Compactness via Henkin's method and ultraproducts

Introductory Model Theory

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1 "Nice" formulas

Definition 1.
- A term $t(\bar{x})$ is "nice" if t is a variable or a constant.
- An atomic formula is "nice" if it has one of the forms

$$t_1(\bar{x}) = t_2(\bar{x})$$
$$R(t_1(\bar{x}), \ldots, t_m(\bar{x}))$$
$$f(t_1(\bar{x}), \ldots, t_m(\bar{x})) = t_{m+1}(\bar{x}),$$

  where $t_1, \ldots, t_{m+1}$ are nice terms and R and f are m-ary relation and function symbols.
- A formula is "nice" if it is built from nice atomic formulas using $\exists, \wedge, \neg$.

For example, in the language of ordered rings $L = \{0, 1, +, \times, -, <\}$, the formulas $x = y$ and $x + 1 = 0$ and $0 < 1$ are nice, but $x + y = z + w$ and $\forall x : (x = y)$ are not.

Exercise 2. Every formula is equivalent to a nice formula.

For example, $x + 1 < y \ \vee \ 0 = 1 + 1$ is equivalent to

$$\neg(\neg(\exists z : (x + 1 = z) \ \wedge \ z < y) \wedge \neg(1 + 1 = 0)).$$

2 Compactness via Henkin's method

These notes roughly follow Section 2.1 of Dave Marker's textbook *Model Theory: An introduction.* Let $A \subseteq_f B$ mean "$A \subseteq B$ and A is finite." Let $T \models \phi$ mean "every model of T satisfies ϕ."

Definition 3. Let T be an L-theory.

- T is *satisfiable* if there is a model $M \models T$.
- T is *finitely satisfiable* if every $T_0 \subseteq_f T$ is satisfiable.

- T is *complete* if for every L-sentence ϕ, either $\phi \in T$ or $\neg\phi \in T$.
- T has the *witness property* if for every L-formula $\phi(x)$, if $(\exists x : \phi(x)) \in T$, then there is a constant symbol c such that $\phi(c) \in T$.

Lemma 4. *Let T be a finitely satisfiable L-theory. Then there is a finitely satisfiable complete L-theory $T' \supseteq T$.*

Proof. By Zorn's lemma there is a $T' \supseteq T$ which is maximal among finitely satisfiable L-theories. We claim T' is complete. Otherwise there is a sentence ϕ with $\phi \notin T'$ and $\neg\phi \notin T'$. By maximality, $T' \cup \{\phi\}$ isn't finitely satisfiable, so there is $T_1 \subseteq_f T'$ with $T_1 \models \neg\phi$. Similarly, $\neg\phi \notin T'$ implies there is $T_2 \subseteq_f T$ with $T_2 \models \phi$. Then $T_1 \cup T_2$ is not satisfiable, a contradiction. □

Lemma 5. *Let T be a finitely satisfiable L-theory. Let $\phi(x)$ be an L-formula such that $(\exists x\ \phi(x)) \in T$. Let $L' = L \cup \{c\}$ where c is a new constant symbol. Then $T \cup \{\phi(c)\}$ is a finitely satisfiable L'-theory.*

Proof. Otherwise, there is $T_0 \subseteq_f T$ with $T_0 \cup \{\phi(c)\}$ unsatisfiable. Without loss of generality $\exists x\ \phi(x) \in T_0$. Take $M \models T_0$. Then $M \models \exists x\ \phi(x)$. There is some $b \in M$ such that $M \models \phi(b)$. Expand M to an L'-structure by taking $c^M = b$. Then $M \models \phi(c)$, so $M \models T_0 \cup \{\phi(c)\}$, a contradiction. □

Proposition 6. *Let T be a finitely satisfiable L-theory. Then there is a language $L' \supseteq L$ and a theory $T' \supseteq T$ such that T' is finitely satisfiable, complete, and has the witness property.*

Proof. Build increasing chains

$$L_0 \subseteq L_1 \subseteq \cdots$$
$$T_0 \subseteq T_1 \subseteq \cdots$$

where T_i is a finitely satisfiable L_i-theory as follows:

- If $n = 0$, take $L_0 = L$ and $T_0 = T$.
- If $n > 0$ and n is odd, take $L_n = L_{n-1}$ and T_n a complete L_n-theory extending T_{n-1}. (Lemma 4.)
- If $n > 0$ and n is even: let $\{\phi_i(x) : i \in I\}$ enumerate the L_{n-1}-formulas such that $(\exists x\ \phi_i(x)) \in T_{n-1}$. Let $L_n = L_{n-1} \cup \{c_i : i \in I\}$ where the c_i are new constant symbols. Let $T_n = T_{n-1} \cup \{\phi_i(c_i) : i \in I\}$. Then T_n is finitely satisfiable by Lemma 5.

Finally, let $L' = \bigcup_{i=0}^{\infty} L_i$ and $T' = \bigcup_{i=0}^{\infty} T_i$. Then T' is finitely satisfiable because each T_i is. T' is complete because of the odd steps. T' has the witness property by the even steps. □

Lemma 7. *Suppose T is finitely satisfiable and complete. Suppose $T_0 \subseteq_f T$ and $T_0 \models \phi$. Then $\phi \in T$.*

Proof. Otherwise $\neg\phi \in T$. But $T_0 \cup \{\neg\phi\}$ isn't satisfiable. $\square$

Lemma 8. *Suppose T is a finitely satisfiable, complete L-theory with the witness property. Let ϕ, ψ be sentences and $\theta(x)$ be a formula.*

1. $\neg\phi \in T \iff \phi \notin T$.
2. $\phi \wedge \psi \in T \iff (\phi \in T$ *and* $\psi \in T)$.
3. $(\exists x\ \theta(x)) \in T \iff$ (*there is a constant symbol c such that $\theta(c) \in T$*).

Proof.

1. $\phi \in T$ or $\neg\phi \in T$ by completeness. If $\phi \in T$ and $\neg\phi \in T$, then T isn't finitely satisfiable.
2. Both directions hold by Lemma 7.
3. The witness property gives $\Rightarrow$, and Lemma 7 gives $\Leftarrow$.

$\square$

Proposition 9. *Suppose T is a finitely satisfiable, complete L-theory with the witness property. Then T has a model.*

Proof. Let $\mathcal{C}$ be the set of constant symbols. For $c, d \in \mathcal{C}$, let $c \sim d$ mean that $(c = d) \in T$.

Claim. $\sim$ is an equivalence relation.

Proof. If $(c = d) \in T$ and $(d = e) \in T$, then $(c = e) \in T$ by Lemma 7. Symmetry and reflexivity are similar. $\square_{\text{Claim}}$

Let $M = \mathcal{C}/\sim$. Let c^* denote the equivalence class of $c \in \mathcal{C}$. Make M an L-structure as follows:

- $c^M = c^*$.
- If R is an n-ary relation symbol, define
$$R^M(c_1^*, \ldots, c_n^*) \iff (R(c_1, \ldots, c_n) \in T).$$
Why is this well-defined? If $c_i^* = d_i^*$ for $1 \le i \le n$, then $c_i \sim d_i$, so $(c_i = d_i) \in T$. Then
$$\{c_1 = d_1, \ldots, c_n = d_n, R(c_1, \ldots, c_n)\} \models R(d_1, \ldots, d_n)$$
so Lemma 7 gives $R(c_1, \ldots, c_n) \in T \implies R(d_1, \ldots, d_n) \in T$.
- If f is an n-ary function symbol, define
$$f(c_1^*, \ldots, c_n^*) = c_{n+1}^* \iff (f(c_1, \ldots, c_n) = c_{n+1}) \in T.$$
Why is this well-defined? The previous argument shows there is a well-defined relation $R_f \subseteq M^{n+1}$ such that
$$R_f(c_1^*, \ldots, c_n^*, c_{n+1}^*) \iff (f(c_1, \ldots, c_n) = c_{n+1}) \in T.$$
Then we need R_f to be the graph of a function $M^n \to M$. Fix some $c_1, \ldots, c_n$.

- $(\exists x : f(c_1, \ldots, c_n) = x) \in T$ by Lemma 7, so the witness property gives some $d \in \mathcal{C}$ with $(f(c_1, \ldots, c_n) = d) \in T$, and then $R_f(c_1^*, \ldots, c_n^*, d^*)$ holds.
- If $R_f(c_1^*, \ldots, c_n^*, d^*)$ and $R_f(c_1^*, \ldots, c_n^*, e^*)$, then

$$(f(c_1, \ldots, c_n) = d) \in T$$
$$(f(c_1, \ldots, c_n) = e) \in T$$

  so $(d = e) \in T$ (Lemma 7), and $d^* = e^*$.

This shows R_f is a function.

Now M is an L-structure.

Claim. $\phi \in T \iff M \models \phi$, for any sentence ϕ.

Proof. By Exercise 2 we may assume ϕ is nice. Proceed by induction on ϕ.

- If ϕ is a nice atomic formula, then $M \models \phi \iff \phi \in T$ by choice of the L-structure on M.
- If ϕ is $\neg\psi$, then

$$\begin{aligned} M \models \phi &\iff M \not\models \psi \\ &\iff \psi \notin T && \text{by induction} \\ &\iff \phi \in T && \text{by Lemma 8(1).} \end{aligned}$$

- If ϕ is $\psi \wedge \theta$, use induction and Lemma 8(2) instead.
- If ϕ is $\exists x\ \psi(x)$, use induction and Lemma 8(3) instead. $\square_{\text{Claim}}$

By the claim, $M \models T$. $\square$

We call the model constructed in the proof of Proposition 9 the "canonical model" of T. It is uniquely characterized up to isomorphism by the fact that every element of M is named by a constant symbol.

Theorem 10 (Compactness theorem)**.** *If T is finitely satisfiable then T is satisfiable.*

Proof. By Proposition 6, we can find $T' \supseteq T$ that is finitely satisfiable, complete, and has the witness property. Then T' has a model by Proposition 9. $\square$

3 Ultraproducts

Let I be a set, and let $\mathcal{P}(I)$ be its powerset.

Definition 11. A *filter* on I is a set $\mathcal{F} \subseteq \mathcal{P}(I)$ satisfying the following:

- If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
- If $X \subseteq Y \subseteq I$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.
- $I \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$.

Definition 12. A family $\mathcal{S}$ has the *finite intersection property* (FIP) if for any $X_1, \ldots, X_n \in \mathcal{S}$, we have $\bigcap_{i=1}^n X_i \neq \varnothing$.

Lemma 13. *If $\mathcal{S} \subseteq \mathcal{P}(I)$ has the FIP, then there is a filter $\mathcal{F} \supseteq \mathcal{S}$.*

Proof. Let $\mathcal{F}$ be the set of $X \subseteq I$ such that there are $Y_1, \ldots, Y_n \in \mathcal{S}$ with $X \supseteq \bigcap_{i=1}^n Y_i$. (Note $\bigcap_{i=1}^n Y_i = I$ when $n = 0$.) Then $\mathcal{F}$ is a filter containing $\mathcal{S}$. $\square$

Definition 14. An *ultrafilter* on I is a filter $\mathcal{U}$ such that for any $X \subseteq I$, $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

Lemma 15. *If $\mathcal{F}$ is a filter on I, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.*

Proof. By Zorn's lemma, there is a maximal filter $\mathcal{U} \supseteq \mathcal{F}$. We claim $\mathcal{U}$ is an ultrafilter. Otherwise, there is $X \subseteq I$ with $X \notin \mathcal{U}$ and $I \setminus X \notin \mathcal{U}$. By maximality, $\mathcal{U} \cup \{X\}$ is not contained in a filter, and does not have FIP. So there are $Y_1, \ldots, Y_n \in \mathcal{U}$ such that $X \cap \bigcap_{i=1}^n Y_i = \varnothing$. As $\mathcal{U}$ is a filter, $Y := \bigcap_{i=1}^n Y_i \in \mathcal{U}$. Then $X \cap Y = \varnothing$, so $Y \subseteq I \setminus X$.

Similarly, $I \setminus X \notin \mathcal{U}$ implies there is $Z \in \mathcal{U}$ with $Z \subseteq X$. Then $Y \cap Z \subseteq (I \setminus X) \cap X = \varnothing$, so $\varnothing = Y \cap Z \in \mathcal{U}$, a contradiction. $\square$

Now suppose that M_i is a non-empty L-structure for every $i \in I$, and $\mathcal{U}$ is an ultrafilter on I. Let

$$P = \{f : I \to \bigcup_{i \in I} M_i \mid \forall i \in I : f(i) \in M_i\}.$$

P is usually denoted $\prod_{i \in I} M_i$. Let $L' = L \cup \{c : c \in P\}$, where the c are new constant symbols. Make M_i into an L'-structure by interpresting $c \in P$ as $c(i) \in M_i$.

For any L'-sentence ϕ, let $[\phi] = \{i \in I : M_i \models \phi\}$. Note:

- $[\phi \wedge \psi] = [\phi] \cap [\psi]$.
- $[\neg\phi] = I \setminus [\phi]$.

Lemma 16. *Let $T = \{\phi \in L' : [\phi] \in \mathcal{U}\}$. Then T is finitely satisfiable and complete, and has the witness property.*

Proof. Finite satisfiability: suppose $\phi_1, \ldots, \phi_n \in T$. Then $[\phi_i] \in \mathcal{U}$ so $S := [\bigwedge_{i=1}^n \phi_i] = \bigcap_{i=1}^n [\phi_i] \in \mathcal{U}$. Then $S \neq \varnothing$. If $i \in S$ then $M_i \models \{\phi_1, \ldots, \phi_n\}$.

Completeness: given ϕ, $[\phi]$ and $[\neg\phi]$ are complementary, so one is in $\mathcal{U}$.

Witness property: suppose $S = [\exists x\ \phi(x)] \in \mathcal{U}$. Define $c \in P$ as follows:

- If $i \in S$, then $M_i \models \exists x\ \phi(x)$. Take $c(i) \in M_i$ so that $M_i \models \phi(c(i))$.
- If $i \notin S$, take $c(i) =$ anything in M_i.

Then $i \in S \implies M_i \models \phi(c)$, so $[\phi(c)] \supseteq S \in \mathcal{U}$. Then $[\phi(c)] \in \mathcal{U}$ and $\phi(c) \in T$. □

Definition 17. Let T be as in Lemma 16. The *ultraproduct* $\prod_{i \in I} M_i/\mathcal{U}$ is defined to be the canonical model of T as constructed in in Proposition 9—the unique model of T in which every element is named by a constant symbol.

Remark 18. If $c, d \in \prod_{i \in I} M_i$, then $c^* = d^*$ if and only if $(c = d) \in T$ if and only if $[c = d] \in \mathcal{U}$ if and only if $\{i \in I : M_i \models c = d\} \in \mathcal{U}$ if and only if $\{i \in I : c(i) = d(i)\} \in \mathcal{U}$.

Therefore, $\prod_{i \in I} M_i/\mathcal{U}$ is $\prod_{i \in I} M_i$ modulo the equivalence relation

$$c \sim d \iff \{i \in I : c(i) = d(i)\} \in \mathcal{U}.$$

Remark 19. Let R be an m-ary relation symbol. Let $c_1, \ldots, c_m \in \prod_{i \in I} M_i$. Then $R(c_1^*, \ldots, c_m^*)$ holds in the ultraproduct if and only if $R(c_1, \ldots, c_m) \in T$ if and only if $\{i \in I : M_i \models R(c_1, \ldots, c_m)\} \in \mathcal{U}$ if and only if $\{i \in I : M_i \models R(c_1(i), \ldots, c_m(i))\} \in \mathcal{U}$.

So the relation R on the ultraproduct is defined by

$$R(c_1, \ldots, c_m) \iff \{i \in I : M_i \models R(c_1(i), \ldots, c_m(i))\}.$$

Remark 20. Let f be an m-ary function symbol. Let $c_1, \ldots, c_m \in \prod_{i \in I} M_i$. Define $d(i) = f^{M_i}(c_1(i), \ldots, c_m(i))$ for $i \in I$. Then $d \in \prod_{i \in I} M_i$, and $M_i \models d = f(c_1, \ldots, c_m)$ for all i. Therefore the ultraproduct thinks $d = f(c_1, \ldots, c_m)$.

The preceding three remarks give the usual *definition* of ultraproducts.

Theorem 21 (Łoś's theorem)**.** *Let M be an ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$. Let $\phi(x_1, \ldots, x_m)$ be an L-formula. Let $c_1, \ldots, c_m$ be elements of $\prod_{i \in I} M_i$. Then*

$$M \models \phi(c_1, \ldots, c_m) \iff \{i \in I : M_i \models \phi(c_1(i), \ldots, c_m(i))\} \in \mathcal{U}.$$

Specializing to the case where ϕ is an L-sentence,

$$M \models \phi \iff \{i \in I : M_i \models \phi\} \in \mathcal{U}.$$

Proof. $M \models \phi(c_1, \ldots, c_m) \iff [\phi(c_1, \ldots, c_m)] \in \mathcal{U} \iff \{i \in I : M_i \models \phi(c_1, \ldots, c_m)\} \in \mathcal{U} \iff \{i \in I : M_i \models \phi(c_1(i), \ldots, c_m(i))\} \in \mathcal{U}$. □

If M is a structure and $\mathcal{U}$ is an ultrafilter on I, the *ultrapower* $M^{\mathcal{U}}$ is $\prod_{i \in I} M/\mathcal{U}$, i.e., the ultraproduct where $M_i = M$ for all i.

Example. Suppose $L = \{+, \times, 0, 1, \leq\}$ and $I = \mathbb{N}$ and $M_i = \mathbb{R}$ for all i. Then the ultrapower $\mathbb{R}^{\mathcal{U}}$ can be described as follows:

- The product $\prod_{i \in \mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$ is the set of functions $f : \mathbb{N} \to \mathbb{R}$. We think of f as an infinite tuple $(f(0), f(1), f(2), f(3), \ldots)$.
- The ultrapower is $\mathbb{R}^{\mathbb{N}}$ modulo the equivalence relation

$$(x_0, x_1, x_2, \ldots) \sim (y_0, y_1, y_2, \ldots)$$
$$\iff \{i \in \mathbb{N} : x_i = y_i\} \in \mathcal{U}.$$

- The order $\leq$ is defined by

$$(x_0, x_1, x_2, \ldots) \leq (y_0, y_1, y_2, \ldots)$$
$$\iff \{i \in \mathbb{N} : x_i \leq y_i\} \in \mathcal{U}.$$

- Addition is given by

$$(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots)$$
$$= (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$$

- Multiplication is given by

$$(x_0, x_1, x_2, \ldots) \cdot (y_0, y_1, y_2, \ldots)$$
$$= (x_0 y_0, x_1 y_1, x_2 y_2, \ldots)$$

- Negation is given by

$$-(x_0, x_1, x_2, \ldots) = (-x_0, -x_1, -x_2, \ldots)$$

- $1 = (1, 1, 1, \ldots)$ and $0 = (0, 0, 0, \ldots)$.

If ϕ is a sentence, then by Łoś's theorem, $\mathbb{R}^{\mathcal{U}} \models \phi$ iff

$$\{i \in \mathbb{N} : \mathbb{R} \models \phi\} \in \mathcal{U}.$$

If $\mathbb{R} \models \phi$, this set is $\mathbb{N} \in \mathcal{U}$, and if $\mathbb{R} \not\models \phi$, this set is $\varnothing \notin \mathcal{U}$. Therefore $\mathbb{R}^{\mathcal{U}} \models \phi \iff \mathbb{R} \models \phi$, or equivalently, $\mathbb{R}^{\mathcal{U}} \equiv \mathbb{R}$.

We can use ultraproducts and Łoś's theorem to give another proof of compactness.

Theorem 22 (Compactness theorem)**.** *If T is a finitely satisfiable L-theory, then T is satisfiable.*

Proof. Let $\{M_i : i \in I\}$ be a collection of L-structures containing at least one representative from every elementary equivalence class. For ϕ an L-sentence, let $[\phi] = \{i \in I : M_i \models \phi\}$. Let $\mathcal{S} = \{[\phi] : \phi \in T\}$. We claim $\mathcal{S}$ has FIP. Otherwise, there are $\phi_1, \ldots, \phi_n \in T$ such that $[\bigwedge_{i=1}^n \phi_i] = \bigcap_{i=1}^n [\phi_i] = \varnothing$. But T is finitely satisfiable, so there is some $N \models \bigwedge_{i=1}^n \phi_i$. There is some $M_i \equiv N$, and then $i \in [\bigwedge_{i=1}^n \phi_i] = \varnothing$, a contradiction.

So there is an ultrafilter $\mathcal{U}$ on I containing $\mathcal{S}$. Let $M = \prod_{i \in I} M_i/\mathcal{U}$. Then for $\phi \in T$, we have

$$\{i \in I : M_i \models \phi\} = [\phi] \in \mathcal{S} \subseteq \mathcal{U},$$

so $M \models \phi$ by Łoś's theorem. Thus $M \models T$. $\square$

4 Applications of compactness

Theorem 23 (Löwenheim-Skolem)**.** *Let T be an L-theory. Suppose T has an infinite model, or that for every $n < \omega$ T has a model of size $> n$. Then for any $\kappa \geq |L|$, T has a model of size κ.*

Proof. Let L' be L plus new constant symbols c_α for $\alpha < \kappa$. Let T' be T plus the sentences $c_\alpha \neq c_\beta$ for $\alpha < \beta < \kappa$.

Claim. T' is finitely satisfiable.

Proof. Let $T_0 \subseteq_f T'$. Then there is $S \subseteq_f \kappa$ such that

$$T_0 \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in S,\ \alpha < \beta\}.$$

Take $M \models T$ with $|M| \geq |S|$. Expand M to an L'-structure by interpreting c_α for $\alpha \in S$ as distinct elements of M (and define c_α randomly for $\alpha \notin S$). Then $M \models T_0$. $\square_{\text{Claim}}$

By compactness, T' has a model M. Then the c_α^M are pairwise distinct, so $|M| \geq \kappa$. By Downward Löwenheim-Skolem (also called Löwenheim's theorem), we can find an elementary substructure $N \preceq M$ with $|N| = \kappa$. Then $N \equiv M$, so $N \models T$. $\square$

Given an L-structure M, let $L(M)$ be L plus a new constant symbol for each element of M. Then M is naturally an $L(M)$-structure.

Definition 24. The *elementary diagram* of M is the set of $L(M)$-sentences true in M. It is denoted $\mathrm{eldiag}(M)$.

Poizat calls this the *diagram* of M, and writes it $T(M)$. Some authors use "diagram" to mean the quantifier-free part of $\mathrm{eldiag}(M)$.

Remark 25. Suppose $N \models \mathrm{eldiag}(M)$. Define $f : M \to N$ to be the map sending $c \in M$ to its interpretation $c^N \in N$. Then

$$M \models \phi(a_1, \ldots, a_n) \iff \phi(a_1, \ldots, a_n) \in \mathrm{eldiag}(M) \iff N \models \phi(f(a_1), \ldots, f(a_n)).$$

So $f : M \to N$ is an elementary embedding. Conversely, if $f : M \to N$ is an elementary embedding then N is naturally a model of $\mathrm{eldiag}(M)$.

Theorem 26. *If $M_1 \equiv M_2$, then there is a structure N and elementary embeddings $M_1 \to N$ and $M_2 \to N$.*

Proof. Note $\mathrm{eldiag}(M_i)$ is closed under conjunction for $i = 1, 2$.

Claim. $\mathrm{eldiag}(M_1) \cup \mathrm{eldiag}(M_2)$ is finitely satisfiable.

Proof. Otherwise, there is $\phi \in \mathrm{eldiag}(M_1)$ and $\psi \in \mathrm{eldiag}(M_2)$ with $\phi \wedge \psi$ being unsatisfiable. We can write ϕ as $\phi(\bar{a})$ for some L-formula ϕ and $\bar{a}$ in M_1. Similarly, we can write ψ as $\psi(\bar{b})$ for some L-formula ψ and $\bar{b}$ in M_2.

Then $M_2 \models \psi(\bar{b})$, so $M_2 \models \exists \bar{x}\ \psi(\bar{x})$, so $M_1 \models \exists \bar{x}\ \psi(\bar{x})$. Take $\bar{c}$ in M_1 with $M_1 \models \psi(\bar{c})$. Expand M_1 to an $(L(M_1) \cup L(M_2))$-structure by interpreting b_i as c_i. Then $M_1 \models \phi(\bar{a}) \wedge \psi(\bar{b})$, a contradiction. $\square_{\text{Claim}}$

By compactness, there is $N \models \mathrm{eldiag}(M_1) \cup \mathrm{eldiag}(M_2)$. Then there are elementary embeddings $M_1 \to N$ and $M_2 \to N$. $\square$