Metric spaces 101

Introduction to Model Theory
(Third hour)

September 16, 2021

Section 1

Metrics and metric spaces

Metrics

$\mathbb{R}_{\geq 0}$ denotes $[0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$.

Definition

A *metric* on a set M is a function $d : M \times M \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Example

$M = \mathbb{R}^2$, $d(x, y) =$ (the distance from x to y).

$$d(x_1, x_2; y_1, y_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

More metrics

Example

The *Manhattan metric* on $\mathbb{R}^2$ is given by

$$d(x_1, x_2; y_1, y_2) = |x_1 - y_1| + |x_2 - y_2|.$$

Idea: how you measure distances in a city grid.

Example

Let M be the set of strings. The *edit distance* from x to y is the minimum number of insertions, deletions, and substitutions to go from x to y.

$$d(drip, rope) = 3$$
$$drip \mapsto drop \mapsto rop \mapsto rope$$

Edit distance is a metric on M.

Metric spaces

Definition

A *metric space* is a pair (M, d), where M is a set and d is a metric space.

- $(\mathbb{R}^n, d_{Euclidean})$, where $d_{Euclidean}$ is the usual Euclidean distance.
- $(\mathbb{R}^2, d_{Manhattan})$, where $d_{Manhattan}$ is the Manhattan distance.

Often we abbreviate (M, d) as M, when d is clear from context.

Section 2

Open sets and interior

Balls and interior

Fix a metric space (M, d).

Definition

If $p \in M$ and $\epsilon > 0$, then

$$B_\epsilon(p) = \{x \in M : d(x,p) < \epsilon\}$$
$$\overline{B}_\epsilon(p) = \{x \in M : d(x,p) \leq \epsilon\}$$

$B_\epsilon(p)$ and $\overline{B}_\epsilon(p)$ are called the *open* and *closed* balls of radius ϵ around p.

Example

In $\mathbb{R}^2$ with the Euclidean metric, the open ball of radius 2 around $(0,0)$ is the open disk

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 2^2\}.$$

Balls and interior

Fix a metric space (M, d).

Definition

If $p \in M$ and $\epsilon > 0$, then

$$B_\epsilon(p) = \{x \in M : d(x, p) < \epsilon\}$$
$$\overline{B}_\epsilon(p) = \{x \in M : d(x, p) \leq \epsilon\}$$

$B_\epsilon(p)$ and $\overline{B}_\epsilon(p)$ are called the *open* and *closed* balls of radius ϵ around p.

Example

In $\mathbb{R}^2$ with the Manhattan metric, the closed ball of radius 1 around $(0, 0)$ is the closed diamond

$$\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}.$$

Balls and interior

Suppose $p \in M$ and $X \subseteq M$.

Definition

p is an *interior point* of X if X contains an open ball of positive radius around p.

In particular, p must be an element of X.

Example

If $X = [-1,1] \times [-1,1]$, then $(0,0)$ is an interior point of X, but $(1,0)$ and $(0,2)$ are not.

Balls and interior

Definition

The *interior* $int(X)$ is the set of interior points.

- The interior of $[-1,1] \times [-1,1]$ is $(-1,1) \times (-1,1)$.
- The interior of $\{(x,y) : x^2 + y^2 \leq 5\}$ is $\{(x,y) : x^2 + y^2 < 5\}$.

Warning

There are metric spaces where the interior of $\overline{B}_\epsilon(p)$ isn't $B_\epsilon(p)$.

Open sets

Definition

A set $X \subseteq M$ is *open* if $X = \mathit{int}(X)$, i.e., every point of X is an interior point of X.

Example (in $\mathbb{R}$)

The set $(-1, 2)$ is open. The sets $[-1, 2]$ and $[-1, 2)$ are not; they have interior $(-1, 2)$.

Fact

The interior $\mathit{int}(X)$ is the unique largest open set contained in X.

Section 3

Limits

Limits of sequences

Let $a_1, a_2, \ldots$ be a sequence in a metric space (M, d), and let p be a point.

Definition

"$\lim_{i\to\infty} a_i = p$" if for every $\epsilon > 0$, there is n such that

$$\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq B_\epsilon(p).$$

Example

Work in $\mathbb{R}$ with the usual distance. Let $a_n = 1/n$. Then $\lim_{n\to\infty} a_n = 0$, but $\lim_{n\to\infty} a_n \neq 1$.

Usually we would write "$\lim_{n\to\infty} 1/n = 0$."

Limits of sequences

Definition

"$\lim_{i\to\infty} a_i = p$" if for every $\epsilon > 0$, there is n such that

$$\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq B_\epsilon(p).$$

Fact

For any sequence $a_1, a_2, a_3, \ldots$ in (M, d), there is at most one point p such that $\lim_{i\to\infty} a_i = p$.

If such a p exists, it is called the *limit*, and written $\lim_{i\to\infty} a_i$. Otherwise, $\lim_{i\to\infty} a_i$ *does not exist* (or *does not converge*).

- $1, -1, 1, -1, 1, -1, \ldots$ does not converge.

Section 4

Closed sets and closure

Accumulation points and closure

Let X be a set and p be a point in a metric space (M, d).

Definition

p is an *accumulation point* of X if $p = \lim_{n\to\infty} a_n$ for some sequence a_n in X.

Terminology is very inconsistent; some people define accumulation point differently.

Equivalently,

Definition

p is an *accumulation point* of X if for every $\epsilon > 0$, we have $B_\epsilon(p) \cap X \neq \varnothing$.

Accumulation points and closure

Definition

The *closure* of X, written $cl(X)$ or $\overline{X}$, is the set of accumulation points.

Example (in $\mathbb{R}$)

The closure of $(-1, 1) \cup \{7\}$ is $[-1, 1] \cup \{7\}$.
The closure of $\mathbb{Q}$ is $\mathbb{R}$.

Closed sets

Definition

A set $X \subseteq M$ is *closed* if $X = cl(X)$, i.e., every accumulation point of X is in X.

Example (in $\mathbb{R}$)

The set $[-1, 2]$ is closed. The sets $(-1, 2)$ and $[-1, 2)$ are not; they have closure $[-1, 2]$.

Fact

The closure $cl(X)$ is the unique smallest closed set containing X.

Terminological warning

Work in $\mathbb{R}$ with the usual distance $d(x, y) = |x - y|$.

Example

The set $\mathbb{Q}$ is neither closed nor open.

Example

The set $\mathbb{R}$ is both closed and open. So is $\varnothing$.

One says that $\mathbb{R}$ and $\varnothing$ are *clopen*.

Warning

In any metric space, open balls are open and closed balls are closed.
There are metric spaces where open balls and closed balls are clopen.

Closed sets and open sets

Let X^c denote the complement $M \setminus X$.

Fact

X is closed iff X^c is open.

Fact

$int(X) = cl(X^c)^c$, *and* $cl(X) = int(X^c)^c$.

Section 5

Continuity

Continuity

Let (M, d) and (M', d) be metric spaces. Let $f : M \to M'$ be a function.

Definition (Unusual)

f is *continuous* if

$$\lim_{n\to\infty} a_n = p \implies \lim_{n\to\infty} f(a_n) = f(p)$$

for any $a_1, a_2, a_3, \ldots, p \in M$.

Idea

f is continuous iff f *preserves limits*

Continuity

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

Then $\lim_{n\to\infty} 1/n = 0$, but

$$\lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 1 = 1 \neq -1 = f(0).$$

So f is *not* continuous.

Continuity: the usual definitions

Fix $f : (M, d) \to (M', d)$. The following are equivalent:

1. f is continuous.
2. For every open set $U \subseteq M'$, the preimage $f^{-1}(U)$ is open.
3. For every $p \in M$, for every $\epsilon > 0$, there is $\delta > 0$, such that for every $x \in M$,
$$d(x, p) < \delta \implies d(f(x), f(p)) < \epsilon.$$

Everything is continuous

Fact

The functions $\sin, \cos, \exp, \sqrt[3]{-}$, *and polynomials are continuous.*

Fact

If $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous, then* $f + g$, $f \cdot g$, $f - g$, $f \circ g$ *are continuous.*

Fact

If $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $f(x) \neq 0$ *for all* x, *then* $1/f(x)$ *is continuous. If* $f(x) \geq 0$ *for all* x, *then* $\sqrt{f(x)}$ *is continuous.*

Example

This function is continuous:

$$h(x) = \exp\left(\frac{1}{1+x^2}\right) - \frac{1}{17 + \sin(\sqrt[3]{x})}$$

Everything is continuous

Definition

A function $f : M \to M'$ is *Lipschitz continuous* if there is $c \in \mathbb{R}$ such that for any $x, y \in M$,

$$d(f(x), f(y)) \leq c \cdot d(x, y).$$

Example (in $\mathbb{R}$)

The function $f(x) = |x| + |x - 1|$ is Lipschitz continuous with $c = 2$.

Fact

If f is Lipschitz continuous, then f is continuous.

Example

The function $f(x) = x^2$ is continuous but not Lipschitz continuous.

Section 6

Subspaces

Subspaces

Let (M, d) be a metric space and $S \subseteq M$ be a set. Then (S, d') is a metric space, where $d'(x, y) = d(x, y)$ for $x, y \in S$.

- d' is the restriction of d to $S \times S$.
- We say that (S, d') is a *subspace* of (M, d).

Subspaces and continuity

Let $(M, d), (M', d)$ be metric spaces, $S \subseteq M$, and $f : S \to M'$ be a function.

Definition

f is *continuous* if f is continuous as a map from the subspace (S, d') to (M', d).

Example (in $\mathbb{R}$)

Let $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ be given by $f(x) = 1/x$. Then f is continuous.

Section 7

Isometry

Isometry

Definition

An *isometry* or *isomorphism* from (M, d) to (M', d') is a bijection $f : M \to M'$ such that for any $x, y \in M$

$$d(x, y) = d'(f(x), f(y)).$$

Example (in $\mathbb{R}^2$)

The map $(x, y) \mapsto (x + 1, y - 7)$ is an isometry.
So is the map $(x, y) \mapsto (3/5x + 4/5y, -4/5x + 3/5y)$.

Isometry

Definition

Two metric spaces (M, d) and (M', d') are *isometric* or *isomorphic* if there is an isometry from one to the other.

Example

These two metric spaces are isometric via the isometry $x \mapsto (x, 0)$.

- $\mathbb{R}$ with the usual distance.
- The subspace $\mathbb{R} \times \{0\}$ inside $\mathbb{R}^2$ with the usual distance.

Isometries of $\mathbb{R}^2$

Fact

The isometries of $\mathbb{R}^2$ are exactly the rotations, translations, reflections, and glide reflections.

Section 8

Cauchy sequences and completeness

Cauchy sequences

Let X be a non-empty set in a metric space.

Definition

The *diameter* of X, written $\mathrm{diam}(X)$, is

$$\sup\{d(p,q) : p,q \in X\}.$$

(Possibly $\mathrm{diam}(X) = +\infty$.)

Example

In $\mathbb{R}^2$ with the usual metric, the diameter of $B_r(p)$ is $2r$.

Cauchy sequences

Work in a metric space M.

Definition

A *Cauchy sequence* is a sequence $a_1, a_2, a_3, \ldots$ such that

$$\lim_{n \to \infty} \mathrm{diam}(\{a_n, a_{n+1}, a_{n+2}, \ldots\}) = 0.$$

Fact

Every sequence which converges to a point in M is a Cauchy sequence.

Cauchy sequences

Cauchy sequences are the sequences which “could” converge in an extension.

Fact

Let $a_1, a_2, a_3, \ldots$ be a sequence in a metric space (M, d). The following are equivalent:

- *The sequence is a Cauchy sequence.*
- *There is some metric space M' such that M is a subspace of M', and $\lim_{n\to\infty} a_n$ converges in M'.*

Cauchy sequences

Fact

In $\mathbb{R}$, every Cauchy sequence converges.

This fails in the subspace $\mathbb{Q}$; the following Cauchy sequence does not converge:

$$3, 3.1, 3.14, 3.141, \ldots$$

Likewise, it fails in the subspace $(-1, 1)$; the following Cauchy sequence does not converge:

$$0.9, 0.99, 0.999, \ldots$$

Completeness

Definition

A metric space (M, d) is *complete* if every Cauchy sequence in M converges (to a point in M).

Example

$\mathbb{R}$ is complete. The subspaces $\mathbb{Q}$ and $(-1, 1)$ are not complete.

Completions

Let (M, d) be a metric space.

Definition

The *completion* of M is a new metric space $\overline{M}$. Objects of $\overline{M}$ are equivalence classes of Cauchy sequences in M. Two Cauchy sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ are equivalent if $\lim_{i\to\infty} d(a_i, b_i) = 0$. The distance in $\overline{M}$ between two Cauchy sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ is $\lim_{i\to\infty} d(a_i, b_i)$.

Fact

This is well-defined, and $\overline{M}$ is complete.

Fact

If we identify $c \in M$ with the constant sequence $c, c, c, c, \ldots$, then M is a dense subspace of $\overline{M}$. If M is complete, then $\overline{M} = M$.

Completions

Example

$\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to its usual metric.

Example

The *p-adic norm* on $\mathbb{Q}$ is defined by

$$|0|_p = 0$$
$$|p^k a/b|_p = p^{-k} \text{ if } a, b \text{ are integers not divisible by } p$$

For example $|1.3|_5 = |5^{-1} \cdot 13/2|_5 = 5^1$.
The *p-adic metric* on $\mathbb{Q}$ is given by $d(x, y) = |x - y|_p$. This is an incomplete metric. The completion is called $\mathbb{Q}_p$, the set of *p-adic numbers*.

Uniform convergence

Definition

$C([0,1])$ is the space of continuous functions $f : [0,1] \to \mathbb{R}$.

Fact

There is a metric on $C([0,1])$ *where* $d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}$. *This makes* $C([0,1])$ *into a complete metric space.*

Section 9

Connectedness

Connected sets

Recall X is *clopen* if it is closed and open.

Definition

A metric space (M, d) is *connected* if the only clopen sets are M and $\varnothing$. Otherwise M is *disconnected*.

Definition

A set $X \subseteq M$ is *connected* (resp. *disconnected*) if the subspace (X, d) is connected or disconnected as a metric space.

Fact

X is disconnected iff there is a non-constant continuous function $f : X \to \{0, 1\}$.

Connected sets

Example

The set $[-10,-1] \cup [1,10]$ is disconnected, as witnessed by
$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$

Example

The set $[-10,10] \setminus \{0\}$ is disconnected, for similar reasons.

Example

The set $\mathbb{Q}$ is disconnected, witnessed by

$$f(x) = \begin{cases} 0 & \text{if } x < \sqrt{2} \\ 1 & \text{if } x > \sqrt{2}. \end{cases}$$

The set $\mathbb{R} \setminus \mathbb{Q}$ is disconnected by a similar argument.

Connected sets in $\mathbb{R}$

Fact

If $X \subseteq \mathbb{R}$ is non-empty, then the following are equivalent:

- *X is connected.*
- *X is convex: if $a, b \in X$, then $[a, b] \subseteq X$.*
- *X is an interval, a set of the form*

$$[a, b], (a, b), (a, b], [a, b),$$
$$(-\infty, a), (-\infty, a], [a, +\infty), (a, +\infty), (-\infty, \infty).$$

Connected sets and continuity

Fact

Let $f : M \to M'$ be continuous. If $X \subseteq M$ is connected, then $f(X) \subseteq M'$ is connected.

Corollary (Intermediate Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < y < f(b)$, then there is $x \in [a, b]$ with $f(x) = y$.

Proof.

$f([a, b])$ is connected, hence convex, so it contains $y \in [f(a), f(b)]$. Therefore there is $x \in [a, b]$ with $f(x) = y$. □

Warning

There are discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the IVT.

Section 10

Compactness

Cluster points

Let $a_1, a_2, a_3, \ldots$ be a sequence in M.

Definition

A point $b \in M$ is a *cluster point* of $a_1, a_2, \ldots$ if for any $\epsilon > 0$, there are infinitely many i such that $d(a_i, b) < \epsilon$.

If the a_i are pairwise distinct, then b is a cluster point iff

$$\forall \epsilon > 0 \ (B_\epsilon(b) \cap \{a_1, a_2, \ldots\} \text{ is infinite})$$

Example

The sequence $1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots$ has two cluster points, -1 and 1.

Cluster points

Fact

If $b = \lim_{i\to\infty} a_i$, then b is the unique cluster point.

Fact

If $a_1, a_2, \ldots$ is a bounded sequence in $\mathbb{R}$, then $\limsup_{i\to\infty} a_i$ is the largest cluster point and $\liminf_{i\to\infty} a_i$ is the smallest cluster point.

Compactness

Definition

A metric space (M, d) is *compact* if every sequence in M has at least one cluster point.

Suppose $X \subseteq M$.

Definition

The set X is *compact* if the subspace (X, d) is compact as a metric space. Equivalently, every sequence in X has a cluster point $\underline{\text{in } X}$.

Compactness

Example

$\mathbb{R}$ is not compact. The sequence $1, 2, 3, 4, \ldots$ has no cluster points.

Example

The subspace $(0, 1) \subseteq \mathbb{R}$ is not compact. The sequence $1, 1/2, 1/3, 1/4, \ldots$ has no cluster point *in* $(0, 1)$.

Fact (Bolzano-Weierstrass)

The set $[0, 1]$ *is compact.*

Compactness

Definition

A set S is *bounded* if $\mathrm{diam}(S) < \infty$.

Fact

A set $S \subseteq \mathbb{R}^d$ is compact iff it is closed and bounded.

Fact

In any metric space, any compact set is closed.

Definition

A set S is *totally bounded* if for any $\epsilon > 0$, we can write S as a *finite* union of sets of diameter $< \epsilon$.

Fact

A metric space is compact iff it is complete and totally bounded.

Compactness: the usual definition

Let M be a metric space.

Definition

A *cover* is a collection of open sets $\{U_i : i \in I\}$ such that $\bigcup_{i \in I} = M$. A *subcover* of $\{U_i : i \in I\}$ is a cover of the form $\{U_i : i \in I_0\}$ for some $I_0 \subseteq I$.

Fact

M is compact iff every cover has a finite subcover.

Compactness and continuity

Fact

If $f : M \to M'$ is continuous and $X \subseteq M$ is compact, then $f(X) \subseteq M'$ is compact.

Fact

If $X \subseteq \mathbb{R}$ is compact and non-empty, then $\max(X)$ and $\min(X)$ exist.

Consequently:

Fact

If X is compact and non-empty and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a minimum and a maximum.

Compactness and continuity

Fact

If X is compact and non-empty and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a minimum and a maximum.

Example (non-example)

The functions $f(x) = x$ and $g(x) = 1/x$ are continuous on $(0, 1)$, but have no maximums or minimums.

Compactness and continuity

Fact

If X is compact and non-empty and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a minimum and a maximum.

Example

If X is a closed, bounded, and non-empty set in $\mathbb{R}^d$, and $f : \mathbb{R}^d \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum on X.

Often we can use calculus to find the points where the maximum and minimum are realized, and then calculate $\max(f), \min(f)$.