Set Theory

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1 Ordinal Numbers

1.1 Linear and Partial Ordering

Definition 1.1. A binary relation $<$ on a set P is a **partial ordering** of P if

1. $p \not< p$ for any $p \in P$
2. if $p < q$ and $q < r$ then $p < r$

$(P, <)$ is called a **partially ordered set**. A partial ordering $<$ of P is a **linear ordering** if moreover

3. $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$

   If $<$ is a partial ordering, then $\leq$ is also a partial ordering

if $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is **order-preserving** if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called **increasing**

1.2 Well-Ordering

Definition 1.2. A linear ordering $<$ of a set P is a **well-ordering** if every nonempty subset of P has a least element

Lemma 1.3. *If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction □

Corollary 1.4. *The only automorphism of a well-ordered set is the identity*

Proof. By Lemma 1.3, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x □

Corollary 1.5. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique*

if W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an **initial segment** of W

Lemma 1.6. *No well-ordered set is isomorphic to an initial segment of itself*

Proof. If $\operatorname{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 1.3 □

Theorem 1.7. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds*

1. *W_1 is isomorphic to W_2*
2. *W_1 is isomorphic to an initial segment of W_2*
3. *W_2 is isomorphic to an initial segment of W_1*

Proof. For $u \in W_i$, $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}$$

Using Lemma 1.6, f is a injective: if $f(x_1) = f(x_2) = y$, then $W_1(x_1) \cong W_2(y) \cong W_1(x_2)$, and $x_1 < x_2$ or $x_2 < x_1$ fail. If h is an isomorphism between $W_1(x)$ and $W_2(y)$, and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case 1 holds

if $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. Thus if $\mathrm{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$, we have $\mathrm{ran}(f) = W_2(y_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where x_0=the least element of $W_1 - \mathrm{dom}(f)$ □

if W_1 and W_2 are isomorphic, we say that they have the same **order-type**.

1.3 Ordinal Numbers

Definition 1.8. A set T is **transitive** if every element of T is a subset of T

Definition 1.9. A set is an **ordinal number** (an **ordinal**) if it is transitive and well-ordered by $\in$

Define

$$\alpha < \beta \quad \text{iff} \quad \alpha \in \beta$$

Lemma 1.10.

1. *$0 = \emptyset$ is an ordinal*
2. *if α is an ordinal and $\beta \in \alpha$, then β is an ordinal*
3. *if $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$*
4. *if α, β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$*

Proof. 1,2 by definition

3. if $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ: for $\eta \in \alpha$, $\eta \neq \gamma$ and $\gamma \notin \eta$, hence $\eta \in \gamma$ since ordinals are well-ordered by $\in$. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.

4. $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$ and $\gamma \in \beta$, by 3. Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a **strict** ordering of α)

$\square$

Using Lemma 1.10 one gets the followings

1. $<$ is a linear ordering of the class Ord
2. for each α, $\alpha = \{\beta : \beta < \alpha\}$
3. if C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$
4. if X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$
5. for every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$. In view of 4, the class Ord is a proper class; otherwise consider $\sup \mathrm{Ord} + 1$

Theorem 1.11. *Every well-ordered set is isomorphic to a unique ordinal number*

Proof. The uniqueness follows from Lemma 1.6: suppose $\alpha \cong \beta$ and $\alpha \neq \beta$. As $\alpha \neq \beta$, either $\alpha \in \beta$ or $\beta \in \alpha$, thus α is isomorphic to an initial segment of β or vice versa. But by Lemma 1.6, we get a contradiction.

Given a well-ordered set W, define $F(x) = \alpha$ is α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exists). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ $\square$

0 is a limit ordinal and define $\sup \emptyset = 0$

Definition 1.12 (Natural Numbers)**.** We denote the least nonzero limit ordinal ω (or $\mathbb{N}$). The ordinals less than ω are call **finite ordinals**, or **natural numbers**

1.4 Induction and Recursion

Theorem 1.13 (Transfinite Induction)**.** *Let C be a class of ordinals and assume that*

1. $0 \in C$

2. if $\alpha \in C$, then $\alpha + 1 \in C$

3. *if α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$*

Then C is the class of all ordinals

Proof. Otherwise, let α be the least $\alpha \notin C$ and apply 1,2 and 3. $\square$

A function whose domain is the set $\mathbb{N}$ is called an (**infinite**) **sequence** (A **sequence in** X is a function $f : \mathbb{N} \to X$). The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

A **finite sequence** is a function s s.t. $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \mathbb{N}$; then s is a **sequence of length** n

A **transfinite sequence** is a function whose domain is an ordinal

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α**-sequence** or a **sequence of length** α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an **enumeration** of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}$$

Sometimes we call a "sequence"

$$\langle a_\alpha : \alpha \in \mathrm{Ord} \rangle$$

a function (a proper class) on Ord

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequence), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

s.t.

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$

Theorem 1.14 (Transfinite Recursion)**.** *Let G be a function (on V), then* (1) *below defines a unique function F on Ord s.t.*

$$F(\alpha) = G(F \restriction \alpha)$$

for each α

In other words, if we let $a_\alpha = F(\alpha)$, then for each α

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

(Note that we tacitly use Replacement: $F \restriction \alpha$ is a set for each α)

Corollary 1.15. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ s.t.* $\operatorname{ran}(G) \subset X$ *there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta \rangle$ in X s.t. $a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$ for every $\alpha < \theta$*

Proof. Let

$$F(\alpha) = x \leftrightarrow \text{there is a sequence } \langle a_\xi : \xi < \alpha \rangle \text{ s.t.:} \tag{1}$$

$$1.\ (\forall \xi < \alpha) a_\xi = G(\langle a_\eta : \eta < \xi \rangle)$$

$$2.\ x = G(\langle a_\xi : \xi < \alpha \rangle)$$

For every α, if there is an α-sequence that satisfies 1, then such a sequence is unique: if $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ are two α-sequences satisfying 1, one shows $a_\xi = b_\xi$ by induction on ξ.Thus $F(\alpha)$ is determined uniquely by 2, and therefore F is a function.

it follows, again by induction, that for each α there is an α-sequence that satisfies 1 (at limit steps, we use Replacement to get the α-sequence as the union of all the ξ-sequences, $\xi < \alpha$). Thus F is defined for all $\alpha \in$ Ord. It obviously satisfies

$$F(\alpha) = G(F \restriction \alpha)$$

If F' is any function on Ord that satisfies

$$F'(\alpha) = G(F' \restriction \alpha)$$

then it follows by induction that $F'(\alpha) = F(\alpha)$ for all α □

Definition 1.16. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a **nondecreasing** sequence of ordinals. We define the **limit** of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in \text{Ord} \rangle$ is **normal** if it is increasing and **continuous**, i.e., for every limit α,$\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$

1.5 Ordinal Arithmetic

Definition 1.17 (Addition)**.** For all ordinal numbers α

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ for all β
3. $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$

Definition 1.18 (Multiplication)**.** For all ordinal numbers α

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β
3. $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$

Definition 1.19 (Exponentiation)**.** For all ordinal numbers α

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β
3. $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$

Lemma 1.20. *For all ordinals α, β and γ*

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Proof. Induction on γ □

Neither + nor $\cdot$ are commutative:

$$1 + \omega = \omega \neq \omega + 1, \quad 2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$$

Definition 1.21. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The **sum** of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ iff

1. $x, y \in A$ and $x <_A y$, or
2. $x, y \in B$ and $x <_B y$, or
3. $x \in A$ and $y \in B$

Definition 1.22. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The **product** of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1, b_1) < (a_2, b_2) \text{ iff either } b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2)$$

Lemma 1.23. *For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are isomorphic to the sum and product of α and β*

Proof. We can define $S(\alpha, \beta) = \{(0, a) : a \in \alpha\} \cup \{(1, b) \in \beta\}$
if $\beta = 0$, then $S(\alpha, \beta) = \alpha$
if $\beta = \eta + 1$, then $S(\alpha, \beta) = S(\alpha, \eta) \cup \{(1, \eta)\}$ □

Lemma 1.24.

1. *if $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$*
2. *if $\alpha < \beta$ then there exists a unique δ s.t. $\alpha + \delta = \beta$*
3. *if $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$*
4. *if $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ s.t. $\gamma = \alpha \cdot \beta + \rho$*
5. *if $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$*

Proof.

1. induction on γ
2. let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$; δ is unique by 1
3. γ
4. let β be the greatest ordinal s.t. $\alpha \cdot \beta \leq \gamma$
5. γ

□

Theorem 1.25 (Cantor's Normal Form Theorem)**.** *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers

Proof. By induction on α. For $\alpha = 1$, we have $1 = \omega^0 \cdot 1$; for arbitrary $\alpha > 0$, let β be the greatest ordinal s.t. $\omega^\beta \leq \alpha$. By Lemma 1.24 (4) there exists a unique δ and a unique $\rho < \omega^\beta$ s.t. $\alpha = \omega^\beta \cdot \delta + \rho$; this δ must necessarily be finite □

1.6 Well-Founded Relations

A binary relation E on a set P is **well-founded** if every nonempty $X \subset P$ has an E**-minimal** element, that is $a \in X$ s.t. there is no $x \in X$ with xEa

Given a well-founded relation E on a set P, we can define the **height** of E, and assign to each $x \in P$ an ordinal number, the **rank** of x in E

Theorem 1.26. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals s.t. for all $x \in P$*

$$\rho(x) = \sup\{\rho(y) + 1 : yEx\}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the **height** of E

Proof. By induction, let

$$\begin{aligned} &P_0 = \emptyset \\ &P_{\alpha+1} = \{x \in P : \forall y(yEx \to y \in P_\alpha)\} \\ &P_\alpha = \bigcup_{\xi<\alpha} P_\xi \quad \text{if } \alpha \text{ is a limit ordinal} \end{aligned}$$

Let θ be the least ordinal s.t. $P_{\theta+1} = P_\theta$ (such θ exists by Replacement, θ is at least α i guess). First $P_\alpha \subset P_{\alpha+1}$ for each α. Thus $P_0 \subset P_1 \subset \cdots \subset P_\theta$. We claim that $P_\theta = P$. Otherwise, let a be an E-minimal element of $P - P_\theta$. It follows that each xEa is an P_θ, and so $a \in P_{\theta+1}$, a contradiction. Now we define $\rho(x)$ as the least α s.t. $x \in P_{\alpha+1}$. The ordinal θ is the height of E.

Uniqueness: let ρ' be another function and consider an E-minimal element of the set $\{x \in P : \rho(x) \neq \rho'(x)\}$. □

2 Cardinal Numbers

2.1 Cardinality

Two sets X, Y have the same **cardinality**

$$|X| = |Y|$$

if there exists a one-to-one mapping of X onto Y

$$|X| \leq |Y|$$

if there exists a one-to-one mapping of X into Y.

Theorem 2.1 (Cantor). *For every set X, $|X| < |P(X)|$*

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ iff $z \notin Y$. Thus f is not a function of X onto $P(X)$. Hence $|P(X)| \neq |X|$

The function $f(x) = \{x\}$ is the required one $\square$

Theorem 2.2 (Cantor-Bernstein). *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$*

Proof. From a nice note

We will write $X \sim Y$ to denote the existence of a bijection from X to Y.

Given injections $f : A \to B$ and $g : B \to A$. Let

$$\begin{array}{ll}
A_0 = A & B_0 = B \\
A_1 = g(B_0) & B_1 = f(A_0) \\
A_2 = g(B_1) & B_2 = f(A_1) \\
\vdots & \vdots \\
A_n = g(B_{n-1}) & B_n = f(A_{n-1}) \\
\vdots & \vdots
\end{array}$$

Then

$$\begin{aligned}
A = A_0 \sim B_1 \sim A_2 \sim B_3 \sim A_4 \cdots \\
B = B_0 \sim A_1 \sim B_2 \sim A_3 \sim B_4 \sim \cdots
\end{aligned}$$

and

$$\begin{aligned}
A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \\
B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots
\end{aligned}$$

If $A_n = A_{n+1}$, then $A \sim B$, hence we assume $A_n \supsetneq A_{n+1}$

Problem here now is that $X_1 \sim Y_1$ and $X_2 \sim Y_2$ do **not** imply $X_1 \cup X_2 \sim Y_1 \cup Y_2$ and therefore $A \cup A_1 \sim B_1 \cup B$ $\square$

Lemma 2.3. *Suppose we have sets $\{X_i\}$ and $\{Y_i\}$ satisfying $X_i \sim Y_i$ for all i. If all the X_i are pairwise disjoint, and all the Y_i are pairwise disjoint, then*

$$\bigcup_i X_i \sim \bigcup_i Y_i$$

Continuation of proof 2.2. Hence for each n, set $A_n^* = A_n - A_{n+1}$. By our assumption, all A_n^* are nonempty, moreover they are pairwise disjoint. Also we get

$$A^* = A_0^* \sim B_1^* \sim A_2^* \sim B_3^* \sim A_4^* \cdots$$
$$B^* = B_0^* \sim A_1^* \sim B_2^* \sim A_3^* \sim B_4^* \sim \cdots$$

Hence we get

$$\tilde{A} := \bigcup_{n\geq 0} A_n^* \sim \tilde{B} := \bigcup_{n\geq 0} B_n^*$$

Let $\bar{A} = \bigcap_{n\geq 0} A_n$ and $\bar{b} = \bigcap_{n\geq 0} B_n$

Claim $A = \bar{A} \cup \tilde{A}$ is a partition of A, and $B = \bar{B} \cup \tilde{B}$ is a partition of B

Now it remains to show that $\bar{A} \sim \bar{B}$, which is immediate as $f(\bar{A}) = \bar{B}$ and $g(\bar{B}) = \bar{A}$ ☐

The arithmetic operations on cardinals are defined as follows

$$\begin{aligned} &\kappa + \lambda = |A \cup B| && \text{where } |A| = \kappa, |B| = \lambda, \text{ and } A, B \text{ are disjoint} \\ &\kappa \cdot \lambda = |A \times B| && \text{where } |A| = \kappa, |B| = \lambda \\ &\kappa^\lambda = \left|A^B\right| && \text{where } |A| = \kappa, |B| = \lambda \end{aligned}$$

Lemma 2.4. *if $|A| = \kappa$, then $|P(A)| = 2^\kappa$*

Proof. For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases}$$

The mapping $f : X \to \chi_X$ is.a one-to-one correspondence between $P(A)$ and $\{0,1\}^A$ ☐

Thus Cantor's Theorem 2.1 can be formulated as

$\kappa < 2^\kappa$ *for every cardinal* κ

Very useful link

Proposition 2.5. *1. $+$ and $\cdot$ is associative, commutative and distributive*

2. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

3. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$

4. $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$

5. *if* $\kappa \leq \lambda$*, then* $\kappa^\mu \leq \lambda^\mu$

6. *if* $0 < \lambda \leq \mu$*, then* $\kappa^\lambda \leq \kappa^\mu$

7. $\kappa^0 = 1$*;* $1^\kappa = 1$*;* $0^\kappa = 0$ *if* $\kappa > 0$

Proof. 1. commutativity of + follows from $A \cup B = B \cup A$, and so is the commutativity of $\cdot$. Similar for associativity

3. Given $f : A \cup B \to C$, we get $f{\restriction}A$ and $f{\restriction}B$. Therefore we have a map $f \mapsto (f{\restriction}A, f{\restriction}B)$

6. let $|A| = \kappa$, $|B| = \lambda$, $|C| = \mu$. Given injection $f : B \to C$, for each $h : B \to A$ we associate a $g(y) : C \to A$ by $g(f(x)) = h(x)$ if $y \in f(B)$, otherwise $g(y)$ can be anything.

□

2.2 Alephs

An ordinal α is called a **cardinal number** if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$.

If W is a well-ordered set, then there exists an ordinal α s.t. $|W| = |\alpha|$. Thus we let

$$|W| = \text{ the least ordinal s.t. } |W| = |\alpha|$$

Every natural number is a cardinal

The infinite ordinal numbers that are cardinals are called **alephs**

Lemma 2.6. *1. For every* α *there is a cardinal number greater than* α

2. if X *is a set of cardinals, then* $\sup X$ *is a cardinal*

for every α, let α^+ be the least cardinal number greater than α, the **cardinal successor** of α

Proof. 1. for any set X, let $h(X)$=the least α s.t. there is no one-to-one function of α into X. There is only a set of possible well-orderings of subsets of X. (But the collection of ordinals is a class) Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

if α is an ordinal, then $|\alpha| < |h(\alpha)|$

2. let $\alpha = \sup X$. if f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be s.t. $\beta < \kappa \leq \alpha$. Then $|\kappa| = |\{f(\xi) : \xi < \kappa\}| \leq \beta$, a contradiction. Thus α is a cardinal

$\square$

Use Lemma 2.6 we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type

$$\begin{aligned}
&\aleph_0 = \omega_0 = \omega \\
&\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+ \\
&\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal}
\end{aligned}$$

Theorem 2.7. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

2.3 The Canonical Well-Ordering of $\alpha \times \alpha$

relevant reading

We define a well-ordering of the class Ord $\times$ Ord of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the **canonical well-ordering** of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class Ord

We define

$$\begin{aligned}
(\alpha, \beta) < (\gamma, \delta) \leftrightarrow &\text{ either } \max\{\alpha, \beta\} < \max\{\gamma, \delta\} \\
&\text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \alpha < \gamma \\
&\text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\}, \alpha = \gamma \text{ and } \beta < \delta
\end{aligned}$$

If $X \subset \text{Ord} \times \text{Ord}$ is nonempty, then X has a least element. For each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha, \beta) = \text{ the order-type of the set } \{(\xi, \eta) : (\xi, \eta) < (\alpha, \beta)\}$$

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(\alpha, \beta) < (\gamma, \delta) \quad \text{iff} \quad \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Note that $\Gamma(\omega \times \omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha \times \alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is also continuous, and so $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrary large α.

Proof of Theorem 2.7. We will show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal s.t. $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be s.t. $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ s.t. $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of Ord $\times$ Ord in the canonical well-ordering and contains (β, γ), we have $\Gamma(\delta \times \delta) \supset \omega_\alpha$, and so $|\delta \times \delta| \geq \aleph_\alpha$. However $|\delta \times \delta| = |\delta| \cdot |\delta|$, and by the minimality of α, $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$. □

As a corollary we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$$

2.4 Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is **cofinal** in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is **cofinal** in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the **cofinality** of α is

$$\begin{aligned} \operatorname{cf} \alpha = &\text{ the least limit ordinal } \beta \text{ s.t. there is an increasing} \\ &\beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha \end{aligned}$$

Lemma 2.8. $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$

Lemma 2.9. *Let $\alpha > 0$ be a limit ordinal*

1. *if $A \subset \alpha$ and* $\sup A = \alpha$, *then the order-type of A is at least* $\operatorname{cf} \alpha$
2. *if $\beta_0 \leq \beta_1 \leq \ldots \leq \beta_\xi \leq \ldots, \xi < \gamma$, is a nondecreasing γ-sequence of ordinals in α and* $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, *then* $\operatorname{cf} \gamma = \operatorname{cf} \alpha$

Proof.
1. the order-type of A is the length of the increasing enumeration of A which is an increasing sequence with limit α
2. if $\gamma = \lim_{\nu \to \operatorname{cf} \gamma} \xi(\nu)$, then $\alpha = \lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle \beta_{\xi(\nu)} : \nu < \operatorname{cf} \gamma \rangle$ has an increasing subsequence of length $\leq \operatorname{cf} \gamma$, with the same limit. Thus $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$

   To show that $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$, let $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \alpha_\nu$. For each $\nu < \operatorname{cf} \alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota)$, $\iota < \nu$, s.t. $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu \to \operatorname{cf} \alpha} \beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu \to \operatorname{cf} \alpha} \xi(\nu) = \gamma$, and so $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$.
   □

An infinite cardinal $\aleph_\alpha$ is **regular** if $\operatorname{cf} \omega_\alpha = \omega_\alpha$. It is **singular** if $\operatorname{cf} \omega_\alpha < \omega_\alpha$

Lemma 2.10. *For every limit ordinal α, $\operatorname{cf}\alpha$ is a regular cardinal*

Proof. if α is not a cardinal, then by an bijection $f : |\alpha| \sim \alpha$, we get a cofinal sequence in α of length $\leq |\alpha|$, therefore $\operatorname{cf}\alpha < \alpha$

if α is a cardinal, $\operatorname{cf}\alpha = \alpha$ by Lemma 2.9 □

Let κ be a limit ordinal. A subset $X \subset \kappa$ is **bounded** if $\sup X < \kappa$, and **unbounded** if $\sup X = \kappa$

Lemma 2.11. *Let κ be an aleph*

1. *If $X \subset \kappa$ and $|X| < \operatorname{cf}\kappa$ then X is bounded*
2. *If $\lambda < \operatorname{cf}\kappa$ and $f : \lambda \to \kappa$ then the range of f is bounded*

it follows from 1 that every unbounded subset of a regular cardinal has cardinality κ

Proof.

1. Lemma 2.9
2. if $X = \operatorname{ran} f$, then $|X| \leq \lambda$, then use 1.

□

There are arbitrary large singular cardinals. For each α, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality ω

Lemma 2.12. *An infinite cardinal κ is singular iff there exists a cardinal $\lambda < \kappa$ and a family $\{S_\xi : \xi < \lambda\}$ of subsets of κ s.t. $|S_\xi| < \kappa$ for each $\xi < \lambda$, and $\kappa = \bigcup_{\xi<\lambda} S_\xi$. The least cardinal λ satisfies the condition is $\operatorname{cf}\kappa$.*

Proof. If κ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \operatorname{cf}\kappa \rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$.

If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ s.t. $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing, and by the minimality of λ, $\beta_\xi < \kappa$ for all $\xi < \lambda$. If not, then $\beta_\xi = \kappa$ and $\bigcup_{\nu<\xi} S_\nu = \kappa$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf}\kappa \leq \lambda$.

Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: if $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where ξ is the least ξ s.t. $\alpha \in S_\xi$ and γ is the order type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$ □

Theorem 2.13. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$*

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find $f : \operatorname{cf}\kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi \to \operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf}\kappa$, let

$$f(\xi) = \text{ least } \gamma \text{ s.t. } \gamma \neq f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi$$

Such γ exists since $\left|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}\right| \leq \left|\alpha_\xi\right| < \kappa$. Obviously, $f \neq f_\alpha$ for all $\alpha < \kappa$ □

Consequently $\kappa^\lambda > \kappa$ whenever $\lambda \geq \operatorname{cf}\kappa$.

3 Real Numbers

Theorem 3.1 (Cantor)**.** *The set of all real numbers is uncountable*

Proof. Suppose not, let $c_0, c_1, \ldots$ be an enumeration of $\mathbb{R}$

Let $a_0 = c_0$ and $b_0 = c_{k_0}$, where k_0 is the least k s.t. $a_0 < c_k$. For each n, let $a_{n+1} = c_{i_n}$ where i_n is the least i s.t. $a_n < c_i < b_n$ and $b_{n+1} = c_{k_n}$ where k_n is the least k s.t. $a_{n+1} < c_k < b_n$. If we let $a = \sup\{a_n : n \in \mathbb{N}\}$, then $a \neq c_k$ for all k. □

3.1 The Cardinality of the Continuum

Let $\mathfrak{c}$ denote the cardinality of $\mathbb{R}$. As the set $\mathbb{Q}$ of all rational numbers is dense in $\mathbb{R}$, every real number r is equal to $\sup\{q \in \mathbb{Q} : q < r\}$ and because $\mathbb{Q}$ is countable, it follows that $\mathfrak{c} \leq |P(\mathbb{Q})| = 2^{\aleph_0}$

4 Question

2 1.23 2.3