

## 1 1.1

2.  $u_{xx} + u_{yy} = (2x \cdot \frac{1}{2}(x^2 + y^2)^{-1})_x + (2y \cdot \frac{1}{2}(x^2 + y^2)^{-1})_y = 2(x^2 + y^2)^{-1} - (2x^2 + 2y^2) \cdot (x^2 + y^2)^{-2} = 0.$

4. The statement of this problem is somewhat unclear in whether they mean  $(u_{xx})^2 + (u_{yy})^2 = 0$  (the more likely one) or  $(u^2)_{xx} + (u^2)_{yy} = 0$ , so either interpretation would be considered correct. With the first interpretation it is obvious that all function in the stated form satisfy that  $u_{xx} = u_{yy} = 0$ . With the second there would need to be additional constraints on  $a, b, c, d$  for it to work.

5. The general solution is  $u = xF(t) + G(t)$ , hence one can let  $u = t^2 + x(1 - t^2)$ .

6.  $u_{tt} = (g(x + ct) + g(x - ct))_t = c(g'(x + ct) - g'(x - ct)), u_{xx} = c^{-1}(g(x + ct) - g(x - ct))_x = c^{-1}(g'(x + ct) - g'(x - ct)).$

7.  $(e^{at} \sin bx)_t = ae^{at} \sin bx, (e^{at} \sin bx)_{xx} = -b^2 e^{at} \sin bx$ , hence  $a = -kb^2$ .

8.  $(u_x)_t = 1 - 3u_x$ , hence  $u_x = \frac{1}{3} + e^{-3t}f(x)$  for some arbitrary function  $f$ , hence  $u(x, t) = \frac{x}{3} + e^{-3t}F(x) + G(t)$  for arbitrary function  $F$  (which is the anti-derivative of  $f$ ) and  $G$ .

12. To sketch wave profile, pick some  $k, A, D$  or  $c$ , sketch  $u(x, t)$  for different values of  $t$ , and if  $u$  is complex-valued you can sketch either the real or imaginary part.

Dispersion relations: a)  $\omega = -iDk^2$ . b)  $\omega = \pm ck$ . c)  $\omega = -k^3$ . d)  $\omega = k^2$ . e)  $\omega = ck$ .

14. Dispersion relation is  $\omega = (-1 + \delta k^2 - k^4)i$  hence this is diffusive. When  $\delta = k^2 + 1/k^2$  the solution has growth rate 0. When  $k^2 + 1/k^2 > \delta$  the solution decays.

## 2 1.2

1. From equation (1.7) in the text we have  $\frac{d}{dt} \int_a^b u A dx = A\phi|_a - A\phi|_b$ . Differentiate with respect to  $b$  (or use some other argument, for example as in the textbook), we have  $Au_t = -A_x\phi - A\phi_x$ , hence  $u_t + \phi_x = -A'\phi/A$ .

3. By chain rule,  $u_x = u_\xi$ ,  $u_t = -cu_\xi + u_\tau$ , hence the equation (1.12) becomes  $u_\tau = -\lambda u$ , hence the general solution is  $u = e^{-\lambda\tau}F(\xi) = e^{-\lambda t}F(x - ct)$ .

4.  $u_t + cu_x = -\lambda u$ . If  $w = ue^{\lambda t}$ ,  $u = we^{-\lambda t}$  hence  $u_t + cu_x = w_t e^{-\lambda t} - \lambda w e^{-\lambda t} + cw_x e^{-\lambda t}$ ,  $-\lambda u = -\lambda w e^{-\lambda t}$ , hence  $w_t + cw_x = 0$ .

5. By method of characteristics  $u_t + xtu_x = 0$  has characteristics  $x = Ce^{t^2/2}$ , hence the general solution is  $u = F(xe^{-t^2/2})$ . Together with the initial value condition we know that  $F = f$  hence  $u = f(xe^{-t^2/2})$ . The general solution of  $u_t + xu_x = e^t$  is  $u = e^t + F(xe^{-t})$ , so with the initial condition, the solution should be  $u = e^t + f(xe^{-t}) - 1$ .

6(b). The characteristics are  $x = Ct$ , and the general solution is  $u = e^{-2t}F(x/t)$ . Use the initial condition we get  $F = e^2 f$ , hence  $u = e^{-2(t-1)}f(x/t)$ .

7. The general solution is  $u = e^{-\lambda t}F(x - ct)$ . The initial-boundary condition tells us that  $F(x) = 0$  for  $x > 0$  and  $e^{-\lambda t}F(-ct) = g(t)$  for  $t > 0$ , hence  $F(x) = \begin{cases} 0 & x > 0 \\ e^{\lambda x/c}g(x/c) & x \leq 0 \end{cases}$ .

12. By the method of characteristics,  $u(x, t) = F(x - ct)e^{(\alpha t - u)/\beta}$ . Set  $t = 0$  we have  $F(x) = f(x)e^{f/\beta}$  hence  $u(x, t) = f(x - ct)e^{(\alpha t - u + f(x - ct))/\beta}$ .

14. Characteristics are  $x = Ce^{-ut}$  hence  $u = F(xe^{ut})$ . Together with the initial condition we get  $u = xe^{ut}$ . A solution does not exist for all  $t$ . For example, there doesn't exist any  $u$  at point  $x = t = 1$  because  $s < e^s$  for all  $s \in \mathbb{R}$ .

### 3 1.3

2.  $\frac{d}{dt} \int_0^l u^2 dx = \int_0^l 2uu_t dx = \int_0^l 2kuu_{xx} dx = 2kuu_x|_0^l - \int_0^l 2k(u_x)^2 dx \leq 0$ , hence  $\int_0^l u^2 dx \leq \int_0^l u_0^2 dx$  for  $t \geq 0$ .

3. Let  $w = u - g + (x/l)(h - g)$ , then  $w(0, t) = w(l, t) = 0$ ,  $u_t = ku_{xx}$  will imply  $w_t = kw_{xx} - g' + (x/l)(h' - g')$ .

4. The steady state satisfy  $0 = ku_{xx} - hu$  and  $u(0) = u(1) = 1$ , hence  $u = \frac{e^{(h/k)^{1/2}(x-1/2)} + e^{(h/k)^{1/2}(1/2-x)}}{e^{(h/k)^{1/2}/2} + e^{-(h/k)^{1/2}/2}}$ .

5.  $u_t = w_t e^{\alpha x - \beta t} - \beta w e^{\alpha x - \beta t} = w_t e^{\alpha x - \beta t} - \beta u$ ,  $u_x = w_x e^{\alpha x - \beta t} + \alpha u$ ,  $u_{xx} = w_{xx} e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha^2 u$ , hence  $0 = u_t - Du_{xx} + cu_x + \lambda u = (w_t - Dw_{xx})e^{\alpha x - \beta t} + (c - 2D\alpha)w_x e^{\alpha x - \beta t} + (\lambda - \beta - D\alpha^2 + c\alpha)u$ , so when  $\alpha = c/(2D)$  and  $\beta = \lambda - D\alpha^2 + c\alpha = \lambda + c^2/(4D)$ ,  $0 = w_t - Dw_{xx}$ .

6. The steady state doesn't depend on the initial condition. It is  $u = \frac{1}{2k}x(1 - x)$ .

10. The flux is  $Du_x + u^2/2$ . Replace  $u = \psi_x$  we have  $\psi_{xt} = D\psi_{xxx} + \psi_x \psi_{xx}$ . Integrate along  $x$  we have  $\psi_t = D\psi_{xx} + (\psi_x)^2/2 + F(t)$ . Replace  $\psi_t$  with  $\psi_t + \int_0^t F(s)ds$  we can get rid of  $F$ . Now let  $\psi = -2D \ln v$  we get  $-2Dv_t/v = -2D^2(v_{xx}v - (v_x)^2)/v^2 + 2D^2(v_x)^2/v^2$ , hence  $v_t = Dv_{xx}$ .

### 4 1.4

3. For  $u_t = Du_{xx} - cu_x$ , the time independent solution satisfies  $0 = Du_{xx} - cu_x$ . So the solution is  $u = C_1 + C_2 e^{cx/D}$ . For  $u_t = Du_{xx} - cu_x + ru$ , the time independent case reduces to  $0 = Du_{xx} - cu_x + ru$ , the characteristic polynomial is  $D\lambda^2 - c\lambda + r = 0$  whose roots are  $r = \frac{c \pm \sqrt{c^2 - 4Dr}}{2D}$ . Hence, when  $c^2 = 4Dr$  the general solution is  $u = (C_1 + C_2 x)e^{\frac{cx}{2D}}$ , when  $c^2 > 4Dr$  the general solution is  $u = C_1 e^{\frac{xc + x\sqrt{c^2 - 4Dr}}{2D}} + C_2 e^{\frac{xc - x\sqrt{c^2 - 4Dr}}{2D}}$ , when  $c^2 < 4Dr$  the general solution is  $u = C_1 e^{\frac{xc}{2D}} \cos(\frac{x\sqrt{4Dr - c^2}}{2D}) + C_2 e^{\frac{xc}{2D}} \sin(\frac{x\sqrt{4Dr - c^2}}{2D})$ .

9.  $u = ax + b$  then  $u_{xx} = 0$ .

$u = a \ln r + b$  then  $u_{xx} + u_{yy} = a(\frac{x}{r^2})_x + a(\frac{y}{r^2})_y = a(\frac{r^2 - 2x^2 + r^2 - 2y^2}{r^4}) = 0$ .

$u = \frac{a}{\rho + b}$  then  $u_{xx} + u_{yy} + u_{zz} = a((\frac{x}{\rho^3})_x + (\frac{y}{\rho^3})_y + (\frac{z}{\rho^3})_z) = 0$ .

12. (a)  $\frac{d}{dt} \int_a^b 2\pi r u dr = 2\pi a(-Du_r|_a) - 2\pi b(-Du_r|_b)$ . Differentiate on  $b$  we get  $bu_t|_b = Db u_{rr}|_b + Du_r|_b$ , hence  $u_t = Du_{rr} + \frac{D}{r}u_r = D\frac{1}{r}(ru_r)_r$ .

(b)  $\frac{d}{dt} \int_a^b 4\pi r^2 u dr = 4\pi a^2(-Du_r|_a) - 4\pi b^2(-Du_r|_b)$ . Differentiate on  $b$  then you get the differential equation.

## 5 1.5

1. You can do it however you want, for example, in the 3rd equation on page 51, add a term  $-\int_a^b \rho_0 g dx$  to the right.

3. Verification is by chain rule. Sketch  $u = \frac{1}{2}(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2})$ .

4. The initial condition is  $u_n(x, 0) = \sin \frac{n\pi x}{l}$ ,  $(u_n)_t(x, 0) = 0$ . The frequency is  $\frac{cn}{2l}$ , they decrease as  $l$  increases and as  $c$  (tension) increases.

5.  $\frac{d}{dt} E = \int_0^l (\rho_0 u_t u_{tt} + \tau_0 u_x u_{tx}) dx = \tau_0 \int_0^1 (u_t u_{xx} + u_x u_{tx}) dx = \tau_0 u_t u_x|_0^l = 0$ .

9.  $I_x + CV_t + GV = 0$ , so  $I_{xx} + CV_{xt} + GV_x = 0$ . Substitute  $V_x = -LI_t + RI$ , we get that  $I$  satisfy the telegraph equation. The fact that  $V$  satisfy telegraph equation follows analogously. When  $R = G = 0$  the speed of wave is  $(LC)^{-1/2}$ .

## 6 1.7

1.  $\text{div}(\text{gradu}) = \text{div}((u_x, u_y, u_z)) = u_{xx} + u_{yy} + u_{zz}$ .

## 7 Quiz 1:

$u_t + (x+1)u_x = 1$ ,  $u(x, 0) = \sin x$ .

Characteristics are  $x = Ce^t - 1$ . Hence  $u = t + F((x+1)e^{-t})$ , hence  $F(x) = \sin(x-1)$  and  $u = t + \sin((x+1)e^{-1} - 1)$ .

## 8 1.7

3. This is divergence theorem. The heat generated in  $\Omega$  equals the heat flowing out at the boundary.

4. Let  $\phi = (\phi_1, \phi_2, \phi_3)$ , then  $\text{div}(w\phi) = (w\phi_1)_x + (w\phi_2)_y + (w\phi_3)_z = (w_x\phi_1 + w_y\phi_2 + w_z\phi_3) + w((\phi_1)_x + (\phi_2)_y + (\phi_3)_z) = \phi \cdot \text{grad} w + w \text{div} \phi$ . Let  $\phi = \text{gradu}$  then Green's identity follows from this and the divergence theorem.

5.  $\lambda = \frac{\int_{\Omega} u \Delta u dV}{\int_{\Omega} u^2 dV} = -\frac{\int_{\Omega} |\text{gradu}|^2 dV}{\int_{\Omega} |u|^2 dV} < 0$ ,

6. Let  $w = u + v$  where  $v$  is 0 at the boundary, then  $\int_{\Omega} |\text{grad} w|^2 dV = \int_{\Omega} |\text{grad} u|^2 dV + \int_{\Omega} |\text{grad} v|^2 dV + 2 \int_{\Omega} \text{grad} u \cdot \text{grad} v dV$ . By 4 and the assumption, the last term is 0, hence  $\int_{\Omega} |\text{grad} w|^2 dV \geq \int_{\Omega} |\text{grad} u|^2 dV$ .

7. Use  $\text{cpu}_t = \text{div} \phi$ .

## 9 1.8

1. Maximum are at  $r = 2$ ,  $\theta = \pi/4, 5\pi/4$ , minimum are at  $r = 2$ ,  $\theta = 3\pi/4, 7\pi/4$ .

2.  $u = (x^2 + y^2)/4 - a^2/4$ .

4. The solution is spherical symmetric because the function and the boundary conditions are both spherical symmetric, i.e.  $u = u(\rho)$ . Hence  $\Delta u = 1$  reduces to  $u_{\rho\rho} + \frac{2}{\rho}u_{\rho} = 1$ , hence  $(\rho^2 u_{\rho})_{\rho} = \rho^2$ , hence  $\rho^2 u_{\rho} = \frac{1}{3}\rho^3 + C_1$ ,  $u' = \frac{1}{3}\rho + \frac{C_1}{\rho^2}$ , hence  $u = \frac{1}{6}\rho^2 - \frac{C_1}{\rho} + C_2$ . Apply the boundary condition one gets  $u = \frac{1}{6}\rho^2 + \frac{b^3}{3\rho} - \frac{1}{6}a^2 - \frac{b^3}{3a}$ .

5.  $u = A \tan(x) + B$ , solve for constants  $A$  and  $B$  using the boundary condition.

6.  $u = A \log r + B$ .  $u = \frac{10}{\log 2} \log r$ .

8. Use chain rule.

9.  $\text{curl} E = 0$  implies that such a potential exists.  $\Delta V = \text{divgrad} V = \text{div} E = 0$ .

## 10 1.9

1. This is a parabolic equation.  $u = F(kx-t) + (x+kt)G(kx-t)$ , or you can write it in other equivalent ways.

2. Let  $p = 2x + t$ ,  $q = t$ , then  $u_x = 2u_p$ ,  $u_{xx} = 4u_{pp}$ ,  $u_t = u_p + u_q$ ,  $u_{xt} = 2u_{pp} + 2u_{pq}$ , hence the equation becomes  $u_p = 4u_{qp}$ , hence  $u = F(2x+t)e^{t/4} + G(t)$ .

3. It is hyperbolic. Under the change of variable, by chain rule,  $u_x = \frac{4}{x}u_{\tau}$ ,  $u_{xx} = \frac{16}{x^2}u_{\tau\tau} - \frac{4}{x^2}u_{\tau}$ ,  $u_{xt} = \frac{4}{x}u_{\tau\xi} + \frac{4}{x}u_{\tau\tau}$ , so  $0 = xu_{xx} + 4u_{xt} = -\frac{4}{x}u_{\tau} - 16u_{\tau\xi}$ , hence  $u = e^{-\xi/4}f(\tau) + g(\xi)$ .

4. Use chain rule and product rule.

5. Elliptic. Find the eigenvalues of matrix  $\begin{bmatrix} 1 & -3 \\ -3 & 12 \end{bmatrix}$ .

6. Parabolic. The general solution calculation is similar to 3 above.

7. a) Elliptic when  $xy > 1$  and hyperbolic when  $xy < 1$ . b) Elliptic.

Midterm 1

1. Solve the following initial or initial/boundary value problems:

- (1)  $u_t = xu_x$ ,  $u(x, 0) = x^2$ . Here  $u$  is a function of  $x$  and  $t$ . (25 points)  
(2)  $u_t + u_x = \sin x$ ,  $u(x, 0) = 0$  for  $x \geq 0$ ,  $u(0, t) = t$  for  $t \geq 0$ . Here  $u$  is a function of  $x$  and  $t$ . (15 points)

Answer: (1) General solution is  $u = F(xe^t)$ , hence  $u = x^2e^{2t}$ .  
(2) General solution is  $u = -\cos x + F(x-t)$ , so  $u = -\cos x - (x-t) + 1$  when  $x \leq t$ , and  $u = -\cos x + \cos(x-t)$  when  $x \geq t$ .

2. (1) Find the general solution of  $u_{tt} = u_{tx}$ . (15 points)  
(2) Find the solution of the initial value problem:  $u_{tt} = u_{tx}$ ,  $u(x, 0) = 0$ ,  $u_t(x, 0) = x$ . (10 points)

Answer: (1)  $u_t = f(x+t)$ , so  $u = F(x+t) + G(x)$  where  $F$  and  $G$  are arbitrary functions.  
(2)  $F(x) + G(x) = 0$ ,  $F'(x) = x$ , so  $u = \frac{1}{2}(x+t)^2 - x^2$ .

3. Consider the 1 dimensional advection-diffusion equation:  $u_t = u_x + u_{xx}$ .  
(1) Use change of coordinate of the form  $p = x - Ct$ ,  $q = t$  to reduce it to the 1 dimensional heat equation. (13 points)  
(2) Recall that the solution of initial value problem of 1-dimensional heat equation:  $v_t = v_{xx}$  when  $t > 0$ ,  $v(x, 0) = f(x)$  can be given by the Poisson integral representation:

$$v(x, t) = \int_{-\infty}^{\infty} f(y)G(x-y, t)dy, \text{ where } G(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Can you write down the analogous formula for the following initial value problem:  $u_t = u_x + u_{xx}$  when  $t > 0$ ,  $u(x, 0) = f(x)$ ? (10 points)

- (3) Consider the following problem with periodic boundary condition:  $u_t = u_x + u_{xx}$  when  $0 < x < 1$ ,  $u(0, t) = u(1, t)$ ,  $u_x(0, t) = u_x(1, t)$ . Show that  $I(t) = \int_0^1 u^2(x, t)dx$  is a non-increasing function by calculating  $\frac{d}{dt}I$ . (7 points)

Answer: (1)  $u_t = -Cu_p + u_q$ ,  $u_x = u_p$ ,  $u_{xx} = u_{pp}$ , hence when  $C = -1$ ,  $u_q = u_{pp}$ .  
(2)  $u(x, t) = \int_{-\infty}^{\infty} f(y)G(x+t-y, t)dy$ .  
(3)  $\frac{d}{dt}I = \int_0^1 2uu_t dx = \int_0^1 2uu_x + 2uu_{xx} dx = u^2|_0^1 + 2uu_x|_0^1 - \int_0^1 2(u_x)^2 dx \leq 0$ .

4. Consider the equation  $u_{xx} + u_{yy} = x^2 + y^2$  on  $\mathbb{R}^2 \setminus (0, 0)$ . Find all radial symmetric solutions (In other words, all solutions of the form  $u(x, y) = g(\sqrt{x^2 + y^2})$ ). You may want to use the fact that the Laplace operator in polar coordinate  $(r, \theta)$  is  $\Delta = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ . (5 points)

Answer:  $u_{rr} + u_r/r = r^2$ , so  $(ru_r)_r = r^3$ ,  $ru_r = A + \frac{1}{4}r^4$ ,  $u_r = A/r + \frac{1}{4}r^3$ , and  $u(r) = B + A \log r + \frac{1}{16}r^4$ .

## 11 2.1

2.  $|u| = |\int_{\mathbb{R}} \phi(y)G(x-y, t)dy| \leq \int_{\mathbb{R}} |\phi(y)G(x-y, t)|dy \leq M \int_{\mathbb{R}} G(x-y)dy = M$ , where  $G$  is the heat kernel.

3.  $u(x_0, t) = \int_{\mathbb{R}} \phi(y)G(x_0 - y, t)dy = u_0 \int_0^\infty G(x_0 - y, t)dy = u_0 \int_{-\infty}^{x_0} G(s, t)ds = u_0(\int_{-\infty}^0 G(s, t)ds + \int_0^{x_0} G(s, t)ds)$ . We know  $\int_{-\infty}^0 G(s, t)ds = 1/2$ ,  $\int_0^{x_0} G(s, t)ds = \int_0^{x_0/\sqrt{4t}} G(s, 1)dt$  which converges to 0 as  $t \rightarrow \infty$ , hence  $\lim_{t \rightarrow \infty} u(x_0, t) = u_0/2 = 1/2$ .

## 12 2.2

3. The solution of the latter Cauchy problem is  $u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(s)ds$ , and the solution of the first Cauchy problem is the partial derivative of the solution of the latter Cauchy problem in  $t$  direction which by fundamental theorem of calculus is  $\frac{1}{2}(\phi(x-ct) + \phi(x+ct))$ .

## 13 Quiz 2

$$u_{tt} = 4u_{xx} + e^{x+t}, \quad u_t(x, 0) = 0, \quad u(x, 0) = \sin x.$$

$$\text{Solution: By Dahamel's principle, } u_{tt} = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \int_0^t \frac{1}{4} \int_{x-2t+2s}^{x+2t-2s} e^{r+s} dr ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4} \int_0^t e^{x+2t-s} - e^{x-2t+3s} ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(e^{x+2t} - e^{x+t}) - \frac{1}{12}(e^{x+t} - e^{x-2t}).$$

## 14 2.3

3.  $|u^1 - u^2| = |(\frac{1}{2}(f^1(x-ct) + f^1(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g^1(s)ds) - (\frac{1}{2}(f^2(x-ct) + f^2(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g^2(s)ds)| \leq |\frac{1}{2}((f^1 - f^2)(x-ct) + (f^1 - f^2)(x+ct))| + |\frac{1}{2c} \int_{x-ct}^{x+ct} (g^1 - g^2)(s)ds| = \delta_1 + \delta_2 T$ . It shows that this Cauchy problem is stable and well posed.

## 15 2.4

2. Do odd extension of the initial condition, one gets  $u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} dy$ .

## 16 2.5

1. By Duhamel's principle,  $u(x, t) = \int_0^t \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds d\tau = -\frac{1}{2c} \int_0^t \cos(x+c(t-\tau)) - \cos(x-c(t-\tau)) d\tau = \frac{1}{2c^2} (\sin(x+ct) + \sin(x-ct) - 2\sin(x))$ .

## 17 2.6

4.  $\mathcal{L}(\int_0^t f(\tau) d\tau) = \int_0^\infty e^{-st} (\int_0^t f(\tau) d\tau) dt = \int_0^\infty (\int_\tau^\infty e^{-st} dt) f(\tau) d\tau = \frac{1}{s} \int_0^\infty e^{-s\tau} f(\tau) d\tau = \frac{\mathcal{L}(f)}{s}$ .

8. Let  $v = Lu$  in the  $t$  direction, we have  $sv - u_0 = v_{xx}$ ,  $v_x(0, s) = v(0, s)$ , because we want bounded solution,  $v = -\frac{u_0}{s(1+\sqrt{s})} e^{-\sqrt{s}x} + \frac{u_0}{s}$ , hence  $u = -u_0 L^{-1}(\frac{1}{s(1+\sqrt{s})} e^{-\sqrt{s}x}) + u_0$ .

Remark: for those who know complex analysis, we can evaluate  $L^{-1}(\frac{1}{s(1+\sqrt{s})} e^{-\sqrt{s}x})$  using the inverse formula on pp. 107 of the textbook. The answer is  $-\frac{1}{2\pi i} \int_0^\infty e^{-st} (\frac{1}{-s(1+i\sqrt{s})} e^{-\sqrt{s}ix} - \frac{1}{-s(1-i\sqrt{s})} e^{\sqrt{s}ix}) ds = -\frac{1}{\pi} \int_0^\infty \frac{\sin(\sqrt{s}ix) + \sqrt{s} \cos(\sqrt{s}ix)}{s(1+s)e^{st}} ds$ .

## 18 2.7

5.  $\mathcal{F}u = \int_{-\infty}^\infty e^{-|x|+i\xi x} dx = \frac{2}{1+\xi^2}$ . So,  $\mathcal{F}^{-1}(\frac{1}{(1+\xi^2)^2}) = \frac{1}{4}u * u = \frac{1}{4} \int_{-\infty}^\infty e^{-|y|-|x-y|} dy = \frac{1}{4}(|x|e^{-|x|} + e^{-|x|})$ .

15. Do Fourier transform in the  $x$  direction, let  $v = \mathcal{F}u$ , we have  $v_t = -Ds^2v + cism$ , so  $v(s, t) = (\mathcal{F}\phi)(s)e^{(-Ds^2+cis)t}$ , and  $u(x, t) = \mathcal{F}^{-1}((\mathcal{F}\phi)(s)e^{(-Ds^2+cis)t}) = \phi * \mathcal{F}^{-1}(e^{(-Ds^2+cis)t}) = \phi * \left(\frac{1}{\sqrt{4\pi Dt}} e^{-(x-ct)^2/(4Dt)}\right)$ .



## 19 Solution of other exercise problems

2.2.6  $u(x, t) = \frac{1}{2}(e^{-|x-ct|} + e^{-|x+ct|}) + \frac{1}{2c}(\sin(x+ct) - \sin(x-ct)).$

2.3.4. I don't see a 2.3.4 in my textbook?

2.5.3.  $u(x, t) = \int_0^t w(x, t - \tau; \tau) d\tau = \int_0^t f(x - ct + c\tau, \tau) d\tau.$

2.5.4. Use 2.5.3,  $u(x, t) = \int_0^t (x - 2t + 2\tau)e^{-\tau} d\tau = (x - 2t)(1 - e^{-t}) - 2te^{-t} + 2 - 2e^{-t} = x - 2t + 2 - xe^{-t} - 2e^{-t}.$

2.6.10. Do Laplace transform in  $t$  direction, let  $v = L(u)$ , then  $s^2v = c^2v_{xx} - gs^{-1}$ , with boundary condition  $v(x, s) \not\rightarrow \infty$  as  $x \rightarrow \infty$ ,  $v(0, s) = 0$ . So  $v(x, s) = \frac{g}{s^3}(e^{-sx/c} - 1)$ ,  $u = \frac{g}{2}(x^2/c^2 - 2tx/c)$  when  $x < tc$  and  $-\frac{g}{2}t^2$  when  $x > tc$ . Note that when you use the table on page 114, all functions are 0 for  $t < 0$  or  $s < 0$ .

2.6.11. Do Laplace transform in  $t$  direction,  $v = Lu$ ,  $sv = v_{yy}$ , so  $v(x, y, s) = C(x, s)e^{-\sqrt{s}y}$ . Use the other boundary conditions one gets  $C(0, s) = 1/s$ ,  $sC(x, s) + C_x(x, s) = 0$ , so  $C(x, s) = e^{-xs}/s$ . The result can now be obtained from Table 2.1.

2.7.11. Use the hint, write  $v$  as in Example 2.18, then integrate the differential form  $vdy$ . You can also use Fourier transform directly to get a solution but without the arbitrary constant  $C$ .

2.7.16. (a)  $\omega = k^3$ .

(b) Let  $v = Fu$ , suppose the initial condition is  $u(x, 0) = f(x)$ , then  $v_t + is^3v = 0$ , so  $v(s, t) = F(f)e^{-is^3t}$ ,  $u = f * K(x, t)$  where  $K(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-is^3t - isx} ds = \frac{1}{2\pi} t^{-1/3} Ai(\frac{x}{t^{1/3}}).$

Midterm 2

1. (1) Find the inverse Fourier transform of  $\cos(x)e^{-|x|}$ . (6 points)
- (2) Find the Laplace transform of  $\cos(x)e^{-x}$ . (6 points)
- (3) Find the convolution between  $e^x$  and  $e^{-x^2}$ . (6 points)

Solution: (1)  $\frac{1}{2\pi} \left( \int_0^\infty \frac{1}{2}(e^{ix} + e^{-ix})e^{-x}e^{-isx}dx + \int_{-\infty}^0 \frac{1}{2}(e^{ix} + e^{-ix})e^xe^{-isx}dx \right) = \frac{1}{2\pi} \left( \frac{2}{1+(s+1)^2} + \frac{2}{1+(s-1)^2} \right)$ .

$$(2) \int_0^\infty \cos xe^{-x-xs}dx = \frac{1+s}{(1+s)^2+1}.$$

$$(3) \int_{-\infty}^\infty e^{x-y}e^{-y^2}dy = e^{x+1/4}\sqrt{\pi}.$$

2. Consider the initial-boundary value problem

$$u_t = u_{xx}, u(x, 0) = \sin x, u_x(0, t) = 1$$

on the region  $x > 0, t > 0$ .

- (1) Reduce it to a problem of the form

$$v_t = v_{xx} + f(x, t), v(x, 0) = g(x), v_x(0, t) = 0$$

by adding a function to  $u$ . (10 points)

- (2) Find the solution of the original initial-boundary value problem about  $u$ . (22 points)

Solution: (1) Let  $v = u - \sin x$ , then  $v_t = v_{xx} - \sin x$ ,  $v(x, 0) = 0$ ,  $v_x(0, t) = 0$ .

$$(2) u(x, t) = \sin x - \int_0^t \int_0^\infty \sin y (G(x+y, t-\tau) + G(x-y, t-\tau)) dy d\tau, \text{ where } G(x, t) = \frac{1}{\sqrt{4\pi}} e^{-x^2/(4t)}.$$

3. Consider the following problem:

$$u_{tt} = u_{xx} - 4u, u(0, t) = u(1, t) = 0, u(x, 0) = f(x)$$

on the region  $0 < x < 1, t > 0$ .

- (1) For any integer  $n$ , find a solution of  $u_{tt} = u_{xx} - 4u$ ,  $u(0, t) = u(1, t) = 0$  of the form  $u = \phi(t) \sin(n\pi x)$ . (10 points)

- (2) Find the solution of the original problem for  $f(x) = \sin(\pi x) - \sin(3\pi x)$ . (10 points)

Solution: (1)  $u = (A \cos(\sqrt{n^2\pi^2 + 4t}) + B \cos(\sqrt{n^2\pi^2 + 4t})) \sin(n\pi x)$ .

$$(2) u = \cos(\sqrt{\pi^2 + 4t}) \sin(\pi x) - \cos(\sqrt{9\pi^2 + 4t}) \sin(3\pi x).$$

4. Find the bounded solution of the following problem:

$$u_{tt} = u_{xx}, u_t(x, 0) = u(x, 0) = 0, u_x(0, t) = u(0, t) + \sin t$$

on the region  $x > 0, t > 0$ . You may want to use the Laplace transform or the general solution of 1-d wave equations. (16 points)

Solution 1:  $u = F(t-x) + G(t+x)$ ,  $u_t(x, 0) = u(x, 0) = 0$  implies that we can set  $F = 0$  on  $(-\infty, 0]$  and  $G = 0$ . The boundary condition is saying that  $F' = -F - \sin t$  so  $F(s) = -\int_0^s \sin re^{r-s} dr$ ,

$$u = \begin{cases} -\int_0^{t-x} \sin r e^{r-t+x} dr & t > x \\ 0 & t \leq x \end{cases}.$$

Solution 2: Laplace transform in  $t$  direction,  $v = Lu$ , then  $s^2 v = v_{xx}$ ,  $v(x, s) = C(s)e^{-xs}$ ,  $-sC = C + L(\sin t)$ , so  $v = -L(\sin t) \frac{1}{1+s} e^{-xs}$ ,  $u = -(\sin t * e^{-t+x})H(t-x)$ .

5. Find the bounded solution of  $u_{xx} + u_{yy} = u$ ,  $u(x, 0) = f(x)$  on the region  $y > 0$ . You may want to use the Fourier or Laplace transform. (14 points)

Solution 1: Do Fourier transform in the  $x$  direction, let  $v = F(u)$ , then  $-s^2 v + v_{yy} = v$ ,  $u(x, y) = f * F^{-1}(e^{-\sqrt{s^2+1}y})$ .

Solution 2: Do Laplace transform in the  $y$  direction, let  $v = L(u)$ , then  $v_{xx} + s^2 v - sf - g = v$ , where  $g(x) = u_y(x, 0)$ . Because  $v$  should decay as  $s \rightarrow \infty$ ,  $v = F^{-1}(\frac{1}{s^2 - \xi^2 - 1}(sF(f) + F(g)))$ . Furthermore, boundedness implies that there shouldn't be a pole when  $s^2 - \xi^2 - 1 = 0$ , which is only possible when  $F(g) = -\sqrt{\xi^2 + 1}F(f)$ , hence  $u = L^{-1}(f * F^{-1}(\frac{1}{s + \sqrt{\xi^2 + 1}}))$ .

## 20 Quiz 3

Find the Laplace transform of  $f(x) = \begin{cases} \sin(\pi x) & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ .

Solution:  $Lf = \int_0^1 \sin(\pi x) e^{-sx} dx = -\frac{i}{2} \int_0^1 (e^{-sx+i\pi x} + e^{-sx-i\pi x}) dx = \frac{i}{2} \left( \frac{e^{-s+i\pi}-1}{s-i\pi} + \frac{e^{-s-i\pi}-1}{s+i\pi} \right).$

## 21 3.1

1. a) b) d) are straightforward.

c)  $a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin x dx.$

e)  $u(x, t) = \sum_n \frac{2}{n\pi} \int_0^\pi f(x) \sin x dx \sin nct \sin nt.$

## 22 4.1

4.  $u(x, t) = \sum_n 2(\int_0^1 f(s) \sin(n\pi s) ds) e^{-t} (\cos(t\sqrt{n^2\pi^2/4 - 1}) + (n^2\pi^2/4 - 1)^{-1/2} \sin(t\sqrt{n^2\pi^2/4 - 1})) \sin(n\pi x).$

5. (a)  $u = 0$ . (b)  $u(x, t) = \sum_n \frac{2}{l} (\int_0^l f(s) \sin(n\pi s/l) ds) e^{-kn^2\pi^2/l^2 + ht} \sin(n\pi x/l).$

## 23 4.2

4. When  $\lambda < 0$ ,  $y(x) = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}$ , so the boundary condition is  $C_1 + C_2 + 2\sqrt{-\lambda}C_1 - 2\sqrt{-\lambda}C_2 = 0$ ,  $3(e^{2\sqrt{-\lambda}}C_1 + e^{-2\sqrt{-\lambda}}C_2) + 2(\sqrt{-\lambda}e^{2\sqrt{-\lambda}}C_1 - 2\sqrt{-\lambda}e^{-2\sqrt{-\lambda}}C_2) = 0$ , so there is a non-zero solution iff  $\sqrt{-\lambda}$  is the solution of  $e^{4t} = 1 + \frac{8t}{3-4t-4t^2}$ . By taking derivatives and intermediate value theorem there is a unique such  $t$  in  $(0, 1/2)$ . 0 is not an eigenvalue. There are infinite positive eigenvalues by Sturm Liouville theory.

8. The boundary condition is not symmetric e.g.  $y_1(a) = y_1(b) = y_2(a) = y_2(b) = y'_2(a) = 2y'_2(b) = 1$ ,  $y'_1(a) = y'_1(b) = 0$ . It is not self adjoint either. For  $u, v$  satisfying the boundary conditions,  $\int_a^b -u''v = v'u - vu'|_a^b + \int_a^b -uv''$ , and  $v'u - vu'|_a^b$  can not be guaranteed to be 0.

## 24 4.3

1. If  $u = X(x)T(t)$ , the first pde can be separated into  $-(x^2X')'/(x^2X) = T''/T = \lambda$ . If one let  $u = X(x)Y(y)$  in the second PDE, it becomes  $X''Y + (x^2 + 2xy + y^2)XY'' = 0$ , and is not separable.

4. If  $y$  is an eigenfunction with eigenvalue  $\lambda$ ,  $\lambda = \frac{-\int_1^\pi y(x^2y')' dx}{\int_1^\pi y^2 dx} = \frac{-\int_1^\pi x^2 y'^2 dx}{\int_1^\pi y^2 dx} \geq 0$ .

The differential equation is the Cauchy-Euler equation, hence eigenfunctions are  $x^{-1/2} \sin(\frac{\sqrt{4\lambda_n - 1}}{2} \log x)$ , and  $\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log \pi)^2}$ .