Tame Topology And O-minimal Structures

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Contents

1 Some Elementary Results

Definition 1.1. A **structure** on a nonempty set R is a sequence $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{N}}$ s.t. for each $m \geq 0$

1. $\mathcal{S}_m$ is a boolean algebra of subsets of R^m
2. if $A \in \mathcal{S}_m$, then $R \times A$ and $A \times R$ belong to $\mathcal{S}_{m+1}$ $(\forall)$

3. $\{(x_1, \dots, x_m) \in R^m : x_1 = x_m\} \in \mathcal{S}_m$
4. if $A \in S_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ where $\pi : R^{m+1} \to R^m$ is the projection map on the first m coordinates ($\exists$)
5. $\{a\} \in \mathcal{S}_1$ for $a \in R$

Fact 1.2. *If $(R, \dots)$ is a model-theoretic structure and $\mathcal{S}_n = \{D \subseteq R^n : D$ is definable$\}$, then $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a structure on R*

Definition 1.3. $X \subseteq \mathbb{C}^n$ is **constructible** if $X = \bigcup_{i=1}^m Y_i$ where each Y_i has the form

$$\{\bar{x} \in \mathbb{C}^n : P_1(\bar{x}) = 0, \dots, P_n(\bar{x}) = 0, Q_1(\bar{x}) \neq 0, \dots, Q_n(\bar{x}) \neq 0\}$$

Fact 1.4. *If $S_m = \{D \subseteq \mathbb{C}^m : D$ constructible$\}$, then $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a structure on $\mathbb{C}$*

Theorem 1.5 (Chevalley's Theorem, Quantifier elimination in $\mathbb{C}$)**.** *Projections works*

Definition 1.6. $X \subseteq \mathbb{R}^n$ is **semialgebraic** if X is a finite union of sets of the form

$$\{\bar{x} \in \mathbb{R}^n : P_1(\bar{x}) = 0, \dots, P_n(\bar{x}) = 0, Q_1(\bar{x}) > 0, \dots, Q_m(\bar{x}) > 0\}$$

Semialgebraic sets are closed under intersection, union, complement, cartesian product, projection

Fact 1.7 (Tarski-Seidenberg)**.** *Semialgebraic sets are a structure on $\mathbb{R}$ (projection)*

Fact 1.8. *If $f : X \to Y$ is definable*

1. *if f^{-1} exists then f^{-1} is definable*
2. *if $g : Y \to Z$ is definable, then $g \circ f$ is definable*
3. *if $A \subseteq X$ is definable, then $f(A)$ is definable*
4. *if $A \subseteq Y$ is definable, then $f^{-1}(A)$ is definable*
5. *If $A \subseteq X$ is definable, then so is $f \restriction A$*

Given functions $f, g : X \to R_\infty$ on a set $X \subseteq R^m$ we put

$$(f, g) := \{(x, r) \in X \times R : f(x) < r < g(x)\}$$
$$[f, g] := \{(x, r) \in X \times R_\infty : f(x) \leq r \leq g(x)\}$$

We consider (f, g) as a subset of R^{m+1}; also $[f, g] \subseteq R^{m+1}$ if f and g are R-valued

Definition 1.9. Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints. An **o-minimal structure** on $(R, <)$ is by definition a structure $\mathcal{S}$ on R s.t.

1. $\{(x, y) \in R^2 : x < y\} \in \mathcal{S}_2$
2. the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points

In $\mathbb{R}$, "definable" = "semialgebraic", in $\mathbb{Q}$, "definable" = "semilinear"

Fact 1.10. *Semialgebraic sets are an o-minimal structure on* $\mathbb{R}$

context

- $(R, \leq)$ dense linear order with no endpoints
- for each n, there's $\mathcal{S}_n$

Fix an o-minimal structure $\mathcal{S}$ on $(R, <)$
Why o-minimality?

1. results results for definable sets
2. a bunch of o-minimal structures exist

Fact 1.11 (Wilkie)**.** *There is an o-minimal structure on* $\mathbb{R}$ *where* $\exp(-)$, $\log(-)$ *are definable*

$\sin(x)$ cannot be definable in o-minimal structure on $\mathbb{R}$

Lemma 1.12. *Let $A \subseteq R$ be definable. Then*

1. $\inf(A)$ *and* $\sup(A)$ *exist in* R_∞ *(dedekind completeness for definable sets)*
2. *the boundary* $bd(A) := \{x \in R :$ *each interval containing x intersects both A and $R - A\}$ is finite, and if $a_1 < \cdots < a_k$ are the points of $bd(A)$ in order, then each interval (a_i, a_{i+1}), where $a_0 = -\infty$ and $a_{k+1} = +\infty$ is either part of A or disjoint from A*

3. *If* $|X| = \infty$ *then* $X \supseteq I$ *for some* I

4. *If* X *is dense in* I*, then* $|X| = \infty$*,* $X \supseteq J$ *(not true in* $\mathbb{Q}/\mathbb{R}$*)* $(X \subseteq I$ *is* ***dense*** *in* I *if* $\forall J \subseteq I(J \cap X \neq \emptyset))$

5. *If* $p \in R$*, then* $\exists b > a$ *s.t.* $(a, b) \subseteq X$ *or* $(a, b) \cap X = \emptyset$*. Locally,*

Proof. 2. $bd(X \cup Y) \subseteq bd(X) \cup bd(Y)$

3. X is a union of interval and points

□

Lemma 1.13. *1. If* $A \subseteq R^m$ *is definable, so are* $cl(A)$ *and* $int(A)$

2. *If* $A \subseteq B \subseteq R^m$ *are definable sets, and* A *is open in* B*, then there is a definable open* $U \subseteq R^m$ *with* $U \cap B = A$

Proof.

$$(x_1, \dots, x_m) \in \mathrm{cl}(A)$$
$$\Leftrightarrow$$
$$(\forall y_1, \dots, y_m \forall z_1, \dots, z_m[y_1 < x < z_1 \wedge \cdots \wedge y_m < x_m < z_m) \rightarrow$$
$$\exists a_1, \dots, a_m(y_1 < a < z_1) \wedge \cdots \wedge y_m < a_m < z_m \wedge (a_1, \dots, a_m) \in A]$$

take U is as the union of all boxes in R^m whose intersection with B is contained in A □

Definition 1.14. A set $X \subseteq R^m$ is **definably connected** if X is definable and X is not the union of two disjoint nonempty definable open subsets of X

Lemma 1.15. *1. the definably connected subsets of* R *are the following: the empty set, the intervals, the sets* $[a, b)$ *with* $-\infty < a < b \leq +\infty$*, the sets* $(a, b]$ *with* $-\infty \leq a < b < +\infty$ *and the sets* $[a, b)$ *with* $-\infty < a \leq b < +\infty$*, and the sets* $[a, b]$ *with* $-\infty < a \leq b < +\infty$

2. *the image of a definably connected set* $X \subseteq R^m$ *under a definable continuous map* $f : X \to R^n$ *is definably connected*

3. *if* X *and* Y *are definable subsets of* R^m*,* $X \subseteq Y \subseteq cl(X)$*, and* X *is definably connected, then* Y *is definably connected*

4. *if* X *and* Y *are definably connected subsets of* R^m *and* $X \cap Y \neq \emptyset$ *, then* $X \cup Y$ *is definably connected*

Proof. 3. suppose $Y = U_1 \cup U_2$ where U_1, U_2 are definably open, then $X \subseteq U_1$ or $X \subseteq U_2$ $\square$

note the following special case of (2):

If the function $f : [a, b] \to R$ is definable and continuous, then f assumes all values between $f(a)$ and $f(b)$

Lemma 1.16. *If $I, J \subseteq R$ intervals, $X \subseteq R$ definable, $I < J$, $|I \setminus X| = \infty = |J \cap X|$, then there is a s.t. $I < a < J$, and there is $c < a < b$ s.t. $(c, a) \cap X = \emptyset$, $(a, b) \subseteq X$*

Proof. take $a = \inf X \setminus bd(X)$ $\square$

1.1 O-minimal ordered groups and rings

Order group is a group equipped with a linear order that is invariant under left and right multiplication:

$$x < y \Rightarrow zx < zy \wedge xz < yz$$

Lemma 1.17. *The only definable subsets of R that are also subgroups are $\{1\}$ and R*

Proof. Given a definable subgroup G we first show that G is convex: if not, then there are $g \in G$, $r \in R - G$ with $1 < r < g$. This gives a sequence

$$1 < r < g < rg < g^2 < rg^2 < g^3 < ...$$

whose terms alternate in being in and out of the definable set G.

So G is convex, hence assuming $G \neq \{1\}$ we have $s := \sup(G) > 1$ with $(1, s) \subseteq G$. If $G = +\infty$, then clearly $R = G$. If $s < +\infty$, then we take any $g \in (1, s)$ and obtain $s = gg^{-1}s \in G$, since $g^{-1}s \in (1, s)$ hence $s < gs \in G$ $\square$

Proposition 1.18. *Suppose $(R, <, \mathcal{S})$ is an o-minimal structure and $\mathcal{S}$ contains a binary operation $\cdot$ on R, s.t. $(R, <, \cdot)$ is an ordered group. Then the group $(R, \cdot)$ is abelian, divisible and torsion-free*

Proof. for each $r \in R$ the centralizer $C_r := \{x \in R : rx = xr\}$ is a definable subgroup containing r, so $C_r = R$ by the lemma. Hence R is abelian. For each $n > 0$ the subgroup $\{x^n : x \in R\}$ is definable, hence equal to R. Every ordered group is torsion free $\square$

Remark. Let $(R, <, +)$ be an ordered abelian group, $R \neq \{0\}$, so $(R, <)$ has no endpoints. Assume also that the linearly ordered set $(R, <)$ is dense. Then the addition operation $+ : R^2 \to R$ and the additive inverse operation $- : R \to R$ are continuous w.r.t. the interval topology, that is, $(R, +)$ is a topological group w.r.t. the interval topology

An **ordered ring** is a ring (associative with 1) equipped with a linear order < s.t.

1. $0 < 1$
2. < is translation invariant
3. < is invariant under multiplication by positive elements

Note that then the additive group of the ring is an ordered group, that the ring has no zero divisors, that $x^2 \geq 0$ for all x, and that $k \mapsto k \cdot 1 : \mathbb{Z} \to$ ring is a strictly increasing ring embedding

suppose our ordered ring is moreover a **division ring**: for each $x \neq 0$ there is y with $x \cdot y = 1$. It is easy to check that such a y is unique, and satisfies $y \cdot x = 1$ and that $x > 0$ implies $y > 0$. It is easy to see that the additive group is divisible, the underlying ordered set is dense without endpoints, and the maps $(x, y) \to xy$ and $x \mapsto x^{-1}$ are continuous w.r.t. the interval topology

An **ordered field** is an ordered division ring with commutative multiplication. Examples: field of reals, field of rational numbers. Define **real closed field** to be an ordered field s.t. if $f(X)$ is a one-variable polynomial with coefficients in the field and $a < b$ are elements in the field with $f(a) < 0 < f(b)$, then there is $c \in (a, b)$ in the field with $f(c) = 0$

Proposition 1.19. *Suppose $(R, <, \mathcal{S})$ is an o-minimal structure and $\mathcal{S}$ contains binary operations $+ : R^2 \to R$ and $\cdot : R^2 \to R$ s.t. $(R, <, +, \cdot)$. Then $(R, <, +, \cdot)$ is a real closed field*

Proof. For each $r \in R$ we have a definable additive subgroup rR of $(R, +)$, hence $rR = R$ if $r \neq 0$. This shows that $(R, <, +, \cdot)$ is an ordered division ring. Let $Pos(R) = \{r \in R : r > 0\}$. Clearly $Pos(R)$ is an ordered multiplicative group. By restricting $\mathcal{S}$ to $Pos(R)$ it follows from the previous proposition that multiplication is commutative on $Pos(R)$, hence on all of R. So $(R, <, +, \cdot)$ is an ordered field. Each one-variable polynomial $f(X) \in R[X]$ gives rise to a definable continuous function $x \mapsto f(x) : R \to R$. Now apply 1.15 $\square$

1.2 Model-theoretic structures

Definition 1.20. A model-theoretic structure $\mathcal{R} = (R, <, \dots)$ where $<$ is a dense linear order without endpoints on R, is called **o-minimal** if $\mathrm{Def}(\mathcal{R}_R)$ is an o-minimal structure on $(R, <)$, in other words, every set $S \subseteq R$ that is definable in $\mathcal{R}$ using constants is a union of finitely many intervals and points

1.3 The simplest o-minimal structures

Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints

We prove below that the model theoretic structure $(R, <)$ is o-minimal

Let $1 \le i \le m$. The function $(x_1, \dots, x_m) \mapsto x_i : R^m \to R$ will be denoted by x_i. The **simple** functions on R^m are by definition these coordinate functions $x_1, \dots, x_m$ and the constant functions $R^m \to R$

Let $f_1, \dots, f_N$ be simple functions on R^m, and let $\epsilon : \{1, \dots, N\}^2 \to \{-1, 0, 1\}$ be given. Then we put

$$\begin{aligned} \epsilon(f_1, \dots, f_N) := \{x \in R^m : & \forall (i,j) \in \{1, \dots, N\}^2 \\ & f_i(x) < f_j(x) \text{ if } \epsilon(i,j) = -1 \\ & f_i(x) = f_j(x) \text{ if } \epsilon(i,j) = 0 \\ & f_i(x) > f_j(x) \text{ if } \epsilon(i,j) = 1\} \end{aligned}$$

If ξ and η are the restrictions of f_i and f_j to $\epsilon(f_1, \dots, f_N)$, then either $\xi < \eta$ or $\xi = \eta$ or $\xi > \eta$. Let $\xi_1 < \dots < \xi_k$ be the restrictions of $f_1, \dots, f_N$ to $\epsilon(f_1, \dots, f_N)$ arranged in increasing order. One checks easily that the sets $\Gamma(\xi_j)$ $(1 \le j \le k)$ and the sets (ξ_j, ξ_{j+1}) $(0 \le j \le k$, where $\xi_0 = -\infty$ and $\xi_{k+1} = +\infty$ by convention) are exactly the nonempty subsets of R^{m+1} of the form $\epsilon'(f_1, \dots, f_N, x_{m+1})$ where

$$\epsilon' : \{1, \dots, N, N+1\}^2 \to \{-1, 0, 1\}$$

is an extension of ϵ. suppose $x_{m+1}(x) = y$, we only need to know the relation among $f_1(x), \dots, f_N(x), y$. And $\bigcup \Gamma(\xi_j) \cup \bigcup (\xi_j, \xi_{j+1}) = \epsilon(f_1, \dots, f_N) \times R$

Define a **simple set** in R^m to be the subset of R^m of the form $\epsilon(f_1, \dots, f_N)$ with $f_1, \dots, f_N$ simple functions on R^m and $\epsilon : \{1, \dots, N\}^2 \to \{-1, 0, 1\}$. We have just proved that if $S \subseteq R^{m+1}$ is simple, then its image under the projection map

$$(x_1, \dots, x_m, x_{m+1}) \mapsto (x_1, \dots, x_m) : R^{m+1} \to R^m$$

is simple in R^m

Proposition 1.21. *The subsets of R^m that are definable in $(R, <)$ using constants are exactly the finite unions of simple sets in R^m*

Proof. Let $\mathcal{S}_m$ be the collection of finite unions of simple sets in R^m. Clearly $\mathcal{S}_m$ is a boolean algebra of subsets of R^m, and each set in $\mathcal{S}_m$ is definable in $(R, <)$ using constants. Texts above show that $\mathcal{S} := (\mathcal{S}_m)_{m \in \mathbb{N}}$ is a structure on the set R, hence the sets in $\mathcal{S}_m$ are exactly the subsets of R^m definable in $(R, <)$ using constants $\square$

Corollary 1.22. *The model-theoretic structure $(R, <)$ is o-minimal*

1.4 Semilinear sets

In this section we show that the sets definable using constants in an ordered vector space over an ordered field are exactly the semilinear sets.
definition

2 Semialgebriac sets

2.1 Thom's lemma and continuity of roots

Lemma 2.1. *Let $\alpha \in \mathbb{C}$ be a zero of the monic polynomial*

$$a_0 + a_1 T + \cdots + a_{d-1} T^{d-1} + T^d \in \mathbb{C}[T], d \geq 1$$

Then $|\alpha| \leq 1 + \max\{|a_i| : i = 0, \ldots, d-1\}$

Proof. Put $M := \max\{|a_i| : i = 0, \ldots, d-1\}$ and suppose $\alpha > 1 + M$. Then $\left|a_0 + a_1\alpha + \cdots + a^{d-1}\alpha^{d-1}\right| \leq M(1 + abs\alpha + \cdots + |\alpha|^{d-1}) = M(|\alpha|^d - 1)/(|\alpha| - 1) < |\alpha|^d$, contradicting $0 = |f(\alpha)|$ $\square$

Lemma 2.2 (Thom)**.** *Let $f_1, \ldots, f_k \in \mathbb{R}[T]$ be nonzero polynomials s.t. if $f_i' \neq 0$, then $f_i' \in \{f_1, \ldots, f_k\}$. Let $\epsilon : \{1, \ldots, k\} \to \{-1, 0, 1\}$, and put*

$$A_\epsilon := \{t \in \mathbb{R} : \operatorname{sgn}(f_i(t)) = \epsilon(i), i = 1, \ldots, k\} \subseteq \mathbb{R}$$

Then A_ϵ is empty, a point, or an interval. If $A_\epsilon \neq \emptyset$, then its closure is given by

$$cl(A_\epsilon) = \{t \in \mathbb{R} : \operatorname{sgn}(f_i(t)) \in \{\epsilon(i), 0\}, i = 1, \ldots, k\}$$

If $A_\epsilon = \emptyset$, then $\{t \in \mathbb{R} : \operatorname{sgn}(f_i(t)) \in \{\epsilon(i), 0\}, i = 1, \ldots, k\}$ is empty or a point

We call ϵ a **sign condition** for $f_1, \ldots, f_k$. The 3^k possible sign conditions ϵ determine 3^K disjoint sets A_ϵ, which together cover the real line $\mathbb{R}$. The second statement of the lemma says that for nonempty A_ϵ its closure can be obtained by relaxing all strict inequalities to weak inequalities

Proof. By induction on k. The lemma holds trivially for $k = 0$. Let $f_1, \ldots, f_k, f_{k+1} \in \mathbb{R}[T] - \{0\}$ be polynomials s.t. if $f_i' \neq 0$, then $f_i' \in \{f_1, \ldots, f_{k+1}\}$. We may assume that $\deg(f_{k+1}) = \max\{\deg(f_i) : 1 \leq i \leq k+1\}$. Let $\epsilon' : \{1, \ldots, k+1\} \to \{-1, 0, 1\}$, and let ϵ be the restriction of ϵ' to $\{1, \ldots, k\}$. By the inductive hypothesis, A_ϵ is empty, a point or an interval. It A_ϵ is empty or a point, so is $A_{\epsilon'} = A_\epsilon \cap \{t \in \mathbb{R} : \operatorname{sgn}(f_{k+1}(t)) = \epsilon'(k+1)\}$, and the other properties to be checked in this case follow easily from the inductive hypothesis on A_ϵ

Suppose A_ϵ is an interval. Since f_{k+1}' has a constant sign on A_ϵ, the function f_{k+1} is either strictly monotone on A_ϵ, or constant. In both cases, it is routine to check that $A_{\epsilon'} = A_\epsilon \cap \{t \in \mathbb{R} : \operatorname{sgn}(f_{k+1}(t)) = \epsilon'(k+1)\}$ has the required properties $\square$

Lemma 2.3 (Continuity of roots)**.** *Let $f(T) = a_0 + a_1 + \cdots + a_d T^d \in \mathbb{C}[T]$ be a polynomial that has no zero on the boundary circle $|z - c| = r$ of a given open disc $|z - c| < r$ in the complex plane ($c \in \mathbb{C}, r > 0$). Then there is $\epsilon > 0$ s.t. if $|a_i - b_i| \leq \epsilon$ for $i = 0, \ldots, d$ then $g(T) := b_0 + b_1 T + \cdots + b_d T_d \in \mathbb{C}[T]$ also has no zero on the circle, and f and g have the same number of zeros in the disc*

3 Cell Decomposition

Fix an arbitrary o-minimal structure $(R, <, \mathcal{S})$. Instead of saying that a set $A \subseteq R^m$ belongs to $\mathcal{S}$, we say that A is definable

3.1 The monotonicity theorem and the finiteness lemma

Theorem 3.1 (Monotonicity theorem)**.** *Let $f : (a, b) \to R$ be a definable function on the interval (a, b). Then there are points $a_1 < \cdots < a_k$ in (a, b) s.t. on each subinterval (a_j, a_{j+1}) with $a_0 = a$, $a_{k+1} = b$, the function is either constant, or strictly monotone and continuous*

We derive this from the threes below. In these lemmas we consider a definable function $f : I \to R$ on an interval I

Lemma 3.2. *There is a subinterval of I on which f is constant or injective*

Lemma 3.3. *If f is injective, then f is strictly monotone on a subinterval of I*

Lemma 3.4. *If f is strictly monotone, then f is continuous on a subinterval of I*

These lemmas imply the monotonicity theorem as follows:
Let

$$X := \{x \in (a,b) : \text{on some subinterval of } (a,b) \text{ containing } x \text{ the function } f \\ \text{is either constant, or strictly monotonicity and continuous}\}$$

Now $(a,b)-X$ must be finite, since otherwise it would contain an interval I; applying successively lemmas 3.2, 3.3, 3.4 we can make I so small that f is either constant, or strictly monotone and continuous on I. But then $I \subseteq X$, a contradiction

Since $(a,b) - X$ is finite, we can reduce the proof of the theorem to the case that $(a,b) = X$, by replacing (a,b) by each of the finitely many intervals of which the open set X consists. In particular, we may assume that f is continuous. By splitting up (a,b) further we can reduce to one of the following three cases

Case 1. For all $x \in (a,b)$, f is constant on some neighborhood of x

Case 2. For all $x \in (a,b)$, f is strictly increasing on some neighborhood of x

Case 3. For all $x \in (a,b)$, f is strictly decreasing on some neighborhood of x

Case 1. Take $x_0 \in (a,b)$ and put

$$s := \sup\{x : x_0 < x < b, f \text{ is constant on } [x_0, x)\}$$

Then $s = b$, since $s < b$ implies that f is constant on some neighborhood of s, contradiction. From $s = b$ it follows that f is constant on $[x_0, b)$. Similarly we prove that f is constant on $(a, x_0]$, therefore f is constant on (a,b)

Case 2. Take $x_0 \in (a,b)$ and put

$$s := \sup\{x : x_0 < x < b, f \text{ is strictly increasing on } [x_0, x)\}$$

Then $s = b$, since $s < b$ leads to a contradiction

We now prove the lemmas

Proof of Lemma 3.2. If some $y \in R$ had infinite preimage $f^{-1}(y)$, then this preimage would contain a subinterval of I and f would take the constant value y on that subinterval. So we may assume that each $y \in R$ has finite preimage. Then $f(I)$ is infinite, and so contains an interval J. Define an "inverse" $g : J \to I$ by

$$g(y) := \min\{x \in I : f(x) = y\}$$

Since g is injective by definition, $g(J)$ is infinite, and hence $g(J)$ contains a subinterval of I, and f is necessarily injective on this subinterval

If $x_1, x_2 \in J' \subseteq g(J)$, $x_i = g(y_i)$, $f(x_1) = f(x_2) \Rightarrow y_1 = y_2 \Rightarrow x_1 = x_2$ and f is injective □

Fix $f : I \to R$, $a \in I$, $\Phi_{-+}(a)$ means $\exists \epsilon$ s.t. if $x \in (a-\epsilon, a)$ then $f(x) < f(a)$, and if $x \in (a, a+\epsilon)$ then $f(x) > f(a)$. "locally increasing"

$\Phi_{+-}(a)$, $\Phi_{++}(a)$, Φ_{--} is similar

$\Phi_{00}(a)$, $\exists \epsilon$, $x \in (a-\epsilon, a+\epsilon) \Rightarrow f$ is increasing

Definition 3.5. $a \in slbd(D)$ if $(a-\epsilon, a) \cap D = \emptyset$, $(a, a+\epsilon) \subseteq D$, strong left boundary

Fact 3.6. *If $X, Y \subseteq R$, $|X| = |Y| = \infty$, $X < Y$, if $D \subseteq R$, $X \cap D = \emptyset$, $Y \subseteq D$ then $\exists a \in slbd(D)$, $X \leq a \leq Y$*

Lemma 3.7. *If $\Phi_{-+}(a)$, $\forall a \in I$, then f is increasing*

Proof. suppose $a, b \in I$, $a < b$, $f(a) \geq f(b)$. there is ϵ s.t. if $x \in (a, a+\epsilon)$ then $f(x) > f(a)$, and if $x \in (b-\epsilon, b)$, $f(x) < f(b) \leq f(a)$.

$D = \{x : f(x) \leq f(a)\}$, $(a, a+\epsilon) \cap D = \emptyset$, $(b-\epsilon, b) \subseteq D$, then $\exists c \in slbd(D)$, $c-\delta, c \cap D = \emptyset$ and $(c, c+\delta) \subseteq D$, so $\Phi_{-+}(c)$ is false □

Lemma 3.8.

1. *If $\forall a \in I$, $\Phi_{+-}(a)$, then f is decreasing*
2. *If $\forall a \in I$, $\Phi_{00}(a)$, then f is constant*

Lemma 3.9. *If $f : I \to R$ injective, $a \in I$, then $\Phi_{++}(a)$ or $\Phi_{+-}(a)$ or $\Phi_{-+}(a)$ or $\Phi_{--}(a)$*

if f is not injective, then there may be 9 cases

Fact 3.10. *If $D \subseteq R$ definable, $a \in R$, then there is ϵ s.t. $(a, a+\epsilon) \subseteq D$ or $(a, a+\epsilon) \cap D = \emptyset$ and $(a-\epsilon, a) \subseteq D$ or $(a-\epsilon, a) \cap D = \emptyset$*

Proof. Let $D = \{x \in I : f(x) > f(a)\}$, then the fact gives 4 cases □

Lemma 3.11. *If $f : I \to R$ is definable*

1. *It can't be that: $\forall a \in I$, $\Phi_{++}(a)$*
2. *It can't be that: $\forall a \in I$, $\Phi_{--}(a)$*

Proof. 1. Assume $\forall x \Phi_{++}(x)$

$\Psi_{+-}(a) \Leftrightarrow \exists y, \epsilon$, if $x \in (a-\epsilon, a)$, then $f(x) > y$, $x \in (a, a+\epsilon)$, $f(x) < y$

Let $I = (a, b)$, $S = \{x \in I \mid \exists x' \in I, x' > x, f(x') < f(x)\}$

Case 1: $(\exists\epsilon)(b-\epsilon, b) \cap S = \emptyset$. Then on the interval $(b-\epsilon, b)$, f is increasing, $\Phi_{++}(x)$ doesn't hold

Case 2: $(\exists\epsilon)(b-\epsilon, b) \subseteq S$

Take $x_0 \in (b-\epsilon, b)$, $x_0 \in S$, and we could get a decreasing sequence

Let $D = \{x \in I : f(x) > f(x_0)\}$. So there are infinitely many points $< x_0$ in D, and infinitely many points $> x_0$ not in D

$\exists c$ s.t. $(c-\epsilon, c) \subseteq D$, $(c, c+\epsilon) \cap D = \emptyset$. So $\Psi_{+-}(c)$ is true

□

Lemma 3.12. $\exists J \subseteq I, \forall x \in J, \Psi_{+-}(x)$,

Proof. $S = \{x \in I : \Psi_{+-}(x)\}$. If S is finite, replace I with $I' \subseteq I \setminus S$, replace f with $f|_{I'}$, apply previous lemma, get $c \in I'$, $\Psi_{+-}(c)$, a contradiction □

Similarly, $\exists J \subseteq I, \forall x \in J, \Psi_{-+}(x)$

Combine these, get $I \supseteq I' \supseteq I''$, $\forall x \in I'$, $\Psi_{+-}(x)$, and $\forall x \in I''$, $\Psi_{+-}(x)$, a contradiction

Lemma 3.13. *If* $f : I \to R$, $\exists a \in I$, $\Phi_{-+}(a)$ *or* $\Phi_{+-}(a)$ *or* $\Phi_{00}(a)$

Proof. By Lemma 3.2, there is $J \subseteq I$, if $f|_J$ is constant, then we are done.

If $f|_J$ is injective, let $S_{+-} = \{a \in J, \Phi_{+-}(a)\}$ and other sets similarly. $J = S_{+-} \cup S_{++} \cup S_{-+} \cup S_{--}$. If $|S_{++}| = \infty$, there is $I' \subseteq S_{++}$, a contradiction. Therefore S_{--} and S_{++} are finite. But $|J|$ is infinite, so S_{+-} or S_{-+} is nonempty □

Lemma 3.14. $f : I \to R, \exists c_0 < c_1 < \cdots < c_n, I = (c_0, c_n)$, $f|_{(c_i, c_{i+1})}$ *is constant or decreasing or increasing*

Proof. Let $E = I \setminus (S_{+-} \cup S_{-+} \cup S_{00})$. If $|E| = \infty$, then $J \subseteq E$ and $f|_E$ contradicts 3.13. Take $\{c_0, \ldots, c_n\} \supseteq E \cup bd(I) \cup bd(S_{+-}) \cup bd(S_{-+}) \cup bd(S_{00})$.

So all the sets respect the partition

$$(c_0, c_1), \{c_1\}, (c_1, c_2), \ldots, (c_{n-1}, c_n)$$

□

Lemma 3.15. *If $f : I \to R$ definable and $S = \{x \in I : f \text{ is not continuous at } x\}$, then S is finite*

Proof. S is definable. If $|S| = \infty$, take $J \subseteq S$, replace f with $f|_J$, we may assume f is nowhere continuous. By Lemma 3.14, there is $J \subseteq I$, $f|_J$ is constant or monotone. Replace f with $f|_J$, now f is monotone (constant is continuous). Assume f is increasing, then f is injective, $|f(I)| = \infty$, take $J \subseteq f(I)$, $[c,d] \subseteq f(I)$, $c = f(a)$, $d = f(b)$, $x \in (a,b) \Rightarrow f(x) \in (c,d)$. f is strictly increasing. if $y \in (c,d) \subseteq f(I)$, so $\exists x \in I$, $y = f(x)$, therefore f is surjective. Also f is order-preserving, thus f is continuous on (a,b) (since we are using order to define the topology). But f is continuous at nowhere, so a contradiction □

Then the monotonicity theorem follows from the proof of Lemma 3.14 (modify the boundary to include the discontinuous points)

Corollary 3.16. *If $f : (a,b) \to R$ definable, $\lim_{x \to a^+} f(x)$ exists in R_∞*

Proof.

1. Take ϵ, $f|_{a,a+\epsilon}$ is continuous and monotone. Then $\lim_{x \to a^+} f(x)$ is $\sup\{f(x) : x \in (a, a+\epsilon)\}$ or $\inf\{f(x) : x \in (a, a+\epsilon)\}$

□

Corollary 3.17. *If $f : [a,b] \to R$ is definable and continuous, then $\max_{x \in [a,b] f(x)}$ and $\min_{x \in [a,b]} f(x)$ exist*

Proof. Take maximum for each piece and combine □

Uniform Finiteness

Suppose $D \subseteq R^n \times R$, for $\bar{a} \in R^n$, $D_{\bar{a}} = \{y \in R : (a,y\} \in D$

Theorem 3.18 (Uniform Finitness)**.** *Suppose $\forall \bar{a}$, $|D_{\bar{a}} < \infty|$. Then $\exists N < \infty \forall \bar{a} |D_{\bar{a}}| < N$*

For now, consider $n = 1$.

Fix $D \subseteq R^2$ definable, $|D_a| < \infty$ for all $a \in R$

Definition 3.19. $(a,b) \subseteq R \times R_\infty$ is **normal** if either

- $(a,b) \notin \mathrm{cl}(D)$, $(\exists \epsilon)(a-\epsilon, a+\epsilon) \times (b-\epsilon, b+\epsilon) \cap D = \emptyset$
- $(a,b) \in D$ and $(\exists \epsilon, \delta) D \cap (a-\epsilon, a+\epsilon) \times (b-\delta, b+\delta)$ is $\Gamma(f)$ for some continuous function f

Otherwise (a,b) is abnormal

Remark. $\{(x, y) \text{ normal}\}$ is open, $\{(x, y) \text{ abnormal}\}$ is closed.

Definition 3.20. $a \in R$ is **good** if $\forall b \in R_\infty$, (a, b) is normal, is **bad** if $\exists b \in R_\infty$, (a, b) is abnormal

This is a definable definition

Lemma 3.21. *$\{x \in R : x \text{ is bad}\}$ is finite*

Proof. Otherwise, take $I \subseteq B$, $\forall x \in I$, $\{y \in R_\infty : (x, y) \text{ abnormal}\}$ is closed, nonempty.

Let $f(x) = \min\{y \in R_\infty : (x, y) \text{ abnormal}\}$, $f : I \to R_\infty$ definable.

$\forall x$, break into cases based on these questions

- $f(x) = -\infty$ vs $f(x) \in R$ vs $f(x) = +\infty$
- $(x, f(x)) \in D$ vs not
- whether $\exists y > f(x)$, $(x, y) \in D$
- whether $\exists y < f(x)$, $(x, y) \in D$

So 24 pieces

Shrink I to make all the answers constant

Assume $\forall x \in I$, $f(x) \in R$, $(x, f(x)) \in D$, $(\exists y < f(x))(x, y) \in D$, $(\exists z > f(x))(x, z) \in D$

Let $g(x) = \max\{y : y < f(x), (x, y) \in D\}$, $h(x) = \min\{y : y > f(x), (x, y) \in D\}$

Idea: apply monotonicity theorem, get f, g, h continuous, then $(x, f(x))$ is normal $\square$

3.2 The cell decomposition theorem

for each definable set X in R^m we put

$$C(X) := \{f : X \to R : f \text{ definable and continuous}\}$$
$$C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$$

where we regard $-\infty$ and $+\infty$ as constant functions on X

For $f, g \in C_\infty(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\}$$

So $(f, g)_X$ is a definable subset of R^{m+1}

Definition 3.22. Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones of length m. An $(i_1, \dots, i_m)$**-cell** is a definable subset of R^m obtained by induction on m as follows:

1. a (0)-cell is a one-element set $\{r\} \subseteq R$, a (1)-cell is an interval $(a, b) \subseteq R$
2. suppose $(i_1, \dots, i_m)$-cells are already defined, then an $(i_1, \dots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where X is an $(i_1, \dots, i_m)$-cell; further, an $(i_1, \dots, i_m, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, \dots, i_m)$-cell and $f, g \in C_\infty(X)$, $f < g$

So a $(0, 0)$-cell is a "point" $\{(r, s)\} \subseteq R^2$, a $(0, 1)$-cell is an "interval" on a vertical line $\{a\} \times R$, and a $(1, 0)$-cell is the graph of a continuous definable function defined on an interval.

Definition 3.23. A **cell in** R^m is an $(i_1, \dots, i_m)$-cell for some (necessarily unique) sequence $(i_1, \dots, i_m)$. Since the $(1, \dots, 1)$-cells are exactly the cells which are open in their ambient space R^m, we call these **open cells**

The non-open cells are "thin":
The union of finitely many non-open cells in R^m has empty interior

Proposition 3.24. *Each cell is locally closed, i.e., open in its closure*

Proof. Let $C \subseteq R^{m+1}$ be a cell. Put $B := \pi(C) \subseteq R^m$ and assume inductively that the cell B is open in its closure $\text{cl}(B)$, so that $\text{cl}(B) - B$ is a closed set. If $C = \Gamma(f)$ with $f : B \to R$ a definable continuous function, then $\text{cl}(C) - C$ is contained in $(\text{cl}(B) - B) \times R$, hence C is open in the closed set $C \cup ((\text{cl}(B) - B) \times R)$

If $C = (f, g)$ with $f, g : B \to R$ definable continuous functions on B, $f < g$, then one verifies that $\text{cl}(C) - C \subseteq \Gamma(f) \cup \Gamma(g) \cup ((\text{cl}(B) - B) \times R)$ and that C is open in the closed set $C \cup \Gamma(f) \cup \Gamma(g) \cup ((\text{cl}(B) - B) \times R)$ $\square$

we consider the point-space R^0 as a cell, or ()-cell, where () is the sequence of length 0

Each cell is homeomorphic under a coordinate projection to an open cell. We now make this explicit. Let $i = (i_1, \dots, i_m)$ be a sequence of zeros and ones

Define $p_i : R^m \to R^k$ as follows: let $\lambda(1) < \cdots < \lambda(k)$ be the indices $\lambda \in \{1, \dots, m\}$ for which $i_\lambda = 1$, so that $k = i_1 + \cdots + i_m$; then

$$p_i(x_1, \dots, x_m) := (x_{\lambda(1), \dots, x_{\lambda(k)}})$$

It is easy to show by induction on m that p_i maps each i-cell A homeomorphically onto an open cell $p_i(A)$ in R^k. We denote $p_i(A)$ also by $p(A)$ and the homeomorphism $p_i|A : A \to p(A)$ by p_A. Clearly $p_A = \mathrm{id}_A$ if A is an open cell

If A is a cell in R^{m+1} then $\pi(A)$ is a cell in R^m, where $\pi : R^{m+1} \to R^m$ is the projection on the first m coordinates. Here is a simple application of this fact

Proposition 3.25. *Each cell is definably connected*

Proof. For intervals and points this is stated in 1.15

If A is a cell in R^{m+1}, then we assume inductively that the cell $\pi(A)$ in R^m is definably connected and use the fact that each fiber $\pi^{-1}(x) \cap A$ is definably connected $\square$

Definition 3.26. A **decomposition** of R^m is a special kind of partition of R^m into finitely many cells. The definition is by induction on m

1. a decomposition of $R^1 = R$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

   where $a_1 < \cdots < a_k$ are points

2. a decomposition of R^{m+1} is a finite partition of R^{m+1} into cells A s.t. the set of projections $\pi(A)$ is a decomposition of R^m

Let $\mathcal{D} = \{A(1), \dots, A(k)\}$ be a decomposition of R^m, $A(i) \neq A(j)$ if $i \neq j$, and let for each $i \in \{1, \dots, k\}$ functions $f_{i1} < \cdots < f_{in(i)}$ in $C(A_i)$ be given

Then

$$\mathcal{D}_i := \{(-\infty, f_{i1}), (f_{i1}, f_{i2}), \dots, (f_{in(i)}, +\infty), \Gamma(f_{i1}), \dots, \Gamma(f_{in(i)})\}$$

is a partition of $A(i) \times R$ and one easily checks that $\mathcal{D}^* := \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k$ is a decomposition of R^{m+1}, and that every decomposition of R^{m+1} arises in this way from a decomposition $\mathcal{D}$ of R^m. We write $\mathcal{D} = \pi(\mathcal{D}^*)$

A decomposition $\mathcal{D}$ of R^m is said to be **partition** a set $S \subseteq R^m$ if each cell in $\mathcal{D}$ is either part of S or disjoint from S, in other words, if S is a union of cells in $\mathcal{D}$.

Theorem 3.27 (Cell Decomposition Theorem)**.**

1. (I_m)*Given any definable sets* $A_1, \dots, A_k \subseteq R^m$ *there is a decomposition of* R^m *partitioning each of* $A_1, \dots, A_k$

2. (II_m)*For each definable function $f : A \to R$, $A \subseteq R^m$, there is a decomposition $\mathcal{D}$ of R^m partitioning A s.t. the restriction $f|B : B \to R$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous*

(I_1) holds by o-minimality, and that (II_1) follows from the monotonicity theorem

We now assume that $\mathrm{I}_1, \dots, \mathrm{I}_m$ and $\mathrm{II}_1, \dots, \mathrm{II}_m$ hold

The proof is lengthy. The first step is to generalize the finiteness lemma of the previous section. Call a set $Y \subseteq R^{m+1}$ **finite over** R^m if for each $x \in R^m$ the fiber $Y_x := \{r \in R : (x, r\} \in Y$ is finite; call Y **uniformly finite over** R^m if there is $N \in \mathbb{N}$ s.t. $|Y_x| \leq N$ for all $x \in R^m$

Lemma 3.28 (Uniform Finitness Property)**.** *Suppose the definable subset Y of R^{m+1} is finite over R^m, then Y is uniformly finite over R^m*

Proof. □

Lemma 3.29. *Let X be a topological space, $(R_1, <)$, $(R_2, <)$ dense linear orderings without endpoints and $f : X \times R_1 \to R_2$ a function s.t. for each $(x, r) \in X \times R_1$*

1. *$f(x, \cdot) : R_1 \to R_2$ is continuous*

2. *$f(\cdot, r) : X \to R_2$ is continuous*

Then f is continuous

Proof. Let $(x, r) \in X \times R_1$ and $f(x, r) \in J$, where J is an interval in R_2. We shall find a neighborhood U of x and an interval I around r s.t. $f(U \times I) \subseteq J$. By (1) there are r_-, r_+ in R_1 s.t. $r_- < r < r_+$ and $f(x, r_-), f(x, r_+) \in J$. Now use (2) to get a neighborhood U of x s.t. $f(U \times \{r_-\}) \subseteq J$ and $f(U \times \{r_+\}) \subseteq J$. We claim that then $f(U \times I) \subseteq J$ for $I = (r_-, r_+)$

Let $x' \in U$ and $r_- < r' < r_+$. Assume $f(x', \cdot)$ is increasing, then $f(x', r_-) \leq f(x', r') \leq f(x', r_+)$ and $f(x', r_-)$, $f(x', r_+)$ are both in J, hence $f(x', r')$ is in J □

A **definably connected component** of a nonempty definable set $X \subseteq R^m$ is by definition a maximal definably connected subset of X

Proposition 3.30. *Let $X \subseteq R^m$ be a nonempty definable set. Then X has only finitely many definably connected components. They are open and closed in X and form a finite partition of X*

Proof. Let $\{C_1, \dots, C_k\}$ be a partition of X into k disjoint cells. For each nonempty set of indices $I \subseteq \{1, \dots, k\}$, put $C_I := \bigcup_{i \in I} C_i$. Among the $2^k - 1$ sets C_I, let C' be maximal w.r.t. being definably connected.

Claim: If a set $Y \subseteq X$ is definably connected and $C' \cap Y \neq \emptyset$, then $Y \subseteq C'$

Put $C_Y := \bigcup\{C_i : C_i \cap Y \neq \emptyset\}$. Since the C_i's cover X we have $Y \subseteq C_Y$, so C_Y is the union of Y with certain cells that intersect Y. Hence C_Y is definably connected . By maximality of C' it follows that $C' \cup C_Y = C'$. Hence $Y \subseteq C_Y \subseteq C'$, which proves the claim.

It follows in particular that C' is a definably connected component of X. Further the claim shows that the sets C' are the only definable connected components of X. Note that because the closure in X of a definably connected subset of X is also definably connected, the definably connected components of X are closed in X. Hence they are open in X $\square$

3.3 Definable families

Let $S \subseteq R^{m+n} = R^m \times R^n$ be definable. For each $a \in R^m$ we put

$$S_a := \{x \in R^n : (a, x) \in S\} \subset R^n$$

We view S as describing the family of sets $(S_a)_{a \in R^m}$. Such a family is called a **definable family** (of subsets of R^n, with parameter space R^m). The sets S_a are also called the **fibers** of the family

Example 3.1. Let $\mathcal{R} := (\mathbb{R}, <, +, \cdot)$ and consider the formula

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

This defines a relation $S \subseteq \mathbb{R}^6 \times \mathbb{R}^2$. For each point $(a, b, c, d, e, f) \in \mathbb{R}^6$ the subset $S_{(a,b,c,d,e,f)} \in \mathbb{R}^2$ consists of the points (x, y) satisfying the equation

In the following $\pi : R^{m+n} \to R^m$ denotes the projection on the first m coordinates

Proposition 3.31. 1. *Let C be a cell in R^{m+n} and $a \in \pi(C)$. Then C_a is a cell in R^n*

2. *Let $\mathcal{D}$ be a decomposition of R^{m+n} and $a \in R^m$. Then the collection*

$$\mathcal{D}_a := \{C_a : C \in \mathcal{D}, a \in \pi(C)\}$$

is a decomposition of R^n

Proof. For $n = 1$ this is immediate from the definitions

Suppose the proposition holds for a certain n, and let C be a cell in $R^{m+(n+1)}$. Let $\pi_1 : R^{m+(n+1)} \to R^{m+n}$ be the obvious projection map, so that $\pi \circ \pi_1 : R^{m+(n+1)} \to R^m$ is the projection on the first m coordinates

If $C = \Gamma(f)$, then $C_a = \Gamma(f_a)$, where $f_a : (\pi_1 C) \to R$ is defined by $f_a(x) = f(a, x)$

If $C = (f, g)_D$ with $D = \pi_1 C$, then $C_a = (f_a, g_a)_E$ where $E = D_a$

In both cases C_a is a cell in R^{n+1} $\square$

Corollary 3.32. *Let $S \subseteq R^m \times R^n$ be definable. Then there is a number $M_S \in \mathbb{N}$ s.t. for each $a \in R^m$ the set $S_a \subseteq R^n$ has a partition into at most M_S cells. In particular, each fiber S_a has at most M_S definably connected components*

Proof. Take a decomposition $\mathcal{D}$ of R^{m+n} partitioning S. Then for each $a \in R^m$ the decomposition $\mathcal{D}_a = \{C_a : C \in \mathcal{D}, a \in \pi C\}$ of R^m consists of at most $|\mathcal{D}|$ cells and partitions S_a. So we can take $M_S = |\mathcal{D}|$ $\square$

Corollary 3.33. *Let $S \subseteq R^m \times R^n$ be definable. Then there is a natural number M_S s.t. for each $a \in R^m$ the set $S_a \subseteq R^n$ has at most M_S isolated points. In particular, each finite fiber S_a has cardinality at most M_S*

4 Definable invariants: dimension and euler characteristic

4.1 Dimension

We define the **dimension** of a nonempty definable set $X \subseteq R^m$ by

$$\dim := \max\{i_1 + \cdots + i_m : X \text{ contains an } (i_1, \ldots, i_m)\text{-cell}\}$$

To the empty set we assign the dimension $-\infty$

Lemma 4.1. *If $A \subseteq R^m$ is an open cell and $f : A \to R^m$ an injective definable map, then $f(A)$ contains an open cell*

Proof. Clearly for $m = 1$. Let $m > 1$ and assume inductively the lemma holds for lower values of m. Taking a decomposition of R^m that partitions $f(A)$ we have

$$f(A) = C_1 \cup \cdots \cup C_k \text{ for cells } C_i \text{ in } R^m$$

Then

$$A = f^{-1}(C_1) \cup \cdots \cup f^{-1}(C_k)$$

so at least one of the $f^{-1}(C_i)$, say $f^{-1}(C_1)$, contains a box B , and by taking B suitably small we may assume that $f|B$ is continuous. We now claim that C_1 is open.

If not, then by composing $f|B : B \to C_1$ with a definable homeomorphism of C_1 with a cell in R^{m-1} we obtain a definable continuous injective map $g : B \to R^{m-1}$. Write $B = B' \times (a, b)$

Take c with $a < c < b$ and consider the map $h : B' \to R^{m-1}$ given by $h(x) = f(x, c)$. By the inductive assumption applied to h we get $h(B') \supseteq D$ for some box D in R^{m-1}. Let y be a point in D and take x in B' with $h(x) = y$

If $c' \neq c$ is sufficiently close to c, then $g(x, c')$ will be in D, so $g(x, c') = h(x') = g(x', c)$ for some $x' \in B'$. This contradicts the injectivity of g $\square$

Box is a cell

Proposition 4.2. *1. If $X \subseteq Y \subseteq R^m$ and X, Y are definable, then* $\dim X \leq \dim Y \leq m$

2. *If $X \subseteq R^m$ and $Y \subseteq R^n$ are definable and there is a definable bijection between X and Y, then* $\dim X = \dim Y$

3. *If $X, Y \subseteq R^m$ are definable, then* $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$

Proof. 2. Let $f : X \to Y$ be a definable bijection and $d = \dim X$, $e = \dim Y$. It is enough to show $d \leq e$.

Let A be an $(i_1, \dots, i_m)$-cell contained in X, with $d = i_1 + \dots + i_m$. Then $f \circ (p_A^{-1}) : p(A) \to Y$ is an injective map and $p(A)$ an open cell. Replacing X by $p(A)$, Y by $f(A)$ and f by $f \circ (p_A^{-1})$ we may as well assume that $d = m$ and that X is an open cell in R^d. Let $Y = C_1 \cup \dots \cup C_k$ be a partition of $Y = f(X)$ into cells. Then $X = f^{-1}(C_1) \cup \dots \cup f^{-1}(C_k)$, so by the cell decomposition theorem $f^{-1}(C_i)$ contains an open cell B since X is open, for some i. Fix such i and B

Let $C_i = C \subseteq R^n$ be a $(j_1, \dots, j_n)$-cell. We shall prove that $d \leq j_1 + \dots + j_n$.

Suppose $d > j_1 + \dots + j_n$, the composition

$$B \xrightarrow{f|B} C \xrightarrow{p_C} p(C) \subseteq R^{j_1 + \dots + j_n}$$

is an injective map. Identify $R^{j_1 + \dots + j_n}$ with a non-open cell $(R^{j_1 + \dots + j_n}) \times \{p\}$ in R^d, where $p \in R^{d-(j_1 + \dots + j_n)}$, we obtain a contradiction with lemma 3.3

3. Let $d = \dim(X \cup Y)$, and let A be an $(i_1, \dots, i_m)$-cell contained in $X \cup Y$ with $d = i_1 + \dots + i_m$. The open cell $pA \subseteq R^d$ is the union of $p_A(A \cap X)$ and $p_A(A \cap Y)$, so by the cell decomposition theorem, one of these sets, say $p_A(A \cap X)$, contains a box B in R^d. Then $p_A^{-1}(B)$ is an $(i_1, \dots, i_m)$-cell contained in X, so that

$$\dim X \geq d \geq \dim X$$

□

The next result says among other things that the dimension of a set from a definable family depends "definably" on its parameters

Proposition 4.3. *Let $S \subseteq R^m \times R^n$ be definable. For $d \in \{-\infty, 0, 1, \dots, n\}$ put*

$$S(d) := \{a \in R^m : \dim S_a = d\}$$

Then $S(d)$ is definable and the part of S above $S(d)$ has dimension given by

$$\dim \left(\bigcup_{a \in S(d)} \{a\} \times S_a \right) = \dim(S(d)) + d$$

Proof. Let $\mathcal{D}$ be a decomposition of R^{m+n} partitioning S □

5 Problems

3.2 3.2