Higher Order Computability

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April 1, 2022

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1 Theory of Computability Models

1.1 Simulations Between Computability Models

1.1.1 Simulations and Transformations

Definition 1.1. Let **C** and **D** be lax computability models over type worlds T, U respectively. A **simulation** γ of **C** in **D** (written in $\gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$) consist of

- a mapping $\sigma \mapsto \gamma\sigma : T \to U$
- for each $\sigma \in T$, a relation $\Vdash^{\gamma}_{\sigma} \subseteq \mathbf{D}(\gamma\sigma) \times \mathbf{C}(\sigma)$

satisfying the following

1. For all $a \in \mathbf{C}(\sigma)$ there exists $a' \in \mathbf{D}(\gamma\sigma)$ s.t. $a' \Vdash^{\gamma}_{\sigma} a$
2. Every operation $f \in \mathbf{C}[\sigma, \tau]$ is **tracked** by some $f' \in \mathbf{D}[\gamma\sigma, \gamma\tau]$, in the sense that whenever $f(a)\downarrow$ and $a' \Vdash^{\gamma}_{\sigma} a$, we have $f'(a) \Vdash^{\gamma}_{\tau} f(a)$

For any **C** we have the **identity** simulation $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{C}$ given by $\mathrm{id}_{\mathbf{C}}\,\sigma = \sigma$ and $a' \Vdash^{\mathrm{id}_{\mathbf{C}}}_{\sigma} a$ iff $a' = a$

Given simulations $\gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$ and $\delta : \mathbf{D} \longrightarrow\!\!\triangleright\ \mathbf{E}$ we have the composite simulation $\delta \circ \gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \gamma\,\mathbf{E}$ defined by $(\delta \circ \gamma)\sigma = \delta(\gamma\sigma)$ and $a' \Vdash_{\sigma}^{\delta\circ\gamma} a$ iff there exists $a'' \in \mathbf{D}(\gamma\sigma)$ with $a'' \Vdash_{\sigma}^{\gamma} a$ and $a' \Vdash_{\gamma\sigma}^{\delta} a''$.

Definition 1.2. Let $\mathbf{C}$, $\mathbf{D}$ be lax computability models and suppose $\gamma, \delta : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$ are simulations. We say γ is **transformable** to δ, and write $\gamma \preceq \delta$, if for each $\sigma \in |\mathbf{C}|$ there is an operation $t \in \mathbf{D}[\gamma\sigma, \delta\sigma]$ s.t.

$$\forall a \in \mathbf{C}(\sigma), a' \in \mathbf{D}(\gamma\sigma). a' \Vdash_{\sigma}^{\gamma} a \Rightarrow t(a') \Vdash_{\sigma}^{\delta} a$$

We write $\gamma \sim \delta$ if both $\gamma \preceq \delta$ and $\delta \preceq \gamma$

Definition 1.3. Models $\mathbf{C}, \mathbf{D}$ are **equivalent** ($\mathbf{C} \simeq \mathbf{D}$) if there exist simulations $\gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$ and $\delta : \mathbf{D} \longrightarrow\!\!\triangleright\ \mathbf{C}$ s.t. $\delta \circ \gamma \sim \mathrm{id}_{\mathbf{C}}$ and $\gamma \circ \delta \sim \mathrm{id}_{\mathbf{D}}$

A model is **essentially untyped** if it is equivalent to a model over the singleton type world O

Exercise 1.1.1. Show that a model $\mathbf{C}$ is essentially untyped iff it contains a **universal type**: that is, a datatype U s.t. for each $A \in |\mathbf{C}|$ there exists operations $e \in \mathbf{C}[A, U]$, $r \in \mathbf{C}[U, A]$ with $r(e(a)) = a$ for all $a \in A$

Proof. $\Leftarrow$: Let $\mathsf{O} = \{U\}$. For each $f \in \mathbf{C}[A, B]$, $\mathbf{D}$ contains $\bar{f} \in \mathbf{D}[U, U]$ s.t. $\bar{f}()$ Let

$$\mathbf{D}[U, U] = \{\bar{f} : e[A] \to e[B] : f \in \mathbf{C}[A, B]\}$$

where $\bar{f}(e(a)) = e(f(a))$

each $A \in |\mathbf{C}|$, let $\gamma(A) = U$ and define $a' \Vdash_{A}^{\gamma} a$ iff $a' = e(a)$ □

Definition 1.4. Suppose $\mathbf{C}, \mathbf{D}$ are lax models with weak products and weak terminals (I, i), (J, j) respectively. A simulation $\gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$ is **cartesian** if

1. for each $\sigma, \tau \in |\mathbf{C}|$ there exists $t \in \mathbf{D}[\gamma\sigma \bowtie \gamma\tau, \gamma(\sigma \bowtie \tau)]$ s.t.

$$\pi_{\gamma\sigma}(d) \Vdash_{\sigma}^{\gamma} a \wedge \pi_{\gamma\tau}(d) \Vdash_{\tau}^{\gamma} b \Rightarrow$$
$$\exists c \in \mathbf{C}(\sigma \bowtie \tau). \pi_{\sigma}(c) = a \wedge \pi_{\tau}(c) = b \wedge t(d) \Vdash_{\sigma\bowtie\tau}^{\gamma} c$$

2. there exists $u \in \mathbf{D}[J, \gamma I]$ s.t. $u(j) \Vdash_{I}^{\gamma} i$

Definition 1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be lax relative TPCAs over the type worlds T, U respectively. An **applicative simulation** $\gamma : \mathbf{A} \longrightarrow\!\!\triangleright\ \mathbf{B}$ consists of

- a mapping $\sigma \mapsto \gamma\sigma : \mathsf{T} \to \mathsf{U}$
- for each $\sigma \in \mathsf{T}$, a relation $\Vdash_{\sigma}^{\gamma} \subseteq \mathbf{B}^{\circ}(\gamma\sigma) \times \mathbf{A}^{\circ}(\sigma)$

satisfying the following

1. For all $a \in \mathbf{A}^{\circ}(\sigma)$ there exists $b \in \mathbf{B}^{\circ}(\gamma\sigma)$ with $b \Vdash^{\gamma}_{\sigma} a$

2. For all $a \in \mathbf{A}^{\sharp}(\sigma)$ there exists $b \in \mathbf{B}^{\sharp}(\gamma\sigma)$ with $b \Vdash^{\gamma}_{\sigma} a$

3. 'Application in **A** is effective in **B**': that is, for each $\sigma, \tau \in \mathsf{T}$, there exists some $r \in \mathbf{B}^{\sharp}(\gamma(\sigma \to \tau) \to \gamma\sigma \to \gamma\tau)$, called a **realizer for** γ **at** σ**,**τ, s.t. for all $f \in \mathbf{A}^{\circ}(\sigma \to \tau)$, $f' \in \mathbf{B}^{\circ}(\gamma(\sigma \to \tau))$, $a \in \mathbf{A}^{\circ}(\sigma)$ and $a' \in \mathbf{B}^{\circ}(\gamma\sigma)$ we have

$$f' \Vdash_{\sigma\to\tau} f \wedge a' \Vdash_{\sigma} a \wedge f \cdot a \downarrow \Rightarrow r \cdot f' \cdot a' \Vdash_{\tau} f \cdot a$$

Theorem 1.6. *Suppose* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are* (*lax*) *weakly cartesian closed models, and suppose* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are the corresponding* (*lax*) *relative TPCAs with pairing via the correspondence of Theorem* **??**. *Then cartesian simulations* $\boldsymbol{C} \longrightarrow\!\!\triangleright\ \boldsymbol{D}$ *correspond precisely to applicative simulations* $\boldsymbol{A} \longrightarrow\!\!\triangleright\ \boldsymbol{B}$

Proof. Suppose first that $\gamma : \mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$ is a cartesian simulation

1. Definition

2. Suppose $a \in \mathbf{A}^{\sharp}(\sigma)$ where $\mathbf{A}^{\circ}(\sigma) = A$. Then we may find $g \in \mathbf{C}[I, A]$ with $g(i) = a$, where (I, i) is a weak terminal in $\mathbf{C}$. Take $g' \in \mathbf{D}[\gamma I, \gamma A]$ tracking g, and compose it with $u \in \mathbf{D}[J, \gamma I]$, we obtain $g'' \in \mathbf{D}[J, \gamma A]$. Then $g''(j) \in \mathbf{B}^{\sharp}(\gamma\sigma)$, and it is easy to see that $g''(j) \Vdash^{\gamma}_{\sigma} a$

3. Let σ, τ be any types; then by the definition of weakly cartesian closedness, we have $app_{\sigma\tau} \in \mathbf{C}[(\sigma \to \tau) \times \sigma, \tau]$ tracked by some $app'_{\sigma\tau} \in \mathbf{D}[\gamma((\sigma \to \tau) \times \sigma), \gamma\tau]$. By definition of cartesian simulation, we have $t \in \mathbf{D}[\gamma(\sigma \to \tau) \times \gamma\sigma, \gamma((\sigma \to \tau) \times \sigma)]$, we have an operation $\mathbf{D}[\gamma(\sigma \to \tau) \times \gamma\sigma, \gamma\tau]$, and hence an operation $\mathbf{D}[\gamma(\sigma \to \tau), \gamma\sigma \to \gamma\tau]$, and then an operation $\mathbf{D}[J, \gamma(\sigma \to \tau) \to \gamma\sigma \to \gamma\tau]$, and hence realizer $r \in \mathbf{B}^{\sharp}(\gamma(\sigma \to \tau) \to \gamma\sigma \to \gamma\tau)$ with the required properties: for all $f \in \mathbf{A}^{\circ}(\sigma \to \tau)$, $f' \in \mathbf{B}^{\circ}(\gamma(\sigma \to \tau))$, $a \in \mathbf{A}^{\circ}(\sigma)$, $a' \in \mathbf{B}^{\circ}(\gamma\sigma)$, and $f' \Vdash_{\sigma\to\tau} f$, $a' \Vdash_{\sigma} a$, $f \cdot a \downarrow$, we have $t(f', a') \Vdash_{(\sigma\to\tau)\times\sigma} (f, a)$. Then $app'_{\sigma\tau}(t(f', a')) \Vdash^{\gamma}_{\tau} app_{\sigma\tau}(f, a)$

Conversely, suppose $\gamma : \mathbf{A} \longrightarrow\!\!\triangleright\ \mathbf{B}$ is an applicative simulation. To see that γ is a simulation $\mathbf{C} \longrightarrow\!\!\triangleright\ \mathbf{D}$, it suffices to show that every operation in $\mathbf{C}$ is tracked by one in $\mathbf{D}$. But given $f \in \mathbf{C}[\sigma, \tau]$, we may find a corresponding element $a \in \mathbf{A}^{\sharp}(\sigma \to \tau)$, whence some $a' \in \mathbf{B}^{\sharp}(\gamma(\sigma \to \tau))$ with $a' \Vdash^{\gamma}_{\sigma\to\tau} a$; by using a realizer $r \in \mathbf{B}^\sharp$ for γ at σ,τ, we have an element $a'' \in \mathbf{B}^\sharp(\gamma\sigma \to \gamma\tau)$ and so a corresponding operation $f' \in \mathbf{D}[\gamma\sigma, \gamma\tau]$.

It remains to show that γ is cartesian. For any types σ,τ, we have by assumption an element $pair_{\sigma\tau} \in \mathbf{A}^\sharp(\sigma \to \tau \to \sigma \times \tau)$, yielding some $p \in \mathbf{C}[\sigma, \tau \to \sigma \times \tau]$. Since γ is a simulation, this is tracked by some $p' \in \mathbf{D}[\gamma\sigma, \gamma(\tau \to \sigma \times \tau)]$. From the weak product structure in $\mathbf{D}$ we may thence obtain an operation

$$p'' \in \mathbf{D}[\gamma\sigma \times \gamma\tau, \gamma(\tau \to \sigma \times \tau) \times \gamma\tau]$$

and together with a realizer for γ at τ and $\sigma \times \tau$, this yields an operation $t \in \mathbf{D}[\gamma\sigma \times \gamma\tau, \gamma(\sigma \times \tau)]$ with the required properties.

$i \in \mathbf{A}^\sharp(I)$, hence there is $b \in \mathbf{B}^\sharp(\gamma I)$ with $b \Vdash_I^\gamma i$. But $b = u(j)$ for some $u \in \mathbf{D}[J, \gamma I]$ $\square$

The notion of a transformation between simulations carries across immediately to the relative TPCA setting: an applicative simulation $\gamma : \mathbf{A} \longrightarrow\!\!\triangleright \mathbf{B}$ is transformable to δ if for each type σ there exists $t \in \mathbf{B}^\sharp(\gamma\sigma \to \delta\sigma)$ s.t. $a' \Vdash_\sigma^\gamma a$ implies $t \cdot a' \Vdash_\sigma^\delta a$

1.2 Examples of Simulations and Transformations

Example 1.1. Suppose $\mathbf{C}$ is any (lax) computability model with weak products, and consider the following variation on the 'product completion' construction described in the proof of Theorem **??**. Let $\mathbf{C}^\times$ be the computability model whose datatypes are sets $A_0 \times \cdots \times A_{m-1}$ where $A_i \in |\mathbf{C}|$, and whose operations $f \in \mathbf{C}^\times[A_0 \times \cdots \times A_{m-1}, B_0 \times \cdots \times B_{n-1}]$ are those partial functions represented by some operation in $\mathbf{C}[A_0 \bowtie \cdots \bowtie A_{m-1}, B_0 \bowtie \cdots \bowtie B_{n-1}]$. Clearly the inclusion $\mathbf{C} \hookrightarrow \mathbf{C}^\times$ and $\mathbf{C}^\times \to \mathbf{C}$ sending $A_0 \times \dots A_{m-1}$ to $A_0 \bowtie \cdots \bowtie A_{m-1}$ are simulations. Moreover, they constitute an equivalence $\mathbf{C} \simeq \mathbf{C}^\times$. This shows that every strict (lax) computability model with weak products is equivalent to one with standard products