1 Probability and random variables

- **Probability**: S sample space (all possible states of the system), $F \subset \mathcal{P}(S)$ a σ-algebra, $P : F \to \mathbb{R}$ a measure, such that $P(S) = 1$.
- **Random variable**: $X : S \to \mathbb{R}$, such that preimages of open sets are in F (i.e. has a well defined probability).
- **Cumulative distribution function** of random variable: $F_X(t) = P(X \leq t)$.
- **Probability distribution** of random variable: g such that $F_X(t) = \sum_{x \leq t, x \in C} g(x)$.
- **Probability density function**: f such that $F_X(t) = \int_{-\infty}^{t} f(s)ds$.
- Two random variables have the **same distribution** if they have the same cdf.

Example: **uniform distribution**:

- S a finite interval $[a, b]$
- F: Set of Borel sets on S (sets with a well defined "length")
- P: Borel measure ("length") divided by $b - a$
- $X = id$.

1.1 Expectation of random variables and their functions

- X is a random variable, the **expectation** of X is $E[X] = \int_S X dP$.
- The **variance** of X is $E[(X - E[X])^2]$.
- The k-th **moment** of X is $E[X^k]$.
- The **moment generating function** of X is $E[e^{Xt}]$ (two sided Laplace transform)
- The **characteristic function** of X is $E[e^{itX}]$ (Fourier transform)

Since expectation is defined via integration, one can use the properties of integration to prove statements regarding expectation.

Example: **Chebyshev's theorem**: $E[X] = 0$, $E[X^2] = 1$, then $P(|X| < k) \geq 1 - \frac{1}{k^2}$.
Proof:

$$1 = E[X^2] = \int_S X^2 dP \geq k^2 \int_{|X| \geq k} 1 dP = k^2(1 - P(|X| < k))$$

Example: If X has p.d.f. f_X, then $E[g(X)] = \int_{-\infty}^{\infty} g f_x dt$. We prove it when $g(X)$ is bounded via Fubini's theorem:

$$E[g(X)] = \int_S g(X) dP$$

$$= \int_{g(X)\geq 0} \int_0^{g(X)} 1 dy dP - \int_{g(X)<0} \int_{g(X)}^0 1 dy dP$$

$$= \int_0^{\infty} \int_{g^{-1}([y,\infty])} f_X(t) dt dy - \int_{-\infty}^0 \int_{g^{-1}([-\infty,y])} f_X(t) dt dy$$

$$= \int_{-\infty}^{\infty} g f_x dt$$

There is a multivariate version of this formula, and one can also write down $E[g(X)]$ when only the c.d.f. of X is known (via Fubini's theorem or integration by parts).

Can you write down a random variable with neither probability distribution nor p.d.f.?

Can you write down a random variable with no expectation?

1.2 Independence and conditional probability for random events

- $A, B \in F$ are **independent** iff $P(A \cap B) = P(A)P(B)$.
- If $P(B) \neq 0$, $P(A \cap B) = P(B)P(A|B)$. Here $P(A|B)$ is the **conditional probability** of A when B is known to happen.

1.3 Joint distribution, marginal distribution, conditional distribution

1.3.1 Joint distribution

- X and Y are two random variables. The **joint cumulative distribution function** is $F(s,t) = P(X \leq s, Y \leq t)$.
- If $F(s,t) = \sum_{(x,y)\in C, x\leq s, y\leq t} g(s,t)$, we call g the **joint probability distribution**.
- If $F(s,t) = \int_{(-\infty,s]\times(-\infty,t]} f(x,y) dx dy$ we call f the **joint probability density function**.
- X and Y are called independent iff the joint c.d.f. is $F(x,y) = F_X(x)F_Y(y)$.
- The **covariance** between X and Y is $E[(X - E[X])(Y - E[Y])]$

Example: X and Y are two independent random variable with uniform distribution on $[0,1]$. What is the joint distribution function of X and Y? How about $max(X,Y)$ and $min(X,Y)$? What are their covariances?

1.3.2 Marginal distribution

Knowing the joint c.d.f. of X and Y, the c.d.f. of X or Y are called the **marginal cumulative distribution function**, their p.d. or p.d.f. the **marginal p.d.** or **marginal p.d.f**.

1.3.3 Conditional distribution

- If A is a set such that $P(Y \in A) > 0$, then the **conditional cumulative distribution function** of X is $F_{X|Y\in A}(t) = P(X \leq t|Y \in A) = P(X \leq t \cap Y \in A)/P(Y \in A)$. The **conditional p.d.f.**, **conditional p.d.** and **conditional expectation** are defined similarly.
- If $P(Y \in A) = 0$ there isn't a definition of conditional distribution that works in all cases. For example, if X, Y has joint p.d.f. $f_{X,Y}$, and the marginal p.d.f. of Y, denoted as $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)dx$, exists and is non zero at y_0, then the conditional p.d.f. at $Y = y_0$ is defined as $f_{X|Y=y_0} = f_{X,Y}(x,y_0)/f_Y(y_0)$. The conditional c.d.f. is its integral.

Remark: The definition of conditional distribution for the case $P(Y \in A) = 0$ depends on Y and not just $Y^{-1}(A)$. For example, if $Z = Ye^X$, $f_{X|Y=0} \neq f_{X|Z=0}$.

Example: X is a random variable with uniform distribution on $[0,1]$, $P(Y = 1|X = p) = p$ (i.e. $P(Y = 1|X \in A) = \int_A p dF_x(p))$, $P(Y = 0|X = p) = 1 - p$. Find the conditional distribution of X when $Y = 1$.

When there are N random variables, $N \geq 3$, the joint/marginal/conditional distributions can be defined analogously.

2 Special probability distributions, central limit theorem

2.1 Special discrete distributions

- **Bernoulli distribution**: $f(1) = \theta$, $f(0) = 1 - \theta$.
- **Binomial distribution** (sum of iid Bernoulli): $f(x) = \begin{pmatrix} n \\ x \end{pmatrix} \theta^x(1-\theta)^{n-x}$, $x = 0, 1, \ldots, n$.

- **Negative Binomial distribution** (waiting time for the k-th success of iid trials): $f(x) = \left(\begin{array}{c} x-1 \\ k-1 \end{array} \right) \theta^k (1-\theta)^{x-k}$, $x = k, k+1, \ldots$. When $k = 1$ it is the **geometric distribution**.

- **Hypergeometric distribution** (randomly pick n elements at random from N elements, the number of elements picked from a fixed subset of M elements) $f(x) = \left(\begin{array}{c} M \\ x \end{array} \right) \left(\begin{array}{c} N-M \\ n-x \end{array} \right) \left(\begin{array}{c} N \\ n \end{array} \right)^{-1}$.

- **Poisson distribution** (limit of binomial as $n \to \infty$, $n\theta \to \lambda$) $f(x) = \lambda^x e^{-\lambda}/x!$.

- **Multinomial distribution** $f(x_1, \ldots x_k) = \left(\begin{array}{c} n \\ x_1, \ldots, x_k \end{array} \right) \theta_1^{x_1} \ldots \theta_k^{x_k}$, $\sum_i x_i = n$, $\theta_i \theta_i = 1$.

- **Multivariate Hypergeometric distribution** $f(x_1, \ldots, x_k) = \prod_i \left(\begin{array}{c} M_i \\ x_i \end{array} \right) \cdot \left(\begin{array}{c} N \\ n \end{array} \right)^{-1}$. $\sum_i x_i = n$, $\sum_i M_i = N$.

2.2 Special continuous distributions

- **Uniform distribution**: $f(x) = \begin{cases} 1/(b-a) & x \in (a,b) \\ 0 & x \notin (a,b) \end{cases}$.

- **Normal distribution**: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

- **Multivariate Normal distribution**: $x \in \mathbb{R}^d$, Σ positive definite $d \times d$ symmetric matrix, $f(x) = (2\pi)^{-d/2} |\Sigma|^{-1/2} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$.

- χ^2 **distribution** d: degrees of freedom. Squared sum of d normal distributions: $f(x) = \begin{cases} \frac{1}{2^{d/2}\Gamma(d/2)} x^{\frac{d-2}{2}} e^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases}$.

- **Exponential distribution** $f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0 \\ 0 & x \leq 0 \end{cases}$.

- **Gamma-distribution**: $f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & x \leq 0 \end{cases}$

- **Beta distribution**: (conjugate prior of Bernoulli distribution) $f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$.

2.3 Central Limit Theorem

- **Convergence in distribution**: cdf pointwise convergence.
- **Convergence almost surely**: $P(\lim_i X_i \neq X) = 0$.

2.3.1 Law of Large Numbers

Law of Large Numbers X_i, $i = 1, 2, \ldots$ i.i.d. (independent with identical distribution) and $E(X_i) = \mu$, then $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges a.s. to constant μ.

Proof: Use Borel-Cantelli Lemma and Chebyshev inequality.

2.3.2 Central Limit Theorem

Levy's continuity theorem: If $\phi_{X_j} \to \phi$ pointwise and ϕ_X is continuous at 0, then X_j converges to X in distribution.

Suppose X_i is a sequence of i.i.d. random variables with finite expectation μ and variance $\sigma^2 > 0$, then $\phi_{\sqrt{\frac{1}{n\sigma^2}}\sum_{i=1}^{n}(X_i-\mu)}$ converges pointwise to the characteristic function of normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Hence:

(Levy's) Central Limit Theorem X_i i.i.d. with expectation μ and variance $\sigma^2 > 0$. $Y_n = \sqrt{\frac{1}{n\sigma^2}}\sum_i (X_i - \mu)$, then Y_n converges in distribution to standard normal distribution (normal distribution with $\mu = 0$ and $\sigma^2 = 1$).

3 Sample statistics

4 Point estimators and their properties

5 Method of moments, Maximum likelihood

6 Maximum a posteriori

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