Topics:

- L^2 Betti numbers, Novikov-Shubin invariant, L^2 torsion.
- Determinant and approximation conjecture.

1 Notations and Reviews

$\mathbb{C}[G]$: group algebra of G. $l^2(G)$: L^2 summable functions on G, can be seen as a L^2 completion of $\mathbb{C}[G]$. $\mathcal{N}(G)$: G-equivariant (with left G action) bounded linear maps on $l^2(G)$. A finite dimensional Hilbert $\mathcal{N}(G)$-module is a G-Hilbert space that G-equivariantly isometrically embedded into $\mathcal{C}^n \otimes l^2(G)$. Unless specified otherwise, all spaces are assumed to be finitely dimensional Hilbert G modules.

1.1 Spectral theory of bounded self adjoint operators

Let H be a Hilbert space. A bounded operator A on H is called self adjoint if $A^* = A$, positive if $(Ax, x) > 0$ for all $x \in H$.

E is called a spectral measure if it sends Borel sets in $\mathbb{R}$ to bounded positive self adjoint maps on H, satisfying the following:

- $E(\emptyset) = 0$, $E(\mathbb{R}) = I$
- $E(A)^2 = E(A)$
- $E(A \cap B) = E(A)E(B)$
- If $A \cap B = \emptyset$, $E(A \cup B) = E(A) + E(B)$
- $\forall x, y \in H$, $(E(\cdot)x, y)$ is a $\mathbb{C}$-measure.

Theorem 1
A is bounded self adjoint on H then there is some spectral measure E such that $A = \int \lambda E(\lambda)$. If f is an analytic function, $f(A) = \int f(\lambda)E(\lambda)$

Remark 1. When H is finite dimensional, $E = \sum \delta_{\lambda_i} v_i^* v_i$.

2 L^2-Betti numbers

2.1 Definitions

- G is a group, f is a G-equivariant self adjoint bounded operator from $\mathbb{C}^n \otimes l^2(G)$. The L^2 **trace** is defined as $tr_G(f) = \sum_i (f(e_i \otimes 1), e_i \otimes 1)$. The L^2-trace of a self map on Hilbert G modules are defined as the composition of projection and this self map.
- Let $M \subset \mathbb{C}^n \otimes l^2(G)$ be a Hilbert G module, pr_M the orthogonal projection on M. Then the L^2**-dimension** is defined as $\dim_G M = tr_G(pr_M)$. (Exercise: prove that the L^2-dimension does not depend on the choice of the embedding.)

- L^2**-chain complex** of finitely dimensional Hilbert G modules is a sequence $\dots \to C_{k+1} \to C_k \to C_{k-1} \to \dots$ such that the composition of two successive boundary maps is 0. The **homology** are $H_k^{(2)} = ker(C_k \to C_{k-1})/\overline{im(C_{k+1} \to C_k)}$.

- Let X be a CW-complex with a free, cellular G left action such that $G\backslash X$ is finite. Then L^2 dimension of the homology of the L^2 completion of the cellular chain complex $(C_\cdot^{(2)})$ are called the L^2 **Betti numbers** $b_\cdot^{(2)}$.

2.2 Examples

Example 2. $X = \mathbb{R}^2$ tiled by unit cubes, $G = \mathbb{Z}^2$ (with generators a, b), $G\backslash X = T^2$. The L^2 chain complex is

$$0 \to l^2 \to (l^2)^2 \to l^2 \to 0$$

Such that $\partial_2(x) = ((xa-x),(x-xb))$, $\partial_1(x,y) = xb-x+ya-y$. By computation we see $H_*^{(2)} = 0$, hence $b_*^{(2)} = 0$.

Example 3. X a double cover of the θ graph unwrapping over one of the two loops, $G = \mathbb{Z}/2$, $G\backslash X$ is the θ-shaped graph. $H_1^{(2)}$ is of dimension 3, $(pr_{H_1}(e), e)$ can be computed explicitly, and $b_1^{(2)} = 3/2$, $b_0^{(2)} = 1/2$.

Remark 4.

- When X is a finite simplicial graphs, $pr_{H_1}(e)$ can be calculated via spanning trees: Let $\mathcal{T}$ be the set of all spanning trees on X. For any $T \in \mathcal{T}$, let $path(T, e^-, e^+)$ be the path on T from e^- to e^+.

$$pr_{H_1}(e) = e - \frac{1}{|\mathcal{T}|}\sum_{T\in\mathcal{T}} path(T, e^-, e^+)$$

- There is also a physical interpretation of $pr_{H_1}(e)$ via electrical currents.

- In general, if G is a finite group, $b_k^{(2)}(X) = b_k(X)/|G|$.

Example 5. X is the universal cover of θ-shaped graph, $G = F_2$ the deck transformation. $pr_{H_1}e$ can be explicitly calculated (hint: first show that the element in $C_1^{(2)}$ of a complete binary tree whose ∂ is at the root that minimizes the norm has norm 1) $(pr_{H_1}e, e) = 1/3$, $b_1^{(2)} = 1$, $b_0^{(2)} = 0$.

Remark 6. There are alternative interpretations of the computation above through electrical currents and random walks.

2.3 Elementary properties of L^2 dimension, L^2 homology and L^2 Betti numbers

Some elementary properties of L^2 trace:

- $f \leq g \implies tr_G(f) \leq tr_G(g)$
- If f_i is increasing and weakly converging to f, $tr_G f = \sup\{tr_G(f_i)\}$.
- $f \geq 0$, $tr_G(f) = 0 \iff f = 0$.
- $tr_G(f + \lambda g) = tr_G(f) + \lambda tr_G(g)$
- f, g and h are self adjoint maps compatible with an exact sequence of Hilbert G modules, then $tr_G(g) = tr_G(f) + tr_G(h)$.
- $f : U \to V$, then $tr_G(f^* f) = tr_G(f f^*)$
- f and g are maps on Hilbert G and H modules, then $tr_{G\times H} f \otimes g = tr_G f \otimes tr_H g$.
- H is a finite index subgroup of G, then $tr_H f = [G : H] tr_G f$

Proof: use definition and functional analysis.
Some elementary properties of L^2-dimensions:

- $\dim_G(V) = 0 \iff V = 0$.
- $0 \to U \to V \to W \to 0$ weakly exact (L^2 homology vanishes), then $\dim_G(V) = \dim_G(U) + \dim_G(W)$.
- V_i increasing, $\dim_G \overline{\cup_i V_i} = \sup_i \dim_G V_i$.
- V_i decreasing, $\dim_G \cap_i V_i = \inf_i \dim_G V_i$.
- U, V are G and H modules respectively, then $\dim_{G\times H} U \otimes V = \dim_G U \dim_H V$.
- $[G : H] < \infty$, $dim_H(V) = [G : H] dim_G V$.

Theorem 2
Let $0 \to C_* \to D_* \to E_* \to 0$ be an exact sequence of chains of G-modules, then there is a long exact sequence which is weakly exact.

Some elementary properties of L^2 Betti numbers

- $f : X \to Y$ cellular G-equiv maps between free G cell complexes, with induced map on homology isomorphism for $p < d$ and surjective for $p = d$, then so are the induced maps on L^2 homology. (proof: chain homotopy, then use long exact sequence)
- X free G cell complex with $G\backslash X$ finite. Then $\chi(G\backslash X) = \sum_k (-1)^k b_k^{(2)}(X)$.

- X is a cocompact d-dimensional manifold, then $b_p^{(2)} = b_{d-p}^{(2)}$.
- Künneth formula for products, formula for wedges, connected sums for manifolds of dimension at least 3, Morse inequalities all same as the usual Betti numbers.
- If X is connected, $b_0^{(2)} = 1/|G|$.
- $[G : H] < \infty$, then X seen as H complex has L^2-Betti numbers $[G : H]$ of it seen as G complex.

Theorem 3
f a cellular map of a finite connected complex, T_f its mapping tori, $\pi_1(T_f) \to G \to \mathbb{Z}$ for some G, then the G-cover of T_f, denoted as $\overline{T_f}$ and seen as G-complex has zero L^2 Betti numbers.

Proof. Let G_n be the preimage of $n\mathbb{Z}$ in $G \to \mathbb{Z}$. Then $\overline{T_f}$ has n-times as much L^2-Betti numbers, however $G_n \backslash \overline{T_f} = T_{f^n}$ has bounded number of cells, hence all Betti number has to be 0. □

3 Approximation for subgroups of finite index

Theorem 4
X is a cell complex with free cellular G action as before, $G\backslash X$ finite. $G \supset G_1 \dots$ normal subgroups such that $\cap_i G_i = 1$, $[G : G_1] < \infty$, then $b_k^{(2)}(X) = \lim_{i\to\infty} b_k^{(2)}(G_i\backslash X)$, the latter as G/G_i complexes.

This can be easily reduced to the following "algebraic" statement:

Proposition 7
Suppose f is a positive self adjoint map on $\mathbb{C}^n \otimes l^2(G)$ induced by a (left) $\mathbb{Z}[G]$ module homomorphism, f_i be the induced maps on $\mathbb{Z}[G/G_i]$, then $dim_G \ker(f) = \lim_{i\to\infty} \dim_{G/G_i} \ker(f_i)$.

Proof. Step 1: Let K be n^2 of the largest sum of all coeff of an entry in the matrix representing f, then it is larger than the operator norm of both f and f_m.

Step 2: The map can be represented as a right multiplication of a $\mathbb{Z}[G]$-matrix, hence $tr_G(f) = tr_{G/G_i}(f_i)$ for large enough i. Furthermore, for any polynomial p, $tr_G(p(f)) = tr_{G/G_i}(p(f_i))$ for large enough i.

Step 3: Let F, F_i be the spectral density function for f and f_i ($F(\lambda) = \dim_G(E([0,\lambda])$, $F(\lambda) = \dim_{G/G_i}(E_i([0,\lambda]))$. Let $\overline{F}$, $\underline{F}$ be the lim sup and lim inf of F_i. We shall prove that $\overline{F} \leq F \leq \underline{F}^+$. Let p_n be polynomials above $\chi([0,\lambda])$ and below $\chi([0,\lambda+1/n] + 1/n\chi([0,K])$ slightly above that. Then

$$\overline{F}(\lambda) \leq tr_G(p_n(f)) \leq \underline{F}(\lambda + 1/n) + 1/n$$

And as $n \to \infty$ the middle term converges to $F(\lambda)$ due to spectral decomposition.

Step 4: We now prove that F_i are uniformly right-continuous at 0. This is due to a fact in linear algebra:

Lemma 8. f: self adjoint positive linear map on $\mathbb{C}^n$. K a bound on operator norm of f, C a lower bound on the first non-zero term of characteristic polynomial of f. Then for $\lambda < 1$,

$$\frac{\text{num. of roots in } (0, \lambda)}{n} \leq \frac{-\log(C)}{n(-\log(\lambda))} + \frac{\log(K)}{-\log(\lambda)}$$

Proof. Count non-zero roots.

Because the matrix is integral C can be chosen uniformly as 1, which finishes the proof.

Example 9. X is the universal cover of closed surfaces, G the deck group.

Example 10. Let Γ be a finite graph, $\Gamma \leftarrow \Gamma_1 \leftarrow \dots$ regular covers, for every edge $e \in \Gamma$, let $d_i(e)$ be the ratio of spanning trees of Γ_i that doesn't contain a specific lift of e. Then d_i converges.

4 Other L^2 invariants

4.1 Definition

F is the spectral density function.

- **Novikov-Shubin invariants** $\alpha(F) = \liminf_{\lambda \to 0^+} \frac{log(F(\lambda) - F(0))}{\log \lambda}$
- **Fuglede-Kadison determinant** $\det = \exp(\int \log(\lambda) dF)$.

Remark 11.

- Determinant conjecture: For any group G, any $\mathbb{Z}[G]$ matrix f, the F-K determinant of $f^* f$ is at least 1.
- Determinant conjecture implies approximation for any sequence of subgroups.