

## 1 1.1

2.  $u_{xx} + u_{yy} = (2x \cdot \frac{1}{2}(x^2 + y^2)^{-1})_x + (2y \cdot \frac{1}{2}(x^2 + y^2)^{-1})_y = 2(x^2 + y^2)^{-1} - (2x^2 + 2y^2) \cdot (x^2 + y^2)^{-2} = 0.$

4. The statement of this problem is somewhat unclear in whether they mean  $(u_{xx})^2 + (u_{yy})^2 = 0$  (the more likely one) or  $(u^2)_{xx} + (u^2)_{yy} = 0$ , so either interpretation would be considered correct. With the first interpretation it is obvious that all function in the stated form satisfy that  $u_{xx} = u_{yy} = 0$ . With the second there would need to be additional constraints on  $a, b, c, d$  for it to work.

5. The general solution is  $u = xF(t) + G(t)$ , hence one can let  $u = t^2 + x(1 - t^2)$ .

6.  $u_{tt} = (g(x + ct) + g(x - ct))_t = c(g'(x + ct) - g'(x - ct)), u_{xx} = c^{-1}(g(x + ct) - g(x - ct))_x = c^{-1}(g'(x + ct) - g'(x - ct)).$

7.  $(e^{at} \sin bx)_t = ae^{at} \sin bx, (e^{at} \sin bx)_{xx} = -b^2 e^{at} \sin bx$ , hence  $a = -kb^2$ .

8.  $(u_x)_t = 1 - 3u_x$ , hence  $u_x = \frac{1}{3} + e^{-3t}f(x)$  for some arbitrary function  $f$ , hence  $u(x, t) = \frac{x}{3} + e^{-3t}F(x) + G(t)$  for arbitrary function  $F$  (which is the anti-derivative of  $f$ ) and  $G$ .

12. To sketch wave profile, pick some  $k, A, D$  or  $c$ , sketch  $u(x, t)$  for different values of  $t$ , and if  $u$  is complex-valued you can sketch either the real or imaginary part.

Dispersion relations: a)  $\omega = -iDk^2$ . b)  $\omega = \pm ck$ . c)  $\omega = -k^3$ . d)  $\omega = k^2$ . e)  $\omega = ck$ .

14. Dispersion relation is  $\omega = (-1 + \delta k^2 - k^4)i$  hence this is diffusive. When  $\delta = k^2 + 1/k^2$  the solution has growth rate 0. When  $k^2 + 1/k^2 > \delta$  the solution decays.

## 2 1.2

1. From equation (1.7) in the text we have  $\frac{d}{dt} \int_a^b u A dx = A\phi|_a - A\phi|_b$ . Differentiate with respect to  $b$  (or use some other argument, for example as in the textbook), we have  $Au_t = -A_x\phi - A\phi_x$ , hence  $u_t + \phi_x = -A'\phi/A$ .

3. By chain rule,  $u_x = u_\xi$ ,  $u_t = -cu_\xi + u_\tau$ , hence the equation (1.12) becomes  $u_\tau = -\lambda u$ , hence the general solution is  $u = e^{-\lambda\tau}F(\xi) = e^{-\lambda t}F(x - ct)$ .

4.  $u_t + cu_x = -\lambda u$ . If  $w = ue^{\lambda t}$ ,  $u = we^{-\lambda t}$  hence  $u_t + cu_x = w_t e^{-\lambda t} - \lambda w e^{-\lambda t} + cw_x e^{-\lambda t}$ ,  $-\lambda u = -\lambda w e^{-\lambda t}$ , hence  $w_t + cw_x = 0$ .

5. By method of characteristics  $u_t + xtu_x = 0$  has characteristics  $x = Ce^{t^2/2}$ , hence the general solution is  $u = F(xe^{-t^2/2})$ . Together with the initial value condition we know that  $F = f$  hence  $u = f(xe^{-t^2/2})$ . The general solution of  $u_t + xu_x = e^t$  is  $u = e^t + F(xe^{-t})$ , so with the initial condition, the solution should be  $u = e^t + f(xe^{-t}) - 1$ .

6(b). The characteristics are  $x = Ct$ , and the general solution is  $u = e^{-2t}F(x/t)$ . Use the initial condition we get  $F = e^2 f$ , hence  $u = e^{-2(t-1)}f(x/t)$ .

7. The general solution is  $u = e^{-\lambda t}F(x - ct)$ . The initial-boundary condition tells us that  $F(x) = 0$  for  $x > 0$  and  $e^{-\lambda t}F(-ct) = g(t)$  for  $t > 0$ , hence  $F(x) = \begin{cases} 0 & x > 0 \\ e^{\lambda x/c}g(x/c) & x \leq 0 \end{cases}$ .

12. By the method of characteristics,  $u(x, t) = F(x - ct)e^{(\alpha t - u)/\beta}$ . Set  $t = 0$  we have  $F(x) = f(x)e^{f/\beta}$  hence  $u(x, t) = f(x - ct)e^{(\alpha t - u + f(x - ct))/\beta}$ .

14. Characteristics are  $x = Ce^{-ut}$  hence  $u = F(xe^{ut})$ . Together with the initial condition we get  $u = xe^{ut}$ . A solution does not exist for all  $t$ . For example, there doesn't exist any  $u$  at point  $x = t = 1$  because  $s < e^s$  for all  $s \in \mathbb{R}$ .

### 3 1.3

2.  $\frac{d}{dt} \int_0^l u^2 dx = \int_0^l 2uu_t dx = \int_0^l 2kuu_{xx} dx = 2kuu_x|_0^l - \int_0^l 2k(u_x)^2 dx \leq 0$ , hence  $\int_0^l u^2 dx \leq \int_0^l u_0^2 dx$  for  $t \geq 0$ .

3. Let  $w = u - g + (x/l)(h - g)$ , then  $w(0, t) = w(l, t) = 0$ ,  $u_t = ku_{xx}$  will imply  $w_t = kw_{xx} - g' + (x/l)(h' - g')$ .

4. The steady state satisfy  $0 = ku_{xx} - hu$  and  $u(0) = u(1) = 1$ , hence  $u = \frac{e^{(h/k)^{1/2}(x-1/2)} + e^{(h/k)^{1/2}(1/2-x)}}{e^{(h/k)^{1/2}/2} + e^{-(h/k)^{1/2}/2}}$ .

5.  $u_t = w_t e^{\alpha x - \beta t} - \beta w e^{\alpha x - \beta t} = w_t e^{\alpha x - \beta t} - \beta u$ ,  $u_x = w_x e^{\alpha x - \beta t} + \alpha u$ ,  $u_{xx} = w_{xx} e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha w_x e^{\alpha x - \beta t} + \alpha^2 u$ , hence  $0 = u_t - Du_{xx} + cu_x + \lambda u = (w_t - Dw_{xx})e^{\alpha x - \beta t} + (c - 2D\alpha)w_x e^{\alpha x - \beta t} + (\lambda - \beta - D\alpha^2 + c\alpha)u$ , so when  $\alpha = c/(2D)$  and  $\beta = \lambda - D\alpha^2 + c\alpha = \lambda + c^2/(4D)$ ,  $0 = w_t - Dw_{xx}$ .

6. The steady state doesn't depend on the initial condition. It is  $u = \frac{1}{2k}x(1 - x)$ .

10. The flux is  $Du_x + u^2/2$ . Replace  $u = \psi_x$  we have  $\psi_{xt} = D\psi_{xxx} + \psi_x \psi_{xx}$ . Integrate along  $x$  we have  $\psi_t = D\psi_{xx} + (\psi_x)^2/2 + F(t)$ . Replace  $\psi_t$  with  $\psi_t + \int_0^t F(s)ds$  we can get rid of  $F$ . Now let  $\psi = -2D \ln v$  we get  $-2Dv_t/v = -2D^2(v_{xx}v - (v_x)^2)/v^2 + 2D^2(v_x)^2/v^2$ , hence  $v_t = Dv_{xx}$ .

### 4 1.4

3. For  $u_t = Du_{xx} - cu_x$ , the time independent solution satisfies  $0 = Du_{xx} - cu_x$ . So the solution is  $u = C_1 + C_2 e^{cx/D}$ . For  $u_t = Du_{xx} - cu_x + ru$ , the time independent case reduces to  $0 = Du_{xx} - cu_x + ru$ , the characteristic polynomial is  $D\lambda^2 - c\lambda + r = 0$  whose roots are  $r = \frac{c \pm \sqrt{c^2 - 4Dr}}{2D}$ . Hence, when  $c^2 = 4Dr$  the general solution is  $u = (C_1 + C_2 x)e^{\frac{cx}{2D}}$ , when  $c^2 > 4Dr$  the general solution is  $u = C_1 e^{\frac{xc + x\sqrt{c^2 - 4Dr}}{2D}} + C_2 e^{\frac{xc - x\sqrt{c^2 - 4Dr}}{2D}}$ , when  $c^2 < 4Dr$  the general solution is  $u = C_1 e^{\frac{xc}{2D}} \cos(\frac{x\sqrt{4Dr - c^2}}{2D}) + C_2 e^{\frac{xc}{2D}} \sin(\frac{x\sqrt{4Dr - c^2}}{2D})$ .

9.  $u = ax + b$  then  $u_{xx} = 0$ .

$u = a \ln r + b$  then  $u_{xx} + u_{yy} = a(\frac{x}{r^2})_x + a(\frac{y}{r^2})_y = a(\frac{r^2 - 2x^2 + r^2 - 2y^2}{r^4}) = 0$ .

$u = \frac{a}{\rho + b}$  then  $u_{xx} + u_{yy} + u_{zz} = a((\frac{x}{\rho^3})_x + (\frac{y}{\rho^3})_y + (\frac{z}{\rho^3})_z) = 0$ .

12. (a)  $\frac{d}{dt} \int_a^b 2\pi r u dr = 2\pi a(-Du_r|_a) - 2\pi b(-Du_r|_b)$ . Differentiate on  $b$  we get  $bu_t|_b = Db u_{rr}|_b + Du_r|_b$ , hence  $u_t = Du_{rr} + \frac{D}{r}u_r = D\frac{1}{r}(ru_r)_r$ .

(b)  $\frac{d}{dt} \int_a^b 4\pi r^2 u dr = 4\pi a^2(-Du_r|_a) - 4\pi b^2(-Du_r|_b)$ . Differentiate on  $b$  then you get the differential equation.

## 5 1.5

1. You can do it however you want, for example, in the 3rd equation on page 51, add a term  $-\int_a^b \rho_0 g dx$  to the right.

3. Verification is by chain rule. Sketch  $u = \frac{1}{2}(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2})$ .

4. The initial condition is  $u_n(x, 0) = \sin \frac{n\pi x}{l}$ ,  $(u_n)_t(x, 0) = 0$ . The frequency is  $\frac{cn}{2l}$ , they decrease as  $l$  increases and as  $c$  (tension) increases.

5.  $\frac{d}{dt} E = \int_0^l (\rho_0 u_t u_{tt} + \tau_0 u_x u_{tx}) dx = \tau_0 \int_0^1 (u_t u_{xx} + u_x u_{tx}) dx = \tau_0 u_t u_x|_0^l = 0$ .

9.  $I_x + CV_t + GV = 0$ , so  $I_{xx} + CV_{xt} + GV_x = 0$ . Substitute  $V_x = -LI_t + RI$ , we get that  $I$  satisfy the telegraph equation. The fact that  $V$  satisfy telegraph equation follows analogously. When  $R = G = 0$  the speed of wave is  $(LC)^{-1/2}$ .

## 6 1.7

1.  $\text{div}(\text{gradu}) = \text{div}((u_x, u_y, u_z)) = u_{xx} + u_{yy} + u_{zz}$ .

## 7 Quiz 1:

$u_t + (x+1)u_x = 1$ ,  $u(x, 0) = \sin x$ .

Characteristics are  $x = Ce^t - 1$ . Hence  $u = t + F((x+1)e^{-t})$ , hence  $F(x) = \sin(x-1)$  and  $u = t + \sin((x+1)e^{-1} - 1)$ .

## 8 1.7

3. This is divergence theorem. The heat generated in  $\Omega$  equals the heat flowing out at the boundary.

4. Let  $\phi = (\phi_1, \phi_2, \phi_3)$ , then  $\text{div}(w\phi) = (w\phi_1)_x + (w\phi_2)_y + (w\phi_3)_z = (w_x\phi_1 + w_y\phi_2 + w_z\phi_3) + w((\phi_1)_x + (\phi_2)_y + (\phi_3)_z) = \phi \cdot \text{grad} w + w \text{div} \phi$ . Let  $\phi = \text{gradu}$  then Green's identity follows from this and the divergence theorem.

5.  $\lambda = \frac{\int_{\Omega} u \Delta u dV}{\int_{\Omega} u^2 dV} = -\frac{\int_{\Omega} |\text{gradu}|^2 dV}{\int_{\Omega} |u|^2 dV} < 0$ ,

6. Let  $w = u + v$  where  $v$  is 0 at the boundary, then  $\int_{\Omega} |\text{grad} w|^2 dV = \int_{\Omega} |\text{grad} u|^2 dV + \int_{\Omega} |\text{grad} v|^2 dV + 2 \int_{\Omega} \text{grad} u \cdot \text{grad} v dV$ . By 4 and the assumption, the last term is 0, hence  $\int_{\Omega} |\text{grad} w|^2 dV \geq \int_{\Omega} |\text{grad} u|^2 dV$ .

7. Use  $\text{cpu}_t = \text{div} \phi$ .

## 9 1.8

1. Maximum are at  $r = 2$ ,  $\theta = \pi/4, 5\pi/4$ , minimum are at  $r = 2$ ,  $\theta = 3\pi/4, 7\pi/4$ .

2.  $u = (x^2 + y^2)/4 - a^2/4$ .

4. The solution is spherical symmetric because the function and the boundary conditions are both spherical symmetric, i.e.  $u = u(\rho)$ . Hence  $\Delta u = 1$  reduces to  $u_{\rho\rho} + \frac{2}{\rho}u_{\rho} = 1$ , hence  $(\rho^2 u_{\rho})_{\rho} = \rho^2$ , hence  $\rho^2 u_{\rho} = \frac{1}{3}\rho^3 + C_1$ ,  $u' = \frac{1}{3}\rho + \frac{C_1}{\rho^2}$ , hence  $u = \frac{1}{6}\rho^2 - \frac{C_1}{\rho} + C_2$ . Apply the boundary condition one gets  $u = \frac{1}{6}\rho^2 + \frac{b^3}{3\rho} - \frac{1}{6}a^2 - \frac{b^3}{3a}$ .

5.  $u = A \tan(x) + B$ , solve for constants  $A$  and  $B$  using the boundary condition.

6.  $u = A \log r + B$ .  $u = \frac{10}{\log 2} \log r$ .

8. Use chain rule.

9.  $\text{curl} E = 0$  implies that such a potential exists.  $\Delta V = \text{divgrad} V = \text{div} E = 0$ .

## 10 1.9

1. This is a parabolic equation.  $u = F(kx-t) + (x+kt)G(kx-t)$ , or you can write it in other equivalent ways.

2. Let  $p = 2x + t$ ,  $q = t$ , then  $u_x = 2u_p$ ,  $u_{xx} = 4u_{pp}$ ,  $u_t = u_p + u_q$ ,  $u_{xt} = 2u_{pp} + 2u_{pq}$ , hence the equation becomes  $u_p = 4u_{qp}$ , hence  $u = F(2x + t)e^{t/4} + G(t)$ .

3. It is hyperbolic. Under the change of variable, by chain rule,  $u_x = \frac{4}{x}u_{\tau}$ ,  $u_{xx} = \frac{16}{x^2}u_{\tau\tau} - \frac{4}{x^2}u_{\tau}$ ,  $u_{xt} = \frac{4}{x}u_{\tau\xi} + \frac{4}{x}u_{\tau\tau}$ , so  $0 = xu_{xx} + 4u_{xt} = -\frac{4}{x}u_{\tau} - 16u_{\tau\xi}$ , hence  $u = e^{-\xi/4}f(\tau) + g(\xi)$ .

4. Use chain rule and product rule.

5. Elliptic. Find the eigenvalues of matrix  $\begin{bmatrix} 1 & -3 \\ -3 & 12 \end{bmatrix}$ .

6. Parabolic. The general solution calculation is similar to 3 above.

7. a) Elliptic when  $xy > 1$  and hyperbolic when  $xy < 1$ . b) Elliptic.

# Midterm 1

1. Solve the following initial or initial/boundary value problems:

- (1)  $u_t = xu_x$ ,  $u(x, 0) = x^2$ . Here  $u$  is a function of  $x$  and  $t$ . (25 points)  
 (2)  $u_t + u_x = \sin x$ ,  $u(x, 0) = 0$  for  $x \geq 0$ ,  $u(0, t) = t$  for  $t \geq 0$ . Here  $u$  is a function of  $x$  and  $t$ . (15 points)

Answer: (1) General solution is  $u = F(xe^t)$ , hence  $u = x^2e^{2t}$ .  
 (2) General solution is  $u = -\cos x + F(x-t)$ , so  $u = -\cos x - (x-t) + 1$  when  $x \leq t$ , and  $u = -\cos x + \cos(x-t)$  when  $x \geq t$ .

2. (1) Find the general solution of  $u_{tt} = u_{tx}$ . (15 points)  
 (2) Find the solution of the initial value problem:  $u_{tt} = u_{tx}$ ,  $u(x, 0) = 0$ ,  $u_t(x, 0) = x$ . (10 points)

Answer: (1)  $u_t = f(x+t)$ , so  $u = F(x+t) + G(x)$  where  $F$  and  $G$  are arbitrary functions.  
 (2)  $F(x) + G(x) = 0$ ,  $F'(x) = x$ , so  $u = \frac{1}{2}(x+t)^2 - x^2$ .

3. Consider the 1 dimensional advection-diffusion equation:  $u_t = u_x + u_{xx}$ .  
 (1) Use change of coordinate of the form  $p = x - Ct$ ,  $q = t$  to reduce it to the 1 dimensional heat equation. (13 points)  
 (2) Recall that the solution of initial value problem of 1-dimensional heat equation:  $v_t = v_{xx}$  when  $t > 0$ ,  $v(x, 0) = f(x)$  can be given by the Poisson integral representation:

$$v(x, t) = \int_{-\infty}^{\infty} f(y)G(x-y, t)dy, \text{ where } G(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Can you write down the analogous formula for the following initial value problem:  $u_t = u_x + u_{xx}$  when  $t > 0$ ,  $u(x, 0) = f(x)$ ? (10 points)

- (3) Consider the following problem with periodic boundary condition:  $u_t = u_x + u_{xx}$  when  $0 < x < 1$ ,  $u(0, t) = u(1, t)$ ,  $u_x(0, t) = u_x(1, t)$ . Show that  $I(t) = \int_0^1 u^2(x, t)dx$  is a non-increasing function by calculating  $\frac{d}{dt}I$ . (7 points)

Answer: (1)  $u_t = -Cu_p + u_q$ ,  $u_x = u_p$ ,  $u_{xx} = u_{pp}$ , hence when  $C = -1$ ,  $u_q = u_{pp}$ .  
 (2)  $u(x, t) = \int_{-\infty}^{\infty} f(y)G(x+t-y, t)dy$ .  
 (3)  $\frac{d}{dt}I = \int_0^1 2uu_t dx = \int_0^1 2uu_x + 2uu_{xx} dx = u^2|_0^1 + 2uu_x|_0^1 - \int_0^1 2(u_x)^2 dx \leq 0$ .

4. Consider the equation  $u_{xx} + u_{yy} = x^2 + y^2$  on  $\mathbb{R}^2 \setminus (0, 0)$ . Find all radial symmetric solutions (In other words, all solutions of the form  $u(x, y) = g(\sqrt{x^2 + y^2})$ ). You may want to use the fact that the Laplace operator in polar coordinate  $(r, \theta)$  is  $\Delta = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ . (5 points)

Answer:  $u_{rr} + u_r/r = r^2$ , so  $(ru_r)_r = r^3$ ,  $ru_r = A + \frac{1}{4}r^4$ ,  $u_r = A/r + \frac{1}{4}r^3$ , and  $u(r) = B + A \log r + \frac{1}{16}r^4$ .

## 11 2.1

2.  $|u| = |\int_{\mathbb{R}} \phi(y)G(x-y, t)dy| \leq \int_{\mathbb{R}} |\phi(y)G(x-y, t)|dy \leq M \int_{\mathbb{R}} G(x-y)dy = M$ , where  $G$  is the heat kernel.

3.  $u(x_0, t) = \int_{\mathbb{R}} \phi(y)G(x_0 - y, t)dy = u_0 \int_0^\infty G(x_0 - y, t)dy = u_0 \int_{-\infty}^{x_0} G(s, t)ds = u_0(\int_{-\infty}^0 G(s, t)ds + \int_0^{x_0} G(s, t)ds)$ . We know  $\int_{-\infty}^0 G(s, t)ds = 1/2$ ,  $\int_0^{x_0} G(s, t)ds = \int_0^{x_0/sqrt{4t}} G(s, 1)dt$  which converges to 0 as  $t \rightarrow \infty$ , hence  $\lim_{t \rightarrow \infty} u(x_0, t) = u_0/2 = 1/2$ .

## 12 2.2

3. The solution of the latter Cauchy problem is  $u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(s)ds$ , and the solution of the first Cauchy problem is the partial derivative of the solution of the latter Cauchy problem in  $t$  direction which by fundamental theorem of calculus is  $\frac{1}{2}(\phi(x-ct) + \phi(x+ct))$ .

## 13 Quiz 2

$u_{tt} = 4u_{xx} + e^{x+t}$ ,  $u_t(x, 0) = 0$ ,  $u(x, 0) = \sin x$ .

Solution: By Dahamel's principle,  $u_{tt} = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \int_0^t \frac{1}{4} \int_{x-2t+2s}^{x+2t-2s} e^{r+s} dr ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4} \int_0^t e^{x+2t-s} - e^{x-2t+3s} ds = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(e^{x+2t} - e^{x+t}) - \frac{1}{12}(e^{x+t} - e^{x-2t})$ .