Chapter 1
Preliminaries

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1 Notation

Definition 1.1. A first-order structure $\mathcal{M} = (M, R_1, R_2, ..., f_1, f_2, ..., c_1, c_2, ...)$ consists of the following data:

- an underlying set M;
- relations $R_1 \subseteq M^{n_1}, R_2 \subseteq M^{n_2}, ...$;
- functions $f_1 : M^{m_1} \to M, f_2 : M^{m_2} \to M, ...$;
- constants $c_1, c_2, ...$

Example 1.2. .

- Group $(G, \cdot, \cdot^{-1},\ e)$;
- Ring $(G, +, \cdot, 0, 1)$;
- Ordered set $(X, \leq)$;
- Graph (V, E);

Definition 1.3. A first-order formula is an expression of the form

$$\psi(y_1, ..., y_m) = \forall x_1 \exists x_2 ... \forall x_{n-1} \exists x_n \phi(x_1, ..., x_n, y_1, ..., y_m),$$

where ϕ is a boolean combination of basic relations and basic functions.

- We denote the set of all formulas by L
- If $B \subseteq M$ is a set of parameters, $L(B) = \{\psi(x, b) |\ \psi \in L, b \in B^{|b|}\}$;
- If $\psi(x) \in L(B)$ is satisfied by $a \in M^{|x|}$, we denote it by $\mathcal{M} \models \psi(a)$ or $a \models \psi(x)$;

- If $\Psi(x)$ is a set of formulas, by $a \models \Psi(x)$, we mean that $a \models \psi(x)$ for each $\psi \in \Psi$;
- If $A \subseteq M^{|x|}$, then $\psi(A) = \{a \in A | \ a \models \psi(x)\}$.
- We say that $X \subseteq M^n$ is B-definable if there is $\psi(x) \in L(B)$ such that $X = \psi(M^n)$.
- A formula has no free variables is called a sentence.
- The theory of $\mathcal{M}$, denoted by $\mathrm{Th}(\mathcal{M})$, is the collection of all sentence that are ture in $\mathcal{M}$.

Example 1.4. $\mathcal{M} = (\mathbb{C}, +, \times, 0, 1)$

- $\mathrm{Th}(\mathcal{M}) = ACF_0$ (algebraicly closed field of char 0);
- ACF_0 has quantifier elimination;
- Definable subsets of $\mathbb{C}^n$ are exactly constructible sets;
- Every definable subset of $\mathbb{C}$ is either finite or cofinite (strongly minimal);

Example 1.5. $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1, <)$

- $\mathrm{Th}(\mathcal{M}) = RCF$ (ordered real closed field);
- RCF has quantifier elimination;
- Definable subsets of $\mathbb{R}^n$ are exactly semialgebraic sets;
- Every definable subset of $\mathbb{R}$ is a finite union of points and intervals (o-minimal);

Example 1.6. $\mathcal{M} = (\mathbb{N}, +, \times, 0, 1)$

- $\mathrm{Th}(\mathcal{M})$ does not have quantifier elimination;
- Undecidable;

Definition 1.7. .

- Let T be a set of sentences, we say that T is consistent if there is $\mathcal{M} \models T$;
- Let T_1, T_2 are two sets of sentences, we say that $T_1 \models T_2$ if every model of T_1 is also a model of T_2;
- A set of sentences T is called a theory if T is consistent and $T \models \sigma \implies \sigma \in T$
- A theory T is complete if for each sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$;
- We say that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, denoted by $\mathcal{M} \equiv \mathcal{N}$, if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Definition 1.8. Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.

- We say that a partial map $f : M \to N$ is an elementary map if for $a_1, ..., a_k \in \mathrm{dom}(f)$ and any formula $\phi \in L$

$$\mathcal{M} \models \phi(a_1, ..., a_k) \iff \mathcal{N} \models \phi(f(a_1), ..., f(a_k))$$

- We say that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, denoted by $\mathcal{M} \prec \mathcal{N}$, if the embedding map is elementary.

If $\mathcal{M}$ is finite, then $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N}) \implies \mathcal{M} \cong \mathcal{N}$.

Fact 1.9. *[Löwenheim-Skolem Theory] Suppose that $\mathcal{M}$ is an infinite structure. Then*

- *for any $A \subseteq M$, there is $M_0 \prec \mathcal{M}$ such that $A \subseteq M_0$ and $|M_0| = |A| + |L|$;*
- *for any cardinal $\kappa > |\mathcal{M}|$, there is $\mathcal{N} \succ \mathcal{M}$ such that $|\mathcal{N}| = |L| + \kappa$.*

Definition 1.10 (interpretation)**.** Let $\mathcal{M}$ be an L-structure and $\mathcal{N}$ be an L'-structure.

- An interpretation of $\mathcal{M}$ in $\mathcal{N}$ is a map f from a subset of N^n onto M such that for every definable subset $X \subseteq M^n$, its preimage $f^{-1}(X)$ is definable in $\mathcal{N}$.
- $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable if there exists an interpretation of $\mathcal{M}$ in $\mathcal{N}$ and an interpretation of $\mathcal{N}$ in $\mathcal{M}$ such that the composite interpretations of $\mathcal{M}$ in itself and of $\mathcal{N}$ in itself are definable in $\mathcal{M}$ and $\mathcal{N}$, respectively.

Example 1.11. .

- (Julia Robison) $(\mathbb{Z}, +, \times)$ is definable in $(\mathbb{Q}, +, \times)$;
- If $(G, \cdot)$ is a group and H is a definable subgroup, then G/H is interpretable in G;
- $(\mathbb{C}, +, \times)$ is interpretable in $(\mathbb{R}, +, \times)$, but not the other way around.
- Every structure in a finite relational language is bi-interpretable with a graph.
- (Mekler's construction) Every structure in a finite relational language is interpretable in a pure group.

Example 1.12. [Morleyzation].

- Let $\mathcal{M}_0$ be an L_0-structure;
- $\mathcal{M}_1$ is an expansion of $\mathcal{M}_0$ in the language

$$L_1 = L_0 \cup \{R_\phi(x) |\ \phi(x) \in L\}$$

  and interpret $R_\phi(M) = \phi(M)$;

- every L_0-formula is equivalent to aquantifier-free L_1-formula;
- Every structure in a finite relational language is bi-interpretable with a graph.
- (Mekler's construction) Every structure in a finite relational language is interpretable in a pure group.
- ...
- We obtain an expansion $\mathcal{M}_\infty$ of $\mathcal{M}_0$ in the language $L_\infty = \bigcup_{n<\omega} L_n$;
- Then $\mathcal{M}_0$ and $\mathcal{M}_\infty$ have the same definable sets and thus they are bi-interpretable.

Fact 1.13. *(Compactness Theorem)Let L be an arbitrary language, and let Φ be a set of L-sentences (of arbitrary size!). Assume that every finite subset $\Phi_0 \subseteq \Psi$ is consistent(i.e.there is some L-structure $\mathcal{M} \models \Psi_0$), then Ψ is consistent.*

2 Saturation, monster models, definable and algebraic closures

Definition 2.1. Let $A \subseteq \mathcal{M}$ be a set of parameters.

- By a partial type $\Phi(x)$ over A we mean a collection of formulas of the form $\phi(x)$ with parameters from A such that every finite subcollection has a common solution in $\mathcal{M}$.
- By a complete type over A we mean a partial type such that for every formula $\phi(x) \in L(A)$, either $\phi(x)$ or $\neg\phi(x)$ is in it.
- For $b \in \mathcal{M}$, $\mathrm{tp}(a/A) = \{\phi(x) \mid b \models \phi(x) \text{ and } \phi \in L(A)\}$, is called the complete type of b over A.

Definition 2.2. Let κ be an infinite cardinal.

- We say that $\mathcal{M}$ is κ-saturated if for any set of parameters $A \subseteq M$ with$|A| < \kappa$, every partial type $\Phi(x)$ over A with $|x| < \kappa$ can be realized in $\mathcal{M}$ (enough to verifyit for 1-types).
- We say that $\mathcal{M}$ is saturated if it is $|\mathcal{M}|$-saturated;
- We say that $\mathcal{M}$ is (strongly) κ-homogenous if any partial elementary map from $\mathcal{M}$ to itself with a domain of size $< \kappa$ can be extended to an automorphism of $\mathcal{M}$.

Fact 2.3. *For any T and κ, there is $\mathcal{M} \models T$ which is κ-saturated and k-homogeneous.*

Proof. .
Claim. For any $\mathcal{M}_0 \models T$, there is $\mathcal{M}_1 \succ \mathcal{M}_0$ such that $\mathcal{M}_1$ is $|\mathcal{M}_0|^+$-saturated. (Exercise)

- Let $\mathcal{M}_0 \prec \mathcal{M}_1 \prec ... \prec \mathcal{M}_\alpha \prec ...$ be a elementary chain of length κ^+
- such that each $\mathcal{M}_{\alpha+1}$ is $|\mathcal{M}_\alpha|^+$-saturated;
- Then $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{M}_\alpha$ is κ-saturated and κ-homogeneous. (Exercise).

□

Remark 2.4. .

- Assume GCH or inaccessible cardinals, each T has a saturated model
- Without these assumption, RCF has no saturated models.

Example 2.5. .

- $(\mathbb{C}, +, \times, 0, 1)$ is saturated;

- $(\mathbb{R}, +, \times, 0, 1)$ is NOT $\aleph_0$-saturated;
- $(\mathbb{N}, +, \times, 0, 1)$ is NOT $\aleph_0$-saturated;

From now on, we fix:

- A complete L-theory T;
- A monster model $\mathbb{M}$ of T;
- $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous for some sufficiently large $\kappa(\mathbb{M})$.
- Every model of T of size$\leq \kappa(\mathbb{M})$ embeds elementarily in to $\mathbb{M}$;
- A model of T means a elementary submodel of $\mathbb{M}$;
- tuples, paramters, and definable sets are from $\mathbb{M}$;
- "small" means "of size $\leq \kappa(\mathbb{M})$";
- Let $\phi(x) \in L(\mathbb{M})$ and $a \in\in \mathbb{M}^{|x|}$, by $\models \phi(a)$, we mean $\mathbb{M} \models \phi(a)$;
- If $\Phi(x)$ and $\Psi(x)$ are two sets of formulas, then $\Phi(x) \vdash \Psi(x)$ means for any $a \models \Phi(x)$, also $a \models \Psi(x)$.

Fact 2.6. *Let $\phi(x) \in L(\mathbb{M})$ and $\Phi(x) \subseteq L(\mathbb{M})$ a small set of formulas. If $\Phi(x) \vdash \phi(x)$, then there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0(x) \vdash \phi(x)$.*

(Counter example: $\mathbb{M} = (\mathbb{R}, +, \times, 0, 1)$)

Fact 2.7. *Let A be a subset of $\mathbb{M}$. Let $\mathrm{Aut}(\mathbb{M}/A)$ be the group of automophism of $\mathbb{M}$ fix A pointwise. If A is small and $a, b \in \mathbb{M}$, then*

$$\mathrm{tp}(a/A) = \mathrm{tp}(b/A) \iff \exists \sigma \in \mathrm{Aut}(\mathbb{M}/A)(\sigma(a) = b).$$

Lemma 2.8. *Let $X \subseteq \mathbb{M}^n$ be definable, $A \subseteq \mathbb{M}$ is small. Then X is A-definable iff $\sigma(X) = X$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.*

Proof. $\Rightarrow$

- Assume that $X = \phi(\mathbb{M}, b)$ for some $b \in A^n$;
- For each $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$

$$a \in X \iff \models \phi(a, b) \iff \phi(\sigma(a), \sigma(b)) \iff \phi(\sigma(a), b) \iff \sigma(a) \in X$$

- So $\sigma(X) = X$.

$\Leftarrow$

- Suppose that $X = \phi(\mathbb{M}, b)$;
- Let $\Sigma(x) = \{\psi(x) \in L(A) |\ \phi(x, b) \vdash \psi(x)\}$;
- Claim: $\Sigma(x) \vdash \phi(x, b)$;
- Otherwise , $\exists c \in \mathbb{M}$ such that $c \models \Sigma(x)$ and $c \notin X$;
- Let $p(x) = \mathrm{tp}(c/A)$, then $p(\mathbb{M}) \cap X = \emptyset$ (why ?);
- By compactness, $\exists \psi_0(x) \in p(x)$ such that $\psi_0(\mathbb{M}) \cap X = \emptyset$;
- So $X \subseteq \neg\psi_0(\mathbb{M}) \implies \phi(x, b) \vdash \neg\psi_0(x) \implies \neg\psi_0(x) \in \Sigma(x)$;
- $p(x) = \mathrm{tp}(a/A) \vdash \Sigma(x)$, $\psi_0 \in p$ and $\neg\psi_0 \in \sigma$;
- Contradiction!

$\square$

A slight generalization of the previous lemma.

Lemma 2.9. *Let $X \subseteq \mathbb{M}^n$ be definable. The following are equivalent:*

1. *X is almost A-definable, i.e. there is an A-definable equivalence relation E on $\mathbb{M}^n$ with finitely many classes, such that X is a union of E-classes.*
2. *The set $\{\sigma(X) |\ \sigma \in \mathrm{Aut}(M/A)\}$ is finite.*
3. *The set $\{\sigma(X) |\ \sigma \in \mathrm{Aut}(M/A)\}$ is small.*

Proof. .
$1 \Rightarrow 2$: Any $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ permits the E-classes.
$2 \Rightarrow 1$:

- Let $X_0 = X, X_1, ..., X_k$ be the A-conjugates of X;
- Define $E(x, y) := \bigwedge_{i=0}^{k} (x \in X_i \leftrightarrow y \in X_i)$
- Then E is $\mathrm{Aut}(\mathbb{M}/A)$-invariant $\implies$ E is A-definable.
- E has finitely many equivalence classes and X is a union of E-classes.

$2 \Rightarrow 3$: Clearly.

$3 \Rightarrow 2$:

- Assume that $X = \phi(\mathbb{M}, b)$ and $p(y) = \mathrm{tp}(a/A)$;

- There are $\lambda < \kappa(\mathbb{M})$ and $(b_i)_{i<\lambda}$ s.t.
- each $b_i \models p(y)$ and $\forall \sigma \in \mathrm{Aut}(\mathbb{M}/A) \exists i < \lambda (\sigma(X) = \phi(\mathbb{M}, b_i))$;
- Let $\theta(y, z) = \forall x(\phi(x, y) \leftrightarrow \phi(x, z))$;
- Then $p(y) \vdash \bigvee_{i<\lambda} \theta(y, b_i)$;
- By compactness $p(y) \vdash \bigvee_{k=0}^{n} \theta(y, b_{i_k})$;
- $\phi(\mathbb{M}, b)$ has finitely many A-conjugates.

$\square$

Definition 2.10. Let A be a set of parameters and b a tuple.

1. We say that b is definable over A if $\{b\}$ is A-definable;
2. We say that b is algebraic over A if $\{b\}$ is almost A-definable;
3. $\mathrm{dcl}(A) = \{a \in \mathbb{M} |\ a$ is definable over $A\}$;
4. $\mathrm{acl}(A) = \{a \in \mathbb{M} |\ a$ is algebraic over $A\}$;

Clearly, $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$, all $\mathrm{Aut}(\mathbb{M}/A)$-invariant.

Corollary 2.11. .

1. *$b \in \mathrm{dcl}(A) \iff \forall \sigma \in \mathrm{Aut}(\mathbb{M}/A)(\sigma(b) = b)$;*
2. *$b \in \mathrm{acl}(A) \iff \mathrm{Aut}(\mathbb{M}/A)$-orbit of b is finite $\iff \mathrm{Aut}(\mathbb{M}/A)$-orbit of b is small;*

Example 2.12. .

1. T= theory of a set: $\mathrm{acl}(A) = \mathrm{dcl}(A) = A$;
2. T= theory of vector space : $\mathrm{acl}(A) = \mathrm{dcl}(A) =$linear span of A;
3. $T = ACF_0$: $\mathrm{dcl}(A) =$ is the field generated by A;
4. $T = ACF_0$: $\mathrm{acl}(A)$ is the usual algebraic closure in the field sense;
5. $T = ACF_p$: $\mathrm{dcl}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{Frob}^{-n}(k)$, where k is the field generated by A;
6. $T = ACF_p$: $\mathrm{acl}(A)$ is the usual algebraic closure in the field sense;

Corollary 2.13. . *For $A \subseteq \mathbb{M}$, $\mathrm{acl}(A) = \bigcap\{\mathcal{M} |\ A \subseteq \mathcal{M}, \mathcal{M} \prec \mathbb{M}\}$.*

Proof. .
$\subseteq$:

- If $a \in \mathrm{acl}(A)$ and $\mathcal{M} \supseteq A$.;
- Then there is $\phi(x) \in L(A)$ s.t. $\models \phi(a)$ and $|\phi(\mathbb{M})| = n < \omega$;
- $\mathcal{M} \prec \mathbb{M} \implies |\phi(\mathcal{M})| = n \implies \phi(\mathbb{M}) = \phi(\mathcal{M})$

$\supseteq$:

- Suppose that $a \notin \mathrm{acl}(A)$, then $\mathrm{Aut}(\mathbb{M}/A)$-orbit of a is NOT small;
- Let $\mathcal{M} \supseteq A$ such that $|\mathcal{M}|$ is small;
- Then there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(a) \notin \mathcal{M} \implies a \notin \sigma^{-1}(M) \supseteq (A)$.

$\square$

3 M^{eq} and strong types

Definition 3.1. Let M be an L-structure and $T = \mathrm{Th}(M)$

- $\mathrm{ER}(T) :=$ all L-forlmulas $E(x,y)$ that defines an equivalence relation on $M^{|x|}$;
- $L^{\text{eq}} := L \cup \{S_E : \ E \in \mathrm{ER}(T)\} \cup \{f_E : \ E \in \mathrm{ER}(T)\}$;
- S_E is a new sort, and f_E is a new function from M^n to S_E;
- $S_=$ denotes the sort for the home universe;
- Expand M to a canonical L^{eq}-structure M^{eq};
- the sort S_E in M^{eq} is given by the set $\{a/E : a \in M^n\}$;
- the function f_E is interpreted by $a \mapsto a/E$;
- For any $\phi(x) \in L$ and $a \in M^n$, $M \models \phi(a) \iff M^{\text{eq}} \models \phi(a)$;
- T^{eq} is the union of the following
  1. T
  2. $\{ \ \big(\forall g \in S_E \exists x \in S_=(f_E(x) = y)\big) \mid E \in \mathrm{ER}(T)\}$
  3. $\{ \ \big(\forall x_1, x_2 \in S_=[f_E(x_1) = f_E(x_2) \leftrightarrow E(x_1,x_2)]\big) \mid E \in \mathrm{ER}(T)\}$.
- For any $N \models T$, $N^{\text{eq}} \models T^{\text{eq}}$.

Lemma 3.2. .

- *Every $\mathcal{M}^* \models T^{eq}$ is of the form M^{eq} for some $M \models T$;*
- *Given $E_1, ..., E_k \in ER$ and $\phi(x_1, ..., x_k) \in L^{eq}$, with x_i living on S_{E_i}, there is $\psi(y_1, ..., y_k) \in L$ s.t.*
$$T^{eq} \vdash \forall y_1 ... y_k \in S_= \Bigg(\psi(y_1, ..., y_k) \leftrightarrow \phi(f_{E_1}(y_1), ..., f_{E_k}(y_k)) \Bigg).$$
- *T^{eq} is complete;*
- *$M^{eq} = \mathrm{dcl}^{eq}(M)$;*
- *$X \subseteq M^n$ is definable in the structure M^{eq} is already definabl;e in M;*
- *If M is κ-saturated (κ-homogeneous), then M^{eq} is κ-saturated (resp. κ-homogeneous);*
- *Every automorphism of M extends in a unique way to an automorphism of M^{eq}.*

Remark 3.3. Every definable $X \subseteq \mathbb{M}^n$ corresponds to an element of $\mathbb{M}^{\text{eq}}$

- Suppose that $X = \phi(\mathbb{M}, b)$;
- Let $E(y_1, y_2) := \forall x(\phi(x, y_1) \leftrightarrow \phi(x, y_2))$;
- $b/E \in S_E$ is in $\mathbb{M}^{\mathrm{eq}}$;
- $\forall \sigma \in \mathrm{Aut}(\mathbb{M})(\sigma(X) = X \iff \sigma(b/E) = b/E$;
- X is b/E-definable in $\mathbb{M}^{\mathrm{eq}}$;
- Let $\psi(x, z) := \exists y(\phi(x, y) \wedge f_E(y) = z)$;
- Then b/E is the unique element of sort S_E such that $X = \psi(\mathbb{M}, b/E)$;
- b/E is a code for X;
- $\forall \sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(b/E)$ is a code for $\sigma(X)$.
- If a and b are codes for X, then $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{dcl}^{\mathrm{eq}}(a)$;

Definition 3.4. We say that T has elemination of imaginaries, or EI, if for any $e \in \mathbb{M}^{\mathrm{eq}}$ there is $c \in \mathbb{M}^n$ s.t. $e \in \mathrm{dcl}^{\mathrm{eq}}(c)$ and $c \in \mathrm{dcl}^{\mathrm{eq}}(e)$. Equivalently, any definable subset $X \subseteq \mathbb{M}^n$ has a code in $\mathbb{M}$.

Exercise 3.5. T has EI iff for any 0-definable equivalent relation E, there is a 0-definable function f s.t.

$$\forall x, y(E(x, y) \leftrightarrow f(x) = f(y)).$$

Lemma 3.6. *T^{eq} eliminates imaginaries. (Exercise)*

Lemma 3.7. *Let $X \subseteq \mathbb{M}^n$ be definable, and $e \in \mathbb{M}^{eq}$ a code for X.*

- *X is A-definable iff $e \in \mathrm{dcl}^{eq}(A)$;*
- *X is almost A-definable iff $e \in \mathrm{acl}^{eq}(A)$.*

Proof. .

- X is A-definable $\Leftrightarrow$ X is $\mathrm{Aut}(\mathbb{M}/A)$-invariant $\Leftrightarrow$ e is $\mathrm{Aut}(\mathbb{M}/A)$-invariant;
- X is almost A-definable $\Leftrightarrow$ its A-conjugates is finite $\Leftrightarrow$ the A- conjugates of e is finite.

□

Corollary 3.8. *Let $X \subseteq \mathbb{M}^n$ be definable, $A \subseteq \mathbb{M}$. Then X is almost A-definable iff X is $\mathrm{acl}^{eq}(A)$-definable in $\mathbb{M}^{eq}$*

Definition 3.9. Let a, b be n-tuples from $\mathbb{M}$. Then they have the same strong type over $A \subseteq \mathbb{M}$, written

$$\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$$

if for any A-definable equivalence relation E with finitely many classes, $E(a, b)$ holds.

Clearly,

$$\operatorname{stp}(a/A) = \operatorname{stp}(b/A) \implies \operatorname{tp}(a/A) = \operatorname{tp}(b/A)$$

Example 3.10. Let T = Theory of an equivalence relation with two infinite class.

- By back-and-forth, T has quantifier elimination;
- Let a, b be two elements in different classes;
- Then $\operatorname{tp}(a/\emptyset) = \operatorname{tp}(b/\emptyset)$ but $\operatorname{stp}(a/\emptyset) \neq \operatorname{stp}(b/\emptyset)$.

Lemma 3.11. *TFAE.*

1. $\operatorname{stp}(a/\emptyset) = \operatorname{stp}(b/\emptyset)$*;*
2. *If $X \subseteq \mathbb{M}^n$ is almost A-definable, then $a \in X \iff b \in X$;*
3. $\operatorname{tp}(a/\operatorname{acl}^{\subseteq}(A)) = \operatorname{tp}(b/\operatorname{acl}^{\subseteq}(A))$.

Proof. $1 \implies 2$

- X is almost A-definable $\Rightarrow$ X is a union of E-classes;
- E is A-definable with finite equiv. classes.;
- $E(a, b) \implies (a \in X \iff b \in X)$.

$2 \implies 1$

- If $\neg E(a, b)$, where E is A-definable with finite equiv. classes;
- E is A-definable with finite equiv. classes.;
- $E(a, b) \implies (a \in X \iff b \in X)$.
- Let $X = E(a, \mathbb{M})$, then X is almost A-definable;
- $a \in X$ and $b \notin X$.

$2 \iff 3$: almost A-definable $\iff$ $\operatorname{acl}^{\mathrm{eq}}(A)$-definable.

□

4 Stone duality and spaces of types

Fact 4.1. *Let B be a Boolean algebra.*

- *Let $S(B)$ be the space of ultrafilter on B;*
- *For $b \in B$, define:*
$$< b >= \{u \in S(B) |\ b \in u\}$$
- $E(a, b) \implies (a \in X \iff b \in X)$.
- *The topology of $S(B)$ is generated by the basis of sets of the form $< b >$;*
- *Each $< b >$ is clopen;*
- *$S(B)$ is a compact totally disconnected Hausdorff space.*
- *Each Boolean algebra B is isomorphic to the algebra of the clopen subsets of its Stone space $S(B)$.*

Definition 4.2. For any $A \subseteq \mathbb{M}$, $\mathrm{Def}_x(A)$ is the Boolean algebra of all A-definable subsets of $\mathbb{M}^{|x|}$.

- We denote the stone space of $\mathrm{Def}_x(A)$ by $S_x(A)$;
- The basis of clopens for $S_x(A)$ is given by the sets of the form
$$< \phi(x) >= \{p \in S_x(A) |\ \phi \in p\}$$
for $\phi \in L_x(A)$.
- $S_x(A)$ is also denoted by $S_n(A)$, where $n = |x|$;
- Elements of $S_n(\mathbb{M})$ are called global types.

Remark 4.3. .

- The embedding $a \mapsto \mathrm{tp}(a/M)$ from $\mathcal{M}$ to its type space $S_1(M)$ makes M a dense subset of $S_1(M)$.
- So one can think of the space of types as a "compactification of the model".

Example 4.4. .

- Let T be the theory of an infinite set, in the language $\{=\}$.
- By QE, $S_1(M) = M \cup \{p^*\}$, where $p^* = \{x \neq a |\ a \in M\}$.

Example 4.5. .

- Let T be the theory of dense linear order without end points(DLO), in the language $\{<\}$.
- By back-and-forth, DLO has QE.
- Let $M \models T$ and $A, B \subseteq M$;
- We say that $C = (A, B)$ is a Dedekind cut if $M = A \cup B$ and $A < B$.
- For any a Dedekind cut $C = (A, B)$, let $p_C = \{a < x < b |\ a \in A, b \in B\}$ (non-realized type).
- Then $S_1(M) = M \cup \{p_C |\ C \text{ is a Dedekind cut}\}$.