A Course in Model Theory

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October 13, 2021

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1 The Basics

1.1 Structures

Definition 1.1. A **language** L is a set of constants, function symbols and relation symbols

Definition 1.2. Let L be a language. An L**-structure** is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$ where

A	if a non-empty set, the **domain** or **universe** of $\mathfrak{A}$
$z^{\mathfrak{A}} \in A$	if Z is a constant
$Z^{\mathfrak{A}} : A^n \to A$	if Z is an n-ary function symbol
$Z^{\mathfrak{A}} \subseteq A^n$	if Z is an n-ary relation symbol

Definition 1.3. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. A map $h : A \to B$ is called a **homomorphism** if for all $a_1, \dots, a_n \in A$

$$\begin{array}{rcl} h(c^{\mathfrak{A}}) & = & c^{\mathfrak{B}} \\ h(f^{\mathfrak{A}}(a_1, \dots, a_n)) & = & f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \\ R^{\mathfrak{A}}(a_1, \dots, a_n) & \Rightarrow & R^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \end{array}$$

We denote this by

$$h : \mathfrak{A} \to \mathfrak{B}$$

If in addition h is injective and

$$R^{\mathfrak{A}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n))$$

for all $a_1, \dots, a_n \in A$, then h is called an (isomorphic) **embedding**. An **isomorphism** is a surjective embedding

Definition 1.4. We call $\mathfrak{A}$ a **substructure** of $\mathfrak{B}$ if $A \subseteq B$ and if the inclusion map is an embedding from $\mathfrak{A}$ to $\mathfrak{B}$. We denote this by

$$\mathfrak{A} \subseteq \mathfrak{B}$$

We say $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$

Lemma 1.5. *Let $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ be an isomorphism and $\mathfrak{B}$ an extension of $\mathfrak{A}$. Then there exists an extension $\mathfrak{B}'$ of $\mathfrak{A}'$ and an isomorphism $g : \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}'$ extending h*

For any family $\mathfrak{A}_i$ of substructures of $\mathfrak{B}$, the intersection of the A_i is either empty or a substructure of $\mathfrak{B}$. Therefore if S is any non-empty subset of $\mathfrak{B}$, then there exists a smallest substructure $\mathfrak{A} = \langle S \rangle^{\mathfrak{B}}$ which contains S. We call the $\mathfrak{A}$ the substructure **generated** by S

Lemma 1.6. *If $\mathfrak{a} = \langle S \rangle$, then every homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is determined by its values on S*

Definition 1.7. Let $(I, \leq)$ be a **directed partial order**. This means that for all $i, j \in I$ there exists a $k \in I$ s.t. $i \leq k$ and $j \leq k$. A family $(\mathfrak{A}_i)_{i \in I}$ of L-structures is called **directed** if

$$i \leq j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j$$

If I is linearly ordered, we call $(\mathfrak{A}_i)_{i \in I}$ a **chain**

If a structure $\mathfrak{A}_1$ is isomorphic to a substructure $\mathfrak{A}_0$ of itself,

$$h_0 : \mathfrak{A}_0 \xrightarrow{\sim} \mathfrak{A}_1$$

then Lemma 1.5 gives an extension

$$h_1 : \mathfrak{A}_1 \xrightarrow{\sim} \mathfrak{A}_2$$

Continuing in this way we obtain a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$ and an increasing sequence $h_i : \mathfrak{A}_i \xrightarrow{\sim} \mathfrak{A}_{i+1}$ of isomorphism

Lemma 1.8. *Let $(\mathfrak{A}_i)_{i \in I}$ be a directed family of L-structures. Then $A = \bigcup_{i \in I} A_i$ is the universe of a (uniquely determined) L-structure*

$$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$$

which is an extension of all $\mathfrak{A}_i$

A subset K of L is called a **sublanguage**. An L-structure becomes a K-structure, the **reduct**.

$$\mathfrak{A} \restriction K = (A, (Z^{\mathfrak{A}})_{Z \in K})$$

Conversely we call $\mathfrak{A}$ an **expansion** of $\mathfrak{A} \restriction K$.

1. Let $B \subseteq A$, we obtain a new language

$$L(B) = L \cup B$$

and the $L(B)$-structure

$$\mathfrak{A}_B = (\mathfrak{A}, b)_{b \in B}$$

Note that **Aut**$(\mathfrak{A}_B)$ is the group of automorphisms of $\mathfrak{A}$ fixing B elementwise. We denote this group by **Aut**$(\mathfrak{A}/B)$

Let S be a set, which we call the set of sorts. An S-sorted language L is given by a set of constants for each sort in S, and typed function and relations. For any tuple $(s_1, \dots, s_n)$ and $(s_1, \dots, s_n, t)$ there is a set of relation symbols and function symbols respectively. An S-sorted structure is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$, where

A	if a family $(A_s)_{s \in S}$ of non-empty sets
$Z^{\mathfrak{A}} \in A_s$	if Z is a constant of sort $s \in S$
$Z^{\mathfrak{A}} : A_{s_1} \times \dots \times A_{s_n} \to A_t$	if Z is a function symbol of type $(s_1, \dots, s_n, t)$
$Z^{\mathfrak{A}} \subseteq A_{s_1} \times \dots \times A_{s_n}$	if Z is a relation symbol of type $(s_1, \dots, s_n)$

Example 1.1. Consider the two-sorted language L_{Perm} for permutation groups with a sort x for the set and a sort g for the group. The constants and function symbols for L_{Perm} are those of L_{Group} restricted to the sort g and an additional function symbol φ of type (x, g, x). Thus an L_{Perm}-structure (X, G) is given by a set X and an L_{Group}-structure G together with a function $X \times G \to X$

1.2 Language

Lemma 1.9. *Suppose $\overrightarrow{b}$ and $\overrightarrow{c}$ agree on all variables which are free in φ. Then*

$$\mathfrak{A} \vDash \varphi[\overrightarrow{b}] \Leftrightarrow \mathfrak{A} \vDash \varphi[\overrightarrow{c}]$$

We define

$$\mathfrak{A} \vDash \varphi[a_1, \dots, a_n]$$

by $\mathfrak{A} \vDash \varphi[\overrightarrow{b}]$, where $\overrightarrow{b}$ is an assignment satisfying $\overrightarrow{b}(x_i) = a_i$. Because of Lemma 1.9 this is well defined.

Thus $\varphi(x_1, \dots, x_n)$ defines an n-ary relation

$$\varphi(\mathfrak{A}) = \{\bar{a} \mid \mathfrak{A} \vDash \varphi[\bar{a}]\}$$

on A, the **realisation set** of φ. Such realisation sets are called **0-definable subsets** of A^n, or 0-definable relations

Let B be a subset of A. A B**-definable** subset of $\mathfrak{A}$ is a set of the form $\varphi(\mathfrak{A})$ for an $L(B)$-formula $\varphi(x)$. We also say that φ are defined **over** B and that the set $\varphi(\mathfrak{A})$ is defined by φ. We call two formulas **equivalent** if in every structure they define the same set.

Atomic formulas and their negations are called **basic**. Formulas without quantifiers are Boolean combinations of basic formulas. It is convenient to allow the empty conjunction and the empty disjunction. For that we introduce two new formulas: the formula $\top$, which is always true, and the formula $\bot$, which is always false. We define

$$\bigwedge_{i<0} \pi_i = \top$$
$$\bigvee_{i<0} \pi_i = \bot$$

A formula is in **negation normal form** if it is built from basic formulas using $\wedge, \vee, \exists, \forall$

Lemma 1.10. *Every formula can be transformed into an equivalent formula which is in negation normal form*

Proof. Let $\sim$ denote equivalence of formulas. We consider formulas which are built using $\wedge, \vee, \exists, \forall, \neg$ and move the negation symbols in front of atomic formulas using

$$\neg(\varphi \wedge \psi) \sim (\neg\varphi \vee \neg\psi)$$
$$\neg(\varphi \vee \psi) \sim (\neg\varphi \wedge \neg\psi)$$
$$\neg\exists x\varphi \sim \forall x\neg\varphi$$
$$\neg\forall x\varphi \sim \exists x\neg\varphi$$
$$\neg\neg\varphi \sim \varphi$$

□

Definition 1.11. A formula in negation normal form which does not contain any existential quantifier is called **universal**. Formulas in negation normal form without universal quantifiers are called **existential**

Lemma 1.12. *Let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then for all existential formulas $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in A$ we have*

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Rightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

For universal φ, the dual holds

$$\mathfrak{B} \vDash \varphi[h(a_1), \dots, h(a_n)] \Rightarrow \mathfrak{A} \vDash \varphi[a_1, \dots, a_n]$$

Let $\mathfrak{A}$ be an L-structure. The **atomic diagram** of $\mathfrak{A}$ is

$$\mathrm{Diag}(\mathfrak{A}) = \{\varphi \text{ basic } L(A)\text{-sentence} \mid \mathfrak{A}_A \vDash \varphi\}$$

Lemma 1.13. *The models of* $\mathrm{Diag}(\mathfrak{A})$ *are precisely those structures* $(\mathfrak{B}, h(a))_{a\in A}$ *for embeddings* $h : \mathfrak{A} \to \mathfrak{B}$

Proof. The structures $(\mathfrak{B}, h(a))_{a\in A}$ are models of the atomic diagram by Lemma **??**. For the converse, note that a map h is an embedding iff it preserves the validity of all formulas of the form

$$(\neg)x_1 \doteq x_2$$
$$c \doteq x_1$$
$$f(x_1, \dots, x_n) \doteq x_0$$
$$(\neg)R(x_1, \dots, x_n)$$

□

Exercise 1.2.1. Every formula is equivalent to a formula in prenex normal form:

$$Q_1 x_1 \dots Q_n x_n \varphi$$

The Q_i are quantifiers and φ is quantifier-free

Proof.

$(\forall x)\phi \wedge \psi \vDash\!\!\!\Dashv \forall x(\phi \wedge \psi)$ if $\exists x \top$(at least one individual exists)

$(\forall x\phi) \vee \psi \vDash\!\!\!\Dashv \forall x(\phi \vee \psi)$

$(\exists x\phi) \wedge \psi \vDash\!\!\!\Dashv \exists x(\phi \wedge \psi)$

$(\exists x\phi) \vee \psi \vDash\!\!\!\Dashv \exists x(\phi \vee \psi)$ if $\exists x \top$

$\neg\exists x\phi \vDash\!\!\!\Dashv \forall x \neg\phi$

$\neg\forall x\phi \vDash\!\!\!\Dashv \exists x \neg\phi$

$(\forall x\phi) \to \psi \vDash\!\!\!\Dashv \exists x(\phi \to \psi)$ if $\exists x \top$

$(\exists x\phi) \to \psi \vDash\!\!\!\Dashv \forall x(\phi \to \psi)$

$\phi \to (\exists x\psi) \vDash\!\!\!\Dashv \exists x(\phi \to \psi)$ if $\exists x \top$

$\phi \to (\forall x\psi) \vDash\!\!\!\Dashv \forall x(\phi \to \psi)$

□

1.3 Theories

Definition 1.14. An L**-theory** T is a set of L-sentences

A theory which has a model is a **consistent** theory. We call a set Σ of L-formulas **consistent** if there is an L-structure and **an assignment** $\overrightarrow{b}$ **s.t.** $\mathfrak{A} \models \Sigma[\overrightarrow{b}]$ for all $\varphi \in \Sigma$

Lemma 1.15. *Let T be an L-theory and L' be an extension of L. Then T is consistent as an L-theory iff T is consistent as a L'-theory*

Lemma 1.16. *1. If $T \models \varphi$ and $T \models (\varphi \to \psi)$, then $T \models \psi$*

2. If $T \models \varphi(c_1, \dots, c_n)$ and the constants $c_1, \dots, c_n$ occur neither in T nor in $\varphi(x_1, \dots, x_n)$, then $T \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$

Proof. 2. Let $L' = L \setminus \{c_1, \dots, c_n\}$. If the L'-structure is a model of T and $a_1, \dots, a_n$ are arbitrary elements, then $(\mathfrak{A}, a_1, \dots, a_n) \models \varphi(c_1, \dots, c_n)$. This means $\mathfrak{A} \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$.

□

S and T are called **equivalent**, $S \equiv T$, if S and T have the same models

Definition 1.17. A consistent L-theory T is called **complete** if for all L-sentences φ

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi$$

Definition 1.18. For a complete theory T we define

$$|T| = \max(|L|, \aleph_0)$$

The typical example of a complete theory is the theory of a structure $\mathfrak{A}$

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$$

Lemma 1.19. *A consistent theory is complete iff it is maximal consistent, i.e., if it is equivalent to every consistent extension*

Definition 1.20. Two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are called **elementary equivalent**

$$\mathfrak{A} \equiv \mathfrak{B}$$

if they have the same theory

Lemma 1.21. *Let T be a consistent theory. Then the following are equivalent*

1. *T is complete*

2. *All models of T are elemantarily equivalent*

3. *There exists a structure $\mathfrak{A}$ with $T \equiv \mathrm{Th}(\mathfrak{A})$*

Proof. $1 \to 3 \to 2 \to 1$ □

2 Elementary Extensions and Compactness

2.1 Elementary substructures

Let $\mathfrak{A}, \mathfrak{B}$ be two L-structures. A map $h : A \to B$ is called **elementary** if for all $a_1, \dots, a_n \in A$ we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

which is actually saying $(\mathfrak{A}, a)_{a \in A} \equiv (\mathfrak{B}, a)_{a \in A}$. We write

$$h : \mathfrak{A} \xrightarrow{\prec} \mathfrak{B}$$

Lemma 2.1. *The models of* $\mathrm{Th}(\mathfrak{A}_A)$ *are exactly the structures of the form* $(\mathfrak{B}, h(a))_{a \in A}$ *for elementary embeddings* $h : \mathfrak{A} \xrightarrow{\prec} \mathfrak{B}$

We call $\mathrm{Th}(\mathfrak{A}_A)$ the **elemantary diagram** of $\mathfrak{A}$

A substructure $\mathfrak{A}$ of $\mathfrak{B}$ is called **elementary** if the inclusion map is elementary. In this case we write

$$\mathfrak{A} \prec \mathfrak{B}$$

Theorem 2.2 (Tarski's Test)**.** *Let $\mathfrak{B}$ be an L-structure and A a subset of B. Then A is the universe of an elementary substructure iff every $L(A)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ can be satisfied by an element of A*

Proof. If $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \models \exists x \varphi(x)$, we also have $\mathfrak{A} \models \exists x \varphi(x)$ and there exists $a \in A$ s.t. $\mathfrak{A} \models \varphi(a)$. Thus $\mathfrak{B} \models \varphi(a)$

Conversely, suppose that the condition of Tarski'test is satisfied. First we show that A is the universe of a substructure $\mathfrak{A}$. The $L(A)$-formula $x \doteq x$ is satisfiable in $\mathfrak{A}$, so A is not empty. If $f \in L$ is an n-ary function symbol $(n \geq 0)$ and $a_1, \dots, a_n$ is from A, we consider the formula

$$\varphi(x) = f(a_1, \dots, a_n) \doteq x$$

Since $\varphi(x)$ is always satisfied by an element of A, it follows that A is closed under $f^{\mathcal{B}}$

Now we show, by induction on ψ, that

$$\mathfrak{A} \models \psi \Leftrightarrow \mathfrak{B} \models \psi$$

for all $L(A)$-sentences ψ.

For $\psi = \exists x \varphi(x)$. If ψ holds in $\mathfrak{A}$, there exists $a \in A$ s.t. $\mathfrak{A} \models \varphi(a)$. The induction hypothesis yields $\mathfrak{B} \models \varphi(x)$, thus $\mathfrak{B} \models \psi$. For the converse suppose ψ holds in $\mathfrak{B}$. Then $\varphi(x)$ is satisfied in $\mathcal{B}$ and by Tarski's test we find $a \in A$ s.t. $\mathfrak{B} \models \varphi(a)$. By induction $\mathfrak{A} \models \varphi(a)$ and $\mathfrak{A} \models \psi$ $\square$

We use Tarski's Test to construct small elementary substructures

Corollary 2.3. *Suppose S is a subset of the L-structure $\mathfrak{B}$. Then $\mathfrak{B}$ has a elementary substructure $\mathfrak{A}$ containing S and of cardinality at most*

$$\max(|S|, |L|, \aleph_0)$$

Proof. We construct A as the union of an ascending sequence $S_0 \subseteq S_1 \subseteq ...$ of subsets of B. We start with $S_0 = S$. If S_i is already defined, we choose an element $a_\varphi \in B$ for every $L(S_i)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ and define S_{i+1} to be S_i together with these a_φ.

An L-formula is a finite sequence of symbols from L, auxiliary symbols and logical symbols. These are $|L| + \aleph_0 = \max(|L|, \aleph_0)$ many symbols and there are exactlymax$(|L|, \aleph_0)$ many L-formulas

Let $\kappa = \max(|S|, |L|, \aleph_0)$. There are κ many $L(S)$-formulas: therefore $|S_1| \leq \kappa$. Inductively it follows for every i that $|S_i| \leq \kappa$. Finally we have $|A| \leq \kappa \cdot \aleph_0 = \kappa$ $\square$

A directed family $(\mathfrak{A}_i)_{i \in I}$ of structures is **elementary** if $\mathfrak{A}_i \prec \mathfrak{A}_j$ for all $i \leq j$

Theorem 2.4 (Tarski's Chain Lemma)**.** *The union of an elementary directed family is an elementary extension of all its members*

Proof. Let $\mathfrak{A} = \bigcup_{i \in I} (\mathfrak{A}_i)_{i \in I}$. We prove by induction on $\varphi(\bar{x})$ that for all i and $\bar{a} \in \mathfrak{A}_i$

$$\mathfrak{A}_i \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \models \varphi(\bar{a})$$

$\square$

Exercise 2.1.1. Let $\mathfrak{A}$ be an L-structure and $(\mathfrak{A}_i)_{i \in I}$ a chain of elementary substructures of $\mathfrak{A}$. Show that $\bigcup_{i \in I} A_i$ is an elementary substructure of $\mathfrak{A}$.

Exercise 2.1.2. Consider a class $\mathcal{C}$ of L-structures. Prove

1. Let $\text{Th}(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \models \varphi \text{ for all } \mathfrak{A} \in \mathcal{C}\}$ be the **theory of** $\mathcal{C}$. Then $\mathfrak{M}$ is a model of $\text{Th}(C)$ iff $\mathfrak{M}$ is elementary equivalent to an ultraproduct of elements of $\mathcal{C}$

2. Show that $\mathcal{C}$ is an elementary class iff $\mathcal{C}$ is closed under ultraproduct and elementary equivalence

3. Assume that $\mathcal{C}$ is a class of finite structures containing only finitely many structures of size n for each $n \in \omega$. Then the infinite models of $\text{Th}(\mathcal{C})$ are exactly the models of

$$\text{Th}_a(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \models \varphi \text{ for all but finitely many } \mathfrak{A} \in \mathcal{C}\}$$

Proof. Chang&Keisler p220 □

2.2 The Compactness Theorem

We call a theory T **finitely satisfiable** if every finite subset of T is consistent

Theorem 2.5 (Compactness Theorem)**.** *Finitely satisfiable theories are consistent*

Let L be a language and C a set of new constants. An $L(C)$-theory T' is called a **Henkin theory** if for every $L(C)$-formula $\varphi(x)$ there is a constant $c \in C$ s.t.

$$\exists x \varphi(x) \to \varphi(c) \in T'$$

The elements of C are called **Henkin constants** of T'

An L-theory T is **finitely complete** if it is finitely satisfiable and if every L-sentence φ satisfies $\varphi \in T$ or $\neg\varphi \in T$

Lemma 2.6. *Every finitely satisfiable L-theory T can be extended to a finitely complete Henkin Theory T^**

Note that conversely the lemma follows directly from the Compactness Theorem. Choose a model $\mathfrak{A}$ of T. Then $\text{Th}(\mathfrak{A}_A)$ is a finitely complete Henkin theory with A as a set of Henkin constants

Proof. We define an increasing sequence $\emptyset = C_0 \subseteq C_1 \subseteq \cdots$ of new constants by assigning to every $L(C_i)$-formula $\varphi(x)$ a constant $c_{\varphi(x)}$ and

$$C_{i+1} = \{c_{\varphi(x)} \mid \varphi(x) \text{ a } L(C_i)\text{-formula}\}$$

Let C be the union of the C_i and T^H the set of all Henkin axioms

$$\exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)})$$

for $L(C)$-formulas $\varphi(x)$. It is easy to see that one can expand every L-structure to a model of T^H. Hence $T \cup T^H$ is a finitely satisfiable Henkin theory. Using the fact that the union of a chain of finitely satisfiable theories is also finite satisfiable, we can apply Zorn's Lemma and get a maximal finitely satisfiable $L(C)$-theory T^* which contains $T \cup T^H$. As in Lemma 1.19 we show that T^* is finitely complete: if neither φ nor $\neg\varphi$ belongs to T^*, neither $T^* \cup \{\varphi\}$ nor $T^* \cup \{\neg\varphi\}$ would be finitely satisfiable. Hence there would be a finite subset Δ of T^* which would be consistent neither with φ nor with $\neg\varphi$. Then Δ itself would be inconsistent and T^* would not be finite satisfiable. This proves the lemma. $\square$

Lemma 2.7. *Every finitely satisfiable L-theory T can be extended to a finitely complete Henkin theory T^**

Lemma 2.8. *Every finitely complete Henkin theory T^* has a model $\mathfrak{A}$ (unique up to isomorphism) consisting of constants; i.e.,*

$$(\mathfrak{A}, a_c)_{c\in C} \models T^*$$

with $A = \{a_c \mid c \in C\}$

Proof. Since T^* is finite complete, every sentence which follows from a finite subset of T^* belongs to T^*

Define for $c, d \in C$

$$c \simeq d \Leftrightarrow c \dot{=} d \in T^*$$

$\simeq$ is an equivalence relation. We denote the equivalence class of c by a_c, and set

$$A = \{a_c \mid c \in C\}$$

We expand A to an L-structure $\mathfrak{A}$ by defining

$$R^{\mathfrak{A}}(a_{c_1}, \dots, a_{c_n}) \Leftrightarrow R(c_1, \dots, c_n) \in T^* \qquad (\star)$$

$$f^{\mathfrak{A}}(a_{c_1}, \dots, a_{c_n}) \Leftrightarrow f(c_1, \dots, c_n) \dot{=} c_0 \in T^* \qquad (\star\star)$$

We have to show that this is well-defined. For $(\star)$ we have to show that

$$a_{c_1} = a_{d_1}, \dots, a_{c_n} = a_{d_n}, R(c_1, \dots, c_n) \in T^*$$

implies $R(d_1, \dots, d_n) \in T^*$, which is obvious.

For $(\star\star)$, we have to show that for all $c_1, \dots, c_n$ there exists c_0 with $f(c_1, \dots, c_n) \doteq c_0 \in T^*$.

Let $\mathfrak{A}^*$ be the $L(C)$-structure $(\mathfrak{A}, a_c)_{c \in C}$. We show by induction on the complexity of φ that for every $L(C)$-sentence φ

$$\mathfrak{A}^* \models \varphi \Leftrightarrow \varphi \in T^*$$

$\square$

Corollary 2.9. *We have $T \models \varphi$ iff $\Delta \models \varphi$ for a finite subset Δ of T*

Corollary 2.10. *A set of formulas $\Sigma(x_1, \dots, x_n)$ is consistent with T if and only if every finite subset of Σ is consistent with T*

Proof. Introduce new constants $c_1, \dots, c_n$. Then Σ is consistent with T is and only if $T \cup \Sigma(c_1, \dots, c_n)$ is consistent. Now apply the Compactness Theorem $\square$

Definition 2.11. Let $\mathfrak{A}$ be an L-structure and $B \subseteq A$. Then $a \in A$ **realises** a set of $L(B)$-formulas $\Sigma(x)$ if a satisfied all formulas from Σ. We write

$$\mathfrak{A} \models \Sigma(a)$$

We call $\Sigma(x)$ **finitely satisfiable** in $\mathfrak{A}$ if every finite subset of Σ is realised in $\mathfrak{A}$

Lemma 2.12. *The set $\Sigma(x)$ is finitely satisfiable in $\mathfrak{A}$ iff there is an elementary extension of $\mathfrak{A}$ in which $\Sigma(x)$ is realised*

Proof. By Lemma 2.1 Σ is realised in an elementary extension of $\mathfrak{A}$ iff Σ is consistent with Th($\mathfrak{A}_A$). So the lemma follows from the observation that a finite set of $L(A)$-formulas is consistent with Th($\mathfrak{A}_A$) iff it is realised in $\mathfrak{A}$ $\square$

Definition 2.13. Let $\mathfrak{A}$ be an L-structure and B a subset of A. A set $p(x)$ of $L(B)$-formulas is a **type** over B if $p(x)$ is maximal finitely satisfiable in $\mathfrak{A}$ (satisfiable in an elementary extension of $\mathfrak{A}$). We call B the **domain** of p. Let

$$S(B) = S^{\mathfrak{A}}(B)$$

denote the set of types over B.

Every element a of $\mathfrak{A}$ determines a type

$$\text{tp}(a/B) = tp^{\mathfrak{A}}(a/B) = \{\varphi(x) \mid \mathfrak{A} \vDash \varphi(a), \varphi \text{ an } L(B)\text{-formula}\}$$

So an element a realises the type $p \in S(B)$ exactly if $p = \text{tp}(a/B)$. If $\mathfrak{A}'$ is an elementary extension of $\mathfrak{A}$, then

$$S^{\mathfrak{A}}(B) = S^{\mathfrak{A}'}(B) \quad \text{and} \quad \text{tp}^{\mathfrak{A}'}(a/B) = \text{tp}^{\mathfrak{A}}(a/B)$$

If $\mathfrak{A}' \vDash p(x)$ then $\mathfrak{A}' \vDash \exists x p(x)$, so $\mathfrak{A} \vDash \exists x p(x)$.

We use the notation tp(a) for tp$(a/\emptyset)$

Maximal finitely satisfiable sets of formulas in $x_1, \dots, x_n$ are called n-**types** and

$$S_n(B) = S_N^{\mathfrak{A}}(B)$$

denotes the set of n-types over B.

$$\text{tp}(C/B) = \{\varphi(x_{c_1}, \dots, x_{c_n}) \mid \mathfrak{A} \vDash \varphi(c_1, \dots, c_n), \varphi \text{ an } L(B)\text{-formula}\}$$

Corollary 2.14. *Every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ in which all types over A are realised*

Proof. We choose for every $p \in S(A)$ a new constant c_p. We have to find a model of

$$\text{Th}(\mathfrak{A}_A) \cup \bigcup_{p \in S(A)} p(c_p)$$

This theory is finitely satisfiable since every p is finitely satisfiable in $\mathfrak{A}$.

Or use Lemma 2.12. Let $(p_\alpha)_{\alpha<\lambda}$ be an enumeration of $S(A)$. Construct an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\beta \prec \dots (\beta \leq \lambda)$$

s.t. each p_α is realised in $\mathfrak{A}_{\alpha+1}$ (by recursion theorem on ordinal numbers)

Suppose that the elementary chain $(\mathfrak{A}_{\alpha'})_{\alpha'<\beta}$ is already constructed. If β is a limit ordinal, we let $\mathfrak{A}_\beta = \bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$, which is elementary by Lemma 2.4. If $\beta = \alpha + 1$ we first note that p_α is also finitely satisfiable in $\mathfrak{A}_\alpha$, therefore we can realise p_α in a suitable elementary extension $\mathfrak{A}_\beta \succ \mathfrak{A}_\alpha$ by Lemma 2.12. Then $\mathfrak{B} = \mathfrak{A}_\lambda$ is the model we were looking for $\square$

2.3 The Löwenheim-Skolem Theorem

Theorem 2.15 (Löwenheim-Skolem)**.** *Let $\mathfrak{B}$ be an L-structure, S a subset of B and κ an infinite cardinal*

1. *If*
$$\max(|S|, |L|) \leq \kappa \leq |B|$$
then $\mathfrak{B}$ has an elementary substructure of cardinality κ containing S

2. *If $\mathfrak{B}$ is infinite and*
$$\max(|\mathfrak{B}|, |L|) \leq \kappa$$
then $\mathfrak{B}$ has an elementary extension of cardinality κ

Proof. 1. Choose a set $S \subseteq S' \subseteq B$ of cardinality κ and apply Corollary 2.3

2. We first construct an elementary extension $\mathfrak{B}'$ of cardinality at least κ. Choose a set C of new constants of cardinality κ. As $\mathfrak{B}$ is infinite, the theory
$$\text{Th}(\mathfrak{B}_B) \cup \{\neg c \doteq d \mid c, d \in C, c \neq d\}$$
is finitely satisfiable. By Lemma 2.1 any model $(\mathfrak{B}'_B, b_c)_{c \in C}$ is an elementary extension of $\mathcal{B}$ with κ many different elements (b_c)

Finally we apply the first part of the theorem to $\mathcal{B}'$ and $S = B$

□

Corollary 2.16. *A theory which has an infinite model has a model in every cardinality $\kappa \geq \max(|L|, \aleph_0)$*

Definition 2.17. Let κ be an infinite cardinal. A theory T is called **κ-categorical** if for all models of T of cardinality κ are isomorphic

Theorem 2.18 (Vaught's Test)**.** *A κ-categorical theory T is complete if the following conditions are satisfied*

1. *T is consistent*

2. *T has no finite model*

3. *$|L| \leq \kappa$*

Proof. We have to show that all models $\mathfrak{A}$ and $\mathfrak{B}$ of T are elemantarily equivalent. As $\mathfrak{A}$ and $\mathfrak{B}$ are infinite, Th($\mathfrak{A}$) and Th($\mathfrak{B}$) have models $\mathfrak{A}'$ and $\mathfrak{B}'$ of cardinality κ. By assumption $\mathfrak{A}'$ and $\mathfrak{B}'$ are isomorphic, and it follows that

$$\mathfrak{A} \equiv \mathfrak{A}' \equiv \mathfrak{B}' \equiv \mathfrak{B}$$

□

Example 2.1. 1. The theory DLO of dense linear orders without endpoints is $\aleph_0$-categorical and by Vaught's test complete. Let $A = \{a_i \mid i \in \omega\}$, $B = \{b_i \mid i \in \omega\}$. We inductively define sequences $(c_i)_{i<\omega}$, $(d_i)_{i<\omega}$ exhausting A and B. Assume that $(c_i)_{i<m}, (d_i)_{i<m}$ have defined so that $c_i \mapsto d_i, i < m$ is an order isomorphism. If $m = 2k$ let $c_m = a_j$ where a_j is the element with minimal index in $\{a_i \mid i \in \omega\}$ not occurring in $(c_i)_{i<m}$. Since $\mathfrak{B}$ is a dense linear order without endpoints there is some element $d_m \in \{b_i \mid i \in \omega\}$ s.t. $(c_i)_{i\leq m}$ and $(d_i)_{i\leq m}$ are order isomorphic. If $m = 2k+1$ we interchange the roles of $\mathfrak{A}$ and $\mathfrak{B}$

2. For any prime p or $p = 0$, the theory ACF_p of algebraically closed fields of characteristic p is κ-categorical for any $\kappa > \aleph_0$

Consider the Theorem 2.18 we strengthen our definition

Definition 2.19. Let κ be an infinite cardinal. A theory T is called **κ-categorical** if it is complete, $|T| \leq \kappa$ and, up to isomorphism, has exactly one model of cardinality κ

3 Quantifier Elimination

3.1 Preservation theorems

Lemma 3.1 (Separation Lemma)**.** *Let T_1, T_2 be two theories. Assume $\mathcal{H}$ is a set of sentences which is closed under $\wedge, \vee$ and contains $\perp$ and $\top$. Then the following are equivalent*

1. *There is a sentence $\varphi \in \mathcal{H}$ which separates T_1 from T_2. This means*

$$T_1 \vDash \varphi \quad \textit{and} \quad T_2 \vDash \neg\varphi$$

2. *All models $\mathfrak{A}_1$ of T_1 can be separated from all models $\mathfrak{A}_2$ of T_2 by a sentence $\varphi \in \mathcal{H}$. This means*

$$\mathfrak{A}_1 \vDash \varphi \quad \textit{and} \quad \mathfrak{A}_2 \vDash \neg\varphi$$

For 1, suppose $T_1 = T \cup \{\psi\}$ and $T_2 = T \cup \{\neg\psi\}$. If $T_1 \models \varphi$ and $T_2 \models \neg\varphi$, then $T \models \psi \to \varphi$ and $T \models \neg\psi \to \neg\varphi$ which is equivalent to $T \models \varphi \to \psi$. Thus we have $T \models \varphi \leftrightarrow \psi$.

Proof. $2 \to 1$. For any model $\mathfrak{A}_1$ of T_1 let $\mathcal{H}_{\mathfrak{A}_1}$ be the set of all sentences from $\mathcal{H}$ which are true in $\mathfrak{A}_1$. (2) implies that $\mathcal{H}_{\mathfrak{A}_1}$ and T_2 cannot have a common model. By the Compactness Theorem there is a finite conjunction $\varphi_{\mathfrak{A}_1}$ of sentences from $\mathcal{H}_{\mathfrak{A}_1}$ inconsistent with T_2. Clearly

$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$

is inconsistent. Again by compactness T_1 implies a disjunction φ of finitely many of the $\varphi_{\mathfrak{A}_1}$ (Corollary 2.10) and

$$T_1 \models \varphi \quad \text{and} \quad T_2 \models \neg\varphi$$

□

For structures $\mathfrak{A}, \mathfrak{B}$ and a map $f : A \to B$ preserving all formulas from a set of formulas Δ, we use the notation

$$f : \mathfrak{A} \to_\Delta \mathfrak{B}$$

We also write

$$\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$$

to express that all sentences from Δ true in $\mathfrak{A}$ are also true in $\mathfrak{B}$

Lemma 3.2. *Let T be a theory, $\mathfrak{A}$ a structure and Δ a set of formulas, closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent*

1. *All sentences $\varphi \in \Delta$ which are true in $\mathfrak{A}$ are consistent with T*

2. *There is a model $\mathfrak{B} \models T$ and a map $f : \mathfrak{A} \to_\Delta \mathfrak{B}$*

Proof. $2 \to 1$. Assume $f : \mathfrak{A} \to_\Delta \mathfrak{B} \models T$. If $\varphi \in \Delta$ is true in $\mathfrak{A}$, it is also true in $\mathfrak{B}$ and therefore consistent with T.

$1 \to 2$. Consider $\mathrm{Th}_\Delta(\mathfrak{A}_A)$, the set of all sentences $\delta(\bar{a})$ ($\delta(\bar{x}) \in \Delta$), which are true in $\mathfrak{A}_A$. The models $(\mathfrak{B}, f(a)_{a\in A})$ of this theory correspond to maps $f : \mathfrak{A} \to_\Delta \mathfrak{B}$. **This means that we have to find a model of** $T \cup \mathrm{Th}_\Delta(\mathfrak{A}_A)$. To show finite satisfiability it is enough to show that $T \cup D$ is consistent for every finite subset D of $\mathrm{Th}_\Delta(\mathfrak{A}_A)$. Let $\delta(\bar{a})$ be the conjunction of the elements of D. Then $\mathfrak{A}$ is a model of $\varphi = \exists\bar{x}\delta(\bar{x})$, so by assumption T has a model $\mathfrak{B}$ which is also a model of φ. This means that there is a tuple $\bar{b}$ s.t. $(\mathfrak{B}, \bar{b}) \models \delta(\bar{a})$ □

Lemma 3.2 applied to $T = \mathrm{Th}(\mathfrak{B})$ shows that $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$ iff there exists a map f and a structure $\mathfrak{B}' \equiv \mathfrak{B}$ s.t. $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$

Theorem 3.3. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *There is a universal sentence which separates T_1 from T_2*
2. *No model of T_2 is a substructure of a model of T_1*

Proof. $1 \to 2$. Let φ be a universal sentence which separates T_1 and T_2. Let $\mathfrak{A}_1$ be a model of T_1 and $\mathfrak{A}_2$ a substructure of $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is a model of φ, $\mathfrak{A}_2$ is also a model of φ. Therefore $\mathfrak{A}_2$ cannot be a model of T_2

$2 \to 1$. Here we add some details for the proof $2 \to 1$. If T_1 and T_2 cannot be separated by a universal sentence, then they have models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which cannot be separated by a universal sentence. That is, for all universal sentence φ, if $\mathfrak{A}_1 \vDash \varphi$ then $\mathfrak{A}_2 \vDash \varphi$. Thus $\mathfrak{A}_1 \Rightarrow_\forall \mathfrak{A}_2$, here $\Rightarrow_\forall$ means for all universal sentence.

Now note that

$$\mathfrak{A}_1 \vDash \varphi \to \mathfrak{A}_2 \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{A}_2 \vDash \neg\varphi \to \mathfrak{A}_2 \vDash \neg\varphi$$

and $\neg\varphi$ is an existential sentence. Hence we have

$$\mathfrak{A}_2 \Rightarrow_\exists \mathfrak{A}_1$$

The reason that we want to use $\exists$ is that it holds in the substructure case and we could imagine that $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$ (I guess this is our intuition). Now by Lemma 3.2 we have $\mathfrak{A}_1' \equiv \mathfrak{A}_1$ and a map $f : \mathfrak{A}_2 \to_\exists \mathfrak{A}_1'$. Apparently $\mathfrak{A}_1' \vDash \mathrm{Diag}(\mathfrak{A}_2)$ and f is an embedding. Hence $\mathfrak{A}_1'$ is a model of T_1 and T_2 $\square$

Definition 3.4. For any L-theory T, the formulas $\varphi(\bar{x}), \psi(\bar{x})$ are said to be **equivalent** modulo T (or relative to T) if $T \vDash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

Corollary 3.5. *Let T be a theory*

1. *Consider a formula $\varphi(x_1, \dots, x_n)$. The following are equivalent*
   (a) *$\varphi(x_1, \dots, x_n)$ is, modulo T, equivalent to a universal formula*
   (b) *If $\mathfrak{A} \subseteq \mathfrak{B}$ are models of T and $a_1, \dots, a_n \in A$, then $\mathfrak{B} \vDash \varphi(a_1, \dots, a_n)$ implies $\mathfrak{A} \vDash \varphi(a_1, \dots, a_n)$*

2. *We say that a theory which consists of universal sentences is universal. Then T is equivalent to a universal theory iff all substructures of models of T are again models of T*

Proof. 1. Assume (2). We extend L by an n-tuple $\bar{c}$ of new constants $c_1, \dots, c_n$ and consider theory

$$T_1 = T \cup \{\varphi(\bar{c})\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi(\bar{c})\}$$

Then (2) says the substructures of models of T_1 cannot be models of T_2. By Theorem 3.3 T_1 and T_2 can be separated by a universal $L(\bar{c})$-sentence $\psi(\bar{c})$. By Lemma 1.16, $T_1 \vDash \psi(\bar{c})$ implies

$$T \vDash \forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$$

and from $T_2 \vDash \neg\psi(\bar{c})$ we see

$$T \vDash \forall \bar{x}(\neg\varphi(\bar{x}) \to \neg\psi(\bar{x}))$$

2. Suppose a theory T has this property. Let φ be an axiom of T. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, it is not possible for $\mathfrak{B}$ to be a model of T and for $\mathfrak{A}$ to be a model of $\neg\varphi$ at the same time. By Theorem 3.3 there is a universal sentence ψ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of T follow from

$$T_\forall = \{\psi \mid T \vDash \psi, \psi \text{ universal}\}$$

□

An $\forall\exists$-formula is of the form

$$\forall x_1 \dots x_n \psi$$

where ψ is existential

Lemma 3.6. *Suppose φ is an $\forall\exists$-sentence, $(\mathfrak{A}_i)_{i\in I}$ is a directed family of models of φ and $\mathfrak{B}$ the union of the $\mathfrak{A}_i$. Then $\mathfrak{B}$ is also a model of φ.*

Proof. Write

$$\varphi = \forall \bar{x} \psi(\bar{x})$$

where ψ is existential. For any $\bar{a} \in B$ there is an A_i containing $\bar{a}$, clearly $\psi(\bar{a})$ holds in $\mathfrak{A}_i$. As $\psi(\bar{a})$ is existential it must also hold in $\mathfrak{B}$ □

Definition 3.7. We call a theory T **inductive** if the union of any directed family of models of T is again a model

Theorem 3.8. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *there is an $\forall\exists$-sentence which separates T_1 and T_2*

2. *No model of T_2 is the union of a chain (or of a directed family) of models of T_1*

Proof. $1 \to 2$. Assume φ is a $\forall\exists$-sentence which separates T_1 from T_2, $(\mathfrak{A}_i)_{i\in I}$ is a directed family of models of φ, by Lemma 3.6 $\mathfrak{B}$ is also a model of φ. Since $\mathfrak{B} \vDash \varphi$, $\mathfrak{B}$ cannot be a model of T_2

$2 \to 1$. If (1) is not true, Suppose $\mathfrak{A} \vDash T_1$ and $\mathfrak{B}^0 \vDash T_2$. Then

$$\mathfrak{A} \Rightarrow_{\forall\exists} \mathfrak{B}^0$$

Again we have

$$\mathfrak{B}^0 \Rightarrow_{\exists\forall} \mathfrak{A}$$

we have a map

$$f' : \mathfrak{B}^0 \to_{\exists\forall} \mathfrak{A}^0$$

where $\mathfrak{A}^0 \equiv \mathfrak{A}$. Since $\forall$-sentences are also $\exists\forall$-sentences, we thus have a map $f : \mathfrak{B}^0 \to_{\forall} \mathfrak{A}^0$.

Here we need to prove that $\mathfrak{B}^0$ is isomorphic to a substructure of $\mathfrak{A}^0$, which is clear since f is an embedding. Then we can assume that $\mathfrak{B}^0 \subseteq \mathfrak{A}^0$ and f is the inclusion map. Then

$$\mathfrak{A}^0_B \Rightarrow_{\exists} \mathfrak{B}^0_B$$

(Here we are talking about existential sentences in the original language. If $\mathfrak{B}^0 \vDash \exists\bar{x}\varphi(\bar{x})$ for some $\varphi(\bar{x})$, then $\mathfrak{B}^0 \vDash \varphi(\bar{b})$. So we can use constants B to talk about existential sentences) Applying Lemma 3.2 again, we obtain an extension $\mathfrak{B}^1_B$ of $\mathfrak{A}^0_B$ with $\mathfrak{B}^1_B \equiv \mathfrak{B}^0_B$, i.e. $\mathfrak{B}^0 \prec \mathfrak{B}^1$. Hence we have an infinite chain

$$\mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq^1 \mathfrak{B}^1 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{B}^2 \subseteq \cdots$$
$$\mathfrak{B}^0 \prec \mathfrak{B}^1 \prec \mathfrak{B}^2 \prec \cdots$$
$$\mathfrak{A}^i \equiv \mathfrak{A}$$

Let $\mathfrak{B}$ be the union of the $\mathfrak{A}^i$. Since $\mathfrak{B}$ is also the union of the elementary chain of the $\mathfrak{B}^i$, it is an elementary extension of $\mathfrak{B}^0$ and hence a model of T_2. But the $\mathfrak{A}^i$ are models of T_1, so (2) does not hold $\square$

Corollary 3.9. *Let T be a theory*

1. *For each sentence φ the following are equivalent*

   (a) *φ is, modulo T, equivalent to an $\forall\exists$-sentence*

   (b) *If*
   $$\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \cdots$$
   and their union $\mathfrak{B}$ are models of T, then φ holds in $\mathfrak{B}$ if it is true in all the $\mathfrak{A}^i$

2. *T is inductive iff it can be axiomatised by $\forall\exists$-sentences*

Proof.

1. Theorem 3.6 shows that $\forall\exists$-formulas are preserved by unions of chains. Hence (a)$\Rightarrow$(b). For the converse consider the theories
$$T_1 = T \cup \{\varphi\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi\}$$
Part (b) says that the union of a chain of models of T_1 cannot be a model of T_2. By Theorem 3.8 we can separate T_1 and T_2 by an $\forall\exists$-sentence ψ. Hence $T \cup \{\varphi\} \vDash \psi$ and $T \cup \{\neg\varphi\} \vDash \neg\psi$

2. Clearly $\forall\exists$-axiomatised theories are inductive. For the converse assume that T is inductive and φ is an axiom of T. Ifpp $\mathfrak{B}$ is a union of models of T, it cannot be a model of $\neg\varphi$. By Theorem 3.8 there is an $\forall\exists$-sentence ψ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of T follows from
$$T_{\forall\exists} = \{\psi \mid T \vDash \psi, \psi \ \forall\exists\text{-formula}\}$$

□

Exercise 3.1.1. Let X be a topological space, Y_1 and Y_2 quasi-compact (compact but not necessarily Hausdorff) subsets, and $\mathcal{H}$ a set of clopen subsets. Then the following are equivalent

1. There is a positive Boolean combination B of elements from $\mathcal{H}$ s.t. $Y_1 \subseteq B$ and $Y_2 \cap B = \emptyset$

2. For all $y_1 \in Y_1$ and $y_2 \in Y_2$ there is an $H \in \mathcal{H}$ s.t. $y_1 \in H$ and $y_2 \notin H$

Proof. $2 \rightarrow 1$. Consider an element $y_1 \in Y_1$ and $\mathcal{H}_{y_1}$, the set of all elements of $\mathcal{H}$ containing y_1. 2 implies that the intersection of the sets in $\mathcal{H}_{y_1}$ is disjoint from Y_2. So a finite intersection h_{y_1} of elements of $\mathcal{H}_{y_1}$ is disjoint from Y_2. The $h_{y_i}, y_1 \in Y_1$, cover Y_1. So Y_1 is contained in the union H of finitely many of the h_{y_i}. Hence H separates Y_1 from Y_2 □

3.2 Quantifier elimination

Definition 3.10. A theory T has **quantifier elimination** if every L-formula $\varphi(x_1, \dots, x_n)$ in the theory is equivalent modulo T to some quantifier-free formula $\rho(x_1, \dots, x_n)$

For $n = 0$, this means that modulo T every sentence is equivalent to a quantifier-free sentence. If L has no constants, $\top$ and $\bot$ are the only quantifier free sentences. Then T is either inconsistent or complete.

It's easy to transform any theory T into a theory with quantifier elimination if one is willing to expand the language: just enlarge L by adding an n-place relation symbol R_φ for every L-formula $\varphi(x_1, \dots, x_n)$ and T by adding all axioms

$$\forall x_1, \dots, x_n (R_\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n))$$

The resulting theory, the **Morleyisation** T^m of T, has quantifier elimination

A **prime structure** of T is a structure which embeds into all models of T

Lemma 3.11. *A consistent theory T with quantifier elimination which possess a prime structure is complete*

Proof. If $\mathfrak{M}, \mathfrak{N} \vDash T$ and $\mathfrak{M} \vDash \varphi$ and $\mathfrak{N} \vDash \neg\varphi$. Suppose prime structure is $\mathfrak{H}$, then $\mathfrak{H} \vDash \varphi$ and $\mathfrak{H} \vDash \neg\varphi$ since we have quantifier elimination ☐

Definition 3.12. A **simple existential formula** has the form

$$\varphi = \exists y \rho$$

for a quantifier-free formula ρ. If ρ is a conjunction of basic formulas, φ is called **primitive existential**

Lemma 3.13. *The theory T has quantifier elimination iff every primitive existential formula is, modulo T, equivalent to a quantifier-free formula*

Proof. We can write every simple existential formula in the form $\exists y \bigvee_{i<n} \rho_i$ for ρ_i which are conjunctions of basic formulas. This shows that every simple existential formula is equivalent to a disjunction of primitive existential formulas, namely to $\bigvee_{i<n}(\exists y \rho_i)$. We can therefore assume that every simple existential formula is, modulo T, equivalent to a quantifier-free formula

We are now able to eliminate the quantifiers in arbitrary formulas in prenex normal form (Exercise 1.2.1)

$$Q_1 x_1 \dots Q_n x_n \rho$$

if $Q_n = \exists$, we choose a quantifier-free formula ρ_0 which, modulo T, is equivalent to $\exists x_n \rho$ and proceed with the formula $Q_1 x_1 \ldots Q_{n-1} x_{n-1} \rho_0$. If $Q_n = \forall$, we find a quantifier-free ρ_1 which is, modulo T, equivalent to $\exists x_n \neg\rho$ and proceed with $Q_1 x_1 \ldots Q_{n-1} x_{n-1} \neg\rho_1$ □

Theorem 3.14. *For a theory T the following are equivalent*

1. *T has quantifier elimination*
2. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T with a common substructure $\mathfrak{A}$ we have*

$$\mathfrak{M}^1_A \equiv \mathfrak{M}^2_A$$

3. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T with a common substructure $\mathfrak{A}$ and for all primitive existential formulas $\varphi(x_1, \ldots, x_n)$ and parameter $a_1, \ldots, a_n$ from A we have*

$$\mathfrak{M}^1 \vDash \varphi(a_1, \ldots, a_n) \Rightarrow \mathfrak{M}^2 \vDash \varphi(a_1, \ldots, a_n)$$

   (*this is exactly the equivalence relation*)

If L has no constants, $\mathfrak{A}$ is allowed to be the empty "structure"

Proof. $1 \to 2$. Let $\varphi(\bar{a})$ be an $L(A)$-sentence which holds in $\mathfrak{M}^1$. Choose a quantifier-free $\rho(\bar{x})$ which is, modulo T, equivalent to $\varphi(\bar{x})$. Then

$$\begin{array}{ccccccc} \mathfrak{M}^1 & \vDash & \varphi(\bar{a}) & \Rightarrow & \mathfrak{M}^1 & \vDash & \rho(\bar{a}) \\ & & & \Rightarrow & \mathfrak{A} & \vDash & \rho(\bar{a})) \quad \Rightarrow \\ \mathfrak{M}^2 & \vDash & \rho(\bar{a})) & \Rightarrow & \mathfrak{M}^2 & \vDash & \varphi(\bar{a}) \end{array}$$

$3 \to 1$. Let $\varphi(\bar{x})$ be a primitive existential formula. In order to show that $\varphi(\bar{x})$ is equivalent, modulo T, to a quantifier-free formula $\rho(\bar{x})$ we extend L by an n-tuple $\bar{c}$ of new constants $c_1, \ldots, c_n$. **We have to show that we can separate $T \cup \{\varphi(\bar{c})\}$ and $T \cup \{\neg\varphi(\bar{c})\}$ by a quantifier free sentence** $\rho(\bar{c})$. Then $T \vDash \varphi(\bar{c}) \to \rho(\bar{c})$ and $T \vDash \neg\varphi(\bar{c}) \to \neg\rho(\bar{c})$. Hence $T \vDash \varphi(\bar{c}) \leftrightarrow \rho(\bar{c})$.

We apply the Separation Lemma ($\mathcal{H}$ hear is the set of quantifier-free sentence). Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be two models of T with two distinguished n-tuples $\bar{a}^1$ and $\bar{a}^2$. Suppose that $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ satisfy the same quantifier-free $L(\bar{c})$-sentences. We have to show that

$$\mathfrak{M}^1 \vDash \varphi(\bar{a}^1) \Rightarrow \mathfrak{M}^2 \vDash \varphi(\bar{a}^2) \tag{$\star$}$$

which says that if T's model $\mathfrak{A}_1, \mathfrak{A}_2$ satisfies the same quantifier-free sentences, then $\mathfrak{M}^1 \Rightarrow_\exists \mathfrak{M}^2$. If $\mathfrak{M}^1 \vDash T \cup \{\varphi(\bar{c})\}$ and $\mathfrak{M}^2 \vDash T \cup \{\neg\varphi(\bar{c})\}$ and satisfy the same quantifier-free $L(\bar{c})$ sentence, then $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$, a contradiction. Thus we finish the proof

Consider the substructure $\mathfrak{A}^i = \langle \bar{a}^i \rangle^{\mathfrak{M}^i}$, generated by $\bar{a}^i$. If we can show that there is an isomorphism

$$f : \mathfrak{A}^1 \to \mathfrak{A}^2$$

taking $\bar{a}$ to $\bar{a}$, we may assume that $\mathfrak{A}^1 = \mathfrak{A}^2 = \mathfrak{A}$ and $\bar{a}^1 = \bar{a}^2 = \bar{a}$. Then $\star$ follows directly from 3.

Every element of $\mathfrak{A}^1$ has the form $t^{\mathfrak{M}^1}[\bar{a}^1]$ for an L-term $t(\bar{x})$. The isomorphism $f$to be constructed must satisfy

$$f(t^{\mathfrak{M}^1}[\bar{a}^1]) = t^{\mathfrak{M}^2}[\bar{a}^2]$$

We define f by this equation and have to check that f is well defined and injective. Assume

$$s^{\mathfrak{M}^1}[\bar{a}^1] = t^{\mathfrak{M}^1}[\overline{af^1}]$$

Then $\mathfrak{M}^1, \bar{a}^1 \vDash s(\bar{c}) \doteq t(\bar{c})$, and by our assumption, $\mathfrak{M}^1$ and $\mathfrak{M}^2$ satisfy the same quantifier-free $L(\bar{c})$-sentence, it also holds in $(\mathfrak{M}^2, \bar{a}^2)$, which means

$$s^{\mathfrak{M}^2}[\bar{a}^2] = t^{\mathfrak{M}^2}[\bar{a}^2]$$

Swapping the two sides yields injectivity.

Surjectivity is clear. It remains to show that f commutes with the interpretation of the relation symbols. Now

$$\mathfrak{M}^1 \vDash R\left[t_1^{\mathfrak{M}^1}[\bar{a}^1], \dots, t_m^{\mathfrak{M}^1}[\bar{a}^1]\right]$$

is equivalent to $(\mathfrak{M}^1, \bar{a}^1) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which is equivalent to $(\mathfrak{M}^2, \bar{a}^2) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which in turn is equivalent to

$$\mathfrak{M}^2 \vDash R\left[t_1^{\mathfrak{M}^2}[\bar{a}^2], \dots, t_m^{\mathfrak{M}^2}[\bar{a}^2]\right]$$

$\square$

Note that (2) of Theorem 3.14 is saying that T is **substructure complete**; i.e., for any model $\mathfrak{M} \vDash T$ and substructure $\mathfrak{A} \subseteq \mathfrak{M}$ the theory $T \cup \text{Diag}(\mathfrak{A})$ is complete

Definition 3.15. We call T **model complete** if for all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T

$$\mathfrak{M}^1 \subseteq \mathfrak{M}^2 \Rightarrow \mathfrak{M}^1 \prec \mathfrak{M}^2$$

T is model complete iff for any $\mathfrak{M} \models T$ the theory $T \cup \text{Diag}(\mathfrak{M})$ is complete

Note that if $\mathfrak{M}_1 \models \text{Diag}(\mathfrak{M})$, then there is an embedding $h : \mathfrak{M} \to \mathfrak{M}_1$ and $\mathfrak{M}_1$ is isomorphic to an extension $\mathfrak{M}_1'$ of $\mathfrak{M}$. Then we have $\mathfrak{M} \subseteq \mathfrak{M}_1'$.

So here we are actually saying that all embeddings are elementary

Lemma 3.16 (Robinson's Test)**.** *Let T be a theory. Then the following are equivalent*

1. *T is model complete*
2. *For all models $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ of T and all existential sentences φ from $L(M^1)$*

$$\mathfrak{M}^2 \models \varphi \Rightarrow \mathfrak{M}^1 \models \varphi$$

3. *Each formula is, modulo T, equivalent to a universal formula*

Proof. $1 \leftrightarrow 3$. Corollary 3.5

(2) and Corollary 3.5 shows that all existential sentences are, modulo T, equivalent to a universal sentence. Then by induction we can show 3. (Details) □

If $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ satisfies (2), we call $\mathfrak{M}^1$ **existentially closed** in $\mathfrak{M}^2$. We denote this by

$$\mathfrak{M}^1 \prec_1 \mathfrak{M}^2$$

Definition 3.17. Let T be a theory. A theory T^* is a **model companion** of T if the following three conditions are satisfied

1. Each model of T can be extended to a model of T^*
2. Each model of T^* can be extended to a model of T
3. T^* is model complete

Theorem 3.18. *A theory T has, up to equivalence, at most one model companion T^**

Proof. If T^+ is another model companion of T, every model of T^+ is contained in a model of T^* and conversely. Let $\mathfrak{A}_0 \models T^+$. Then $\mathfrak{A}_0$ can be embedded in a model $\mathfrak{B}_0$ of T^*. In turn $\mathfrak{B}_0$ is contained in a model $\mathfrak{A}_1$ of T^+. In this way we find two elementary chains $(\mathfrak{A}_i)$ and $(\mathfrak{B}_i)$, which have a common union $\mathfrak{C}$. Then $\mathfrak{A}_0 \prec \mathfrak{C}$ and $\mathfrak{B}_0 \prec \mathfrak{C}$ implies $\mathfrak{A}_0 \equiv \mathfrak{B}_0$ since T are all sentences. Thus $\mathfrak{A}_0$ is a model of T^* □

Existentially closed structures and the Kaiser hull

Let T be an L-theory. It follows from 3.3 that the models of $T_\forall = \{\varphi \mid T \models \varphi$ where φ is universal$\}$ are the substructures of models of T. The conditions (1) and (2) in the definition of "model companion" can therefore be expressed as

$$T_\forall = T^*_\forall$$

(1 and 2 says $\mathrm{Mod}(T_\forall) = \mathrm{Mod}(T^*_\forall)$) Hence the model companion of a theory T depends only on $T_\forall$.

Definition 3.19. An L-structure $\mathfrak{A}$ is called T**-existentiallay closed** (or T-**ec**) if

1. $\mathfrak{A}$ can be embedded in a model of T
2. $\mathfrak{A}$ is existentially closed in every extension which is a model of T

A structure $\mathfrak{A}$ is T-ec exactly if it is $T_\forall$-ec. Since every model of $\mathfrak{B}$ of $T_\forall$ can be embedded in a model $\mathfrak{M}$ of T and $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$ and $\mathfrak{A} \prec_1 \mathfrak{M}$ implies $\mathfrak{A} \prec_1 \mathfrak{B}$

Lemma 3.20. *Every model of a theory T can be embedded in a T-ec structure*

Proof. Let $\mathfrak{A}$ be a model of $T_\forall$. We choose an enumeration $(\varphi_\alpha)_{\alpha<\kappa}$ of all existential $L(A)$-sentences and construct an ascending chain $(\mathfrak{A}_\alpha)_{\alpha\leq\kappa}$ of models of $T_\forall$. We begin with $\mathfrak{A}_0 = \mathfrak{A}$. Let $\mathfrak{A}_\alpha$ be constructed. If φ_α holds in an extension of $\mathfrak{A}_\alpha$ which is a model of T we let $\mathfrak{A}_{\alpha+1}$ be such a model. Otherwise we set $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$. For limit ordinals λ we define $\mathfrak{A}_\lambda$ to be the union of all $\mathfrak{A}_\alpha$. $\mathfrak{A}_\lambda$ is again a model of $T_\forall$

The structure $\mathfrak{A}^1 = \mathfrak{A}_\kappa$ has the following property: every existential $L(A)$-sentence which holds in an extension of $\mathfrak{A}^1$ that is a model of T holds in $\mathfrak{A}^1$. Now in the same manner, we construct $\mathfrak{A}^2$ from $\mathfrak{A}^1$, etc. The union $\mathfrak{M}$ of the chain $\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \ldots$ is the desired T-ec structure $\square$

Every elementary substructure $\mathfrak{N}$ of a T-ec structure $\mathfrak{M}$ is again T-ec: Let $\mathfrak{N} \subseteq \mathfrak{A}$ be a model of T. Since $\mathfrak{M}_N \Rightarrow_\exists \mathfrak{A}_N$, there is an embedding of $\mathfrak{M}$ in an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ which is the identity on N. Since $\mathfrak{M}$ is existentially closed in $\mathfrak{B}$, it follows that $\mathfrak{N}$ is existentially closed in $\mathfrak{B}$ and therefore also in $\mathfrak{A}$

$$\begin{tikzcd}
& \mathfrak{B} & \\
\mathfrak{A} \arrow[ur, "\prec"] & & \mathfrak{M} \arrow[ul, "\prec_1"'] \\
& \mathfrak{N} \arrow[ul, "\prec_1"] \arrow[ur, "\prec"'] &
\end{tikzcd}$$

Lemma 3.21. *Let T be a theory. Then there is a biggest inductive theory T^{KH} with $T_\forall = T^{\mathrm{KH}}_\forall$. We call T^{KH} the **Kaiser hull** of T*

Proof. Let T^1 and T^2 be two inductive theories with $T^1_\forall = T^2_\forall = T_\forall$. We have to show that $(T^1 \cup T^2)_\forall = T_\forall$. Note that for every model $\mathfrak{A} \models T^1$ and $\mathfrak{B} \models T^2$ we have $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$ and vice versa. Then we have the embeddings just like model companions. Let $\mathfrak{M}$ be a model of T, as in the proof of 3.18 we extend $\mathfrak{M}$ by a chain $\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \cdots$ of models of T^1 and T^2. The union of this chain is a model of $T^1 \cup T^2$ $\square$

Lemma 3.22. *The Kaiser hull T^{KH} is the $\forall\exists$-part of the theory of all T-ec structures*

Proof. Let T^* be the $\forall\exists$-part of the theory of all T-ec structures. Since T-ec structures are models of $T_\forall$, we have $T_\forall \subseteq T^*_\forall$. It follows from 3.20 that $T^*_\forall \subseteq T_\forall$. Hence T^* is contained in the Kaiser Hull.

It remains to show that every T-ec structure $\mathfrak{M}$ is a model of the Kaiser hull. Choose a model $\mathfrak{N}$ of T^{KH} which contains $\mathfrak{M}$. Then $\mathfrak{M} \prec_1 \mathfrak{N}$. This implies $\mathfrak{N} \Rightarrow_{\forall\exists} \mathfrak{M}$ and therefore $\mathfrak{M} \models T^{KH}$ $\square$

This implies that T-ec strctures are models of $T_{\forall\exists}$

Theorem 3.23. *For any theory T the following are equivalent*

1. *T has a model companion T^**
2. *All models of K^{KH} are T-ec*
3. *The T-ec structures form an elementary class.*

If T^* exists, we have

$$T^* = T^{\mathrm{KH}} = \text{ theory of all } T\text{-ec structures}$$

Proof. $1 \to 2$: let T^* be the model companion of T. As a model complete theory

$3 \to 1$: Assume that the T-ec structures are exactly the models of the theory T^+. By 3.20 we have $T_\forall = T_\forall^+$. Criterion 3.16 implies that T^+ is model complete. So T^+ is the model companion of T. □

Exercise 3.2.1. Let L be the language containing a unary function f and a binary relation symbol R and consider the L-theory $T = \{\forall x \forall y (R(x,y) \to (R(x, f(y))))\}$. Showing the follow

1. For any T-structure $\mathfrak{M}$ and $a, b \in M$ with $b \notin \{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ we have $\mathfrak{M} \vDash \exists z (R(z,a) \wedge \neg R(z,b))$
2. Let $\mathfrak{M}$ be a model of T and a an element of M s.t. $\{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ is infinite. Then in an elementary extension $\mathfrak{M}'$ there is an element b with $\mathfrak{M}' \vDash \forall z (R(z,a) \to R(z,b))$
3. The class of T-ec structures is not elementary, so T does not have a model companion

Exercise 3.2.2. A theory T with quantifier elimination is axiomatisable by sentences of the form

$$\forall x_1 \dots x_n \psi$$

where ψ is primitive existential formula

3.3 Examples

Infinite sets. The models of the theory Infset of **infinite sets** are all infinite sets without additional structure. The language $L_\emptyset$ is empty, the axioms are (for $n = 1, 2, \dots$)

- $\exists x_0 \dots x_{n-1} \bigwedge_{i<j<n} \neg x_i \doteq x_j$

Theorem 3.24. *The theory Infset of infinite sets has quantifier elimination and is complete*

Proof. Since the language is empty, the only basic formula is $x_i = x_j$ and $\neg(x_i = x_j)$. By Lemma 3.13 we only need to consider primitive existential formulas. Then every sentence is actually saying there is n different elements. Then for any $\mathfrak{M}^1, \mathfrak{M}^2 \vDash \mathsf{Infset}$, they have a common substructure $\mathfrak{A}$ with ω different elements. Visibly, $\mathfrak{M}^1_A \equiv \mathfrak{M}^2_A$ □

Dense linear orderings.

$$\forall a, b(a \leq b \wedge b \leq a \rightarrow a \dot{=} b)$$
$$\forall a, b, c(a \leq b \wedge b \leq c \rightarrow a \leq c)$$
$$\forall a, b(a \leq b \vee b \leq a)$$
$$\forall a, b \exists c(a < b \rightarrow a < c < b)$$

Theorem 3.25. *DLO has quantifier elimination*

Proof. Let A be a finite common substructure of the two models O_1 and O_2. We choose an ascending enumeration $A = \{a_1, \dots, a_n\}$. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence, which is true in O_1 and assume $O_1 \vDash \rho(b_1)$. We want to extend the order preserving map $a_i \mapsto a_i$ to an order preserving map $A \cup \{b_1\} \rightarrow O_2$. For this we have an image b_2 of b_1. There are four cases

1. $b_1 \in A$, we set $b_2 = b_1$
2. $b_1 \in (a_i, a_{i+1})$. We choose b_2 in O_2 with the same property
3. b_1 is smaller than all elements of A. We choose a $b_2 \in O_2$ of the same kind
4. b_1 is bigger than all a_i. Choose b_2 in the same manner

This defines an isomorphism $A \cup \{b_1\} \rightarrow A \cup \{b_2\}$, which show that $O_2 \vDash \rho(b_2)$ □

Modules. Let R be a (possibly non-commutative) ring with 1. An R-module

$$\mathfrak{M} = (, 0, +, -, r)_{r \in R}$$

is an abelian group $(M, 0, +, -)$ together with operations $r : M \rightarrow M$ for every ring element $r \in R$. We formulate the axioms in the language $L_{Mod}(R) = L_{AbG} \cup \{r \mid r \in R\}$. The theory $\mathsf{Mod}(R)$ of R-modules consists of

$$\mathsf{AbG}$$
$$\forall x, y\ r(x+y) \dot{=} rx + ry$$
$$\forall x\ (r+s)x \dot{=} rx + sx$$
$$\forall x\ (rs)x \dot{=} r(sx)$$
$$\forall x\ 1x \dot{=} x$$

for all $r, s \in R$. Then $\mathsf{Infset} \cup \mathsf{Mod}(R)$ is the theory of all infinite R-modules

A module over fields is a vector space

Theorem 3.26. *Let K be a field. Then the theory of all infinite K-vector spaces has quantifier elimination and is complete*

Proof. Let A be a common finitely generated substructure (i.e., a subspace) of the two infinite K-vector spaces V_1 and V_2. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence which holds in V_1. Choose a b_1 from V_1 which satisfies $\rho(y)$. If b_1 belongs to A, we finished. If not, we choose a $b_2 \in V_2 \setminus A$. Possibly we have to replace V_2 by an elementary extension. The vector spaces $A + Kb_1$ and $A + Kb_2$ are isomorphic by an isomophism which maps b_1 to b_2 and fixes A elementwise. Hence $V_2 \models \rho(b_2)$ □

Definition 3.27. An **equation** is an $L_{Mod}(R)$-formula $\gamma(\bar{x})$ of the form

$$r_1 x_1 + \dots + r_m x_m = 0$$

A **positive primitive** formula (**pp**-formula) is of the form

$$\exists \bar{y} (\gamma_1 \wedge \dots \wedge \gamma_n)$$

where the $\gamma_i(\overline{xy})$ are equations

Theorem 3.28. *For every ring R and any R-module M, every $L_{Mod}(R)$-formula is equivalent (modulo the theory of M) to a Boolean combination of positive primitive formulas*

Remark. 1. We assume the class of positive primitive formulas to be closed under $\wedge$

2. A pp-formula $\varphi(x_1, \dots, x_n)$ defines a subgroup $\varphi(M^n)$ of M^n:

$$M \models \varphi(0) \quad \text{and} \quad M \models \varphi(x) \wedge \varphi(y) \to \varphi(x - y)$$

Lemma 3.29. *Let $\varphi(x, y)$ be a pp-formula and $a \in M$. Then $\varphi(M, a)$ is empty or a coset of $\varphi(M, 0)$*

Proof. $M \models \varphi(x, a) \to (\varphi(y, 0) \leftrightarrow \varphi(x + y, a))$
Or, if $x, y \in \varphi(M, a)$, then $\varphi(x - y, 0)$. □

Corollary 3.30. *Let $a, b \in M$, $\varphi(x, y)$ a pp-formula. Then (in M) $\varphi(x, a)$ and $\varphi(x, b)$ are equivalent or contradictory*

Lemma 3.31 (B. H. Neumann)**.** *Let H_i denote subgroups of some abelian group. If $H_0 + a_0 \subseteq \bigcup_{i=1}^n H_i + a_i$ and $H_0/(H_0 \cap H_i)$ is infinite for $i > k$, then $H_0 + a_0 \subseteq \bigcup_{i=1}^k H_i + a_i$*

Lemma 3.32. *Let $A_i, i \leq k$, be any sets. If A_0 is finite, then $A_0 \subseteq \bigcup_{i=1}^{k} A_i$ iff*

$$\sum_{\Delta \subseteq \{1,\ldots,k\}} (-1)^{|\Delta|} \left| A_0 \cap \bigcap_{i \in \Delta} A_i \right| = 0$$

Algebraically closed fields.

Theorem 3.33 (Tarski)**.** *The theory ACF of algebraically closed fields has quantifier elimination*

Proof. Let K_1 and K_2 be two algebraically closed fields and R a common subring. Let $\exists y \rho(y)$ be a simple existential sentence with parameters in R which hold in K_1. We have to show that $\exists y \rho(y)$ is also true in K_2.

Let F_1 and F_2 be the quotient fields of R in K_1 and K_2, and let $f : F_1 \to F_2$ be an isomorphism which is the identity on R. Then f extends to an isomorphism $g : G_1 \to G_2$ between the relative algebraic closures G_i of F_i in K_i. Choose an element $b_1 \in K_1$ which satisfies $\rho(y)$

$$\begin{tikzcd}
K_1 \arrow[d, dash, dashed] & K_2 \arrow[dd, dash] \\
G_1(b_1) \arrow[d, dash, dashed] & \\
G_1 \arrow[r, "g"] \arrow[d, dash] & G_2 \arrow[d, dash] \\
F_1 \arrow[r, "f"] \arrow[d, dash] & F_2 \arrow[d, dash] \\
R \arrow[r, "id"] & R
\end{tikzcd}$$

There are two cases

Case 1: $b_1 \in G_1$. Then $b_2 = g(b_1)$ satisfies the formula $\rho(y)$ in K_2

Case 2: $b_1 \notin G_1$. Then b_1 is transcendental over G and the field extension $G_1(b_1)$ is isomorphic to the rational function field $G_1(X)$. If K_2 is a proper extension of G_2, we choose any element from $K_2 \setminus G_2$ for b_2. Then g extends to an isomorphism between $G_1(b_1)$ and $G_2(b_2)$ which maps b_1 to b_2. Hence b_2 satisfies $\rho(y)$ in K_2. In case that $K_2 = G_2$ we take a proper elementary extension K_2' of K_2 (Such a K_2' exists by 2.15 since K_2 is infinite). Then $\exists y \rho(y)$ holds in K_2' and therefore in K_2 □

Corollary 3.34. *ACF is model complete*

ACF is not complete: for prime numbers p let

$$\mathsf{ACF}_p = \mathsf{ACF} \cup \{p \cdot 1 \doteq 0\}$$

be the theory of algebraically closed fields of characteristic p and

$$\mathsf{ACF}_0 = \mathsf{ACF} \cup \{\neg n \cdot 1 \doteq 0 \mid n = 1, 2, \dots\}$$

be the theory of algebraically closed fields of characteristic 0.

Corollary 3.35. *The theories ACF_p and ACF_0 are complete*

Proof. This follows from Lemma 3.11 since the prime fields are prime structures for these theories □

Corollary 3.36 (Hilbert's Nullstellensatz)**.** *Let K be a field. Then any proper ideal I in $K[X_1, \dots, X_n]$ has a zero in the algebraic closure acl(K)*

Proof. As a proper ideal, I is contained in a maximal ideal P. Then $L = K[X_1, \dots, X_n]/P$ is an extension field of K in which the cosets of the X_i are a zero of I. □

Real closed fields. It is axiomatised in the language L_{ORing} of ordered rings

Theorem 3.37 (Tarski-Seidenberg)**.** *RCF has quantifier elimination and is complete*

Proof. Let $(K_1, <)$ and $(K_2, <)$ be two real closed field with a common subring R. Consider an $L_{ORing}(R)$-sentence $\exists y \rho(y)$ (for a quantifier-free ρ), which holds in $(K_1, <)$. We have to show $\exists y \rho(y)$ also holds in $(K_2, <)$

We build first the quotient fields F_1 and F_2 of R in K_1 and K_2. By **??** there is an isomorphism $f : (F_1, <) \to (F_2, <)$ which fixes R. The relative algebraic closure G_i of F_i in K_i is a real closure of $(F_i, <)$. By **??** f extends to an isomorphism $g : (G_1, <) \to (G_2, <)$

Let $b_1 \in K_1$ which satisfies $\rho(y)$. There are two cases

Case 1: $b_1 \in G_1$: Then $b_2 = g(b_1)$ satisfies $\rho(y)$ in K_2

Case 2: $b_1 \notin G_1$. Then b_1 is transcendental over G_1 and the field extension $G_1(b_1)$ is isomorphic to the rational function field $G_1(X)$. Let G_1^l be the set of all elements of G_1 which are smaller than b_1, and G_1^r be the set of all elements of G_1 which are larger than b_1. Then all elements of $G_2^l = g(G_1^l)$ are smaller than all elements of $G_2^r = g(G_1^r)$. Since fields are densely ordered, we find in an elementary extension $(K_2', <)$ of $(K_2, <)$ an element b_2

which lies between the elements of G_2^l and the elements of G_2^r. Since b_2 is not in G_2, it is transcendental over G_2. Hence g extends to an isomorphism $h : G_1(b_1) \to G_2(b_2)$ which maps b_1 to b_2

In order to how that h is order preserving it suffices to show that h is order preserving on $G_1[b_1]$. Let $p(b_1)$ be an element of $G_1[b_1]$. Corollary **??** gives us a decomposition

$$p(X) = \epsilon \prod_{i<m}(X - a_i) \prod_{j<n}((X - c_j)^2 + d_j)$$

with positive d_j. The sign of $p(b_1)$ depends only on the signs of the factors $\epsilon, b_1 - a_0, \dots, b_1 - a_{m-1}$. The sign of $h(p(b_1))$ depends in the same way on the signs of $g(\epsilon), b_2 - g(a_0), \dots, b_2 - g(a_{m-1})$. But b_2 was chosen in such a way that

$$b_1 < a_i \iff b_2 < g(a_i)$$

Hence $p(b_1)$ is positive iff $h(p(b_1))$ is positive

Finally we have

$$\begin{aligned}(K_1, <) \vDash \rho(b_1) \Rightarrow (G_1(b_1), <) \vDash \rho(b_1) \Rightarrow (G_2(b_2), <) \vDash \rho(b_2) \Rightarrow \\ \Rightarrow (K_2', <) \vDash \exists y \rho(y) \quad \Rightarrow (K_2, <) \vDash \exists y \rho(y)\end{aligned}$$

RCF is complete since the ordered field of the rationals is a prime structure

□

Corollary 3.38 (Hilbert's 17th Problem). *Let $(K, <)$ be a real closed field. A polynomial $f \in K[X_1, \dots, X_n]$ is a sum of squares*

$$f = g_1^2 + \cdots + g_k^2$$

of rational functions $g_i \in K(X_1, \dots, X_n)$ iff

$$f(a_1, \dots, a_n) \geq 0$$

for all $a_1, \dots, a_n \in K$

Proof. Clearly a sum of squares cannot have negative values. For the converse, assume that f is not a sum of squares. Then by Corollary **??**, $K(X_1, \dots, X_n)$ has an ordering in which f is negative. Since in K the positive elements are squares, this ordering , which we denote by $<$, extends the ordering of K. Let $(L, <)$ be the real closure of $(K(X_1, \dots, X_n), <)$. In $(L, <)$, the sentence

$$\exists x_1, \dots, x_n f(x_1, \dots, x_n) < 0$$

is true. Hence it is also true in $(K, <)$ □

Exercise 3.3.1. Let Graph be the theory of graphs. The theory RG of the **random graph** is the extension of Graph by the following axiom scheme

$$\forall x_0 \ldots x_{m-1} y_1 \ldots y_{n-1} \Big(\bigwedge_{i \neq j} \neg x_i \doteq y_j \rightarrow$$
$$\exists z (\bigwedge_{i<m} z R x_i) \wedge (\bigwedge_{j<n} \neg z R y_j \wedge \neg z \doteq y_j) \Big)$$

From here, some definitions of random graphs

Let $p \in [0, 1]$ denote the probability with which a given pair is included. We assume all the edges have the same probability of occurrence. We denote the set of graphs constructed in this manner by $\mathcal{G}(n, p)$, where n is the number of elements in the vertex set.

Definition 3.39. A graph G has property $\mathcal{P}_{i,j}$ with $i, j = 0, 1, 2, 3, \ldots$ if, for any disjoint vertex sets V_1 and V_2 with $|V_1| \leq i$ and $|V_2| \leq j$, there exists a vertex $v \in G$ that satisfies three conditions

1. $v \notin V_1 \cup V_2$
2. $v \leftrightarrow x$ for every $x \in V_1$ and
3. $v \not\leftrightarrow y$ for every $y \in V_2$

Lemma 3.40. *An infinite graph $G \in \mathcal{G}(\aleph_0, p)$ has all the properties $\mathcal{P}_{i,j}$ with probability 1*

4 Countable Models

4.1 The omitting types theorem

Definition 4.1. Let T be an L-theory and $\Sigma(x)$ a set of L-formulas. A model $\mathfrak{A}$ of T not realizing $\Sigma(x)$ is said to **omit** $\Sigma(x)$. A formula $\varphi(x)$ **isolates** $\Sigma(x)$ if

1. $\varphi(x)$ is consistent with T
2. $T \models \forall x(\varphi(x) \rightarrow \sigma(x))$ for all $\sigma(x) \in \Sigma(x)$

A set of formulas is often called a **partial type**.

Theorem 4.2 (Omitting Types)**.** *If T is countable and consistent and if $\Sigma(x)$ is not isolated in T, then T has a model which omits $\Sigma(x)$*

If $\Sigma(x)$ is isolated by $\varphi(x)$ and $\mathfrak{A}$ is a model of T, then $\Sigma(x)$ is realised in $\mathfrak{A}$ by all realisations $\varphi(x)$. Therefore the converse of the theorem is true for **complete** theories T: if $\Sigma(x)$ is isolated in T, then it is realised in every model of T

Proof. We choose a countable set C of new constants and extend T to a theory T^* with the following properties

1. T^* is a Henkin theory: for all $L(C)$-formulas $\psi(x)$ there exists a constant $c \in C$ with $\exists x \psi(x) \to \psi(c) \in T^*$

2. for all $c \in C$ there is a $\sigma(x) \in \Sigma(x)$ with $\neg\sigma(c) \in T^*$

We construct T^* inductively as the union of an ascending chain

$$T = T_0 \subseteq T_1 \subseteq T_1 \subseteq \ldots$$

of consistent extensions of T by finitely many axioms from $L(C)$, in each step making an instance of (1) or (2) true.

Enumerate $C = \{c_i \mid i < \omega\}$ and let $\{\psi_i(x) \mid i < \omega\}$ be an enumeration of the $L(C)$-formulas

Assume that T_{2i} is the already constructed. Choose some $c \in C$ which doesn't occur in $T_{2i} \cup \{\psi_i(x)\}$ and set $T_{2i+1} = T_{2i} \cup \{\exists x \psi_i(x) \to \psi_i(c)\}$.

Up to equivalence T_{2i+1} has the form $T \cup \{\delta(c_i, \bar{c})\}$ for an L-formula $\delta(x, \bar{y})$ and a tuple $\bar{c} \in C$ which doesn't contain c_i. Since $\exists \bar{y} \delta(x, \bar{y})$ doesn't isolate $\Sigma(x)$, for some $\sigma \in \Sigma$ the formula $\exists \bar{y} \delta(x, \bar{y}) \wedge \neg\sigma(x)$ is consistent with T. Thus $T_{2i+2} = T_{2i+1} \cup \{\neg\sigma(c_i)\}$ is consistent

Take a model $(\mathfrak{A}', a_c)_{c \in C}$ of T^*. Since T^* is a Henkin theory, Tarski's Test 2.2 shows that $A = \{a_c \mid c \in C\}$ is the universe of an elementary substructure $\mathfrak{A}$ (Lemma 2.7). By property (2), $\Sigma(x)$ is omitted in $\mathfrak{A}$ □

Corollary 4.3. *Let T be countable and consistent and let*

$$\Sigma_0(x_0, \ldots, x_{n_0}), \Sigma_1(x_1, \ldots, x_{n_1}), \ldots$$

be a sequence of partial types. If all Σ_i are not isolated, then T has a model which omits all Σ_i

Proof. If $\Sigma_0(x), \Sigma_1(x), \ldots$. Then $T_{2i+2} = T_{2i+1} \cup \{\neg\sigma_m(c_{mn})\}$

If $\Sigma(x_1, \ldots, x_n)$, then $T_{2i+1} = T_{2i} \cup \{\exists \bar{x} \psi_i(\bar{x}) \to \psi_i(\bar{c})\}$.

Combine the two case □

4.2 The space of types

Fix a theory T. An n**-type** is a maximal set of formulas $p(x_1,\dots,x_n)$ consistent with T. We denote by $S_n(T)$ the set of all n-types of T. We also write $S(T)$ for $S_1(T)$. $S_0(T)$ is all complete extensions of T

If B is a subset of an L-structure $\mathfrak{A}$, we recover $S_n^{\mathfrak{A}}(B)$ as $S_n(\mathrm{Th}(\mathfrak{A}_B))$. In particular, if T is complete and $\mathfrak{A}$ is any model of T, we have $S^{\mathfrak{A}}(\emptyset)=S(T)$

For any L-formula $\varphi(x_1,\dots,x_n)$, let $[\varphi]$ denote the set of all types containing φ.

Lemma 4.4. *1. $[\varphi]=[\psi]$ iff φ and ψ are equivalent modulo T*

2. The sets $[\varphi]$ are closed under Boolean operations. In fact $[\varphi]\cap[\psi]=[\varphi\wedge\psi]$, $[\varphi]\cup[\psi]=[\varphi\vee\psi]$, $S_n(T)\setminus[\varphi]=[\neg\varphi]$, $S_n(T)=[\top]$ and $\emptyset=[\bot]$

It follows that the collection of sets of the form $[\varphi]$ is closed under finite intersection and includes $S_n(T)$. So these sets form a basis of a topology on $S_n(T)$

In this book, compact means finite cover and Hausdorff

Lemma 4.5. *The space $S_n(T)$ is 0-dimensional and compact*

Proof. Being 0-dimensional means having a basis of clopen sets. Our basic open sets are clopen since their complements are also basic open

If p and q are two different types, there is a formula φ contained in p but not in q. It follows that $[\varphi]$ and $[\neg\varphi]$ are open sets which separate p and q. This shows that $S_n(T)$ is Hausdorff

To prove compactness, we need to show that any collection of closed subsets of X with the finite intersection property has nonempty intersection. Could check this

Consider a family $[\varphi_i]$ $(i\in I)$, with the finite intersection property.This means that $\varphi_{i_i}\wedge\dots\wedge\varphi_{i_k}$ are consistent with T. So Corollary 2.10 $\{\varphi_i\mid i\in I\}$ is consistent with T and can be extended to a type p, which then belongs to all $[\varphi_i]$. □

Lemma 4.6. *All clopen subsets of $S_n(T)$ has the form $[\varphi]$*

Proof. Closed subset of a compact space is compact. It follows from Exercise 3.1.1 that we can separate any two disjoint closed subsets of $S_n(T)$ by a basic open set. □

The Stone duality theorem asserts that the map

$$X\mapsto\{C\mid C\text{ clopen subset of }X\}$$

yields an equivalence between the category of 0-dimensional compact spaces and the category of Boolean algebras. The inverse map assigns to every Boolean algebra to its **Stone space**

Definition 4.7. A map f from a subset of a structure $\mathfrak{A}$ to a structure $\mathfrak{B}$ is **elementary** if it preserves the truth of formulas; i.e., $f : A_0 \to B$ is elementary if for every formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} \in A_0$ we have

$$\mathfrak{A} \vDash \varphi(\bar{a}) \Rightarrow \mathfrak{B} \vDash \varphi(f(\bar{a}))$$

Lemma 4.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures, A_0 and B_0 subsets of A and B, respectively. Any elementary map $A_0 \to B_0$ induces a continuous surjective map $S_n(B_0) \to S_n(A_0)$*

Proof. If $q(\bar{x}) \in S_n(B_0)$, we define

$$S(f)(q) = \{\varphi(x_1, \dots, x_n, \bar{a}) \mid \bar{a} \in A_0, \varphi(x_1, \dots, x_n, f(\bar{a})) \in q(\bar{x})\}$$

If $\varphi(\bar{x}, f(\bar{a})) \notin q(\bar{x})$, then $\mathfrak{B} \nvDash \varphi(\bar{x}, \bar{a})$. Therefore $\mathfrak{A} \nvDash \varphi(\bar{x}, \bar{a})$. $S(f)$ defines a map from $S_n(B_0)$ to $S_n(A_0)$. Moreover, it is surjective since $\{\varphi(x_1, \dots, x_n, f(\bar{a})) \mid \varphi(x_1, \dots, x_n, a) \in p\}$ is finitely satisfiable for all $p \in S_n(A_0)$. And $S(f)$ is continuous since $[\varphi(x_1, \dots, x_n, f(\bar{a}))]$ is the preimage of $[\varphi(x_1, \dots, x_n, \bar{a})]$ under $S(f)$ □

There are two main cases

1. An elementary bijection $f : A_0 \to B_0$ defines a homeomorphism $S_n(A_0) \to S_n(B_0)$. We write $f(p)$ for the image of p

2. If $\mathfrak{A} = \mathfrak{B}$ and $A_0 \subseteq B_0$, the inclusion map induces the **restriction** $S_n(B_0) \to S_n(A_0)$. We write $q{\restriction}A_0$ for the restriction of q to A_0. We call q an extension of $q{\restriction}A_0$)

Lemma 4.9. *A type p is isolated in T iff p is an isolated point in $S_n(T)$. In fact, φ isolates p iff $[\varphi] = \{p\}$. That is, $[\varphi]$ is an **atom** in the Boolean algebra of clopen subsets of $S_n(T)$*

Proof. p being an isolated point means that $\{p\}$ is open, that is, $\{p\} = [\varphi]$.

The set $[\varphi]$ is a singleton iff $[\varphi]$ is non-empty and cannot be divided into two non-empty clopen subsets $[\varphi \wedge \psi]$ and $[\varphi \wedge \neg\psi]$. This means that for all ψ either ψ or $\neg\psi$ follows from φ modulo T. So $[\varphi]$ is a singleton iff φ generates the type

$$\langle\varphi\rangle = \{\psi(\bar{x}) \mid T \vDash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$$

which is the only element of $[\varphi]$

This shows that $[\varphi] = \{p\}$ implies that φ isolates p.

Conversely, φ isolates p, this means that $\langle\varphi\rangle$ is consistent with T and contains p. Since p is a type, we have $p = \langle\varphi\rangle$ $\square$

We call a formula $\varphi(x)$ **complete** if

$$\{\psi(\bar{x}) \mid T \models \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$$

is a type.

Corollary 4.10. *A formula isolates a type iff it is complete*

Exercise 4.2.1.

1. Closed subsets of $S_n(T)$ have the form $\{p \in S_n(T) \mid \Sigma \subseteq p\}$, where Σ is any set of formulas
2. Let T be countable and consistent. Then any meagre[1] subset X of $S_n(T)$ can be omitted, i.e., there is a model which omits all $p \in X$

Proof.

1. The sets $[\varphi]$ are a basis for the closed subsets of $S_n(T)$. So the closed sets of $S_n(T)$ are exactly the intersections $\bigcap_{\varphi\in\Sigma}[\varphi] = \{p \in S_n(T) \mid \Sigma \subseteq p\}$
2. The set X is the union of a sequence of countable nowhere dense sets X_i. We may assume that X_i are closed, i.e., of the form $\{p \in S_n(T) \mid \Sigma_i \subseteq p\}$. That X_i has no interior means that Σ_i is not isolated. The claim follows now from Corollary 4.3

$\square$

Exercise 4.2.2. Consider the space $S_\omega(T)$ of all complete types in variables $v_0, v_1, \dots$. Note that $S_\omega(T)$ is again a compact space and therefore not meagre by Baire's theorem

1. Show that $\{\text{tp}(a_0, a_1, \dots) \mid \text{ the } a_i \text{ enumerate a model of } T\}$ is comeagre in $S_\omega(T)$

Exercise 4.2.3. Let B be a subset of $\mathfrak{A}$. Show that the **restriction** (restriction of variables) map $S_{m+n}(B) \to S_n(B)$ is open, continuous and surjective. Let a be an n-tuple in A. Show that the fibre over $\text{tp}(a/B)$ is canonically homeomorphic to $S_m(aB)$.

Consider the restriction map $\pi : S_{m+1}(B) \to S_1(B)$. Then $\pi^{-1}(\text{tp}(a/B)) \cong S_m(aB)$

[1] A subset of a topological space is **nowhere dense** if its closure has no interior. A countable union of nowhere dense sets is meagre

Proof. We define the restriction map $f : S_{m+n}(B) \to S_n(B)$ as: for $q(\bar{x}, \bar{y}) \in S_{m+n}(B)$, we let $f(q(\bar{x}, \bar{y})) = \{\varphi(\bar{y}) : \varphi(\bar{y}) \in q(\bar{x}, \bar{y})\}$, where $\bar{x}$ and $\bar{y}$ are of size m and n respectively.

continuous is easy

Now given an open set $[\phi(\bar{v}, \overline{w})] \subseteq S_{m+n}(B)$. We need to prove $f([\phi(\bar{v}, \overline{w})]) = [\exists \bar{v} \phi(\bar{v}, \overline{w})]$ which is clear

Now the problem is, for $\text{tp}(a/B) \subset q(\bar{y}, x, \bar{b}) \in \pi^{-1}(\text{tp}(a/B))$, is it realized by $\bar{c}a$ in some $\mathfrak{A} \prec \mathfrak{B}$?

So what will happen if $\text{tp}(a/B) = \text{tp}(c/B)$ for some $c \in A$.

For any $\mathfrak{M} \models q(\bar{c}, d, \bar{b})$, for any $\phi(\bar{y}, x, \bar{b}) \in q(\bar{y}, x, \bar{b})$, $\mathfrak{M} \models \exists \bar{y}\ \phi(\bar{y}, d, \bar{b})$ and $\mathfrak{M} \models \exists \bar{y}\ \phi(\bar{y}, d, \bar{b}) \leftrightarrow \exists \bar{y}\ \phi(\bar{y}, a, \bar{b})$. Hence we have $\mathfrak{M} \models q(\bar{c}', a, \bar{b})$ for some $\bar{c}'$.

Hence for any $q(\bar{y}, x, \bar{b}) \in \pi^{-1}(\text{tp}(a/B))$, we can assume $\mathfrak{M} \models (\bar{c}, a, \bar{b})$. Hence a is fixed as $\bar{b}$. Thus, in fact, we are talking about some types in $S_m(aB)$. □

Exercise 4.2.4. A theory T has quantifier elimination iff every type is implied by its quantifier-free part

Exercise 4.2.5. Consider the structure $\mathfrak{M} = (\mathbb{Q}, <)$. Determine all types in $S_1(\mathbb{Q})$. Which of these types are realised in $\mathbb{R}$? Which extensions does a type over $\mathbb{Q}$ have to a type over $\mathbb{R}$?

Proof. □

4.3 $\aleph_0$-categorical theories

Theorem 4.11 (Ryll-Nardzewski)**.** *Let T be a countable complete theory. Then T is $\aleph_0$-categorical iff for every n there are only finitely many formulas $\varphi(x_1, \dots, x_n)$ up to equivalence relative to T*

Definition 4.12. An L-structure $\mathfrak{A}$ is **ω-saturated** if all types over finite subsets of A are realised in $\mathfrak{A}$

The types in the definition are meant to be 1-types. On the other hand, it is not hard to see that an ω-saturated structure realises all n-types over finite sets (Exercise 4.3.3) for all $n \geq 1$. The following lemma is a generalisation of the $\aleph_0$-categoricity of DLO.

Lemma 4.13. *Two elementarily equivalent, countable and ω-saturated structures are isomorphic*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are as in the lemma. We choose enumerations $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$. Then we construct an ascending sequence $f_0 \subseteq f_1 \subseteq \cdots$ of finite elementary maps

$$f_i : A_i \to B_i$$

between finite subsets of $\mathfrak{A}$ and $\mathfrak{B}$. We will choose the f_i in such a way that A is the union of the A_i and B the union of the B_i. The union of the f_i is then the desired isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$

The empty map $f_0 = \emptyset$ is elementary since $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. Assume that f_i is already constructed. There are two cases:

$i = 2n$; We will extend f_i to $A_{i+1} = A_i \cup \{a_n\}$. Consider the type

$$p(x) = \text{tp}(a_n/A_i) = \{\varphi(x) \mid \mathfrak{A} \vDash \varphi(a_n), \varphi(x) \text{ a } L(A_i)\text{-formula}\}$$

Since f_i is elemantarily, $f_i(p)(x)$ is in $\mathfrak{B}$ a type over B_i. (note that f_i is elementary iff $\mathfrak{A}_{A_i} \equiv \mathfrak{B}_{B_i}$) Since $\mathfrak{B}$ is ω-saturated, there is a realisation b' of this type. So for $\bar{a} \in A_i$

$$\mathfrak{A} \vDash \varphi(a_n, \bar{a}) \Rightarrow \mathfrak{B} \vDash \varphi(b', f_i(\bar{a}))$$

Given b', then the type that it realises is fixed. Hence

$$\mathfrak{B} \vDash \varphi(b', f_i(\bar{a})) \Rightarrow \mathfrak{A} \vDash \varphi(a_n, \bar{a})$$

This shows that $f_{i+1}(a_n) = b'$ defines an elementary extension of f_i

$i = 2n + 1$; we exchange $\mathfrak{A}$ and $\mathfrak{B}$ □

Proof of Theorem 4.11. Assume that there are only finitely many $\varphi(x_1, \dots, x_n)$ relative to T for every n. By Lemma 4.13 it suffices to show that all models of T are ω-saturated. Let $\mathfrak{M}$ be a model of T and A an n-element subset. If there are only N many formulas, up to equivalence, in the variable $x_1, \dots, x_{n+1}$, there are, up to equivalence in $\mathfrak{M}$, at most N many $L(A)$-formulas $\varphi(x)$. Thus, each type $\varphi(x) \in S(A)$ is isolated (w.r.t. $\text{Th}(\mathfrak{M}_A)$) by a smallest formula $\varphi_p(x)$ ($\bigwedge p(x)$). Each element of M which realises $\varphi_p(x)$ also realises $p(x)$, so $\mathfrak{M}$ is ω-saturated.

Conversely, if there are infinitely many $\varphi(x_1, \dots, x_n)$ modulo T for some n, then - as the type space $S_n(T)$ is compact - there must be some non-isolated type p (if p is isolated, then $\{p\}$ is open). Then by Lemma 4.9 p is not isolated in T. By the Omitting Types Theorem there is a countable model of T in which this type is not realised. On the other hand, there also exists a countable model of T realizing this type. So T is not $\aleph_0$-categorical □

The proof shows that a countable complete theory with infinite models is $\aleph_0$-categorical iff all countable models are ω-saturated

given a variables $\varphi_i(a_i)$ where $a_i \in A$, we can consider $\bigwedge \exists x_i \varphi_i(x_i)$.

Definition 4.14. An L-structure $\mathfrak{M}$ is **ω-homogeneous** if for every elementary map f_0 defined on a finite subset A of M and for any $a \in M$ there is some element $b \in M$ s.t.

$$f = f_0 \cup \{\langle a, b\rangle\}$$

is elementary

$f = f_0 \cup \{\langle a, b\rangle\}$ is elementary iff b realises $f_0(\mathrm{tp}(a/A))$

Corollary 4.15. *Let $\mathfrak{A}$ be a structure and $a_1, \dots, a_n$ elements of $\mathfrak{A}$. Then $\mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-categorical iff $\mathrm{Th}(\mathfrak{A}, a_1, \dots, a_n)$ is $\aleph_0$-categorical*

Proof. If $\mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-categorical, then for any $m+n$ there is only finitely many formulas $\varphi(x_1, \dots, x_{m+n})$ up to equivalence relative to $\mathrm{Th}(\mathfrak{A})$, hence there is only finitely many $\varphi(x_1, \dots, x_m, a_1, \dots, a_n)$ up to equivalence relative to $\mathrm{Th}(\mathfrak{A}, a_1, \dots, a_n)$

For the converse, $\mathrm{Th}(\mathfrak{A}) \subset \mathrm{Th}(\mathfrak{A}, a_1, \dots, a_n)$ □

Example 4.1. The following theories and $\aleph_0$-categorical

1. Infset (saturated)

2. For every finite field $\mathbb{F}_q$, the theory of infinite $\mathbb{F}_q$-vector spaces. (Vector spaces over the same field and of the same dimension are isomorphic)

3. The theory DLO of dense linear orders without endpoints. This follows from Theorem 4.11 since DLO has quantifier elimination: for every n there are only finitely many (say N_n) ways to order n elements. Each of these possibility corresponds to a complete formula $\psi(x_1, \dots, x_n)$. Hence there are up to equivalence, exactly 2^{N_n} many formulas $\varphi(x_1, \dots, x_n)$

Definition 4.16. A theory T is **small** if $S_n(T)$ are at most countable for all $n < \omega$

Lemma 4.17. *A countable complete theory is small iff it has a countable ω-saturated model*

Proof. If T has a finite model $\mathfrak{A}$, T is small and $\mathfrak{A}$ is ω-saturated: since T is complete, for any type $p(x) \in S_n(T)$, $T \models p(x)$. For finite model $\mathfrak{A}$, there are only finitely many assignments. If we have two distinct types $p(x), q(x) \in S_n(T)$, then there is $\phi(x) \in p(x)$ and $\phi(x) \notin q(x)$. Since they are maximally consistent, $q(x) \models \neg\phi(x)$ hence $p(x)$ and $q(x)$ cannot be realised by the same element. So we may assume that T has infinite models

If all types can be realised in a single countable model, there can be at most countably many types.

if conversely all $S_{n+1}(T)$ are at most countable, then over any n-element subset of a model of T there are at most countably many types. We construct an elementary chain

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \dots$$

of models of T. For $\mathfrak{A}_0$ we take any countable model. if $\mathfrak{A}_i$ is already constructed, we use Corollary 2.14 and Theorem 2.15 to construct a countable model $\mathfrak{A}_{i+1}$ in such a way that all types over finite subsets of A_i are realised in $\mathfrak{A}_{i+1}$. This can be done since there are only countable many such types. The union $\mathfrak{A} = \bigcup_{i\in\omega} \mathfrak{A}_i$ is countable and ω-saturated since every type over a finite subset B of $\mathfrak{A}$ is realised in $\mathfrak{A}_{i+1}$ if $B \subseteq A_i$ □

Theorem 4.18 (Vaught). *A countable complete theory cannot have exactly two countably models*

Proof. We can assume that T is small and not $\aleph_0$-categorical (if T is not small, then it has no countable model). We will show that T has at least three non-isomorphic countable models. First, T has an ω-saturated countable model $\mathfrak{A}$ and there is a non-isolated type $p(\bar{x})$ which can be omitted in a countable model $\mathfrak{B}$. Let $p(\bar{x})$ be realised in $\mathfrak{A}$ by $\bar{a}$. Since $\mathrm{Th}(\mathfrak{A}, \bar{a})$ is not $\aleph_0$-categorical as $T \subset \mathrm{Th}(\mathfrak{A}, \bar{a})$, $\mathrm{Th}(\mathfrak{A}, \bar{a})$ has a countable model $(\mathfrak{C}, \bar{c})$ which is not ω-saturated. Then $\mathfrak{C}$ is not ω-saturated and therefore not isomorphic to $\mathfrak{A}$. But $\mathfrak{C}$ realises $p(\bar{x})$ and is therefore not isomorphic to $\mathfrak{B}$ □

Exercise 4.3.1. Show that T is $\aleph_0$-categorical iff $S_n(T)$ is finite for all n

Exercise 4.3.2. Show that for every $n > 2$ there is a countable complete theory with exactly n countable models

Proof. StackExchange □

Exercise 4.3.3. If $\mathfrak{A}$ is ω-saturated, all n-types over finite sets are realised.

Proof. Assume that $\mathfrak{A}$ is κ-saturated, B a subset of A of smaller cardinality than κ and $p(x, \bar{y})$ a $(n+1)$-type over B. Let $\bar{b} \in A$ be a realisation of $q(\bar{y}) = p\restriction\bar{y}$ and $a \in A$ a realisation of $p(x, \bar{b})$. Then $(a, \bar{b})$ realises p. □

4.4 The amalgamation method

Definition 4.19. For any language L, the **skeleton** $\mathcal{K}$ of an L-structure $\mathfrak{M}$ is the class of all finitely-generated L-structures which are isomorphic to a substructure of $\mathfrak{M}$. We say that an L-structure $\mathfrak{M}$ is $\mathcal{K}$**-saturated** if its skeleton is $\mathcal{K}$ and if for all $\mathfrak{A}$, $\mathfrak{B}$ in $\mathcal{K}$ and all embeddings $f_0 : \mathfrak{A} \to \mathfrak{M}$ and $f_1 : \mathfrak{A} \to \mathfrak{B}$ there is an embedding $g_1 : \mathfrak{B} \to \mathfrak{M}$ with $f_0 = g_1 \circ f_1$

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "f_0"] \arrow[dr, "f_1"'] & & \mathfrak{M} \\ & \mathfrak{B} \arrow[ur, "g"'] & \end{tikzcd}$$

Theorem 4.20. *Let L be a countable language. Any two countable $\mathcal{K}$-saturated structures are isomorphic*

Proof. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable L-structures with the same skeleton $\mathcal{K}$, and assume that $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{K}$-saturated. As in the proof of Lemma 4.13 we construct an isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ as the union of an ascending sequence of isomorphisms between finitely-generated substructures of M and N.

If $f_1 : \mathfrak{A} \to \mathfrak{N}$ is an embedding of a finitely-generated substructure of $\mathfrak{A}$ of $\mathfrak{M}$ into $\mathfrak{N}$, and a is an element of $\mathfrak{M}$, then by $\mathcal{K}$-saturation f_1 can be extended to an embedding $g_1 : \mathfrak{A}' \to \mathfrak{N}$ where $\mathfrak{A}' = \langle Aa \rangle^{\mathfrak{M}}$. Now interchange the roles of $\mathfrak{M}$ and $\mathfrak{N}$. $\square$

The proof shows that any countable $\mathcal{K}$-saturated structure $\mathfrak{M}$ is **ultrahomogeneous** i.e., any isomorphism between finitely generated substructure extends to an automorphism of $\mathfrak{M}$.

Theorem 4.21. *Let L be a countable language and $\mathcal{K}$ a countable class of finitely-generated L-structures. There is a countable $\mathcal{K}$-saturated L-structure $\mathfrak{M}$ iff*

1. (*Heredity*) *if $\mathfrak{A}_0 \in \mathcal{K}$, then all elements of the skeleton of $\mathfrak{A}_0$ also belongs to $\mathcal{K}$*

2. (*Joint Embedding*) *for $\mathfrak{B}_0, \mathfrak{B}_1 \in \mathcal{K}$ there are some $\mathcal{D} \in \mathcal{K}$ and embeddings $g_i : \mathfrak{B}_i \to \mathfrak{D}$*

3. (*Amalgamation*) *if $\mathfrak{A}, \mathfrak{B}_0, \mathfrak{B}_1 \in \mathcal{K}$ and $f_i : \mathfrak{A} \to \mathfrak{B}_i$, $(i = 0, 1)$ are embeddings, there is some $\mathcal{D} \in \mathcal{K}$ and two embeddings $g_i : \mathfrak{B}_i \to \mathfrak{D}$ s.t. $g_0 \circ f_0 = g_1 \circ f_1$*

$$\begin{tikzcd}
 & \mathfrak{D} & \\
\mathfrak{B}_0 \arrow[ur, "g_0"] & & \mathfrak{B}_1 \arrow[ul, "g_1"'] \\
 & \mathfrak{A} \arrow[ul, "f_0"] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

in this case, $\mathfrak{M}$ is unique up to isomorphism and is called the Fraïssé limit of $\mathcal{K}$

Proof. Let $\mathcal{K}$ be the skeleton of a countably $\mathcal{K}$-saturated structure $\mathfrak{M}$. Clearly, $\mathcal{K}$ has the hereditary property (substructure of a substructure is still a substructure). To see that $\mathcal{K}$ has the Amalgamation Property, let $\mathfrak{A}, \mathfrak{B}_0, \mathfrak{B}_1, f_0$ and f_1 be as in 3. We may assume that $\mathfrak{B}_0 \subseteq \mathfrak{M}$ and f_0 is the inclusion map. Furthermore we can assume $\mathfrak{A} \subseteq \mathfrak{B}_1$ and that f_1 is the inclusion map. Now the embedding $g_1 : \mathfrak{B}_1 \to \mathfrak{M}$ is the extension of the isomorphism $f_0 : \mathfrak{A} \to f_0(\mathfrak{A})$ to $\mathfrak{B}_1$ and satisfies $f_0 = g_1 \circ f_1$. For $\mathfrak{D}$ we choose a finitely-generated substructure of $\mathfrak{M}$ which contains $\mathfrak{B}_0$ and the image of g_1. For $g_0 : \mathfrak{B}_0 \to \mathfrak{D}$ take the inclusion map. For Joint Embedding Property take $\langle B_0 B_1 \rangle^{\mathfrak{M}}$

For the converse assume that $\mathcal{K}$ has properties 1, 2 and 3. Choose an enumeration $(\mathfrak{B}_i)_{i \in \omega}$ of all isomorphism types in $\mathcal{K}$ (they are not isomorphic). We construct $\mathfrak{M}$ as the union of an ascending chain

$$\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \cdots \subseteq \mathfrak{M}$$

of elements of $\mathcal{K}$. Suppose that $\mathfrak{M}_i$ is already constructed. If $i = 2n$, we choose $\mathfrak{M}_{i+1}$ as the top of a diagram

$$\begin{tikzcd}
 & \mathfrak{M}_{i+1} & \\
\mathfrak{M}_i \arrow[ur, "g_0"] & & \mathfrak{B}_n \arrow[ul, "g_1"']
\end{tikzcd}$$

where we can assume that g_0 is the inclusion map. if $i = 2n + 1$, let $\mathfrak{A}$ and $\mathfrak{B}$ from $\mathcal{K}$ and two embeddings $f_0 : \mathfrak{A} \to \mathfrak{M}_i$ and $f_1 : \mathfrak{A} \to \mathfrak{B}$ be given.

$$\begin{tikzcd}
& \mathfrak{M}_{i+1} & \\
\mathfrak{M}_i \arrow[ur, "g_0"] & & \mathfrak{B} \arrow[ul, "g_1"'] \\
& \mathfrak{A} \arrow[ul, "f_0"] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

To ensure that $\mathfrak{M}$ is $\mathcal{K}$-saturated we have in the odd steps to make the right choice of $\mathfrak{A}, \mathfrak{B}, f_0$ and f_1. Assume that we have $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and embeddings $f_0 : \mathfrak{A} \to \mathfrak{M}$ and $f_1 : \mathfrak{A} \to \mathfrak{B}$. For large j the image of f_0 will be contained in $\mathfrak{M}_j$. During the construction of the $\mathfrak{M}_i$, in order to guarantee the $\mathcal{K}$-saturation of $\mathfrak{M}$, we have to ensure that eventually, for some odd $i \geq j$, the embeddings $f_0 : \mathfrak{A} \to \mathfrak{M}_i$ and $f_1 : \mathfrak{A} \to \mathfrak{B}$ were used in the construction of $\mathfrak{M}_{i+1}$. This can be done since for each j there are - up to isomorphism - at most countably many possibilities. Thus there exists an embedding $g_1 : \mathfrak{B} \to \mathfrak{M}_{i+1}$ with $f_0 = g_1 \circ f_1$.

$\mathcal{K}$ is the skeleton of $\mathfrak{M}$: the finitely-generated substructure are the substructures of the $\mathfrak{M}_1$. Since $\mathfrak{M}_i \in \mathcal{K}$, their finitely-generated substructure also belong to $\mathcal{K}$. On the other hand each B_n is isomorphic to a substructure of $\mathfrak{M}_{2n+1}$

Uniqueness follows from Theorem 4.20 □

For finite relational languages L, any non-empty finite subset is itself a (finitely-generated) substructure. For such languages, the construction yields $\aleph_0$-categorical structures. We now take a look at $\aleph_0$-categorical theories with quantifier elimination in a **finite relational language**

Remark. A complete theory T in a finite relational language with quantifier elimination is $\aleph_0$-categorical. So all its models are ω-homogeneous

Proof. For every n there is only a finite number of non-equivalent quantifier free formulas $\rho(x_1, \dots, x_n)$. If T has quantifier elimination, this number is also the number of all formulas $\varphi(x_1, \dots, x_n)$ modulo T and so T is $\aleph_0$-categorical by Theorem 4.11 □

Lemma 4.22. *Let T be a complete theory in a finite relational language and $\mathfrak{M}$ an infinite model of T. TFAE*

1. *T has quantifier elimination*
2. *Any isomorphism between finite substructures is elementary*

3. *the domain of any isomorphism between finite substructures can be extended to any further element*

Proof. $2 \to 1$. if any isomorphism between finite substructure of $\mathfrak{M}$ is elementary, all n-tuples $\bar{a}$ which satisfy in $\mathfrak{M}$ the same quantifier-free type

$$\text{tp}_{\text{qf}}(\bar{a}) = \{\rho(\bar{x}) \mid \mathfrak{M} \vDash \rho(\bar{a}), \rho(\bar{x}) \text{ quantifier-free}\}$$

satisfy the same simple existential formulas. We will show from this that every simple existential formula $\varphi(x_1, \dots, x_n) = \exists y \rho(x_1, \dots, x_n, y)$ is, modulo T, equivalent to a quantifier-free formula. Let $r_1(\bar{x}), \dots, r_{k-1}(\bar{x})$ be the quantifier-free types of all n-tuples in $\mathfrak{M}$ which satisfy $\varphi(\bar{x})$. Let $\rho_i(\bar{x})$ be equivalent to the conjunction of all formulas from $r_i(\bar{x})$. Then

$$T \vDash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \bigvee_{i<k} \rho_i(\bar{x}))$$

$1 \to 3$ the theory T is $\aleph_0$-categorical and hence all models are ω-homogeneous. Since any isomorphism between finite substructures is elementary, 3 follows.

$3 \to 2$. If the domain of any finite isomorphism can be extended to any further element, it is easy to see that every finite isomorphism is elementary. Here we can only consider $\exists x \varphi(x)$. ☐

Theorem 4.23. *Let L be a finite relational language and $\mathcal{K}$ a class of finite L-structures. If the Fraïssé limit of $\mathcal{K}$ exists, its theory is $\aleph_0$-categorical and has quantifier elimination*

4.5 Prime Models

Let T be a countable complete theory with infinite models

Definition 4.24. Let T be a countable theory with infinite models, not necessarily complete

1. We call $\mathfrak{A}_0$ a **prime model** of T if $\mathfrak{A}_0$ can be elementarily embedded into all models of T

2. A structure $\mathfrak{A}$ is called **atomic** if all n-tuples $\bar{a}$ of elements of $\mathfrak{A}$ are atomic. This means that the types $\text{tp}(\bar{a})$ are isolated in $S_n^{\mathfrak{A}}(\emptyset) = S_n(T)$

Prime models need not exists. By Corollary 4.10, a tuple $\bar{a}$ is atomic iff it satisfies a complete formula.

Since T has countable models, prime models must be countable and since non-isolated types can be omitted in suitable models by Theorem 4.2, only isolated types can be realised in prime models. So any tp$(\bar{a})$ of a prime model must be isolated.

Theorem 4.25. *A model of T is prime iff it is countable and atomic*

Proof. As just noted, a prime model has to be countable and atomic.

Let $\mathfrak{M}_0$ be a countable and atomic model of T and $\mathfrak{M}$ any model of T. We construct an elementary embedding of $\mathfrak{M}_0$ to $\mathfrak{M}$ as a union of an ascending sequence of elementary maps

$$f : A \to B$$

between finite subsets A of M_0 and B of M. The empty map is elementary since T is complete and $\mathfrak{M}_0 \equiv \mathfrak{M}$

We show that f can be extended to any given $A \cup \{a\}$. Let $p(x) = \text{tp}(a/A)$ and $f(p) = f(p(x))$. We show that $f(p)$ has a realisation $b \in M$

Let $\bar{a}$ be a tuple which enumerates the elements of A and $\varphi(x, \bar{x})$ an L-formula which isolates the tp$(a\bar{a}/A)$ since $\mathfrak{M}_0$ is atomic. Then $p(x)$ is isolated by $\varphi(x, \bar{a})$: clearly $\varphi(x, \bar{a}) \in \text{tp}(a/\bar{a})$ and if $\rho(x, \bar{a}) \in \text{tp}(a/\bar{a})$ we have $\rho(x, y) \in \text{tp}(a, \bar{a})$. This implies that $\mathfrak{M}_0 \vDash \forall x, y(\varphi(x, y) \to \rho(x, y))$ and $\mathfrak{M} \vDash \forall x(\varphi(x, \bar{a}) \to \rho(x, \bar{a}))$. Thus $f(p)$ is isolated by $\varphi(x, f(\bar{a}))$ and since $\varphi(x, f(\bar{a}))$ can be realised in $\mathfrak{M}$, so can be $f(p)$. $\square$

Definition 4.26. The isolated types are **dense** in T if every consistent L-formulas $\psi(x_1, \dots, x_n)$ belongs to an isolated type $p(x_1, \dots, x_n) \in S_n(T)$

Example 4.2. Let T be the language having a unary predicate P_s for every finite 0-1-sequence $s \in 2^{<\omega}$. The axioms of Tree say that the $P_s, s \in 2^{<\omega}$, form a binary decomposition of the universe

- $\forall x \; P_\emptyset(x)$
- $\exists x \; P_s(x)$
- $\forall x \; ((P_{s0}(x) \vee P_{s1}(x)) \leftrightarrow P_s(x))$
- $\forall x \; \neg(P_{s0}(x) \wedge P_{s1}(x))$

Tree is complete and has quantifier elimination. There are no complete formulas and no prime model

See Marker to see the full content

Definition 4.27. A family of formulas $\varphi_s(\bar{x})$, $s \in 2^{<\omega}$ is a **binary tree** if for all $s \in 2^{<\omega}$ the following holds

1. $T \models \forall \bar{x}\ ((\varphi_{s0}(\bar{x}) \vee \varphi_{s1}(\bar{x})) \rightarrow \varphi_s(\bar{x}))$
2. $T \models \forall \bar{x}\ \neg(\varphi_{s0}(\bar{x}) \wedge \varphi_{s1}(\bar{x}))$

Theorem 4.28. *Let T be a complete theory*

1. *If T is small, it has no binary tree of consistent L-formulas. If T is countable, the converse holds as well*
2. *If T has no binary tree of consistent L-formulas, the isolated types are dense*

Proof. 1. Let $(\varphi_s(x_1, \dots, x_n))$ be a binary tree of consistent formulas. Then, for all $\eta \in 2^\omega$, the set

$$\{\varphi_s(\bar{x}) \mid s \subseteq \eta\}$$

is consistent and therefore is contained in some type $p_\eta(\bar{x}) \in S_n(T)$. The $p_\eta(\bar{x})$ are all different showing that T is not small.

□

Exercise 4.5.1. Countable theories without a binary tree of consistent formulas are small

Proof. If countable theory T is not small. □

Exercise 4.5.2. Show that isolated types being dense is equivalent to isolated types being (topologically) dense in the Stone space $S_n(T)$.

Proof. Let $S = \{$the isolated types in $S_n(T)\}$. S is dense in $S_n(T)$ iff $\bar{S} = S_n(T)$. For any $p \in S_n(T) \setminus S$, p is non-isolated. For any $p \in [\phi]$, ϕ belongs to an isolated type q. Thus $q \in S_n(T) \cap S$. Hence $\bar{S} = S_n(T)$. □

5 $\aleph_1$-categorical Theories

5.1 Indiscernibles

Definition 5.1. Let I be a linear order and $\mathfrak{A}$ an L-structure. A family $(a_i)_{i \in I}$ of elements of A is called a **sequence of indiscernibles** if for all L-formulas $\varphi(x_1, \dots, x_n)$ and all $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ from I

$$\mathfrak{A} \models \varphi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n})$$

if two of the a_i are equal, all a_i are the same. Therefore it is often assumed that the a_i are distinct

Sometimes sequences of indiscernibles are also called **order indiscernible** to distinguish them from **totally indiscernible** sequences in which the ordering of the index set does not matter.

Definition 5.2. Let I be an infinite linear order and $\mathcal{I} = (a_i)_{i\in I}$ a sequence of k-tuples in $\mathfrak{M}$, $A \subseteq M$. The **Ehrenfeucht-Mostowski type EM**$(\mathcal{I}/A)$ of $\mathcal{I}$ over A is the set of $L(A)$-formulas $\varphi(x_1, \dots, x_n)$ with $\mathfrak{M} \vDash \varphi(a_{i_1}, \dots, a_{i_n})$ for all $i_1 < \dots < i_n \in I$, $n < \omega$

Lemma 5.3 (The Standard Lemma)**.** *Let I and J be two infinite linear orders and $\mathcal{I} = (a_i)_{i\in I}$ a sequence of elements of a structure $\mathfrak{M}$. Then there is structure $\mathfrak{N} \equiv \mathfrak{M}$ with an indiscernible sequences $(b_j)_{j\in J}$ realizing the Ehrenfeucht-Mostowski type **EM**$(\mathcal{I})$ of $\mathcal{I}$*

Corollary 5.4. *Assume that T has an infinite model. Then for any linear order I, T has a model with a sequence $(a_i)_{i\in I}$ of distinct indiscernibles*

Let $[A]^n$ denote the set of all n-element subsets of A

Theorem 5.5 (Ramsey)**.** *Let A be infinite and $n \in \omega$. Partition the set of n-elements subsets $[A]^n$ into subsets $C_1, \dots, C_k$. Then there is an infinite subset of A whose n-element subsets all belong to the same subset C_i*

Proof. Thinking of the partition as a colouring on $[A]^n$, we are looking for an infinite subset B of A s.t. $[B]^n$ is monochromatic. We prove the theorem by induction on n. For $n = 1$, the statement is evident from the pigeonhole principle since there are infinite elements and finite colors.

Assuming the theorem is true for n, we now prove it for $n+1$. Let $a_0 \in A$. Then any colouring of $[A]^{n+1}$ induces a colouring of the n-element subsets of $A' = A \setminus \{a_0\}$: just colour $x \in [A']^n$ by the colour of $\{a_0\} \cup x \in [A]^{n+1}$. By the induction hypothesis, there exists an infinite monochromatic subset B_1 of A' in the induced colouring. Thus, all the $(n+1)$-element subsets of A consisting of a_0 and n elements of B_1 have the same colour but $\{a_0\} \cup B$ is not our desired set.

Now pick any $a_1 \in B_1$. By the same argument we obtain an infinite subset $B_2 \subseteq B_1$ with the same properties. We thus construct an infinite sequence $A = B_0 \supset B_1 \supset B_2 \supset \dots$ and elements $a_i \in B_i \setminus B_{i+1}$ s.t. the colour of each $(n+1)$-element subset $\{a_{i(0)}, \dots, a_{i(n)}\}$ with $i(0) < i(1) < \dots < i(n)$ depends only on the value of $i(0)$.

$$a_0, a_1, a_2, \dots, a_n, \dots$$

Again by the pigeonhole principle there are infinitely many values of $i(0)$ for which this colour will be the same and we take $\{a_{i(0)}\}$. These $a_{i(0)}$ then yields the desired monochromatic set. $\square$

Proof of Lemma **??**. Choose a set C of new constants with an ordering isomorphic to J. Consider the theories

$$T' = \{\varphi(\bar{c}) \mid \varphi(\bar{x}) \in \mathbf{EM}(\mathcal{J})\}$$
$$T'' = \{\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}) \mid \bar{c}, \bar{d} \in C\}$$

Here the $\varphi(\bar{x})$ are L-formulas and $\bar{c}, \bar{d}$ tuples in increasing order. We have to show that $T \cup T' \cup T''$ is consistent. It is enough to show that

$$T_{C_0,\Delta} = T \cup \{\varphi(\bar{c}) \in T' \mid \bar{c} \in C_0\} \cup \{\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}) \mid \varphi(\bar{x}) \in \Delta, \bar{c}, \bar{d} \in C_0\}$$

is consistent for finite sets C_0 and Δ. Note that $\mathrm{Diag}_{\mathrm{el}}(\mathfrak{M}) \subseteq T$. We can assume that the elements of Δ are formulas with free variables $x_1, \dots, x_n$ and that all tuples $\bar{c}$ and $\bar{d}$ have the same length

for notational simplicity we assume that all a_i are different. So we may consider $A = \{a_i \mid i \in I\}$ as an ordered set, which is the interpretation of C. We define an equivalence relation on $[A]^n$ by

$$\bar{a} \sim \bar{b} \iff \mathfrak{M} \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b}) \text{ for all } \varphi(x_1, \dots, x_n) \in \Delta$$

where $\bar{a}, \bar{b}$ are tuples in increasing order. Since this equivalence has at most $2^{|\Delta|}$ many classes, by Ramsey's Theorem there is an infinite subset $B \subseteq A$ with all n-element subsets in the same equivalence class. We interpret the constants $c \in C_0$ by elements b_c in B ordered in the same way as the c. Then $(\mathfrak{M}, b_c)_{c \in C_0}$ is a model of $T_{C_0,\Delta}$. $\square$

Lemma 5.6. *Assume L is countable. If the L-structure $\mathfrak{M}$ is generated by a well-ordered sequence (a_i) of indiscernibles, then $\mathfrak{M}$ realises only countably many types over every countable subset of M*

Proof. : need more time to think

If $A = \{a_i \mid i \in I\}$, then every element $b \in M$ has the form $b = t(\bar{a})$, where t is an L-term and $\bar{a}$ is a tuple from A since $\mathfrak{M}$ is generated by (a_i)

Consider a countable subset S of M. Write

$$S = \{t_n^{\mathcal{M}}(\bar{a}^n) \mid n \in \omega\}$$

Let $A_0 = \{a_i \mid i \in I_0\}$ be the (countable) set of elements of A which occur in the $\bar{a}^n$. Then every type $\mathrm{tp}(b/S)$ is determined by $\mathrm{tp}(b/A_0)$ since every $L(S)$-formula

$$\varphi(x, t^{\mathcal{M}}_{n_1}(\bar{a}^n), \dots)$$

can be replaced by the $L(A_0)$-formula $\varphi(x, t_{n_1}(\bar{a}^{n_1}), \dots)$

$\mathrm{tp}(b/A_0) = \mathrm{tp}(t(\bar{a})/A_0) = \{\varphi(\bar{x})\ \mathcal{L}_{A_0}\text{-formula} : \mathfrak{M} \vDash \varphi(t(\bar{a}))\}$.

Now the type of $b = t(\bar{a})$ over A_0 depends only on $t(\bar{x})$ (countably many possibilities) and the type $\mathrm{tp}(\bar{a}/A_0)$ (really?). Write $\bar{a} = a_{\bar{i}}$ for a tuple $\bar{i}$ from I. Since the a_i are indiscernible, the type depends only on the quantifier-free type $\mathrm{tp}_{\mathrm{qf}}(\bar{i}/I_0)$ in the structure $(I, <)$ since it has quantifier elimination. This type again depends on $\mathrm{tp}_{\mathrm{qf}}(\bar{i})$ (finitely many possibilities) and on the types $p(x) = \mathrm{tp}_{\mathrm{qf}}(i/I_0)$ of the elements i (Note the quantifier elimination, then we only need to Booleanly combine these things to get $\mathrm{tp}_{\mathrm{qf}}(\bar{i}/I_0)$) of $\bar{i}$. There are three kinds of such types:

1. i is bigger than all elements of I_0
2. i is an element i_0 of I_0
3. For some $i_0 \in I_0$, i is smaller than i_0 but bigger than all elements of $\{j \in I_0 \mid j < i_0\}$

There is only one type in the first case, in the other case the type is determined by i_0. This results in countably many possibilities for each component of $\bar{i}$ □

Definition 5.7. Let L be a language. A **Skolem theory** $\mathsf{Skolem}(L)$ is a theory in a bigger language L_{Skolem} with the following properties

1. $\mathsf{Skolem}(L)$ has quantifier elimination
2. $\mathsf{Skolem}(L)$ is universal
3. Every L-structure can be expanded to a model of $\mathsf{Skolem}(L)$
4. $|L_{\mathsf{Skolem}}| \leq \max(|L|, \aleph_0)$

Theorem 5.8. *Every language L has a Skolem theory.*

Proof. Nice slide. We have

1. $\exists x P(x)$ is a consequence of $P(a)$

2. $P(a)$ is not a consequence of $\exists x P(x)$, but a model of $\exists x P(x)$ **provides** a model of $P(a)$

Skolemization eliminates existential quantifiers and transforms a closed formula A to a formula B such that :

- A is a consequence of B, $B \vDash A$
- every model of A **provides** a model of B

Hence, A has a model if and only if B has a model : *skolemization preserves the existence of a model*, in other words *it preserves satisfiability*.

We define an ascending sequence of languages

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$

by introducing for every quantifier-free L_i-formula $\varphi(x_1, \dots, x_n, y)$ a new n-place **Skolem function** f_φ (if $n = 0$, f_φ is a constant) and defining L_{i+1} as the union of L_i and the set of these function symbols. The language L_{Skolem} is the union of all L_i. We now define the Skolem theory as

$$\mathsf{Skolem} = \{\forall \bar{x}(\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, f_\varphi(\bar{x}))) \mid \varphi(\bar{x}, y) \text{ q.f. } L_{\mathsf{Skolem}}\text{-formula}\}$$

□

Corollary 5.9. *Let T be a countable theory with an infinite model and let κ be an infinite cardinal. Then T has a model of cardinality κ which realises only countably many types over every countable subset.*

Proof. Consider the theory $T^* = T \cup \mathsf{Skolem}(L)$. Then T^* is countable, has an infinite model and quantifier elimination

Claim. T^* is equivalent to a universal theory

Proof. Modulo $\mathsf{Skolem}(L)$ every axiom φ of T is equivalent to a quantifier-free L_{Skolem}-sentence φ^*. Therefore T^* is equivalent to the universal theory

Let I be a well-ordering of cardinality κ and $\mathfrak{N}^*$ a model of T^* with indiscernibles $(a_i)_{i \in I}$ (Existence by the Standard Lemma 5.3). The claim implies that the substructure $\mathfrak{M}^*$ generated by the a_i is a model of T^* and $\mathfrak{M}^*$ has cardinality κ (As we can't control the size of an elementary extension and Corollary 3.5). Since T^* has quantifier elimination, $\mathfrak{M}^*$ is an elementary substructure of $\mathfrak{N}^*$ and (a_i) is indiscernible in $\mathfrak{M}^*$. By Lemma 5.6, there are only countably many types over every countable set realised in $\mathfrak{M}^*$. The same is then true for the reduct $\mathfrak{M} = \mathfrak{M}^*\big|_L$ □

Exercise 5.1.1. A sequence of elements in $(\mathbb{Q}, <)$ is indiscernible iff it is either constant, strictly increasing or strictly decreasing

Proof. For any formula $\varphi(x_1, x_2, \dots, x_n)$,

$$\mathbb{Q} \models \varphi(x_1, x_2, \dots, x_n) \leftrightarrow$$

$\square$

5.2 ω-stable theories

In this section we fix a complete theory T with infinite models

In the previous section we saw that we may add indiscernible elements to a model without changing the number of realised types. We will now use this to show that $\aleph_1$-categorical theories a small number of types, i.e., they are ω-stable. Conversely, with few types it is easier to be saturated and since saturated structures are unique we find the connection to categorical theories.

Definition 5.10. Let κ be an infinite cardinal. We say T is **κ-stable** if in each model of T, over every set of parameters of size at most κ, and for each n, there are at most κ many n-types, i.e.,

$$|A| \leq \kappa \Rightarrow |S_n(A)| \leq \kappa$$

Note that if T is κ-stable, then - up to logical equivalence - we have $|T| \leq \kappa$ (Exercise 5.2.3)

Lemma 5.11. *T is κ-stable iff T is κ-stable for 1-types, i.e.,*

$$|A| \leq \kappa \Rightarrow |S(A)| \leq \kappa$$

Proof. Assume that T is κ-stable for 1-types. We show that T is κ-stable for n-types by induction on n. Let A be a subset of the model $\mathfrak{M}$ and $|A| \leq \kappa$. We may assume that all types over A are realised in $\mathfrak{M}$ (otherwise we take some elementary extensions by Corollary 2.14). Consider the restriction map $\pi : S_n(A) \to S_1(A)$. By assumption the image $S_1(A)$ has cardinality at most κ. Every $p \in S_1(A)$ has the form tp(a/A) for some $a \in M$ since all types over A are realized in $\mathfrak{M}$. By Exercise 4.2.3, the fibre $\pi^{-1}(p)$ is in bijection with $S_{n-1}(aA)$ and so has cardinality at most κ by induction. This shows $|S_n(A)| < \kappa$. $\square$

Example 5.1 (Algebraically closed fields)**.** The theories ACF_p for p a prime or 0 are κ-stable for all κ

Note that by Theorem 5.14 below it would suffice to prove that the theories ACF_p are ω-stable

Proof. Let K be a subfield of an algebraically closed field. By quantifier elimination, the type of an element a over K is determined by the isomorphism type of the extension $K[a]/K$. If a is transcendental over K, $K[a]$ is isomorphic to the polynomial ring $K[X]$. If a is algebraic with minimal polynomial $f \in K[X]$, then $K[a]$ is isomorphic to $K[X]/(f)$. So there is one more 1-type over K than there are irreducible polynomials □

That ACF_p is κ-stable for n-types has a direct algebraic proof: the isomorphism type of $K[a_1, \dots, a_n]/K$ is determined by the vanishing ideal P of $a_1, \dots, a_n$. By :((((

Theorem 5.12. *A countable theory T which is categorical in an uncountable cardinal κ is ω-stable*

Proof. Let $\mathfrak{N}$ be a model and $A \subseteq N$ countable with $S(A)$ uncountable. Let $(b_i)_{i \in I}$ be a sequence of $\aleph_1$ many elements with pairwise distinct types over A. (Note that we can assume that all types over A are realised in $\mathfrak{N}$) We choose first an elementary substructure $\mathfrak{M}_0$ of cardinality $\aleph_1$ which contains A and all b_i. Then we choose an elementary extension $\mathfrak{M}$ of $\mathfrak{M}_0$. The model $\mathfrak{M}$ is of cardinality κ and realises uncountably many types over the countable set A. By Corollary 5.9, T has another model where this is not the case. So T cannot be κ-categorical □

Definition 5.13. A theory T is **totally transcendental** if it has no model $\mathfrak{M}$ with a binary tree of consistent $L(M)$-formulas

Theorem 5.14.

1. *ω-stable theories are totally transcendental*
2. *Totally transcendental theories are κ-stable for all $\kappa \geq |T|$*

It follows that a countable theory T is ω-stable iff it is totally transcendental

Proof.

1. Let $\mathfrak{M}$ be a model with a binary tree of consistent $L(M)$-formulas with free variables among $x_1, \dots, x_n$. The set A of parameters which occur in the tree's formulas is countable but $S_n(A)$ has cardinality $2^{\aleph_0}$
2. Assume that there are there are more than κ many n-types over some set A of cardinality κ. Let us call an $L(A)$-formula **big** if it belongs to more than κ many types over A ($|[\phi]| > \kappa$) and **thin** otherwise.

By assumption the true formula is big. If we can show that each big formula decomposes into two big formulas, we can construct a binary tree of big formulas, which finishes the proof.

So assume that φ is big. Since each thin formula belongs to at most κ types and since there are at most κ formulas, there are at most κ types which contain thin formulas. Therefore φ belongs to two distinct types p and q which contain only big formulas. If we separate p and q by $\psi \in p$ and $\neg\psi \in q$, we decompose φ into the big formulas $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$.

□

The general case follows from Exercise 5.2.2

Definition 5.15. Let κ be an infinite cardinal. An L-structure $\mathfrak{A}$ is **κ-saturated** if in $\mathfrak{A}$ all types over sets of cardinality less than κ are realised. An infinite structure $\mathfrak{A}$ is **saturated** if it is $|\mathfrak{A}|$-saturated

Lemma 4.13 generalises to sets

Lemma 5.16. *Elementarily equivalent saturated structures of the same cardinality are isomorphic*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementary equivalent saturated structures each of cardinality κ. We choose enumerations $(a_\alpha)_{\alpha<\kappa}$ and $(b_\alpha)_{\alpha<\kappa}$ of A and B and construct an increasing sequence of elementary maps $f_\alpha : A_\alpha \to B_\alpha$. Assume that the f_β are constructed for all $\beta < \alpha$. The union of the f_β is an elementary map $f^*_\alpha : A^*_\alpha \to B^*_\alpha$. The construction will imply that A^*_α and B^*_α have cardinality at most $|\alpha|$, which is smaller than κ

We write $\alpha = \lambda + n$, and distinguish two cases

$n = 2i$: In this case, we consider $p(x) = \mathrm{tp}(a_{\lambda+i}/A^*_\alpha)$. Realise $f^*_\alpha(p)$ by $b \in B$ and define

$$f_\alpha = f^*_\alpha \cup \{\langle a_{\lambda+i}, b\rangle\}$$

$n = 2i + 1$: Similarly, we find an extension

$$f_\alpha = f^*_\alpha \cup \{\langle a, b_{\lambda+i}\rangle\}$$

Thus $\bigcup_{\alpha<\kappa} f_\alpha$ is the desired isomorphism □

Lemma 5.17. *If T is κ-stable, then for all regular $\lambda \leq \kappa$, there is a model of cardinality κ which is λ-saturated*

Proof. By Exercise 5.2.3 we may assume that $|T| \leq \kappa$. Consider a model $\mathfrak{M}$ of cardinality κ. Since $S(M_\alpha)$ has cardinality κ, Corollary 2.14 and the Löwenheim–Skolem theorem give an elementary extension of cardinality κ in which all types over $\mathfrak{M}$ are realised. So can construct a continuous elementary chain

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \cdots \prec \mathfrak{M}_\alpha \prec \cdots (\alpha < \lambda)$$

of models of T with cardinality κ s.t. all $p \in S(M_\alpha)$ are realised in $\mathfrak{M}_{\alpha+1}$. Then $\mathfrak{M}$ is λ-saturated. In fact, if $|A| < \lambda$ and if $a \in A$ is contained in $M_{\alpha(a)}$ then $\Lambda = \bigcup_{a \in A} \alpha(a)$ is an initial segment of λ of smaller cardinality than λ. So Λ has an upper bound $\mu < \lambda$. It follows that $A \subseteq \mathfrak{M}_\mu$ and all types over A are realised in $\mathfrak{M}_{\mu+1}$ □

Theorem 5.18. *A countable theory T is κ-categorical iff all models of cardinality κ are saturated*

Proof. If all models of cardinality κ are saturated, it follows from Lemma 5.16 that T is κ-categorical

Assume, for the converse that T is κ-categorical. For $\kappa = \aleph_0$ the theorem follows from Theorem 4.11. So we may assume that κ is uncountable. Then T is totally transcendental by Theorem 5.12 and 5.14 and therefore κ-stable by Theorem 5.14.

By Lemma 5.17, all models of T of cardinality κ are μ^+-saturated for all $\mu < \kappa$. i.e., κ-saturated □

Exercise 5.2.1. Show that the theory of an equivalence relation with two infinite classes has quantifier elimination and is ω-stable. Is it $\aleph_1$-categorical?

Exercise 5.2.2. If T is an L-theory and K is a sublanguage of L, the **reduct** $T\restriction K$ is the set of all K-sentences which follow from T. Show that T is totally transcendental iff $T\restriction K$ is ω-stable for all at most countable $K \subseteq L$

Proof. □

Exercise 5.2.3. If T is κ-stable, then *essentially* (i.e., up to logical equivalence) $|T| \leq \kappa$

Proof. If T is κ-stable, then $|S_n(\emptyset)| \leq \kappa$. Choose for any two n-types over the empty set a separating formula. Then any formula is logically equivalent to a finite Boolean combination of these κ-many formulas □

5.3 Prime extensions

Definition 5.19. Let $\mathfrak{M}$ be a model of T and $A \subseteq M$.

1. $\mathfrak{M}$ is a **prime extension** of A (or **prime over** A) if every elementary map $A \to \mathfrak{N}$ extends to an elementary map $\mathfrak{M} \to \mathfrak{N}$

$$\begin{tikzcd} \mathfrak{M} \arrow[r] & \mathfrak{N} \\ A \arrow[u, "id_A"] \arrow[ur] & \end{tikzcd}$$

2. $B \subseteq M$ is **constructible** over A if B has an enumeration

$$B = \{b_\alpha \mid \alpha < \lambda\}$$

   where each b_α is atomic over $A \cup B_\alpha$ ($\mathrm{tp}(b_\alpha / A \cup B_\alpha)$ is isolated), with $B_\alpha = \{b_\mu \mid \mu < \alpha\}$

   b_α is atomic on the sense of $\mathrm{Th}(\mathcal{M}_A)$, maybe $S_1(\mathrm{Th}(\mathcal{M}_A))$

So $\mathfrak{M}$ is a prime extension of A iff $\mathfrak{M}_A$ is a prime model of $\mathrm{Th}(\mathfrak{M}_A)$

Lemma 5.20. *If a model M is constructible over A, then $\mathfrak{M}$ is prime over A*

Proof. Let $(m_\alpha)_{\alpha<\lambda}$ an enumeration of M, s.t. each m_α is atomic over $A \cup M_\alpha$. Let $f : A \to \mathfrak{N}$ be an elementary map. We define inductively an increasing sequence of elementary maps $f_\alpha : A \cup M_{\alpha+1} \to \mathfrak{N}$ with $f_0 = f$. Assume that f_β is defined for all $\beta < \alpha$. The union of these f_β is an elementary map $f'_\alpha : A \cup M_\alpha \to \mathfrak{N}$. Since $p(x) = \mathrm{tp}(a_\alpha / A \cup M_\alpha)$ is isolated, $f'_\alpha(p) \in S(f'_\alpha(A \cup M_\alpha))$ is also isolated and has a realisation b in $\mathfrak{N}$. We set $f_\alpha = f'_\alpha \cup \{\langle a_\alpha, b\rangle\}$

Finally, the union of all f_α ($\alpha < \lambda$) is an elementary embedding $\mathfrak{M} \to \mathfrak{N}$. □

Theorem 5.21. *If T is totally transcendental, every subset of a model of T has a constructible prime extension*

Lemma 5.22. *If T is totally transcendental, the isolated types are dense over every subset of any model*

Proof. Consider a subset A of a model $\mathfrak{M}$. Then $\mathrm{Th}(\mathfrak{M}_A) \supset T$ has no binary tree of consistent formulas. Then $\mathrm{Th}(\mathfrak{M}_A)$ has no binary tree of consistent formulas. By Theorem 4.28 □

Proof of Theorem 5.21. By Lemma 5.20 it suffices to construct an elementary substructure $\mathfrak{M}_0 \prec \mathfrak{M}$ which contains A and is constructible over A. □

5.4 Vaughtian pairs

A crucial fact about uncountably categorical theories is the absence of definable sets whose size is independent of the size of the model in which they live

In this section, T is a countable complete theory with infinite models

Definition 5.23. We say that T has a **Vaughtian pair** if there are two models $\mathfrak{M} \prec \mathfrak{N}$ and an $L(M)$-formula $\varphi(x)$ s.t.

1. $\mathfrak{M} \neq \mathfrak{N}$
2. $\varphi(\mathfrak{M})$ is infinite
3. $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$

If $\varphi(x)$ doesn't contain parameters, we say that T has a Vaughtian pair for $\varphi(x)$

Remark. Notice that T does not have a Vaughtian pair iff every model $\mathfrak{M}$ is a minimal extension of $\varphi(\mathfrak{M}) \cup A$ for any formula $\varphi(x)$ with parameters in $A \subseteq M$ which defines an infinite set in $\mathfrak{M}$. Löwenheim–Skolem Theorem.

Let $\mathfrak{N}$ be a model of T where $\varphi(\mathfrak{N})$ is infinite but has smaller cardinality than $\mathfrak{N}$. The Löwenheim–Skolem Theorem yields an elementary substructure $\mathfrak{M}$ of $\mathfrak{N}$ which contains $\varphi(\mathfrak{N})$ and has the same cardinality as $\varphi(\mathfrak{N})$. Then $\mathfrak{M} \prec \mathfrak{N}$ is a Vaughtian pair for $\varphi(x)$. The next theorem shows that a converse of this observation is also true

Theorem 5.24 (Vaught's Two-cardinal Theorem)**.** *If T has a Vaughtian pair, it has a model $\overline{\mathfrak{M}}$ of cardinality $\aleph_1$ with $\varphi(\overline{\mathfrak{M}})$ countable for some formula $\varphi(x) \in L(\overline{M})$*

Lemma 5.25. *Let T be complete, countable and with infinite models*

1. *Every countable model of T has a countable ω-homogeneous elementary extension*
2. *The union of an elementary chain of ω-homogeneous models is ω-homogeneous*
3. *Two ω-homogeneous countable models of T realizing the same n-types for all $n < \omega$ are isomorphic*

Proof. 1. Let $\mathfrak{M}_0$ be a countable model of T. We realise the countably many types

$$\{f(\text{tp}(a/A)) \mid a, A \subseteq M_0, A \text{ finite}, f : A \to M_0 \text{ elementary}\}$$

in a countable elementary extension $\mathfrak{M}_1$. By iterating this process we obtain an elementary chain

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \cdots$$

whose union is ω-homogeneous

2. Clear

3. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are ω-homogeneous, countable and realise the same n-types. We show that we can extend any finite elementary map $f : \{a_1, \dots, a_i\} \to \{b_1, \dots, b_i\}$; $a_j \mapsto b_j$ to any $a \in A \setminus A_i$. Realise the type $\text{tp}(a_1, \dots, a_i, a)$ by some tuples $\overline{b'} = b'_1, \dots, b'_{i+1}$ in B. Note that $\text{tp}(\bar{a}) = \text{tp}(\overline{b''} = \text{tp}(\bar{b}))$. Using the ω-homogeneity of B we may extend the finite partial isomorphism $g = \{(b'_j, b_j) \mid 1 \leq j \leq i\}$ by (b'_{i+1}, b) for some $b \in B$. Then $f_{i+1} = f_i \cup \{(a, b)\}$ is the required extension. Reverse the roles of B and A we construct the desired isomorphism.

□

Proof of Theorem 5.24. Suppose that the Vaughtian pair is witnessed (in certain models) by some formula $\varphi(x)$. For simplicity we assume that $\varphi(x)$ does not contain parameters (see Exercise 5.4.1). Let P be a new unary predicate. It is easy to find an $L(P)$-theory T_{VP} whose models $(\mathfrak{N}, M)$ consist of a model $\mathfrak{N} \models T$ and a subset M defined by the new predicate P which is the universe of an elementary substructure $\mathfrak{M}$ which together with $\mathfrak{N}$ forms a Vaughtian pair for $\varphi(x)$. As in Marker's p152. Let $\mathfrak{M}$ be the elementary substructure of $\mathfrak{N}$ by Löwenheim–Skolem Theorem . The Löwenheim–Skolem Theorem applied to T_{VP} yields a Vaughtian pair $\mathfrak{M}_0 \prec \mathfrak{N}_0$ for $\varphi(x)$ with $\mathfrak{M}_0, \mathfrak{N}_0$ countable

We first construct an elementary chain

$$(\mathfrak{N}_0, M_0) \prec (\mathfrak{N}_1, M_1) \prec \cdots$$

of countable Vaughtian pairs, with the aim that both components of the union pair

$$(\mathfrak{N}, M)$$

are ω-homogeneous and realise the same n-types. If $(\mathfrak{N}_i, M_i)$ is given, we first choose a countable elementary extension $(\mathfrak{N}', M')$ s.t. $\mathfrak{M}'$ realises all n-types which are realised in $\mathfrak{N}_i$. Add N_i to language and add all $\mathrm{tp}_n^P(\bar{n})$ to T_{VP}. Then we choose as in the proof of Lemma 5.25 a countable elementary extension $(\mathfrak{N}_{i+1}, \mathfrak{M}_{i+1})$ of $(\mathfrak{N}', M')$ for which $\mathfrak{N}_{i+1}$ and $\mathfrak{M}_{i+1}$ are ω-homogeneous

It follows from Lemma 5.25 (3) that $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic.

Next we construct a continuous elementary chain

$$\mathfrak{M}^0 \prec \mathfrak{M}^1 \prec \cdots \prec \mathfrak{M}^\alpha \prec \cdots \quad (\alpha < \omega_1)$$

with $(\mathfrak{M}^{\alpha+1}, \mathfrak{M}^\alpha) \cong (\mathfrak{N}, M)$ for all α. We start with $\mathfrak{M}^0 = \mathfrak{M}$. If $\mathfrak{M}^\alpha$ is constructed, we choose an isomorphism $\mathfrak{M} \to \mathfrak{M}^\alpha$ and extend it to an isomorphism $\mathfrak{N} \to \mathfrak{M}^{\alpha+1}$ (Lemma 1.5). For a countable limit ordinal λ, $\mathfrak{M}^\lambda$ is the union of the $\mathfrak{M}^\alpha$ $(\alpha < \lambda)$. So $\mathfrak{M}^\lambda$ is isomorphic to $\mathfrak{M}$ by Lemma 5.25 (2) and 5.25 (3)

Finally we set

$$\overline{\mathfrak{M}} = \bigcup_{\alpha < \omega_1} \mathfrak{M}^\alpha$$

$\overline{\mathfrak{M}}$ has cardinality $\aleph_1$ while $\varphi(\overline{\mathfrak{M}}) = \varphi(\mathfrak{M}^\alpha) = \varphi(\mathfrak{M}^0)$. If $\overline{\mathfrak{M}} \models \varphi(\bar{a})$, then there is some $\alpha < \omega_1$ s.t. $\bar{a} \in M^\alpha \cong M^0$. □

Corollary 5.26. *If T is categorical in an uncountable cardinality, it does not have a Vaughtian pair*

Proof. If T has a Vaughtian pair, then by Theorem 5.24 it has a model $\mathfrak{M}$ of cardinality $\aleph_1$ s.t. for some $\varphi(x) \in L(M)$ the set $\varphi(\mathfrak{M})$ is countable. On the other hand, if T is categorical in an uncountable cardinal, it is $\aleph_1$-categorical by Corollary **??** and by Theorem **??**, all models of T of cardinality $\aleph_1$ are saturated. In particular, each formula is either satisfied by a finite number or by $\aleph_1$ many elements, a contradiction. □

Corollary 5.27. *Let T be categorical in an uncountable cardinal, $\mathfrak{M}$ a model, and $\varphi(\mathfrak{M})$ infinite and definable over $A \subseteq M$. Then $\mathfrak{M}$ is the unique prime extension of $A \cup \varphi(\mathfrak{M})$*

Proof. By Corollary 5.26, T does not have a Vaughtian pair, so $\mathfrak{M}$ is minimal over $A \cup \varphi(\mathfrak{M})$. If $\mathfrak{N}$ is a prime extension □

Definition 5.28. We say that T **eliminates the quantifier** $\exists^\infty x$ **(there are infinitely many** x), if for every L-formula $\varphi(x, \bar{y})$ there is a finite bound n_φ

s.t. in all models $\mathfrak{M}$ of T and for all parameters $\bar{a} \in M$

$$\varphi(\mathfrak{M}, \bar{a})$$

is either infinite or has at most n_φ elements

Remark. This means that for all $\varphi(x, \bar{y})$ there is a $\psi(\bar{y})$ s.t. in all models $\mathfrak{M}$ of T and for all $\bar{a} \in M$

$$\mathfrak{M} \models \exists^\infty x \varphi(x, \bar{a}) \iff \mathfrak{M} \models \psi(\bar{a})$$

We denote this by

$$T \models \forall \bar{y} \, (\exists^\infty x \varphi(x, \bar{y}) \leftrightarrow \psi(\bar{y}))$$

Proof. If n_φ exists, we can use $\psi(\bar{y}) = \exists^{>n_\varphi} x \varphi(x, \bar{y})$. If conversely $\psi(\bar{y})$ is a formula which is implied by $\exists^\infty x \varphi(x, \bar{y})$, a compactness argument shows that there must be a bound n_φ s.t.

$$T \models \exists^{>n_\varphi} x \varphi(x, \bar{y}) \to \psi(\bar{y})$$

First note that T is complete. If there is no such bound, then for any $n \in \mathbb{N}$, $T \not\models \exists^{>n} x \varphi(x, \bar{y}) \to \psi(\bar{y})$, which is $T \models \exists^{>n} x \varphi(x, \bar{y}) \wedge \neg \psi(\bar{y})$. Thus by compactness $T \models \exists^\infty x \varphi(x, \bar{y}) \wedge \neg\psi(\bar{y})$, a contradiction. □

Lemma 5.29. *A theory T without Vaughtian pair eliminates the quantifier $\exists^\infty x$*

Proof. Let P be a new unary predicate and $c_1, \dots, c_n$ new constants. Let T^* be the theory of all $L \cup \{P, c_1, \dots, c_n\}$-structures

$$(\mathfrak{M}, N, a_1, \dots, a_n)$$

where $\mathfrak{M}$ is a model of T, N is the universe of a proper elementary substructure, $a_1, \dots, a_n$ elements of N and $\varphi(\mathfrak{M}, \bar{a}) \subseteq N$. Suppose that the bound n_φ does not exists. Then, for any n, there is a model $\mathfrak{N}$ of T and $\bar{a} \in N$ s.t. $\varphi(\mathfrak{N}, \bar{a})$ is finite, but has more than n elements. Let $\mathfrak{M}$ be a proper elementary extension of $\mathfrak{N}$. Then $\varphi(\mathfrak{M}, \bar{a}) = \varphi(\mathfrak{N}, \bar{a})$ (as $\varphi(\mathfrak{N}, \bar{a})$ is finite, we can add formulas to ensure this) and the pair $(\mathfrak{M}, N, \bar{a})$ is a model of T^*. This shows that the theory

$$T^* \cup \{\exists^{>n} x \varphi(x, \bar{c}) \mid n = 1, 2, \dots\}$$

is finitely satisfiable. A model of this theory gives a Vaughtian pair for T. □

Exercise 5.4.1.

5.5 Algebraic formulas

Definition 5.30. Let $\mathfrak{M}$ be a structure and A a subset of M. A formula $\varphi(x) \in L(A)$ is called **algebraic** if $\varphi(\mathfrak{M})$ is finite. An element $a \in M$ is algebraic over A if it realizes an algebraic $L(A)$-formula. We call an element algebraic if it is algebraic over the empty set. The **algebraic closure** of A, acl(A), is the set of all elements of $\mathfrak{M}$ algebraic over A, and A is called **algebraically closed** if it equals its algebraic closure

Remark. Note that the algebraic closure of A does not grow in elementary extensions of $\mathfrak{M}$ because an $L(A)$-formula which defines a finite set in $\mathfrak{M}$ defines the same set in every elementary extension We can express there are exactly m solutions in formula.

By Theorem 2.15

$$|\mathrm{acl}(A)| \leq \max(|T|, |A|)$$

In algebraically closed fields, an element a is algebraic over A precisely if a is algebraic (in the field-theoretical sense) over the field generated by A. This follows from quantifier elimination in ACF

We call a type $p(x) \in S(A)$ algebraic iff p contains an algebraic formula. Any algebraic type p is isolated by an algebraic formula $\varphi(x) \in L(A)$, namely by any $\varphi \in p$ having the minimal number of solutions in $\mathfrak{M}$. If φ has m solutions and ψ has n $(n > m)$ solutions, then $\varphi \to \psi$ as they are consistent. This number is called the **degree** deg(p) of p. As isolated types are realised in every model, the algebraic types over A are exactly of the form tp(a/A) where a is algebraic over A. The **degree** of a over A deg(a/A) is the degree of tp(a/A).

Lemma 5.31. *Let $p \in S(A)$ be non-algebraic and $A \subseteq B$. Then p has a non-algebraic extension $q \in S(B)$.*

Proof. The extension $q_0(x) = p(x) \cup \{\neg\psi(x) \mid \psi(x)$ algebraic $L(B)$-formula$\}$ is finitely satisfiable. For otherwise there are $\varphi(x) \in p(x)$ and algebraic $L(B)$-formulas $\psi_1(x), \dots, \psi_n(x)$ with

$$\mathfrak{M} \vDash \forall x(\varphi(x) \to \psi_1(x) \vee \cdots \vee \psi_n(x))$$

But then $\varphi(x)$ ($\varphi(x)$ has finitely many solutions) and hence $p(x)$ is algebraic. So we can take for q any type containing q_0. □

Remark. Since algebraic types are isolated by algebraic formulas, an easy compactness argument shows that a type $p \in S(A)$ is algebraic iff p has only finitely many realisations (namely deg(p) many) in all elementary extensions of $\mathfrak{M}$.

Proof. $\Rightarrow$. Obvious.
$\Leftarrow$. Ofc it only has finitely many solutions $\square$

Lemma 5.32. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two structures and $f : A \to B$ an elementary bijection between two subsets. Then f extends to an elementary bijection between acl(A) and acl(B)*

Proof. Let $g : A' \to B'$ a maximal extension of f to two subsets of acl(A) and acl(B). Let $a \in$ acl(A). Since a is algebraic over A', a is atomic over A'. We can therefore realise the type $g(\text{tp}(a/A'))$ in $\mathfrak{N}$ - by an element $b \in$ acl(B) - and obtain an extension $g \cup \{\langle a, b\rangle\}$ of g. It follows that $a \in A'$. So g is defined on the whole acl(A). Interchanging A and B shows that g is surjective $\square$

Definition 5.33. A **pregeometry** (or **matroid**) (X, cl) is a set X with a closure operator cl : $\mathcal{P}(X) \to \mathcal{P}(X)$ where $\mathcal{P}$ denotes the power set, s.t. for all $A \subseteq X$ and $a, b \in X$

1. (REFLEXIVITY) $A \subseteq \text{cl}(A)$
2. (FINITE CHARACTER) cl(A) is the union of all cl(A'), where the A' range over all finite subsets of A
3. (TRANSITIVITY) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$
4. (EXCHANGE) $a \in \text{cl}(Ab) \setminus \text{cl}(A) \Rightarrow b \in \text{cl}(Aa)$

A set A is called **closed** if $A = \text{cl}(A)$.

Lemma 5.34. *If X is the universe of a structure, acl satisfies REFLEXIVITY, FINITE CHARACTER and TRANSITIVITY*

5.6 Strongly minimal sets

We fix a complete theory T with infinite models.

Definition 5.35. Let $\mathfrak{M}$ be a model of T and $\varphi(\bar{x})$ a non-algebraic $L(M)$-formula

1. The set $\varphi(\mathfrak{M})$ is called **minimal in** $\mathfrak{M}$ if for all $L(M)$-formulas $\psi(\bar{x})$ the intersection $\varphi(\mathfrak{M}) \wedge \psi(\mathfrak{M})$ is either finite or cofinite in $\varphi(\mathfrak{M})$
2. The formula $\varphi(\bar{x})$ is **strongly minimal** if $\varphi(\bar{x})$ defines a minimal set in all elementary extensions of $\mathfrak{M}$. In this case, we also call the definable set $\varphi(\mathfrak{M})$ strongly minimal. A non-algebraic type containing a strongly minimal formula is called strongly minimal

3. A theory T is strongly minimal if the formula $x \doteq x$ is strongly minimal

Strong minimality is preserved under definable bijections; i.e., if A and B are definable subsets of $\mathfrak{M}^k$, $\mathfrak{M}^m$ defined by φ and ψ, respectively, s.t. there is a definable bijection between A and B, then if φ is strongly minimal so is ψ. Suppose bijection $f(\bar{a}) = \bar{b}$ iff $\gamma(\bar{a}, \bar{b})$. Then $\psi(\bar{x})$ iff $\exists \bar{y}(\varphi(\bar{y}) \wedge \gamma(\bar{y}, \bar{x}))$.

Example 5.2. 1. The following theories are strongly minimal, which is easily seen in each case using quantifier elimination

- Infset. The sets which are definable over a parameter set A in a model M are the finite subsets S of A and their completements $M \setminus S$
- For a field K, the theory of infinite K-vector spaces. The sets definable over a set A are the finite subsets of the subspace spanned by A and their complements
- The theories ACF_p. The definable sets of any model K are Boolean combinations of zero-sets

$$\{a \in K \mid f(a) = 0\}$$

of polynomials $f(X) \in K[X]$. Zero-sets are finite, or if $f = 0$, all of K. $f(x) \neq 0$ is cofinite .

2. If $K \vDash \mathsf{ACF}_p$, for any $a, b \in K$, the formula $ax_1 + b = x_2$ defining an affine line A in K^2 is strongly minimal as there is a definable bijection between A and K. The formula defines a map

3. For any strongly minimal formula $\varphi(x_1, \dots, x_n)$, the **induced theory** $T \restriction \varphi$ is strongly minimal. Here, for any $\mathfrak{M} \vDash T$, the induced theory is the theory of $\varphi(\mathfrak{M})$ with the structure given by all intersections of 0-definable subsets of M^{nm} with $\varphi(\mathfrak{M})^m$ for all $m \in \omega$. This theory depends only on T and φ, not on $\mathfrak{M}$.

Whether $\varphi(\bar{x}, \bar{a})$ is strongly minimal depends only on the type of the parameter tuple $\bar{a}$ and not on the actual model: observe that $\varphi(\bar{x}, \bar{a})$ is strongly minimal iff for all L-formulas $\psi(\bar{x}, \bar{z})$ the set

$$\Sigma_\psi(\bar{z}, \bar{a}) = \{\exists^{>k}\bar{x}(\varphi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{z})) \wedge \exists^{>k}\bar{x}(\varphi(\bar{x}, \bar{a}) \wedge \neg\psi(\bar{x}, \bar{z})) \mid k = 1, 2, \dots\}$$

cannot be realised in any elementary extension. This means that for all $\psi(\bar{x}, \bar{z})$ there is a bound k_ψ s.t.

$$\mathfrak{M} \models \forall \bar{z}(\exists^{\leq k_\psi} \bar{x}(\varphi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{z})) \vee \exists^{\leq k_\psi}(\varphi(\bar{x}, \bar{a}) \wedge \neg\psi(\bar{x}, \bar{z})))$$

This is an **elementary property** of $\bar{a}$, i.e., expressible by a first-order formula. So it makes sense to call $\varphi(\bar{x}, \bar{a})$ a strongly minimal formula without specifying a model

Lemma 5.36. *If $\mathfrak{M}$ is ω-saturated, or if T eliminates the quantifier $\exists^\infty$, any minimal formula is strongly minimal. If T is totally transcendental, every infinite definable subset of $\mathfrak{M}^n$ contains a minimal set $\varphi(\mathfrak{M})$.*

Proof. If $\mathfrak{M}$ is ω-saturated and $\varphi(\bar{x}, \bar{a})$ not strongly minimal, then for some L-formula $\psi(\bar{x}, \bar{z})$ the set $\Sigma_\psi(\bar{z}, \bar{a})$ is realised in $\mathfrak{M}$ ($\Sigma_\psi(\bar{z}, \bar{a})$ is consistent as it can be realized in some elementary extensions) , so φ is not minimal.

If on the other hand $\varphi(\bar{x}, \bar{a})$ is minimal and T eliminates the quantifier $\exists^\infty$ □

A Fields

A.1 Ordered fields

Let R be an integral domain. A linear $<$ ordering on R is **compatible** with the ring structure if for all $x, y, z \in R$

$$x < y \to x + z < y + z$$
$$x < y \wedge 0 < z \to xz < yz$$

A field $(K, <)$ together with a compatible ordering is an **ordered field**

Lemma A.1. *Let R be an integral domain and $<$ a compatible ordering of R. Then the ordering $<$ can be uniquely extended to an ordering of the quotient field of R*

It is easy to see that in an ordered field sums of squares can never be negative. In particular, 1, 2, . . . are always positive and so the characteristic of an ordered field is 0. A field K in which −1 is not a sum of squares is called **formally real**.

B TODO Don't understand

Lemma 3.22

Exercise 3.2.2

theorem 4.11 need to enhance my TOPOLOGY and ALGEBRA!!!

5.1

5.2

5.4

5.27

1

3