Source: On Iterated Maps Of The Interval
Milnor & Thurston 1981 (- 1983)

Motivation

the logistic growth model

Goal: Model a population of bunnies

i.e. cook up a function $F : \mathbb{N} \longrightarrow I := [0,1]$ s.t.

$F(n) = x$ means the # of bunnies after n years is $100 \cdot x$ % of the practical maximum (carrying capacity) see ③ below

Assumptions :

① the # of bunnies next year depends only on the # of bunnies this year

② the bunnies "are fruitful & MULTIPLY"

③ overpopulation is not sustainable

The rigorization :

$$\underbrace{F(n+1)}_{\text{“next year”}} = b \cdot \underbrace{F(n)\,\overbrace{(1-F(n))}^{\text{Controls overpopulation ③}}}_{\text{“this year”}}$$

growth rate ②

$\boxed{b \in [1,4]}$ ②

$\boxed{■ = g(■)}$ ①

On Iterated Maps Of The Interval

↳ the above equation implies the map $g(x) := bx(1-x)$ is iterated beginning with some initial population $F(0)$. This iteration produces a sequence $\{x_n\}_{n\in\mathbb{N}}$ of values in I.

$$x_n := g^n(F(0))$$

The value of the parameter b (i.e. bunny reproduction rate) has a <u>major impact</u> on the behavior of the sequence $\{x_n\}$.

<u>e.g.</u>

Sometimes...

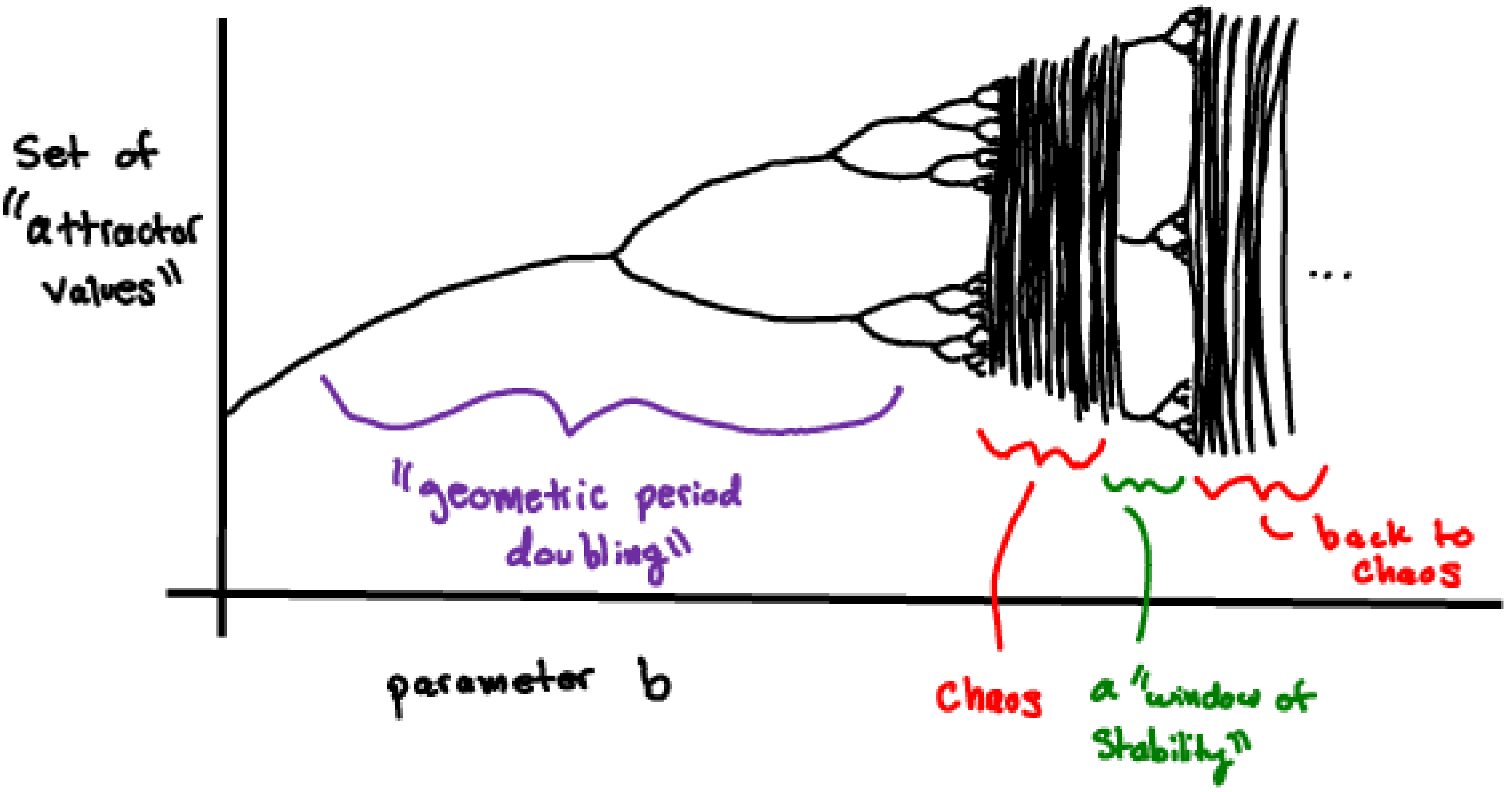
- x_n converges to a constant value

  i.e. $\exists N$ s.t. $\forall n \geq N$ $x_{n+1} = x_n$
- x_n is eventually periodic of period 2

  i.e. $\exists N$ s.t. $\forall n \geq N$ $x_{n+2} = x_n$
- x_n can be (& has been) used as a pseudorandom number generator

  i.e. x_n is chaotic

Rough sketch of orbit diagram



<u>def:</u> "attractor values" are sets of states toward which a system tends to evolve (wiki)

For particular values of b the associated sequence $\{x_n\}$ Stabilizes. As b increases, the period eventually doubles, then doubles again, and again ... the ratio of the lengths of the intervals of adjacent periods limits to a famous (and mysterious) constant known as the "Feigenbaum constant"

$$\approx \underbrace{4.6692}_{\frac{621}{133}}01609\ldots$$

$\frac{621}{133}$ ~ rational approximation

→ <u>Fun Fact:</u> the ratio between the diameters of Successive circles on the real axis of the Mandelbrot set (in $\mathbb{C}$) is also the Feigenbaum constant & this is not the only connection between logistic growth & the Mandelbrot set

Summary: On Iterated Maps Of The Interval

(i) $g : I \longrightarrow I$
$x \longmapsto bx(1-x)$ — a map of the interval depending on a parameter $b \in [1,4]$

(ii) $\{g^n(x)\}_{n\in\mathbb{N}} =: \{x_n\}_{n\in\mathbb{N}}$ — gives rise to a sequence of iterates applied to some initial value

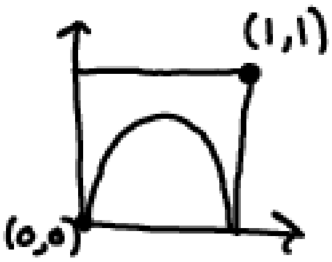
(iii) $x_{n+1} = bx_n(1-x_n)$ — by definition the sequence satisfies this recursive formula

$$\begin{tikzcd}
{[0,1]} \arrow[r, "g"] \arrow[d, "\wr"'] & {[0,1]} \arrow[d, "\wr"] \\
{[-b,b]} \arrow[r, "f"] & {[-b,b]}
\end{tikzcd}$$

$x \longmapsto bx(1-x)$

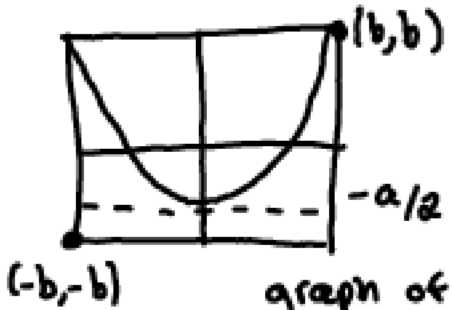
$x \mapsto b(1-2x)$

$x \longmapsto \dfrac{x^2-a}{2}$



graph of g

vertical arrows are topological conjugacy (homeomorphism) representing canonical linear change of coordinates given by replacing argument by the derivative of g



graph of f

$b \in [1,4]$ $\quad b \mapsto b^2-2b \quad$ $\exists a \in [-1,8]$

$1+\sqrt{1+a} \longleftarrow a$

the maps relating the parameters a & b are bijective & mutually inverse behavior of iterates of $f(x) := \frac{x^2 - a}{2}$ as a varies

single stable fixed point @ $1-\sqrt{1+a}$ & no periodic points of period > 1

the stable fixed point bifurcates into a stable orbit $-1 \pm \sqrt{a-3}$ (period 2)

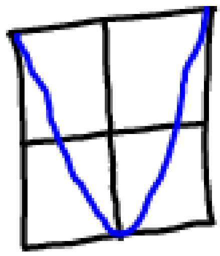
stable orbit of period 4

stable orbit of period 8

bifurcations happen as often as this limit is approached

\# of turning points of iterates of f^n grows exponentially in n

graph of f



Q: What does it mean for a fixed point to be "stable?"

A: "Invariant coordinate" nearby this point agree w/ its invariant coord

The General Story

Setting:

(i) $f : I \longrightarrow I$ smooth map of the interval

i.e. $f^{(n)}(x)$ exists $\forall n \in \mathbb{N}, x \in I$

$\Longleftrightarrow$ we can take derivatives whenever we like

(ii) f piecewise monotone

the domain

i.e. $\exists$ decomposition of $I = (\bigcup_{i=1}^{\ell} I_i)$ s.t.

"Laps"

$$I_i \cap I_j = \begin{cases} \emptyset & \text{if } j \neq j \pm 1 \\ c_i & \text{if } j = i+1 \end{cases}$$

"turning points"

$c_0 := 0$, $c_\ell := 1$ so that $I = [0,1] = [c_0, c_\ell]$

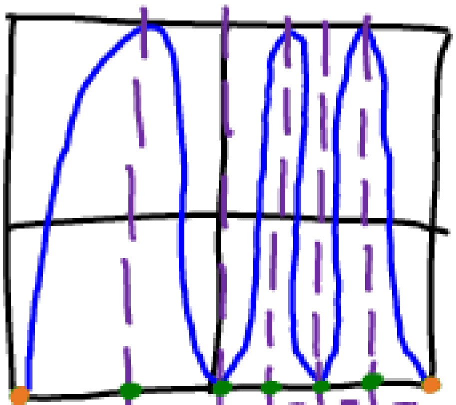
& $\forall i \in \{1, \dots, l\}$ either

f is monotone decreasing on this Lap → $f(x) \geq f(y)$ for all $x, y \in I_i$

f is monotone increasing on this Lap → or $f(x) \leq f(y)$ for all $x, y \in I_i$

& this decomposition is maximal (each I_i is a Lap - cannot be extended to any larger monotone interval $\Leftrightarrow$ intersections $I_i \cap I_{i+1}$ are local extrema of f

e.g.



6 Laps := maximal monotone intervals

5 turning points := intersections of two laps $\Updownarrow$ local min/max

2 end points

Outline:

(I) Study the Laps of f & its iterates

↳ $\ell(-) : \{\text{maps } f: I \to I \text{ satisfying (i)\&(ii)}\} \longrightarrow \mathbb{Z}_{>0}$

$g \longmapsto \ell(g) :=$ # laps of g

↳ $\ell(f^*) := \{\ell(f^n)\}_{n \in \mathbb{N}}$ sequence of positive integers composed of lap # of iterates of f.

↳ $\lim_{n \to \infty} \ell(f^n)^{1/n}$:= "Growth # of f"

(II) Study the behavior of a single point under iterates of f

$$\hookrightarrow A_f(-,-): \mathbb{N} \times I \longrightarrow \left(\bigcup_{i=1}^{\ell} I_i\right) \cup \left(\bigcup_{j=0}^{\ell} c_j\right)$$

$$(n,x) \longmapsto A(f^n(x)) := \text{"Address of } x \text{ under } f^n\text{"}$$

$\approx$ indicator function for laps & turning points

$$:= \begin{cases} I_i & \text{if } f^n(x) \in I_i^{int} \\ c_j & \text{if } f^n(x) = c_j \end{cases}$$

$$\hookrightarrow A(f^*(-)): I \longrightarrow \{\text{sequences of addresses}\}$$

$$x \longmapsto A(f^*(x)) := \text{"Itinerary of the point } x\text{"}$$

$$:= (A(f^0(x)), A(f^1(x)), \dots)$$

$$\hookrightarrow \theta: I \longrightarrow \underbrace{(\overbrace{\mathrm{Vect}_{\mathbb{Q}}\{I_1, \dots, I_\ell\}}^{\vee}) [[t]]}_{\text{Formal power series ring in } t \text{ over a } \mathbb{Q}\text{-vector space w/ the laps of } f \text{ as formal basis vectors}}$$

"Invariant coordinate function"

⚠ The codomain of this map seems to come from nowhere & is very a very strange object

Q1: Where does it come from?

Q2: What does it afford us?

A1: this map serves as a refinement of the itinerary of a point. Rather than thinking of itineraries as sequences of Laps & turning points we can view the n^{th} term of the sequence (i.e. the $n+1^{st}$ address $\lambda(f^n(x))$) as the coefficient of t^n in some formal power series. These coefficients are now elements of a vector space, where

$$I_i \longleftrightarrow I_i$$

$$C_i \longleftrightarrow \frac{I_i + I_{i+1}}{2}$$

this is why the field needs to be $\mathbb{Q}$

Finally, we can keep track of the local behavior of f^n @ each point using + & − signs. (i.e. this allows us to encode information about derivatives of iterates of f at points too)

<u>A2:</u> There are 3 main advantages to this approach

a) The map $\theta : I \longrightarrow V[[t]]$ satisfies $\theta(x) \leq \theta(y)$ whenever

$$x < y$$

(i.e. θ is non-decreasing or weakly order preserving w.r.t. an appropriate ordering on $V[[t]]$)

b) there are 2 different ways of thinking about $V[[t]]$

c) with some work we can use θ to construct power series in $\mathbb{Z}[[t]]$ that converge in some disk of positive radius in $\mathbb{C}$ w/ <u>smallest real zero</u>

coinciding w/ the topological entropy of the map f.

Woah! How?

(III) Study Neighborhoods of turning points via continuity (or more precisely discontinuity) properties of the map θ

↳ Left limits

$$\theta(c_i^-) := \lim_{\substack{x \to c_i \\ x < c_i}} \theta(x)$$

Right limits

$$\theta(c_i^+) := \lim_{\substack{x \to c_i \\ x > c_i}} \theta(x)$$

↳ "Kneading increments" - measure the discontinuity of θ @ turning points c_i

$$V_i := \theta(c_i^+) - \theta(c_i^-)$$

recall left lim = right lim

$\Updownarrow$

continuous

hence $V_i = 0 \Leftrightarrow \theta$ continuous @ c_i

↳ "Kneading Matrix" - matrix of kneading increments

$$[N_{ij}] \in M_{(\ell-1)\times \ell}(\mathbb{Z}[[t]])$$

$$[N_{ij}] = \begin{bmatrix} | & | & & | \\ V_1 & V_2 & \cdots & V_{\ell-1} \\ | & | & & | \end{bmatrix}$$

Kneading increments are columns of kneading matrix

↳ "Kneading Determinant"

Note: the method of scaling makes this statement non-vacuous

Delete Any column & take the determinant when scaling these determinants in a particular way, we always get the same answer - a power series in $\mathbb{Z}[[t]]$

(IV) Study the properties of the kneading determinant as a function $\mathbb{C} \longrightarrow \mathbb{C}$ (i.e drop the "formality" assumption on our power series)

↳ the radius of convergence is

$$\frac{1}{\text{growth}} \; *$$

↳ Smallest real root on $[0,1)$ is topological entropy.

Summary:

For a piecewise monotone, smooth map of the interval f we have the following topological conjugacy invariants

the tail of purple arrows encodes the tip

- Lap numbers of iterates
- Growth number $\left(= e^{\text{topological entropy}}\right)$
- Itineraries of points
- Invariant coordinates of points
- Kneading increments
- the kneading matrix
- the kneading determinant

Claim: The Itinerary of a point retains a decent amount of information about the point & its images under f^n