

# 1 9/5 Matrices, vectors, and their applications

Algebra: study of objects and operations on them.

Linear algebra: object: matrices and vectors. operations: addition, multiplication etc.

Algorithms/Geometric intuition/sets and maps

$m \times n$  matrix: numbers forming a rectangular grid,  $m$  rows and  $n$  columns. Motivation: coefficients of a system of linear equations. Data tables in statistics.

$(i, j)$ -th entry of a matrix.

Vectors: matrices with one row/column. Motivation: coordinates in plane and space.

Operations: (1) Addition. (2) Scalar multiplication. (3) Matrix-vector multiplication. (4) Transpose.

Example:  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .  $Ax$ ,  $A(Ax)$ .

Example: Averaging over columns. Covariance? Other statistical concepts?

Laws: The usual laws one may expect. e.g.  $A(x + y) = Ax + Ay$ ,  $(A + B)^T = A^T + B^T$ ,  $(A^T)^T = A$ .  
Note:  $A(Bx) \neq B(Ax)$ !

Zero and one matrix. Standard vectors.

Example: Rotation by 60 degrees (or  $\pi/3$ ).

Consequence: Matrix is completely determined by its action on the standard vectors! Matrix-matrix multiplication.

Example:  $2 \times 2$  case.

The concept of linear combination. Relationship with matrix-vector multiplication.

Example: Rotation and Translation.

Example: Random walk on graphs.

## 2 9/8 Linear equations

Review:

- Matrix multiplications
- Transposes
- Standard vectors
- Identity Matrix
- Rotation matrix

- Stochastic matrix

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Linear systems as matrix equations. *Coefficient matrix* and *augmented matrices*

*Elementary row operations*: swap, multiply, add. Property: reversible, and preserves solution set.

*Row echelon form*: The first non-zero entry (called pivot) of each row is to the right of the previous.

*Reduced row echelon form*: The first non-zero entry is 1 and is the only non-zero entry in that column. Uniqueness under row operations.

Algorithm (Gaussian elimination):

- Write augmented matrix.
- Use row operations, turn it into reduced echelon form.
- General solution from RREF (Example:  $x_1 + 2x_2 + x_3 + x_4 = 3$ ,  $x_1 + 3x_3 - x_4 = 8$ ).

		Pivot at last col.	No pivot at last col.
Structure of solutions:	All coefficient col. have pivot	None	One
	Some coeff. col. have no pivot	None	Inf
Examples of the 4 cases.			

True or false:

- A system of 3 linear equations with 6 variables can not have just one solution.
- A system of 3 linear equations with 6 variables must have infinitely many solutions.

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Counting: number of arbitrary constants and the number of pivots. Rank and dimension.

Explicit algorithm from RREF to general solutions.

## 3 9/12 Linear equations cont.

### 3.1 Review

- Augmented matrix, row operations.
- RREF.
- Condition for no/one/infinitely many solutions.
- General solution: write *basic variables* in terms of *free variables*, or the *vector form*.

### 3.2 Gaussian elimination

Augment matrices to REF or RREF through finitely many elementary row operations.

For  $r=1, 2, \dots, n$ :

- Find the left-most non-zero entry among the  $r, r+1, \dots, n$  rows. If there aren't any, terminate.
- Exchange rows to move this entry to the  $r$ -th row.

Multiply the  $r$ -th row and add it to the  $r+1, \dots$  rows to eliminate all entries on the left-most non-zero column.

To Further turn it into a RREF (backward pass):

Multiply to each non-zero row to make the first entry 1.

For each non-zero row, multiply and add it to each of the rows above it to turn the entries on pivot columns 0.

Reason for distinguish forward/backward passes: forward pass is a permutation matrix with a lower triangular matrix with 1 on the diagonals, backward pass is a upper triangular matrix. Row pivoting.

Example:  $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 4 & 7 \\ 2 & 0 & 1 & 0 \end{pmatrix}$ . RREF? General solution?

### 3.3 Uniqueness of RREF

Key idea: read the RREF from matrix using linear combinations of rows or columns!

Appendix E uses columns. One can also use rows as follows: Let  $R$  be the space of linear combination (span) of the row vectors. The last non-zero row in RREF is the one in  $R$  with the most number of 0 entries on the left and the first non-zero entry 1. Let the index of the first non-zero entry be  $c_1$ . The preceding row in RREF is the one in  $R$  with  $c_1$ -th entry 0, first non-zero entry 1, and the most possible number of 0 on the left, etc.

### 3.4 Rank and nullity

Rank of  $A$ : num. of pivots in  $A$ =num. of non-zero rows in REF of  $A$ =num of basic variables in  $Ax = b$

Nullity of  $A$ : num of non-pivot columns in  $A$ =num. of columns of  $A$ -rank of  $A$ =num of free variables in  $Ax = b$

True or false:

- The rank of  $[A \ B]$  must be no smaller than the sum of the ranks of  $A$  and  $B$ .
- The nullity of  $[A \ B]$  must be no smaller than the sum of the ranks of  $A$  and  $B$ .
- The RREF of a square matrix of no nullity must be the identity matrix
- The nullity of  $A$  is non-zero iff some row of  $A$  is a linear combination of the others.
- $[A \ B]$  has the same rank as  $B$  iff the columns of  $A$  are linear combinations of the columns of  $B$ .

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Structure of the general solution in terms of rank or nullity:

If  $rank(A) < rank([A, b])$ :

No solution.

Else:

If  $nullity(A) = 0$ :

One solution.

Else:

Infinitely many solutions.

Example:  $\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$ .

## 4 9/15 Span

Review:

- Augmented matrix and row operations
- REF, RREF, pivot
- free and basic variables
- Rank and Nullity

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Linear combination:  $S$  is a set of matrices of the same size,  $v$  is called a linear combination of  $S$  iff there exist finitely many matrices  $A_1 \dots A_n$  in  $S$ , and scalars  $a_1, \dots, a_n$ , so that  $v = \sum_k a_k A_k$ .

Span: The span of a set is the set of all linear combinations of that set.  $S$  is called a generating set of the set  $\text{Span}(S)$ .

Example: span of the standard vectors.

Span closed under addition and scalar multiplication.

Transitivity.

$b$  is in the span of columns of  $A$  iff  $Ax = b$  has a solution.

$\mathcal{R}^n$ : all vectors of  $n$  entries. Span is  $\mathcal{R}^n$  iff matrix is *full rank* iff ...

Example: use linear equation to detect spans.

Implication on the rank of the matrices while adding columns.

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Algorithm for minimal generating set. Example.

True or false:

Row operation changes the span of the column vectors.

A matrix is in REF, then the span of the columns are the span of some standard vectors.

## 5 9/18 Linear dependency

### 5.1 Review

Notation: when  $A$  and  $B$  has the same number of rows, by  $[A \ B]$  we mean a larger matrix formed by stacking them together horizontally.

relationship between matrices, system of equations  $Ax = b$ , and the column vectors:

The followings are equivalent:

- $Ax = b$  has a solution (is **consistent**).
- $b$  lies in the **span** of the columns vectors of  $A$ .

- The **span** of the columns of  $A$  is the same as the span of the columns of  $A$  and  $b$ .
- $\text{rank}([A \ b]) = \text{rank}(A)$ .
- $\text{Nullity}([A \ b]) = \text{Nullity}(A) + 1$ .
- In the **RREF** of  $[A \ b]$ , the last column does not contain a **pivot**.

Examples.

The followings are equivalent:

- $Ax = b$  has a solution (is **consistent**) for all  $b$ .
- The **span** of the columns vectors of  $A$  is  $\mathcal{R}^m$ .
- $\text{rank}(A) = m$ .
- $\text{Nullity}(A) = n - m$ .
- In the **RREF** of  $A$ , every row contain a **pivot**.
- The **RREF** of  $A$  does not contain zero rows.

Examples.

## 5.2 Linear dependence/independence

A set  $S$  is called **linearly independent**, if for any sequence of distinct elements  $x_1, \dots, x_k \in S$ ,  $c_1x_1 + \dots + c_kx_k = 0$  implies that  $c_1 = c_2 = \dots = 0$ . If a set is not linearly independent it is linearly dependent.

$a_1 \dots a_n$  are linearly dependent if and only if  $[a_1 \dots a_n]x = 0$  (the **homogeneous eq.**) has one (hence infinitely many) non-zero solutions. (hence  $[a_1 \dots a_n]x = b$  has infinitely many solutions for some  $b$ , hence has free variables, hence the nullity of  $A$  is non-zero).

Example: 1 or 2 vectors.

Linear dependency in standard vectors.

Linear dependency in RREF.

Linear dependency in vector form of the general solution.

## 5.3 Number of rows and columns

$m > n$ : column vectors may or may not be linearly dependent, but can never span  $\mathcal{R}^m$ .

$m < n$ : column vectors may or may not span  $\mathcal{R}^m$ , but can never be linearly independent.

$m = n$ : column vectors span  $\mathcal{R}^m$  iff they are linearly independent.

## 5.4 Adding and removing vectors

If  $S$  is linearly independent, any subset of  $S$  is linearly independent and has a smaller span,  $S \cap \{v\}$  is linearly independent iff  $v$  is in the span of  $S$ .

If  $S$  is linearly dependent, so is any set larger than  $S$ .

Examples.

\*\*\*\*\*Optional\*\*\*\*\*

Row vectors under row operation.

Rank=num. of linearly independent column vectors.

Vertical stacks of matrices.

Relationship between homogeneous and non-homogeneous equations.

## 6 9/22 Review of Chapter 1, more on matrix multiplication

Important concepts to remember:

- Matrix
  - Identity matrix
  - Zero matrix
  - Scalar multiplication
  - Addition
  - Linear combination
  - Span
  - Linear independence
  - Transpose
  - Symmetric matrix
  - Row operation
  - REF, RREF
  - Pivot
  - Rank
  - Nullity
- Vector
  - Standard vectors
  - $\mathcal{R}^n$
- System of linear equation
  - Homogeneous equation

- consistence
- Augmented matrix
- Coefficient matrix
- Free variable
- Basic variable
- General solution
- General solution in vector form

True or false:

A set of 3 vectors in  $\mathcal{R}^3$  is either linearly dependent or spans  $\mathcal{R}^3$ .

If the nullity of  $A$  is greater than 0, then  $Ax = b$  has infinitely many solutions.

If  $Ax = b$  has a unique solution, then the nullity of the augmented matrix is 1.

Fibonacci series.

Unique circle passing through 3 points.

$$x + ay = b, cx + dy = e.$$

## 7 9/26 Matrix algebra

Review:

Relationship between homogeneous and non-homogeneous system: if  $Ax = b$  is consistent,  $x_0$  is a solution, then any solution can be written as  $x_0 + x_1$  where  $x_1$  is a solution of  $Ax = 0$ .

Finding minimal generating set: Put into matrix, find pivot columns.

Matrix multiplication: three equivalent ways of defining it:

- row-column rule
- multiple matrix-vector multiplication
- composition:  $AB = [A(Be_1), A(Be_2), \dots]$ .

Example using 2-by-2 matrices

Properties: the usual one, except

- No longer commutative.
- relationship with transposes.

Multiplication by identity matrix and diagonal matrix.

Example: matrix algebra and complex numbers. The idea of linear representation.

Example: non-commutativity of 3-d rotation.

## 8 9/29 Elementary matrix, inverses

Elementary row operation is left-multiplication by elementary matrices.

Column correspondence property.

Applications:

- Read linear relation from RREF.
- General solution of homogeneous equations.
- Row operation doesn't change solution.
- General solution of non-homogeneous equation.
- Uniqueness of RREF

Definition of the Inverse of a matrix. Elementary matrices are invertable.

## 9 10/3 Invertibility

A matrix is invertable iff  $rank = \#rows = \#columns$ .

Algorithm for  $A^{-1}$ .

Algorithm for solving  $AX = B$ .

Solution of  $Ax = b$  when  $A$  is invertable:  $x = A^{-1}b$ .



## 10 Midterm I review

The midterm exam will cover up to section 2.3. Please make sure you know the following:

- Able to calculate the product between matrices and vectors.
- Able to solve system of linear equations with Gaussian elimination.
- Know the meaning of the following terms: matrix, identity matrix, zero matrix, symmetric matrix, diagonal matrix, elementary matrix, transpose, linear combination, span, linear dependency, row operation, pivot, rank, nullity, free variable, basic variable.
- Able to translate statements about matrices to statements about linear equations and vice versa. For example, the columns of matrix  $A$  are linear independent, then  $Ax = 0$  has a unique solution.

Practice problems:

- (1) Find  $t$  so that the vectors  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \\ t \end{bmatrix}$  are linearly dependent.

Solution: This is asking when  $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & t \end{bmatrix}$  has rank smaller than 3. By Gaussian elimination you can see that it has rank smaller than 3 iff  $t = 5/2$ .

- (2) True or false:

- a) Elementary row operations does not change the span of column vectors.
- b) If  $Ax = b$  has at least two solutions, then the column vectors of  $A$  are linearly dependent.

Solution: a) is false. For example,  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  can be turned into  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  by exchanging the two rows, and the span of the columns are not the same. b) is true, because  $Ax = b$  has more than one solution means that  $Ax = 0$  has non-zero solution, hence the columns of  $A$  are linearly dependent by the definition of matrix-vector multiplication.

- (3) Find matrix  $E$  so that  $E \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

Solution: You can solve a system of linear equation, or alternatively, recognize that to turn  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  into  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  one can first add the first row to the second then exchange the two rows, hence  $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ .

- (4) Show that the transpose of elementary matrices are also elementary matrices.

Solution: write down the three types of elementary matrices and see that their transposes are elementary matrices of the same type.