1 9/5 Matrices, vectors, and their applications

Algebra: study of objects and operations on them.

Linear algebra: object: matrices and vectors. operations: addition, multiplication etc.

Algorithms/Geometric intuition/sets and maps

$m \times n$ matrix: numbers forming a rectangular grid, m rows and n columns. Motivation: coefficients of a system of linear equations. Data tables in statistics.

(i, j)-th entry of a matrix.

Vectors: matrices with one row/column. Motivation: coordinates in plane and space.

Operations: (1) Addition. (2) Scalar multiplication. (3) Matrix-vector multiplication. (4) Transpose.

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Ax, $A(Ax)$.

Example: Averaging over columns. Covariance? Other statistical concepts?

Laws: The usual laws one may expect. e.g. $A(x + y) = Ax + Ay$, $(A + B)^T = A^T + B^T$, $(A^T)^T = A$. Note: $A(Bx) \neq B(Ax)$!

Zero and one matrix. Standard vectors.

Example: Rotation by 60 degrees (or $\pi/3$).

Consequence: Matrix is completely determined by its action on the standard vectors! Matrix-matrix multiplication.

Example: 2×2 case.
The concept of linear combination. Relationship with matrix-vector multiplication.

Example: Rotation and Translation.

Example: Random walk on graphs.

2 9/8 Linear equations

Review:

- Matrix multiplications
- Transposes
- Standard vectors
- Identity Matrix
- Rotation matrix

- Stochastic matrix

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Linear systems as matrix equations. *Coefficient matrix* and *augmented matrices*

Elementary row operations: swap, multiply, add. Property: reversible, and preserves solution set.

Row echelon form: The first non-zero entry (called pivot) of each row is to the right of the previous. *Reduced row echelon form*: The first non-zero entry is 1 and is the only non-zero entry in that column. Uniqueness under row operations.

Algorithm (Gaussian elimination):

- Write augmented matrix.
- Use row operations, turn it into reduced echelon form.
- General solution from RREF (Example: $x_1 + 2x_2 + x_3 + x_4 = 3$, $x_1 + 3x_3 - x_4 = 8$).

Structure of solutions:

	Pivot at last col.	No pivot at last col.
All coefficient col. have pivot	None	One
Some coeff. col. have no pivot	None	Inf

Examples of the 4 cases.

True or false:

- A system of 3 linear equations with 6 variables can not have just one solution.
- A system of 3 linear equations with 6 variables must have infinitely many solutions.

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Counting: number of arbitrary constants and the number of pivots. Rank and dimension.

Explicit algorithm from RREF to general solutions.

3 9/12 Linear equations cont.

3.1 Review

- Augmented matrix, row operations.
- RREF.
- Condition for no/one/infinitely many solutions.
- General solution: write *basic variables* in terms of *free variables*, or the *vector form*.

3.2 Gaussian elimination

Augment matrices to REF or RREF through finitely many elementary row operations.

For r=$1, 2, \dots n$:

Find the left-most non-zero entry among the $r, r+1, \dots n$ rows. If there aren't any, terminate.
Exchange rows to move this entry to the r-th row.

Multiply the r-th row and add it to the $r+1, \ldots$ rows to eliminate all entries on the left-most non-zero column.

To Further turn it into a RREF (backward pass):

Multiply to each non-zero row to make the first entry 1.
For each non-zero row, multiply and add it to each of the rows above it to turn the entries on pivot columns 0.

Reason for distinguish forward/backward passes: forward pass is a permutation matrix with a lower triangular matrix with 1 on the diagonals, backward pass is a upper triangular matrix. Row pivoting.

Example: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 4 & 7 \\ 2 & 0 & 1 & 0 \end{pmatrix}$. RREF? General solution?

3.3 Uniqueness of RREF

Key idea: read the RREF from matrix using linear combinations of rows or columns!
Appendix E uses columns. One can also use rows as follows: Let R be the space of linear combination (span) of the row vectors. The last non-zero row in RREF is the one in R with the most number of 0 entries on the left and the first non-zero entry 1. Let the index of the first non-zero entry be c_1. The preceding row in RREF is the one in R with c_1-th entry 0, first non-zero entry 1, and the most possible number of 0 on the left, etc.

3.4 Rank and nullity

Rank of A: num. of pivots in A=num. of non-zero rows in REF of A=num of basic variables in $Ax = b$
Nullity of A: num of non-pivot columns in A=num. of columns of A-rank of A=num of free variables in $Ax = b$

True or false:

- The rank of $[A \ \ B]$ must be no smaller than the sum of the ranks of A and B.
- The nullity of $[A \ \ B]$ must be no smaller than the sum of the ranks of A and B.
- The RREF of a square matrix of no nullity must be the identity matrix
- The nullity of A is non-zero iff some row of A is a linear combination of the others.
- $[A \ \ B]$ has the same rank as B iff the columns of A are linear combinations of the columns of B.

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Structure of the general solution in terms of rank or nullity:
If $rank(A) < rank([A, b])$:
  No solution.
Else:
  If $nullity(A) = 0$:
    One solution.
  Else:
    Infinitely many solutions.

Example: $\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$.

4 9/15 Span

Review:

- Augmented matrix and row operations
- REF, RREF, pivot
- free and basic variables
- Rank and Nullity

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Linear combination: S is a set of matrices of the same size, v is called a linear combination of S iff there exist finitely many matrices $A_1 \dots A_n$ in S, and scalars $a_1, \dots a_n$, so that $v = \sum_k a_k A_k$.

Span: The span of a set is the set of all linear combinations of that set. S is called a generating set of the set $Span(S)$.

Example: span of the standard vectors.

Span closed under addition and scalar multiplication.

Transitivity.

b is in the span of columns of A iff $Ax = b$ has a solution.
$\mathcal{R}^n$: all vectors of n entries. Span is $\mathcal{R}^n$ iff matrix is *full rank* iff ...

Example: use linear equation to detect spans.

Implication on the rank of the matrices while adding columns.

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Algorithm for minimal generating set. Example.

True or false:

Row operation changes the span of the column vectors.

A matrix is in REF, then the span of the columns are the span of some standard vectors.

5 9/18 Linear dependency

5.1 Review

Notation: when A and B has the same number of rows, by $[A \;\; B]$ we mean a larger matrix formed by stacking them together horizontally.

relationship between matrices, system of equations $Ax = b$, and the column vectors:

The followings are equivalent:

- $Ax = b$ has a solution (is **consistent**).
- b lies in the **span** of the columns vectors of A.

- The **span** of the columns of A is the same as the span of the columns of A and b.
- **rank**$([A\ \ b]) = rank(A)$.
- **Nullity**$([A\ \ b]) = Nullity(A) + 1$.
- In the **RREF** of $[A\ \ b]$, the last column does not contain a **pivot**.

Examples.

The followings are equivalent:

- $Ax = b$ has a solution (is **consistent**) for all b.
- The **span** of the columns vectors of A is $\mathcal{R}^m$.
- **rank**$(A) = m$.
- **Nullity**$(A) = n - m$.
- In the **RREF** of A, every row contain a **pivot**.
- The **RREF** of A does not contain zero rows.

Examples.

5.2 Linear dependence/independence

A set S is called **linearly independent**, if for any sequence of distinct elements $x_1, \ldots x_k \in S$, $c_1 x_1 + \ldots c_k x_k = 0$ implies that $c_1 = c_2 = \cdots = 0$. If a set is not linearly independent it is linearly dependent.

$a_1 \ldots a_n$ are linearly dependent if and only if $[a_1 \ldots a_n]x = 0$ (the **homogeneous eq.**) has one (hence infinitely many) non-zero solutions. (hence $[a_1 \ldots a_n]x = b$ has infinitely many solutions for some b, hence has free variables, hence the nullity of A is non-zero).

Example: 1 or 2 vectors.

Linear dependency in standard vectors.

Linear dependency in RREF.

Linear dependency in vector form of the general solution.

5.3 Number of rows and columns

$m > n$: column vectors may or may not be linearly dependent, but can never span $\mathcal{R}^m$.

$m < n$: column vectors may or may not span $\mathcal{R}^m$, but can never be linearly independent.

$m = n$: column vectors span $\mathcal{R}^m$ iff they are linearly independent.

5.4 Adding and removing vectors

If S is linearly independent, any subset of S is linearly independent and has a smaller span, $S \cap \{v\}$ is linearly independent iff v is in the span of S.

If S is linearly dependent, so is any set larger than S.

Examples.

******Optional*******

Row vectors under row operation.

Rank=num. of linearly independent column vectors.

Vertical stacks of matrices.

Relationship between homogeneous and non-homogeneous equations.

6 9/22 Review of Chapter 1, more on matrix multiplication

Important concepts to remember:

- Matrix
  - Identity matrix
  - Zero matrix
  - Scalar multiplication
  - Addition
  - Linear combination
  - Span
  - Linear independence
  - Transpose
  - Symmetric matrix
  - Row operation
  - REF, RREF
  - Pivot
  - Rank
  - Nullity
- Vector
  - Standard vectors
  - $\mathcal{R}^n$
- System of linear equation
  - Homogeneous equation

- consistence
- Augmented matrix
- Coefficient matrix
- Free variable
- Basic variable
- General solution
- General solution in vector form

True or false:
A set of 3 vectors in $\mathcal{R}^3$ is either linearly dependent or spans $\mathcal{R}^3$.
If the nullity of A is greater than 0, then $Ax = b$ has infinitely many solutions.
If $Ax = b$ has a unique solution, then the nullity of the augmented matrix is 1.

Fibonacci series.

Unique circle passing through 3 points.

$x + ay = b$, $cx + dy = e$.

7 9/26 Matrix algebra

Review:

Relationship between homogeneous and non-homogeneous system: if $Ax = b$ is consistent, x_0 is a solution, then any solution can be written as $x_0 + x_1$ where x_1 is a solution of $Ax = 0$.

Finding minimal generating set: Put into matrix, find pivot columns.

Matrix multiplication: three equivalent ways of defining it:

- row-column rule
- multiple matrix-vector multiplication
- composition: $AB = [A(Be_1), A(Be_2), \ldots]$.

Example using 2-by-2 matrices
Properties: the usual one, except

- No longer commutative.
- relationship with transposes.

Multiplication by identity matrix and diagonal matrix.
Example: matrix algebra and complex numbers. The idea of linear representation.

Example: non-commutativity of 3-d rotation.

8 9/29 Elementary matrix, inverses

Elementary row operation is left-multiplication by elementary matrices.

Column correspondence property.

Applications:

- Read linear relation from RREF.
- General solution of homogeneous equations.
- Row operation doesn't change solution.
- General solution of non-homogeneous equation.
- Uniqueness of RREF

Definition of the Inverse of a matrix. Elementary matrices are invertable.

9 10/3 Invertibility

A matrix is invertable iff $rank = \#rows = \#columns$.

Algorithm for A^{-1}.

Algorithm for solving $AX = B$.

Solution of $Ax = b$ when A is invertable: $x = A^{-1}b$.

10 Midterm I review

The midterm exam will cover up to section 2.3. Please make sure you know the following:

- Able to calculate the product between matrices and vectors.
- Able to solve system of linear equations with Gaussian elimination.
- Know the meaning of the following terms: matrix, identity matrix, zero matrix, symmetric matrix, diagonal matrix, elementary matrix, transpose, linear combination, span, linear dependency, row operation, pivot, rank, nullity, free variable, basic variable.
- Able to translate statements about matrices to statements about linear equations and vice versa. For example, the columns of matrix A are linear independent, then $Ax = 0$ has a unique solution.

Practice problems:

(1) Find t so that the vectors $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ t \end{bmatrix}$ are linearly dependent.

Solution: This is asking when $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & t \end{bmatrix}$ has rank smaller than 3. By Gaussian elimination you can see that it has rank smaller than 3 iff $t = 5/2$.

(2) True or false:

a) Elementary row operations does not change the span of column vectors.

b) If $Ax = b$ has at least two solutions, then the column vectors of A are linearly dependent.

Solution: a) is false. For example, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ can be turned into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by exchanging the two rows, and the span of the columns are not the same. b) is true, because $Ax = b$ has more than one solution means that $Ax = 0$ has non-zero solution, hence the columns of A are linearly dependent by the definition of matrix-vector multiplication.

(3) Find matrix E so that $E \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: You can solve a system of linear equation, or alternatively, recognize that to turn $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ into $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ one can first add the first row to the second then exchange the two rows, hence $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(4) Show that the transpose of elementary matrices are also elementary matrices.

Solution: write down the three types of elementary matrices and see that their transposes are elementary matrices of the same type.

11 10/13 Block multiplication cont. Linear transformation and matrices

11.1 Block multiplication

AB can be calculated by dividing A and B into rectangular blocks so that the block numbers and the sizes of blocks matches.

Example: Divide A and B into row/column vectors.

11.2 Linear transformation

Linear transformation: A map T from R^n to R^m is a linear transformation iff $T(x+y) = T(x) + T(y)$, $T(ax) = aT(x)$.

A is a $m \times n$ matrix, then $x \mapsto Ax$ is a linear transformation.

Write down the matrix of linear transformations: $[T(e_1), \ldots T(e_n)]$.

Composition, identity, surjectivity, injectivity, and inverse.

Examples.

Translation between the 3 viewpoints:

Equations¡—¿Matrices¡—¿Spaces and maps

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Real-life application of block multiplication: BLUP in linear mixed models.

Determinant in 2×2.

12 10/17 Determinant

Recall: det: Defined on square matrices. the simplest poly. which characterizes invertability. Geometrically related to volumns.

Definition: A square matrix, A_{ij} A with i-th row, j-th column removed, $c_{ij} = (-1)^{i+j} det(A_{ij})$ the *cofactor*, then $\det([a]) = a$, $det(A) = \sum_i a_{1i}c_{1i}$.

Properties:

1. Linear for each column.
2. $det(I) = 1$.
3. Negative when switching columns.
4. Linear for each row.

5. Negative when switching rows.
6. Cofactor expansion for other rows/columns.
7. Invariant under transposes.
8. $det(EA) = det(E)det(A)$.
9. $det(AB) = det(A)det(B)$.
10. Inverse
11. Cramer's law
12. For diagonal matrices, the *det* is the product of entries on the diagonal

Examples: 2-by-2, 3-by-3, 4-by-4 computed using LU decomp.