ENTROPY AND ARTIN MAZUR ζ FUNCTION FOR NON-ARCHIMEDEAN SUBHYPERBOLIC DYNAMICS

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Abstract. We study the topological entropy $h_{\mathrm{top}}(K,\phi)$ for subhyperbolic rational maps $\phi \in K(z)$ defined over discrete valued non-archimedean fields K. We establish that the numbers $\exp(h_{\mathrm{top}}(K,\phi))$, as ϕ varies over all subhyperbolic rational maps in $K(z)$ with compact Julia sets, form the set of all weak Perron numbers. The main ingredients are the surgery construction for rational maps and a characterization of the Julia-set dynamics via a finite type Markov partition. As a byproduct of the characterization of the Julia dynamics, we show that the Artin-Mazur zeta function for ϕ on its Julia set is rational. Our results extend Thurston's work on the entropy for postcritically finite self-maps of the unit interval to the p-adic setting.

1. introduction

In dynamical systems, the topological entropy measures the complexity of the orbit structure, which is invariant under topological conjugacy. A classical problem in dynamics is to determine which number can arise as the topological entropy for a dynamical system. In one-dimensional complex or non-archimedean dynamics, the study of topological entropy for rational maps has been well explored when the space is algebraically closed, see [1,6,8]. While when the space is not algebraically closed, the situation is more complicated, see [7,13,15]. In [15] Thurston studied the topological entropy for postcritically finite self-maps of the unit interval as well as finite graphs, showing that in both cases the set of possible entropies is identical to the set:

$$\{\log(\lambda) : \lambda \text{ is weak Perron}\}$$

Recall that a real positive algebraic integer λ is called *weak Perron*, for any of its Galois conjugate λ', we have $|\lambda'| \leq \lambda$.

The goal of our paper is to explore the topological entropy for rational maps on locally compact non-archimedean fields. Recall that we say a field K is *non-archimedean*, if there is a norm $|\cdot|$ on K such that $|a+b| \leq \max(|a|,|b|)$. We say it has *discrete valuation* if $|\cdot|$ takes value in $\{0\} \cup \{\alpha^n : n \in \mathbb{Z}\}$ for some real number $0 < \alpha < 1$.

Definition 1.1. Let K be a non-archimedean field with non-trivial discrete valuation, let $\phi \in K(z)$ be a rational map of degree at least 2.

- The *K-Fatou set* of ϕ, denoted as $F_K(\phi)$, is the domain of equicontinuity of $\{\phi^n\}$ on $\mathbb{P}^1(K)$. The *K-Julia set* is then defined as $J_K(\phi) = \mathbb{P}^1(K) \backslash F_K(\phi)$. Here $\phi^n = \phi^{\circ n}$ is the n-times composition of ϕ.
- By a *repelling periodic point of period n*, we mean a fixed point p of ϕ^n, such that for any q sufficiently close to p, $|p - \phi^n(q)| > |p - q|$.
- Following [2] and [4], let $\mathrm{Crit_K}(\phi)$ be the set of critical points of ϕ in $\mathbb{P}^1(K)$. We say that ϕ is *hyperbolic* if $J_K(\phi) \cap \mathrm{Crit}_K(\phi) = \emptyset$, and say that ϕ is *subhyperbolic* if all points in $J_K(\phi) \cap \mathrm{Crit}_K(\phi) = \emptyset$ are have *eventually periodic orbit* for every critical point c there exists integers $n' > n > 0$ such that $\phi^{n'}(c) = \phi^n(c)$.

Remark 1.2. By [10], when K is complete and algebrically complete, an equivalent, alternative definition for $J_K(\phi)$ is that it consists of points p such that for any neighborhood U of p, $\mathbb{P}^1(K)\backslash\bigcup_n \phi^n(U)$ has at most one point. When $J_K(\phi)$ is compact, a subhyperbolic rational map exhibits expanding property with respect some natural (singular) metric, see [5].

Remark 1.3. If ϕ is subhyperbolic, a critical point of ϕ either has eventually periodic orbit, or has a neighborhood within which there are no repelling periodic points.

Our main result is a characterization of the topological entropy of subhyperbolic maps in $K(z)$, analogous to the main result in [15]:

Theorem 1.4. *Let K be a locally compact non-Archimedean field with discrete valuation (for example $\mathbb{Q}_p$ or its finite extension). Then the set*

$$\mathcal{H}_K = \{\exp(h_\phi) : \phi \in K(z) \textit{ subhyperbolic}, J_K(\phi) \subsetneq \mathbb{P}^1(K)\}$$

equals the set of weak Perron numbers.

Here by topological entropy of a map f on a metric space, we mean the number

$$h_f = \limsup_{n\to\infty} \frac{N_n(f)}{n}$$

where $N_n(f)$ is the number of repelling fixed points of $f^{\circ n}$.

Remark 1.5. Let K be a locally compact non-Archimedean field with discrete valuation, and $\phi \in K(z)$. It is easy to see that both $\mathbb{P}^1(K)$ and $J_K(\phi)$ are compact, and the topological entropy of ϕ on $\mathbb{P}^1(K)$ equals the topological entropy of $\phi_{J_K(\phi)}$, and 0 if $J_K(\phi) = \emptyset$.

1.1. **Outline of Proof.** Recall that if X is a metric space, $Y \subseteq X$ (which may be equal to X), $f : Y \to X$ a continuous map, then the Artin-Mazur ζ function f is defined as

$$\zeta_f(t) = \exp\left(\sum_{n\geq 1} \frac{N_n(f)t^n}{n}\right)$$

where $N_n(f)$ is the number of repelling fixed points of f^n. It is easy to see that the radius of convergence of ζ_f at 0 is $\exp(-h_f)$, where $h_f = \limsup_{n\to\infty} \frac{N_n(f)}{n}$ as defined above.

Theorem 1.4 follows immediately from the following two theorems:

Theorem 1.6. *Let K be any locally compact non-Archimedean field with discrete valuation, λ a weak Perron number. Then there is a* hyperbolic *rational map $\phi \in K(z)$ such that $J_K(\phi) \subseteq K$, and the topological entropy of ϕ on $\mathbb{P}^1(K)$ is* $\log(\lambda)$.

Theorem 1.7. *Let K be a locally compact non-Archimedean field with discrete valuation. Suppose $\phi \in K(z)$ is subhyperbolic, then the Artin-Mazur ζ function of ϕ on $\mathbb{P}^1(K)$ is a rational function of the form $\frac{\prod_i(1-t^{l_i})}{\det(I-tA)}$, where A is a non negative integer matrix.*

Proof of Theorem 1.4 given Theorems 1.6 and 1.7. $\supseteq$ is due to Theorem 1.6, $\subseteq$ due to Theorem 1.7. ∎

Theorem 1.6 is proved via the classical result of [12] and the "non Archimedean surgery" in [14]. To prove Theorem 1.7, we will define a concept of subhyperbolicity for self maps of a compact non-Archimedean metric space with discrete valuation (or alternatively, proper maps on locally finite infinite trees, see Theorem 5.4). Roughly speaking, a *subhyperbolic map well behaved at the critical points* on such a metric space is a continuous map f, such that

(1) For all but finitely many points (called *critical points*) in the domain, the map is locally a bijective scaling.
(2) The critical points which are in the closure of repelling periodic points must be eventually sent to a repelling periodic map under iterations of the map, and there is a scaling map in a neighborhood of the critical point conjugate to the map f^p on a neighborhood of this repelling periodic point, where p is the period.

More precise definition of subhyperbolicity in non Archimedean metric space will be given in Section 5. Then, we will prove Theorem 5.3, which says that the ζ function for these subhyperbolic continuous maps satisfies the conclusion of Theorem 1.7, and finally show that Theorem 1.7 can be reduced to a special case of Theorem 5.3.

Remark 1.8. There are maps with compact Julia sets defined over discrete valued non-archimedean fields with infinite residue fields, see [9, 11, 16]. For many of these maps the conclusion of Theorem 1.7 remains true. See Remark 6.3.

1.2. **Structure of the Paper.** In Section 2 we will recall the relationship between Artin-Mazur ζ function and Markov decompositions, in Section 3 we will prove Theorem 1.6. In Section 4 we will develop the tools for finding Markov decompositions, which will be used in Section 5 to prove Theorem 5.3. Finally in Section 6 we will prove Theorem 1.7 from Theorem 5.3 which would finish the proof of Theorem 1.4. In Section 7 we will illustrate the proof of Theorem 1.7 via an example. In Section 8 we will discuss some further questions.

2. Scaling Discs, *zeta*unctions and Topological Entropies

In this section we will establish some notations and also recall some basic facts about the Artin-Mazur ζ function and topological entropies (cf. e.g. [3]) which will be used throughout the paper.

Definition 2.1. By a *non-Archimedean metric space*, we mean a metric space (X, d), such that for any $a, b, c \in X$, $d(a, c) \leq \max(d(a, b), d(b, c))$. We say a non-Archimedean metric space has *discrete valuation*, if d takes value in $\{0\} \cup \{\alpha^n : n \in \mathbb{Z}\}$ for some $0 < \alpha < 1$.

Definition 2.2. Let (X, d) be a metric space, $Y \subseteq X$, $f : Y \to X$ a map. We say a disc $D = D(x, r) = \{a \in X : d(x, a) \leq r\}$ in X is a *scaling disc with scaling factor L*, if

(1) $D \subseteq Y$.
(2) $f(D)$ is a disc in X.
(3) f is a bijection from D to $f(D)$.
(4) For every $a, b \in D$, $d(f(a), f(b)) = Ld(a, b)$.

Remark 2.3. If K is a locally compact non-Archimedean field with discrete valuation, the projective line $\mathbb{P}^1(K)$ is a compact non-Archimedean metric space. By Hensel's Lemma, if f is a rational map, p a regular point of f (i.e. $f'(p) \neq 0$), then there is a scaling disc of f centered at p.

Remark 2.4. Here the reason we set the domain of the map as $Y \subseteq X$ and not X itself is because we want it to be an open neighborhood of the Julia set. This way we do not need to know the exact shape of the Julia set, and will also be able to use tools that work only on open sets like Hensel's Lemma.

Definition 2.5. Let (X, d) be a metric space, $Y \subseteq X$, $f : Y \to X$. We call a set of scaling discs $\mathcal{D}$ on metric space (X, d) *Markov*, if

(1) Y is a disjoint union of elements of $\mathbb{D}$.
(2) Every $D \in \mathcal{D}$ must be in one of the following three situations:

(a) For every natural number n, there is some $D_n \in \mathcal{D}$ such that $f^n(D) \subseteq D_n$.
(b) $f(D) \cap Y$ is a union of elements of $\mathcal{D}$. Here we allow the case when $f(D) \cap Y$ is a union of zero elements of $\mathcal{D}$, i.e. $f(D) \cap Y = \emptyset$.

Given f and a Markov set of scaling discs $\mathcal{D}$, we can construct a directed graph $G(f, \mathcal{D})$ as follows:

(1) For every $D \in \mathcal{D}$ we associated it with a vertex.
(2) If D is not in Case (a) above and only in Case (b), in other words if $f(D) \cap Y$ is a union of elements of $\mathcal{D}$, and it is not true that for every n, $f^n(D)$ is a subset of some disc in $\mathcal{D}$, then add an directed edge from the vertex associated with D to the vertices associated with elements of $\mathcal{D}$ which are subsets of $f(D)$.

Proposition 2.6. *Let (X, d) be a complete metric space, $Y \subseteq X$, $f : Y \to X$, and f admits a Markov set of scaling discs $\mathcal{D}$. Let $n > 0$ be any positive natural number. Then there is a one-to-one correspondence between repelling fixed points of f^n and closed paths on the graph $G(f, \mathcal{D})$ of length n. Here a closed path on a directed graph G of length n is a path on $G = G(f, \mathcal{D})$ consisting of n consecutive edges, where the tail of the first edge equals the head of the last.*

Proof. Let p be a repelling fixed point of f^n, D_k, $k = 0, 1, \dots, n$, be the disc in $\mathcal{D}$ containing $f^k(p)$, then one gets a closed path in G starting with the vertex associated with D_0, passing through the vertices associated with D_1, D_2, $\dots$ and ends at the vertex associated with $D_n = D_0$. Such a path exists because p being a repelling periodic point implies that non of the D_k can be in Case (b) above.

To show that this map is a bijection, we only need to show that for any closed path γ with length D on G, we can find such a p. Let D_0, D_1, $\dots$, $D_n = D_0$ be the discs in $\mathcal{D}$ associated with the vertices γ passes through. Then $(f|_{D_0})^{-1} \circ (f|_{D_1})^{-1} \circ \cdots \circ (f|_{D_{n-1}})^{-1}$ is a scaling map from $D_n = D_0$ to D_0. By construction of G we know that this scaling map has scaling factor less than 1, so it has a unique fixed point in D_0 due to completeness of the metric. It is now easy to see that this fixed point is the desired point p. $\square$

When $\mathcal{D}$ is finite, $G(f, \mathcal{D})$ is a finite graph, the counting of closed paths can be done via the following well known fact from linear algebra:

Proposition 2.7. *Let G be a finite directed graph with n vertices. The* transition matrix *A_G is a $n \times n$ matrix, where the (i, j)-th entry is the number of directed edges from vertex i to vertex j. Let $N(G, n)$ be the number of closed paths on G of length n. Then:*

$$\exp\left(\sum_{n \geq 1} \frac{N(G, n) t^n}{n}\right) = \exp\left(\sum_{n \geq 1} \frac{tr(tA_G^n)}{n}\right) = (\det(I_n - tA_G))^{-1}$$

$\square$

As a consequence, if the Markov set of scaling discs $\mathcal{D}$ is finite, we have

$$\zeta_f(t) = (\det(I - tA_{G(f,\mathcal{D})}))^{-1}$$

and the topological entropy equals log of the leading eigenvalue of $A_{G(f,\mathcal{D})}$.

3. Hyperbolic rational maps from weak Perron numbers

In this section we will give a proof of Theorem 1.6.

Definition 3.1. Given a $k \times k$ matrix A, we build a G_A as follows:

(1) There are k vertices in G_A, labeled as $1, \dots, k$.
(2) The number of edges from vertex i to vertex j equals the (i, j)-th entry of A.

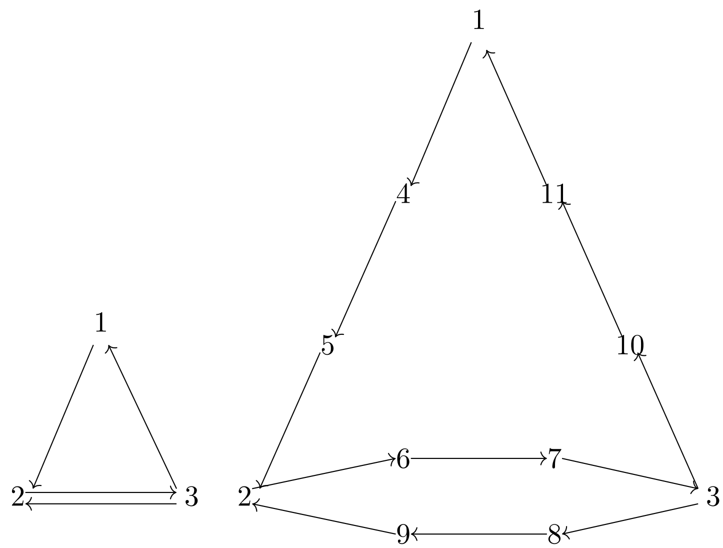

FIGURE 1. The left hand side is G, right hand side is G', $nj = 3$.

In other words, G_A is a finite directed graph whose transition matrix, as defined in the statement of Proposition 2.7, equals A. We call G_A the *associated graph* of A.

Proof of Theorem 1.6. Let λ be a weak Perron number, then there is some natural number $n > 1$, such that λ^n is a Perron number. Lind's theorem [12] implies that there is a non-negative integer matrix A, such that there is some natural number $j > 0$, A^j has all entries greater than 0, and λ^n is the Perron-Frobenious eigenvalue of A. As a consequence, A^j has Perron-Frobenious eigenvalue λ^{nj}.

Now let $G = G_{A^j}$ be the directed graph associated with A^j. For every directed edge of G, replace it with a path of length nj, adding $nj-1$ extra vertices in between, and denote the resulting graph G', as illustrated in Figure 1. It is easy to see from this construction that the number of closed paths on G and G' are related by

$$N(G', L) = \begin{cases} njN(G, L/(nj)) & (nj)|L \\ 0 & (nj) \not| L \end{cases}$$

which implies that the Perron-Frobenious eigenvalue of G' equals $(\lambda^{nj})^{1/(nj)} = \lambda$. So we only need to find a hyperbolic rational map f, a subset $Y \subseteq \mathbb{P}^1(K)$ that covers all repelling periodic points, and a Markov set of scaling discs $\mathcal{D}$ such that $G(f, \mathcal{D}) = G'$.

Suppose G' has m vertices, label them with $1, \dots, m$. By construction, between any two vertices there are at most 1 arrow, and if there is a vertex w that has two arrows to two distinct vertices v, v', then no other vertex can have an arrow to either v or v'. Let m' be the number of vertices of G' with outgoing arrows, let $j_1, \dots, j_{m'}$ be their indices. On $\mathbb{P}^1(K)$ pick m' disjoint discs $D'_1, \dots, D'_{m'}$ that do not contain 0 or ∞, and for each $i = 1, \dots, m$, if there is an arrow from vertex labeled with j_r to the vertex labeled with i, make a proper subdisc D_i in D'_r which is disjoint from all D_j where $j < i$. If no arrows end at the vertex labeled with i, make a D_i outside the $D'_1, \dots, D'_m$ which also does not contain 0 or ∞. Now for each $j_1, \dots, j_{m'}$, define f_{j_i} as a linear map sending D_{j_i} to D'_i. For all other $j \notin \{j_1, \dots, j_{m'}\}$ let f_j be the constant 0 function on D_j. Follow the ideas of [14, Theorem 4.2], suppose the diameter of D_i is α^{n_i}, pick $a_i \in D_i$ and $c_i \in K$ such that $|c_i| = \alpha^{2n_i-1}$, and $M >> 1$ sufficiently large, define:

$$f_M(z) = \sum_{i=1}^{m} \frac{f_i(z)}{1 - \frac{(z-a_i)^{2M}}{c_i^M}}$$

By construction, as M increases, $\frac{1}{1-\frac{(z-a_i)^{2M}}{c_i^M}}$ converges uniformly to 1 on D_i, uniformly to 0 on $\mathbb{P}^1(K)\backslash D_i$. Now set M sufficiently large so that the complement of $\bigcup_{i=1}^m D_i$ get send to a small disc centered at 0 where f is a contraction (which guaranteed that the complement of $\bigcup_{i=1}^m D_i$ has no repelling periodic points and is contained in the Fatou set), and D_j are all scaling discs of f_M, which are sent to D'_j bijectively by f_M. Now we can set $Y = \bigcup_{i=1}^m D_i$, and $\mathcal{D} = \{D_1, \ldots, D_m\}$. To show that such a $f = f_M$ is hyperbolic, note that the Julia set is contained in $\bigcup_{i=1}^m D_i$, hence can be covered by scaling discs, which implies that it would not contain any critical point. □

4. Splitting of the Scaling Discs

In this section we will discuss a procedure to turn a set of scaling discs to a Markov set, which was introduced in [4] and we made minor adaptations for our purpose.

Firstly, we list some elementary properties of non-Archimedean metric spaces which will be used later:

Proposition 4.1. *Let (X, d) be a non Archimedean metric space with discrete valuation*

1. *Two discs in X are either disjoint or one contains the other.*
2. *Every disc with non zero radius is both open and closed.*
3. *Let D be a disc, $D' \subsetneq D$ a proper subdisc, then $D\backslash D'$ is both open and closed.*
4. *Let U be an open set in X. A disc D in X is called* maximal *in U, if $D \subseteq U$, and a disc D' satisfies $D \subseteq D'$ and $D' \subseteq U$ implies that $D = D'$. Then:*
   (a) *Every disc contained in U is a subset of one of these maximal discs.*
   (b) *U can be written as a disjoint union of maximal discs in U.*
   (c) *U is compact if and only if it has only finitely many maximal discs.*

Proof. (1) follows the definition of "non-archimedean", (2) follows from discrete valuation, and (3) follows from (2). (4) (a) follows from discrete valuation and compactness of X (which implies that d has an upper bound). When U is open, every point in U is contained in some disc in U which is a subset of some maximal disc, and if two maximal discs are not disjoint, one is contained in the other as a proper subdisc, which contradicts with the definition of "maximal", hence (4) (b) follows. Lastly, if U is compact, the cover $\{D : D \text{ is a disc in } X, D \subseteq U\}$ has a finite subcover, and each element in the subcover belongs to a unique maximal disc, which together covers U, hence there can only be finitely many maximal discs. If U has only finitely many maximal discs, each of them is closed hence compact, hence U is the union of finitely many compact sets hence compact. □

Let X be a non-Archimedean metric space with discrete valuation, $Y \subseteq X$, $f : Y \to X$ a map. We further assume that Y is a union of scaling discs of f. Let $\mathcal{D}_0$ be a set of scaling discs whose union equals Y. Because in a non Archimedean metric space, two discs are either disjoint or one contains the other, one can remove from $\mathcal{D}_0$ all discs that are subdiscs of another, to get a disjoint set of scaling discs $\mathcal{D}$ whose union equals Y.

Now we describe an operation on sets of disjoint scaling discs, called *splitting*:

Definition 4.2. Let X be a non-Archimedean metric space.

1. Let D be a disc in X, $D' \subsetneq D$ a proper subdisc. By a *splitting of D via D'*, we mean a set of subdiscs $\{D'\} \cup \{D_\alpha : D_\alpha \text{ is a maximal disc in } D\backslash D'\}$.

(2) Let $\mathcal{D}$ be a set of disjoint discs in X. We say a set of disjoint discs $\mathcal{D}'$ is *obtained from $\mathcal{D}$ via a splitting*, if $\mathcal{D}'$ can be obtained by replacing a single element $D \in \mathcal{D}'$ with its splitting via a proper subdisc. We say $\mathcal{D}'$ is *obtained from $\mathcal{D}$ via finite splitting*, if one can obtain $\mathcal{D}'$ from $\mathcal{D}$ by carrying out the procedure of replacing a disc with its splitting finitely many times.

Remark 4.3. It is easy to see that if $\mathcal{D}$ is a set of disjoint scaling discs, $\mathcal{D}'$ a set of discs obtained from $\mathcal{D}$ via finite splitting, then $\mathcal{D}'$ is also a set of disjoint scaling discs. If X is locally finite then all but finitely many discs in $\mathcal{D}'$ are in $\mathcal{D}$.

Now we state the main theorem for this Section:

Theorem 4.4. *Let X be a non archimedean metric space with discrete valuation, $Y \subseteq X$, $f : Y \to X$, and Y can be decomposed into a set $\mathcal{D}$ of disjoint scaling discs. If for all but finitely many discs in $\mathcal{D}$, the image under f intersecting with Y is a union of discs in $\mathcal{D}$, then there is a Markov set of scaling discs $\mathcal{D}'$ obtained from $\mathcal{D}$ via finite splitting.*

Proof. Let $D_1, \ldots, D_l$ be the discs in $\mathcal{D}$ which does not satisfies the Markov condition. In other words, there are natural numbers $n_j > 1$, such that $f^{n_j}(D_j) \cap Y$ is a union of discs but $f^{n'}(D_j)$ is contained in some disc in $\mathcal{D}$ as a proper subdisc. Let $\mathcal{N}(\mathcal{D}) = \sum_j (n_j - 1)$, then $\mathcal{N} > 0$.

Now consider a disc $D_1 \in \mathcal{D}$. Let D' be the disc in $\mathcal{D}$ which contains $f^{n_1-1}(D_1)$ as a proper subdisc, do a splitting of $\mathcal{D}$ by replacing D' with $f^{n_1-1}(D_1)$ as well as the maximal subdiscs of $D' \backslash f^{n_1-1}(D_1)$, as in Definition 4.2. Now in this new set of scaling discs, n_1 would go down by at least 1, and there would not be any other discs that fail the Markov condition. To see this, we only need to show that all the newly added discs can not be sent to a proper subdisc of some disc in $\mathcal{D}$ under f:

(1) For the disc $f^{n_1-1}(D_1)$, by definition of n_1, $f(f^{n_1-1}(D_1)) = f^{n_1}(D_1)$ whose intersection with Y is a union of discs in $\mathcal{D}$.
(2) For any maximal disc D_m in $D' \backslash f^{n_1-1}(D_1)$, because $f(D')$ is a disc that contains $f^{n_1}(D_1) \cap Y$, its intersection with Y is a disjoint union of discs in $\mathcal{D}$. Suppose $f(D_m)$ is a proper subdisc of some $D'' \in \mathcal{D}$, then $D'' \cap f(D') \neq \emptyset$, hence $D'' \subseteq f(D')$ or $f(D') \subseteq D''$. The latter is not possible because discs in $\mathcal{D}$ are disjoint. In the former case, $(f|_{D'})^{-1}(D'')$ is a subdisc of $D' \backslash f^{n_1}(D_1)$ hence must be contained in some D_m, a contradiction.

Now repeat the splitting process described above, because $\mathcal{N}(\mathcal{D})$ is finite, it would get to 0 after finitely many steps, which would make the set of scaling discs Markov. This proved the theorem. $\square$

5. Subhyperbolic maps on non-Archimedean metric spaces

In this section we will define a concept of "subhyperbolic" for compact non-Archimedean metric spaces with discrete valuation, and show that for such maps the Artin-Mazur ζ function is rational and in the desired form.

Definition 5.1. Let X be a compact non-Archimedean metric space with discrete valuation, $Y \subseteq X$ a subset which is both open and closed. $f : Y \to X$ a continuous map.

(1) We say that f is *hyperbolic*, if for every $p \in Y$, there is a scaling disc D_y of f such that $y \in D_y$.
(2) We say that f is *subhyperbolic*, if:
   (a) For all but finitely many $p \in Y$, there are scaling discs D_y such that $y \in D_y$. These points in Y are called *regular points*, and the finitely many points in Y that are not in scaling discs are called *critical points*.

(b) Every critical point c is in either one of the two situations:
   (i) There is some neighborhood U_c containing c where there are no repelling periodic points of f.
   (ii) There are positive integers k_c, m_c such that $f^{k_c}(c)$ is a repelling periodic point of f which is regular and has minimal period p_c, and $f^{k_c+n}(c)$ are all regular for all $n > 0$. $f^{n'}(c)$ for any $0 \leq n' \leq k_c - 1$ are not periodic point of f.

(3) If furthermore for each critical point in case (b) (ii) above, we can find integers p_c such that for $j = 0, \ldots, k_c - 1$, there are disjoint small discs $V_{c,j}$ around $f^j(c)$, and scaling maps $T_{c,j}$ defined on $V_{c,j}$ which fixes $f^j(c)$, such that on each $V_{c,j}$ where $0 \leq j < k_c - 1$, $T_{c,j+1} \circ f = f \circ T_{c,j}$, and on V_{c,k_c-1}, $f \circ T_{k_c-1} = f^{1+p_c m_c}$, we say f is *subhyperbolic and well behaved at the critical points.*

Remark 5.2. It is easy to see that hyperbolic implies subhyperbolic. If f is hyperbolic, compactness implies that one can cover Y with finitely many disjoint scaling discs. Apply Theorem 4.4, we see that there is a finite set of disjoint scaling discs that are Markov. Now Proposition 2.6 and Proposition 2.7 imply that the Artin-Mazur ζ function is of the form $(\det(I - tA))^{-1}$ for some non negative integer matrix A.

The main theorem for this section is a geometrical version of Theorem 1.7:

Theorem 5.3. *Let X be a compact non-Archimedean metric space with discrete valuation, $f : Y \to X$ a subhyperbolic map which is well behaved at the critical points, then the Artin-Mazur ζ function of f on is a rational funcition of the form $\zeta_f(t) = \frac{\prod_i (1-t^{l_i})}{\det(I-tA)}$, where A is a non negative integer matrix.*

Proof. Because removing all the U_c in Definition 5.1 Part (2) (b) (i) from Y would not change the number of repelling periodic points, we can assume that all critical points in Y are in Case (ii) of Part (2) (b) of the definition, without loss of generality.

Now let $C_1, \ldots, C_l$ be the finitely many repelling periodic orbit. By assumption they all consist only of regular points. Pick some point q_i in each C_i. Let p_i be the length of C_i. For every critical point c which is not in the forward orbit of another critical point, let i_c, $0 \leq b_c < p_{i_c}$ be integers such that $f^{k_c}(c) = f^{b_c}(q_{i_c})$. Let $m_i = lcm\{m_c : i = i_c\}$, then by replacing m_c with m_{i_c} and $T_{c,j}$ with $T_{c,j}^{m_{i_c}/m_c}$ we can assume without loss of generality that $m_i = m_c$ for all c such that $i_c = i$.

Now for each q_i, pick small scaling disc $D_{i,0}$ in Y centered at it, and for every critical point c which is not in the forward orbit of another, pick $V_{c,j}$ as in Definition 5.1 Part (3) which are small enough, such that:

(1) The discs $D_{i,j} = f^j(D_{j,0})$ centered at $f^j(q_i)$, for $0 \leq j < p_i$, are all scaling discs in Y. f^{m_i} is a scaling map on each $D_{i,j}$.
(2) f sends $V_{c,j}$ to $V(c, j+1)$, and V_{c,k_c-1} to D_{i_c,b_c}.
(3) All the $D_{i,j}$ and $V_{c,j}$ are disjoint.

For each q_i, let $A_{i,0,0}$ be the annulus $D_{i,0} \backslash (f^{p_i}|_{D_{i,0}})^{-1}(D_{i,0})$, and let $A_{i,0,n+1}$ be the preimage of $A_{i,0,n}$ under the scaling map $f^{p_i}|_{D_{i,0}}$. Then $D_{i,0} = \{q_i\} \cup \bigcup_{n=0}^{\infty} A_{i,0,n}$. Let $A_{i,j,n} = f^j(A_{i,j,0})$, then $D_{i,j} = \{f^j(q_i)\} \cup \bigcup_{n=0}^{\infty} A_{i,j,n}$. For every c where $i_c = i$, every $0 \leq j < k_c$, let $A'_{c,j,0}$ be the annulus $V_{c,j} \backslash (T_{c,j})^{-1}(V_{c,j})$, and $A'_{c,j,n+1}$ be the preimage of $A'_{c,j,n}$ under $T_{c,j}$.

Now define a map F on $\bigcup_{c,j} A'_{c,j,0} \cup \bigcup_{i,0 \leq n < m_i, 0 \leq j < p_i} A_{i,j,n}$ as follows:

(1) If $j < k_c - 1$, $x \in A'_{c,j,0}$, let t_x be an integer such that $f(x) \in A'_{c,j+1,t_x}$, and $F(x) = T_{c,j+1}^{t_x}(f(x))$.

(2) If $x \in A'_{c,k_c-1,0}$, let t_x be an integer such that $f(x) \in A_{i_c,b_c,n+t_x m_x}$ for some $0 \leq n < m_x$, and $F(x) = f^{m_x t_x}(f(x))$.
(3) If $x \in A_{i,j,n}$, if $j = p_i - 1$ and $n = 0$, then $F(x) = (f^{m_i}|_{D_{i,0}})^{-1} f(x)$, otherwise $f(x) = F(x)$.

Now it is easy to see that F is a locally scaling map defined on a compact set. By Proposition 4.4, the $A'_{c,j,0}$ and $A_{i,j,n}$ (where $0 \leq n < m_i$) can be decomposed in finitely many scaling discs such that F send each to a union of other scaling discs. Let $D'_{c,j,s,0}$ be such scaling discs in $A'_{c,j,0}$, $D_{i,j,s,n,0}$ be such scaling discs in $A_{i,j,n}$, and let $D'_{c,j,s,h}$ and $D_{i,j,s,n,h}$ be their image under $T_{c,j}^{-h}$ and $(f^{m_i}|_{D_{i,j}})^{-h}$ respectively.

The complement of $D_{i,j}$ and $V_{c,j}$ does not contain any critical point and is compact hence can also be decomposed into the disjoint union of finitely many scaling discs D_m. By the construction of F, for all but finitely many h, f sends $D'_{c,j,s,h}$ as well as $D_{i,j,s,n,h}$ to disjoint unions of discs of the form $D'_{c,j,s,h}$ or $D_{i,j,s,n,h}$, hence the countable set of disjoint scaling discs $\{D'_{c,j,s,h}, D_{i,j,h}, D_m\}$ satisfies the assumption of Proposition 4.4, so can be made Markov after finitely many splittings. Let $M >> 0$ be large enough such that $\bigcup_{h \geq M} D_{i,j,s,n,h}$, as well as their iterated preimages in all the $V_{c,j}$ where $i = i_c$, are not affected by any of the splittings. Then these $\bigcup_{h \geq M} D_{i,j,s,n,h}$, their preimages in the $V_{c,j}$s, the finitely many critical points and their forward images, as well as the remaining finitely many discs in the Markov set of scaling discs which are not in any of these unions of discs, form a finite disjoint decomposition $\mathcal{D}$ of the compact set Y, such that f send each of these finitely many subsets to a disjoint union of other subsets in this list. We call the $\bigcup_{h \geq M} D_{i,j,s,n,h}$ and their preimages in $V_{c,j}$ the *non compact* elements of $\mathcal{D}$ and the others the *compact* elements.

Now carry out the same argument as in Proposition 2.6, let $G_0 = G(f, \mathcal{D})$ as in Definition 2.5. Note that because points in the forward orbit of critical points are sent to other points in the forward orbit, those points would correspond to isolated vertices in G_0, so we modify G_0 into a directed graph G by adding back the directed edges correspond to maps between these elements of $\mathcal{D}$ that consists of single points. Hence, similar to the proof of Proposition 2.6, now any repelling periodic point of period n either corresponds to a closed path on G of length n. On the other hand, if a closed path on G of length n passes through at least one non compact element, then contracting mapping theorem implies that it would correspond to a unique repelling periodic point of period n, while if it only passes through non compact elements it would not correspond to any periodic point of f. By construction, the only possible closed paths consisting of only vertices corresponding to non compact elements are those that only involve the $\bigcup_{h \geq M} D_{i,j,s,n,h}$s. So $N_n(f) = N(G, n) - \sum_{i, m_i | n} N_i$ where N_i is the cardinality of $\{\bigcup_{h \geq M} D_{i,0,s,n,h}\}$, hence

$$\zeta_f(t) = \exp\left(\sum_{n \geq 1} \frac{N_n(f) t^n}{n}\right) = \exp\left(\sum_{n \geq 1} \frac{N(G,n) t^n}{n}\right) \prod_i \left(\exp\left(\sum_{n \geq 1} \frac{t^{n m_i}}{m_i}\right)\right)^{-N_i}$$

$$= \frac{\prod_i (1 - t^{m_i})^{N_i}}{\det(I - t A_G)}$$

□

5.1. **Restatement of Theorem 5.3 in terms of trees.** Let T be a locally finite infinite simplicial tree, then ∂T can be made into a compact non Archimedean metric space, by setting all edges to have length 1, picking a vertex v, then set $d(x, y) = e^{-n}$ if the rays from v to x and from v to y overlaps on an interval of length n.

Theorem 5.3 can then be restated via the language of infinite simplicial trees as follows:

Theorem 5.4. *Let T be a locally finite, infinite simplicial tree. By a* full subtree *we mean a connected component of T with one edge deleted. Let Y a finite union of full subtrees, $f : Y \to T$ a proper, continuous map, which induces $\partial_f : \partial Y \to \partial T$. If f is* subhyperbolic, *by which we mean the following:*

1. *For all but finitely many $p \in \partial Y$, there are full subtrees $T_y \subseteq Y$ such that $y \in \partial T_y$, and f send T_y homeomorphically to a full subtree of T. These points in ∂Y are called* regular points*, and the finitely many points in ∂Y that are not in scaling discs are called* critical points*.*
2. *For every critical point c, there are positive integers k_c, p_c, m_c such that*
   - *(a) $(\partial f)^{k_c}(c)$ is a fixed point of $(\partial f)^{p_c}$, and there is a full subtree whose boundary contains $(\partial f)^{k_c}(c)$ which is sent homeomophically to a larger full subtree by f^{p_c}.*
   - *(b) For $j = 0, \dots, k_c - 1$, there are disjoint full subtrees $V_{c,j}$ whose boundary contains $(\partial f)^j(c)$, and homeomorphisms $T_{c,j}$ from $V_{c,j}$ to a larger full subtree whose induced map on the boundary $(\partial f)^j(c)$, such that on each $V_{c,j}$ where $0 \le j < k_c - 1$, $T_{c,j+1} \circ f = f \circ T_{c,j}$, and on V_{c,k_c-1}, $f \circ T_{k_c-1} = f^{1+p_c m_c}$.*

Then the Artin-Mazur ζ function of ∂f on is a rational funcition of the form $\zeta_{\partial f}(t) = \frac{\prod_i (1-t^{l_i})}{\det(I-tA)}$, where A is a non negative integer matrix.

□

6. ζ−function for subhyerbolic rational maps

Now we finish the proof of Theorem 1.7 from Theorem 5.3, which finishes the proof of Theorem 1.4.

Proof of Theorem 1.7 from Theorem 5.3. We only need to show that every critical point in the Julia set of a subhyperbolic rational function ϕ satisfies the condition in Definition 5.1 Part (3). This follows immediately from the following fact:

Lemma 6.1. *If f and g are analytic functions over some non Archimedean field K with discrete valuation, $f(0) = g(0) = 0$ and they have the same degree at 0, and there is some $a \in K$ such that $\frac{d}{dx} f(ax)|_{x=0} = \frac{d}{dx} g(x)|_{x=0}$, then there is an analytic function h, $h(0) = 0$, $\frac{d}{dx} h(x)|_{x=0} = a$, such that $g = f \circ h$.*

Proof. Write both g and $f \circ h$ as power series and compare the coefficients, one get a formal power series with the first term cx. g and f converges in a neighborhood of 0 implies that so is h. □

Now for every critical point c in the Julia set, let m_c be the degree of ϕ^{k_c}, $a = \frac{d}{dx}\phi^{p_c}(x)|_{\phi^{k_c}(c)}$, $f(x) = \phi^{k_c}(x + c) - \phi^{k_c}(c)$, and $g(x) = \phi^{k_c + m_c p_c}(x + c) - \phi^{k_c}(c)$, apply the Lemma above, we get the function h, then $T_{c,0}(z) = h(z - c) + c$. One can define the other $T_{c,j}$ similarly. From construction we have the $f^j \circ T_{k_c-j} = f^{j+p_c m_c}$. To show $T_{c,j+1} \circ f = f \circ T_{c,j}$, note that to solve for $T_{c,j}$ from $T_{c,j+1}$, one has finitely many solutions which are distinguished by their linear terms, and one can choose the c in the proof of the Lemma above to make them consistent. □

Remark 6.2. It is easy to see that in Theorem 1.7, one can replace "rational" with "piecewise analytic".

Remark 6.3. When K is not known to be locally compact but $J_K(\phi)$ is, the conclusion of Theorem 1.7 remain valid. To show that, we can modify the statement and proof of 5.3 as follows:

(1) Instead of requiring X to be compact, we require $\bigcap_{i=0}^{\infty} f^{-i}(Y)$ to be compact.
(2) We need to modify the disc splitting process for F. Instead of applying Proposition 4.4 we will now need to do it "manually".
(3) When we build the directed graph G at the end of the proof of Theorem 5.3, we can no longer assume it to be a finite graph. However, if a vertex of G consists of a single disc, the disc is a scaling disc of a closed subset of Y and all such discs are disjoint, so only finitely many of them will hit $\bigcap_{i=0}^{\infty} f^{-i}(Y)$. Now replace G with the largest strongly connected subgraph containing these finitely many vertices, which is finite by construction, and one can proceed with the argument.

7. An Example for Theorem 1.7

To illustrate the proof of Theorem 1.7 and Theorem 5.3, we consider $K = \mathbb{Q}_3$, $\phi(z) = z(z-1)^2/3$. It is easy to see that the Julia set is contained in $\mathbb{Z}_3$, and the only critical point in the Julia set is 1, whose image under ϕ is 0 which is a repelling fixed point, so it satisfies the assumption of Theorem 1.7.

Now we follow the notation as in the proof of Theorem 5.3. $c = 1$, $k_1 = 1$, $p_1 = 1$, $m_1 = 2$, $l = 1$, $q_1 = 0$, $b_1 = 0$, $V_{1,0} = 1 + 9\mathbb{Z}_3$, $D_{1,0} = 27\mathbb{Z}_3$, $A_{1,0,n} = \pm 3^{n+3} + 3^{n+4}\mathbb{Z}_3$, $A'_{1,0,n} = 1 + \pm 3^{n+2} + 3^{n+3}\mathbb{Z}_3$. The map F sends $\pm 27 + 81\mathbb{Z}_3$ to $\pm 81 + 243\mathbb{Z}_3$ and vice versa via a scaling, and sends both $1 + \pm 9 + 27\mathbb{Z}_3$ to $27 + 81\mathbb{Z}_3$. Apply Proposition 4.4, we get a set of scaling discs and their images under F in Table 1:

Table 1

Scaling Disc	Image under F
$27 + 81\mathbb{Z}_3$	$81 + 243\mathbb{Z}_3$
$81 + 243\mathbb{Z}_3$	$27 + 81\mathbb{Z}_3$
$54 + 81\mathbb{Z}_3$	$162 + 243\mathbb{Z}_3$
$162 + 243\mathbb{Z}_3$	$54 + 81\mathbb{Z}_3$
$1 + 9 + 27\mathbb{Z}_3$	$27 + 81\mathbb{Z}_3$
$1 + 18 + 27\mathbb{Z}_3$	$27 + 81\mathbb{Z}_3$

The set $\mathbb{Z}_3 \backslash (D_{1,0} \cup V_{1,0})$ can be decomposed into 7 scaling discs: $2 + 3\mathbb{Z}_3$, $1 + \pm 3 + 9\mathbb{Z}_3$, $\pm 3 + 9\mathbb{Z}_3$ and $\pm 9 + 27\mathbb{Z}_3$. Apply Proposition 4.4 to them as well as the preimage of scaling discs in Table 1, we get a Markov set of scaling discs as below:

Table 2. Decomposition of $\mathbb{Z}_2 \backslash \{0,1\}$ into Markov set of scaling discs, n non negative integers

Scaling Disc	Image under f
$2 + 3\mathbb{Z}_3$	Outside $\mathbb{Z}_3$
$\pm 3^{n+1} + 3^{n+2}\mathbb{Z}_3$	$\pm 3^n + 3^{n+1}\mathbb{Z}_3$
$1 \pm 3^{n+1} + 3^{n+2}\mathbb{Z}_3$	$3^{2n+2} + 3^{2n+3}\mathbb{Z}_3$

By calculation, the Artin-Mazur ζ function equals $\zeta_\phi(t) = \frac{1-t^2}{(1-t)(1-t^2-2t^3)}$.

8. Further Questions

We are working on extending our algorithm for Artin-Mazur ζ function and topological entropy may be generalizable to non subhyperbolic rational maps, and also on studying the dynamics in the whole Berkovich analytic space.

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