Homework 6: The fundamental order and non-forking extensions

Advanced Model Theory

Due April 7, 2022

Work in a monster model $\mathbb{M}$ of a complete theory T. Don't assume stability.

1. Let $p \in S_1(M)$ be a non-constant type. Show that $[p]$ is not a minimal element in the fundamental order. *Hint:* take an extension q of p over a bigger model N, such that q is constant, and show $[q] < [p]$.

2. Let $p \in S_1(M)$ be a constant type and let $q \in S_1(N)$ be an extension of p. Show that $[q] = [p]$.

3. Let $p \in S_1(M)$ be a constant type. Show that $[p]$ is a minimal element in the fundamental order. *Hint:* if $[q] < [p]$, embed p into an ultrapower of q.

4. Suppose the theory T is DLO. Let M, N be small models. Let a, b be elements of $\mathbb{M}$. Suppose $a \notin M$, and $\mathrm{tp}(a/M)$ is not the type at $+\infty$ or $-\infty$. Suppose $b \notin N$, and $\mathrm{tp}(b/N)$ is not the type at $+\infty$ or $-\infty$. Let $\varphi(x, \bar{c})$ be a formula in $\mathrm{tp}(a/M)$. Show there is $\bar{c}' \in N$ such that $\varphi(x, \bar{c}')$ is a formula in $\mathrm{tp}(b/N)$. *Hint:* let $S = \{c_1, \ldots, c_n\} \subseteq M$ where $\bar{c} = (c_1, \ldots, c_n)$. Find $S' \subseteq N$ and a partial elementary map $S \cup \{a\} \to S' \cup \{b\}$.

(Problem 4 shows that $\mathrm{tp}(a/M)$ and $\mathrm{tp}(b/N)$ have the same class in the fundamental order.)