Some questions regarding the dynamics and analysis on graphs

Project Summary

Overview

The aim of this project is to investigate questions on graphs and graph maps using ideas from complex dynamics, symbolic dynamics as well as complex analysis. They will be mostly about the following three related topics:

- Entropy of maps on intervals and Hubbard trees. The PI and his collaborators conjectured some properties on the distribution of Galois conjugates of the exponent of core entropy of quadratic maps as well as an analogy of the Julia-Mandelbrot correspondence for the setting of interval maps as well as maps on the Hubbard tree. A main goal of this project is to hopefully establish these conjectured properties in certain special cases.
- Estimating the asymptotic translation length on spherical complex of train track maps on finite graphs. The PI and his collaborators are working on generalizing their prior works on the asymptotic translation length of pseudo-Anosov maps on curve complex to the setting of train track maps on finite graphs. It is hoped that these new results will be useful for understanding the dynamics of both graph maps and surface maps.
- Dynamics and analysis on metric graphs. Motivated by the analogy between graphs and surfaces above, the PI and his collaborators are also working on generalizing certain results on the geometry and analysis on Riemann surfaces to the setting of metric graphs.

Intellectual Merit

The study of the entropy of maps on Hubbard tree and translation lengths on curve complex are important questions that have been extensively studied, and it is hoped that the results in this project will shed new lights on these topics. Furthermore, it is hoped that the results obtained in this project will be a special case of more general results, for example generalized Milnor-Thurston kneading theory that applies to maps on (finite or infinite) trees or graphs, Julia-Mandelbrot type correspondences between the dynamic plane and parameter plane for general iterated function systems or similar systems associated with a symbolic dynamics, or generalized McMullen's polynomial that can encode the translation length on curve complexes.

Broader Impacts

Most of the questions in this project can provide many opportunities for undergraduate research, and can be used to engage students in lectures for both undergraduate and high school level. The PI has previously used some materials from the third topic above for a short course in the KAIX (KAIST Advanced Institute for Science) summer school with an audience of undergraduate and beginning graduate students in both math and applied math majors. Furthermore, some algorithms that can potentially be obtained from this project may be useful for real-life purposes like surface matching or dimension reduction.

Project Description

A common theme of this project is on applying ideas in dynamics and analysis to the study of graphs and graph maps. The project can be divided into three related topics which will be further described below:

- Topological entropy of interval maps and maps on Hubbard tree.
- Asymptotic translation length on curve complex and sphere complex.
- Graph-theoretic analogy of theorems on Riemann surfaces.

The first motivation for this project is the connection between dynamics on surfaces and graphs. For example, dynamics on pseudo-Anosov maps can be encoded by their action on the invariant train track [62, 56], and dynamics of polynomial maps on the Riemann sphere with finite postcritical orbits can be encoded by maps on the Hubbard tree [21], which is the tree connecting the orbit of the critical points within the Julia set. In the former case, the topological entropy can be calculated via the action on the train track, and the train track provides a Markov decomposition of the surface. In the latter case, the topological entropy of the map on the Hubbard tree is called the "core entropy", which, in the quadratic case, is related to the Hausdorff dimension of the set of external rays landing on Mandelbrot set [58] and is also known to be continuous [59], and both train tracks and Hubbard trees provides useful combinatorial encoding for the original two-dimensional map. Many questions in this project are about the dynamics on train tracks as well as Hubbard trees.

Another motivation for this project is the well studied anology between compact Riemann surfaces and metric graphs which has results in many interesting and important results in geometry, topology, dynamics and number theory. For example, Culler and Vogtmann [20] proposed the concept of outer space which serves the same purpose of Teichmüller space for the study of $Out(F_n)$. The outer space on which $Out(F_n)$ acts on consists of all finite metric graphs with n loops and no degree 1 vertices, together with a marking map which is a homotopy equivalence between this metric graph and the wedge of n circles, where $Out(F_n)$ acts by precomposition with the marking. Similarly, the Teichmüller space consists of Riemann surfaces with prescribed topological type, together with a marking map which is a homotopy equivalence from a model surface to the Riemann surface. Both the outer space and the Teichmüller space are contractable, and the group action described above are proper with finite stablizers. This analogy made it possible to generalize many concepts and properties on Riemann surfaces and Teichmuller spaces, e.g. Nielsen–Thurston classification, train tracks, measured laminations, Teichmuller metrics, and curve complexes. Also, by reducing the problem to the topology and geometry of outer space, this analogy provides a new and fruitful approach to study the algebraic and topological properties of the outer automorphism group which is a cell complex consisting of simplicial cells and are assigned an asymmetric metric, resulting in many important results by Bestvina, Feign, Handel, Dowdall, Kapovich, Leininger (cf. [14, 13, 23]) etc.

Another way to understand the analogy between Riemann surfaces and metric graphs is through tropical geometry and Berkovich spaces (cf. [11]). Just as Riemann surfaces can be seen as analytic spaces of complex algebraic curves, the Berkovich spaces of dimension one serve as an analytic space for curves on non-Archimedean algebraic field. They have a natural geometric structure and by collapsing $\mathbb{R}$-trees can be turned into finite metric graphs. Motivated this analogy, it is possible to define the analogy of harmonic functions and maps, Jacobians (which are real tori, unlike the Jacobian of Riemann surfaces), and line bundles, and show that they satisfies many of the same properties as in the case of Riemann surfaces, cf. the works of Baker, Chinberg, Rumely, de Jong, Shokrieh etc. (cf. [31, 6]). Many of the questions in this project is related to further investigation and utilization of this analogy for the study of both graphs and Riemann surfaces.

1 Topological entropy of interval maps and maps on Intervals and on Hubbard tree.

Topological entropy is a way to characterize the complexity of a dynamical system. In the case when the system admits a Markov decomposition, the exponent of the topological entropy is the eigenvalue of a Perron-Frobenious matrix hence must be an algebraic integer.

The study of dynamics on intervals is related to continuous fractions and β-expansions [29, 43, 25, 49, 53], complex dynamics [21, 33, 44, 41, 54, 58, 59], and via Milnor-Thurston kneading theory [42], the study of symbolic dynamics and iterated function systems [9, 8, 50, 19]. In the case when the map is continuous and the forward-orbit of critical points are finite, the union of these orbits provides a Markov partition of the interval. In his last paper [55], Willian Thurston proposed the *master teapot*, which is the set

$$T := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

Here h_{top} is the topological entropy, $\mathcal{F}$ is the set of unimocal maps with periodic critical orbit. Here an interval map is unimodal if it has a single critical point c in the interior, it has periodic critical orbit if there is some $n > 0$ such that $f^{\circ n}(c) = c$. The projection of the Master teapot on $\mathbb{C}$ is called the *Thurston set*. The study of Thurston teapot and Thurston set can be seen as providing necessary conditions for an algebraic integer λ to be the exponent of the entropy of a unimodal map with periodic critical orbit: all its Galois conjugates must be in the Thurston set, and if z is a Galois conjugate of λ, (z, λ) must be in T.

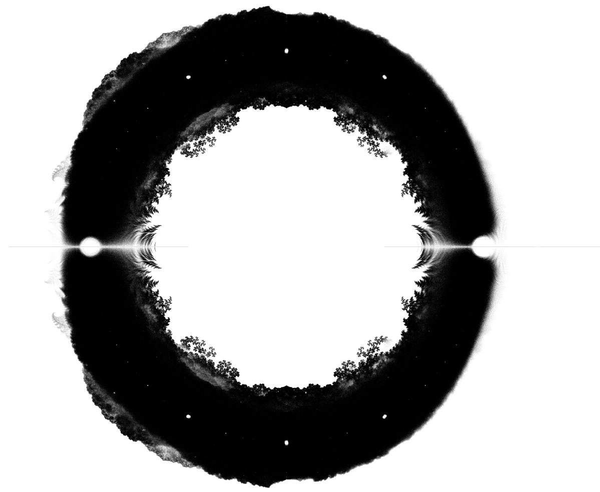
In [60], a characterization of the Thurston set is given and the part of it inside the unit circle are shown to be the closure of the set of roots of all Littlewood polynomials (polynomials with all coefficients ± 1) [35], and [19] showed that there are holes in the Thurston set other than the trivial one around zero. Figure 1 and Figure 1 are finite approximations of the Thurston set and Thurston teapot respectively.

In some prior works by the PI and his collaborators [18, 34], a characterization of the Thurston teapot inside the "unit cylinder" $\{(z, \lambda) : |z| < 1\}$ analogous to the characterization of Thurston set in [60] is found, and an algorithm is given that can be used to certify a point not belonging to the Thurston teapot. In particular, given any $\lambda \in (1, 2)$, the PI and his collaborators found a subset M_λ of $\{0, 1\}^{\mathbb{N}}$ invariant under shift, non decreasing as λ increases, such that for any $|z| < 1$, $(z, \lambda) \in T$ iff there is some $w = w_1 w_2 \cdots \in M_\lambda$, such that

$$G(w, z) := \lim_{n \to \infty} f_{w_1, z} \circ \ldots \circ f_{w_n, z}(1) = 1$$

Here $f_{0,z}(x) = zx$ and $f_{1,z}(x) = 2 - zx$.

As a consequence,



Theorem 1. [18] If $(z, \lambda) \in T$, $|z| < 1$, then for any other $\lambda' \in [\lambda, 2]$, $(z, \lambda') \in T$.

For part of the teapot outside the unit cylinder a characterization can be easily derived from observations in [55, 60], which was also reviewed in [18, 34]. In particular,

Theorem 2. The set $\{z : (z, \lambda) \in T \text{ or } |z| \leq 1\}$ changes continuously with λ under the Hausdorff topology.

When the unimodal maps are chosen to be quadratic, the interval can be seen as a special case of the Hubbard tree [21]. Here, the *Hubbard tree* is a finite simplicial tree whose vertices are orbits of the critical point and is contained in the Julia set. The quadratic map on the Julia set induces a simplicial map on the Hubbard tree, and the topological entropy of this simplicial map is called the *core entropy*. Now a critically periodic unimodal map can be seen as a Misiurewicz point ($c \in \mathbb{C}$ such that $f_c : z \mapsto z^2 + c$ satisfies $f_c^{\circ n}(0) = 0$ for some $n > 0$) on the real slice of the Mandelbrot sets, and the study of the topological entropy of these interval maps is the same as the study of the core entropy on these points.

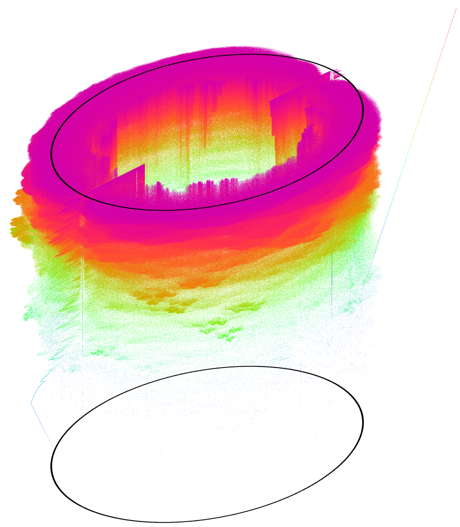
Motivated by evidences from numerical experiments, the PI and his collaborators proposed the following conjectures

Conjecture 1.

1. When one replace $\mathcal{F}$ in the definition of T with interval maps with at most 2, critical values, there is an analogous characterization of points in T. In particular, both Theorem 1 and Theorem 2 remains true.

2. The set of Galois conjugates of the exponent of the core entropy union with the unit disk changes continuously on the Mandelbrot set. This is a strengthening of the main result in [59].

3. Along principal veins of the Mandelbrot set there is an analogy for Theorem 1 regarding the closure of Galois conjugates of the exponent of core entropy inside the unit circle.



4. [34] For any complex number $|z| < 1$, any $\lambda \in (1,2)$, the set $X_z := \{x - z : G(w,x) = 1 \text{ for some } w \in M_\lambda\}$ is asymptotically similar to the set
$$J_z = \{G(w,z) - 1 : w \in M_\lambda\}.$$
Here two sets A and B are asymptotically similar means that there exists a real number $r > 0$ and sequences $(t_n), (t'_n) \in \mathbb{C}$ with $t_n, t'_n \to \infty$ such that, denoting Hausdorff distance by d_{Haus},
$$\lim_{n\to\infty} d_{\text{Haus}}\left(\overline{B_r(0)} \cap (t_n A), \overline{B_r(0)} \cap (t'_n B)\right) = 0.$$
This is analogous to the Julia-Mandelbrot correspondence [32], where the set X_z is analogous to the Mandelbrot set while the set J_z is analogous to the filled Julia set.

A question for this project is the following:

Question 1. Can the four conjectures above be proved?

At the moment the PI and his collaborators are able to show Conjecture 1 provided the kneading determinants (cf. [42]) is irreducible for sufficiently many interval maps, and Conjecture 2 and 3 provided the denominator of the power series Δ in [58] is irreducible for sufficiently many Misiurewicz points.

2 Asymptotic translation length on curve complex and sphere complex

A pseudo-Anosov map on a closed surface S is a self-homeomorphism that no power of it sends a simple closed curve that does not bound a disc to itself up to isotopy. Up to homotopy, one can make it preserve a pair of transverse singular measured foliations, and the stretch factors on these measured foliations is the exponent of the topological entropy of this map. These stretch factors are always bi-Perron algebraic integers, and it is an open problem if all bi-Perron algebraic integers can be realized as a stretch factor. For pseudo-Anosov maps that have homeomorphic mapping torus and lie in the same *fibered cone*, their stretch factors are related to each other via the Teichmüller polynomial by McMullen [39], which are related to the Alexander polynomial of the mapping torus. Here the concept of *fibered cone* was introduced in [57]:

Definition 2. [57]

- Let M be a hyperbolic 3-manifold, for every integer cohomology class $\alpha \in H^1(M;\mathbb{Z})$, the *Thurston norm* of α is defined as

  $$\|\alpha\| = \min_S \max\{0, -\chi(S_i)\}$$

  Where $S = \bigcup_i S_i$ is an embedded surface that represents α. The Thurston norm can be extended to $H^1(M;\mathbb{R})$ as a piecewise linear function with rational coefficients.

- If M is homeomorphic to a mapping torus of a surface map ϕ (seen as a surface bundle over the circle), let the cohomology class associated with ϕ, denoted as α_ϕ, be the pullback of the generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called *fibered cones*, and the map associated with a primitive integer class α in the fibered cone are denoted as ϕ_α.

Besides stretch factors, another way of characterizing the topological complexity of a pseudo-Anosov map is through its asymptotic translation length on the curve graph of S. The curve graph $\mathcal{C}(S)$ is a graph where each vertex represent an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. It is easy to see that the mapping class group of S acts on curve graph and curve complex by isometry. Masur-Minsky [37] show that $\mathcal{C}(S)$ is δ-hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserves a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. The study of curve graphs are also related to the hirerachical hyperbolic structure on the mapping class group [38, 10]. The asymptotic translation length of a pseudo-Anosov map g on $\mathcal{C}(S)$ can now be defined as

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$.

It is shown in [37] that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov. Furthermore, the technique in [37] in showing the positivity of $l_{\mathcal{C}}$, which is based on studying the incidence matrix on the induced map on invariant traintracks, have been used by [24, 61, 3] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases. Furthermore, in [17] the asymptotic translation length is shown to be a rational number, and in [47, 15] algorithms for its computation are described.

In [30], a sequence of pseudo-Anosov maps in different genus are constructed that realized the asymptotic lower bound. The PI and his collaborators generalized the argument in [30] and proved the following:

Theorem 3. [4] Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

Balázs Strenner [51] also proved a stronger result for the asymptotic translation length of arc complexes.

In [2], the PI and his collaborators have shown that this asymptotic upper bound is sharp when $r \leq 3$. Furthermore, in [2], the PI and his collaborators uses techniques similar to [4] to show the following:

Theorem 4. Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.

Motivated by the relationship between pseudo-Anosov maps and the induced map on the invariant train track as well as the analogy between $Out(F_n)$ and the mapping class group c.f. [14, 13], the PI and his collaborators are working on generalizing Theorem 3 to the case of the metric graph. In an upcoming paper, the PI and his collaborators will prove an analogy of the above theorem in the setting of asymptotic translation lengths on the sphere complex for train track maps.

Let G be a finite simplicial graph. A cellular map $\psi : G \to G$ is called a train track map if the restriction of ψ^n to e for each $n \geq 1$ and each edge e is an immersion (no back-tracking condition). We further assume ψ to be irreducible as an element of $Out(F_n)$. One can make a 3-manifold M_G from G by replacing every edge with $S^2 \times I$ and every vertex with $\mathbb{S}^3$. In the case when ψ is a train track map, ψ induces a homeomorphism ψ_1 on M_G. Let $\mathcal{S}(G)$ be the simplicial graph whose vertices are isotopy classes of embedded spheres in M_G, and there is an edge of length 1 between two vertices if and only if they are disjoint up to isotopy, then it is easy to see that ψ_1 is an isometry of $\mathcal{S}$ and we can define the concept of asymptotic translation length of ψ_1 analogously.

In [23, 22], the argument in [57] and [39] are generalized to the case of maps on finite graphs as follows:

Definition 3. [23, 22] Suppose ψ is an irreducible train track map, let γ_ψ be a folding path of ψ in the Culler-Vogtmann outer space. The *folded mapping torus* N is a 2-d cell complex built from γ_ψ, which has a surjection over the circle and the fibers are the graphs in the folding path. A flow on N is defined such that any flow line is the orbit of a point on the graph under folding, and an analogy for the fibered face containing ϕ is $\mathcal{S}$ which consists of first cohomology classes whose dual are transverse to all flow lines. This is a rational cone call the "cone of sections" or "McMullen cone" in [22].

The PI and his collaborators are able to show the following:

Theorem 5. Given any finite graph G and any irreducible train track map ψ on G, let $\mathcal{C}$ be any proper subcone of the McMullen cone in [39] containing ψ, then any primitive integer element α in $\mathcal{C}$ must satisfies

$$l(\psi_\alpha) \lesssim n_\alpha^{-1-1/d}$$

Where d is the dimension of the fibered cone, n_α the genus of the fiber corresponding to α and ψ_α the corresponding monodromy, and $l(\cdot)$ the translation length on the spherical complex obtained by thickening the graph G.

Some further questions the PI and his collaborators are working on are the following:

Question 2.

- What is the relationship between the translation length on the sphere complex of the thickened invariant train track and the curve complex? The PI and his collaborators hope that this can be useful for generalizing Theorem 3 to families of pseudo Anosov maps that do not lie in the same fibered cone, for example, those arising from maps on a fibered cone under a subsurface projection.
- Can similar results be proved for other complexes related to $Out(F_n)$, like the free splitting complex [27], free factor complex [12], or the cyclic splitting complex [36]?
- Can there be a lower bound for the asymptotic translation lengths in the case of train track maps or pseudo-Anosov map that shows that the upper bound is asymmtotically optimal?
- Can there be an analogy of Theorem 4 in the case of train track graphs?

3 Graph-theoretic analogy of theorems on Riemann surfaces

By a metric graph, we mean an undirected simplicial graph with a positive edge length function l defined on the edges, and where each edge e is made into an interval of length $l(e)$. When the graph has finitely many edges we call it a finite metric graph. The following definitions are standard:

Definition 4. [5] Let G be metric graph.

1. By a 1-form on G we mean an element in the 1-simplicial cochain $C^1(G)$. Let $d : C^0(G;\mathbb{R}) \to C^1(G;\mathbb{R})$ be the coboundary map.
2. Let G be a metric graph with edge set $E(G)$, w a 1-form, we define the norm of w as $\|w\| = (\sum_{e\in E(G)} w(e)^2/l(e))^{1/2}$. We call a 1-form L^2 if it has a finite norm. Under this norm, denote the dual of d δ.
3. We call a 1-form α harmonic, if $\delta(\alpha) = 0$. In other words, for every vertex v, all outgoing edges e_i from v, $\sum_i w(e_i)/l(e_i) = 0$.

The analogy between Riemann surfaces and finite metric graphs allow many important concepts and properties on Riemann surfaces to be generalized to the graph-theoretic setting. For example, there is a graph-theoretic Riemann-Roch theorem [7] which is related to the question of chip firing, which is as follows:

Theorem 6. [7] Let G be a finite simplicial graph with no loops, all edges are assumed to have length 1. A *divisor* on G is an integer linear combination of vertices of G, i.e. an element in $C^0(G;\mathbb{Z})$, the degree is the sum of all the coefficients in this linear combination. A divisor D is called *effective* if all its coefficients are non negative, two divisors D and D' are *linearly equivalent* if $\delta d(D - D') = 0$, and $r(D)$ is the largest natural number such that $D - E$ is linearly equivalent to an effective divisor for any effective divisor of degree r, or, if no such natural number exists, -1. Then

$$r(D) - r(K-D) = deg(D) + 1 - g$$

Here g is the first betti number of G, K is the *canonical divisor* $K = \sum_v (\deg(v) - 2)(v)$.

The proof however is very different from the Riemann Roch theorem on surfaces, as r can no longer be interpreted as the dimension of some vector space.

Also, analogous to the Arakelov metric on Riemann surfaces, there is the "canonical metric" on finite metric graphs [63, 6], which is defined as

Definition 5. The canonical metric[63, 6] on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

This is shown by [1, 45] to be related to the distribution of Weistrass points of line bundles on finite metric graphs.

Graph theoretic analogies of differential operators and their spectral theory have also been extensively studied, for example [5, 28, 52].

In a prior work of the PI and his collaborator, an analogous result on a property of the Arakelov metric on Riemann surfaces in [40, Alppendix] was found, which shows that when passing to larger and larger normal covers the canonical metric on metric graphs converges:

Theorem 7. [48] Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.

In the case when $\cap_i \pi_1(G_i)$ is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of G. Because in [40, Alppendix] the limit of the Arakelov metrics for a closed Riemann surface of genus ≥ 2 under a tower of normal covers that converges to the universal cover would converge to the hyperbolic metric, one can see the limiting metric in Theorem 7 as a discrete analogy of the hyperbolic metric. However it is unclear what the relationship between this limiting metrics and other approaches of discrete uniformization [16, 26, 46] are.

Currently, the PI and his collaborators are working on the following questions in this direction:

Question 3.

- Let Δ_μ be the Laplace operator on metric graph composed with projection to the orthogonal complement of the canonical measure. What are the eigenvalues? What can we say about the ζ function $\zeta(z) = \sum_i \lambda_i^{-z}$?
- Can the Riemann Roch theorem be explained as the index theorem of some differential operator?

Broader Impacts

Many results and techniques used in this project are accessible to an undergraduate audience while illustrating important concepts in analysis and dynamics, hence would serve as good examples that can be used in a classroom setting or outreach activities and can also facilitate possible undergraduate research. For example, questions inspired by all three topics in this project have been used as undergraduate research problems by collaborators of the PI. The PI also discussed some of his prior results related to this project in a summer course in KAIX summer school in Daejeon, Korea with an audience consisting of undergraduate and lower level graduate students from Korea, Thailand and Vietnam, and received active participation and feedback from the students. Furthermore, techniques developed in this project may serve as tools for the study of real-life dynamical systems and might also be useful for applications like surface matching and discrete uniformization.

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References Cited

[1] Omid Amini. Equidistribution of weierstrass points on curves over non-archimedean fields. *arXiv preprint arXiv:1412.0926*, 2014.

[2] Hyungryul Baik, Eiko Kin, Hyunshik Shin, and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone. *arXiv:1909.00974*, 2019.

[3] Hyungryul Baik and Hyunshik Shin. Minimal Asymptotic Translation Lengths of Torelli Groups and Pure Braid Groups on the Curve Graph. *International Mathematics Research Notices*, 12 2018. https://doi.org/10.1093/imrn/rny273.

[4] Hyungryul Baik, Hyunshik Shin, and Chenxi Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces. *arXiv preprint arXiv:1801.06638, to appear in IUMJ*, 2018.

[5] Matt Baker and Robert Rumely. Harmonic analysis on metrized graphs. *Canadian Journal of Mathematics*, 59(2):225–275, 2007.

[6] Matthew Baker and Xander Faber. Metric properties of the tropical abel–jacobi map. *Journal of Algebraic Combinatorics*, 33(3):349–381, 2011.

[7] Matthew Baker and Serguei Norine. Riemann–roch and abel–jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788, 2007.

[8] Christoph Bandt. On the mandelbrot set for pairs of linear maps. *Nonlinearity*, 15(4):1127, 2002.

[9] Michael F Barnsley and Andrew N Harrington. A mandelbrot set for pairs of linear maps. *Physica D: Nonlinear Phenomena*, 15(3):421–432, 1985.

[10] Jason Behrstock, Mark Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, i: Curve complexes for cubical groups. *Geometry & Topology*, 21(3):1731–1804, 2017.

[11] Vladimir G Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Number 33. American Mathematical Soc., 2012.

[12] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *Advances in Mathematics*, 256:104–155, 2014.

[13] Mladen Bestvina, Mark Feighn, and Michael Handel. Laminations, trees, and irreducible automorphisms of free groups. *Geometric & Functional Analysis GAFA*, 7(2):215–244, 1997.

[14] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Annals of Mathematics*, pages 1–51, 1992.

[15] Joan Birman, Dan Margalit, and William Menasco. Efficient geodesics and an effective algorithm for distance in the complex of curves. *Mathematische Annalen*, 366(3-4):1253–1279, 2016.

[16] Alexander I Bobenko, Nikolay Dimitrov, and Stefan Sechelmann. Discrete uniformization of polyhedral surfaces with non-positive curvature and branched covers over the sphere via hyper-ideal circle patterns. *Discrete & Computational Geometry*, 57(2):431–469, 2017.

[17] Brian H Bowditch. Tight geodesics in the curve complex. *Inventiones mathematicae*, 171(2):281–300, 2008.

[18] Harrison Bray, Diana Davis, Kathryn Lindsey, and Chenxi Wu. The shape of thurston's master teapot. *arXiv preprint arXiv:1902.10805*, 2019.

[19] Danny Calegari, Sarah Koch, and Alden Walker. Roots, schottky semigroups, and a proof of bandt's conjecture. *Ergodic Theory and Dynamical Systems*, 37(8):2487–2555, 2017.

[20] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae*, 84(1):91–119, 1986.

[21] A DOUADY. Etude dynamique des polynomes complexes. *Publ. Math. Orsay*, pages 1984–1985.

[22] Spencer Dowdall, Ilya Kapovich, and Chris Leininger. Mcmullen polynomials and Lipschitz flows for free-by-cyclic groups. *J. Eur. Math. Soc.*, (11):3253–3353, 2017.

[23] Spencer Dowdall, Ilya Kapovich, and Christopher J Leininger. Dynamics on free-by-cyclic groups. *Geometry & Topology*, 19(5):2801–2899, 2015.

[24] Vaibhav Gadre and Chia-Yen Tsai. Minimal pseudo-anosov translation lengths on the complex of curves. *Geometry & Topology*, 15(3):1297–1312, 2011.

[25] Paweł Góra. Invariant densities for generalized β-maps. *Ergodic Theory and Dynamical Systems*, 27(5):1583–1598, 2007.

[26] Xianfeng David Gu, Feng Luo, Jian Sun, and Tianqi Wu. A discrete uniformization theorem for polyhedral surfaces. *Journal of differential geometry*, 109(2):223–256, 2018.

[27] Michael Handel and Lee Mosher. The free splitting complex of a free group, i hyperbolicity. *Geometry & Topology*, 17(3):1581–1670, 2013.

[28] JM Harrison, Klaus Kirsten, and Christophe Texier. Spectral determinants and zeta functions of schrödinger operators on metric graphs. *Journal of Physics A: Mathematical and Theoretical*, 45(12):125206, 2012.

[29] Shunji Ito and Taizo Sadahiro. Beta-expansions with negative bases. *Integers*, 9(3):239–259, 2009.

[30] Eiko Kin and Hyunshik Shin. Small asymptotic translation lengths of pseudo-anosov maps on the curve complex. *Groups, Geometry, and Dynamics*, 2019.

[31] Motoko Kotani and Toshikazu Sunada. Jacobian tori associated with a finite graph and its abelian covering graphs. *Advances in Applied Mathematics*, 24(2):89–110, 2000.

[32] Tan Lei. Similarity between the mandelbrot set and julia sets. *Communications in mathematical physics*, 134(3):587–617, 1990.

[33] Tao Li. *A monotonicity conjecture for the entropy of Hubbard trees.* PhD thesis, Stony Brook University, 2007.

[34] Kathryn Lindsey and Chenxi Wu. A characterization of thurston's master teapot. *arXiv preprint arXiv:1909.10675*, 2019.

[35] JE Littlewood. On polynomials $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{0i}$. *Journal of the London Mathematical Society*, 1(1):367–376, 1966.

[36] Brian Mann. Hyperbolicity of the cyclic splitting graph. *Geometriae Dedicata*, 173(1):271–280, 2014.

[37] Howard A Masur and Yair N Minsky. Geometry of the complex of curves i: Hyperbolicity. *Inventiones mathematicae*, 138(1):103–149, 1999.

[38] Howard A Masur and Yair N Minsky. Geometry of the complex of curves ii: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000.

[39] Curtis T McMullen. Polynomial invariants for fibered 3-manifolds and teichmüller geodesics for foliations. In *Annales scientifiques de l'Ecole normale supérieure*, volume 33, pages 519–560. Elsevier, 2000.

[40] Curtis T McMullen. Entropy on riemann surfaces and the Jacobians of finite covers. *Commentarii Mathematici Helvetici*, 88(4):953–964, 2013.

[41] Philipp Meerkamp and Dierk Schleicher. Hausdorff dimension and biaccessibility for polynomial julia sets. *Proceedings of the American Mathematical Society*, 141(2):533–542, 2013.

[42] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems*, pages 465–563. Springer, 1988.

[43] William Parry. On the β-expansions of real numbers. *Acta Mathematica Academiae Scientiarum Hungarica*, 11(3-4):401–416, 1960.

[44] Alfredo Poirier. Critical portraits for postcritically finite polynomials. *Fundamenta Mathematicae*, 203:107–163, 2009.

[45] David Harry Richman. Equidistribution of weierstrass points on tropical curves. *arXiv preprint arXiv:1809.07920*, 2018.

[46] Rohan Sawhney and Keenan Crane. Boundary first flattening. *ACM Transactions on Graphics (ToG)*, 37(1):5, 2018.

[47] Kenneth J Shackleton. Tightness and computing distances in the curve complex. *Geometriae Dedicata*, 160(1):243–259, 2012.

[48] Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem. *arXiv:1711.02609, accepted by Invent. Math.*, 2017.

[49] Boris Solomyak. Conjugates of beta-numbers and the zero-free domain for a class of analytic functions. *Proceedings of the London Mathematical Society*, 3(3):477–498, 1994.

[50] Boris Solomyak and Hui Xu. On the "Mandelbrot set" for a pair of linear maps and complex Bernoulli convolutions. *Nonlinearity*, 16(5):1733, 2003.

[51] Balázs Strenner. Fibrations of 3-manifolds and asymptotic translation length in the arc complex. *arXiv preprint arXiv:1810.07236*, 2018.

[52] Christophe Texier. ζ-regularized spectral determinants on metric graphs. *Journal of Physics A: Mathematical and Theoretical*, 43(42):425203, 2010.

[53] Daniel J Thompson. Generalized β-transformations and the entropy of unimodal maps. *Commentarii Mathematici Helvetici*, 92(4):777–800, 2017.

[54] Dylan P Thurston. From rubber bands to rational maps: a research report. *Research in the Mathematical Sciences*, 3(1):15, 2016.

[55] William Thurston. Entropy in dimension one. 2014.

[56] William P Thurston. *The geometry and topology of three-manifolds.* Princeton University Princeton, NJ, 1979.

[57] William P Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 339:99–130, 1986.

[58] Giulio Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the mandelbrot set. *Advances in Mathematics*, 273:651–715, 2015.

[59] Giulio Tiozzo. Continuity of core entropy of quadratic polynomials. *Inventiones mathematicae*, 203(3):891–921, 2016.

[60] Giulio Tiozzo. Galois conjugates of entropies of real unimodal maps. *International Mathematics Research Notices*, 2020(2):607–640, 2020.

[61] Aaron D Valdivia. Asymptotic translation length in the curve complex. *New York Journal of Mathematics*, 20, 2014.

[62] Robert F Williams. One-dimensional non-wandering sets. *Topology*, 6(4):473–487, 1967.

[63] Shouwu Zhang. Admissible pairing on a curve. *Inventiones mathematicae*, 112(1):171–193, 1993.

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(a) Professional Preparation

- Cornell University, Ithaca NY, Mathematics, PhD 2016
- Peking University, Beijing, China, Mathematics BS 2010

(b) Appointments

- University of Wisconsin at Madison, Assistant Professor, 2020-
- Rutgers University, Hill Assistant Professor, 2017-2020
- Max-Planck Institute of Mathematics, Postdoc, 2016-2017

(c) Publications

- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805, conditionally accepted by *Advances in Mathematics*, 2020
- Kathryn Lindsey and Chenxi Wu, Characterization of the Shape of Thurston's Teapot *arXiv:*1909.10675
- Hyungryul Baik, Hyunshik Shin and Chenxi Wu. An upper bound on the asymptotic translation length on the curve graph and fibered faces *arXiv:* 1801.06638, accepted by *Indiana University Math Journal*, 2020
- Hyungryul Baik, Eiko Kin, Hyunshik Shin and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone *arXiv:*1909.00974, submitted.
- Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019. doi: 10.1007/s00222-018-0838-5

Other Significant Publications

- Hyungryul Baik, Farbod Shokrieh, Chenxi Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019. doi: 10.1515/crelle-2019-0007.
- Hyungryul Baik, Ahmad Rafiqi and Chenxi Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019. doi: 10.1017/etds.2017.109
- Chenxi Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics*, 214(2), 733-740, 2016. doi: 10.1007/s11856-016-1357-y

- Chenxi Wu. The relative cohomology of abelian covers of the flat pillowcase. *Journal of Modern Dynamics*, 9, 123-140, 2015. doi:10.3934/jmd.2015.9.123

- Hyungryul Baik, Chenxi Wu, KyeongRo Kim, and TaeHyouk Jo. An algorithm to compute Teichmüller polynomial from matrices *Geometriae Dedicata* 2019 doi: 10.1007/s10711-019-00450-4

(d) Synergistic Activities

- Gave lectures in KAIX (KAIST Advanced Institute for Science) Summer School in Daejeon, Korea, in August 2019

- Volunteered for F.E.M.M.E.S. (Women+ Excelling More in Math, Engineering and the Sciences) event at University of Michigan in November 2017

- Gave lectures in Math Explorer's Club at Cornell in March 2016