Math 856 presentation

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Reference: Dave Witte Morris, *Ratner's Theorems on Unipotent Flows*

1 Oppenheim conjecture

Theorem. *(Oppenheim's conjecture, Margulis' theorem) Let Q be a nondegenerate indefinite quadratic form in $n \geqslant 3$ variables. Then either Q is proportional to a form with integer coefficients, or $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

History. The statement was conjectured by Oppenheim for quadratic form in $n \geqslant 5$ variables in 1929, and was extended to $n \geqslant 3$ by Davenport. The conjecture was finally proved by Margulis in 1987. Ratner proved more general results on unipotent flows in 1990s.

Remark.
- Suppose all of coefficients are integers. Then $Q(\mathbb{Z}^n) \subset \mathbb{Z}$, which is discrete in $\mathbb{R}$. More generally, if Q is a scalar multiple of rational form, then $Q(\mathbb{Z}^n)$ is still not dense in $\mathbb{R}$.
- If Q is definite, then Q can only be positive or it can only be negative, so the statement is false.
- The requirement of $n \geqslant 3$ is necessary. In the proof, we use unipotent flows, but the group $\mathrm{SO}(1,1)$ is multiplicative, which is not generated by unipotent, so we can not apply Ratner's theorem in this case.

  Counterexample: Suppose β is a quadratic irrational, there exists $\epsilon > 0$ such that if p/q is any rational, then $\left|\frac{p}{q} - \beta\right| > \frac{\epsilon}{q^2}$. The quadratic form $Q(x_1, x_2) = x_1^2 - \beta^2 x_2^2$ does not satisfy the conjecture because
  $$|Q(x_1, x_2)| = |(x_1 - \beta x_2)(x_1 + \beta x_2)| = |x_2^2| \left|\frac{x_1}{x_2} - \beta\right| \left|\frac{x_1}{x_2} + \beta\right| > \epsilon.$$
  For example, we can let $\beta = (1 + \sqrt{5})/2$

The proof can be reduced to $n = 3$. If Q is irrational with $n \geqslant 3$, then there is a 3-dimensional rational subspace so that the restriction of Q is still irrational, indefinite and nondegenerate. More precisely, choose any vectors v_1 and v_2 in $\mathbb{Z}^n$ satisfying $Q(v_1)/Q(v_2)$ is irrational and negative. Then choose some general $v_3 \in \mathbb{Z}^n$ so that $Q'(x_1, x_2, x_3) = Q(x_1 v_1, x_2 v_2, x_3 v_3)$ is nondegenerate.

To prove the case $n = 3$, let $Q(x_1, x_2, x_3)$ be a nondegenerate indefinite quadratic form. Let $G = \mathrm{SL}_3(\mathbb{R})$, $H = \mathrm{SO}(Q)^\circ$ and $\mathrm{SO}(Q) = \{h \in G | Q(hx) = Q(x) \text{ for all } x \in \mathbb{R}^3\}$.

Since Q is indefinite, nondegenerate, $H \simeq \mathrm{SO}(2,1)^\circ = \mathrm{PSL}_2(\mathbb{R})$, so it is generated by unipotent elements. Then we apply Ratner's orbit closures theorem for G, H and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$.

Theorem. *(Ratner orbit closure theorem) Let G be a connected Lie group, and let Γ be a lattice in G. Let H be one-parameter unipotent group (more generally, a connected Lie subgroup of G generated by unipotent one-parameter groups). Then for any $x \in G/\Gamma$ there exists a closed connected subgroup $P \subset G$ containing H such that $\overline{Hx} = Px$ and Px admits a P-invariant probability measure ($P \cap x\Gamma x^{-1}$ is a lattice of P).*

Example. Let's see two examples where the theorem holds.

- $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$, $\mathbb{T}^n = G/\Gamma$ is an n-dimensional torus. Any vector $v \in \mathbb{R}^n$ determines a unipotent flow on the torus. The theorem says the closure of the orbit of each point of $\mathbb{T}^n$ is a subtorus of $\mathbb{T}^n$.
- $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, N the subgroup of upper triangular matrices with diagonal 1. The subgroups containing N can only be N, B (group of upper triangular matrices) and G. B is not unimodular so it does not contain any lattices. We conclude that this horocycle flow on G/Γ is either close or dense.

Remark. There are some orbits of geodesic flows whose closures are nowhere close to being a submanifold.

Lemma. *The only connected subgroups of G containing H are H and G.*

Proof. By the Lie group–Lie algebra correspondence, suffices to prove $\mathfrak{so}(2,1)$ is a maximal subalgebra of $\mathfrak{sl}_3(\mathbb{R})$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{sl}_3(\mathbb{R})$ containing $\mathfrak{so}(2,1)$, then $\mathfrak{h}$ is a representation of $\mathfrak{so}(2,1)$ under adjoint action. Because the quotient representation $\mathfrak{sl}_3(\mathbb{R})/\mathfrak{so}(2,1)$ of the Lie algebra $\mathfrak{so}(2,1)$ is irreducible, $\mathfrak{h}$ can only be $\mathfrak{so}(2,1)$ or $\mathfrak{sl}_3(\mathbb{R})$. □

Side. Some computation:

Suppose our $\mathfrak{so}(2,1)$ is preserving the quadratic form with the matrix whose entries are 1's on the antidiagonal and equal 0 otherwise, then $\mathfrak{so}(2,1) = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$. A complement of $\mathfrak{so}(2,1)$ is the subspace $\left\{ \begin{pmatrix} d & e & g \\ f & -2d & e \\ h & f & d \end{pmatrix} \right\}$. The subalgebra $\mathfrak{t} = \{\mathrm{diag}(a, 0, -a)\}$ is a Cartan subalgebra.

By the lemma, the orbit-cloaure theorem tells us that either:

- $H\,\mathrm{SL}_3(\mathbb{Z})$ is closed, and $\Gamma_Q = \mathrm{SL}_3(\mathbb{Z}) \cap H$ is a lattice in H, or
- $H\,\mathrm{SL}_3(\mathbb{Z})$ is dense in $\mathrm{SL}_3(\mathbb{R})$.

Theorem. *(Borel Density Theorem) Suppose $G \subset \mathrm{SL}_l(\mathbb{R})$ is a linear connected semisimple group and has no compact factors, then the lattice Γ is Zariski dense in G. That is, if $F \in \mathbb{R}[x_{1,1}, \dots, x_{l,l}]$ is a polynomial function on $\mathrm{Mat}_{l\times l}(\mathbb{R})$, such that $F(\Gamma) = 0$, then $F(G) = 0$.*

In the first case, by the Borel density theorem, H is contained in the Zariski closure of Γ_Q, so the group $\mathrm{SO}(Q)$ is exactly the Zariski closure of Γ_Q, so it is defined over $\mathbb{Q}$ (because the solutions of the polynomials all have integral entries) and thus Q must be a scalar multiple of a rational form.

In the second case, because $H = \mathrm{SO}(Q)^\circ$ and $\mathrm{SL}_3(\mathbb{Z})\mathbb{Z}^3 = \mathbb{Z}^3$,

$$Q(\mathbb{Z}^3) = Q(H\mathbb{Z}^3) = Q(H\,\mathrm{SL}_3(\mathbb{Z})\mathbb{Z}^3).$$

Since $H\,\mathrm{SL}_3(\mathbb{Z})$ is dense in $\mathrm{SL}_3(\mathbb{R})$,

$$\overline{Q(\mathbb{Z}^3)} = Q\left(\overline{H\,\mathrm{SL}_3(\mathbb{Z})\mathbb{Z}^3}\right) = Q(\mathrm{SL}_3(\mathbb{R})\mathbb{Z}^3) = Q(\mathbb{R}^3) = \mathbb{R}.$$

The last identity is because Q is indefinite.

2 Ratner's theorems

Theorem. *(Ratner measure classification theorem)*

Let G be a connected Lie group, and let Γ be a lattice in G. Let U be a one-parameter unipotent subgroup. Then any ergodic U-invariant measure μ on G/Γ is algebraic. That is, there exists $x \in G/\Gamma$ and a subgroup $F \subset G$ containing U and generated by unipotents such that Fx is closed and μ is the F-invariant probability measure on Fx.

This theorem implies the equidistribution theorem and the orbit closure theorem.

Theorem. *(Ratner equidistribution thorem)*

Let G be a connected Lie group, and let Γ be a lattice in G. Let $U = \{\varphi_t\}_{t\in\mathbb{R}}$ be a one-parameter unipotent subgroup. Then for any $x \in G/\Gamma$ there exists a closed connected subgroup $P \subset G$ containing U such that $\overline{Ux} = Px$ and Px admits a P-invariant probability measure μ_P (which is unique). For every $f \in C_0(G/\Gamma)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x))dt = \int_{Px} f d\mu_P.$$