Chapter 2
Stability

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1 Historic remarks and motivations

- Can a first order theory T determine its models;
- Any theory T with an infinite model has models of arbitrary infinite cardinalities (L-S-T);
- For a fixed infinite cardinal κ, how many models of T has cardinal κ?;
- Consider the function $I_T(-) : \kappa \mapsto \#\{\text{models of } T \text{ of card. } \kappa\}$.
- Then $1 \leq I_T(\kappa) \leq 2^\kappa$;
- $\#\{L\text{-structures of card. } \kappa\} \leq 2^\kappa$;

Fact 1.1. *[Morley's Theorem] Let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable cardinal κ, then $I_T(\kappa) = 1$ for all uncountable cardinal κ. (Categoricity)*

Example 1.2. .

- The Theory of infinite sets;
- The theory of vector space over a fixed coundtable field;
- The theory of algebraicly closed fields with fixed char;
- The theory of $(\mathbb{Z}, S, 0)$.

Shelah's stability theory intended to generalize Morley's Theory and classify the complete first order theories.

Conjecture 1.3. *[Morley] Let T be countable, then the function $I_T(\kappa)$ is non-decreasing on uncountable cardinals.*

Fact 1.4. *[Shelah's Main gap theorem] Let T be a countable first order complete theory T. then one of these situations holds:*

- $\forall\alpha,\ I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$
- $\forall\alpha,\ I_T(\aleph_\alpha) < \beth_{\aleph_1}(|\alpha|)$.

Here, $\beth_0(\kappa) = \kappa$, $\beth_\alpha(\kappa) = 2^{\beth_{\alpha+1}(\kappa)}$, and $\beth_\nu(\kappa) = \sup\{\beth_\alpha(\kappa)|\ \alpha < \nu\}$ for limit ordinals ν.

Remark 1.5. .

- The name "Main Gap" refers to the gap between $\beth_{\aleph_1}(|\alpha|)$ and $2^{\aleph_\alpha}$ $(\alpha \geq \omega)$
- Depending on α this may be no gap at all;
- But in general $\beth_{\aleph_1}(|\alpha|)$ goes only moderately compared to $2^{\aleph_\alpha}$;
- The case $I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$ is called thenon-structure case, we have a kind of chaos.
- The second case, namely, the case where there are relatively few non-isomorphic models, is called the structure case;
- In this case every model can be characterized up to isomorphism in terms of certain invariants;
- The most important "dividing lines" on the space of first-order theories is "stability";
- Main gap theorem says that: "If T is a first-order theory and is stable and . . . , then the class of models looks like . . . . Otherwise, theres no hope".

2 Counting types and stability'

Definition 2.1. For a complete first order theory T, let $f_T : \mathrm{Card} \to \mathrm{Card}$ be defined by

$$f_T(\kappa) = \sup\{\ |S_1 M|\ :\ M \models T,\ |M| = \kappa\},$$

for κ an infinite cardinal.

It is esay to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$.

Fact 2.2. *Let T be an arbitrary complete theory in a first order language. The $f_T(\kappa)$ is one of the following functions*

$$\kappa, \kappa + 2^{\aleph_0},\ \operatorname{ded} k,\ (\operatorname{ded} k)^{\aleph_0},\ 2^\kappa$$

Here

$$\operatorname{ded}\kappa = \sup\{\ |I| :\ I \textit{ is a linear order with a dense subset of size } \kappa\}$$

$$\operatorname{ded}\kappa = \sup\{\ \lambda :\ \textit{There is a linear order of size } \kappa \textit{ with } \lambda \textit{ cuts}\}$$

Lemma 2.3. $\kappa < \operatorname{ded} \kappa \leq 2^{\kappa}$.

Proof. $\kappa < \operatorname{ded} \kappa$:

- Let μ be minimal such that $2^{\mu} > \kappa$;
- Consider 2^{μ} as a set of $0-1$ sequence of length μ;
- then $2^{<\mu}$ is a dense subset of 2^{μ};
- $\mu \leq \kappa \implies 2^{<\mu} \leq \kappa$;
- so $\operatorname{ded} \kappa \geq \mu > \kappa$.

$\operatorname{ded} \kappa \leq 2^{\kappa}$:

- Every cut is determined by the subset of elements in its lower half.

□

Definition 2.4. Let $M \models T$.

- A formula $\phi(x, y)$, with its variables partitioned into two groups x, y, has the k-order property, $k \in \omega$, if there are some $a_i \in M_x$, $b_j \in M_y$ for $i, j < k$ such that

$$M \models \phi(a_i, b_j) \iff i < j$$

- $\phi(x, y)$ has the order property if it has the k-order property for all $k \in \omega$;
- We say that a formula $\phi(x, y) \in L$ is stable if there is some $k \in \omega$ such that it does not have the k-order property.
- A theory is stable if it implies that all formulas are stable (note that this is indeed a property of a theory).

Proposition 2.5. *Assume that T is unstable, then $f_T(\kappa) \geq \operatorname{ded} \kappa$ for all cardinals $\kappa \geq |T|$.*

Proof.

- Fix a cardinal κ. Let $\phi(x, y) \in L$ be a formula has the k-order property for all $k \in \omega$;
- Let $(I, <)$ be a dense linear order order of size κ;
- Let $a_{i \in I}$ and $b_{i \in I}$ be two sequences of new constants;
- Then $\{\phi(a_i, b_j) | \ i < j\} \cup \{\neg\phi(a_i, b_j) | \ i \geq j\}$ is consistent with T;
- By compactness, there is a model $\mathcal{M} \models T$ and $a_{i \in I}$, $b_{i \in I}$ from M such that

$$\mathcal{M} \models \phi(a_i, b_j) \iff i < j$$

- By L-S-T, we may assume that $|M| = \kappa$;
- For any cut $C = (A, B)$ in I

$$\Phi_C(x) = \{\phi(x, b_j)|\ i \in B\} \cup \{\neg\phi(x, b_j)|\ j \in A\}$$

  is a partial type over M;
- It is easy to see that $C_1 \neq C_2 \implies \Phi_{C_1} \cup \Phi_{C_2}$ is inconsistent;
- Let $p_C(x) \in S_x(M)$ be a complete extension of $\Phi_C(x)$;
- Then $|S_x(M)| \geq$ number of cuts in I;
- As I is arbitrary,

$$f_T(\kappa) = \sup\{|S_x(M)|\ M \models T, |M| = \kappa\} \geq \operatorname{ded}\kappa$$

□

Recall

Fact 2.6 (Ramsey Theory)**.** $\aleph_0 \to (\aleph_0)^n_k$ *holds for all* $n, k \in \omega$ *(i.e.for any coloring of subsets of* N *of size* n *in* k *colors, there is some infinite subset* I *of* N *such that all* n*-element subsets of* I *have the same color).*

Lemma 2.7. *Let* $\phi(x, y)$, $\psi(x, z)$ *be stable formulas (where* y, z *are not necessarily disjoint tuples of variables). Then:*

1. *Let* $\phi^*(y, x) = \phi(x, y)$, *i.e. we switch the roles of the variables. Then* $\phi^*(y, x)$ *is stable.*
2. $\neg\phi(y, x)$ *is stable.*
3. $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ *and* $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ *are stable.*
4. *If* $y = uv$ *and* $c \in M_v$ *then* $\theta(x, u) := \phi(x, uc)$ *is stable.*
5. *If* T *is stable, then every* L^{eq}*-formula is stable as well.*

Proof. .
(1) and (2) are trivial.
(3):

- Suppose that $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ is unstable;
- there are $(a_i, b_i, c_i|\ i \in \mathbb{N})$ such that

$$\phi(a_i, b_j) \vee \psi(a_i, c_j) \iff i < j$$

- Let $f : [\mathbb{N}]^2 \to \{0,1\}$ defined by: for each $i < j \in \mathbb{N}$

$$f(i,j) = 1 \iff \models \psi(a_i, c_j) \text{ and } f(i,j) = 0 \iff \models \neg\psi(a_i, c_j)$$

- By Ramsey, there is a infinite subset $I \subseteq J$ such that
- f is constant on I;
- If $f(I) = 1$, then $\forall i, j \in I(\psi(a_i, b_j) \iff i < j)$
- If $f(I) = 0$, then $\forall i, j \in I(\phi(a_i, b_j) \iff i < j)$
- So either ϕ or ψ is unstable.

(4): Trivial. □

Theorem 2.8 (Erdös-Makkai)**.** *Let B be an infinite set, $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(c_{i<\omega}) \subseteq B$ and $(S_{i<\omega}) \subseteq \mathcal{F}$ such that one of the following holds:*

1. $c_i \in S_j \iff j < i (\forall i, j \in \omega)$,

2. $c_i \in S_j \iff i < j (\forall i, j \in \omega)$.

We need the following lemma:

Lemma 2.9. *Let X be a set and $Y_1, ..., Y_n$ are subsets of X. Define:*

$$E(x,y) := \bigwedge_{i=1}^{n} (x \in X_i \iff y \in X_i).$$

Then E is an equivalence relation on X and $Z \subseteq X$ is a boolean combination of X_i's iff

$$E(x,y) \implies (x \in Z \iff y \in Z)$$

Proof. Exercise. □

We now proof the Theorem 2.8

Proof.
- Choose $\mathcal{F}' \subseteq \mathcal{F}$ such that
  - $|\mathcal{F}'| = |B|$;
  - For any finite $B_0, B_1 \subseteq B$,

$$\exists S \in \mathcal{F}(B_1 \subseteq S \wedge B_2 \subseteq B \backslash S) \implies \exists S' \in \mathcal{F}'(B_1 \subseteq S' \wedge B_2 \subseteq B \backslash S').$$

- $\mathcal{F}'$ exists as there are at most $|B|$-many different pairs of finite subsets of B;

- $|\mathcal{F}| > |\mathcal{F}'| \implies \exists S^* \in \mathcal{F}$ which is not a boolean combination of elements of $\mathcal{F}'$;
- Let $a_0 \in S^*$ and $b_0 \notin S^*$;
- There is $S_0 \in \mathcal{F}'$ s.t. $a_0 \in S_0$ and $b_0 \notin S_0$;
- Since S^* is NOT a b. c. of $\{S_0\}$, there are a_1, b_1 s.t. :
  - $a_1 \in S_0 \iff b_1 \in S_0$, and ;
  - $a_1 \in S^*$ but $b_1 \notin S^*$.
- Now $\{a_0, a_1\} \subseteq S^*$ and $\{b_0, b_1\} \subseteq B \backslash S^*$;
- By the assumption of $\mathcal{F}'$, $\exists S_1 \in \mathcal{F}'(\{a_0, a_1\} \subseteq S_1 \wedge \{b_0, b_1\} \subseteq B \backslash S_1)$;
- Since S^* is NOT a b. c. of $\{S_0, S_1\}$, there are a_2, b_2 s.t. :
  - $a_2 \in S_i \iff b_2 \in S_i$, for $i < 2$, and ;
  - $a_2 \in S^*$ but $b_2 \notin S^*$.
- ...
- Inductively, we have infinite sequences $(a_{i<\omega}) \subseteq S^*$ and $(b_{i<\omega}) \subseteq B \backslash S^*$ s.t.
  - $a_n \in S_i \iff b_n \in S_i$, for $i < n$;
  - $\{a_0, ..., a_n\} \subseteq S_n$, $\{b_0, ..., b_n\} \subseteq B \backslash S_n$

  By Ramsey, there is an infinite $I \subseteq \omega$ such that
  - either $\forall i < j \in I(a_i \in S_j) \implies \forall i < j \in I(b_i \in S_j \iff i > j)$,
  - or $\forall i < j \in I(a_i \notin S_j) \implies \forall i < j \in I(a_i \in S_j \iff i \leq j)$
- In the first case we set $c_i = b_i$;
- In the second case we set $c_i = a_{i+1}$;

$\square$

Definition 2.10. Let $\phi(x, y)$ be a formula, by a complete ϕ-type over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x, b), \neg\phi(x, b)$ where b ranges over A. Let $S_\phi(A)$ be the space of all complete ϕ-types over A.

Proposition 2.11. *Assume that $|S_\phi(B)| > |B|$ for some infinite set of parameters B. Then $\phi(x, y)$ is unstable.*

Proof. .

- For $a \in \mathbb{M}_x$, $\text{tp}_\phi(a/B)$ is determined by $\phi(a, B) = \{b \in B | \models \phi(a, b)\}$;

- $|S_\phi(B)| > |B| \implies |\{\phi(a,B) |\ a \in \mathbb{M}_x\}| > |B|$;
- By Erdös-Makkai, there are sequences $(a_{i<\omega})$ and $(a_{i<\omega})$ s.t.

$$\text{either } \models \phi(a_i, b_j) \iff i < j, \text{ or } \models \phi(a_i, b_j) \iff j < i.$$

$\square$

3 Local ranks and definability of types

Definition 3.1. We define Shelahs local 2-rank taking values in $\{-\infty\} \cup \omega \cup \{+\infty\}$ by induction on $n \in \omega$. Let Δ be a set of L-formulas, and $\theta(x)$ a partial type over $\mathbb{M}$.

- $R_\Delta(\theta(x)) \geq 0 \iff \theta$ is consistent (and $-\infty$ otherwise);
- $R_\Delta(\theta(x)) \geq n+1$ if $\exists \phi(x,y) \in \Delta$ and $a \in \mathbb{M}_y$ s.t.

$$R_\Delta(\theta(x) \wedge \phi(x,a)) \geq n \text{ and } R_\Delta(\theta(x) \wedge \neg\phi(x,a)) \geq n$$

- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and $R_\Delta(\theta(x)) \not\geq n+1$
- $R_\Delta(\theta(x)) = +\infty$ if $R_\Delta(\theta(x)) \geq n$ for all $n \in \omega$.

If ϕ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$.

Proposition 3.2. *$\phi(x,y)$ is stable iff $R_\phi(x=x)$ is finite.*

Proof. Assume that $\phi(x,y)$ is unstable:

- By compactness, there is a sequence $(a_i b_i | \ i \in [0,1])$ such that

$$\models \phi(a_i, b_j) \iff i < j$$

- Both $\phi(x, b_{\frac{1}{2}})$ and $\neg\phi(x, b_{\frac{1}{2}})$ contain dense subsequences of a_i's.
- Each of these sets can be split again, by $\phi(x, b_{\frac{1}{4}})$ and $\phi(x, b_{\frac{3}{4}})$;
- ...

Conversely, assume that the rank is infinite:

- We can find a infinity tree of parameters

$$B = \{b_\eta | \ \eta \in 2^{<\omega}\}$$

  such that
- for each $\eta \in 2^\omega$, let

$$\Phi_\eta = \{\phi^{\eta(n)}(x, b_{\eta|_n}) | \ n \in \omega\},$$

  where $\phi^1 = \phi$ and $\phi^0 = \neg\phi$;
- Then each Φ_η is consistent;
- Different Φ_η's are inconsistent;
- $|S_\phi(B)| \geq 2^{|B|} \implies \phi(x,y)$ is unstable.

$\square$

Definition 3.3. .

- Let $\phi(x,y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is definable over B if there is some $L(B)$-formula $\psi(y)$ such that for all $a \in A$

$$\phi(x,a) \in p \iff \models \psi(a)$$

- A type $p \in S_x(A)$ is definable over B if $p|_\phi$ is definable over B forall $\phi(x,y) \in L$.
- A type is definable if it is definable over its domain.
- We say that types in T are uniformly definable if for every $\phi(x,y)$ there is some $\psi(y,z)$ such that every type can be defined by an instance of $\psi(y,z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that

$$\phi(x,a) \in p \iff \models \psi(a,b),$$

  forall $a \in A$.

Remark 3.4. .

- Let $A \subseteq \mathbb{M}_x$, and $B \subseteq A$. We say that B is externally definable if there is some $\mathbb{M}$-definable set X such that $B = X \cap A$
- If $X = \phi(\mathbb{M}, c)$. Then $\mathrm{tp}_\phi(c/A)$ is definable iff B is in fact internally definable.
- A set is called stably embedded if for every externally definable subset of it is internally definable.

Example 3.5. Consider $(\mathbb{Q}, <) \models DLO$, and let $p = \mathrm{tp}(\pi/\mathbb{Q})$. Then $x < y \in p(x) \iff x < \pi$. By QE, p is not definable.

Lemma 3.6. .

1. *The set $\{e \in \mathbb{M}^k |\ R_\phi(\theta(x,e)) \geq n\}$ is definable for all $n \in \omega$;*
2. *If $R_\phi(\theta(x)) = n$, then for any $a \in \mathbb{M}_y$, at most one of $\theta(x) \wedge \phi(x,a)$, $\theta(x) \wedge \neg\phi(x,a)$ has R_ϕ-rank n.*

Proof. (1):

- Induction on n.
- $n = 0 \implies R_\phi(\theta(x,e)) \geq 0 \iff \models \exists x(\theta(x,e))$;

- $n = k+1 \implies$

$$R_\phi(\theta(x,e)) \geq k{+}1 \iff \exists y(\Big(R_\phi(\theta(x,e)\wedge\phi(x,y)) \geq k\Big)\wedge\Big(R_\phi(\theta(x,e)\wedge\neg\phi(x,y)) \geq k\Big))$$

(2): Trivial. □

Proposition 3.7. *Let $\phi(x,y)$ be a stable formula. Then all ϕ-types are uniformly definable.*

Proof. .

- Suppose that $R_\phi(x=x)$ is $n \in \omega$;
- Let $p \in S_\phi(A)$;
- Then there is $\chi(x) \in p$ such that $R_\phi(\chi(x)) = \min\{R_\phi(\varphi(x))|\ \varphi \in p\}$;
- For each $b \in A_y$, either $\phi(x,b) \in p$ or $\neg\phi(x,b) \in p$;
- either $R_\phi(\chi(x) \wedge \phi(x,b)) < n$ or $R_\phi(\chi(x) \wedge \neg\phi(x,b)) < n$;
- $R_\phi(\chi(x))$ is minimal $\implies$ $\big(\phi(x,b) \in p \iff R_\phi(\chi(x) \wedge \phi(x,b)) = n\big)$.

□

Theorem 3.8. *The following are equivalent for a formula $\phi(x,y)$.*

1. *$\phi(x,y)$ is stable;*
2. *$R_\phi(x=x) < \omega$;*
3. *All ϕ-types are uniformly definable;*
4. *All ϕ-types over models are uniformly definable;*
5. *$S_\phi(M) \leq \kappa$ for all $\kappa \geq |L|$ and $M \models T$ with $|M| = \kappa$;*
6. *There is some κ such that $|S_\phi(M)| < \operatorname{ded}\kappa$ for all $M \models T$ with $|M| = \kappa$.*

Proof. .

- (1) $\iff$ (2) by Proposition 3.2;
- (2) $\implies$ (3) by Proposition 3.7;
- (3) $\implies$ (4) is obvious;
- (4) $\implies$ (5), each ϕ-type is determined by a $L(M)$-formula, its own definition;
- (5) $\implies$ (6) is obvious;

- (6) $\implies$ (1) is by Proposition 2.5.

$\square$

Global case:

Theorem 3.9. *Let T be a complete theory. Then the following are equivalent.*

1. *T is stable;*
2. *There is NO sequence of tuples $(c_i|\ i \in \omega)$ from $\mathbb{M}$ and formula $\phi(z_1, z_2) \in L(M)$ such that*
$$\models \phi(c_i, c_j) \iff i < j;$$
3. *$f_T(\kappa) \leq \kappa^{|T|}$ for all infinite cardinals κ;*
4. *There is some κ such that $f_T(\kappa) \leq \kappa$;*
5. *There is some κ such that $f_T(\kappa) < \operatorname{ded} \kappa$;*
6. *All formulas of the form $\phi(x, y)$ where x is a singleton variable, are stable;*
7. *All types over models are definable.*

Proof. .

- (1) $\implies$ (2) by definition;
- (2) $\implies$ (1):
  - Let $\psi(x, y)$ be a formula with order property witnessed by sequence
$$\{(a_i, b_i)|\ i < \omega\};$$
  - Let $\phi(x_1y_1;\ x_2y_2) := \psi(x_1, y_2)$ and $c_i := a_ib_i$;
  - Then $\models \phi(c_i, c_j) \iff i < j$.
- (1) $\implies$ (3) :$S_x(M) \to \prod_{\phi \in L} S_\phi(M)$ is injective;
- (3) $\implies$ (4) is obvious;
- (4) $\implies$ (5) is obvious;
- (5) $\implies$ (1) is by Proposition 2.5.
- (6) $\iff$ $(1-5)$: Fix some κ, then $S_1(M) \leq \kappa$ for all M with $|M| = \kappa$ iff $S_n(M) \leq \kappa$ for all M with $|M| = \kappa$;
- (7) $\iff$ $(1-5)$ by Theorem 3.9

$\square$

Example 3.10.

- Stability $\iff$ all types over all models are definable;
- Some unstable theories have certain special models over which all types are definable;
- $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$, all types over $\mathbb{R}$ are uniformly definable;
- $\mathcal{M} = (\mathbb{Q}_p, +, \times, 0, 1)$, all types over $\mathbb{Q}_p$ are uniformly definable.