χ^2 distribution

Notation:

$$X \sim \chi^2_k \quad \text{with} \quad k \in \mathbb{R}^{>0}$$

The χ^2 distribution is a continuous distribution with k *degrees of freedom*. Expected value and variance are given by:

$$\mathrm{E}(X) = k \qquad \text{and} \qquad \mathrm{Var}(X) = 2k$$

Density function

The density function is given by:

$$f(x) = \begin{cases} \frac{x^{\frac{k}{2}-1} \exp - \frac{x}{2}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} & \text{for} \quad x > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is the gamma function for x.

We can motivate the χ^2 distribution via the standard normal distribution: a sum of stochastically independant, quadratic standard normally distributed random variables follows a χ^2 distribution with k degrees of freedom where k is the number of terms of the sum. Therefore, we have:

$$\sum_{i=1}^{k} z_i^2 \sim \chi^2_k \quad \text{with} \quad z_i \sim N(0,1) \quad \text{and} \quad \mathrm{Cov}(z_i, z_j) = 0 \quad \forall i \neq j$$

Cumulative distribution function

The cumulative distribution function (cdf) is given by:

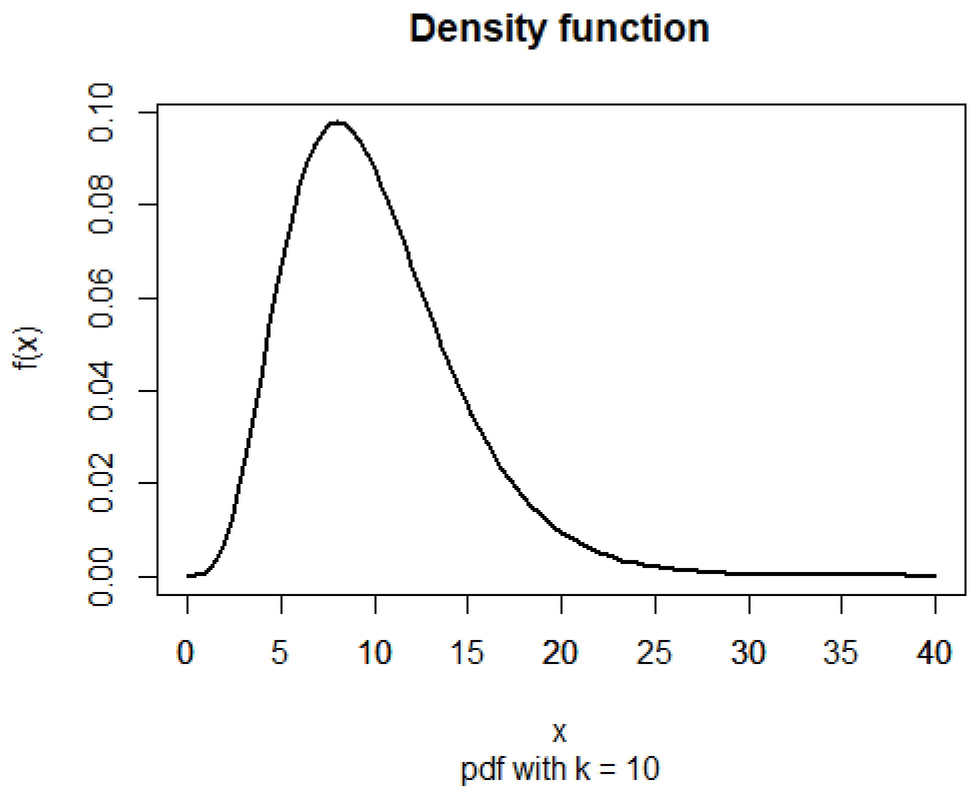
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) dt$$

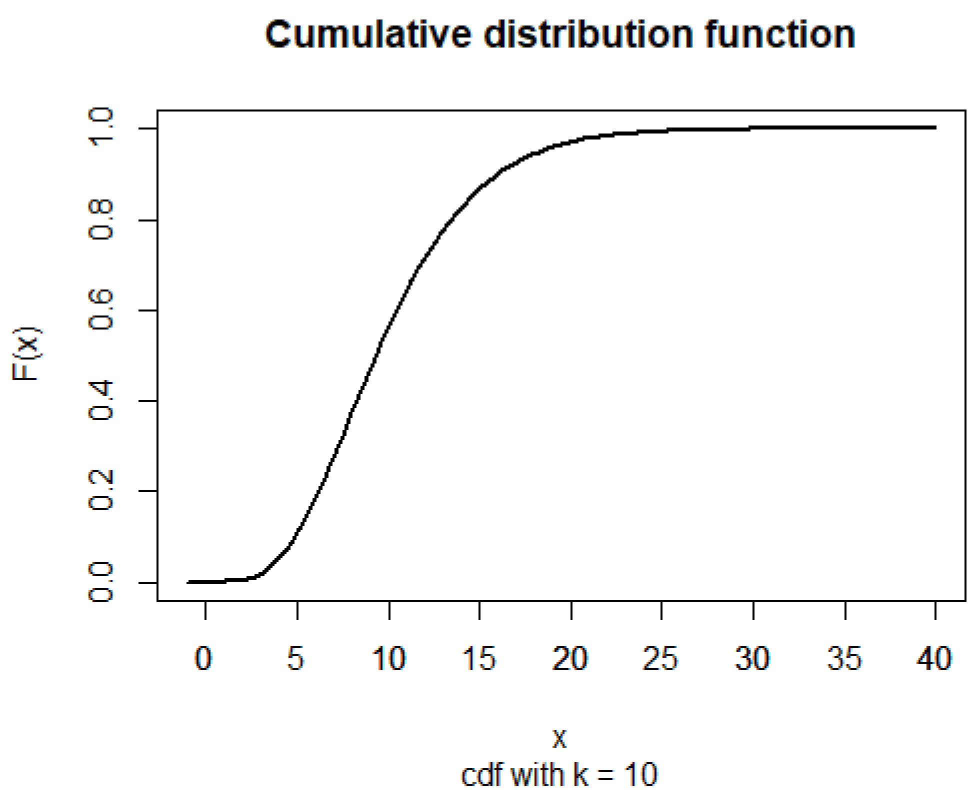
The value of the cumulative distribution function is the probability that the random variable X is less than or equal to x.

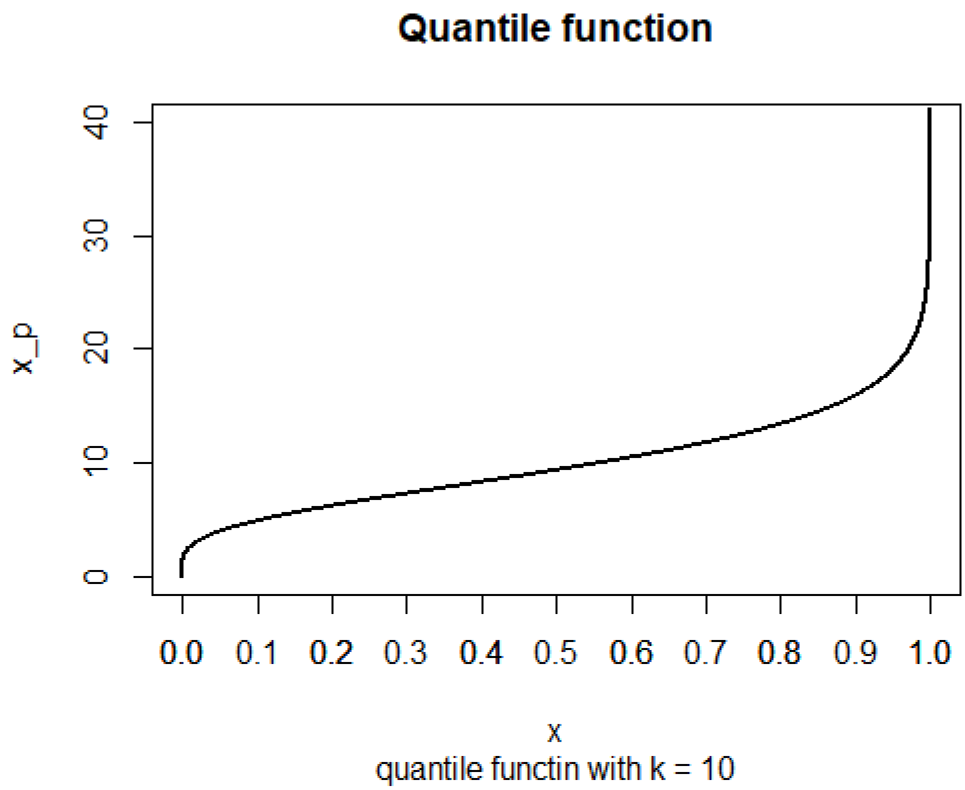
Quantile function

The quantile function returns the value x_p under which is p% of the probability mass. Formally, the quantile function is the inverse of the distribution function:

$$x_p = F^{-1}(p) = F^{-1}[P(X \leq x_p)]$$


Density function
f(x)
0.00
0.02
0.04
0.06
0.08
0.10
0
5
10
15
20
25
30
35
40
x
pdf with k = 10


Cumulative distribution function
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
35
40
x
cdf with k = 10



Excel commands

Density function and distribution function of the χ^2 distribution

- =CHIQU.VERT(x; k; **kumuliert**)
  - x := The value x the function should be evaulated at
  - k := Degrees of freedom
  - kumuliert $= 1$:= Value of the distribution function (a probability)
  - kumuliert $= 0$:= Value of the density function (not a probability!)

Right-tail of the χ^2 distribution

- =CHIQU.VERT.RE(x; k)
  - x := The value x the function should be evaulated at
  - k := Number of degrees of freedom

The function CHIQU.VERT.RE calculates: $P(X \geq x)$

Quantile function of the χ^2 distribution

- =CHIQU.INV(p; k_1; k_2)
  - p := Probability
  - k := Degrees of freedom

Two-sided quantile of the χ^2 distribution

- `=CHIQU.INV.RE`(p; k)
  - $p :=$ Probability
  - $k :=$ Degrees of freedom

The function `CHIQU.INV.RE` calculates: $x = F^{-1}[P(X > x)] = F^{-1}[1 - P(X \leq x)]$