CAREER: Dynamics and geometry of graph maps and surfaces maps via handlebody groups

Project Summary

Overview

There is a well known analogy between closed Riemann surfaces and finite metric graphs, as well as mapping classes on them, from the perspective of Teichmüller spaces vs Culler-Vogtmann outer space, from Berkovich analytic spaces of curves, or from train-track maps induced by a pseudo-Anosov mapping class. The main goal of the project is to study the connection between graph maps and surface maps, in particular from the perspective of handlebody groups. The boundary of a handlebody is a surface and the handlebody has the homotopy type of a graph, hence their self homeomorphisms provides another way to relate mapping classes on surfaces and on graphs, and an open question is how to relate the dynamics of resulting surface and graph mapping classes. The PI has done prior works on the geometry and dynamics of both graph maps and surface maps, and is planning on combining insights from prior work to provide a better understanding of this problem as well as investigate possible applications.

The PI will also carry out educational and outreach activities in connection with the research project. The PI plans to incorporate the results and problems from the research project into teaching and mentoring in both undergraduate and graduate levels, and also in new outreach programs aiming at engaging underprivileged K-12 students.

Intellectual Merit

Handlebodies and handlebody groups are important tools in low dimensional topology and geometry. Through this research project the PI hopes to shed new lights on the connection between 3-dimensional topology, symbolic dynamics and Teichmüller theory, and it is hoped that the results will be helpful to the study of certain open problems in these area, e.g. the approximation conjecture for L^2 torsions, determining whether or not an algebraic integer is a pseudo-Anosov stretch factor, and the various conjectures on the shape of Mandelbrot set and its higher degree analogies.

Broader Impacts Of The Proposed Work

The project involves questions at varying levels of abstraction and therefore should be useful for integration into teaching and outreach activities. Furthermore, algorithms obtained in this project has potential real life applications in areas like machine learning and numerical analysis, and has the potential of furthering partnership between academia and industries. The teaching and outreach activities proposed will also increase mathematical literacy among the general public, as well as improve STEM education, particularly in underserved communities.

Project Description

Background And Overview

The study of mapping class groups (the group of homotopy classes of homeomorphisms from a closed oriented surface to itself) as well as the $Out(F_n)$ (the group of outer automorphism groups on free groups are important topics in geometric group theory, low dimensional topology and dynamics. The mapping class group acts on the Teichmüller space (which is the space of marked complex structures on the given surface, and is contractible and admits a complex structure and a number of geometrically important metrics) [36] as well as the curve complex of the surface [44] (which is Gromov hyperbolic), and a generic element in the mapping class group is pseudo-Anosov [28] (can be represented by a homeomorphism that preserves a pair of transverse singular foliations). Analogously, the group $Out(F_n)$ acts on the Culler-Vogtmann outer space [22], as well as the free factor [34] and free splitting [33] complexes, and one possible analogy of being pseudo-Anosov would be admitting an irreducible train-track representation [15].

One can go from a pseudo-Anosov map to a train track map on graph by simply taking its invariant traintrack [16]. This construction can be (fully or partially) reversed for certain $Out(F_n)$ elements [32, 24]. Another way to establish connections between graph and surface maps is via the concept of handlebody groups [35]. Furthermore, the analogy between mapping class groups and $Out(F_n)$, between closed Riemann surfaces and finite metric graphs, can also be understood via the perspective of Berkovich spaces [12]. The main goal of the research program of the PI is to deepen the understanding of these connections and apply them to important questions regarding the geometry and dynamics of surfaces and graphs.

The research project can be roughly divided into three interconnected components:

(i) **Understanding the complexity of graph maps**. Which may be characterized by topology entropy, translation length on free factor or free splitting complexes, or other quantities.

(ii) **Find good “approximations” of graph maps by surface maps**. For example, start with a graph map we can find infinitely many handlebody group elements corresponding to it, and we want one whose “complexity”, as measured by entropy or translation length on curve complex, is closed to the original graph map. The results in component 1 above would be useful for understanding when good approximations are possible.

(iii) **Applications to questions regarding the geometry and dynamics of graph and surface maps**. For example, the PI will use the relationship established above to study better estimates for the translation length on curve complexes, as well as to understand a conjecture by Koberda on the gap between stretch factor of a traintrack map on graph and the spectral radius on its homology.

For teaching and outreach, the PI will incorporate the results from research into my course designs in undergraduate and graduate level, as well as collaborating with Wisconsin Discovery Institute for outreach programs in the summer.

Research Plan

Below is a more detailed description of the three components of the research plan:

Understanding The Complexity Of Graph Maps

The first part of the research concerns with simplicial maps on finite graphs. Here a finite graph is a connected finite 1-dimensional simplicial complex, and a simplicial map is a continuous map that sends vertices to vertices and edges to a collection of edges. The PI plan to investigate the complexity of such maps via the following measurements:

- Topological entropy.
- A graph can be "thickened" into a double handlebody, and one can then investigate the translation length on the sphere complex.
- One can also measure the "complexity" of a graph by the smallest integer k such that there is an edge whose image under the k-th iterates contain itself. This is motivated by the study of lower bound of curve complex translation length in [44, 29, 6].

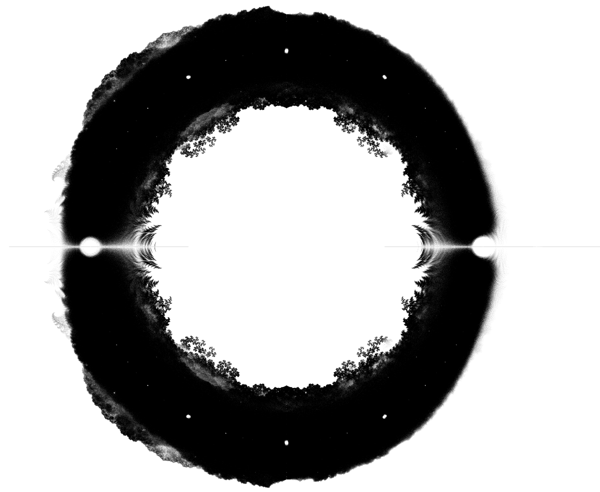
The study of simplicial maps on metric graphs has many applications in topology and dynamics: the self maps on intervals are related to conttinuous fractions and β-expansions [37, 50, 30, 57, 60], symbolic dynamics and iterative function systems [10, 9, 58, 21], point processes [51], and the study of similarity structure on surfaces; dynamics of post-critically finite polynomial maps can be related to simplicial maps on trees via the Hubbard tree [25, 40, 52, 48, 61, 64, 65]; pseudo-Anosov maps on surfaces can be related to maps on traintrack graphs [16]. Also, since the fundamental group of graphs is a free group, graph maps are related to outer endormorphisms or automorphisms of the free group.

The study of topological entropy A first example of simplicial maps on graphs is post-critically finite maps on intervals. Here by *post-critically finite* we mean the forward orbit of all critical points consists of finitely many points, and by *critically periodic* we mean the forward orbit of all critical points are periodic. A key tool in the study of self maps on intervals is the Milnor-Thurston kneading theory [49] which has also been generalized to the case of trees and more general graphs, as well as higher dimensional objects [1, 23].

When the map is critically periodic or postcritically finite, the dynamical system admits a Markov decomposition, the exponent of the topological entropy is the eigenvalue of a Perron-Frobenious matrix hence must be an algebraic integer, hence one can study the set of Galois conjugates of this exponent. In his last paper [62], Willian Thurston proposed the *master teapot*, which is the set

$$T := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

Here h_{top} is the topological entropy, $\mathcal{F}$ is the set of unimocal maps with periodic critical orbit. Here an interval map is unimodal if it has a single critical point c in the interior, it has periodic critical orbit if there is some $n > 0$ such that $f^{\circ n}(c) = c$. The projection of the Master teapot on $\mathbb{C}$ is called the *Thurston set*. One application of the Thurston set and Thurston teapot is that they provide necessary conditions on when an algebraic integer can be the exponent of topological entropy of a critically periodic unimodal map, as such an integer λ must have $(z, \lambda) \in T$ for all Galois conjugate z of λ. The properties of these sets have been extensively studied. In particular, [66] gives a characterization of the Thurston set and relate it to the roots of Littlewood polynomials (polynomials with all coefficients ± 1) [42]. Figure 2.2.1 and Figure 2.2.1 are finite approximations of the Thurston set and Thurston teapot respectively.



In some prior works by the PI and his collaborators [20, 41], an analogous characterization of the Thurston teapot is found, and an algorithm is given that can be used to certify a point not belonging to the Thurston teapot. In particular, given any $\lambda \in (1, 2)$, the PI and his collaborators found a subset M_λ of $\{0, 1\}^{\mathbb{N}}$ invariant under shift, non decreasing as λ increases, such that for any $|z| < 1$, $(z, \lambda) \in T$ iff there is some $w = w_1 w_2 \cdots \in M_\lambda$, such that

$$G(w, z) := \lim_{n\to\infty} f_{w_1,z} \circ \ldots \circ f_{w_n,z}(1) = 1$$

Here $f_{0,z}(x) = zx$ and $f_{1,z}(x) = 2 - zx$.

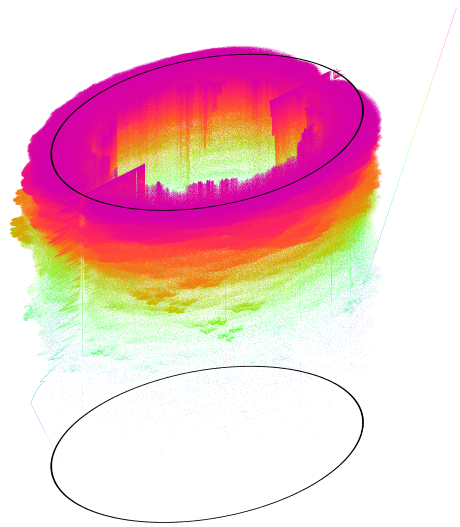
As a consequence, the PI and his collaborators found this surprising fact:

Theorem 2.1. *If $(z, t) \in T$, $|z| < 1$, then $(z, y) \in T$ for all $t \leq y \leq 2$.*

In a upcoming paper with Kathryn Lindsey and Ethan Farber, the PI will be able to generalize the above results to the multimodal case where there are exactly two critical values whose forward orbits are both periodic.

The interval can be seen as a special case of the Hubbard tree [25]. Here, the *Hubbard tree* is a finite simplicial tree whose vertices are orbits of the critical point and is contained in the Julia set. The quadratic map on the Julia set induces a simplicial map on the Hubbard tree, and the topological entropy of this simplicial map is called the *core entropy*. Now a critically periodic unimodal map can be seen as a superattracting point ($c \in \mathbb{C}$ such that $f_c : z \mapsto z^2 + c$ satisfies $f_c^{\circ n}(0) = 0$ for some $n > 0$) on the real slice of the Mandelbrot sets. In an upcoming paper with Kathryn Lindsey and Guilio Tiozzo, the PI and his collaborators, the characterization of the Thurston teapot in [20, 41] have generalized to the principal veins of the Mandelbrot sets. More precisely, for every superattracting parameter c, the Hubbard tree admits a Markov decomposition, so one can write down the incidence matrices and find all the eigenvalues. Now consider the vein with each hyperbolic component collapsed into one point, which becomes an interval I, and consider the product $I \times \mathbb{C}$. Now the analogy of the Thurston teapot T' is the closure of pairs (p, z) where p is a superattracting parameter in I and z an eigenvalue of the incidence matrix. We are able to prove:

Theorem 2.2.
- *The set $O_p = \{z : |z| \leq 1 \text{ or } (p, z) \in T'\}$ changes continuously with p under Hausdorff topology.*



- *The set $I_p = \{z : |z| < 1$ and $(p, z) \in T'\}$ is monotone increasing as p moves towards the tip of the principal veins.*

The first question for this project, which the PI and his collaborators have been working on, is:

Research question 1. *Can the characterization of the Thurston set above be generalized to more general veins, or to the case of Misiurewicz points?*

The key step for solving this question is to extend the "combinatorial surgery" by Tiozzo [64] to more general setting using the kneading theory on graphs as described in [1].

Also, motivated by evidences from numerical experiments, the PI and his collaborators proposed the following conjecture [41]:

Conjecture 2.3. *For any complex number $|z| < 1$, any $\lambda \in (1, 2)$, the set $X_z := \{x - z : G(w, x) = 1$ for some $w \in M_\lambda\}$ is asymptotically similar to the set*

$$J_z = \{G(w, z) - 1 : w \in M_\lambda\}.$$

This is analogous to the Julia-Mandelbrot correspondence [39], where the set X_z is analogous to the Mandelbrot set while the set J_z is analogous to the filled Julia set.

Here two sets A and B are asymptotically similar means that there exists a real number $r > 0$ and sequences $(t_n), (t'_n) \in \mathbb{C}$ with $t_n, t'_n \to \infty$ such that, denoting Hausdorff distance by d_{Haus},

$$\lim_{n\to\infty} d_{\mathrm{Haus}}\left(\overline{B_r(0)} \cap (t_n A), \overline{B_r(0)} \cap (t'_n B)\right) = 0.$$

Another question for the project is the following:

Research question 2. *Possibly via a strategy similar to [39], prove or disprove the conjecture for certain λ and z.*

Furthermore, because kneading theory can be generalized to maps on more general graphs, a third question the PI is currently working on is:

Research question 3. *Can the characterization of the Thurston set as well as Thurston teapots in [41] be generalized to train track maps on graphs?*

A well known question in Teichmuller dynamics is the description of all possible stretch factors for pseudo-Anosov maps. It is conjectured that the set of all stretch factors are exactly the set of bi-Perron algebraic integers (i.e. real algebraic integers $\lambda > 1$ whose Galous conjugates all lie between λ and $1/\lambda$). Similar to the unimodal case, here a characterization of the "Thurston set" or "Thurston teapot", or any set that contain them, for train track maps on a given graph would provide a necessary condition for an algebraic integer to be the stretch factor for pseudo-Anosovs of a surface of some given genus, and it is hoped that these necessary conditions would be helpful for the study of this conjecture or some weakened versions of it.

Translation length on sphere complexes The study of sphere complex translation length is motivated by the study of curve complex translation length for pseudo-Anosov maps. For pseudo-Anosov maps that have homeomorphic mapping torus and lie in the same *fibered cone*, their stretch factors are related to each other via the Teichmüller polynomial by McMullen [46], which are related to the Alexander polynomial of the mapping torus. Here the concept of *fibered cone* was introduced in [63]:

Definition 2.4. *[63]*

- *Let M be a hyperbolic 3-manifold, for every integer cohomology class $\alpha \in H^1(M;\mathbb{Z})$, the* Thurston norm *of α is defined as*

$$\|\alpha\| = \min_S \max\{0, -\chi(S_i)\}$$

  Where $S = \bigcup_i S_i$ is an embedded surface that represents α. The Thurston norm can be extended to $H^1(M;\mathbb{R})$ as a piecewise linear function with rational coefficients.

- *If M is homeomorphic to a mapping torus of a surface map ϕ (seen as a surface bundle over the circle), let the cohomology class associated with ϕ, denoted as α_ϕ, be the pullback of the generator of the first cohomology of the surface. There are faces of the unit ball of the Thurston norm such that any primitive integer cohomology class in the cone over them are associated with a surface map. The cones over these faces are called* fibered cones*, and the map associated with a primitive integer class α in the fibered cone are denoted as ϕ_α.*

Besides stretch factors, another way of characterizing the topological complexity of a pseudo-Anosov map is through its asymptotic translation length on the curve graph of S. The curve graph $\mathcal{C}(S)$ is a graph where each vertex represent an isotopy class of simple closed curve, and two vertices are connected by an edge (which we assume to be of length 1) if the corresponding curves can become disjoint under isotopy. It is easy to see that the mapping class group of S acts on curve graph and curve complex by isometry. Masur-Minsky [44] show that $\mathcal{C}(S)$ is δ-hyperbolic, and that the mapping class group elements that are pseudo-Anosov (i.e. those that preserves a pair of transverse singular measured foliations), are loxodromic isometries in the curve graph. The study of curve graphs are also related to the hirerachical hyperbolic structure on the mapping class group [45, 11]. The asymptotic translation length of a pseudo-Anosov map g on $\mathcal{C}(S)$ can now be defined as

$$l_{\mathcal{C}}(g) = \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n\gamma, \gamma)}{n}$$

where γ is any vertex in $\mathcal{C}$.

It is shown in [44] that $l_{\mathcal{C}}$ is well defined and non-zero if g is pseudo-Anosov. Furthermore, the technique in [44] in showing the positivity of $l_{\mathcal{C}}$, which is based on studying the incidence matrix on the induced map on invariant traintracks, have been used by [29, 67, 5] and others to provide asymptotics of the lower bound on $l_{\mathcal{C}}$ as the genus and number of punctures on S increases. Furthermore, in [19] the asymptotic translation length is shown to be a rational number, and in [55, 17] algorithms for its computation are described.

In [38], a sequence of pseudo-Anosov maps in different genus are constructed that realized the asymptotic lower bound. The PI and his collaborators generalized the argument in [38] and proved the following:

Theorem 2.5. *[7] Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion r that intersects with P. For every primitive element $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$,*

$$l_{\mathcal{C}}(\phi_\alpha) \lesssim \|\alpha\|^{-(1+1/(r-1))}$$

Balázs Strenner [59] also proved a stronger result for the asymptotic translation length of arc complexes.

In [4], the PI and his collaborators have shown that this asymptotic upper bound is sharp when $r \leq 3$. Furthermore, in [4], the PI and his collaborators uses techniques similar to [7] to show the following:

Theorem 2.6. *Suppose M is a closed hyperbolic 3-manifold and P a fibered cone in $H^1(M)$, L a rational subspace of $H^1(M)$ of dimendion 2 that intersects with P. Then for all but finitely many primitive elements $\alpha \in P \cap L \cap H^1(M;\mathbb{Z})$, ϕ_α is a normal generator of the corresponding mapping class group.*

Motivated by the relationship between pseudo-Anosov maps and the induced map on the invariant train track as well as the analogy between $Out(F_n)$ and the mapping class group c.f. [15, 14], the PI and his collaborators are working on generalizing Theorem 2.5 to the case of the metric graph. In an upcoming paper, the PI and his collaborators will prove an analogy of the above theorem in the setting of asymptotic translation lengths on the sphere complex for train track maps.

Let G be a finite simplicial graph. A cellular map $\psi : G \to G$ is called a train track map if the restriction of ψ^n to e for each $n \geq 1$ and each edge e is an immersion (no back-tracking condition). We further assume ψ to be irreducible as an element of $Out(F_n)$. One can make a 3-manifold M_G from G by replacing every edge with $S^2 \times I$ and every vertex with $\mathbb{S}^3$. In the case when ψ is a train track map, ψ induces a homeomorphism ψ_1 on M_G. Let $\mathcal{S}(G)$ be the simplicial graph whose vertices are isotopy classes of embedded spheres in M_G, and there is an edge of length 1 between two vertices if and only if they are disjoint up to isotopy, then it is easy to see that ψ_1 is an isometry of $\mathcal{S}$ and we can define the concept of asymptotic translation length of ψ_1 analogously.

In [27, 26], the argument in [63] and [46] are generalized to the case of maps on finite graphs as follows:

Definition 2.7. *[27, 26] Suppose ψ is an irreducible train track map, let γ_ψ be a folding path of ψ in the Culler-Vogtmann outer space. The* folded mapping torus *N is a 2-d cell complex built from γ_ψ, which has a surjection over the circle and the fibers are the graphs in the folding path. A flow on N is defined such that any flow line is the orbit of a point on the graph under folding, and an analogy for the fibered face containing ϕ is $\mathcal{S}$ which consists of first cohomology classes whose dual are transverse to all flow lines. This is a rational cone call the "cone of sections" or "McMullen cone" in [26].*

The Hyungryul Baik, Dongryul Kim and the PI are able to show the following:

Theorem 2.8. *[3] Given any finite graph G and any irreducible train track map ψ on G, let $\mathcal{C}$ be any proper subcone of the intersection of the McMullen cone in [46] containing ψ and the negative of the McMullen cone containing ψ^{-1}, then any primitive integer element α in $\mathcal{C}$ must satisfies*

$$l(\psi_\alpha) \lesssim n_\alpha^{-1-1/d}$$

Where d is the dimension of the fibered cone, n_α the genus of the fiber corresponding to α and ψ_α the corresponding monodromy, and $l(\cdot)$ the translation length on the spherical complex obtained by thickening the graph G.

The sphere complex is related to the free splitting [33] and free factor complexes [13], hence the theorem above gives us estimates on these complexes as well.

Some further questions the PI and his collaborators are working on are the following:

Research question 4.

- *What is the relationship between the translation length on the sphere complex of the thickened invariant train track and the curve complex? The PI and his collaborators hope that this can be useful for generalizing Theorem 2.5 to families of pseudo Anosov maps that do not lie in the same fibered cone, for example, those arising from maps on a fibered cone under a subsurface projection.*
- *Can there be a lower bound for the asymptotic translation lengths in the case of train track maps that shows that the upper bound is asymmtotically optimal?*
- *Can similar results be proved for other complexes related to $Out(F_n)$, like the cyclic splitting complex [43]?*
- *Can there be an analogy of Theorem 2.6 in the case of train track graphs?*

Other characterizations of combinatorial complexity The key step in [29, 6] for finding lower bound of the asymptotic translation length on curve complexes relies on the estimation of this quantity: let τ be the invariant train track of some pseudo-Anosov map ψ, ψ induces a train-track map ψ' on τ. Let k be the smallest integer where there is a real edge e of τ contained in ${\psi'}^k(e)$. This is also the key estimate for the lower bound in a paper by PI and his collaborators [38]. Furthermore, in an upcoming paper with Hyugryul Baik and Dongryul Kim, the PI is able to use similar estimate to generalize a result in [6] and prove that:

Theorem 2.9. *There is some $C > 0$, such that any pseudo-Anosov map that preserves a k-dimensional subspace of a genus g surface has asymptotic translation length on curve complex no smaller than $\frac{C}{g(2g+1-k)}$.*

The PI and his collaborators will work on the following questions:

Research question 5.

- *Can one find the relationship between the different k for train-track maps in the same fibered cone? If this can be found then one may be able to prove that the bound in [7] is optimal in general.*
- *Can the lower bound in Theorem 2.9 be improved?*

Approximating Graphs Maps With Surface Maps

A *handlebody* is a closed ball with finitely many "handles" of the form $D_2 \times I$, where D_2 is a 2-disk and I is a closed interval, being glued to its surface. The homotopy type of a handlebody with g handles is a ribbon graph $(S^1)^{\vee g}$, and the surface of the handlebody is a closed oriented surface with genus g. The *handlebody group* is the group of orientation preserving self homeomorphisms of the handlebody to itself up to isotopy. As a consequence, any handlebody group element gives rise to an element in $Out(F_n)$ (due to the homotopy equivalence between handlebodies and graphs), as well as an element in the mapping class group of genus g (by restriction to the boundary). It is known (cf. [35]) that any $Out(F_n)$ element can be associated with infinitely many handlebody group elements.

With Hyungryul Baik and Sebastian Hensel, the PI is working on the following questions:

Research question 6. *(i) Given an irreducible train-track map ψ, what is the associated handlebody group element with the least "complexity"? Here complexity can be measured by topological entropy (when restricted to the boundary) or by disk complex translation length.*

(ii) What is the gap between the topological entropy of ψ and the corresponding "optimal" handlebody group element restricted to the boundary?

(iii) When the "optimal" handlebody group element is pseudo-Anosov, let ψ' be its restriction to the boundary of the handlebody. What's the relationship between ψ and the induced map of ψ' on its invariant train-track? In particular, can they ever be the same? When?

For optimizing the topological entropy, at the moment the PI and his collaborators are looking into two potential approaches:

(i) Use Teichmüller geometry to study lengths of geodesics on moduli space of curves corresponding to these pseudo-Anosov maps.

(ii) Find and fix a metric on the handlebody, then estimate the resulting Lipschitz constant.

Also, the answer in Question 3 should provide a necessary condition of when the second question would have a non-zero answer.

Application To The Study Of Graph Maps

The analogy between Riemann surfaces and finite metric graphs allow many important concepts and properties on Riemann surfaces to be generalized to the graph-theoretic setting. For example, analogous to the Arakelov metric on Riemann surfaces, there is the "canonical metric" on finite metric graphs [68, 8], which is defined as

Definition 2.10. *The canonical metric[68, 8] on a metric graph is defined as follows: for every edge e, the length of e under the new metric is:*

$$l_{can}(e) = \sup_{\|w\|=1, w \text{ harmonic}} |w(e)| \ .$$

This is shown by [2, 53] to be related to the distribution of Weistrass points of line bundles on finite metric graphs.

In a prior work of the PI and his collaborator, an analogous result on a property of the Arakelov metric on Riemann surfaces in [47, Alppendix] was found, which shows that when passing to larger and larger normal covers the canonical metric on metric graphs converges:

Theorem 2.11. *[56] Let $G \leftarrow G_1 \leftarrow G_2 \dots$ be a tower of finite regular covers of a finite metric graph G, then the canonical metric on G_i are pullbacks of metrics d_i defined on G, and d_i converges uniformly to some limiting metric that depends only on G and $\cap_i \pi_1(G_i)$. More precisely, let $G \leftarrow G'$ be the regular cover defined by $\cap_i \pi_i(G_i)$, then the limiting metric pulls back to the canonical metric on G'.*

In the case when $\cap_i \pi_1(G_i)$ is the identity the limiting metric can be obtained explicitly as the roots of some algebraic equations. There is also an alternative characterization of it in terms of equilibrium measures on the Gromov boundary of the universal cover of G. Because in [47, Alppendix] the limit of the Arakelov metrics for a closed Riemann surface of genus ≥ 2 under a tower of normal covers that converges to the universal cover would converge to the hyperbolic metric, one can see the limiting metric in Theorem 2.11 as a discrete analogy of the hyperbolic metric. However it is unclear what the relationship between this limiting metrics and other approaches of discrete uniformization [18, 31, 54] are. The key idea in the proof of Theorem 2.11 is Lück's approximation theorem on L^2 betti numbers.

The main theorem in [47] is the following:

Theorem 2.12. *Let ψ be a pseudo-Anosov map on a closed surface S. Let ρ be the stretch factor (i.e. exponent of the topological entropy) of ψ. Then, either there is a finite cover of S, where ψ lifts and the induced map on homology has spectral radius ρ, or there exists $\epsilon > 0$ such that for any finite cover of S where ψ lifts, the spectral radius on the homology is no more than $\rho - \epsilon$.*

It is conjectured by Koberda that the same is true for irreducible train-track maps:

Conjecture 2.13. *Let ψ be an irreducible traintrack map on a finite graph G. Let ρ be the stretch factor (i.e. exponent of the topological entropy) of ψ. Then, either there is a finite cover of G, where ψ lifts and the induced map on homology has spectral radius ρ, or there exists $\epsilon > 0$ such that for any finite cover of S where ψ lifts, the spectral radius on the homology is no more than $\rho - \epsilon$.*

Furthermore, the PI and his collaborators conjectured the following:

Conjecture 2.14. *When the lifting of ψ on universal cover $\tilde{G}$ acting on the space of L^2 integrable harminoc 1-forms has spectral radius equals to ρ, there is a finite cover of the graph G preserved by ψ, and the action of the lifting of ψ on its homology has spectral radius ρ. When the spectral radius on the space of L^2 integrable harmonic 1-forms on universal cover is strictly less than ρ, there is a gap between ρ and the spectral radius of the lifting of ψ on the homology of any finite cover of G.*

A major question for this project is:

Research question 7. *Prove or disprove Conjectures 2.13 and 2.14 for some families of irreducible train track maps.*

It is possible to show that Conjecture 2.13 is true when ψ is geometric or parageometric [32]. The next step the PI and his collaborators are looking into is the case when the inverse of ψ is parageometric. It is also hoped that the "optimal handlebody group element" in the previous component can be a useful tool for dealing with certain cases.

Teaching And Outreach

Below is a more detailed description of the different parts of the teaching and outreach plan:

New Graduate And Undergraduate Courses

The PI will be starting a new graduate level course which would be an introductory course on dynamical systems. The PI plans to incorporate his research topic into the course, in particular, use the study of unimodal maps on intervals, train-track maps on graphs, pseudo-Anosov maps on intervals, as examples when illustrating important concepts in hyperbolic dynamics, symbolic dynamics and ergodic theory. The PI will also incorporate some of these materials in his future teaching in the undergraduate level, especially in courses related to differential geometry, dynamics, analysis and PDE. Previously, while at Rutgers University, the PI has used the dynamics of unimodal maps as an example when teaching a course which is an informal introduction of dynamical system and qualitative theory of non linear ODEs and PDEs. And the PI also used contents related to his research for a summer course at KAIST in Korea.

Mentoring Undergraduate And Graduate Students

The PI is currently mentoring an undergraduate student under the Undergraduate Research Scholars program, on some problems related to combinatorial puzzles and optimizations, and has encouraged the student to learn more about various important topics in mathematics and computer science as well as to practice the skill of literature search and review. In the current ongoing research projects described above there has been an undergraduate student from KAIST and a graduate student from Boston College as well. The PI plan to continue mentoring undergraduate research as well as start mentoring graduate students and include them in the research activities.

Summer Dynamics Workshops

Previously at Rutgers the PI had served as informal mentor to several undergraduate students and helped them navigating their undergraduate level math studies. At the moment the PI is has been the organizer for the dynamics seminar at UW Madison for a year, aiming at graduate students in the math department as well as faculty members who are in dynamics or related groups. In the future the PI plans to organize week-long summer workshops on dynamics aiming at undergraduate and beginning graduate students. The PI plan to solicit applications about 5 months in advance of the workshop and send invitations about 4 months in advance, and the main target audience would be junior and senior undergraduates as well as grad students in their first 2 years. The workshop will consist of about 5 expository minicourses, taught by dynamics group members of UW Madison as well as faculty elsewhere who are working on dynamics. The PI has already received two commitments from outside faculty members who would be willing to give a course should the workshops happen. Other than these, the workshop would also have problem sessions for the exchange of potential research problems and for the participants to start new collaborative research projects.

Outreach Activities With WARF

The PI will participate in the Wisconsin IDEA STEM fellowship program by Wisconsin Alumni Research Foundation, and use their resources to participate in the monthly Saturday Science program in Wisconsin Institutes for Discovery, as well as organize summer workshops for middle school and high school students, especially those from underprivileged background, as well as K-12 teachers in the community. The PI will use examples related to the research as motivation to discuss interesting topics in geometry, combinatorics and analysis, pose some more elementary research questions as “math puzzles” to students, and recruit interested students for further research activities.

Broader Impacts

The PI has prior experience in various outreach activities that has a positive impact to the wider community. While at Cornell the PI volunteered in the math club of local high school, as well as in the Math Explorer's Club which is a series of weekend lectures aiming at middle school and high school students. Currently the PI is working in the Undergraduate Research Scholars program with an undergraduate student from non math background.

The project involves questions at varying levels of abstraction and therefore should be useful for integration into teaching and outreach activities. Furthermore, algorithms obtained in this project has potential real life applications in areas like machine learning and numerical analysis, and has the potential of furthering partnership between academia and industries. The mentoring of undergraduate and graduate students, as well as summer dynamics workshop, would be helpful for the academic and career development of math students and contribute to building a more diverse and competitive STEM workforce. The other teaching and outreach activities proposed will also increase mathematical literacy among the general public, as well as improve STEM education, particularly in underserved communities.

Results From Prior NSF Support

The PI hasn't received any prior NSF support.

References Cited

[1] JF Alves and J Sousa Ramos. Kneading theory for tree maps. *Ergodic Theory and Dynamical Systems*, 24(4):957–985, 2004.

[2] Omid Amini. Equidistribution of weierstrass points on curves over non-archimedean fields. *arXiv preprint arXiv:1412.0926*, 2014.

[3] Hyungryul Baik, Dongryul M Kim, and Chenxi Wu. On the asymptotic translation lengths on the sphere complexes and the generalized fibered cone. *arXiv preprint arXiv:2011.08034*, 2020.

[4] Hyungryul Baik, Eiko Kin, Hyunshik Shin, and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone. *arXiv:1909.00974*, 2019.

[5] Hyungryul Baik and Hyunshik Shin. Minimal Asymptotic Translation Lengths of Torelli Groups and Pure Braid Groups on the Curve Graph. *International Mathematics Research Notices*, 12 2018. https://doi.org/10.1093/imrn/rny273.

[6] Hyungryul Baik and Hyunshik Shin. Minimal asymptotic translation lengths of torelli groups and pure braid groups on the curve graph. *International Mathematics Research Notices*, 2020(24):9974–9987, 2020.

[7] Hyungryul Baik, Hyunshik Shin, and Chenxi Wu. An upper bound on the asymptotic translation lengths on the curve graph and fibered faces. *arXiv preprint arXiv:1801.06638, to appear in IUMJ*, 2018.

[8] Matthew Baker and Xander Faber. Metric properties of the tropical Abel-Jacobi map. *Journal of Algebraic Combinatorics*, 33(3):349–381, 2011.

[9] Christoph Bandt. On the mandelbrot set for pairs of linear maps. *Nonlinearity*, 15(4):1127, 2002.

[10] Michael F Barnsley and Andrew N Harrington. A mandelbrot set for pairs of linear maps. *Physica D: Nonlinear Phenomena*, 15(3):421–432, 1985.

[11] Jason Behrstock, Mark Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, i: Curve complexes for cubical groups. *Geometry & Topology*, 21(3):1731–1804, 2017.

[12] Vladimir G Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields.* Number 33. American Mathematical Soc., 2012.

[13] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *Advances in Mathematics*, 256:104–155, 2014.

[14] Mladen Bestvina, Mark Feighn, and Michael Handel. Laminations, trees, and irreducible automorphisms of free groups. *Geometric & Functional Analysis GAFA*, 7(2):215–244, 1997.

[15] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Annals of Mathematics*, pages 1–51, 1992.

[16] Mladen Bestvina and Michael Handel. Train-tracks for surface homeomorphisms. *Topology*, 34(1):109–140, 1995.

[17] Joan Birman, Dan Margalit, and William Menasco. Efficient geodesics and an effective algorithm for distance in the complex of curves. *Mathematische Annalen*, 366(3-4):1253–1279, 2016.

[18] Alexander I Bobenko, Nikolay Dimitrov, and Stefan Sechelmann. Discrete uniformization of polyhedral surfaces with non-positive curvature and branched covers over the sphere via hyper-ideal circle patterns. *Discrete & Computational Geometry*, 57(2):431–469, 2017.

[19] Brian H Bowditch. Tight geodesics in the curve complex. *Inventiones mathematicae*, 171(2):281–300, 2008.

[20] Harrison Bray, Diana Davis, Kathryn Lindsey, and Chenxi Wu. The shape of thurston's master teapot. *arXiv preprint arXiv:1902.10805*, 2019.

[21] Danny Calegari, Sarah Koch, and Alden Walker. Roots, schottky semigroups, and a proof of bandt's conjecture. *Ergodic Theory and Dynamical Systems*, 37(8):2487–2555, 2017.

[22] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae*, 84(1):91–119, 1986.

[23] André de Carvalho and Toby Hall. Pruning theory and thurston's classification of surface homeomorphisms. *Journal of the European Mathematical Society*, 3(4):287–333, 2001.

[24] André de Carvalho and Toby Hall. Braid forcing and star-shaped train tracks. *Topology*, 43(2):247–287, 2004.

[25] A DOUADY. Etude dynamique des polynomes complexes. *Publ. Math. Orsay*, pages 1984–1985.

[26] Spencer Dowdall, Ilya Kapovich, and Chris Leininger. Mcmullen polynomials and Lipschitz flows for free-by-cyclic groups. *J. Eur. Math. Soc.*, (11):3253–3353, 2017.

[27] Spencer Dowdall, Ilya Kapovich, and Christopher J Leininger. Dynamics on free-by-cyclic groups. *Geometry & Topology*, 19(5):2801–2899, 2015.

[28] Viveka Erlandsson, Juan Souto, and Jing Tao. Genericity of pseudo-anosov mapping classes, when seen as mapping classes. *L'Enseignement Mathématique*, 66(3):419–439, 2021.

[29] Vaibhav Gadre and Chia-Yen Tsai. Minimal pseudo-anosov translation lengths on the complex of curves. *Geometry & Topology*, 15(3):1297–1312, 2011.

[30] Paweł Góra. Invariant densities for generalized β-maps. *Ergodic Theory and Dynamical Systems*, 27(5):1583–1598, 2007.

[31] Xianfeng David Gu, Feng Luo, Jian Sun, and Tianqi Wu. A discrete uniformization theorem for polyhedral surfaces. *Journal of differential geometry*, 109(2):223–256, 2018.

[32] Michael Handel and Lee Mosher. Parageometric outer automorphisms of free groups. *Transactions of the American Mathematical Society*, 359(7):3153–3183, 2007.

[33] Michael Handel and Lee Mosher. The free splitting complex of a free group, i hyperbolicity. *Geometry & Topology*, 17(3):1581–1670, 2013.

[34] Allen Hatcher and Karen Vogtmann. The complex of free factors of a free group. *Quarterly Journal of Mathematics*, 49(196):459–468, 1998.

[35] Sebastian Hensel. A primer on handlebody groups. *Handbook of Group Actions, to appear*, 14, 2018.

[36] John H Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics*, volume 2. Matrix editions, 2016.

[37] Shunji Ito and Taizo Sadahiro. Beta-expansions with negative bases. *Integers*, 9(3):239–259, 2009.

[38] Eiko Kin and Hyunshik Shin. Small asymptotic translation lengths of pseudo-anosov maps on the curve complex. *Groups, Geometry, and Dynamics*, 2019.

[39] Tan Lei. Similarity between the mandelbrot set and julia sets. *Communications in mathematical physics*, 134(3):587–617, 1990.

[40] Tao Li. *A monotonicity conjecture for the entropy of Hubbard trees.* PhD thesis, Stony Brook University, 2007.

[41] Kathryn Lindsey and Chenxi Wu. A characterization of thurston's master teapot. *arXiv preprint arXiv:1909.10675*, 2019.

[42] JE Littlewood. On polynomials $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{0i}$. *Journal of the London Mathematical Society*, 1(1):367–376, 1966.

[43] Brian Mann. Hyperbolicity of the cyclic splitting graph. *Geometriae Dedicata*, 173(1):271–280, 2014.

[44] Howard A Masur and Yair N Minsky. Geometry of the complex of curves i: Hyperbolicity. *Inventiones mathematicae*, 138(1):103–149, 1999.

[45] Howard A Masur and Yair N Minsky. Geometry of the complex of curves ii: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000.

[46] Curtis T McMullen. Polynomial invariants for fibered 3-manifolds and teichmüller geodesics for foliations. In *Annales scientifiques de l'Ecole normale supérieure*, volume 33, pages 519–560. Elsevier, 2000.

[47] Curtis T McMullen. Entropy on riemann surfaces and the Jacobians of finite covers. *Commentarii Mathematici Helvetici*, 88(4):953–964, 2013.

[48] Philipp Meerkamp and Dierk Schleicher. Hausdorff dimension and biaccessibility for polynomial julia sets. *Proceedings of the American Mathematical Society*, 141(2):533–542, 2013.

[49] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems*, pages 465–563. Springer, 1988.

[50] William Parry. On the β-expansions of real numbers. *Acta Mathematica Academiae Scientiarum Hungarica*, 11(3-4):401–416, 1960.

[51] Robin Pemantle and Igor Rivin. The distribution of zeros of the derivative of a random polynomial. In *Advances in combinatorics*, pages 259–273. Springer, 2013.

[52] Alfredo Poirier. Critical portraits for postcritically finite polynomials. *Fundamenta Mathematicae*, 203:107–163, 2009.

[53] David Harry Richman. Equidistribution of weierstrass points on tropical curves. *arXiv preprint arXiv:1809.07920*, 2018.

[54] Rohan Sawhney and Keenan Crane. Boundary first flattening. *ACM Transactions on Graphics (ToG)*, 37(1):5, 2018.

[55] Kenneth J Shackleton. Tightness and computing distances in the curve complex. *Geometriae Dedicata*, 160(1):243–259, 2012.

[56] Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem. *arXiv:1711.02609, accepted by Invent. Math.*, 2017.

[57] Boris Solomyak. Conjugates of beta-numbers and the zero-free domain for a class of analytic functions. *Proceedings of the London Mathematical Society*, 3(3):477–498, 1994.

[58] Boris Solomyak and Hui Xu. On the "Mandelbrot set" for a pair of linear maps and complex Bernoulli convolutions. *Nonlinearity*, 16(5):1733, 2003.

[59] Balázs Strenner. Fibrations of 3-manifolds and asymptotic translation length in the arc complex. *arXiv preprint arXiv:1810.07236*, 2018.

[60] Daniel J Thompson. Generalized β-transformations and the entropy of unimodal maps. *Commentarii Mathematici Helvetici*, 92(4):777–800, 2017.

[61] Dylan P Thurston. From rubber bands to rational maps: a research report. *Research in the Mathematical Sciences*, 3(1):15, 2016.

[62] William Thurston. Entropy in dimension one. 2014.

[63] William P Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 339:99–130, 1986.

[64] Giulio Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the mandelbrot set. *Advances in Mathematics*, 273:651–715, 2015.

[65] Giulio Tiozzo. Continuity of core entropy of quadratic polynomials. *Inventiones mathematicae*, 203(3):891–921, 2016.

[66] Giulio Tiozzo. Galois conjugates of entropies of real unimodal maps. *International Mathematics Research Notices*, 2020(2):607–640, 2020.

[67] Aaron D Valdivia. Asymptotic translation length in the curve complex. *New York Journal of Mathematics*, 20, 2014.

[68] Shouwu Zhang. Admissible pairing on a curve. *Inventiones mathematicae*, 112(1):171–193, 1993.

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Publications

Five Selected Publications

- Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu. The shape of Thurston's Master Teapot *arXiv:*1902.10805, conditionally accepted by *Advances in Mathematics*, 2020
- Kathryn Lindsey and Chenxi Wu, Characterization of the Shape of Thurston's Teapot *arXiv:*1909.10675
- Hyungryul Baik, Dongryul M. Kim, Chenxi Wu. On the asymptotic translation lengths on the sphere complexes and the generalized fibered cone *arXiv:*2011.08034
- Hyungryul Baik, Hyunshik Shin and Chenxi Wu. An upper bound on the asymptotic translation length on the curve graph and fibered faces *arXiv:* 1801.06638, accepted by *Indiana University Math Journal*, 2020
- Farbod Shokrieh and Chenxi Wu. Canonical measures on metric graphs and a Kazhdan's theorem *Invent. Math.* 215(3), 819-862, 2019. doi: 10.1007/s00222-018-0838-5

Other Significant Publications

- Hyungryul Baik, Farbod Shokrieh, Chenxi Wu. Limits of canonical forms on towers of Riemann surfaces *Crelle* 2019. doi: 10.1515/crelle-2019-0007.
- Hyungryul Baik, Ahmad Rafiqi and Chenxi Wu. Is a typical bi-Perron number a pseudo-Anosov dilatation? *Ergodic Theory and Dynamical Systems* 39(7), 1745-1750, 2019. doi: 10.1017/etds.2017.109
- Chenxi Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel Journal of Mathematics*, 214(2), 733-740, 2016. doi: 10.1007/s11856-016-1357-y
- Hyungryul Baik, Eiko Kin, Hyunshik Shin and Chenxi Wu. Asymptotic translation length and normal generation for the fibered cone *arXiv:*1909.00974, submitted.