

1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example: $u_y = x$.

Example: $(yu)_y = 1$.

General solution of a PDE.

Formally: PDE: $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde

Linear PDE.

Linear homogeneous PDE.

What are the order and linearity of the following PDEs?

$u_x + u_{yyx} = 1$, $uu_x + u = 0$, $u_x + (x^2 + y^2)u_{yy} = 1$.

Some PDEs we will focus on later:

Heat: $u_t = u_{xx}$: (heat transmission, diffusion)

Laplace: $u_{xx} + u_{yy} = 0$: (static electric field, Newton's gravity, equilibrium of random walk)

Wave: $u_{tt} = u_{xx}$: (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations: $u_{tt} = u_{xx} - ku_{xxxx}$ (stiff string)

Cauchy-Riemann equation: $u_x = v_y$, $u_y = -v_x$

Non-linear PDEs you may see in later classes:

Navier-Stokes

Nonlinear Schrodinger: $iu_t = -\Delta u + k|u|^2 u$

KdV: $u_t + u_{xxx} + 6uu_x = 0$, etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow) $u_t = A \frac{\nabla u}{|\nabla u|} \cdot \nabla u + B|\nabla u| \nabla \cdot \frac{\nabla u}{|\nabla u|}$.

Types of problems:

Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.

Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:

Existence

Uniqueness

Regularity

Continuous dependency on boundary

Typical strategy: integral transform: $(Tu)(y) = \int u(x)K(x,y)dx$, then $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$, assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)

Example: solve $u_{xx} + u_{yy} = 1$, where $u = 0$ on the unit circle.

Example: $u_x = u_t$, $u_x = u_t + 1$.

2 9/7 Review of ODE, Advection and Diffusion

Review of ODE & multivariable calculus topics:

- $u' + p(t)u + q(t) = 0$
- $u''' + Au'' + Bu' + Cu = 0$
- Chain rule: Example: $u_{xx} = u_{tt}$, what happens with change-of-variable $y = x + t$, $w = x - t$?
- Fubini's theorem.
- Differentiating an integral. Example: $\frac{d}{dt} \int_0^{t^2} \sin(ts) ds$.
Solution: Let $x = t$, $y = t$, then $\frac{d}{dt} \int_0^{t^2} e^{-ts^2} ds = \frac{d}{dt} \int_0^{x^2} e^{-ys^2} ds = (\int_0^{x^2} e^{-ys^2} ds)_x + (\int_0^{x^2} e^{-ys^2} ds)_y = 2x \cdot e^{-y(x^2)^2} + \int_0^{x^2} (e^{-ys^2})_y ds = 2xe^{-y(x^2)^2} - \int_0^{x^2} s^2 e^{-ys^2} ds = 2te^{-t^5} - \int_0^{t^2} s^2 e^{-ts^2} ds$.
- Example: $u_{tt} = u_{xx} + u_{yy}$, $u(x, y, t) = \sin(x \cos \theta + y \sin \theta + t)$ are solutions, hence $\int_0^{2\pi} \sin(x \cos \theta + y \sin \theta + t) d\theta$ is also a solution.

PDE from conservation laws, 1-dimensional case:

Consider the flow of some material whose total quantity remain unchanged, along a thin tube with section area $A(x)$. Then, conservation means:

$$\frac{d}{dt} \int_a^b u(x, t) A(x) dx = A(a) \phi(a, t) - A(b) \phi(b, t) + \int_a^b f(x, t) A(x) dx$$

ϕ : flux. f : source.

Differentiate w.r.t. b one gets: $Au_t = -A\phi_x - A'\phi + fA$.

- $\phi = u$: e.g. cars which travels at the same speed, age distribution etc.
- $\phi = -u_x$: heat conduction etc.
- $\phi = u - u_x$: contaminated flow etc.
- $f = -u$: decay.

Relationship with random motion: see $u(\cdot, t)$ as the probability distribution.

Example: $u_t = u_x - u$. Decay vs. "widening".

Example: u has two components (e.g. mass, momentum): wave equation.