1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example: $u_y = x$.
Example: $(xu)_y = 0$.

General solution of a PDE.

Formally: PDE: $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde
Linear PDE.
Linear homogeneous PDE.

What are the order and linearality of the following PDEs?
$u_x + u_{yyx} = 1$, $uu_x + u = 0$, $u_x + (x^2 + y^2)u_{yy} = 1$.

Some PDEs we will focus on later:

Heat: $u_t = u_{xx}$: (heat transmission, diffusion)
Laplace: $u_{xx} + u_{yy} = 0$: (static electric field, Newton's gravity, equilibrium of random walk)
Wave: $u_{tt} = u_{xx}$: (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations: $u_{tt} = u_{xx} - ku_{xxxx}$ (stiff string)
Cauchy-Riemann equation: $u_x = v_y$, $u_y = -v_x$

Non-linear PDEs you may see in later classes:
Navier-Stokes
Nonlinear Schrodinger: $iu_t = -\Delta u + k|u|^2 u$
KdV: $u_t + u_{xxx} + 6uu_x = 0$, etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow) $u_t = A\frac{\nabla u}{|\nabla u|} \cdot \nabla u + B|\nabla u|\nabla \cdot \frac{\nabla u}{|\nabla u|}$.

Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.

Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:
Existence
Uniqueness
Regularity
Continuous dependency on boundary

Typical strategy: integral transform: $(Tu)(y) = \int u(x)K(x,y)dx$, then $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$, assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)
Example: solve $u_{xx} + u_{yy} = 1$, where $u = 0$ on the unit circle.