

Homework4

Qi'ao Chen
21210160025

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Exercise 1. Let $\mathcal{L} = \{P\}$, a language with only one unary relation symbol. Classify complete theories with \mathcal{L} , i.e. determine all complete theories with only one unary symbol

Proof. Suppose $\mathfrak{M} = (M, R)$ and $\mathfrak{N} = (N, S)$.

1. If the universe of \mathfrak{M} is finite. Then $|\mathfrak{M}| = a$ and $|P(\mathfrak{M})| = b$ for some $a, b \in \mathbb{N}$ and $b \leq a$.

For $n \in \mathbb{N}$, let

$$\begin{aligned}\varphi_n &= \exists x_1 \dots x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \forall x \left(\bigwedge_{i=1}^n x = x_i \right) \right) \\ \varphi_{n,P} &= \exists x_1 \dots x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \forall x \left(\bigwedge_{i=1}^n x = x_i \right) \wedge \bigwedge_{i=1}^n P(x_i) \right)\end{aligned}$$

Then φ_n says there are exactly n elements and $\varphi_{n,P}$ says there are exactly n elements satisfying P . Let $\theta_{a,b} = \varphi_a \wedge \varphi_{b,P}$, then $\mathfrak{M} \models \theta_{a,b}$.

Claim $\mathfrak{M} \cong \mathfrak{N} \Leftrightarrow \mathfrak{M} \models \theta_{a,b}$ and $\mathfrak{N} \models \theta_{a,b}$

Left to right is obvious. Now suppose both \mathfrak{M} and \mathfrak{N} both satisfy $\theta_{a,b}$. Then they have same number of elements and have same number of elements satisfying P . Let $f : P(\mathfrak{M}) \rightarrow P(\mathfrak{N})$ be a bijection as they have same cardinality and let $f' : \mathfrak{M} \rightarrow \mathfrak{N}$ be the bijection such that $f' \supseteq f$. Thus for any $a \in M$, $Ra \Leftrightarrow S f'(a)$ and f' is an isomorphism.

Thus $\mathfrak{M} \equiv \mathfrak{N}$ and any complete theory consisting of $\theta_{a,b}$ is unique.

2. If the universe of \mathfrak{M} is infinite.

For each $n \in \mathbb{N}$, let

$$\psi_n = \exists x_1 \dots x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$$

and $S = \{\psi_i : i \in \mathbb{N}\}$.

(a) If $|P(\mathfrak{M})| = a$.

Let $\Gamma_a = S \cup \{\varphi_{a,P}\}$. For any $p \in \mathbb{N}$, we show that Duplicator wins $\text{EF}_p(R, S)$: if Spoiler chooses an element from $M \setminus P(\mathfrak{M}) (N \setminus P(\mathfrak{N}))$, then Duplicator chooses a new element from $N \setminus P(\mathfrak{N}) (M \setminus P(\mathfrak{M}))$; if Spoiler chooses an element from $P(\mathfrak{M}) (P(\mathfrak{N}))$, then Duplicator chooses a element from $P(\mathfrak{N}) (P(\mathfrak{M}))$. Then we get a map s and $Ra \Leftrightarrow Ss(a)$ for $a \in \text{dom}(s)$. Thus s is a local isomorphism and $\mathfrak{M} \equiv \mathfrak{N}$. Hence any complete theory that contains Γ_a is unique

(b) if $|P(\mathfrak{M})|$ is infinite.

Let $\Gamma_\omega = S \cup \{\varphi_{n,P} : n \in \mathbb{N}\}$. Similarly we can that prove any complete theory containing Γ_ω is unique

Thus we prove that there is three kinds of complete theory

1. For each $n \in \omega$ and $m \leq n$, there is a unique complete theory containing $\theta_{m,n}$
2. For each $n \in \omega$ there is a unique complete theory containing Γ_n
3. Unique complete theory containing Γ_ω

□

Exercise 2. Show that there is a structure $(M, +, \cdot, <, 0, 1)$ elementarily equivalent to $(\mathbb{R}, +, \cdot, <, 0, 1)$ s.t. the order on M is not complete: there is a bounded set with no supremum

Proof. Take $\mathbb{Q} \subseteq \mathbb{R}$, then by Löwenheim's theorem, we can find an elementarily restriction \mathfrak{Q} of $(\mathbb{R}, +, \cdot, <, 0, 1)$ whose domain Q contains \mathbb{Q} and $|\mathfrak{Q}| = \aleph_0$. Thus by Theorem 1.7 \mathfrak{Q} is also a dense linear order. As every countable dense linear set is isomorphic to \mathbb{Q} , $\mathfrak{Q} \cong \mathbb{Q}$. Since \mathbb{Q} is not complete, \mathfrak{Q} is also not complete. □

Exercise 3. Show that the open interval $((0, 1), <)$ is an elementary substructure of $(\mathbb{R}, <)$

Proof. First $(\mathbb{R}, <) \cong ((0, 1), <)$ as we have the isomorphic function $f(x) = \arctan(x)$.

We first show that $((0, 1), <) \sim_\omega (\mathbb{R}, <)$. For any $p \in \mathbb{N}$ and game $\text{EF}_p(((0, 1), <), (\mathbb{R}, <))$

- If Spoiler chooses $x \in \mathbb{R}$, then Duplicator chooses $y = f(x) \in (0, 1)$
- If Spoiler chooses $y \in (0, 1)$, then Duplicator chooses $x = f^{-1}(y)$

The induced map s is a local isomorphism as $s \subset f$. Thus Duplicator wins.

Thus $((0, 1), <)$ is an elementary substructure of $(\mathbb{R}, <)$ \square

Exercise 4. Show that every formula is equivalent to a “nice” formula.

Proof. We prove this by induction on the complexity of φ

1. If φ is atomic formula

(a) If φ is of the form $t_1(\bar{x}) = t_2(\bar{x})$

If both of t_1 and t_2 are not good, then we replace φ with $\exists y(t_1(\bar{x}) = y \wedge t_2(\bar{x}) = y)$. Obviously $\models \varphi(\bar{x}) \leftrightarrow \exists y(t_1(\bar{x}) = y \wedge t_2(\bar{x}) = y)$

(b) If φ is of the form $R(t_1(\bar{x}), \dots, t_m(\bar{x}))$

If among $t_1(\bar{x}), \dots, t_m(\bar{x}), t_{r_1}, \dots, t_{r_n}$ where $n \leq m$ are not good. Then we replace φ with

$$\psi = \exists y_1 \dots y_n \left((R(t_1(\bar{x}), \dots, t_m(\bar{x})))_{y_1, \dots, y_n}^{t_{r_1}(\bar{x}), \dots, t_{r_n}(\bar{x})} \wedge \bigwedge_{i=1}^n t_{r_i}(\bar{x}) = y_i \right)$$

and we have $\models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$

2. If φ is of the form $\neg\psi, \psi \wedge \theta, \psi \vee \theta$ or $\exists x\psi$. As we can transform ψ and θ into nice formulas, φ is equivalent to a “nice” formula

\square

Exercise 5. Let T be the set of $\mathcal{L}_{\text{ring}}$ -sentences true in $(\mathbb{R}, +, \cdot, 0, 1)$. Show that T is finitely satisfiable and complete, but does not have the witness property

Proof. Let $\mathfrak{M} = (\mathbb{R}, +, \cdot, 0, 1)$. Then $T = \text{Th}(\mathfrak{M})$. As for every $\mathcal{L}_{\text{ring}}$ -sentence φ , either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \neg\varphi$, thus either $\varphi \in T$ or $\neg\varphi \in T$. Hence T is complete.

For every finite subset S of $\text{Th}(\mathfrak{M})$, we can enumerate them as $\varphi_1, \dots, \varphi_n$. Let $\psi = \bigwedge_{i=1}^n \varphi_i$. As $\mathfrak{M} \models \psi$, S is satisfiable and thus $\text{Th}(\mathfrak{M})$ is finitely satisfiable.

Sentence $\varphi = \exists x(x \cdot x = 1 + 1)$ doesn't have the witness property. \square