1 First order logic

1.1 Basic concepts

- A **deduction** consists of a sequence of sentences or other deductions.
- There are two types of sentences: some are true or false, while others contain unknown quantities (free variables) and its truthness depends on the choice of those quantities. The first type are called **propositions**, the latter called **predicates**.
- For example, "we are all going to die some day", or "the earth is flat", is a proposition, "$x + 1 > 2$" is a predicate.
- When we add ("deduce") a proposition in the deduction, we mean that if all the preceding propositions in its context is true, then this new proposition is true. When we add a predicate, we mean if its free variables are chosen so that all the preceding sentences are true, then it is true.
- For example, from "the square of any real number is non negative" and "2 is a real number" we can get "2^2 is non negative", while from "the square of any real number is non negative" and "x is a real number" we can get "x^2 is non negative"

1.2 The elements of the first order language, and the rules for deduction

- Predicates and functions
  - $A(x)$ denotes a predicate which gives a truth value for any values of x. The truth value depends only on what x is.
  - $f(x)$ denotes a function which sends a value x to another value. The result depends only on the value of x.
  - The usual practice is to use upper case letters to denote propositions and predicates, lower case letters from the beginning of the alphabet (like a, b, c, f, g...) to denote functions or constants, and lower case letters from the end of the alphabet (w, x, y, z) to denote variables. However this is not followed at all time and the exact meaning of a letter should be determined by context.
- : $\wedge$ And
  - To show $A \wedge B$, need to show that both A and B are true.
  - If $A \wedge B$ is known to be true, then A is true, B is also true.
  - Truth table:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

- $\vee$ Or
  - To show $A \vee B$, one can either show A is true, or show B is true.
  - If $A \vee B$ is known to be true, both A and B implies C, then C is true.
  - Truth table:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

- $\neg$ Not, $\perp$ Contradiction
  - A can be replaced by $\neg\neg A$ and vice versa.
  - To show $\neg A$, one can assume A and deduce a contradiction. (proof by contradiction)
  - If A and $\neg A$ are both known to be true, there must be a contradiction.
  - If there is a contradiction, one can deduce anything from it.
  - Truth table:

A	$\neg A$
T	F
F	T

- $\implies$ Implies
  - To show $A \implies B$, assume A, try to deduce B from it.
  - If $A \implies B$ is known to be true, and A is true, then B is also true.
  - Truth table:

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

- $\iff$ If and only if
  - $A \iff B$ is the same as $(A \implies B) \wedge (B \implies A)$.
  - $A \iff B$ is the same as $(A \wedge B) \vee (\neg A \wedge \neg B)$.
  - Truth table:

A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

- $\forall$ For all
  - To show $\forall x P(x)$, need to deduce $P(x)$, here the variable x can not appear in any assumptions as a free variable, i.e. one can not assume anything on x. (for example, one can not say "assume $P(x)$, then $\forall x P(x)$")

  - If $\forall x P(x)$ is known to be true, then $P(t)$ is true for any term t that does not contain bounded variable in P. (for example, one can not say $\forall x \exists y P(x,y)$ implies $\exists y P(y,y)$)

- $\exists$ Exists
  - To show $\exists x P(x)$, need to show that $P(t)$ is true for some t that does not contain bounded variable in P. (for example, one can not say $\forall y P(y,y)$ implies $\exists x \forall y P(x,y)$)
  - If $\exists x P(x)$, and the fact that $P(y)$ is true for some y would induce B (which does not contain y), then B can be deduced. Here y must be a distinct variable. (for example, if we know $\exists x A(x)$, we can not say "let y be such that $A(y)$, then $A(y)$, then $\forall y A(y)$")

- $=$ Equals
  - $=$ satisfies the usual qualities one should expect, like $a = a$, if $a = b$, $b = c$ then $a = c$, if $a = b$ then $b = a$.

1.3 Examples for deduction in first order logic

Here we do not distinguish "A" and "A is true", and sometimes just add "is true" to make the sentence more readable. In more formal treatment of logic however these two statements would need be distinguished.

All the results of the examples here can be used in HW or exams without needing to prove them yourself.

Indentations here are just to clarify the logical orders of the assumptions, you do not need to write proofs in lines or with indentations in HW or exams.

Example 0 $(A \implies B) \iff (\neg B \implies \neg A)$

Proof strategy: This is an iff statement, so assume one side, try to deduce the other side, and vice versa. The negatives in the statement would need to be dealt with using double negatives or proof by contradiction.

Proof:

```
Assume $A \implies B$
 Suppose $\neg B$
  Suppose $A$
   Then $B$ must be true
   Contradiction
  Hence $\neg A$
 Hence $\neg B \implies \neg A$
Hence $(A \implies B) \implies (\neg B \implies \neg A)$
Assume $\neg B \implies \neg A$
 Suppose $A$
  Suppose $\neg B$
   Then $\neg A$
   Contradiction
```

```
    Hence $\neg\neg B$, i.e. $B$
  Hence $A \implies B$
So $(\neg B \implies \neg A) \implies (A \implies B)$
$(A \implies B) \iff (\neg B \implies \neg A)$
```

Example 1 $A \vee \neg A$
Proof strategy: This is an or statement so one would need to show either A or $\neg A$. However, in general neither proposition can be guaranteed to be true, so a possible way around it is to use proof by contradiction.
Proof:

```
Assume $\neg(A \vee \neg A)$
  Assume $A$ is true
    $A \vee \neg A$ is true
    This contradicts with the assumption
  Hence $\neg A$ is true
  Hence $A \vee \neg A$ is true
  Contradiction
Hence $\neg\neg(A \vee \neg A)$, i.e. $A \vee \neg A$.
```

1.3.1 One can define some of the 5 logical symbols $\neg, \wedge, \vee, \implies, \iff$ by the other symbols

Example 2 $A \wedge B \iff \neg(A \implies \neg B)$
Proof:

```
Assume $A \wedge B$
  Assume further that $A \implies \neg B$
    From the first assumption, $A$ is true
    Hence $\neg B$ from the second assumption
    However also from the first assumption, $B$ is true
    Contradiction
  Hence $\neg(A \implies \neg B)$
Hence $A \wedge B \implies \neg(A \implies \neg B)$
Assume $\neg(A \implies \neg B)$
  Assume $\neg A$
    Further assume $A$ is true
      There is a contradiction
      Hence $\neg B$ is true
    Hence $A \implies \neg B$
    This contradicts with the assumption that $\neg(A \implies \neg B)$
  Hence $\neg\neg A$, i.e. $A$ is true
  Assume $\neg B$
    Further assume $A$
    Because $\neg B$ is already known to be true, we have $A \implies \neg B$
    A contradiction
  Hence $\neg\neg B$, i.e. $B$
  This implies $A \wedge B$
```

```
So $\neg(A \implies \neg B) \implies A \wedge B$
Hence $A \wedge B \iff \neg(A \implies \neg B)$
```

Example 3 $A \vee B \iff (\neg A \implies B)$

```
Proof:
Assume $A \vee B$
  Assume $A$ is true
    Assume further that $\neg A$ is true
      There is a contradiction, hence $B$ is true
    Hence $\neg A \implies B$
  Hence $A \implies (\neg A \implies B)$
  Assume $B$ is true
    Assume $\neg A$ is true
    Because $B$ is already known to be true, $\neg A \implies B$
  Hence $B \implies (\neg A \implies B)$
Hence $A \vee B \implies B \implies (\neg A \implies B)$
Assume $\neg A \implies B$
  From example 1, we have $A \vee \neg A$
  Suppose $A$
    Then $A \vee B$
  So $A \implies A \vee B$
  Suppose $\neg A$
    Then $B$
    Hence $A \vee B$
  Hence $\neg A \implies A \vee B$
  Hence $(A \vee \neg A) \implies A \vee B$
  Hence $A \vee B$
Hence $(\neg A \implies B) \implies A \vee B$
Hence $A \vee B \iff (\neg A \implies B)$.
```

Example 4 $(A \implies B) \iff \neg A \vee B$

Proof:
By Example 3, $\neg A \vee B \iff (\neg\neg A \implies B)$
Hence Example 4 follows, because $\neg\neg A$ is just A.

1.3.2 Negating a proposition

Example 5 $\neg(A \wedge B) \iff (\neg A \vee \neg B)$

```
Proof: Assume $\neg(A \wedge B)$
  From Example 1, $A \vee \neg A$
  Suppose $A$
    Further suppose $B$
      Then $A \wedge B$, a contradiction
    Hence $\neg B$
    Hence $\neg A \vee \neg B$
```

```
 Hence $A \implies \neg A \lor \neg B$
 Suppose $\neg A$
  Hence $\neg A \lor \neg B$
 Hence $\neg A \implies \neg A \lor \neg B$
 Hence $(A \lor \neg A) \implies \neg A \lor \neg B$
 Hence $\neg A \lor \neg B$
$\neg(A \land B) \implies \neg A \lor \neg B$
Suppose $\neg A \lor \neg B$
 Suppose $A \land B$
  Then $A$ and $B$ are both true
  Suppose $\neg A$
   There is a contradiction
  So $\neg A$ implies a contradiction
  Suppose $\neg B$
   There is a contradiction
  So $\neg B$ implies a contradiction
  Hence $\neg A \lor \neg B$ implies a contradiction
  Hence there must be a contradiction
 Hence $\neg(A \land B)$
Hence $\neg A \lor \neg B \implies \neg(A \land B)$
$\neg(A \land B) \iff \neg A \lor \neg B$
```

Example 6 $\neg(A \lor B) \iff (\neg A \land \neg B)$

Proof:

```
Suppose $\neg(A \lor B)$
 Suppose $\neg(\neg A \land \neg B)$
  From Example 5, we have $\neg\neg A \lor \neg\neg B$, i.e. $A \lor B$
  A contradiction
 Hence $\neg A \land \neg B$
Hence $\neg(A \lor B) \implies \neg A \land \neg B$
Suppose $\neg A \land \neg B$
 Suppose $A \lor B$
  Then $\neg\neg A \lor \neg\neg B$
  Then from Example 5, $\neg(\neg A \land \neg B)$
  Contradiction
 Hence $\neg(A \lor B)$
Hence $\neg A \land \neg B \implies \neg(A \lor B)$
Hence $\neg A \land \neg B \iff \neg(A \lor B)$
```

Example 7 $\neg(A \implies B) \iff \neg B \land A$

Proof:

```
Suppose $\neg(A \implies B)$
 Then $\neg(A \implies \neg\neg B)$
 From Example 2, we have $A \land \neg B$
```

```
Hence $\neg(A \implies B) \implies A \wedge \neg B$
Suppose $\neg B \wedge A$
 From example 2, we have $\neg(A \implies \neg\neg B)$, i.e. $\neg(A \implies B)$
Hence $\neg B \wedge A \implies \neg(A \implies B)$
Hence $\neg(A \implies B) \iff \neg B \wedge A$
```

Example 8 $\neg(A \iff B) \iff (\neg B \wedge A) \vee (\neg A \wedge B)$
Proof: This follows from Example 5 and Example 7.

Example 9 $\neg\forall x P(x) \iff \exists x \neg P(x)$
Proof strategy: When we assume left hand side and try to deduce the right hand side, we need to prove an existence statement. This could be done by using examples, but such an example is not obvious, so we try proof by contradiction.
Proof:

```
Suppose $\neg\forall x P(x)$
 Further assume that $\neg\exists x \neg P(x)$
  Suppose for some $y$, $\neg P(y)$
   Then $\exists x \neg P(x)$
   Contradiction
  So $\neg\neg P(y)$, i.e. $P(y)$
  Hence $\forall x P(x)$
  Contradiction
 Hence $\exists x \neg P(x)$
Hence $\neg\forall x P(x) \implies \exists x \neg P(x)$.
Suppose $\exists x \neg P(x)$
 Let $y$ be such that $\neg P(y)$
 Suppose $\forall x P(x)$
  Then $P(y)$
  Contradiction
Hence $\neg\forall x P(x)$
Hence $\exists x \neg P(x) \implies \neg\forall x P(x)$
$\neg\forall x P(x) \iff \exists x \neg P(x)$
```

Example 10 $\neg\exists x P(x) \iff \forall x \neg P(x)$
Proof:

```
Suppose $\neg\exists x P(x)$
 Suppose $P(y)$ for some $y$
  Then $\exists x P(x)$
  Contradiction
 Hence $\neg P(y)$
 Hence $\forall x P(x)$
Hence $\neg\exists x P(x) \implies \forall x P(x)$
Suppose $\forall x \neg P(x)$
 Suppose $\exists x P(x)$
  Let $y$ be such that $P(y)$
```

By the prior assumption that $\forall x \neg P(x)$, we have $\neg P(y)$
Contradiction
Hence $\neg \exists x P(x)$
Hence $\forall x \neg P(x) \implies \neg \exists x P(x)$
$\neg \exists x P(x) \iff \forall x \neg P(x)$

1.3.3 Injections

Example 11 $(\forall x \forall y \neg(x = y) \implies \neg(f(x) = f(y))) \implies (\forall x \forall y \neg(x = y) \implies \neg(f(f(x)) = f(f(y))))$
Proof:
Suppose $\forall x \forall y \neg(x = y) \implies \neg(f(x) = f(y))$
Consider some z, w so that $\neg(z = w)$
Then by assumption, $\neg(z = w) \implies \neg(f(z) = f(w))$
Hence $\neg(f(z) = f(w))$
Also by assumption, $\neg(f(z) = f(w)) \implies \neg(f(f(z)) = f(f(w)))$
Hence $\neg(f(f(z)) = f(f(w)))$
Hence $\neg(z = w) \implies \neg(f(f(z)) = f(f(w)))$
Hence $\forall x \forall y \neg(x = y) \implies \neg(f(f(x)) = f(f(y)))$
$(\forall x \forall y \neg(x = y) \implies \neg(f(x) = f(y))) \implies (\forall x \forall y \neg(x = y) \implies \neg(f(f(x)) = f(f(y))))$

1.3.4 More tautologies in proposition logic

Example 12 $((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$
Proof:
Assume $((A \implies B) \wedge (B \implies C))$
Then $A \implies B$
Assume A
Then because $A \implies B$, B is true
Also from the first assumption, $B \implies C$
So C is true
So $A \implies C$
So $((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$

Example 13 $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$
Proof:
Assume $(A \wedge B) \wedge C$
Then both $A \wedge B$ and C are true
Hence A, B, C are all true
Hence $B \wedge C$ is true
So $A \wedge (B \wedge C)$ is true
This shows that $(A \wedge B) \wedge C \implies A \wedge (B \wedge C)$
Assume $A \wedge (B \wedge C)$
Both A and $B \wedge C$ are true
A, B, C are all true

  Hence $(A \wedge B)$ is true
  Hence $(A \wedge B) \wedge C$ is true
Hence $A \wedge (B \wedge C) \implies (A \wedge B) \wedge C$

Example 14 $(A \vee B) \vee C \iff A \vee (B \vee C)$
Proof:
Assume $(A \vee B) \vee C$
  Assume $A \vee B$
    Assume A
      Then $A \vee (B \vee C)$
    Hence $A \implies A \vee (B \vee C)$
    Assume B
      Then $B \vee C$, which implies $A \vee (B \vee C)$
    Hence $B \implies A \vee (B \vee C)$
    Hence $A \vee (B \vee C)$
  Hence $A \vee C \implies A \vee (B \vee C)$
  Assume C
    Then $B \vee C$
    Hence $A \vee (B \vee C)$
  This implies that $A \vee (B \vee C)$ is true
Hence $(A \vee B) \vee C \implies A \vee (B \vee C)$
Assume $A \vee (B \vee C)$
  Assume $B \vee C$
    Assume C
      Then $(A \vee B) \vee C$
    Hence $C \implies A \vee (B \vee C)$
    Assume B
      Then $A \vee B$, which implies $(A \vee B) \vee C$
    Hence $B \implies (A \vee B) \vee C$
    Hence $(A \vee B) \vee C$
  Hence $B \vee C \implies (A \vee B) \vee C$
  Assume A
    Then $A \vee B$
    Hence $(A \vee B) \vee C$
  This implies that $(A \vee B) \vee C$ is true
Hence $A \vee (B \vee C) \implies (A \vee B) \vee C$
Together with the results of the first half, we get $(A \vee B) \vee C \iff A \vee (B \vee C)$

Example 15 $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$
Proof:
Suppose $A \vee (B \wedge C)$
  Assume A
    Then $A \vee B$, $A \vee C$ are both true
    So $(A \vee B) \wedge (A \vee C)$
  So $A \implies (A \vee B) \wedge (A \vee C)$

Assume $B \wedge C$
Then B and C are both true
Hence $A \vee B$, $A \vee C$ are both true
So $(A \vee B) \wedge (A \vee C)$
So $B \wedge C \implies (A \vee B) \wedge (A \vee C)$
So $(A \vee B) \wedge (A \vee C)$
This shows $A \vee (B \wedge C) \implies (A \vee B) \wedge (A \vee C)$
Suppose $(A \vee B) \wedge (A \vee C)$
Then $A \vee B$ and $A \vee C$ are both true
From Example 1, we have $A \vee \neg A$
Suppose A is true
Then $A \vee (B \wedge C)$ is true
Suppose $\neg A$ is true
Since $A \vee B$ is known, we consider the two cases, which is when A is true and when B is true
Suppose A is true
There is a contradiction, hence B
Suppose B is true, we get the same result
Hence B is true
Do the same for $A \vee C$, we get that C is true
So $B \wedge C$ is true, which implies $A \vee (B \wedge C)$
So $A \vee (B \wedge C)$
This shows that $(A \vee B) \wedge (A \vee C) \implies A \vee (B \wedge C)$
Hence $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$

Example 16 $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$
Proof:
Suppose $A \wedge (B \vee C)$
Then A is true
And $B \vee C$ is true
Suppose B is true
Then $A \wedge B$ is true, which implies $(A \wedge B) \vee (A \wedge C)$
Hence $B \implies (A \wedge B) \vee (A \wedge C)$
Suppose C is true
Then $A \wedge C$ is true, which implies $(A \wedge B) \vee (A \wedge C)$
Hence $C \implies (A \wedge B) \vee (A \wedge C)$
Hence $(A \wedge B) \vee (A \wedge C)$ is true
This shows that $A \wedge (B \vee C) \implies (A \wedge B) \vee (A \wedge C)$
Suppose $(A \wedge B) \vee (A \wedge C)$
Suppose $A \wedge B$
Then both A and B are true
Hence $B \vee C$ is true
Hence $A \wedge (B \vee C)$
Suppose $A \wedge C$
Then both A and C are true

Hence $B \vee C$ is true
Hence $A \wedge (B \vee C)$
This shows that $A \wedge (B \vee C)$ is true
Hence $(A \wedge B) \vee (A \wedge C) \implies A \wedge (B \vee C)$
This shows that $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$

1.4 A few more commonly seen logic symbols

- $a \neq b$ is short hand for $\neg(a = b)$
- $\nexists x P(x)$ is short hand for $\neg \exists x P(x)$
- $\exists! x P(x)$ means $(\exists x P(x)) \wedge (\forall x \forall y (P(x) \wedge P(y) \implies (x = y)))$

2 First order theory of natural numbers (First order Peano Arithmetics)

To remind ourselves and others that we are doing deduction in the universe of natural numbers, we replace $\forall x$ with $\forall x \in \mathbb{N}$, $\exists x$ with $\exists x \in \mathbb{N}$.

2.1 New symbols and rules

- We introduces 4 more symbols and 7 more rules associated to them. The symbols are: 0, $s(\cdot)$ (successor, intuitively, $s(n) = n + 1$), $+$, $\times$. Here 0 is a constant and the other 3 are functions.
- Rule 1: s is an injection: $\forall x \in \mathbb{N} \forall y \in \mathbb{N} \neg(x = y) \implies \neg(s(x) = s(y))$
- Rule 2: 0 is the first natural number: $\neg \exists x \in \mathbb{N}(0 = s(x))$
- Rule 3: Mathematical induction: $(P(0) \wedge \forall \in \mathbb{N} x (P(x) \implies P(s(x)))) \implies \forall \in \mathbb{N} x P(x)$
- Rule 4: First rule for addition: $\forall x \in \mathbb{N} x + 0 = x$
- Rule 5: Second rule for addition: $\forall x \in \mathbb{N} \forall y \in \mathbb{N} x + s(y) = s(x + y)$
- Rule 6: First rule for multiplication: $\forall x \in \mathbb{N} x \times 0 = 0$
- Rule 7: Second rule for multiplication: $\forall x \in \mathbb{N} \forall y \in \mathbb{N} x \times s(y) = x \times y + x$

A shortened format for writing proofs using mathematical induction is:
(what needs to be proved is $\forall x \in \mathbb{N} P(x)$)
Induction on x
Prove $P(0)$
Suppose $P(x)$
...
$P(s(x))$

Hence by induction, $\forall x \in \mathbb{N} P(x)$.

The numbers are defined as $1 = s(0)$, $2 = s(s(0))$, $\ldots$. $s(x)$ can also be written as $x + 1$.

2.2 Some examples

Example 17 $1 + 2 = 3$
Proof:
$s(0) + s(s(0)) = s(s(0) + s(0)) = s(s(s(0) + 0)) = s(s(s(0))) = 3$.
Here we repeated use the properties of $=$ in logic and the rule 5 and 6 for natural numbers.

Example 18 $\forall x \in \mathbb{N} \neg(x = s(x))$
Proof:
Induction on x.
Suppose $0 = s(0)$
Then $\exists x \in \mathbb{N} 0 = s(x)$, which contradicts with rule 2
So $\neg(0 = s(0))$.
Suppose $\neg(x = s(x))$
Suppose $(s(x) = s(s(k)))$
Then by Rule 1, $k = s(k)$, a contradiction
Hence $\neg(s(x) = s(s(x)))$
By induction, $\forall x \in \mathbb{N} \neg(x = s(x))$

2.3 Properties of arithmetics

The followings are true for all natural numbers $a.b, c \ldots$:

- $0 + a = a$
- $a + b = b + a$
- $(a + b) + c = a + (b + c)$
- $0 \times a = 0$
- $a \times b = b \times a$
- $(a \times b) \times c = a \times (b \times c)$
- $a \times (b + c) = a \times b + a \times c$
- $a + b = c + b \iff a = c$
- $a \times b = c \times b \iff (b = 0 \vee a = c)$
- $\ldots$

They can all be easily proven via mathematical induction.

Example 19 $\forall x \in \mathbb{N} 0 + x = x$
Proof:
Induction on x
$0 + 0 = 0$
Suppose $0 + x = x$
$0 + s(x) = s(0 + x) = s(x)$
Hence by induction, the proposition is proved.

Example 20 $\forall x \in \mathbb{N} \forall y \in \mathbb{N} x + y = y + x$
Proof:
Induction on x
$\forall y \in \mathbb{N} 0 + y = y = y + 0$ by Example 19
Suppose $\forall y \in \mathbb{N} x + y = y + x$
Induction on y
$s(x) + 0 = s(x) = 0 + s(x)$
Suppose $s(x) + y = y + s(x)$
$s(x) + s(y) = s(s(x) + y) = s(y + s(x)) = s(s(y + x)) = s(s(x + y))$, and $s(y) + s(x) = s(s(y) + x) = s(x + s(y)) = s(s(x + y))$, these two are the same
Hence by induction, $\forall y \in \mathbb{N} s(x) + y = y + s(x)$
By induction, the proposition is proved.

2.4 Divisibility and comparison

Definition: $a|b$ iff $\exists c \in \mathbb{N} b = a \times c$

Definition: $a \leq b$ iff $\exists c \in \mathbb{N} b = a + c$, $a \geq b$ iff $b \leq a$, $a < b$ iff $a \leq b$ and $a \neq b$, $a > b$ iff $b < a$.

Definition: The power function is defined as $m^0 = 1$ when $m \neq 0$, $m^{n+1} = m^n \times m$. The factorial function is defined as $0! = 1$, $(n + 1)! = n!(n + 1)$. The summation symbol is defined as $\sum_{i=a}^{a} f(i) = f(a)$, $\sum_{i=a}^{b+1} f(a) = \sum_{i=a}^{b} f(a) + f(b + 1)$. The product symbol is defined as $\prod_{i=a}^{a} f(i) = f(a)$, $\prod_{i=a}^{b+1} f(a) = \prod_{i=a}^{b} f(a) + f(b + 1)$.

Remark Strictly speaking, the concept of functions in first order logic as described in the previous section must be defined everywhere, so if one wants to be completely rigorous one should extend those functions to where they are undefined, e.g. let $0^0 = 1$.

More properties:

- $a = b \vee a < b \vee a > b$
- $\neg(a < b \wedge a > b)$

- $(a \leq b \wedge b \leq c) \implies a \leq c$
- $a|b \implies a|b \times c$
- $a|b \wedge a > 0 \implies a \leq b$
- $c > 0 \implies (a < b \iff a + c < b + c \iff a \times c < b \times c)$
- $\ldots$

All these can be proven from the rules, definitions and the properties of × and + in the previous subsection.

2.5 Formal and informal proofs

Formal proofs: every step must be an assumption, or follows from prior steps using one of the prescribed rules (rules of first order logic, first order theory of natural numbers, etc).

Guideline for informal proofs:

- Write down enough steps so that a reader that is mathematically literate can fill in the rest and get a formal proof.
- For the current class, write down as much detail as the examples I do in class/put in lecture notes.
- When you're not sure, err on the side of more details.

Examples of formal vs informal proofs:

Example 21 $\forall x \in \mathbb{N}(x = 0 \vee \exists y \in \mathbb{N} x = y + 1)$
Formal Proof:
Induction on x.
$0 = 0$
Which implies $0 = 0 \vee \exists y \in \mathbb{N} 0 = y + 1$
Suppose $x = 0 \vee \exists y \in \mathbb{N} x = y + 1$
$x + 1 = x + 1$
$\exists y \in \mathbb{N} x + 1 = y + 1$
$x + 1 = 0 \vee \exists y \in \mathbb{N} x + 1 = y + 1$
By induction, $\forall x \in \mathbb{N}(x = 0 \vee \exists y \in \mathbb{N} x = y + 1)$
Informal proof:
We prove it by induction on x. When $x = 0$, $0 = 0$. Suppose the statement $x = 0 \vee \exists y \in \mathbb{N} x = y + 1$ is known for some x, because $x + 1 = x + 1$ it is also true for $x + 1$. Hence it is true for all x.

Example 22 $\forall x \in \mathbb{N} \forall y \in \mathbb{N}(x \leq y \vee y \leq x)$
Informal Proof:
Induction on x. $0 \leq y$ for all y. Suppose this is known for some value x, then, for each y, either $x \leq y$ or $y \leq x$. In the latter case $y \leq x+1$, while in the former case, let z be such that $y = x+z$, then from the previous example $z = 0$ or $z \geq 1$. If $z = 0$ then $x = y$ and $y \leq x+1$, while if $z \geq 1$ then $x+1 \leq y$. Hence in all cases the statement is true for $x+1$, the proposition is proved.
Formal Proof:
Firstly include the proofs of Example 19, Example 20, Example 21, and the associativity rule of addition.
Induction on x
By Example 19, $0 + y = y$
$0 \leq y \vee y \leq 0$
$\forall y \in \mathbb{N}(0 \leq y \vee y \leq 0)$
Now suppose $\forall y \in \mathbb{N}(x \leq y \vee y \leq x)$
$x \leq y \vee y \leq x$
Suppose $x \leq y$
$\exists z \in \mathbb{N}(x + z = y)$
By Example 21, $z = 0 \vee \exists w \in \mathbb{N} z = w + 1$
Suppose $z = 0$
$x = x + 0 = y$
$y + 1 = x + 1$
$y \leq x + 1$
$x + 1 \leq y \vee y \leq x + 1$
Suppose $\exists w \in \mathbb{N} z = w + 1$
Let w be such that $z = w + 1$
$(x+1) + w = x + (1 + w)$ by associativity rule of addition.
$1 + w = w + 1$ by Example 20.
$(x+1) + w = x + (1+w) = x + (w+1) = x + z = y$
$x + 1 \leq y$
Hence $x + 1 \leq y \vee y \leq x + 1$
By $\vee$ rule, $x + 1 \leq y \vee y \leq x + 1$
Suppose $y \leq x$
$\exists z \in \mathbb{N} y + z = x$
Let z be such that $y + z = x$
$y + (z + 1) = x + 1$
$y \leq x + 1$
$x + 1 \leq y \vee y \leq x + 1$
By $\vee$ rule again, $x + 1 \leq y \vee y \leq x + 1$
By induction, the Example is proved.

Starting from now we will stop requiring that all deduction steps must follow the rules of first order logic or Peano arithmetics. In other words, we will freely use properties about numbers we learned in grade schools and in your previous classes.

Example 23 $\forall x \in \mathbb{N} \neg(\exists y \in \mathbb{N} x = 2y \wedge \exists y \in \mathbb{N} x = 2y + 1)$
Proof:
Suppose $\exists y \in \mathbb{N} x = 2y \wedge \exists y \in \mathbb{N} x = 2y + 1$
 Let z be such that $x = 2z$
  Let w be such that $x = 2w + 1$
   (By Example 22) $z \leq w \vee w \leq z$
   Suppose $z \leq w$
    $\exists c \in \mathbb{N} w = z + c$
    Let c satisfy $w = z + c$
     $2(z + c) + 1 = 2z$, hence $2c + 1 = 0$, contradiction.
   Suppose $w \leq z$
    $\exists c \in \mathbb{N} z = w + c$
    Let c satisfy $z = w + c$
     $2w + 1 = 2(w + c)$, hence $1 = 2c$
     Whether $c = 0$ or $c > 0$, there is a contradiction.
Hence $\neg \exists y \in \mathbb{N} x = 2y \wedge \exists y \in \mathbb{N} x = 2y + 1$
$\forall x \neg \exists y \in \mathbb{N} x = 2y \wedge \exists y \in \mathbb{N} x = 2y + 1$

2.6 Further examples on induction and proof writing

- Unless specified, in an informal proof you are allowed to use simple tautologies in first order logic (similar to the examples in this notes) as well as things you learn prior to this course (e.g. arithmetic, Euclidean geometry, calculus etc).
- Clearly distinguish comments (“we are going to show...”, “This is because of ...”), assumptions (“suppose..”, “Let x satisfy...”) and other regular statements in the proof.
- It’s never a bad idea to write more details, but the “details” have to be correct
- For now, it is recommended that you write down the reasoning of every step in parenthesis when writing proofs.

Example 24 Problem 4 in Workshop 2:
Suppose $x^2 = 2y^2$
 By Problem 3, $\exists x' \in \mathbb{N}(x = 2x')$
 $4x'^2 = 2y^2$
 $2x'^2 = y^2$
 By Problem 3, $\exists y' \in \mathbb{N}(y = 2y')$
 $\exists x' \in \mathbb{N} \exists y' \in \mathbb{N}(x'^2 = 2y'^2 \wedge x = 2x' \wedge y = 2y')$
$x^2 = 2y^2 \implies \exists x' \in \mathbb{N} \exists y' \in \mathbb{N}(x'^2 = 2y'^2 \wedge x = 2x' \wedge y = 2y')$

This indicate to us that we should try induction on x, because what happens to a larger x can be reduced to what happens to a smaller x. Yet attempts of simple

induction doesn't work. What is needed is first strengthen the proposition that needs to be proved then use induction, as follows:
Induction on N to show that $\forall x \in \mathbb{N}((x \leq N) \implies \forall y \in \mathbb{N}(x^2 = 2y^2 \implies x = 0))$ When $N = 0$, $x \leq N$ implies $x = 0$, hence this predicate is true. (here we used tautology $(A \implies C) \implies (A \implies (B \implies C))$, and the fact that $a \leq 0 \implies a = 0$)
Suppose $\forall x \in \mathbb{N}((x \leq N) \implies \forall y \in \mathbb{N}(x^2 = 2y^2 \implies x = 0))$
Suppose $\neg\forall x \in \mathbb{N}((x \leq N+1) \implies \forall y \in \mathbb{N}(x^2 = 2y^2 \implies x = 0))$
Then $\exists x \in \mathbb{N}((x \leq N+1) \wedge \exists y \in \mathbb{N}(x^2 = 2y^2) \wedge \neg(x = 0))$
Let x, y satisfy $x^2 = 2y^2$ and $x \neq 0$.
Due to the earlier argument, $\exists x' \in \mathbb{N}(x = 2x')$, $\exists y' \in \mathbb{N}(y = 2y')$, and $x'^2 = 2y'^2$.
Because $x' < x$, $x' \leq N$, a contradiction.
Hence $\forall x \in \mathbb{N}((x \leq N+1) \implies \forall y \in \mathbb{N}(x^2 = 2y^2 \implies x = 0))$
The Example is proved due to induction.

Example 25, an example of recursive definition $\forall n \in \mathbb{N}(2^n \geq 2n)$
Proof:
Induction on n
$2^0 = 1 \geq 0 = 2 \times 0$
Suppose $2^n \geq 2n$
$2^{n+1} \geq 4n$
We know $n = 0 \vee n \geq 1$
Suppose $n = 0$
$2^{n+1} = 2 \geq 2 = 2 \times (0+1)$
Suppose $n \geq 1$
$4n \geq 2(n+1)$, hence $2^{n+1} \geq 2(n+1)$
By induction, $\forall n \in \mathbb{N}(2^n \geq 2n)$

Generally, if one need to make use of recursive definitions (define a function using the same function, but acts on different values), one use mathematical induction.

2.7 The remainder theorem

Example 26, Remainder theorem $\forall x \in \mathbb{N}((x > 0) \implies (\forall y \in \mathbb{N}\exists! r \in \mathbb{N}\exists! q \in \mathbb{N} r < x \wedge y = xq + r))$
Proof:
Suppose $x > 0$
For existence: prove by induction on y
When $y = 0$, $0 = x \times 0 + 0$, so we can let $r = q = 0$
Suppose $\exists r \in \mathbb{N}\exists q \in \mathbb{N} r < x \wedge y = xq + r$
We have $r + 1 < x$ or $r + 1 = x$
Suppose $r + 1 < x$
Then $r + 1 < x \wedge y + 1 = xq + r + 1$
so $\exists r \in \mathbb{N}\exists q \in \mathbb{N} y + 1 = xq + r$

Suppose $r+1=x$
Then $0<x \wedge y+1=x(q+1)$
so $\exists r \in \mathbb{N} \exists q \in \mathbb{N} y+1=xq+r$
By induction, $\forall y \exists r \in \mathbb{N} \exists q \in \mathbb{N} r<x \wedge y=xq+r$
Now for uniqueness: suppose $xq+r=xq'+r'$, $r<x$, $r'<x$
$q=q'$ or $q<q'$ or $q>q'$
Suppose $q=q'$, then $xq=xq'$, hence $r=r'$, which shows uniqueness of both q and r
If $q<q'$, let $q'=p+q$, so $r=xp+r'$. Because $p>1$, $xp+r'>x$, a contradiction, hence uniqueness is also true in this case.
The situation for $q>q'$ is similar.

2.8 Alternatives to induction

We have an alternative presentation of induction:

Example 27 (Any non empty set of natural numbers has a minimum element)
$(\exists x \in \mathbb{N} P(x)) \implies (\exists x \in \mathbb{N}(P(x) \wedge (\forall y \in \mathbb{N}(P(y) \implies x \leq y))))$.
Proof:
We prove the contrapositive.
Assume $\forall x \in \mathbb{N}(\neg P(x) \vee \exists y \in \mathbb{N}(P(y) \wedge y<x))$.
We will use induction on x to prove $\forall x \in \mathbb{N} \neg \exists y \in \mathbb{N}(P(y) \wedge y<x)$
Suppose $\exists y \in \mathbb{N}(P(y) \wedge y<0)$
Yet $\neg \exists y \in \mathbb{N}(y<0)$
Contradiction
Hence $\neg \exists y \in \mathbb{N}(P(y) \wedge y<0)$
Suppose $\neg \exists y \in \mathbb{N}(P(y) \wedge y<x)$
Suppose $\exists y \in \mathbb{N}(P(y) \wedge y<x+1)$
Let z be such that $P(z) \wedge z<x+1$
Suppose $z<x$
$\exists y \in \mathbb{N}(P(y) \wedge y<x)$, a contradiction
Hence $z=x$
Because of the initial assumption, and that $P(z)$ is true, $\exists y \in \mathbb{N}(P(y) \wedge y<z)$
Hence $\exists y \in \mathbb{N}(P(y) \wedge y<x)$, a contradiction.
Hence $\neg \exists y \in \mathbb{N}(P(y) \wedge y<x+1)$
By induction, $\forall x \in \mathbb{N} \neg \exists y \in \mathbb{N}(P(y) \wedge y<x)$
Hence $\forall x \in \mathbb{N} \neg P(x)$
The Example is proved.

This provides an alternative proof of Example 24 and Example 26.