Week3

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Exercise 1. Show that the collection of formulas $x > a$ for $a \in M$ generates a complete type $\tau_M(x) \in S_1(M)$. In other words, show that the partial type $\{(x > a) : a \in M\}$ has a unique completion

Proof. Let $\Sigma(x) = \{x > a : a \in M\}$, $p, q \in S_1(M)$, $p, q \supseteq \Sigma(x)$, $p \neq q$. Since DLO has quantifier elimination, there is a quantifier free formula $\varphi(x) \in p \setminus q$. φ has the form

$$\bigwedge_{a \in A} x > a \wedge \bigwedge_{b \in B} x \leq b \wedge \bigwedge_{c \in C} x \neq c \wedge \bigwedge_{d \in D} x = d$$

where A, B, C, D are finite. But since $p, q \supseteq \Sigma(x)$, $B = D = \emptyset$. Also $x \neq c$ is implied by $\Sigma(x)$. Thus if we choose $a' = \max\{a : a \in A\}$, then

$$M \vdash \varphi \leftrightarrow x > a'$$

Thus $\varphi \in q$, a contradiction. Hence $p = q$ □

Exercise 2. Show that τ_M is definable

Proof. By exercise 3 for each $N \succeq M$, τ_M has a unique heir and thus τ_M is definable □

Exercise 3. Suppose $N \succeq M$. Show that τ_N is an heir of τ_M

Proof. Let $q \in S_1(N)$ be an heir of τ_M and suppose $x \leq a \in q(x)$ for some $a \in N$. Then there is $a' \in M$ s.t. $x \leq a' \in \tau_M$, which is impossible. Thus $\{x > a : a \in N\} \subseteq q$ and $q = \tau_N$. Hence τ_N is the unique heir of τ_M by Exercise 1 □

Exercise 4. Suppose $N \succeq M$ and N is $|M|^+$-saturated. Show that τ_N is not a coheir of τ_M

Proof. Since N is $|M|^+$-saturated, there is $c \in N$ s.t. $N \models \tau_M(c)$. Then there is no $a \in M$ satisfying $x > c$. $\square$

Exercise 5. If $N \succeq M$, show that τ_M has a unique coheirs over N

Proof. Suppose τ_M has two different coheirs $p, q \in S_1(N)$. Because p and q are the same thing as cuts, we may assume that there is $c \in N$ s.t. $x < c \in p$ and $x > c \in q$. But for any x realizing, $x > m$ for all $m \in M$. Thus there is no $m \in M$ satisfying $x > c$. Hence p is not a coheir of τ_M, a contradiction $\square$

Exercise 6. Give an example of models $M \preceq N$ of DLO where τ_N is a coheir of τ_M

Proof. $M = \mathbb{Q}$, $N = \mathbb{R}$ $\square$