Compulsory Assignment 3

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1 Problem Description

We consider a game league playing a tournament with N teams, numbered by $1, 2, \ldots, N$. Each game is played by two team. Winning team gets 2 points, while losing team gets 0. Both teams get 1 point in case of a tie.

Assume we are in the middle of the tournament, i.e., we have a score list for each team. Besides, we have a future match list.

We want to find out if it is possible for team 1 to win the tournament, i.e., team 1's point strictly larger than any other team.

2 Preparation for ILP Modelling

We assume that team 1 wins all its future matches. Then we get the final score, denoted as T, for team 1.

Denote by L the list of remaining matches, i.e., future matches that team 1 is not involved. There is no need to consider the matches involving team 1, as all team 1 will get 0 points by our assumption.

Denote by s_i the current score of team i for $i = 2, 3, \ldots, N$.

We introduce decision variables $x_{m,t} \in \{0, 1, 2\}$, standing for the score that team t got in the match m where $m \in L$ and t is one of the two teams in m. For convienince, we denote $t_1(m)$ and $t_2(m)$ as the two teams in m. We can treat t_1 as a map from L to $\{2, 3, \ldots, N\}$, and the same goes for t_2.

Then we have team i's final score is $s_i + \sum_{m:t_1(m)=i \vee t_2(m)=i} x_{m,i}$, for $i = 2, 3, \ldots, N$.

3 Completion for the ILP

To guarantee team 1 to be the unique winner, its final score must be largest. Then we get contraints

$$s_i + \sum_{m:t_1(m)=i} x_{m,i} + \sum_{m:t_2(m)=i} x_{m,i} \leq T - 1$$

for $i = 2, 3, \ldots, N$.

It's easy to see for each match $m \in L$, we get contraint

$$x_{m,t_1(m)} + x_{m,t_2(m)} = 2$$

as team $t_1(m)$ and team $t_2(m)$ will get total score of 2 no matter what the result of the match will be.

To sum up, we model the ILP as following,

$$
\begin{aligned}
i \in \{2, \ldots, N\} : & s_i + \sum_{m:t_1(m)=i} x_{m,i} + \sum_{m:t_2(m)=i} x_{m,i} \leq T - 1 \\
m \in L : & x_{m,t_1(m)} + x_{m,t_2(m)} = 2 \\
m \in L : & x_{m,t_1(m)}, x_{m,t_2(m)} \in \{0, 1, 2\}
\end{aligned}
$$

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As we focus on the feasiblity of the problem, there is no need to consider the "maximum" function when we look at the standard form,

$$
\begin{aligned}
i \in \{2, \ldots, N\} : & \sum_{m:t_1(m)=i} x_{m,i} + \sum_{m:t_2(m)=i} x_{m,i} \leq T - 1 - s_i \\
m \in L : & x_{m,t_1(m)} + x_{m,t_2(m)} \leq 2 \\
m \in L : & - x_{m,t_1(m)} - x_{m,t_2(m)} \leq -2 \\
m \in L : & x_{m,t_1(m)}, x_{m,t_2(m)} \geq 0 \\
m \in L : & x_{m,t_1(m)}, x_{m,t_2(m)} \in \mathbb{Z}
\end{aligned}
$$

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Observe that the coefficient of any variable is 1, 0, or -1. Hence the problem is totally unimodular. Then the basic solutions of the problem consists of integers. The problem can be solved by simplex algorithm.