



# Artificial Neural Networks

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# Outline

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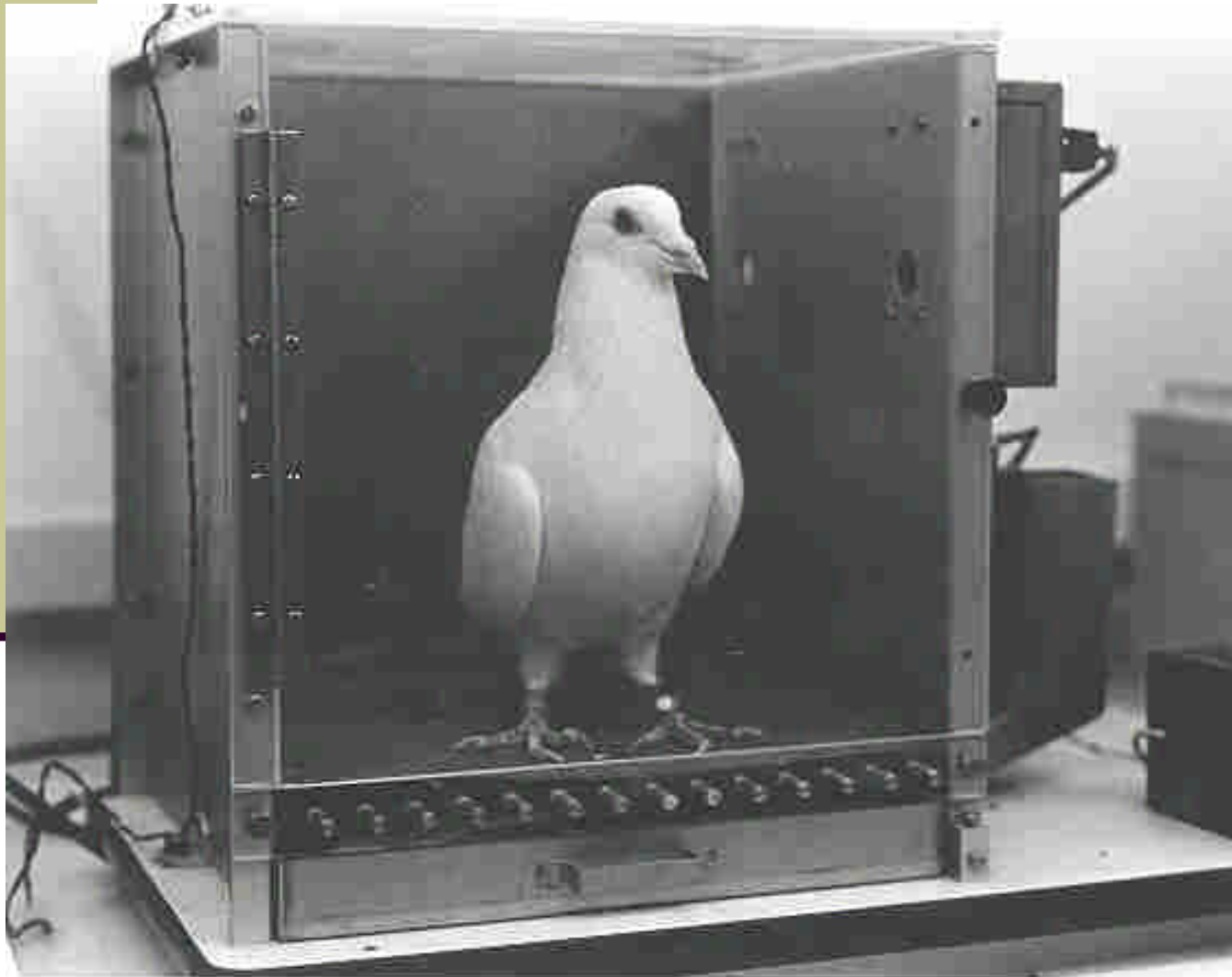
- Perceptrons
- Gradient descent
- Multilayer Feedforward networks
- Backpropagation
- Hidden layer representations
- Examples
- Advanced topics

# What is an Artificial Neural Network?

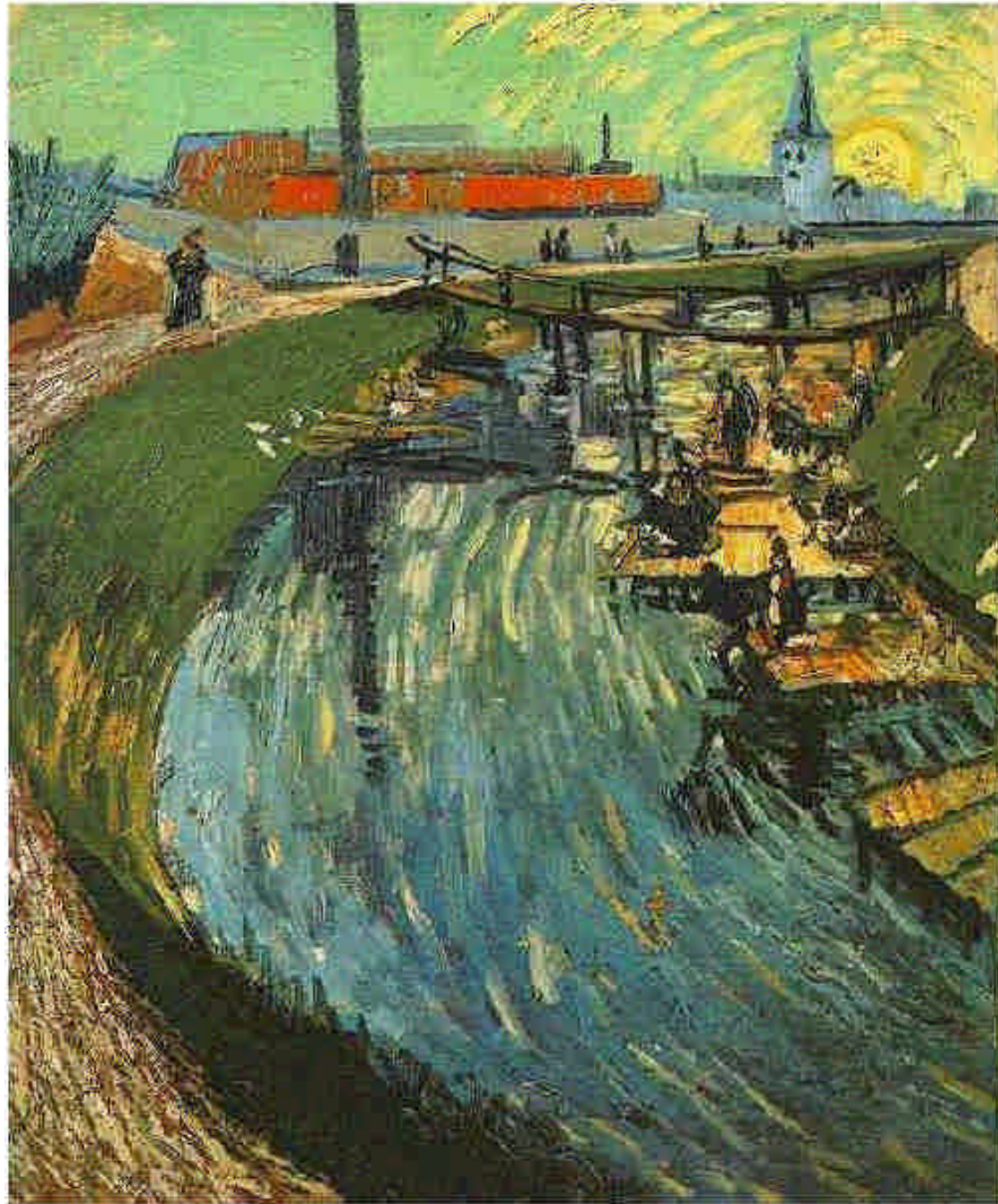
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- It is a formalism for representing functions **inspired** from biological learning systems
- The network is composed of **parallel computing units** which each **computes a simple function**
- Some useful computations taking place in **Feedforward Multilayer** Neural Networks are
  - Summation
  - Multiplication
  - Threshold (e.g.,  $1/(1 + e^{-x})$ , the sigmoidal threshold function). Other functions are also possible

# Pigeons as art experts



- Pigeons as art experts (Watanabe *et al.* 1995)
- Experiment:
  1. Pigeon in Skinner box
  2. Present paintings of two different artists (e.g. Chagall / Van Gogh)
  3. Reward for pecking when presented a particular artist (e.g. Van Gogh)





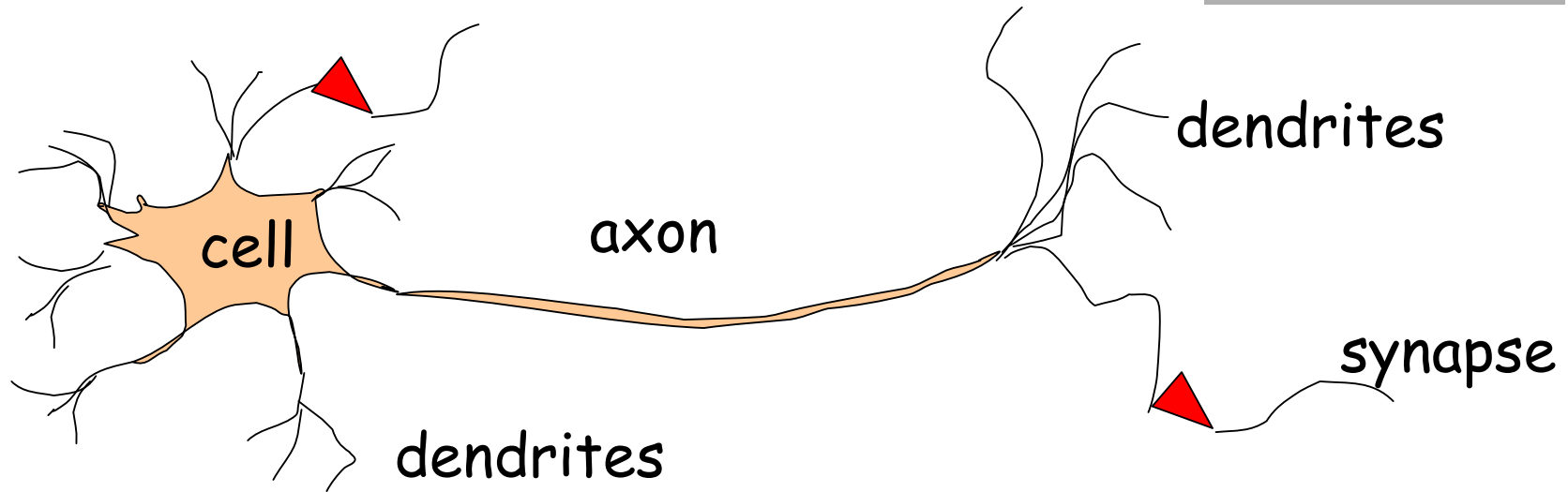


# Pigeons as art experts (cont.)

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- Pigeons were able to discriminate between Van Gogh and Chagall with 95% accuracy (when presented with pictures they had been trained on)
- Discrimination still 85% successful for previously unseen paintings of the artists
- Pigeons do not simply memorise the pictures
- They generalize from the already seen to make predictions

# Biological Motivation



- Biological Learning Systems are built of very complex webs of interconnected neurons
- Information-processing abilities of biological neural systems must follow from **highly parallel processes** operating on representations that are distributed over many neurons
- ANNs attempt to capture this mode of computation



# Biological Neural Systems

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- Neuron switching time :  $> 10^{-3}$  secs
  - Computer takes  $10^{-10}$  secs
- Number of neurons in the human brain:  $\sim 10^{11}$
- Connections (synapses) per neuron:  $\sim 10^4$ - $10^5$
- Face recognition :  $\sim 0.1$  secs
  - 100 inference steps? Brain must be parallel!
- High degree of parallel computation
- Distributed representations

# Properties of Artificial Neural Nets (ANNs)

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- Many simple neuron-like threshold switching units
- Many weighted interconnections among units
- Highly parallel, distributed processing
- Learning by tuning the connection weights
- ANNs are motivated by biological neural systems; but not as complex as biological systems
  - For instance, individual units in ANN output a single constant value instead of a complex time series of spikes

# A Brief History of Neural Networks (Pomerleau)

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- **1943**: McCulloch and Pitts proposed a model of a neuron → Perceptron (Mitchell, section 4.4)
- **1960s**: Widrow and Hoff explored Perceptron networks (which they called “Adelines”) and the delta rule.
- **1962**: Rosenblatt proved the convergence of the perceptron training rule.
- **1969**: Minsky and Papert showed that the Perceptron cannot deal with nonlinearly-separable data sets --- even those that represent simple function such as X-OR.
- **1975**: Werbos’ ph.D. thesis at Harvard (beyond regression) defines backpropagation.
- **1985**: PDP book published that ushers in modern era of neural networks.
- **1990’s**: Neural networks enter mainstream applications.

# Appropriate Problem Domains for Neural Network Learning

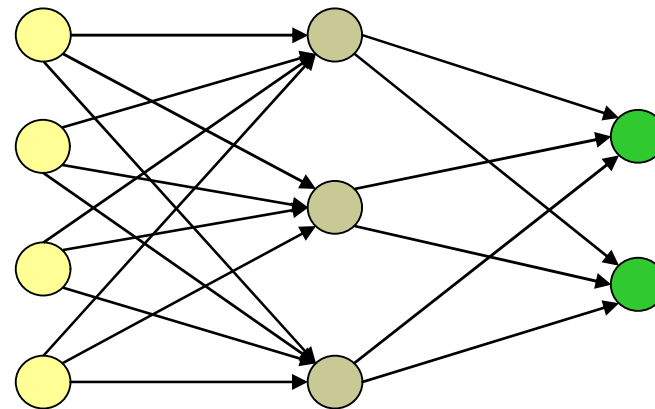
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- Input is **high-dimensional** discrete or real-valued (e.g. raw sensor input)
- **Output is discrete or real valued**
- Output is a vector of values
- **Form of target function is unknown**
- Humans do not need to interpret the results (black box model)
- **Training examples may contain errors (ANN are robust to errors)**
- Long training times acceptable

# Prototypical ANN Approach

- Mostly, network structure is fixed. Therefore, learning = weight adjustment
  - deciding the structure is an art!
- Units interconnected in **layers**
  - directed, acyclic graph (DAG)
  - skip-layer connection is possible
  - the network can be sparse, with not all possible connections within a layer being present (convolutional networks)

input layer   hidden layer   output layer





# Types of ANNs

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- **Feedforward**: Links are unidirectional, and there are no cycles, i.e., the network is a directed acyclic graph (DAG). Units are arranged in layers, and **each unit is linked only to units in the next layer**. **There is no internal state other than the weights**
- **Recurrent**: Links can form arbitrary topologies. **Cycles can implement memory**. Behavior can become *unstable, oscillatory, or chaotic*

# ALVINN: training and performance

Drives 70 mph on a public highway, by  $\sim 5$  mins training

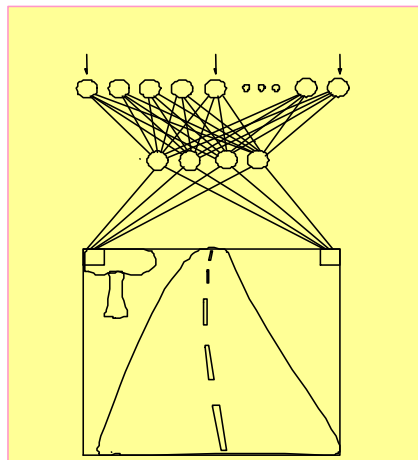
Camera  
image



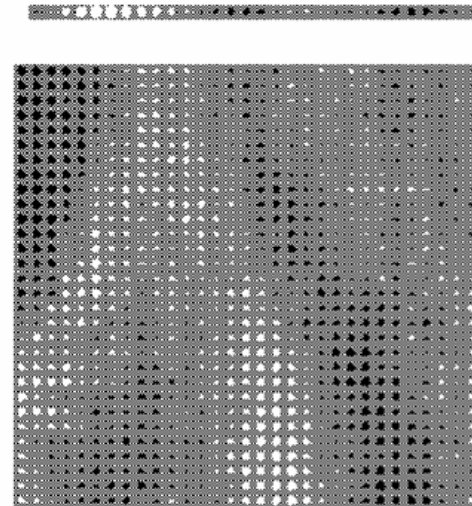
30 outputs  
for steering

4 hidden  
units

30x32 pixels  
as inputs



The weights from  
a hidden unit to  
30 output units



30x32 weights  
into one out of  
four hidden  
unit. A white box  
indicates a  
positive weight  
and a black box  
a negative  
weight

# ALVINN: training and performance

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- Trained with computer-generated road images
- Involved 1200 different combinations of scenes, curvatures, lighting conditions and distortion levels
- Entire driver implemented on an on-board computer and a modified Chevy van!
- Performed comparably to the best traditional vision-based navigation systems evaluated under similar conditions
- Training was done in half-an-hour!
  - Was training done on board?
- For comparison -- Algorithm-based drivers take months for algorithm development

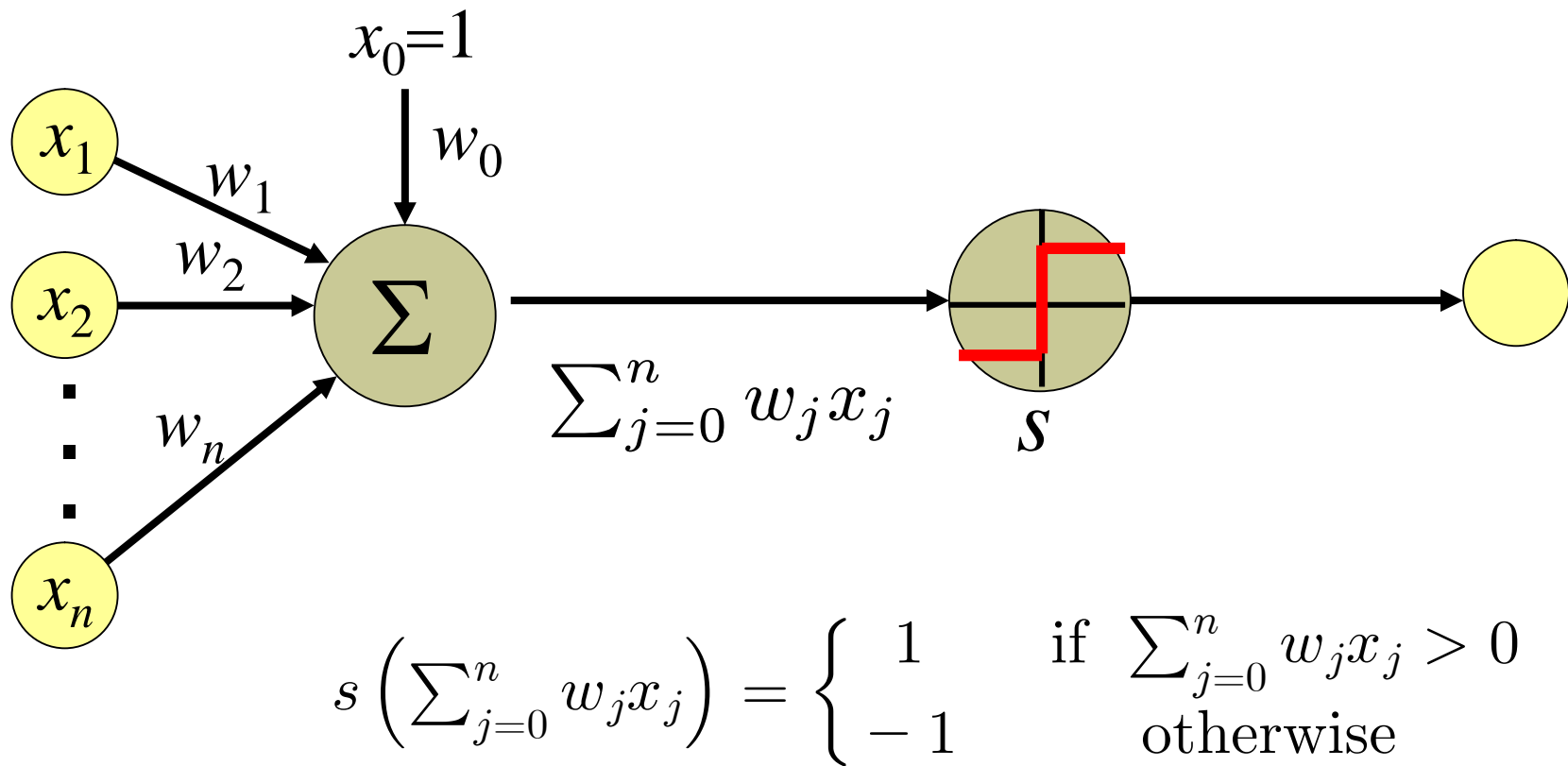
# Perceptrons

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- Structure & function
  - inputs, weights, threshold
  - hypotheses in weight vector space
- Representational power
  - defines a hyperplane decision surface
  - linearly separable problems
  - most boolean functions
  - $m$  of  $n$  functions
    - Output “1” if  $m$  of  $n$  inputs are “1”s

# Perceptron

- Linear threshold unit (LTU)



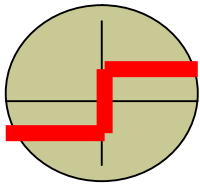


# Purpose of the Activation Function $s$

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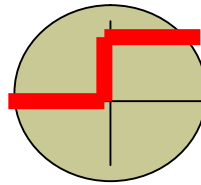
- We want the unit to be “active” (near +1) when the “right” inputs are given
- We want the unit to be “inactive” (near -1) when the “wrong” inputs are given.
- It’s preferable for  $s$  to be nonlinear. Otherwise, the entire neural network collapses into a simple linear function.

# Possibilities for function $s$



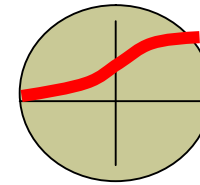
Sign function

$$\text{sign}(x) = +1, \text{ if } x > 0 \\ -1, \text{ if } x \leq 0$$



Step function

$$\text{step}(x) = 1, \text{ if } x > \text{threshold} \\ 0, \text{ if } x \leq \text{threshold} \\ (\text{in picture above, threshold} = 0)$$

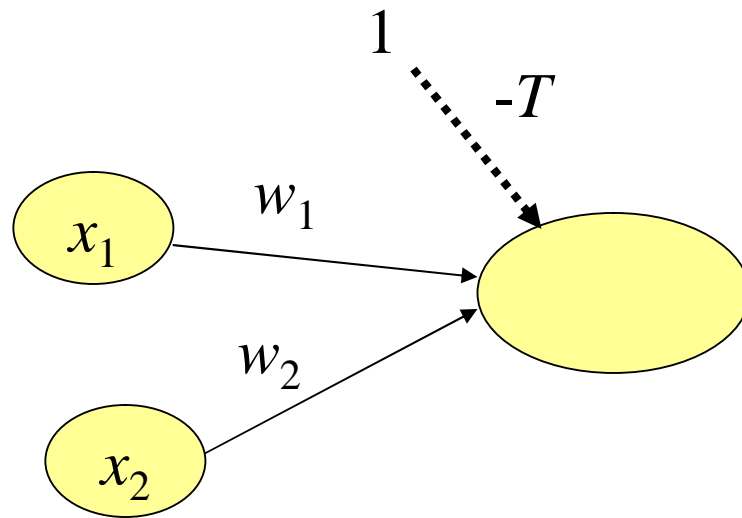


Sigmoid (logistic) function

$$\text{sigmoid}(x) = 1/(1+e^{-x})$$

Adding an extra input with activation  $x_0 = 1$  and weight  $w_0 = -T$  (called the *bias weight*) is equivalent to having a threshold at  $T$ . This way we can always assume a 0 threshold.

# Using a Bias Weight to Standardize the Threshold

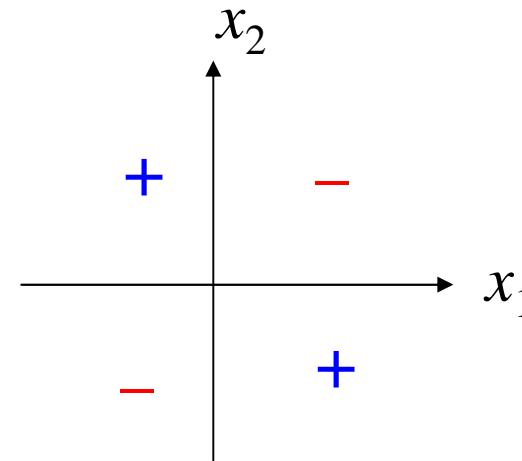
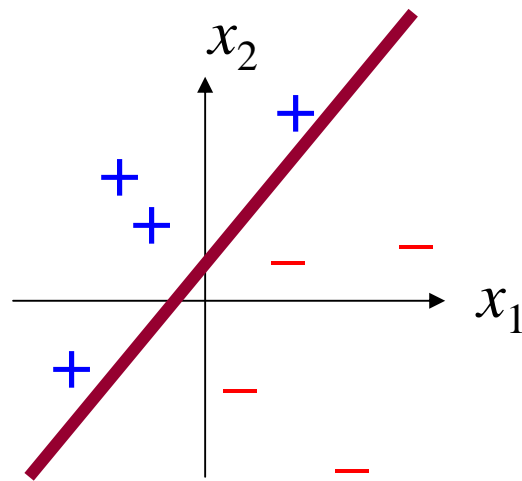


$$w_1x_1 + w_2x_2 < T$$



$$w_1x_1 + w_2x_2 - T < 0$$

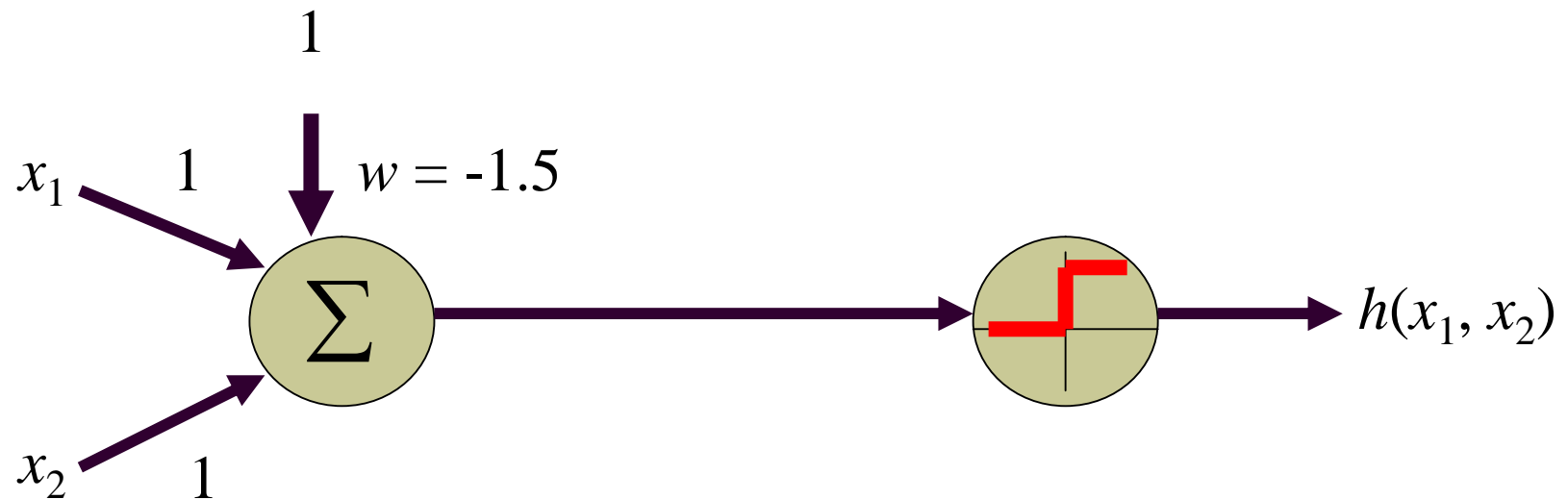
# Decision Surface of a Perceptron



- Perceptron is able to represent some useful functions and  $(x_1, x_2)$ : choose weights  $w_0 = -1.5$ ,  $w_1 = 1$ ,  $w_2 = 1$
- But functions that are not linearly separable (e.g. XOR) are not representable

# Implementing AND

Assume Boolean (0/1) input values...

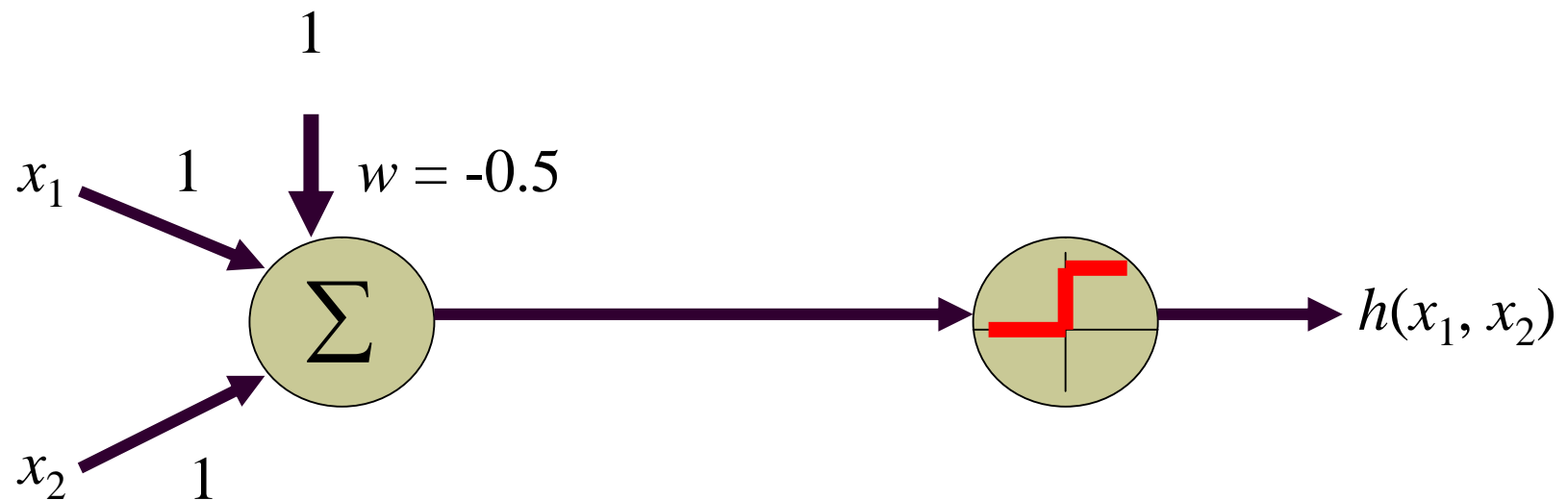


$$h(x_1, x_2) = \begin{cases} 1 & \text{if } -1.5 + x_1 + x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$



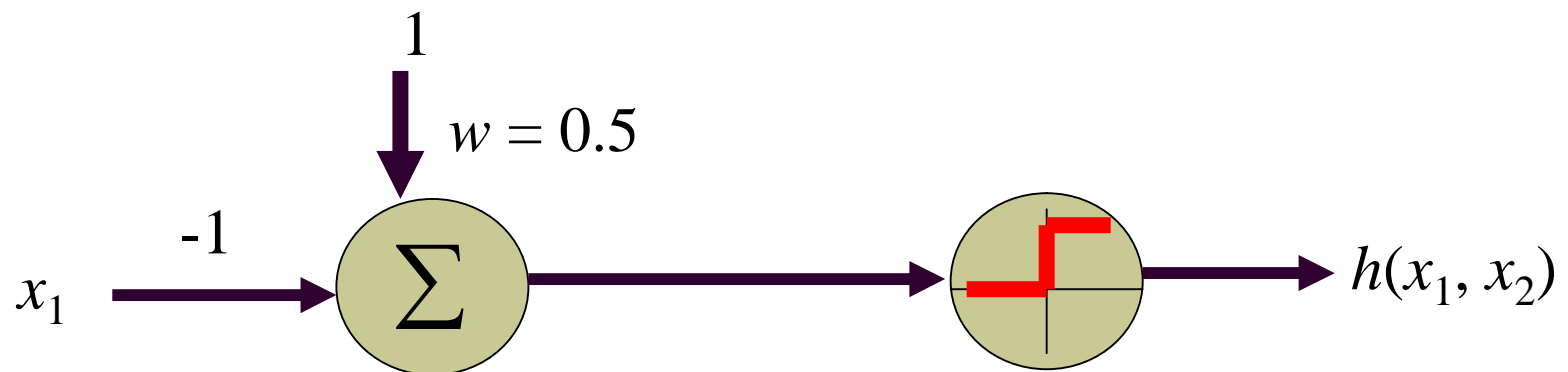
# Implementing OR

Assume Boolean (0/1) input values...



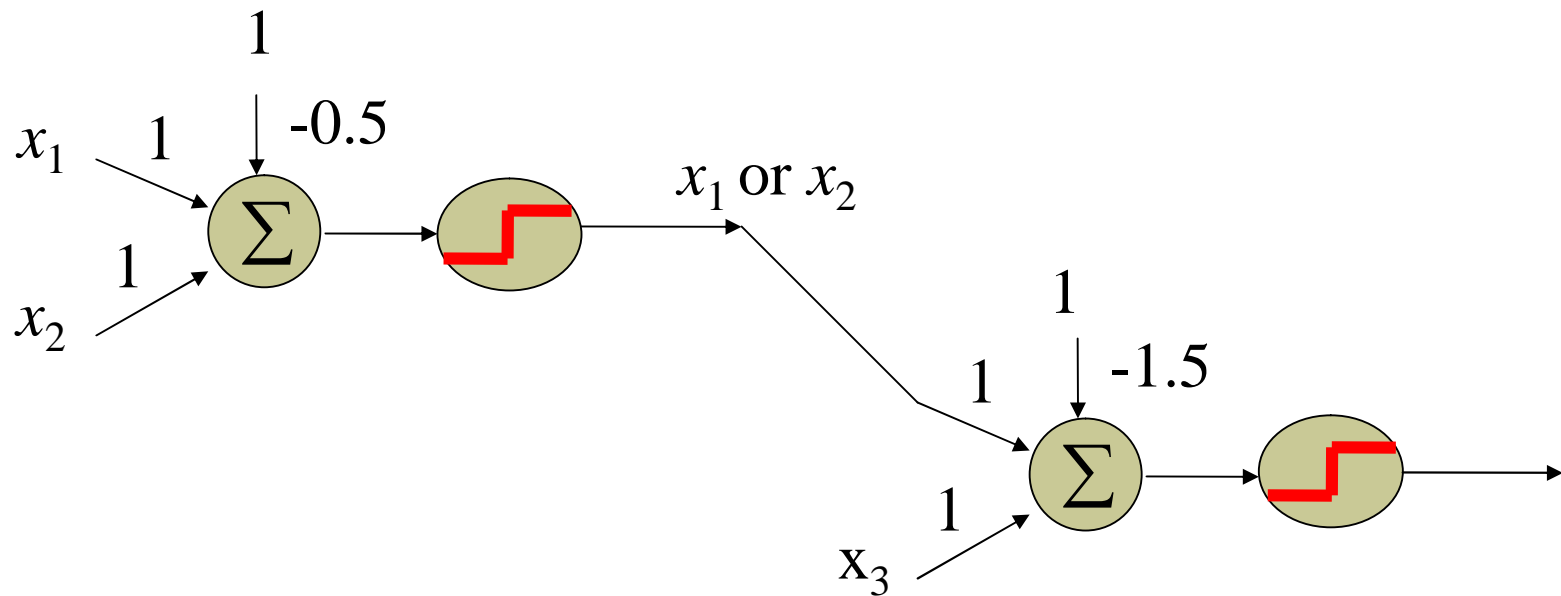
$$h(x_1, x_2) = \begin{cases} 1 & \text{if } -0.5 + x_1 + x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

# Implementing NOT



$$h(x_1) = \begin{cases} 1 & \text{if } 0.5 - x_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

# Implementing more complex Boolean functions



$(x_1 \text{ or } x_2) \text{ and } x_3$

# Perceptron Learning Rule

$$w_j \leftarrow w_j + \Delta w_j$$

$$\Delta w_j = \eta (y - h(\mathbf{x})) x_j$$

$y$  is the target output for the current training example

$h(\mathbf{x})$  is the **perceptron output**

$\eta$  is a small constant (e.g. 0.1) called *learning rate*

- Start with some random weights (usually small values)
- If the output is correct ( $y = h(\mathbf{x})$ ) the weights  $w_j$  are not changed
- If the output is incorrect ( $y \neq h(\mathbf{x})$ ) the weights  $w_j$  are changed such that the output of the perceptron for the new weights is *closer* to  $y$ .
- The algorithm converges to the correct classification
  - if the training data is linearly separable
  - and  $\eta$  is sufficiently small

# Perceptron Learning Rule

$$w = [0.25 \ -0.1 \ 0.5]$$

$$x_2 = 0.2 x_1 - 0.5$$

$$(x, y) = ([2, 1], -1)$$

$$(x, y) = ([1, 1], 1)$$

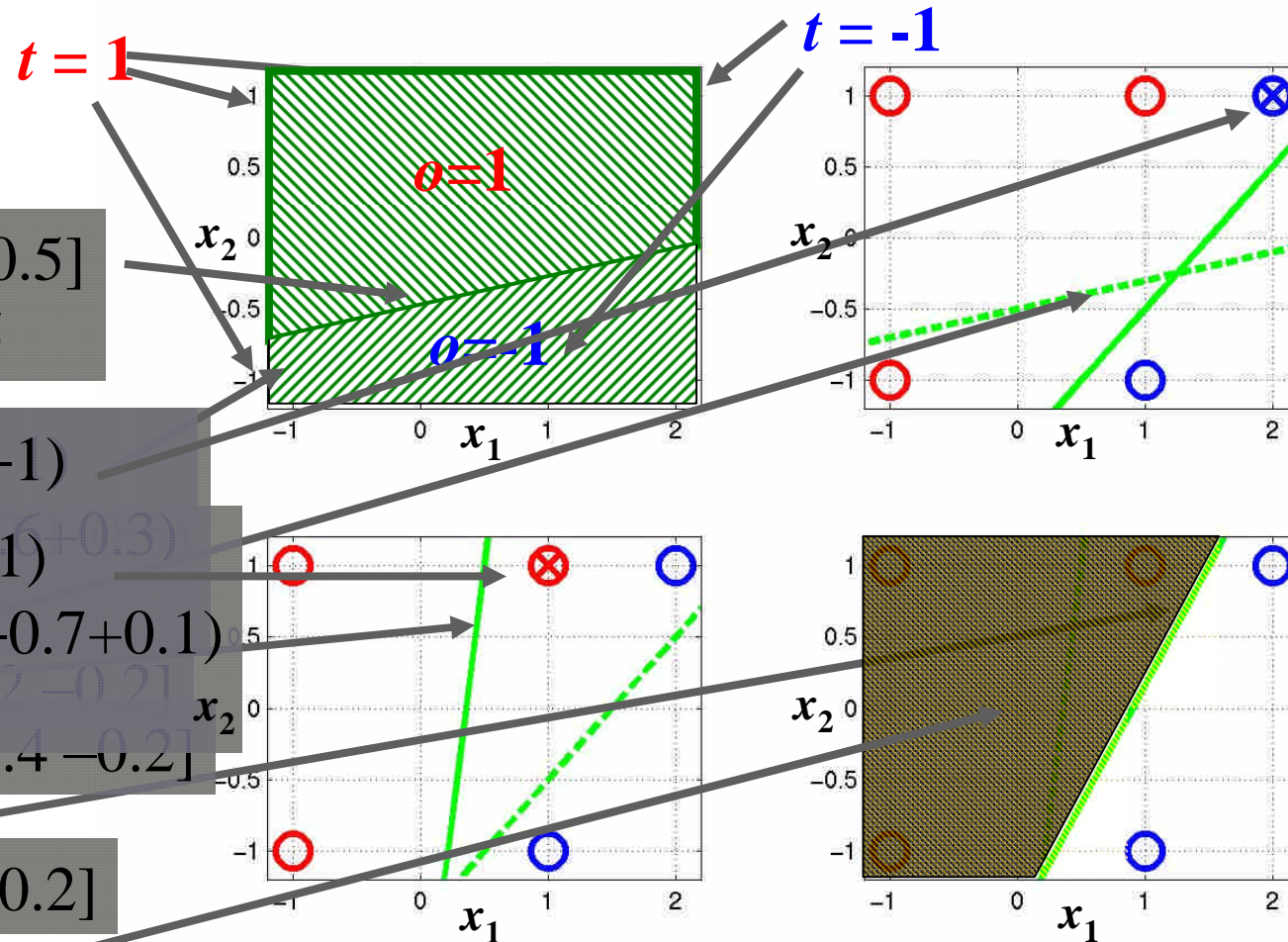
$$h = \text{sgn}(0.25 - 0.7 + 0.1)$$

$$= -1$$

$$\Delta w = [-0.2 \ -0.4 \ -0.2]$$

$$\Delta w = [0.2 \ 0.2 \ 0.2]$$

$$-0.5x_1 + 0.3x_2 + 0.45 > 0 \Rightarrow h = 1$$





# Perceptron Convergence Theorem

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- If the training data are linear separable, the perceptron learning algorithm is guaranteed to find an exact solution in a **finite** number of steps (by many – Rosenblatt (1962), Block, Nilsson, Minsky and Pappert, Duda and Hart, etc.)
- Proof:
- Many different solutions can be found with the different initialization of the parameters or the order of presentation of data points

# Gradient Descent Learning Rule

- Perceptron learning rule **fails to converge** if examples are **not linearly separable**
- Consider linear unit **without threshold** and continuous output  $h$  (not just  $-1, 1$ )
  - $h = w_0 + w_1 x_1 + \dots + w_n x_n$
- Train the  $w_j$ 's such that they minimize the squared error
  - $E[w_1, \dots, w_n] = \frac{1}{2} \sum_{i \in D} (y_i - h(x_i))^2$   
where  $D$  is the set of training examples

# Gradient Descent

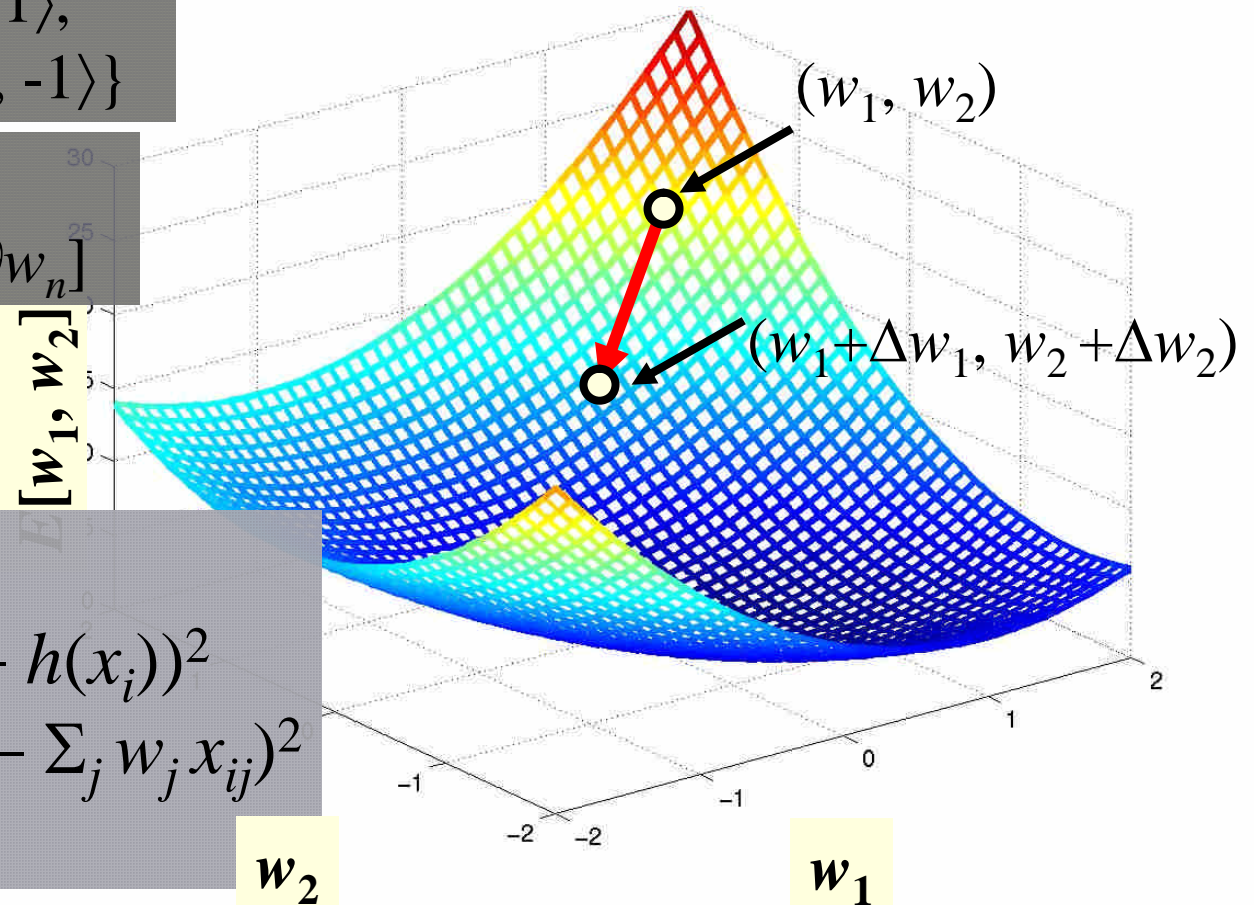
$$D = \{ \langle (1,1), 1 \rangle, \langle (-1,-1), 1 \rangle, \\ \langle (1,-1), -1 \rangle, \langle (-1,1), -1 \rangle \}$$

Gradient:

$$\nabla E[w] = [\partial E / \partial w_0, \dots, \partial E / \partial w_n]$$

$$\Delta w = -\eta \nabla E[w]$$

$$\begin{aligned} \Delta w_j &= -\eta \partial E / \partial w_j \\ &= -\eta \partial / \partial w_j \frac{1}{2} \sum_i (y_i - h(x_i))^2 \\ &= -\eta \partial / \partial w_j \frac{1}{2} \sum_i (y_i - \sum_j w_j x_{ij})^2 \\ &= \eta \sum_i (y_i - h(x_i)) x_{ij} \end{aligned}$$



# Gradient Descent

- Train the  $w_j$ 's such that they minimize the squared error

- $E[w_1, \dots, w_n] = 1/2 \sum_{i \in D} (y_i - h(x_i))^2$

Gradient:

$$\nabla E[w] = [\partial E / \partial w_0, \dots, \partial E / \partial w_n]$$

$$\Delta w = -\eta \nabla E[w]$$

$$\Delta w_j = -\eta \partial E / \partial w_j$$

$$= -\eta \partial / \partial w_j 1/2 \sum_i (y_i - h(x_i))^2$$

$$= -\eta \partial / \partial w_j 1/2 \sum_i (y_i - \sum_j w_j x_{ij})^2$$

$$= -\eta \sum_i (y_i - h(x_i))(-x_{ij})$$

# Gradient Descent

Gradient-Descent(*training\_examples*,  $\eta$ )

Each training example is a pair of the form  $\langle (x_1, \dots, x_n), y \rangle$  where  $(x_1, \dots, x_n)$  is the vector of input values, and  $y$  is the target output value,  $\eta$  is the learning rate (e.g. 0.1)

- Initialize each  $w_j$  to some small random value
- Until the termination condition is met, Do
  - Initialize each  $\Delta w_j$  to zero
  - For each  $\langle (x_1, \dots, x_n), y \rangle$  in *training\_examples* Do
    - Input the instance  $(x_1, \dots, x_n)$  to the linear unit and compute the output  $h$
    - For each linear unit weight  $w_j$  Do
      - $\Delta w_j = \Delta w_j + \eta (y - h(\mathbf{x})) x_j$
  - For each linear unit weight  $w_j$  Do
    - $w_j = w_j + \Delta w_j$
- Termination condition – error falls under a given threshold

# Perceptron Learning (Thresholded Version)

1. Initialize weights and threshold: Set weights  $w_j$  to small random values
2. Present Input and Desired Output: Set the inputs to the example values  $x_j$  and let the desired output be  $y$
3. Calculate **Actual Output**

$$h = \text{sgn}(\vec{w} \cdot \vec{x})$$

4. Adapt Weights: If actual output is different from desired output, then

$$w_j \leftarrow w_j + \eta(y - h(\mathbf{x}))x_j$$

where  $0 < \eta < 1$  is the learning rate

5. Repeat from Step 2 until done

# Gradient Descent Learning (Unthresholded Version)

1. Initialize weights and threshold: Set weights  $w_j$  to small random values
2. Present Input and Desired Output: Set the inputs to the example values  $x_j$  and let the desired output be  $y$
3. Calculate **Unthresholded Output**

$$h = \vec{w} \cdot \vec{x}$$

4. Adapt Weights: If actual output is different from desired output, then

$$w_j \leftarrow w_j + \eta \sum_{i \in D} (y_i - h(\mathbf{x}_i)) x_{ij}$$

where  $0 < \eta < 1$  is the learning rate

5. Repeat from Step 2 until done

# Incremental Stochastic Gradient Descent

- Batch mode : gradient descent

$w = w - \eta \nabla E_D[w]$  over the **entire** data  $D$

$$E_D[w] = \frac{1}{2} \sum_i (y_i - h(\mathbf{x}_i))^2$$

- Incremental mode: gradient descent

$w = w - \eta \nabla E_i[w]$  over **individual training examples  $i$**

$$E_i[w] = \frac{1}{2} (y_i - h(\mathbf{x}_i))^2$$

- **Incremental Gradient Descent can approximate Batch Gradient Descent arbitrarily closely if  $\eta$  is small enough**



# Comparison Perceptron and Gradient Descent Rule

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Perceptron learning rule guaranteed to succeed  
(**converge in finite steps**) if

- Training examples are **linearly separable**
- Sufficiently small learning rate  $\eta$

Gradient descent learning rules uses gradient descent

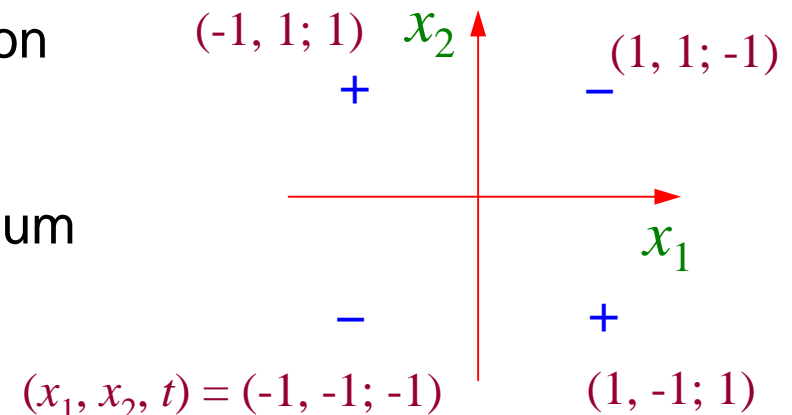
- Guaranteed to **converge** to **hypothesis with minimum squared error asymptotically**
- Given sufficiently small learning rate  $\eta$
- Even when training data contains **noise**
- Even when training data **not linearly separable**

# XOR

$$h(\vec{x}) = \vec{w} \cdot \vec{x}$$

$$\begin{aligned} E(\vec{w}) &= \frac{1}{2} \sum_{i \in D} (y_i - h(\mathbf{x}_i))^2 \\ &= \frac{1}{2} \left[ (-1 - w_0 - w_1 - w_2)^2 + (1 - w_0 + w_1 - w_2)^2 \right. \\ &\quad \left. + (-1 - w_0 + w_1 + w_2)^2 + (1 - w_0 - w_1 + w_2)^2 \right] \\ &= 2(1 + w_0^2 + w_1^2 + w_2^2) \end{aligned}$$

- The error will reach the minimum 2 when  $w_0 = w_1 = w_2 = 0$
- For perceptron learning, the iteration will not stop!
- For gradient descent learning, process will converge to the minimum even the dataset is not linearly-separable!



# Limitations of Threshold and Perceptron Units

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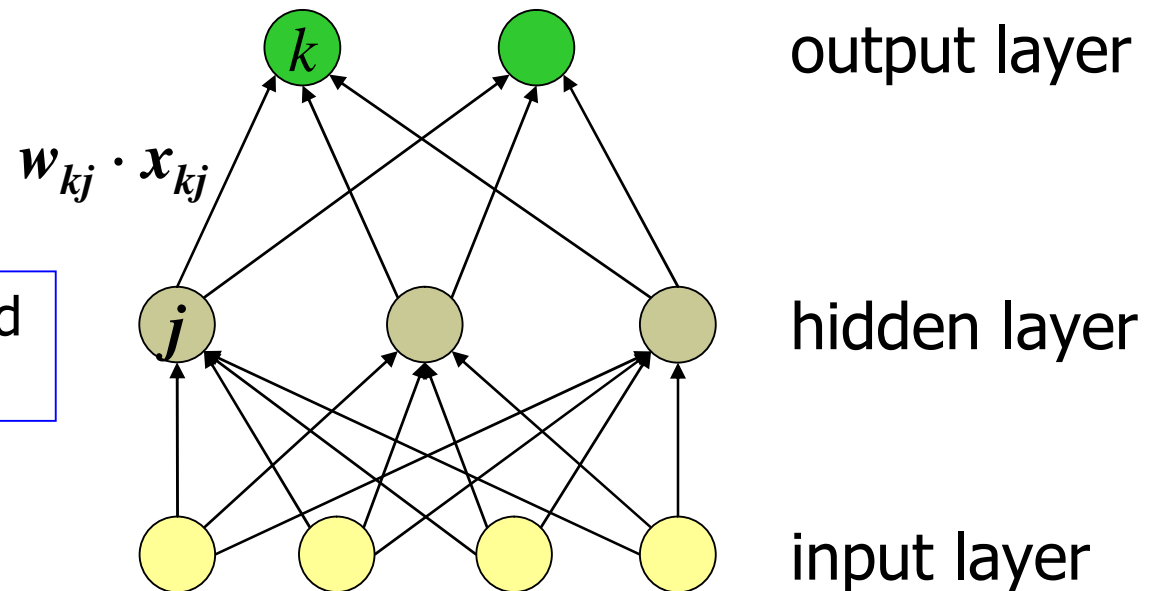
## Limitations of Threshold and Perceptron Units

- Perceptrons can only learn linearly separable classes
- Perceptrons cycle if classes are not linearly separable
- Threshold units converge always to MSE hypothesis
- Network of perceptrons – how to train?
- Network of threshold units – not necessary! (why?)

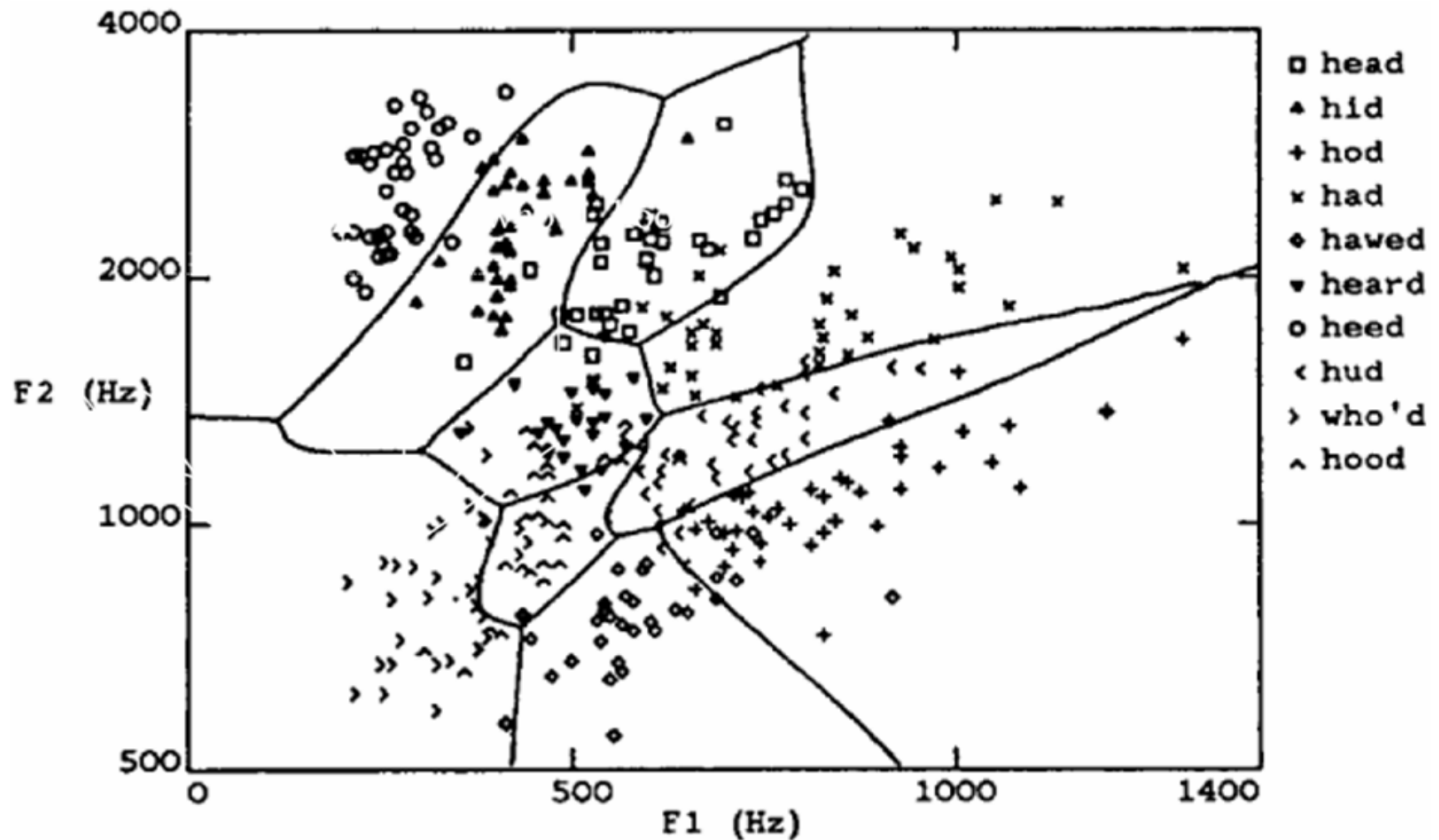
# Multilayer Networks

- Single perceptrons can only express linear decision surfaces
- On the other hand, multilayer networks are capable of expressing a rich variety of nonlinear decision surfaces

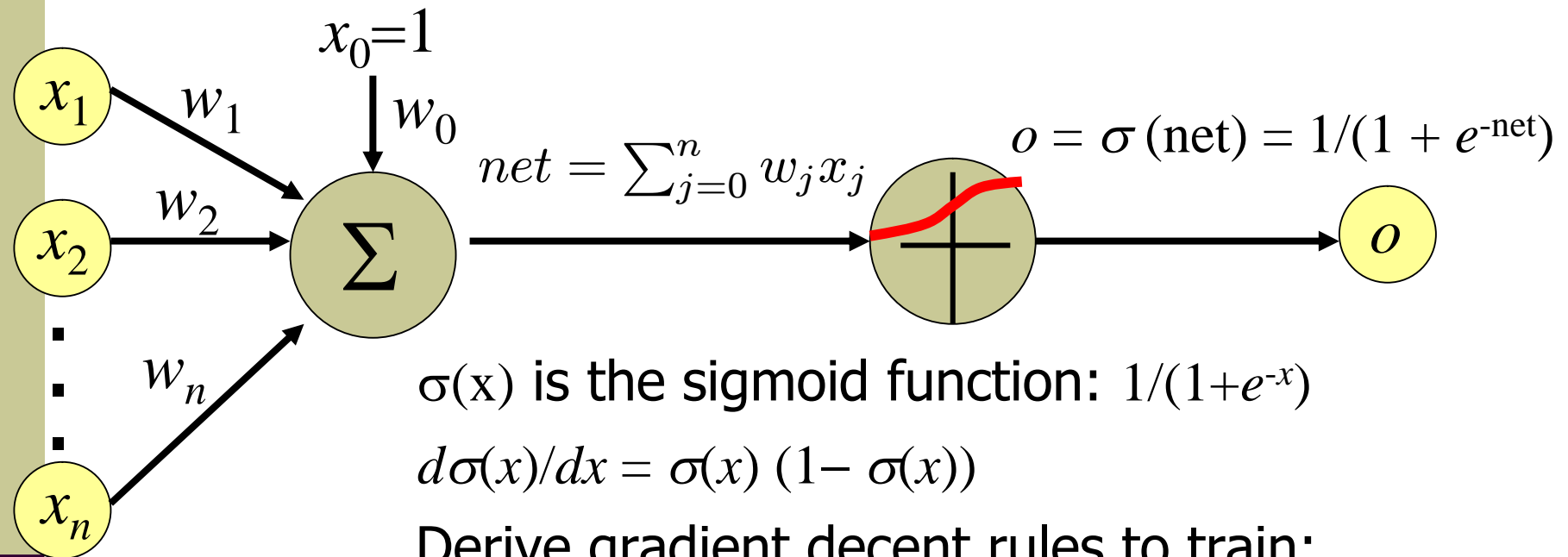
conventionally, it's called a two-layer network



# A Speech Recognition Task



# Sigmoid Threshold Unit



- one sigmoid function

$$\partial E / \partial w_j = -\sum_i (y_i - h(\mathbf{x}_i)) h(\mathbf{x}_i) (1 - h(\mathbf{x}_i)) x_{ij}$$

- Multilayer networks of sigmoid units  
backpropagation:

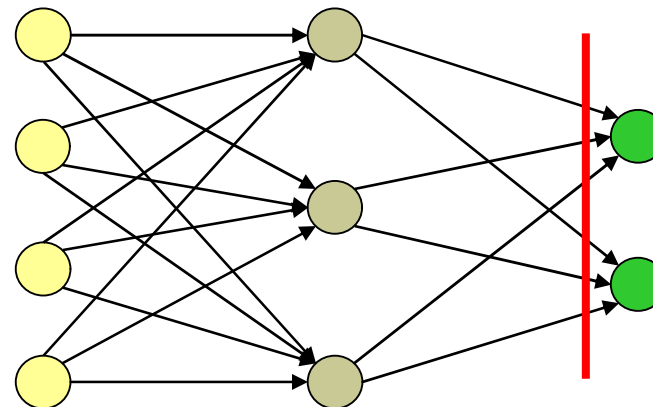
# Designation of Output Units

- Regression: identity function
- Binary classification: e.g., sigmoid function
- Multiclass classification: softmax function

$$h(\mathbf{x}, \mathbf{w}) = \frac{\exp(\text{net}_k)}{\sum_{\ell} \exp(\text{net}_{\ell})}$$

- $K$  binary classification problem:  $K$  sigmoid function for each problem

input layer    hidden layer    output layer



# BACKPROPAGATION Algorithm

Initialize each  $w_j$  to some small random value

Until the termination condition is met, Do

For each training example  $\langle (x_1, \dots, x_n), y \rangle$  Do

Input the instance  $(x_1, \dots, x_n)$  to the network and compute the network outputs  $h_k$  for every output unit  $k$

For each output unit  $k$

$$\delta_k = h_k(1-h_k)(y_k-h_k)$$

For each hidden unit  $j$

$$\delta_j = h_j(1-h_j) \sum_k w_{kj} \delta_k$$

For each network weight  $w_{kj}$  Do

$$w_{kj} = w_{kj} + \Delta w_{kj} \quad \text{where}$$

$$\Delta w_{kj} = \eta \delta_k x_{kj}$$



# Derivation of the **BACKPROPAGATION** Rule I

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$$E_i(\vec{w}) \equiv \frac{1}{2} \sum_{k \in \text{outputs}} (y_k - h_k)^2$$

$$\Delta w_{kj} = -\eta \frac{\partial E_i}{\partial w_{kj}}$$

- $x_{kj}$ : the  $j$ th input to unit  $k$
- $w_{kj}$ : the weight associated with the  $j$ th input to unit  $k$
- $\text{net}_k = \sum_j w_{kj} x_{kj}$  (the weighted sum of inputs for unit  $k$ )
- $h_k$ : the output computed by unit  $k$
- $y_k$ : the target output for unit  $k$
- $\sigma$ : the sigmoid function
- outputs: the set of units in the final layer of the network
- $\text{Downstream}(k)$ : the set of units whose immediate inputs include the output of unit  $k$

# Derivation of the **BACKPROPAGATION** Rule II

$$\begin{aligned}\frac{\partial E_i}{\partial w_{kj}} &= \frac{\partial E_i}{\partial \text{net}_k} \frac{\partial \text{net}_k}{\partial w_{kj}} \\ &= \frac{\partial E_i}{\partial \text{net}_k} x_{kj}\end{aligned}$$

Training rule for  
output unit weights:

$$\begin{aligned}\frac{\partial E_i}{\partial \text{net}_k} &= \frac{\partial E_i}{\partial h_k} \frac{\partial h_k}{\partial \text{net}_k} \\ \frac{\partial E_i}{\partial h_k} &= \frac{\partial}{\partial h_k} \frac{1}{2} \sum_{\ell \in \text{outputs}} (y_\ell - h_\ell)^2\end{aligned}$$

$$\begin{aligned}\frac{\partial E_i}{\partial h_k} &= \frac{\partial}{\partial h_k} \frac{1}{2} (y_k - h_k)^2 \\ &= \frac{1}{2} 2(y_k - h_k) \frac{\partial (y_k - h_k)}{\partial h_k} \\ &= -(y_k - h_k)\end{aligned}$$

$$\begin{aligned}\frac{\partial h_k}{\partial \text{net}_k} &= \frac{\partial \sigma(\text{net}_k)}{\partial \text{net}_k} \\ &= h_k(1 - h_k)\end{aligned}$$

$$\frac{\partial E_i}{\partial \text{net}_k} = -(y_k - h_k) h_k(1 - h_k)$$

$$\Delta w_{kj} = -\eta \frac{\partial E_i}{\partial w_{kj}} = \eta (y_k - h_k) h_k(1 - h_k) x_{kj}$$

# Derivation of the **BACKPROPAGATION** Rule III

$$\frac{\partial E_i}{\partial \text{net}_k} = \sum_{\ell \in \text{Downstream}(k)} \frac{\partial E_i}{\partial \text{net}_\ell} \frac{\partial \text{net}_\ell}{\partial \text{net}_k} \quad \leftarrow \text{Training rule for hidden unit weights}$$

$$= \sum_{\ell \in \text{Downstream}(k)} -\delta_\ell \frac{\partial \text{net}_\ell}{\partial \text{net}_k}$$

$$= \sum_{\ell \in \text{Downstream}(k)} -\delta_\ell \frac{\partial \text{net}_\ell}{\partial h_k} \frac{\partial h_k}{\partial \text{net}_k}$$

$$= \sum_{\ell \in \text{Downstream}(k)} -\delta_\ell w_{\ell k} \frac{\partial h_k}{\partial \text{net}_k}$$

$$= \sum_{\ell \in \text{Downstream}(k)} -\delta_\ell w_{\ell k} h_k (1 - h_k)$$

$$\delta_k = h_k (1 - h_k) \sum_{\ell \in \text{Downstream}(k)} \delta_\ell w_{\ell k}$$

$$\Delta w_{kj} = \eta \delta_k x_{kj}$$

# Backpropagation

- Gradient descent over entire *network* weight vector
- Easily generalized to arbitrary directed graphs
- Will find a local, not necessarily global error minimum
  - in practice often works well (can be invoked multiple times with different initial weights)
- Often include weight *momentum* term
$$\Delta w_{kj}(n) = \eta \delta_k x_{kj} + \alpha \Delta w_{kj}(n-1)$$
- Minimizes error training examples
  - Will it generalize well to unseen instances (overfitting)?
- Training can be slow typical 1000-10000 iterations  
(use Levenberg-Marquardt instead of gradient descent)
- Using network after training is fast

# Learning in Arbitrary Acyclic Networks

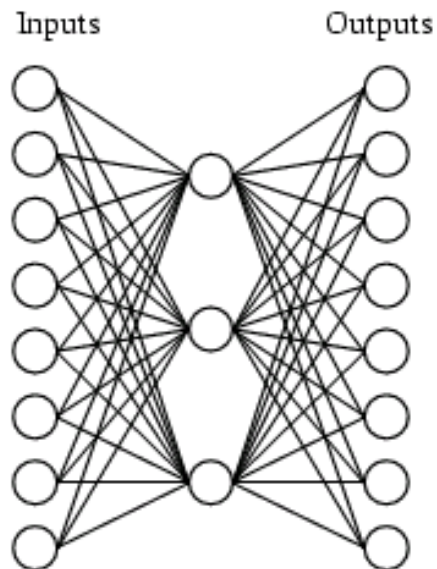
- For networks of more than two layers
  - The  $\delta_r$  value for a unit  $r$  in layer  $m$  is computed from the  $d$  values at the next deeper layer  $m+1$  according to

$$\delta_r = o_r(1 - o_r) \sum_{s \in \text{layer } m+1} w_{sr} \delta_s$$

- For networks where nodes are not arranged in uniform layers
  - The  $\delta_r$  value for any internal unit

$$\delta_r = o_r(1 - o_r) \sum_{s \in \text{Downstream } m+1} w_{sr} \delta_s$$

# Learning Hidden Layer Representations



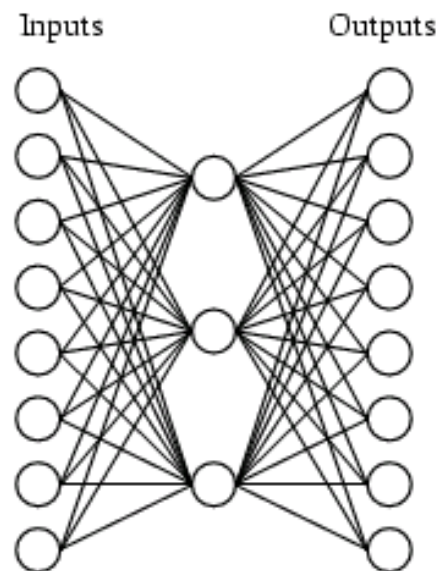
A target function:

Input		Output
10000000	→	10000000
01000000	→	01000000
00100000	→	00100000
00010000	→	00010000
00001000	→	00001000
00000100	→	00000100
00000010	→	00000010
00000001	→	00000001

Can this be learned??

# Learning Hidden Layer Representations

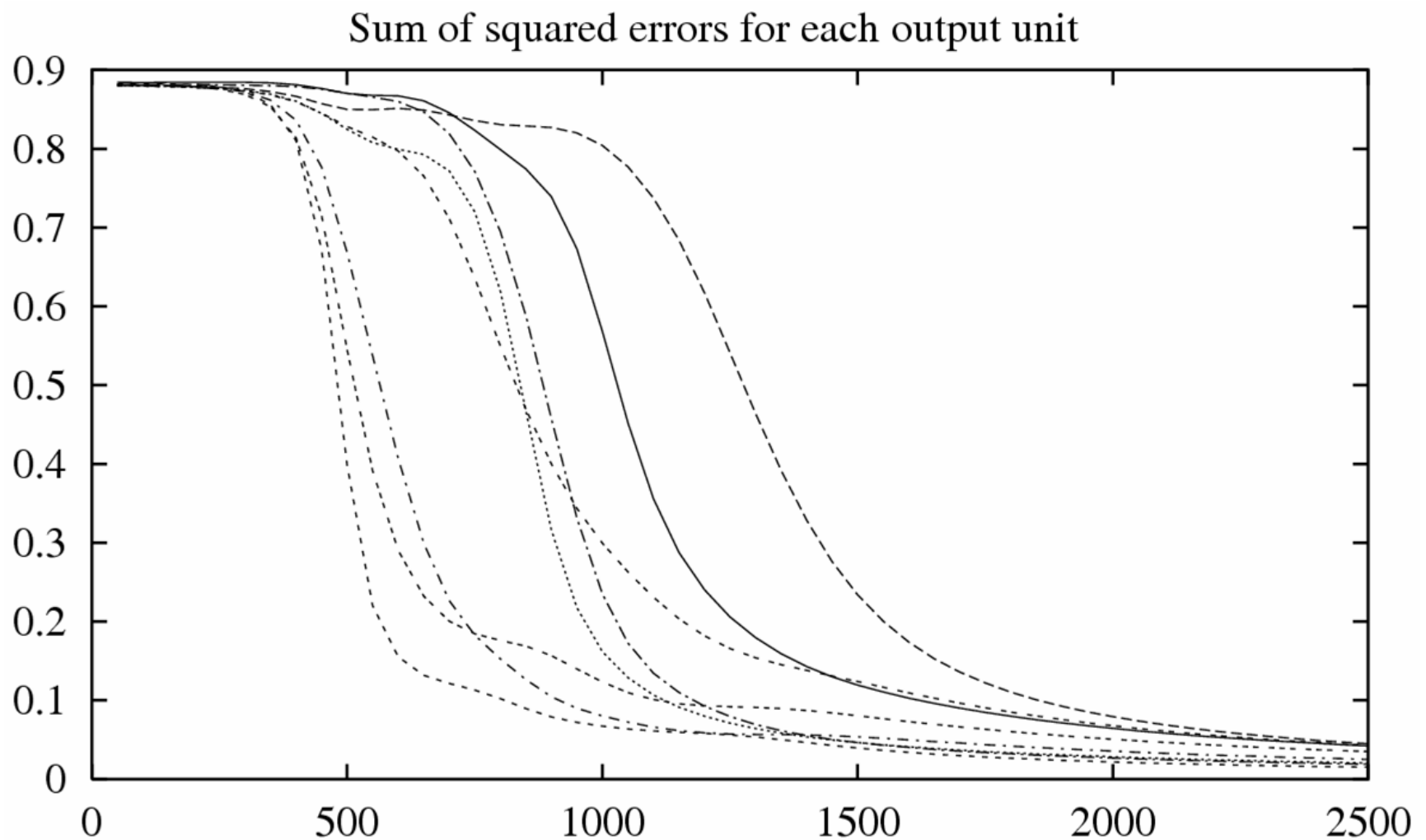
A network:



Learned hidden layer representation:

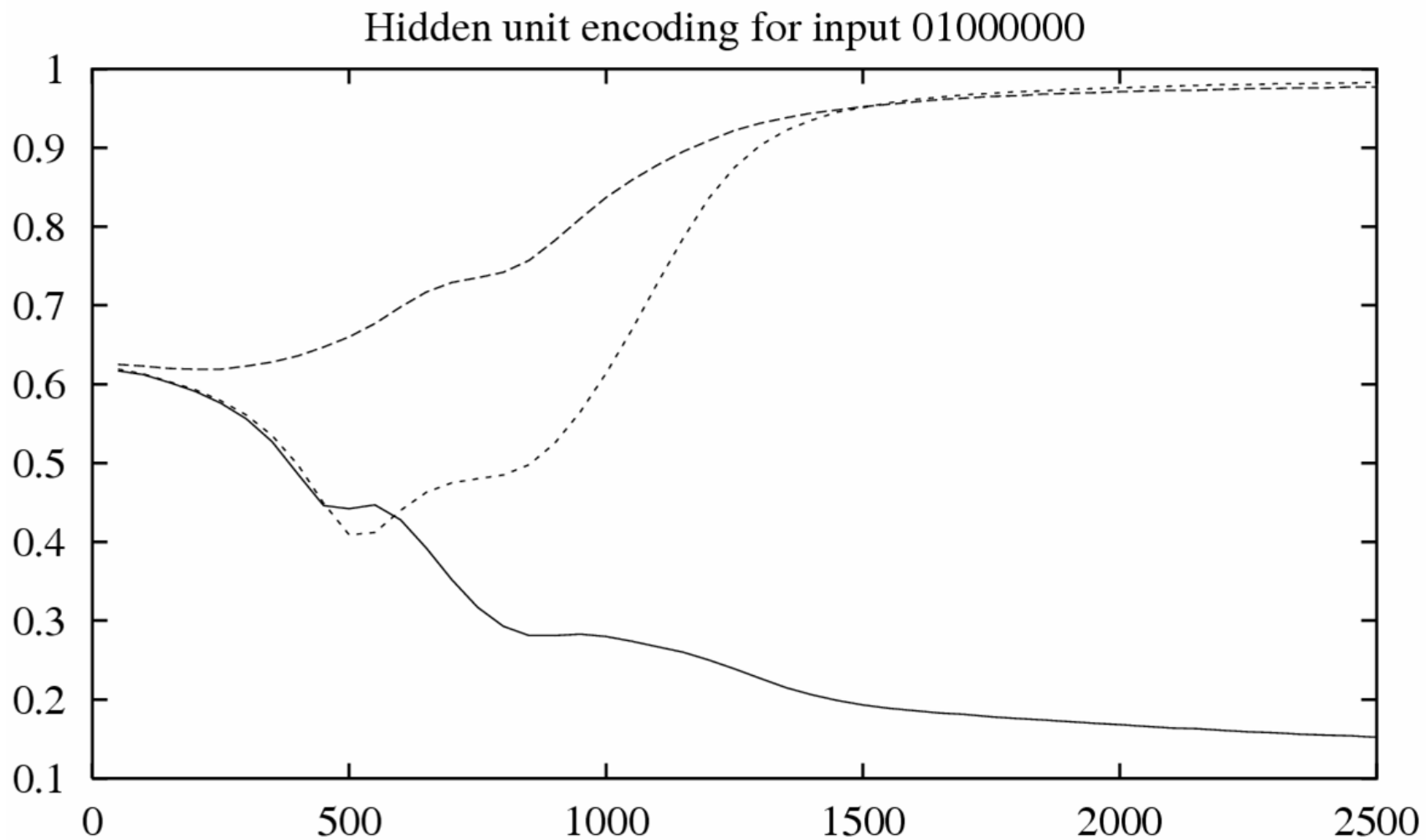
Input		Hidden Values			Output	
10000000	→	.89	.04	.08	→	10000000
01000000	→	.15	.99	.99	→	01000000
00100000	→	.01	.97	.27	→	00100000
00010000	→	.99	.97	.71	→	00010000
00001000	→	.03	.05	.02	→	00001000
00000100	→	.01	.11	.88	→	00000100
00000010	→	.80	.01	.98	→	00000010
00000001	→	.60	.94	.01	→	00000001

# Training

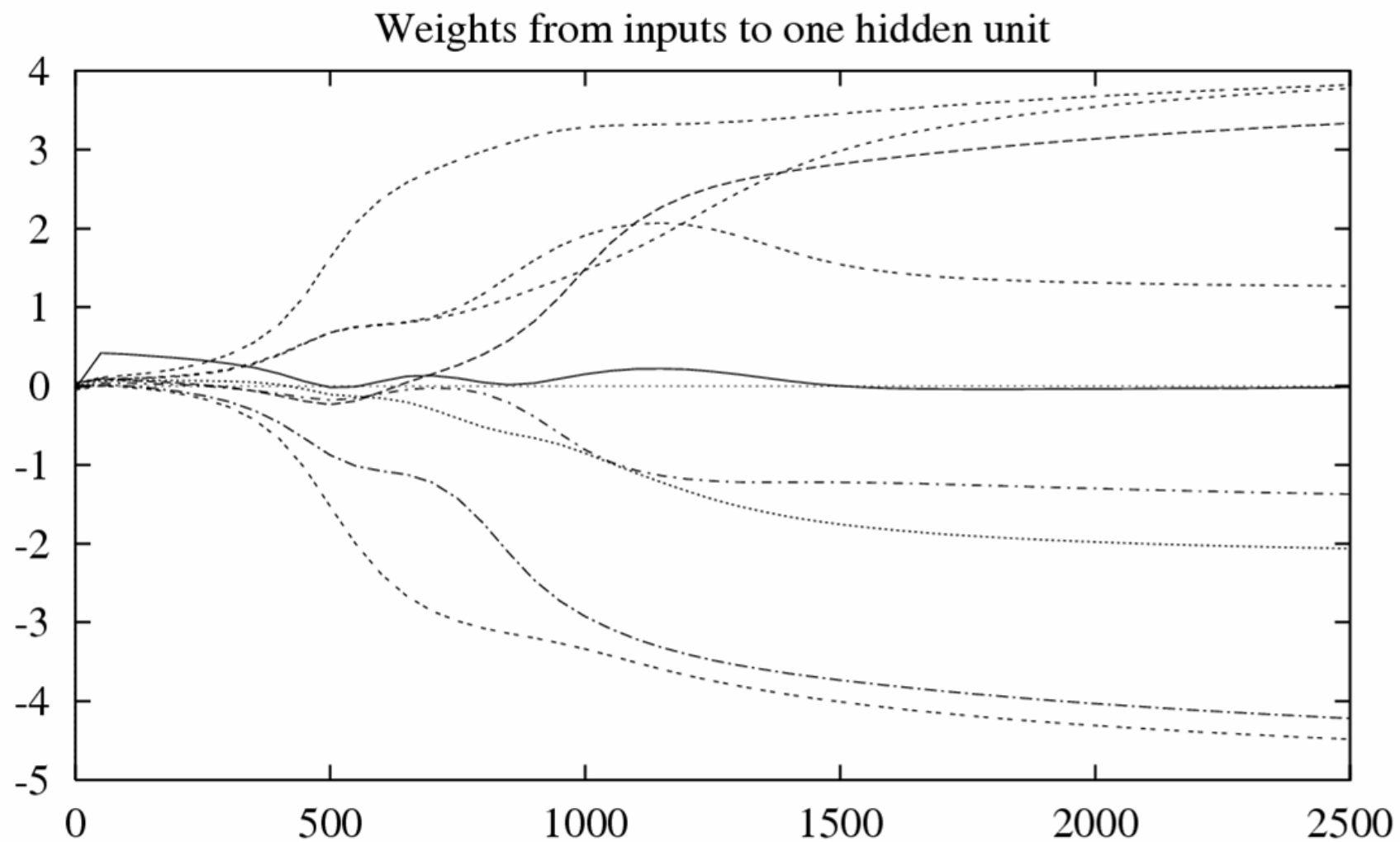




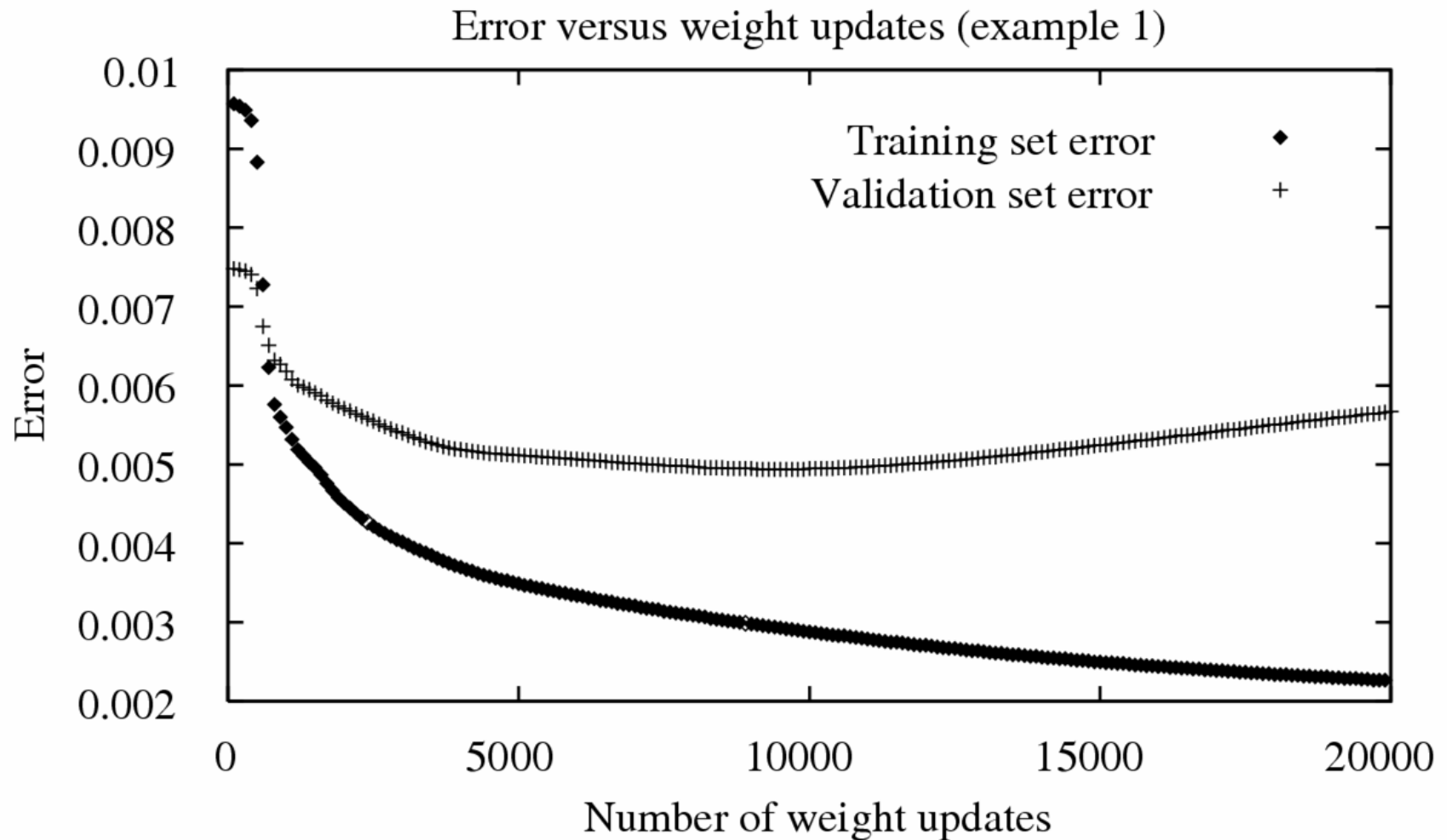
# Training



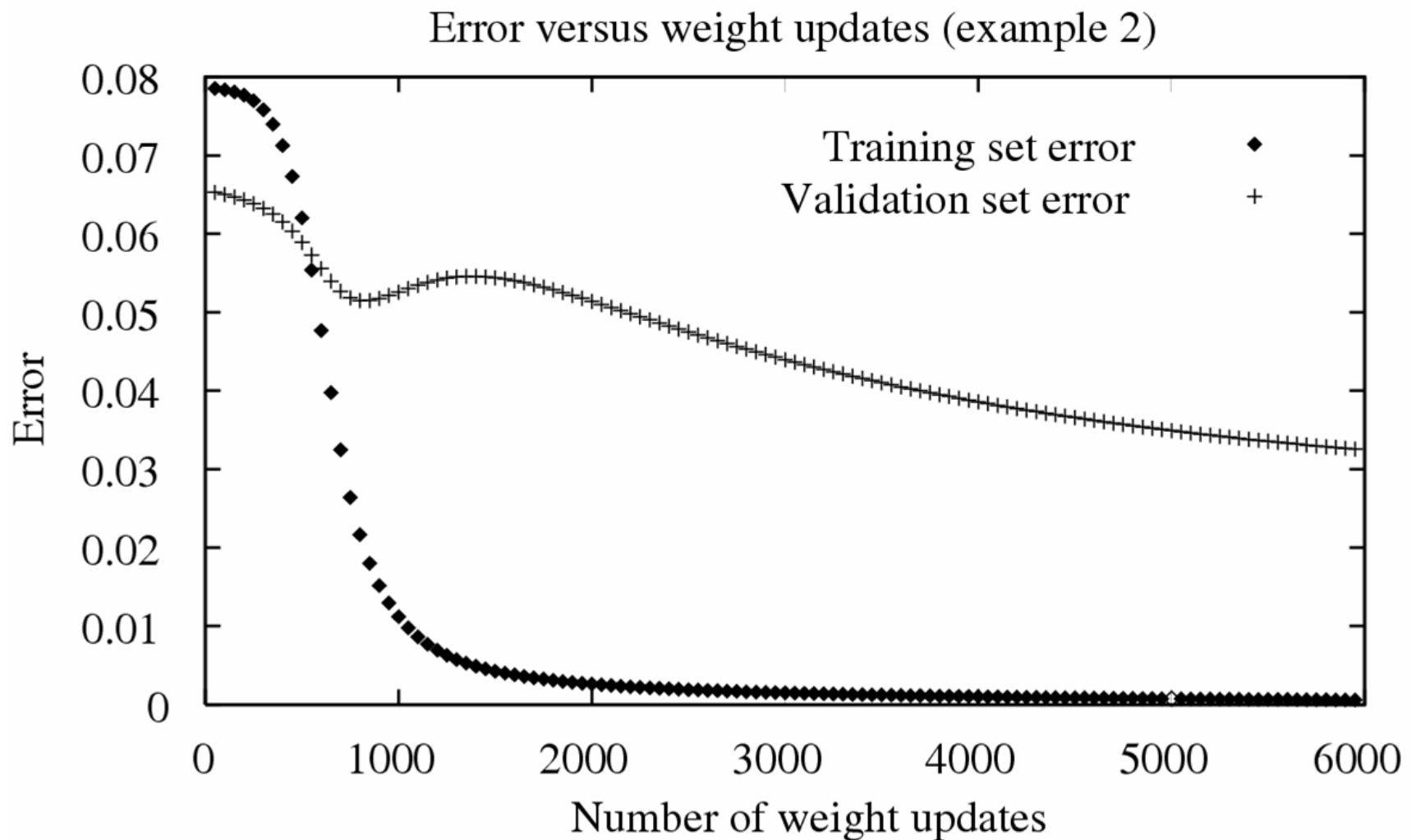
# Training



# Overfitting: case I



# Overfitting: case II



# Convergence of Backprop

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Gradient descent to some local minimum

- Perhaps not global minimum (because the function is nonlinear!)

Nature of convergence

- Initialize weights near zero
- Therefore, initial networks near-linear
- Increasingly nonlinear functions possible as training progresses
- Close enough to the global min. if only a local minimum

# Avoid the Local Minimum

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- Add momentum (through smooth area)
- **Stochastic** gradient descent
- Train multiple nets with different initial weights
  - Choose the best one by validation
  - Using the result from “committee”

# Avoid ANN Overfitting

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## 1. Weight decay

- Decrease each weight by a small factor during each iteration
- Plays the role of a penalty term
- [Keep weight values small]

## 2. Use a different validation set

- Use the number of iterations that leads to the lowest error on the validation set

# Expressive Capabilities of ANN

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## Boolean functions

- Every boolean function can be represented by network with single hidden layer
- But might require exponential (in number of inputs) hidden units

## Continuous functions

- Every bounded continuous function can be approximated with arbitrarily small error, by network with one hidden layer [Cybenko 1989, Hornik 1989]
- Any function can be approximated to arbitrary accuracy by a network with two hidden layers [Cybenko 1988]



# Literature & Resources

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## ■ Textbook:

- “Neural Networks for Pattern Recognition”, Ch. 5, C. M. Bishop, 1996
- “Machine Learning”, Ch. 4, T. M. Mitchell, 1997

## ■ Software:

- Neural Networks for Face Recognition  
<http://www.cs.cmu.edu/afs/cs.cmu.edu/user/mitchell/ftp/faces.html>
- SNNS Stuttgart Neural Networks Simulator  
<http://www-ra.informatik.uni-tuebingen.de/SNNS>
- Neural Networks at your fingertips  
<http://www.stats.gla.ac.uk/~ernest/files/NeuralAppl.html>