

# Project Summary

## Overview

The aim of this project is to investigate graphs and graph maps using ideas from complex dynamics, symbolic dynamics and complex analysis. The research concerns the following three related topics:

- **Entropy of maps on intervals and Hubbard trees** The PI will investigate some properties of the distribution of Galois conjugates of the exponent of core entropy of quadratic maps as well as an analogy of the Julia-Mandelbrot correspondence for the setting of interval maps as well as maps on the Hubbard tree.
- **Estimating the asymptotic translation length on the sphere complexes of train track maps on finite graphs** The PI will work on finding analogies of some prior results by the PI and his collaborators on the asymptotic translation length of pseudo-Anosov maps on the curve complex in the setting of train track maps on finite graphs, in particular, for those train track maps with the same mapping torus.
- **Divisors and differential operators on metric graphs** The PI will also work on finding the analogies of certain results on the geometry and analysis on Riemann surfaces in the setting of metric graphs, for example, those about the  $\zeta$ -functions and regularized determinants of differential operators.

## Intellectual Merit

Entropy of maps on Hubbard tree and translation lengths on curve complexes are important questions that have been extensively studied, and it is hoped that the results in this project will shed new lights on these topics. The results of this project will have applications in other areas of mathematics like geometric group theory, combinatorics and tropical geometry. Furthermore, it is hoped that the results obtained in this project will be a special case of more general results: for example, some kind of generalized Milnor-Thurston kneading theory that applies to maps on (finite or infinite) trees or graphs, Julia-Mandelbrot type correspondences between the dynamic plane and parameter plane for general iterated function systems or similar systems associated with a symbolic dynamics, or a kind of analogy of the McMullen polynomials that can encode the asymptotic translation length on curve complexes.

## Broader Impacts

The PI will mentor students in the Undergraduate Research Scholars program based on some questions from this project. The PI have, and will continue to disseminate the results through talks to both mathematical and non mathematical audience, and will use examples from this project to illustrate ideas in lectures for courses like dynamics, numerical analysis and PDE in order to engage students more effectively. Furthermore, some algorithms that can potentially be obtained from this project may be useful for real-life purposes like surface matching or dimension reduction.

# Project Description

The main motivations of this project are to investigate the relationship between dynamics in two dimensions and one dimension, via traintracks (introduced by Williams [71], Thurston [65] and Hubbard trees (introduced by Douady-Hubbard [21])), and the analogy between compact Riemann surfaces and finite metric graphs. The project can be divided into three related topics which will be further described below:

- Topological entropy of interval maps and maps on Hubbard tree.
- Asymptotic translation length on curve complex and sphere complex.
- Divisors and differential operators on metric graphs.

The next three sections will be an outline of background and prior results of these three topic, the questions the PI will work on and partial results towards the resolution of these questions.

## 1 Topological entropy of maps on Intervals and on Hubbard tree.

### 1.1 Background and Prior Work

#### 1.1.1 Interval maps

The study of dynamics of maps on intervals is related to continued fractions and  $\beta$ -expansions (see Ito-Sadahiro [33], Parry [50], Góra [27], Solomyak [57], Thompson [62] etc), complex dynamics (see Douady-Hubbard [21], Li [39], Poirier [51], Meerkamp-Schleicher[47], Thurston [63], Tiozzo [67, 68] etc), and, via Milnor-Thurston kneading theory [49], to the study of symbolic dynamics and iterated function systems (see Barnsley-Harrington [9], Bandt [8], Solomyak-Xu [58], Calegari-Koch-Walker [19] etc).

Topological entropy is a way to characterize the complexity of a dynamical system. In the case when there is a Markov decomposition, the exponent of the topological entropy is the leading eigenvalue of a Perron-Frobenius matrix, and hence must be an algebraic integer. In the case when the map is continuous and *post-critically finite* (the forward orbit of all critical points is a finite set), the union of these orbits determines a Markov partition of the interval, hence it is possible to investigate the distribution of the Galois conjugate of the exponents of the entropies. In his last paper [64], Willian Thurston defined the *master teapot*, which is the set

$$T := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

Here  $h_{top}$  is the topological entropy, and  $\mathcal{F}$  is the set of unimodal, critically periodic interval self-maps. Here an interval map is called *unimodal* if it has a single critical point  $c$  in the interior of the interval, and *critically periodic* if there is some  $n > 0$  such that  $f^{\circ n}(c) = c$ . The orthogonal projection of the Master teapot to  $\mathbb{C}$  is called the *Thurston set*. The study of Thurston's master teapot and Thurston set can be seen as providing necessary conditions for an algebraic integer  $\lambda$  to be the exponent of the entropy of a critically periodic unimodal map: if  $\lambda$  is the exponent of such entropy, all its Galois conjugates must be in the Thurston set, and if  $z$  is a Galois conjugate of  $\lambda$ ,  $(z, \lambda)$  must be in  $T$ .

Tiozzo [69] gave a characterization of the Thurston set, where the part of it inside the unit circle is shown to be the closure of the set of roots of all Littlewood polynomials [41] (polynomials with all coefficients  $\pm 1$ ) , and Calegari-Koch-Walker [19] showed that there are holes in the Thurston set other than the trivial one around zero. Figure 1 and Figure 2 are finite approximations of the Thurston set and Thurston teapot respectively.

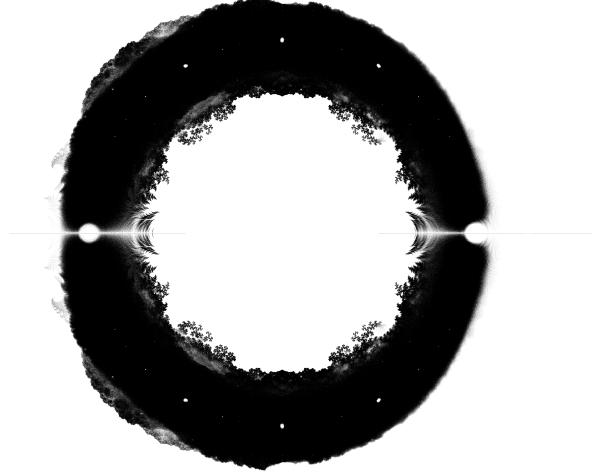


Figure 1:

In [18, 40], the PI and his collaborators proved a characterization of the part of the Thurston teapot inside the “unit cylinder”  $\{(z, \lambda) : |z| < 1\}$ . This characterization is analogous to the characterization of the Thurston set by Tiozzo [69]. Furthermore, the second paper by the PI and his collaborator [40] presents an algorithm that can be used to certify that a point does not belong to the Thurston teapot. In particular, given any real number  $\lambda \in (1, 2)$ , the PI and his collaborators found a subset  $M_\lambda \subseteq \{0, 1\}^{\mathbb{N}}$  that is invariant under shift, non-decreasing with  $\lambda$ , and for any  $|z| < 1$ ,  $(z, \lambda) \in T$  iff there is some  $w = w_1 w_2 \dots \in M_\lambda$ , such that

$$G(w, z) := \lim_{n \rightarrow \infty} f_{w_1, z} \circ \dots \circ f_{w_n, z}(1) = 1$$

Here  $f_{0,z}(x) = zx$  and  $f_{1,z}(x) = 2 - zx$ .

As a consequence, the PI and his collaborators showed that

*Theorem 1.* [18] If  $(z, \lambda) \in T$ ,  $|z| < 1$ , then for any  $\lambda' \in [\lambda, 2]$ ,  $(z, \lambda') \in T$ .

In other words, the intersection of the teapot with horizontal unit discs  $\{(\lambda, z) : |z| \leq 1\}$  grows monotonously with the height  $\lambda$ .

For part of the teapot outside the unit cylinder a characterization can be derived from observations by Thurston [64] and Tiozzo [69], see also [18, 40]. In particular,

*Theorem 2.* The set  $\{z : (z, \lambda) \in T \text{ or } |z| \leq 1\}$  varies continuously with  $\lambda$  under the Hausdorff topology.

In other words, the part of the teapot outside the unit cylinder consists of countably many curves transversal to horizontal planes.

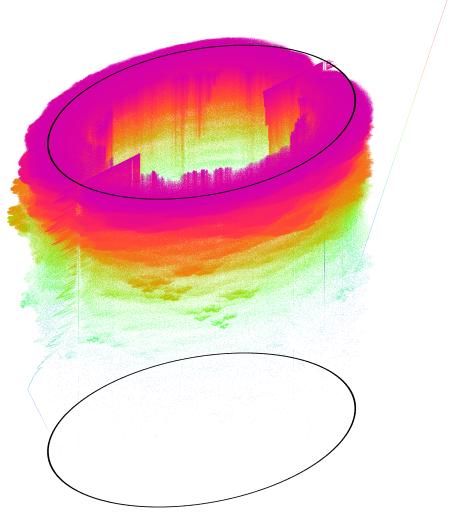


Figure 2:

### 1.1.2 Maps on Hubbard Trees

The dynamics of unimodal interval maps can be seen as a special case of the dynamics on Hubbard trees of quadratic maps. Consider complex quadratic maps  $f_c : z \mapsto z^2 + c$ , the *filled Julia set* is the set of initial values  $z$  whose orbit does not converge to infinity, the *Julia set* is the boundary of filled Julia set, the *Mandelbrot set* is the set of parameters  $c$  such that the corresponding Julia set is connected, or, equivalently, the set of parameters  $c$  where  $f_c^{\circ n}(0)$  does not go to infinity as  $n \rightarrow \infty$ . There is a unique Riemann map  $\Phi$  from the complement of the unit disc to the complement of the Mandelbrot set, such that  $\Phi(\infty) = \infty$  and  $\Phi'(\infty) > 0$ , and the ray  $\{\Phi(\rho e^{i\theta}) : \rho \geq 1\}$  is called the *external ray* with angle  $\theta$ . The *external ray* for simply connected filled Julia set can be defined analogously. Similar to the case of interval maps,  $f_c$  is said to be *post-critically finite* if the forward orbit of 0 is finite, and *critically periodic*, or *superattracting*, if the forward orbit of 0 is periodic.

A way to completely encode the dynamics of postcritically finite quadratic maps is through the concept of the *Hubbard tree* [21], which is a tree connecting the forward orbit of 0 that lies within the filled Julia set. The quadratic map induces a continuous map on the Hubbard tree. Hubbard trees can also be defined for higher degree polynomial maps, and infinite Hubbard trees can be defined for some non post-critically finite maps. The topological entropy of the map on the Hubbard tree is called by Thurston the *core entropy*. Thurston proposed a combinatorial algorithm [25] to calculate the core entropy from the direction of the corresponding external rays landing on the Mandelbrot set. Furthermore, Tiozzo showed in [67] that the core entropy is related to the Hausdorff dimension of the set of external rays landing on the corresponding *principal vein* in the Mandelbrot set (which is an arc connecting origin to a point  $c_{p/q}$ , where the quadratic map sends the critical point to a fixed point where the external ray of the filled Julia set has angle 0,

in exactly  $q$  steps), and in [68] that the core entropy is continuous as a function of the angle of the external ray associated with it, resolving a conjecture by Thurston.

Now a critically periodic unimodal map can be seen as a superattracting parameter on the real slice of the Mandelbrot sets, and the study of the topological entropy of these interval maps is the same as the study of the core entropy on these points.

## 1.2 Questions and partial results

Motivated by prior work of the PI and others, as well as by evidence from numerical experiments, the PI and his collaborators conjectured the followings:

*Question 1.* [40] Is it true that for any complex number  $|z| < 1$ , any  $\lambda \in (1, 2)$ , the set  $X_z := \{x - z : G(w, x) = 1 \text{ for some } w \in M_\lambda\}$  is asymptotically similar to the set

$$J_z = \{G(w, z) - 1 : w \in M_\lambda\} ?$$

Here two sets  $A$  and  $B$  are asymptotically similar if there exists a real number  $r > 0$  and sequences  $(t_n), (t'_n) \in \mathbb{C}$  with  $t_n, t'_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} d_{\text{Haus}} \left( \overline{B_r(0)} \cap (t_n A), \overline{B_r(0)} \cap (t'_n B) \right) = 0 ,$$

where  $d_{\text{Haus}}$  is the Hausdorff dimension.

This is analogous to the Julia-Mandelbrot correspondence proved by Lei [38], where the set  $X_z$  plays the role of the Mandelbrot set while the set  $J_z$  plays the role of the filled Julia set. The case when  $\lambda = 2$  was conjectured by Baez, Christensen, Derbyshire and Egan. Techniques for solving this question might be applicable to more general parametrized families of iterative function systems.

*Question 2.* When one replace  $\mathcal{F}$  in the definition of  $T$  with interval maps with at most 2, critical values, is an analogous characterization of points in  $T$  as in the PI's prior work [40]?

The PI and his collaborators are able to adapt most of the argument in [40] to this new setting to get a partial result for the closure of the roots of the kneading determinants, which already provides a non-trivial sufficient condition for an algebraic integer to be the exponent of the entropy of such a critically periodic interval map. To fully resolve this question it only remains to show that the kneading determinants (cf. Milnor-Thurston [49]) is irreducible for sufficiently many interval maps.

*Question 3.* Is it true that the set of Galois conjugates of the exponent of the core entropy of post-critical parameters union with the unit disk changes continuously with the angle of corresponding external ray?

If true, this would be a strengthening of the main result in Tiozzo's work [68] and could provide a necessary condition for a number to be the core entropy of a quadratic map. Furthermore, techniques for solving this problem could be applicable to the dynamics of pseudo Anosov maps on infinite translation surfaces.

*Question 4.* Along principal veins of the Mandelbrot set, is there an analogy for Theorem 1 regarding the closure of Galois conjugates of the exponent of core entropy of superattracting parameters?

A positive answer to this question will lead to a stronger necessary condition for determining if an algebraic number can be a core entropy of some superattracting point on a principal vein. Similarly to the case of Question 2, the PI and his collaborators are able to adapt most of the argument in [40] to this setting to get a partial result, which can be used to establish a non trivial necessary condition for an algebraic integer to be the exponent of the core entropy of a superattracting map on a specific principal vein. Also similar to Question 2, the full answer can be established if the denominator of the power series  $\Delta$  in Tiozzo's paper [67] can be shown to be irreducible for sufficiently many superattracting points.

## 2 Asymptotic translation length on the curve complex and the sphere complex

### 2.1 Background and prior work

A *pseudo-Anosov map* on a closed surface  $S$  is a self-homeomorphism with no finite order isotopy class of curves. Up to homotopy, one can make such a map preserve a pair of transverse singular measured foliations. The stretch factors on these measured foliations is the exponent of the topological entropy of this map. These stretch factors are always bi-Perron algebraic integers. However, it is an open problem if all bi-Perron algebraic integers can be realized as stretch factors of pseudo-Anosov maps. McMullen [45] proved that pseudo-Anosov maps that have homeomorphic mapping tori and lie in the same *fibered cone* have stretch factors that are related to each other via a Teichmüller polynomial, and the Alexander polynomial of the mapping torus is a factor of the Teichmüller polynomial. The definition of *fibered cone* first introduced by Thurston [66] is as follows:

*Definition 5.* [66]

- Let  $M$  be a hyperbolic 3-manifold, for every integer cohomology class  $\alpha \in H^1(M; \mathbb{Z})$ , the *Thurston norm* of  $\alpha$  is defined as

$$\|\alpha\| = \min_S \max_i \{\chi(S_i)\},$$

where  $S = \bigcup_i S_i$  is an embedded surface that represents  $\alpha$ . The Thurston norm can be extended to  $H^1(M; \mathbb{R})$  as a piecewise linear function with rational coefficients.

- If  $M$  is homeomorphic to a mapping torus of a surface map  $\phi$  (seen as a surface bundle over the circle), let the cohomology class associated with  $\phi$ , denoted as  $\alpha_\phi$ , be the pullback of the generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called *fibered cones*, and the map associated with a primitive integer class  $\alpha$  in the fibered cone are denoted as  $\phi_\alpha$ .

Besides stretch factors, another way of characterizing the topological complexity of a pseudo-Anosov map is through its asymptotic translation length on the curve graph of  $S$ . The *curve graph*  $\mathcal{C}(S)$  is a graph where each vertex represents an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. It is easy to see that the mapping class group of  $S$  acts on the curve graph by isometry. Masur-Minsky [43] showed that  $\mathcal{C}(S)$  is  $\delta$ -hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserve a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. The study of curve graphs are also related to the hierarchical hyperbolic structure on the mapping class group (Masur-Minsky [44], Behrstock-Hagen-Sisto [10]). The asymptotic translation length of a pseudo-Anosov map  $g$  on  $\mathcal{C}(S)$  can now be defined as

$$l_{\mathcal{C}}(g) = \lim_{n \rightarrow \infty} \frac{d_{\mathcal{C}}(g^n \gamma, \gamma)}{n},$$

where  $\gamma$  is any vertex in  $\mathcal{C}$ .

It was shown by Masur-Minsky [43] that  $l_{\mathcal{C}}$  is well defined and non-zero if  $g$  is pseudo-Anosov. The technique in [43] for showing the positivity of  $l_{\mathcal{C}}$ , which is based on studying the incidence matrix on the induced map on invariant traintracks, has been used by Gadre-Tsai [24], Vldivia [70], Baik-Shin [3] and others to provide asymptotics of the lower bound on  $l_{\mathcal{C}}$  as the genus and number of punctures on  $S$  increases. Furthermore, Bowditch [17] showed that the asymptotic translation length is a rational number, and Shackleton [55], Birman-Margalit-Menasco [15] described algorithms for its computation.

Kin-Shin [36] found a sequence of pseudo-Anosov maps in different genera that realize the asymptotic lower bound. The PI and his collaborators generalized the argument by Kin-Shin [36] and proved the following:

*Theorem 3.* [4] Suppose  $M$  is a closed hyperbolic 3-manifold and  $P$  a fibered cone in  $H^1(M)$ ,  $L$  a rational subspace of  $H^1(M)$  of dimension  $r$  that intersects  $P$ . For every primitive element  $\alpha \in P \cap L \cap H^1(M; \mathbb{Z})$ ,

$$l_{\mathcal{C}}(\phi_{\alpha}) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

Balázs Strenner [59] also proved a stronger result for the asymptotic translation length of arc complexes.

In [2], the PI and his collaborators proved that this asymptotic upper bound is sharp when  $r \leq 3$ . Furthermore, in [2], the PI and his collaborators used techniques similar to [4] to show the following:

*Theorem 4.* Suppose  $M$  is a closed hyperbolic 3-manifold and  $P$  a fibered cone in  $H^1(M)$ ,  $L$  a rational subspace of  $H^1(M)$  of dimension 2 that intersects with  $P$ . Then for all but finitely many primitive elements  $\alpha \in P \cap L \cap H^1(M; \mathbb{Z})$ ,  $\phi_{\alpha}$  is a normal generator of the corresponding mapping class group.

A key tool for proving the above results is the encoding of pseudo-Anosov dynamics by the induced map on the invariant *traintracks* [71, 65], which are finite graphs smoothly embedded into the surface and where each vertex is a “switch”, and the associated edges arrives at two opposite directions. The topological entropy of the pseudo-Anosov map can be calculated via the

action on the train track. A way to extend this relationship between two dimensional and one dimensional dynamics is through the Culler-Vogtmann Outer space [20], which serves the same purpose as Teichmüller space does for the study of  $Out(F_n)$ . The outer space on which  $Out(F_n)$  acts on consists of all finite metric graphs with  $n$  loops and no degree 1 vertices, together with a marking map which is a homotopy equivalence between this metric graph and the wedge of  $n$  circles, where  $Out(F_n)$  acts by precomposition with the marking. Similarly, the Teichmüller space consists of Riemann surfaces with prescribed topological type, together with a marking map which is a homotopy equivalence from a model surface to the Riemann surface. Both the outer space and the Teichmüller space are contractable, and the group action described above are proper with finite stabilizers. This analogy made it possible to generalize many concepts and properties on Riemann surfaces and Teichmüller spaces, e.g. Nielsen-Thurston classification, train tracks (see Bestvina-Handel [14]), measured laminations (see Bestvina-Feighn-Handel [13]), Teichmüller metrics, fibered cones, McMullen's polynomials (see Dowdall-Kapovich-Leininger [23, 22]), and curve complexes (see Handel-Mosher [29], Hatcher-Vogtmann [31]), and is a fruitful approach for the study of the properties of outer automorphism group of free groups.

## 2.2 Questions and partial results

Motivated by the relationship between pseudo-Anosov maps and the induced map on the invariant train track as well as the analogy between  $Out(F_n)$  and the mapping class group, the PI will work on finding the analogy of Theorem 3 in the setting of finite metric graph. In an upcoming paper, the PI and his collaborators will prove an analogy of the above theorem in the setting of asymptotic translation lengths on the sphere complex for train track maps, which is an analogous characterization of the topological complexity of graph maps, which will be described below. The PI hopes that this can be used to strengthen the bounds in their prior work [4, 2] or to generalize them to other families of pseudo-Anosov maps.

Let  $G$  be a finite simplicial graph. A cellular map  $\psi : G \rightarrow G$  is called a **train track map** if the restriction of  $\psi^n$  to  $e$  for each  $n \geq 1$  and each edge  $e$  is an immersion (no back-tracking condition). We further assume  $\psi$  to be irreducible as an element of  $Out(F_n)$ . Following Hatcher-Vogtmann [31], one can make a 3-manifold  $M_G$  from  $G$  by replacing every edge with  $S^2 \times I$  and every vertex with  $\mathbb{S}^3$ . In the case when  $\psi$  is a train track map,  $\psi$  induces a homeomorphism  $\psi_1$  on  $M_G$ . Let  $\mathcal{S}(G)$  be the simplicial graph whose vertices are isotopy classes of embedded spheres in  $M_G$ , and there is an edge of length 1 between two vertices if and only if they are disjoint up to isotopy, then it is easy to see that  $\psi_1$  is an isometry of  $\mathcal{S}(G)$  and we can define the concept of asymptotic translation length of  $\psi_1$  analogously.  $\mathcal{S}(G)$  is the 1-skeleton of what was first proposed by Hatcher-Vogtmann [31] as the *sphere complex* (whose barycentric subdivision is related to the *free factor complex*), which is an analogy of the curve complex in the setting of the finite graphs and is also the simplicial closure of the Culler-Vogtmann Outer space, and has been shown to be hyperbolic by Bestvina-Feighn [12].

Dowdall-Kapovich-Leininger [23, 22] found analogies for arguments by Thurston [66] and McMullen [45] in the setting of maps on finite graphs as follows:

*Definition 6.* [23, 22] Suppose  $\psi$  is an irreducible train track map, let  $\gamma_\psi$  be a folding path of  $\psi$  in the Culler-Vogtmann outer space. The *folded mapping torus*  $N$  is a 2-d cell complex built from  $\gamma_\psi$ , which has a surjection over the circle and the fibers are the graphs in the folding path. A flow on  $N$  is defined such that any flow line is the orbit of a point on the graph under folding,

and an analogy for the fibered face containing  $\phi$  is  $\mathcal{S}$  which consists of first cohomology classes whose dual are transverse to all flow lines. This is furthermore a rational cone in  $H^1(N)$  call the *cone of sections* or *McMullen cone* in [22], defined in a procedure analogous to the one used by McMullen [45].

In an upcoming paper, the PI and his collaborators prove the following:

*Theorem 5.* Given any finite graph  $G$  and any irreducible train track map  $\psi$  on  $G$ , let  $\mathcal{C}$  be any proper subcone of the McMullen cone in [22] containing  $\psi$ , then any primitive integer element  $\alpha$  in  $\mathcal{C}$  must satisfies

$$l(\psi_\alpha) \lesssim n_\alpha^{-1-1/(d-1)}$$

Where  $d$  is the dimension of the McMullen cone,  $n_\alpha$  the genus (i.e. first betti number) of the fiber  $G_\alpha$  corresponding to  $\alpha$  and  $\psi_\alpha$  the corresponding monodromy, and  $l(\cdot)$  the translation length on the sphere graph  $\mathcal{S}(G_\alpha)$ .

Some further questions the PI will work on are the following:

*Question 7.* Can there be any lower bound for the asymptotic translation lengths in the case of traintrack maps?

It is conjectured that there might be a lower bound at the order of  $O(g^{-2})$  where  $g$  is the first betti number of the graph, but so far none is known.

*Question 8.* Can there be an analogy of Theorem 4 in the case of traintrack maps on graphs?

*Question 9.* Can similar results be proved for other complexes related to  $Out(F_n)$ , like the free splitting complex by Handel-Mosher [29] or the cyclic splitting complex by Mann [42]?

A positive answer to these questions would result in better understanding of the properties of  $Out(F_n)$  and their relationship with mapping class groups.

*Question 10.* What is the relationship between the translation length on the sphere complex of the thickened invariant train track and the curve complex?

The PI hopes that this can be useful for generalizing Theorem 3 to families of pseudo Anosov maps that do not lie in the same fibered cone, for example, those arising from maps on a fibered cone under a subsurface projection.

Furthermore, the PI hopes to generalize their construction in [2] to get a better lower bound in more settings, and hopefully show that their upper bound is asymptotically optimal.

### 3 Divisors and differential operators on metric graphs

#### 3.1 Background and prior work

In addition to Culler-Vogtmann outer space, another way to understand the analogy between Riemann surfaces and metric graphs is through tropical geometry and Berkovich spaces [11]. Just as Riemann surfaces can be seen as analytic spaces of complex algebraic curves, the Berkovich

spaces of dimension one serve as an analytic space for curves on non-Archimedean fields. They have a natural geometric structure and, by collapsing  $\mathbb{R}$ -trees, can be turned into finite metric graphs. It is possible to construct, by analogy, harmonic functions and maps, Jacobians, and divisors, and show that they satisfies many of the same properties as in the case of Riemann surfaces, cf. the works of Baker, Chinberg, Rumely, de Jong, Shokrieh etc. ([37, 5]).

By a metric graph, we mean an undirected simplicial graph with a positive edge length function  $l$ , where each edge  $e$  is made into an interval of length  $l(e)$ . When the graph has finitely many edges we call it a finite metric graph. The following definitions are standard:

*Definition 11.* [7, 56] Let  $G$  be metric graph.

1. By a 1-form on  $G$  we mean an element in the 1-simplicial cochain  $C^1(G)$ . Let  $d : C^0(G; \mathbb{R}) \rightarrow C^1(G; \mathbb{R})$  be the coboundary map.
2. Let  $G$  be a metric graph with edge set  $E(G)$ ,  $w$  a 1-form, we define the norm of  $w$  as  $\|w\| = (\sum_{e \in E(G)} w(e)^2 / l(e))^{1/2}$ . We call a 1-form  $L^2$  if it has a finite norm. Under this norm, denote the dual of  $d$   $\delta$ .
3. We call a 1-form  $\alpha$  harmonic, if  $\delta(\alpha) = 0$ . In other words, for every vertex  $v$ , all outgoing edges  $e_i$  from  $v$ ,  $\sum_i w(e_i) / l(e_i) = 0$ .

The analogy between Riemann surfaces and finite metric graphs shows that many important concepts and properties on Riemann surfaces corresponds to results in the graph-theoretic setting. For example, there is a graph-theoretic Riemann-Roch theorem by Baker-Norine [6], which is related to the question of chip-firing games on graphs, which is as follows:

*Theorem 6.* [6] Let  $G$  be a finite simplicial graph with no loops, all edges are assumed to have length 1. A divisor on  $G$  is an integer linear combination of vertices of  $G$ , i.e. an element in  $C^0(G; \mathbb{Z})$ , the degree is the sum of all the coefficients in this linear combination. A divisor  $D$  is called *effective* if all its coefficients are non negative, two divisors  $D$  and  $D'$  are *linearly equivalent* if  $\delta(D - D') = 0$ , and  $r(D)$  is the largest natural number  $r$  such that  $D - E$  is linearly equivalent to an effective divisor for any effective divisor of degree  $r$ , or, if no such natural number exists,  $-1$ . Then

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

Here  $g$  is the first betti number of  $G$ ,  $K$  is the *canonical divisor*  $K = \sum_v (\deg(v) - 2)(v)$ .

The proof however is very different from the Riemann-Roch theorem on surfaces, as  $r$  can no longer be interpreted as the dimension of some vector space. The theorem can be generalized to arbitrary metric graphs as well, see Mikhalkin-Zharkov [48], Hladký-Král-Norine [32], Gathmann-Kerber [26] etc.

Also, analogous to the Arakelov metric on Riemann surfaces, there is the “canonical metric” on finite metric graphs introduced by Zhang [72], also see [53], which is defined as follows:

*Definition 12.* The canonical metric [72, 53] on a metric graph is defined so that for every edge  $e$ , the length of  $e$  under the new metric is:

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| .$$

This is shown by Amini [1] and Richman [52] to be related to the distribution of Weierstrass points of divisors on finite metric graphs.

Graph theoretic analogies of differential operators and their spectral theory have also been extensively studied, see for example Baker-Rumely [7], Harrison-Kirsten-Texier [30], Texier [61].

In a prior work of the PI and his collaborator, an analogous result on a property by Kazhdan [35] (see also [34], or McMullen's paper [46, Appendix]) of the Arakelov metric on Riemann surfaces was found, which shows that when passing to larger and larger normal covers the canonical metric on metric graphs converges:

*Theorem 7.* [56] Let  $G \leftarrow G_1 \leftarrow G_2 \dots$  be a tower of finite regular covers of a finite metric graph  $G$ , then the canonical metric on  $G_i$  are pullbacks of metrics  $d_i$  defined on  $G$ , and  $d_i$  converges uniformly to some limiting metric that depends only on  $G$  and  $\bigcap_i \pi_1(G_i)$ . More precisely, let  $G \leftarrow G'$  be the regular cover defined by  $\bigcap_i \pi_i(G_i)$ , then the limiting metric pulls back to the canonical metric on  $G'$ .

In the case when  $\bigcap_i \pi_1(G_i)$  is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of  $G$ . Because in [46, Appendix] the limit of the Arakelov metrics for a closed Riemann surface of genus  $\geq 2$  under a tower of normal covers that converges to the universal cover would converge to the hyperbolic metric, one can see the limiting metric in Theorem 7 as a discrete analogy of the hyperbolic metric. However it is unclear what the relationship between this limiting metrics and other approaches of discrete uniformization by Bobenko-Dimitrov-Sechelmann [16], Gu-Luo-Sun-Wu [28], Sawhney-Crane [54] is.

### 3.2 Questions and Partial results

Following Baker-Rumely [7], let  $\Delta_\mu$  be the Laplace operator defined on the set of piecewise  $C^2$  functions with  $L^1$  second derivative, whose average regarding the canonical measure  $\mu = \mu_{can}$  is zero, which is the usual graph Laplacian composed with subtraction of average under  $\mu_{can}$ . There are infinitely such eigenvalues, and their only limit point is infinity [7].

*Question 13.* What else can we say about these eigenvalues, and, for example, the  $\zeta$  function  $\zeta(z) = \sum_i \lambda_i^{-z}$  and the regularized determinant  $e^{-\zeta'(0)}$ ?

The PI hopes that values of the  $\zeta$  function or regularized determined can be used to recover topological data from the graph. Furthermore, if  $\mu$  is replaced with the admissible metric correspond to a divisor, the PI hopes to recover  $r(D)$  from these eigenvalues and  $\zeta$  functions, which can potentially provide new proofs of the Riemann-Roch theorem on graphs, which may lead to new ways of finding tropical analogies of index theory and K-theory. So far the PI has done calculation in some cases and will work on formulating a more specific conjecture.

*Question 14.* What is the relationship with other  $\zeta$  functions on graphs, for example those in Terras [60]? What is the relationship with the determinant of discrete Laplacian, which is related to spanning tree counting?

Spanning tree counting is related to the determinant of the graph-theoretic Laplacian, so they can potentially be related to the regularized determinant by some kind of limiting process via subdivision of the edges.

## **Broader Impacts**

The PI will mentor undergraduate students in the Undergraduate Research Scholars (URS) program in University of Wisconsin based on some questions from this project. Currently the PI is a mentor in the URS project. The PI will use results and techniques from this project to illustrate concepts in analysis in dynamics during lectures and talks aiming at an undergraduate audience. For example, harmonic analysis on metric graphs can serve as a good example for concepts in courses like PDE or numerical analysis, while the dynamics of interval maps as well as maps on Hubbard tree will be used to illustrate techniques in an introductory dynamics class. The PI have, and will continue to disseminate the results through talks to both mathematical and non mathematical audience. For example, the PI has previously used some materials from the third topic above for a short course in the KAIX (KAIST Advanced Institute for Science) summer school with an audience of undergraduate and beginning graduate students in both math and applied math majors, from Korea, Vietnam and several other East and Southeast Asian countries. Furthermore, some algorithms that can potentially be obtained from this project may be useful for real-life purposes like surface matching or dimension reduction. For example, at the moment the PI is working on a new surface matching algorithm based on discrete uniformization, and results on the analysis on metric graphs may offer new ways to do discrete uniformization. The PI is also organizing the dynamics seminar in university of Wisconsin which is attended by grad students.

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## (a) Professional Preparation

- Cornell University, Ithaca NY, Mathematics, PhD 2016
- Peking University, Beijing, China, Mathematics BS 2010

## (b) Appointments

- University of Wisconsin at Madison, Assistant Professor, 2020-
- Rutgers University, Hill Assistant Professor, 2017-2020
- Max-Planck Institute of Mathematics, Postdoc, 2016-2017

## (c) Publications

### Five selected publications

- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:1902.10805*, conditionally accepted by *Advances in Mathematics*, 2020
- Kathryn Lindsey and Chenxi Wu, Characterization of the Shape of Thurston's Teapot *arXiv:1909.10675*
- Hyungryul Baik, Hyunshik Shin and Chenxi Wu. An upper bound on the asymptotic translation length on the curve graph and fibered faces *arXiv: 1801.06638*, accepted by *Indiana University Math Journal*, 2020
- Hyungryul Baik, Eiko Kin, Hyunshik Shin and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone *arXiv:1909.00974*, submitted.
- Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019. doi: 10.1007/s00222-018-0838-5

### Other Significant Publications

- Hyungryul Baik, Farbod Shokrieh, Chenxi Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019. doi: 10.1515/crelle-2019-0007.
- Hyungryul Baik, Ahmad Rafiqi and Chenxi Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019. doi: 10.1017/etds.2017.109

- Chenxi Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics*, 214(2), 733-740, 2016. doi: 10.1007/s11856-016-1357-y
- Chenxi Wu. The relative cohomology of abelian covers of the flat pillowcase. *Journal of Modern Dynamics*, 9, 123-140, 2015. doi:10.3934/jmd.2015.9.123
- Hyungryul Baik, Chenxi Wu, KyeongRo Kim, and TaeHyouk Jo. An algorithm to compute the Teichmüller polynomial from matrices *Geometriae Dedicata* 2019 doi: 10.1007/s10711-019-00450-4

#### **(d) Synergistic Activities**

- Mentor at Undergraduate Research Scholars program in the University of Wisconsin at Madison, from September 2020
- Gave lectures in KAIX (KAIST Advanced Institute for Science) Summer School in Daejeon, Korea, to undergraduate students from East and Southeast Asia, in August 2019
- Served as classroom assistant for F.E.M.M.E.S. (Women+ Excelling More in Math, Engineering and the Sciences) event at University of Michigan, to high school students, in November 2017
- Gave lectures in Math Explorer's Club at Cornell, to high school and middle school students, in March 2016

## **Data Management Plan**

The results of this project will be put on arXiv and submit for publication. There will not be any other data produced from this proposal.