Coheir independence II

Advanced Model Theory

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This document continues the previous notes on coheir independence.

1. We give an alternative proof that any two A-invariant types commute in a stable theory.
2. We show that the relation $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{u}_{M} \bar{b}$ is closely related to Morley products.

1 Finitely satisfiable types commute with definable types

Work in a monster model $\mathbb{M}$ of a complete theory T, not necessarily stable.

Recall that if $M \preceq N \preceq \mathbb{M}$, then

$$N \mathop{\smile\!\!\!\!\!\!|}^{u}_{M} \bar{a} \iff \mathrm{tp}(\bar{a}/N) \sqsupseteq \mathrm{tp}(\bar{a}/M).$$

Therefore, the following lemma generalizes the fact that definable types have unique heirs (Proposition 15 in the February 24th notes).

Lemma 1. *Let M be a small model. Suppose $\mathrm{tp}(\bar{a}/M)$ is definable and $\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{u}_{M} \bar{a}$. Then $\mathrm{tp}(\bar{a}/M\bar{b})$ is $p \restriction M\bar{b}$, where p is the M-definable global type extending $\mathrm{tp}(\bar{a}/M)$ (see Proposition 15 in the March 10th notes).*

Proof. Similar to Proposition 15 in the February 24th notes. But for completeness, here is the proof. We must show that for any L-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and any $\bar{c} \in M$,

$$\varphi(\bar{x}, \bar{b}, \bar{c}) \in \mathrm{tp}(\bar{a}/M\bar{b}) \iff \mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c}).$$

Otherwise, these things are true:

$$\big(\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c})\big) \not\Longleftrightarrow \big(\mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})\big)$$
$$\mathbb{M} \models \big(\varphi(\bar{a}, \bar{b}, \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})\big)$$
$$(\varphi(\bar{a}, \bar{y}, \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{y}, \bar{c})) \in \mathrm{tp}(\bar{b}/M\bar{a}).$$

As $\bar{b} \mathop{\smile\!\!\!\!\!\!\;|}^{u}_{M} \bar{a}$, the type $\operatorname{tp}(\bar{b}/M\bar{a})$ is finitely satisfiable in M, so there is $\bar{b}' \in M$ such that these things are true:

$$\begin{gathered}
\mathbb{M} \models \big(\varphi(\bar{a}, \bar{b}', \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})\big) \\
\big(\mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c})\big) \iff\!\!\!\!\!\!\!/\;\; \big(\mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})\big) \\
\big(\varphi(\bar{x}, \bar{b}', \bar{c}) \in \operatorname{tp}(\bar{a}/M)\big) \iff\!\!\!\!\!\!\!/\;\; \big(\mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})\big).
\end{gathered}$$

This contradicts the choice of the formula $(d_p\bar{x})\varphi(\bar{x}, \bar{y}, \bar{z})$. $\square$

Lemma 2. *Let $p \in S_n(\mathbb{M})$ be finitely satisfiable in a small model M. If $\bar{a} \models p \restriction (M\bar{b})$, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{u}_{M} \bar{b}$.*

Proof. Trivial. $\square$

Theorem 3. *Let p, q be global types. Suppose p is definable over some small set A.*[1] *Suppose q is finitely satisfiable in some small set B.*[2] *Then p and q commute: $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$.*

Proof. Otherwise, there is an $L(\mathbb{M})$-formula $\varphi(\bar{x}, \bar{y})$ such that

$$\begin{gathered}
(p \otimes q)(\bar{x}, \bar{y}) \vdash \varphi(\bar{x}, \bar{y}) \\
(q \otimes p)(\bar{y}, \bar{x}) \vdash \neg\varphi(\bar{x}, \bar{y}).
\end{gathered}$$

The formula $\varphi(\bar{x}, \bar{y})$ uses only finitely many parameters $\bar{c}$ from $\mathbb{M}$. By Löwenheim-Skolem there is a small model M containing $AB\bar{c}$. Then $\varphi(\bar{x}, \bar{y})$ is an $L(M)$-formula. Also, p is M-definable (a weaker condition than being A-definable) and q is finitely satisfiable in M (a weaker condition than being finitely satisfiable in B). Note that p, q, and the products $p \otimes q$ and $q \otimes p$ are M-invariant global types. Take $(\bar{a}, \bar{b}) \models (p \otimes q) \restriction M$. By definition of $\otimes$, this means that $\bar{a} \models p \restriction M$ and $\bar{b} \models q \restriction M\bar{a}$. By Lemma 2,

$$\bar{b} \models q \restriction M\bar{a} \implies \bar{b} \mathop{\smile\!\!\!\!\!\!\;|}^{u}_{M} \bar{a}.$$

Now $\operatorname{tp}(\bar{a}/M)$ is the definable type $p \restriction M$, so by Lemma 1,

$$\bar{a} \models p \restriction M\bar{b}.$$

So $\bar{b} \models q \restriction M$ and $\bar{a} \models p \restriction M\bar{b}$, which means

$$(\bar{b}, \bar{a}) \models (q \otimes p) \restriction M.$$

It follows that $(q \otimes p)(\bar{y}, \bar{x})$ and $(p \otimes q)(\bar{x}, \bar{y})$ have the same restriction to M. (Both restrictions are $\operatorname{tp}(\bar{a}, \bar{b}/M)$.) But $\varphi(\bar{x}, \bar{y})$ is an $L(M)$-formula that is in $(p \otimes q)(\bar{x}, \bar{y})$ but not $(q \otimes p)(\bar{y}, \bar{x})$, a contradiction. $\square$

[1] In particular, p is A-invariant by Remark 14 in the March 10th notes.

[2] In particular, q is B-invariant by (the proof of) Theorem 17(1) in the March 10th notes.

2 Types commute in stable theories

Assume the theory T is stable.

Proposition 4 (Assuming stability)**.** *Let $p \in S_n(\mathbb{M})$ be a global type and M be a small model. The following are equivalent:*

1. *p is finitely satisfiable in M.*
2. *p is M-invariant.*
3. *p is M-definable.*

Proof. (1) $\implies$ (2): Theorem 17(1) in the March 10th notes.

(2) $\implies$ (3): Lemma 19 in the March 10th notes.

(3) $\implies$ (1). Suppose p is M-definable. By Proposition 15 in the March 10th notes, p is the heir of some definable type $q \in S_n(M)$. By Corollary 21 in the March 10th notes, p is a coheir of q, which means p is finitely satisfiable in M. $\square$

Theorem 5 (Assuming stability)**.** *Let $p(\bar{x}), q(\bar{y})$ be two invariant global types. Then p and q commute: $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$.*

Proof. The types p and q are invariant over small sets A and B, respectively. Take a small model M containing $A \cup B$. Then p and q are M-invariant. By Proposition 4, p is M-definable and q is finitely satisfiable in M. Therefore p and q commute by Theorem 3. $\square$

3 Morley products and $\mathop{\smile\!\!\!\!\!\mid}^{u}$

Let M be a small model. If p, q are M-definable types, then the Morley product $p \otimes q$ is also M-definable by Proposition 26 in the March 10th notes. Since M-definable global types correspond to (M-)definable types over M (Proposition 15 in the March 10th notes), we can regard $\otimes$ as an operation on definable types over M.

If T is stable, then *all* types over M are definable, and we get an operation

$$\begin{aligned} S_n(M) \times S_n(M) &\to S_{m+n}(M) \\ (p, q) &\mapsto p \otimes q \end{aligned}$$

The following theorem shows that, at least in stable theories, there is a very close connection between the Morley product $p \otimes q$ and the coheir independence relation $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{u}_{M} \bar{b}$.

Theorem 6. *Assume T is stable. Let $M \preceq \mathbb{M}$ be a small model and $\bar{a}, \bar{b}$ be tuples in $\mathbb{M}$. Then*

$$\left(\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{u}_{M} \bar{b}\right) \iff \left(\operatorname{tp}(\bar{b}, \bar{a}/M) = \operatorname{tp}(\bar{b}/M) \otimes \operatorname{tp}(\bar{a}/M)\right)$$

Proof. First suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^u_M \bar{b}$. Then $\operatorname{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M. By Lemma 4 in the March 10th notes, there is a global type p which is finitely satisfiable in M and extends $\operatorname{tp}(\bar{a}/M\bar{b})$. By Proposition 4 above, p is M-definable. Then p is the unique M-definable global extension of the definable type $\operatorname{tp}(\bar{a}/M)$. Let q be the unique M-definable global extension of the definable type $\operatorname{tp}(\bar{b}/M)$. Then

$$\bar{b} \models q \restriction M \text{ and } \bar{a} \models p \restriction M\bar{b}$$

because p extends $\operatorname{tp}(\bar{a}/M\bar{b})$. Therefore

$$(\bar{b}, \bar{a}) \models (q \otimes p) \restriction M,$$

or equivalently, $\operatorname{tp}(\bar{b}, \bar{a}/M) = (q \otimes p) \restriction M$. By how we defined $\otimes$ on types over M, this means

$$\operatorname{tp}(\bar{b}, \bar{a}/M) = \operatorname{tp}(\bar{b}/M) \otimes \operatorname{tp}(\bar{a}/M).$$

Conversely, suppose $\operatorname{tp}(\bar{b}, \bar{a}/M) = \operatorname{tp}(\bar{b}/M) \otimes \operatorname{tp}(\bar{a}/M)$. Let q be the unique M-definable global extension of the definable type $\operatorname{tp}(\bar{b}/M)$ and let p be the unique M-definable global extension of the definable type $\operatorname{tp}(\bar{a}/M)$. Then

$$(\bar{b}, \bar{a}) \models (q \otimes p) \restriction M,$$

or equivalently,

$$\bar{b} \models q \restriction M \text{ and } \bar{a} \models p \restriction M\bar{b}.$$

By Proposition 4, p is finitely satisfiable in M, and so

$$\bar{a} \models p \restriction M\bar{b} \implies \bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^u_M \bar{b}$$

by Lemma 2. □

In stable theories, any two types commute, either by Theorem 16 in the March 17th notes or Theorem 5 above. Then

$$\operatorname{tp}(\bar{a}, \bar{b}/M) = \operatorname{tp}(\bar{a}/M) \otimes \operatorname{tp}(\bar{b}/M) \iff \operatorname{tp}(\bar{b}, \bar{a}/M) = \operatorname{tp}(\bar{b}/M) \otimes \operatorname{tp}(\bar{a}/M).$$

By Theorem 6, this means

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^u_M \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!\,|}^u_M \bar{a}.$$

This gives another proof of symmetry of $\mathop{\smile\!\!\!\!\!\!\,|}^u$ in stable theories (Theorem 17 in the previous notes on coheir independence).

4 Onwards

Suppose T is stable and A is a small set. Recall that the *algebraic closure* of A, written $\operatorname{acl}(A)$, is the union of all finite A-definable sets. It turns out that any type over $\operatorname{acl}(A)$ has a unique $\operatorname{acl}(A)$-definable global extension.[3] This yields a bijection between types over $\operatorname{acl}(A)$ and $\operatorname{acl}(A)$-definable global types. Analogous to what happens with models, this gives an operation $\otimes$ on types over $\operatorname{acl}(A)$. The general definition of *non-forking independence* ($\mathop{\smile\!\!\!\!\!\!\mid}$) in stable theories is that

$$\left(\bar{a} \underset{A}{\mathop{\smile\!\!\!\!\!\!\mid}} \bar{b}\right) \iff \left(\operatorname{tp}(\bar{a},\bar{b}/\operatorname{acl}(A)) = \operatorname{tp}(\bar{a}/\operatorname{acl}(A)) \otimes \operatorname{tp}(\bar{b}/\operatorname{acl}(A))\right),$$

by analogy to Theorem 6. (In particular, $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_M \bar{b} \iff \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{b}$, when M is a small model.)

And if $A \subseteq B \subseteq M$ and $\bar{c} \in \mathbb{M}^n$, then $\operatorname{tp}(\bar{c}/B)$ is a *non-forking extension* of $\operatorname{tp}(\bar{c}/A)$, written $\operatorname{tp}(\bar{c}/B) \sqsupseteq \operatorname{tp}(\bar{c}/A)$, iff $\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}_A B$. This is analogous to how for $M \preceq N \preceq \mathbb{M}$ and $\bar{c} \in \mathbb{M}^n$,

- $\operatorname{tp}(\bar{c}/N)$ is a coheir of $\operatorname{tp}(\bar{c}/M)$ if and only if $\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}^u_M N$.
- $\operatorname{tp}(\bar{c}/N)$ is an heir of $\operatorname{tp}(\bar{c}/M)$ if and only if $N \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{c}$.

In particular, if $q \in S_n(N)$ and $p \in S_n(M)$, then q is an heir of p iff q is a coheir of p iff q is a non-forking extension of p. Non-forking generalizes the (co)heir relation from types over models to types over arbitrary sets.

[3]Technically this only works if T has elimination of imaginaries, and we need to pass to T^{eq} otherwise.