Homework 9: forking calculus

Advanced Model Theory

Due May 12 [sic], 2022

1 Setting

Let $\mathbb{M}$ be a monster model of some theory. Let $A \mathop{\smile\!\!\!\!\!\!|}_C B$ be a ternary relation between small subsets $A, B, C \subseteq \mathbb{M}$. (The theory is not necessarily stable and $\mathop{\smile\!\!\!\!\!\!|}$ is not necessarily forking independence.) Suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the following properties:

(Symmetry) $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$.

(Monotonicity) If $A' \subseteq A$ and $B' \subseteq B$ then $A \mathop{\smile\!\!\!\!\!\!|}_C B \implies A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.

(Normality) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then $A \mathop{\smile\!\!\!\!\!\!|}_C BC$.

(Transitivity) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{CB} B'$, then $A \mathop{\smile\!\!\!\!\!\!|}_C BB'$.

(Base monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}_C BB'$, then $A \mathop{\smile\!\!\!\!\!\!|}_{CB} B'$.

(Invariance) If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(B)$.

(Extension) Given A, B, C, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(A) \mathop{\smile\!\!\!\!\!\!|}_C B$.

In these statements, "AB" is an abbreviation for $A \cup B$. (You may use this abbreviation in your solutions.) See the next page for the problems.

2 Problems

There are six problems. **Please only do problems 3–6. The solutions of problems 1–2 are given as examples later in this document** though I suppose you could try to do them on your own for practice. If $n < m$, you may assume the conclusion of Problem n in your solution to Problem m, including when $n \leq 2$. For help typesetting, see the last page.

1. Suppose $C \subseteq C'$ and $AB \mathop{\smile\!\!\!\!\!\!\!\;|}_C C'$. Show that $A \mathop{\smile\!\!\!\!\!\!\!\;|}_C B \iff A \mathop{\smile\!\!\!\!\!\!\!\;|}_{C'} B$.
2. Show that $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B \operatorname{acl}(B)$ for any A, B.
3. Show that $A \mathop{\smile\!\!\!\!\!\!\!\;|}_C B \iff A \mathop{\smile\!\!\!\!\!\!\!\;|}_{\operatorname{acl}(C)} \operatorname{acl}(BC)$ for any A, B, C. *Hint:* use Problem 2 and things related to transitivity.

For the remaining problems, assume $B_1 \subseteq B_2$, and A is some set.

4. Show there is a model $M_1 \supseteq B_1$ such that $M_1 \mathop{\smile\!\!\!\!\!\!\!\;|}_{B_1} AB_2$, and a model $M_2 \supseteq M_1 \cup B_2$ such that $M_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{M_1 B_2} A$. *Hint:* use Extension.

Fix M_1, M_2 as in the previous problem.

5. Show that $A \mathop{\smile\!\!\!\!\!\!\!\;|}_{B_1} M_1$ and $A \mathop{\smile\!\!\!\!\!\!\!\;|}_{B_2} M_2$. *Hint:* play around with Monotonicity, Symmetry, Base Monotonicity, and Transitivity. Eventually things will work out.
6. Show that $A \mathop{\smile\!\!\!\!\!\!\!\;|}_{B_1} B_2 \iff A \mathop{\smile\!\!\!\!\!\!\!\;|}_{M_1} M_2$. *Hint:* use Problem 5 and things related to Transitivity.

See the next page for more hints.

When writing your solutions, please try to be clear about which statements are logically connected to which statements using what rules. Try to avoid writing something like the "confusing solution" to Problem 1 given below.

3 General hints

Base monotonicity says

$$A \mathop{\downarrow}_{C} BB' \implies A \mathop{\downarrow}_{CB} B'.$$

Here is another way of describing base monotonicity, which is useful in certain contexts. Suppose $C_1 \subseteq C_2 \subseteq C_3$. Then $A \downarrow_{C_1} C_3$ implies $A \downarrow_{C_2} C_3$, because

$$A \mathop{\downarrow}_{C_1} C_3 \iff A \mathop{\downarrow}_{C_1} C_2C_3 \xRightarrow{\text{Base Monotonicity}} A \mathop{\downarrow}_{C_1C_2} C_3 \iff A \mathop{\downarrow}_{C_2} C_3.$$

In fact, when $C_1 \subseteq C_2 \subseteq C_3$, we have

$$A \mathop{\downarrow}_{C_1} C_3 \iff (A \mathop{\downarrow}_{C_1} C_2 \text{ and } A \mathop{\downarrow}_{C_2} C_3).$$

The $\Leftarrow$ direction is transitivity: if $A \downarrow_{C_1} C_2$ and $A \downarrow_{C_1C_2} C_3$, then $A \downarrow_{C_1} C_2C_3$. The $\Rightarrow$ direction is base monotonicity (in the form just described) plus monotonicity.

By Symmetry, there are "left" versions of Normality, Transitivity, Base monotonicity, and Extension:

(Left normality) If $A \downarrow_C B$, then $AC \downarrow_C B$.

(Left transitivity) If $A \downarrow_C B$ and $A' \downarrow_{CA} B$, then $AA' \downarrow_C B$.

(Left base monotonicity) If $AA' \downarrow_C B$, then $A' \downarrow_{CA} B$.

(Right extension) Given A, B, C, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $A \downarrow_C \sigma(B)$.

None of the problems actually use normality or left normality, by the way.

4 Solutions to problem 1 and 2

1. Suppose $C \subseteq C'$ and $AB \downarrow_C C'$. Show that $A \downarrow_C B \iff A \downarrow_{C'} B$.

   Solution. Assume
   $$AB \mathop{\downarrow}_{C} C'. \tag{1}$$
   First suppose
   $$A \mathop{\downarrow}_{C} B. \tag{2}$$
   Then (1) and left base monotonicity gives
   $$A \mathop{\downarrow}_{BC} C'. \tag{3}$$

Next, (2), (3), and transitivity give

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} BC'. \tag{4}$$

Finally, (4) and base monotonicity[1] give

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B.$$

Conversely, suppose

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B. \tag{5}$$

Then (1) and monotonicity give

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} C'. \tag{6}$$

Next, (5), (6) and transitivity[2] give

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} C'B. \tag{7}$$

Finally, (7) and monotonicity give

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B. \qquad \square$$

Confusing solution. [3]

$\Rightarrow$: By left base monotonicity, $A \mathop{\smile\!\!\!\!\!\!\mid}_{BC} C'$. By transitivity, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} BC'$. By base monotonicity, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B$.

$\Leftarrow$: By monotonicity, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} C'$. By transitivity, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} C'B$. So $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$. $\square$

2. Show that $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} \operatorname{acl}(B)$ for any A, B.

   Solution. By right extension, there is $\sigma \in \operatorname{Aut}(\mathbb{M}/B)$ such that

   $$A \mathop{\smile\!\!\!\!\!\!\mid}_{B} \sigma(\operatorname{acl}(B)).$$

   But $\sigma(\operatorname{acl}(B)) = \operatorname{acl}(\sigma(B)) = \operatorname{acl}(B)$, so $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} \operatorname{acl}(B)$. $\square$

[1] $A \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B$ is equivalent to $A \mathop{\smile\!\!\!\!\!\!\mid}_{CC'} B$ because $C \subseteq C'$.

[2] $A \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B$ is equivalent to $A \mathop{\smile\!\!\!\!\!\!\mid}_{CC'} B$ because $C \subseteq C'$.

[3] This is the same solution, written in a way that is hard to follow.

5 Typesetting

You might find it easier to do this assignment by hand rather than using LaTeX. If you choose to use LaTeX, here are some useful things to know.

The symbol $\perp\!\!\!\smile$ doesn't exist in LaTeX, but you can get it by putting

```
\DeclareMathOperator*{\ind}{\raise0.2ex\hbox{\ooalign{
\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}
```

(without the line break) in the preamble of your LaTeX document—the part of the document before

```
\begin{document}
```

Then you can get $\perp\!\!\!\smile$ by writing `\ind` in math mode.[4] If you want $A \perp\!\!\!\smile_C B$ or $A \perp\!\!\!\smile_{CD} B$, you can write `A \ind_C B` or `A \ind_{CD} B`.

To put a slash through the $\perp\!\!\!\smile$ sign, indicating "not independent", you can write `A \not\ind_C B`, which gives $A \not\perp\!\!\!\smile_C B$, and people will know what you mean. If you want the slash to not look ridiculously, put

```
\usepackage{centernot}
```

in the preamble and write `A \centernot\ind_C B` which gives $A \not\perp\!\!\!\smile_C B$.

To get numbered equations, use

```
\begin{equation}
\end{equation}
```

or

```
\begin{gather}
\end{gather}
```

rather than the `equation*`, `gather*` environments. (The "equation" environment can only do a single line. The "gather" environment can do several lines, separated by \\.) If you write something like

```
\begin{equation}
  2 = 1 + 1, \label{mylabel}
\end{equation}
```

it will look like

$$2 = 1 + 1, \tag{8}$$

and you can refer to (8) in the text by writing `(\ref{mylabel})`. (Give each numbered line its own label; don't call them all "mylabel".)

[4]For other better ways to typeset $\perp\!\!\!\smile$, see `https://tex.stackexchange.com/questions/42093/what-is-the-latex-symbol-for-forking-independent-model-theory`