RESEARCH STATEMENT

CHENXI WU

My research has so far been mostly on geometric group theory, low dimension topology, symbolic dynamics, and the geometry of translation surfaces. The followings are my main research projects at the moment:

1. Some geometrical and dynamical problems on finite metric graphs under covering. (Section 1)
2. Estimation of asymptotic translation lengths of surface maps on the curve graph, and related problems. (Section 2)
3. Characterization of the Galois conjugate of entropies of certain intetrval maps with periodic critical orbit. (Section 3)
4. Compactification of the moduli space of flat surfaces into a real orbifold with corners. (Section 4)
5. Invertable Bowen-Series like coding of elements in Kleinian groups, and its application on ergodic theory. (Section 5)

1. Geometry and dynamics on metric graphs

The main motivation of this project is the analogy between Riemann surface and finite metric graphs. For example, both closed surfaces and finite simplicial graphs are Eilenberg-MacLane spaces, hence just like the mapping class group acts on the Teichmüller space, the outer automorphism group of free groups acts on the moduli space of finite metric graph with marking i.e. the Culler-Vogtmann outer space [CV]. Another example of this analogy comes from the concept of Berkovich spaces [B] in tropical geometry, which is a kind of analytic space for algebraic curves over non-Archimedean fields. Motivated by these analogies, it is conceivable that many classical results on Riemann surfaces should have their analogy in the setting of metric graphs, and Farbod Shokrieh and I have been working on proving some of them.

1.1. Canonical metric on graphs.

Let G be a finite simplicial graph and l the length function on its edges. The following definitions are standard:

Definition 1.1.

1. *By a* 1*-form on* G *we mean a map from the directed edges of* G *to* $\mathbb{R}$*, where an edge and its inverse takes on opposite values.*
2. *Let* G *be a metric graph with edge set* $E(G)$*,* w *a* 1*-form, we define the norm of* w *as* $\|w\| = (\sum_{e\in E(G)} w(e)^2/l(e))^{1/2}$*. We call a* 1*-form* L^2 *if it has a finite norm.*
3. *We call a* 1*-form harmonic, if for every vertex* v*, for all the outgoing edges* e_i *from* v*,* $\sum_i w(e_i)/l(e_i) = 0$*.*

4. The canonical, or Arakelov, metric [Z, BF] *on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:*

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

Motivated by the following classical result by Kazhdan:

Theorem 1. [K]*, also cf.* [M, Appendix] *Let $S \leftarrow S_1 \leftarrow S_2 \dots$ be a tower of finite regular covers of a compact Riemann surface S, $\cap_i \pi_1(S_i) = \{1\}$, then the canonical metric on S_i are pullbacks of metric d_i defined on S, and d_i converges uniformly to a multiple of the hyperbolic metric. Here, the canonical metric on a compact Riemann surface S is defined as:*

$$\|v\| = \sup_{\|w\|=1, w \text{ holomorphic 1-form on } S} w(v)$$

Where v is any tangent vector.

Farbod Shokrieh and I proved the following:

Theorem 2. [SW] *Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.*

The main ingredients of the argument is Lück's approximation theorem [LS]. The argument can be easily generalized to other settings e.g. Riemann surfaces, compact Riemannian manifolds or simplicial complexes with piecewise Euclidean structures, with suitably defined concept of canonical metric. One such generalization is outlined in [BSW].

Since for Riemann surfaces, the limit of the canonical metric for a tower of covers that converges to the universal cover is the hyperbolic metric, one may want to see the limit of the canonical metric of a tower of covers as the analogy of the hyperbolic metric for a metric graph. However, it is not known what kind of metrics may be such a limiting metric, and when would the limiting metric be identical to the original metric, although it has been observed by Huiping Pan that if one use the limiting metric as the new l to go through the process of calculating canonical metric on a tower of regular cover again, and repeat this process on and on, eventually the metric will converge. Farbod Shokrieh and I are working on answering these remaining questions, as well as generalizing the result to the case of complex manifolds.

As a related question, there is a tau-invariant defined in [C, CR, Z] which is conjectured to be bounded from below by 1/108 of the the total edge lengths of the graph [C]. Many cases for this conjecture have been shown by Cinkir, and Farbod Shokrieh and I are currently working on proving or disproving this conjecture.

1.2. **Spectral radius of the train track map on homology.** In [M], McMullen proved the following:

Theorem 3. [M] *Let ϕ be a pseudo-Anosov map on Riemann surface. Let λ be the stretch factor of ϕ on the invariant measured foliations, ρ be the size of the leading eigenvalue of the action of ϕ on the homology of the surface. Then, either $\lambda = \phi$ when passing to a double cover, or there is some $\epsilon > 0$ so that when passing to any finite cover of the surface, $\lambda - \rho > \epsilon$.*

It is conjectured by Thomas Koberda that there is an analogous result for graph homotopy equivalences that admits an irreducible train-track representation. More precisely, Let ψ be a train track map on a finite graph, with irreducible incidence matrix. Then, either after passing to some finite cover the stretch factor is identical to the spectral radius of the induced map on homology, or there is some $\epsilon > 0$ such that when passing to arbitrary finite cover, the stretch factor is at least ϵ above the spectral radius of the induced map on homology. Farbod Shokrieh and I are working on proving or disproving this conjecture.

2. Asymptotic translation lengths on curve complexes

Let S be a surface of finite type. The curve graph of S, denoted as $\mathcal{C}(S)$, is a graph where the vertices are isotopy classes of simple closed curves on S and there is an edge between two vertices if they have disjoint representations. A metric can be assigned to the curve graph by setting all edge lengths to be 1. The mapping class group of S acts isometrically on S, and the asymptotic translation length of a mapping class g on $\mathcal{C} = \mathcal{C}(S)$ is

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$. The motivating question of this project is to understand the relationship between the asymptotic translation lengths on curve graphs and group theoretic properties of elements in the mapping class group.

It is shown in [MM] that the curve graph is δ-hyperbolic, and that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov (not of finite order, and doesn't preserve any finite collection of disjoint simple closed curves up to isotopy). Furthermore, the technique in [MM] have been used by [GT, V, BSh] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases.

In [KS], a sequence of pseudo-Anosov maps in different genus are constructed that realized the asymptotic lower bound. The construction arises from an arithmetic sequence in a fibered cone of a hyperbolic 3-manifold. Here, the concept of fibered cones is introduced in [T2]: let M be a closed hyperbolic 3-manifold. If M fibers over a circle as a surface bundle (with possibly disconnected fibers), then the pull-back of the generator of $H^1(S^1;\mathbb{Z})$ is an element in $H^1(M;\mathbb{Z})$. It is proven in [T2] that all integer cohomology classes of M arising from this process are those lying in the interior of some finite collection of rational cones (cones in $H^1(M;\mathbb{R})$ defined by linear inequalities with rational coefficients), which are called the fibered cones.

Generalizing the construction in [KS], Hyungryul Baik, Hyunshik Shin and I proved the following:

Theorem 4. [BShiW] *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone* [T2] *in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,*

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

In an upcoming paper, Hyungryul Baik, Eiko Kin, Hyunshik Shin, and I will show that this asymptotic upper bound is sharp when $r \leq 3$.

There is also the question on how constraints on the dimension of invariant cohomology may affect the minimal asymptotic translation length. For example, in [BSh], it is shown that

Theorem 5. [BSh] *The minimal asymptotic translation length on the curve graph of a pseudo-Anosov map in the Torelli group for closed surfaces of genus g grows at $O(1/g)$ as genus g increases.*

In a second upcoming paper, together with my collaborators, we are able to get the following:

Theorem 6. *Let $L_c(k, g)$ be the minimal asymptotic translation length on curve graph of a pseudo-Anosov map on closed surface of genus at most g that preserved a subset of cohomology of dimension at least k. Then*

$$\frac{1}{g(2g-k+1)} \lesssim L_c(k,g) \lesssim \frac{k+1}{g^2}$$

With my collaborators I am currently working on further strengthening these results, and possibly generalizing them to the case of $Out(F_n)$ action on free factor complex [BFe] or the setting of hierarchical hyperbolic spaces [BHS, MM2].

Furthermore, in the upcoming paper with Baik, Kin and Shin, we proved the following:

Theorem 7. *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.*

We are currently working on removing the condition on dimension of L being 2.

3. Dynamics of interval maps of constant slope

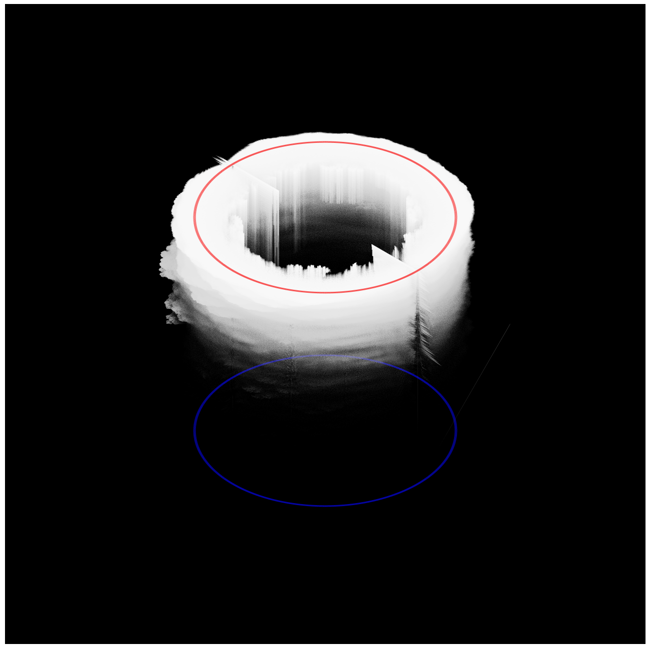
The dynamics of unimodal maps on intervals is a classical subject that has important connections with ergodic theory, symbolic dynamics and complex dynamics. A main tool for studying interval maps is the kneading theory of Milnor-Thurston [MT], which says that the dynamic of such a map is semiconjugate to both a real quadratic map and a subshift. For any $\lambda \in (1,2)$, let the tent map $f_\lambda : [0,1] \to [0,1]$ be defined as

$$f_\lambda(x) = \begin{cases} \lambda x & x \le 1/\lambda \\ 2 - \lambda x & x > 1/\lambda \end{cases}$$

In [T], Thurston proposed the "teapot" which is a closed subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ which is defined as follows:

$$T = \overline{\{(z,\lambda) : z \text{ is a Galois conjugate of } \lambda, \exists n \in \mathbb{N} f_\lambda^{\circ n}(1) = 1\}}$$

This is a picture of the Thurston's Teapot:



Using kneading theory and the idea of "dominant words" in [Ti], Harrison Bray, Diana Davis, Kathryn Lindsey and I showed the following:

Theorem 8. [BDLS] *If $|z| = 1$ then $z \times [1, 2] \subset T$. If $(z, \lambda) \in T$, $|z| \leq 1$, then $z \times [\lambda, 2] \subset T$.*

Furthermore, in an upcoming paper, Kathryn Lindsey and I will show this full characterization of Thurston's teapot:

Definition 3.1.

(i) Let $\lambda \in (1, 2)$. The itinerary of $x \in [0, 1]$ under f_λ, denoted as $it_\lambda(x)$, is the infinite string of 0 and 1, where the i-th entry is 1 iff the $i-1$-th iteration of x under f_λ is in $(1/\lambda, 1]$.
(ii) If s is an infinite string, $\sigma(s)$ is s with the first letter removed.
(iii) If $w = w_1w_2 \dots w_n$ is a finite word of length n consisting of 0 and 1, $F(z, w) = F_z^{w_1} \cdot F_z^{w_2} \dots F_z^{w_n}(1)$, here $F_z^0(x) = x/z$, $F_z^1(x) = (2 - x)/z$.
(iv) We call a finite word w to be λ-suitable, if
  (a) For any $k \in \mathbb{N}$, $\sigma^k(w^\infty)$ is the itinerary of some $x \in [0, 1]$ under f_λ.
  (b) If a suffix of w is the same as a prefix of $it_\lambda(1)$, there are odd number of 1s in this suffix.

Theorem 9. *$(z, \lambda) \in T$ iff one of the following is true:*

(i) $|z| = 1$, $\lambda \in [1, 2]$
(ii) $|z| > 1$, and $1/z$ is a root of the kneading power series corresponding to λ.
(iii) $|z| < 1$, $\lambda \in [\sqrt{2}, 2]$, and there is some $R > 0$, such that for any $\lambda' > \lambda$ there are infinitely many λ'-suitable words w such that $|F(z, w)| < R$
(iv) There is some k such that (z^{2^k}, λ^{2^k}) is in one of the three previous cases.

Kathryn Lindsey and I will be working on generalizing this description to the case of more complicated interval maps. In particular, we will be considering this kind of interval maps (here $\lambda > 1$:

$$h_\lambda = \begin{cases} \lambda x - 2n & x \in [2n/\lambda, (2n+1)/\lambda] \\ 2n + 2 - \lambda x & x \in [(2n+1)/\lambda, (2n+2)/\lambda] \end{cases}$$

Based on numerical evidence, we conjectured that the analogy of the above two theorems should be true for this new set

$$U = \overline{\{(z, \lambda) : z \text{ is a Galois conjugate of } \lambda, \exists n \in \mathbb{N} h_\lambda^{\circ n}(1) = 1\}}$$

Furthermore, Kathryn Lindsey and I will be working on proving the analogy of Julia-Mandebrot correspondance in the setting of Thurston teapots, i.e. give a better explaination of the shape of the part of T and U inside the unit cylinder $\mathbb{D}_1 \times \mathbb{R}$,

4. Compactification of the moduli space of translation surfaces

A finite translation surface is a holomorphic differential on a closed Riemann surface. The set of translation surfaces where the degrees of zeros are fixed form an affine manifold under the "period coordinate", denoted as $\mathcal{H}(k)$, where k is the vector indicating the degrees of the zeros. cf. [Wr]. There is a $GL(2, \mathbb{R})$ action on $\mathcal{H}(k)$ defined by

$$\begin{pmatrix} Real(Aw) \\ Imag(Aw) \end{pmatrix} = A \begin{pmatrix} Real(w) \\ Imag(w) \end{pmatrix}$$

Which is called the "affine action". The action of the subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is called the "horocycle action". The study of these two actions are connected to many questions in dynamics, for example the study of polygonal billiards and interval exchange maps. It is known in the seminal paper [EMM] that affine invariant closed submanifolds of $\mathcal{H}(k)$ are affine submanifolds, and the recent work by Chaika-Smillie-Weiss shows that the horocycle closures are generally not submanifolds at all.

Many approaches have been proposed so far to give a compractification for the moduli space of finite translation surfaces $\mathcal{H}(k)$. For example, the "WYSIWYG" bordification in [MW] has been used to study the dynamics of affine invariant subspaces in such moduli spaces, and there is a compactification proposed in [BCGGM] that is known to have good properties and is particularly useful in the study of the geometry of such moduli spaces. Motivated by the Borel-Serre compactification [BS] and the Bestvina-Horbez compactification of outer space [BH], John Smillie and I are working on a bordification of the moduli space of translation surfaces as a (real) orbifold with corners. A points on this bordifications can be described with the following data:

- A "component diagram", which is a finite graph G with some half edges which is divided into levels, such that every edge between levels is assigned a non-zero integer, every edge within a level is assigned 0, and the half edges are assigned numbers which are the entries of the vector k plus 1.
- A set of meromorphic differentials on possibly disjoint Riemann surfaces assigned to each level of G, such that each connected component of the Riemann surface corresponds to a vertex in G, each zero, pole or marked

point correspond to an edge or a half edge, with turning angles determined by the assigned numbers, and the residue of the poles satisfy certain linear conditions.

- For each pair of matching poles or pole and zero, a gluing map along the boundary created via real oriented blowups.

Together with Prof. Smillie, we are working on understanding the properties of this bordification and use it to study the fundamental group of moduli spaces of translation surfaces as well as the geometry and dynamics of horocycle flows.

5. Bowen-Series type coding for Kleinian groups

Bufetov-Klemenko-Series [BKS] described a coding of elements of a Fuchsian group by admissible sequences of a geometric Markov chain (i.e. paths on a finite directed graph from a given set of vertices (the initial states) to a given set of vertices (the final states)) motivated by Bowen-Series [BowS] and Wroten [Wro], which is bijective and where the finite directed graph admits an involution that exchanges the inital and final states and is compatible with the map $g \mapsto g^{-1}$. They used this coding map to show the a.s. and L^2 convergence of sphereical averages of the Fuchsian group action on functions on Lebesgue probability spaces. Together with Bufetov and Klemenko, we are able to generalize this Wroten-style coding to the case of cocompact Kleinian groups and prove the following:

Theorem 10. *If G is a Kleinian group with a compact, polygonal fundamental domain D, such that the 2-skeleton of the dual complex of the tiling of $\mathbb{H}^3$ by the G-orbit of D is a $C(4)-T(4)-P$ small cancellation complex* [GS], *then elements of G admits a Wroten-style coding with an involution as in* [BKS]. *Let G_D be the graph whose vertices are the faces of D, where two are connected iff the corresponding face are not adjacent. Let F_D be the graph whose vertices are the faces of D, where two are connected iff the corresponding face are adjacent. If G_D is connected, and the 1-neighborhood of every subset of F_D of diameter 1 does not cover all vertices, then the graph representing the geometric Markov chain used in this coding is strongly connected.*

We are working on relaxing the assumptions on G in the theorem above, as well as showing further properties of this coding in order to obtain results on ergodicity as in [BKS].

6. Prior Projects

During grad school and postdoc I have done and finished the following research projects:

- Calculation of the affine diffeomorphism group action on relative homology of abelian covers of the flat pillowcase. [Wu1]
- Calculation of the smallest triangle and virtual triangle of lattice surfaces. [Wu2]
- Non-uniform discreteness of the holonomies of saddle connections. [Wu3]
- Examples of rotational components of infinite flat surfaces. [CRW]
- Construction of pseudo-Anosov maps from interval maps, and calculation of their Teichmüller polynomials. [BRW, BW]
- Estimates on the proportion of bi-Perron numbers being a pseudo-Anosov stretch factor in low genus. [BRW2]

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