

# 1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example:  $u_y = x$ .

Example:  $(yu)_y = 1$ .

General solution of a PDE.

Formally: PDE:  $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde

Linear PDE.

Linear homogeneous PDE.

What are the order and linearity of the following PDEs?

$u_x + u_{yyx} = 1$ ,  $uu_x + u = 0$ ,  $u_x + (x^2 + y^2)u_{yy} = 1$ .

Some PDEs we will focus on later:

Heat:  $u_t = u_{xx}$ : (heat transmission, diffusion)

Laplace:  $u_{xx} + u_{yy} = 0$ : (static electric field, Newton's gravity, equilibrium of random walk)

Wave:  $u_{tt} = u_{xx}$ : (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations:  $u_{tt} = u_{xx} - ku_{xxxx}$  (stiff string)

Cauchy-Riemann equation:  $u_x = v_y$ ,  $u_y = -v_x$

Non-linear PDEs you may see in later classes:

Navier-Stokes

Nonlinear Schrodinger:  $iu_t = -\Delta u + k|u|^2 u$

KdV:  $u_t + u_{xxx} + 6uu_x = 0$ , etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow)  $u_t = A \frac{\nabla u}{|\nabla u|} \cdot \nabla u + B|\nabla u| \nabla \cdot \frac{\nabla u}{|\nabla u|}$ .

Types of problems:

Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.

Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:

Existence

Uniqueness

Regularity

Continuous dependency on boundary

Typical strategy: integral transform:  $(Tu)(y) = \int u(x)K(x,y)dx$ , then  $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$ , assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)

Example: solve  $u_{xx} + u_{yy} = 1$ , where  $u = 0$  on the unit circle.

Example:  $u_x = u_t$ ,  $u_x = u_t + 1$ .

## 2 9/7 Review of ODE, Advection and Diffusion

Review of ODE & multivariable calculus topics:

- $u' + p(t)u + q(t) = 0$
- $u''' + Au'' + Bu' + Cu = 0$
- Chain rule: Example:  $u_{xx} = u_{tt}$ , what happens with change-of-variable  $y = x + t$ ,  $w = x - t$ ?
- Fubini's theorem.
- Differentiating an integral. Example:  $\frac{d}{dt} \int_0^{t^2} \sin(ts) ds$ .  
Solution: Let  $x = t$ ,  $y = t$ , then  $\frac{d}{dt} \int_0^{t^2} e^{-ts^2} ds = \frac{d}{dt} \int_0^{x^2} e^{-ys^2} ds = (\int_0^{x^2} e^{-ys^2} ds)_x + (\int_0^{x^2} e^{-ys^2} ds)_y = 2x \cdot e^{-y(x^2)^2} + \int_0^{x^2} (e^{-ys^2})_y ds = 2xe^{-y(x^2)^2} - \int_0^{x^2} s^2 e^{-ys^2} ds = 2te^{-t^5} - \int_0^{t^2} s^2 e^{-ts^2} ds$ .
- Example:  $u_{tt} = u_{xx} + u_{yy}$ ,  $u(x, y, t) = \sin(x \cos \theta + y \sin \theta + t)$  are solutions, hence  $\int_0^{2\pi} \sin(x \cos \theta + y \sin \theta + t) d\theta$  is also a solution.

PDE from conservation laws, 1-dimensional case:

Consider the flow of some material whose total quantity remain unchanged, along a thin tube with section area  $A(x)$ . Then, conservation means:

$$\frac{d}{dt} \int_a^b u(x, t) A(x) dx = A(a) \phi(a, t) - A(b) \phi(b, t) + \int_a^b f(x, t) A(x) dx$$

$\phi$ : flux.  $f$ : source.

Differentiate w.r.t.  $b$  one gets:  $Au_t = -A\phi_x - A'\phi + fA$ .

- $\phi = u$ : e.g. cars which travels at the same speed, age distribution etc.
- $\phi = -u_x$ : heat conduction etc.
- $\phi = u - u_x$ : contaminated flow etc.
- $f = -u$ : decay.

Relationship with random motion: see  $u(\cdot, t)$  as the probability distribution.

Example:  $u_t = u_x - u$ . Decay vs. "widening".

Example:  $u$  has two components (e.g. mass, momentum): wave equation.

### 3 9/12 Method of characteristics

Question: first order linear PDE in 2 dimension:  $u_t + fu_x + gu + h = 0$

First consider the case when  $g = h = 0$ . Recall that for 1st order ODE, there is a concept of *first integral*: the solution of  $x'F_x + F_t = 0$  are the level curves of  $F(x, t)$ . Hence, the level curves of  $u$  are exactly the solutions of  $x' = f$ , which are called *characteristics*.

Example:  $u_t = xu_x - u$ .

Example:  $u_t = u_x + u_y$ .

Example:  $u_t = \sin t u_x + 1$ .

Non-linear advection:  $u_t = f(u)u_x$ : level curves are straight lines of slope  $f(c)$ . Breaking time.

Example:  $u_t = (1 - u)u_x$ .