Competitive programming

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Contents

1 Dynamic Programming

1.1 General

Problem 1.1.1 (LeetCode: Find All Good Indices). You are given a *0-indexed* integer array nums of size n and a positive integer k.

We call an index i in the range $k \leq i < n - k$ good if the following conditions are satisfied:

- The k elements that are just before the index i are in non-increasing order.
- The k elements that are just after the index i are in non-decreasing order.

Return an array of all good indices sorted in increasing order.

Solution. For j, suppose the non-increasing elements before j (including j) is $left_j$, the non-decreasing elements after j (including j) is $right_j$, then i is good iff $left_{i-1} \geq k$ and $right_{i+1} \geq k$ □

Problem 1.1.2 (LeetCode: Get Kth Magic Number). Design an algorithm to find the kth number such that the only prime factors are 3, 5, and 7. Note that 3, 5, and 7 do not have to be factors, but it should not have any other prime factors. For example, the first several multiples would be (in order) 1, 3, 5, 7, 9, 15, 21.

Solution. We can use heap: for each element x took out, add $3x, 5x, 7x$ into the heap. Also we need to eliminate the duplicates

Define $dp[i]$ is the ith number, so $dp[1] = 1$, and let $p_3 = p_5 = p_7 = 1$ initially, then for $2 \leq i \leq k$

$$dp[i] = \min(dp[p_3] \cdot 3, dp[p_5] \cdot 5, dp[p_7] \cdot 7)$$

and increment the corresponding p_k where $k \in \{3, 5, 7\}$ □

Problem 1.1.3 (LeetCode: Remove Boxes). You are given several boxes with different colors represented by different positive numbers.

You may experience several rounds to remove boxes until there is no box left. Each time you can choose some continuous boxes with the same color (i.e., composed of k boxes, $k \geq 1$), remove them and get k^2 points.

Return the maximum points you can get.

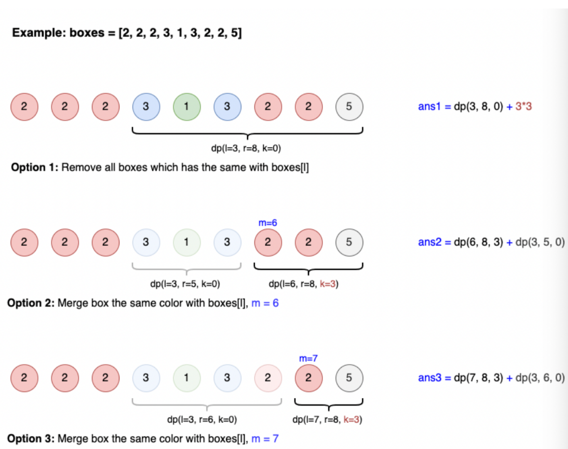
Solution. Let $dp(l, r, k)$ denote the maximum points we can get in boxes $[l, r]$ if we have extra k boxes which is the same color with $boxes[l]$ in the left side.

For example, if `boxes=[3,3,1,3,3]`, then $dp(3, 4, 2)$ is the maximum we can get in `boxes[3,4]` if we have extra 2 boxes the same color with `boxes[3]` in the left side

Since $(a+b)^2 > a^2+b^2$ where $a > 0, b > 0$, it's better to greedy to remove all contiguous boxes of the same color, instead of split them. So we increase both l and k while `boxes[l+1]==boxes[l]`

Now we have some options:

- remove all boxes which has the same color with boxes l, total points we can get is $dp(l + 1, r, 0) + (k + 1)^2$
- merge non-contiguous boxes of the same color together, by
  - find the index j where $l+1 \le j \le r$ so that `boxes[j]==boxes[l]`
  - total points we can get is $dp(j, r, k + 1) + dp(l + 1, j - 1, 0)$



□

Problem 1.1.4 (LeetCode: K Inverse Pairs Array). For an integer array nums, an **inverse pair** is a pair of integers (i, j) where $0 \le i < j < len(nums)$ and $nums[i] > nums[j]$

Given two integers n and k, return the number of different arrays consist of numbers from 1 to n such that there are exactly k inverse pairs. Since the answer can be huge, return it modulo $10^9 + 7$.

Solution. Let $f(i, j)$ denote the number of different arrays consisting of numbers from 1 to i s.t. there are exactly j inverse pairs.

Suppose we fix k as the last element of the array, then the number of inverse pairs is the sum of

- the inverse pairs between k and other numbers
- the inverse pairs among other numbers

The first part is $i - k$, therefore the second part should be $j - (i - k)$.

$$f(i, j) = \sum_{k=1}^{i} f(i-1, j-(i-k)) = \sum_{k=0}^{i-1} f(i-1, j-k)$$

But the above formula's complexity is $O(n^2k)$.

Note that

$$f(i, j-1) = \sum_{k=0}^{i-1} f(i-1, j-1-k)$$
$$f(i, j) = \sum_{k=0}^{i-1} f(i-1, j-k)$$

Therefore

$$f(i, j) = f(i, j-1) - f(i-1, j-i) + f(i-1, j)$$

□

Problem 1.1.5 (LeetCode: Minimum Swaps To Make Sequences Increasing). You are given two integer arrays of the same length $nums1$ and $nums2$. In one operation, you are allowed to swap $nums1[i]$ with $nums2[i]$.

For example, if $nums1 = [1, 2, 3, 8]$, and $nums2 = [5, 6, 7, 4]$, you can swap the element at $i = 3$ to obtain $nums1 = [1, 2, 3, 4]$ and $nums2 = [5, 6, 7, 8]$.

Return the minimum number of needed operations to make $nums1$ and $nums2$ strictly increasing. The test cases are generated so that the given input always makes it possible.

An array arr is strictly increasing if and only if $arr[0] < arr[1] < arr[2] < \cdots < arr[arr.length - 1]$

Solution. For each i, one of the following is true

1. $nums_1[i] > nums_1[i-1]$ and $nums_2[i] > nums_2[i-1]$
2. $nums_1[i] > nums_2[i-1]$ and $nums_2[i] > nums_1[i-1]$

Use $dp[i][0]$ to denote the minimum number of needed operations for $[0, i]$ and we don't do the exchange at i. Use $dp[i][1]$ to denote the number that we exchange at i.

Case $1 \wedge \neg 2$:

$$\begin{cases} dp[i][0] = dp[i-1][0] \\ dp[i][1] = dp[i-1][1] + 1 \end{cases}$$

Case $\neg 1 \wedge 2$:

$$\begin{cases} dp[i][0] = dp[i-1][1] \\ dp[i][1] = dp[i-1][0] + 1 \end{cases}$$

Case $1 \wedge 2$:

$$\begin{cases} dp[i][0] = \min\{dp[i-1][0], dp[i-1][1]\} \\ dp[i][1] = \min\{dp[i-1][1], dp[i-1][0]\} + 1 \end{cases}$$

and we set $dp[0][0] = dp[0][1] = 1$ □

Problem 1.1.6 (LeetCode: Count Ways to Build Rooms i nan Ant Colony). You are an ant tasked with adding n new rooms numbered 0 to $n-1$ to your colony. You are given the expansion plan as a 0-indexed integer array of length n, $prevRoom$, where $prevRoom[i]$ indicates that you must build room $prevRoom[i]$ before building room i, and these two rooms must be connected directly. Room 0 is already built, so $prevRoom[0] = -1$. The expansion plan is given such that once all the rooms are built, every room will be reachable from room 0.

You can only build one room at a time, and you can travel freely between rooms you have already built only if they are connected. You can choose to build any room as long as its previous room is already built.

Return the number of different orders you can build all the rooms in. Since the answer may be large, return it modulo $10^9 + 7$.

Every room is reachable from room 0 once all the rooms are built.

Problem 1.1.7. Suppose there are a_0 0's, a_1 1's, ..., a_{n-1} $n-1$'s, then the number of different plan to put them in a row is

$$\frac{(a_0 + \cdots + a_{n-1})!}{a_0! \cdot a_1! \cdot \cdots \cdot a_{n-1}!}$$

$a^{\varphi(m)} \equiv 1 \mod m$, when p is prime, we have $a^{p-1} \equiv 1 \mod p$

Define $f(i)$ to be the number of different topological sort for the subtree with the root i. Suppose i has child $c_{i,0}, \dots, c_{i,k}$, and let $cnt(i)$ denote the number of nodes in the subtree, then

$$f(i) = \prod_c f(c) \times \frac{(cnt(i)-1)!}{\prod_c cnt(c)!}$$

Problem 1.1.8 (LeetCode: Valid Permutations for DI sequence). You are given a string s of length n where $s[i]$ is either:

- 'D' means decreasing, or
- 'I' means increasing.

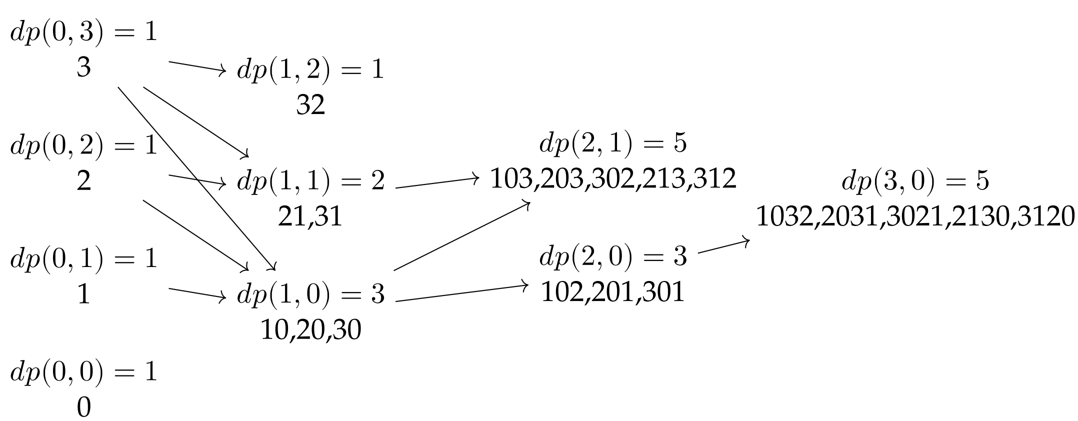
A permutation perm of $n+1$ integers of all the integers in the range $[0, n]$ is called a valid permutation if for all valid i:

- If $s[i] =' D'$, then $perm[i] > perm[i+1]$, and
- If $s[i] ==' I'$, then $perm[i] < perm[i+1]$.

Return the number of valid permutations perm. Since the answer may be large, return it modulo $10^9 + 7$.

Solution. Define o$dp(i, j)$ to be the number of possible permutations of first $i+1$ digits where the $i+1$th digit is $j+1$th smallest in the rest of unused digits

Let's see an example of "DID"



□

Problem 1.1.9 (LeetCode: Distinct Subsequences II). Given a string s, return the number of distinct non-empty subsequences of s. Since the answer may be very large, return it modulo $10^9 + 7$.

A subsequence of a string is a new string that is formed from the original string by deleting some (can be none) of the characters without disturbing the relative positions of the remaining characters. (i.e., "ace" is a subsequence of "abcde" while "aec" is not.

Solution. Define $F(i)$ to be the subsequences ends in character $s[i]$, and $f(i) = |F(i)|$

How do we eliminate the duplicates? Note that if $s[i] = s[j]$ and $i < j$, then $F(i) \subseteq F(j)$, therefore for each character, we only need to care about the last appearance.

$$f(i) = 1 + \sum_{0 \le k < 26, last[k] \neq -1} f(last[k])$$

Note that we only needs 26 values, define $g[26]$, then

$$g(c) = 1 + \sum_{i=0}^{25} g(i)$$

Then, we only need to store $total = \sum g(i)$. □

1.2 Digit DP

Problem 1.2.1 (LeetCode 788: Rotated Digits). An integer x is a **good** if after rotating each digit individually by 180 degrees, we get a valid number that is different from x. Each digit must be rotated - we cannot choose to leave it alone.

A number is valid if each digit remains a digit after rotation. For example:

- 0, 1, and 8 rotate to themselves,
- 2 and 5 rotate to each other (in this case they are rotated in a different direction, in other words, 2 or 5 gets mirrored)
- 6 and 9 rotate to each other, and

- the rest of the numbers do not rotate to any other number and become invalid.

Given an integer n, return the number of good integers in the range [1, n].

Solution. Given n. Let $f(pos, bound, diff)$ be the number of good numbers satisfying

1. Only consider posth digit and pos starts from left, which means 0th digit is the highest digit. And we assume the first $pos - 1$ digits are fixed
2. If digits in $[0, pos - 1]$ are first pos digits of n, then $bound$ is `true`
3. If digits in $[0, pos - 1]$ has at least one 2/5/6/9, then $diff$ is `true`

Therefore the answer is $f(0, true, false)$, and the transition formula is

$$f(pos, bound, diff) = \sum f(pos + 1, bound', diff')$$

- $bound'$ is true iff $bound$ is true and the digit we choose is the posth digit of n
- $diff'$ is true iff $diff$ is true or we chose 2/5/6/9

□

Problem 1.2.2 (LeetCode: Numbers at most N given digit set). Given an array of digits which is sorted in non-decreasing order. You can write numbers using each $digits[i]$ as many times as we want. For example, if digits = ['1','3','5'], we may write numbers such as '13', '551', and '1351315'.

Return the number of positive integers that can be generated that are less than or equal to a given integer n.

Solution. Suppose we have m digits.

Define $dp(i, 0)$ to be the number of different number in i digits that is less than the first i digits of n, and $dp(i, 1)$ to be the number of i digits that is equal to the first i digits of n.

Let $C(i)$ to be the number of digits that is less than ith digit of n. Then

$$dp(i, 0) = \begin{cases} C(i) & i = 1 \\ m + dp(i-1, 0) \times m + dp(i-1, 1) \times C(i) & i > 1 \end{cases}$$

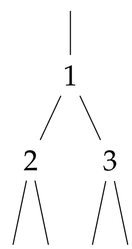
where $dp(0, 0) = 0$ and $dp(0, 1) = 1$ □

2 Graph

2.1 Tree

Problem 2.1.1 (LeetCode: Navigation Device). Given a binary tree T, find minimum number of nodes (device) $n_1, \ldots, n_k$ s.t. for each node m in tree, $(d_1, \ldots, d_k)$ is unique where d_i is the distance between m and n_i.

Solution. Observation: Given a subtree of the form



Then there are at least two device on subtree 2, subtree 3 and parent of 1.

Therefore if we know that ancestors of 1 have device and 1's two childs don't have device, we should put a device in either 2 or 3.

Now since we are assuming the root has device, we need to check whether it needs.

1. the left and right subtree has device: don't need

2. only one of the subtree has device: if the subtree has two device, then we don't need

3. none of the subtree has device: need

□

Problem 2.1.2 (LeetCode: Smallest Missing Genetic Value in Each Subtree). There is a family tree rooted at 0 consisting of n nodes numbered 0 to $n-1$. You are given a 0-indexed integer array parents, where $parents[i]$ is the parent for node i. Since node 0 is the root, $parents[0] = -1$.

There are 10^5 genetic values, each represented by an integer in the inclusive range $[1, 10^5]$. You are given a 0-indexed integer array nums, where $nums[i]$ is a distinct genetic value for node i.

Return an array ans of length n where $ans[i]$ is the smallest genetic value that is missing from the subtree rooted at node y.

The subtree rooted at a node x contains node x and all of its descendant nodes.

Solution. Start from the node with num 1 and go up. □

2.2 Union find

Problem 2.2.1 (LeetCode: Number of Good Paths). There is a tree (i.e. a connected, undirected graph with no cycles) consisting of n nodes numbered from 0 to $n-1$ and exactly $n-1$ edges.

You are given a 0-indexed integer array `vals` of length n where `vals[i]` denotes the value of the ith node. You are also given a 2D integer array edges where `edges[i] = [ai,bi]` denotes that there exists an undirected edge connecting nodes a_i and b_i.

A good path is a simple path that satisfies the following conditions:

1. The starting node and the ending node have the same value.
2. All nodes between the starting node and the ending node have values less than or equal to the starting node (i.e. the starting node's value should be the maximum value along the path).

Return the number of distinct good paths.

Note that a path and its reverse are counted as the same path. For example, `0 -> 1` is considered to be the same as `1 -> 0`. A single node is also considered as a valid path.

Solution. First, to solve the problem, we can enumerate the paths from the nodes with largest `vals`, and then delete these nodes and continue; this requires $O(n^2)$ time

If we reverse the direction, we are merging nodes with values from low to high, so what comes to our mind? Union find.

For each node s and its neighbor t:

1. if `vals[s]<vals[t]`, then pass
2. if `vals[s]=vals[find[t]]`, then add `size[find[s]]*size[find[t]]`
3. merge `s` and `t`

□

Problem 2.2.2 (LeetCode: Bricks Falling When Hit). You are given an $m \times n$binary grid, where each 1 represents a brick and 0 represents an empty space. A brick is stable if:

- It is directly connected to the top of the grid, or
- At least one other brick in its four adjacent cells is stable.

You are also given an array $hits$, which is a sequence of erasures we want to apply. Each time we want to erase the brick at the location $hits[i] = (row_i, col_i)$. The brick on that location (if it exists) will disappear. Some other bricks may no longer be stable because of that erasure and will fall. Once a brick falls, it is immediately erased from the grid (i.e., it does not land on other stable bricks).

Return an array result, where each $result[i]$ is the number of bricks that will fall after the ith erasure is applied.

Note that an erasure may refer to a location with no brick, and if it does, no bricks drop.

Solution. In essence, think the problem in reverse direction

Method 1: union find

Method 2: dfs □

3 Greedy

Problem 3.0.1 (LeetCode: Course Schedule III). There are n different online courses numbered from 1 to n. You are given an array courses where $courses[i] = [duration_i, lastDay_i]$ indicate that the ith course should be taken continuously for $duration_i$ days and must be finished before or on $lastDay_i$

You will start on the 1st day and you cannot take two or more courses simultaneously.

Return the maximum number of courses that you can take.

Solution. For any two courses (t_1, d_1) and (t_2, d_2), if $d_1 \leq d_2$, then it's optimal to study the first before the latter. Then "we can study 2 and then 1" always implies "we can study 1 and then 2"

Now we prove by induction.

Given i courses, sort them by lastDay. Suppose we choose k courses $(t_{x_1}, d_{x_1}), (t_{x_2}, d_{x_2}), \dots, (t_{x_k}, d_{x_k})$ where $x_1 < x_2 < \cdots < x_k$ from the first $i-1$ courses which is optimal for the first $i-1$ courses. Then

$$\begin{cases} t_{x_1} \leq d_{x_1} \\ t_{x_1} + t_{x_2} \leq d_{x_2} \\ \vdots \\ t_{x_1} + \cdots + t_{x_k} \leq d_{x_k} \end{cases}$$

Then we can build the optimal plan for the first i courses based on this and (t_i, d_i)

- if $t_{x_1} + \cdots + t_{x_k} + t_i \leq d_i$, then we can put (t_i, d_i) into our plan, which is optimal.
- if $t_{x_1} + \cdots + t_{x_k} + t_i > d_i$

□

4 General

4.1 Intervals

Problem 4.1.1 (LeetCode: Count Days Spent Together). Alice and Bob are traveling to Rome for separate business meetings.

You are given 4 strings `arriveAlice`, `leaveAlice`, `arriveBob`, and `leaveBob`. Alice will be in the city from the dates `arriveAlice` to `leaveAlice` (inclusive), while Bob will be in the city from the dates `arriveBob` to `leaveBob` (inclusive). Each will be a 5-character string in the format "MM-DD", corresponding to the month and day of the date.

Return the total number of days that Alice and Bob are in Rome together.

You can assume that all dates occur in the same calendar year, which is not a leap year. Note that the number of days per month can be represented as: [31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31].

Solution. First, we can convert the string to ith day of the year, then Alice's interval is $[a, b]$, Bob's interval is $[c, d]$, then we need to calculate the intersection of these intervals.

$[a, b] \cap [c, d] \neq \emptyset$ iff $b \geq c \land d \geq a$.

$[a, b] \cap [c, d] = \min(b, d) - \max(a, c) + 1$ □

4.2 Binary search

Problem 4.2.1 (LeetCode: Maximum Running Time of N Computers). You have n computers. You are given the integer n and a 0-indexed integer array batteries where the ith battery can run a computer for $batteries[i]$ minutes. You are interested in running all n computers simultaneously using the given batteries.

Initially, you can insert at most one battery into each computer. After that and at any integer time, you can remove a battery from a computer and insert another battery any number of times. The inserted battery can be a new battery or a battery from another computer. You may assume that the removing and inserting processes take no time.

Note that the batteries cannot be recharged.

Return the maximum number of minutes you can run all the n computers simultaneously.

Solution. Suppose the maximum is k, then we can draw a $n \times k$ matrix A where $A[i, j] \in [0, m)$ and m is the length of the batteries, meaning the jth computer uses battery $A[i, j]$ in ith minute.

By the problem constraints, each row's elements are distinct, and each $x \in [0, m)$ should appear less than $batteries[x]$ times. Therefore in overall, the number of appearance of x should less than or equal to $\min(batteries[x], k)$

We can use binary search to determine the k since if we can run them for k minutes, we can run them for $k-1, k-2, ...$ minutes. Therefore there is k' s.t. $\leq k'$ works and $> k'$ doesn't.

In each step, suppose we need to check mid. Then the appearance time of each x shouldn't be greater than $occ(x) = \min(batteries[x], mid)$. If $\sum_{x=0}^{m-1} occ(x) < mid \times n$, then it is impossible.

Otherwise, we have a strategy to fill the matrix: fill the column one by one by contiguous block of x. □

Problem 4.2.2 (LeetCode: Kth Smallest Number in Multiplication Table). Nearly everyone has used the Multiplication Table. The multiplication table of size $m \times n$ is an integer matrix mat where $mat[i][j] == i \times j$ (1-indexed).

Given three integers m, n, and k, return the kth smallest element in the $m \times n$ multiplication table.

Solution. Consider: given x, how small is it?

There are

$$\sum_{i=1}^{m} \min(\lfloor \frac{x}{i} \rfloor, n)$$

numbers less than x. Since $i \leq \lfloor \frac{x}{n} \rfloor \Rightarrow \lfloor \frac{x}{i} \rfloor \geq n$, we can simplify the above equation to

$$\lfloor \frac{x}{n} \rfloor \cdot n + \sum_{i=\lfloor \frac{x}{n} \rfloor + 1}^{m} \lfloor \frac{x}{i} \rfloor$$

□

Now let's see a generalization of the above problem:

Problem 4.2.3 (LeetCode: Kth Smallest Product of Two Sorted Arrays). Given two sorted 0-indexed integer arrays $nums1$ and $nums2$ as well as an integer k, return the kth (1-based) smallest product of $nums1[i] \times nums2[j]$ where $0 \leq i < nums1.length$ and $0 \leq j < nums2.length$

Solution. □

4.3 Bit operation

Get all subsets of a bit mask:

```
for (int subset=state; subset>0; subset=(subset-1)&state) {}
```

Problem 4.3.1 (Leetcode: Missing Two LCCI). You are given an array with all the numbers from 1 to N appearing exactly once, except for two number that is missing. How can you find the missing number in $O(N)$ time and $O(1)$ space?

You can return the missing numbers in any order.

Input	Output
`[1]`	`[2,3]`
`[2,3]`	`[1,4]`

`nums.length <= 30000`

Solution. Suppose the missing two numbers are x_1 and x_2, and if we add $1, \ldots, N$ to the end of the array A, then $x = \bigoplus A = x_1 \oplus x_2$.

By `x&-x` we can get the lowest bit of x, assume it's in lth bit. Then we can assume x_1's lth bit is 0, and x_2's lth bit is 1, and we can partition A into A_1 and A_2 by whether the elements' lth bit is 1, then $\bigoplus A_1 = x_1$ and $\bigoplus A_2 = x_2$ □

Problem 4.3.2 (LeetCode: Find a Value of a Mysterious Function Closest to Target).

```
func(arr, l, r) {
  if (r < l) {
    return -1000000000;
  }
  ans = arr[l];
  for (i = l + 1; i <= r; i++) {
    ans = ans & arr[i];
  }
  return ans;
}
```

Winston was given the above mysterious function `func`. He has an integer array `arr` and an integer `target` and he wants to find the values `l` and `r` that make the value `|func(arr, l, r) - target|` minimum possible.

Return the minimum possible value of `|func(arr, l, r) - target|`.

Notice that func should be called with the values l and r where `0 <= l, r < arr.length`.

Constraints:

- `1 <= arr.length <= 10^5`
- `1 <= arr[i] <= 10^6`
- `0 <= target <= 10^7`

Solution. If we fix r

- f is a non-decreasing function
- there is at most 20 different values for $f(arr, l, r)$ as $arr[r] \leq 10^6 < 2^{20}$, since from right to left, 0 won't be transformed into 1

□

Problem 4.3.3 (LeetCode: Smallest Subarrays With Maximum Bitwise OR). You are given a *0-indexed* array nums of length n, consisting of non-negative integers. For each index i from 0 to $n-1$, you must determine the size of the minimum sized non-empty subarray of nums starting at i (inclusive) that has the maximum possible bitwise OR.

Return an integer array answer of size n where answer[i] is the length of the minimum sized subarray starting at i with maximum bitwise OR.

A subarray is a contiguous non-empty sequence of elements within an array.

Solution. Induction and we build a new array $A = \{a_i : a_i = nums[i]\}$. In the ith round, for each $j < i$, check whether $a_j | a_i > a_j$. If so, $a_j | a_i$ is the new possible maximum for a_j and the possible $answer[j] \geq i - j + 1$.

If $a_j | a_i = a_j$, then $a_i \subseteq a_j$ in the sense of bits and for each $k < j$, $a_k | a_i = a_k | a_j$. So we don't need to consider $k < j$ □

Problem 4.3.4 (LeetCode: Triples with Bitwise AND Equal to Zero). Given an integer array nums, return the number of AND triples.

An AND triple is a triple of indices (i, j, k) such that:

- `0 <= i < nums.length`
- `0 <= j < nums.length`
- `0 <= k < nums.length`

- `nums[i] & nums[j] & nums[k] == 0`, where & represents the bitwise-AND operator.

Solution. □

4.4 Math

Problem 4.4.1 (Leetcode: Reach a Number). You are standing at position 0 on an infinite number line. There is a destination at position target.

You can make some number of moves numMoves so that:

- On each move, you can either go left or right.
- During the ith move (starting from `i = 1` to `i = numMoves`), you take i steps in the chosen direction.

Given the integer target, return the minimum number of moves required (i.e., the minimum numMoves) to reach the destination.

Solution. Let s be the target and $k = \min\{t \mid \sum_{i=1}^{t} i > s \wedge s - \sum_{i=1}^{t} i \equiv 0 \mod 2\}$. If $\sum_{i=1}^{k} i = s$, we are done. Otherwise, we need to change the sign of some numbers whose sum is half of the difference, which could be done obviously. □

4.5 Hard to say

Problem 4.5.1 (LeetCode: Minimum Money Required Before Transactions). You are given a 0-indexed 2D integer array transactions, where `transactions[i] = [costi, cashbacki]`.

The array describes transactions, where each transaction must be completed exactly once in some order. At any given moment, you have a certain amount of money. In order to complete transaction i, `money >= cost_i` must hold true. After performing a transaction, money becomes `money-cost_i+cashback_i`.

Return the minimum amount of money required before any transaction so that all of the transactions can be completed regardless of the order of the transactions.

Solution. The worst case is, we put money-losing transaction first and then put the transaction with highest cost after it (erase the transaction before if necessary, and assume its index is i)

Suppose $total$ is the total lose, then if the transaction is money-losing, then the money we need is

$$total - (cost[i] - cashback[i]) + cost[i] = total + cashback[i]$$

Otherwise

$$total + cost[i]$$

$\square$

Problem 4.5.2 (LeetCode: Sparse Similarity). The similarity of two documents (each with distinct words) is defined to be the size of the intersection divided by the size of the union. For example, if the documents consist of integers, the similarity of $\{1, 5, 3\}$ and $\{1, 7, 2, 3\}$ is 0.4, because the intersection has size 2 and the union has size 5. We have a long list of documents (with distinct values and each with an associated ID) where the similarity is believed to be "sparse". That is, any two arbitrarily selected documents are very likely to have similarity 0. Design an algorithm that returns a list of pairs of document IDs and the associated similarity.

Input is a 2D array docs, where docs[i] is the document with id i. Return an array of strings, where each string represents a pair of documents with similarity greater than 0. The string should be formatted as {id1},{id2}: {similarity}, where id1 is the smaller id in the two documents, and similarity is the similarity rounded to four decimal places. You can return the array in any order.

return in any order.

Solution. Assume we have D documents and each document has at most W words

Brute force: given two documents A, B, answer is $(|A|+|B|-|A \cup B|)/|A \cup B|$, $O(D^2W)$

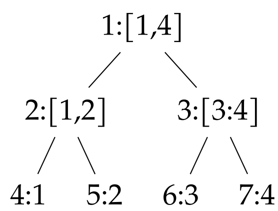
We use inverted index to optimize D^2. We can build a hash table with key the elements of documents and the value the index of the document.

Then to find the document with similarity > 0 with A, we only need to check the hash value for each element of A $\square$

5 Miscellaneous Topics

5.1 Bottom-up segment tree

Suppose we are given interval $[1,3]$, we treat it as $[1,4]$, and then the tree is like



Then the elements starting from $m = 4$ is the leaf nodes. And we can initial our leaf nodes there and climb the tree.

Here is an example

Problem 5.1.1 (CSES: List Removals). You are given a list consisting of n integers. Your task is to remove elements from the list at given positions, and report the removed elements.

The first input line has an integer n: the initial size of the list. During the process, the elements are numbered $1, 2, \dots, k$ where k is the current size of the list.

The second line has n integers $x_1, \dots, x_n$: the contents of the list.

The last line has n integers $p_1, \dots, p_n$: the positions of the elements to be removed.

Solution.

```
int seg[1000000];

// x is the sum we want
int query(int x, const int &m) {
  int i = 1;
  while (i < m) {
    i <<= 1;
    // ensures we find the minimum index satisfying the condition
    if (seg[i] < x) {
      x -= seg[i];
      i++;
    }
    // since the leaf nodes -1
    seg[i]--;
```

```
  }
  return i - m;
}

int main(void) {
  ios_base::sync_with_stdio(0);
  cin.tie(0);
  int n, x, p;
  cin >> n;
  int m = 1;
  int arr[n];
  while (m < n) {
    m <<= 1;
  }

  // initialize leaf nodes
  for (int i = 0; i < n; i++) {
    cin >> arr[i];
    seg[i+m] = 1;
  }

  // initialize internal and root nodes
  for (int i = m-1; i; i--) {
    seg[i] = seg[i*2] + seg[i*2+1];
  }

  for (int i = 0; i < n; i++) {
    cin >> p;
    cout << arr[query(p, m)] << "\n";
  }
```

□

5.2 Tricks

Compute the modular multiplicative inverse:

By Fermat's little theorem, $a^{-1} \equiv 1 \mod p$ for prime p.

Note that

$$aba^{-1}b^{-1} \equiv 1 \mod p$$

therefore

$$((n-1)!)^{-1} \equiv (n!)^{-1} \cdot n \mod p$$

By `x&-x` we can get the lowest bit of x.