1 1.1

2. $u_{xx} + u_{yy} = (2x \cdot \frac{1}{2}(x^2+y^2)^{-1})_x + (2y \cdot \frac{1}{2}(x^2+y^2)^{-1})_y = 2(x^2+y^2)^{-1} - (2x^2+2y^2)\cdot(x^2+y^2)^{-2} = 0.$

4. The statement of this problem is somewhat unclear in whether they mean $(u_{xx})^2 + (u_{yy})^2 = 0$ (the more likely one) or $(u^2)_{xx} + (u^2)_{yy} = 0$, so either interpretation would be considered correct. With the first interpretation it is obvious that all function in the stated form satisfy that $u_{xx} = u_{yy} = 0$. With the second there would need to be additional constraints on a, b, c, d for it to work.

5. The general solution is $u = xF(t) + G(t)$, hence one can let $u = t^2 + x(1-t^2)$.

6. $u_{tt} = (g(x+ct) + g(x-ct))_t = c(g'(x+ct) - g'(x-ct))$, $u_{xx} = c^{-1}(g(x+ct) - g(x-ct))_x = c^{-1}(g'(x+ct) - g'(x-ct))$.

7. $(e^{at}\sin bx)_t = ae^{at}\sin bx$, $(e^{at}\sin bx)_{xx} = -b^2e^{at}\sin bx$, hence $a = -kb^2$.

8. $(u_x)_t = 1 - 3u_x$, hence $u_x = \frac{1}{3} + e^{-3t}f(x)$ for some arbitrary function f, hence $u(x,t) = \frac{x}{3} + e^{-3t}F(x) + G(t)$ for arbitrary function F (which is the anti-derivative of f) and G.

12. To sketch wave profile, pick some k, A, D or c, sketch $u(x,t)$ for different values of t, and if u is complex-valued you can sketch either the real or imaginary part.

Dispersion relations: a) $\omega = -iDk^2$. b) $\omega = \pm ck$. c) $\omega = -k^3$. d) $\omega = k^2$. e) $\omega = ck$.

14. Dispersion relation is $\omega = (-1 + \delta k^2 - k^4)i$ hence this is diffusive. When $\delta = k^2 + 1/k^2$ the solution has growth rate 0. When $k^2 + 1/k^2 > \delta$ the solution decays.

2 1.2

1. From equation (1.7) in the text we have $\frac{d}{dt}\int_a^b uA dx = A\phi|_a - A\phi|_b$. Differentiate with respect to b (or use some other argument, for example as in the textbook), we have $Au_t = -A_x\phi - A\phi_x$, hence $u_t + \phi_x = -A'\phi/A$.

3. By chain rule, $u_x = u_\xi$, $u_t = -cu_\xi + u_\tau$, hence the equation (1.12) becomes $u_\tau = -\lambda u$, hence the general solution is $u = e^{-\lambda\tau}F(\xi) = e^{-\lambda t}F(x - ct)$.

4. $u_t + cu_x = -\lambda u$. If $w = ue^{\lambda t}$, $u = we^{-\lambda t}$ hence $u_t + cu_x = w_te^{-\lambda t} - \lambda we^{-\lambda t} + cw_xe^{-\lambda t}$, $-\lambda u = -\lambda we^{-\lambda t}$, hence $w_t + cw_x = 0$.

5. By method of characteristics $u_t + xtu_x = 0$ has characteristics $x = Ce^{t^2/2}$, hence the general solution is $u = F(xe^{-t^2/2})$. Together with the initial value condition we know that $F = f$ hence $u = f(xe^{-t^2/2})$. The general solution of $u_t + xu_x = e^t$ is $u = e^t + F(xe^{-t})$, so with the initial condition, the solution should be $u = e^t + f(xe^{-t}) - 1$.

6(b). The characteristics are $x = Ct$, and the general solution is $u = e^{-2t}F(x/t)$. Use the initial condition we get $F = e^2 f$, hence $u = e^{-2(t-1)}f(x/t)$.

7. The general solution is $u = e^{-\lambda t}F(x - ct)$. The initial-boundary condition tells us that $F(x) = 0$ for $x > 0$ and $e^{-\lambda t}F(-ct) = g(t)$ for $t > 0$, hence $F(x) = \begin{cases} 0 & x > 0 \\ e^{\lambda x/c}g(x/c) & x \le 0 \end{cases}$.

12. By the method of characteristics, $u(x,t) = F(x-ct)e^{(\alpha t-u)/\beta}$. Set $t=0$ we have $F(x) = f(x)e^{f/\beta}$ hence $u(x,t) = f(x-ct)e^{(\alpha t-u+f(x-ct))/\beta}$.

14. Characteristics are $x = Ce^{-ut}$ hence $u = F(xe^{ut})$. Together with the initial condition we get $u = xe^{ut}$. A solution does not exist for all t. For example, there doesn't exist any u at point $x=t=1$ because $s < e^s$ for all $s \in \mathbb{R}$.

3 1.3

2. $\frac{d}{dt}\int_0^l u^2dx = \int_0^l 2uu_tdx = \int_0^l 2kuu_{xx}dx = 2kuu_x|_0^l - \int_0^l 2k(u_x)^2dx \le 0$, hence $\int_0^l u^2dx \le \int_0^l u_0^2dx$ for $t \ge 0$.

3. Let $w = u - g + (x/l)(h-g)$, then $w(0,t) = w(l,t) = 0$, $u_t = ku_{xx}$ will imply $w_t = kw_{xx} - g' + (x/l)(h'-g')$.

4. The steady state satisfy $0 = ku_{xx} - hu$ and $u(0) = u(1) = 1$, hence $u = \frac{e^{(h/k)^{1/2}(x-1/2)} + e^{(h/k)^{1/2}(1/2-x)}}{e^{(h/k)^{1/2}/2} + e^{-(h/k)^{1/2}/2}}$.

5. $u_t = w_te^{\alpha x-\beta t} - \beta we^{\alpha x-\beta t} = w_te^{\alpha x-\beta t} - \beta u$, $u_x = w_xe^{\alpha x-\beta t} + \alpha u$, $u_{xx} = w_{xx}e^{\alpha x-\beta t} + \alpha w_xe^{\alpha x-\beta t} + \alpha w_xe^{\alpha x-\beta t} + \alpha^2u$, hence $0 = u_t - Du_{xx} + cu_x + \lambda u = (w_t - Dw_{xx})e^{\alpha x-\beta t} + (c-2D\alpha)w_xe^{\alpha x-\beta t} + (\lambda - \beta - D\alpha^2 + c\alpha)u$, so when $\alpha = c/(2D)$ and $\beta = \lambda - D\alpha^2 + c\alpha = \lambda + c^2/(4D)$, $0 = w_t - Dw_{xx}$.

6. The steady state doesn't depend on the initial condition. It is $u = \frac{1}{2k}x(1-x)$.

10. The flux is $Du_x + u^2/2$. Replace $u = \psi_x$ we have $\psi_{xt} = D\psi_{xxx} + \psi_x\psi_{xx}$. Integrate along x we have $\psi_t = D\psi_{xx} + (\psi_x)^2/2 + F(t)$. Replace ψ_t with $\psi_t + \int_0^t F(s)ds$ we can get rid of F. Now let $\psi = -2D\ln v$ we get $-2Dv_t/v = -2D^2(v_{xx}v - (v_x)^2)/v^2 + 2D^2(v_x)^2/v^2$, hence $v_t = Dv_{xx}$.

4 1.4

3. For $u_t = Du_{xx} - cu_x$, the time independent solution satisfies $0 = Du_{xx} - cu_x$. So the solution is $u = C_1 + C_2e^{cx/D}$. For $u_t = Du_{xx} - cu_x + ru$, the time independent case reduces to $0 = Du_{xx} - cu_x + ru$, the characteristic polynomial is $D\lambda^2 - c\lambda + r = 0$ whose roots are $r = \frac{c \pm \sqrt{c^2-4Dr}}{2D}$. Hence, when $c^2 = 4Dr$ the general solution is $u = (C_1 + C_2x)e^{\frac{cx}{2D}}$, when $c^2 > 4Dr$ the general solution is $u = C_1e^{\frac{xc+x\sqrt{c^2-4Dr}}{2D}} + C_2e^{\frac{xc-x\sqrt{c^2-4Dr}}{2D}}$, when $c^2 < 4Dr$ the general solution is $u = C_1e^{\frac{xc}{2D}}\cos(\frac{x\sqrt{4Dr-c^2}}{2D}) + C_2e^{\frac{xc}{2D}}\sin(\frac{x\sqrt{4Dr-c^2}}{2D})$.

9. $u = ax+b$ then $u_{xx} = 0$.

$u = a\ln r + b$ then $u_{xx} + u_{yy} = a(\frac{x}{r^2})_x + a(\frac{y}{r^2})_y = a(\frac{r^2-2x^2+r^2-2y^2}{r^4}) = 0$.

$u = \frac{a}{\rho+b}$ then $u_{xx} + u_{yy} + u_{zz} = a((\frac{x}{\rho^3})_x + (\frac{y}{\rho^3})_y + (\frac{z}{\rho^3})_z) = 0$.

12. (a) $\frac{d}{dt}\int_a^b 2\pi rudr = 2\pi a(-Du_r|_a) - 2\pi b(-Du_r|_b)$. Differentiate on b we get $bu_t|_b = Dbu_{rr}|_b + Du_r|_b$, hence $u_t = Du_{rr} + \frac{D}{r}u_r = D\frac{1}{r}(ru_r)_r$.

(b) $\frac{d}{dt}\int_a^b 4\pi r^2udr = 4\pi a^2(-Du_r|_a) - 4\pi b^2(-Du_r|_b)$. Differentiate on b then you get the differential equation.

5 1.5

1. You can do it however you want, for example, in the 3rd equation on page 51, add a term $-\int_a^b \rho_0 g dx$ to the right.

3. Verification is by chain rule. Sketch $u = \frac{1}{2}(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2})$.

4. The initial condition is $u_n(x,0) = \sin \frac{n\pi x}{l}$, $(u_n)_t(x,0) = 0$. The frequency is $\frac{cn}{2l}$, they decrease as l increases and as c (tension) increases.

5. $\frac{d}{dt}E = \int_0^l (\rho_0 u_t u_{tt} + \tau_0 u_x u_{tx})dx = \tau_0 \int_0^1 (u_t u_{xx} + u_x u_{tx})dx = \tau_0 u_t u_x|_0^l = 0$.

9. $I_x + CV_t + GV = 0$, so $I_{xx} + CV_{xt} + GV_x = 0$. Substitute $V_x = -LI_t + RI$, we get that I satisfy the telegraph equation. The fact that V satisfy telegraph equation follows analogously. When $R = G = 0$ the speed of wave is $(LC)^{-1/2}$.

6 1.7

1. $div(gradu) = div((u_x, u_y, u_x)) = u_{xx} + u_{yy} + u_{zz}$.