Short Lecture on Mapping Class Groups

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1 Introduction

Let S be a surface. We will usually take S to be compact, connected, and orientable, though this can be relaxed (in particular, we may have reason to puncture S). Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation-preserving homeomorphisms of S which restrict to the identity on ∂S, and let $\mathrm{Homeo}_0^+(S, \partial S)$ be the connected component of the identity in this group. The mapping class group of S, denoted $\mathrm{Mod}(S)$, is defined to be

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0^+(S, \partial S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

It turns out that homotopic homeomorphisms of a compact surface are isotopic (the isotopy may be taken to be relative to the boundary if the original homotopy is), with two exceptions:

- An orientation-reversing homeomorphism of D^2 is homotopic to the identity by contractibility, but it fails to be isotopic to the identity since its restriction to the boundary has degree $-$
- The map $f : S^1 \times I \to S^1 \times I$ given by $f(z, t) = (z, 1-t)$ is homotopic, but not isotopic, to the identity.

If we allow punctures, we obtain counterexamples in the case of the sphere with one or two punctures. In all these cases, the counterexamples are orientation-reversing, so we may consider orientation-preserving homeomorphisms up to homotopy.

In addition, every homeomorphism of a compact surface is isotopic to a diffeomorphism, and two isotopic diffeomorphisms of a compact surface are smoothly isotopic. Thus, we may take $\mathrm{Mod}(S)$ to be the group of diffeomorphisms of S up to smooth isotopy.

2 Some Computations

Since we consider compact, connected, orientable surfaces, possible with punctures. These are classified by genus, number of boundary components, and number of punctures. Write $S_{g,n}$ for the surface with genus g, empty boundary, and n punctures, and write $S_g = S_{g,0}$.

Example 1. $\mathrm{Mod}(D_2) = 1$. By the discussion in the previous section, it's enough to show that a homeomorphism $\phi : D^2 \to D^2$ is homotopic to the identity rel ∂D^2. The straight line homotopy $F_t(z) = zt + \phi(z)(1-t)$ does the trick. For an explicit example of an isotopy, consider

$$F_t(z) = \begin{cases} (1-t)\phi\left(\frac{z}{1-t}\right) & 0 \le |z| < 1-t \\ z & 1-t \le |z| \le 1 \end{cases}$$

A similar argument shows that the one-puncutred disk has trivial mapping class group.

Example 2. $\mathrm{Mod}(S_{0,1}) = 1$. Identifying $S_{0,1}$ with $\mathbb{R}^2$ by stereographic projection, we may homotope any homeomorphism to the identity by a straight line.

Example 3. $\mathrm{Mod}(S^2) = 1$. Given a homeomorphism $\phi : S^2 \to S^2$, we may rotate the sphere to find an isotopic $\widetilde{\phi}$ which fixes some point p_0. Then $\widetilde{\phi}$ restricts to a homeomorphism $S^2 - \{p_0\} \to S^2 - \{p_0\}$, so we have reduced to the case of the previous example.

We now arrive at our first non-trivial example.

Example 4. We may think of $S_{0,3}$ either as a sphere with 3 punctures, or with 3 marked points. A homeomorphism of $S_{0,3}$ is then a homeomorphism of the sphere which permutes those marked points, which will be preserved by isotopy. Thus, we have a map $\mathrm{Mod}(S_{0,3}) \to \Sigma_3$, which turns out to be an isomorphism. Similarly, $\mathrm{Mod}(S_{0,2}) = \mathbb{Z}/2\mathbb{Z}$.

Example 5. Let S be the annulus. Then $\mathrm{Mod}(S) = \mathbb{Z}$

Proof. We identify the universal cover of S with $\widetilde{S} = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq 1\}$, with deck group $\mathbb{Z}$ acting by $z \mapsto z+n$. Then $\mathbb{R} \subset \mathbb{C}$ covers one boundary circle, and $\mathbb{R}+i$ covers the other. We may lift a homeomorphism $\phi : S \to S$ to a map $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ such that $\widetilde{\phi}(0) = 0$. Then $\widetilde{F} \mid_{\mathbb{R}}$ is the identity, while $\widetilde{F}(x+i) = x+i+n_F$ for some $n_F \in \mathbb{Z}$. The map $F \mapsto n_F$ is an isomorphism $\mathrm{Mod}(S) \to \mathbb{Z}$. □

Remark 6. The generator of this group is called a *Dehn Twist* (draw a picture), which will be important for us later.

An example which we will often reference in later sections is the torus.

Example 7. $\mathrm{Mod}(T^2) = SL_2(\mathbb{Z})$

Sketch. A homeomorphism of T^2 induces an isomorphism of $H_1 \cong \mathbb{Z}^2$, so we obtain a map $\mathrm{Mod}(T^2) \to GL_2(\mathbb{Z})$. Because orientation-preserving homeomorphisms must preserve intersection numbers, the image of this map is $SL_2(\mathbb{Z})$. Injectivity of this map follows form a similar covering idea as in the case of the annulus. □

Finally, consider the disk with n punctures. Its mapping class group is isomorphic to the *braid group* on n strands, denoted B_n. This is also the fundamental group of the nth unordered configuration space of $\mathbb{R}^2$, and it is important in topology and algebraic geometry.

3 Special Subgroups of Mod(S)

The Symplectic Representation and the Torelli Group

We now consider the case of the surface S_g. Recall that $H_1(S_g;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$. Given homology classes h_1 and h_2, we choose representing cycles c_1 and c_2 which intersect transversely. Now we define the intersection form $i : H_1(S_g;\mathbb{Z}) \wedge H_1(S_g;\mathbb{Z}) \to \mathbb{Z}$ by:

$$i(h_1, h_2) = \sum_{p \in c_1 \cap c_2} \mathrm{sign}_{c_1,c_2}(p)$$

This is a *symplectic* (i.e. bilinear, alternating, non-degenerate) form on $H_1(S_g;\mathbb{Z})$. Given some free $\mathbb{Z}$-module M of rank $2g$ with a symplectic form ω, we define

$$\mathrm{Sp}(M,\omega) = \{A \in \mathrm{GL}(M) \mid \omega(Av, Aw) = \omega(v,w)\}$$

We may take a symplectic basis for M, in which case the group $\mathrm{Sp}(M,\omega)$ is identified with the group $\mathrm{Sp}_{2g}(\mathbb{Z}) = \{A \in GL_{2g}(\mathbb{Z}) = A^T J_g A = A\}$ where J_g is the $2g \times 2g$ matrix:

$$J_g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus g}$$

If $\phi \in \text{Homeo}^+(S_g)$, then $\phi_* : H_1(S_g;\mathbb{Z}) \to H_1(S_g;\mathbb{Z})$ is an automorphism which respects the intersection form. Since ϕ_* is a homotopy invariant, it must be the identity if $\phi \in \text{Homeo}_0^+(S_g)$. Thus, we have a map $\text{Mod}(S) \to \text{Sp}_{2g}(H_1(S_g;\mathbb{Z}), i)$. This map happens to be surjective, and its kernel is known as its *Torelli group* $\mathcal{I}(S_g)$. Thus, we have a short exact sequence:

$$0 \to \mathcal{I}(S_g) \to \text{Mod}(S_g) \to \text{Sp}_{2g}(\mathbb{Z}) \to 0$$

The above discussion also holds for surfaces with one boundary component. In general, the intersection form is degenerate, so it is not clear how to define the Torelli group. In the settings where it does apply, it is often the case that the study of the mapping class group reduces to the study of the Torelli group, since symplectic groups are well-understood. For example, this apparently can be used to show that mapping class groups are residually finite.

Congruence Subgroups of Mod(S)

For an integer N, consider the group $\text{Sp}_{2g}(\mathbb{Z}, N)$ of matrices which are the identity mod N. The preimage of this group under the symplectic representation is known as the *level N congruence subgroup* $\text{Mod}(S_g, N)$ of $\text{Mod}(S_g)$. Equivalently, these are the mapping classes that act trivially on $H_1(S_g;\mathbb{Z}/N\mathbb{Z})$. When $g = 0$, these are the familiar congruence subgroups of $SL_2(\mathbb{Z})$ that arise in number theory. These groups also have important applications. For instance, the first Chern class of a line bundle on the moduli space of genus g curves with a level N structure lies in $H^2(\text{Mod}(S_g, N))$.

4 Mod$^{\pm}$(S) and Dehn-Nielsen-Baer

Let S be a surface without boundary. We define the *extended* mapping class group $\text{Mod}^{\pm}(S)$ to be the group of isotopy classes of homeomorphisms of S (including those which are orientation-reversing).
Fix a basepoint $p \in S$. Given a homeomorphism $\phi : S \to S$ and a path γ from p to $\phi(p)$, the map $\phi_{\gamma,*} : [\alpha] \to [\gamma * \phi(\alpha) * \gamma^{-1}]$ is an automorphism of $\pi_1(S, p)$. Given two such paths γ and γ', we see that $\phi_{\gamma',*}([\alpha]) = [\gamma' * \gamma^{-1} * \phi_{\gamma,*}([\alpha]) * \gamma * (\gamma')^{-1}]$. Thus, ϕ defines an element of $\text{Aut}(\pi_1(S, p))$ up to *inner automorphism*, meaning we have a map

$$\text{Homeo}^{\pm}(S) \to \text{Out}(\pi_1(S))$$

Assume $\chi(S) \leq 0$, so S is a $K(\pi_1(S), 1)$. Then any homomorphism $\pi_1(S, p) \to \pi_1(S, p)$ is induced by a map $(S, p) \to (S, p)$ which is unique up to homotopy (see Hatcher Proposition 1B.9).
FIGURE OUT THE KERNEL

Theorem 8 (Dehn-Nielsen-Baer)**.** *For $g \geq 1$, the homomorphism*

$$\sigma : \mathit{Mod}^{\pm}(S_g) \to \mathit{Out}(\pi_1(S_g))$$

is an isomorphism.

5 Teichmüller Space and Moduli

Let S be a compact, connected, orientable surface, possibly with punctures. We define a hyperbolic structure on S to be a diffeomorphism $\phi : S \to X$ where X is a complete, finite-area hyperbolic surface with totally geodesic boundary. Two hyperbolic structures $\phi_1 : S \to X_1$ and $\phi_2 : S \to X_2$ are homotopic if there's an isometry $I : X_1 \to X_2$ such that $I \circ \phi_1$ is homotopic to ϕ_2. We could have also considered diffeomorphisms from X to a Riemann surface, with the role of the isotopy replaced by that of a biholomorphic map.

Definition 9. The Teichmüller Space $\text{Teich}(S)$ is defined to be the space of hyperbolic structures (or Riemann structures) on S up to homotopy.

Example 10. The example that this seeks to replicate is that of the torus. Any complex structure on the torus is of the form $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some $\tau \in \mathbb{H}^2$, giving us Teich$(T^2) = \mathbb{H}^2$

We may understand Teichmüller spaces of higher genus surfaces in a similar way by noting that $\mathbb{H}^2 = PSL_2(\mathbb{R})/PSO(2)$.

Proposition 11. *Consider the action of $PGL_2(\mathbb{R})$ on the space of discrete, faithful representations $\rho : \pi_1(S_g) \to PSL_2(\mathbb{R})$ by conjugation. Then there is a natural bijection between the quotient space*

$$DF(\pi_1(S_g), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$$

and Teich(S_g).

This allows us endow Teich(S_g) with a topology: we consider $\pi_1(S_g)$ to be discrete, and then take the usual topology on $PSL_2(\mathbb{R})$. Then we consider the space $DF(\pi_1(S_g), PSL_2(\mathbb{R})$ with the compact-open topology, and then the orbit space carries a quotient topology. With this topology, Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$, so it is in particular contractible

Action of the mapping class group

Given a hyperbolic structure $[\phi]$ where $\phi : S \to X$ and a mapping class $[f]$ where $f : S \to S$, we write $[f] \cdot [\phi] = [\phi \circ f]$. This action is proper discontinuous, and some finite-index subgroupf of Mod(S) acts freely. The quotient is $\mathcal{M}_g$, the moduli space of genus g curves. Since Teichmüller space is contractible, we consider the long exact sequence of a fibration, which becomes

$$1 \to \pi_1(S_g) \to \pi_1(\text{Teich}(S_g)) \to \pi_1(\mathcal{M}_g)$$

(these are orbifold fudnamental groups). By Dehn-Nielsen-Baer, this tells us that $\pi_1(\mathcal{M}_g) \cong \text{Mod}(S)$, demonstrating how mapping class groups arise in algebraic geometry. (Some chit chat about Deligne-Mumford stacks here if there's time).
Maybe talk about how if B is some space, we (almost) have a bijection between S_g bundles over B, maps $B \to \mathcal{M}_g$, and conjugacy classes of representations $B \to \text{Mod}(S_g)$.

6 Dehn Twists

Given a simple closed curve in a surface, we can find some tubular neighborhood which is an annulus. We may extend Dehn twist on this embedded annulus (see example 5 and remark 6) by the identity on the complement to obtain a global homeomorphism of the surface, which we also refer to as a Dehn twist. For example, consider the following matrices in $SL_2(\mathbb{Z})$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

These are Dehn twists around the $(1,0)$ and $(0,1)$ curves in T^2. These two matrices actually generate $SL_2(\mathbb{Z})$, and in fact this holds in general:

Theorem 12. *The mapping class group of a surface is generated by some finite set of Dehn twists.*

7 Nielsen-Thurston Classification

While we already know that the mapping class group of the torus is isomorphic to $SL_2(\mathbb{Z})$, there is more to know than just the pure group theoretic properties. We want an explicit description of its elements. Think about the distinction between knowing the group theory of $GL_2(\mathbb{C})$, and knowing that every matrix can be

put into Jordan Normal Form.
Every mapping class in $\mathrm{Mod}(T^2)$ is represented by some matrix A, which has the map $v \mapsto Av \mod \mathbb{Z}^2$ as a representative. In general, for $SL_2(\mathbb{R})$, we have a classification of elements into elliptic, parabolic, and hyperbolic, depending on whether their associated fractional linear transformations fix 0, 1, or 2 points on $\partial\mathbb{H}^2$ (equivalently, whether $|\mathrm{tr}(A)|$ is less than, equal to, or greater than 2). We can understand this classification through how these elements act on the torus:

- Elliptic elements are precisely those of finite order.
- Parabolic elements fix a curve an isotopy class of curves.
- Hyperbolic elements act by Anosov diffeomorphisms.

The Nielsen-Thurston classification mirrors this in general. Reducible mapping classes are those which fix some isotopy classes of curves which have pairwise 0 intersection number (generalizing parabolic elements on the torus). Finally, pseudo-Anosov maps preserve a transverse pair of measured foliationns, scaling the measure of one up and of the other down, in much the same way that hyperbolic maps of the torus preserve the foliation given by positive and negative eigenspaces projected down to it. This classification is mirrored in general

Theorem 13 (Nielsen-Thurston)**.** *Every mapping class $[f] \in Mod(S)$ is either finite order, reducible, or pseudo-Anosov*

Pseudo-Anosov diffeomorphisms are particularly interesting. For one, they share various dynamical properties with Anosov diffeomorphisms, such as the existence of a dense orbit, and density of the sets of periodic points. In addition, the mapping torus of a diffeomorpism of S_g, which will be some S_g-bundle fibered over the circle, carries a hyperbolic structure if and only if the diffeomorphism is pseudo-Anosov.

8 Recent Work

We may consider mapping class groups of higher-dimensional spaces, rather than just surfaces. Here we have to worry about the distinction between diffeomorphisms and homeomorphisms, as well as the distinction between homotopy and isotopy. We will now define the mapping class group of any manifold M as

$$\mathrm{Mod}(M) = \pi_0(\mathrm{Diff}^+(M))$$

Before Nielsen-Thurston classification, we mostly knew that mapping class groups were generated by Dehn twists. Our current understanding of mapping class groups of 4-manifolds is in similar shape. Recall the intersection pairing $i : H_1(S_g; \mathbb{Z}) \times H_1(S_g; \mathbb{Z}) \to \mathbb{Z}$ for surfaces. In the same way, if M is a closed, connected, orientable 4-manifold, we have an intersection form

$$i : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to \mathbb{Z}$$

In this case, the form is symmetric rather than alternating. As before, a diffeomorphism $M \to M$ induces an automorphism of H_2 which respects the intersection form, and one which is smoothly isotopic to the identity induces the identity.

Theorem 14 (Freedman-Quinn)**.** *Let M be a simply connected 4-manifold. Then the natural map*

$$Mod(M) \to Aut(H_2(M; \mathbb{Z}), i)$$

is an isomorphism[1]*.*

[1] Note in particular that $\mathrm{Mod}(S)$ is an arithmetic group

But can we get something closer to Nielsen-Thurston, where we have an explicit understanding of the elements?

Definition 15. A *K3 surface* is a closed, simply connected complex surface with a nowhere-vanishing holomorphic 2-form.

For example, a smooth degree 4 hypersurface in $\mathbb{P}^3$ is a K3 surface. K3 surfaces are diffeomorphic, so they are simply connected and their intersection form is given by the *K3 lattice*: $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. These are the simplest complex surfaces, after tori and things coming from curves. The program by Farb and Looijenga is to explicitly understand mapping classes on these spaces (note that we already know what $\mathrm{Mod}(M)$ is as an abstract group). One of the first results along these lines involves the *Nielsen Realization Problem.*

Question 16. *The Nielsen Realization Problem asks whether a finite subgroup G of the mapping class group of a smooth manifold comes from some finite group of diffeomorphisms of M.*

For surfaces, this is true. In fact, there is some hyperbolic metric on the surface for which our finite group is precisely the group of isometries of this metric. Recent work by Farb and Margalit resolves this problem for K3 surfaces.