Homework 8: forking and related topics

Advanced Model Theory

Due April 28, 2022

Let T be the theory of structures $(M, \approx)$ where $\approx$ is an equivalence relation with two classes, and both classes are infinite.

1. Show that T is $\aleph_0$-categorical and complete.

In the remaining problems, fix a monster model $\mathbb{M} \models T$. In these problems, $\mathrm{acl}(A)$ means algebraic closure in the original structure $\mathbb{M}$, and $\mathrm{acl}^{\mathrm{eq}}(A)$ means algebraic closure in $\mathbb{M}^{\mathrm{eq}}$.

2. Show that T is $\aleph_0$-stable. *Hint:* it suffices to show that $S_1(M)$ is countable for countable $M \preceq \mathbb{M}$.
3. Show that $S_1(\varnothing)$ has a single point: if $a, b \in \mathbb{M}$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$.
4. Show that $\mathrm{acl}(\varnothing) = \varnothing$.
5. Let X be one of the equivalence classes. Show that X is $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$-definable but not $\mathrm{acl}(\varnothing)$-definable.
6. Let p be the unique 1-type over $\varnothing$. Let q be a global non-forking extension. Show that q is $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$-definable but not $\mathrm{acl}(\varnothing)$-definable.

As usual, you may assume the results of problems $1, \ldots, n-1$ in problem n.

Hint/advice: All of these problems can be solved without proving quantifier elimination, instead using symmetries in the models. For example, the countable model has lots of symmetries. If you do use quantifier elimination, please prove it.