1 9/5 Matrices, vectors, and their applications

Algebra: study of objects and operations on them.

Linear algebra: object: matrices and vectors. operations: addition, multiplication etc.

Algorithms/Geometric intuition/sets and maps

$m \times n$ matrix: numbers forming a rectangular grid, m rows and n columns. Motivation: coefficients of a system of linear equations. Data tables in statistics.

(i, j)-th entry of a matrix.

Vectors: matrices with one row/column. Motivation: coordinates in plane and space.

Operations: (1) Addition. (2) Scalar multiplication. (3) Matrix-vector multiplication. (4) Transpose.

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Ax, $A(Ax)$.

Example: Averaging over columns. Covariance? Other statistical concepts?

Laws: The usual laws one may expect. e.g. $A(x + y) = Ax + Ay$, $(A + B)^T = A^T + B^T$, $(A^T)^T = A$. Note: $A(Bx) \neq B(Ax)$!

Zero and one matrix. Standard vectors.

Example: Rotation by 60 degrees (or $\pi/3$).

Consequence: Matrix is completely determined by its action on the standard vectors! Matrix-matrix multiplication.

Example: 2×2 case.
The concept of linear combination. Relationship with matrix-vector multiplication.

Example: Rotation and Translation.

Example: Random walk on graphs.

2 9/8 Linear equations

Review:

- Matrix multiplications
- Transposes
- Standard vectors
- Identity Matrix
- Rotation matrix

- Stochastic matrix

*****

Linear systems as matrix equations. *Coefficient matrix* and *augmented matrices*

Elementary row operations: swap, multiply, add. Property: reversible, and preserves solution set.

Row echelon form: The first non-zero entry (called pivot) of each row is to the right of the previous. *Reduced row echelon form*: The first non-zero entry is 1 and is the only non-zero entry in that column. Uniqueness under row operations.

Algorithm (Gaussian elimination):

- Write augmented matrix.
- Use row operations, turn it into reduced echelon form.
- General solution from RREF (Example: $x_1 + 2x_2 + x_3 + x_4 = 3$, $x_1 + 3x_3 - x_4 = 8$).

Structure of solutions:

	Pivot at last col.	No pivot at last col.
All coefficient col. have pivot	None	Inf
Some coeff. col. have no pivot	None	One

Examples of the 4 cases.

True or false:

- A system of 3 linear equations with 6 variables can not have just one solution.
- A system of 3 linear equations with 6 variables must have infinitely many solutions.

*****

Counting: number of arbitrary constants and the number of pivots. Rank and dimension.

Explicit algorithm from RREF to general solutions.

3 9/12 Linear equations cont.

3.1 Review

- Augmented matrix, row operations.
- RREF.
- Condition for no/one/infinitely many solutions.
- General solution: write *basic variables* in terms of *free variables*, or the *vector form*.

3.2 Gaussian elimination

Augment matrices to REF or RREF through finitely many elementary row operations.

For r=$1, 2, \ldots n$:

Find the left-most non-zero entry among the $r, r+1, \ldots n$ rows. If there aren't any, terminate.
Exchange rows to move this entry to the r-th row.

Multiply the r-th row and add it to the $r+1, \dots$ rows to eliminate all entries on the left-most non-zero column.

Multiply the r-th row so that the leading.

To Further turn it into a RREF (backward pass):

Multiply to each non-zero row to make the first entry 1.
For each non-zero row, multiply and add it to each of the rows above it to turn the entries on pivot columns 0.

Reason for distinguish forward/backward passes: forward pass is a permutation matrix with a lower triangular matrix with 1 on the diagonals, backward pass is a upper triangular matrix. Row pivoting.

Example: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 4 & 7 \\ 2 & 0 & 1 & 0 \end{pmatrix}$. RREF? General solution?

3.3 Uniqueness of RREF

Key idea: read the RREF from matrix using linear combinations of rows or columns!

Appendix E uses columns. One can also use rows as follows: Let R be the space of linear combination (span) of the row vectors. The last non-zero row in RREF is the one in R with the most number of 0 entries on the left and the first non-zero entry 1. Let the index of the first non-zero entry be c_1. The preceding row in RREF is the one in R with c_1-th entry 0, first non-zero entry 1, and the most possible number of 0 on the left, etc.

3.4 Rank and nullity

Rank of A: num. of pivots in A=num. of non-zero rows in REF of A=num of basic variables in $Ax = b$
Nullity of A: num of non-pivot columns in A=num. of columns of A-rank of A=num of free variables in $Ax = b$

True or false:

- The rank of $[A \;\; B]$ must be no smaller than the sum of the ranks of A and B.
- The nullity of $[A \;\; B]$ must be no smaller than the sum of the ranks of A and B.
- The RREF of a square matrix of no nullity must be the identity matrix
- The nullity of A is non-zero iff some row of A is a linear combination of the others.
- $[A \;\; B]$ has the same rank as B iff the columns of A are linear combinations of the columns of B.

******

Structure of the general solution in terms of rank or nullity:
If $rank(A) < rank([A, b])$:
  No solution.
Else:
  If $nullity(A) = 0$:
    One solution.
  Else:
    Infinitely many solutions.

Example: $\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$.