Content in blue will only be covered in class if time permits.

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- Please use Piazza for math questions!
- Lecture notes and discussion worksheets will be uploaded.

Topics in Math 222

- Some techniques for integration.
- Differential Equations
- Sequences and Series
- Analytic Geometry

Advice for succeed in this course:

- Understand the concept before working through problems.
- Problem solving via trial and errors.

Contents

1 Integration by parts

Recall that: $F' = f$, f continuous, then:

$$\int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

There is connection between rules of differentiation and rules of integration.

- Chain Rule for differentiation:

$$\frac{d}{dx} f(g(x)) = g'(x) f'(g(x))$$

- Substitution Rule for integration:

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

- We can write $g'(x)dx$ as $dg(x)$, then we have

$$\int f'(g)dg = f(g(x)) + C$$

- Product (Leibniz) Rule for differentiation:

$$(uv)' = u'v + uv'$$

- **Integration by Parts**

$$u(x)v(x) + C = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

- Let $dv = v'(x)dx$, $du = u'(x)dx$, we have

$$\int udv = uv - \int vdu$$

Example 1:

$$\int xe^x dx$$

$$= \int xde^x$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Here we let $u = x$, $v = e^x$.
Here the integration would become more complicated if $v' = x$, $u = e^x$.
Need to pick the right u and v.

Example 2:

$$\int x\sin(x)dx$$

$$= \int xd(-\cos(x))$$

$$= -x\cos(x) + \int \cos(x)dx$$

$$= -x\cos(x) + \sin(x) + C$$

Here we let $u = x$, $v = -\cos(x)$.

Example 3:

$$\int \ln(x)dx$$

$$= x\ln(x) - \int x d\ln(x)$$

$$= x\ln(x) - \int \frac{x}{x}dx$$

$$= x\ln(x) - x + C$$

Here we let $u = \ln(x)$, $v = x$.

integration of inverse functions, WILL NOT BE IN THE EXAM!
In general, if $f(x) = F'(x)$, and f has inverse f^{-1} (i.e. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$), then:

$$\int f^{-1}(x)dx$$

$$= \int f^{-1}(f(s))df(s)$$

$$= \int sf'(s)ds = \int sdf(s)$$

$$= sf(s) - \int f(s)ds$$

$$= sf(s) - F(s) + C = xf^{-1}(x) - F(f^{-1}(x)) + C$$

The second line uses substitution $x = f(s)$, $s = f^{-1}(x)$.
The fourth line uses integration by parts, $u = s$, $v = f(s)$.
The answer has a geometric meaning.

Example 4

$$\int e^x\cos(x)dx$$

$$\int e^x\cos(x)dx = \int e^x d\sin(x)$$

$$= e^x\sin(x) - \int \sin(x)de^x$$

$$= e^x\sin(x) + \int (-\sin(x))e^x dx$$

$$= e^x \sin(x) + \int e^x d\cos(x)$$

$$= e^x \sin(x) + e^x \cos(x) - \int \cos(x) e^x dx$$

Here we let $u = e^x$, $v = \sin(x)$, then let $u = e^x$, $v = \cos(x)$.
Suppose $\int e^x \cos(x) dx = I(x) + C$, then:

$$I(x) + C = e^x \sin(x) + e^x \cos(x) - (I(x) + C')$$

$$I(x) = \frac{1}{2}(e^x \sin(x) + e^x \cos(x)) + C$$

Sometimes we need to do integration by parts multiple times. Sometimes integration can be obtained by setting up an algebraic equation.

Definite integral version of integration rules

- Because

  $$\int f'(g) dg = f(g(x)) + C$$

  We have:

  $$\int_{g(a)}^{g(b)} f'(g) dg = f(g(b)) - f(g(a)) =: (f \circ g)|_a^b$$

- Because

  $$\int u(x) v'(x) dx = u(x) v(x) - \int u'(x) v(x) dx$$

  We have:

  $$\int_a^b u(x) v'(x) dx = u(b) v(b) - u(a) v(a) - \int_a^b u'(x) v(x) dx$$

  $$=: uv|_a^b - \int_a^b u'(x) v(x) dx$$

Example 5

$$\int_0^{\frac{1}{2}} \cos^{-1}(x) dx$$

$$= x \cos^{-1}(x)|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{6} - \frac{1}{2} \int_1^{\frac{3}{4}} \frac{1}{\sqrt{u}} du$$

$$= \frac{\pi}{6} - (\frac{\sqrt{3}}{2} - 1)$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$$

Sometimes we need to combine integration by parts with substitution rule.

2 Trig Integrals

Example 1, trig integrals

$$\int \cos^3(x)dx$$

Method 1:

$$= \int \cos^2(x)d\sin(x)$$

$$= \cos^2(x)\sin(x) + 2\int \sin^2(x)\cos(x)dx$$

$$= \cos^2(x)\sin(x) + 2\int (1-\cos^2(x))\cos(x)dx$$

So

$$\int \cos^3(x)dx = \frac{\cos^2(x)\sin(x)}{3} + \frac{2}{3}\int \cos(x)dx$$

$$= \frac{\cos^2(x)\sin(x)}{3} + \frac{2}{3}\sin(x) + C$$

This can be summarized into a **reduction formula**:

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$$

Integration by parts is often used to obtain reduction formula, for example, for positive integer n,

$$\int x^n e^{-x}dx = x^n(-e^{-x}) + n\int x^{n-1}e^{-x}dx$$

Method 2: Let $s = \sin(x)$, then

$$\int \cos^3(x)dx$$

$$= \int (1-s^2)ds$$

$$= s - \frac{s^3}{3} + C$$

$$= \sin(x) - \frac{\sin^3(x)}{3} + C$$

Generally these are some common strategies for trig integration:

1. **To integrate $\int \sin^{2n+1}(x)\cos^m(x)dx$, where n, m are integers, $n \geq 0$, do substitution $u = \cos(x)$, and then it becomes $-\int (1-u^2)^n u^m du$.**

2. **To integrate $\int \sin^m(x)\cos^{2n+1}(x)dx$, where n, m are integers, $n \geq 0$ do substitution $u = \sin(x)$, and then it becomes $\int u^m (1-u^2)^n du$.**

3. **To integrate $\int \tan^m(x)\sec^{2n+2}(x)dx$, when n, m are integers, $n \geq 0$, do substitution $u = \tan(x)$, and then it becomes $\int u^m (u^2+1)^n du$.**

4. **To integrate $\int \tan^{2n+1}(x)\sec^m(x)dx$, and then it becomes $\int (u^2 - 1)^n u^{m-1} du$.**

5. **To integrate products of $\sin$ and $\cos$, we can also make use of the following formula:**

$$\sin^2(a) = \frac{1-\cos(2a)}{2}$$

$$\cos^2(a) = \frac{\cos(2a)+1}{2}$$

$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$$

$$\sin(a)\sin(b) = \frac{\cos(a+b)-\cos(a-b)}{2}$$

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$$

Method 3:

$$\int \cos^3(x)dx$$

$$= \frac{1}{2}\int \cos(x)(\cos(2x)+1)dx$$

$$= \frac{1}{2}\int \frac{\cos(3x)+\cos(x)}{2} + \cos(x)dx$$

$$= \frac{1}{12}\sin(3x) + \frac{3}{4}\sin(x) + C$$

Example 2:

$$\int \cos^4(x)dx$$

$$= \frac{1}{4}\int (\cos(2x)+1)^2 dx$$

$$= \frac{1}{4}\int \cos^2(2x)dx + \frac{1}{2}\int \cos(2x)dx + \frac{1}{4}\int 1dx$$

$$= \frac{1}{8}\int \cos(4x)dx + \frac{1}{8}\int 1dx + \frac{1}{2}\int \cos(2x)dx + \frac{1}{4}\int 1dx$$

$$= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + C$$

Example 3:

$$\int \sec(x)dx$$

$$= \int \frac{d\sin(x)}{\cos^2(x)}$$

$$= \int \frac{du}{1-u^2}$$

$$= \int \frac{1}{2}(\frac{1}{1-u} + \frac{1}{1+u})du$$

$$= -\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) + C$$

$$= \frac{1}{2}\ln(\frac{1+\sin(x)}{1-\sin(x)}) + C$$

$$= \ln|\sec(x) + \tan(x)| + C$$

Here $u = \sin(x)$.

3 Trig Substitution

Recall that we have these two methods for integration so far:

- Substitution Rule
  - Indefinite integral:

$$\int f'(g(x))g'(x)dx = \int f'(g)dg = f(g(x)) + C$$

  - Definite integral:

$$\int_a^b f'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f'(g)dg = f(g(x)) + C$$

- Integration by Parts
  - Indefinite integral:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

  - Definite integral:

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx$$

Here $dg = g'(x)dx$.

Now we will learn more strategies for using the substitution rule:

- If the function being integrated has $\sqrt{a^2 - x^2}$, let $x = a\sin(u)$, $u \in [-\pi/2, \pi/2]$.
- If the function being integrated has $\sqrt{x^2 + a^2}$, let $x = a\tan(u)$, $u \in (-\pi/2, \pi/2)$.
- If the function being integrated has $\sqrt{x^2 - a^2}$, let $x = a\sec(u)$, $u \in (0, \pi/2) \cup (\pi/2, \pi)$.

The latter two cases can be done via sinh and cosh as well.

Example 1

$$\int x^2\sqrt{4 - x^2}dx$$

$$= \int (2\sin(u))^2 \cdot 2\cos(u)d(2\sin(u))$$

$$= 16\int \cos^2(u)\sin^2(u)du$$

$$= 4\int \sin^2(2u)du$$

$$= 2\int (1 - \cos(4u))du$$

$$= 2u - \frac{1}{2}\sin(4u) + C$$

$$= 2\sin^{-1}(x/2) - \frac{1}{2}\sin(4\sin^{-1}(x/2)) + C$$

$$= 2\sin^{-1}(x/2) - x(1 - x^2/2)\sqrt{1 - x^2/4} + C$$

Here $x = 2\sin(u)$, $u = \sin^{-1}(x/2)$. Because $u \in [-\pi/2, \pi/2]$, $\cos(u) \geq 0$, hence $\sqrt{4 - x^2} = |2\cos(u)| = 2\cos(u)$.

When there is inverse trig composed with trig functions we need to simplify them.

Example 2

$$\int \frac{\sqrt{9 - x^2}}{x^2}dx$$

$$= \int \frac{3\cos(u)}{9\sin^2(u)}d(3\sin(u))$$

$$= \int \frac{\cos^2(u)}{\sin^2(u)}du$$

$$= \int \frac{1}{\sin^2(u)} du - \int 1 du$$
$$= -\cot(u) - u + C$$
$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(x/3) + C$$

Here $x = 3\sin(u)$, $u = \sin^{-1}(x/3)$.

Sometimes we need to do a "completing the square" to turn a quadratic polynomial into $\pm a^2 \pm x^2$:

Example 3

$$\int \frac{1}{(\sqrt{x^2+2x+5})^3} dx$$
$$= \int \frac{1}{(\sqrt{(x+1)^2+4})^3} dx$$
$$= \frac{1}{4} \int \frac{1}{\sec^3(u)} d\tan(u)$$
$$= \frac{1}{4} \int \frac{1}{\sec(u)} du$$
$$= \frac{1}{4} \int \cos(u) du$$
$$= \frac{1}{4} \sin(u) + C$$
$$= \frac{1}{4} \sin(\tan^{-1}(\frac{x+1}{2})) + C$$
$$= \frac{x+1}{4\sqrt{4+(x+1)^2}} + C$$

Here $x+1 = 2\tan(u)$, i.e. $x = 2\tan(u)-1$. Because $u \in (-\pi/2, \pi/2)$, $\sec(u) > 0$, hence $\sqrt{(x+1)^2+4} = 2\sec(u)$.

4 Integration by partial fractions

Often we need to integrate **rational functions**, i.e. functions of the form $P(x)/Q(x)$, where P and Q are **polynomials**. In many cases we can reduce integration problem to the case of rational functions via appropriate substitution e.g. for trig integrals.

There is a method for integrating all rational functions called **integration by partial fractions**. Any rational function

$$\frac{P(x)}{\prod_i (x-a_i)^{n_i} \prod_j (x^2+c_jx+d_j)^{m_j}}$$

where $c_j^2 < 4d_j$, can be written into

$$q(x) + \sum_i \sum_{k=0}^{n_i} \frac{A_{ik}}{(x-a_i)^k} + \sum_j \sum_{k=0}^{m_j} \frac{B_{jk}x + C_{jk}}{(x^2 + c_j x + d_j)^k}$$

Where A, B and C are constants. This is called **partial fraction decomposition**.

Method of Partial Fractions:

- Calculate q by long division.
- Factorize the denominator.
- Find the constants A_{ik}, B_{jk} and C_{jk} by linear equations, establishing the partial fraction decomposition.
- Integrate each term according to formula.

Example 1

$$\int \frac{x^3+2}{x^2-1} dx$$

$$= \int \frac{x^3 - x + x + 2}{x^2 - 1} dx$$

$$= \int x + \frac{x+2}{(x+1)(x-1)} dx$$

To write $\frac{x+2}{(x+1)(x-1)}$ into partial fractions, suppose

$$\frac{x+2}{(x+1)(x-1)} = \frac{A_1}{x+1} + \frac{A_2}{x-1}$$

Then

$$(A_1 + A_2)x + (A_2 - A_1) = x + 2$$

Hence

$$A_2 = 3/2, A_1 = -1/2$$

The original integral is

$$\int x dx + \frac{3}{2} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x-1} dx$$

$$= \frac{x^2}{2} + \frac{3}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C$$

Example 2:

$$\int \frac{1}{x(x^2+2)} dx$$

Firstly, write the function into partial fractions:

$$\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

Hence

$$(A+B)x^2 + Cx + 2A = 1$$

$$A = 1/2, B = -1/2, C = 0$$

The original integral becomes

$$\frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{x}{x^2+2}dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| + C$$

4.1 Simplification of compositions of trigs and inverse trig functions

When there is composition between trig and inverse trig functions we need to simplify them. There are two methods which we shall illustrated with an example below:

Example: $\cot(\sin^{-1}(x))$

Method 1: Using trig identities.

$$\cot(\sin^{-1}(x)) = \frac{\cos(\sin^{-1}(x))}{\sin(\sin^{-1}(x))} = \frac{\cos(\sin^{-1}(x))}{x}$$

By definition $\sin^{-1}(x) \in [-\pi/2, \pi/2]$, hence $\cos(\sin^{-1}(x)) \geq 0$, hence

$$\cos(\sin^{-1}(x)) = \sqrt{\cos^2(\sin^{-1}(x))} = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$$

Hence

$$\cot(\sin^{-1}(x)) = \frac{\sqrt{1-x^2}}{x}$$

Method 2: Right triangle method.

Consider right triangle ΔABC, angle C is $\pi/2$, edge AB has length 1, edge BC has length x, then angle A is by definition $\sin^{-1}(x)$. Hence

$$\cot(\sin^{-1}(x)) = \frac{AC}{BC} = \frac{\sqrt{AB^2 - BC^2}}{BC} = \frac{\sqrt{1-x^2}}{x}$$

5 Summary of techniques for integrations

- Rules for integration:
  - Linearity
  - Substitution
  - Integration by Parts
- Common strategies for different kinds of functions:
  - Trignometric functions: substitution, trig formula
  - Roots of quadratic functions: Trig substitution
  - Rational functions: Partial fractions.
  - Polynomial times sin, cos, e^x: Integration by parts.
- Generally try substitution first before integration by parts.
- Sometimes needs multiple steps and multiple methods/strategies.

6 Improper integrals

Recall that:

- If $a < b$ are two real numbers, $f(x)$ is continuous on $[a, b]$, then the **definite integral** $\int_a^b f(x)dx$ is the **signed area between the graph of** f**,** x **axis, and the vertical lines** $x = a$ **and** $x = b$.
- (Fundamental theorem of calculus) If $\int f(x)dx = F(x)+C$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Improper integral is a way to generalize this to the case when the interval being integrated or the function being integrated is unbounded. The motivation is that to define "area" for unbounded region, we approximate it by bounded regions and then take a limit.

Definition:

- If a is a real number, f is continuous on $[a, \infty)$, we define

$$\int_a^\infty f(x)dx = \lim_{M\to\infty} \int_a^M f(x)dx$$

- If b is a real number, f is continuous on $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)dx = \lim_{M\to-\infty} \int_M^b f(x)dx$$

- If $a < b$ are real numbers, f is continuous on $[a, b)$, we define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

- If $a < b$ are real numbers, f is continuous on $(a, b]$, we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

- In general, if the function f is continuous on interval I except for finitely many points, we decompose I into subintervals of the above four cases and define the improper integral as the sum of integrals of f on these subintervals.

- If the limits above does not exist we call the integral to be **divergent**, otherwise we call it **convergent**.

Example 1:

$$\int_0^\infty e^{-x}dx = \lim_{M \to \infty} \int_0^M e^{-x}dx = \lim_{M \to \infty} -e^{-M} - (-e^0) = 1$$

Example 2:

$$\int_1^\infty x^{-3} = \lim_{M \to \infty} \int_1^M x^{-3}dx = \lim_{M \to \infty} -\frac{1}{2}x^{-2}|_1^M = \frac{1}{2}$$

Example 3:

$$\int_0^1 \ln(x)dx = \lim_{c \to 0^+} \int_c^1 \ln(x)dx = \lim_{c \to 0^+} (x\ln(x) - x)|_c^1 = -1$$

Example 4:

$$\int_{-1}^1 \frac{1}{x}dx = \int_{-1}^0 \frac{1}{x}dx + \int_0^1 \frac{1}{x}dx$$

$$\int_{-1}^0 \frac{1}{x}dx = \lim_{c \to 0^-} \int_{-1}^c \frac{1}{x}dx = \lim_{c \to 0^-} (\ln(c) - \ln(1))$$

Which does not exist, hence this integration diverges. This example also shows that there is no "fundamental theorem of calculus" for improper integrals:

$$\int \frac{1}{x}dx = \ln|x|$$

But

$$\int_{-1}^1 \frac{1}{x}dx \neq \ln|x||_{-1}^1$$

Example 5:

$$\int_{-1}^1 |x|^{-1/2}dx = \int_{-1}^0 |x|^{-1/2}dx + \int_0^1 |x|^{-1/2}dx = \lim_{c \to 0^-} \int_{-1}^c |x|^{-1/2}dx + \lim_{d \to 0^+} \int_d^1 |x|^{-1/2}dx$$

$$= \lim_{c \to 0^-} (-2|x|^{1/2})|_{-1}^c + \lim_{d \to 0^+} (2|x|)|_d^1 = 4$$

6.1 Comparison test

Comparison Theorem: If $0 \leq f \leq g$ on some interval I, improper integral of g on I converges, then improper integral of f on I converges. If improper integral of f on I diverges, then improper integral of g on I diverges.

Strategy of using this theorem: Cut the interval we are integrating into subintervals, then apply comparison theorem using appropriate functions.

Example 6: Check if $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{0} e^{-x^2} dx + \int_{0}^{\infty} e^{-x^2} dx$$
$$\int_{-\infty}^{0} e^{-x^2} dx = \int_{-1}^{0} e^{-x^2} dx + \int_{-\infty}^{-1} e^{-x^2} dx$$

On $(-\infty, -1]$, $0 \leq e^{-x^2} \leq e^{x}$, and $\int_{-\infty}^{-1} e^{-x} dx = e^{-1}$, hence this improper integral converges. Similarly, $\int_0^{\infty} e^{-x^2} dx$ converges, hence the improper integral $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges.

Remark: The indefinite integral of e^{-x^2} is not an elementary function. However in future classes we will learn how to calculate $\int_{-\infty}^{\infty} e^{-x^2} dx$.

The convergence and divergence of some important improper integrals:

- $\int_0^1 x^a dx$ converges iff $a > -1$.
- $\int_1^{\infty} x^a dx$ converges iff $a < -1$.
- $\int_0^{\infty} e^{ax}$ converges iff $a < 0$.
- $\int_0^1 \frac{1}{\ln(x)} dx$ converges.

Example 7: Check if $\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^4}} dx$ converges.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^4}} dx = \int_{-\infty}^{-1} \frac{1}{\sqrt{1+x^4}} dx + \int_{-1}^{1} \frac{1}{\sqrt{1+x^4}} dx + \int_{1}^{\infty} \frac{1}{\sqrt{1+x^4}} dx$$

On $(-\infty, -1]$, $0 \leq \frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$, and $\int_{-\infty}^{-1} \frac{1}{x^2} = 1$, hence $\int_{-\infty}^{-1} \frac{1}{\sqrt{1+x^4}} dx$ converges.

On $[1, \infty)$, $0 \leq \frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$, and $\int_1^{\infty} \frac{1}{x^2} = 1$, hence $\int_1^{\infty} \frac{1}{\sqrt{1+x^4}} dx$ converges.

$\int_{-1}^{1} \frac{1}{\sqrt{1+x^4}}$ is definite integral, or proper integral. Hence $\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^4}} dx$ converges.

Example 8: Check if $\int_0^1 \frac{1}{\sin(x)} dx$ converges.

Answer: On interval $(0, 1]$ we have $0 \leq \sin(x) \leq x$, hence $0 \leq \frac{1}{x} \leq \frac{1}{\sin(x)}$. $\int_0^1 \frac{1}{x} dx$ diverges, hence so is $\int_0^1 \frac{1}{\sin(x)} dx$.

7 Differential Equations

- A **differential equation** is an equation relating a function and its derivatives.
- The highest order of derivatives is called the **order** of the differential equation.
- A **solution** is a function that satisfies the equation.
- An **initial condition** is a condition on the value and derivatives of the unknown function at a point.

Example 1: A ball is falling in vacuum towards earth, let $h(t)$ be the distance between the ball to the center of earth, then

$$h''(t) = -\frac{C}{h^2}$$

Example 2: Let $y = f(x)$ be the graph of a function f. If we want the tangent at any point (x, y) has slope $x + y$. Then

$$y'(x) = x + y$$

8 Direction fields and Euler's Method

For equation of the form $y' = f(x, y)$, on the $x - y$ plane, for every point (x, y), draw a short segment at slope $f(x, y)$. This is called the **direction field**.

If $y = y(x)$ is a solution, then the graph of y passing through (x, y) implies that f is tangent to the segment at that point.

A way to approximate the solution of initial value problem

$$y' = f(x, y), y(a) = b$$

is as follows: let $h > 0$ be a small number, then y can be approximated by $y(x_0 + nh) \approx y_n$, $y_0 = b$, $y_n = y_{n-1} + hf(a + (n-1)h, y_{n-1})$.

Example:

$$y' = x + y, y(0) = 0$$

Here $a = b = 0$, $f(x, y) = x + y$, hence

$$y_0 = 0$$

$$y_n = y_{n-1} + h((n-1)h + y_{n-1})$$

The calculation can be done via "table method": for example, if $h = 0.1$, we have:

n	$a + nh$	y_n	$f(a + nh, y_n) = a + nh + y_n$	$y_n + hf(a + nh, y_n) = y_{n+1}$
0	0	0	$0 + 0 = 0$	$0 + 0.1 \times 0 = 0$
1	0.1	0	$0.1 + 0 = 0.1$	$0 + 0.1 \times 0.1 = 0.01$
2	0.2	0.01	$0.2 + 0.01 = 0.21$	$0.01 + 0.1 \times 0.21 = 0.031$
3	0.3	0.031	$0.3 + 0.031 = 0.331$	$0.031 + 0.1 \times 0.331 = 0.0641$
...	...	...	...	...

Add $1 + nh$ to both sides, we get:

$$y_n+1+nh = y_{n-1}+1+(n-1)h+h((n-1)h+y_{n-1}+1) = (y_{n-1}+1+(n-1)h)(1+h)$$

Let $z_n = y_n + 1 + nh$, then:

$$z_0 = 1$$

$$z_n = (1 + h)z_{n-1}$$

Hence

$$z_n = (1 + h)^n$$

$$y(nh) \approx y_n = (1 + h)^n - 1 - nh$$

Let $h \to 0$, $nh = x$, then:

$$\lim_{h \to 0} y_{\frac{x}{h}} = \lim_{h \to 0} (1 + h)^{x/h} - 1 - x = e^x - 1 - x$$

It is easy to check that $y(x) = e^x - 1 - x$ is the solution of the original initial value problem.

9 Separation of Variables

Consider first order differential equation of the form:

$$y' = \frac{f(x)}{g(y)}$$

Multiply both sides by $g(y)$, we get:

$$g(y)y' = f(x)$$

View both sides as function of x, take indefinite integral:

$$\int g(y)y'dx = \int f(x)dx$$

By substitution rule, the left hand side becomes:

$$\int g(y)dy$$

Suppose $\int g(y)dy = G(y) + C$, $\int f(x)dx = F(x) + C$, then

$$G(y) = F(x) + C$$

This method is called **Separation of Variables**.

Example 1: $y' = y\sin(x)$.

$$\frac{y'}{y} = \sin(x)$$

$$\int \frac{1}{y}dy = \int \sin(x)dx$$

This assumed that $y \neq 0$.

$$\ln|y| = -\cos(x) + C$$

Now rename $\pm e^C$ as C, we have:

$$y = Ce^{-\cos(x)}$$

If $y = 0$, $y' = 0 = y\sin(x)$, hence $y(x) = 0$ is a solution. However this is already included in $y(x) = Ce^{-\cos(x)}$ as the case $C = 0$.

As a notation, we can write y' as $\frac{dy}{dx}$, and move dx to the right hand side.

Example 2: $y' = \frac{1}{y^2}$

$$\frac{dy}{dx} = \frac{1}{y^2}$$

$$y^2dy = dx$$

$$\int y^2dy = \int 1dx$$

$$\frac{y^3}{3} = x + C$$

Rename $3C$ as C, we have:

$$y = (x + C)^{1/3}$$

Example 3: $y' = y(1 - y)$

$$\frac{dy}{dx} = y(1 - y)$$

$$\frac{dy}{y(1-y)} = dx$$

The line above assumed that $y(1-y) \neq 0$

$$\int \frac{dy}{y(1-y)} = \int 1dx$$

Use partial fraction:

$$\int \frac{dy}{y} + \int \frac{dy}{1-y} = \int 1dx$$

$$\ln|y| - \ln|1-y| = x + C$$

$$\frac{y}{1-y} = Ce^x$$

$$y = \frac{Ce^x}{1+Ce^x}$$

Now deal with the case when $y(1-y) = 0$:

- If $y = 0$, $y' = 0 = y(1-y)$, hence $y = 0$ is a solution. This is already included in $y = \frac{Ce^x}{1+Ce^x}$ as the case $C = 0$.
- If $y = 1$, $y' = 0 = y(1-y)$, hence $y = 1$ is also a solution. This is NOT yet included in $y = \frac{Ce^x}{1+Ce^x}$.

Hence the solution of the differential equation should be:

$$y = \frac{Ce^x}{1+Ce^x} \text{ or } y = 1$$

10 Examples of differential equation models

10.1 Curves orthogonal to a family of curves

Consider a family of curves $x = ky^2$. Find the equation of a curve passing through $(1,1)$ and orthogonal to all curves in this family that intersects to it.

- Step 1: **Formulate the equation**: Given any point (a,b), the curve in the family that passes through it is $x = \frac{a}{b^2}y^2$. The tangent vector at (a,b) is $(2a,b)$, hence the normal vector is $(b,-2a)$, the equation is

$$y' = -\frac{2x}{y}$$

- Step 2: **Solve the equation**:

$$ydy = -2xdx$$

$$\frac{1}{2}y^2 = -x^2 + C$$

$$y^2 + 2x^2 = C$$

- Step 3: **Plug in the initial value condition**: Because $(1, 1)$ is on the curve, we know that $C = 3$.

10.2 Mixing problem

A tank of volume V has solution of some substance with concentration u_0, solution of the same substance but with concentration u_1 enters at rate r, and the mixture exit at the same rate. How would the concentration change?

Answer: $u'(t) = \frac{u_1 r - ur}{V}$

10.3 Population models

- Unconstrained, "natural" population growth: $p' = kp$
- Natural growth with fixed rate of predation: $p' = kp - c$
- Logistic model (with environmental constraint): $p' = kp(1 - p/M)$
- Logistic model with fixed rate of predation: $p' = kp(1 - p/M) - c$
- Logistic model, die out when population is too low: $p' = kp(1 - p/M)(1 - m/p)$

11 Linear Equations

We now consider another family of differential equations, which we call **first order differential equations**:

$$\frac{dy}{dx} + p(x)y = q(x)$$

We want to find two functions $I(x)$ and $J(x)$ such that

$$(I(x)y)' = J(x)(\frac{dy}{dx} + p(x)y)$$

The left hand side equals $I'y + Iy'$, hence

$$I = J$$

$$I' = Jp$$

Hence one solution is

$$I = J = e^{P(x)}$$

where $P' = p$. Hence we have:

$$(e^{P(x)}y(x))' = p(x)e^{P(x)}y(x) + e^{P(x)}y'(x) = e^{P(x)}q(x)$$

Hence

$$y(x) = e^{-P(x)} \int e^{P(x)}q(x)dx$$

Example 1: $y' = ky - h$.

$$y' - ky + h = 0$$

Hence

$$p(x) = -k$$

$$q(x) = -h$$

$$P(x) = -kx$$

$$y(x) = e^{kx} \int e^{-kx}(-h)dx = e^{kx}\frac{h}{k}(e^{-kx} + C) = \frac{h}{k}(1 + Ce^{kx})$$

Example 2: $y' + 2y = \sin(x)$.

$$p(x) = 2$$

$$q(x) = \sin(x)$$

$$P(x) = 2x$$

$$y(x) = e^{-2x} \int e^{2x} \sin(x)dx$$

$$\int e^{2x} \sin(x)dx = -e^{2x} \cos(x) + 2 \int e^{2x} \cos(x)dx$$

$$= -e^{2x} \cos(x) + 2e^{2x} \sin(x) - 4 \int \sin(x)e^{2x}dx$$

So

$$\int e^{2x} \sin(x)dx = -\frac{1}{5}e^{2x} \cos(x) + \frac{2}{5}e^{2x} \sin(x) + C$$

$$y(x) = -\frac{1}{5} \cos(x) + \frac{2}{5} \sin(x) + Ce^{-2x}$$

12 Sequences and their limits

Definition: A **sequence** is a function defined on the set of positive integers, denoted as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$. Here a_n is the function evaluated at n, which is also called the n-th term of the sequence. Sometimes when the pattern is clear we can also write a sequence informally as $\{a_1, a_2, a_3, a_4, \ldots\}$.

Example 1: $\{e^n\}$, $\{n^2\}$ and $\{\sin(n)\}$ are sequences.

Definition: We say the **Limit** of a sequence $\{a_n\}$ is b, denoted as

$$\lim_{n\to\infty} a_n = b$$

if for any $\epsilon > 0$, there is $N > 0$ such that for any $n > N$, $|a_n - b| < \epsilon$.

Example 2: $\lim_{n\to\infty} \frac{1}{n} = 0$, because for any $\epsilon > 0$, we can pick N as any integer greater than $\frac{1}{\epsilon}$, then $n > N$ implies

$$|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

We will not ask you to check this definition in exams (like Example 2). That would be the requirement of an analysis course.

Definition: We say $\lim_{n\to\infty} a_n = \infty$ if for any $M > 0$, there is $N > 0$ such that $n > N$ implies $a_n > M$. We say $\lim_{n\to\infty} a_n = -\infty$, if for any $M > 0$, there is $N > 0$ such that $n > N$ implies $a_n < -M$. If $\lim_{n\to\infty} a_n$ exists we call it **converges**, otherwise **diverges**.

Example 3: $\lim_{n\to\infty} -n^2 = -\infty$, $\lim_{n\to\infty} e^n = \infty$.

Methods for calculating limit:

1. If $a_n = f(n)$, f is a function defined on positive real numbers and $\lim_{x\to\infty} f(x) = b$, then $\lim_{n\to\infty} a_n = b$.

   Example 4: By L'Hospital's rule $\lim_{x\to\infty} \frac{x}{e^x} = 0$, hence $\lim_{n\to\infty} \frac{n}{e^n} = 0$.

   Remark: $\lim_{x\to\infty} f(x)$ does not exist does not imply that $\lim_{n\to\infty} a_n$ does not exist. For example, $\lim_{x\to\infty} \sin(\pi x)$ does not exist but $\lim_{n\to\infty} \sin(n\pi) = 0$.

2. If $\lim_{n\to\infty} a_n = c$, $\lim_{n\to\infty} b_n = d$, then:
   - $\lim_{n\to\infty}(a_n \pm b_n) = c \pm d$
   - $\lim_{n\to\infty}(a_n b_n) = cd$

- $\lim_{n\to\infty}(ka_n) = kc$
- If $d \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{c}{d}$.
- If $a_n > 0$, $p > 0$ or $c \neq 0$, then $\lim_{n\to\infty} a_n^p = c^p$.
- If f is a continuous function, then $\lim_{n\to\infty} f(a_n) = f(c)$.

Example 5: Find $\lim_{n\to\infty} \frac{n^2-n}{n^2+3}$.

$$\frac{n^2-n}{n^2+3} = \frac{1-\frac{1}{n}}{1+\frac{3}{n^2}}$$

$$\lim_{n\to\infty}(1-\frac{1}{n}) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1-0 = 1$$

$$\lim_{n\to\infty}(1+\frac{3}{n^2}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{3}{n^2} = 1+0 = 1 \neq 0$$

$$\lim_{n\to\infty}\frac{n^2-n}{n^2+3} = \lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{3}{n^2}} = \frac{1}{1} = 1$$

Example 6: Find $\lim_{n\to\infty} \frac{\sqrt{n-1}}{n+2}$.

$$\frac{\sqrt{n-1}}{n+2} = \frac{\sqrt{\frac{1}{n}-\frac{1}{n^2}}}{1+\frac{2}{n}}$$

$$\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{1}{n^2} = 0-0 = 0$$

$$\lim_{n\to\infty}\sqrt{\frac{1}{n}-\frac{1}{n^2}} = \sqrt{0} = 0$$

$$\lim_{n\to\infty}(1+\frac{2}{n}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{2}{n} = 1+0 = 1 \neq 0$$

Hence the original limit is $\frac{0}{1} = 0$.

Example 7: $\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = e^0 = 1$.

3. **Squeeze theorem**: If $a_n \leq b_n \leq c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = d$, then $\lim_{n\to\infty} b_n = d$ as well.

   Example 8: $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$, and $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0$, by squeeze theorem, $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$.

   Example 9: $-\frac{1}{e^n} \leq \frac{\sin(n)}{e^n} \leq \frac{1}{e^n}$, and $\lim_{n\to\infty}\left(-\frac{1}{e^n}\right) = \lim_{n\to\infty}\frac{1}{e^n} = 0$, by squeeze theorem, $\lim_{n\to\infty}\frac{\sin(n)}{e^n} = 0$.

Remark: In general, if $|a_n| \le b_n$, $\lim_{n\to\infty} b_n = 0$, then $-b_n \le a_n \le b_n$ and by squeeze theorem $\lim_{n\to\infty} a_n = 0$.

Example 10: $a_n = \frac{(-3)^n}{n!}$. When $n > 4$,

$$|a_n| = \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \frac{3}{4} \cdots \frac{3}{n}$$
$$\le \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \frac{3}{n} = \frac{27}{2n}$$

Because $\lim_{n\to\infty} \frac{27}{2n} = 0$, $\lim_{n\to\infty} a_n = 0$.

Example 11: $\lim_{n\to\infty} \sin(\frac{1}{n^2})\cos(n) = 0$, because

$$0 \le \lim_{n\to\infty} |\sin(\frac{1}{n^2})\cos(n)| \le \lim_{n\to\infty} \sin(\frac{1}{n^2}) = \sin(\lim_{n\to\infty} \frac{1}{n^2}) = \sin(0) = 0$$

4. Existence of limit for bounded monotone sequences:
   - If $a_{n+1} \ge a_n$ for all n, and there is some M such that $M \ge a_n$ for all n, then $\lim_{n\to\infty} a_n$ exists.
   - If $a_{n+1} \le a_n$ for all n, and there is some M such that $a_n \ge M$ for all n, then $\lim_{n\to\infty} a_n$ exists.

   Example 12: $a_n = \sum_{j=1}^{n} \frac{1}{j^2}$. This sequence satisfies $a_{n+1} > a_n$ for all n, i.e. it is **monotone increasing**. Also,

$$a_n = \sum_{j=1}^{n} \frac{1}{j^2} \le 1 + \sum_{j=2}^{n} \frac{1}{j(j-1)} = 1 + \sum_{j=2}^{n} (\frac{1}{j-1} - \frac{1}{j}) \le 2$$

   Hence $\lim_{n\to\infty} a_n$ exists. (The actual limit is $\frac{\pi^2}{6}$.)

13 Series

Let $\{a_n\}$ be a sequence. One can build another sequence called **partial sums**, $\{s_n\}$, where $s_n = \sum_{i=1}^{n} a_i$. Then $\lim_{n\to\infty} s_n$ is called an infinite series, denoted as $\sum_{n=1}^{\infty} a_n$. If the limit exists we call the series **converges**, otherwise we call it **diverges**.

Basic methods for studying series:

1. For some simpler series, we can find explicit formula for s_n. For example:
   Example 1:
   - $\sum_{n=1}^{\infty} r^n$ for $|r| < 1$. The partial sum

$$s_n = \sum_{i=1}^{n} r^i = \frac{r - r^{n+1}}{1 - r}$$

     So $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The partial sum

$$s_n = \sum_{i=1}^{n} \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

  So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

2. If $\lim_{n\to\infty} a_n$ does not exist or is not zero, then $\sum_{n=1}^{\infty} a_n$ diverges. For example, $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Example 2: Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This series diverges, because

$$\frac{1}{n} = \int_n^{n+1} \frac{1}{n} dx \geq \int_n^{n+1} \frac{1}{x} dx = \ln(n+1) - \ln(n)$$

Hence

$$s_n = \sum_{i=1}^{n} \frac{1}{i} \geq \sum_{i=1}^{n} (\ln(i+1) - \ln(i)) = \ln(n+1) - \ln(2)$$

which goes to infinity as $n \to \infty$.

In Example 1 we made use of the following **summation formula of geometric series**: if $p \neq 1$, then:

$$\sum_{i=1}^{n} ap^{i-1} = \frac{a - ap^n}{1-p}$$

To prove it, let

$$S = \sum_{i=1}^{n} ap^{i-1}$$

then

$$(p-1)S = pS - S = \sum_{i=1}^{n} ap^i - \sum_{i=1}^{n} ap^{i-1} = \sum_{i=2}^{n+1} ap^{i-1} - \sum_{i=1}^{n} ap^{i-1} = ap^n - a$$

So

$$S = \frac{a - ap^n}{1-p}$$

14 Integral test

Motivated by the harmonic series example above, we have:

Lemma: If f is continuous and decreasing on $[1, \infty)$, then

$$\int_1^{n+1} f(x)dx \leq \sum_{i=1}^{n} f(i) \leq f(1) + \int_1^{n} f(x)dx$$

Proof:

$$\begin{aligned}\sum_{i=1}^{n} f(i) &= \int_{i=1}^{n}\int_{i}^{i+1} f(i)dx \\ &\geq \sum_{i=1}^{n}\int_{i}^{i+1} f(x)dx \\ &= \int_{1}^{n+1} f(x)dx\end{aligned}$$

The $\geq$ is because when $x \in [i, i+1]$, $x \geq i$, hence $f(x) \leq f(i)$ due to f being decreasing.

$$\begin{aligned}\sum_{i=1}^{n} f(i) = f(1) + \sum_{i=2}^{n} f(i) &= f(1) + \sum_{i=2}^{n}\int_{i-1}^{i} f(i)dx \\ &\leq f(1) + \sum_{i=2}^{n}\int_{i-1}^{i} f(x)dx \\ &= f(1) + \int_{1}^{n} f(x)dx\end{aligned}$$

The $\leq$ is because when $x \in [i-1, i]$, $x \leq i$, hence $f(x) \geq f(i)$ due to f being decreasing.

Theorem (Integral test): If f is a continuous, positive and decreasing function on $[1, \infty)$, then:

- If $\int_1^\infty f(x)dx = \infty$, then $\sum_{n=1}^{\infty} f(n) = \infty$.
- If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges.

Proof:

- By the Lemma above,

  $$s_n = \sum_{j=1}^{n} f(j) \geq \int_{1}^{n+1} f(x)dx$$

  Hence if $\int_1^\infty f(x)dx = \infty$ then so is $\lim_{n\to\infty} s_n$.

- By the Lemma above, if $\int_1^\infty f(x)dx = L$, then

  $$s_n = \sum_{j=1}^{n} f(j) \leq f(1) + \int_{1}^{n} f(x)dx \leq f(1) + L$$

  Hence s_n is an increasing sequence bounded by $f(1)+L$ and must converge.

Theorem (Remainder estimate): If f is continuous, positive and decreasing on $[1, \infty)$, and $\sum_{n=1}^{\infty} f(n)$ converges, then the N-th remainder $R_N = \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n)$ satisfies:

$$\int_{N+1}^{\infty} f(x)dx \leq R_N \leq \int_{N}^{\infty} f(x)dx$$

Proof:

$$R_N = \lim_{M\to\infty} \sum_{n=1}^{M} f(n) - \sum_{n-1}^{N} f(n) = \lim_{M\to\infty} \sum_{N+1}^{M} f(n)$$

Now consider $g(x) = f(N + x)$, $h(x) = f(N - 1 + x)$, then

$$R_N = \sum_{n=1}^{\infty} g(n) = \sum_{n=1}^{\infty} h(n) - f(N)$$

Now apply Lemma to both g and h.

Example 1: $\sum_{n=1}^{\infty} x^{-p}$ converges iff $p > 1$.

Example 2: $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ diverges, because

$$\int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} = \lim_{M\to\infty} \int_1^{M} \frac{dx}{(x+1)\ln(x+1)}$$
$$= \lim_{M\to\infty} (\ln(\ln(M + 1)) - \ln(\ln(2)))$$

which does not exist.

Example 3: Check convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{n}{2^n} = \frac{1}{2} + \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{n+1}{2^{n+1}} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{n+1}{2^{n+1}}$$

Let $f(x) = \frac{x+1}{2^{x+1}}$, then f is continuous, positive and decreasing on $[1, \infty)$, and $\int_1^{\infty} f(x)dx$ exists, hence the series converges.

Example 4: The remainder formula can be used to estimate the value of some infinite series. For example, let $S = \sum_{i=1}^{\infty} \frac{1}{i^2}$. Then

$$s_3 = 1 + 1/4 + 1/9 = 49/36$$
$$\int_4^{\infty} \frac{1}{x^2} dx \leq R_3 \leq 1/16 + \int_4^{\infty} \frac{1}{x^2} dx$$

Hence

$$1/4 \leq R_3 \leq 5/16$$
$$29/18 \leq S = s_3 + R_3 \leq 241/144$$

15 Comparison Test

Comparison test: If $a_n \geq b_n \geq 0$, then

- If $\sum_{n=1}^{\infty} a_n$ converges then so is $\sum_{n=1}^{\infty}$.
- If $\sum_{n=1}^{\infty} b_n$ diverges then so is $\sum_{n=1}^{\infty} a_n$.
- When $\sum_{n=1}^{\infty} a_n$ converges, let R_n, R'_n be the remainder of a_n and b_n respectively, then $R_n \geq R'_n$.

The proof is by definition of limit and the bounded monotone convergence theorem.

Example 1: Check convergence of $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$.

$$\frac{1+\sin(n)}{n^2} \leq \frac{2}{n^2}$$

And $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges due to Example 1 of the previous section.

Example 2: Check convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Solution 1:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}$$

Because

$$(n+1)! \geq (n+1)n \geq n^2$$

$$0 \leq \frac{1}{(n+1)!} \leq \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges hence so is $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Solution 2:

$$n! = 1 \times 2 \times 3 \times \cdots \times n \geq 1 \times 2 \times 2 \cdots \times 2 = 2^{n-1}$$

Hence

$$0 \leq \frac{1}{n!} \leq 2^{1-n}$$

And

$$\sum_{n=1}^{\infty} 2^{1-n} = \lim_{n\to\infty} \frac{1-2^{-n}}{1-\frac{1}{2}} = 2$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

A consequence of the comparison test is the **Limit comparison test**: if $\lim_{n\to\infty} \frac{a_n}{b_n} = C \neq 0$, $a_n > 0$, $b_n > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}$ either both converge or both diverge.

Example 3: Check convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^6+n^4+1}}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{\sqrt{n^6+n^4+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+n}{\sqrt{n^6+n^4+1}}$$

$$= \lim_{n\to\infty} \frac{1+1/n^2}{\sqrt{1+1/n^2+1/n^6}} = \frac{1+0}{\sqrt{1+0+0}} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, hence by limit comparison test the original series diverges as well.

16 More methods to test convergence of series

1. For any integer $m \geq 1$, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=m}^{\infty} a_n := \sum_{i=1}^{\infty} a_{i+m-1}$ converges.

2. **Theorem: (alternating series test)** If $a_n = (-1)^{n-1} b_n$ (or $(-1)^n b_n$), $0 \leq b_{n+1} \leq b_n$, and $\lim_{n\to\infty} b_n = 0$, then

   - $\sum_{n=1}^{\infty} a_n$ converges.
   - The n-th remainder $|R_n| = |\sum_{i=n+1}^{\infty} a_i| \leq a_{n+1}$.

   Proof: We only do the case $a_n = (-1)^{n-1} b_n$, the other case is analogous. Let $s_n = \sum_{i=1}^{\infty} a_i$, then $s_n = p_n + q_n$, where:

   $$p_n = \begin{cases} s_n & n \text{ even} \\ s_{n-1} & n \text{ odd} \end{cases}$$

   $$q_n = \begin{cases} 0 & n \text{ even} \\ b_n & n \text{ odd} \end{cases}$$

   Then p_n is an nondecreasing sequence which is bounded by b_1, hence converges and $0 \leq \lim_{n\to\infty} p_n \leq p_1$. $\lim_{n\to\infty} q_n = 0$, hence

   $$\sum_{n=1}^{\infty} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} p_n + \lim_{n\to\infty} q_n = \lim_{n\to\infty} p_n$$

   The remainder estimate follows.

   Example 1: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, the sum is between -1 and 0.

Example 2: The convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n}$ can not be verified using alternating series test, because the absolute value of the terms does not converge to 0. Actually it diverges because $\sum_{n=1}^{\infty} a_n$ converges implies $\lim_{n\to\infty} a_n = 0$.

3. **Theorem: (absolute convergence implies convergence)** If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges. In this case we call the original series $\sum_{n=1}^{\infty} a_n$ **absolutely converges**.

   Proof:

   $$p_n = \sum_{i=1}^{n} \max(a_i, 0)$$

   $$q_n = \sum_{i=1}^{n} \min(a_i, 0)$$

   Then the n-th partial sum $s_n = p_n + q_n$, and p_n is monotone nondecreasing bounded by $\sum_{n=1}^{\infty} |a_n|$, q_n monotone nonincreasing bounded by $-\sum_{n=1}^{\infty} |a_n|$.

   Example 3: $\sum_{n=1}^{\infty} \frac{\cos(n)}{(n+1)(\ln(n+1))^2}$ converges absolutely, because

   $$0 \leq \left| \frac{\cos(n)}{(n+1)(\ln(n+1))^2} \right| \leq \frac{1}{(n+1)(\ln(n+1))^2}$$

   And

   $$\sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln(n+1))^2}$$

   exists via integral test.

   Combine absolute convergence theorem and comparison test we can get:

   (a) Ratio test: If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If the limit is greater than 1 then the series diverges.

   (b) Root test: If $\lim_{n\to\infty} |a_n|^{1/n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If the limit is greater than 1 then the series diverges.

   Example 4: $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$ converges, because

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10}{n+1} = 0 < 0$$

   Example 5: $\sum_{n=1}^{\infty} \frac{1\times 3\times \cdots \times (2n-1)}{1\times 3\times \cdots \times 3n}$ converges, because

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n+1}{3(n+1)} = \frac{2}{3} < 0$$

Example 6: $\sum_{n=1}^{\infty}\left(\frac{n-1}{2n-1}\right)^n$ converges, because

$$\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\frac{n-1}{2n-1}=\frac{1}{2}<1$$

Example 7: $\sum_{n=1}^{\infty}\left(\frac{n-1}{n}\right)^{n^2}$ converges, because

$$\begin{aligned}\lim_{n\to\infty}|a_n|^{1/n}&=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n\\&=\lim_{n\to\infty}e^{n\ln(1-\frac{1}{n})}\\&=e^{\lim_{x\to 0}\frac{\ln(1-x)}{x}}\\&=e^{-1}<1\end{aligned}$$

17 Power Series

A **power series centered at** a is a series of the form

$$\sum_{n=0}^{\infty}c_n(x-a)^n$$

where a, c_n are real numbers.

The set of possible x where a power series converges is called the **interval or domain of convergence**.

Theorem: The interval of convergence of a power series centered at a must be in one of the three cases:

- $\{a\}$.
- $\mathbb{R}$.
- $(a-R,a+R)$ or $[a-R,a+R)$ or $(a-R,a+R]$ or $[a-R,a+R]$.

We call the **radius of convergence** to be 0, ∞ and R in the above three cases.

Generally we use root test or ratio test to find radius of convergence. The end points in the third case will need to be determined via other tests.

Example 1:

- Radius of convergence of $\sum_{n=0}^{\infty} n!x^n$ is 0.
- Radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is ∞.
- Interval of convergence of $\sum_{n=0}^{\infty} x^n$ is $(-1, 1)$.
- Interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ is $[-1, 1)$.
- Interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ is $(-1, 1]$.
- Interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ is $[-1, 1]$.

Example 2: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n}{n+n^2+1}x^n$.

First do ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 2|x|\left|\frac{n+n^2+1}{(n+1)+(n+1)^2+1}\right| = 2|x|$$

So if $|x| < \frac{1}{2}$ the series converges, if $|x| > \frac{1}{2}$ it diverges.

When $x = \pm\frac{1}{2}$,

$$\sum_{n=0}^{\infty}\left|\frac{2^n}{n+n^2+1}x^n\right| = \sum_{n=0}^{\infty}\frac{1}{n+n^2+1}$$

Because

$$\frac{1}{n+n^2+1} \leq \frac{1}{n^2}$$

And $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n+n^2+1}$ converges, hence $\sum_{n=0}^{\infty} \frac{1}{n+n^2+1}$ converges. Because absolute convergence implies convergence, $x = \pm\frac{1}{2}$ are both in the interval of convergence. Hence the answer is $[-\frac{1}{2}, \frac{1}{2}]$.

Example 3: Find interval of convergence of $\sum_{n=0}^{\infty} \sin(n)x^n$.

When $|x| < 1$, because $|\sin(n)x^n| \leq |x|^n$, $\sum_{n=0}^{\infty} |\sin(n)x^n|$ convergences, hence $\sum_{n=0}^{\infty} \sin(n)x^n$ converges.

When $|x| \geq 1$, we shall now show that the series diverges. We prove it by contradiction. Suppose the series converges, we must have $\lim_{n\to\infty} |\sin(n)x^n| = 0$. $|\sin(n)| = |\sin(n)x^n| \cdot \frac{1}{|x|^n}$, and $\lim_{n\to\infty} \frac{1}{|x|^n}$ is 1 when $|x| = 1$ and 0 when $|x| > 1$, hence

$$\lim_{n\to\infty} \sin(n) = \lim_{n\to\infty} |\sin(n)| = 0$$

However,

$$\sin(n+2) - \sin(n) = 2\sin(1)\cos(n)$$

Hence

$$\lim_{n\to\infty} \cos(n) = \lim_{n\to\infty} \frac{\sin(n+2) - \sin(n)}{2\sin(1)} = \frac{0-0}{2\sin(1)} = 0$$

However

$$1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty} (\sin^2(n) + \cos^2(n)) = 0^2 + 0^2 = 0$$

Which is a contradiction. Hence the series can not converge when $|x| \geq 1$, the interval of convergence is $(-1, 1)$.

The main application of power series is that it can be used to represent certain families of functions (sometimes "locally", i.e. only on the interval of convergence). Functions that can be represented by power series are called **analytic functions**.

Example 4: Polynomial function are power series themselves.

Example 5: $\sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax}$.

One can obtain power series representations of functions from known representations via **addition, multiplication, composition, differentiation and integration**.

Example 6:

- Addition:

$$\frac{1}{x+1} + \frac{1}{x-1} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} (-1) x^n = \sum_{n=0}^{\infty} (-2) x^{2n+1}$$

- Multiplication:

$$\frac{x^2+1}{1-x} = (x^2+1) \sum_{n=0}^{\infty} x^n = 1 + x + \sum_{n=2}^{\infty} 2x^n$$

- Composition:

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$$

- Differentiation:

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \sum_{n=0}^{\infty} (n+1) x^n$$

- Integration:

$$\ln(1-x) = -\int_0^x \frac{1}{1-x} dx = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$$

18 Taylor Series

The differentiation rule implies that:

Theorem 1: If f can be written as power series centered at a around a, then the power series must be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall that $0! = 1$. When $a = 0$ it is called the **Maclaurin series**.

Example 1:

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{\sin(a+\frac{n\pi}{2})}{n!}(x-a)^n$$

More examples of writing function into power series:

Example 2: When p is not an integer, $f(x) = (1+x)^p$, then $f^{(n)}(x) = p(p-1)\ldots(p-n+1)(1+x)^{p-n}$, hence

$$(1+x)^p = 1 + \sum_{n=0}^{\infty} \frac{p(p-1)\ldots(p-n+1)}{n!}x^n$$

We can introduce the **generalized binomial coefficient** $\binom{p}{n}$ as

$$\binom{p}{0} = 1$$

$$\binom{p}{n} = \frac{p(p-1)\ldots(p-n+1)}{n!} \text{ when } n \geq 1$$

Then

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n$$

Example 3:

$$\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n} dt = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{2n+1}$$

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{n=0}^{\infty} \binom{-1/2}{n}(-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1} x^{2n+1}$$

There are many ways to estimate the remainder of Taylor series, one of which is as follows:

Theorem 2: Suppose $|f^{(n+1)}| \leq M$ on the interval between a and x, then

$$|f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

19 More on products and quotients of power series

We can formalize the calculation of products and quotients of power series in a way analogous to the long multiplication and long division of decimal numbers as well as polynomials. For example:

Example:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Now we can use Long Multiplication to find the first few terms of the Maclaurin series of $\sin(x)e^x$:

$$\begin{array}{cccccccc}
 & & x & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & \dots \\
\times & 1 & +x & +\frac{x^2}{2!} & +\frac{x^3}{3!} & +\frac{x^4}{4!} & +\frac{x^5}{5!} & \dots \\
\hline
 & & x & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & \dots \\
 & & & x^2 & & -\frac{x^4}{3!} & & \dots \\
 & & & & \frac{x^3}{2!} & & -\frac{x^5}{2!3!} & \dots \\
 & & & & & \frac{x^4}{3!} & & \dots \\
 & & & & & & \frac{x^5}{4!} & \dots \\
 & \dots & & & & & & \\
\hline
 & & x & +x^2 & +\frac{x^3}{3} & & -\frac{x^5}{30} & \dots
\end{array}$$

We can also use Long Division to find the first few terms of the Maclaurin series of $\frac{\sin(x)}{e^x}$:

$$\begin{array}{r|ccccccc}
 & x & -x^2 & +\frac{x^3}{3} & & -\frac{x^5}{30} & \dots \\
\hline
1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots & x & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & \dots \\
 & x & +x^2 & +\frac{x^3}{2} & +\frac{x^4}{6} & +\frac{x^5}{24} & \dots \\
\hline
 & & -x^2 & -\frac{2}{3}x^3 & -\frac{x^4}{6} & -\frac{x^5}{30} & \dots \\
 & & -x^2 & -x^3 & -\frac{x^4}{2} & -\frac{x^5}{6} & \dots \\
\hline
 & & & \frac{x^3}{3} & +\frac{x^4}{3} & +\frac{2}{15}x^5 & \dots \\
 & & & \frac{x^3}{3} & +\frac{x^4}{3} & +\frac{x^5}{6} & \dots \\
\hline
 & & & & & -\frac{x^5}{30} & \dots
\end{array}$$

20 Approximation with power series

If a function can be represented as a power series, its value can be approximated by partial sum, (which are also called **Taylor polynomials**).

The degree 1 case is the "linear approximation" you may have seen in Calc 1.

This approximation can be used in numerics or theoretically (perturbation analysis).

Example 1:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$1-1+1/2-1/6+1/24-1/120 \leq e^{-1} \leq 1-1+1/2-1/6+1/24-1/120+1/720$$

Example 2:

$$\sqrt{1-x^2} = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{(-1)^n}{n!} x^{2n}$$

$$\frac{\pi}{12} + \frac{\sqrt{3}}{4} = \int_0^{1/2} \sqrt{1-t^2}dt = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{(-1)^n}{(2n+1)n!} 2^{-2n-1}$$

Take first three terms we get:

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} \approx \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{8} - \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{1}{32} = \frac{1837}{3840}$$

$$\pi \approx \frac{1837}{320} - \frac{3\sqrt{3}}{2} \approx 3.14254879$$

The error term can be approximated by higher order derivatives via error formula of Taylor series, or via other remainder estimates e.g. those of the alternating or integral test.

21 3-d analytic geometry

Key idea of analytic geometry:

- Represent points as numbers.
- A geometric shape is a set of points. It can then be represented by the conditions these points have to satisfy, often via equalities or inequalities.
- Under an orthogonal coordinate system (Cartesian coordinate chart), any point in 3-d space can be represented by 3 numbers, obtained via orthogonal projection to the three axis.

- Distance between two points (a_1, a_2, a_3) and (b_1, b_2, b_3) is $\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.
- Equation for planes, balls, cylinders
  - Plane: $Ax + By + Cz = D$, A, B, C not all zero.
  - Balls: $(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$
  - Spheres: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$
  - Cylinder centered at z-axis: $x^2 + y^2 = r^2$.

22 Vectors

A **vector** is a (2 or 3)-tuple of real numbers.

- Vector notation can often be used to simplify equations and formulas.
- When there are two points in 2-d or 3-d space, the **directed line segment** from one to another can be represented as **displacement vector**, which are the differences of the different coordinates on a given Cartesian coordinate chart.
- Any point P in 2-d or 3-d space (with a Cartesian coordinate chart) can be identified with a vector from origin to P, called **position vector**.

Definition 1: The sum of two vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$

The scalar multiplication between a vector (a_1, a_2, a_3) and $c \in \mathbb{R}$ is

$$c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Let a, b be two vectors, we define $a - b$ as $a + (-1)b$.

- Properties (commutativity, associativity, distribution law etc)
- Geometric meaning: sum as parallel law, scalar multiplication by positive (negative) number means keeping (reversing) the direction and change the length.

Definition 2: Dot product:

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$$

- Properties

- $a \cdot a \geq 0$, $a \cdot a = 0$ iff $a = 0$.
- $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$.
- For $r \in \mathbb{R}$, $a \cdot rb = ra \cdot b = r(a \cdot b)$.
- $a \cdot b = b \cdot a$.

- Geometric meaning:
  - When seen as displacement vector, $x \cdot x$ is the square of the length of x.
  - In general, $a \cdot b = |a||b|\cos(\theta)$ where θ is the angle between a and b. To show this, consider ΔABC, let a be displacement from A to B, b be displacement from A to C, then displacement from B to C is $a - b$.

    $$|BC|^2 = (a - b) \cdot (a - b) = a \cdot a + b \cdot b - 2a \cdot b = |AB|^2 + |AC|^2 - 2a \cdot b$$

    But from cosine theorem in trigonometry, we know that

    $$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos(\angle A)$$

    So

    $$a \cdot b = |a||b|\cos(\angle A)$$

Example 1: The angle between $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$ is

$$\cos^{-1}\left(\frac{\langle 1, 1, 0\rangle \cdot \langle 0, 1, 1\rangle}{|\langle 1, 1, 0\rangle||\langle 0, 1, 1\rangle|}\right) = \pi/3$$

Example 2: The equation of a sphere centered at (a, b, c) of radius r can be written as

$$(\langle x, y, z\rangle - \langle a, b, c\rangle) \cdot (\langle x, y, z\rangle - \langle a, b, c\rangle) = r^2$$

Definition 3: Let $a = \langle a_1, a_2, a_3\rangle$.

- The numbers a_i are called the **component** of a
- The **length** of a is $|a| = \sqrt{a \cdot a}$.
- a is called a **unit vector** if $|a| = 1$.
- The **standard basis vectors** are $i = \langle 1, 0, 0\rangle$, $j = \langle 0, 1, 0\rangle$, $k = \langle 0, 0, 1\rangle$.
- The **direction angles** α, β, γ are the three angles between a and the three standard basis vectors. Their cosines are called **direction cosines**.
- The **projection** from a to b is $\frac{a \cdot b}{b \cdot b} b$.

Calculations of direction cosines:

- $$\cos(\alpha) = \frac{a \cdot i}{|a|} = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$
  $$\cos(\beta) = \frac{a \cdot j}{|a|} = \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$
  $$\cos(\gamma) = \frac{a \cdot k}{|a|} = \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$
- $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$.
- The unit vector in the same direction of a is
  $$\frac{a}{|a|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

Example 3: Let $v = \langle 1, 2, 3 \rangle$, find the direction cosines and the unit vector in the opposite direction of v.

Answer: $\cos(\alpha) = 1/\sqrt{14}$, $\cos(\beta) = 2/\sqrt{14}$, $\cos(\gamma) = 3/\sqrt{14}$. The unit vector is $\langle -1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14}$.

Example 4: Let A be the point $(1, 0, 0)$, B be the point $(1, 2, 3)$, C be the point $(2, 1, 1)$, find the three angles of ΔABC.

Answer: The displacement vector from A to B is $\langle 0, 2, 3 \rangle$, the displacement vector from A to C is $\langle 1, 1, 1 \rangle$, hence the angle A is

$$\cos^{-1}\left(\frac{0 \times 1 + 2 \times 1 + 3 \times 1}{\sqrt{0^2 + 2^2 + 3^3}\sqrt{1^2 + 1^2 + 1^2}}\right) = \cos^{-1}(5/\sqrt{39})$$

Similarly, angle B is $\cos^{-1}(8/\sqrt{78})$, angle C is $\cos^{-1}(-\sqrt{2}/3)$.

Definition 4: Cross product:

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

- Properties:
  - $a \times b = b \times a$
  - $a \times (b + c) = a \times b + a \times c$, $(a + b) \times c = a \times c + b \times c$
  - $a \times b$ is orthogonal to both a and b.
  - $i \times j = k$, $j \times k = i$, $k \times i = j$.
  - $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$.

– $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

- Geometric meaning of cross product: $a \times b$ is orthogonal to both a and b, direction satisfies the right-hand rule, and length is $|a||b|\sin(\theta)$ where θ is the angle between a and b.
- Triple product $(a \cdot (b \times c))$ is the signed volume of the parallelotope spanned by a, b and c.

23 Different kinds of equations of lines and planes

23.1 Lines

- **Parametric equation**: $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$. $(a, b, c$ called **direction numbers**.
- **Vector equation**: $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$.
- **Symmetric equation**: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. If knowing passing through (x_0, y_0, z_0) and (x_1, y_1, z_1), can let $a = x_1 - x_0$, $b = y_1 - y_0$, $c = z_1 - z_0$.
- Two lines are **parallel** if the corresponding direction vectors $\langle a, b, c\rangle$ are parallel (one is a multiple of another). Two lines **intersect** if the corresponding system of linear equations has a solution. If two lines are not parallel and do not intersect they are called **skew lines**.

23.2 Planes

- Linear equation: $ax + by + cz = d$. Knowing a point (x_0, y_0, z_0) on the plane, we can write it as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- Vector equation: $\langle a, b, c\rangle\langle x, y, z\rangle = d$. Knowing a point (x_0, y_0, z_0) on the plane, we can write it as $\langle a, b, c\rangle\langle x, y, z\rangle = \langle a, b, c\rangle\langle x_0, y_0, z_0\rangle$
- Two planes are **parallel** if the normal vectors $\langle a, b, c\rangle$ are parallel. Otherwise they **intersects** at a line.
- To study intersection of planes, solve systems of linear equations.

23.3 Distances between points, lines and planes

Definition: Distance between two sets is the minimal distances between one point in the first set and another point in the second set.

Distance between points, lines or planes can be obtained via dot products or cross products.

- Let P, P' be two points. Let r be the displacement vector from P to P', then
$$dist(P, P') = |r|$$

- Let P be a point, L a line passing through P' and has direction vector v. Let r be the displacement from P to P', then
$$dist(P, L) = \frac{|r \times v|}{|v|}$$

- Let P be a point, H a plane passing through P' and has normal vector n. Let r be the displacement from P to P', then
$$dist(P, H) = \frac{|r \cdot n|}{|n|}$$

- Let L be a line passing through P and has direction vector v, L' another line passing through P' and has direction vector v'. Let r be the displacement from P to P', then
$$dist(L, L') = \begin{cases} \frac{|r \times v|}{|v|} & |v \times v'| = 0 \\ \frac{|r \cdot (v \times v')|}{|v \times v'|} & |v \times v'| \neq 0 \end{cases}$$

- Let L be a line passing through P and has direction vector v, H a plane passing through P' and has normal vector n. Let r be the displacement vector from P to P', then
$$dist(L, H) = \begin{cases} 0 & v \cdot n \neq 0 \\ \frac{|r \cdot n|}{|n|} & v \cdot n = 0 \end{cases}$$

- Let H be a plane passing through P with normal vector n, H' another plane passing through P' with normal vector n'. Let r be the displacement vector from H to H', then
$$dist(H, H') = \begin{cases} 0 & |n \times n'| \neq 0 \\ \frac{|r \cdot n|}{|n|} & |n \times n'| = 0 \end{cases}$$

Example 1: Find the symmetric equation of the line passing through $(0, 1, 0)$ and $(1, 0, 1)$.

Answer: The direction vector can be chosen as the displacement vector from the first point to the second point, i.e. $\langle 1, -1, 1 \rangle$. Hence the symmetric equation can be written as
$$\frac{x}{1} = \frac{y-1}{-1} = \frac{z}{1}$$

Example 2: Find coordinate of the intersection point between line $x = t, y = 2t, z = 1 - t$ and $x = 2s, y = s + 1, z = 1 - 2s$.

Answer: Solve system of equations:

$$t = 2s$$

$$2t = s + 1$$

$$1 - t = 1 - 2s$$

We get $s = 1/3$, $t = 2/3$. So the intersection point is $(2/3, 4/3, 1/3)$.

Example 3: Find the parametric equation of the intersection between $x + y + z = 1$ and $x + 2y = 0$.

Answer: Solve the system of equation:

$$x + y + z = 1$$

$$x + 2y = 0$$

We get the solution: $y = -x/2$, $z = 1 - x/2$, x can be any number. So we can write the intersection line as $x = t, y = -t/2, z = 1 - t/2$.

Example 4: Find a line passing through $(1, 1, 1)$ and is parallel to both $x + y = 0$ and $x + z = 0$.

Answer: The direction vector must be orthogonal to both normal vectors of the two planes, hence it can be chosen as their cross product

$$\langle 1, 1, 0\rangle \times \langle 1, 0, 1\rangle = \langle 1, -1, -1\rangle$$

So the equation of the line can be written as $x - 1 = -(y - 1) = -(z - 1)$.

Example 5: Find the distance between plane $x + y + z = 1$ and line $x = 2t, y = 2 - t, z = 2 - t$.

Answer: The direction vector of the line $\langle 2, -1, -1\rangle$ is orthogonal to the normal vector of the plane $\langle 1, 1, 1\rangle$. Hence one just need to calculate the distance from one point on the line to the plane. Pick the point $(0, 2, 2)$, we get that the distance is

$$\frac{|0 + 2 + 2 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \sqrt{3}$$