The shape of Thurston's Master Teapot

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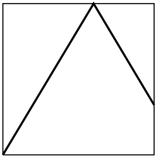
Collaboration with Harrison Bray, Diana Davis, Kathryn Lindsey

Motivating question: Which algebraic numbers can be the exp of the entropy of a unimodal map with periodic critical orbit?

Tent map

- $f_\lambda(x) = \begin{cases} \lambda x & x \in [0, 1/\lambda] \\ 2 - \lambda x & x \in [1/\lambda, 1] \end{cases}.$
- f_λ has **periodic critical orbit** iff $f_\lambda^{\circ n}(1) = 1$.



Theorem [Milnor-Thurston]

Any unimodal map on an interval is semiconjugate to a tent map.

- The topological entropy of f_λ is $\log(\lambda)$.
- If f_λ has periodic critical orbit, $f_\lambda^{\circ n}(1) - 1 = 0$, hence λ is an algebraic integer.

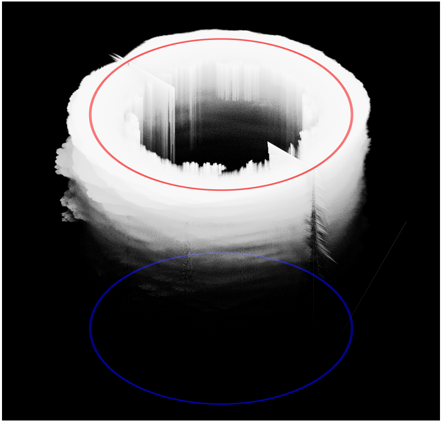
Some remarks:

- Finite critical orbit = superattracting
- When f_λ has finite critical orbit, there is a Markov decomposition and λ is weak Perron (algebraic integers with norm no less than any Galois conjugates)
- For more general interval maps, Thurston proved that all weak Perron numbers can be exp of the entropy.
- For pseudo-Anosov maps, this is conjectured but not known.

Thurston's Teapot

- The Master Teapot is

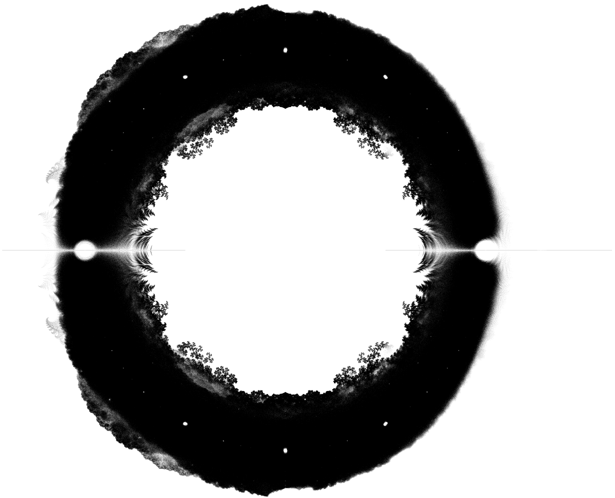
$$T = \overline{\{(z, \lambda) \in \mathbb{C} \times [1,2] : f_\lambda \in \mathcal{P}, z \text{ is a Galois conjugate of } \lambda\}}$$



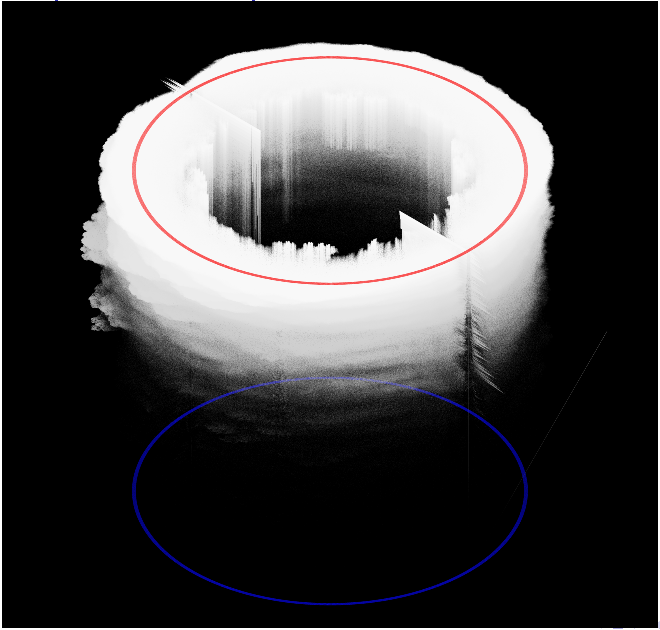
Overview of some prior works

- The "Thurston set" is the projection of the teapot onto $\mathbb{C}$.
- Tiozzo gave a description of the Thurston set. In particular, the part of the Thurston set inside the unit disc is the closure of roots of Littlewood polynomials (polynomials with all coefficients ± 1).
- Many other works have been done on Thurston set or related concepts by Bandt, Lyubich, Parry, Solomyak, Steiner, Thompson, Verger-Gaugry etc.
- Calegai, Koch and Walker proved that there are holes in the closure of roots of all Littlewood polynomials, which provides holes in Thurston set.

Thurston set



Shape of the teapot



Some observed properties of the teapot

1. **Period doubling**: $(z,\lambda) \in T \iff (\sqrt{z}, \sqrt{\lambda}) \in T$
2. **Unit cylinder**: $\{(z,\lambda) : |z| = 1, 1 \le \lambda \le 2\} \subset T$.
3. **“Icicles” inside unit cylinder**: If $|z| < 1$,
   $(z,\lambda) \in T \implies \{z\} \times [\lambda, 2] \subset T$
4. **“hairs” outside unit cylinder**: T outside unit cylinder is the union of countably many curves.
5. **Asymmetry**: Some horizontal slices of T has no left-right symmetry even when restricted to the unit disc.
6. **Non connectedness of the slices**: Some horizontal slices of T are not connected.

1 is classical, 4 is known by Thurston et al, we proved 3 and 5, 2 follows from 1 and 3, and we conjectured that 6 is true.

Statement of the results

Theorem A [Bray-Davis-Lindsey-W]

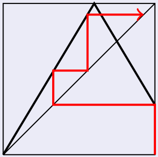
If
$$(z, \lambda) \in T, |z| < 1$$
Then
$$\{z\} \times [\lambda, 2] \subset T$$

- Main Tool: Milnor-Thurston kneading theory + the concept of "dominance strings" by Tiozzo.
- Theorem A together with the path connectedness of Thurston set shows that T is path connected.

Definition

- The **itinerary** of a point x under f_λ is an infinite string $it_\lambda(x) = (a_i) : i \in \mathbb{Z}_{\geq 0}$, such that $a_i = 0$ if $f^{\circ i}(x) \in [0, 1/\lambda)$, $a_i = 1$ if $f^{\circ i}(x) \in (1/\lambda, 1]$.
- k-**prefix** of $a = (a_i) : i \in \mathbb{Z}_{\geq 0}$ is $Pre(a, k) = (a_i) : 0 \leq i \leq k - 1$
- k-**suffix** of $b = (b_i) : i \in \mathbb{Z}_{\leq 0}$ is $Suf(b, k) = (b_i) : -k + 1 \leq i \leq 0$

Example



$it_\lambda(1) = 1001\ldots$

Definition, cont.

- Given $\lambda \in (1,2]$, $b = (b_i) : i \in \mathbb{Z}_{\leq 0}$ is called λ-**suitable**, iff
  - For any k, $Sur(b,k)$ is identical to some $Pre(it_\lambda(x),k)$.
  - If $Sur(b,k) = Pre(it_\lambda(1),k)$, then $\sum_{-k+1\leq i\leq 0} b_i$ is odd.
- Given finite word $w = (w_i)_{a\leq i\leq b}$, $f_{w,\lambda} := f_{w_a,\lambda} \circ \cdots \circ f_{w_b,\lambda}$. Here $f_{0,\lambda}(x) = \lambda x$, $f_{1,\lambda}(x) = 2 - \lambda x$

Theorem B [Lindsey-W]

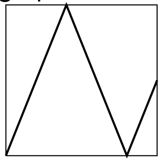
- If $|z| > 1$, $(z,\lambda) \in T$ iff $\lim_{k\to\infty} \frac{1}{z^k} f_{Pre(it_\lambda(1),k),z} = 0$.
- If $|z| < 1$, $\lambda > \sqrt{2}$, $(z,\lambda) \in T$ iff there is some λ-suitable $b = \{b_i\} : i \in \mathbb{Z}_{\leq 0}$, such that $\lim_{k\to\infty} f_{Sur(b,k),z}(1) = 1$

Remarks on Theorem B

- Theorem B together with period doubling gives us an algorithm to determine if a point is not in T. We used this algorithm to show the lack of left-right symmetry of horizontal slices.
- The first part of Theorem B implies the "hairs" outside unit cylinder.
- The second part of Theorem B, together with Milnor-Thurston Kneading theory, implies the "icicles" inside unit cylinder.

Generalization to $\lambda > 2$

Consider f_λ with the following graph:



- We can define T analogously as

$$T = \overline{\{(z, \lambda) : f_\lambda^k(1) = 1, z \text{ is Galois conjugate of } \lambda\}}$$

- Numerical evidence shows that Theorem A and B are likely true in both cases.

Conjectured Julia-Mandebrot correspondence

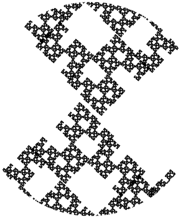
For any $\lambda \in (1,2)$, let

$$T_\lambda = \{z : (z,\lambda) \in T\}$$

be the horizontal slice at height λ. Numerical evidence shows that:

Conjecture

For any $|z| < 1$, The sets $T_\lambda - z$ and $\{\lim_{k\to\infty} f_{Sur(b,k),z}(1) : b \text{ is } \lambda-\text{suitable}\} - 1$ are asymptotically similar at 0.



References

- John Milnor and William P. Thurston. On iterated maps of the interval. Dynamical Systems, 1988
- Giulio Tiozzo. Galois conjugates of entropies of real unimodal maps. IMRN, 2018
- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805
- Kathryn Lindsey and Chenxi Wu. A characterization of Thurston's Master Teapot *arXiv:*1909.10675