Stable Translation Length on Sphere graphs and applications

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Sphere graphs and stable translation lengths

- Let M be a d-dim compact manifold, the **sphere graph** S_M of M is the simplicial graph where vertices are isotopy classes of embedded spheres of dimension $d-1$, an edge between two vertices if they do not intersect. We can give a metric to S_M by setting all edge lengths to be 1.
- Let f be a diffeomorphism from M to itself, it induces an isometry f_* from S_M to itself. The **stable translation length** of f_* is defined as

$$l(f) = \lim \inf_{n\to\infty} \frac{d(f^n(x), x)}{n}$$

Examples

- M closed oriented surface of genus $g \geq 2$: S_M is the curve graph.
  - It is δ-hyperbolic, $l(f) > 0$ iff f is pseudoanosov.(Mazur-Minsky)
  - $l(f)$ is rational (Bowditch) and can be calculated by an algorithm.
  - When genus is fixed as g, minimal non-zero $l(f) \sim g^{-2}$ (Gadre-Tsai), when f is required to be Torelli then minimal $l(f) \sim g^{-1}$. (Baik-Shin)
- When M is doubled handlebody then S_M is the free splitting graph (1-skeleton of the simplicial completion of the Culler-Vogtmann outer space), and $l(f) > 0$ iff f has filling invariant lamination. (Handel-Mosher)

Cone of homological directions

- Let f be a diffeo from M to itself, $\tilde{M}$ a free abelian cover with deck group Γ, where f lifts to $\tilde{f}$. Then this induces a $\Gamma \oplus \mathbb{Z}$-cover of the mapping torus N, denoted as $\tilde{N}$.
- Directions in $(\Gamma \oplus \mathbb{Z}) \otimes \mathbb{R}$ are called **homological direction** (Fried) iff it can be approximated by flow lines.
- $\Gamma \oplus \mathbb{Z}$ is a quotient of $H_1(N)$, hence its dual is a subspace of $H^1(N)$.
- A primitive integer classe α in the dual of the homological direction corresponds to other ways of writing N as fibering over the circle, with fiber M_α and monodromy f_α.
- The cone of homological direction depends on the diffeo, not just its homotopy type.

Main Theorem

- Theorem (Baik-Kim-W): If Γ has rank d, C a proper subcone of the dual cone of the fibered cone, $||$ any norm on the space $((\Gamma \oplus \mathbb{Z}) \otimes \mathbb{R})^*$ induced by a quadratic form, α a primitive integer element in C, then $l(f_\alpha) \lesssim |\alpha|^{-1-1/d}$
- When M is surface and f is pseudo-Anosov, the dual cone is Thurston's fibered cone, result was shown in previous paper by Baik-Shin-W.
- If $d = 1$, this bound optimal. If $d = 2$ in at least one case optimal. (Baik-Kin-Shin-W)

Applications, cont.

- When M is doubled handlebody, dual cone can be made to contain the positive cone $\mathcal{A}$ by Dowdall-Kapovich-Leininger.
- Similar bound can be proved to free factor complexes as well.
- There is an analogous theorem for disc complexes, applying that to handlebody, we can show that minimal non-zero $l(f)$ is $\sim g^{-2}$ if f is required to be a handlebody group element.

Analogy in simplicial setting

- Goal: generalization to certain homoemorphisms on simplicial complexes.
- Semiflow: continuous action on additive semigroup (i. e. a "flow" where flow lines are allowed to merge).
- If X is a Δ-complex, $f : X \to X$ a homotopy equivalence. If the mapping torus N of f can be modified by homology into a Δ-complex where the 1-skeleton is all non-horizontal and where there is a vertical semiflow. Let A be the subset of $H^1(N; \mathbb{R})$ where the elements can be represented by a 1-cochain where all upward edges of N are positive, which we shall call the **positive cone**. Then a primitive integer element α of A correspond to sections of the positive flow which are also embedded Δ-complexes of N. Let f_α be the first return map.

- Theorem: Let C be a proper subcone of A, H a d-dim rational subspace of $H^1(N)$, $||$ a norm induced by a quadratic form on H, $\alpha \in H \cap C$, then the minimal number of iterates of f_α that make the image of every facet intersects with every other facet $\gtrsim |\alpha|^{1+1/(d-1)}$.
- This provides an alternative proof of the bound on stable translation length on free splitting complexes that does not require the use of double handlebody.

Further directions

- Application to outer space of raag (Charney-Stambaugh-Vogtmann)
- End periodic graph and surface maps?
- Lower bound?

References

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