# 1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example: $u_y = x$.
Example: $(yu)_y = 1$.

General solution of a PDE.

Formally: PDE: $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde
Linear PDE.
Linear homogeneous PDE.

What are the order and linearality of the following PDEs?
$u_x + u_{yyx} = 1$, $uu_x + u = 0$, $u_x + (x^2 + y^2)u_{yy} = 1$.

Some PDEs we will focus on later:

Heat: $u_t = u_{xx}$: (heat transmission, diffusion)
Laplace: $u_{xx} + u_{yy} = 0$: (static electric field, Newton's gravity, equilibrium of random walk)
Wave: $u_{tt} = u_{xx}$: (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations: $u_{tt} = u_{xx} - ku_{xxxx}$ (stiff string)
Cauchy-Riemann equation: $u_x = v_y$, $u_y = -v_x$

Non-linear PDEs you may see in later classes:
Navier-Stokes
Nonlinear Schrodinger: $iu_t = -\Delta u + k|u|^2 u$
KdV: $u_t + u_{xxx} + 6uu_x = 0$, etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow) $u_t = A\frac{\nabla u}{|\nabla u|} \cdot \nabla u + B|\nabla u|\nabla \cdot \frac{\nabla u}{|\nabla u|}$.

Types of problems:
Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.
Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:
Existence
Uniqueness
Regularity
Continuous dependency on boundary

Typical strategy: integral transform: $(Tu)(y) = \int u(x)K(x,y)dx$, then $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$, assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)
Example: solve $u_{xx} + u_{yy} = 1$, where $u = 0$ on the unit circle.

Example: $u_x = u_t$, $u_x = u_t + 1$.

## 2 9/7 Review of ODE, Advection and Diffusion

Review of ODE & multivatiable calculus topics:

- $u' + p(t)u + q(t) = 0$
- $u''' + Au'' + Bu' + Cu = 0$
- Chain rule: Example: $u_{xx} = u_{tt}$, what happens with change-of-variable $y = x + t$, $w = x - t$?
- Fubini's theorem.
- Differentiating an integral. Example: $\frac{d}{dt}\int_0^{t^2} \sin(ts)ds$.
  Solution: Let $x = t, y = t$, then $\frac{d}{dt}\int_0^{t^2} e^{-ts^2}ds = \frac{d}{dt}\int_0^{x^2} e^{-ys^2}ds = (\int_0^{x^2} e^{-ys^2}ds)_x + (\int_0^{x^2} e^{-ys^2}ds)_y = 2x \cdot e^{-y(x^2)^2} + \int_0^{x^2}(e^{-ys^2})_y ds = 2xe^{-y(x^2)^2} - \int_0^{x^2} s^2 e^{-ys^2}ds = 2te^{-t^5} - \int_0^{t^2} s^2 e^{-ts^2}ds$.
- Example: $u_{tt} = u_{xx} + u_{yy}$, $u(x, y, t) = \sin(x\cos\theta + y\sin\theta + t)$ are solutions, hence $\int_0^{2\pi}\sin(x\cos\theta + y\sin\theta + t)d\theta$ is also a solution.

PDE from conservation laws, 1-dimensional case:

Consider the flow of some material whose total quantity remain unchanged, along a thin tube with section area $A(x)$. Then, conservation means:

$$\frac{d}{dt}\int_a^b u(x,t)A(x)dx = A(a)\phi(a,t) - A(b)\phi(b,t) + \int_a^b f(x,t)A(x)dx$$

$\phi$: *flux*. $f$: *source*.

Differentiate w.r.t. $b$ one gets: $Au_t = -A\phi_x - A'\phi + fA$.

- $\phi = u$: e.g. cars which travels at the same speed, age distribution etc.
- $\phi = -u_x$: heat conduction etc.
- $\phi = u - u_x$: contaminated flow etc.
- $f = -u$: decay.

Relationship with random motion: see $u(\cdot, t)$ as the probability distribution.

Example: $u_t = u_x - u$. Decay vs. "widening".

Example: $u$ has two components (e.g. mass, momentum): wave equation.

## 3 9/12 Method of characteristics

Question: first order linear PDE in 2 dimension: $u_t + fu_x + gu + h = 0$

First consider the case when $g = h = 0$. Recall that for 1st order ODE, there is a concept of *first integral*: the solution of $x'F_x + F_t = 0$ are the level curves of $F(x, t)$. Hence, the level curves of $u$ are exactly the solutions of $x' = f$, which are called *characteristics*.

Example: $u_t = xu_x - u$.

Example: $u_t = u_x + u_y$.

Example: $u_t = \sin tu_x + 1$.

Non-linear advection: $u_t = f(u)u_x$: level curves are straight lines of slope $f(c)$. Breaking time.
Example: $u_t = (1 - u)u_x$.

## 4 9/14 Diffusion, fundamental solutions

Review of method of characteristics: $u_t + cu_x = x$.

Fick's law: $\phi = -Du_x$, which results in $u_t = Du_{xx}$. Simple observation:

1. Steady state solution: $u = ax + b$.
2. Loss of information: should study initial value problem: $u_t = u_{xx}$, $u(x, 0) = f(x)$ on region $t > 0$.
3. Time scale: remains unchanged under $t = c^2t'$, $x = cx'$.
4. Conservation of the "total heat": $\int udx$ remain unchanged.

One could expect solution whose "shape" remain unchanged as one scales as in (3). However the integral in (4) changes under this scaling, so one should expect a factor of $t^{-1/2}$. Let $u = t^{-1/2}v(x^2/t)$, then $v$ can be chosen as $v = Ce^{-s/4}$. One can normalize it into $u = \frac{1}{4\pi Dt}e^{-x^2/4t}$.

This is called the *fundamental solution* of heat equation in one dimension. $\delta$ distribution.

Alternative interpretation of the fundamental solution: discretize, then use central limit theorem. General solution: Convolution.

Fundamental solution of heat equations in higher dimensions?

$u_t = u_x + u_{xx}$

Method of mirrors: IBV problem.