Asymptotic translation lengths on curve complex and free factor complex

Chenxi Wu

September 30, 2020

- Curve complex and asymptotic translation lengths
- An upper bound on the fibered cone
- Analogy on free factor complex
- Proof for the upper bound
- Remaining questions

Curve graph and Curve complex

Let S be a closed surface with genus > 1.

- Curve graph:
  - Vertices are isotopy classes of simple closed curves
  - Two vertices are connected by an edge of length 1 if the corresponding curves are disjoint.
- Curve complex:
  - Vertices are isotopy classes of simple closed curves
  - Vertices form a simplex iff the corresponding curves are disjoint.
- Curve graphs are Gromov hyperbolic. (Masur-Minsky)
- The mapping class group acts on it by isometry.
- The action is hyperbolic (has 2 fixed points at the boundary) iff the mapping class is pseudo-Anosov. (Masur-Minsky)

Asymptotic translation length on curve graphs

- Asymptotic translation length, or stable length: $l(g) := \lim_{n\to\infty} \frac{d(v, g^n v)}{n}$, where v is a vertex of the curve graph and d the distance in curve graph.
- $l(g) > 0$ iff g pseudo-Anosov.
- $l(g)$ are rational numbers, with denominator bounded by some number depending only on the genus (Bowditch). Hence $l(g)$ can be calculated by finding geodesics on the curve graphs.
- $l(g) \gtrsim g(S)^{-2}$, where $g(S)$ is the genus, and this lower bound is optimal. (Gadre-Tsai)
- $l(g) \gtrsim g(S)^{-1}$, when g is in the Torelli group, and the lower bound is optimal. (Baik-Shin)

Thurston's norm and fibered cone

- Kin-Shin: the example in Gadre-Tsai for pseudo-Anosov maps with small asymptotic translation lengths can be made to be within a single fibered cone.
- Baik-Shin-Wu: one can further find an upper bound for all maps within the same fibered cone.
- Thurston's fibered cone:
  - $\psi : S \to S$ pseudo-Anosov, $M = S \times [0,1]/\sim$, $(\psi(x), 0) \sim (x, 1)$: mapping torus of ψ. $\alpha \in H^1(M;\mathbb{Z})$ pullback from the projection on S^1.
  - Thurston norm: $\beta \in H^1(M;\mathbb{Z})$, $\|\beta\| = \min \max\{0, -\sum_i \chi(S)\}$ where S is a dual of β.
  - Thurston norm can be extended to $H^1(M;\mathbb{R})$ as PL norm, the unit ball is a rational polytope. The cone over the fact which contains α is the fibered face containing α, in which any primitive integer class β represents a fiber of M over the circle, hence a pseudo-Anosov map ψ_β on the fiber S_β.

- Theorem (Baik-Shin-W) Let L be a rational slice of a proper subcone of the fibered cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $I(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$.

Analogy for free factor complex

Let F_n be the free group with n generators.

- Free factor complex FF_n:
  - Vertices: conjugacy class of free factors of F_n.
  - Faces: sequences of free factors arranged by containment.
- It is the simplicial completion of the Culler-Vogtmann outer space.
- It is Gromov hyperbolic (Bestvina-Feighn)
- $Out(F_n)$ acts on it by isometry.
- The element of $Out(F_n)$ that acts hyperbolically are the ones that are fully irreducible (has no periodic conjugcy class of proper free factor, i.e. can be represented by irreducible train track maps).

- ψ: a graph map representing a fully irreducible element in $Out(F_n)$, M: its mapping torus.
- Dowdall-Kapovich-Leininger: there is a "cone of sections" or "McMullen cone" containing the pullback of generator of $H^1(S^1)$, where every primitive integer class β represent a fully irreducible outer automorphism ψ_β. Let the negative Euler characteristic of the fiber be $\|\beta\|$.
- Theorem (Baik-Kim-W) Let L be a rational slice of a proper subcone of the McMullen cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$, l is the asymptotic translation length on $FF_n^{(1)}$.

Proof for curve complex case, $d = 2$

- $\psi : S \to S$. Lift it to an invariant $\mathbb{Z}$ fold cover $\tilde{S} \to S$ as $\tilde{\psi}$. Let h be the deck transformation.
- Let $\tilde{M}$ be the $\mathbb{Z}^2$ cover of M, deck transformation group is generated by $\{\Psi, H\}$. Let $\{e_1, e_2\}$ be the dual basis. Then $\psi^p_{(p,q)} = \tilde{\psi}$, $S_{(p,q)} = \tilde{S}/\langle \tilde{\psi}^q h^{-p} \rangle$.
- When p is large, let D be a fundamental domain of $\tilde{S}$, then h^{-p} can be far from $\tilde{\psi}^q$, hence, if c is a curve in one fundamental domain in between, it would take many iterations of $\tilde{\psi}^q$ till it covers the whole $S_{(p,q)}$.
- When can h^{-p} be far from $\tilde{\psi}^q$? Use McMullen's "Teichmuller polynomials".

General case and Remaining questions

- Higher d can be proved analogously.
- For FF_n, reduce it to sphere complex.

Remaining questions:

- Is the upper bound optimal? (known in only a single example)
- Can one get lower bound for the FF_n case?

References

- H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I hyperbolicity
- B. H. Bowditch. Tight geodesics in the curve complex.
- V. Gadre and CY Tsai. Minimal pseudo-Anosov translation lengths on the complex of curves
- H. Baik, H. Shin and C. Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces
- A. Hatcher and K. Vogtmann. The complex of free factors of a free group.
- M. Bestvina and M. Feighn. Hyperbolicity of the complex of free factors.
- S. Dawdall, I. Kapovich and C. J. Leininger. McMullen polynomials and Lipschitz fows for free-by-cyclic group.