1. Let $A = 4$. Let $x_1 = a$, $a > 0$. $x_{n+1} = (x_n + A/x_n)/2$
   (a) Show that if $x_n = x_n + 1$, then $a = \pm 2$.
   (b) If $a = 1$, write down the first few x_n.
   (c) Can you see any pattern?
2. Let $A = 4$. For each n, let a_n be an integer between 0 and $3^n - 1$, such that:
   - $x_1 = a$, where $a = 1$ or 2.
   - For every n, let q_n be an integer between 0 and 3^{n+1}, such that $x_n q_n - 1$ is a multiple of 3^{n+1}.
   - Now x_{n+1} is chosen such that $2x_{n+1} - x_n - Aq_n$ is a multiple of 3^{n+1}.

   (a) Let $a = 1$ or 2, write down the next few terms.
   (b) Can youshow that the process can always continue?
   (c) Can you see any pattern?

1. We can show that x_n has a limit as follows: it is easy to see that for all $n \geq 2$, $x_n > \sqrt{A}$. Now

$$|x_{n+1}^2 - A| = \left|\frac{(x_n^2 - A)^2}{4x_n^2}\right| \leq \frac{1}{4}|x_n^2 - A|$$

2. To show that this process can always continue on, induction on n to show that all x_n exists and $x_n - 1$ is a multiple of 3. Let $s_n = x_n + A(1 - (x_n - 1) + (x_n - 1)^2 - \cdots + (-1)^n (x_n - 1)^n$, then x_{n+1} equals the remainder of $s_n/2$ divided by 3^{n+1} if s_n is even, the remainder of $(s_n + 3^{n+1})/2$ divided by 3^{n+1} if s_n is odd. We can verify that $x_{n+1} - 1$ is a multiple of 3 as well.

   We can also prove by induction on n that $x_n^2 - A$ is a multiple of 3^n.

If A is not a perfect power, say equals 7, neither approach will converge to a fixed integer, so we need to introduce broader classes of numbers as limits of such sequences, which we call **completion**. In the first case we carry out completion and get the set of real numbers $\mathbb{R}$, in the second we get the set of p adic integers $\mathbb{Z}_p$.