

# Kazhdan's Theorem for Canonical Metric on Graphs

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November 23, 2019

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- Motivation: Kazhdan's theorem on Riemann surface
- Canonical metric on graphs
- Statement of our theorem
- Generalization
- Sketch of proof

- A Riemann surface is a connected complex manifold of dimension 1.
- Any connected oriented surface with Riemannian metric is a Riemann surface
- **Conformal** means that angles are unchanged.
- $\Omega^1$  is the space of holomorphic 1-forms.
- $X' \rightarrow X$  is called a **regular covering**, if there is some group  $G$  acting freely on  $X'$  such that  $X = X'/G$ .
- If  $X$  is a connected manifold, there is a regular covering which is simply connected, called the **universal cover**.

## Uniformization theorem [Poincaré, Koebe, 1907]:

Any simply connected Riemann surface is conformal to  $\mathbb{C}$ ,  $\overline{\mathbb{C}}$ , or the unit disc.

- Any Riemann surface  $S$  has a conformal metric with constant curvature.
- When  $S$  is other than  $\mathbb{C}$ ,  $\overline{\mathbb{C}}$ ,  $\mathbb{C} \setminus \{p\}$ , annulus or torus, the uniformization metric has constant negative curvature. We call such  $S$  **hyperbolic**
- Kazhdan's theorem gives an explicit way to obtain this uniformization metric, via the canonical metric for some regular coverings.

# Kazhdan's Theorem for Riemann Surfaces

- $S$ : a compact Riemann surface
- $\Omega^1(S)$ : Space of holomorphic 1-forms on  $S$
- $\{\omega_i\}$ : Orthonormal basis of  $\Omega^1(S)$  ( $\langle u, v \rangle = \frac{1}{2\sqrt{-1}} \int \bar{u} \wedge v$ )
- $d_c^S = \sum_i |\omega_i|^2$ ;  $d_c^S$  is called the **Canonical** or **Arakelov** metric;
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$ : infinite tower of finite regular covers,  
 $\cap_i \pi_1(S_i) = 1$
- $d_i$ : Riemannian metrics on  $S$  whose pull-back on  $S_i$  are the  $d_c^{S_i}$

## Theorem [Kazhdan, 70s]

If  $S$  is hyperbolic,  $d_i$  converges uniformly to a multiple of the uniformization metric.

# Canonical Metric on Graphs

$G = \{V(G), E(G), l\}$ : a finite metric graph

- $E(G)$ : directed edges.
- $\bar{e}$ : the opposite of  $e \in E(G)$
- $e \in E(G) \iff \bar{e} \in E(G)$
- $l : E(G) \rightarrow \mathbb{R}^+$ : the edge-length function.
- $l(e) = l(\bar{e})$  for all  $e \in E(G)$

- $C^1(G) = \{\alpha \in \text{Map}(E(G), \mathbb{R}) : \alpha(e) = -\alpha(\bar{e})\}$ : space of simplicial 1-cochains.
- $C^0(G) = \text{Map}(V(G), \mathbb{R})$ : space of simplicial 0-cochains.
- Inner product on  $C^1(G)$ :  $(\alpha, \beta) = \frac{1}{2} \sum_{e \in \mathcal{O}} \frac{\alpha(e)\beta(e)}{l(e)}$ .
- Inner product on  $C^0(G)$ :  $(\alpha, \beta) = \sum_{v \in V(G)} \alpha(v)\beta(v)$ .
- $d : C^0(G) \rightarrow C^1(G)$ : the coboundary map,  
 $d(\alpha)(e) = \alpha(e^+) - \alpha(e^-)$  for all  $e \in E(G)$ .
- $\delta = d^* : C^1(G) \rightarrow C^0(G)$

All these definitions works for infinite graphs when  $C^1$  and  $C^0$  are replaced by  $L^2$  summable forms.

- $\mathcal{H}(G) = \{\alpha \in C^1(G) : \delta\alpha = 0\}$ : the space of harmonic-forms on  $G$ .
- Explicit description:  $\forall v \in V(G), \sum_{e \in \mathcal{O}, e^+ = v} \frac{\alpha(e)}{l(e)} = 0$ .

### Definition of canonical metric for a finite metric graph

(Zhang 93, Baker-Farber 11, Chinburg-Rumely 93 et al): Given finite metric graph  $G = (E(G), V(G), l)$ ,

$$l_c^G(e) = \frac{1}{l(e)} \sup_{\|\alpha\| \leq 1, \alpha \in \mathcal{H}} |\alpha(e)|$$

$$= \sum_i \omega_i^2(e)$$

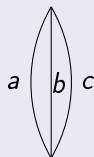
Where  $\{\omega_i\}$  is orthonormal basis of  $\mathcal{H}$ .



# Interpretation: Counting of Spanning Tree

- $\mathcal{T} = \{T\}$ : Set of spanning tree of  $G$ . A spanning tree is a subgraph with no loops and contains all vertices of  $G$ .
- Weight of a tree:  $w(T) = \prod_{e \in T} l(e)$
- $I_c^G(e) = \frac{\sum_{T \in \mathcal{T}, e \notin T} w(T)}{\sum_{T \in \mathcal{T}} w(T)}$  (Foster)

Example:

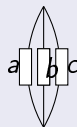


$$l(a) = 2, l(b) = l(c) = 1$$

$$I_c^G(a) = \frac{l(a)l(c) + l(a)l(b)}{l(a)l(c) + l(a)l(b) + l(b)l(c)} = \frac{4}{5}, I_c(b) = I_c(c) = \frac{3}{5}$$

# Interpretation: Network of Resistors

## Example, cont.



Turn the graph into resistor network:

Let  $R$  be effective resistance between the two vertices, by parallel law:

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{1} + \frac{1}{1} = \frac{5}{2}$$

$$I_c(a) = 1 - R/2 = 4/5, I_c(b) = I_c(c) = 1 - R/1 = 3/5$$

- For every  $e \in E(G)$ ,  $R(e)$  is the effective resistance between  $e^+$  and  $e^-$ .
- $I_c^G = 1 - \frac{R(e)}{I(e)}$ , which is also called **Foster's coefficient**.

- The equivalence between effective resistance interpretation and harmonic 1-form interpretation:

Harmonic analysis on graph	Resistor network
1-form	Current distribution
$\alpha(e)$	Potential between $e^+$ and $e^-$
Norm on $C^1$	Energy
Harmonicity of 1-form	Kirchhoff's first law

- There are other interpretations of canonical metric of finite graphs, for example:
  - As pull back from graph Jacobian metric (Baker-Farber)
  - As limits of Weierstrass points in Berkovich spaces (Amini).

# Statement of Our Result

## Theorem [Shokrieh, W]

- $G$ : finite connected metric graph
- $G \leftarrow G_1 \leftarrow G_2 \leftarrow \dots$ : tower of finite regular covers  
( $\pi_1(G_i) \triangleleft \pi_1(G)$ )
- $\pi_i : G_i \rightarrow G$ : covering map.
- $l_i : E(G) \rightarrow \mathbb{R}$ , such that  $\pi_i^* l_i = l_c^{G_i}$

Then  $\lim_{n \rightarrow \infty} l_i$  exists, which depends only on  $G$  and  $\cap_i(\pi_1(G_i))$ .

When  $\cap_i(\pi_1(G_i)) = \{1\}$ , we can think of the limiting metric  $l_\infty$  on  $G$  as a candidate of the uniformization metric.  $l_\infty$  may be 0 on some edges.

# Kazhdan's Theorem for Riemann Surfaces

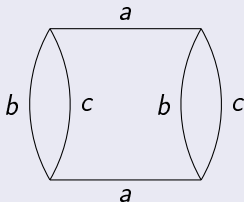
- $S$ : a compact Riemann surface
- $\Omega^1(S)$ : Space of holomorphic 1-forms on  $S$
- $\{\omega_i\}$ : Orthonormal basis of  $\Omega^1(S)$  ( $\langle u, v \rangle = \frac{1}{2\sqrt{-1}} \int \bar{u} \wedge v$ )
- $d_c^S = \sum_i |\omega_i|^2$
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$ : infinite tower of finite regular covers,  $\cap_i \pi_1(S_i) = 1$
- $d_i$ : Riemannian metrics on  $S$  whose pull-back on  $S_i$  are the  $d_c^{S_i}$

## Theorem [Kazhdan, 70s]

If  $S$  is hyperbolic,  $d_i$  converges uniformly to a multiple of the uniformization metric.

## Example, cont.

Consider this double cover  $G_1$ :



$l_c^{G_1}(a) = 2/5$ ,  $l_c^{G_1}(b) = l_c^{G_1}(c) = 11/20$ . If we make a tower of coverings such that  $\cap_i \pi_1(G_i) = \{1\}$ , the limiting metric is  $l_\infty(a) = \frac{11-\sqrt{41}}{10}$ ,  $l_\infty(b) = l_\infty(c) = \frac{\sqrt{41}-1}{20}$ , which **doesn't depend on the choice of the tower of coverings**.

# Generalization and Remaining Problems

- Our proof is based on  $L^2$  techniques, hence our theorem can be easily generalized to the following cases
  - Compact Riemann surfaces
  - Riemannian manifolds
  - Compact flat surfaces with Delaunay triangulation
  - ...
- In the graph case, when  $\cap_i \pi_1(G_i) = \{1\}$ , we have an algorithm to calculate the limiting metric, and the limiting metric can be interpreted via equilibrium measure on  $\partial$  of universal cover  $\overline{G}$ .
- It is unknown how to calculate the limiting metric efficiently in other cases, or what properties they would have.

- $S$ : a compact Riemann surface
- $\Omega^1(S)$ : Space of holomorphic 1-forms on  $S$
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- $d_c^S = \sum_i |\omega_i|^2$
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$ : infinite tower of finite regular covers.
- $d_i$ : Riemannian metrics on  $S$  whose pull-back on  $S_i$  are the  $d_c^{S_i}$

## Theorem [Baik-Shokrieh-W]

$d_i$  converges uniformly as a tensor.



# Riemannian Manifolds

- $M$ : a compact Riemannian manifold
- $\mathcal{H}^1(M)$ : Space of harmonic 1-forms on  $M$
- $\{\omega_i\}$ : Orthonormal basis of  $\mathcal{H}^1(M)$
- $d_c^M = \sum_i |\omega_i|^2$
- $M \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$ : infinite tower of finite regular covers.
- $d_i$ : Riemannian metrics on  $M$  whose pull-back on  $M_i$  are the  $d_c^{M_i}$

## Theorem

$d_i$  converges uniformly as a tensor.

# Piecewise Euclidean Surfaces

- $X$ : a closed flat surface with finitely many cone points, with a Delaunay triangulation  $\mathcal{T}$ .
- $\mathcal{H}(X)$ : Space of discrete harmonic 1-forms on  $X$ , defined using the cotangent formula.
- $l_c^X(e) = \frac{1}{l(e)} \sup_{\|\alpha\| \leq 1, \alpha \in \mathcal{H}} |\alpha(e)|$ , the norm is also defined using cotangent formula.
- $X \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$ : infinite tower of finite regular covers.
- $l_i$ : Functions on edges of  $X$  whose pull-back to  $X_i$  are the  $l_c^{X_i}$ .

## Theorem

$l_i$  converges on all edges, and the limit satisfies the triangle inequality.

# Sketch of Proof

- Step 1: Find the limiting metric
- Step 2: Upper bound
- Step 3: Lower bound via Lück's approximation

## Step 1: Find the Limiting Metric

Let  $G'$  be the cover of  $G$  corresponding to  $\cap_i(\pi_1(G_i))$ .

Define:

$$l_c^{G'} = \frac{1}{l(e)} \sup_{\|\alpha\| \leq 1, \alpha \in \mathcal{H}_{L^2}} |\alpha(e)|$$

Here  $\mathcal{H}_{L^2}$  is the space of harmonic 1-forms with finite  $L^2$  norm.  
Then  $l_\infty$  pulls back to  $l_c^{G'}$  on  $G'$ .

## Step 2: Upper Bound

For each  $e \in E(G)$ , let  $e_i$  be a lift on  $G_i$ ,  $e'$  a lift on  $G'$ .

Let  $B_{G_i}(e_i, R)$  and  $B_{G'}(e', R)$  be the  $R$ -neighborhood of  $e_i$  and  $e'$ .

Because  $\pi_1(G') = \cap_i \pi_1(G_i)$ , for large enough  $i$  these two are isometric. Let  $B'(R)$  be these two neighborhoods with their boundaries collapsed to a single point.

By the effective resistance interpretation,

$$l_i(e) = l_c^{G_i}(e_i) \leq l_c^{B'(R)}(e_i) = l_c^{B'(R)}(e')$$

However

$$\lim_{R \rightarrow \infty} l_c^{B'(R)}(e') = l_c^{G'}(e') = l_\infty(e)$$

Hence

$$\limsup_{i \rightarrow \infty} l_i(e) \leq l_\infty(e)$$

# Lower Bound via Lück's Approximation

## Theorem [Lück]

$X$ : CW complex with group  $\Gamma$ -action which is free and cellular,  
 $X/\Gamma$  finite,  $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots$  finite index normal subgroups of  $\Gamma$ ,  
 $\cap_i \Gamma_i = 1$ . Then

$$\lim_{i \rightarrow \infty} \frac{b_j(X/\Gamma_i)}{[\Gamma : \Gamma_i]} = b_j^{L^2}(X)$$

( $b_j$ :  $j$ -th Betti number.  $[\Gamma : \Gamma_i]$  index of  $\Gamma_i$  as a subgroup of  $\Gamma$ .)

$$\sum_{e \in E(G)} l_i(e) = \frac{2b_1(G_i)}{[\pi_1(G) : \pi_1(G_i)]}$$

$$\sum_{e \in E(G)} l_\infty(e) = 2b_1^{L^2}(G')$$

# Proof of Our Theorem

From step 2:

$$\limsup_{i \rightarrow \infty} l_i(e) \leq l_\infty(e)$$






From step 3:

$$\lim_{i \rightarrow \infty} \sum_{e \in E(G)} l_i(e) = \sum_{e \in E(G)} l_\infty(e)$$

Because  $E(G)$  is finite,

$$\lim_{i \rightarrow \infty} l_i(e) = l_\infty(e)$$

# References

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