

1 1.1

2. $u_{xx} + u_{yy} = (2x \cdot \frac{1}{2}(x^2 + y^2)^{-1})_x + (2y \cdot \frac{1}{2}(x^2 + y^2)^{-1})_y = 2(x^2 + y^2)^{-1} - (2x^2 + 2y^2) \cdot (x^2 + y^2)^{-2} = 0.$

4. The statement of this problem is somewhat unclear in whether they mean $(u_{xx})^2 + (u_{yy})^2 = 0$ (the more likely one) or $(u^2)_{xx} + (u^2)_{yy} = 0$, so either interpretation would be considered correct. With the first interpretation it is obvious that all function in the stated form satisfy that $u_{xx} = u_{yy} = 0$. With the second there would need to be additional constraints on a, b, c, d for it to work.

5. The general solution is $u = xF(t) + G(t)$, hence one can let $u = t^2 + x(1 - t^2)$.

6. $u_{tt} = (g(x + ct) + g(x - ct))_t = c(g'(x + ct) - g'(x - ct)), u_{xx} = c^{-1}(g(x + ct) - g(x - ct))_x = c^{-1}(g'(x + ct) - g'(x - ct)).$

7. $(e^{at} \sin bx)_t = ae^{at} \sin bx, (e^{at} \sin bx)_{xx} = -b^2 e^{at} \sin bx$, hence $a = -kb^2$.

8. $(u_x)_t = 1 - 3u_x$, hence $u_x = \frac{1}{3} + e^{-3t}f(x)$ for some arbitrary function f , hence $u(x, t) = \frac{x}{3} + e^{-3t}F(x) + G(t)$ for arbitrary function F (which is the anti-derivative of f) and G .

12. To sketch wave profile, pick some k, A, D or c , sketch $u(x, t)$ for different values of t , and if u is complex-valued you can sketch either the real or imaginary part.

Dispersion relations: a) $\omega = -iDk^2$. b) $\omega = \pm ck$. c) $\omega = -k^3$. d) $\omega = k^2$. e) $\omega = ck$.

14. Dispersion relation is $\omega = (-1 + \delta k^2 - k^4)i$ hence this is diffusive. When $\delta = k^2 + 1/k^2$ the solution has growth rate 0. When $k^2 + 1/k^2 > \delta$ the solution decays.

2 1.2

1. From equation (1.7) in the text we have $\frac{d}{dt} \int_a^b u A dx = A\phi|_a - A\phi|_b$. Differentiate with respect to b (or use some other argument, for example as in the textbook), we have $Au_t = -A_x\phi - A\phi_x$, hence $u_t + \phi_x = -A'\phi/A$.

3. By chain rule, $u_x = u_\xi$, $u_t = -cu_\xi + u_\tau$, hence the equation (1.12) becomes $u_\tau = -\lambda u$, hence the general solution is $u = e^{-\lambda\tau}F(\xi) = e^{-\lambda t}F(x - ct)$.

4. $u_t + cu_x = -\lambda u$. If $w = ue^{\lambda t}$, $u = we^{-\lambda t}$ hence $u_t + cu_x = w_t e^{-\lambda t} - \lambda w e^{-\lambda t} + cw_x e^{-\lambda t}$, $-\lambda u = -\lambda w e^{-\lambda t}$, hence $w_t + cw_x = 0$.

5. By method of characteristics $u_t + xtu_x = 0$ has characteristics $x = Ce^{t^2/2}$, hence the general solution is $u = F(xe^{-t^2/2})$. Together with the initial value condition we know that $F = f$ hence $u = f(xe^{-t^2/2})$. The general solution of $u_t + xu_x = e^t$ is $u = e^t + F(xe^{-t})$, so with the initial condition, the solution should be $u = e^t + f(xe^{-t}) - 1$.

6(b). The characteristics are $x = Ct$, and the general solution is $u = e^{-2t}F(x/t)$. Use the initial condition we get $F = e^2 f$, hence $u = e^{-2(t-1)}f(x/t)$.

7. The general solution is $u = e^{-\lambda t}F(x - ct)$. The initial-boundary condition tells us that $F(x) = 0$ for $x > 0$ and $e^{-\lambda t}F(-ct) = g(t)$ for $t > 0$, hence $F(x) = \begin{cases} 0 & x > 0 \\ e^{\lambda x/c}g(x/c) & x \leq 0 \end{cases}$.

12. By the method of characteristics, $u(x, t) = F(x - ct)e^{(\alpha t - u)/\beta}$. Set $t = 0$ we have $F(x) = f(x)e^{f/\beta}$ hence $u(x, t) = f(x - ct)e^{(\alpha t - u + f(x - ct))/\beta}$.

14. Characteristics are $x = Ce^{-ut}$ hence $u = F(xe^{ut})$. Together with the initial condition we get $u = xe^{ut}$. A solution does not exist for all t . For example, there doesn't exist any u at point $x = t = 1$ because $s < e^s$ for all $s \in \mathbb{R}$.

3 1.3

2. $\frac{d}{dt} \int_0^l u^2 dx = \int_0^l 2uu_t dx = \int_0^l 2kuu_{xx} dx = 2kuu_x|_0^l - \int_0^l 2k(u_x)^2 dx \leq 0$, hence $\int_0^l u^2 dx \leq \int_0^l u_0^2 dx$ for $t \geq 0$.

3. Let $w = u - g + (x/l)(h - g)$, then $w(0, t) = w(l, t) = 0$, $u_t = ku_{xx}$ will imply $w_t = kw_{xx} - g' + (x/l)(h' - g')$.

4. The steady state satisfy $0 = ku_{xx} - hu$ and $u(0) = u(1) = 1$, hence $u = \frac{e^{(h/k)^{1/2}(x-1/2)} + e^{(h/k)^{1/2}(1/2-x)}}{e^{(h/k)^{1/2}/2} + e^{-(h/k)^{1/2}/2}}$.

5. $u_t = w_t e^{\alpha x - \beta t} - \beta w e^{\alpha x - \beta t} = w_t e^{\alpha x - \beta t} - \beta u$, $u_x = w_x e^{\alpha x - \beta t} + \alpha u$, $u_{xx} = w_{xx} e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha^2 u$, hence $0 = u_t - Du_{xx} + cu_x + \lambda u = (w_t - Dw_{xx})e^{\alpha x - \beta t} + (c - 2D\alpha)w_x e^{\alpha x - \beta t} + (\lambda - \beta - D\alpha^2 + c\alpha)u$, so when $\alpha = c/(2D)$ and $\beta = \lambda - D\alpha^2 + c\alpha = \lambda + c^2/(4D)$, $0 = w_t - Dw_{xx}$.

6. The steady state doesn't depend on the initial condition. It is $u = \frac{1}{2k}x(1 - x)$.

10. The flux is $Du_x + u^2/2$. Replace $u = \psi_x$ we have $\psi_{xt} = D\psi_{xxx} + \psi_x \psi_{xx}$. Integrate along x we have $\psi_t = D\psi_{xx} + (\psi_x)^2/2 + F(t)$. Replace ψ_t with $\psi_t + \int_0^t F(s)ds$ we can get rid of F . Now let $\psi = -2D \ln v$ we get $-2Dv_t/v = -2D^2(v_{xx}v - (v_x)^2)/v^2 + 2D^2(v_x)^2/v^2$, hence $v_t = Dv_{xx}$.

4 1.4

3. For $u_t = Du_{xx} - cu_x$, the time independent solution satisfies $0 = Du_{xx} - cu_x$. So the solution is $u = C_1 + C_2 e^{cx/D}$. For $u_t = Du_{xx} - cu_x + ru$, the time independent case reduces to $0 = Du_{xx} - cu_x + ru$, the characteristic polynomial is $D\lambda^2 - c\lambda + r = 0$ whose roots are $r = \frac{c \pm \sqrt{c^2 - 4Dr}}{2D}$. Hence, when $c^2 = 4Dr$ the general solution is $u = (C_1 + C_2 x)e^{\frac{cx}{2D}}$, when $c^2 > 4Dr$ the general solution is $u = C_1 e^{\frac{xc + x\sqrt{c^2 - 4Dr}}{2D}} + C_2 e^{\frac{xc - x\sqrt{c^2 - 4Dr}}{2D}}$, when $c^2 < 4Dr$ the general solution is $u = C_1 e^{\frac{xc}{2D}} \cos(\frac{x\sqrt{4Dr - c^2}}{2D}) + C_2 e^{\frac{xc}{2D}} \sin(\frac{x\sqrt{4Dr - c^2}}{2D})$.

9. $u = ax + b$ then $u_{xx} = 0$.

$u = a \ln r + b$ then $u_{xx} + u_{yy} = a(\frac{x}{r^2})_x + a(\frac{y}{r^2})_y = a(\frac{r^2 - 2x^2 + r^2 - 2y^2}{r^4}) = 0$.

$u = \frac{a}{\rho + b}$ then $u_{xx} + u_{yy} + u_{zz} = a((\frac{x}{\rho^3})_x + (\frac{y}{\rho^3})_y + (\frac{z}{\rho^3})_z) = 0$.

12. (a) $\frac{d}{dt} \int_a^b 2\pi r u dr = 2\pi a(-Du_r|_a) - 2\pi b(-Du_r|_b)$. Differentiate on b we get $bu_t|_b = Dbu_{rr}|_b + Du_r|_b$, hence $u_t = Du_{rr} + \frac{D}{r}u_r = D\frac{1}{r}(ru_r)_r$.

(b) $\frac{d}{dt} \int_a^b 4\pi r^2 u dr = 4\pi a^2(-Du_r|_a) - 4\pi b^2(-Du_r|_b)$. Differentiate on b then you get the differential equation.

5 1.5

1. You can do it however you want, for example, in the 3rd equation on page 51, add a term $-\int_a^b \rho_0 g dx$ to the right.

3. Verification is by chain rule. Sketch $u = \frac{1}{2}(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2})$.

4. The initial condition is $u_n(x, 0) = \sin \frac{n\pi x}{l}$, $(u_n)_t(x, 0) = 0$. The frequency is $\frac{cn}{2l}$, they decrease as l increases and as c (tension) increases.

5. $\frac{d}{dt} E = \int_0^l (\rho_0 u_t u_{tt} + \tau_0 u_x u_{tx}) dx = \tau_0 \int_0^1 (u_t u_{xx} + u_x u_{tx}) dx = \tau_0 u_t u_x|_0^l = 0$.

9. $I_x + CV_t + GV = 0$, so $I_{xx} + CV_{xt} + GV_x = 0$. Substitute $V_x = -LI_t + RI$, we get that I satisfy the telegraph equation. The fact that V satisfy telegraph equation follows analogously. When $R = G = 0$ the speed of wave is $(LC)^{-1/2}$.

6 1.7

1. $\text{div}(\text{gradu}) = \text{div}((u_x, u_y, u_z)) = u_{xx} + u_{yy} + u_{zz}$.

7 Quiz 1:

$u_t + (x+1)u_x = 1$, $u(x, 0) = \sin x$.

Characteristics are $x = Ce^t - 1$. Hence $u = t + F((x+1)e^{-t})$, hence $F(x) = \sin(x-1)$ and $u = t + \sin((x+1)e^{-1} - 1)$.

8 1.7

3. This is divergence theorem. The heat generated in Ω equals the heat flowing out at the boundary.

4. Let $\phi = (\phi_1, \phi_2, \phi_3)$, then $\text{div}(w\phi) = (w\phi_1)_x + (w\phi_2)_y + (w\phi_3)_z = (w_x\phi_1 + w_y\phi_2 + w_z\phi_3) + w((\phi_1)_x + (\phi_2)_y + (\phi_3)_z) = \phi \cdot \text{grad} w + w \text{div} \phi$. Let $\phi = \text{gradu}$ then Green's identity follows from this and the divergence theorem.

5. $\lambda = \frac{\int_{\Omega} u \Delta u dV}{\int_{\Omega} u^2 dV} = -\frac{\int_{\Omega} |\text{gradu}|^2 dV}{\int_{\Omega} |u|^2 dV} < 0$,

6. Let $w = u + v$ where v is 0 at the boundary, then $\int_{\Omega} |\text{grad} w|^2 dV = \int_{\Omega} |\text{grad} u|^2 dV + \int_{\Omega} |\text{grad} v|^2 dV + 2 \int_{\Omega} \text{grad} u \cdot \text{grad} v dV$. By 4 and the assumption, the last term is 0, hence $\int_{\Omega} |\text{grad} w|^2 dV \geq \int_{\Omega} |\text{grad} u|^2 dV$.

7. Use $\text{cpu}_t = \text{div} \phi$.

9 1.8

1. Maximum are at $r = 2$, $\theta = \pi/4, 5\pi/4$, minimum are at $r = 2$, $\theta = 3\pi/4, 7\pi/4$.

2. $u = (x^2 + y^2)/4 - a^2/4$.

4. The solution is spherical symmetric because the function and the boundary conditions are both spherical symmetric, i.e. $u = u(\rho)$. Hence $\Delta u = 1$ reduces to $u_{\rho\rho} + \frac{2}{\rho}u_{\rho} = 1$, hence $(\rho^2 u_{\rho})_{\rho} = \rho^2$, hence $\rho^2 u_{\rho} = \frac{1}{3}\rho^3 + C_1$, $u' = \frac{1}{3}\rho + \frac{C_1}{\rho^2}$, hence $u = \frac{1}{6}\rho^2 - \frac{C_1}{\rho} + C_2$. Apply the boundary condition one gets $u = \frac{1}{6}\rho^2 + \frac{b^3}{3\rho} - \frac{1}{6}a^2 - \frac{b^3}{3a}$.

5. $u = A \tan(x) + B$, solve for constants A and B using the boundary condition.

6. $u = A \log r + B$. $u = \frac{10}{\log 2} \log r$.

8. Use chain rule.

9. $\text{curl} E = 0$ implies that such a potential exists. $\Delta V = \text{divgrad} V = \text{div} E = 0$.

10 1.9

1. This is a parabolic equation. $u = F(kx-t) + (x+kt)G(kx-t)$, or you can write it in other equivalent ways.

2. Let $p = 2x + t$, $q = t$, then $u_x = 2u_p$, $u_{xx} = 4u_{pp}$, $u_t = u_p + u_q$, $u_{xt} = 2u_{pp} + 2u_{pq}$, hence the equation becomes $u_p = 4u_{qp}$, hence $u = F(2x+t)e^{t/4} + G(t)$.

3. It is hyperbolic. Under the change of variable, by chain rule, $u_x = \frac{4}{x}u_{\tau}$, $u_{xx} = \frac{16}{x^2}u_{\tau\tau} - \frac{4}{x^2}u_{\tau}$, $u_{xt} = \frac{4}{x}u_{\tau\xi} + \frac{4}{x}u_{\tau\tau}$, so $0 = xu_{xx} + 4u_{xt} = -\frac{4}{x}u_{\tau} - 16u_{\tau\xi}$, hence $u = e^{-\xi/4}f(\tau) + g(\xi)$.

4. Use chain rule and product rule.

5. Elliptic. Find the eigenvalues of matrix $\begin{bmatrix} 1 & -3 \\ -3 & 12 \end{bmatrix}$.

6. Parabolic. The general solution calculation is similar to 3 above.

7. a) Elliptic when $xy > 1$ and hyperbolic when $xy < 1$. b) Elliptic.

Midterm 1

1. Solve the following initial or initial/boundary value problems:

- (1) $u_t = xu_x$, $u(x, 0) = x^2$. Here u is a function of x and t . (25 points)
(2) $u_t + u_x = \sin x$, $u(x, 0) = 0$ for $x \geq 0$, $u(0, t) = t$ for $t \geq 0$. Here u is a function of x and t . (15 points)

Answer: (1) General solution is $u = F(xe^t)$, hence $u = x^2e^{2t}$.
(2) General solution is $u = -\cos x + F(x-t)$, so $u = -\cos x - (x-t) + 1$ when $x \leq t$, and $u = -\cos x + \cos(x-t)$ when $x \geq t$.

2. (1) Find the general solution of $u_{tt} = u_{tx}$. (15 points)
(2) Find the solution of the initial value problem: $u_{tt} = u_{tx}$, $u(x, 0) = 0$, $u_t(x, 0) = x$. (10 points)

Answer: (1) $u_t = f(x+t)$, so $u = F(x+t) + G(x)$ where F and G are arbitrary functions.
(2) $F(x) + G(x) = 0$, $F'(x) = x$, so $u = \frac{1}{2}(x+t)^2 - x^2$.

3. Consider the 1 dimensional advection-diffusion equation: $u_t = u_x + u_{xx}$.
(1) Use change of coordinate of the form $p = x - Ct$, $q = t$ to reduce it to the 1 dimensional heat equation. (13 points)
(2) Recall that the solution of initial value problem of 1-dimensional heat equation: $v_t = v_{xx}$ when $t > 0$, $v(x, 0) = f(x)$ can be given by the Poisson integral representation:

$$v(x, t) = \int_{-\infty}^{\infty} f(y)G(x-y, t)dy, \text{ where } G(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Can you write down the analogous formula for the following initial value problem: $u_t = u_x + u_{xx}$ when $t > 0$, $u(x, 0) = f(x)$? (10 points)

- (3) Consider the following problem with periodic boundary condition: $u_t = u_x + u_{xx}$ when $0 < x < 1$, $u(0, t) = u(1, t)$, $u_x(0, t) = u_x(1, t)$. Show that $I(t) = \int_0^1 u^2(x, t)dx$ is a non-increasing function by calculating $\frac{d}{dt}I$. (7 points)

Answer: (1) $u_t = -Cu_p + u_q$, $u_x = u_p$, $u_{xx} = u_{pp}$, hence when $C = -1$, $u_q = u_{pp}$.
(2) $u(x, t) = \int_{-\infty}^{\infty} f(y)G(x+t-y, t)dy$.
(3) $\frac{d}{dt}I = \int_0^1 2uu_t dx = \int_0^1 2uu_x + 2uu_{xx} dx = u^2|_0^1 + 2uu_x|_0^1 - \int_0^1 2(u_x)^2 dx \leq 0$.

4. Consider the equation $u_{xx} + u_{yy} = x^2 + y^2$ on $\mathbb{R}^2 \setminus (0, 0)$. Find all radial symmetric solutions (In other words, all solutions of the form $u(x, y) = g(\sqrt{x^2 + y^2})$). You may want to use the fact that the Laplace operator in polar coordinate (r, θ) is $\Delta = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. (5 points)

Answer: $u_{rr} + u_r/r = r^2$, so $(ru_r)_r = r^3$, $ru_r = A + \frac{1}{4}r^4$, $u_r = A/r + \frac{1}{4}r^3$, and $u(r) = B + A \log r + \frac{1}{16}r^4$.

11 2.1

2. $|u| = |\int_{\mathbb{R}} \phi(y)G(x-y, t)dy| \leq \int_{\mathbb{R}} |\phi(y)G(x-y, t)|dy \leq M \int_{\mathbb{R}} G(x-y)dy = M$, where G is the heat kernel.

3. $u(x_0, t) = \int_{\mathbb{R}} \phi(y)G(x_0 - y, t)dy = u_0 \int_0^\infty G(x_0 - y, t)dy = u_0 \int_{-\infty}^{x_0} G(s, t)ds = u_0(\int_{-\infty}^0 G(s, t)ds + \int_0^{x_0} G(s, t)ds)$. We know $\int_{-\infty}^0 G(s, t)ds = 1/2$, $\int_0^{x_0} G(s, t)ds = \int_0^{x_0/\sqrt{4t}} G(s, 1)dt$ which converges to 0 as $t \rightarrow \infty$, hence $\lim_{t \rightarrow \infty} u(x_0, t) = u_0/2 = 1/2$.

12 2.2

3. The solution of the latter Cauchy problem is $u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(s)ds$, and the solution of the first Cauchy problem is the partial derivative of the solution of the latter Cauchy problem in t direction which by fundamental theorem of calculus is $\frac{1}{2}(\phi(x-ct) + \phi(x+ct))$.

13 Quiz 2

$$u_{tt} = 4u_{xx} + e^{x+t}, \quad u_t(x, 0) = 0, \quad u(x, 0) = \sin x.$$

$$\text{Solution: By Dahamel's principle, } u_{tt} = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \int_0^t \frac{1}{4} \int_{x-2t+2s}^{x+2t-2s} e^{r+s} dr ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4} \int_0^t e^{x+2t-s} - e^{x-2t+3s} ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(e^{x+2t} - e^{x+t}) - \frac{1}{12}(e^{x+t} - e^{x-2t}).$$

14 2.3

3. $|u^1 - u^2| = |(\frac{1}{2}(f^1(x-ct) + f^1(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g^1(s)ds) - (\frac{1}{2}(f^2(x-ct) + f^2(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g^2(s)ds)| \leq |\frac{1}{2}((f^1 - f^2)(x-ct) + (f^1 - f^2)(x+ct))| + |\frac{1}{2c} \int_{x-ct}^{x+ct} (g^1 - g^2)(s)ds| = \delta_1 + \delta_2 T$. It shows that this Cauchy problem is stable and well posed.

15 2.4

2. Do odd extension of the initial condition, one gets $u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} dy$.

16 2.5

1. By Duhamel's principle, $u(x, t) = \int_0^t \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds d\tau = -\frac{1}{2c} \int_0^t \cos(x+c(t-\tau)) - \cos(x-c(t-\tau)) d\tau = \frac{1}{2c^2} (\sin(x+ct) + \sin(x-ct) - 2\sin(x))$.

17 2.6

4. $\mathcal{L}(\int_0^t f(\tau) d\tau) = \int_0^\infty e^{-st} (\int_0^t f(\tau) d\tau) dt = \int_0^\infty (\int_\tau^\infty e^{-st} dt) f(\tau) d\tau = \frac{1}{s} \int_0^\infty e^{-s\tau} f(\tau) d\tau = \frac{\mathcal{L}(f)}{s}$.

8. Let $v = Lu$ in the t direction, we have $sv - u_0 = v_{xx}$, $v_x(0, s) = v(0, s)$, because we want bounded solution, $v = -\frac{u_0}{s(1+\sqrt{s})} e^{-\sqrt{s}x} + \frac{u_0}{s}$, hence $u = -u_0 L^{-1}(\frac{1}{s(1+\sqrt{s})} e^{-\sqrt{s}x}) + u_0$.

Remark: for those who know complex analysis, we can evaluate $L^{-1}(\frac{1}{s(1+\sqrt{s})} e^{-\sqrt{s}x})$ using the inverse formula on pp. 107 of the textbook. The answer is $-\frac{1}{2\pi i} \int_0^\infty e^{-st} (\frac{1}{-s(1+i\sqrt{s})} e^{-\sqrt{s}ix} - \frac{1}{-s(1-i\sqrt{s})} e^{\sqrt{s}ix}) ds = -\frac{1}{\pi} \int_0^\infty \frac{\sin(\sqrt{s}ix) + \sqrt{s} \cos(\sqrt{s}ix)}{s(1+s)e^{st}} ds$.

18 2.7

5. $\mathcal{F}u = \int_{-\infty}^\infty e^{-|x|+i\xi x} dx = \frac{2}{1+\xi^2}$. So, $\mathcal{F}^{-1}(\frac{1}{(1+\xi^2)^2}) = \frac{1}{4}u * u = \frac{1}{4} \int_{-\infty}^\infty e^{-|y|-|x-y|} dy = \frac{1}{4}(|x|e^{-|x|} + e^{-|x|})$.

15. Do Fourier transform in the x direction, let $v = \mathcal{F}u$, we have $v_t = -Ds^2v + cism$, so $v(s, t) = (\mathcal{F}\phi)(s)e^{(-Ds^2+cis)t}$, and $u(x, t) = \mathcal{F}^{-1}((\mathcal{F}\phi)(s)e^{(-Ds^2+cis)t}) = \phi * \mathcal{F}^{-1}(e^{(-Ds^2+cis)t}) = \phi * \left(\frac{1}{\sqrt{4\pi Dt}} e^{-(x-ct)^2/(4Dt)}\right)$.

19 Solution of other exercise problems

2.2.6 $u(x, t) = \frac{1}{2}(e^{-|x-ct|} + e^{-|x+ct|}) + \frac{1}{2c}(\sin(x+ct) - \sin(x-ct)).$

2.3.4. I don't see a 2.3.4 in my textbook?

2.5.3. $u(x, t) = \int_0^t w(x, t - \tau; \tau) d\tau = \int_0^t f(x - ct + c\tau, \tau) d\tau.$

2.5.4. Use 2.5.3, $u(x, t) = \int_0^t (x - 2t + 2\tau)e^{-\tau} d\tau = (x - 2t)(1 - e^{-t}) - 2te^{-t} + 2 - 2e^{-t} = x - 2t + 2 - xe^{-t} - 2e^{-t}.$

2.6.10. Do Laplace transform in t direction, let $v = L(u)$, then $s^2v = c^2v_{xx} - gs^{-1}$, with boundary condition $v(x, s) \not\rightarrow \infty$ as $x \rightarrow \infty$, $v(0, s) = 0$. So $v(x, s) = \frac{g}{s^3}(e^{-sx/c} - 1)$, $u = \frac{g}{2}(x^2/c^2 - 2tx/c)$ when $x < tc$ and $-\frac{g}{2}t^2$ when $x > tc$. Note that when you use the table on page 114, all functions are 0 for $t < 0$ or $s < 0$.

2.6.11. Do Laplace transform in t direction, $v = Lu$, $sv = v_{yy}$, so $v(x, y, s) = C(x, s)e^{-\sqrt{s}y}$. Use the other boundary conditions one gets $C(0, s) = 1/s$, $sC(x, s) + C_x(x, s) = 0$, so $C(x, s) = e^{-xs}/s$. The result can now be obtained from Table 2.1.

2.7.11. Use the hint, write v as in Example 2.18, then integrate the differential form vdy . You can also use Fourier transform directly to get a solution but without the arbitrary constant C .

2.7.16. (a) $\omega = k^3$.

(b) Let $v = Fu$, suppose the initial condition is $u(x, 0) = f(x)$, then $v_t + is^3v = 0$, so $v(s, t) = F(f)e^{-is^3t}$, $u = f * K(x, t)$ where $K(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-is^3t - isx} ds = \frac{1}{2\pi} t^{-1/3} Ai(\frac{x}{t^{1/3}}).$

Midterm 2

1. (1) Find the inverse Fourier transform of $\cos(x)e^{-|x|}$. (6 points)
- (2) Find the Laplace transform of $\cos(x)e^{-x}$. (6 points)
- (3) Find the convolution between e^x and e^{-x^2} . (6 points)

Solution: (1) $\frac{1}{2\pi} \left(\int_0^\infty \frac{1}{2}(e^{ix} + e^{-ix})e^{-x}e^{-isx}dx + \int_{-\infty}^0 \frac{1}{2}(e^{ix} + e^{-ix})e^xe^{-isx}dx \right) = \frac{1}{2\pi} \left(\frac{2}{1+(s+1)^2} + \frac{2}{1+(s-1)^2} \right)$.

$$(2) \int_0^\infty \cos xe^{-x-xs}dx = \frac{1+s}{(1+s)^2+1}.$$

$$(3) \int_{-\infty}^\infty e^{x-y}e^{-y^2}dy = e^{x+1/4}\sqrt{\pi}.$$

2. Consider the initial-boundary value problem

$$u_t = u_{xx}, u(x, 0) = \sin x, u_x(0, t) = 1$$

on the region $x > 0, t > 0$.

- (1) Reduce it to a problem of the form

$$v_t = v_{xx} + f(x, t), v(x, 0) = g(x), v_x(0, t) = 0$$

by adding a function to u . (10 points)

- (2) Find the solution of the original initial-boundary value problem about u . (22 points)

Solution: (1) Let $v = u - \sin x$, then $v_t = v_{xx} - \sin x$, $v(x, 0) = 0$, $v_x(0, t) = 0$.

$$(2) u(x, t) = \sin x - \int_0^t \int_0^\infty \sin y (G(x+y, t-\tau) + G(x-y, t-\tau)) dy d\tau, \text{ where } G(x, t) = \frac{1}{\sqrt{4\pi}} e^{-x^2/(4t)}.$$

3. Consider the following problem:

$$u_{tt} = u_{xx} - 4u, u(0, t) = u(1, t) = 0, u(x, 0) = f(x)$$

on the region $0 < x < 1, t > 0$.

- (1) For any integer n , find a solution of $u_{tt} = u_{xx} - 4u$, $u(0, t) = u(1, t) = 0$ of the form $u = \phi(t) \sin(n\pi x)$. (10 points)

- (2) Find the solution of the original problem for $f(x) = \sin(\pi x) - \sin(3\pi x)$. (10 points)

Solution: (1) $u = (A \cos(\sqrt{n^2\pi^2 + 4t}) + B \cos(\sqrt{n^2\pi^2 + 4t})) \sin(n\pi x)$.

$$(2) u = \cos(\sqrt{\pi^2 + 4t}) \sin(\pi x) - \cos(\sqrt{9\pi^2 + 4t}) \sin(3\pi x).$$

4. Find the bounded solution of the following problem:

$$u_{tt} = u_{xx}, u_t(x, 0) = u(x, 0) = 0, u_x(0, t) = u(0, t) + \sin t$$

on the region $x > 0, t > 0$. You may want to use the Laplace transform or the general solution of 1-d wave equations. (16 points)

Solution 1: $u = F(t-x) + G(t+x)$, $u_t(x, 0) = u(x, 0) = 0$ implies that we can set $F = 0$ on $(-\infty, 0]$ and $G = 0$. The boundary condition is saying that $F' = -F - \sin t$ so $F(s) = -\int_0^s \sin re^{r-s} dr$,

$$u = \begin{cases} -\int_0^{t-x} \sin r e^{r-t+x} dr & t > x \\ 0 & t \leq x \end{cases}.$$

Solution 2: Laplace transform in t direction, $v = Lu$, then $s^2 v = v_{xx}$, $v(x, s) = C(s)e^{-xs}$, $-sC = C + L(\sin t)$, so $v = -L(\sin t) \frac{1}{1+s} e^{-xs}$, $u = -(\sin t * e^{-t+x})H(t-x)$.

5. Find the bounded solution of $u_{xx} + u_{yy} = u$, $u(x, 0) = f(x)$ on the region $y > 0$. You may want to use the Fourier or Laplace transform. (14 points)

Solution 1: Do Fourier transform in the x direction, let $v = F(u)$, then $-s^2 v + v_{yy} = v$, $u(x, y) = f * F^{-1}(e^{-\sqrt{s^2+1}y})$.

Solution 2: Do Laplace transform in the y direction, let $v = L(u)$, then $v_{xx} + s^2 v - sf - g = v$, where $g(x) = u_y(x, 0)$. Because v should decay as $s \rightarrow \infty$, $v = F^{-1}(\frac{1}{s^2 - \xi^2 - 1}(sF(f) + F(g)))$. Furthermore, boundedness implies that there shouldn't be a pole when $s^2 - \xi^2 - 1 = 0$, which is only possible when $F(g) = -\sqrt{\xi^2 + 1}F(f)$, hence $u = L^{-1}(f * F^{-1}(\frac{1}{s + \sqrt{\xi^2 + 1}}))$.

20 Quiz 3

Find the Laplace transform of $f(x) = \begin{cases} \sin(\pi x) & 0 < x < 1 \\ 0 & x > 1 \end{cases}$.

$$\text{Solution: } Lf = \int_0^1 \sin(\pi x) e^{-sx} dx = -\frac{i}{2} \int_0^1 (e^{-sx+i\pi x} + e^{-sx-i\pi x}) dx = \frac{i}{2} \left(\frac{e^{-s+i\pi}-1}{s-i\pi} + \frac{e^{-s-i\pi}-1}{s+i\pi} \right).$$

21 Quiz 4

$$u_{tt} + u_{xx} = u_t, \quad u_x(0, t) = u(1, t) = 0, \quad u(x, 0) = f(x).$$

Solution: Eigenfunctions are $\cos(\pi(n+1/2)x)$, so the result is $2 \sum_{n=0}^{\infty} \left(\int_0^1 f(s) \cos(\pi(n+1/2)s) ds \right) \cos(\pi(n+1/2)x) e^{(1/2 - \sqrt{\pi^2(1/2+n)^2 + 1/4})t}$.

22 3.1

1. a) b) d) are straightforward.

c) $a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin x dx.$

e) $u(x, t) = \sum_n \frac{2}{n\pi} \int_0^\pi f(x) \sin x dx \sin nct \sin nt.$

23 4.1

4. $u(x, t) = \sum_n 2(\int_0^1 f(s) \sin(n\pi s) ds) e^{-t} (\cos(t\sqrt{n^2\pi^2/4 - 1}) + (n^2\pi^2/4 - 1)^{-1/2} \sin(t\sqrt{n^2\pi^2/4 - 1})) \sin(n\pi x).$

5. (a) $u = 0$. (b) $u(x, t) = \sum_n \frac{2}{l} (\int_0^l f(s) \sin(n\pi s/l) ds) e^{-kn^2\pi^2/l^2 + ht} \sin(n\pi x/l).$

24 4.2

4. When $\lambda < 0$, $y(x) = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}$, so the boundary condition is $C_1 + C_2 + 2\sqrt{-\lambda}C_1 - 2\sqrt{-\lambda}C_2 = 0$, $3(e^{2\sqrt{-\lambda}}C_1 + e^{-2\sqrt{-\lambda}}C_2) + 2(\sqrt{-\lambda}e^{2\sqrt{-\lambda}}C_1 - 2\sqrt{-\lambda}e^{-2\sqrt{-\lambda}}C_2) = 0$, so there is a non-zero solution iff $\sqrt{-\lambda}$ is the solution of $e^{4t} = 1 + \frac{8t}{3-4t-4t^2}$. By taking derivatives and intermediate value theorem there is a unique such t in $(0, 1/2)$. 0 is not an eigenvalue. There are infinite positive eigenvalues by Sturm Liouville theory.

8. The boundary condition is not symmetric e.g. $y_1(a) = y_1(b) = y_2(a) = y_2(b) = y'_2(a) = 2y'_2(b) = 1$, $y'_1(a) = y'_1(b) = 0$. It is not self adjoint either. For u, v satisfying the boundary conditions, $\int_a^b -u''v = v'u - vu'|_a^b + \int_a^b -uv''$, and $v'u - vu'|_a^b$ can not be guaranteed to be 0.

25 4.3

1. If $u = X(x)T(t)$, the first pde can be separated into $-(x^2X')'/(x^2X) = T''/T = \lambda$. If one let $u = X(x)Y(y)$ in the second PDE, it becomes $X''Y + (x^2 + 2xy + y^2)XY'' = 0$, and is not separable.

4. If y is an eigenfunction with eigenvalue λ , $\lambda = \frac{-\int_1^\pi y(x^2y')' dx}{\int_1^\pi y^2 dx} = \frac{-\int_1^\pi x^2 y'^2 dx}{\int_1^\pi y^2 dx} \geq 0$.

The differential equation is the Cauchy-Euler equation, hence eigenfunctions are $x^{-1/2} \sin(\frac{\sqrt{4\lambda_n - 1}}{2} \log x)$, and $\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log \pi)^2}$.