

1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example: $u_y = x$.

Example: $(yu)_y = 1$.

General solution of a PDE.

Formally: PDE: $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde

Linear PDE.

Linear homogeneous PDE.

What are the order and linearity of the following PDEs?

$u_x + u_{yyx} = 1$, $uu_x + u = 0$, $u_x + (x^2 + y^2)u_{yy} = 1$.

Some PDEs we will focus on later:

Heat: $u_t = u_{xx}$: (heat transmission, diffusion)

Laplace: $u_{xx} + u_{yy} = 0$: (static electric field, Newton's gravity, equilibrium of random walk)

Wave: $u_{tt} = u_{xx}$: (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations: $u_{tt} = u_{xx} - ku_{xxxx}$ (stiff string)

Cauchy-Riemann equation: $u_x = v_y$, $u_y = -v_x$

Non-linear PDEs you may see in later classes:

Navier-Stokes

Nonlinear Schrodinger: $iu_t = -\Delta u + k|u|^2 u$

KdV: $u_t + u_{xxx} + 6uu_x = 0$, etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow) $u_t = A \frac{\nabla u}{|\nabla u|} \cdot \nabla u + B |\nabla u| \nabla \cdot \frac{\nabla u}{|\nabla u|}$.

Types of problems:

Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.

Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:

Existence

Uniqueness

Regularity

Continuous dependency on boundary

Typical strategy: integral transform: $(Tu)(y) = \int u(x)K(x,y)dx$, then $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$, assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)

Example: solve $u_{xx} + u_{yy} = 1$, where $u = 0$ on the unit circle.

Example: $u_x = u_t$, $u_x = u_t + 1$.

2 9/7 Review of ODE, Advection and Diffusion

Review of ODE & multivariable calculus topics:

- $u' + p(t)u + q(t) = 0$
- $u''' + Au'' + Bu' + Cu = 0$
- Chain rule: Example: $u_{xx} = u_{tt}$, what happens with change-of-variable $y = x + t$, $w = x - t$?
- Fubini's theorem.
- Differentiating an integral. Example: $\frac{d}{dt} \int_0^{t^2} \sin(ts) ds$.
Solution: Let $x = t$, $y = t$, then $\frac{d}{dt} \int_0^{t^2} e^{-ts^2} ds = \frac{d}{dt} \int_0^{x^2} e^{-ys^2} ds = (\int_0^{x^2} e^{-ys^2} ds)_x + (\int_0^{x^2} e^{-ys^2} ds)_y = 2x \cdot e^{-y(x^2)^2} + \int_0^{x^2} (e^{-ys^2})_y ds = 2xe^{-y(x^2)^2} - \int_0^{x^2} s^2 e^{-ys^2} ds = 2te^{-t^5} - \int_0^{t^2} s^2 e^{-ts^2} ds$.
- Example: $u_{tt} = u_{xx} + u_{yy}$, $u(x, y, t) = \sin(x \cos \theta + y \sin \theta + t)$ are solutions, hence $\int_0^{2\pi} \sin(x \cos \theta + y \sin \theta + t) d\theta$ is also a solution.

PDE from conservation laws, 1-dimensional case:

Consider the flow of some material whose total quantity remain unchanged, along a thin tube with section area $A(x)$. Then, conservation means:

$$\frac{d}{dt} \int_a^b u(x, t) A(x) dx = A(a) \phi(a, t) - A(b) \phi(b, t) + \int_a^b f(x, t) A(x) dx$$

ϕ : flux. f : source.

Differentiate w.r.t. b one gets: $Au_t = -A\phi_x - A'\phi + fA$.

- $\phi = u$: e.g. cars which travels at the same speed, age distribution etc.
- $\phi = -u_x$: heat conduction etc.
- $\phi = u - u_x$: contaminated flow etc.
- $f = -u$: decay.

Relationship with random motion: see $u(\cdot, t)$ as the probability distribution.

Example: $u_t = u_x - u$. Decay vs. "widening".

Example: u has two components (e.g. mass, momentum): wave equation.

3 9/12 Method of characteristics

Question: first order linear PDE in 2 dimension: $u_t + fu_x + gu + h = 0$

First consider the case when $g = h = 0$. Recall that for 1st order ODE, there is a concept of *first integral*: the solution of $x'F_x + F_t = 0$ are the level curves of $F(x, t)$. Hence, the level curves of u are exactly the solutions of $x' = f$, which are called *characteristics*.

Example: $u_t = xu_x - u$.

Example: $u_t = u_x + u_y$.

Example: $u_t = \sin tu_x + 1$.

Non-linear advection: $u_t = f(u)u_x$: level curves are straight lines of slope $f(c)$. Breaking time.

Example: $u_t = (1 - u)u_x$.

4 9/14 Diffusion, fundamental solutions

Review of method of characteristics: $u_t + cu_x = x$.

Fick's law: $\phi = -Du_x$, which results in $u_t = Du_{xx}$. Simple observation:

1. Steady state solution: $u = ax + b$.
2. Loss of information: should study initial value problem: $u_t = u_{xx}$, $u(x, 0) = f(x)$ on region $t > 0$.
3. Time scale: remains unchanged under $t = c^2t'$, $x = cx'$.
4. Conservation of the "total heat": $\int u dx$ remain unchanged.

One could expect solution whose "shape" remain unchanged as one scales as in (3). However the integral in (4) changes under this scaling, so one should expect a factor of $t^{-1/2}$. Let $u = t^{-1/2}v(x^2/t)$, then v can be chosen as $v = Ce^{-s/4}$. One can normalize it into $u = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4t}$.

This is called the *fundamental solution* of heat equation in one dimension. δ distribution.

Alternative interpretation of the fundamental solution: discretize, then use central limit theorem. General solution: Convolution.

Fundamental solution of heat equations in higher dimensions?

$$u_t = u_x + u_{xx}$$

Method of mirrors: IBV problem.

5 9/18 Wave equation

$$u_{tt} = u_{xx}$$

Model 1: String vibration: u_{tt} proportional to force which is characterized by u_{xx} .

Model 2: Sound wave in 1-dimension: $\rho_t = -(\rho v)_x$, $(\rho v)_t = -(\rho v^2)_x - p_x$, $p = k\rho^\gamma$.

Review: general solution.

Solution for initial value problem.

Sound speed.

Initial-boundary value problems with one boundary (mirror), initial-boundary value problems with 2 boundaries, periodicity.

(Optional) Spherical waves in higher dimensions.

6 9/21 Wave equation, boundary conditions, review of multivariable calculus

Correction: derivation of the general solution of 1-D wave equation:

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\(\partial_t + c\partial_x)(\partial_t - c\partial_x)u &= 0 \\(\partial_t + c\partial_x)u &= f(x + ct) \\u &= G_1(x - ct) + \int_0^t f(cs + (x - ct) + cs)ds \\F'_1 &= f \\u &= G_1(x - ct) + (F_1(x + ct) - F_1(x - ct))/c = (G_1 - F_1/c)(x - ct) + (F_1/c)(x + ct)\end{aligned}$$

Now let $G = G_1 - F_1/c$, $F = F_1/c$.

Boundary conditions: Dirichlet, Neumann, Robin.

Homogeneous boundary condition.

Example: $u_{tt} = u_{xx}$, $u(0, t) = 0$, $u_X(1, t) = 0$, general solution?

Example: non-homogeneous boundary and non-homogeneous equations

Example: $u_{tt} = u_{xx} + \sin x$.

Vector field in 3 dimension: $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. *grad*, *div* and *curl*. Stokes theorem in \mathbb{R} , \mathbb{R}^2 , \mathbb{R}^3 .

7 9/26 Heat equation in high dimension, Laplace equation

Mass balance in high dimension: $u_t + \text{div}\phi = 0$. Heat: $\phi = -k\text{grad}(u)$.

Steady-state: Laplace equation.

Maximal principle, uniqueness.

Example of solutions. Fundamental solution.

Variational principle.

Laplacian in spherical coordinates. Spherical harmonics.

8 9/28 Types of PDEs

Consider 2nd order equation $Au_{xx} + Bu_{xy} + Cu_{yy} + f(u, u_x, u_y, x, y) = 0$. It is called elliptic/parabolic/hyperbolic iff $Ax^2 + Bxy + Cy^2$ is positive or negative definite/degenerate/indefinite.

Canonical forms: $u_{xx} + u_{yy} + \dots = 0$, $u_{xy} + \dots = 0$, $u_{xx} + \dots = 0$

Example: different types at different places.

Example: type remains unchanged under coordinate change: polar coordinate.

9 10/3 Heat equation

Formula for the Green's function/fundamental solution $G(x, t)$.

Properties: $\int_{-\infty}^{\infty} G(x, t) dx = 1$, $\lim_{t \rightarrow 0^+} \int_{|x| > c > 0} G(x, t) dx = 0$, $G_t = kG_{xx}$.

Poisson integration formula: is a solution: linearity; initial condition: the properties above.

Non-uniqueness of the solution: Tychonov 1935

Higher dimension.

Theorem (Poisson integration): If f is a bounded continuous function, then a solution of $u_t = ku_{xx}$ when $t > 0$, $u(x, 0) = f(x)$ is:

$$u = \int_{\mathbb{R}} f(y) G(x - y, t) dy$$

Proof: By computation we know that:

1. $\int_{\mathbb{R}} G(x, t) dx = 1$
2. For any $c > 0$, $\int_{x \notin [-c, c]} G(x, t) dx \rightarrow 0$ as $t \rightarrow 0$.
3. $G_t = kG_{xx}$

$u_t = ku_{xx}$ follows from 3. and the fact that all infinite integrals involves converges absolutely. Now we need to show the initial condition, i.e. that $u(x, t) \rightarrow f(x)$ as $t \rightarrow 0^+$. Let M be a bound of $|f(x)|$.

For any $c > 0$,

$$|u(x, t) - f(x)|$$

$$\begin{aligned}
&\leq \left| \int_{x-c}^{x+c} f(x)G(x-y, t)dy - f(x) \right| + \left| \int_{x-c}^{x+c} (f(y) - f(x))G(x-y, t)dy \right| + \left| \int_{y \notin [x-c, x+c]} f(y)G(x-y, t)dy \right| \\
&\leq |f(x)| \int_{y \notin [-c, c]} G(y, t)dy + \sup_{x-c < y < x+c} |f(y) - f(x)| + M \left| \int_{y \notin [-c, c]} G(y, t)dy \right|
\end{aligned}$$

Now, for any $\epsilon > 0$, let c be small enough so that $\sup_{x-c < y < x+c} |f(y) - f(x)| < \epsilon/2$, t be small enough so that $\left| \int_{y \notin [-c, c]} G(y, t)dy \right| < \epsilon/4M$, then $|u(x, t) - f(x)| < \epsilon$. Hence $u(x, t) \rightarrow f(x)$ as $t \rightarrow 0$. Furthermore, because any continuous function is absolutely continuous when restricted to a bounded closed neighborhood, the convergence is uniform when x is restricted to any bounded interval. Hence u is continuous on $t = 0$.

10 10/5 Examples, Poisson problem for wave equation

$$u_t = u_{xx}, u(x, 0) = \chi_{[-1, 1]}$$

$$u_t = u_{xx}, u(x, 0) = e^{-x^2}$$

$$\text{erf function: } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

d'Alembert from change of variable: $u_{tt} = k^2 u_{xx}$, $p = x + kt$, $q = x - kt$, then $u_{pq} = 0$, $u = F(p) + G(q)$. Now $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, which in p, q -coordinate means $F(x) + G(x) = f$, $kF'(x) - kG'(x) = g$. Solve for F and G then one gets the d'Alembert formula.

Negative and positive characteristics, domain of influence and domain of dependence

11 Review for Midterm I

The following may appear in the first midterm:

- Simplify PDE by substitution
- Prove properties of the solution by chain rules, fundamental theorem of calculus, and divergence theorem
- Solve PDE by reducing it to ODE either through restriction to a curve or through the use of symmetry.
- Obtain particular solution from the general solution by applying boundary condition.
- Method of characteristics
- General solution of 1-dimensional wave equations
- Poisson integration representation for initial value problem of the heat equation
- Can recognize elliptic, parabolic and hyperbolic 2nd-order equations

Practice problems:

1. Solve the initial value problem $u_t + \sin t u_x = 1$, $u(x, 0) = \sin x$.

Solution: By method of characteristics, the general solution is $u(x, t) = t + F(x + \cos t)$, so $u(x, t) = t + \sin(x + \cos t - 1)$.

2. Find the steady state solution of $u_t = u_{xx} + xu_x$.

Solution: The steady state solution satisfies $u_{xx} + xu_x = 0$, hence $u = A \int_0^x e^{-t^2/2} dt + B$. You can also write it using the *erf* function.

3. Consider the equation: $u_{tt} = u_{xx} + u_{yy}$. If a solution satisfy $u = \sin tv(x, y)$, what is the PDE v satisfies? Can you find a solution when v depends only on y ?

Solution: By product law, we get $v_{xx} + v_{yy} + v = 0$. If v depends only on y then $v = A \cos y + B \sin y$.

4. Consider the boundary value problem $u_{tt} = u_{xx} - u_t$, $u(0, t) = u(1, t) = 0$. Show that the function $\int_0^1 u_t^2 + u_x^2 dx$ is decreasing. What's the limit of u as $t \rightarrow \infty$?

Solution: $\frac{d}{dt} \int_0^1 u_t^2 + u_x^2 dx = \int_0^1 2u_t u_{tt} + 2u_x u_{xt} dx = 2(u_t u_x)|_0^1 - 2 \int_0^1 u_t^2 dx \leq 0$. As $t \rightarrow \infty$, the energy $\int_0^1 u_t^2 + u_x^2 dx$ will decay towards 0, and the limit will be 0.

12 10/10 Well posed problem, review

Some known solutions of IVP:

- $u_t = u_x, u(x, 0) = f(x)$
Answer: $u(x, t) = f(x + t)$.
- $u_{tt} = u_{xx}, u(x, 0) = f(x), u_t(x, 0) = g(x)$
Answer: $u(x, t) = \frac{1}{2}(f(x + t) + f(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds$.
- $u_t = u_{xx}, u(x, 0) = f(x), u$ bounded. (or $\leq Ce^{Cx^2}$)
Answer: $u(x, t) = \int_{\mathbb{R}} f(s) G(x - s) ds$.

In all cases, we have: (1) solution exist. (2) solution is unique. (3) solution depends on the initial condition continuously. Hence we call them **well posed** problems.

Example of non-well-posed problems:

Nonlinear advection.

Reverse heat equation.

$$u_{xx} + u_{tt} = 0.$$

Review:

1. $u_t = tu_x, u(x, 0) = x^2$.

2. $u_{tt} = u_{xx} - u$: steady state?

13 10/17 Semi-infinite domain, Dahamel's Principle

Example 1: $u_t = u_{xx}, u(x, 0) = f, u(0, t) = 0$: $u = \int G(x - y, t) \phi(y) dy$, so $\phi(x) = f(x)$ when $x > 0$ and $-f(-x)$ when $x < 0$.

Example 2: $u_{tt} = u_{xx}, u(x, 0) = f, u_t(x, 0) = g, u_x(0, t) = 0, x \geq 0, t \geq 0$: $u = \frac{1}{2}(\phi(x - t) + \phi(x + t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds$. So ϕ and ψ are even extension of f and g respectively.

Example 3: L linear operator in the space of functions on x . $u_t = Lu, u(0) = \alpha$ has solution $u(t, \alpha)$. Then, $u_t = Lu + f(t), u(0) = \alpha$ has solution $u(t) = u(t, \alpha) + \int_0^t u(s, f(t - s)) ds$.

Example 4: $u_{tt} = u_{xx} + \sin(x + t), u_t(x, 0) = u(x, 0) = 0$. Let $U = [u, u_t]^T$, use the principle above.

Example 5: $u_t = u_{xx}, u(0, t) = t$. Solution: combine ideas from problem 1 and 3.

14 10/19 Laplace Transform and Fourier Transform

Review: Homogeneous boundary: mirroring; Non-homogeneous equation: $w(t, \alpha)$ being the solution of $w_t = Tw, w(0) = \alpha$, then $u_t = Tu + f(t), u(0) = b$ has solution $u = w(t, b) + \int_0^t w(t - s, f(s)) ds$. Hence, to solve non-homogeneous equations, first solve for w then put it in the formula.

Laplace transform: $L(f) = \int_0^\infty e^{-st} f(t) dt$.

Properties: $L(f') = sL(f) - f(0)$, $L(f * g) = L(f)L(g)$. Here f and g are 0 on $(-\infty, 0)$.

$L(f) = 0$ iff f a.e. 0. When f is analytic, $L^{-1}(f) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) e^{st} ds$, but we won't use this.

Formulas we will use:

$$(1): L\left(\frac{1}{\sqrt{4\pi t}} e^{-a^2/(4t)}\right) = \frac{1}{\sqrt{4s}} e^{-|a|\sqrt{s}}.$$

$$(2): L\left(\frac{a}{2t^{3/2}} e^{-a^2/(4t)}\right) = \sqrt{\pi} e^{-a\sqrt{s}}.$$

Example 1: $u_t = u_{xx}$, $u(x, 0) = f(x)$, f compactly supported (or have similar decay condition)

$sL(u) - f(x) = (Lu)_{xx}$, hence $(Lu)(x, s) = \frac{1}{2\sqrt{s}} \left(e^{-\sqrt{s}x} \int_{-\infty}^x e^{\sqrt{s}r} f(r) dr + e^{\sqrt{s}x} \int_x^\infty e^{-\sqrt{s}r} f(r) dr \right) = \frac{1}{\sqrt{4s}} \int_{-\infty}^\infty e^{-\sqrt{s}|x-r|} f(r) dr = L\left(\int_{-\infty}^\infty G(x-r, t) y(r) dr\right)$. Here we use (1), and also the formula for solving non-homogeneous 2nd order ODE: $y = y_2 \int_a^x (y_1 f/W) ds - y_1 \int_a^x (y_2 f/W) ds$.

Example 2: $u_t = u_{xx}$, $u(x, 0) = 0$, $u(0, t) = f(t)$.

$$sL(u) = (Lu)_{xx}, \text{ so } (Lu)(x, s) = L(f) e^{-\sqrt{s}x} \text{ so } u = L^{-1}(L(f)) * \frac{x}{\sqrt{4\pi t^3}} e^{-\frac{x^2}{4t}} = \int_0^t f(\tau) \frac{x}{\sqrt{4\pi(t-\tau)^3}} e^{-\frac{x^2}{4(t-\tau)}} d\tau.$$

How about $f = 1$?

15 10/24 Laplace and Fourier transform

Steps for solving PDEs using integration transform:

1. Do transform, turn it into ODE.
2. Apply initial/boundary conditions.
3. Solve ODE, take the inverse transform.

Example 1: $u_t = u_x$, $u(x, 0) = f(x)$, use Laplace transform on t .

$sLu - f(x) = (Lu)_x$, so $Lu = F(s) + \int_x^\infty f(r) e^{s(x-r)} dr = F(s) + L(f(x + \cdot))$. So $u = L^{-1}(F) + f(x + t)$, by initial condition $F = 0$.

Example 2: (PIP) $u_t = Ku_{xx}$, $u(x, 0) = 0$, $u(0, t) = f$, find K from $u_x(t, 0)$.

$u = \int_0^t f(\tau) \frac{x}{\sqrt{4K\pi(t-\tau)^3}} e^{-\frac{x^2}{4K(t-\tau)}} d\tau = -2K \int_0^t G_x(x, t-\tau) f(\tau) d\tau = -2 \int_0^t G(x, t-\tau) f'(\tau) d\tau = \dots$. Do everything for x small then take limit.

Fourier transform: $F(f) = \int_{\mathbb{R}} e^{ist} f(t) dt$. Properties: $F(f') = -isF(f)$. $F^{-1}(f) = \frac{1}{2\pi} e^{-ist} f(t) dt$. $F(f * g) = F(f) * F(g)$. $(F^{-1}(f * g)) = \frac{1}{2\pi} F^{-1}(f) F^{-1}(g)$

Example 3: $u_t = u_{xx}$, $u(x, 0) = f$. F on x : $(Fu)_t = -y^2(Fu)$, $Fu = e^{-ty^2} F(f)$, $u = F^{-1}(e^{-ty^2}) * f = \dots$. Here, one uses that $\int_{\mathbb{R}} e^{(-x+iy)^2} dx$ does not depend on y .

16 10/26 Fourier transform

Review: Definition, derivatives, convolution, inverse.

Example 1: $u_{tt} = 4u_{xx} + f(x, t)$, $u(x, 0) = g(x)$, $u_t(x, 0) = 0$.

Fourier transform on x , $v = F(u)$: $v_{tt} = -4s^2v + F(f)$, $v(s, 0) = Fg$, $v_t(s, 0) = 0$. So $v(x, t) = (Fg)(s) \cos(2st) + \int_0^t \frac{1}{2s} \sin(2s(t-r))(Ff)(s, r)dr$. Now by the inverse formula, we have $F^{-1}(\cos(2st) \cdot Fg)(x, t) = \frac{1}{2}(g(x-2t) + g(x+2t))$, and $F^{-1}(\frac{1}{2s} \sin(2s(t-r)) \cdot Ff) = F^{-1}(\frac{1}{4is}(F(f(x+2t-2r, r) - f(x-2t+2r, r)))) = \frac{1}{4} \int_{x-2t+2r}^{x+2t-2r} f(y, r)dy$. Hence the solution is $u = \frac{1}{2}(g(x-2t) + g(x+2t)) + \frac{1}{4} \int_0^t \int_{x-2t+2r}^{x+2t-2r} f(y, r)dy$.

Example 2: $u_{tt} + u_{xx} = 0$, $u(x, 0) = f(x)$, u bounded on $t > 0$. (a model for electric potential, current field, Newtonian gravity etc.)

Fourier transform on x : $v = F(u)$, then $v_{tt} = s^2v$, $v(s, t) = F(f)(s)e^{-|s|t}$, $u = F^{-1}(F(f)(s)e^{-|s|t}) = f * \frac{t}{\pi(t^2 + x^2)}$.

Example 3: 3-dimensional wave equation: $u_{tt} = \Delta u$, $u_t(x, 0) = f(x)$, $u(x, 0) = 0$.

Multi-variable Fourier transform on x , $v = F(u)$, we get $v_{tt} = |s|^2v$. $v = \frac{\sin(|s|t)}{|s|}F(f)$. Calculate $\frac{F^{-1}(\frac{\sin(|s|t)}{|s|})}{|s|}$ in coordinate system (r, h, θ) where $h = s \cdot x$, one gets that it is a distribution concentrated at $|x| = t$. Huygen's principle.

Example 4: $u_{tt} = u_{xx} - u_t$, $u(x, 0) = 0$, $u_t(x, 0) = f(x)$.

Do Fourier transform in x direction, one gets $\hat{u} = -\hat{f}(s) \cdot (1 - 4s^2)^{-1/2} (e^{-\frac{1+\sqrt{1-4s^2}}{2}t} - e^{-\frac{1-\sqrt{1-4s^2}}{2}t})$. So $u = f * \Phi$, $\Phi(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}} (1 - 4s^2)^{-1/2} (e^{-\frac{1+\sqrt{1-4s^2}}{2}t} - e^{-\frac{1-\sqrt{1-4s^2}}{2}t}) e^{-isx} ds$.