1 1.1

2. $u_{xx} + u_{yy} = (2x \cdot \frac{1}{2}(x^2+y^2)^{-1})_x + (2y \cdot \frac{1}{2}(x^2+y^2)^{-1})_y = 2(x^2+y^2)^{-1} - (2x^2+2y^2) \cdot (x^2+y^2)^{-2} = 0$.

4. The statement of this problem is somewhat unclear in whether they mean $(u_{xx})^2 + (u_{yy})^2 = 0$ (the more likely one) or $(u^2)_{xx} + (u^2)_{yy} = 0$, so either interpretation would be considered correct. With the first interpretation it is obvious that all function in the stated form satisfy that $u_{xx} = u_{yy} = 0$. With the second there would need to be additional constraints on a, b, c, d for it to work.

5. The general solution is $u = xF(t) + G(t)$, hence one can let $u = t^2 + x(1-t^2)$.

6. $u_{tt} = (g(x+ct) + g(x-ct))_t = c(g'(x+ct) - g'(x-ct))$, $u_{xx} = c^{-1}(g(x+ct) - g(x-ct))_x = c^{-1}(g'(x+ct) - g'(x-ct))$.

7. $(e^{at} \sin bx)_t = ae^{at} \sin bx$, $(e^{at} \sin bx)_{xx} = -b^2 e^{at} \sin bx$, hence $a = -kb^2$.

8. $(u_x)_t = 1 - 3u_x$, hence $u_x = \frac{1}{3} + e^{-3t} f(x)$ for some arbitrary function f, hence $u(x,t) = \frac{x}{3} + e^{-3t}F(x) + G(t)$ for arbitrary function F (which is the anti-derivative of f) and G.

12. To sketch wave profile, pick some k, A, D or c, sketch $u(x,t)$ for different values of t, and if u is complex-valued you can sketch either the real or imaginary part.

Dispersion relations: a) $\omega = -iDk^2$. b) $\omega = \pm ck$. c) $\omega = -k^3$. d) $\omega = k^2$. e) $\omega = ck$.

14. Dispersion relation is $\omega = (-1 + \delta k^2 - k^4)i$ hence this is diffusive. When $\delta = k^2 + 1/k^2$ the solution has growth rate 0. When $k^2 + 1/k^2 > \delta$ the solution decays.

2 1.2

1. From equation (1.7) in the text we have $\frac{d}{dt}\int_a^b uA dx = A\phi|_a - A\phi|_b$. Differentiate with respect to b (or use some other argument, for example as in the textbook), we have $Au_t = -A_x\phi - A\phi_x$, hence $u_t + \phi_x = -A'\phi/A$.

3. By chain rule, $u_x = u_\xi$, $u_t = -cu_\xi + u_\tau$, hence the equation (1.12) becomes $u_\tau = -\lambda u$, hence the general solution is $u = e^{-\lambda\tau}F(\xi) = e^{-\lambda t}F(x-ct)$.

4. $u_t + cu_x = -\lambda u$. If $w = ue^{\lambda t}$, $u = we^{-\lambda t}$ hence $u_t + cu_x = w_t e^{-\lambda t} - \lambda w e^{-\lambda t} + cw_x e^{-\lambda t}$, $-\lambda u = -\lambda w e^{-\lambda t}$, hence $w_t + cw_x = 0$.

5. By method of characteristics $u_t + xtu_x = 0$ has characteristics $x = Ce^{t^2/2}$, hence the general solution is $u = F(xe^{-t^2/2})$. Together with the initial value condition we know that $F = f$ hence $u = f(xe^{-t^2/2})$. The general solution of $u_t + xu_x = e^t$ is $u = e^t + F(xe^{-t})$, so with the initial condition, the solution should be $u = e^t + f(xe^{-t}) - 1$.

6(b). The characteristics are $x = Ct$, and the general solution is $u = e^{-2t}F(x/t)$. Use the initial condition we get $F = e^2 f$, hence $u = e^{-2(t-1)}f(x/t)$.

7. The general solution is $u = e^{-\lambda t}F(x-ct)$. The initial-boundary condition tells us that $F(x) = 0$ for $x > 0$ and $e^{-\lambda t}F(-ct) = g(t)$ for $t > 0$, hence $F(x) = \begin{cases} 0 & x > 0 \\ e^{\lambda x/c} g(x/c) & x \le 0 \end{cases}$.

12. By the method of characteristics, $u(x,t) = F(x-ct)e^{(\alpha t-u)/\beta}$. Set $t=0$ we have $F(x) = f(x)e^{f/\beta}$ hence $u(x,t) = f(x-ct)e^{(\alpha t-u+f(x-ct))/\beta}$.

14. Characteristics are $x = Ce^{-ut}$ hence $u = F(xe^{ut})$. Together with the initial condition we get $u = xe^{ut}$. A solution does not exist for all t. For example, there doesn't exist any u at point $x=t=1$ because $s < e^s$ for all $s \in \mathbb{R}$.

3 1.3

2. $\frac{d}{dt}\int_0^l u^2 dx = \int_0^l 2uu_t dx = \int_0^l 2kuu_{xx} dx = 2kuu_x|_0^l - \int_0^l 2k(u_x)^2 dx \le 0$, hence $\int_0^l u^2 dx \le \int_0^l u_0^2 dx$ for $t \ge 0$.

3. Let $w = u - g + (x/l)(h-g)$, then $w(0,t) = w(l,t) = 0$, $u_t = ku_{xx}$ will imply $w_t = kw_{xx} - g' + (x/l)(h'-g')$.

4. The steady state satisfy $0 = ku_{xx} - hu$ and $u(0) = u(1) = 1$, hence $u = \frac{e^{(h/k)^{1/2}(x-1/2)} + e^{(h/k)^{1/2}(1/2-x)}}{e^{(h/k)^{1/2}/2} + e^{-(h/k)^{1/2}/2}}$.

5. $u_t = w_t e^{\alpha x-\beta t} - \beta w e^{\alpha x-\beta t} = w_t e^{\alpha x-\beta t} - \beta u$, $u_x = w_x e^{\alpha x-\beta t} + \alpha u$, $u_{xx} = w_{xx}e^{\alpha x-\beta t} + \alpha w_x e^{\alpha x-\beta t} + \alpha w_x e^{\alpha x-\beta t} + \alpha^2 u$, hence $0 = u_t - Du_{xx} + cu_x + \lambda u = (w_t - Dw_{xx})e^{\alpha x-\beta t} + (c - 2D\alpha)w_x e^{\alpha x-\beta t} + (\lambda - \beta - D\alpha^2 + c\alpha)u$, so when $\alpha = c/(2D)$ and $\beta = \lambda - D\alpha^2 + c\alpha = \lambda + c^2/(4D)$, $0 = w_t - Dw_{xx}$.

6. The steady state doesn't depend on the initial condition. It is $u = \frac{1}{2k}x(1-x)$.

10. The flux is $Du_x + u^2/2$. Replace $u = \psi_x$ we have $\psi_{xt} = D\psi_{xxx} + \psi_x\psi_{xx}$. Integrate along x we have $\psi_t = D\psi_{xx} + (\psi_x)^2/2 + F(t)$. Replace ψ_t with $\psi_t + \int_0^t F(s)ds$ we can get rid of F. Now let $\psi = -2D\ln v$ we get $-2Dv_t/v = -2D^2(v_{xx}v - (v_x)^2)/v^2 + 2D^2(v_x)^2/v^2$, hence $v_t = Dv_{xx}$.

4 1.4

3. For $u_t = Du_{xx} - cu_x$, the time independent solution satisfies $0 = Du_{xx} - cu_x$. So the solution is $u = C_1 + C_2 e^{cx/D}$. For $u_t = Du_{xx} - cu_x + ru$, the time independent case reduces to $0 = Du_{xx} - cu_x + ru$, the characteristic polynomial is $D\lambda^2 - c\lambda + r = 0$ whose roots are $r = \frac{c \pm \sqrt{c^2-4Dr}}{2D}$. Hence, when $c^2 = 4Dr$ the general solution is $u = (C_1 + C_2 x)e^{\frac{cx}{2D}}$, when $c^2 > 4Dr$ the general solution is $u = C_1 e^{\frac{xc + x\sqrt{c^2-4Dr}}{2D}} + C_2 e^{\frac{xc - x\sqrt{c^2-4Dr}}{2D}}$, when $c^2 < 4Dr$ the general solution is $u = C_1 e^{\frac{xc}{2D}}\cos(\frac{x\sqrt{4Dr-c^2}}{2D}) + C_2 e^{\frac{xc}{2D}}\sin(\frac{x\sqrt{4Dr-c^2}}{2D})$.

9. $u = ax + b$ then $u_{xx} = 0$.

$u = a\ln r + b$ then $u_{xx} + u_{yy} = a(\frac{x}{r^2})_x + a(\frac{y}{r^2})_y = a(\frac{r^2 - 2x^2 + r^2 - 2y^2}{r^4}) = 0$.

$u = \frac{a}{\rho + b}$ then $u_{xx} + u_{yy} + u_{zz} = a((\frac{x}{\rho^3})_x + (\frac{y}{\rho^3})_y + (\frac{z}{\rho^3})_z) = 0$.

12. (a) $\frac{d}{dt}\int_a^b 2\pi r u dr = 2\pi a(-Du_r|_a) - 2\pi b(-Du_r|_b)$. Differentiate on b we get $bu_t|_b = Dbu_{rr}|_b + Du_r|_b$, hence $u_t = Du_{rr} + \frac{D}{r}u_r = D\frac{1}{r}(ru_r)_r$.

(b) $\frac{d}{dt}\int_a^b 4\pi r^2 u dr = 4\pi a^2(-Du_r|_a) - 4\pi b^2(-Du_r|_b)$. Differentiate on b then you get the differential equation.

5 1.5

1. You can do it however you want, for example, in the 3rd equation on page 51, add a term $-\int_a^b \rho_0 g dx$ to the right.

3. Verification is by chain rule. Sketch $u = \frac{1}{2}(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2})$.

4. The initial condition is $u_n(x,0) = \sin\frac{n\pi x}{l}$, $(u_n)_t(x,0) = 0$. The frequency is $\frac{cn}{2l}$, they decrease as l increases and as c (tension) increases.

5. $\frac{d}{dt}E = \int_0^l (\rho_0 u_t u_{tt} + \tau_0 u_x u_{tx})dx = \tau_0 \int_0^1 (u_t u_{xx} + u_x u_{tx})dx = \tau_0 u_t u_x|_0^l = 0$.

9. $I_x + CV_t + GV = 0$, so $I_{xx} + CV_{xt} + GV_x = 0$. Substitute $V_x = -LI_t + RI$, we get that I satisfy the telegraph equation. The fact that V satisfy telegraph equation follows analogously. When $R = G = 0$ the speed of wave is $(LC)^{-1/2}$.

6 1.7

1. $div(gradu) = div((u_x, u_y, u_x)) = u_{xx} + u_{yy} + u_{zz}$.

7 Quiz 1:

$u_t + (x+1)u_x = 1$, $u(x,0) = \sin x$.

Characteristics are $x = Ce^t - 1$. Hence $u = t + F((x+1)e^{-t})$, hence $F(x) = \sin(x-1)$ and $u = t + \sin((x+1)e^{-1} - 1)$.

8 1.7

3. This is divergence theorem. The heat generated in Ω equals the heat flowing out at the boundary.

4. Let $\phi = (\phi_1, \phi_2, \phi_3)$, then $div(w\phi) = (w\phi_1)_x + (w\phi_2)_y + (w\phi_3)_z = (w_x\phi_1 + w_y\phi_2 + w_z\phi_3) + w((\phi_1)_x + (\phi_2)_y + (\phi_3)_z) = \phi \cdot gradw + wdiv\phi$. Let $\phi = gradu$ then Green's identity follows from this and the divergence theorem.

5. $\lambda = \frac{\int_\Omega u\Delta u dV}{\int_\Omega u^2 dv} = -\frac{\int_\Omega ||gradu||^2 dV}{\int_\Omega |u|^2 dV} < 0$,

6. Let $w = u + v$ where v is 0 at the boundary, then $\int_\Omega |gradw|^2 dV = \int_\Omega |gradu|^2 dV + \int_\Omega |gradv|^2 dV + 2\int_\Omega gradu \cdot gradv dV$. By 4 and the assumption, the last term is 0, hence $\int_\Omega |gradw|^2 dV \geq \int_\Omega |gradu|^2 dV$.

7. Use $c\rho u_t = div\phi$.

9 1.8

1. Maximum are at $r = 2$, $\theta = \pi/4$, $5\pi/4$, minimum are at $r = 2$, $\theta = 3\pi/4$, $7\pi/4$.

2. $u = (x^2 + y^2)/4 - a^2/4$.

4. The solution is spherical symmetric because the function and the boundary conditions are both spherical symmetric, i.e. $u = u(\rho)$. Hence $\Delta u = 1$ reduces to $u_{\rho\rho} + \frac{2}{\rho} u_\rho = 1$, hence $(\rho^2 u_\rho)_\rho = \rho^2$, hence $\rho^2 u_\rho = \frac{1}{3}\rho^3 + C_1$, $u' = \frac{1}{3}\rho + \frac{C_1}{\rho^2}$, hence $u = \frac{1}{6}\rho^2 - \frac{C_1}{\rho} + C_2$. Apply the boundary condition one gets $u = \frac{1}{6}\rho^2 + \frac{b^3}{3\rho} - \frac{1}{6}a^2 - \frac{b^3}{3a}$.

5. $u = Aatan(x) + B$, solve for constants A and B using the boundary condition.

6. $u = A \log r + B$. $u = \frac{10}{\log 2} \log r$.

8. Use chain rule.

9. $curlE = 0$ implies that such a potential exists. $\Delta V = divgradV = divE = 0$.

10 1.9

1. This is a parabolic equation. $u = F(kx-t) + (x+kt)G(kx-t)$, or you can write it in other equivalent ways.

2. Let $p = 2x + t$, $q = t$, then $u_x = 2u_p$, $u_{xx} = 4u_{pp}$, $u_t = u_p + u_q$, $u_{xt} = 2u_{pp} + 2u_{pq}$, hence the equation becomes $u_p = 4u_{qp}$, hence $u = F(2x+t)e^{t/4} + G(t)$.

3. It is hyperbolic. Under the change of variable, by chain rule, $u_x = \frac{4}{x} u_\tau$, $u_{xx} = \frac{16}{x^2} u_{\tau\tau} - \frac{4}{x^2} u_\tau$, $u_{xt} = \frac{4}{x} u_{\tau\xi} + \frac{4}{x} u_{\tau\tau}$, so $0 = x u_{xx} + 4 u_{xt} = -\frac{4}{x} u_\tau - 16 u_{\tau\xi}$, hence $u = e^{-\xi/4} f(\tau) + g(\xi)$.

4. Use chain rule and product rule.

5. Elliptic. Find the eigenvalues of matrix $\begin{bmatrix} 1 & -3 \\ -3 & 12 \end{bmatrix}$.

6. Parabolic. The general solution calculation is similar to 3 above.

7. a) Elliptic when $xy > 1$ and hyperbolic when $xy < 1$. b) Elliptic.

Midterm 1

1. Solve the following initial or initial/boundary value problems:

(1) $u_t = xu_x$, $u(x,0) = x^2$. Here u is a function of x and t. (25 points)
(2) $u_t + u_x = \sin x$, $u(x,0) = 0$ for $x \geq 0$, $u(0,t) = t$ for $t \geq 0$. Here u is a function of x and t. (15 points)

Answer: (1) General solution is $u = F(xe^t)$, hence $u = x^2 e^{2t}$.
(2) General solution is $u = -\cos x + F(x-t)$, so $u = -\cos x - (x-t) + 1$ when $x \leq t$, and $u = -\cos x + \cos(x-t)$ when $x \geq t$.

2. (1) Find the general solution of $u_{tt} = u_{tx}$. (15 points)
(2) Find the solution of the initial value problem: $u_{tt} = u_{tx}$, $u(x,0) = 0$, $u_t(x,0) = x$. (10 points)

Answer: (1) $u_t = f(x+t)$, so $u = F(x+t) + G(x)$ where F and G are arbitrary functions.
(2) $F(x) + G(x) = 0$, $F'(x) = x$, so $u = \frac{1}{2}(x+t)^2 - x^2$.

3. Consider the 1 dimensional advection-diffusion equation: $u_t = u_x + u_{xx}$.
(1) Use change of coordinate of the form $p = x - Ct$, $q = t$ to reduce it to the 1 dimensional heat equation. (13 points)
(2) Recall that the solution of initial value problem of 1-dimensional heat equation: $v_t = v_{xx}$ when $t > 0, v(x,0) = f(x)$ can be given by the Poisson integral representation:

$$v(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)dy, \text{ where } G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Can you write down the analogous formula for the following initial value problem: $u_t = u_x + u_{xx}$ when $t > 0, u(x,0) = f(x)$? (10 points)
(3) Consider the following problem with periodic boundary condition: $u_t = u_x + u_{xx}$ when $0 < x < 1, u(0,t) = u(1,t), u_x(0,t) = u_x(1,t)$. Show that $I(t) = \int_0^1 u^2(x,t)dx$ is a non-increasing function by calculating $\frac{d}{dt}I$. (7 points)

Answer: (1) $u_t = -Cu_p + u_q$, $u_x = u_p$, $u_{xx} = u_{pp}$, hence when $C = -1$, $u_q = u_{pp}$.
(2) $u(x,t) = \int -\infty^{\infty} f(y)G(x+t-y,t)dy$.
(3) $\frac{d}{dt}I = \int_0^1 2uu_t dx = \int_0^1 2uu_x + 2uu_{xx} dx = u^2|_0^1 + 2uu_x|_0^1 - \int_0^1 2(u_x)^2 dx \leq 0$.

4. Consider the equation $u_{xx} + u_{yy} = x^2 + y^2$ on $\mathbb{R}^2 \backslash (0,0)$. Find all radial symmetric solutions (In other words, all solutions of the form $u(x,y) = g(\sqrt{x^2+y^2}))$. You may want to use the fact that the Laplace operator in polar coordinate (r,θ) is $\Delta = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. (5 points)

Answer: $u_{rr} + u_r/r = r^2$, so $(ru_r)_r = r^3$, $ru_r = A + \frac{1}{4}r^4$, $u_r = A/r + \frac{1}{4}r^3$, and $u(r) = B + A\log r + \frac{1}{16}r^4$.

11 2.1

2. $|u| = |\int_{\mathbb{R}} \phi(y)G(x-y,t)dy| \leq \int_{\mathbb{R}} |\phi(y)G(x-y,t)|dy \leq M \int_{\mathbb{R}} G(x-y)dy = M$, where G is the heat kernel.

3. $u(x_0,t) = \int_{\mathbb{R}} \phi(y)G(x_0-y,t)dy = u_0 \int_0^\infty G(x_0-y,t)dy = u_0 \int_{-\infty}^{x_0} G(s,t)ds = u_0(\int_{-\infty}^{0} G(s,t)ds + \int_0^{x_0} G(s,t)ds)$. We know $\int_{-\infty}^{0} G(s,t)ds = 1/2$, $\int_0^{x_0} G(s,t)ds = \int_0^{x_0/sqrtt} G(s,1)dt$ which converges to 0 as $t \to \infty$, hence $\lim_{t\to\infty} u(x_0,t) = u_0/2 = 1/2$.

12 2.2

3. The solution of the latter Cauchy problem is $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(s)ds$, and the solution of the first Cauchy problem is the partial derivative of the solution of the latter Cauchy problem in t direction which by fundamental theorem of calculus is $\frac{1}{2}(\phi(x-ct)+\phi(x+ct))$.

13 Quiz 2

$u_{tt} = 4u_{xx} + e^{x+t}$, $u_t(x,0) = 0$, $u(x,0) = \sin x$.

Solution: By Dahamel's principle, $u_{tt} = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \int_0^t \frac{1}{4}\int_{x-2t+2s}^{x+2t-2s} e^{r+s} dr ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}\int_0^t e^{x+2t-s} - e^{x-2t+3s} ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(e^{x+2t} - e^{x+t}) - \frac{1}{12}(e^{x+t} - e^{x-2t})$.

14 2.3

3. $|u^1-u^2| = |(\frac{1}{2}(f^1(x-ct)+f^1(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct} g^1(s)ds)-(\frac{1}{2}(f^2(x-ct)+f^2(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct} g^2(s)ds)| \leq |\frac{1}{2}((f^1-f^2)(x-ct)+(f^1-f^2)(x+ct))|+|\frac{1}{2c}\int_{x-ct}^{x+ct}(g^1-g^2)(s)ds| = \delta_1+\delta_2 T$. It shows that this Cauchy problem is stable and well posed.

15 2.4

2. Do odd extension of the initial condition, one gets $u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_0^\infty e^{\frac{-(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}} dy$.

16 2.5

1. By Duhamel's principle, $u(x,t) = \int_0^t \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds d\tau = -\frac{1}{2c}\int_0^t \cos(x+c(t-\tau))-\cos(x-c(t-\tau))d\tau = \frac{1}{2c^2}(\sin(x+ct)+\sin(x-ct)-2\sin(x))$.

17 2.6

4. $\mathcal{L}(\int_0^t f(\tau)d\tau) = \int_0^\infty e^{-st}(\int_0^t f(\tau)d\tau)dt = \int_0^\infty(\int_\tau^\infty e^{-st}dt)f(\tau)d\tau = \frac{1}{s}\int_0^\infty e^{-s\tau}f(\tau)d\tau = \frac{\mathcal{L}(f)}{s}$.