Omitting types and ω-categoricity

Introductory Model Theory

December 16, 2021

Recommended reading: Section 10.1 and Theorem 10.11, though both use a slightly different approach from what we do here. Section 10.1 uses Henkin enumerations; we do something simpler. Theorem 10.11 uses atomic models; we use saturated models instead.

1 Topology on type-spaces

Suppose M is an L-structure and $A \subseteq M$. Fix $n \leq \omega$ [sic].

Definition 1. If $\phi(x_1, \ldots, x_n)$ is an $L(A)$-formula, then $[\phi] = \{p \in S_n(A) : \phi \in p\}$.

If $N \succeq M$ and $b \in N^n$, then

$$\mathrm{tp}^N(b/A) \in [\phi] \iff \phi \in \mathrm{tp}^N(b/A) \iff N \models \phi(b).$$

Thus

$$\begin{aligned}
[\phi \wedge \psi] &= [\phi] \cap [\psi] \\
[\phi \vee \psi] &= [\phi] \cup [\psi] \\
[\neg\phi] &= S_n(A) \setminus [\phi].
\end{aligned}$$

Definition 2. A set $X \subseteq S_n(A)$ is *clopen* if $X = [\phi]$ for some $\phi(\bar{x}) \in L(A)$. A set $X \subseteq S_n(A)$ is *open* if $X = \bigcup_{i \in I} Y_i$ where the Y_i are clopen.

Lemma 3. *If X, Y are open, then $X \cap Y$ is open.*

Proof. Let $X = \bigcup_{i \in I} X_i$ and $Y = \bigcup_{j \in J} Y_j$, where the X_i and Y_j are clopen. Then $X \cap Y = \bigcup_{(i,j) \in I \times J} (X_i \cap Y_j)$, where the $X_i \cap Y_j$ are clopen. □

Lemma 4. *Let X_i be clopen for $i \in I$. Suppose $\{X_i : i \in I\}$ has the FIP: for any $I_0 \subseteq_f I$, $\bigcap_{i \in I_0} X_i \neq \varnothing$. Then $\bigcap_{i \in I} X_i \neq \varnothing$.*

Proof. Let $X_i = [\phi_i]$. Let $\Sigma = \{\phi_i : i \in I\}$. Then Σ is finitely satisfiable, so there is a type $p \in S_n(A)$ with $p \supseteq \Sigma$. Then $p \in \bigcap_{i \in I} [\phi_i]$. □

Lemma 5. *If $S_n(A) = \bigcup_{i \in I} X_i$ and each X_i is clopen, then there is $I_0 \subseteq_f I$ such that $S_n(A) = \bigcup_{i \in I_0} X_i$.*

Proof. Let $Y_i = S_n(A) \setminus X_i$. By assumption, $\bigcap_{i \in I} Y_i = \varnothing$, so there must be $I_0 \subseteq_f I$ such that $\bigcap_{i \in I_0} Y_i = \varnothing$, or equivalently, $\bigcup_{i \in I_0} X_i = S_n(A)$. $\square$

Definition 6. ϕ *isolates* p if for any $N \succeq M$ and $\bar{b} \in N^n$,

$$\mathrm{tp}(\bar{b}/A) = p \iff N \models \phi(\bar{b})$$

or equivalently

$$\mathrm{tp}(\bar{b}/A) = p \iff \mathrm{tp}(\bar{b}/A) \in [\phi]$$

or equivalently

$$\mathrm{tp}(\bar{b}/A) \in \{p\} \iff \mathrm{tp}(\bar{b}/A) \in [\phi].$$

Equivalently, ϕ isolates p if $\{p\} = [\phi]$.

A type p is *isolated* if some formula isolates it, or equivalently, $\{p\}$ is clopen.

Lemma 7. *$S_n(A)$ is finite iff all types in $S_n(A)$ are isolated.*

Proof. Suppose $S_n(A) = \{p_1, \ldots, p_m\}$. For $1 < i \leq m$, take $\phi_i \in p_1, \phi_i \notin p_i$. Then $p_1 \in [\phi_i]$, $p_i \notin [\phi_i]$. Therefore $[\bigwedge_{i=1}^m \phi_i] = \bigcap_{i=1}^m [\phi_i] = \{p_1\}$, and p_1 is isolated. Similarly, p_i is isolated.

Suppose each $p \in S_n(A)$ is isolated. The family $\{\{p\} : p \in S_n(A)\}$ covers $S_n(A)$, so there is a finite subcover. This is impossible unless $S_n(A)$ is finite. $\square$

Definition 8. A set $X \subseteq S_n(A)$ is *dense* if X intersects any non-empty clopen set Y.

Taking $Y = [\top] = S_n(A)$, we see dense sets are non-empty.

Theorem 9 (Baire Category Theorem for $S_n(A)$)**.** *Let $U_1, U_2, U_3, \ldots$ be dense open sets. Then $\bigcap_{i=1}^\infty U_i$ is dense (hence non-empty).*

Proof. Let V_0 be a non-empty clopen set. Then $V_0 \cap U_1$ is a non-empty open set. It contians a non-empty clopen set V_1. Continuing, we can build a descending chain of clopen sets

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

with $V_i \subseteq U_i$. The family $\{V_0, V_1, V_2, \ldots\}$ has the FIP, so there is $p \in \bigcap_{i=0}^\infty V_i \subseteq \bigcap_{i=1}^\infty U_i$. $\square$

Definition 10. A set $X \subseteq S_n(A)$ is *comeager* if $X \supseteq \bigcap_{i=1}^\infty U_i$ for some dense open sets U_i.

In other words, a set is comeager if it contains a countable intersection of dense open sets. By Theorem 9, any comeager set is dense.

Lemma 11. *If $X_1, X_2, \ldots$ are comeager, then $\bigcap_{i=1}^\infty X_i$ is comeager.*

Proof. Take dense open sets $U_{i,1}, U_{i,2}, \ldots$ with $X_i \supseteq \bigcap_{j=1}^\infty U_{i,j}$. Then $\bigcap_{i=1}^\infty X_i \supseteq \bigcap_{i=1}^\infty \bigcap_{j=1}^\infty U_{i,j}$, so $\bigcap_{i=1}^\infty X_i$ is comeager. $\square$

2 Half of Ryll-Nardzewski

Fix a countable language L, and a complete L-theory T with infinite models.

Lemma 12. *If $p \in S_n(A)$ is isolated, then p is realized in M.*

Proof. Take ϕ isolating p. Then $\{\phi\} \subseteq_f p$, so ϕ is satisfied by some $b \in M^n$. (Types are finitely satisfiable.) Then $\mathrm{tp}(b/A) \in [\phi] = \{p\}$, so $\mathrm{tp}(b/A) = p$. $\square$

In contrast, we'll see below in §3 that non-isolated types are not necessarily realized—they are *omitted* in certain models.

Lemma 13. *If $|A| = m$, then $|S_n(A)| \leq |S_{n+m}(\varnothing)|$.*

Proof. See Lemmas 11 and 13 in last week's notes. It's because if $A = \{a_1, \ldots, a_m\}$, then there is an injection $S_n(A) \to S_{n+m}(\varnothing)$ sending $\mathrm{tp}(\bar{b}/A)$ to $\mathrm{tp}(\bar{b}, \bar{a}/\varnothing)$. $\square$

Recall that T is *ω-categorical* if T has a unique model of size ω. If T is a complete theory, then $S_n(T)$ denotes $S_n(\varnothing)$ in any model of T.

Theorem 14. *Let T be a complete theory. Suppose $S_n(T)$ is finite for all $n < \omega$. Then T is ω-categorical.*

Proof. If $M \models T$ and $A \subseteq_f M$, then $S_n(A)$ is finite by Lemma 13. By Lemma 7, every type in $S_n(A)$ is isolated. By Lemma 12, every type in $S_n(A)$ is realized.

So every model of T is ω-saturated. Every countable model is saturated. There is a unique countable saturated model, so there is a unique countable model. $\square$

3 Omitting types

Work in $S_\omega(T)$, the space of types in ω-many variables $x_0, x_1, x_2, x_3, \ldots$. We will show that if $\mathrm{tp}(c_0, c_1, c_2, \ldots/\varnothing)$ is "generic," then $\{c_0, c_1, c_2, \ldots\}$ is a model of T omitting whatever non-isolated types we want.

Lemma 15. *For any formula $\phi(x_0, \ldots, x_n, y)$, there is a dense open set Z_ϕ such that if $M \models T$ and $\bar{c} \in M^\omega$ and $\mathrm{tp}^M(\bar{c}) \in Z_\phi$ and $M \models \exists y\ \phi(c_0, \ldots, c_n, y)$, then there is $i < \omega$ such that $M \models \phi(c_0, \ldots, c_n, c_i)$.*

Proof. Take $A = [\neg\exists y\ \phi(x_0, \ldots, x_n, y)]$ and $B_i = [\phi(x_0, \ldots, x_n, x_i)]$ for $i < \omega$. Let $Z_\phi = A \cup \bigcup_{i=0}^\infty B_i$, which is open. If $p = \mathrm{tp}^M(\bar{c}) \in Z_\phi$ and $M \models \exists y\ \phi(c_0, \ldots, c_n, y)$, then $p \notin A$, so there is $i < \omega$ such that $p \in B_i$, meaning $M \models \phi(c_0, \ldots, c_n, c_i)$.

It remains to show that Z_ϕ is dense. Take non-empty $[\psi] \subseteq S_\omega(T)$; we claim $Z_\phi \cap [\psi] \neq \varnothing$. Take $p = \mathrm{tp}^M(\bar{e}) \in [\psi]$. We may assume $p \notin Z_\phi$, or we are done. Then $p \notin A$, so $M \models \exists y\ \phi(e_0, \ldots, e_n, y)$. Take $b \in M$ such that $M \models \phi(e_0, \ldots, e_n, b)$. Take $i > n$ so large that x_i doesn't appear in ϕ. Let $\bar{c} = (e_0, \ldots, e_{i-1}, b, e_{i+1}, e_{i+2}, \ldots)$. We have $M \models \psi(\bar{e})$ because $\mathrm{tp}(\bar{e}) \in [\psi]$, and therefore $M \models \psi(\bar{c})$, so $\mathrm{tp}(\bar{c}) \in [\psi]$. Also, $M \models \phi(c_0, \ldots, c_n, c_i)$, so $\mathrm{tp}(\bar{c}) \in B_i \subseteq Z_\phi$, showing $Z_\phi \cap [\psi] \ni \mathrm{tp}(\bar{c})$. $\square$

Proposition 16. *There is a comeager set $W \subseteq S_\omega(T)$ such that if $\mathrm{tp}^M(\bar{c}) \in W$, then $\{c_i : i < \omega\} \preceq M$.*

Proof. Let $W = \bigcap_\phi Z_\phi$. Suppose $\mathrm{tp}^M(\bar{c}) \in W$. Then for any $\phi(x_0, \dots, x_n, y)$, if $M \models \exists y\ \phi(c_0, \dots, c_n, y)$, then there is $i < \omega$ such that $M \models \phi(c_0, \dots, c_n, c_i)$. By Tarski-Vaught, $\{c_i : i < \omega\} \preceq M$. ☐

Lemma 17. *Let $p \in S_n(T)$ be non-isolated. For any $(j_1, \dots, j_n) \in \mathbb{N}^n$, there is a dense open set $V_{p,\bar{j}} \subseteq S_\omega(T)$ such that $\mathrm{tp}^M(\bar{c}) \in V_{p,\bar{j}} \iff \mathrm{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$.*

Proof. Let $V_{p,\bar{j}} = V = \bigcup_{\phi \in p} [\neg\phi(x_{j_1}, \dots, x_{j_n})]$. If $\mathrm{tp}^M(\bar{c}) \in V$, then there is some $\phi \in p$ such that $M \models \neg\phi(c_{j_1}, \dots, c_{j_n})$, and so $\mathrm{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$. Conversely, if $\mathrm{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$, there is $\phi \in p$ such that $M \models \neg\phi(c_{j_1}, \dots, c_{j_n})$, and then $\mathrm{tp}^M(\bar{c}) \in V$.

It remains to show that V is dense. Suppose $[\psi] \subseteq S_\omega(T)$ is non-empty; we claim $V \cap [\psi] \neq \varnothing$. Take $q = \mathrm{tp}^M(\bar{e}) \in [\psi]$. We may assume $q \notin V$. By choice of V, $\mathrm{tp}(e_{j_1}, \dots, e_{j_n}) = p$. Take m so large that $m \geq \max(j_1, \dots, j_n)$ and ψ is a formula in $x_0, \dots, x_m$. Let $\phi(y_1, \dots, y_n)$ be

$$\exists x_0, \dots, x_m\ \psi(x_0, \dots, x_m)\ \wedge\ \bigwedge_{i=1}^{n} (y_i = x_{j_i}).$$

Then $(e_{j_1}, \dots, e_{j_n})$ satisfies ϕ (take $(x_0, \dots, x_m) = (e_0, \dots, e_m)$), and so $\phi \in p$. As p is not isolated, there is $N \models \phi(d_1, \dots, d_n)$ with $\mathrm{tp}(d_1, \dots, d_n) \neq p$. By definition of ϕ there are $c_0, \dots, c_m \in N$ with $N \models \psi(c_0, \dots, c_m)$ and $(d_1, \dots, d_n) = (c_{j_1}, \dots, c_{j_n})$. Choose $c_{m+1}, c_{m+2}, \dots \in N$ arbitrarily. Then $\bar{c} = (c_i : i < \omega) \in N^\omega$, and $\mathrm{tp}(\bar{c}) \in [\psi]$, and $\mathrm{tp}(c_{j_1}, \dots, c_{j_n}) = \mathrm{tp}(d_1, \dots, d_n) \neq p$, so $\mathrm{tp}(\bar{c}) \in V$, showing $V \cap [\psi] \neq \varnothing$. ☐

Proposition 18. *Let $p \in S_n(T)$ be non-isolated. There is a comeager set $V_p \subseteq S_\omega(T)$ such that if $\mathrm{tp}^M(\bar{c}) \in V_p$, then p is not realized by a tuple in $\{c_i : i < \omega\}$.*

Proof. Let $V_p = \bigcap_{\bar{j} \in \mathbb{N}^n} V_{p,\bar{j}}$. If $\mathrm{tp}^M(\bar{c}) \in V_p$, then for any $j_1, \dots, j_n \in \mathbb{N}$,

$$\mathrm{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p.$$

☐

Theorem 19 (Omitting types theorem)**.** *Let Π be a countable set of pairs (p, n), where $n < \omega$ and p is a non-isolated type in $S_n(T)$. There is a countable model $M \models T$ omitting p for every $(p, n) \in \Pi$.*

Proof. The set $Q = W \cap \bigcap_{(p,n) \in \Pi} V_p$ is comeager, hence non-empty. Take $\mathrm{tp}^N(\bar{c}) \in Q$. Then $M := \{c_i : i < \omega\} \preceq N$ because $\mathrm{tp}^N(\bar{c}) \in W$. For $(p, n) \in \Pi$, M omits p because $\mathrm{tp}(\bar{c}) \in V_p$. ☐

Theorem 20 (Ryll-Nardzewski)**.** *Let T be a complete theory in a countable language. Then T is ω-categorical iff $S_n(T)$ is finite for every $n < \omega$.*

Proof. One direction was Theorem 14. For the other, suppose $S_n(T)$ is infinite for some n. By Lemma 7 there is non-isolated $p \in S_n(T)$. By Theorem 19 there is a countable model $M_0 \models T$ omitting p. Take an elementary extension $M_1 \succeq M_0$ in which p is realized by $\bar{a} \in M_1^n$. By downward Löwenheim-Skolem we may assume M_1 is countable. Then $M_1 \not\cong M_0$ because M_1 realizes p and M_0 does not. ☐