1 1.1

1. $\begin{bmatrix} 8 & -4 & 20 \\ 12 & 16 & 4 \end{bmatrix}$.

3. $\begin{bmatrix} 6 & -4 & 24 \\ 8 & 10 & -4 \end{bmatrix}$.

5. $\begin{bmatrix} 2 & 4 \\ 0 & 6 \\ -4 & 8 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{bmatrix}$.

17. $A - B$ is undefined.

19. $\begin{bmatrix} 7 & 1 \\ -3 & 0 \\ 3 & -3 \\ 4 & -4 \end{bmatrix}$.

23. $\begin{bmatrix} -7 & -1 \\ 3 & 0 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$.

25. -2.

37-56. (T=True, F=False) TTTFFTFFTFTTTFTTTTTT

71. For example, the zero and identity matrices of size 2×2 and 3×3 are both symmetric.

75. $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$.

79. The (i, i)-th entry of A^T is the same as the (i, i)-th entry of A. By skew-symmetry, it is also the negative of the (i, i)-th entry of A, hence it must be 0.

81. For any 3×3 matrix A, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

82. (a) This is because the (i, i)-th entry of $A + B$ is the sum of the (i, i)-th entry of A and the (i, i)-th entry of B.
(b) This is because the (i, i)-th entry of cA is c times the (i, i)-th entry of A.
(b) This is because the (i, i)-th entry of A^T equals the (i, i)-th entry of A.

2 1.2

1. $\begin{bmatrix} 12 \\ 14 \end{bmatrix}$.

3. $\begin{bmatrix} 11 \\ 0 \\ 10 \end{bmatrix}$.

9. $\begin{bmatrix} as \\ bt \\ cu \end{bmatrix}$.

15. $\begin{bmatrix} 21 \\ 13 \end{bmatrix}$.

17. $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$.

19. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}-1}{2} \\ \frac{3+\sqrt{3}}{2} \end{bmatrix}$.

29. $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

31. u is not a linear combination of elements of $\mathcal{S}$.

35. $u = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

37. The answer is not unique, e.g. $u = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

39. u is not a linear combination of elements of $\mathcal{S}$.

45-63. TFTTTFFFTFTTFTFTFFT

67. $A_\theta(A_\beta v) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} (\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})$

$$= \begin{bmatrix} (\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta))v_1 - (\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta))v_2 \\ (\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta))v_2 + (\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta))v_1 \end{bmatrix} = A_{\theta+\beta}v.$$

68. $A_\theta^T = A_{-\theta}$, hence by 67. both are u.

75. $Au = \begin{bmatrix} a \\ 0 \end{bmatrix}$.

76. $A(Au) = A \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = Au$.

77. Such a vector v must be of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$, hence $Av = \begin{bmatrix} a \\ 0 \end{bmatrix} = v$.

78. $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

3 1.3

1. $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ -3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ -3 & 4 & 1 \end{bmatrix}$.

7. $\begin{bmatrix} 0 & 2 & -4 & 4 & 2 \\ -2 & 6 & 3 & -1 & 1 \\ 1 & -1 & 0 & 2 & -3 \end{bmatrix}$.

9. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ 0 & 4 & 3 & 3 & -5 \\ 0 & 2 & -4 & 4 & 2 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ -2 & 6 & 3 & -1 & 1 \\ 0 & 1 & -2 & 2 & 1 \end{bmatrix}$.

23. Yes.

25. No.

39. $x_1 = 2 + x_2$, x_2 free.

41. $x_1 = 2x_2 + 6$, x_2 free.

43. Inconsistent.

45. $x_1 = 4 + 2x_2$, $x_3 = 1/3$, x_2 free.

47. $x_4 \begin{bmatrix} 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$.

49. $\begin{bmatrix} -3 \\ -4 \\ 5 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

51 $\begin{bmatrix} 6 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

53 Inconsistent.

55. $n - k$, because a variable is either free or basic.

57-76. FFTFTTFTTFTTFTTTFTFT

81. There are 3 cases when the last row is non-zero, 3 when the last row is 0 and the first row isn't, and 1 when the matrix is zero, so 7 in total.

4 1.4

1 $x_1 = -2 - 3x_2$, x_2 free.

3 $x_2 = -5$, $x_1 = 4$.

5 Inconsistent.

7 $x_3 = 2$, $x_1 = 2x_2 - 1$, x_2 free.

11 $x_1 = -3x_2 + x_4 - 4$, $x_3 = 3 - 2x_4$, x_2, x_4 free.

13 Inconsistent.
17 -12.

19 Anything non-zero.

23 By row reduction one gets $\begin{bmatrix} -1 & r & 2 \\ 0 & r^2-9 & 6+2r \end{bmatrix}$. Hence 3.

27 When r is not 2 it has exactly one solution, when r is 2 and s is 15 it has infinitely many solutions, when r is 2 and s is not 15 it has no solution.

35 Rank 3, nullity 1.

37 Rank 2, nullity 3.

43 (a) Mine 1: 10 days, Mine 2: 20 days, Mine 3: 25 days. (b) The system of equations has a unique solution which is not non-negative, hence no.

53-72. TFTTTTFFTTTFFTTFTTFT

74. 0. 0 matrix has rank 0.

75. 4. There can be at most one pivot per row.

76. 4. There can be at most one pivot per column.

77. 3. Because of problem 75.

78. 0. Because of problem 76.

81. No. Do row reduction of A, the last row must be 0. Do the reverse of the row reduction to the vector e_4, then it is a b for which $Ax = b$ has no solution.

82. The rank of A must be n so that there aren't any free variable.

83. It can never have just one solution.

84. (a) $x_1 = 1$, $x_1 = 2$. (b) $x_1 = 1$, $2x_1 = 2$. (c) $x_1 + x_2 = 0$, $2x_1 + 2x_2 = 0$, $3x_1 + 3x_2 = 0$.

87. Yes. Because $A(cu) = c(Au) = c0 = 0$.

88. Yes. Because $A(u + v) = Au + Av = 0 + 0 = 0$.

89. $A(u - v) = Au - Av = b - b = 0$.

90. $A(u + v) = Au + Av = 0 + b = b$.

91. If there is some v so that $Av = b$, then $A(cv) = cb$ hence $Ax = cb$ is consistent.

5 1.6

1. Yes, $-1\begin{bmatrix}1\\0\\1\end{bmatrix}+2\begin{bmatrix}-1\\1\\1\end{bmatrix}+2\begin{bmatrix}1\\1\\3\end{bmatrix}=\begin{bmatrix}-1\\4\\7\end{bmatrix}$.

3. No, write the system of linear equation and you can see that it is inconsistent.

17. This is equivalent to finding r so that the system of equations with augmented matrix $\begin{bmatrix}1 & 01 & 2\\0 & 3 & r\\-1 & 2 & -1\end{bmatrix}$ is consistent. By Gaussian elimination, $r = 3$.

19. Same approach as 17, $r = -6$.

21. No, because for example $\begin{bmatrix}1\\0\end{bmatrix}$ is not in the span.

25. Yes. Form a matrix with the three vectors as columns, do Gaussian elimination, one sees that there is a pivot at each row.

29. Yes. There is a pivot at each row when turn it into row echelon form.

31. No. There is only one pivot in its row echelon form.

39. Use them as columns one sees that there are pivots on the first and third columns. Hence $\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\}$.

43. Same approach as 39. $\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\}$.

The solution of 39 and 43 are not unique. What are other possible answers?

45-64. TTTFTTTFFFTTTTTTTTTT

70. $u+v$ and $u-v$ are both linear combinations of u and v, hence the span of $u+v$ and $u-v$ must be contained in the span of u and v. On the other hand, $u = \frac{1}{2}(u+v)+\frac{1}{2}(u-v)$, $v = \frac{1}{2}(u+v)-\frac{1}{2}(u-v)$, so the span of u and v are contained in the span of $u+v$ and $u-v$.

72. Follow the same argument as 70, use $u_1 = (u_1 + cu_2) - cu_2$.

6 1.7

1. Yes, they are linearly dependent.

5. No.

13. $\{\begin{bmatrix}1\\-2\\3\end{bmatrix}\}$.

15. $\left\{\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}\right\}$.

23. No.

25. Yes.

29. No.

33. $\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

39. -4.

41. -2.

51. $x_2\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

53. $x_2\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 0 \\ 6 \\ 1 \end{bmatrix}$.

57. $x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

63-82. TFTTTTFTFFFTTTFTFTTT

87. If $c_1(u+v) + c_2(u-v) = 0$, because u, v are linearly independent, $c_1 + c_2 = c_1 - c_2 = 0$, hence $c_1 = c_2 = 0$.

89. Same argument as 87.

7 2.1

5. $\begin{bmatrix} 22 \\ -18 \end{bmatrix}$.

7. $\begin{bmatrix} 14 & -2 \\ 21 & -3 \end{bmatrix}$.

9. Undefined.

11. $\begin{bmatrix} 5 & 0 \\ 25 & 20 \end{bmatrix}$.

13. $\begin{bmatrix} 29 & 56 & 23 \\ 7 & 8 & 9 \end{bmatrix}$.

15. Undefined.

17. $\begin{bmatrix} -35 & -30 \\ 45 & 10 \end{bmatrix}$.

19. Undefined.

22. Both are $\begin{bmatrix} 15 & 40 & 5 \\ 115 & 200 & 105 \end{bmatrix}$.

23. Both are $\begin{bmatrix} 5 & 25 \\ 0 & 20 \end{bmatrix}$.

25. $-3 * 0 + (-2) * 1 + 0 * (-2) = -2$.

27. $4 * 3 + 3 * 4 + (-2) * 0 = 24$.

29. $\begin{bmatrix} -4 \\ -9 \\ -2 \end{bmatrix}$.

31. $\begin{bmatrix} 7 \\ 16 \end{bmatrix}$.

33-50; FFFTFFTTFTTTFTFTTT

8 2.3:

1. No.

3. Yes.

9. Use $(A^T)^{-1} = (A^{-1})^T$.

11. Use $(AB)^{-1} = B^{-1}A^{-1}$.

13. Use $(AB^T)^{-1} = (B^{-1})^T A^{-1}$.

17. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

19. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

23. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

25. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$.

31. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Midterm 1

1. Solve the following system of linear equations and write the general solution in vector form. (22 points)

$$\begin{cases} x_2 + x_3 + x_4 = 2 \\ x_1 + x_3 + x_4 = 3 \\ x_1 + x_2 + 2x_4 = 0 \end{cases}$$

Solution: The augmented matrix is $\begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 3 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}$. Do Gaussian elimination, the resulting RREF is $\begin{bmatrix} 1 & 0 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 5/2 \end{bmatrix}$, and the general solution is $\begin{bmatrix} 1/2 \\ -1/2 \\ 5/2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(1) Calculate $B^T AB$. (20 points)
(2) Calculate the rank and nullity of $B^T AB$. (10 points)

Solution: (1) Following the row-column rule for matrix multiplication, it is $\begin{bmatrix} 8 & 14 & 2 \\ 14 & 24 & 4 \\ 2 & 4 & 0 \end{bmatrix}$.

(2)Use Gaussian elimination to turn the matrix in (1) into row echelon form, one see that the rank is 2 and nullity is 1.

3. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} t \\ 0 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} s \\ -1 \\ 1 \end{bmatrix}$.

(1) Find all possible t so that v_1, v_2 and v_3 are linearly dependent. (15 points)
(2) For each of the t you found in (1), find all possible s so that b is in the span of $\{v_1, v_2, v_3\}$. (15 points)
(3) For each of the t you found in (1), find a set of linearly independent vectors with the same span as $span\{v_1, v_2, v_3\}$. (10 points)

Solution: (1) Use v_1, v_2 and v_3 as column vectors to form a 3×3 matrix A, repeatedly do row operations, one gets $\begin{bmatrix} 1 & 0 & -t^2 \\ 0 & 1 & t \\ 0 & 0 & t + 2t^2 \end{bmatrix}$. So $t = 0$ or $t = -1/2$.

(2) When $t = 0$, we want the system of linear equation with augmented matrix $\begin{bmatrix} 1 & 0 & 0 & s \\ 2 & 0 & 0 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$ to be consistent. By Guassian elimination, $s = -1/2$. When $t = -1/2$, we want the system of linear equation with augmented matrix $\begin{bmatrix} 1 & -1/2 & 0 & s \\ 2 & 0 & -1/2 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$ to be consistent. By Guassian elimination, $s = 1/2$.

(3) For both $t = 0$ and $t = -1/2$, v_1, v_2 are linearly independent and has the same span of v_1, v_2, v_3, because the first two columns are the pivot columns of A. There are many other valid answers to this question.

4. True or false (8 points, no need to explain your reasoning)

(1) Any non-zero 4×1 matrix can be turned into any other non-zero 4×1 matrix by a sequence of elementary row operations.

True. There are only two possible RREF for 4×1 matrices and one is 0.

(2) The row vectors of elementary matrices are all standard vectors.

False. For example, the first row of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(3) The product of two elementary matrices can never be an elementary matrix.

False. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ multiplies with itself is still elementary.

(4) If A is a 2×4 matrix, the rank of $A^T A$ can not be greater than 2.

True. Because the columns of $A^T A$ are linear combinations of the two columns of A^T, by the definition of matrix-matrix and matrix-vector multiplications.

(5) a, b and c are vectors, then the span of $\{a, b, c\}$ is the same as the span of $\{a, a+b, a+b+c\}$.

True. $b = (a+b) - a$, $c = (a+b+c) - (a+b)$.

(6) If $A^T A = I$ then $A = I$. Here I is the identity matrix.

False. For example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(7) If A is a diagonal matrix, then $AB = BA$.

False. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(8) If A is a diagonal matrix, B is in row echelon form, then BA is in row echelon form.

False. For example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

9 2.4

19. $\begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}$.

27. $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $P = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

35-54: TFTTTTTTTTTTTFTFFTTT

10 2.5

3. $\begin{bmatrix} -2 \\ 7 \end{bmatrix}$.

11 2.6

3. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

12 Quiz 2

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find A^{-1}.

Do Gaussian elimination: $\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -2 & 1 \end{bmatrix} \mapsto$
$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/4 & -1/2 & 1/4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 & -3/4 & 3/2 & -1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 3/4 & -1/2 & 1/4 \end{bmatrix} \mapsto$
$\begin{bmatrix} 1 & 0 & 0 & 1/4 & -1/2 & 3/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 3/4 & -1/2 & 1/4 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} 1/4 & -1/2 & 3/4 \\ -1/2 & 1 & -1/2 \\ 3/4 & -1/2 & 1/4 \end{bmatrix}$.

13 2.6

33-41: FTFFFTFFT

43. Let $V = [v_{ij}]$ be U^{-1}. Firstly, because U is invertible, the first column of U can not be 0 hence $u_{11} \neq 0$. Now use the row-column rule to evaluate the first column of $VU = I$. one gets that $v_{11}u_{11} = 1$ and $v_{j1}u_{11} = 0$ for $j > 1$, hence $v_{11} = 1/u_{11}$ and $v_{j1} = 0$ for $j > 1$.

Now suppose we already know that $u_{ii} \neq 0$ and $v_{ji} = 0$ for all $j > i$ and $i < k$, we shall show that $u_{kk} \neq 0$, $v_{kk} = 1/u_{kk}$, and $v_{jk} = 0$ for all $j > k$. To do that, evaluate the k-th column of $VU = I$ under the row-column rule. The k-th entry of that column is $v_{kk}u_{kk} = 1$ hence $u_{kk} \neq 0$ and $v_{kk} = 1/u_{kk}$, and for each $j > k$, the j-th entry of that column is $v_{jk}u_{kk} = 0$ so $v_{jk} = 0$. By induction we get that the inverse of U is an upper diagonal matrix and the entries on the diagonal are $1/u_{ii}$.

Alternatively, you can just write down the matrix $V = [v_{ij}] = U^{-1}$ explicitly, which is:

$$v_{ij} = \begin{cases} 0 & j > i \\ u_{ij}^{-1} & i = j \\ -u_{ii}^{-1}(\sum_{j<k\leq i} v_{jk}u_{ki} & j < i \end{cases}$$

14 3.1

1. 0.

15 3.1

55. False. For example, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

16 3.2

27. The determinant is $c(c+4) - 12$, so when $c = 2$ or $c = -6$ the determinant is 0 and the matrix is not invertible.

74. If n is odd, $det(A) = det(A^T) = det(-A) = (-1)^n det(A) = -det(A)$ so $det(A) = 0$, so A is not invertible. This is not true when n is even, for example $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

17 4.1

5. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$.

62. True.

18 4.2

7. The RREF is $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So a basis of the col space consists of the first three column vectors, a basis of the null space is $\{ \begin{bmatrix} -4 \\ -4 \\ -1 \\ 1 \end{bmatrix} \}$.

50. False. For example, the null space in problem 7 does not contain any standard vectors.

19 4.3

5. 1; 3; 1; 0.

42. True. If V is not R^n, there must be some vector $v \in R^n \backslash V$, so a basis of V union with $\{v\}$ is a linearly independent set of R^n with $n+1$ elements which is impossible.

83. (a) $u = [u_i]$, then $u^T u = \sum u_i^2$, so $u^T u = 0$ iff $u_i = 0$ for all i.

(b) If $v \in NullA$, $Av = 0$, hence for any row vector r of A, $r^T v = 0$. Because u is a linear combination of the row vectors, $u^T v = 0$ by distribution law of matrix multiplication.

(c) This follows from (a) and (b).

20 Quiz 3

Find basis of row, col and null space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

The RREF is $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So a basis of the column space is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$, a basis of the row space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$, and a basis of the null space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$. The solution is obviously not unique.

21 5.1

46. False. It's the null space.

60. True.

63. True.

22 5.2

13. Eigenvalues are 1, -4, $V_1 = span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, $V_{-4} = span\{\begin{bmatrix} 3 \\ -5 \end{bmatrix}\}$.

65. True.

72. True.

85. $A = [a_{ij}]$, then $a_{12} = a_{21}$, the characteristic polynomial is $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}^2)$ which has discriminant $(a_{11} + a_{22})^2 - 4(a_{11} + a_{22}) + 4a_{12}^2 = (a_{11} - a_{22})^2 + 4a_{12}^2 \geq 0$.

Midterm II

1. (1) Find the number t so that the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & t & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ is not invertible. (10 points)

(2) When $t = 1$, find the inverse of A. (10 points)

Answer: (1) $t = 0$. (2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & 1/2 & 0 \\ -4 & 2 & -1 & 1 \end{bmatrix}$.

2. (1) Find the basis for the row space, column space and null space of the matrix $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

(20 points)

(2) Find a vector $b \in \mathcal{R}^4$ so that $Mx = b$ does not have a solution. (5 points)

Answer: (1) Row space has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, column space has a basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and nul space has a basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$. (2) There are many, for example, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

3. Let $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(1) Calculate $B^T B$. (5 points)

(2) Find the characteristic polynomial of $B^T B$. (15 points)

(3) Diagonalize $B^T B$, in other words, find invertible matrix P and diagonal matrix D, so that $B^T B = PDP^{-1}$. (15 points)

(4) Find the $(1,2)$-th entry of $(B^T B)^{10}$. (10 points)

Answer: (1) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(2) $\lambda^2(\lambda - 2)^2$.

(3) $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. $P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$.

(4) 0.

4. True or false (10 points, no need to explain your reasoning)

(1) A is a diagonalizable square matrix with characteristic polynomial $\lambda^3(\lambda^3 - 1)$, then the rank of A is 3. True. The null space, which is the eigenspace for 0, has dimension 3, while the matrix is 6×6.

(2) Two matrices with the same characteristic polynomials are similar. False. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(3) Let A be a square matrix, then $det(-A^T) = -det(A^T)$. False. For example, if A is 2×2, then $det(-A^T) = det(A^T)$.

(4) Let A be a $n \times n$ matrix, $f(\lambda)$ be the characteristic polynomial of A^3, then $f(x^3)$ has a factor of degree $2n$. True. $f(x^3) = det(A^3 - x^3I) = det(A - xI)det(A^2 + xA + x^2I)$.

(5) If the row space and column space of A are identical, then A is symmetric. False. For example, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(6) If A and B are both 3×3 matrices that are diagonalizable, then AB is diagonalizable. False.

(7) The null space of a 3×6 matrix has dimension at least 3. True. The rank is at most 3.

(8) If a vector x is in both the null space of $A^3 - I$ and A, then $x = 0$. True. $A^3x - x = 0$, $Ax = 0$, then $x = A^2(Ax) = 0$.

(9) If $v \in \mathcal{R}^4$ is a non-zero column vector, then the matrix vv^T is diagonalizable. True. There are two eigenspaces, with dimension 3 and 1 respectively.

(10) If A and B are both 3×3 matrices that have LU decompositions, then AB has an LU decomposition. False. For example, $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

23 5.3

5. $P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

19. This is not diagonalizable, because 0 has multiplicity 2 as a root of characteristic polynomial but the corresponding eigenspace has dimension 1.

40. False. Should be nullity instead.

47. False. For example, the matrix in problem 19.

85. (a) A is diagonalizable then $A = QDQ^{-1}$ where D is diagonal, hence $B = (P^{-1}Q)D(p^{-1}Q)^{-1}$. The other direction is similar. (b) The same, because characteristic polynomial is similarity invariant. (c) If x is an eigenvector of A, then $Ax = \lambda x$ for some λ, so $B(P^{-1}x) = P^{-1}APP^{-1}x = \lambda P^{-1}x$, hence $P^{-1}x$ is an eigenvector of B.

24 6.1

5. $\|u\| = \sqrt{11}$, $\|v\| = \sqrt{5}$, $d(u, v) = \sqrt{14}$.

29. $\|u\| = \sqrt{21}$, $\|v\| = \sqrt{11}$, $d(u, v) = \sqrt{34}$.

43. The orthogonal projection is $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$. Distance is $\frac{7\sqrt{2}}{2}$.

66. True. This follows from repeatedly using the fact that $u \perp v$ then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

25 6.2

13. $$\left\{ \begin{bmatrix} 0 \\ \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix}, \begin{bmatrix} \sqrt{3/5} \\ -\sqrt{4/15} \\ \sqrt{1/15} \\ \sqrt{1/15} \end{bmatrix} \begin{bmatrix} \sqrt{9/35} \\ \sqrt{9/35} \\ -\sqrt{16/35} \\ \sqrt{1/35} \end{bmatrix} \right\}.$$

26 Quiz 4

Do orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto the null space of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

Solution: The null space is spanned by $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ so the answer is $\begin{bmatrix} 2/5 \\ -1/5 \\ -1/5 \\ 2/5 \end{bmatrix}$.