The dichotomy property and the fundamental order

Advanced Model Theory

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Note: this document builds off the earlier set of notes `dichotomy-and-definable-types.pdf` [1], which is posted on eLearning on the "Extra Notes" page, and was previously posted in the WeChat thread.

Suppose $N \succeq M$ and $q \in S_n(N)$ is an extension of $p \in S_n(M)$. Then $[q] \le [p]$, where $[p]$ denotes the class of p in the fundamental order. Recall the notation $q \sqsupseteq p$ means that q is an heir of p. In *any* theory,

$$q \sqsupseteq p \implies [q] = [p]. \qquad \text{(March 31, Proposition 5(3))}$$

In a *stable* theory, the converse holds

$$[q] = [p] \implies q \sqsupseteq p. \qquad \text{(March 31, Proposition 9)}$$

The proof in class used ultrapowers to build a tree of types. Here is an alternative proof, not using ultrapowers, but instead using the $R_{\varphi,2}$-ranks defined in the document [1, Definition 11].

As in the proof in class, one reduces to showing that any type $p \in S_n(M)$ has *at most* one extension which is equivalent in the fundamental order:

Lemma 1. *Assume stability. Suppose $M \preceq N \preceq \mathbb{M}$, $p \in S_n(M)$, and $q_1, q_2 \in S_n(N)$ are extensions of p. If $[q_1] = [q_2] = [p]$, then $q_1 = q_2$.*

We will need the following fact, which is similar to [1, Remark 16]:

Fact 2. *If $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$ are formulas and $n < \omega$, then*

$$\{\bar{b} \in \mathbb{M} : R_{\varphi,2}(\{\psi(\bar{x}, \bar{b})\} \ge n\}$$

is definable without parameters.

Like [1, Remark 16], Fact 2 can either be proved by induction using the recursive definition of $R_{\varphi,2}$ [1, Definition 11], or directly using the explicit description of $R_{\varphi,2}$ in terms of trees in [1, Lemma 12].

Proof (of Lemma 1). If $q_1 \neq q_2$, then there is a formula $\varphi(\bar{x}, \bar{y})$ and parameter $\bar{b} \in N$ with

$$\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x})$$
$$\varphi(\neg\bar{x}, \bar{b}) \in q_2(\bar{x}).$$

By stability, φ doesn't have the dichotomy property, so $R_{\varphi,2}(-)$ is always finite [1, Remark 15]. Take $\Sigma(\bar{x})$ a finite subtype of $p(\bar{x})$ minimizing $R_{\varphi,2}(\Sigma(\bar{x}))$. Let $k = R_{\varphi,2}(\Sigma(\bar{x}))$.[1] Then

$$\Sigma(\bar{x}) \subseteq p(\bar{x})$$
$$\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\} \subseteq q_1(\bar{x})$$
$$\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\} \subseteq q_2(\bar{x}).$$

By monotonicity [1, Remark 14],

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \leq R_{\varphi,2}(\Sigma(\bar{x})) = k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) \leq R_{\varphi,2}(\Sigma(\bar{x})) = k.$$

One of the inequalities is strict; otherwise $R_{\varphi,2}(\Sigma(\bar{x})) \geq k + 1$ by definition of $R_{\varphi,2}$ [1, Definition 11]. Suppose

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) < k.$$

(The other case is similar.) Let $\sigma(\bar{x}, \bar{z})$ be an L-formula and $\bar{c}$ be a tuple in M such that $\sigma(\bar{x}, \bar{c}) \equiv \bigwedge \Sigma(\bar{x})$. (This makes sense as $\Sigma(\bar{x})$ is a finite set of $L(M)$-formulas.) Then

$$\sigma(\bar{x}, \bar{c}) \wedge \varphi(\bar{x}, \bar{b}) \in q_1(\bar{x})$$
$$R_{\varphi,2}(\{\sigma(\bar{x}, \bar{c}) \wedge \varphi(\bar{x}, \bar{b})\}) < k.$$

So some formula in $q_1(\bar{x})$ has $R_{\varphi,2}$-rank less than k. Now the idea is to get similar behavior inside p (contradicting the choice of Σ and k) using $[q_1] = [p]$ and the definability of rank from Fact 2.

First, Fact 2 shows that the set

$$U = \{(\bar{b}', \bar{c}') \in \mathbb{M} : R_{\varphi,2}(\{\sigma(\bar{x}, \bar{c}') \wedge \varphi(\bar{x}, \bar{b}')\}) < k\}$$

is definable without parameters, and therefore defined by some formula $\psi(\bar{y}, \bar{z})$. Then $(\bar{b}, \bar{c}) \in U$, so $\mathbb{M} \models \psi(\bar{b}, \bar{c})$. This implies $\psi(\bar{b}, \bar{c}) \in q_1(\bar{x})$. (A complete type over N contains all true $L(N)$-sentences.) Then

$$\sigma(\bar{x}, \bar{c}) \wedge \varphi(\bar{x}, \bar{b}) \wedge \psi(\bar{b}, \bar{c}) \in q_1(\bar{x}).$$

As $[q_1] = [p]$, there must be $\bar{b}', \bar{c}' \in M$ such that

$$\sigma(\bar{x}, \bar{c}') \wedge \varphi(\bar{x}, \bar{b}') \wedge \psi(\bar{b}', \bar{c}') \in p(\bar{x}).$$

But then $\mathbb{M} \models \psi(\bar{b}', \bar{c}')$, which means $(\bar{b}', \bar{c}') \in U$, which means

$$R_{\varphi,2}(\{\sigma(\bar{x}, \bar{c}') \wedge \varphi(\bar{x}, \bar{b}')\}) < k = R_{\varphi,2}(\Sigma(\bar{x})).$$

As $\sigma(\bar{x}, \bar{c}') \wedge \varphi(\bar{x}, \bar{b}') \in p(\bar{x})$, this contradicts the choice of $\Sigma(\bar{x})$. $\square$

[1] Actually $k = R_{\varphi,2}(p(\bar{x}))$ by [1, Fact 17], but we don't need this.

With Lemma 1 in hand, we then prove

$$(q \supseteq p \text{ and } [q] = [p]) \implies q \sqsupseteq p$$

as in class: Let q' be the heir of p. Then $[q'] = [p] = [q]$, so $q = q'$ by Lemma 1, and then q is the heir of p.

References

[1] WJ. The dichotomy property and definability of types. `dichotomy-and-definable-types.pdf`, posted on eLearning and WeChat, March 10, 2022.