ω-saturated models and quantifier elimination

Introductory Model Theory

November 11, 2021

Recommended reading: Poizat's *Course in Model Theory*, Chapter 5.

Definition 1. A formula φ is *quantifier-free* if it has no quantifiers.

Definition 2. A theory T has *quantifier elimination* if for every formula $\varphi(\bar{x})$, there is a quantifier-free formula $\psi(\bar{x})$ such that $T \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Definition 3. If $\bar{a} \in M^n$, then $\mathrm{qftp}^M(\bar{a})$ is the set of quantifier-free L-formulas $\varphi(\bar{x})$ such that $M \models \varphi(\bar{b})$.

Theorem 4. *Let T be a theory. Suppose that*

$$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b}) \implies \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b}) \tag{$*$}$$

whenever $M, N \models T$ and $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then T has quantifier elimination.

Proof. Suppose quantifier elimination fails for $\varphi(\bar{x})$. Let Q be the set of quantifier-free formulas $\psi(\bar{x})$ such that $T \vdash \psi \to \varphi$. If $\psi_1, \ldots, \psi_m \in Q$, then $T \vdash \bigvee_{i=1}^m \psi_i \to \varphi$, so $T \not\vdash \varphi \to \bigvee_{i=1}^m \psi_i$. (Otherwise, φ is equivalent to the quantifier-free formula $\bigvee_{i=1}^m \psi_i$.) Therefore there is $M \models T$ and $\bar{a} \in M^n$ with $M \models \varphi(\bar{a}) \wedge \bigwedge_{i=1}^m \neg\psi_i(\bar{a})$.

By compactness, there is $M \models T$ and $\bar{a} \in M^n$ such that $M \models \varphi(\bar{a})$, but $M \models \neg\psi(\bar{a})$ for all $\psi \in Q$.

Now suppose $\theta_1, \ldots, \theta_m \in \mathrm{qftp}(\bar{a})$. Then $\psi := \bigwedge_{i=1}^m \theta_i \in \mathrm{qftp}(\bar{a})$, so $\psi \notin Q$. Therefore $T \not\vdash \psi \to \varphi$. So there is $N \models T$ and $\bar{b} \in N^n$ such that $N \models \psi(\bar{b}) \wedge \neg\varphi(\bar{b})$. Equivalently, $N \models \bigwedge_{i=1}^m \theta_i(\bar{b})$ but $N \models \neg\varphi(\bar{b})$.

By compactness there is $N \models T$ and $\bar{b} \in N^n$ such that $N \models \neg\varphi(\bar{b})$, and $N \models \theta(\bar{b})$ for all $\theta \in \mathrm{qftp}(\bar{a})$. Then $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. On the other hand, $M \models \varphi(\bar{a})$ and $N \models \neg\varphi(\bar{b})$, so $\mathrm{tp}(\bar{a}) \neq \mathrm{tp}(\bar{b})$, a contradiction. $\square$

Conversely, quantifier elimination implies $(*)$. Suppose $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. For any formula $\varphi(\bar{x})$ there is an equivalent quantifier-free formula $\psi(\bar{x})$, and then

$$M \models \varphi(\bar{a}) \iff M \models \psi(\bar{a}) \iff N \models \psi(\bar{b}) \iff N \models \varphi(\bar{b}).$$

Definition 5. Let M be a structure and A be a subset. The substructure *generated by* A is

$$\langle A\rangle = \{t(\bar{a}) : t(x_1, \ldots, x_n) \text{ is an } L\text{-term, } \bar{a} \in A^n\}.$$

The structure $\langle A\rangle$ is the smallest substructure of M containing A.

Theorem 6. *Let M, N be L-structures. If $\bar{a} \in M^n$ and $\bar{b} \in N^n$, then the following are equivalent:*

1. *There is an isomorphism $f : \langle\bar{a}\rangle_M \to \langle\bar{b}\rangle_N$ such that $f(\bar{a}) = \bar{b}$.*
2. $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$.

Here, $f(\bar{a}) = (f(a_1), \ldots, f(a_n))$.

Proof sketch. (1) $\implies$ (2): If $t(\bar{x})$ is an L-term, then $f(t^M(\bar{a})) = t^N(\bar{b})$ by induction on $t(\bar{x})$. If $\varphi(\bar{x})$ is a quantifier-free formula, then $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b})$ by induction on $\varphi(\bar{x})$.

(2) $\implies$ (1): Define $f : \langle\bar{a}\rangle \to \langle\bar{b}\rangle$ by sending $t(\bar{a})$ to $t(\bar{b})$.

- This is well-defined: if $M \models t(\bar{a}) = t'(\bar{a})$, then $N \models t(\bar{b}) = t'(\bar{b})$ because $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$.
- This is a bijection: if we define $g : \langle\bar{b}\rangle \to \langle\bar{a}\rangle$ similarly, then $g = f^{-1}$.
- This is an isomorphism: for example, if R is a 2-ary relation symbol, then $M \models R(t(\bar{a}), t'(\bar{a})) \iff N \models R(t(\bar{b}), t'(\bar{b}))$ because $R(t(\bar{x}), t'(\bar{x}))$ is quantifier-free. $\square$

Theorem 7. *The following are equivalent for a theory T:*

1. *Let M, N be ω-saturated models. Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. For any $\alpha \in M$ there is $\beta \in N$ such that $\mathrm{qftp}(\bar{a}, \alpha) = \mathrm{qftp}(\bar{b}, \beta)$.*
2. *T has quantifier elimination.*

Proof. (1) $\implies$ (2): Assume (1).

Claim. If M, N are ω-saturated and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$ and $\varphi(\bar{x})$ is a formula, then $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b})$.

Proof. By induction on $\varphi(\bar{x})$.

- $\varphi(\bar{x})$ is atomic: then it's quantifier-free, and it follows because $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$.
- $\varphi(\bar{x})$ is $\neg\psi(\bar{x})$. Easy.
- $\varphi(\bar{x})$ is $\psi \wedge \theta$. Easy.
- $\varphi(\bar{x})$ is $\exists y\ \psi(\bar{x}, y)$. If $M \models \varphi(\bar{a})$ then there is α such that $M \models \psi(\bar{a}, \alpha)$. Take $\beta \in N$ such that $\mathrm{qftp}(\bar{a}, \alpha) = \mathrm{qftp}(\bar{b}, \beta)$. By induction, $N \models \psi(\bar{b}, \beta)$. Thus $N \models \varphi(\bar{b})$. This shows $M \models \varphi(\bar{a}) \implies N \models \varphi(\bar{b})$, and the converse follows by swapping M and N. $\square_{\text{Claim}}$

Now we prove (2). Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. We must show that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Replacing M and N with elementary extensions (which doesn't change the types), we may assume M and N are ω-saturated. Then the claim shows $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, proving (2).

(2) $\implies$ (1): Assume (2). In the set up of (1), $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ by quantifier elimination. Let $p(\bar{x}, y) = \mathrm{tp}^M(\bar{a}, \alpha)$. For any $\varphi(\bar{x}, y) \in p(\bar{x}, y)$, we have $M \models \exists y\, \varphi(\bar{a}, y)$ and therefore $N \models \exists y\, \varphi(\bar{b}, y)$. The partial type $p(\bar{b}, y)$ is therefore finitely satisfiable in N. By ω-saturation, it is satisfied by some $\beta \in N$. Then $\mathrm{tp}(\bar{b}, \beta) = p = \mathrm{tp}(\bar{a}, \alpha)$, so $\mathrm{qftp}(\bar{b}, \beta) = \mathrm{qftp}(\bar{a}, \alpha)$. $\square$

Example. Let T be the theory of discrete linear orders without endpoints. For each $n < \omega$, add a binary relation $R_n(x, y)$ saying that $d(x, y) = n$. We will prove quantifier elimination. Suppose M, N are ω-saturated. Let $A \subseteq M$ and $B \subseteq N$ be finitely generated (= finite) substructures and $f : A \to B$ be an isomorphism. Given $\alpha \in M$, we must find $\beta \in N$ and an isomorphism $g : A \cup \{\alpha\} \to B \cup \{\beta\}$ extending f. The local isomorphism f looks like

$$f : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$$
$$f(a_i) = b_i$$

where $a_1 < a_2 < \cdots < a_n$ in M and $b_1 < \cdots < b_n$ in N, and $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$. Let $x = d(a_i, \alpha)$ and $y = d(\alpha, a_{i+1})$. We need β with $b_i < \beta < b_{i+1}$ such that $d(b_i, \beta) = x$ and $d(\beta, b_{i+1}) = y$. There are four cases:

- $x, y < \infty$. Take $\beta \in [b_i, b_{i+1}]$ such that $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = d(b_{i+1}, b_i) - x = d(a_{i+1}, a_i) - x = (x + y) - x = y$.

- $x < \infty = y$. Then $d(b_i, b_{i+1}) = d(a_i, a_{i+1}) = x + y = \infty$. Take $\beta \in [b_i, b_{i+1}]$ such that $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = \infty = y$.

- $x = \infty > y$. Similar.

- $x = y = \infty$. Let $\Sigma(x)$ be the partial type over b_i, b_{i+1} saying

  $$\{b_i < x < b_{i+1}\} \cup \{d(b_i, x) \neq n : n < \omega\} \cup \{d(x, b_{i+1}) \neq n : n < \omega\}.$$

  Then $\Sigma(x)$ is finitely satisfiable, so it is satisfied by some β by ω-saturation. Then $d(b_i, \beta) = \infty = x$ and $d(\beta, b_{i+1}) = \infty = y$.

Therefore T has quantifier-elimination, after adding the symbols R_n.

Theorem 8. *Let T be a theory with quantifier elimination. Let M, N be models of T. Then $M \equiv N$ iff $\langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$.*

Proof. $M \equiv N \iff \mathrm{tp}^M() = \mathrm{tp}^N() \iff \mathrm{qftp}^M() = \mathrm{qftp}^N() \iff \langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$. $\square$

For example, any two discrete linear orders without endpoints are elementarily equivalent. The theory of discrete linear orders without endpoints is complete.