Project Summary

Overview

The aim of this project is to investigate graphs and graph maps using ideas from complex dynamics, symbolic dynamics and complex analysis. The research goals will be focus on the following three related topics:

- **Entropy of maps on intervals and Hubbard trees** The PI and his collaborators conjectured some properties on the distribution of Galois conjugates of the exponent of core entropy of quadratic maps as well as an analogy of the Julia-Mandelbrot correspondence for the setting of interval maps as well as maps on the Hubbard tree.
- **Estimating the asymptotic translation length on sphere complexes of train track maps on finite graphs** The PI and his collaborators are working on finding analogies of their prior works on the asymptotic translation length of pseudo-Anosov maps in the setting of train track maps on finite graphs.
- **Divisors and differential operators on metric graphs** Motivated by the analogy between graphs and surfaces above, the PI and his collaborators are also working on finding the analogies of certain results on the geometry and analysis on Riemann surfaces in the setting of metric graphs, for example, those about the ζ-functions and regularized determinants of differential operators.

Intellectual Merit

Entropy of maps on Hubbard tree and translation lengths on curve complexes are important questions that have been extensively studied, and it is hoped that the results in this project will shed new lights on these topics. Furthermore, it is hoped that the results obtained in this project will be a special case of more general results, for example generalized Milnor-Thurston kneading theory that applies to maps on (finite or infinite) trees or graphs, Julia-Mandelbrot type correspondences between the dynamic plane and parameter plane for general iterated function systems or similar systems associated with a symbolic dynamics, or generalized McMullen's polynomial that can encode the translation length on curve complexes.

Broader Impacts

The PI is currently mentoring students in the Undergraduate Research Scholars program based on some questions from this project. The PI have, and will continue to disseminate the results through talks to both mathematical and non mathematical audience. For example, the PI has previously used some materials from the third topic above for a short course in the KAIX (KAIST Advanced Institute for Science) summer school with an audience of undergraduate and beginning graduate students in both math and applied math majors. Furthermore, some algorithms that can potentially be obtained from this project may be useful for real-life purposes like surface matching or dimension reduction.

Project Description

The main goal of this project is to apply ideas in dynamics and analysis to the study of graphs and graph maps. The project can be divided into three related topics which will be further described below:

- Topological entropy of interval maps and maps on Hubbard tree.
- Asymptotic translation length on curve complex and sphere complex.
- Graph-theoretic analogies of some theorems on Riemann surfaces.

The main motivations of this project are to investigate the relationship between dynamics in two dimensions and one dimension, via traintracks [70, 64] and Hubbard trees [21], and the analogy between compact Riemann surfaces and finite metric graphs. The next three sections will be an outline of background and prior results of these three topic, the questions the PI will work on and partial results towards the resolution of these questions.

1 Topological entropy of maps on Intervals and on Hubbard tree.

1.1 Background and Prior Works

1.1.1 Interval maps

The study of dynamics on intervals is related to continuous fractions and β-expansions [32, 49, 27, 56, 61], complex dynamics [21, 38, 50, 46, 62, 66, 67], and via Milnor-Thurston kneading theory [48], to the study of symbolic dynamics and iterated function systems [9, 8, 57, 19].

Topological entropy is a way to characterize the complexity of a dynamical system. In the case when the system admits a Markov decomposition, the exponent of the topological entropy is the eigenvalue of a Perron-Frobenious matrix, and hence must be an algebraic integer. In the case when the map is continuous and *post-critically finite* (the forward orbit of all critical points is a finite set), the union of these orbits determines a Markov partition of the interval, hence it is possible to investigate the distribution of the Galois conjugate of the exponents of the entropies. In his last paper [63], Willian Thurston proposed the *master teapot*, which is the set

$$T := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

Here h_{top} is the topological entropy, and $\mathcal{F}$ is the set of unimodal, critically periodic interval self-maps. Here an interval map is called *unimodal* if it has a single critical point c in the interior of the interval, and *critically periodic* if there is some $n > 0$ such that $f^{\circ n}(c) = c$. The orthogonal projection of the Master teapot to $\mathbb{C}$ is called the *Thurston set*. The study of Thurston's master teapot and Thurston set can be seen as providing necessary conditions for an algebraic integer λ to be the exponent of the entropy of a critically periodic unimodal map: if λ is the exponent of such entropy, all its Galois conjugates must be in the Thurston set, and if z is a Galois conjugate of λ, (z, λ) must be in T.

In [68], a characterization of the Thurston set is given and the part of it inside the unit circle is shown to be the closure of the set of roots of all Littlewood polynomials (polynomials with all coefficients ± 1) [40], and [19] showed that there are holes in the Thurston set other than the trivial one around zero. Figure 1 and Figure 2 are finite approximations of the Thurston set and Thurston teapot respectively.

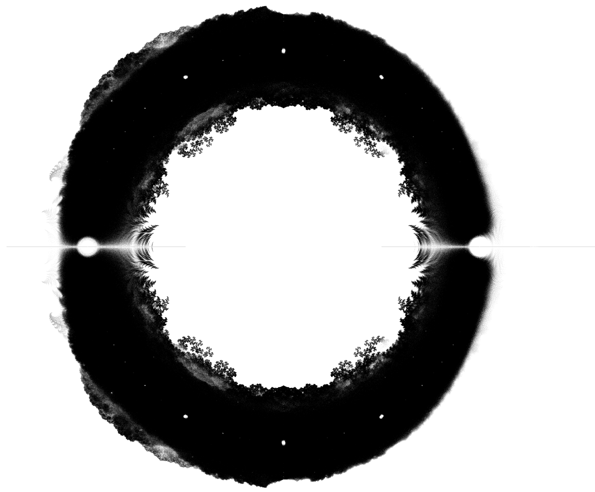

Figure 1:

In [18, 39], the PI and his collaborators proved a characterization of the part of the Thurston teapot inside the "unit cylinder" $\{(z, \lambda) : |z| < 1\}$. This characterization is analogous to the characterization of the Thurston set in [68]. [39] presents an algorithm that can be used to certify that a point does not belong to the Thurston teapot. In particular, given any $\lambda \in (1, 2)$, the PI and his collaborators found a subset $M_\lambda \subseteq \{0, 1\}^{\mathbb{N}}$ that is invariant under shift, non-decreasing with λ, and for any $|z| < 1$, $(z, \lambda) \in T$ iff there is some $w = w_1 w_2 \cdots \in M_\lambda$, such that

$$G(w, z) := \lim_{n\to\infty} f_{w_1,z} \circ \ldots \circ f_{w_n,z}(1) = 1$$

Here $f_{0,z}(x) = zx$ and $f_{1,z}(x) = 2 - zx$.

As a consequence,

Theorem 1. [18] If $(z, \lambda) \in T$, $|z| < 1$, then for any $\lambda' \in [\lambda, 2]$, $(z, \lambda') \in T$.

In other words, the intersection of the teapot with horizontal unit discs $\{(\lambda, z) : |z| \leq 1\}$ grows monotonously with the height λ.

For part of the teapot outside the unit cylinder a characterization can be easily derived from observations in [63, 68], which was also reviewed in [18, 39]. In particular,

Theorem 2. The set $\{z : (z, \lambda) \in T \text{ or } |z| \leq 1\}$ varies continuously with λ under the Hausdorff topology.

In other words, the part of the teapot outside the unit cylinder consists of countably many curves transversal to horizontal planes.

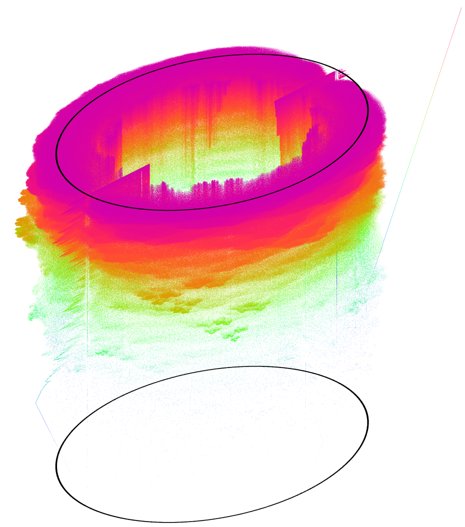

Figure 2:

1.1.2 Maps on Hubbard Trees

The dynamics of unimodal interval maps can be seen as a special case of the dynamics on Hubbard trees of quadratic maps. Consider complex quadratic maps $f_c : z \mapsto z^2 + c$, the *filled Julia set* is the set of initial values z whose orbit does not converge to infinity, the *Julia set* is the boundary of filled Julia set, the *Mandelbrot set* is the set of parameters c such that the corresponding Julia set is connected, or, equivalently, the set of parameters c where $f_c^{\circ n}(0)$ does not go to infinity as $n \to \infty$. There is a unique Riemann map Φ from the complement of the unit disc to the complement of the mandelbrot set, such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, and the ray $\{\Phi(\rho e^{i\theta}) : \rho \geq 1\}$ is called the *external ray* with angle θ. Similar to the case of interval maps, f_c is said to be *post-critically finite* if the forward orbit of 0 is finite, and *critically periodic*, or *superattracting*, if the forward orbit of 0 is periodic.

A way to completely encode the dynamics of postcritically finite quadratic maps is through the concept of the *Hubbard tree* [21], which is a tree connecting the forward orbit of 0 that lies within the filled Julia set. The quadratic map induces a continuous map on the Hubbard tree. Hubbard trees can also be defined for higher degree polynomial maps, and infinite Hubbard trees can be defined for some non post-critically finite maps. The topological entropy of the map on the Hubbard tree is called by Thurston the *core entropy*. Thurston proposed a combinatorial algorithm [25] via the direction of the external rays landing on the Mandelbrot set. Furthermore, Tiozzo showed in [66] that the core entropy is related to the Hausdorff dimension of the rays landing on the corresponding vein in the Mandelbrot set, and in [67] that the core entropy is continuous with respect to the angle of the external ray associated with it, resolving a conjecture by Thurston.

Now a critically periodic unimodal map can be seen as a superattracring parameter on the real slice of the Mandelbrot sets, and the study of the topological entropy of these interval maps is the same as the study of the core entropy on these points.

1.2 Questions and partial results

Motivated by prior works by the PI and others, as well as by evidences from numerical experiments, the PI and his collaborators proposed the following conjectures:

Conjecture 1.

1. When one replace $\mathcal{F}$ in the definition of T with interval maps with at most 2, critical values, there is an analogous characterization of points in T. In particular, both Theorem 1 and Theorem 2 remains true.

2. The set of Galois conjugates of the exponent of the core entropy of postcritical parameters union with the unit disk changes continuously with the angle of corresponding external ray. This is a strengthening of the main result in [67].

3. Along principal veins of the Mandelbrot set there is an analogy for Theorem 1 regarding the closure of Galois conjugates of the exponent of core entropy of superattracting parameters.

4. [39] For any complex number $|z| < 1$, any $\lambda \in (1, 2)$, the set $X_z := \{x - z : G(w, x) = 1 \text{ for some } w \in M_\lambda\}$ is asymptotically similar to the set

$$J_z = \{G(w, z) - 1 : w \in M_\lambda\}.$$

Here two sets A and B are asymptotically similar means that there exists a real number $r > 0$ and sequences $(t_n), (t'_n) \in \mathbb{C}$ with $t_n, t'_n \to \infty$ such that, denoting Hausdorff distance by d_{Haus},

$$\lim_{n\to\infty} d_{\text{Haus}}\left(\overline{B_r(0)} \cap (t_n A), \overline{B_r(0)} \cap (t'_n B)\right) = 0.$$

This is analogous to the Julia-Mandelbrot correspondence [37], where the set X_z is analogous to the Mandelbrot set while the set J_z is analogous to the filled Julia set. The case when $\lambda = 2$ was conjectured by Baez, Christensen, Derbyshire and Egan.

At the moment the PI and his collaborators are able to show Conjecture 1 provided the kneading determinants (cf. [48]) is irreducible for sufficiently many interval maps, and Conjecture 3 provided the denominator of the power series Δ in [66] is irreducible for sufficiently many superattracting points. Furthermore, the PI and his collaborators hope that the techniques in the resolution of Conjectures 2 and 3 can be applied to pseudo Anosov maps on infinite translation surfaces.

2 Asymptotic translation length on curve complex and sphere complex

2.1 Background and prior works

A pseudo-Anosov map on a closed surface S is a self-homeomorphism such that no power of the map sends a simple closed curve that does not bound a disc to itself up to isotopy. Up to homotopy, one can make such a map preserve a pair of transverse singular measured foliations. The stretch factors on these measured foliations is the exponent of the topological entropy of this map. These stretch factors are always bi-Perron algebraic integers, and it is an open problem if all bi-Perron algebraic integers can be realized as a stretch factor. McMullen [44] proved that, for pseudo-Anosov maps that have homeomorphic mapping torus and lie in the same *fibered cone*, their stretch factors are related to each other via a Teichmüller polynomial, one of its factors is the Alexander polynomial of the mapping torus. The definition of *fibered cone* first introduced in [65] is as follows:

Definition 2. [65]

- Let M be a hyperbolic 3-manifold, for every integer cohomology class $\alpha \in H^1(M;\mathbb{Z})$, the *Thurston norm* of α is defined as

  $$\|\alpha\| = \min_S \max\{0, -\chi(S_i)\} \ ,$$

  where $S = \bigcup_i S_i$ is an embedded surface that represents α. The Thurston norm can be extended to $H^1(M;\mathbb{R})$ as a piecewise linear function with rational coefficients.

- If M is homeomorphic to a mapping torus of a surface map ϕ (seen as a surface bundle over the circle), let the cohomology class associated with ϕ, denoted as α_ϕ, be the pullback of the generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called *fibered cones*, and the map associated with a primitive integer class α in the fibered cone are denoted as ϕ_α.

Besides stretch factors, another way of characterizing the topological complexity of a pseudo-Anosov map is through its asymptotic translation length on the curve graph of S. The curve graph $\mathcal{C}(S)$ is a graph where each vertex represents an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. It is easy to see that the mapping class group of S acts on the curve graph and curve complex by isometry. Masur-Minsky show [42] that $\mathcal{C}(S)$ is δ-hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserve a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. The study of curve graphs are also related to the hirerachical hyperbolic structure on the mapping class group ([43, 10]). The asymptotic translation length of a pseudo-Anosov map g on $\mathcal{C}(S)$ can now be defined as

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n} \ ,$$

where γ is any vertex in $\mathcal{C}$.

It is shown in [42] that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov. Furthermore, the technique in [42] for showing the positivity of $l_{\mathcal{C}}$, which is based on studying the incidence matrix on the induced map on invariant traintracks, has been used by [24, 69, 3] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases. Furthermore, in [17] the asymptotic translation length is shown to be a rational number, and in [54, 15] algorithms for its computation are described.

In [35], a sequence of pseudo-Anosov maps in different genus were constructed that realized the asymptotic lower bound. The PI and his collaborators generalized the argument in [35] and proved the following:

Theorem 3. [4] Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

Balázs Strenner [58] also proved a stronger result for the asymptotic translation length of arc complexes.

In [2], the PI and his collaborators proved that this asymptotic upper bound is sharp when $r \leq 3$. Furthermore, in [2], the PI and his collaborators used techniques similar to [4] to show the following:

Theorem 4. Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.

A key tool for proving the above results is the encoding of pseudo-Anosov dynamics by the induced map on the invariant *traintracks* [70, 64], which are finite graphs smoothly embedded into the surface and where each vertex is a "switch", and the associated edges arrives at two opposite directions. The topological entropy of the pseudo-Anosov map can be calculated via the action on the train track. A way to extend this relationship between two dimensional and one dimensional dynamics is through the Culler-Vogtmann Outer space [20], which serves the same purpose as Teichmüller space does for the study of $Out(F_n)$. The outer space on which $Out(F_n)$ acts on consists of all finite metric graphs with n loops and no degree 1 vertices, together with a marking map which is a homotopy equivalence between this metric graph and the wedge of n circles, where $Out(F_n)$ acts by precomposition with the marking. Similarly, the Teichmüller space consists of Riemann surfaces with prescribed topological type, together with a marking map which is a homotopy equivalence from a model surface to the Riemann surface. Both the outer space and the Teichmüller space are contractable, and the group action described above are proper with finite stablizers. This analogy made it possible to generalize many concepts and properties on Riemann surfaces and Teichmuller spaces, e.g. Nielsen–Thurston classification, train tracks [14], measured laminations [13], Teichmuller metrics, fibered cones, McMullen's polynomials [23, 22], and curve complexes [29], and is a fruitful approach for the study of the properties of outer automorphism group of free groups.

2.2 Questions and partial results

Motivated by the relationship between pseudo-Anosov maps and the induced map on the invariant train track as well as the analogy between $Out(F_n)$ and the mapping class group, the PI and his collaborators are working on finding the analogy of Theorem 3 in the setting of finite metric graph. The PI and his collaborators proved an analogy of the above theorem in the setting of asymptotic translation lengths on the sphere complex for train track maps, which is an analogous characterization of the topological complexity of graph maps, and will show up in an upcoming paper. The PI and his collaborators hope that this can be used to strengthen the bounds in [4, 2] or to generalize them to other families of pseudo-Anosov maps.

Let G be a finite simplicial graph. A cellular map $\psi : G \to G$ is called a train track map if the restriction of ψ^n to e for each $n \geq 1$ and each edge e is an immersion (no back-tracking condition).

We further assume ψ to be irreducible as an element of $Out(F_n)$. One can make a 3-manifold M_G from G by replacing every edge with $S^2 \times I$ and every vertex with $\mathbb{S}^3$. In the case when ψ is a train track map, ψ induces a homeomorphism ψ_1 on M_G. Let $\mathcal{S}(G)$ be the simplicial graph whose vertices are isotopy classes of embedded spheres in M_G, and there is an edge of length 1 between two vertices if and only if they are disjoint up to isotopy, then it is easy to see that ψ_1 is an isometry of $\mathcal{S}$ and we can define the concept of asymptotic translation length of ψ_1 analogously.

In [23, 22], the arguments in [65] and [44] are found to have analogies in the case of maps on finite graphs as follows:

Definition 3. [23, 22] Suppose ψ is an irreducible train track map, let γ_ψ be a folding path of ψ in the Culler-Vogtmann outer space. The *folded mapping torus* N is a 2-d cell complex built from γ_ψ, which has a surjection over the circle and the fibers are the graphs in the folding path. A flow on N is defined such that any flow line is the orbit of a point on the graph under folding, and an analogy for the fibered face containing ϕ is $\mathcal{S}$ which consists of first cohomology classes whose dual are transverse to all flow lines. This is a rational cone call the “cone of sections” or “McMullen cone” in [22].

In an upcoming paper, the PI and his collaborators proved the following:

Theorem 5. Given any finite graph G and any irreducible train track map ψ on G, let $\mathcal{C}$ be any proper subcone of the McMullen cone in [44] containing ψ, then any primitive integer element α in $\mathcal{C}$ must satisfies

$$l(\psi_\alpha) \lesssim n_\alpha^{-1-1/d}$$

Where d is the dimension of the fibered cone, n_α the genus of the fiber corresponding to α and ψ_α the corresponding monodromy, and $l(\cdot)$ the translation length on the sphere complex obtained by thickening the graph G.

Some further questions the PI and his collaborators are working on are the following:

Question 1.

- What is the relationship between the translation length on the sphere complex of the thickened invariant train track and the curve complex? The PI and his collaborators hope that this can be useful for generalizing Theorem 3 to families of pseudo Anosov maps that do not lie in the same fibered cone, for example, those arising from maps on a fibered cone under a subsurface projection.
- Can similar results be proved for other complexes related to $Out(F_n)$, like the free splitting complex [29], free factor complex [12], or the cyclic splitting complex [41]?
- Can there be a lower bound for the asymptotic translation lengths in the case of train track maps or pseudo-Anosov map that shows that the upper bound is asymmtotically optimal?
- Can there be an analogy of Theorem 4 in the case of train track graphs?

3 Graph-theoretic analogies of some theorems on Riemann surfaces

3.1 Background and prior works

In addition to Culler-Vogtmann outer space, another way to understand the analogy between Riemann surfaces and metric graphs is through tropical geometry and Berkovich spaces (cf. [11]). Just as Riemann surfaces can be seen as analytic spaces of complex algebraic curves, the Berkovich spaces of dimension one serve as an analytic space for curves on non-Archimedean fields. They have a natural geometric structure and, by collapsing $\mathbb{R}$-trees, can be turned into finite metric graphs. It is possible to construct, by analogy, harmonic functions and maps, Jacobians, and divisors, and show that they satisfies many of the same properties as in the case of Riemann surfaces, cf. the works of Baker, Chinberg, Rumely, de Jong, Shokrieh etc. ([36, 6]).

By a metric graph, we mean an undirected simplicial graph with a positive edge length function l defined on the edges, and where each edge e is made into an interval of length $l(e)$. When the graph has finitely many edges we call it a finite metric graph. The following definitions are standard:

Definition 4. [5, 55] Let G be metric graph.

1. By a 1-form on G we mean an element in the 1-simplicial cochain $C^1(G)$. Let $d : C^0(G;\mathbb{R}) \to C^1(G;\mathbb{R})$ be the coboundary map.
2. Let G be a metric graph with edge set $E(G)$, w a 1-form, we define the norm of w as $\|w\| = (\sum_{e\in E(G)} w(e)^2/l(e))^{1/2}$. We call a 1-form L^2 if it has a finite norm. Under this norm, denote the dual of d δ.
3. We call a 1-form α harmonic, if $\delta(\alpha) = 0$. In other words, for every vertex v, all outgoing edges e_i from v, $\sum_i w(e_i)/l(e_i) = 0$.

The analogy between Riemann surfaces and finite metric graphs allow many important concepts and properties on Riemann surfaces to have analogies in the graph-theoretic setting. For example, there is a graph-theoretic Riemann-Roch theorem [7] which is related to the question of chip-firing games on graphs, which is as follows:

Theorem 6. [7] Let G be a finite simplicial graph with no loops, all edges are assumed to have length 1. A *divisor* on G is an integer linear combination of vertices of G, i.e. an element in $C^0(G;\mathbb{Z})$, the degree is the sum of all the coefficients in this linear combination. A divisor D is called *effective* if all its coefficients are non negative, two divisors D and D' are *linearly equivalent* if $\delta d(D - D') = 0$, and $r(D)$ is the largest natural number r such that $D - E$ is linearly equivalent to an effective divisor for any effective divisor of degree r, or, if no such natural number exists, -1. Then

$$r(D) - r(K - D) = deg(D) + 1 - g$$

Here g is the first betti number of G, K is the *canonical divisor* $K = \sum_v (\deg(v) - 2)(v)$.

The proof however is very different from the Riemann-Roch theorem on surfaces, as r can no longer be interpreted as the dimension of some vector space. The theorem can be generalized to arbitrary metric graphs as well [47, 31, 26].

Also, analogous to the Arakelov metric on Riemann surfaces, there is the “canonical metric” on finite metric graphs [71, 52], which is defined as

Definition 5. The canonical metric [71, 52] on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

This is shown by [1, 51] to be related to the distribution of Weierstrass points of divisors on finite metric graphs.

Graph theoretic analogies of differential operators and their spectral theory have also been extensively studied, see for example [5, 30, 60].

In a prior work of the PI and his collaborator, an analogous result on a property by Kazhdan [34] (see also [33], [45, Alppendix]) of the Arakelov metric on Riemann surfaces was found, which shows that when passing to larger and larger normal covers the canonical metric on metric graphs converges:

Theorem 7. [55] Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.

In the case when $\cap_i \pi_1(G_i)$ is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of G. Because in [45, Alppendix] the limit of the Arakelov metrics for a closed Riemann surface of genus ≥ 2 under a tower of normal covers that converges to the universal cover would converge to the hyperbolic metric, one can see the limiting metric in Theorem 7 as a discrete analogy of the hyperbolic metric. However it is unclear what the relationship between this limiting metrics and other approaches of discrete uniformization [16, 28, 53] are.

3.2 Questions

Currently, the PI and his collaborators are working on the following questions in this direction:

Question 2.

- Following [5], let Δ_μ be the Laplace operator defined on the set of piecewise C^2 functions with L^1 second derivative, whose average regarding the canonical measure μ_{can} is zero, which is the usual graph Laplacian composed with subtraction of average under μ_{can}. There are infinitely such eigenvalues whose only limit point is infinity [5]. What else can we say about these eigenvalues? What can we say about the ζ function $\zeta(z) = \sum_i \lambda_i^{-z}$ and the regularized determinant $e^{-\zeta'(0)}$? What is the relationship with other ζ functions on graphs [59]? What is the relationship with the determinant of discrete Laplacian, which is related to spanning tree counting?

- Can the Riemann Roch theorem be explained as the index theorem of some differential operator? Is it related to the ζ function and regularized determined in the question above?

Broader Impacts

Many results and techniques used in this project are accessible to an undergraduate audience while illustrating important concepts in analysis and dynamics, hence would serve as good examples that can be used in a classroom setting or outreach activities and can also facilitate possible undergraduate research. The PI is currently mentoring students in the Undergraduate Research Scholars program based on some questions from this project. The PI is organizing the dynamics seminar in university of Wisconsin which is attended by grad students. The PI have, and will continue to disseminate the results through talks to both mathematical and non mathematical audience. For example, the PI has previously used some materials from the third topic above for a short course in the KAIX (KAIST Advanced Institute for Science) summer school with an audience of undergraduate and beginning graduate students in both math and applied math majors. Furthermore, some algorithms that can potentially be obtained from this project may be useful for real-life purposes like surface matching or dimension reduction.

Results From Prior NSF Support

The PI has not previously been supported by the NSF.

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(a) Professional Preparation

- Cornell University, Ithaca NY, Mathematics, PhD 2016
- Peking University, Beijing, China, Mathematics BS 2010

(b) Appointments

- University of Wisconsin at Madison, Assistant Professor, 2020-
- Rutgers University, Hill Assistant Professor, 2017-2020
- Max-Planck Institute of Mathematics, Postdoc, 2016-2017

(c) Publications

- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805, conditionally accepted by *Advances in Mathematics*, 2020
- Kathryn Lindsey and Chenxi Wu, Characterization of the Shape of Thurston's Teapot *arXiv:*1909.10675
- Hyungryul Baik, Hyunshik Shin and Chenxi Wu. An upper bound on the asymptotic translation length on the curve graph and fibered faces *arXiv:* 1801.06638, accepted by *Indiana University Math Journal*, 2020
- Hyungryul Baik, Eiko Kin, Hyunshik Shin and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone *arXiv:*1909.00974, submitted.
- Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019. doi: 10.1007/s00222-018-0838-5

Other Significant Publications

- Hyungryul Baik, Farbod Shokrieh, Chenxi Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019. doi: 10.1515/crelle-2019-0007.
- Hyungryul Baik, Ahmad Rafiqi and Chenxi Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019. doi: 10.1017/etds.2017.109
- Chenxi Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics*, 214(2), 733-740, 2016. doi: 10.1007/s11856-016-1357-y

- Chenxi Wu. The relative cohomology of abelian covers of the flat pillowcase. *Journal of Modern Dynamics*, 9, 123-140, 2015. doi:10.3934/jmd.2015.9.123
- Hyungryul Baik, Chenxi Wu, KyeongRo Kim, and TaeHyouk Jo. An algorithm to compute Teichmüller polynomial from matrices *Geometriae Dedicata* 2019 doi: 10.1007/s10711-019-00450-4

(d) Synergistic Activities

- Mentor at Undergraduate Research Scholars program in the University of Wisconsin at Madison, from September 2020
- Gave lectures in KAIX (KAIST Advanced Institute for Science) Summer School in Daejeon, Korea, in August 2019
- Volunteered for F.E.M.M.E.S. (Women+ Excelling More in Math, Engineering and the Sciences) event at University of Michigan in November 2017
- Gave lectures in Math Explorer's Club at Cornell in March 2016