Universal theories, heirs, and ultrapowers

Advanced Model Theory

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Remember those weird theorems saying that some type embeds into an ultrapower of another type? (March 3, Proposition 2, and March 31, Proposition 7.) Let's get a better conceptual understanding of why they're true. Along the way, we'll learn some useful model theory.

1 Universal theories

Definition 1. A *universal formula* is a formula of the form $\forall \bar{y}\ \varphi(\bar{x}, \bar{y})$, where φ is quantifier-free. A universal sentence is a sentence that is universal, or equivalently, a sentence of the form $\forall \bar{x}\ \varphi(\bar{x})$ with φ quantifier-free.

Proposition 2. *Let M be a substructure of N.*

1. *If $\varphi(\bar{x})$ is a quantifier-free formula and $\bar{a} \in M$, then*

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}).$$

2. *If $\varphi(\bar{x})$ is a universal formula and $\bar{a} \in M$, then*

$$N \models \varphi(\bar{a}) \implies M \models \varphi(\bar{a})$$

Proof. "Obvious," but here are the details for completeness:

1. If φ is atomic, this holds by definition of "substructure." Otherwise, proceed by induction on the complexity of φ.

2. Write $\varphi(\bar{x})$ as $\forall \bar{y}\ \psi(\bar{x}, \bar{y})$ for quantifier-free $\psi(\bar{x}, \bar{y})$. If $N \models \varphi(\bar{a})$, then

$$\begin{aligned}
&N \models \forall \bar{y}\ \psi(\bar{a}, \bar{y}) \\
&\forall \bar{b} \in N\ (N \models \psi(\bar{a}, \bar{b})) \\
&\forall \bar{b} \in M\ (N \models \psi(\bar{a}, \bar{b})) \qquad \text{as } M \subseteq N \\
&\forall \bar{b} \in M\ (M \models \psi(\bar{a}, \bar{b})) \qquad \text{by part (1)} \\
&M \models \forall \bar{y}\ \psi(\bar{a}, \bar{y}) \\
&M \models \varphi(\bar{a}).
\end{aligned}$$

□

Definition 3. A *universal theory* is a set of universal sentences.

Proposition 4. *Let T be a universal theory.*

1. *If $N \models T$ and M is a substructure of N, then $M \models T$.*
2. *If $f : M \to N$ is an embedding and $N \models T$, then $M \models T$.*

Proof.

1. By Proposition 2(2), the axioms of T transfer from N to M.
2. If M' is the image of f, then $M \cong M'$ and M' is a substructure of N, so $N \models T \implies M' \models T \implies M \models T$. □

Example. A *semigroup* is a structure $(G, \cdot)$ satisfying the universal sentence

$$\forall x, y, z \ (x \cdot (y \cdot z) = (x \cdot y) \cdot z).$$

Proposition 4(1) implies that any substructure of a semigroup is a semigroup (which is obvious).

Definition 5. If T is a theory, then $T_\forall$ is the set of universal sentences φ such that $T \vdash \varphi$.

Lemma 6. *Suppose $N \models T$. If M is a substructure of N, then $M \models T_\forall$. More generally, if there is an embedding $M \to N$, then $M \models T_\forall$.*

Proof. If $N \models T$ then $N \models T_\forall$, because the axioms of $T_\forall$ are logical consequences of the axioms of T. Then $M \models T_\forall$ by Proposition 4. □

Using compactness, we can prove a converse of Lemma 6.

Lemma 7. *Suppose $M \models T_\forall$. Then there is a model $N \models T$ and an embedding $M \to N$. We may even take N to be an extension of M.*

Proof. Consider the language $L(M)$ where we add a constant symbol for each element of M. Let D be the set of quantifier-free $L(M)$-sentences true in M. (D is usually called the *diagram* of M.)

Claim. $D \cup T$ is consistent.

Proof. Otherwise, by compactness there are $\varphi_1, \ldots, \varphi_n \in D$ such that $\{\varphi_1, \ldots, \varphi_n\} \cup T$ is inconsistent. Let $\varphi = \bigwedge_{i=1}^n \varphi_i$. Then $\varphi \in D$ and $\{\varphi\} \cup T$ is inconsistent. Note φ is a quantifier-free $L(M)$-sentence. Write φ as $\theta(\bar{c})$ for some quantifier-free L-formula θ and some tuple $\bar{c} \in M$.[1] As $M \models \theta(\bar{c})$ and $M \models T_\forall$,

$$M \not\models \forall \bar{x}\ \neg\theta(\bar{x})$$
$$(\forall \bar{x}\ \neg\theta(\bar{x})) \notin T_\forall$$
$$T \not\vdash \forall \bar{x}\ \neg\theta(\bar{x}).$$

Therefore there is a model $N \models T$ with

$$N \models \neg\forall \bar{x}\ \neg\theta(\bar{x})$$
$$N \models \exists \bar{x}\ \theta(\bar{x})$$
$$N \models \theta(\bar{e})$$

for some $\bar{e} \in N$. Expand the L-structure N to an $L(M)$-structure by interpreting $c_i \in M$ as e_i and interpreting the other elements of M randomly.[2] Then $N \models \theta(\bar{c})$. Consequently, $N \models \{\varphi\} \cup T$, contradicting the fact that $\{\varphi\} \cup T$ is inconsistent. $\square_{\text{Claim}}$

By the Claim and compactness, there is some $L(M)$-structure N with $N \models D \cup T$. Then N is a model of T. Let $f : M \to N$ be the map sending $c \in M$ to its interpretation in N (which is an $L(M)$-structure). The fact that $N \models D$ means that f is an embedding. For example:

- If $c, d \in M$ and $c \neq d$, then $M \models c \neq d$, so $(c \neq d) \in D$, so $N \models c \neq d$, which really means $f(c) \neq f(d)$.
- If R is a binary relation and $M \models R(c, d)$, then $R(c, d) \in D$, so $N \models R(c, d)$, which really means $N \models R(f(c), f(d))$.

So we have embedded M into a model of T.

For the final clause of Lemma 7, once we have an embedding $f : M \to N$ with $N \models T$, we can find an isomorphism $g : N \xrightarrow{\cong} N'$ such that $g(f(x)) = x$ for $x \in M$. Then $N' \models T$ and the embedding $M \to N'$ is an inclusion, meaning that M is a substructure of N'. $\square$

For future reference, we note the following Fact:

Fact 8. *Let M, N be L-structures. Let T be the complete L-theory of N. Suppose $M \models T_\forall$. (In other words, M satisfies all the universal sentences that N satisfies.)*

[1] For technical reasons, we need to make two conditions hold. (1) The coordinates of $\bar{c}$ should be pairwise distinct. (2) If $M \neq \varnothing$, then $\bar{c}$ should have length at least one. These two conditions can be ensured as follows. Let $c_1, \ldots, c_n$ enumerate *without repeats* the elements of M mentioned in φ. But if M is non-empty and φ mentions no elements of M, take $n = 1$ and take $c_1 \in M$ arbitrary. Every element of M mentioned in φ is one of the c_i, so we can write φ as $\theta(\bar{c})$ for some θ.

[2] There are two technical details to check. (1) If $c_i = c_j$, we need e_i to equal e_j. This works because we made $c_i \neq c_j$ for $i \neq j$. (2) If there are elements of M outside of the c_i, we need N to be non-empty. This works because when $M \neq \varnothing$, we arranged for the length of $\bar{c}, \bar{x}, \bar{e}$ to be positive, and then $\bar{e} \in N \implies N \neq \varnothing$.

1. *There is $N' \equiv N$ and an embedding $M \to N'$.*

2. *There is an ultrapower $N^{\mathcal{U}}$ and an embedding $M \to N^{\mathcal{U}}$.*

(1) is a direct consequence of Lemma 7. (2) should sound plausible, because Lemma 7 was proved using compactness, and compactness was proved using ultraproducts.

Combining Lemmas 6 and 7, we get the following useful fact:

Theorem 9. *Let T be a theory. Then $M \models T_\forall$ iff M is a substructure of a model of T.*

Fact 10. *Here are two examples:*

1. $\mathrm{DLO}_\forall$ *is the theory of linear orders. A structure $(M, \leq)$ is a linear order iff $(M, \leq)$ embeds into a model of* DLO*.*

2. $\mathrm{ACF}_\forall$ *is the theory of integral domains. A structure $(R, +, \cdot, -, 0, 1)$ is an integral domain iff $(R, +, \cdot, -, 0, 1)$ embeds into a model of* ACF*.*

If you're curious, here is a proof of (1). If $(M, \leq) \models \mathrm{DLO}_\forall$, then $(M, \leq)$ is a substructure of a dense linear order, and so $(M, \leq)$ is a linear order. Conversely, suppose $(M, \leq)$ is a linear order. We claim $(M, \leq) \models \mathrm{DLO}_\forall$. Otherwise, there is a universal sentence $\forall \bar{x}\ \varphi(\bar{x})$ such that

$$\mathrm{DLO} \vdash \forall \bar{x}\ \varphi(\bar{x})$$
$$M \not\models \forall \bar{x}\ \varphi(\bar{x}).$$

Take $\bar{a} \in M$ with $M \models \neg\varphi(\bar{a})$. Let $A = \{a_1, \ldots, a_n\}$, viewed as a substructure of M. By Proposition 2,

$$M \models \neg\varphi(\bar{a}) \implies A \models \neg\varphi(\bar{a}).$$

But then

$$A \models \neg\varphi(\bar{a}) \implies A \not\models \forall \bar{x}\ \varphi(\bar{x}) \implies A \not\models \mathrm{DLO}_\forall.$$

By Lemma 6, A does not embed into any model of DLO. But finite linear orders embed into $(\mathbb{Q}, \leq) \models \mathrm{DLO}$, a contradiction.

Here is another useful variant of Theorem 9:

Theorem 11. *Let T be a theory. The following are equivalent:*

1. *T can be axiomatized by universal sentences.*

2. *If $N \models T$ and M is a substructure of N, then $M \models T$.*

Proof. (1) $\implies$ (2): Proposition 4.

(2) $\implies$ (1): Assume (2). We claim that T and $T_\forall$ are logically equivalent. Certainly $M \models T \implies M \models T_\forall$. Conversely, suppose $M \models T_\forall$. By Lemma 7, there is an extension $N \supseteq M$ with $N \models T$. By (2), $M \models T$. So T is axiomatized by the universal theory $T_\forall$. $\square$

Remark 12. An $\forall\exists$-sentence is a sentence of the form $\forall \bar{x}\ \exists \bar{y}\ \varphi(\bar{x}, \bar{y})$ where φ is quantifier-free. The following fact is similar to Theorem 11 and worth knowing. (But the proof is a little harder.)

Fact 13. *If T is a theory, the following are equivalent:*

1. *T can be axiomatized by $\forall\exists$-sentences.*
2. *If $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ is an increasing chain of models of T, then $\bigcup_{i=1}^{\infty} M_i$ is a model of T.*

2 The fundamental order and heirs

Recall the structures (M, dp) from the first lecture on February 24: if $M \models T$ and $p \in S_n(M)$, then (M, dp) is the expansion of M by new symbols $d\varphi$ for each L-formula $\varphi(\bar{x}, \bar{y})$, where

$$d\varphi = \{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}.$$

You can check that the following equivalences hold in (M, dp) for formulas $\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})$:

$$(d\varphi) \wedge (d\psi) \iff d(\varphi \wedge \psi) \tag{1}$$
$$(d\varphi) \vee (d\psi) \iff d(\varphi \vee \psi) \tag{2}$$
$$\neg(d\varphi) \iff d(\neg\varphi) \tag{3}$$
$$\varphi \iff d\varphi \qquad \text{(when } \bar{x} \text{ has length 0.)} \tag{4}$$

Let L_d be the language of structures like $(M, dp), (N, dq)$. Using (1)–(4), one can see the following:

Fact 14. *Any quantifier-free L_d-formula $\psi(\bar{y})$ is equivalent to $d\varphi(\bar{y})$ for some L-formula $\varphi(\bar{x}, \bar{y})$.*

Also, Equation (4) means we secretly added a relation symbol for every L-formula $\varphi(\bar{y})$ on M, and so

Fact 15. *Any L-formula is equivalent to a quantifier-free L_d-formula.*

Combining these facts and the previous section, we can get a clearer proof of something mentioned in class:

Proposition 16. *If $p \in S_n(M)$ and $q \in S_n(N)$ and $[q] \leq [p]$, then there is an ultrapower $N^{\mathcal{U}}$ and an elementary embedding $f : M \to N^{\mathcal{U}}$ making $q^{\mathcal{U}}$ an extension of p.*

Proof.

Claim. (M, dp) satisfies all the universal L_d-sentences that (N, dq) satisfies.

Proof. Suppose $(N, dq) \models \forall \bar{y}\ \varphi(\bar{y})$ for some quantifier-free L_d-formula $\varphi(\bar{y})$. By Fact 14, $\varphi(\bar{y})$ is equivalent to $d\theta$ for some L-formula $\theta(\bar{x}, \bar{y})$. Then

$$\begin{gathered}(N, dq) \models \forall \bar{y}\ d\theta(\bar{y}) \\ \forall \bar{b} \in N : \theta(\bar{x}, \bar{b}) \in q(\bar{x}) \\ \forall \bar{b} \in N : (\neg\theta(\bar{x}, \bar{b})) \notin q(\bar{x}).\end{gathered}$$

So q omits the formula $\neg\theta(\bar{x}, \bar{y})$. As $[q] \leq [p]$, we see that p omits the formula $\neg\theta(\bar{x}, \bar{y})$. Then

$$\begin{gathered}\forall \bar{b} \in M : (\neg\theta(\bar{x}, \bar{b})) \notin p(\bar{x}) \\ \forall \bar{b} \in M : \theta(\bar{x}, \bar{b}) \in p(\bar{x}) \\ (M, dp) \models \forall \bar{y}\ d\theta(\bar{y}) \\ (M, dp) \models \forall \bar{y}\ \varphi(\bar{y}).\end{gathered}$$

$\square_{\text{Claim}}$

By Fact 8(2), there is an ultrafilter $\mathcal{U}$ and embedding $(M, dp) \to (N, dq)^{\mathcal{U}} = (N^{\mathcal{U}}, dq^{\mathcal{U}})$. The embedding $M \to N^{\mathcal{U}}$ preserves all quantifier-free L_d-formulas, so it preserves *all* L-formulas by Fact 15. That is, $M \to N^{\mathcal{U}}$ is an elementary embedding. The fact that $(M, dp) \to (N^{\mathcal{U}}, dq^{\mathcal{U}})$ is an embedding implies that $q^{\mathcal{U}}$ extends p. $\square$

What about heirs?

Proposition 17. *Suppose $N \succeq M$ and $q \in S_n(N)$ is an heir of $p \in S_n(M)$. Then there is an ultrapower $M^{\mathcal{U}}$ and an elementary embedding $N \to M^{\mathcal{U}}$ over M, making $p^{\mathcal{U}}$ extend q.*

Here, "over M" means that $N \to M^{\mathcal{U}}$ fixes the elements of M. If we're willing to move $M^{\mathcal{U}}$ by an isomorphism, this lets us arrange

$$\begin{gathered}M \preceq N \preceq M^{\mathcal{U}} \\ p \subseteq q \subseteq p^{\mathcal{U}}.\end{gathered}$$

Proof. Note that the following are equivalent:

1. q is an heir of p.
2. If $\varphi(\bar{x}, \bar{y})$ is an $L(M)$-formula and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$ for some $\bar{x} \in N$, then there is $\bar{b}' \in M$ with $\varphi(\bar{x}, \bar{b}') \in p(\bar{x})$.
3. If we expand to the language $L(M)$, then $[p] \leq [q]$ in the fundamental order.

Expand to $L(M)$. Then $[p] \leq [q]$. By Proposition 16, there is an ultrapower $M^{\mathcal{U}}$ and an elementary embedding $N \to M^{\mathcal{U}}$ making $p^{\mathcal{U}}$ extend q. This is an elementary embedding *of $L(M)$-structures*, so it fixes the elements of M. $\square$