STATEMENT OF RECENT RESEARCH

My research explores various topics in geometric topology and dynamics, including geometric group theory, discrete geometry, symbolic dynamics, and the study of translation surfaces. The following are my recently finished research projects. A longer version of my research statement can be found at https://wuchenxi.github.io.

Generalizing Kazhdan's theorem to metric graphs and other settings. Let S be any closed hyperbolic Riemann surface, the *canonical, or Arakelov, metric* on S is defined as $\|v\| = \sup_{\|w\|=1, w\in\Omega^1(S)} |w(v)|$. A classical result by Kazhdan [K] (also c.f. [M]) says that when passing to a tower of finite regular covers of S, the canonical metrics converges uniformly to a multiple of the hyperbolic metric. Using the L^2 technique by Lück, Farbod Shokrieh (UW Seattle) and I found an analogue for it in the setting of the metric graphs.

It has been known (see [Z, BF, CR]) that the correct analogue for canonical metric in the setting of metric graphs should be Zhang's canonical measure, which can be defined in one of the following ways:

- Let $e \in E(G)$ be any edge of G, $\mu_G(e) = \frac{1}{l(e)} \cdot \sup_w |w(e)|^2$ where supremum goes through all piecewise constant harmonic 1-forms on G with norm no more than 1.
- Viewing G as a resistive electric circuit, $\mu_G(e)$ is the Foster's coefficient, which is $1-r(e^+, e^-)/l(e)$, where $l(e)$ is the edge length of e, and $r(e^+, e^-)$ is the effective resistance between the end points of e.
- Let $e \in E(G)$ be any edge of G, $\mu_G(e) = \frac{\sum_{T \text{ spanning tree of } G, e\notin T} \prod_{e'\notin T} l(e')}{\sum_{T \text{ spanning tree of } G} \prod_{e'\notin T} l(e')}$, where l is the edge length.

Theorem 1. [SW] *Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G. Let $e \in E(G)$ be any edge of G, e_i be a preimage of e on G_i, then the limit $\lim_{i\to\infty} \mu_{G_i}(e_i)$ always exists.*

The technique we used is very general, in [BSW] we stated the generalization on Riemann surfaces, and there are also generalizations on higher dimensional Riemannian manifolds as well as piecewise Euclidean surfaces via the Delaunay triangulation. Our result provide a new approach for getting analogies of the "hyperbolic metric" for simplicial complexes with metric and Berkovich spaces, and gives a new potential idea for discrete uniformization which might be useful in image processing.

Asymptotic translation lengths on curve complexes and sphere complexes. Let S be a surface of finite type. The *curve graph* of S, denoted as $\mathcal{C}(S)$, is the graph where the vertices are isotopy classes of simple closed curves on S and there is an edge of length 1 between two vertices if they have disjoint representations. The classical work of Mazur-Minsky [MM] showed that the curve graph is hyperbolic and pseudo-Anosov group elements act on it loxodromically, i.e. with non-zero asymptotic translation lengths. This asymptotic translation length can be seen as a characterization of the "combinatorial complexity" of the pseudo-Anosov maps, and a combinatorial analogy of the stretch factor. The technique in [MM] was also used by [GT] to find a lower bound for these asymptotic translation lengths for S with a given genus. A main motivation for this project is to investigate the relationship between the group-theoretic properties of the pseudo-Anosov map and its asymptotic translation length on the curve complex.

Building upon the works of [BS], Hyungryul Baik, Hyunshik Shin and I [BShiW] found an upper bound for asymptotic translation lengths for pseudo-Anosovs in the same fibered cone of the Thurston norm of the mapping torus, which is a partial analogue of the result regarding stretch factors by McMullen [M2]:

Theorem 2. [BShiW] *Let M be a closed hyperbolic 3-manifold, P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ that intersects with P. For any primitive integer elements α in $P\cap L$, the asymptotic translation length of the pseudo Anosov map induced by α is bounded above by $C\|\alpha\|^{-(1+1/(\dim(L)-1)}$, where C is a constant depending on L and P, $\|\cdot\|$ is the Thurston norm.*

In a follow-up paper [BKSW] we proved that this upper bound is sharp when $\dim(L) \leq 3$, and also showed that when $\dim(L) = 2$, all but finitely many primitive integer elements in $P \cap L$ induces pseudo-Anosov maps that are normal generators of the corresponding mapping class group. Lastly, in a recently posted paper [BKW], we are able to apply our technique to the study of homeomorphisms of double handlebodies, and obtained a similar bound in the setting of sphere complexes of 3-manifolds, which implies some upper bound of translation lengths of irreducible train track maps on the free factor [HV] and free splitting [HM] complexes.

Thurston's teapot and core entropy. In William Thurston's last paper [T], he described a shape called "master teapot", which is the closure in $\mathbb{C} \times \mathbb{R}$ of pairs of points (z, λ), where z is a Galois conjugate of λ and λ is the exponent of the entropy of a unimodal interval map (a continuous map from an interval to itself, with a unique maximal or minimal point) where the orbit of critical point is periodic. The teapot gives a necessary condition for checking if an algebraic integer can be the exponent of the entropy of such an interval map: for any Galois conjugate z of this algebraic integer λ, the point (z, λ) must be contained in the teapot. The study of interval maps and their entropies also have extensive connections to Diophantine approximation, Littlewood polynomials, iterative function systems, and complex dynamics.

With Harrison Bray, Diana Davis and Kathryn Lindsey, I found an algorithmic description of this set by finding an algorithm that will terminate for points not inside the "teapot" [LW], and as a consequence, showed the following:

Theorem 3. [BDLW] *Let T be the Thurston's master teapot. Then:*

- *If $|z| = 1$, then $(z, \lambda) \in T$ iff $\lambda \in [1, 2]$.*
- *If $|z| < 1$, then $(z, \lambda) \in T$ implies $(z, y) \in T$ for all $y \in [\lambda, 2]$.*
- *Let D be the unit disc in $\mathbb{C}$, $\Omega_\lambda = \{z \in \mathbb{C} : (z, \lambda) \in T\}$. Then $D \cup \Omega_\lambda$ moves continuously under Hausdorff topology.*

The algorithmic characterization of the master teapot gives interesting examples of sets defined via parametrized iterative function systems. Numerical experiments showed the similarity of these sets with the corresponding limit sets of iterative functions systems, which provides a rich family of conjectures that are IFS-analogues of the Julia-Mandelbrot correspondence, that generalized the one by Baez, Christensen, Derbyshire and Egan. Furthermore, as stated above, the algorithm we contained provides a necessary condition for an algebraic integer to be the exponent of the entropy of a unimodal interval map where the critical point has periodic orbit.

Recently, with Kathryn Lindsey and Giulio Tiozzo, I was able to generalize some of these results to the setting of quadratic core entropy (the entropy of the induced map on the quadratic Hubbard tree). We are also able to get similar necessary conditions for some families of interval maps with more than one critical points, by combining our approach for Thurston teapot and techniques in [Ti, Ti2].

Other recent research projects that are relevant. During grad school, together with Hyungryul Baik and Ahmad Rafiqi, I found new constructions for pseudo-Anosov maps which comes from "thickening" of some interval maps [BRW], and calculated their Teichmüller polynomials [BWKJ]. In the process we also found families of such maps with explicit Markov decomposition that approximates an end-periodic map.

Recently, with Elisenda Grigsby, Kathryn Lindsey and Robert Meyerhoff, I found an upper and lower bounds for the maximal degrees of freedom of fully connected feed-forward neural networks with ReLU activation, which will appear in an upcoming paper. In particular, in the case when the size of different layers are decreasing, the bounds we found become the same, which gives a precise estimate for this quantity.

References

[BDLW] H. Bray, D. Davis, K. Lindsey, and C. Wu. The shape of Thurston's Master Teapot *Advances in Mathematics*, vol. 377, 2021

[BF] M. Baker and X. Faber,Metric properties of the tropical Abel-Jacobi map, *J. Algebraic Combin.* 33(3):349–381, 2011

[BKSW] H. Baik, E. Kin, H. Shin, and C. Wu. Asymptotic translation lengths and normal generations of pseudo-Anosov monodromies for fibered 3-manifolds. *arXiv*:1909.00974.

[BKW] H. Baik, D. Kim and C. Wu. On the asymptotic translation lengths on the sphere complexes and the generalized fibered cone *arXiv*: 2011.08034

[BRW] H. Baik, A. Rafiqi, and C. Wu. Constructing pseudo-Anosov maps with given dilatations. *Geometriae Dedicata* 180(1):39-48, 2016

[BS] H. Baik and H. Shin. Minimal Asymptotic Translation Lengths of Torelli Groups and Pure Braid Groups on the Curve Graph. *International Mathematics Research Notices*, 2018.

[BSW] H. Baik, F. Shokrieh, and C. Wu. Limits of canonical forms on towers of Riemann surfaces *J. Reine Angew. Math.* 2020(764):287-304, 2020.

[BShiW] H. Baik, H. Shin, and C. Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces. *arXiv*:1801.06638, 2018, to appear in *Indiana University Math Journal.*

[BWKJ] H. Baik, C. Wu, K. Kim, and T. Jo. An algorithm to compute the Teichmüller polynomial from matrices. *Geometriae Dedicata*, 2019.

[CR] T. Chinburg and R. Rumely, The capacity pairing, *J. reine angew. Math* 434:1-44, 1993

[GT] V. Gadre and C. Tsai. Minimal pseudo-anosov translation lengths on the complex of curves. *Geometry & Topology*, 15(3):1297-1312, 2011.

[HV] Allen Hatcher and Karen Vogtmann. The complex of free factors of a free group, 1998.

[HM] M. Handel and L. Mosher. The free splitting complex of a free group I, Hyperbolicity. *Geometry & Topology*, 17(3):1581-1670, 2013.

[K] D. Kazhdan. On arithmetic varieties. Lie groups and their representations, *Halsted*, pp. 151-216, 1975.

[LW] K. Lindsey and C. Wu. A characterization of Thurston's Master Teapot. *arXiv*: 1909.10675, 2019.

[M] CT. McMullen. Entropy on Riemann surfaces and the Jacobians of finite covers. *Commentarii Mathematici Helvetici* 88(4):953-964, 2013.

[M2] CT. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. *Annales scientifiques de l'Ecole normale sup/'erieure.* 33(4), 2000.

[MM] HA Masur and YN Minsky. Geometry of the complex of curves i: Hyperbolicity. *Inventiones mathematicae* 138(1):103-149. 1999.

[SW] F. Shokrieh and C. Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019.

[T] W. Thurston. Entropy in dimension one. *arXiv*:1402.2008, 2014.

[Ti] G. Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the Mandelbrot set. *Advances in Mathematics*, 273:651-715, 2015.

[Ti2] G. Tiozzo. Continuity of core entropy of quadratic polynomials. *Inventiones mathematicae*, 203(3):891-921, 2016.

[Z] S. Zhang, Admissible pairing on a curve, *Invent. Math.* 112(1): 171-193, 1993

STATEMENT OF TRAVEL AND INVITATION GOALS

For the next year, I plan on working on the following collaborative research projects, and visit my collaborators during the summer and winter breaks if possible:

1. With Kathryn Lindsay (Boston College) and Giulio Tiozzo (Toronto), we are working on improving our results on the properties of Galois conjugates of quadratic core entropies, building on our previous works [T, T2, BDLW, LW] and also looking into the core entropies of higher degree as well as investigating the Julia-Mandelbrot correspondence for iterative function systems. We are also hoping that some techniques we developed would be useful for the study of pseudo-Anosov map, in particular for the conjecture that all bi-Perron numbers are pseudo-Anosov stretch factors.
2. With Karhryn Lindsey and Robert Meyerhoff (Boston College), we are working on finishing our paper on the degrees of freedoms of artificial neural networks and trying to understand them from the perspective of tropical techniques.
3. With Farbod Shokrieh, I plan to work on two main projects: firstly, it is conjectured by Thomas Koberda, that there is a graph-theoretic analogy for McMullen's theorem [M] that the gap between stretch factor of a pseudo-Anosov map, and the spectrum radius of the induced map on homology, when passing to finite regular covers, is either uniformly bounded away from zero or disappear at a double cover. We conjectured that in the graph setting it is related to the study of the spectral radius of the induced map on L^2-harmonic 1-forms on the universal cover, and are currently trying to apply tropical techniques to it. Furthermore, there is a graph-theoretic analogy of Riemann-Roch theorem by Baker-Norine [BN] which is related to the question of chip firing on graphs. We are working on finding an alternative proof for it using heat equation techniques. We hope this would provide new insights on the analogy between finite metric graphs and closed Riemann surfaces.
4. Baker-Rumely [BN] showed that the measure on metric graphs G as described in [Z] is an "equilibrium measure" in the sense that it is the signed measure that maximizes $\int_{G\times G} r(x,y)(d\mu)^2$, where r is the "effective resistance" between two points where G is considered as a resistive electric circuit. However, the measure, instead of the signed measure, that maximizes this integral is still unknown. Farbod Shokrieh, Harry Richman (UW-Seattle) and I did some calculations for this measure in some cases and will investigate this problem further.
5. Building on our previous works, Hyungryul Baik (KAIST) and I are working on two projects related to estimating the asymptotic translation lengths of pseudo-Anosov maps in the curve complex: firstly, we found some upper and lower bounds for the minimal asymptotic translation length when the pseudo-Anosov map preserves a subspace of the homology of given dimension. This provides an analogy for a result by Agol-Leininger-Margalit [ALM]; secondly, we are trying to see if techniques in [DKL] can be applied to the study of invariant train tracks which would gives a better lower bound for this asymptotic translation length when the pseudo-Anosov maps are in the same fibered cone, generalizing our previous work with Eiko Kin and Hyunshik Shin [BKSW].
6. Hyungryul Baik, Sebastian Hensel (University of Munich) and I found an approach to construct singular foliations for manifolds with one-vertex triangulation and left-orderable fundamental groups. We are investigating the set of foliations that can be found in the case of surfaces, and also to see if this technique can be used to get partial results for the well known conjecture [D] of the equivalence of left orderable fundamental group, non-minimality of Heegaard Floer homology and the existence of taut foliation.
7. A surprisingly effective technique for studying the geometry and dynamics of flat surfaces is to relate it to the action of $SL(2,\mathbb{R})$ on the moduli space (or strata) of flat surfaces. To complete this project one need the classification of $SL(2,\mathbb{R})$-orbit closures (which has been studied by Eskin-Mirzakhani-Mohammadi), understanding the volumes of these moduli spaces as well as the horocycle invariant measures, and the compactification by Bainbridge-Chen-Gendron-Grushevsky-Möller [BCGGM] is a useful technique for these purposes. With John Smillie (Warwick), we are working on finishing our work on a related Borel-Serre type compactification of the strata, in particular its relationship to geometry of the strata under the the Avila-Gouezel-Yoccoz [AGY] "sup norm".

References

[AGY] A. Avila, S Gouëzel, and J-C Yoccoz. Exponential mixing for the Teichmüller flow. *Publications Math/'ematiques de l'IHÉS* 104: 143-211, 2006.

[ALM] I. Agol, C. J. Leininger, D. Margalit. Pseudo-Anosov stretch factors and homology of mapping tori. *Journal of the London Mathematical Society*, 93(3):664-682, 2016.

[BCGGM] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Moeller. The moduli space of multi-scale differentials. *arXiv:*1910.13492, 2019.

[BDLW] H. Bray, D. Davis, K. Lindsey, and C. Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805, to appear in *Advances in Mathematics*

[BKSW] H. Baik, E. Kin, H. Shin, and C. Wu. Asymptotic translation lengths and normal generations of pseudo-Anosov monodromies for fibered 3-manifolds. *arXiv*:1909.00974.

[BN] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766-788, 2007.

[D] N. M. Dunfield. Floer homology, group orderability, and taut foliations of hyperbolic 3–manifolds. *Geometry & Topology* 24(4):2075-2125, 2020.

[DKL] S. Dowdall, I. Kapovich, and C. Leininger. McMullen polynomials and Lipschitz flows for free-by-cyclic groups. *J. Eur. Math. Soc.*, (11):3253-3353, 2017.

[LW] K. Lindsey and C. Wu. A characterization of Thurston's Master Teapot. *arXiv*: 1909.10675, 2019.

[M] CT. McMullen. Entropy on riemann surfaces and the Jacobians of finite covers. *Commentarii Mathematici Helvetici* 88(4):953-964, 2013.

[T] G. Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the Mandelbrot set. *Advances in Mathematics*, 273:651-715, 2015.

[T2] G. Tiozzo. Continuity of core entropy of quadratic polynomials. *Inventiones mathematicae*, 203(3):891-921, 2016.

[Z] S. Zhang. Admissible pairing on a curve. *Inventiones mathematicae*, 112(1):171-193, 1993.