RESEARCH STATEMENT

CHENXI WU

My research explores topics in geometric group theory, low dimensional topology, symbolic dynamics, and translation surfaces. The following are my current research projects:

1. Generalizing Kazhdan's result on canonical metric, and some other results on Riemann surfaces, to the setting of graphs (Section 1)
2. Relating the topology of hyperbolic 3-manifolds and the asymptotic translation length of its monodromies on curve complexes. (Section 2)
3. Thurston's "master teapot" for interval maps of constant slope. (Section 3)
4. Borel-Serre type compactification of the moduli space of translation surfaces into a real orbifold with corners. (Section 4)
5. Developing a Bowen-Series type coding for Kleinian groups. (Section 5)

A goal for many of these projects is to understand the connections between questions in dynamics, topology and combinatorics.

1. Generalizing Kazhdan's theorem and other results on Riemann surfaces to metric graphs

Farbod Shokrieh and I are exploring analogies between closed Riemann surfaces and finite metric graphs. We generalized a classical theorem by Kazhdan on the convergence of canonical metric on normal covers of the Riemann surface to the case of finite metric graphs by using Lück's approximation of L^2 betti numbers, and are working on discovering more such analogies.

1.1. Canonical metric on graphs.

Let G be a finite simplicial graph and l the length function on its edges. We have the following definitions:

Definition 1.1.

1. *By a* 1*-form on* G *we mean an element in the space of first cellular cochains* $C^1(G)$*. A norm on the space of* 1*-forms can be defined as* $\|w\| = \left(\sum_{e\in E(G)} \frac{w(e)^2}{l(e)}\right)^{1/2}$.
2. *We call a* 1*-form harmonic, if it is locally the coboundary of a graph harmonic function. More specifically, if for every vertex* v *in* G*, for all the outgoing edges* e_i *from* v*,* $\sum_i \frac{w(e_i)}{l(e_i)} = 0$.
3. *The canonical, or Arakelov, metric* [Z, BF] *on a metric graph is defined as follows: for every edge* e*, the length of* e *under the new metric is:*

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

One of our motivation is the following classical result by Kazhdan:

Theorem 1. [K], *also cf.* [M, Appendix] *Let* $S \leftarrow S_1 \leftarrow S_2 \dots$ *be a tower of finite regular covers of a compact Riemann surface* S*,* $\cap_i \pi_1(S_i) = \{1\}$*, then the canonical metric on* S_i *are pullbacks of metric* d_i *defined on* S*, and* d_i *converges uniformly to a multiple of the hyperbolic metric. Here, the canonical metric on a compact Riemann surface* S *is defined as:*

$$\|v\| = \sup_{\|w\|=1, w \text{ holomorphic 1-form on } S} w(v)$$

Where v *is any tangent vector.*

Farbod Shokrieh and I proved the following analogy of Kazhdan's theorem in the case of finite metric graphs:

Theorem 2. [SW] *Let* $G \leftarrow G_1 \leftarrow G_2 \dots$ *be a tower of finite regular covers of a finite metric graph* G*, then the canonical metric on* G_i *are pullbacks of metrics* d_i *defined on* G*, and* d_i *converges uniformly to some limiting metric that depends only on* G *and* $\cap_i \pi_1(G_i)$*. More precisely, let* $G \leftarrow G'$ *be the regular cover defined by* $\cap_i \pi_i(G_i)$*, then the limiting metric pulls back to the canonical metric on* G'.

The main ingredients of the argument is Lück's approximation theorem [LS]. The argument can be easily generalized to other settings - e.g. Riemann surfaces, compact Riemannian manifolds or simplicial complexes with piecewise Euclidean structures, with suitably defined concept of canonical metric. One such generalization is outlined in [BSW].

For Riemann surfaces, the limit of the canonical metric for a tower of covers that converges to the universal cover is the hyperbolic metric. Hence, one may want to see the limit of the canonical metric of a tower of covers as the analogy of the hyperbolic metric for a metric graph. However, it is not known what kind of metrics may be such a limiting metric, and also when would the limiting metric be identical to the original metric. An observation by Huiping Pan is that if one use the limiting metric as the new l, and go through the process of calculating canonical metric on a tower of regular cover again and again, eventually the metric will converge. My collaborators and I are working on answering these remaining questions, as well as generalizing the result to the case of complex manifolds.

As a related question, there is a τ-invariant defined in [C, CR, Z] which is conjectured to be bounded from below by $1/108$ of the the total edge lengths of the graph [C]. Many cases for this conjecture have been shown by Cinkir, and Farbod Shokrieh and I are currently working on proving or disproving this conjecture.

1.2. **Spectral radius of the train track map on homology.** In [M], McMullen proved the following:

Theorem 3. [M] *Let ϕ be a pseudo-Anosov map on Riemann surface. Let λ be the stretch factor of ϕ on the invariant measured foliations, ρ be the size of the leading eigenvalue of the action of ϕ on the homology of the surface. Then, either $\lambda = \phi$ when passing to a double cover, or there is some $\epsilon > 0$ so that when passing to any finite cover of the surface, $\lambda - \rho > \epsilon$.*

A conjecture by Thomas Koberda asserts that there is an analogous result for graph homotopy equivalences that admits an irreducible train-track representation. More precisely, let ψ be a train track map on a finite graph, with irreducible incidence matrix. Then, either after passing to some finite cover the stretch factor is identical to the spectral radius of the induced map on homology, or there is some $\epsilon > 0$ such that when passing to arbitrary finite cover, the stretch factor is at least ϵ above the spectral radius of the induced map on homology. Farbod Shokrieh and I are working on proving or disproving this conjecture.

2. Relating the topology of hyperbolic 3-manifolds and the asymptotic translation length of its monodromies on curve complexes

Let S be a surface of finite type. The *curve graph* of S, denoted as $\mathcal{C}(S)$, is the graph where the vertices are isotopy classes of simple closed curves on S and there is an edge between two vertices if they have disjoint representations. A metric can be assigned to the curve graph by setting all edge lengths to be 1. The mapping class group of S acts isometrically on S, and the *asymptotic translation length* of a mapping class g on $\mathcal{C} = \mathcal{C}(S)$ is

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$. $l_{\mathcal{C}}$ can be seen as a possible way to characterize the "combinatorial complexity" of the pseudo-Anosov map.

A motivating question of this project is to understand the relationship between the asymptotic translation lengths on curve graphs and group theoretic properties of elements in the mapping class group.

It is shown in [MM] that the curve graph is δ-hyperbolic, and that $\ell_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov (not of finite order, and doesn't preserve any finite collection of disjoint simple closed curves up to isotopy). The technique in [MM] have been used by [GT, V, BSh] and others to provide asymptotics of the lower bound on $\ell_{\mathcal{C}}$ as the genus and number of punctures on S increases. Furthermore, in [BLM] it is shown that there is a relationship between $\ell_{\mathcal{C}}$ and the translation length $\ell_{\mathcal{T}}$ on the Teichmüller space under Teichmüller metric, which is that when $exp(\ell_{\mathcal{T}}) \leq g - 1/2$,

$$\ell_{\mathcal{C}} \leq \frac{4\ell_{\mathcal{T}}}{\log(g - 1/2)}$$

In [KS], a sequence of pseudo-Anosov maps in different genus is constructed that realizes the asymptotic lower bound. The construction arises from an arithmetic sequence in a fibered cone of a hyperbolic 3-manifold. The concept of fibered cones is introduced in [T2]:

Definition 2.1. [T2] *Let M be a closed hyperbolic 3-manifold. A* fibered cone *is a rational cone (cone defined by linear inequalities with rational coefficients) of $H^1(M;\mathbb{Z})$, such that all integer cohomology* *class in it are pull backs of the generator of $H^1(S^1;\mathbb{Z})$ for some map $M \to S^1$ which is a surface bundle over the circle, which is maximal under containment.*

Generalizing the construction in [KS], Hyungryul Baik, Hyunshik Shin and I proved the following:

Theorem 4. [BShiW] *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone* [T2] *in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,*

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

The main tool for this theorem is McMullen's Teichmüller polynomial [M2] which encodes the stretch factors of the monodromies of different fiberations of the same 3-manifold.

In a follow-up paper [BKSW], Hyungryul Baik, Eiko Kin, Hyunshik Shin, and I showed that this asymptotic upper bound is sharp when $r \leq 3$.

There is also the question of how constraints on the dimension of invariant cohomology may affect the minimal asymptotic translation length. For example, in [BSh], it is shown that

Theorem 5. [BSh] *The minimal asymptotic translation length on the curve graph of a pseudo-Anosov map in the Torelli group for closed surfaces of genus g grows at $O(1/g)$ as genus g increases.*

In a upcoming paper, together with my collaborators, we proved the following:

Theorem 6. *Let $L_c(k,g)$ be the minimal asymptotic translation length on curve graph of a pseudo-Anosov map on closed surface of genus at most g that preserved a subset of cohomology of dimension at least k. Then*

$$\frac{1}{g(2g-k+1)} \lesssim L_c(k,g) \lesssim \frac{k+1}{g^2}$$

With my collaborators, I am working on strengthening the above-mentioned results in [BKSW] as well as Theorem 6, and possibly generalizing them to the case of $Out(F_n)$ acting on free factor complex [BFe] or the setting of hierarchical hyperbolic spaces [BHS, MM2].

Furthermore, in [BKSW], we proved the following:

Theorem 7. [BKSW] *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimension 2 that has non empty intersection with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.*

We are currently working to remove the condition that the dimension of L be 2.

3. Thurston's "master teapot" for interval maps of constant slope.

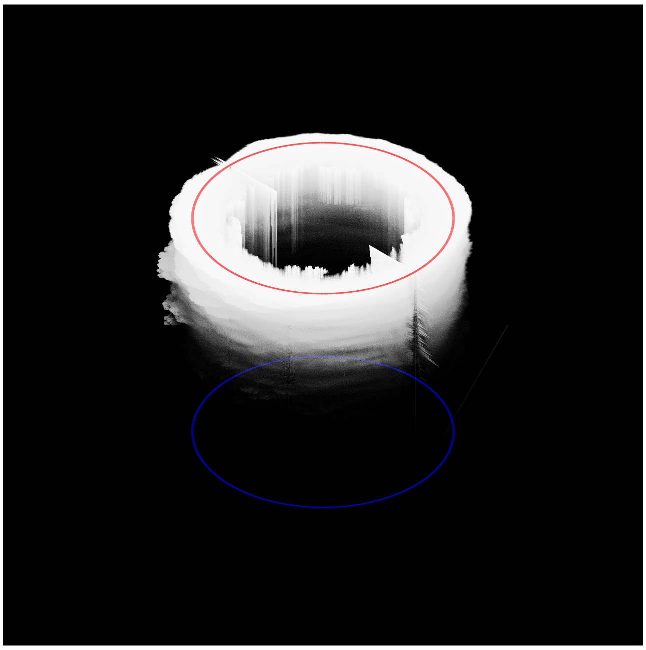
The dynamics of unimodal maps on intervals is a classical subject that has important connections with ergodic theory, symbolic dynamics and complex dynamics. A main tool for studying interval maps is the kneading theory of Milnor-Thurston [MT], which says that the dynamic of such a map is semiconjugate to both a real quadratic map and a subshift. For any $\lambda \in (1,2)$, let the tent map $f_\lambda : [0,1] \to [0,1]$ be defined as

$$f_\lambda(x) = \begin{cases} \lambda x & x \leq 1/\lambda \\ 2-\lambda x & x > 1/\lambda \end{cases}$$

In [T], Thurston proposed the "master teapot" which is a closed subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, defined as follows:

$$T = \overline{\{(z,\lambda) : z \text{ is a Galois conjugate of } \lambda, \text{ for which there exists } n \in \mathbb{N},\ f_\lambda^{\circ n}(1) = 1\}}$$

This is a picture of the Thurston's Teapot:



Using kneading theory and the idea of "dominant words" in [Ti], Harrison Bray, Diana Davis, Kathryn Lindsey and I showed the following:

Theorem 8. [BDLS] *If $|z| = 1$ then $z \times [1, 2] \subset T$. If $(z, \lambda) \in T$, $|z| \leq 1$, then $z \times [\lambda, 2] \subset T$.*

In order to state our characterization of the master teapot, we introduce the following definitions:

Definition 3.1.

(i) Let $\lambda \in (1, 2)$. The itinerary of $x \in [0, 1]$ under f_λ, denoted as $it_\lambda(x)$, is the infinite string of 0 and 1, where the i-th entry is 1 iff the $i-1$-th iteration of x under f_λ is in $(1/\lambda, 1]$.
(ii) If s is an infinite string, $\sigma(s)$ is s with the first letter removed.
(iii) If $w = w_1w_2 \dots w_n$ is a finite word of length n consisting of 0 and 1, $F(z, w) = F_z^{w_1} \cdot F_z^{w_2} \dots F_z^{w_n}(1)$, here $F_z^0(x) = x/z$, $F_z^1(x) = (2 - x)/z$.
(iv) We call a finite word w to be λ-suitable, if
  (a) For any $k \in \mathbb{N}$, $\sigma^k(w^\infty)$ is the itinerary of some $x \in [0, 1]$ under f_λ.
  (b) If a suffix of w is the same as a prefix of $it_\lambda(1)$, there are odd number of 1s in this suffix.

In a paper which will be posted soon, Kathryn Lindsey and I will show this full characterization of Thurston's teapot:

Theorem 9. *$(z, \lambda) \in T$ iff one of the following is true:*

(i) $|z| = 1$, $\lambda \in [1, 2]$
(ii) $|z| > 1$, and $1/z$ is a root of the kneading power series corresponding to λ.
(iii) $|z| < 1$, $\lambda \in [\sqrt{2}, 2]$, and there is some $R > 0$, such that for any $\lambda' > \lambda$ there are infinitely many λ'-suitable words w such that $|F(z, w)| < R$
(iv) There is some k such that (z^{2^k}, λ^{2^k}) is in one of the three previous cases.

Kathryn Lindsey and I are currently working on to generalize this description to the case of more complicated interval maps. In particular, we will be considering this kind of interval maps (here $\lambda > 1$):

$$h_\lambda = \begin{cases} \lambda x - 2n & x \in [2n/\lambda, (2n + 1)/\lambda] \\ 2n + 2 - \lambda x & x \in [(2n + 1)/\lambda, (2n + 2)/\lambda] \end{cases}$$

Based on numerical evidence, we conjectured that the analogy of Theorems 8 and 9 should be true for this new set

$$U = \overline{\{(z, \lambda) : z \text{ is a Galois conjugate of } \lambda, \text{ for which there exists } n \in \mathbb{N}, h_\lambda^{\circ n}(1) = 1\}}$$

Kathryn Lindsey and I are currently work to prove the analogy of Julia-Mandelbrot correspondance (which assers that locally, the Mandelbrot set resembles the corresponding Julia set) in the setting of Thurston teapots give a better explanation of the shape of the part of T and U inside the unit cylinder $\mathbb{D}_1 \times \mathbb{R}$.

4. Borel-Serre type compactification of the moduli space of translation surfaces

A finite translation surface is a holomorphic differential on a closed Riemann surface. The set of translation surfaces where the degrees of zeros are fixed, indicated by a vector k, form an affine manifold under the "period coordinates", denoted as $\mathcal{H}(k)$ cf. [Wr]. There is a $GL(2,\mathbb{R})$ action on $\mathcal{H}(k)$ defined by

$$\begin{pmatrix} Re(Aw) \\ Im(Aw) \end{pmatrix} = A \begin{pmatrix} Re(w) \\ Im(w) \end{pmatrix}$$

This $GL(2,\mathbb{R})$ action is called the *affine action*. The action of the subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is called the *horocycle action*. The study of these two actions are connected to many questions in dynamics, such as the study of polygonal billiards and interval exchange maps. It was established in the seminal paper [EMM] that $GL(2,\mathbb{R})$ orbit closures of $\mathcal{H}(k)$ are affine submanifolds. In contrast, the recent work by Chaika-Smillie-Weiss shows that the horocycle closures are generally not submanifolds at all.

Many approaches have been proposed so far to give a compactification for the moduli space of finite translation surfaces $\mathcal{H}(k)$. For example, the "WYSIWYG" bordification in [MW] has been used to study the dynamics of affine invariant subspaces in such moduli spaces, and there is a compactification proposed in [BCGGM] that is known to have good properties and is particularly useful in the study of the geometry of such moduli spaces. Motivated by the Borel-Serre compactification [BS] and the Bestvina-Horbez compactification of outer space [BH], John Smillie and I are working on a bordification of the moduli space of translation surfaces as a (real) orbifold with corners (a space where around every point, there is a local coordinate chart which is a product of open and half open intervals). A points on this bordifications can be described with the following data:

- A *component diagram*, which is a finite graph G with some half edges which is divided into levels, such that every edge between levels is assigned a non-zero integer, every edge within a level is assigned 0, and the half edges are assigned numbers which are the entries of the vector k plus 1.
- A set of meromorphic differentials on possibly disjoint Riemann surfaces assigned to each level of G, such that each connected component of the Riemann surface corresponds to a vertex in G, each zero, pole or marked point correspond to an edge or a half edge, with turning angles determined by the assigned numbers, and the residue of the poles satisfy certain linear conditions.
- For each pair of matching poles or pole and zero, a gluing map along the boundary created via real oriented blowups.

Together with John Smillie, I am investigating the properties of this bordification and its application to study the fundamental groups of moduli spaces of translation surfaces and the geometry and dynamics of horocycle flows.

5. Developing a Bowen-Series type coding for Kleinian groups

Bufetov-Klemenko-Series [BKS] described a coding of elements of a Fuchsian group by admissible sequences of a geometric Markov chain (i.e. paths on a finite directed graph from a given set of vertices (the initial states) to a given set of vertices (the final states)), which is motivated by Bowen-Series [BowS] and Wroten [Wro]. This coding is bijective, and the finite directed graph used in the coding admits an involution that exchanges the inital and final states and is compatible with the map $g \mapsto g^{-1}$. Bufetov, Klemenko, and Series used this coding map to show the a.s. and L^2 convergence of spherical averages of the Fuchsian group action on functions on Lebesgue probability spaces. Together with Bufetov and Klemenko, we are able to generalize this Wroten-style coding to the case of cocompact Kleinian groups and prove the following:

Theorem 10. *If G is a Kleinian group with a compact, polygonal fundamental domain D, such that the 2-skeleton of the dual complex of the tiling of $\mathbb{H}^3$ by the G-orbit of D is a $C(4)-T(4)-P$ small cancellation complex* [GS]*, then elements of G admits a Wroten-style coding with an involution as in* [BKS]*. Let G_D be the graph whose vertices are the faces of D, where two are connected iff the corresponding face are not adjacent. Let F_D be the graph whose vertices are the faces of D, where two are connected iff the corresponding face are adjacent. If G_D is connected, and the 1-neighborhood of every subset of F_D of diameter 1 does not cover all vertices, then the graph representing the geometric Markov chain used in this coding is strongly connected.*

We are working on relaxing the assumptions on G in the theorem above, as well as showing further properties of this coding in order to obtain results on ergodicity as in [BKS].

6. Prior Publications

The followings are the publications resulting from my prior projects.

- C. Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics* 214(2):733-740, 2016.
- C. Wu. Lattice surfaces and smallest triangle. *Geom Dedicata*, 2016.
- C. Wu. The relative cohomology of abelian covers of the flat pillowcase. *Journal of Modern Dynamics* 9:123-140, 2015.
- L. Clavier, A. Randecker, and C. Wu. Rotational component spaces for infinite-type translation surfaces. *Geometriae Dedicata* 201(1), 57-80, 2019.
- F. Shokrieh and C. Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019.
- H. Baik, A. Rafiqi, and C. Wu. Constructing pseudo-Anosov maps with given dilatations. *Geometriae Dedicata* 180(1):39-48, 2016
- H. Baik, A. Rafiqi, and C. Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019.
- H. Baik, F. Shokrieh, and C. Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019.

References

[B] V. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. Number 33. American Mathematical Soc, 2012.

[BCGGM] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and Möller. Compactification of strata of abelian differentials. *Duke Mathematical Journal* 167(12):2347-2416, 2018.

[BDLS] H. Bray, D. Davis, K. Lindsey, and C. Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805

[BF] M. Baker and X. Faber. Metric properties of the tropical Abel-Jacobi map. *Journal of Algebraic Combinatorics*, 33(3):349–381, 2011.

[BFe] M. Bestvina and M. Feighn. Hyperbolicity of the complex of free factors. *Advances in Mathematics* 256:104–155, 2014.

[BH] M. Bestvina and C. Horbez. A compactification of outer space which is an absolute retract. *arXiv*:1512.02893, to appear in Annales de l'Institut Fourier.

[BHS] J. Behrstock, M. Hagen, and A. Sisto. Hierarchically hyperbolic spaces, i: Curve complexes for cubical groups. *Geometry & Topology* 21(3):1731–1804, 2017.

[BKS] AI. Bufetov, A. Klimenko, and C. Series. Convergence of spherical averages for actions of Fuchsian Groups. *arXiv*:1805.11743, 2018.

[BKSW] H. Baik, E. Kin, H. Shin, and C. Wu. Asymptotic translation lengths and normal generations of pseudo-Anosov monodromies for fibered 3-manifolds. *arXiv*:1909.00974.

[BLM] B. Farb, CJ. Leininger, and D. Margalit. The lower central series and pseudo-Anosov dilatations. *American journal of mathematics* 130(3): 799-827, 2008.

[BRW] H. Baik, A. Rafiqi, and C. Wu. Constructing pseudo-Anosov maps with given dilatations. *Geometriae Dedicata* 180(1):39-48, 2016

[BRW2] H. Baik, A. Rafiqi, and C. Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019.

[BS] A. Borel and JP. Serre. Corners and arithmetic groups. *Commentarii Mathematici Helvetici* 48(1):436-91, 1973.

[BSh] H. Baik and H. Shin. Minimal Asymptotic Translation Lengths of Torelli Groups and Pure Braid Groups on the Curve Graph. *International Mathematics Research Notices*, 2018.

[BSW] H. Baik, F. Shokrieh, and C. Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019.

[BShiW] H. Baik, H. Shin, and C. Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces. *arXiv*:1801.06638, 2018.

[BowS] R. Bowen and C. Series. Markov maps associated with Fuchsian groups. *Publications Mathematiques de l'IHES* 50:153-70, 1979

[BW] H. Baik, C. Wu, K. Kim, and T. Jo. An algorithm to compute the Teichmüller polynomial from matrices. *Geometriae Dedicata*, 2019.

[C] Z. Cinkir. The tau constant of a metrized graph and its behavior under graph operations. *The Electronic Journal of Combinatorics* 18(1):81, 2011.

[CR] T. Chinburg and R. Rumely. The capacity pairing. *J. reine angew. Math*, 434(1):44, 1993.

[CV] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae* 84(1):91–119, 1986.

[EMM] A. Eskin, M. Mirzakhani, and A. Mohammadi. Isolation, equidistribution, and orbit closures for the $SL(2,\mathbb{R})$ action on Moduli space. *Journal of modern dynamics*, 5(2):319–353, 2011.

[GS] SM. Gersten and HB. Short. Small cancellation theory and automatic groups. *Inventiones mathematicae* 102(1):305-34, 1990.

[GT] V. Gadre and C. Tsai. Minimal pseudo-anosov translation lengths on the complex of curves. *Geometry & Topology*, 15(3):1297–1312, 2011.

[K] D. Kazhdan. On arithmetic varieties. Lie groups and their representations, *Halsted*, pp. 151–216, 1975.

[KS] E. Kin and H. Shin. Small asymptotic translation lengths of pseudo-anosov maps on the curve complex. *Groups, Geometry, and Dynamics* 2019.

[LS] W. Lück and T. Schick. Approximating L^2-signatures by their compact analogues. *Forum Mathematicum*, 17: 31–65, 2005.

[M] CT. McMullen. Entropy on riemann surfaces and the Jacobians of finite covers. *Commentarii Mathematici Helvetici* 88(4):953–964, 2013.

[M2] CT. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. *Annales scientifiques de l'Ecole normale supérieure.* 33(4):519-560. 2000.

[MM] HA Masur and YN Minsky. Geometry of the complex of curves i: Hyperbolicity. *Inventiones mathematicae* 138(1):103–149. 1999.

[MM2] HA Masur and YN Minsky. Geometry of the complex of curves ii: Hierarchical structure. *Geometric and Functional Analysis* 10(4):902–974, 2000.

[MT] J. Milnor and W. Thurston. On iterated maps of the interval. *Dynamical systems* pp. 465-563, 1988.

[MW] M. Mirzakhani and A. Wright. The boundary of an affine invariant submanifold. *Inventiones mathematicae* 209(3):927-84, 2017.

[SW] F. Shokrieh and C. Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019.

[Ti] G. Tiozzo, Galois conjugates of entropies of real unimodal maps, *International Mathematics Research Notices*, 2018

[T] W. Thurston. Entropy in dimension one. *arXiv*:1402.2008, 2014.

[T2] W. Thurston. A norm for the homology of 3-manifolds *Mem. Amer. Math. Soc.* 59 (33): 99–130, 1986.

[V] AD Valdivia. Asymptotic translation length in the curve complex. *New York Journal of Mathematics*, 20, 2014.

[Wr] A. Wright. Translation surfaces and their orbit closures: An introduction for a broad audience. *EMS Surveys in Mathematical Sciences* 2(1):63-108, 2015.

[Wro] M. Wroten. The eventual Gaussian distribution for self-intersection numbers on closed surfaces. *arXiv*:1405.7951.

[Z] S. Zhang. Admissible pairing on a curve. *Inventiones mathematicae*, 112(1):171–193, 1993.