Fuzziness-Based Three-Way Decision With Neighborhood Rough Sets Under the Framework of Shadowed Sets

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***Abstract*—Currently, three-way decision with neighborhood rough sets (3WDNRS) is widely used in many fields. The core of 3WDNRS is to calculate threshold pairs to divide a neighborhood space into three pairwise disjoint regions. The majority of research on 3WDNRS mainly aims to calculate thresholds with the given risk parameters to minimize the misclassification cost. However, in practical applications, risk parameters are often subjectively determined based on expert experience. This makes it challenging to accurately obtain the thresholds in 3WDNRS. To solve this problem, fuzziness is introduced into 3WDNRS to provide a new perspective on 3WD theory. First, a shadowed set framework is constructed, named three-way approximations based on shadowed sets (3WA-SS). Based on 3WA-SS, a data-driven adapted neighborhood (DAN) is constructed. Then, an improved fuzziness-based 3WDNRS (F′-3WDNRS) is further proposed and optimized by minimizing uncertainty change to obtain a more reasonable threshold pair based on DAN. Finally, extensive experiments are conducted on our proposed model, and the results show that F′-3WDNRS is effective and reliable for making decisions.**

***Index Terms*—Fuzziness, neighborhood rough sets (NRS), shadowed sets, three-way decision (3WD), uncertainty.**

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I. Introduction

Granular computing (GrC) [1], [2], [3] provides solutions to complex problems by emulating human cognition, which is an essential part of machine learning and data mining. From different perspectives, rough sets [4], fuzzy sets [5], quotient spaces [6], and ganular balls [7] are the four main models of GrC. The rough sets theory [4], proposed by Pawlak, is a data analysis theory for dealing with uncertain information. At present, this approach has been successfully used in decision analysis [8], machine learning [9], data mining [10], cloud computing [11], etc. The classical rough sets is defined on the basis of equivalence relation. To process continuous data, neighborhood rough sets (NRS) [12] adopt the concept of neighborhood granulation and metrics to transform the equivalence relation into a tolerance relation. Equivalence classes are used to construct the basic granules in classical rough sets, while neighborhood relations are used to construct the basic granules in NRS. Xu et al. [13] defined a supporting inclusion function to construct a locally generalized multigranulation NRS model. Sun et al. [14] combined fuzzy multi-NRS with binary whale optimization algorithm for unbalanced data to construct a new two-stage feature selection method. Bai et al. [15] introduced NRS into multivariate variational mode decomposition, and presented a novel multiattribute prediction method. Pan and Xu [16] created the weighted dominance-based NRS by assigning different weights to conditional attributes.

The three-way decision (3WD) theory proposed by Yao [17] is a new method for dealing with uncertain problems. The fundamental principle of 3WD is the division of a universe into three distinct regions, which is a refinement study of the classical two-way decision theory. Recently, by integrating related theories, i.e., formal concept analysis [18], rough sets [19], hesitant fuzzy sets [20], and cognitive computing [21], considerable achievements have been made with the 3WD theory. Presently, 3WD is widely applied in a variety of domains, including recommender systems [22], medical treatment [23], clustering analysis [24], face recognition [25], multiattribute decision-making [26], [27], [28], etc. Many studies have investigated 3WD with NRS (3WDNRS) in a wide range of areas. Huang et al. [29] built a generalized three-way neighborhood decision model by providing each of the target object distribution interval value loss function. Chu et al. [24] proposed a three-way clustering algorithm with NRS to improve decision-making efficiency. Yang et al. [30] investigated a sequential 3WD strategy with integrative multi-granularity in neighborhood system. Zhang et al. [31] improved the conditional neighborhood entropy and established three-way neighborhood entropies by means of granular structures with three levels.

In actual decision-making environments [32], [33], decision information is generally uncertain. Current approaches to 3WD presented by Yao [34] are summed up in three types: 1) minimum cost, 2) minimum distance, and 3) uncertainty invariance under the general framework of the 3WD theory. The threshold pair (α, β) in 3WD is used to divide a universe into three pairwise disjoint regions, namely, the positive region, negative region, and boundary region, respectively. The core of 3WD-NRS is to calculate threshold pair to divide a neighborhood space into three pairwise disjoint regions. Current research on 3WDNRS focuses on determining thresholds through utilizing provided risk parameters for decision-making to minimize misclassification cost. However, in practical applications, the risk parameters are determined subjectively by the judgement of experts. This makes it extremely challenging to accurately acquire risk parameters using known decision information. In addition, some decision-making situations are not required cost concerns. To address the above issues, it is beneficial to introduce fuzziness into 3WDNRS, which provides a novel viewpoint on the 3WD theory. In this article, a shadowed set framework is constructed from the perspective of fuzziness, named three-way approximations based on shadowed sets (3WA-SS). Based on the 3WA-SS framework, a data-driven adapted neighborhoods (DAN) is constructed. Then, a novel fuzziness-based 3WDNRS model is proposed that minimizes uncertainty change to obtain more reasonable threshold pairs without subjective parameters. Moreover, improved fuzziness-based 3WDNRS is further proposed, and is optimized based on DAN.

The major contributions of this article are summarized as follows.

1) Three-way approximations based on the shadowed sets framework are proposed with an iterative optimization method, which provides interpretability for α and β.
2) Based on the idea of a shadowed neighborhood, an adapted neighborhood is constructed, that is, DAN.
3) The fuzziness-based 3WDNRS model with the DAN model is proposed and optimized to maintain uncertainty invariance based on the 3WA-SS framework to obtain more reasonable (α, β).

The rest of this article is organized as follows. Section II reviews related preliminary definitions. In Section III, the 3WA-SS framework is constructed by minimizing uncertainty change. In Section IV, based on the 3WA-SS framework, the DAN is constructed. Then, the fuzziness-based 3WDNRS with DAN is proposed to obtain a more reasonable (α, β) based on the 3WA-SS framework, named F′-3WDNRS. The relevant experiments for verifying the viability and rationality of F′-3WDNRS are described in Section V. Finally, Section VI concludes this article.

II. Preliminaries

In this section, some required concepts related to shadowed sets and NRS are presented to simplify the structure of full article. Let $\mathrm{NS} = (U, C \cup D, V, f, \delta^*)$ denote a neighborhood decision system, where $U = \{x_1, x_2, \ldots, x_n\}$ denotes a collection of objects named universe, C and D are the nonempty finite sets of conditional attribute and decision attribute, respectively, V denotes the value collection of all attributes and $f : U \times C \to V$ denotes an information function, and δ^* denotes the neighborhood.

Definition 1. (Neighborhood) [12]: Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system, $\forall x_j \in U$, the neighborhood $\delta^*(x_j)$ of x_j is denoted as follows:

$$\delta^*(x_j) = \{x_i | x_i \in U, d(x_i, x_j) \leq \delta\} \tag{1}$$

where $d(x_i, x_j)$ is the distance between x_i and x_j, and δ $(\delta \geq 0)$ is the radius of the neighborhood.

Definition 2. (Neighborhood rough sets) [12]: Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system, $X \subseteq U$, the lower and upper approximations of X are denoted as follows:

$$\underline{N}X = \{x \in U | \delta^*(x) \subseteq X\} \tag{2}$$

$$\overline{N}X = \{x \in U | \delta^*(x) \cap X \neq \emptyset\}. \tag{3}$$

Then, the three regions of X are denoted as follows:

$$POS(X) = \underline{N}X \tag{4}$$

$$NEG(X) = \left(\overline{N}X\right)^{\mathrm{C}} = U - \overline{N}X \tag{5}$$

$$BND(X) = \overline{N}X - \underline{N}X. \tag{6}$$

Definition 3. (Probabilistic neighborhood rough sets) [35]: Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system. $\forall x_i \in U$, $\mu(x_i)$ is the neighborhood membership degree of x_i, and (α, β) denotes a threshold pair, where $0 \leq \beta \leq \alpha \leq 1$. The lower and upper approximations of X are denoted as follows:

$$\underline{N}X^{(\alpha,\beta)} = \{x_i \in U | \mu(x_i) \geq \alpha\} \tag{7}$$

$$\overline{N}X^{(\alpha,\beta)} = \{x_i \in U | \mu(x_i) > \beta\}. \tag{8}$$

Then, the three regions of X are denoted as follows:

$$POS^{(\alpha,\beta)}(X) = \underline{N}X^{(\alpha,\beta)} = \{x_i \in U | \mu(x_i) \geq \alpha\} \tag{9}$$

$$NEG^{(\alpha,\beta)}(X) = \left(\overline{N}X^{(\alpha,\beta)}\right)^{\mathrm{C}} = \{x_i \in U | \mu(x_i) \leq \beta\} \quad (10)$$

$$BND^{(\alpha,\beta)}(X) = \overline{N}X^{(\alpha,\beta)} - \underline{N}X^{(\alpha,\beta)} = \{x_i \in U | \alpha < \mu(x_i) < \beta\}. \quad (11)$$

Definition 4. (Shadowed sets) [36]: Given two real numbers α and β which satisfy $0 \leq \beta < \alpha \leq 1$. The mapping from the nonempty finite universe U to the set $\{0, [0, 1], 1\}$ is used to define the shadow set S, and the mapping $S : U \to \{0, [0, 1], 1\}$ is denoted as follows:

$$S_\mu(x) = \begin{cases} 1, & \mu(x) \geq \alpha \\ [0, 1], & \beta < \mu(x) < \alpha \\ 0, & \mu(x) \leq \beta. \end{cases} \quad (12)$$

Several common shadowed sets are shown in Section S1 of Supplementary file, which will be used in the comparative experiments in Section V.

Definition 5. (Fuzziness) [37]: Let $F(U)$ be the family of all fuzzy sets on U, A, and B be any two distinct fuzzy sets on U. A mapping $f : F(U) \to [0, 1]$ conforms to the expressions listed as follows.

1) $f(A) = 0$, iff $A \in P(U)$.
2) $f(A) = 1$, iff $A(x_i) = 1/2$, $\forall x_i \in U$.
3) If $B(x_i) \geq A(x_i) \geq 1/2$ or $B(x_i) \leq A(x_i) \leq 1/2$, $\forall x_i \in U$, then $f(B) \leq f(A)$,

where $f(\cdot)$ is the fuzziness of a fuzzy set.

Definition 6. (Average fuzziness) [38]: Let X be a target fuzzy set on U, and the average fuzziness of fuzzy sets is denoted as follows.

The average fuzziness is expressed with a discrete universe U

$$\bar{f}(X) = \frac{1}{|U|} \sum_{x \in U} f(x). \quad (13)$$

The average fuzziness is expressed with a continuous universe U

$$\bar{f}(X) = \frac{1}{|b-a|} \int_a^b f(x). \quad (14)$$

III. Construction of the Three-Way Approximations Based on Shadowed Sets

In this section, a framework of shadowed sets is proposed by transforming $\mu(x)$ to set $\{\mu_{\min}, m, \mu_{\max}\}$ from the perspective of uncertainty and the derivation process of $\{\mu_{\min}, m, \mu_{\max}\}$ is detailed.

Definition 7. (Three-way approximations based on shadowed sets, 3WA-SS): Given two different numbers α and β with $0 \leq \beta < \alpha \leq 1$. A mapping from the target concept X to the set $\{\mu_{\min},\ m,\ \mu_{\max}\}$ is used to define a shadowed set S_μ^T. The mapping $S_\mu^T : X \to \{\mu_{\min}, m, \mu_{\max}\}$ is denoted as follows:

$$S_\mu^T(X) = \begin{cases} \mu_{\max}, & \mu(x) \geq \alpha \\ m, & \beta < \mu(x) < \alpha \\ \mu_{\min}, & \mu(x) \leq \beta. \end{cases} \quad (15)$$

The 3WA-SS framework is constructed to approximate a fuzzy target concept:

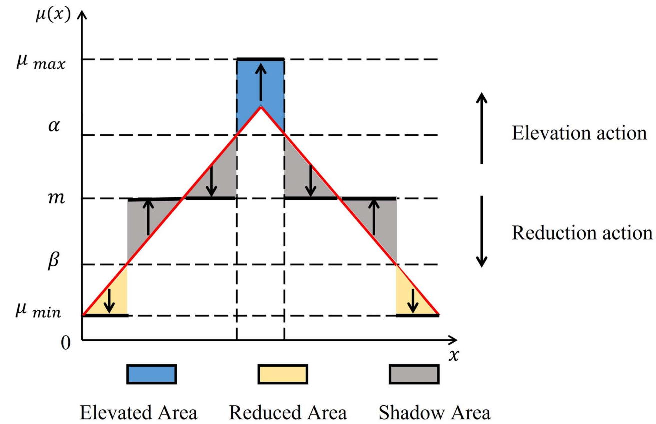

Fig. 1. Three-way approximations based on shadowed sets.

1) when the membership degree satisfies $\mu(x) \leq \beta$, $\mu(x)$ is reduced to $\mu_{\min}$, which implies that x can be assigned to the negative region;
2) when the membership degree satisfies $\mu(x) \geq \alpha$, $\mu(x)$ is elevated to $\mu_{\max}$, which implies that x can be assigned to the positive region;
3) when the membership degree satisfies $\beta < \mu(x) < \alpha$, $\mu(x)$ is transformed into m, which implies that x can be assigned to the boundary region.

Fig. 1 shows the 3WA-SS. The upward arrow indicates the elevation action, and the downward arrow indicates the reduction action. The fuzziness of the 3WA-SS framework is equivalent to the total fuzziness of these areas, that is,

$$\begin{aligned} f(S_\mu^T(X)) &= f(\text{Elevated area}) + f(\text{Reduced area}) \\ &\quad + f(\text{Shadow area}) \\ &= \sum_{\mu(x)\geq\alpha} 4\mu_{\max}(1-\mu_{\max}) + \sum_{\mu(x)\leq\beta} 4\mu_{\min}(1-\mu_{\min}) \\ &\quad + \sum_{\beta<\mu(x)<\alpha} 4m(1-m) \\ &= 4\mu_{\max}(1-\mu_{\max})\,|\{x|\mu(x)\geq\alpha\}| \\ &\quad + 4\mu_{\min}(1-\mu_{\min})\,|\{x|\mu(x)\leq\beta\}| \\ &\quad + 4m(1-m)\,|\{x|\beta<\mu(x)<\alpha\}| \end{aligned}$$

where $|\cdot|$ is the cardinality of a set.

The calculation of $\mu_{\min}$ and $\mu_{\max}$ mainly adopts the idea of optimal variance [39] proposed in our previous works, which is expressed by the mathematical properties of a numerical collection. The optimal variance is defined in Definition S1 of Supplementary file, and Example S1 shows the calculation process of $\mu_{\min}$ and $\mu_{\max}$, which is shown in Fig. S1.

In the process of constructing 3WA-SS, there are two corresponding decision actions for each object x that is uncertain, that is, elevation and reduction. For any object x, when $\mu(x) \geq \alpha$, $\mu(x)$ is elevated to $\mu_{\max}$, when $\mu(x) \leq \beta$, $\mu(x)$ is reduced to $\mu_{\min}$. When $\beta < \mu(x) < \alpha$, $\mu(x)$ is elevated or reduced to m. Therefore, different decision actions will produce uncertainty

TABLE I
UNCERTAINTY CHANGE FOR FOUR DECISION ACTIONS

Actions	Membership degree	Shadowed sets	Uncertainty change
a_e	$\mu(x) \geq \alpha$	$\mu_{\max}$	Ul_e
a_r	$\mu(x) \leq \beta$	$\mu_{\min}$	Ul_r
a_{se}	$\beta < \mu(x) < m$	m	Ul_{se}
a_{sr}	$m < \mu(x) < \alpha$	m	Ul_{sr}

change. In this subsection, the idea of minimizing uncertainty change is used to determine (α, β).

Let Actions $= \{a_e, a_r, a_{se}, a_{sr}\}$ represent the collection of each available decision action for an uncertain object. Uncertainty change generated by each decision action is shown in Table I. $Ul = \{Ul_e, Ul_r, Ul_{se}, Ul_{sr}\}$ denotes uncertainty change caused by the corresponding decision actions.

For $x \in U$, uncertainty change function is obtained as follows:

$$Ul(a|x) = |U_F - U_I| \tag{16}$$

where U_I denotes the initial uncertainty before making a decision action for x, and U_F denotes the final uncertainty after making a decision action for x. Different decision actions are indicated by the parameter $a \in$ Actions. In addition, the total uncertainty change of each object in U is $\sum_{x\in U} Ul(a|x)$. By constructing the minimization function of the total change, the threshold pair (α, β) is calculated as follows:

$$\arg\min_{(\alpha,\beta)} \sum_{x\in U} |U_F - U_I|. \tag{17}$$

In the 3WA-SS framework, the total uncertainty change can be minimized according to the minimization of uncertainty change of each object in U. Assume that an object in a fuzzy set has nonnegative uncertainty change when the 3WA-SS framework is constructed, i.e., $Ul(a_r|x) > 0$, $Ul(a_e|x) > 0$, $Ul(a_{sr}|x) > 0$, and $Ul(a_{se}|x) > 0$.

The derivation of α and β for $\mu(x) \geq m$ is provided in Section S3 in the Supplementary file. According to the derivation, α and β are calculated as follows:

$$\alpha = \frac{1 + \sqrt{1 - 2\left(\mu_{\max}(1-\mu_{\max}) + m(1-m)\right)}}{2} \tag{18}$$

$$\beta = \frac{1 - \sqrt{1 - 2\left(\mu_{\min}(1-\mu_{\min}) + m(1-m)\right)}}{2}. \tag{19}$$

According to Definition 6, the average fuzziness of the target concept X and its three-way approximation $S_\mu^T(X)$ are computed as follows:

$$\bar{f}(X) = \frac{4\sum_{x\in U}\mu(x)(1-\mu(x))}{|U|}$$

$$\begin{aligned}\bar{f}(S_\mu^T(X)) &= \overline{f}(\text{Elevated area}) + \overline{f}(\text{Reduced area}) \\ &\quad + \overline{f}(\text{Shadow area}) \\ &= \frac{4\sum_{\mu(x)\geq\alpha}\mu_{\max}(1-\mu_{\max})}{|\{x|\mu(x)\geq\alpha\}|} + \frac{4\sum_{\mu(x)\leq\beta}\mu_{\min}(1-\mu_{\min})}{|\{x|\mu(x)\leq\beta\}|} \\ &\quad + \frac{4\sum_{\alpha<\mu(x)<\beta} m(1-m)}{|\{x|\beta<\mu(x)<\alpha\}|}.\end{aligned}$$

To minimize the fuzziness change, the following derivation is performed to calculate m:

$$\begin{aligned}&\bar{f}(X) - \bar{f}(S_{\underline{\mu}}^T(X)) = 0 \\ &\Leftrightarrow \bar{f}(X) = \bar{f}(S_\mu^T(X)) \\ &\Leftrightarrow \frac{4\sum_{x\in U}\mu(x)(1-\mu(x))}{|U|} = \frac{4\sum_{\mu(x)\geq\alpha}\mu_{\max}(1-\mu_{\max})}{|\{x|\mu(x)\geq\alpha\}|} \\ &+ \frac{4\sum_{\mu(x)\leq\beta}\mu_{\min}(1-\mu_{\min})}{|\{x|\mu(x)\leq\beta\}|} + \frac{4\sum_{\alpha<\mu(x)<\beta} m(1-m)}{|\{x|\beta<\mu(x)<\alpha\}|} \\ &\Leftrightarrow m(1-m) = \frac{\overline{f}(X)}{4} - \mu_{\max}(1-\mu_{\max}) - \mu_{\min}(1-\mu_{\min}) \\ &\Leftrightarrow m_1 = \frac{1-\sqrt{1-\overline{f}(X)-4(\mu_{\max}(1-\mu_{\max})+\mu_{\min}(1-\mu_{\min}))}}{2} \text{ and} \\ &m_2 = \frac{1+\sqrt{1-\overline{f}(X)-4(\mu_{\max}(1-\mu_{\max})+\mu_{\min}(1-\mu_{\min}))}}{2}.\end{aligned}$$

The two constant values m_1 and m_2 obviously satisfy $0 < m_1 \leq 0.5$, $0.5 \leq m_2 < 1$, and $m_1 + m_2 = 1$. Therefore, $\alpha = \frac{1+\sqrt{1-2(\mu_{\max}(1-\mu_{\max})+m_1m_2)}}{2}$ and $\beta = \frac{1-\sqrt{1-2(\mu_{\min}(1-\mu_{\min})+m_1m_2)}}{2}$.

When first computing $\overline{f}(X)$, objects with membership degree may be extremely close to 0 or 1; these objects can be considered as certain but are ultimately viewed as uncertain objects due to the lack of partitioning conditions. This leads to an inadequate representation of most uncertain objects, which affects the calculation of (α, β) and results in fuzziness change. Therefore, selecting a reasonable m to properly depict the majority of uncertain objects is a key issue in optimizing (α, β) through an iterative method. Objects with membership degree that are extremely close to 0 or 1 are assigned to the negative region or the positive region by an iterative method, providing more reasonable m and (α, β), denoted as m^* and (α^*, β^*). Hence, the 3WA-SS framework further has a lower fuzziness change after the iterations.

The iterative process for obtaining m^* and (α^*, β^*) is displayed in Algorithm S1 in Supplementary file, where Un_j denotes the collection of uncertain objects, $\overline{f}_j(X)$ denotes the average fuzziness, $m_{i,j}$ denotes the constant membership value, $\mu_{\min}$ and $\mu_{\max}$ denote the elevated and reduced membership value, and (α_j, β_j) denotes a pair of thresholds; i indicates the ith constant membership value, j indicates the jth iteration of 3WA-SS framework, $1 \leq j < n$, and n indicates the total number of iterations.

Theorems 1 and 2 reveal the change processes of m and (α, β) in the jth and $j+1$th iterations.

Theorem 1: For a fuzzy set $X = \frac{\mu(x_1)}{x_1} + \frac{\mu(x_2)}{x_2} + \ldots + \frac{\mu(x_n)}{x_n}$, $m_{i,j}$ $(1 \leq j < n \; and \; i = 1, 2)$ denotes a constant value, where j represents the jth iteration of the 3WA-SS framework, $0 < m_{1,j} \leq 0.5$, and $0.5 \leq m_{2,j} < 1$. Then, $m_{1,j} \leq m_{1,j+1}$ and $m_{2,j} \geq m_{2,j+1}$ hold.

Proof: Based on the given information, we have

$$\begin{aligned}m_{1,j+1} - m_{1,j} &= \frac{\sqrt{1-\overline{f}_j(X)-4(\mu_{\max}(1-\mu_{\max})+\mu_{\min}(1-\mu_{\min}))}}{2} \\ &- \frac{\sqrt{1-\overline{f}_{j+1}(X)-4(\mu_{\max}(1-\mu_{\max})+\mu_{\min}(1-\mu_{\min}))}}{2}.\end{aligned}$$

Let $h_j = \bar{f}_j(X) + 4(\mu_{\max}(1-\mu_{\max}) + \mu_{\min}(1-\mu_{\min}))$.

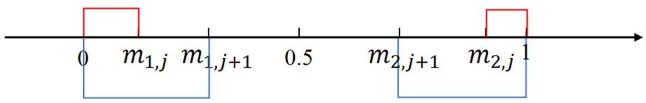

Fig. 2. Change of m in the jth and $j+1$th iteration of 3WA-SS framework.

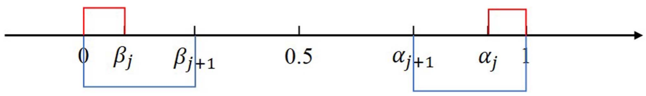

Fig. 3. Change of (α, β) in the jth and $j+1$th iteration of the 3WA-SS framework.

Then, $m_{1,j+1} - m_{1,j} = \frac{\sqrt{1-h_j}}{2} - \frac{\sqrt{1-h_{j+1}}}{2}$. For the jth iteration, if $\mu(x) \geq \alpha_{j-1}$ or $\mu(x) \leq \beta_{j-1}$, the membership degree will increase to $\mu_{\max}$ or decrease to $\mu_{\min}$ in the $j+1$th iteration, which will lead to some uncertain objects being converted to certain objects in the $j+1$th iteration. Therefore, $\text{Card}(Un_{j+1}) \leq \text{Card}(Un_j)$, $\overline{f}_{j+1}(X) \geq \overline{f}_j(X)$,and $h_{j+1} \geq h_j$; thus $m_{1,j+1} - m_{1,j} \geq 0$. Accordingly, $m_{1,j} \leq m_{1,j+1}$ holds.

Similarly, $m_{2,j} \geq m_{2,j+1}$ can be proved; therefore, there is no more tautology here. Fig. 2 shows the change of m in the jth and $j+1$th iteration of the 3WA-SS framework. □

Theorem 2: For a fuzzy set $X = \frac{\mu(x_1)}{x_1} + \frac{\mu(x_2)}{x_2} + \dots + \frac{\mu(x_n)}{x_n}$, and (α_j, β_j) $(1 \leq j < n)$ denotes a threshold pair, where j represents the jth iteration of the 3WA-SS framework. Then, $\alpha_j \geq \alpha_{j+1} \geq 0.5$ and $\beta_j \leq \beta_{j+1} \leq 0.5$ hold.

Proof: Based on the given information, we have

$$\alpha_j - \alpha_{j+1} = \frac{\sqrt{1 - 2\left(\mu_{\max}(1-\mu_{\max}) + m_{1,j}(1-m_{1,j}\right)}}{2} - \frac{\sqrt{1 - 2\left(\mu_{\max}(1-\mu_{\max}) + m_{1,j+1}(1-m_{1,j+1}\right)}}{2}.$$

According to Theorem 1, $m_{1,j} \leq m_{1,j+1} \leq 0.5$ holds. When $m_{1,j} \leq m_{1,j+1} \leq 0.5$, using the monotonicity of the mathematical formula $y = x(1-x)$, it can be obtained that $m_{1,j}(1-m_{1,j}) \leq m_{1,j+1}(1-m_{1,j+1})$. Obviously, $\alpha_j - \alpha_{j+1} \geq 0$ holds. Accordingly, $\alpha_j \geq \alpha_{j+1} \geq 0.5$ holds.

Similarly, $\beta_j \leq \beta_{j+1} \leq 0.5$ can be proved; therefore, there is no tautology here. Fig. 3 shows the change of (α, β) in the jth and $j+1$th iteration of 3WA-SS framework.

IV. Construction of the 3WDNRS Under 3WA-SS Framework

In this section, under the 3WA-SS framework, on one hand, the idea of shadow neighborhoods is implemented to obtain an adaptive neighborhood that is not based on the subjective radius. On the other hand, the fuzziness-based 3WD model with an adapted thresholds is constructed.

A. Data-Driven Adapted Neighborhood

The main concern in constructing neighborhood is the determination of an appropriate neighborhood radius. Normally, the neighborhood radius is determined by setting the parameters [12], [40], which is inherently subjective. To solve this problem, a data-driven approach for calculating the neighborhood radius is proposed in this subsection.

Definition 8. (Neighborhood membership degree): Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system. $\forall x_i, x_j \in U$, where $\delta^*(x_j)$ is the neighborhood of x_j, the membership degree of x_i belonging to $\delta^*(x_j)$ is defined as follows:

$$\xi(x_i^j) = e^{-\left(\frac{d(x_i, x_j)}{dc}\right)^2} \tag{20}$$

where $d(x_i, x_j)$ is the distance between x_i and the neighborhood center x_j, $x_i \neq x_j$, and dc is an adjustable parameter, which is set to $dc = 0.01|U|$. The neighborhood membership degree decreases as the distance between x_i and x_j increases.

In a neighborhood decision system, the objects away from a neighborhood center are less significant. To reduce the redundant calculation and obtain a reasonable range for neighborhood, combined with the idea of optimal variance, the concept of DAN is proposed based on the shadowed neighborhood. The union of the shadowed neighborhoods creates a tripartitioned approximation of the data distribution for three-way classification. $\forall x_i, x_j \in X$, $\xi(x_i^j)$ is the neighborhood membership of x_i and $\delta^*(x_j)$ is the neighborhood of x_j. $(\alpha^*, \ \beta^*)$ denotes the thresholds obtained under 3WA-SS framework after the iterations, $0 \leq \beta^* < \alpha^* \leq 1$. The shadowed neighborhood is denoted by setting a shadowed set mapping $N_{\xi(x_i^j)}(x_j)$ of the neighborhood membership as follows:

$$N_{\xi(x_i^j)}(x_j) = \begin{cases} \mu_{\max}, & \xi(x_i^j) \geq \alpha^* \\ m^*, & \beta^* < \xi(x_i^j) < \alpha^* \\ \mu_{\min}, & \xi(x_i^j) \leq \beta^*. \end{cases} \tag{21}$$

The mapping $N_{\xi(x_i^j)}(x_j)$ uses a step function to describe $\xi(x_i^j)$ and divides the universe into three domains; that is, $\mu_{\max}$ denotes the positive region, $\mu_{\min}$ denotes the negative region, and m^* denotes the boundary region, which constitutes the neighborhood shadow. The three regions of $N_{\xi(x_i^j)}(x_j)$ are represented by:

$$\text{POS}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j)) = \left\{x_i \in U \,\middle|\, \xi(x_i^j) \geq \alpha^*\right\} \tag{22}$$

$$\text{NEG}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j)) = \left\{x_i \in U \,\middle|\, \xi(x_i^j) \leq \beta^*\right\} \tag{23}$$

$$\text{BND}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j)) = \left\{x_i \in U \,\middle|\, \beta^* < \xi(x_i^j) < \alpha^*\right\}. \tag{24}$$

Here, $\text{POS}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$ denotes the objects which certainly belong to $\delta^*(x_j)$, $\text{NEG}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$ denotes the objects which are outside $\delta^*(x_j)$, and $\text{BND}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$ denotes the objects which belong to the shadowed neighborhood of $\delta^*(x_j)$. Fig. 4 shows the shadowed neighborhood of $\delta^*(x_j)$ from the viewpoint of fuzziness.

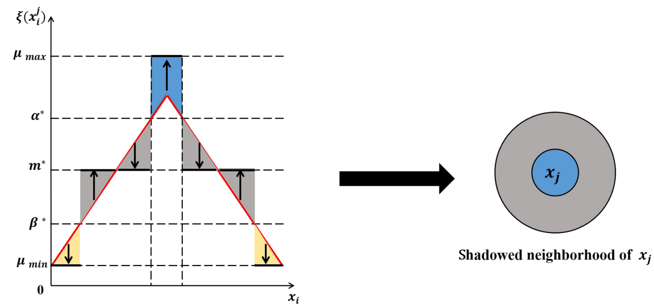

Fig. 4. Shadowed neighborhood of $\delta^*(x_j)$ from the perspective of fuzziness.

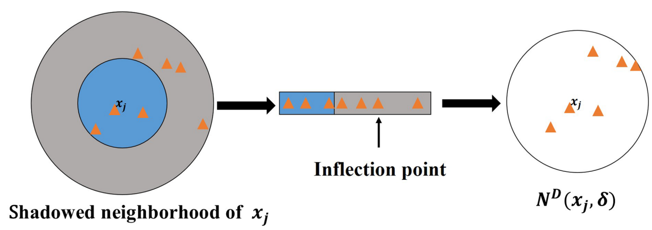

Fig. 5. DAN of x_j.

Based on the shadowed neighborhood, combined with the idea of optimal variance, the concept of DAN is proposed as follows:

Definition 9. (Data-driven adapted neighborhood): Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system. $\forall x_i, x_j \in U$, $\xi(x_i^j)$ is the membership degree of x_i, $\delta^*(x_j)$ denotes the neighborhood of x_j, $POS_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$ and $BND_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$ denote the positive region and boundary region obtained by the shadowed neighborhood. The DAN $N^D(x_j,\delta)$ is denoted as follows:

$$\begin{aligned} &N^D(x_j,\delta)\\ &= \{x_i \in POS_N^{(\alpha^*,\beta^*)}(\delta^*(x_j)) \cup BND_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))\\ &\quad |\xi(x_i^j) > \xi(x_t^j)\} \end{aligned} \tag{25}$$

where $t = \arg\max\{\Delta^I\}$ and Δ^I is the set of optimal deviation of $BND_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$.

Definition 10. (Data-driven membership degree): Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system, $X \subseteq U$. $\forall x_i, x_j \in X$, $\xi(x_i^j)$ is the membership degree of x_i and $\delta^*(x_j)$ denotes the neighborhood of x_j. $\mu_X^D(x_j)$ is the data-driven membership degree of x_j, which is denoted as follows:

$$\mu_X^D(x_j) = \frac{\sum\limits_{x_i \in N^D(x_j,\delta)} \xi(x_i^j)}{Card(N^D(x_j,\delta))} \tag{26}$$

$G_{\mu_X^D} = \{\mu_X^D(x_1), \mu_X^D(x_2), \ldots, \mu_X^D(x_n)\}$ denotes the set of data-driven membership degree.

Fig. 5 shows the construction process for the DAN. Based on the shadowed neighborhood, combined with the idea of optimal variance, the DAN is constructed. The triangles represent the elements in a neighborhood. In the neighborhood of x_j, the blue part represents $\mathrm{POS}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$, and the gray part represents $\mathrm{BND}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$. The inflection point is obtained by using the idea of optimal variance in $\mathrm{BND}_N^{(\alpha^*,\beta^*)}(\delta^*(x_j))$. The objects with neighborhood membership degree are greater than or equal to the inflection point belong to the DAN.

Algorithm S2 is provided in Supplementary file, which shows the construction of DAN. Step 2–6 calculate the optimal threshold according to the idea of the 3WA-SS framework, then combines the optimal variance and Definition 9 to construct the neighborhood. The runtime of the algorithm is $O(n)$.

B. Fuzziness-Based 3WDNRS With Data-Driven Neighborhood

The 3WA-SS framework is used to provide a more reasonable threshold pair for the 3WD via an iterative method. In this subsection, 3WDNRS is constructed under the 3WA-SS framework, which is a data-driven 3WD model from the perspective of uncertainty. $\forall x_i \in X$, $\mu_X^D(x_i)$ is the data-driven membership degree of x_i, and $(\alpha^*,\ \beta^*)$ denotes the thresholds obtained under 3WA-SS framework after the iterations, $0 \le \beta^* < \alpha^* \le 1$. A mapping $S_\mu^\delta : X \to \{\mu_{\min}, m^*, \mu_{\max}\}$ on universe U is denoted as follows:

$$S_\mu^\delta(X) = \begin{cases} \mu_{\max} & \mu_X^D(x_i) \ge \alpha^* \\ m^* & \beta^* < \mu_X^D(x_i) < \alpha^* \\ \mu_{\min} & \mu_X^D(x_i) \le \beta^*. \end{cases} \tag{27}$$

Corresponding to any membership degree $\mu_X^D(x_i)$, three-way approximations for the target concept can be constructed as follows.

1) When the membership degree satisfies $\mu_X^D(x_i) \le \beta^*$, $\mu_X^D(x_i)$ is reduced to $\mu_{\min}$, which implies that x can be assigned to the negative region.

2) When the membership degree satisfies $\mu_X^D(x_i) \ge \alpha^*$, $\mu_X^D(x_i)$ is elevated to $\mu_{\max}$, which implies that x can be assigned to the positive region.

3) When the membership degree satisfies $\beta^* < \mu_X^D(x_i) < \alpha^*$, $\mu_X^D(x_i)$ is transformed into m^*, which implies that x can be assigned to the boundary region.

Based on the 3WA-SS framework, fuzziness-based 3WDNRS (F-3WDNRS) is presented, and then the three regions are obtained. The detailed definitions are as follows.

Definition 11. (F-3WDNRS): Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system. $\forall x_i \in U$, $\mu(x_i)$ denotes the neighborhood membership of x_i, $(\alpha^*,\ \beta^*)$ denotes the threshold pair obtained under the 3WA-SS framework after the iterations, $0 \le \beta^* < \alpha^* \le 1$, the lower and upper approximations of X are denoted as follows:

$$\underline{N}X^I = \{x_i \in U | \mu(x_i) \ge \alpha^*\} \tag{28}$$

$$\overline{N}X^I = \{x_i \in U | \mu(x_i) > \beta^*\}. \tag{29}$$

Then, the three region of X are denoted as follows:

$$\mathrm{POS}^I(X) = \underline{N}X^I = \{x_i \in U | \mu(x_i) \ge \alpha^*\} \tag{30}$$

$$\mathrm{NEG}^I(X) = \left(\overline{N}X^I\right)^{\mathrm{C}} = \{x_i \in U | \mu(x_i) \le \beta^*\} \tag{31}$$

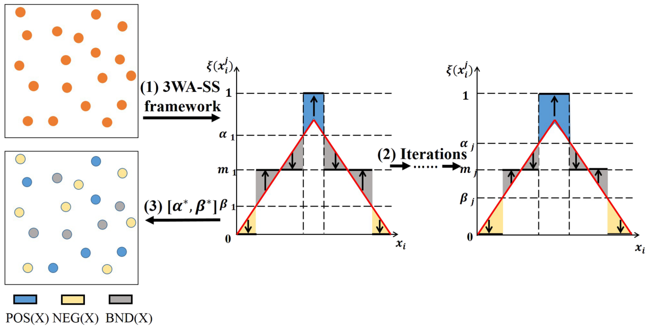

Fig. 6. Construction process of F-3WDNRS.

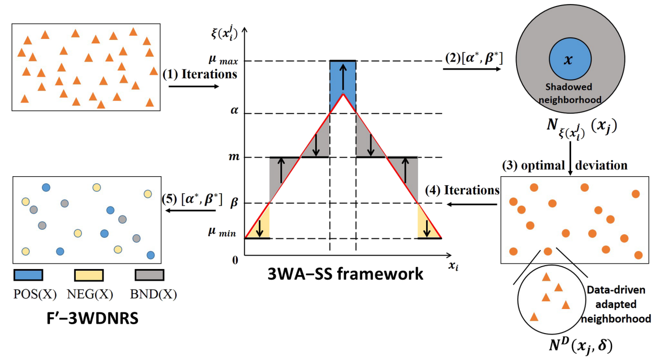

Fig. 7. Procedure for the construction of F′-3WDNRS.

$$\mathrm{BND}^I(X) = \overline{N}X^I - \underline{N}X^I = \{x_i \in U | \alpha^* < \mu(x_i) < \beta^*\}. \tag{32}$$

Fig. 6 shows the construction process of F-3WDNRS. The circles represent the different neighborhood granules. First, the thresholds are obtained by our iterative method to minimize uncertainty change between neighborhood space and its 3WA-SS framework. Second, the obtained thresholds are used to divide the neighborhood space into three decision regions. The 3WA-SS framework is used to provide a more reasonable threshold pair for 3WDNRS, which is a data-driven 3WD model.

The improved fuzziness-based 3WDNRS is further proposed by incorporate DAN into F-3WDNRS, named F′-3WDNRS, then the three corresponding regions are obtained. The detailed definitions of F′-3WDNRS are as follows:

Definition 12. (F′-3WDNRS): Let $NS = (U, C \cup D, V, f, \delta^*)$ be a neighborhood decision system. $\forall x_i \in U$, $\mu_X^D(x_i)$ denotes the DAN membership of x_i, $(\alpha^*, \ \beta^*)$ denotes the thresholds obtained under the 3WA-SS framework after the iterations, $0 \le \beta^* < \alpha^* \le 1$, the lower and upper approximations of X are denoted as follows:

$$\underline{N}X^{II} = \{x_i \in U | \mu_X^D(x_i) \ge \alpha^*\} \tag{33}$$

$$\overline{N}X^{II} = \{x_i \in U | \mu_X^D(x_i) > \beta^*\}. \tag{34}$$

Then, the positive, negative, and boundary regions of X are denoted as follows:

$$POS^{II}(X) = \underline{N}X^{II} = \{x_i \in U | \mu_X^D(x_i) \ge \alpha^*\} \tag{35}$$

$$NEG^{II}(X) = \left(\overline{N}X^{II}\right)^{\mathrm{C}} = \{x_i \in U | \mu_X^D(x_i) \le \beta^*\} \tag{36}$$

$$BND^{II}(X) = \overline{N}X^{II} - \underline{N}X^{II} = \{x_i \in U | \alpha^* < \mu_X^D(x_i) < \beta^*\}. \tag{37}$$

Algorithm S3 is provided in Supplementary file, it describes the construction process of F′-3WDNRS, including the following four phases. In the first phase, the three regions of a shadowed neighborhood can be obtained. In the second phase, according to Algorithm S2, the DAN is constructed and $G_{\mu_X^D}$ is calculated. In the third phase, based on $G_{\mu_X^D}$, m_1^*, m_2^*, and (α^*, β^*) can be obtained through the 3WA-SS framework, which is established to supply thresholds for F′-3WDNRS. In the final phase, the positive, negative, and boundary regions are constructed. The time complexity of Algorithm S3 is $O(n^2)$, where n denotes the cardinality of the universe U, that is, $n = \mathrm{Card}(U)$.

To further explain of F′-3WDNRS, Fig. 7 shows the procedure for the construction of F′-3WDNRS. The triangles represent the elements in a neighborhood, and circles represent the different neighborhood granules. First, the thresholds are obtained by the iterative method of the 3WA-SS framework. Second, the DAN can be obtained by optimal deviation. Then, the 3WA-SS framework is used again on the neighborhood space consisting of the DAN to obtain the thresholds by an iterative method. Finally, the three decision regions of the neighborhood space can be constructed by the thresholds obtained above. Herein, the 3WA-SS framework is established to supply thresholds for F′-3WDNRS, which can be implemented directly by Algorithm S3.

TABLE II
INFORMATION OF TWELVE DATASETS

No.	Datasets	Attribute Characteristics	Instances	Attributes
1	HCV	Integer, Real	596	11
2	Breast Cancer Wisconsin	Real	699	10
3	Raisin	Real, Integer	900	8
4	Cloud	Real	1024	10
5	Concrete Compressive Strength	Real	1030	9
6	Website Phishing	Integer	1353	10
7	Banknote	Real	1371	5
8	Contraceptive Method Choice	Categorical, Integer	1473	9
9	Airfoil Self-Noise	Real	1503	6
10	Iranian Churn	Integer	3150	13
11	Spambase	Integer, Real	4601	57
12	Mushroom	Categorical	8124	22

V. EXPERIMENTS

In this section, the reliability and reasonability of the proposed model are examined through illustrative experiments. The experiments are performed on a computer, which runs Windows 10 operating system with 16.0 GB of RAM and a 3.10 GHz CPU. MATLAB R2022a is used for programming, and IBM SPSS Statistics 27 is used for data analysis. The experiments are conducted on 12 UCI datasets [41] listed in Table II.

A. Comparison of the 3WA-SS Framework on Preiteration and Postiteration

In Section IV, the 3WA-SS framework is used to obtain more reasonable m and (α, β), and the 3WA-SS framework

TABLE III
FUZZINESS CHANGE OF 3WA-SS FRAMEWORK ON PREITERATION AND POSTITERATION

No.	$Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{PRE}$	$Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{POST}$
1	1.9656	0.6896
2	8.1469	1.5523
3	1.4070	0.9985
4	2.7214	1.6399
5	6.0542	2.1398
6	1.2290	0.9988
7	1.2353	0.6827
8	2.4085	1.0716
9	0.0610	0.0602
10	1.6774	0.9995
11	0.2538	0.0800
12	0.7989	0.3714

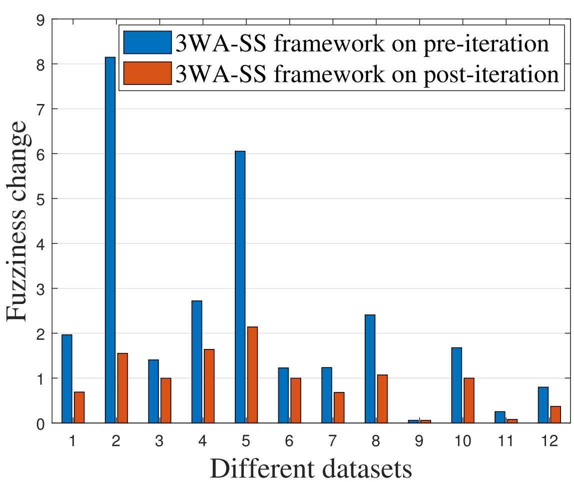

Fig. 8. Fuzziness change of the 3WA-SS framework on preiteration and postiteration.

has less fuzziness change after the iterations. Based on the principle of uncertainty invariance, the optimization principle $Q(\mu_X, T)$ [34] is proposed by Yao, which is used to measure the difference in degree between a fuzzy target concept X and its three-way approximate description T. From the viewpoint of fuzziness change between a neighborhood decision system NS and its three-way approximation, $Q(\mu_{\text{NS}}, T){=}Q(\mu_{\text{NS}}, S_\mu^\delta) = |f(\text{NS}) - f(S_\mu^\delta)|$. In this subsection, the effect of iteration will be verified by comparing $Q(\mu_{\text{NS}}, T)$ of the WA-SS framework on preiteration and postiteration, named $Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{PRE}$ and $Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{POST}$, respectively. To simplify the calculation, let $\mu_{\min} = 0, \mu_{\max} = 1$. Table III shows the fuzziness change of the 3WA-SS framework on preiteration and postiteration. To make comparison more intuitive, the results are presented in Fig. 8. Obviously, $Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{POST}$ is significantly smaller than $Q(\mu_{\text{NS}}, S_\mu^\delta) - \text{PRE}$ on each dataset. This is because objects with membership degree extremely close to 0 or 1 are assigned to the negative region or the positive region by an iterative method, providing more reasonable m and (α, β). Therefore, it is effective to use the iterative method to optimize the construction of the 3WA-SS framework to make less fuzziness change.

TABLE IV
FIVE SETS OF COST PARAMETERS

No.	λ_e	$\lambda_{s\downarrow}$	λ_r	$\lambda_{s\uparrow}$
1	0.7602	0.0888	0.9772	0.1972
2	0.6055	0.7751	0.2294	0.4608
3	0.3500	0.2000	0.2800	0.1700
4	0.6371	0.1502	0.5491	0.4888
5	0.8217	0.1866	0.6591	0.7995

B. Validity and Rationality of 3WA-SS Framework

In this subsection, 0.5 shadowed sets (0.5SS) [42], interval shadowed sets (ISS) [43], membership mean shadowed sets (MMSS) [44] and uncertainty invariance-based shadowed sets (UISS) [45] models are introduced into NRS, then the $Q(\mu_{\text{NS}}, T)$ of 0.5SS, MMSS, ISS, UISS models and 3WA-SS framework are compared on 12 UCI datasets. From the perspective of fuzziness change between a neighborhood decision system NS and its three-way approximation, $Q(\mu_{\text{NS}}, T){=}Q(\mu_{\text{NS}}, S_\mu^*) = |f(\text{NS}) - f(S_\mu^*)|$, where S_μ^* denotes different shadowed sets models mentioned above. To simplify the calculation, let $\mu_{\min} = 0, \mu_{\max} = 1$. For 0.5SS, MMSS models and 3WA-SS framework, in the shadow area, the membership degree $\mu_X^D(x)$ is transformed into the constant value c, thus, the fuzziness of shadowed sets $S_\mu^*(X)$ is the total amount of the fuzziness of each object. Therefore, $f(S_\mu^*(X)){=}\sum_{\beta<\mu_X^D(x)<\alpha} 4\mu_X^D(x)(1{-}\mu_X^D(x)) = 4c(1-c)|x|\beta<\mu_X^D(x)<\alpha|$. For 0.5SS and MMSS models, the two thresholds α and β can be calculated by $\alpha{=}\frac{\lambda_e+\delta\lambda_{s\downarrow}}{\lambda_e+\lambda_{s\downarrow}}, \beta = \frac{\delta\lambda_{s\uparrow}}{\lambda_r+\lambda_{s\uparrow}}$, where the values $\lambda_e, \lambda_r, \lambda_{s\downarrow}, \lambda_{s\uparrow}$ are shown in Table IV, which are provided artificially in [42], for the proposed 3WA-SS framework, the two thresholds α and β can be obtained by the iteration in Section IV, that is, $\alpha = \alpha^*$, $\beta = \beta^*$. For the 0.5SS model, $c = 0.5$; for the MMSS model, $c = \frac{\sum_{x\in U}\mu(x)}{\text{Card}(U)}$; for the proposed 3WA-SS framework, $c = m^*$ can be obtained by the iteration in Section IV. For the ISS model, according to the concept of interval fuzziness proposed by Zhang et al. [43], $f(S_\mu^{[\beta,\alpha]}(X)) = \frac{|x|\beta<\mu_X^D(x)<\alpha|}{\alpha-\beta} \cdot \int_\beta^\alpha 4\mu_X^D(x)(1-\mu_X^D(x))dx$. For the UISS model, the optimal thresholds α and β are computed through the optimal function $\arg\min_{(\alpha,\beta)} Q(\mu_{\text{NS}}, S_\mu^U)$, and $f(S_\mu^U(X)) = \frac{|x|\beta<\mu_X^D(x)<\alpha|}{\alpha-\beta} \cdot \int_\beta^\alpha 4\mu_X^D(x)(1-\mu_X^D(x))dx$.

Tables S1– S5 is shown in Section S5 of Supplementary file, which shows the fuzziness change of five models with different cost parameters in Table IV. Fig. 9 shows the fuzziness change of 12 datasets under different cost parameters. Obviously, $Q(\mu_{\text{NS}}, S_\mu^\delta)$ is significantly smaller than the fuzziness change of other models. The main reason is that the thresholds of the 3WA-SS framework are based on a data-driven method with objectivity, and it uses the iterative method to optimize the construction to minimize fuzziness change. Using the 3WA-SS framework, F′-3WDNRS will possess better performance in many applications, i.e., three-way diagnostic system, three-way recommendation, etc.

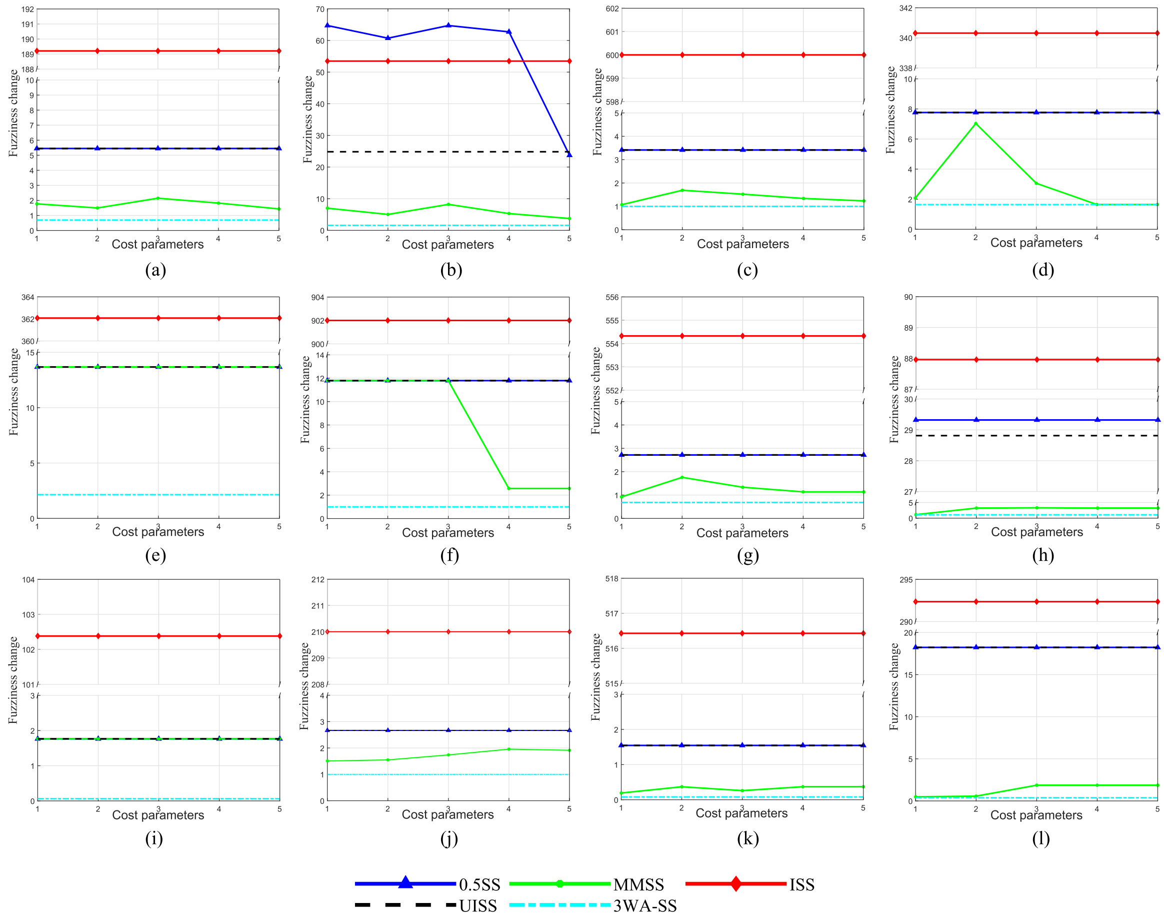

Fig. 9. Fuzziness change of twelve datasets under different cost parameters. (a) HCV. (b) Breast Cancer Wisconsin. (c) Raisin. (d) Cloud. (e) Concrete Compressive Strength. (f) Website Phishing. (g) Banknote. (h) Contraceptive Method Choice. (i) Airfoil Self-Noise. (j) Iranian Churn. (k) Spambase. (l) Mushroom.

C. Analysis of the Efficiency for F′-3WDNRS

To overall evaluate our proposed model, the measures of Precision, Accuracy, Recall, and $F-$ measure (F) are adopted among five models, where TP $=$ POS$(X)\cap X$, $FP=$ POS$(X)\cap X^C$, TN=NEG $(X)\cap X^C$, FP=NEG$(X)\cap X$, FN=NEG$(X)\cap X^C$. The computations of these indicators are presented as follows:

$$\text{Precision} = \frac{\text{TP}}{\text{TP +FP}} \tag{38}$$

$$\text{Accuracy} = \frac{\text{TP +TN}}{\text{TP +TN+FP+FN}} \tag{39}$$

$$\text{Recall} = \frac{\text{TP}}{\text{TP +FN}} \tag{40}$$

$$F = \frac{2\cdot\text{Precision}\cdot\text{Recall}}{\text{Precision}+\text{Recall}}. \tag{41}$$

Table V depicts the Precision, Accuracy, Recall, and $F-$ measure (F) resulting from F′-3WDNRS and 3WDNRS on 0.5SS, MMSS, ISS, and UISS models, which denote as 3WDNRS-0.5SS, 3WDNRS-MMSS, 3WDNRS-ISS, and 3WDNRS-UISS. Table V shows that F′-3WDNRS is superior to the other models concerning Precision, Accuracy, Recall, and $F-$ Measure (F). The main reason is that F′-3WDNRS enhances the decision-making performance by assigning objects with membership degree are extremely close to 0 or 1 to the certainty region under the 3WA-SS framework, and the thresholds of the 3WA-SS framework are based on a data-driven method with objectivity. Furthermore, a statistical analysis of the results is performed, including Friedman test, Wilcoxon rank-sum test, and mean ranking analysis. Win/Loss represents the ratio of wins to losses after pairwise comparison between F′-3WDNRS and the other models. $P-$ value reflects the difference between F′-3WDNRS and the other models. If $P-$ value < 0.05, there is a significant difference between F′-3WDNRS and the other models. Otherwise, there is no significant difference between F′-3WDNRS and the other models. Rank refers to the average rank. A higher rank value indicates that the algorithm is more effective. As shown in Table V, for Win/Loss ratio, compared with the other models, F′-3WDNRS wins 179 times and loses 13 times in a total number of 192 comparisons. For $P-$ value, F′-3WDNRS has no significant differences with the other models. For Rank, F′-3WDNRS achieves the highest overall score, as can be seen in the bold entities in Table V, thus it is ranked first. The main reason is that the thresholds and neighborhood radius of F′-3WDNRS are both based on a data-driven

TABLE V
COMPARISON OF PRECISION, ACCURACY, RECALL, AND $F-$ MEASURE (F) OF 3WDNRS-0.5SS, 3WDNRS-MMSS, 3WDNRS-ISS, 3WDNRS-UISS, AND F′-3WDNRS MODELS

No.	Metrics	3WDNRS-0.5SS	3WDNRS-MMSS	3WDNRS-ISS	3WDNRS-UISS	F′-3WDNRS
1	Precision	0.8985	0.8985	0.8985	0.9031	0.9077
	Accuracy	0.8960	0.8985	0.8960	0.9005	0.9051
	Recall	0.9962	1.0000	0.9962	0.9962	0.9962
	F	0.9448	0.9465	0.9448	0.9474	0.9499
2	Precision	0.3443	0.3443	0.3443	0.3443	0.6512
	Accuracy	0.3448	0.3443	0.3448	0.3448	0.6444
	Recall	0.9959	1.0000	0.9959	0.9959	0.9655
	F	0.5117	0.5123	0.5117	0.5117	0.7778
3	Precision	0.5006	0.5000	0.5000	0.5000	0.5114
	Accuracy	0.5006	0.5000	0.5000	0.5000	0.5114
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.6672	0.6667	0.6667	0.6667	0.6767
4	Precision	0.9853	0.9853	0.9853	0.9853	0.9892
	Accuracy	0.9834	0.9853	0.9834	0.9834	0.9862
	Recall	0.9980	1.0000	0.9980	0.9980	0.9980
	F	0.9916	0.9926	0.9916	0.9916	0.9931
5	Precision	0.3063	0.3063	0.3063	0.3063	0.4591
	Accuracy	0.3427	0.3427	0.3427	0.3427	0.4982
	Recall	0.8871	0.8871	0.8871	0.8871	0.8487
	F	0.4553	0.4553	0.4553	0.4553	0.5959
6	Precision	0.4680	0.4050	0.4050	0.4050	0.4680
	Accuracy	0.4680	0.4050	0.4050	0.4050	0.4680
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.6376	0.5765	0.5765	0.5765	0.6376
7	Precision	0.4446	0.4612	0.4446	0.4446	0.4437
	Accuracy	0.4446	0.4612	0.4446	0.4446	0.4437
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.6155	0.6313	0.6155	0.6155	0.6147
8	Precision	0.3469	0.4006	0.3469	0.3469	0.4006
	Accuracy	0.3469	0.4006	0.3469	0.3469	0.4006
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.5151	0.5720	0.5151	0.5151	0.5720
9	Precision	0.6730	0.6730	0.6730	0.6730	0.7164
	Accuracy	0.6700	0.6730	0.6700	0.6700	0.7113
	Recall	0.9851	1.0000	0.9851	0.9851	0.9817
	F	0.7997	0.8046	0.7997	0.7997	0.8284
10	Precision	0.1682	0.1571	0.1571	0.1571	0.1682
	Accuracy	0.1682	0.1571	0.1571	0.1571	0.1682
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.2879	0.2716	0.2716	0.2716	0.2879
11	Precision	0.3926	0.3925	0.3926	0.3926	0.7910
	Accuracy	0.3919	0.3917	0.3919	0.3919	0.7649
	Recall	0.9928	0.9928	0.9928	0.9928	0.9556
	F	0.5627	0.5625	0.5627	0.5627	0.8655
12	Precision	0.4820	0.3112	0.4820	0.4820	0.5105
	Accuracy	0.4820	0.3112	0.4820	0.4820	0.5105
	Recall	1.0000	1.0000	1.0000	1.0000	1.0000
	F	0.6505	0.4747	0.6505	0.6505	0.6759
Statistics	Win/Loss	2/46	7/41	2/46	2/46	179/13*
	$P-$value	0.174	0.070	0.125	0.122	–
	Rank	2.9375	2.9479	2.5000	2.6458	**3.9688**(1)

* The total Win/Loss cases of F′-3WDNRS

method, which is out of the influence of subjective. This indicates that F′-3WDNRS possesses better performance than the other models in applications.

Table VI shows the comparison results of F′-3WDNRS and F-3WDNRS under different radius $\delta= 0.2$, $\delta= 0.4$, and $\delta= 0.7$. For the Win/Loss ratio, compared with the F-3WDNRS model under different radius, F′-3WDNRS wins 129 times and loses 15 times in a total number of 144 comparisons. For $P-$ value, F′-3WDNRS has no significant differences with F-3WDNRS under different radius. For Rank, F′-3WDNRS achieves the highest overall score, as can be seen in the bold entities in Table VI, thus it is ranked first. The main reason is that F′-3WDNRS is a further optimization of F-3WDNRS based on DAN on the basis of F-3WDNRS. DAN is an adaptive neighborhood obtained by a data-driven method, that is not subjective. Therefore, F′-3WDNRS outperforms the F-3WDNRS which needs to subjectively specify the neighborhood radius.

From the above experimental analysis, F′-3WDNRS is reasonable and efficient. Moreover, it has the benefit of less fuzziness change. F′-3WDNRS is a reasonable improvement for ensuring more precise decision-making, regardless of variations in dataset volumes or distributions.

TABLE VI
COMPARISON OF PRECISION, ACCURACY, RECALL, AND $F-$ MEASURE (F) OF F-3WDNRS WITH DIFFERENT RADIUS AND F′-3WDNRS

No.	Metrics	F-3WDNRS			F′-3WDNRS
		$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.7$	
1	Precision	0.9107	0.9077	0.9077	0.9077
	Accuracy	0.9107	0.9049	0.9051	0.9051
	Recall	1.0000	0.9962	0.9962	0.9962
	F	0.9532	0.9499	0.9499	0.9499
2	Precision	0.2000	0.5357	0.6512	0.6512
	Accuracy	0.2000	0.5357	0.6444	0.6444
	Recall	1.0000	1.0000	0.9655	0.9655
	F	0.3333	0.6977	0.7778	0.7778
3	Precision	0.9809	0.9796	0.9803	0.9892
	Accuracy	0.9809	0.9796	0.9803	0.9862
	Recall	1.0000	1.0000	1.0000	0.9980
	F	0.9904	0.9897	0.9901	0.9931
4	Precision	0.2717	0.4057	0.4591	0.4591
	Accuracy	0.4138	0.4649	0.4982	0.4982
	Recall	0.5814	0.7978	0.8487	0.8487
	F	0.3704	0.5379	0.5959	0.5959
5	Precision	0.3393	0.3920	0.4036	0.4437
	Accuracy	0.3393	0.3920	0.4036	0.4437
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.5067	0.5632	0.5750	0.6147
6	Precision	0.3435	0.3762	0.3762	0.4006
	Accuracy	0.3435	0.3762	0.3762	0.4006
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.5114	0.5468	0.5468	0.5720
7	Precision	0.7085	0.7103	0.7164	0.7164
	Accuracy	0.7085	0.7103	0.7113	0.7113
	Recall	1.0000	1.0000	0.9817	0.9817
	F	0.8294	0.8306	0.8284	0.8284
8	Precision	0.7143	0.6317	0.6317	0.7910
	Accuracy	0.7143	0.6317	0.6317	0.7649
	Recall	1.0000	1.0000	1.0000	0.9556
	F	0.8333	0.7743	0.7743	0.8655
9	Precision	0.3112	0.5105	0.5105	0.5105
	Accuracy	0.3112	0.5105	0.5105	0.5105
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.4747	0.6759	0.6759	0.6759
10	Precision	0.4207	0.5114	0.5114	0.5114
	Accuracy	0.4207	0.5114	0.5114	0.5114
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.5922	0.6376	0.6376	0.6376
11	Precision	0.2911	0.2911	0.4680	0.4680
	Accuracy	0.2911	0.2911	0.4680	0.4680
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.4509	0.4509	0.6376	0.6376
12	Precision	0.1614	0.1682	0.1682	0.1682
	Accuracy	0.1614	0.1682	0.1682	0.1682
	Recall	1.0000	1.0000	1.0000	1.0000
	F	0.2779	0.2879	0.2879	0.2879
Statistics	Win/Loss	8/40	5/43	2/46	129/15*
	$P-$value	0.005	0.428	0.908	–
	Rank	1.9479	2.2813	2.7083	**3.0625**(1)

* The total Win/Loss cases of F′-3WDNRS

Recommendation systems [46] are currently a popular research topic, aiming to bridge the gap between users and projects by analyzing user preferences through information modeling. This approach has demonstrated success in various commercial applications, such as Amazon, Facebook, and Taobao. To validate the effectiveness of F′-3WDNRS in practical applications, F′-3WDNRS is introduced into recommendation model, called fuzziness-based three-way recommendation with NRS (F-3WRNRS). The effectiveness and reasonableness of F-3WRNRS are validated by contrasting its results with those of two classical collaborative filtering recommendation methods, i.e., user-item collaborative filtering (User-Item CF) [47] and item–item collaborative filtering (Item–Item CF) [48]. The adopted similarity measure is Euclidean distance, and contrasting with an up-to-date three-way recommendation, that is, three-way recommender systems on random forests (3WR-RF) [49], which arranges the threshold pair as $(\alpha,\ \beta) = (0.6,\ 0.5)$. Each of the 12 models is divided into a training set and a test set. The $\mathrm{Precision},\ \mathrm{Accuracy},\ \mathrm{Recall},$ and $F-$ measure (F) obtained from the 12 datasets via the four models are depicted in Table S6, which is shown in Section S6 of Supplementary file. As shown in Table S6, for $Win/Loss$ ratio, compared with other models, the F-3WRNRS model wins 130 times and loses 14 times in a total number of 144 comparisons.

For $P - value$, the F-3WRNRS model has statistically significant differences with 3WR-RF and no significant differences with Item–Item CF and User–Item CF. For Rank, F-3WRNRS achieves the highest overall score, as can be seen in the bold entities in Table S6, thus it is ranked first. The main reason is that F-3WRNRS is a three-way recommendation model based on the 3WA-SS framework, and the thresholds of the 3WA-SS framework is obtained by a data-driven method with objective. These results demonstrate that our proposed model presents a viable approach for obtaining recommendations in certain applications.

VI. Conclusion

Combined with the 3WA-SS framework, this article proposed an improved fuzziness-based F′-3WDNRS from the perspective of uncertainty. The results of the comparative experiments verified that the 3WA-SS framework guarantees less fuzziness change on 12 public benchmark datasets. Moreover, F′-3WDNRS ensured more precise decision-making, regardless of variations in dataset volumes or distributions. The main reason is that the thresholds and neighborhood radius of F′-3WDNRS are both based on the data-driven method with objectivity. These works will be enrich the NRS from a new perspective and contribute to the development of 3WD models.

Our model is exploratory and still has certain limitations as follows.

1) The inputs of our model are individual neighborhood granules, which reduces the decision efficiency and generality.

2) Our model is based on NRS, which has poor fault tolerance to noisy data.

The focus of our future work will be on two aspects as follows.

1) A multilevel framework of sequential F′-3WDNRS and its optimal neighborhood space selection mechanism will be designed by considering the user requirements for decision precision and computational expense, respectively.

2) We will introduce granular ball computing [7] into 3WD-NRS to construct a novel granular-ball NRS from the perspective of uncertainty, resulting in greater generality, robustness, and efficiency to make decisions.

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