1 9/5 Matrices, vectors, and their applications

Algebra: study of objects and operations on them.

Linear algebra: object: matrices and vectors. operations: addition, multiplication etc.

Algorithms/Geometric intuition/sets and maps

$m \times n$ matrix: numbers forming a rectangular grid, m rows and n columns. Motivation: coefficients of a system of linear equations. Data tables in statistics.

(i, j)-th entry of a matrix.

Vectors: matrices with one row/column. Motivation: coordinates in plane and space.

Operations: (1) Addition. (2) Scalar multiplication. (3) Matrix-vector multiplication. (4) Transpose.

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Ax, $A(Ax)$.

Example: Averaging over columns. Covariance? Other statistical concepts?

Laws: The usual laws one may expect. e.g. $A(x + y) = Ax + Ay$, $(A + B)^T = A^T + B^T$, $(A^T)^T = A$. Note: $A(Bx) \neq B(Ax)$!

Zero and one matrix. Standard vectors.

Example: Rotation by 60 degrees (or $\pi/3$).

Consequence: Matrix is completely determined by its action on the standard vectors! Matrix-matrix multiplication.

Example: 2×2 case.
The concept of linear combination. Relationship with matrix-vector multiplication.

Example: Rotation and Translation.

Example: Random walk on graphs.

2 9/8 Linear equations

Review:

- Matrix multiplications
- Transposes
- Standard vectors
- Identity Matrix
- Rotation matrix

- Stochastic matrix

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Linear systems as matrix equations. *Coefficient matrix* and *augmented matrices*

Elementary row operations: swap, multiply, add. Property: reversible, and preserves solution set.

Row echelon form: The first non-zero entry (called pivot) of each row is to the right of the previous. *Reduced row echelon form*: The first non-zero entry is 1 and is the only non-zero entry in that column. Uniqueness under row operations.

Algorithm (Gaussian elimination):

- Write augmented matrix.
- Use row operations, turn it into reduced echelon form.
- General solution from RREF (Example: $x_1 + 2x_2 + x_3 + x_4 = 3$, $x_1 + 3x_3 - x_4 = 8$).

Structure of solutions:

	Pivot at last col.	No pivot at last col.
All coefficient col. have pivot	None	Inf
Some coeff. col. have no pivot	None	One

Examples of the 4 cases.

True or false:

- A system of 3 linear equations with 6 variables can not have just one solution.
- A system of 3 linear equations with 6 variables must have infinitely many solutions.

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Counting: number of arbitrary constants and the number of pivots. Rank and dimension.

Explicit algorithm from RREF to general solutions.