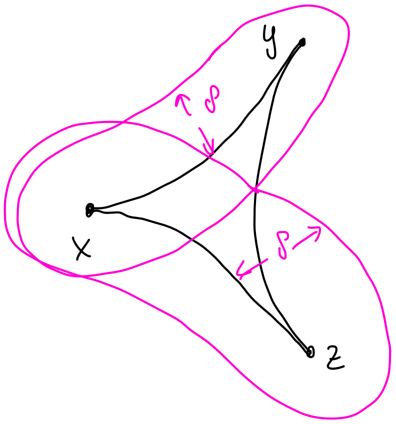
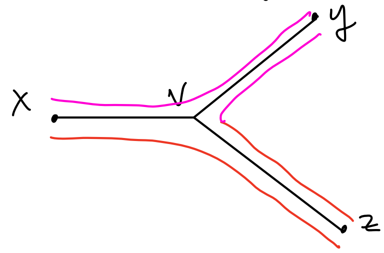
SYMBOLIC DYNAMICS OF HYPERBOLIC GROUPS

I. Preliminaries : δ-hyperbolic : if xyz is a geodesic $\triangle$ in X, each side of triangle is contained in the union of δ-nbhds of other sides of triangle



yz is in δ-nbhd of both xy and xz.

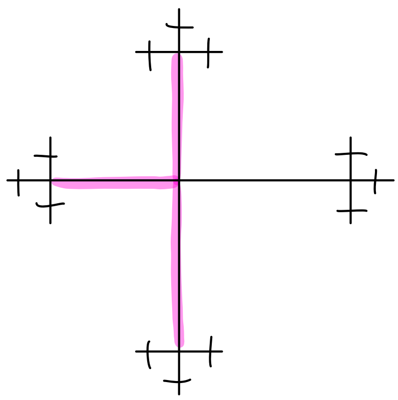
Ex : Tree is 0-hyperbolic



Triangle: tripod
yz is distance 0 from xy, xz.

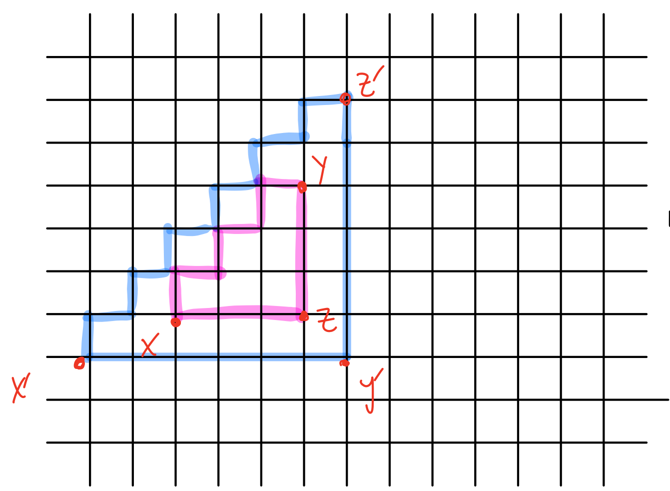
A group G is δ-hyperbolic if some (all) Cayley graph w/ generating set S is δ-hyperbolic.

Ex : $\mathbb{F}_2 = \langle a, b \rangle$



Tree $\Rightarrow$ 0-hyperbolic

Non ex. $\mathbb{Z}^2$



Can find bigger and bigger similar triangles, no bound on δ.

word length : Fix generating set S for group G. Let $g \in G$, $g = s_1 s_2 \dots s_k$.

Then $|g| = \inf \{ k \text{ st } g = s_1 s_2 \dots s_k \}$.

word metric : $\forall\, g, h \in G$, $d_G(g, h) = |g^{-1}h|$.

S = finite set (e.g. generating set for f.g. group G) S^* = set of finite words in alphabet S.

Automaton(Γ) : finite directed graph w/ distinguished initial vertex
edges labeled by elements in S st ea. vertex has at most one outgoing edge w/ a given label

accept states : some subset of vertices

word in S^* determines simplicial path in Γ; start at initial vertex, read letters of word L to R and moving along corresponding edge if it exists, halting if not

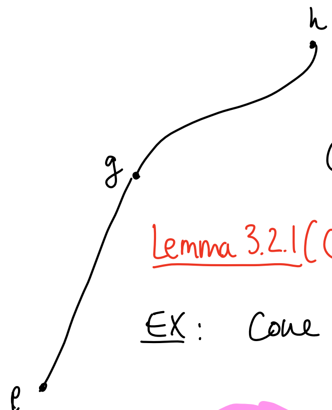
$L \subset S^*$: words read in their entirety w/o halting and for which terminal vertex of associated path ends at an accept state. (associated to Γ)

We say "L is parametrized by (paths in) Γ".

Regular Language: A subset $L \subset S^*$ is a regular language if $\exists$ a finite directed graph Γ as above that parametrizes L. (Γ is not unique.)

prefix-closed language: $\forall\, w \in L$, any prefix of w is also in L (empty word = prefix of every word)

cone type of g ($\mathrm{cone}(g)$): $h \in G$ st some geodesic from id to gh passes through g. (geodesic for h passes through g)



n-level of g: set of h in ball $B_n(id)$ st $|gh| < |g|$

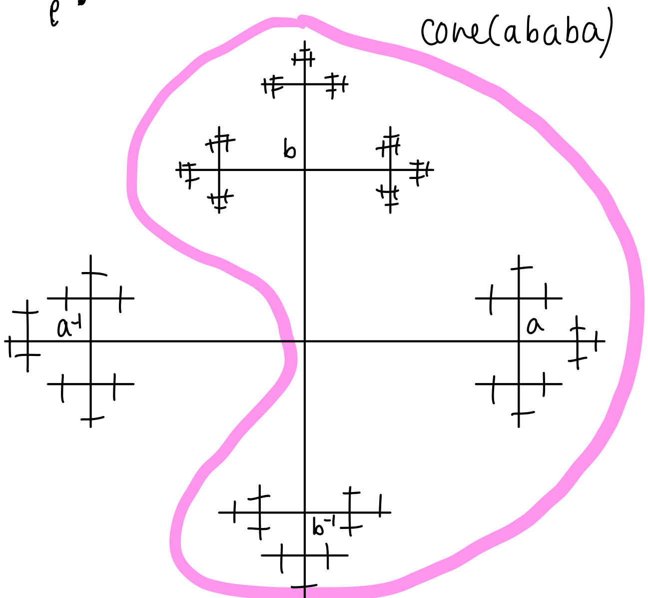
Cannon came up w/ relationship b/w cone types & n-levels:

Lemma 3.2.1 (Cannon): The $2\delta+1$ level of an element determines its cone type. (so $\exists$ finitely many cone types.)

EX: Cone types in F_2

$\delta = 0 \Rightarrow 2\delta+1 = 1$

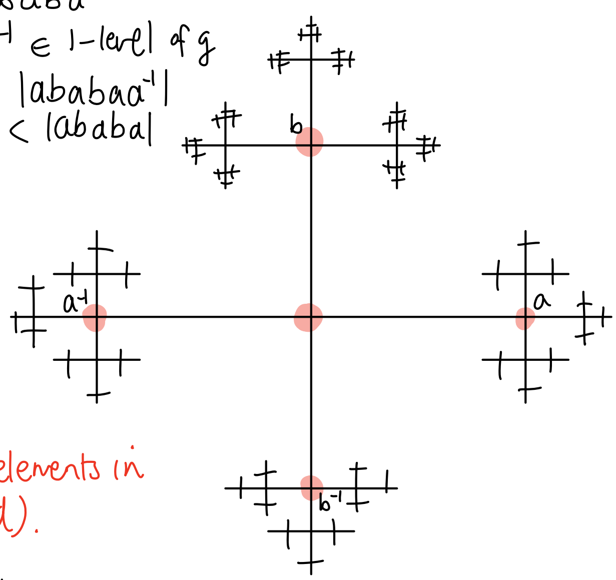
so 1-levels determine cone types.



$g = ababa$

$\Rightarrow a^{-1} \in$ 1-level of g

since $|ababaa^{-1}| < |ababa|$



only 5 elements in $B_1(id)$.

cone(ababa):

words that do not start w/ a^{-1}; $gh = ababaa^{-1}w = ababw$; does not pass through ababa

So there are 5 cone types:

- cone(a): words that end in a (1-level $= a^{-1}$)
- cone(b): words that end in b (1-level $= b^{-1}$)
- cone(a^{-1}): words that end in a^{-1} (1-level $= a$)
- cone(b^{-1}): words that end in b (1-level $= b$)
- cone(e): always its own cone type

Note: there is a slightly different definition for cone(g), which we will use for the rest of the notes. This defn is easier to work with: (can check gives rise to same 5 cone types in example)

cone(g) := set $h \in G$ st a geodesic for h in combing passes through g.

Unique lexicographic (dictionary) order on S^*: A total order $\prec$ on S extends to a unique dictionary order on S^*: (1) the empty word precedes everything;
(2) If u, v are nonempty and start w/ different letters $s, t \in S$, then $u < v$ iff $s \prec t$; and
(3) if $u \prec v$ and ω is arbitrary, then $wu < wv$.

- geodesics on Cayley graph can be lexicographically ordered. $\forall g \in V(\Gamma)$, there is a geodesic that precedes all others. This is the "lexicographically first geodesic" corresponding to g.

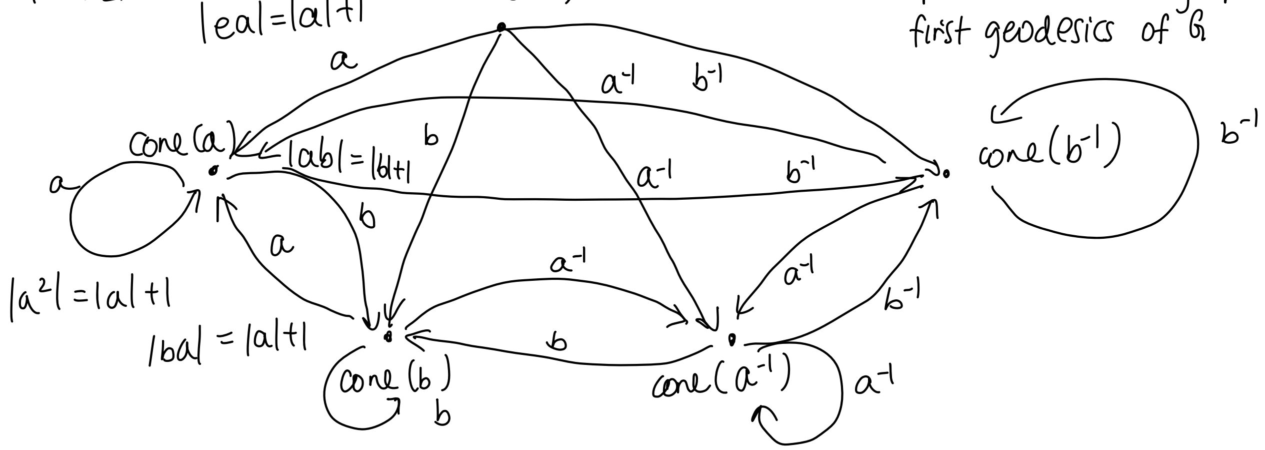
Theorem 3.2.2 (Cannon): Let G be a hyperbolic grap, S a symmetric generating set. Fix a total order $\prec$ on S. Then the language of "lexicographically first geodesics" is prefix-closed, regular.

Constructing Γ: vertices = cone types (all accept states)

edge labeled s from cone(g) to cone(gs) if $|gs| = |g|+1$.

EX: $G = \mathbb{F}_2$

Γ parametrizes lexicographically first geodesics of G



II. Patterson-Sullivan Measures

Theorem 4.12 (Coonaert): Let G be a (non-elementary) hyperbolic group w/ generating set S. G_n = set of elements of word length n. Then $\exists$ constants $K \geq 1$, $\lambda > 1$ st

$$\frac{1}{K}\lambda^n \leq |G_n| \leq K\lambda^n \quad \forall \; n \in \mathbb{Z}_{>0}.$$

Patterson-Sullivan measures: $\forall$ $n \in \mathbb{Z}_{>0}$, there is a probability measure on G defined by:

$$\nu_n := \frac{\sum_{|g|\leq n} \lambda^{-|g|}\delta_g}{\sum_{|g|\leq n} \lambda^{-|g|}} \qquad \text{where } \delta_g(E) := \begin{cases} 0 & \text{if } g \notin E \\ 1 & \text{if } g \in E \end{cases}, \; \lambda \text{ is from Thm. 4.12.}$$

The Patterson-Sullivan measure ν associated with S is: $\displaystyle \underset{n\to\infty}{\lim} \frac{\sum_{|g|\leq n} \lambda^{-|g|}\delta_g}{\sum_{|g|\leq n} \lambda^{-|g|}}$

weak limit

$\mathrm{Supp}(\nu)$ contained in ∂G.

Closely related measure (easier to work with): $\hat{\nu}_n = \frac{1}{n}\sum_{|g|\leq n} \lambda^{-|g|}\delta_g$, $\hat{\nu} = \lim_{n\to\infty} \hat{\nu}_n$

Fact: Any 2 weak limits $\hat{\nu}$, $\hat{\nu}_n$ on ∂G are absolutely cont. wrt one another, satisfy $\frac{1}{K} \leq \frac{d\hat{\nu}}{d\nu} \leq K$.

Proof

$$0 = \nu_n(E) \Leftrightarrow \sum_{|g|\leq n} \lambda^{-|g|}\delta_g(E) = 0 \Leftrightarrow \hat{\nu}_n(E) = 0 \Rightarrow \hat{\nu}, \hat{\nu}_n \text{ absolutely cont. wrt ea. other}$$

$$\hat{\nu}_n(E) = \left(\frac{\sum_{|g|\leq n} \lambda^{-|g|}\delta_g(E)}{\sum_{|g|\leq n}\lambda^{-|g|}}\right)\left(\frac{\sum_{|g|\leq n}\lambda^{-|g|}}{n}\right) = \left(\frac{\sum_{|g|\leq n} \lambda^{-|g|}\delta_g(E)}{\sum_{|g|\leq n}\lambda^{-|g|}}\right)\underbrace{\left(\frac{\sum_{i=0}^{n}|G_i|/\lambda^i}{n}\right)}_{\text{Radon-Nikodym Derivative } d\hat{\nu}_n/d\nu_n}$$

$$\frac{1}{k_i} \leq \frac{|G_i|}{\lambda^i} \leq k_i \Rightarrow n\frac{1}{\max\{k_i\}} \leq \sum_{i=0}^{n}\frac{1}{k_i} \leq \sum_{i=0}^{n}\frac{|G_i|}{\lambda^i} \leq \sum_{i=0}^{n} k_i \leq n\max\{k_i\}$$

So $\forall$ $\hat{\nu}_n$, $\hat{\nu}$, $\exists$ $K \geq 1$ st $\frac{1}{K} \leq \frac{d\hat{\nu}_n}{d\nu_n} \leq K$.

$\Rightarrow$ Fact is true for $\hat{\nu}_n$, $\hat{\nu}$ $\Rightarrow$ true for their limits as well. ∎

- Relation b/w λ and Γ: Γ is a topological Markov chain

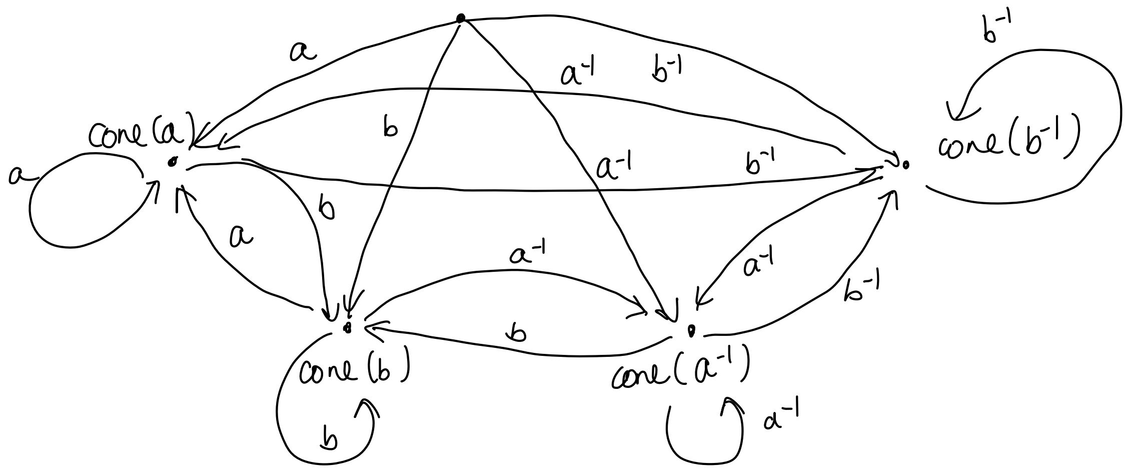
Component of Γ: subset C of Γ st. $\forall$ $v_i, v_j \in C$, $\exists$ directed path from each to the other.

adjacency matrix M associated w/ Γ : $(M^n)_{ij}$ = # of directed paths in Γ from v_i to v_j.

- V = real vector space spanned by $v_i \in V(\Gamma)$ equipped w/ inner product $\langle \cdot, \cdot \rangle$ for which vertices are an orthonormal basis.

- Fact: $\langle v_i, M v_j \rangle = M_{ij}$.

Ex: Γ from before.



Fix:

v_1 = cone(e)
v_2 = cone(a)
v_3 = cone(a^{-1})
v_4 = cone(b)
v_5 = cone(b^{-1})

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \text{eg} \quad M v_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = v_1 + v_2 + v_4 + v_5$$

$$\Rightarrow \langle v_i, M v_2 \rangle = \begin{cases} 0 & \text{if } i = 3 \text{ (no edges from cone(a) to cone}(a^{-1})) \\ 1 & \text{if } i \neq 3 \text{ (1 edge from cone(a) to cone(s) for } s \neq a^{-1}) \end{cases}$$

$M\left(\vec{1}\right)$ also captures information:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

4 outgoing paths of length 1 from cone(e), 3 outgoing paths of length 1 from other verts.

Lemma 4.2: For any v_i, v_j, the # of directed paths in Γ of length n from v_i to v_j is $\langle v_i, M^n v_j \rangle = (M^n)_{ij}$.

Proof: True for $n=1$. (see example) Suppose true for $n=m$. Then the # of paths of length $m+1$ from v_i to v_j is equal to:

$$\sum_k \begin{pmatrix} \text{\# of paths of length } m \\ \text{from } v_i \text{ to } v_k \end{pmatrix} \begin{pmatrix} \text{\# of paths of length } 1 \\ \text{from } v_k \text{ to } v_j \end{pmatrix} = \sum_k \langle v_i, M^m v_k \rangle \langle v_k, M v_j \rangle$$

$$= \sum_k (M^m)_{ik} (M)_{kj} = (M^{m+1})_{ij} \quad \blacksquare$$

- Let $|v| := \sum_i |\langle v, v_i \rangle| \Rightarrow |(M^T)^n v_1| = \sum_i \langle (M^T)^n v_1, v_i \rangle \left(= \sum_i \langle v_1, M^n v_i \rangle \right)$

# of paths in Γ of length n starting at v_1.

Lemma 4.4 : If Γ satisfies : (1) $\exists$ initial vertex v_1;
(2) $\forall\, i \neq 1$, $\exists$ directed path from v_1 to v_i;
(3) $\exists$ constants $\lambda > 1$, $K \geq 1$ st $\frac{1}{K}\lambda^n \leq |(M^T)^n v_1| \leq K\lambda^n \ \forall n \in \mathbb{Z}_{\geq 0}$.

Then Γ is "almost semisimple" and λ (from Lemma 4.12) is the largest real eigenvalue of M and it has multiplicity 1. All other eigenvalues ξ satisfy $|\xi| < \lambda$.

Proof uses Perron-Frobenius. Details omitted.

Note that Γ is almost-semisimple; $|(M^T)^n v_1| = $ # of paths of length n in Γ from $v_1 = \text{cone}(e)$
$= $ # of words of length n in $G = |G_n|$

So by Theorem 4.12 (Coornaert), (3) holds.

Even stronger statement: If M is irreducible ($(M^n)_{ij} > 0$ for some n, or alternatively, $\forall\, v_i, v_j \in V(\Gamma)$, $\exists$ path from v_i to v_j in Γ) then ea. eigenvalue is of form $e^{2\pi i/k}\lambda$, and eigenvalue is simple.

Note that M itself is not irreducible, but the adjacency matrix of each of components of Γ is irreducible so we can exploit that property to get the same result for M.

Ex: M for earlier example ($G = \mathbb{F}_2$):

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

M_C, adjacency matrix of component $= \Gamma \setminus \{\text{cone}(e)\}$

Say ξ is eigenvalue of M_C, ev. $v(\xi) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Then:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + x_3 + x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 + x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_4 \end{bmatrix}$$

same

Set $x_0 = \dfrac{x_1 + x_2 + x_3 + x_4}{\xi}$ $\Rightarrow$ can see ξ is eigenvalue for M as well.

III. Shift Space

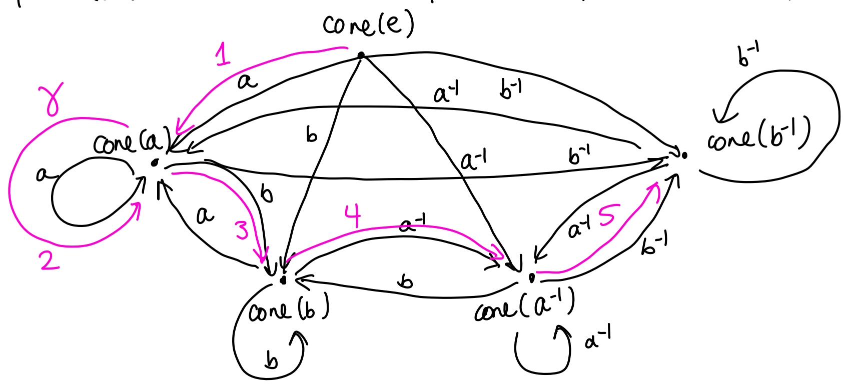
$\forall\, n$, let $Y_n :=$ set of paths in Γ of length n starting at initial vertex

$X_n :=$ set of all paths in Γ of length n .

- Note $X_0 \Leftrightarrow$ vertices of Γ, $\cdots \to X_n \to X_{n-1} \cdots \to X_1 \to X_0$ is an inverse system,

$X_\infty = \varprojlim_{i \in \mathbb{Z}_{\geq 0}} X_i$ is the space of (right) infinite paths; define Y_∞ similarly.

- Shift map $T: X_\infty \to X_\infty$ takes an infinite path to its suffix which is complement of its initial vertex.



- Idea: can think of paths in Γ as elements of G.

eg: $a^2 b a^{-1} b^{-1}$

path in pink, γ

(Cs(G))

- Evaluation map: $E: Y_n \to G$ takes path in Γ to endpoint of geodesic path in G starting at id.
  eg γ in example above evaluates to $a^2 b a^{-1} b^{-1}$ in G.

- $E_\infty : Y \to \partial G$ takes infinite path to endpt. of geodesic in G.

- $p: Y \to Y_n$ takes infinite path to prefix of length n;

- cylinder sets: $p^{-1}(y)$, define Borel algebra on Y.

- let $v \in V$. Then define:

$$\rho(v) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} M^i v, \qquad \ell(v) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} (M^T)^i v$$

Lemma 4.5: $\forall$ vector $v \in V$, let $v = \sum_{\xi} v(\xi)$ where $v(\xi) \in$ generalized eigenspace of ξ.

Then $\rho(v) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} M^i v = v(\lambda)$.

specifically, $\xi = (e^{2\pi i/k})\lambda$ for some $k \geq 3$ and simple

Proof: let ξ be an eigenvalue of M. By Lemma 4.4, either ① $|\xi| < \lambda$ or ② $v(\xi)$ is an eigenvector.

$$① : \lim_{n\to\infty} \rho_n(v(\xi)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \lambda^{-i} M^i v(\xi) \overset{\xi \text{ is simple}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \Big(\underbrace{\frac{\xi}{\lambda}}_{<1}\Big)^i v(\xi) = 0.$$

$$② \lim_{n\to\infty} \rho_n(v(\xi)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \cancel{\lambda^{-i} \lambda^{i}} v(\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} v(\lambda) = v(\lambda).$$ ■

In other words, $\rho_n(v) =$ projection of v onto right λ-eigenspace of M.

Lemma 4.6: $\ell(v)$ is equal to projection of v onto the left λ-eigenspace of M.

Proof is similar.

Lemma 4.7: $\forall\ v, w \in V$, there is the identity $\langle \ell(v), w\rangle = \langle \ell(v), \rho(w)\rangle = \langle v, \rho(w)\rangle$.

Proof: π: projection onto right eigenspace, π^T: projection onto right eigenspace.

$$\rho(v) = \pi v, \quad \ell v = \pi^T(v) \text{ and } \pi\circ\pi = \pi, \quad \pi^T\circ\pi^T = \pi. \quad \text{So:}$$

$$\underbrace{\langle \pi^T v, w\rangle}_{\langle \ell(v), w\rangle} = \langle (\pi^T\circ\pi^T)v, w\rangle = \underbrace{\langle \pi^T v, \pi w\rangle}_{\langle \ell(v), \rho(w)\rangle} = \langle v, (\pi\circ\pi)w\rangle = \underbrace{\langle v, \pi w\rangle}_{\langle v, \rho(w)\rangle} \quad \blacksquare$$

- Define measure on vertices $v \in V(\Gamma)$ (X_0):

$$N_{ij} = \begin{cases} \dfrac{M_{ij}(\rho(\bar{1}))_j}{\lambda(\rho(\bar{1}))_i} & \text{if } \rho(1)_i \neq 0 \\ N_{ii} = 1 \text{ and } N_{ij} = 0 \text{ for } i \neq j. & \text{otherwise} \end{cases}, \quad \mu_i' = (\rho(1))_i\, \ell(v_1)_i$$

interpret N_{ij} as probability that given a path $y \in Y_\infty$ is at v_i, it will go to v_j next.

Lemma 4.9: The matrix N is a stochastic matrix (ie: non-negative, $\forall\ i\ \sum_j N_{ij} = 1$) and preserves the measure μ.

Proof: $\rho(1)_i = 0 \Rightarrow \sum_j N_{ij} = N_{ii} = 1$.

$\rho(1)$ in eigenspace; use Lemma 4.5

$$\rho(1)_i \neq 0: \quad \sum_j N_{ij} = \sum_j \frac{M_{ij}(\rho(1))_j}{\lambda(\rho(1))_i} = \frac{1}{\lambda(\rho(1))_i} \underbrace{\sum_j M_{ij}\,\rho(1)_j}_{(M\rho(1))_i} = \frac{(M\rho(1))_i}{\lambda(\rho(1))_i} = 1$$

$\Rightarrow N$ is stochastic.

N preserves μ':
$$(\mu N)_j = \sum_i \mu_i' N_{ij} = \sum_i \cancel{(\rho(1))_i}\, \ell(v_1)_i \left(\frac{M_{ij}(\rho(1))_j}{\lambda \cancel{\rho(1)_i}}\right) = \sum_i \left(\frac{\ell(v_1)_i M_{ij}}{\lambda}\right)(\rho(1))_j$$
$$= (\rho(1))_j \sum_i \frac{1}{\lambda}\ell(v_1)_i M_{ij} = (\rho(1))_j \left(\frac{1}{\lambda}\right)\underbrace{(\ell(v_1)\, M)_j}_{\lambda\, \ell(v_1)_j}$$
$$= (\rho(1))_j (\ell(v_1))_j = \mu_j'. \quad \blacksquare$$

To construct μ_i, scale μ_i' to a probability measure
μ_i is probability a point on path $y \in Y_\infty$ will be at $v_i \in \Gamma$

- probability measure on X_n: $\mu'(v_{i_0} v_{i_1} \dots v_{i_n}) = \mu'(v_{i_0}) N_{i_0 i_1} N_{i_1 i_2} \cdots N_{i_{n-1} i_n}$ defines measure on ea. cylinder

- stationary measure μ on X_∞: Scale μ' to be probability measure, extend to measure on X_∞.

μ is invariant under shift map: for $w \in T^{-1}(x)$, $x \in X_\infty$ where $w = v_{i_0'} v_{i_0} v_{i_1} \cdots v_{i_n} \cdots$ and $x = v_{i_0} v_{i_1} \cdots v_{i_n}$ by Lemma 4.9:

$$\mu(w) = \mu(v_{i_0'} v_{i_0} v_{i_1} \cdots v_{i_n} \cdots) = \overbrace{\mu(v_{i_0'}) N_{i_0' i_0}}^{\mu_{i_0'} N_{i_0' i_0} = \mu_{i_0}} N_{i_0 i_1} N_{i_1 i_2} \cdots N_{i_{n-1}} N_{i_n} \cdots$$
$$= \mu_{i_0} N_{i_0 i_1} N_{i_1 i_2} \cdots N_{i_{n-1} i_n} \cdots = \mu(v_{i_0} v_{i_1} \cdots v_{i_n} \cdots) = \mu(x).$$

- Relationship b/w μ, N and Patterson-Sullivan measure:

define $\hat{\nu}(\underbrace{p^{-1}(y)}_{\text{all paths w/ prefix } y \in Y_n}) := \lim_{n\to\infty} \hat{\nu}_n(\text{cone}(g))$

$E(y)$

- Explicit formula for $\hat{\nu}_m(\text{cone}(g))$ $\forall g \in G$:

$v_g \in \Gamma$ = vertex that is endpt. of path for $y \in Y_n$ st $E(y) = g$, recall $\hat{\nu}_m$ = "Patterson-Sullivan like measure"

$$g \in G_n,\ m \geq n \Rightarrow \hat{\nu}_m(\text{cone}(g)) \overset{\text{Defn}}{=} \frac{1}{m} \sum_{|h| \leq m} \lambda^{-|h|} \delta_h(\text{cone}(g)) = \frac{1}{m} \sum_{\substack{h \in \text{cone}(g) \\ |h| \leq m}} \lambda^{-|h|}$$
$$= \frac{1}{m} \sum_{i=0}^{m-n} \lambda^{-|n+i|} \left(\#\text{ of paths of length } i \text{ starting at } v_g\right)$$
$$= \frac{1}{m} \lambda^{-n} \sum_{i=0}^{m-n} \lambda^{-i} \sum_k \langle (M^T)^i v_g, v_k \rangle = \frac{1}{m} \lambda^{-n} \sum_{i=0}^{m-n} \lambda^{-i} \langle (M^T)^i v_g, \mathbb{1} \rangle$$

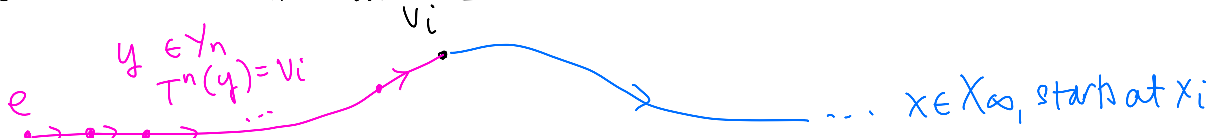
$$\Rightarrow \lim_{m\to\infty} \hat{\nu}_m(\text{cone}(g)) = \lim_{m\to\infty} \frac{1}{m} \lambda^{-n} \sum_{i=0}^{m-n} \lambda^{-i} \langle (M^T)^i v_g, \mathbb{1} \rangle = \lambda^{-n} \Big\langle \underbrace{\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-n} \lambda^{-i} (M^T)^i v_g}_{\ell(v_g)}, \mathbb{1} \Big\rangle$$
$$= \lambda^{-n} \langle \ell(v_g), \mathbb{1} \rangle \overset{\text{Thm 4.7}}{=} \lambda^{-n} \langle v_g, \rho(\mathbb{1}) \rangle = \lambda^{-n} (\rho(\mathbb{1}))_g \quad (*)$$

0 on all entries except gth index

Define $T_*(\hat{\nu}(B)) := \hat{\nu}(T^{-1}(B))$, Borel set $B \subset X$.

Lemma 4.19: The measure $\lim_{n\to\infty} \sum_{i=1}^{\infty} \left(\frac{1}{n}\right) T_*^i \hat{\nu}$ on X is equal to the measure μ on X defined on cylinders using μ on Γ and N. (from previous page.)

Proof:
elements in $T^{-n}(p^{-1}(v_i))$ in Γ:



$$T_*^n \hat{\nu}(p^{-1}(v_i)) = \hat{\nu}(T^{-n}(p^{-1}(v_i))) \overset{\text{(see pic)}}{=} \sum_{\substack{y \in Y_n \\ T^n(y) = v_i}} \hat{\nu}(p^{-1}(y)) \overset{(*)}{=} \sum_{\substack{y \in Y_n \\ T^n(y) = v_i}} \lambda^{-n} \langle v_i, \rho(\mathbb{1}) \rangle$$

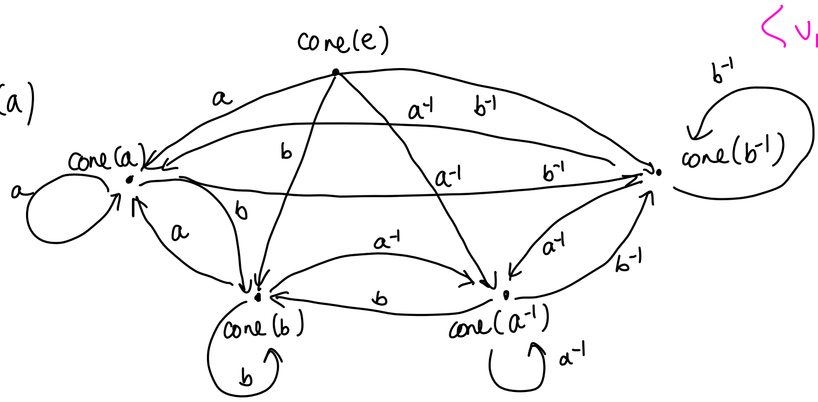
$\left(\# \text{ of } y \in Y_n \text{ st } T^n y = v_i\right) = \underline{\# \text{ of directed paths in } \Gamma \text{ of length } n \text{ that start at } v_1, \text{ end at } v_i}.$

$\langle v_1, M^n v_i \rangle$

eg: choose $v_2 = \text{cone}(a)$

$n = 2$

$T^2(a^3)$
$= T^2(ba^2) = T^2(b^{-1}a^2)$
$= T^2(b^2 a) = T^2(b^{-2}a) = a$



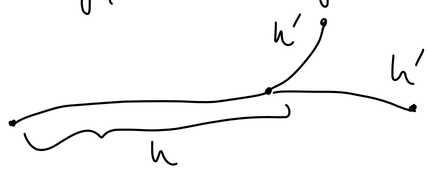
$$\text{So } \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} T_*^i \hat{\nu}\left(p^{-1}(v_i)\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \Bigg(\sum_{\substack{y \in Y_i \\ T^i(y) = y_i}} \lambda^{-i} \langle v_i, \rho(1) \rangle \Bigg)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \lambda^{-k} \langle v_i, \rho(1) \rangle \langle v_1, M^n v_i \rangle = \frac{1}{n} \langle v_i, \rho(1) \rangle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lambda^{-k}}{n} \langle v_1, M^n v_i \rangle$$

$$= \langle v_i, \rho(1) \rangle \langle v_1, \underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda^{-k} M^n v_i}_{\rho(v_i)} \rangle = \langle \underbrace{v_i, \rho(1)}_{\rho(1)_i} \rangle \langle \underbrace{v_1, \rho(v_i)}_{\langle \ell(v_1), v_i \rangle = (\ell(v_1))_i} \rangle = \mu_i$$

$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} T_*^i \hat{\nu}$, μ_i agree on cylinder sets $p^{-1}(v_i)$, for $v_i \in \Gamma = X_0$.

- y_n = prefix of length n of y, last vertex of $y_n = v_i$ and $E(y_n) = h \in G_n$.
- $h' \in G_{n+1} \cap \text{cone}(h)$



say $E(v_j) = h'$.

- $\#$ of paths of length 1 from v_i to $v_j = M_{ij}$

So: $\left(\hat{\nu} \text{ probability } (n+1)^{st} \text{ vertex of } y \in Y_\infty \text{ is } v_j, \text{ given } n^{th} \text{ vertex is } v_i\right) = \lim_{n\to\infty} \dfrac{\sum_{h'} \hat{\nu}_m(\text{cone}(h'))}{\hat{\nu}_m(\text{cone}(h))} \overset{\text{Formula } (*)}{=} \dfrac{\sum_{h'} \lambda^{-(n+1)} (\rho(1))_j}{\lambda^{-n} (\rho(1))_i}$

$$= \frac{\cancel{\frac{1}{n}} \left(\lambda^{-(n+1)} (\rho(1))_j\right)(M_{ij})}{\cancel{\frac{1}{n}} \left(\lambda^{-n} (\rho(1))_i\right)} = \frac{M_{ij} (\rho(1))_j}{\lambda (\rho(1))_i} = N_{ij}.$$

References: "The Ergodic Theory of Hyperbolic Groups" Danny Calegari

"Combable Fxns., quasimorphisms, and the central limit theorem" Danny Calegari, Koji Fujiwara