Generic Properties of Groups

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1 Preliminaries

If $p(\bar{x})$ is a type over A, then we call the set of realizations of p in M

$$p(M^n) = \{\bar{a} \in M^n : (\forall \varphi(\bar{x}) \in p(\bar{x})) M \vDash \varphi(\bar{a})\} \vDash \bigcap_{\varphi(\bar{x}) \in p(\bar{x})} \varphi(M^n)$$

type definable over A. If V is a 0-type-definable subset of M^n, then we sometimes identify V with the set

$$[V] = \{\mathrm{tp}(\bar{a}) : \bar{a} \in V\} \subseteq S_n(\emptyset)$$

A first order structure M is κ-saturated if for any $A \subseteq M$ with $|A| < \kappa$, $n < \omega$ and $p \in S_n(A)$, p has a realization in M.

A type $p(\bar{x})$ is complete over A if for any formula $\varphi(\bar{x}) \in L$ we have that $p(\bar{x}) \vdash \varphi(\bar{x})$ or $p(\bar{x}) \vdash \neg\varphi(\bar{x})$.

A group $(G, \cdot)$ is definable in a structure M if G is a definable subset of M^n for some $n < \omega$ and the group action $\cdot : G \times G \to G$ is a definable function in M. If $p(x)$ is a type over G and $g \in G$, then

We call a first order structure $(M, \cdot, \dots)$ a group if $(M, \cdot)$ satisfies group axioms. We usually denote it by $(G, \cdot, \dots)$. A structure of the form $(G, \cdot)$ is called a **pure** group.

$$g \cdot p(x) = \{g \cdot \varphi(x) : \varphi(x) \in p(x)\} = \{\varphi(g^{-1} \cdot x) : \varphi(x) \in p(x)\}$$

A group $(G, \cdot)$ is definable in a structure M if G is a definable

An infinite totally ordered first order structure $(M, <, \dots)$ is **o-minimal** if every definable subset of M is a union of finitely many intervals and points.

Let $(M, <, \dots)$ be an o-minimal structure. We usually say "ultimately" instead of "for all sufficiently large $a \in M$". We denote an open interval with endpoints a and b by (a, b) and a closed one by $[a, b]$. In contrast, $\langle a, b \rangle$ denotes the pair of elements a and b.

If $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$, $a < b$ and $f : (a, b) \to M$ is a definable function, then there are $a = a_1 < \cdots < a_n = b$ s.t. each interval (a_i, a_{i+1}) of f is either constant or strictly monotone and continuous in the order topology. In particular, every definable function $f : M \to M$ is ultimately continuous and monotone

2 Weak generic types

2.1 Introduction

Definition 2.1. A set $X \subseteq G$ is **(left) generic** if some finitely many left G-translates of X cover G. We say that a formula $\varphi(x)$ is **(left) generic** if the set $\varphi(G)$ of elements of G realizing φ is **(left) generic**. Finally, we say that a type $p(x)$ of elements of G is **(left) generic** if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is (left) generic

In the stable case left generic = right generic

and each partial generic type extends to a complete generic type (since type is definable)

Definition 2.2. A set $A \subseteq G$ is **weak generic**, if for some non-generic $B \subseteq G$ we have that $A \cup B$ is generic. A formula $\varphi(x)$ is **weak generic** if the set $\varphi(G)$ is weak generic. A type $p(x)$ of elements of G is weak generic if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is weak generic

2.2 Basic properties of weak generic sets and types

Lemma 2.3. *Assume that G is a group and X is a definable subset of G. TFAE*

1. *the set X is weak generic*
2. *for some finitely many elements $a_1, \dots, a_n \in G$ the set $G \setminus \bigcup_{i=1}^n a_i \cdot X$ is not generic*
3. *for some definable non-generic set $Y \subseteq G$ the set $X \cup Y$ is generic*

Proof. $1 \Rightarrow 2$: Assume X is weak generic, then there is non-generic set $Y \subseteq G$ s.t. $X \cup Y$ is generic. Then there are $a_1, \dots, a_n \in G$ s.t.

$$\bigcup_{i=1}^n a_i \cdot (X \cup Y) = \bigcup_{i=1}^n a_i \cdot X \cup \bigcup_{i=1}^n a_i \cdot Y = G$$

This means that

$$G \setminus \bigcup_{i=1}^n a_i \cdot X \subseteq \bigcup_{i=1}^n a_i \cdot Y$$

$2 \Rightarrow 3$: Let $Y = G \setminus \bigcup_{i=1}^n a_i \cdot X$. Then Y is definable and not generic so putting $a_{n+1} = e$. Then $G = \bigcup_{i=1}^{n+1} a_i \cdot (X \cup Y)$ □

Lemma 2.4.

1. *If $X, Y \subseteq G$ are not weak generic, then $X \cup Y$ is not weak generic*
2. *If $p(x)$ is a (partial) weak generic type over $A \subseteq G$, then $p(x)$ may be extended to a complete weak generic type over A*

Proof.

1. Let $Z \subseteq G$ be non-generic. Y is not weak generic so $Y \cup Z$ is not generic, so $Y \cup Z \cup X$ is not generic
2. non weak generics form an ideal

   Let $q(x) = \{\varphi(x) \in L(A) : p(x) \cup \{\neg\varphi(x)\}$ is not weak generic$\}$. Then $p \subseteq q$. We shall show that q is a consistent partial type over A. If not, then

   $$G \vDash \neg\exists x \bigwedge_{k=1}^n \varphi_k(x)$$

   for some $n < \omega$ and $\varphi_1, \dots, \varphi_n \in q$. By compactness, for each $k \in \{1, \dots, n\}$ we can find a finite set of formulas $p_k \subseteq p$ s.t. the type $p_k(x) \cup \{\neg\varphi_k(x)\}$ is not weak generic. Let $\psi(x) = \bigwedge\{p_k(x) : 1 \leq k \leq n\}$ and note that for every $k \in \{1, \dots, n\}$ the set $\psi(G) \cap \neg\varphi_k(G)$ is not weak generic. By 1, neither is the union

$$\bigcup_{k=1}^{n}(\psi(G) \cap \neg\varphi_k(G)) = \psi(G) \cap \bigcup_{k=1}^{n} \neg\varphi_k(G) = \psi(G) \cap G = \psi(G)$$

contradicting the fact that $p(x) \vdash \psi(x)$. Finally we take any $r(x) \in S(A)$ with $r \supseteq q$ and the proof is complete

□

We see that (complete) weak generic types exist. By Lemma 2.4, the set

$$WGEN(A) = \{p \in S(A) : p \text{ is weak generic}\}$$

is closed and non-empty in $S(A)$

Lemma 2.5. *Assume G is a group and $A \subseteq G$*

1. *If some weak generic type $p(x) \in S(G)$ is generic, then all weak generic types $q(x) \in S(A)$ are generic*

2. *If for every $p, q \in WGEN(G)$ there is $g \in G$ s.t. $g \cdot p = q$, then all weak generic types $q(x) \in S(A)$ are generic*

3. *If there is just one weak generic type in $S(A)$, then it is generic*

Proof.
1. Suppose that some weak generic type $q(x) \in S(A)$ is not generic. Then some definable generic set $X \subseteq G$ may be divided into two non-generic definable sets X_1, X_2. Since X is generic, some left G-translates X' of X belongs to $p(x)$. Then the corresponding translates X_1', X_2' of X_1, X_2 are also non-generic and one of them belongs to $p(x)$. Hence $p(x)$ is not generic, a contradiction

2. If not, then we can find a formula $\varphi(x) \in L(A)$ which is weak generic but not generic. Note that $\{\neg g \cdot \varphi(x) : g \in G\}$ is a partial weak generic type over G: for each $m < \omega$ and $g_1, \dots, g_m \in G$, the set $\bigcup_{i=1}^m g_i \cdot \varphi(G)$ is not generic, which implies that the set $\bigcap_{i=1}^m (G \setminus g_i \cdot \varphi(G))$ is weak generic. Extend the type $\{\neg g \cdot \varphi(x) : g \in G\}$ to some $q(x) \in WGEN(G)$. Next extend $\varphi(x)$ to $p(x) \in WGEN(G)$. Then $\forall g \in G\ g \cdot p \neq q$, a contradiction

3. by 2, immediately

□

By Lemma 2.5 (1), in the stable case weak generic = generic

As an example note that if $G = (G, <, +, \dots)$ is o-minimal, then there are exactly two complete weak generic types, corresponding to $-\infty$ and $+\infty$, and they are not generic

Lemma 2.6. *Assume that $G \prec H$ and $\varphi(x) \in L(G)$*

1. *If $\varphi(G)$ is weak generic in G, then $\varphi(H)$ is weak generic in H*
2. *If G is $\aleph_0$-saturated and $\varphi(H)$ is weak generic in H, then $\varphi(G)$ is weak generic in G*

Proof.
1. There is a non-generic formula $\psi(x) \in L(G)$ s.t. $\varphi(G) \cup \psi(G)$ is generic in G, therefore $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. Thus $\varphi(H)$ is weak generic in H
2. There is a formula $\psi(x) \in L(H)$ s.t. $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. We have that $\psi(x) = \psi(x, b)$ where $b \subset H$. Let $A \subseteq G$ be a finite set containing all parameters of $\varphi(x)$. By $\aleph_0$-saturation of G, we are able to find in G a tuple $a \subset G$ s.t. $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Then $\psi(x, a) \in L(G)$ has properties needed to deduce the weak genericity of the set $\varphi(G)$ in G. Namely $\psi(G, a)$ is not generic in G and $\varphi(G) \cup \psi(G, a)$ is generic in G. If $\psi(G, a)$ is generic in G, then for some $0 < n < \omega$ we have that

$$G \models \exists x_1, \dots, x_n \forall y \exists z (\psi(z, a) \wedge \bigvee_{k=1}^{n} y = x_k \cdot z)$$

and the same holds in H since $G \prec H$, which would lead to a contradiction

□

All lemmas in this section remain true if we consider a group $(G, \cdot)$ definable in a first order structure M. Then G is a definable subset of M^n for some $n < \omega$ and for every $A \subseteq M$ we define the set $WGEN(A)$ of complete weak generic types over A as the set

$$\{p \in S_n(A) : \forall \varphi(x_1, \dots, x_n) \in p, G \cap \varphi(M^n) \text{ is weak generic in } G\}$$

2.3 Characterizations of weak genericity

Definition 2.7. Let $Y \subset X \subset M^n$, Y is **large** in X is $\dim(X - Y) < \dim(X)$.

Lemma 2.8 (Lemma 2.4 of [Pil88])**.** *Let X be a large definable subset of G. Then finitely many translates of Xcover G.*

Proposition 2.9. *Assume G is a definable group in an o-minimal structure M and X is a definable weak generic subset of G. Then* $\dim(X) = \dim(G)$

Proof. Suppose $\dim(X) < \dim(G)$. Take a generic set A and a non-generic set B s.t. $A = B \cup X$ (where A and B are definable subsets of G, apply Lemma 2.3) Choose a finite $S \subseteq G$ with $S \cdot A = G$. Then $G \setminus (S \cdot B) \subseteq S \cdot X$ and

$$\dim(G \setminus (S \cdot B)) \leq \dim(S \cdot X) = \dim(X) < \dim(G)$$

Hence the set $S \cdot B$ is large in the sense of [Pil88] and it must be generic by Lemma 2.8. But then B is also generic, a contradiction. $\square$

Assume G is a group and $X, Y \subseteq G$. We say that the set X is **translation disjoint** from the set Y if for some $a \in G$, $a \cdot X \cap Y = \emptyset$

Lemma 2.10. *Assume G is a group and X is a weak generic subset of G. Then for some finite $A \subseteq G$ there is no finite covering of X by sets that are translation disjoint from $A \cdot X$*

Proof. take $Y \supseteq X$ generic and $Y \setminus X$ not generic. We have that $G = A \cdot Y$ for some finite $A \subseteq G$. We shall prove that A meets conditions of the lemma.

Suppose for some $X_0, \ldots, X_{n-1} \subseteq G$ and $a_0, \ldots, a_{n-1} \in G$ we have that

$$X = \bigcup_{i<n} X_i \text{ and } \bigwedge_{i<n} (a_i \cdot X_i) \cap (A \cdot X) = \emptyset$$

Then for each $i < n$, $a_i \cdot X_i \subseteq G \setminus A \cdot X \subseteq A \cdot (Y \setminus X)$. So for each $i < n$, $X_i \subseteq a_i^{-1} \cdot A \cdot (Y \setminus X)$, which implies that $X \subseteq \{a_0^{-1}, \ldots, a_{n-1}^{-1}\} \cdot A \cdot (Y \setminus X)$ and finally

$$G = A \cdot Y = A \cdot (Y \setminus X) \cup A \cdot X \subseteq (A \cup (A \cdot \{a_0^{-1}, \ldots, a_{n-1}^{-1}\} \cdot A)) \cdot (Y \setminus X)$$

Then G is covered by finitely many things $\square$

Corollary 2.11. *Assume G is a group and X is a weak generic subset of G. Then the set $X \cdot X^{-1}$ is generic in G*

Proof. Take a finite $A \subseteq G$ as in Lemma 2.10. Then for each $a \in G$, $a \cdot X \cap A \cdot X \neq \emptyset$, which implies that $a \in A \cdot X \cdot X^{-1}$. So $G = A \cdot X \cdot X^{-1}$ $\square$

From now on, let $(G,<,+,\dots)$ be an o-minimal expansion of an ordered group $(G,<,+)$. Then the group G is commutative, divisible and torsion-free. By $(G^n,+)$ we mean the product of groups $(G,+)\times\dots\times(G,+)$ (n times). The ordering of G is dense since for every $a,b\in G$ with $a<b$ we have that $a<\frac{a+b}{2}<b$

Theorem 2.12 (3.3.4)**.** *Assume that $(G,<,+,\dots)$ is an o-minimal expansion of an ordered group $(G,<,+)$, $0<n<\omega$ and $\varphi(x_1,\dots,x_n)\in L(G)$. TFAE*

1. *$\varphi(x_1,\dots,x_n)$ is weak generic in $(G^n,+)$*
2. *$\neg\varphi(x_1,\dots,x_n)$ is not generic in $(G^n,+)$*
3. *the set $\varphi(G^n)$ contains arbitrarily large n-dimensional boxes*

$$(\forall R>0)(\exists a_1,\dots,a_n\in G)[a_1,a_1+R]\times\dots\times[a_n,a_n+R]\subseteq\varphi(G^n)$$

Proof. $3\Rightarrow 2$: suppose there is $k<\omega$ and $\langle g_1^1,\dots,g_n^1\rangle,\dots,\langle g_1^k,\dots,g_n^k\rangle\in G^n$ we have that

$$G^n=\bigcup_{j=1}^{k}(\langle g_1^j,\dots,g_n^j\rangle+(G^n\setminus\varphi(G^n))$$

Put $M=\max\{\left|g_i^j\right|:1\le i\le n,1\le j\le k\}$. Using 3 we are able to find $a_1,\dots,a_n\in G$ s.t.

$$[a_1-M,a_1+M]\times\dots\times[a_n-M,a_n+M]\subseteq\varphi(G^m)$$

Then

$$\langle a_1,\dots,a_n\rangle\notin\bigcup_{j=1}^{k}(\langle g_1^j,\dots,g_n^j\rangle+(G^n\setminus\varphi(G^n)))$$

a contradiction

$2\Rightarrow 1$: since the set $G^n=\varphi(G^n)\cup(G^n\setminus\varphi(G^n))$ is generic in $(G^n,+)$ and the set $G^n\setminus\varphi(G^n)$ is not generic

$1\Rightarrow 3$: W.L.O.G., $n\ge 2$. Using Lemma 2.4 (2) find $p(x_1,\dots,x_n)\in S_n(G)$ s.t. p is a weak generic type in $(G^n,+)$ and $\varphi\in p$. Extend G to a $|G|^+$-saturated group $H\succ G$. Take $\langle a_1,\dots,a_n\rangle\in H^n$ realizing p and fix a positive $R\in G$. We shall show that the follwing condition holds

$$(\forall a\in H)(a_n\le a\le a_n+R\Rightarrow \text{tp}(a/Ga_{<n})=\text{tp}(a_n/Ga_{<n}))\qquad(\star)$$

For the sake of contradiction assume that for some $a\in[a_n,a_n+R]_H$ the types $\text{tp}(a/Ga_{<n})$ and $\text{tp}(a_n/Ga_{<n})$ are distinct. By the o-minimality

of H, we can find $b \in [a_n, a_n + R]_H$ with $b \in \mathrm{dcl}(Ga_{<n})$ (dense). Let $\psi(x_1, \dots, x_{n-1}, y) \in L(G)$ be s.t. $H \vDash \psi(a_{<n}, b) \wedge \exists! y \psi(a_{<n}, y)$. As $b - R \leq a_n \leq b$, we have that $\chi \in p$ where

$$\chi(x_1, \dots, x_n) = \exists! y \psi(x_{<n}, y) \wedge \forall y (\psi(y_{<n}, y) \to (y - R \leq x_n \leq y))$$

Since $\chi \in p$, the set $\chi(G^n)$ is weak generic in $(G^n, +)$

We define $f : G^{n-1} \to G$ as:

$$f(c_{<n}) = \begin{cases} c_n - R & G \vDash \chi(\bar{c}) \\ 0 & \text{otherwise} \end{cases}$$

Take $\langle c_1, \dots, c_{n-1} \rangle \in G^{n-1}$. If there is $c_n \in G$ s.t. $G \vDash \chi(c_1, \dots, c_n)$, then there exists just one $d \in G$ with $G \vDash \psi(c_1, \dots, c_{n-1}, d)$ and we put $f(c_1, \dots, c_{n-1}) = d - R$. Otherwise we put $f(c_1, \dots, c_{n-1}) = 0$. Then the function f is definable over G and we consider the following formula over G:

$$\delta(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1}) \leq x_n \leq f(x_1, \dots, x_{n-1}) + R$$

Since $\chi(G^n) \subseteq \delta(G^n) \subseteq G^n$, the set $\delta(G^n)$ is weak generic in $(G^n, +)$. Let $A \subseteq G^n$ be a finite set chosen for $\delta(G^n)$ as in Lemma 2.10. Consider an arbitrary $\langle h_1, \dots, h_{n-1} \rangle \in H^{n-1}$. Choose $M_{h_{<n}} \in G$ s.t.

$$\{\langle h_1, \dots, h_n \rangle : f(h_{<n}) + M_{h_{<n}} \leq h_n \leq f(h_{<n}) + M_{h_{<n}} + R\} \cap (A + \delta(H^n)) = \emptyset$$

(exists since is bounded and A is finite) If $\mathrm{tp}(h_{<n}/G) = \mathrm{tp}(h'_{<n}/G)$, then $M_{h_{<n}}$ is good also for $h'_{<n}$. By compactness, for each $q(x_1, \dots, x_{n-1}) \in S_{n-1}(G)$ we can find a formula $\varphi_q(x_1, \dots, x_{n-1}) \in L(G)$ and $M_q \in G$ s.t. for every $h_{<n} \in H^{n-1}$ with $H \vDash \varphi_q(h_{<n})$ we have

$$\{\langle h_1, \dots, h_n \rangle : f(h_{<n}) + M_q \leq h_n \leq f(h_{<n}) + M_q + R\} \cap (A + \delta(H^n)) = \emptyset$$

Again by compactness, $S_{n-1}(G) = [\varphi_{q_1}] \cup \dots \cup [\varphi_{q_k}]$ for some $k < \omega$ and $q_1, \dots, q_k \in S_{n-1}(G)$. If not, then $\forall n \in \omega$, $G \vDash \bigwedge_{i=1}^{n} \neg \varphi_q i$, that is, $\{\neg \varphi_{q_i} : i \in \omega\}$ is consistent with G, then realized by H, which leads to a contradiction. For $i \in \{1, \dots, k\}$ put $X_i = (\varphi_{q_i}(G^{n-1}) \times G) \cap \delta(G^n)$ and $e_i = \langle 0, \dots, 0, M_{q_i} \rangle \in G^n$. Then $\delta(G^n) = X_1 \cup \dots \cup X_k$ and for every $i \in \{1, \dots, k\}$ we have that $(e_i + X_i) \cap (A + \delta(G^n)) = \emptyset$. This contradicts the choice of A and finishes the proof of $(\star)$

By $(\star)$, we have that

$$H \vDash \forall y ((a_n \leq y \wedge y \leq a_n + R) \to \varphi(a_1, \dots, a_{n-1}, y))$$

Therefore the formula $\forall y((x_n \leq y \leq x_n + R \to \varphi(x_1, \dots, x_{n-1}, y)))$ belongs to p. In general, for each formula $\psi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$, $k \in \{1, \dots, n\}$ and positive $R \in G$ the formula

$$\forall y((x_k \leq y \leq x_k + R) \to \psi(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n))$$

belongs to p. We inductively create formulas $\varphi_k(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$, $k = \{1, \dots, n\}$,. Namely, provided that $\varphi_1(x_1, \dots, x_n), \dots, \varphi_{k-1}(x_1, \dots, x_n)$ have already been defined, let $\varphi_k(x_1, \dots, x_n)$ be the formula

$$\forall y((x_k \leq y \leq x_k + R) \to (\varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_{k-1}(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n)))$$

Finally, we take any $\bar{g} \in (\varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_n)(G^n)$ and see that

$$[g_1, g_1 + R] \times \dots \times [g_n, g_n + R] \subseteq \varphi(G^n)$$

□

Corollary 2.13 (3.3.5)**.** *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$, $0 < n, k < \omega$ and $\varphi(x_1, \dots, x_n, y_1, \dots, y_k) \in L$*

1. *there is $\psi_1(y_1, \dots, y_k)$ s.t. for every $\langle a_1, \dots, a_k \rangle \in G^k$ we have that $G \models \psi_1(a)$ iff $\varphi(G^n, a)$ is weak generic in $(G^n, +)$*
2. *There is $\psi_2(y_1, \dots, y_k)$ s.t. for every $\langle a_1, \dots, a_k \rangle \in G^k$ we have that $G \models \psi_2(a)$ iff $\varphi(G^n, a)$ is generic in $(G^n, +)$*
3. *there is a natural number N s.t. for every φ-definable $X \subseteq G^n$ the set X is generic in $(G^n, +)$ iff G^n may be covered by at most N left translates of X*

Proof. 1. let $\psi_1(y_1, \dots, y_k)$ be

$$\forall r \exists z_1, \dots, z_n \forall x_1, \dots, x_n((\bigwedge_{i=1}^{n} z_i \leq x_i \wedge x_i \leq z_i + r) \to \varphi(x_1, \dots, x_n, y_1, \dots, y_k))$$

3. Assume that $n = 1$. Let $\psi_2(y_1, \dots, y_k)$ be such as 2. Suppose for every $N < \omega$ we can find $\langle a_1, \dots, a_k \rangle \in G^k$ s.t. the set $\varphi(G, a_1, \dots, a_k)$ is generic in G but not N-generic. Then the set of formulas

$$\bigcup_{N < \omega} \{\psi_2(y_1, \dots, y_k) \wedge \forall z_1, \dots, z_N \exists t \forall x(\varphi(x, y_1, \dots, y_k) \to \bigwedge_{i=1}^{N} t \neq z_i \cdot x)\}$$

is a type in variables $y_1, \dots, y_k$ and has a realization $\langle b_1, \dots, b_k \rangle \in H^k$ in some $\aleph_0$-saturated elementary extension H of G. Then we reach a contradiction as the set $\varphi(H, b_1, \dots, b_k)$ is simultaneously generic and not generic in H

□

Corollary 2.14. *Assume that* $(G, <, +, \dots)$ *is an o-minimal expansion of an ordered group* $(G, <, +)$, $0 < n < \omega$, *and* $p(x_1, \dots, x_n) \in S_n(G)$. *TFAE*

1. $p(x_1, \dots, x_n)$ *is weak generic in* $(G^n, +)$
2. $\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) = p(x_1, \dots, x_n)$ *for every* $\langle g_1, \dots, g_n \rangle \in G^n$

Proof. $1 \Rightarrow 2$: suppose

$$\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) \neq p(x_1, \dots, x_n)$$

for some $\langle g_1, \dots, g_n \rangle \in G^n$. Then for some $\varphi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$ we have that $(\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$. $\varphi(G^n)$ is weak generic in $(G^n, +)$ and hence contains arbitrarily large boxes. Take any $R > \max(|g_1|, \dots, |g_n|)$ and choose $a_1, \dots, a_n \in G$ s.t.

$$B = [a_1, a_1 + R] \times \cdots \times [a_n, a_n + R] \subseteq \varphi(G^n)$$

we obtain

$$\emptyset \neq (\langle g_1, \dots, g_n \rangle + B) \cap B \subseteq (\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$$

a contradiction

$2 \Rightarrow 1$: we shall prove a more general fact. Namely if G is a group and $p(x) \in S(G)$ is s.t. for every $g \in G$ we have that $g \cdot p = p$, then p is weak generic in G

If not, then we can find a formula $\varphi(x) \in p(x)$ which is not weak generic in G. Then $\neg\varphi(x)$ is generic in G so there are $m < \omega$ and $g_1, \dots, g_m \in G$ s.t $G = \bigcup_{i=1}^m g_i(G \setminus \varphi(G))$. Thus $\bigcap_{i=1}^m g_i \cdot \varphi(G) = \emptyset$, which contradicts the fact that the formulas $g_1 \cdot \varphi, \dots, g_m \cdot \varphi$ belong to the consistent type $p(x)$ $\square$

2.4 Stationary

In this section we assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$

Recall that in stable group all weak generic types are generic. Moreover, all of them are stationary over any model M. This means that every (weak) generic type $p \in S(M)$ has a unique extension to a (weak) generic type $q \in S(A)$ for each $A \supseteq M$

Definition 2.15. We call a weak generic type p over a set A **stationary** if for every $B \supseteq A$ the type p has just one extension to a complete weak generic type over B

In general weak generic types do not need to be stationary

Example 2.1. we shall prove that the types $p_1(x) = \{x < a : a \in G\}$ and $p_2(x) = \{x > a : a \in G\}$ are the only two weak generic types in $(G, +)$ complete over G and that both of them are stationary

By the o-minimality of $(G, <, +, \dots)$, every definable subset of G is a union of finitely many points and intervals. For every $a, b \in G$ the interval (a, b) is not weak generic in $(G, +)$ by Lemma 2.3 (2). Thus no type in $S_1(G)$ but p_1 and p_2 is weak generic in $(G, +)$

On the other hand, all intervals of the form $(-\infty, a)$ or $(b, +\infty)$ are weak generic in $(G, +)$ since their complements in G are not generic in $(G, +)$. This gives us the weak genericity of the types p_1 and p_2

If H is any elementary extension of G, then there are also two complete (over H) weak generic types in $(H, +)$. This means that p_1 and p_2 are stationary

Definition 2.16. We call an o-minimal structure $(M, <, \dots)$ **stationary** if for every elementary extension N of M and N-definable function $g : N \to N$ there exists an M-definable function $f : N \to N$ s.t. $g(x) \leq f(x)$ for all sufficiently large $x \in N$

Theorem 2.17. *Assume $(M, <, \dots)$ is a stationary o-minimal structure and $N \succ M$. For every N-definable map $g : N \to N$ with $\lim_{x\to+\infty} g(x) = +\infty$ we can find an M-definable map $f : N \to N$ s.t. $\lim_{x\to+\infty} f(x) = +\infty$ and $f(x) \leq g(x)$ for all sufficiently large $x \in N$*

Proof. First of all, assume that g is a bijection. Then g^{-1} exists and by the stationary of $(M, <, \dots)$ we can find an M-definable function $f : N \to N$ s.t. ultimately $g^{-1} \leq f$. We have that $\lim_{x\to+\infty} g^{-1}(x) = +\infty$, which implies that $\lim_{x\to+\infty} f(x) = +\infty$. Since f is M-definable, we can choose $a \in M$ s.t. f is strictly increasing on $(a, +\infty)$ (monotonicity theorem). We define a function $f_1 : N \to N$ as follows

$$f_1(x) = \begin{cases} f(x) & x > a \\ f(a) + x - a & x \leq a \end{cases}$$

Then f_1 is an M-definable bijection, hence f_1^{-1} exists and also is M-definable. Moreover, $\lim_{x\to+\infty} f_1^{-1}(x) = +\infty$ and ultimately $f_1^{-1} \leq g$ so f_1^{-1} has the desired properties

If g is not a bijection, then proceeding as above we can find an N-definable bijection $g_1 : N \to N$ s.t. ultimately $g_1 = g$ □

By the o-minimality of $(G, <, +, \dots)$, every definable subset of the set $G \times G$ is a union of finitely many cells of dimension 0,1,2. By Proposition 2.9, we are interested only in cells of dimension 2 (since we are interested in weak generic subsets). They are of the form

$$C_{a,b}^{f,g} = \{\langle x, y\rangle \in G \times G : a < x < b \wedge f(x) < y < g(x)\}$$

where $\{-\infty\} \cup G \ni a < b \in G \cup \{\infty\}$ and $f, g : (a, b) \to G \cup \{-\infty, \infty\}$ are definable maps s.t. $f(x) < g(x)$ for each $x \in (a, b)$. If $a, b \in G$, then the cell $C_{a,b}^{f,g}$ is not weak generic in $(G, +) \times (G, +)$ by Theorem 2.12. Since we shall consider only weak generic types $p(x, y)$ in $(G, +) \times (G, +)$ s.t. $\{x > a : a \in G\} \subseteq p(x, y)$, we shall be interested only in weak generic cells of the form $C_{a,b}^{f,g}$ where $a \in G$ and $b = +\infty$

Definition 2.18. Assume that functions $f, g : G \to G$ are definable

1. $f \ll g$ if $f(x) < g(x)$ for all sufficiently large $x \in G$ and the set

   $$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

   is weak generic in $(G, +) \times (G, +)$ $(C_{0,+\infty}^{f,g})$

2. $f \sim g$ if

   $$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

   is not weak generic in $(G, +) \times (G, +)$

$\sim$ is an equivalence relation on the set of all definable functions from G to G and that equivalence classes of $\sim$ are convex (i.e., if $f, g, h : G \to G$ are definable, $f \sim h$ and ultimately $f(x) \leq g(x) \leq h(x)$, then $f \sim g$ and $g \sim h$)

Definition 2.19. Let $f : G \to G$ be a definable function

1. Let $p_f^+(x, y)$ denote the only extension of the type

   $$\{x > a : a \in G\} \cup \{y > f(x)\} \cup \{y < g(x) : g \gg f\}$$

   to a type which is complete over G and weak generic in $(G, +) \times (G, +)$

2. Let $p_f^-(x, y)$ denote the only extension of the type

   $$\{x > a : a \in G\} \cup \{y < f(x)\} \cup \{y > g(x) : g \ll f\}$$

   to a type which is complete over G and weak generic in $(G, +) \times (G, +)$

3. Let $p_{+\infty}(x, y)$ denote the weak generic type

$$\{x > a : a \in G\} \cup \{y > g(x) : g : G \to G \text{ definable}\}$$

4. Let $p_{-\infty}(x, y)$ denote the weak generic type

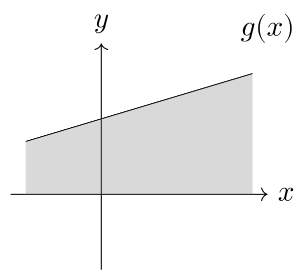
$$\{x > a : a \in G\} \cup \{y < g(x) : g : G \to G \text{ definable}\}$$

Theorem 2.20. *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$. TFAE*

1. *$p_f^+(x, y)$ and $p_f^-(x, y)$ are stationary for each definable function $f : G \to G$*
2. *$p_{+\infty}(x, y)$ and $p_{-\infty}(x, y)$ are stationary*
3. *$(G, <, +, \dots)$ are stationary*

Proof. $1 \Rightarrow 2$: Let $f : G \to G$ be a map constantly equal to 0. Then $p_{+\infty}(x, y) = p_f^+(y, x)$ and therefore $p_{+\infty}$ is stationary

Below is an illustration of $p_f^+(x, y) = p_{+\infty}(y, x)$



where $g(x) \gg f(x)$.

$2 \Rightarrow 3$: Suppose the structure $(G, <, +, \dots)$ is not stationary. Then there exist an $H \succ G$ and a H-definable function $g : H \to H$ s.t. no G-definable map $f : H \to H$ dominates g

Consider the following partial types over H:

$$p_1(x, y) = p_{+\infty}(x, y) \cup \{y < g(x)\}$$
$$p_2(x, y) = p_{+\infty}(x, y) \cup \{y > g(x)\}$$

To reach a contradiction, it is enough to prove that both of them are weak generic in $(H, +) \times (H, +)$, and therefore $p_+(x, y)$ is not stationary. We begin with p_1.

Goal:

$$(\bigwedge_{i=1}^{m} x > a_i) \wedge (\bigwedge_{i=1}^{n} y > f_i(x)) \wedge y < g(x)$$

is weak generic in $(H,+) \times (H,+)$ where $a_1, \dots, a_m \in G$ and $f_1, \dots, f_n$ are functions from H to H definable over G.

Take $a = \max(a_1, \dots, a_n)$ and $f = \max(f_1, \dots, f_n)$ we can confine our attention to the sets X of the form

$$X = \{\langle x, y\rangle \in H \times H : x > a \wedge y > f(x) \wedge y < g(x)\}$$

where $a \in G$ and $f : H \to H$ is definable over G. W.L.O.G., we can assume that f is ultimately non-decreasing

Consider a map $h : H \to H$ defined as follows: $h(a) = f(2a) + a$ for each $a \in H$. Since h is G-definable, g dominates h . Non-stationarity means $\forall x \in N \exists x < y \in N$ s.t. $g(y) > h(y)$ and we can assume g is ultimately increasing. Therefore we can define g' to be

$$g'(x) = \min\{g(y) : x < y \wedge g(y) > h(y)\}$$

Since h is non-decreasing, g' dominates h. Note that for each large enough $M \in H$ the area between the graphs of f and g in $H \times H$ contains the square whose vertices are

$$\langle M, f(2M)\rangle, \langle M, f(2M) + M\rangle, \langle 2M, f(2M)\rangle, \langle 2M, f(2M) + M\rangle$$

By Theorem 2.12, X is weak generic in $(H,+) \times (H,+)$. As a result, the type p_1 is weak generic in $(H,+) \times (H,+)$

$3 \Rightarrow 1$: Take any definable $f : G \to G$. We shall show that both p_f^+ and p_f^- are stationary weak generic types

By the o-minimality of G, f is ultimately non-negative or ultimately non-positive. It is easy to see that p_f^+ is stationary iff p_{-f}^- is stationary and p_f^- is stationary iff p_{-f}^+ is stationary. Therefore, W.L.O.G, we can assume that f is ultimately non-negative. Moreover, f is ultimately non-increasing or ultimately non-decreasing. If f is ultimately non-increasing, then $p_f^+ = p_z^+$ and $p_f^- = p_z^-$ where $z : G \to G$ is constantly equal to 0. So we can assume that f is ultimately non-decreasing (this includes the constant case)

Consider definable sets:

$$A = \{a \in G : (\exists b > a)(\forall c \in (a,b)) f(c) - f(a) \leq c - a\}$$
$$B = \{a \in G : (\exists b > a)(\forall c \in (a,b)) f(c) - f(a) > c - a\}$$

Note that by the o-minimality of G, we have that $G = A \cup B$ and for some $M \in G$ either $(M, +\infty) \subseteq A$ or $(M, +\infty) \subseteq B$. Enlarge M in order to ensure that f is continuous on $(M, +\infty)$

Case 1: $(M, +\infty) \subseteq A$. Then f grows "slowly" on $(M, +\infty)$:

$$(\forall a > M)(\exists b > 0)(\forall c \in (0, b)) f(a + c) \leq f(a) + c \qquad (\star)$$

By $(\star)$ and the continuity of f

$$(\forall a > M)(\forall c > 0) f(a + c) \leq f(a) + c \qquad (\star\star)$$

Because if not, then the opposite holds: $(\exists a > M)(\exists c > 0) f(a+c) > f(a)+c$. Let $C = \{c > 0 : f(a + c) > f(a) + c\}$ and $c_0 = \inf(C)$. Assertion $(\star)$ implies that $c_0 > 0$. Since f is continuous at c_0, $c_0 \notin C$. Choose $d > c_0$ s.t. $(c_0, d) \subseteq C$. Since $c_0 \notin C$, $f(a + c_0) \leq f(a) + c_0$. On the other hand, by the continuity of f at $a + c_0$, we have that $f(a + c_0) \geq f(a) + c_0$. Thus $f(a + c_0) = f(a) + c_0$ and for every $e \in (0, d - c_0)$ we have that

$$f(a + c_0 + e) > f(a) + c_0 + e = f(a + c_0) + e$$

which implies that $a + c_0 \notin A$. But $a + c_0 \in (M, +\infty) \subseteq A$, a contradiction. So ($\star\star$ holds)

For the sake of contradiction assume that p_f^+ is not stationary. Then for some $H \succ G$ and H-definable $g : H \to H$ we have that $f \ll g$ and $g \ll h$ for each G-definable $h : H \to H$ with $f \ll h$. Use the same technique above. If p_f^+ is not stationary, then there is some H-definable function g s.t. both

$$p_f^+(x, y) \cup \{y > g(x)\}$$
$$p_f^+(x, y) \cup \{y \leq g(x)\}$$

are weak generic, which implies that $f \ll g \ll h$ for each G-definable $h : H \to H$ with $f \ll h$. Since $\lim_{x \to +\infty}(g(x) - f(x)) = +\infty$, there exists an increasing to $+\infty$ G-definable function $h : H \to H$ s.t. ultimately $h \leq g - f$ by 2.17. Enlarging M we can assume that h is increasing on $(M, +\infty)$.

Now fix any positive $R \in H$ and find $a > M$ with $h(a) \geq 2R$. By $(\star\star)$, we have that $f(a + R) \leq f(a) + R$. So the area between the graphs of f and $f + h$ contains the square whose vertices are

$$\langle a, f(a) + R\rangle, \langle a, f(a) + 2R\rangle, \langle a + R, f(a) + R\rangle, \langle a + R, f(a) + 2R\rangle$$

As R was arbitrary, we can use Theorem 2.12 to conclude that the area between the graphs of f and $f + h$ is weak generic in $(H, +) \times (H, +)$. So

$f \ll f+h$ and therefore $g \ll f+h$, which contradicts the fact that ultimately $g \geq f+h$. So the type p_f^+ is stationary

Case 2: $(M,+\infty) \subseteq B$. Then f grows "quickly" on $(M,+\infty)$, which implies that $\lim_{x\to+\infty} f(x) = +\infty$. As in 2.17 find a definable bijection $f_1 : G \to G$ s.t. $f_1(a) = f(a)$ for each $a \in (M,+\infty)$. If $g = f_1^{-1}$, then g grows "slowly" on $(M,+\infty)$ and from the previous case we know that the types p_g^+ and p_g^- are stationary. The proof is complete since $p_f^+(x,y) = p_{f_1}^+(x,y) = p_g^-(y,x)$ and $p_f^-(x,y) = p_{f_1}^-(x,y) = p_g^+(x,y)$ $\square$

Corollary 2.21 (1.7.6 of [vdD98]). *Let $\mathcal{S}_m$ be the boolean algebra of semilinear subsets of R^m. Then $\mathcal{S} := (\mathcal{S}_m)_{m\in\mathbb{N}}$ is an o-minimal structure on the ordered set $(R,<)$. Each function in $\mathcal{S}$ is piecewise affine, more precisely, given a function $f : A \to R$ in $\mathcal{S}$ with $A \subseteq R^m$, there is a partition of A into basic semilinear sets A_i $(1 \leq i \leq k)$ s.t. $f|A_i$ is the restriction to A_i of an affine function on R^m, for each $i \in \{1,\dots,k\}$*

Example 2.2. If $(G,<,+)$ is an o-minimal ordered group, then every definable function $f : G \to G$ is ultimately equal to $f_q(x) + a$ for some $a \in G$ and $q \in \mathbb{Q}$ where $f_q(x) = q \cdot x$ (2.21) by considering G as a $\mathbb{Q}$-vector space. Below we list all weak generic types in $(G,+) \times (G,+)$ that are complete over G and contain the formula $x > 0$

1. $p_{-\infty}(x,y)$ and $p_{+\infty}(x,y)$
2. $p_{f_q}^-(x,y)$ and $p_{f_q}^+(x,y)$, $q \in \mathbb{Q}$
3. $\{x > a : a \in G\} \cup \{y > q\cdot x : q \in \mathbb{Q} \wedge q < r\} \cup \{y < q \cdot x : q \in \mathbb{Q} \wedge q > r\}$, $r \in \mathbb{R} \setminus \mathbb{Q}$

The structure $(G,<,+)$ is stationary since its elementary extensions are all linearly bounded. Thus by Theorem 2.20, weak generic types of the form (1) and (2) are stationary.

2.5 Expansions of real closed fields

In this section, $(R,<,+,\cdot,0,1,\dots)$ is an o-minimal expansion of an ordered ring $(R,<,+,\cdot,-,0,1)$. Such a ring must be a real closed field. Since $(R,<,+,\cdot,0,1,\dots)$ is an o-minimal expansion of the ordered group $(R,<,+)$, all results obtained in the previous section apply

Definition 2.22. We call a structure $(R,<,+,\cdot,\dots)$ **polynomially bounded** if for every definable function $f : R \to R$ there is $n \in \mathbb{N}^+$ s.t. $|f(x)| \leq x^n$ for all sufficiently large $x \in R$

Remark. If a real closed field $(R, <, +, \cdot, \dots)$ is polynomially bounded and o-minimal, then for every definable $f : R \to R$ with $\lim_{x \to +\infty} f(x) = +\infty$ we can find $n \in \mathbb{N}_+$ s.t. $f(x) \geq \sqrt[n]{x}$ for all sufficiently large $x \in R$

Proof. We proceed as in the proof of 2.17. Since f is ultimately increasing, we are able to find a definable bijection $g : R \to R$ s.t. $f(x) = g(x)$ for all sufficiently large $x \in R$. We know that the inverse map g^{-1} is ultimately dominated by the polynomial function $x \mapsto x^n$ for some $n \in \mathbb{N}_+$. And this implies $f(x) = g(x) \geq \sqrt[n]{x}$ for sufficiently large x □

Proposition 2.23 (2.6.1 of [BCR13])**.** *Given a real closed field R, let f be a semi-algebraic function (not necessarily continuous) from $(a, +\infty) \subset R$ to R. There exists $r, c \in R$, $r > a$, and $p \in \mathbb{N}$, s.t., for every $x \geq r$, we have $|f(x)| \leq cx^p$. Moreover, if $h \in R[X, Y]$ is a nonzero polynomial, s.t. $h(x, f(x)) = 0$ on $(a, +\infty)$, we can take p to be the degree of h w.r.t. X*

Assume $(R, <, +, \cdot)$ is a pure real closed field. Since every definable map $f : R \to R$ is semi-algebraic, it follows from 2.23 that the structure $(R, <, +, \cdot)$ is polynomially bounded.

Corollary 2.24. *Every pure real closed field $(R, <, +, \cdot)$ is stationary and so are the weak generic types $p_f^-(x, y)$ and $p_f^+(x, y)$ for each definable $f : R \to R$*

Proof. Consider an arbitrary elementary extension S of R and any definable map $f : S \to S$. Since the real closed field $(S, <, +, \cdot)$ is polynomially bounded, there exists $n \in \mathbb{N}_+$ s.t. ultimately $|f(x)| \leq x^n$. This gives us the stationary of the structure $(R, <, +, \cdot)$ □

On the other hand, the structure $(\mathbb{R}, <, +, \cdot, e^x)$ is not polynomially bounded but it is still an o-minimal expansion of the ordered field of real numbers ([Wil96])

Definition 2.25. Assume $(R, +, \cdot, 0, 1)$ is a field, $f, g : R \to R$ and $g(x) \neq 0$ for all sufficiently large $x \in R$. We write $f \approx g$ iff

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$$

Lemma 2.26 (2.5.2 of [BCR13])**.** *Let $A \subset R$ be a semi-algebraic set and $\varphi : A \to R$ a semialgebraic function. There exists a nonzero polynomial $f \in R[X, Y]$ s.t. for every $x \in A$, $f(x, \varphi(x)) = 0$.*

Lemma 2.27 (3.5.5)**.** *Assume $(R, <, +, \cdot)$ is a pure real closed field. If a function $f : R \to R$ is definable and ultimately non-zero, then for some $q \in \mathbb{Q}$ and $c \in R \setminus \{0\}$ we have that $f(x) \approx c \cdot x^q$*

Proof. Let S be an arbitrary $|R|^+$-saturated elementary extension of R. We can find $a \in S$ s.t. $a > r$ for every $r \in R$. Let

$$T = \{s \in S : |s| < r \text{ for some } r \in R\}$$

Then T is a convex subring of S,

$$T^* = \{s \in S : \frac{1}{r} < |s| < r \text{ for some } r \in R\}$$

and $(T^*, \cdot)$ is a subgroup of the multiplicative group $(S^*, \cdot)$.

The quotient group $(S^*/T^*, *, \mathbf{1})$ may be ordered in the following way:

$$s_1/T^* \leq s_2/T^* \Leftrightarrow \frac{s_1}{s_2} \in T$$

We define a function $\nu : S \to S^*/T^* \cup \{-\infty\}$ (where for every $s \in S^*$, $-\infty < s/T^*$ and $(-\infty) * s/T^* = -\infty$) as follows:

$$\nu(s) = \begin{cases} -\infty & s = 0 \\ s/T^* & \text{otherwise} \end{cases}$$

Then ν is a valuation of the field S, i.e., $\forall x, y \in S$,

1. $\nu(x \cdot y) = \nu(x) * \nu(y)$
2. $\nu(x + y) \leq \max(\nu(x), \nu(y))$
   Suppose $x, y \neq 0$ and $\nu(x) \leq \nu(y)$, then $\frac{x}{y} \in T$, $1 + \frac{x}{y} \in T$ and therefore $\nu(x + y) \leq \nu(y)$. Since $\frac{\frac{x}{y}}{\frac{x}{y}+1} \in T$, $\frac{x}{x+y} \in T$ and so $\nu(x + y) \geq \nu(y)$.
3. $\nu(x) \neq \nu(y) \Rightarrow \nu(x + y) = \max(\nu(x), \nu(y))$

Since f is semi-algebraic, by Lemma 2.26, there exists a non-zero polynomial $P(X, Y) \in R[X, Y]$ s.t. $R \models \forall x(P(x, f(x)) = 0)$. So $S \models \forall x(P(x, f(x)) = 0)$ and, in particular, $P(a, f(a)) = 0$. The polynomial $P(X, Y)$ is of the form

$$P(X, Y) = \sum_{i=1}^{n} r_i \cdot X^{k_i} \cdot Y^{l_i}$$

for some $n \in \mathbb{N}_+$, $r_i \in R \setminus \{0\}$ and $k_i, l_i < \omega$ s.t. $\langle k_i, l_i \rangle \neq \langle k_j, l_j \rangle$ for every $i \neq j \in \{1, \dots, n\}$. Thus

$$0 = \sum_{i=1}^{n} r_i \cdot a^{k_i} \cdot f(a)^{l_i}$$

and for some $i \neq j \in \{1, \dots, n\}$ we have that

$$\nu(r_i \cdot a^{k_i} \cdot f(a)^{l_i}) = \nu(r_j \cdot a^{k_j} \cdot f(a)^{l_j}) \neq -\infty$$

since $f(a) \neq 0$ (if $f(a) = 0$, then $f : R \to R$ would be ultimately equal to 0)

This implies that $\nu(\frac{r_i}{r_j} \cdot a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = \mathbf{1}$ and $\nu(a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = \mathbf{1}$. So $a^{k_i - k_j} \cdot f(a)^{l_i - l_j} \in T^*$. If $l_i = l_j$, then $k_i \neq k_j$ and $a^{k_i - k_j} \in T^*$, which implies that $a \in T^*$, a contradiction.

So $l_i \neq l_j$. Let $q = -\frac{k_i - k_j}{l_i - l_j} \in \mathbb{Q}$ we obtain $\frac{f(a)}{a^q} \in T^*$. Therefore $\frac{1}{r} < \left|\frac{f(a)}{a^q}\right| < r$ for some $r \in R$. If $b \in S$ and $b > a$, then $\mathrm{tp}(a/R) = \mathrm{tp}(b/R)$. Hence for every $b > a$ we have that $\frac{1}{r} < \left|\frac{f(b)}{b^q}\right| < r$ and consequently

$$S \vDash \exists y \forall x (x > y \to \frac{1}{r} < \left|\frac{f(x)}{x^q}\right| < r)$$

As $R \prec S$, this implies that $\frac{1}{r} < \left|\frac{f(x)}{x^q}\right| < r$ for all sufficiently large $x \in R$. By the o-minimality of R, for some $c \in R$ with $\frac{1}{r} \leq |c| \leq r$ we have that $\lim_{x \to +\infty} \frac{f(x)}{x^q} = c$, which finishes the proof □

Theorem 2.28 (3.5.6)**.** *Assume $(R, <, +, \cdot)$ is a pure real closed field. Let*

$$f(x) = \sum_{i=1}^{m} a_i \cdot x^{p_i} \quad \textit{and} \quad g(x) = \sum_{j=1}^{n} b_j \cdot x^{q_j}$$

where $m, n \in \mathbb{N}_+$, $a_1, \dots, a_m, b_1, \dots, b_n \in R$, $a_1, b_1 > 0$, $p_1 > \dots > p_m \in \mathbb{Q}$ and $q_1 > \dots > q_n \in \mathbb{Q}$. TFAE

1. $f \ll f + g$
2. $q_1 > \max(0, p_1 - 1)$

Proof. Goal: Ultimately, for each $M \in R$, $x^{p_1} + x^{q_1} \geq (x + M)^{p_1} + M$. Then ultimately, $x^{q_1} \geq k(M) \cdot x^{p_1 - 1} + M$ for some constant k relevant to M.

We define a rate of growth $gr(f)$ of a definable map $f : R \to R$ as follows: if $f(x) \approx c \cdot x^q$ for some $c \in R \setminus \{0\}$ and $q \in \mathbb{Q}$, then $gr(f) = q$

(Lemma 2.27 implies that $gr(f)$ is well defined for each ultimately non-zero definable function $f : R \to R$) and $gr(f) = 0$ otherwise.

$$gr(f) = \begin{cases} q & \exists c \in R \setminus \{0\} \exists q \in \mathbb{Q} \text{ s.t. } f(x) \approx c \cdot x^q \\ 0 & \text{otherwise} \end{cases}$$

Then $gr(f \cdot g) = gr(f) + gr(g)$ and $gr(f + g) = \max(gr(f), gr(g))$ provided $gr(f) \neq gr(g)$

First, we prove that $(x+c)^q - x^q \approx c \cdot q \cdot x^{q-1}$ for every $c \in R \setminus \{0\}$ and $q \in \mathbb{Q}_+$.

Let $q = \frac{p}{p'}$ where $p, p' \in \mathbb{Z}_+$. For each $x \in R_+$ let $\Delta(x) = (x+c)^q - x^q$ and note that $\lim_{x \to +\infty}(\Delta(x) \cdot x^{-q}) = 0$, which implies that $gr(\Delta(x)) < q$. Since $(x+c)^p = (\Delta(x) + x^q)^{p'}$, we have that

$$\sum_{i=0}^{p} \binom{p}{i} \cdot x^i \cdot c^{p-i} = \sum_{i=0}^{p'} \binom{p'}{i} \cdot \Delta(x)^i \cdot (x^q)^{p'-i}$$

and

$$L(x) = \sum_{i=0}^{p-1} \binom{p}{i} \cdot x^i \cdot c^{p-i} = \sum_{i=1}^{p'} \binom{p'}{i} \cdot \Delta(x)^i \cdot (x^q)^{p'-i} = R(x)$$

Obviously, $gr(L(x)) = gr(\binom{p}{p-1} \cdot x^{p-1} \cdot c) = p - 1$. On the other hand, since $gr(\Delta(x)) < q$, we have that

$$gr(R(x)) = gr(\binom{p'}{1} \cdot \Delta(x) \cdot (x^q)^{p'-1}) = gr(\Delta(x)) + q \cdot (p' - 1)$$

Thus $gr(\Delta(x)) = p - 1 - q \cdot (p' - 1) = q - 1$ and $\Delta(x) \approx c \cdot q \cdot x^{q-1}$

$1 \Rightarrow 2$: We see that $q_1 > 0$ because otherwise for some $c \in R$ we would have that $|g(x)| \leq c$ for large $x \in R$ and consequently $f \sim f + g$. Now if $p_1 - 1 \leq 0$, then $q_1 > p_1 - 1$, which finishes the proof. So we can assume $p_1 > 1$

We know that $f(x) < f(x) + g(x)$ for all sufficiently large $x \in R$ and the set

$$A_f^{f+g} = \{\langle x, y \rangle \in R \times R : x > 0 \wedge f(x) < y \wedge y < f(x) + g(x)\}$$

is weak generic in $(R \times R, +)$. By Theorem 2.12, for every $M \in R_+$ there exist $x_M, y_M \in R$ s.t.

$$\{\langle x, y \rangle \in R \times R : x_M < x \wedge x < x_M + M \wedge y_M < y \wedge y < y_M + M\} \subseteq A_f^{f+g}$$

This implies that $f(x_M)+g(x_M) \geq f(x_M+M)+M$ for all sufficiently large $M \in R$. Note that $\lim_{M\to+\infty} x_M = +\infty$

Put $M_0 = \frac{b_1+1}{a_1 p_1}$. Then still for all sufficiently large $M \in R$ we have that $f(x_M)+g(x_M) \geq f(x_M+M_0)+M_0$ and by the o-minimality of $(R,<,+,\cdot)$, $f(x)+g(x) \geq f(x+M_0)+M_0$ for all sufficiently large $x \in R$. So ultimately

$$\sum_{i=1}^{m} a_i \cdot x^{p_i} + \sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M_0 + \sum_{i=1}^{m} a_i \cdot (x+M_0)^{p_i}$$

and

$$\sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M_0 + \sum_{i=1}^{m} a_i \cdot ((x+M_0)^{p_i} - x^{p_i})$$

Finally, comparing the ingredients of the sums with the biggest value of gr we see that ultimately

$$b_1 \cdot x^{q_1} \geq a_1 \cdot ((x+M_0)^{p_1} - x^{p_1}) \approx a_1 \cdot M_0 \cdot p_1 \cdot x^{p_1-1} = (b_1+1)\cdot x^{p_1-1}$$

Hence $q_1 > p_1 - 1$

$2 \Rightarrow 1$: Fix $M \in R_+$. Since $q_1 > \max(0, p_1-1)$, similar as above, we can show that for all sufficiently large $x \in R$

$$\sum_{i=1}^{m} a_i \cdot x^{p_i} + \sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M + \sum_{i=1}^{m} a_i \cdot (x+M)^{p_i}$$

This means that ultimately $f(x)+g(x) \geq f(x+M)+M$. Choose $x_M \in R_+$ satisfying the latter inequality and s.t. f and g are increasing on the interval $(x_M, +\infty)$. Then for $y_M = f(x_M+M)$ we have that

$$\{\langle x,y\rangle \in R\times R : x_M < x \wedge x < x_M+M \wedge y_M < y \wedge y < y_M+M\} \subseteq A_f^{f+g}$$

□

Example 2.3 (3.5.7)**.** Let $(R,<,+,\cdot)$ be a pure real closed field and for $a \in \mathbb{R}_+ \setminus \mathbb{Q}$ let

$$p(x,y) = \{x > r : r \in R\} \cup \{y > x^q : a > q \in \mathbb{Q}\} \cup \{y < x^q : a < q \in \mathbb{Q}\}$$

We shall prove that p is a stationary (complete) weak generic type in the group $(R,+)\times(R,+)$ and p is not of the form p_f^- or p_f^+ for any definable $f : R \to R$

The weak genericity of p follows from Theorem 2.28. Indeed, the set

$$\{\langle x,y\rangle \in R\times R : x>r \wedge y>x^{q_1} \wedge y<x^{q_2}\}$$

is weak generic in $(R,+)\times(R,+)$ since $q_2>\max(0,q_1-1)$

The stationary (and the completeness) of p follows from Lemma 2.27. Namely, if p were non-stationary, then for some $S\succ R$ and definable $f: S\to S$ we would have that ultimately $f(x)>x^q$ for each $q\in\mathbb{Q}\cap(-\infty,a)$ and ultimately $f(x)<x^q$ for each $q\in\mathbb{Q}\cap(a,+\infty)$. By Lemma 2.27, $f(x)\approx c\cdot x^q$ for some $q\in\mathbb{Q}$ and $c\in S_+$ ($c>0$ since f is ultimately increasing). Assume that $q>a$ and take any $r\in\mathbb{Q}\cap(a,q)$. Then ultimately $f(x)>x^r$ (since $q>r$). On the other hand, ultimately $f(x)<x^r$ (since $r\in\mathbb{Q}\cap(a,+\infty)$). If $q<a$, then we reach a contradiction in a similar way. As a result, p is stationary (and complete)

Suppose $p(x,y)=p'(x,y)$ for some definable function $f:R\to R$ and a type $p'(x,y)\in\{p_f^-(x,y),p_f^+(x,y)\}$. Then $f(x)\approx c\cdot x^q$ for some $q\in\mathbb{Q}$ and $c\in R_+$. W.L.O.G., $q>a$. Take any $r\in\mathbb{Q}\cap(a,q)$. Then $(y<x^r)\in p(x,y)$ and $(y>x^r)\in p'(x,y)$, a contradiction.

Example 2.4 (3.5.8)**.** Let $(R,<,+,\cdot,\dots)$ be an o-minimal polynomially bounded expansion of a real closed field $(R,<,+,\cdot)$ and for $q_0\in\mathbb{Q}_+$ let

$$p(x,y)=\{x>r : r\in R\}\cup\{y>r\cdot x^{q_0} : r\in R\}\cup\{y<x^q : q_0<q\in\mathbb{Q}\}$$

We shall prove that p is a non-stationary complete weak generic type in the group $(R,+)\times(R,+)$

It p were not complete, then we could find a definable function $f:R\to R$ s.t. ultimately $f(x)<x^q$ for each $q\in\mathbb{Q}\cap(q_0,+\infty)$ and ultimately $f(x)>r\cdot x^{q_0}$ for each $r\in R$ (thus $\lim_{x\to+\infty}\frac{f(x)}{x^{q_0}}=+\infty$). By Remark 2.5, $\frac{f(x)}{x^{q_0}}\geq\sqrt[n]{x}$ for some $n\in\mathbb{N}_+$ and all sufficiently large $x\in R$. But then ultimately $f(x)\geq x^{q_0+\frac{1}{n}}$, a contradiction.

To obtain non-stationarity of p, let S be an $|R|^+$-saturated elementary extension of R and choose any $a\in S$ s.t. $a>r$ for every $r\in R$. Then

$$\{x>r : r\in R\}\cup\{y>r\cdot x^{q_0} : r\in R\}\cup\{y<x^q : q_0<q\in\mathbb{Q}\}\cup\{y<a\cdot x^{q_0}\}$$

and

$$\{x>r : r\in R\}\cup\{y>r\cdot x^{q_0} : r\in R\}\cup\{y<x^q : q_0<q\in\mathbb{Q}\}\cup\{y>a\cdot x^{q_0}\}$$

are two distinct extensions of p to weak generic types in $(S,+)\times(S,+)$

In RCF, we can approximate the definable function .

2.5.1 Weak generic types in $(\mathbb{R},+) \times (\mathbb{R},+)$

Now we give a description of complete (over $\mathbb{R}$) weak generic types in the group $(\mathbb{R},+) \times (\mathbb{R},+)$ derived in the theory $\mathrm{Th}(\mathbb{R},<,+,\cdot)$.

Let S be a $(2^{\aleph_0})^+$-saturated elementary extension of the field of reals. Choose $a \in S$ s.t. $a > r$ for every $r \in \mathbb{R}$. Let $b_0 \in S$ be s.t. $b_0 \neq \sum_{i=1}^n r_i \cdot a^{q_i}$ for all $n \in \mathbb{N}_+$, $r_i \in \mathbb{R}$ and $q_i \in \mathbb{Q}$ (in this case we say that b_0 is non-polynomial over a). We describe a recursive procedure of defining $b_1, b_2, \dots \in S \setminus \{0\}$, $r_1, r_2, \dots \in \mathbb{R} \setminus \{0\}$ and $q_1, q_2, \dots \in \mathbb{Q}_+$ so that $q_1 > q_2 > \dots$ and $b_n = b_0 - r_1 \cdot a^{q_1} - \dots - r_n \cdot a^{q_n}$ for every $n \in \mathbb{N}_+$.

First we define b_1, r_1 and q_1. We consider two cases, depending on whether b_0 is positive or negative.

- **Case P**. $b_0 > 0$. Consider the following subsets of $\mathbb{Q}_+$:

  $$A = \{q \in \mathbb{Q}_+ : (\forall r \in \mathbb{R}_+) b_0 > r \cdot a^q\}$$
  $$B = \{q \in \mathbb{Q}_+ : (\forall r \in \mathbb{R}_+) b_0 < r \cdot a^q\}$$

  The sets A and B are disjoint and there is a unique $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ s.t. $A \subseteq (0,c]$, $B \subseteq [c,+\infty)$ and $\mathbb{Q}_+ \cup \{c\} = A \cup B \cup \{c\}$. We define b_1, r_1, q_1 only in the case where the following condition holds:

  $$c \in \mathbb{Q}_+, A = \mathbb{Q}_+ \cap (0,c), B = \mathbb{Q}_+ \cap (c,+\infty) \qquad (\dagger)$$

  Otherwise the procedure stops and no b_1, r_1, q_1 are defined

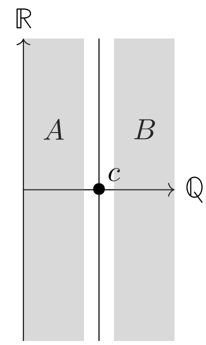
  If $(\dagger)$ holds, then we put $q_1 = c$. We have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$ for some $r' < r'' \in \mathbb{R}_+$. Since the ordering $(\mathbb{R},<)$ is Dedekind complete, there exists a unique $r \in \mathbb{R}_+$ s.t. for every $r', r'' \in \mathbb{R}_+$ with $r' < r < r''$ we have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$. We put $r_1 = r$ and $b_1 = b_0 - r_1 \cdot a^{q_1}$.

- **Case N**. $b_0 < 0$. Here we proceed similarly. Consider the following subsets of $\mathbb{Q}_+$:

  $$A = \{q \in \mathbb{Q}_+ : (\forall r \in \mathbb{R}_-) b_0 < r \cdot a^q\}$$
  $$B = \{q \in \mathbb{Q}_+ : (\forall r \in \mathbb{R}_-) b_0 > r \cdot a^q\}$$

  The sets A and B are disjoint and there is a unique $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ s.t. $A \subseteq (0,c]$, $B \subseteq [c,+\infty)$ and $\mathbb{Q}_+ \cup \{c\} = A \cup B \cup \{c\}$. We define b_1, r_1 and q_1 only in the case where $(\dagger)$ holds. Otherwise the procedure stops and no b_1, r_1 and q_1 are defined.

  If $(\dagger)$ holds, then we put $q_1 = c$. We have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$ for some $r' < r'' \in \mathbb{R}_-$. Since the ordering $(\mathbb{R},<)$ is Dedekind complete, there exists a unique $r \in \mathbb{R}_-$ s.t. for every $r', r'' \in \mathbb{R}_-$ with $r' < r < r''$ we have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$. We put $r_1 = r$ and $b_1 = b_0 - r_1 \cdot a^{q_1}$.



Suppose b_i, r_i and q_i have been defined so that $b_i \neq 0$. Again we consider two cases, depending on whether b_i is positive or negative.

- **Case P**. $b_i > 0$. We define the sets A, B and $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ as in the case of $b_0 > 0$. Again, if (†) fails, then the procedure stops and b_j, r_j, q_j are not defined for any $j > i$. If (†) holds, then we put $q_{i+1} = c$ and define r_{i+1}, b_{i+1} analogously as in the case of b_0.

- **Case N**. $b_i < 0$. Similar.

If $b_1, \dots, b_i, q_1, \dots, q_i$ and $r_1, \dots, r_i$ are defined, then $q_1 > \dots > q_i$. We shall only prove that $q_1 > q_2$. We have that

$$b_2 = b_1 - r_2 \cdot a^{q_2} = b_0 - r_1 \cdot a^{q_1} - r_2 \cdot a^{q_2}$$

W.L.O.G., we can assume that $r_2 > 0$. Choose any real number $r \in (0, r_2)$. Then $b_1 > r \cdot a^{q_2}$. If $q_1 \leq q_2$, then also $b_1 > r \cdot a^{q_1}$ and consequently $b_0 = b_1 + r_1 \cdot a^{q_1} > (r_1 + r) \cdot a^{q_1}$, which contradicts the definition of r_1. Hence $q_1 > q_2$.

Secondly, note that $b_k \neq 0$ for every $k \in \{1, \dots, i\}$. Otherwise we would have that

$$b_k = b_0 - r_1 \cdot a^{q_1} - \dots - r_n \cdot a^{q_n} = 0$$

and b_0 would be polynomial over a, a contradiction.

Now we are able to give a description of complete weak generic types in the group $(\mathbb{R}, +) \times (\mathbb{R}, +)$ (which implies that b_0 is non-polynomial over a) and $a > 0$ (hence $a > r$ for every $r \in \mathbb{R}$). Denote the type tp$(\langle a, b_0 \rangle / \mathbb{R})$ by $p(x, y)$ and note that $\{x > r : r \in \mathbb{R}\} \subseteq p(x, y)$.

- **Case A**. Assume that no b_i are defined for $i > 0$. This happens only if (†) fails. We shall consider one by one all possible cases. It turns out that each of these cases determines uniquely the weak generic type tp$(\langle a, b_0\rangle/\mathbb{R})$.

  First, we consider the situation where $b_0 > 0$. Let A, B be as in Case P.

  - **Case 1**. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

    $$\{x > r : r \in \mathbb{R}\} \cup \{y > x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \{y < x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

    to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is stationary. (Example 2.3)
  - **Case 2**. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

    $$\{x > r : r \in \mathbb{R}\} \cup \{y > r \cdot x^q : r \in \mathbb{R}_+\} \cup \{y < x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

    to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is non-stationary. (Example 2.4)
  - **Case 3.** *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

    $$\{x > r : r \in \mathbb{R}\} \cup \{y > x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup \{y < r \cdot x^q : r \in \mathbb{R}_+\}$$

    to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is non-stationary.
  - **Case 4**. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Since $p(x, y)$ is weak generic and $b_0 > 0$, we also have that $b_0 > r$ for every $r \in \mathbb{R}$. Therefore $p(x, y) = p_z^+(x, y)$ where $z : \mathbb{R} \to \mathbb{R}$, $z(r) = 0$ for all $r \in \mathbb{R}$. By Theorem 2.20, $p(x, y)$ is stationary.
  - **Case 5**. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y) = p_+(x, y)$. By Theorem 2.20, $p(x, y)$ is stationary.

  If $b_0 < 0$, then we get the following cases.

  - **Case 1'**. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

    $$\{x > r : r \in \mathbb{R}\} \cup \{y < -x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \{y > -x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

    to a complete weak generic type over $\mathbb{R}$.

– **Case 2′**. $A = \mathbb{Q}_+ \cap (0, q]$ *and* $B = \mathbb{Q}_+ \cap (q, +\infty)$ *for some* $q \in \mathbb{Q}_+$.
Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y < r \cdot x^q : r \in \mathbb{R}_-\} \cup \{y > -x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$.

– **Case 3′.** $A = \mathbb{Q}_+ \cap (0, q)$ *and* $B = \mathbb{Q}_+ \cap [q, +\infty)$ *for some* $q \in \mathbb{Q}_+$.
Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y < -x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup \{y > r \cdot x^q : r \in \mathbb{R}_-\}$$

to a complete weak generic type over $\mathbb{R}$.

– **Case 4′**. $A = \emptyset$ *and* $B = \mathbb{Q}_+$.
Since $p(x, y)$ is weak generic and $b_0 < 0$, we also have that $b_0 < r$ for every $r \in \mathbb{R}$. Therefore $p(x, y) = p_z^-(x, y)$ where $z : \mathbb{R} \to \mathbb{R}$ is constantly equal to 0.

– **Case 5′**. $A = \mathbb{Q}_+$ *and* $B = \emptyset$.
Then $p(x, y) = p_{-\infty}(x, y)$

- **Case B**. Now assume that for a and b_0 with $\mathrm{tp}(\langle a, b_0\rangle/\mathbb{R})$ weak generic the procedure breaks down at some finite step so that b_i, r_i and q_i are defined only for $i \in \{1, \dots, n\}$. This means that in the condition (†) fails at step n. Let $f(x) = \sum_{i=1}^n r_i \cdot x^{q_i}$ and recall that $r_i \in \mathbb{R} \setminus \{0\}$, $q_i \in \mathbb{Q}_+$ and $q_1 > \dots > q_n$.

  First we consider the situation where $b_n > 0$. Let A, B be as in Case P.

  – **Case 1**. $A = \mathbb{Q}_+ \cap (0, c)$ *and* $B = \mathbb{Q}_+ \cap (c, +\infty)$ *for some* $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.
  Then $p(x, y)$ is the only extension of the type

$$\begin{aligned} r(x, y) =& \{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \\ & \{y - f(x) < x^q : q \in \mathbb{Q} \cap (c, +\infty)\} \end{aligned}$$

  to a complete weak generic type over $\mathbb{R}$. Moreover, $c > q_1 - 1$ and $c < q_n$. For each $p_1 \in \mathbb{Q} \cap (c, +\infty)$ and $p_2 \in \mathbb{Q} \cap (0, c)$, ultimately, for each $M \in \mathbb{R}_+$, we want $x^{p_1} + x^{q_1} \geq (x + M)^{q_1} + (x + M)^{p_2} + M$, which is equivalent to $x^{p_1} \geq k(M) x^{q_1 - 1}$ and $p_1 > q_1 - 1$. Therefore $c > q_1 - 1$.

  – **Case 2**. $A = \mathbb{Q}_+ \cap (0, q]$ *and* $B = \mathbb{Q}_+ \cap (q, +\infty)$ *for some* $q \in \mathbb{Q}_+$.

Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > r \cdot x^q : r \in \mathbb{R}_+\} \cup$$
$$\{y - f(x) < x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q \geq q_1 - 1$ since $q' > q_1 - 1$ for any $q' > q$ and $q < q_n$. If $q = q_1 - 1$, then $p(x, y)$ is stationary (since then $p(x, y) = p_f^+(x, y)$). If $q > q_1 - 1$, then $p(x, y)$ is non-stationary. (Example 2.4)

– **Case 3**. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup$$
$$\{y - f(x) < r \cdot x^q : r \in \mathbb{R}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q > q_1 - 1$ and $q < q_n$. Every weak generic type of this form is non-stationary.

– **Case 4**. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > 0\} \cup \{y - f(x) < x^q : q \in \mathbb{Q}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q_1 \leq 1$ and $p(x, y) = p_f^+(x, y)$. By Theorem 2.20, $p(x, y)$ is stationary.

– **Case 5.** *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y)$ contains the following set of formulas

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^q : q \in \mathbb{Q}_+\}$$

Take any rational $q > q_1$. Then $(y - f(x) > x^q) \in p(x, y)$, which implies that $b_0 - f(a) > a^q$. Hence

$$b_0 > a^q + \sum_{i=1}^{n} \cdot a^{q_i}$$

which contradicts the choice of q_1. So case 5 can not hold.

If $b_n < 0$, then we get the following cases: Let A, B be as in Case N.

- **Case 1′**. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < -x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \\ \{y - f(x) > -x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $c > q_1 - 1$ and $c < q_n$.

- **Case 2′**. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < r \cdot x^q : r \in \mathbb{R}_-\} \cup \\ \{y - f(x) > -x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q \geq q_1 - 1$ and $q < q_n$. If $q = q_1 - 1$, then $p(x, y) = p_f^-(x, y)$.

- **Case 3′**. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < -x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup \\ \{y - f(x) > r \cdot x^q : r \in \mathbb{R}_-\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q > q_1 - 1$ and $q < q_n$.

- **Case 4′**. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < 0\} \cup \{y - f(x) > -x^q : q \in \mathbb{Q}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q_1 \leq 1$ and $p(x, y) = p_f^-(x, y)$.

- **Case 5′**. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Impossible.

- **Case C**. Now assume that for a and b_0 with $\mathrm{tp}(\langle a, b_0 \rangle / \mathbb{R})$ weak generic the procedure never stops and we have defined b_i, r_i and q_i for all $i \in \mathbb{N}_+$. Recall that $b_i \in S \setminus \{0\}$, $r_i \in \mathbb{R} \setminus \{0\}$, $q_i \in \mathbb{Q}_+$, $q_1 > q_2 > \dots$ and for every $i \in \mathbb{N}_+$ we have that $b_i = b_0 - r_1 \cdot a^{q_1} - \dots - r_i \cdot a^{q_i}$.

  Let f be the formal power series $\sum_{i=1}^{+\infty} r_i \cdot x^{q_i}$ and for each $n \in \mathbb{N}$ let $f_n(x) = \sum_{i=1}^{n} r_i \cdot x^{q_i}$ (in particular, $f_0 = 0$). We shall prove that the type $p(x,y) = \text{tp}(\langle a, b_0\rangle/\mathbb{R})$ is the only extension of the type $p_f(x,y)$ to a complete weak generic type over $\mathbb{R}$ where the type $p_f(x,y)$ is defined as follows:

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f_n(x) > r' \cdot x^{q_{n+1}} : n \in \mathbb{N} \wedge r' \in \mathbb{R} \wedge r' < r_{n+1}\} \cup$$
$$\{y - f_n(x) < r'' \cdot x^{q_{n+1}} : n \in \mathbb{N} \wedge r'' \in \mathbb{R} \wedge r'' > r_{n+1}\}$$

Then $p_f(x,y) \subseteq p(x,y)$. Assume that the weak generic types $p(x,y) = \text{tp}(\langle a, b_0\rangle/\mathbb{R})$ and $p'(x,y) = \text{tp}(\langle a', b_0'\rangle/\mathbb{R})$ $(a', b_0' \in S)$ are two distinct extensions of the type $p_f(x,y)$ (hence $\langle a, b_0\rangle$ and $\langle a', b_0'$ produce the same formal power series $f = \sum_{i=1}^{+\infty} r_i \cdot x^{q_i}$). W.L.O.G., we can assume that $a = a'$.

Since $p \neq p'$, there exists a definable (and thus semi-algebraic) function $g : \mathbb{R} \to \mathbb{R}$ s.t. $(y < g(x)) \in p$ and $(y > g(x)) \in p'$. We shall prove that the map g can be replaced with the map of the form $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ where $n \in \mathbb{N}_+$, $s_i \in \mathbb{R}$ and $p_i \in \mathbb{Q}$.

Random links: link1

We know that g has a Puiseux expansion of the form

$$g(x) = \sum_{i=k}^{+\infty} c_i \cdot x^{\frac{-i}{d}} \tag{$\star$}$$

where $k \in \mathbb{Z}$, $d \in \mathbb{Z}_+$ and $c_i \in \mathbb{R}$ for $i \geq k$. Equality $(\star)$ holds for all sufficiently large $x \in \mathbb{R}$. One can prove that for some $M \in \mathbb{R}_+$ we obtain ultimately

$$\left| g(x) - \sum_{i=k}^{-1} c_i \cdot x^{\frac{-i}{d}} \right| \leq M$$

Let $h(x) = \sum_{i=k}^{-1} c_i \cdot x^{\frac{-i}{d}}$. Then still $(y < h(x)) \in p$ and $(y > h(x)) \in p'$, which means that $b_0 < h(a)$ and $b_0' > h(a)$.

Choose unique $n \in \mathbb{N}_+$, $s_i \in \mathbb{R} \setminus \{0\}$ and $p_i \in \mathbb{Q}_+$ s.t. (ultimately) $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ and $p_1 > \cdots > p_n$. Then for each $i \in \{1, \ldots, n\}$ we have that $p_i = q_i$ and $s_i = r_i$. We shall only prove that $p_1 = q_1$ and $s_1 = r_1$.

In order to show that $p_1 = q_1$, we shall consider several possible cases.

- If $p_1 < q_1$ and $r_1 > 0$, then

$$h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} < \frac{r_1}{2} \cdot a^{q_1} < b_0 < h(a)$$

a contradiction. The first equality is because $a > \mathbb{R}$, the second comes from $q_1 > p_1$ and $r_1 > 0$, the third holds since b_0 is close to $r_1 \cdot a^{q_1}$.

- If $p_1 < q_1$ and $r_1 < 0$, then

$$h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} > \frac{r_1}{2} \cdot a^{q_1} > b_0' > h(a)$$

  a contradiction.

- If $p_1 > q_1$ and $s_1 > 0$, then

$$(r_1 + 1) \cdot a^{q_1} < s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} < b_0' < (r_1 + 1) \cdot a^{q_1}$$

- If $p_1 > q_1$ and $s_1 < 0$ then

$$(r_1 - 1) \cdot a^{q_1} > s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} = h(a) > b_0 > (r_1 - 1) \cdot a^{q_1}$$

  a contradiction.

Since $r_1 \neq 0$ and $s_1 \neq 0$, the contradictions above shows that $p_1 = q_1$. Suppose $s_1 \neq r_1$:

- If $s_1 < r_1$, then

$$b_0 < h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} < \frac{s_1 + r_1}{2} \cdot a^{p_1} = \frac{s_1 + r_1}{2} \cdot a^{q_1} < b_0$$

  a contradiction.

- If $s_1 > r_1$, then

$$b_0' > h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} > \frac{s_1 + r_1}{2} \cdot a^{p_1} = \frac{s_1 + r_1}{2} \cdot a^{q_1} > b_0'$$

  a contradiction. Hence $s_1 = r_1$.

The proof that $p_i = q_i$ and $s_i = r_i$ for each $i \in \{2, \dots, n\}$ is analogous.

As a result, we obtain

$$(y < \sum_{i=1}^{n} r_i \cdot x^{q_i}) \in p(x,y)$$
$$(y > \sum_{i=1}^{n} r_i \cdot x^{q_i}) \in p'(x,y)$$

Thus $b_0 < \sum_{i=1}^{n} r_i \cdot a^{q_i}$ and $b_0' > \sum_{i=1}^{n} r_i \cdot a^{q_i}$. We have that

$$b_n = b_0 - \sum_{i=1}^{n} r_i \cdot a^{q_i} < 0$$

which implies that $r_{n+1} < 0$, and

$$b_n' = b_0' - \sum_{i=1}^{n} r_i \cdot a^{q_i} > 0$$

which implies that $r_{n+1} > 0$, a contradiction. Hence $p_f(x,y) \vdash p(x,y)$.

By Theorem 2.28, the sequence of the rational exponents $(q_n)_{n\in\mathbb{N}^+}$ has the following property: $q_n > q_1 - 1$ for every $n > 1$. Otherwise the formal power series f could not be obtained as a final result of the recursive procedure described above (since the type $\text{tp}(\langle a, b_0\rangle/\mathbb{R}) = p(x,y) \supseteq p - f(x,y)$ is weak generic and so must be $p_f(x,y)$).

Finally, proceeding as above we can shows that the weak generic type $p(x,y)$ is stationary. The crucial fact needed to prove this is as follows: for every definable (thus semi-algebraic) function $g : S \to S$ there exist a map $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ (where $n \in \mathbb{N}$, $s_i \in \mathbb{R}$ and $p_i \in \mathbb{Q}_+$) and a constant $M \in S_+$ s.t. $|g(x) - h(x)| \leq M$ for all sufficiently large x.

If we are given an arbitrary formal power series $f = \sum_{i=1}^{\alpha} r_i \cdot x^{q_i}$ (where $\alpha \in \mathbb{N} \cup \{+\infty\}$, $r_i \in \mathbb{R} \setminus \{0\}$, $q_i \in \mathbb{Q}_+$ and $q_1 > q_2 > \ldots$), then in each of the above cases f determines a unique complete (over $\mathbb{R}$) weak generic type in the group $(\mathbb{R},+) \times (\mathbb{R},+)$. In this way we have described complete weak generic types in $(\mathbb{R},+) \times (\mathbb{R},+)$ derived in the theory $\text{Th}(\mathbb{R},<,+,\cdot)$.

2.5.2 Weak generic types in $(\mathbb{R},\cdot) \times (\mathbb{R},\cdot)$

Assume $(R,<,+,\cdot,\ldots)$ is an o-minimal expansion of an ordered field $(R,<,+,\cdot)$. A **power function** is a definable endomorphism of the group $(R_+,\cdot)$.

Every power function is differentiable on R_+. For each $r \in R$ there is at most one power function f with $f'(1) = r$. We denote such a map by x^r and write a^r for $f(a)$. The field

$$K = \{f'(1) : f \text{ is a power function}\} \subseteq R$$

is called **the field of exponents** of R. We say that the structure R is **power bounded** if for every definable $f : R \to R$ there exists an $r \in K$ s.t. ultimately $|f(x)| \leq x^r$. An **exponential function** is an isomorphism of the structures $(R, <, +, 0)$ and $(R_+, <, \cdot, 1)$.

The main result of [Mil96] says that either R defines (without parameters) an exponential function or R is power bounded and for each ultimately non-zero definable function $f : R \to R$ there exist an $a \in R \setminus \{0\}$ and a 0-definable power function x^r s.t. $f(x) \approx a \cdot x^r$

Theorem 2.29. *If $R = (R, <, +, \cdot, \dots)$ is an o-minimal expansion of a real closed field R, TFAE*

1. *all complete (over R) weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ are stationary,*
2. *the structure R is power bounded.*

Proof. $1 \Rightarrow 2$: For the sake of contradiction assume that R is not power bounded. As we mentioned above, this implies that the exponential function $\exp : R \to R_+$ is 0-definable in R. Thus the map

$$(\exp, \exp) : (R, +) \times (R, +) \to (R_+, \cdot) \times (R_+, \cdot)$$

is a 0-definable isomorphism of groups. Hence the groups $(S, +) \times (S, +)$ and $(S, \cdot) \times (S, \cdot)$ are definably isomorphic for every $S \succ R$ and it suffices to show that some weak generic type in $(R, +) \times (R, +)$ is not stationary. To do this, take an arbitrary $S \succ R$, $a \in S \setminus R$ and let $f : S \to S$ be s.t. $f(x) = a \cdot x$ for every $x \in S$. We shall prove that the weak generic types p_f^- and p_f^+ are extensions of the same complete weak generic type over R.

Since the structure R does not need to be $\aleph_0$-saturated, Lemma 2.6 itself is not sufficient to ensure that the restrictions of the types p_f^- and p_f^+ to the complete types over R are weak generic in $(R, +) \times (R, +)$. Nevertheless, this follows from the corollary 2.13

It is enough to show that $f \not\sim g$ for each $g : S \to S$ definable over R. Suppose otherwise, then for some R-definable $g : S \to S$ we have that $S \models g \sim f$ (note that there is a first order formula $\varphi \in L(S)$ expressing the fact that $g \sim f$; namely, φ says that the area defined by the formula

$(x > 0 \wedge f(x) < y \wedge y < g(x))$ doesn't contain arbitrarily large squares). So $S \vDash \exists c(g(x) \sim c \cdot x)$ and $R \vDash \exists c(g(x) \sim c \cdot x)$. Take $b \in R$ s.t. $g(x) \sim b \cdot x$ in R. Then $g(x) \sim b \cdot x$ in S, hence $f(x) \sim b \cdot x$ and $a \cdot x \sim b \cdot x$, a contradiction.

$2 \Rightarrow 1$: Note that it is enough to examine those weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ which contain the formula $(x \geq 1 \wedge y \geq 1)$. To prove this, consider $F, G : R_+ \times R_+ \to R_+ \times R_+$ defined as: $F(x, y) = \langle x, \frac{1}{y} \rangle$ and $G(x, y) = \langle \frac{1}{x}, y \rangle$ for every $x, y \in R_+$. We see that F, G and $F \circ G$ are 0-definable automorphisms of the group $(R_+, \cdot) \times (R_+, \cdot)$ that map the set $\{\langle x, y \rangle : x \geq 1 \wedge y \geq 1\}$ respectively onto the sets

1. $\{\langle x, y \rangle : x \geq 1 \wedge 0 < y \leq 1\}$
2. $\{\langle x, y \rangle : 0 < x \leq 1 \wedge y \leq 1\}$
3. $\{\langle x, y \rangle : 0 < x \leq 1 \wedge 0 < y \leq 1\}$

The same holds for any elementary extension S of R, which enables us to "translate" an example of a non-stationary weak generic type to the set of types $[x \geq 1 \wedge y \geq 1] = \{p \mid x \geq 1 \wedge y \geq 1 \in p\}$.

In order to prove that each complete weak generic type in $(R_+, \cdot) \times (R_+, \cdot)$ containing the formula $(x \geq 1 \wedge y \geq 1)$ is stationary, we are going to show that for every $S \succ R$ and every definable function $f : S \to S \cap [1, +\infty)$ we are able to find an R-definable map $g : S \to S$ s.t. the set

$$\{\langle x, y \rangle \in S \times S : x \geq 1 \wedge y \geq 1 \wedge (f(x) \leq y \leq g(x) \vee f(x) \geq y \geq g(x))\}$$

is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$. So take such S and f. Let $a, r \in S$ be s.t. $f(x) \approx a \cdot x^r$. Then $a > 0$ and $r \geq 0$. The power function $x^r : S \to S$ is R-definable and we put $g = x^r$.

Choose any $c \in S_+$ s.t. $\frac{1}{c} \cdot x^r \leq f(x) \leq c \cdot x^r$ for all sufficiently large $x \in S$. Without loss of generality, we can assume that f is on the whole interval $[1, +\infty]$ since for every $M \geq 1$ the set $X_M = [1, M] \times [1, +\infty]$ is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$ (otherwise, by Corollary 2.11, the set $X_M \cdot X_M^{-1} = [\frac{1}{M}, M] \times S_+$ would be generic in $(S_+, \cdot) \times (S_+, \cdot)$, which is not the case)

Now it suffices to prove that the set

$$X = \{\langle x, y \rangle \in S \times S : x \geq 1 \wedge y \geq 1 \wedge \frac{1}{c} \cdot x^r \leq y \wedge y \leq c \cdot x^r\}$$

is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$. Suppose otherwise, then the set $X \cdot X^{-1}$ is generic in $(S_+, \cdot) \times (S_+, \cdot)$ by Corollary 2.11. We claim that

$$X \cdot X^{-1} \subseteq Y = \{\langle x, y \rangle \in S \times S : x > 0 \wedge \frac{1}{c^2} \cdot x^r \leq y \wedge y \leq c^2 \cdot x^r\}$$

To see this, take any $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in X$. We have that $\frac{1}{c} \cdot x_1^r \leq y_1 \leq c \cdot x_2^r$ and $\frac{1}{c} \cdot x_2^r \leq y_2 \leq c \cdot x_2^r$. So

$$\frac{1}{c^2} \cdot \left(\frac{x_1}{x_2}\right)^r \leq \frac{y_1}{y_2} \leq c^2 \cdot \left(\frac{x_1}{x_2}\right)^r$$

and $\langle x_1, y_1\rangle \cdot \langle x_2, y_2\rangle^{-1} = \langle u, v\rangle$ where $u = \frac{x_1}{x_2}$ and $\frac{1}{c^2} \cdot u^r \leq v \leq c^2 \cdot u^r$. Thus $u = \frac{x_1}{x_2}$ and $\frac{1}{c^2} \cdot u^r \leq v \leq c^2 \cdot u^r$. Thus $\langle u, v\rangle \in Y$ and $X \cdot X^{-1} \subseteq Y$. In turn, since for every $x \in (0, 1)$, $c^2 \cdot x^r \leq c^2$, we have that

$$Y \subseteq Z = (S_+ \times S_+) \setminus ((0, 1) \times (c^2, +\infty))$$

This implies that the set Z is generic in $(S_+, \cdot) \times (S_+, \cdot)$, a contradiction. $\square$

Corollary 2.30. *If $R = (R, <, +, \cdot, \dots)$ is an o-minimal expansion of an archimedean real closed field R, then TFAE:*

1. *all complete (over R) weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ are stationary,*
2. *the structure R is polynomially bounded*

Proof. By [Mil96], R is polynomially bounded iff R is power bounded and K is archimedean. But the field K is archimedean as a subfield of the archimedean field R. The assertion of the corollary immediately follows from Theorem 2.29 $\square$

3 Problems

2.1 2.4

ref	problem	status
2.4		done
2.4		done
2.4		done
2.3	why complete?	done
2.2	why there only 3?	done
2.5.1	condition	done

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