Saturated and strongly homogeneous models

Introductory Model Theory

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Recommended reading: This material is covered in Sections 9.1–9.3 of the textbook, but using a different approach involving p-equivalence and resplendent models.

1 κ-saturation

Let κ be an infinite cardinal.

Definition 1. A structure M is *κ-saturated* if for every $A \subseteq M$, if $|A| < \kappa$ and $p \in S_1(A)$, then p is realized in M.

Lemma 2. *If M is a structure, there is $N \succeq M$ such that every type in $S_1(M)$ is realized in N.*

Proof. Add a new constant symbol c_p for each $p \in S_1(M)$. Then $T(M) \cup \{p(c_p) : p \in S_1(M)\}$ is finitely satisfiable, because each type $p(x)$ is finitely satisfiable in M. Take $N \models T(M) \cup \{p(c_p) : p \in S_1(M)\}$. Then $N \succeq M$ (up to isomorphism), and the interpretation $c_p^N \in N$ realizes p for each $p \in S_1(M)$. □

Lemma 3. *Let $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq \cdots$ be an increasing chain of sets indexed by $\alpha < \kappa$ for some regular cardinal κ. If $A \subseteq \bigcup_{\alpha<\kappa} S_\alpha$ and $|A| < \kappa$, then $A \subseteq S_\alpha$ for some $\alpha < \kappa$.*

Proof. Define $f : A \to \kappa$ by $f(x) = \min\{\alpha : x \in S_\alpha\}$. Then $|f(A)| \le |A| < \kappa$, so $\alpha := \sup f(A) < \kappa$. For any $x \in A$, we have $f(x) \le \alpha$, and so $x \in S_{f(x)} \subseteq S_\alpha$. Thus $A \subseteq S_\alpha$. □

Theorem 4. *If M is a structure and κ is a cardinal, there is a κ-saturated $N \succeq M$.*

Proof. Build an elementary chain

$$M_0 \preceq M_1 \preceq \cdots \preceq M_\alpha \preceq \cdots$$

of length κ^+, where

1. $M_0 = M$.

2. $M_{\alpha+1}$ is an elementary extension of M_α realizing every type in $S_1(M_\alpha)$.

3. If α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

Let $N = \bigcup_{\alpha<\kappa^+} M_\alpha$. Then $N \succeq M_0 = M$ by the Tarski-Vaught theorem on elementary chains. If $A \subseteq N$ and $|A| < \kappa$, then $A \subseteq M_\alpha$ for some $\alpha < \kappa^+$. Any $p \in S_1(A)$ extends to a $p' \in S_1(M_\alpha)$ which is realized in $M_{\alpha+1} \subseteq N$. □

2 Partial elementary maps

Definition 5. Let M, N be two L-structures. Let $A \subseteq M$ and $B \subseteq N$ be subsets. Let $f : A \to B$ be a bijection. Then f is a *partial elementary map* if for any $\bar{a} \in A^n$ and L-formula $\varphi(x_1, \ldots, x_n)$, we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})),$$

where $f(\bar{a}) = (f(a_1), \ldots, f(a_n))$.

Lemma 6. *Let M, N be L-structures. Suppose $A \subseteq M$ and $B \subseteq N$ and $f : A \to B$ is a partial elementary map. Suppose N is $|A|^+$-saturated. For any $\alpha \in A$ there is $\beta \in B$ such that $f \cup \{(\alpha, \beta)\}$ is a partial elementary map.*

Proof. Let $p(x) = \{\varphi(x, f(\bar{a})) : \bar{a} \in A^n,\ \varphi(x, \bar{a}) \in \mathrm{tp}(\alpha/A)\}$. Then p is finitely satisfiable in N. Otherwise, there is $\bar{a} \in A^n$ and $\varphi(x, \bar{a}) \in \mathrm{tp}(\alpha/A)$ such that $N \models \neg\exists x\ \varphi(x, f(\bar{a}))$. Then $M \models \neg\exists x\ \varphi(x, \bar{a})$, contradicted by α.

Then $p \in S_1(B)$, so p is realized by some $\beta \in N$. The map $f \cup \{(\alpha, \beta)\}$ is a partial elementary map by choice of $p(x)$. □

Proposition 7. *Let M, N be L-structures. Suppose $A \subseteq A_1 \subseteq M$ and $B \subseteq N$ and $f : A \to B$ is a partial elementary map. Suppose N is κ-saturated, $|A| < \kappa$, and $|A_1| \leq \kappa$. Then there is a partial elementary map $g : A_1 \to B_1$ extending f.*

Proof. Let $A_1 = \{a_\alpha : \alpha < \lambda\}$ where $\lambda = |A_1|$. For $\alpha < \lambda$, recursively define partial elementary maps f_α as follows:

- $f_0 = f$.
- $f_{\alpha+1}$ is a partial elementary map of the form $f_\alpha \cup \{(a_\alpha, b)\}$ for some $b \in N$.
- If β is a limit ordinal, then $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$.

This works because at each step, $\mathrm{dom}(f_\alpha) \leq |A| + |\alpha| < \lambda \leq \kappa$, so Lemma 6 applies.

Let $g = \bigcup_{\alpha<\lambda}$. Then g is a partial elementary map with $\mathrm{dom}(g) = A_1$. □

Theorem 8 (κ-universality)**.** *Suppose M is κ-saturated. If $N \equiv M$ and $|N| \leq \kappa$, then there is an elementary embedding $N \to M$.*

Proof. $\varnothing$ is a partial elementary map from N to M. By Proposition 7 we can extend it to g with $\mathrm{dom}(g) = N$. Then $g : N \to M$ is an elementary embedding. $\square$

Theorem 9. *Suppose M is κ-saturated. If $A \subseteq M$ and $|A| < \kappa$, then every $p \in S_n(A)$ is realized in M.*

Proof. Take $N \succeq M$ containing a realization $\bar{a}$ of p. By Proposition 7 we can extend the partial elementary map $\mathrm{id}_A : A \to A$ to $f : A \cup \{a_1, \ldots, a_n\} \to B$ where $B \subseteq M$. Then $\mathrm{tp}^M(f(\bar{a})/A) = \mathrm{tp}^N(\bar{a}/A) = p$, so $f(\bar{a})$ realizes p in M. $\square$

Theorem 10 (κ-compactness)**.** *Suppose M is κ-saturated. Let $\Sigma(\bar{x})$ be a set of $L(M)$-formulas. Suppose $|\Sigma| < \kappa$. If $\Sigma(\bar{x})$ is finitely satisfiable in M, then $\Sigma(\bar{x})$ is realized in M.*

Proof. Let $A \subseteq M$ be the set of parameters used in Σ. Then $|A| \leq |\Sigma| < \kappa$, and $\Sigma(\bar{x})$ is a partial A-type. We can extend $\Sigma(\bar{x})$ to a complete A-type $p(\bar{x})$, which is realized in M by Theorem 9. $\square$

3 Strong κ-homogeneity

If M is a structure, then $\mathrm{Aut}(M)$ denotes the set of automorphisms of M, i.e., isomorphisms from M to M. If $A \subseteq M$, then

$$\begin{aligned}\mathrm{Aut}(M/A) &:= \{\sigma \in \mathrm{Aut}(M) : \forall x \in A\ (\sigma(x) = x)\} \\ &= \{\sigma \in \mathrm{Aut}(M) : \sigma \supseteq \mathrm{id}_A\}.\end{aligned}$$

Definition 11. M is *strongly κ-homogeneous* if for any partial elementary map $f : A \to B$ with $A, B \subseteq M$, if $|A| < \kappa$ then there is $\sigma \in \mathrm{Aut}(M)$ with $\sigma \supseteq f$.

Theorem 12. *Suppose M is strongly κ-homogeneous. Suppose $\bar{a}, \bar{b} \in M^n$, $C \subseteq M$, and $|C| < \kappa$. Then the following are equivalent:*

1. $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$.
2. *There is $\sigma \in \mathrm{Aut}(M/C)$ such that $\sigma(\bar{a}) = \bar{b}$.*

Proof. (1) $\Longrightarrow$ (2): if $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$ then there is a partial elementary map

$$f : C \cup \{a_1, \ldots, a_n\} \to C \cup \{b_1, \ldots, b_n\}$$
$$f(x) = \begin{cases} x & \text{if } x \in C \\ b_i & \text{if } x = a_i. \end{cases}$$

Then f extends to an automorphism $\sigma \in \mathrm{Aut}(M)$. Then $\sigma \supseteq f \supseteq \mathrm{id}_C$, so $\sigma \in \mathrm{Aut}(M/C)$, and $\sigma(\bar{a}) = f(\bar{a}) = \bar{b}$.

(2) $\Longrightarrow$ (1): isomorphisms preserve all formulas. $\square$

Lemma 13. *For any M there is an elementary extension $N \succeq M$ with the following properties:*

- *Every type over M is realized in N.*
- *If $A, B \subseteq M$ and $f : A \to B$ is a partial elementary map, then there is $\sigma \in \mathrm{Aut}(N)$ with $\sigma \supseteq f$.*

Proof. Build an elementary chain

$$M = M_0 \preceq M_1 \preceq \cdots$$

of length ω, where M_{i+1} is $|M_i|^+$-saturated (using Theorem 4). Every $p \in S_n(M)$ is realized in M_1.

For the second point, let $f : A \to B$ be given. Recursively build an increasing chain of partial elementary maps f_n with $\mathrm{dom}(f_n), \mathrm{im}(f_n) \subseteq M_n$, as follows:

- $f_0 = f$.
- If $n > 0$ is odd, then f_n is a partial elementary map extending f_{n-1}, with $\mathrm{dom}(f_n) = M_{n-1}$ and $\mathrm{im}(f_n) \subseteq M_n$.
- If $n > 0$ is even, then f_n is a partial elementary map extending f_{n-1}, with $\mathrm{dom}(f_n) \subseteq M_n$ and $\mathrm{im}(f_n) = M_{n-1}$.

Each step is possible by Proposition 7. Take $g = \bigcup_{n=0}^{\infty} f_n$. Then $\mathrm{dom}(g) \supseteq \bigcup M_{2n} = N$ by the odd steps, and $\mathrm{im}(g) \supseteq \bigcup M_{2n+1} = N$ by the even steps, so g is an automorphism. $\square$

Theorem 14. *If M is a structure and κ is a cardinal, there is a strongly κ-homogeneous κ-saturated $N \succeq M$.*

Proof. Build an elementary chain

$$M_0 \preceq M_1 \preceq \cdots \preceq M_\alpha \preceq \cdots$$

of length κ^+, where

1. $M_0 = M$.
2. $M_{\alpha+1}$ is an extension of M_α like in Lemma 13.

As in the proof of Theorem 4, $N \succeq M$ and N is κ-saturated.

Suppose $f : A \to B$ is a partial elementary map on N, with $|A| = |B| < \kappa$. By Lemma 3 we have $A \cup B \subseteq M_\alpha$ for some $\alpha < \kappa^+$.

By Lemma 13, we can extend f to $\sigma_{\alpha+1} \in \mathrm{Aut}(M_{\alpha+1})$, and then $\sigma_{\alpha+2} \in \mathrm{Aut}(M_{\alpha+2})$, and so on. Take $\sigma = \bigcup_{\beta=\alpha+1}^{\kappa^+} \sigma_\beta$. Then $\sigma \in \mathrm{Aut}(N)$ and $\sigma \supseteq f$. $\square$