Galois Conjugates of Exponents of Core Entropies

Chenxi Wu
UW-Madison

September 8, 2022

Collaboration with: Harrison Bray, Diana Davis, Ethan Farber, Kathryn Lindsey, Giulio Tiozzo

Thurston's "Master Teapot"

- Let f be a unimodal map on finite interval $I = [0, 1]$.
- We say it has **periodic critical orbit** if the critical value c has $f^{\circ n}(c) = c$ for some $n > 0$.
- Thurston's "Master Teapot" is defined as:

$$T_2 = \overline{\{(z, \lambda) : \lambda = e^h, h \text{ entropy of unimodal map}}$$

$$\overline{f \text{ with periodic critical orbit}, z \text{ Galois conjugate of } \lambda\}}$$

Remarks

- Motivation: This set provides a **necessary condition** for a number h to be the entropy of a unimodal map with periodic critical orbit: the Galois conjugates z of e^h should all satisfy $(z, h) \in T_2$.
- The orthogonal projection of $T_2 \subset \mathbb{C} \times \mathbb{R}$ to the first factor is called the **Thurston Set** Ω, its shape has been characterized fully by Tiozzo.
- In particular, let D be the unit disc, then $\Omega \cap D$ consists of all roots of all power series of coefficients ± 1.
- Calegari-Koch-Walker showed that there are non-trivial holes in Ω.

- Let $\lambda \in (1,2)$, $\Omega_\lambda = \{z \in \mathbb{C}, (z,\lambda) \in T_2\}$, D be the unit disc.
- $\Omega_\lambda \cap D$ has the following characterization:
  - Consider the **tent map**:

$$f_\lambda = \begin{cases} \lambda x & x \leq 1/\lambda \\ 2 - \lambda x & x > 1/\lambda \end{cases}$$

  - Let $I_0 = [0, 1/\lambda]$, $I_1 = [1/\lambda, 1]$.
  - Let subshift $M_\lambda \subset \{0,1\}^{\mathbb{N}}$ be

$$M_\lambda = \{(w_0, w_1, \dots) : \forall n, \exists x \in [0,1], \forall i \leq n, f_\lambda^{\circ i}(x) \in I_{w_{n-i}}\}$$

    When f_λ has finite critical orbit M_λ is a subshift of finite type.
  - Let $F_{0,z}(x) = zx$, $F_{1,z}(x) = 2 - zx$.
  - **Theorem**: (Lindsey-W)

$$\Omega_\lambda \cap D = \{z \in D : \exists w \in M_\lambda, \lim_{n \to \infty} F_{w_0,z} \circ F_{w_1,z} \circ \dots F_{w_n,z}(1) = 1\}$$

Corollaries

- **Theorem**: (Bray-Davis-Lindsey-W) If $z \in D$, $(z, h) \in T_2$, then for any $y \in [h, 2]$, $(z, y) \in T_2$.
- Alternative characterization of T_2: In definition of T_2, one can replace "z is Galois conjugate of e^h" with "z is an eigenvalue of the Markov incidence matrix of a unimodal map/tent map with entropy h".

Teapot for Veins in the Mandelbrot Set

- Let $z \mapsto z^2 + c$ be a post-critically finite complex quadratic map. There is a unique way to connect the points in the critical orbit within the filled Julia set such that the quadratic map sends this finite tree to itself, called the **Hubbard Tree**. The entropy on this tree is called **Core Entropy**.
- When $c \in \mathbb{R}$, the Hubbard tree is an interval, and the map is unimodal.
- A **vein** in the Mandelbrot set is a path from main cardoid to a tip. The p/q **principal vein** corresponds to tips $c_{p/q}$, where the Julia set is a star with q branches, and 0 is sent to a fixed point which is at the tip of the same branch in q steps. The $1/2$ principal vein is the real axis.

- Let P_q be the set of all superattracting parameters on a given p/q principal vein. Let

$$T_q = \overline{\{(z, e^h) : h \text{ core entropy of some } f \in P_q,}$$
$$\overline{z \text{ eigenvalue of Markov incidence matrix of } f}$$
$$\overline{\text{restricted to Hubbard tree}\}}$$

- **Theorem**: (Lindsey-Tiozzo-W) There is a similar characterization of $\Omega_{\lambda,q} = \{z : (z, \lambda) \in T_q\}$ inside unit disc D.
- In an upcoming paper, Farber-Lindsey-W will do the same for real multiples of Chebyshev polynomials. (https://vimeo.com/260494302)

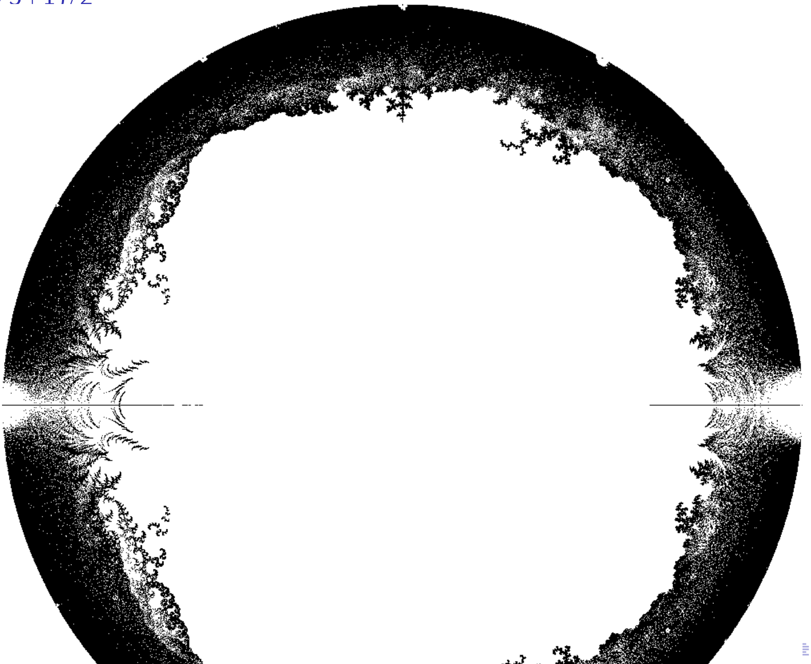
Julia-Mandelbrot correspondence

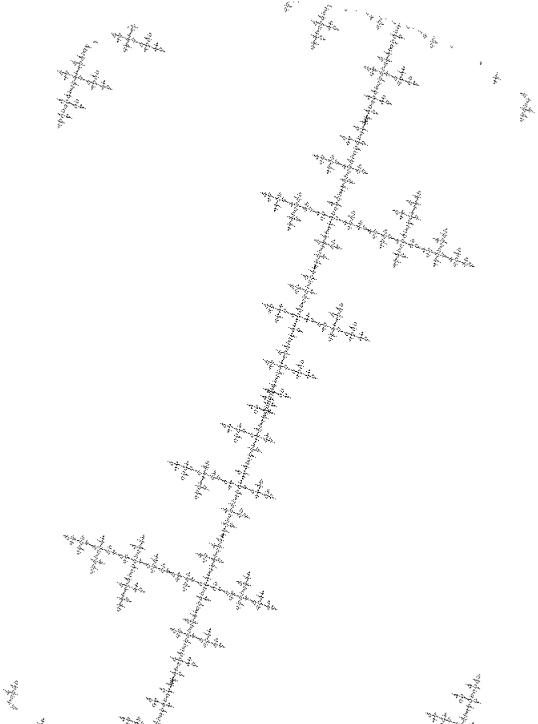
- "Mandelbrot set": $\Omega_\lambda \cap D$.
- "Julia set" for $z \in D$:

$$J_z = \{ \lim_{n\to\infty} F_{w_0,z} \circ F_{w_1,z} \circ \dots F_{w_n,z}(1) - 1 : w \in M_\lambda \}$$

- Their asymptotic similarity can be obtained similar to the case of "classical" Julia-Mandelbrot correspondence (cf. Tan Lei).

$\Omega_{(\sqrt{5}+1)/2}$





Further Questions

- Calculation of various dimensions & local dimensions.
- Interesting measures on the set.
- Generalization to the post-critically finite case.
- Generalization to other veins, or higher degree polynomials.
- Generalization to p-adic polynomial maps on various Berkovich spaces.

References

- Giulio Tiozzo, Topological Entropy of Quadratic Polynomials and Dimension of Sections of the Mandelbrot Set. *Adv. Math.* 273, pp. 651-715, 2015
- Harrison Bray, Diana Davis, Kathryn Lindsey, Chenxi Wu. The Shape of Thurston's Master Teapot, *Adv. Math.* 377, 137148, 2021
- Kathryn Lindsey and Chenxi Wu. A Characterization of Thurston's Master Teapot, arXiv:1909.10675, to appear in *Ergodic Theory and Dynamical Systems*, 2022
- Kathryn Lindsey, Giulio Tiozzo and Chenxi Wu. Master Teapots and Entropy Algorithms for the Mandelbrot Set, arXiv:2112.13726