Kazhdan's theorem for canonical metric on graphs

Chenxi Wu

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Joint work with Farbod Shokrieh

Notations

- A Riemann surface is a connected complex manifold of dimension 1.
- Any connected oriented surface with Riemannian metric is a Riemann surface
- **Conformal** means that angles are unchanged.
- Ω^1 is the space of holomorphic 1-forms.
- $X' \to X$ is called a **regular covering**, if there is some group G acting freely on X' such that $X = X'/G$.
- If X is a connected manifold, there is a regular covering which is simply connected, called the **universal cover**.

Motivation

Uniformization theorem [Poincaré, Koebe, 1907]:

Any simply connected Riemann surface is conformal to $\mathbb{C}$, $\overline{\mathbb{C}}$, or the unit disc.

- Any Riemann surface S has a conformal metric with constant curvature.
- When S is other than $\mathbb{C}$, $\overline{\mathbb{C}}$, $\mathbb{C}\backslash\{p\}$, annulus or torus, the uniformization metric has constant negative curvature. We call such S **hyperbolic**
- Kazhdan's theorem gives an explicit way to obtain this uniformization metric, via the canonical metric for some regular coverings.

Kazhdan's theorem for Riemann surfaces

- S: a compact Riemann surface
- $\Omega^1(S)$: Space of holomorphic 1-forms on S
- $\{\omega_i\}$: Orthonormal basis of $\Omega^1(S)$ ($\langle u, v\rangle = \frac{1}{2\sqrt{-1}}\int \overline{u} \wedge v$)
- $d_c^S = \sum_i |\omega_i|^2$; d_c^S is called the **Canonical** or **Arakelov** metric;
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$: infinite tower of finite regular covers, $\cap_i \pi_1(S_i) = 1$
- d_i: Riemannian metrics on S whose pull-back on S_i are the $d_c^{S_i}$

Theorem [Kazhdan, 70s]

If S is hyperbolic, d_i converges uniformly to a multiple of the uniformization metric.

Canonical metric on graphs

$G = \{V(G), E(G), l\}$: a finite metric graph

- $E(G)$: directed edges.
- $\overline{e}$: the opposite of $e \in E(G)$
- $e \in E(G) \iff \overline{e} \in E(G)$
- $l : E(G) \to \mathbb{R}^+$: the edge-length function.
- $l(e) = l(\overline{e})$ for all $e \in E(G)$

- $C^1(G) = \{\alpha \in Map(E(G), \mathbb{R}) : \alpha(e) = -\alpha(\overline{e})\}$: space of simplicial 1-cochains.
- $C^0(G) = Map(V(G), \mathbb{R})$: space of simplicial 0-cochains.
- Inner product on $C^1(G)$: $(\alpha, \beta) = \frac{1}{2} \sum_{e \in \mathcal{O}} \frac{\alpha(e)\beta(e)}{l(e)}$.
- Inner product on $C^0(G)$: $(\alpha, \beta) = \sum_{v \in V(G)} \alpha(v)\beta(v)$.
- $d : C^0(G) \to C^1(G)$: the coboundary map, $d(\alpha)(e) = \alpha(e^+) - \alpha(e^-)$ for all $e \in E(G)$.
- $\delta = d^* : C^1(G) \to C^0(G)$

All these definitions works for infinite graphs when C^1 and C^0 are replaced by L^2 summable forms.

- $\mathcal{H}(G) = \{\alpha \in C^1(G) : \delta\alpha = 0\}$: the space of harmonic-forms on G.
- Explicit description: $\forall v \in V(G)$, $\sum_{e\in\mathcal{O}, e^+=v} \frac{\alpha(e)}{l(e)} = 0$.

Definition of canonical metric for a finite metric graph

(Zhang 93, Baker-Farber 11, Chinburg-Rumely 93 et al): Given finite metric graph $G = (E(G), V(G), l)$,

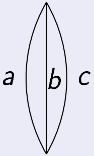
$$\begin{aligned} l_c^G(e) &= \frac{1}{l(e)} \sup_{\|\alpha\|\le 1, \alpha\in\mathcal{H}} |\alpha(e)| \\ &= \sum_i \omega_i^2(e) \end{aligned}$$

Where $\{\omega_i\}$ is orthonomal basis of $\mathcal{H}$.

Interpretation: counting of spanning tree

- $\mathcal{T} = \{T\}$: Set of spanning tree of G. A spanning tree is a subgraph with no loops and contains all vertices of G.
- Weight of a tree: $w(T) = \prod_{e \notin T} l(e)$
- $l_c^G(e) = \frac{\sum_{T \in \mathcal{T}, e \notin T} w(T)}{\sum_{T \in \mathcal{T}} w(T)}$ (Foster)

Example:



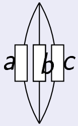
$$l(a) = 2,\ l(b) = l(c) = 1$$

$$l_c^G(a) = \frac{l(a)l(c) + l(a)l(b)}{l(a)l(c) + l(a)l(b) + l(b)l(c)} = \frac{4}{5}, l_c(b) = l_c(c) = \frac{3}{5}$$

Interpretation: Network of resistors

Example, cont.

Turn the graph into resistor network:



Let R be effective resistance between the two vertices, by parallel law:

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{1} + \frac{1}{1} = \frac{5}{2}$$

$$I_c(a) = 1 - R/2 = 4/5, I_c(b) = I_c(c) = 1 - R/1 = 3/5$$

- For every $e \in E(G)$, $R(e)$ is the effective resistance between e^+ and e^-.
- $I_c^G = 1 - \frac{R(e)}{l(e)}$, which is also called **Foster's coefficient**.

- The equivalence between effective resistance interpretation and harmonic 1-form interpretation:

Harmonic analysis on graph	Resistor network
1-form	Current distribution
$\alpha(e)$	Potential between e^+ and e^-
Norm on C^1	Energy
Harmonicity of 1-form	Kirchhoff's first law

- There are other interpretations of canonical metric of finite graphs, for example:
  - As pull back from graph Jacobian metric (Baker-Farber)
  - As limits of Weierstrass points in Berkovich spaces (Amini).

Statement of our result

Theorem [Shokrieh, W]

- G: finite connected metric graph
- $G \leftarrow G_1 \leftarrow G_2 \leftarrow \dots$: tower of finite regular covers $(\pi_1(G_i) \lhd \pi_1(G))$
- $\pi_i : G_i \to G$: covering map.
- $l_i : E(G) \to \mathbb{R}$, such that $\pi_i^* l_i = l_c^{G_i}$

Then $\lim_{n\to\infty} l_i$ exists, which depends only on G and $\cap_i(\pi_1(G_i))$.

When $\cap_i(\pi_1(G_i)) = \{1\}$, we can think of the limiting metric l_∞ on G as a candidate of the uniformization metric. l_∞ may be 0 on some edges.

Kazhdan's theorem for Riemann surfaces

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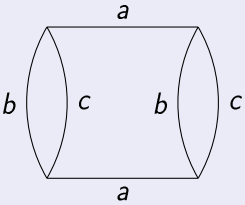
- $\{\omega_i\}$: Orthonormal basis of $\Omega^1(S)$ ($\langle u, v\rangle = \frac{1}{2\sqrt{-1}} \int \overline{u} \wedge v$)
- $d_c^S = \sum_i |\omega_i|^2$
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \ldots$: infinite tower of finite regular covers, $\cap_i \pi_1(S_i) = 1$
- d_i: Riemannian metrics on S whose pull-back on S_i are the $d_c^{S_i}$

Theorem [Kazhdan, 70s]

If S is hyperbolic, d_i converges uniformly to a multiple of the uniformization metric.

Example, cont.

Consider this double cover G_1:



$l_c^{G_1}(a) = 2/5$, $l_c^{G_1}(b) = l_c^{G_1}(c) = 11/20$. If we make a tower of coverings such that $\cap_i \pi_1(G_i) = \{1\}$, the limiting metric is $l_\infty(a) = \frac{11-\sqrt{41}}{10}$, $l_\infty(b) = l_\infty(c) = \frac{\sqrt{41}-1}{20}$, which **doesn't depend on the choice of the tower of coverings**.

Generalization and remaining problems

- Our proof is based on L^2 techniques, hence our theorem can be easily generalized to the following cases
  - Compact Riemann surfaces
  - Riemannian manifolds
  - Compact flat surfaces with Delaunay triangulation
  - ...
- In the graph case, when $\cap_i \pi_1(G_i) = \{1\}$, we have an algorithm to calculate the limiting metric, and the limiting metric can be interpreted via equilibrium measure on ∂ of universal cover $\overline{G}$.
- It is unknown how to calculate the limiting metric efficiently in other cases, or what properties they would have.

Riemann surfaces

- S: a compact Riemann surface
- $\Omega^1(S)$: Space of holomorphic 1-forms on S
- $\{\omega_i\}$: Orthonormal basis of $\Omega^1(S)$ ($\langle u, v\rangle = \frac{1}{2\sqrt{-1}} \int \overline{u} \wedge v$)
- $d_c^S = \sum_i |\omega_i|^2$
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$: infinite tower of finite regular covers.
- d_i: Riemannian metrics on S whose pull-back on S_i are the $d_c^{S_i}$

Theorem [Baik-Shokrieh-W]

d_i converges uniformly as a tensor.

Riemannian manifolds

- M: a compact Riemannian manifold
- $\mathcal{H}^1(M)$: Space of harmonic 1-forms on M
- $\{\omega_i\}$: Orthonormal basis of $\mathcal{H}^1(M)$
- $d_c^M = \sum_i |\omega_i|^2$
- $M \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$: infinite tower of finite regular covers.
- d_i: Riemannian metrics on M whose pull-back on M_i are the $d_c^{M_i}$

Theorem

d_i converges uniformly as a tensor.

Piecewise Euclidean surfaces

- X: a closed flat surface with finitely many cone points, with a Delaunay triangulation $\mathcal{T}$.
- $\mathcal{H}(X)$: Space of discrete harmonic 1-forms on X, defined using the cotangent formula.
- $l_c^X(e) = \frac{1}{l(e)} \sup_{\|\alpha\| \leq 1, \alpha \in \mathcal{H}} |\alpha(e)|$, the norm is also defined using cotangent formula.
- $X \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$: infinite tower of finite regular covers.
- l_i: Functions on edges of X whose pull-back to X_i are the $l_c^{X_i}$.

Theorem

l_i converges on all edges, and the limit satisfies the triangle inequality.

Step 1: Find the limiting metric

Let G' be the cover of G corresponding to $\cap_i(\pi_1(G_i))$.
Define:

$$l_c^{G'} = \frac{1}{l(e)} \sup_{\|\alpha\| \leq 1, \alpha \in \mathcal{H}_{L^2}} |\alpha(e)|$$

Here $\mathcal{H}_{L^2}$ is the space of harmonic 1-forms with finite L^2 norm.
Then l_∞ pulls back to $l_c^{G'}$ on G'.

Step 2: Upper bound

For each $e \in E(G)$, let e_i be a lift on G_i, e' a lift on G'.
Let $B_{G_i}(e_i, R)$ and $B_{G'}(e', R)$ be the R-neighborhood of e_i and e'. Because $\pi_1(G') = \cap_i \pi_1(G_i)$, for large enough i these two are isometric. Let $B'(R)$ be these two neighborhoods with their boundaries collapsed to a single point.
By the effective resistance interpretation,

$$l_i(e) = l_c^{G_i}(e_i) \leq l_c^{B'(R)}(e_i) = l_c^{B'(R)}(e')$$

However

$$\lim_{R \to \infty} l_c^{B'(R)}(e') = l_c^{G'}(e') = l_\infty(e)$$

Hence

$$\limsup_{i \to \infty} l_i(e) \leq l_\infty(e)$$

Lower bound via Lück's approximation

Theorem [Lück]

X: CW complex with group Γ-action which is free and cellular, X/Γ finite, $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots$ finite index normal subgroups of Γ, $\cap_i \Gamma_i = 1$. Then

$$\lim_{i\to\infty} \frac{b_j(X/\Gamma_i)}{[\Gamma : \Gamma_i]} = b_j^{L^2}(X)$$

(b_j: j-th Betti number. $[\Gamma : \Gamma_j]$ index of Γ_j as a subgroup of Γ.)

$$\sum_{e\in E(G)} l_i(e) = \frac{2b_1(G_i)}{[\pi_1(G) : \pi_1(G_i)]}$$

$$\sum_{e\in E(G)} l_\infty(e) = 2b_1^{L^2}(G')$$

Proof of our theorem

From step 2:

$$\limsup_{i\to\infty} l_i(e) \le l_\infty(e)$$

From step 3:

$$\lim_{i\to\infty} \sum_{e\in E(G)} l_i(e) = \sum_{e\in E(G)} l_\infty(e)$$

Because $E(G)$ is finite,

$$\lim_{i\to\infty} l_i(e) = l_\infty(e)$$

References

S. Zhang, Admissible pairing on a curve, Invent. Math. 112(1993), no. 1, 171-193

Curtis T. McMullen,Entropy on Riemann surfaces and the Jacobians of finite covers, Comment. Math. Helv. 88(2013), no. 4, 953-964

Wolfgang Lück,Approximating L2-invariants by their finite-dimensional analogues, Geom. Funct. Anal.4(1994), no. 4, 455-481

Farbod Shokrieh and Chenxi Wu, Canonical measures on metric graphs and a Kazhdan's theorem, Invent. Math. (2019), no. 3, 819-862

Hyungryul Baik, Farbod Shokrieh and Chenxi Wu, Limits of canonical forms on towers of Riemann surfaces. Crelle (2019)