Proof Theory
The First Step Into Impredicativity

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August 1, 2021

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2.1 Some Basic Facts about Ordinals

$$\alpha \in On \quad :\Leftrightarrow \quad Tran(\alpha) \wedge (\alpha, \in) \text{ is well-ordered}$$

where

$$Tran(M) \quad :\Leftrightarrow \quad (\forall x \in M)(\forall y \in x)[y \in M]$$

Proposition 2.1. $\alpha \in On \Rightarrow Tran(\alpha) \wedge (\forall x \in \alpha)[Tran(x)]$

Proof. If $z \in y \in x \in \alpha$, then $z \in x$ because α is well-ordered by $\in$. $\square$

so we have

$$\alpha \in On \wedge x \in \alpha \Rightarrow x \in On, \text{ i.e., } Tran(On)$$

and obtain

$$\alpha \in On \Rightarrow \alpha = \{\beta \mid \beta < \alpha\}$$

We assume that an ordinal is a transitive set
α is **hereditarily transitive** iff $Tran(\alpha) \wedge (\forall x \in \alpha)[Tran(x)]$

Lemma 2.2. *Assume that the membership relation $\in$ well-founded. Then α is an ordinal iff α is a hereditarily transitive set*

Proof. Assume α is hereditarily transitive. By the foundation scheme $\in$ is irreflexive and well-founded on α. Since α is hereditarily transitive, it's also transitive. Assume β is also hereditarily transitive, we show

if β is well-ordered by $\in$ and $\alpha \subseteq \beta$ then $\alpha = \beta \vee \alpha \in \beta$

□

2.2 Fundatmentals of Ordinal Arithmetic