomy vectors on translation surfaces [W].
- In an upcoming paper with John Smillie, we will describe a new Borel-Serre like compactification for the strata of translation surfaces and use it to study the sup norm on strata.
- In an upcoming paper with Kathryn Lindsey and Robert Meyerhoff, we will calculate the maximal degrees of freedom for a large family of neural networks with relu activation.
- I'm currently working with Hyungryul Baik and Sebastian Hensel on connection between left orderability of 3-manifold groups and foliations. [D]

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