

# 1 9/5 PDE terminology & philosophy

PDE: equation for a multivariate function that involves its partial derivatives.

Example:  $u_y = x$ .

Example:  $(yu)_y = 1$ .

General solution of a PDE.

Formally: PDE:  $F(u, x_i, u_{x_i}, u_{x_i x_j}, \dots) = 0$

Order of a pde

Linear PDE.

Linear homogeneous PDE.

What are the order and linearity of the following PDEs?

$u_x + u_{yyx} = 1$ ,  $uu_x + u = 0$ ,  $u_x + (x^2 + y^2)u_{yy} = 1$ .

Some PDEs we will focus on later:

Heat:  $u_t = u_{xx}$ : (heat transmission, diffusion)

Laplace:  $u_{xx} + u_{yy} = 0$ : (static electric field, Newton's gravity, equilibrium of random walk)

Wave:  $u_{tt} = u_{xx}$ : (sound wave, other waves in physics)

Other important linear PDEs:

Dispersive wave equations:  $u_{tt} = u_{xx} - ku_{xxx}$  (stiff string)

Cauchy-Riemann equation:  $u_x = v_y$ ,  $u_y = -v_x$

Non-linear PDEs you may see in later classes:

Navier-Stokes

Nonlinear Schrodinger:  $iu_t = -\Delta u + k|u|^2 u$

KdV:  $u_t + u_{xxx} + 6uu_x = 0$ , etc.

Example: growth of bacteria. Baseline: GMCF (geodesic mean curvature flow)  $u_t = A \frac{\nabla u}{|\nabla u|} \cdot \nabla u + B|\nabla u| \nabla \cdot \frac{\nabla u}{|\nabla u|}$ .

Types of problems:

Evolution model (with time): Boundary condition. Initial condition. Initial value problem. Initial-boundary value problem.

Steady state model (no time): boundary value problem.

Typical questions in the theory of PDE:

Existence

Uniqueness

Regularity

Continuous dependency on boundary

Typical strategy: integral transform:  $(Tu)(y) = \int u(x)K(x,y)dx$ , then  $T(u_x) = \int u_x(x)K(x,y)dx = -\int u(x)K_x(x,y)dx$ , assume some decay conditions on the boundary (or infinity).

Problem: Is such a transform well defined?

Connection with harmonic analysis.

Use of symmetry (method of mirror images, spherical symmetry etc.)

Example: solve  $u_{xx} + u_{yy} = 1$ , where  $u = 0$  on the unit circle.

Example:  $u_x = u_t$ ,  $u_x = u_t + 1$ .

## 2 9/7 Review of ODE, Advection and Diffusion

Review of ODE & multivariable calculus topics:

- $u' + p(t)u + q(t) = 0$
- $u''' + Au'' + Bu' + Cu = 0$
- Chain rule: Example:  $u_{xx} = u_{tt}$ , what happens with change-of-variable  $y = x + t$ ,  $w = x - t$ ?
- Fubini's theorem.
- Differentiating an integral. Example:  $\frac{d}{dt} \int_0^{t^2} \sin(ts) ds$ .  
Solution: Let  $x = t$ ,  $y = t$ , then  $\frac{d}{dt} \int_0^{t^2} e^{-ts^2} ds = \frac{d}{dt} \int_0^{x^2} e^{-ys^2} ds = (\int_0^{x^2} e^{-ys^2} ds)_x + (\int_0^{x^2} e^{-ys^2} ds)_y = 2x \cdot e^{-y(x^2)^2} + \int_0^{x^2} (e^{-ys^2})_y ds = 2xe^{-y(x^2)^2} - \int_0^{x^2} s^2 e^{-ys^2} ds = 2te^{-t^5} - \int_0^{t^2} s^2 e^{-ts^2} ds$ .
- Example:  $u_{tt} = u_{xx} + u_{yy}$ ,  $u(x, y, t) = \sin(x \cos \theta + y \sin \theta + t)$  are solutions, hence  $\int_0^{2\pi} \sin(x \cos \theta + y \sin \theta + t) d\theta$  is also a solution.

PDE from conservation laws, 1-dimensional case:

Consider the flow of some material whose total quantity remain unchanged, along a thin tube with section area  $A(x)$ . Then, conservation means:

$$\frac{d}{dt} \int_a^b u(x, t) A(x) dx = A(a) \phi(a, t) - A(b) \phi(b, t) + \int_a^b f(x, t) A(x) dx$$

$\phi$ : flux.  $f$ : source.

Differentiate w.r.t.  $b$  one gets:  $Au_t = -A\phi_x - A'\phi + fA$ .

- $\phi = u$ : e.g. cars which travels at the same speed, age distribution etc.
- $\phi = -u_x$ : heat conduction etc.
- $\phi = u - u_x$ : contaminated flow etc.
- $f = -u$ : decay.

Relationship with random motion: see  $u(\cdot, t)$  as the probability distribution.

Example:  $u_t = u_x - u$ . Decay vs. "widening".

Example:  $u$  has two components (e.g. mass, momentum): wave equation.

### 3 9/12 Method of characteristics

Question: first order linear PDE in 2 dimension:  $u_t + fu_x + gu + h = 0$

First consider the case when  $g = h = 0$ . Recall that for 1st order ODE, there is a concept of *first integral*: the solution of  $x'F_x + F_t = 0$  are the level curves of  $F(x, t)$ . Hence, the level curves of  $u$  are exactly the solutions of  $x' = f$ , which are called *characteristics*.

Example:  $u_t = xu_x - u$ .

Example:  $u_t = u_x + u_y$ .

Example:  $u_t = \sin tu_x + 1$ .

Non-linear advection:  $u_t = f(u)u_x$ : level curves are straight lines of slope  $f(c)$ . Breaking time.

Example:  $u_t = (1 - u)u_x$ .

### 4 9/14 Diffusion, fundamental solutions

Review of method of characteristics:  $u_t + cu_x = x$ .

Fick's law:  $\phi = -Du_x$ , which results in  $u_t = Du_{xx}$ . Simple observation:

1. Steady state solution:  $u = ax + b$ .
2. Loss of information: should study initial value problem:  $u_t = u_{xx}$ ,  $u(x, 0) = f(x)$  on region  $t > 0$ .
3. Time scale: remains unchanged under  $t = c^2t'$ ,  $x = cx'$ .
4. Conservation of the "total heat":  $\int u dx$  remain unchanged.

One could expect solution whose "shape" remain unchanged as one scales as in (3). However the integral in (4) changes under this scaling, so one should expect a factor of  $t^{-1/2}$ . Let  $u = t^{-1/2}v(x^2/t)$ , then  $v$  can be chosen as  $v = Ce^{-s/4}$ . One can normalize it into  $u = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4t}$ .

This is called the *fundamental solution* of heat equation in one dimension.  $\delta$  distribution.

Alternative interpretation of the fundamental solution: discretize, then use central limit theorem. General solution: Convolution.

Fundamental solution of heat equations in higher dimensions?

$$u_t = u_x + u_{xx}$$

Method of mirrors: IBV problem.

### 5 9/18 Wave equation

$$u_{tt} = u_{xx}$$

Model 1: String vibration:  $u_{tt}$  proportional to force which is characterized by  $u_{xx}$ .

Model 2: Sound wave in 1-dimension:  $\rho_t = -(\rho v)_x$ ,  $(\rho v)_t = -(\rho v^2)_x - p_x$ ,  $p = k\rho^\gamma$ .

Review: general solution.

Solution for initial value problem.

Sound speed.

Initial-boundary value problems with one boundary (mirror), initial-boundary value problems with 2 boundaries, periodicity.

(Optional) Spherical waves in higher dimensions.

## 6 9/21 Wave equation, boundary conditions, review of multivariable calculus

Correction: derivation of the general solution of 1-D wave equation:

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\(\partial_t + c\partial_x)(\partial_t - c\partial_x)u &= 0 \\(\partial_t + c\partial_x)u &= f(x + ct) \\u &= G_1(x - ct) + \int_0^t f(cs + (x - ct) + cs)ds \\F'_1 &= f \\u &= G_1(x - ct) + (F_1(x + ct) - F_1(x - ct))/c = (G_1 - F_1/c)(x - ct) + (F_1/c)(x + ct)\end{aligned}$$

Now let  $G = G_1 - F_1/c$ ,  $F = F_1/c$ .

Boundary conditions: Dirichlet, Neumann, Robin.

Homogeneous boundary condition.

Example:  $u_{tt} = u_{xx}$ ,  $u(0, t) = 0$ ,  $u_X(1, t) = 0$ , general solution?

Example: non-homogeneous boundary and non-homogeneous equations

Example:  $u_{tt} = u_{xx} + \sin x$ .

Vector field in 3 dimension:  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . *grad*, *div* and *curl*. Stokes theorem in  $\mathbb{R}$ ,  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ .

## 7 9/26 Heat equation in high dimension, Laplace equation

Mass balance in high dimension:  $u_t + \text{div}\phi = 0$ . Heat:  $\phi = -k\text{grad}(u)$ .

Steady-state: Laplace equation.

Maximal principle, uniqueness.

Example of solutions. Fundamental solution.

Variational principle.

\*\*\*\*\*

Laplacian in spherical coordinates. Spherical harmonics.

## 8 9/28 Types of PDEs

Consider 2nd order equation  $Au_{xx} + Bu_{xy} + Cu_{yy} + f(u, u_x, u_y, x, y) = 0$ . It is called elliptic/parabolic/hyperbolic iff  $Ax^2 + Bxy + Cy^2$  is positive or negative definite/degenerate/indefinite.

Canonical forms:  $u_{xx} + u_{yy} + \dots = 0$ ,  $u_{xy} + \dots = 0$ ,  $u_{xx} + \dots = 0$

Example: different types at different places.

Example: type remains unchanged under coordinate change: polar coordinate.

## 9 10/3 Heat equation

Formula for the Green's function/fundamental solution  $G(x, t)$ .

Properties:  $\int_{-\infty}^{\infty} G(x, t) dx = 1$ ,  $\lim_{t \rightarrow 0^+} \int_{|x| > c > 0} G(x, t) dx = 0$ ,  $G_t = kG_{xx}$ .

Poisson integration formula: is a solution: linearity; initial condition: the properties above.

Non-uniqueness of the solution: Tychonov 1935

Higher dimension.

Theorem (Poisson integration): If  $f$  is a bounded continuous function, then a solution of  $u_t = ku_{xx}$  when  $t > 0$ ,  $u(x, 0) = f(x)$  is:

$$u = \int_{\mathbb{R}} f(y) G(x - y, t) dy$$

Proof: By computation we know that:

1.  $\int_{\mathbb{R}} G(x, t) dx = 1$
2. For any  $c > 0$ ,  $\int_{x \notin [-c, c]} G(x, t) dx \rightarrow 0$  as  $t \rightarrow 0$ .
3.  $G_t = kG_{xx}$

$u_t = ku_{xx}$  follows from 3. and the fact that all infinite integrals involves converges absolutely. Now we need to show the initial condition, i.e. that  $u(x, t) \rightarrow f(x)$  as  $t \rightarrow 0^+$ . Let  $M$  be a bound of  $|f(x)|$ .

For any  $c > 0$ ,

$$|u(x, t) - f(x)|$$

$$\begin{aligned}
&\leq \left| \int_{x-c}^{x+c} f(x)G(x-y, t)dy - f(x) \right| + \left| \int_{x-c}^{x+c} (f(y) - f(x))G(x-y, t)dy \right| + \left| \int_{y \notin [x-c, x+c]} f(y)G(x-y, t)dy \right| \\
&\leq |f(x)| \int_{y \notin [-c, c]} G(y, t)dy + \sup_{x-c < y < x+c} |f(y) - f(x)| + M \left| \int_{y \notin [-c, c]} G(y, t)dy \right|
\end{aligned}$$

Now, for any  $\epsilon > 0$ , let  $c$  be small enough so that  $\sup_{x-c < y < x+c} |f(y) - f(x)| < \epsilon/2$ ,  $t$  be small enough so that  $\left| \int_{y \notin [-c, c]} G(y, t)dy \right| < \epsilon/4M$ , then  $|u(x, t) - f(x)| < \epsilon$ . Hence  $u(x, t) \rightarrow f(x)$  as  $t \rightarrow 0$ . Furthermore, because any continuous function is absolutely continuous when restricted to a bounded closed neighborhood, the convergence is uniform when  $x$  is restricted to any bounded interval. Hence  $u$  is continuous on  $t = 0$ .

## 10 10/5 Examples, Poisson problem for wave equation

$$u_t = u_{xx}, u(x, 0) = \chi_{[-1, 1]}$$

$$u_t = u_{xx}, u(x, 0) = e^{-x^2}$$

$$\text{erf function: } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

d'Alembert from change of variable:  $u_{tt} = k^2 u_{xx}$ ,  $p = x + kt$ ,  $q = x - kt$ , then  $u_{pq} = 0$ ,  $u = F(p) + G(q)$ . Now  $u(x, 0) = f(x)$ ,  $u_t(x, 0) = g(x)$ , which in  $p, q$ -coordinate means  $F(x) + G(x) = f$ ,  $kF'(x) - kG'(x) = g$ . Solve for  $F$  and  $G$  then one gets the d'Alembert formula.

Negative and positive characteristics, domain of influence and domain of dependence

## 11 Review for Midterm I

The following may appear in the first midterm:

- Simplify PDE by substitution
- Prove properties of the solution by chain rules, fundamental theorem of calculus, and divergence theorem
- Solve PDE by reducing it to ODE either through restriction to a curve or through the use of symmetry.
- Obtain particular solution from the general solution by applying boundary condition.
- Method of characteristics
- General solution of 1-dimensional wave equations
- Poisson integration representation for initial value problem of the heat equation
- Can recognize elliptic, parabolic and hyperbolic 2nd-order equations

Practice problems:

1. Solve the initial value problem  $u_t + \sin t u_x = 1$ ,  $u(x, 0) = \sin x$ .

Solution: By method of characteristics, the general solution is  $u(x, t) = t + F(x + \cos t)$ , so  $u(x, t) = t + \sin(x + \cos t - 1)$ .

2. Find the steady state solution of  $u_t = u_{xx} + xu_x$ .

Solution: The steady state solution satisfies  $u_{xx} + xu_x = 0$ , hence  $u = A \int_0^x e^{-t^2/2} dt + B$ . You can also write it using the *erf* function.

3. Consider the equation:  $u_{tt} = u_{xx} + u_{yy}$ . If a solution satisfy  $u = \sin tv(x, y)$ , what is the PDE  $v$  satisfies? Can you find a solution when  $v$  depends only on  $y$ ?

Solution: By product law, we get  $v_{xx} + v_{yy} + v = 0$ . If  $v$  depends only on  $y$  then  $v = A \cos y + B \sin y$ .

4. Consider the boundary value problem  $u_{tt} = u_{xx} - u_t$ ,  $u(0, t) = u(1, t) = 0$ . Show that the function  $\int_0^1 u_t^2 + u_x^2 dx$  is decreasing. What's the limit of  $u$  as  $t \rightarrow \infty$ ?

Solution:  $\frac{d}{dt} \int_0^1 u_t^2 + u_x^2 dx = \int_0^1 2u_t u_{tt} + 2u_x u_{xt} dx = 2(u_t u_x)|_0^1 - 2 \int_0^1 u_t^2 dx \leq 0$ . As  $t \rightarrow \infty$ , the energy  $\int_0^1 u_t^2 + u_x^2 dx$  will decay towards 0, and the limit will be 0.

## 12 10/10 Well posed problem, review

Some known solutions of IVP:

- $u_t = u_x, u(x, 0) = f(x)$   
Answer:  $u(x, t) = f(x + t)$ .
- $u_{tt} = u_{xx}, u(x, 0) = f(x), u_t(x, 0) = g(x)$   
Answer:  $u(x, t) = \frac{1}{2}(f(x + t) + f(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds$ .
- $u_t = u_{xx}, u(x, 0) = f(x), u$  bounded. (or  $\leq Ce^{Cx^2}$ )  
Answer:  $u(x, t) = \int_{\mathbb{R}} f(s) G(x - s) ds$ .

In all cases, we have: (1) solution exist. (2) solution is unique. (3) solution depends on the initial condition continuously. Hence we call them **well posed** problems.

Example of non-well-posed problems:

Nonlinear advection.

Reverse heat equation.

$$u_{xx} + u_{tt} = 0.$$

Review:

1.  $u_t = tu_x, u(x, 0) = x^2$ .

2.  $u_{tt} = u_{xx} - u$ : steady state?

## 13 10/17 Semi-infinite domain, Dahamel's Principle

Example 1:  $u_t = u_{xx}, u(x, 0) = f, u(0, t) = 0$ :  $u = \int G(x - y, t) \phi(y) dy$ , so  $\phi(x) = f(x)$  when  $x > 0$  and  $-f(-x)$  when  $x < 0$ .

Example 2:  $u_{tt} = u_{xx}, u(x, 0) = f, u_t(x, 0) = g, u_x(0, t) = 0, x \geq 0, t \geq 0$ :  $u = \frac{1}{2}(\phi(x - t) + \phi(x + t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds$ . So  $\phi$  and  $\psi$  are even extension of  $f$  and  $g$  respectively.

Example 3:  $L$  linear operator in the space of functions on  $x$ .  $u_t = Lu, u(0) = \alpha$  has solution  $u(t, \alpha)$ . Then,  $u_t = Lu + f(t), u(0) = \alpha$  has solution  $u(t) = u(t, \alpha) + \int_0^t u(s, f(t - s)) ds$ .

Example 4:  $u_{tt} = u_{xx} + \sin(x + t), u_t(x, 0) = u(x, 0) = 0$ . Let  $U = [u, u_t]^T$ , use the principle above.

Example 5:  $u_t = u_{xx}, u(0, t) = t$ . Solution: combine ideas from problem 1 and 3.