End Periodic Maps on Graphs

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Motivation

- Question: Nielsen-Thurston classification of big mapping class groups.
- Known Cases:
  - Tame maps (Bestvina-Fanoni-Tao): any two simple closed curves α, β, $i(f^n(\alpha), \beta)$ is bounded. These are big MCG version of periodic mapping classes.
  - End periodic maps: all periodic ends (ends that get sent to itself by some f^n) are either attracting or repelling (has a neighborhood U on which f^n is a subshift).
  - Example: $\mathbb{Z}$-cover some closed map, deck transformation composed with a map with compact support.

Properties of End Periodic Surface Maps

- Handel-Miller lamination (Cantwell-Conlon-Fenley)
  - Definition of Handel-Miller lamination: iterating the boundary of ends
  - End periodic maps that have no reducing curves and preserves a pair of transverse Handel-Miller laminations are called **pseudo-Anosov**, any end periodic map can be decomposed into finitely many pieces by reducing corves, on each it is either a shift or pseudo-Anosov.
  - Dynamics of pseudo-Anosov end periodic map at the core is a two sided finite Markov shift (i.e. can calculate entropy and inveriant transverse measures of Handel-Miller laminations).

Properties of End Periodic Surface Maps (cont.)

- Thurston's fibered face and depth-1 foliations (Fenley)
  - Let f be a homeomorphism from closed surface S to itself which can be lifted to a $\mathbb{Z}^d$-cover of S, then the mapping torus T_f of S admits a $\mathbb{Z}^{d+1}$-cover. The dual of the deck group Γ is a $d+1$-dimensional subspace $L \subseteq H^1(T_f)$, and the S-bundle over S^1 with monodromy f gives an element $\alpha_f \in L$.
  - Thurston showed that there is a rational cone, called **fibered cone** in L containing α_f, where every primite integer element β in the interior corresponds to a way of fibering T_f over the circle, with monodromy f_β.
  - If f is pseudo-Anosov, so are all f_β. McMullen showed that there is a way to calculate all their stretch factors, via **McMullen's polynomial**, and it is a real analytic function that blows up at the boundary of the fibered cone.

Properties of End Periodic Surface Maps (cont.)

- Thurston's fibered face and depth-1 foliation (cont.)
  - The limit of an arithmetic sequence in the fibered cone that is parallel to the boundary gives a depth-1 foliation of T_f, the monodromy on the non-compact leaf is end-periodic.
  - Example: $\mathbb{Z}$ fold cover of some compact surface S, deck transformation composed with a map of compact support.
  - When T_f is compact hyperbolic, entropy of end periodic map on the core can be calculated via McMullen's polynomial.
- Spun pseudo-anosov theory (Landry-Minsky-Taylor)
  - Reversing Thurston for atoridal end periodic maps
  - Entropy relationship (even in non-pseudo Anosov case, i.e. when McMullen's polynomial doesn't work.

End periodic graph maps and traintrack maps

Let G be a locally finite simplicial graph with no leaves and finitely many ends, all accumulated by genus. $f : G \to G$ a simplicial map which is also a homotopy equivalence.

- If there is some d such that f^d restricted to the neighborhood of each end is a half-shift, we call f **end-periodic**
- If furthermore f is a train-track map (iterated image of every edge has no back-tracking), we call f an **end-periodic train track map**

Examples and Properties

- Example 1: $\mathbb{Z}$-cover of some finite graph, deck transformation composed with a homotopy equivalence of compact support.
- Example 2: The compact support map can be chosen suitable to get end periodic train track maps.

"Theorem:" One can carry out the Bestvina-Handel algorithm to maps in Example 1, which will end up in one of the following cases:

- Getting some non-trivial subgraph fixed by the map f.
- Getting a train-track map but the resulting graph is no longer locally finite.
- Getting end-periodic traintrack maps.

Remark: For finite graph, Bestvina-Handel algorithm result in a relative train-track, which can be used to tell if an $Out(F_n)$ element has non-zero, realizable minimal translation length on the Culler-Vogtmann Outer Space.

Application: Entropy and Cone of homological direction

- If f is an end-periodic simplicial map in Example 1, replacing the $\mathbb{Z}$-cover with $\mathbb{Z}/N$-cover for large N, denoted as f_N. Then it is easy to see that f_N has homotopic mapping torus, denoted as T_f.
- f_N are monodromies corresponding to an arithmetic sequence on the "fibered cone" ("$\mathcal{A}$-cone", or "positive cone" by Dowdall-Kapovich-Leininger, or alternatively, the dual cone of the "cone of homological directions" by Fried). Their entropy can be calculated by the "McMullen's polynomial" defined by Dowdall-Kapovich-Leininger.
- When f_N are train-track maps, so is f, and the limiting entropy equals the entropy of f.
- One can use this to show the identification between cone of homological direction and the McMullen cone (the cone where the real-analytic entropy function can be analytically extended to).

Cone of homological direction

- Let X be a topological space, f a homeomorphism from X to itself that can be lifted to a map f' on $\mathbb{Z}^d$-cover X'. This induces a $\mathbb{Z}^{d+1}$-cover of the mapping torus T_f, the deck group is $\mathbb{Z} \times \Gamma$ where Γ is the deck group of X'.
- Let D be a fundamental domain on X', consider all elements of the form (d, h) such that $D \cap hf'^d(D) \neq \emptyset$, these points are within bounded Hausdorff distance to a cone in $(\mathbb{Z} \times \Gamma) \otimes \mathbb{R}$, called the "cone of homological directions". Its dual cone can be seen as a "generalized" fibered cone, i.e. primitive integer elements corresponds to fibering of T_f over the circle.
- When X is a surface, this cone is generally smaller than the fibered cone. When f is the pseudo-Anosov representation they are identical.

Cone of homological direction

- When X is a finite graph, this theory can be extended to homotopy equivalences, and is compatible with the "positive cone" by Dowdall-Kapovich-Leininger (Baik-Kim-W).
- Dowdall-Kapovich-Leininger showed that this cone is generally smaller than the McMullen cone, because the McMullen cone of f^{-1} can be smaller than the McMullen cone of f. Our result provided a way to generate many examples where they are identical.

Further Questions

- More general Bestvina-Handel algorithm?
- Nielsen-Thurston classification? Bers type argument for train track?
- How about the group generated by lifts of finite graph maps and compactly supported graph maps?
- (Wirh Farbod Shokrieh) Mapping torus as curves on moduli space of tropical curves.