

# Stability

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## 1 Preface

A combination of various notes from [Pillay(2018)] [Chernikov(2019)] [Tent and Ziegler(2012)]  
A monster model  $\mathfrak{C}$   
[Pillay(2018)] has many typos 😊

## 2 Preliminaries

## 2.1 Indiscernibles

**Definition 2.1.** Let  $I$  be a linear order and  $\mathfrak{A}$  an  $L$ -structure. A family  $(a_i)_{i \in I}$  of elements of  $A$  is called a **sequence of indiscernibles** if for all  $L$ -formulas  $\varphi(x_1, \dots, x_n)$  and all  $i_1 < \dots < i_n$  and  $j_1 < \dots < j_n$  from  $I$

$$\mathfrak{A} \models \varphi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n})$$

or

$$\text{tp}(a_{i_1}, \dots, a_{i_n}) = \text{tp}(a_{j_1}, \dots, a_{j_n})$$

**Theorem 2.2.** Compactness let us “stretch” indiscernibles. Let  $(a_i : i \in \omega)$  be indiscernibles in  $\mathfrak{C}$ , and  $(I, <)$  an ordering. Then there exists an indiscernible  $(b_i : i \in I)$  in  $\mathfrak{C}$  s.t.  $\forall i_1 < \dots < i_n \in I$

$$\text{tp}(a_1, \dots, a_n) = \text{tp}(b_{i_1}, \dots, b_{i_n})$$

Indiscernible sequence are a fundamental tool of model theory, and there are many ways to obtain them.

**Theorem 2.3** (Ramsey, extended). Let  $n_1, \dots, n_r < \omega$ . For each  $i = 1, \dots, r$ , let  $X_{i,1}, X_{i,2}$  be a partition of  $[\omega]^{n_i}$ . Then there is an infinite subset  $Y \subseteq \omega$  which is homogeneous, i.e.,  $\forall i = 1, \dots, r$ , either  $[Y]^{n_i} \subseteq X_{i,1}$  or  $[Y]^{n_i} \subseteq X_{i,2}$

**Proposition 2.4.** For each  $n \in \omega$ , let  $\Sigma_n(x_1, \dots, x_n)$  be a collection of  $L$ -formulas in variables  $x_1, \dots, x_n$ . Suppose that there are  $a_1, a_2, \dots \in \mathfrak{C}$  s.t.

$$\models \Sigma_n(a_{i_1}, \dots, a_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

Then there is an indiscernible  $(b_i : i \in \omega)$  in  $\mathfrak{C}$  s.t.

$$\models \Sigma_n(b_{i_1}, \dots, b_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

we can expand  $\bigcup_{n \in \omega} \Sigma_n$  and obtain the Ehrenfeucht-Mostowski type  $\text{EM}((a_i)_{i \in \omega})$ . This is just the Standard Lemma in Tent

**Example 2.1.** Suppose  $\Sigma_2 = \{x_1 \neq x_2\}$ . Then the proposition yields the existence of infinite indiscernible sequences

*Proof.* Consider

$$\begin{aligned} \Gamma(x_1, x_2, \dots) = \{ & \varphi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \varphi(x_{j_1}, \dots, x_{j_n}) : \\ & i_1 < \dots < i_n, j_1 < \dots < j_n \in \omega, \varphi \in L \} \\ & \cup \bigcup_n \Sigma(x_1, \dots, x_n) \end{aligned}$$

Let  $\Gamma'(x_1, \dots, x_n) \subseteq_f \Gamma$ . Let  $\varphi_1, \dots, \varphi_r$  be the  $L$ -formulas appearing in  $\Gamma'$ . For  $i = 1, \dots, r$ , let

$$\begin{aligned} X_{i,1} &= \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \models \varphi_i(a_{j_1}, \dots, a_{j_n})\} \\ X_{i,2} &= \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \models \neg \varphi_i(a_{j_1}, \dots, a_{j_n})\} \end{aligned}$$

By Ramsey's theorem, there exists an infinite  $Y \subseteq \mathbb{N}$  s.t.  $\forall i = 1, \dots, r, [Y]^{n_i}$  is either contained in  $X_{i,1}$  or in  $X_{i,2}$ . Write  $Y = \{k_1 < k_2 < \dots\}$ . Interpret each  $x_i$  as  $a_{k_i}$  to satisfy  $\Gamma'$   $\square$

**Definition 2.5.** Let  $M < N < \mathfrak{C}$  be models, and  $p(\bar{x}) \in S_{\bar{x}}(N)$ . We say  $p$  is finitely satisfiable in  $M$ , or  $p(\bar{x})$  is a **coheir** of  $p \upharpoonright M \in S_{\bar{x}}(M)$ , if every  $\varphi(\bar{x}) \in p(\bar{x})$  is satisfied by some  $\bar{a} \in M$

*Remark.*  $p(\bar{x}) \in S_{\bar{x}}(N)$  is finitely satisfiable (f.s.) in  $M$  iff  $p(\bar{x})$  is in the topological closure of  $\{\text{tp}(\bar{a}/N) : \bar{a} \in M\} \subseteq S_{\bar{x}}(N)$

**Lemma 2.6.** Suppose  $p(\bar{x}) \in S_{\bar{x}}(M)$  and  $M < N$ , then there is  $p'(\bar{x}) \in S_{\bar{x}}(N)$  s.t.  $p \subseteq p'$  and  $p'$  is f.s. in  $M$

*Proof.* Consider  $\Gamma(\bar{x}) = p(\bar{x}) \cup \{\neg \varphi(\bar{x}) : \varphi(\bar{x}) \in L_N \text{ and not realized in } M\}$ . Let  $\Gamma \supseteq_f \Gamma' = \{\Psi(\bar{x}), \neg \varphi_1(\bar{x}), \dots, \neg \varphi_r(\bar{x})\} \in p$ . Then any solution  $\bar{a}$  of  $\Psi$  in  $M$  satisfies  $\Gamma'$  as  $M \models \forall \bar{x} (\neg \varphi_i(\bar{x}))$   $\square$

*Remark.* Let  $i_M : M^{\bar{x}} \rightarrow S_{\bar{x}}(M)$  s.t.  $m \mapsto \text{tp}(m/M)$ . Define  $i_N : M^{\bar{x}} \rightarrow S_{\bar{x}}(N)$  similarly. Let  $r : S_{\bar{x}}(N) \rightarrow S_{\bar{x}}(M)$ . Note that  $r \circ i_N = i_M$  and the set of types in  $S_{\bar{x}}(N)$  that are f.s. in  $M$  is exactly the closure of  $i_N(M^{\bar{x}})$  in  $S_{\bar{x}}(N)$ . Hence its image under  $r$  is closed. However the image must contain  $i_M(M^{\bar{x}})$  which is dense in  $S_{\bar{x}}(M)$ . Therefore it must be onto, which proves the desired result

*$r$  is continuous and  $r(\overline{i_N(M^n)}) \supseteq i_M(M^n)$  is closed.  $\overline{i_M(M^n)} = S_n(M)$ . Then  $r$  is onto? Then its preimage of  $p$  is what we want*

**Proposition 2.7.** Let  $p(\bar{x}) \in S_{\bar{x}}(M)$ ,  $N > M$  be  $|M|^+$ -saturated, and  $p'(\bar{x}) \in S_{\bar{x}}(N)$  a coheir of  $p$ . Let  $\bar{a}_1, \bar{a}_2, \dots \in N$  be defined as follows

$$\begin{aligned} \bar{a}_1 &\text{ realises } p(\bar{x}) \\ \bar{a}_2 &\text{ realises } p'(\bar{x}) \upharpoonright (M, \bar{a}_1) \\ \bar{a}_3 &\text{ realises } p'(\bar{x}) \upharpoonright (M, \bar{a}_1, \bar{a}_2) \\ &\dots \end{aligned}$$

Then  $(\bar{a}_i : i \in \omega)$  is indiscernible over  $M$

*Proof.* We prove by induction on  $k$  that for any  $n \leq k$  and  $i_1 < \dots < i_n \leq k$  and  $j_1 < \dots < j_n \leq k$ , we have

$$\text{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/M) = \text{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/M)$$

Assume this is true for  $k$  and consider  $k+1$ . Let  $i_1 < \dots < i_n \leq k$ ,  $j_1 < \dots < j_n \leq k$ . We need to show that

$$\text{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}_{k+1}/M) = \text{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}_{k+1}/M)$$

Consider a formula  $\varphi(\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1}) \in L_M$ . Assume by contradiction that

$$M \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}_{k+1}) \wedge \neg \varphi(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}_{k+1})$$

But  $\text{tp}(\bar{a}_{k+1}/M, \bar{a}_1, \dots, \bar{a}_k)$  is f.s. in  $M$ , so there is  $\bar{a}' \in M$  s.t.

$$M \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}') \wedge \neg \varphi(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}')$$

contradicting IH □

## 2.2 Definability and Generalizations

**Definition 2.8.**  $X \subseteq \mathfrak{C}^n$  is **definable almost over**  $A$  if there is an  $A$ -definable equivalence relation  $E$  on  $\mathfrak{C}^n$  with finitely many classes and  $X$  is a union of some  $E$ -classes

**Lemma 2.9.** Let  $\mathbb{D}$  be a definable class and  $A$  a set of parameters. T.F.A.E.

1.  $\mathbb{D}$  is definable over  $A$
2.  $\mathbb{D}$  is invariant under all automorphisms of  $\mathfrak{C}$  which fix  $A$  pointwise

$$S \subseteq K^{\text{alg}} \Rightarrow M \setminus S \subseteq K^{\text{alg}}$$

*Proof.*  $\Rightarrow$  is easy as for any  $F \in \text{Aut}(\mathfrak{C}/A)$  and  $\mathbb{D} = \varphi(\mathfrak{C}, \bar{a})$ ,  $\mathfrak{C} \models \varphi(\bar{s}, \bar{a})$  iff  $\mathfrak{C} \models \varphi(F(\bar{s}), \bar{a})$ . StackExchange

$$x \in \mathbb{D} \Leftrightarrow \varphi(x, \bar{a}) \Leftrightarrow \varphi(F(x), F(\bar{a})) \Leftrightarrow \varphi(F(x), \bar{a}) \Leftrightarrow F(x) \in \mathbb{D}$$

$\Leftarrow$ . Another proof from Chernikov. Assume that  $\mathbb{D} = \varphi(\mathfrak{C}, b)$  where  $b \in \mathfrak{C}$ , and let  $p(y) = \text{tp}(b/A)$

**Claim 1.**  $p(y) \vdash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$ , which says that for any realisations  $b'$ ,  $\varphi(\mathfrak{C}, b) = \varphi(\mathfrak{C}, b')$

Indeed, let  $b' \models p(y)$  be arbitrary. Then  $\text{tp}(b/A) = \text{tp}(b'/A)$  so there is some  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  with  $\sigma(b) = b'$ . Then  $\sigma(X) = \varphi(\mathfrak{C}, b')$  and by assumption  $\sigma(X) = X$ , thus  $\varphi(\mathfrak{C}, b) = X = \varphi(\mathfrak{C}, b')$ .

There is some  $\psi(y) \in p$  (there is a finite subset of  $p(y)$  that does the job and we take the conjunction) s.t.

$$\psi(y) \models \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$$

Let  $\theta(x)$  be the formula  $\exists y(\psi(y) \wedge \varphi(x, y))$ . Note that  $\theta(x)$  is an  $L(A)$ -formula, as  $\psi(y)$  is

**Claim 2.**  $X = \theta(\mathfrak{C})$

If  $a \in X$ , then  $\models \varphi(a, b)$ , and as  $\psi(y) \in \text{tp}(b/A)$  we have  $\models \theta(a)$ . Conversely, if  $\models \theta(a)$ , let  $b'$  be s.t.  $\models \psi(b') \wedge \varphi(a, b')$ . But by the choice of  $\psi$  this implies that  $\models \varphi(a, b)$

$\Leftarrow$  Let  $\mathbb{D}$  be defined by  $\varphi$ , defined over  $B \supset A$ . Consider the maps

$$\mathfrak{C} \xrightarrow{\tau} S(B) \xrightarrow{\pi} S(A)$$

where  $\tau(c) = \text{tp}(c/B)$  and  $\pi$  is the restriction map. Let  $Y$  be the image of  $\mathbb{D}$  in  $S(A)$ . Since  $Y = \pi[\varphi]$ .  $Y$  is closed. **Note that  $\tau(\mathbb{D}) = [\varphi]$ .  $\tau(\mathbb{D}) = \{\text{tp}(c/B) : \mathfrak{C} \models \varphi(c)\} \subseteq [\varphi]$ . For any  $q(x) \in [\varphi]$ , as  $\mathfrak{C}$  is saturated,  $\mathfrak{C} \models q(d)$  and  $d \in \mathbb{D}$ . Thus  $q \in \tau(\mathbb{D})$ .  $\pi$  is continuous**

Assume that  $\mathbb{D}$  is invariant under all automorphisms of  $\mathfrak{C}$  which fix  $A$  pointwise. Since elements which have the same type over  $A$  are conjugate by an automorphism of  $\mathfrak{C}$ , this means that  $\mathbb{D}$ -membership depends only on the type over  $A$ , i.e.,  $\mathbb{D} = (\pi\tau)^{-1}(Y)$ . **For any  $\text{tp}(c/A) = \text{tp}(d/A)$  and  $c \in \mathbb{D}$ , as  $c$  and  $d$  are conjugate,  $d \in \mathbb{D}$ .**

**For any  $c \notin \mathbb{D}$ ,  $\pi\tau(c) \in Y$  iff  $\text{tp}(c/A) \in \pi[\varphi]$  iff there is  $d \in \mathbb{D}$  s.t.  $\text{tp}(c/A) = \text{tp}(d/A)$  but then  $c \in \mathbb{D}$ .**

This implies that  $[\varphi] = \pi^{-1}(Y)$   $\tau(\mathbb{D}) = [\varphi] = \tau(\tau^{-1}\pi^{-1})(Y) = \pi^{-1}(Y)$ , or  $S(A) \setminus Y = \pi[\neg\varphi]$ ; hence  $S(A) \setminus Y$  is also closed and we conclude that  $Y$  is clopen. By Lemma ??  $Y = [\psi]$  for some  $L(A)$ -formula  $\psi$ . This  $\psi$  defines  $\mathbb{D}$ . **For any  $d \in \mathfrak{C}$**

$$\models \psi(d) \Leftrightarrow \text{tp}(d/A) \Leftrightarrow d \in \mathbb{D}$$

□

A slight generalization of the previous lemma

**Lemma 2.10.** *Let  $X \subseteq \mathfrak{C}^n$  be definable. TFAE*

1.  *$X$  is almost  $A$ -definable, i.e., there is an  $A$ -definable equivalence relation  $E$  on  $\mathfrak{C}^n$  with finitely many classes, s.t.  $X$  is a union of  $E$ -classes*
2. *The set  $\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}$  is finite*

3. The set  $\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}$  is small

*Proof.*  $1 \rightarrow 2$ . Let  $\varphi(x_1, x_2) \in L(A)$  be the  $A$ -definable equivalence relation  $E$ , and let  $b_1, \dots, b_n \in M$  be representatives in each equivalence class so that each class can be written as  $[b_i] = \varphi(\mathfrak{C}, b_i)$ . Given  $\sigma \in \text{Aut}(\mathfrak{C}/A)$ , since  $\varphi(x_1, x_2) \leftrightarrow \varphi(\sigma(x_1), \sigma(x_2))$ , the image of each  $[b_i]$  under  $\sigma$  will be

$$\sigma([b_i]) = \{\sigma(x) : \varphi(x, b_i)\} = \{x' : \varphi(x', \sigma(b_i))\} = \{x : \varphi(x, b_{j_i})\} = [b_{j_i}]$$

for some  $j_i \leq n$ . Now  $X$  is a disjoint union of some  $[b_i]$ 's, so  $\sigma(X)$  is a disjoint union of some  $[b_j]$ 's. Since there are only finitely many equivalence classes, there can only be finitely many possibilities for disjoint unions of these classes

$2 \rightarrow 1$ . Let  $X = \varphi(\mathfrak{C}, b)$  and  $p(y) = \text{tp}(b/A)$ . Given  $\sigma \in \text{Aut}(\mathfrak{C}/A)$ , we have  $\sigma(X) = \varphi(\mathfrak{C}, \sigma(b))$ . Then from assumption, there must be distinct  $b_1, \dots, b_n$  s.t.

$$\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\} = \{\varphi(\mathfrak{C}, b_i) : i \leq n\}$$

Now if  $\text{tp}(b'/A) = \text{tp}(b/A)$ , then strong homogeneity yields some  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  s.t.  $\sigma(b) = b'$ . Then the above argument again shows that  $\varphi(x, b')$  defines  $\sigma(X)$  for some  $\sigma \in \text{Aut}(\mathfrak{C}/A)$ . Thus  $\sigma(X) = \varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$  for some  $i \leq k$ . Therefore  $p(y) \vdash \bigvee_{i \leq k} \forall x (\phi(x, y) \leftrightarrow \phi(x, b_i))$ . By compactness there is some  $\psi(y) \in p$  s.t.  $\psi(y) \vdash \bigvee_{i \leq k} \forall x (\phi(x, y) \leftrightarrow \phi(x, b_i))$ . Now define  $E(x_1, x_2)$  as

$$\forall y (\psi(y) \rightarrow (\phi(x_1, y) \leftrightarrow \phi(x_2, y)))$$

so it is  $A$ -definable. It is easy to check that  $E$  is an equivalence relation with finitely many classes, and that  $X$  is a union of  $E$ -classes ( $a_1 E a_2$  iff they agree on  $\phi(x, b_i)$  for all  $i \leq k$ , and so  $X = \phi(\mathfrak{C}, b_0)$  is given by the union of all possible combinations intersected with it)

$3 \rightarrow 1$  Assume for contradiction that

$$|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| = \lambda \geq \omega$$

we can find  $\lambda$ -many elements  $(b_i : i < \lambda) \subset \mathfrak{C}$  to represent the distinct images under automorphisms. Then the set

$$q(y) = p(y) \cup \{\neg \forall x (\varphi(x, y) \leftrightarrow \varphi(x, b_i)) : i < \lambda\}$$

will be finitely satisfiable. Thus  $q(y)$  is realised by some  $b'$ . But such  $b'$  has the same type as  $b$  over  $A$  and so strong homogeneity yields some  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  s.t.  $\sigma(b) = b'$ . Applying such  $\sigma$  on  $X$  gives the image  $\varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$  for some  $i < \lambda$ , a contradiction  $\square$

**Proposition 2.11.** *We can identify definable sets with continuous functions in a certain settings*

1. Formulas  $\varphi(\bar{x}), \psi(\bar{x}) \in L_A$  are equivalent iff  $[\varphi(\bar{x})] = [\psi(\bar{x})]$
2. The clopen subsets of  $S_{\bar{x}}(A)$  are precisely the basic clopen sets
3. Clopen subsets  $X$  of  $S_{\bar{x}}(A)$  correspond exactly to continuous functions  $f : S_{\bar{x}}(A) \rightarrow 2$  (with discrete topology) where  $f(p(\bar{x})) = 1$  if  $p(\bar{x}) \in X$  and 0 otherwise
4. The definable subsets of  $\mathfrak{C}^c$  are in one-to-one correspondence with continuous functions from  $S_{\bar{x}}(A)$  to 2

*Proof.* 3. If  $X$  is clopen, then  $f^{-1}(2) = S_{\bar{x}}(A)$ ,  $f^{-1}(0) = \emptyset$ ,  $f^{-1}(\{1\}) = X$ ,  $f^{-1}(\{0\}) = X^c$

4. By 1, definable sets are in one-to-one correspondence with basic clopen subsets. By 2, basic clopen sets are exactly all of the clopen subsets, so definable sets are in one-to-one correspondence with clopen sets. By 3, clopen sets are in one-to-one correspondence with continuous functions  $f : S_{\bar{x}}(A) \rightarrow 2$

□

### 2.3 Imaginaries and $T^{\text{eq}}$

**Definition 2.12.** The **definable closure**  $\text{dcl}(A)$  of  $A$  is the set of elements  $c$  for which there is an  $L(A)$ -formula  $\varphi(x)$  s.t.  $c$  is the unique element satisfying  $\varphi$ . Elements or tuples  $a$  and  $b$  are said to be **interdefinable** if  $a \in \text{dcl}(b)$  and  $b \in \text{dcl}(a)$ .

**Lemma 2.13.** *Assume  $A \subseteq \mathfrak{C}$  and  $\bar{b} \in \mathfrak{C}$*

1.  $\bar{b} \in \text{acl}(A)$  iff  $\{f(\bar{b}) : f \in \text{Aut}(\mathfrak{C}/A)\}$  is finite
2.  $\bar{b} \in \text{dcl}(A)$  iff  $f(\bar{b}) = \bar{b}$  for all  $f \in \text{Aut}(\mathfrak{C}/A)$

*Proof.* 1. Suppose  $\bar{b} \in \text{acl}(A)$  with witness  $\exists^{\leq k} \varphi(\bar{x})$ . Then  $\varphi(\mathfrak{C})$  is  $A$ -definable and hence is  $\text{Aut}(\mathfrak{C}/A)$ -invariant by Lemma 2.9

Suppose the finiteness. Since the composition of automorphisms is an automorphism, this set is  $\text{Aut}(\mathfrak{C}/A)$ -invariant and therefore  $A$ -definable by some  $\varphi(\bar{x})$ .

2.  $\{\bar{b}\}$  is  $\text{Aut}(\mathfrak{C}/A)$ -invariant

□

The first motivation to develop  $T^{\text{eq}}$  is dealing with quotient objects, without leaving the context of first order logic. That is, if  $E$  is some definable equivalence relation on some definable set  $X$ , we want to view  $X/E$  as a definable set

We work in the setting of multi-sorted languages. Let  $L$  be a 1-sorted language and let  $T$  be a (complete)  $L$ -theory. We shall build a many-sorted language  $L^{\text{eq}}$ -theory  $T^{\text{eq}}$ . We will ensure that in natural sense,  $L^{\text{eq}}$  contains  $L$  and  $T^{\text{eq}}$  contains  $T$

First we define  $L^{\text{eq}}$ . Consider the set  $L$ -formula  $\varphi(x, y)$ , up to equivalence, such that  $T$  models that  $\varphi$  is an equivalence relation. For each  $\varphi$ , define  $s_\varphi$  to be a new sort in  $L^{\text{eq}}$ . Of particular importance is  $s_=$ , the sort given by the formula “ $x = y$ ”. **= is an equivalence relation** This sort  $s_=$  will yield, in each model of  $T^{\text{eq}}$ , a model of  $T$

Also define  $f_\varphi$  to be a function symbol with domain sort  $s_\varphi^n$  (where  $\varphi$  has  $n$  free variables) and codomain sort  $s_\varphi$

For each  $m$ -place relation symbol  $R \in L$ , make  $R^{\text{eq}}$  an  $m$ -place relation symbol in  $L^{\text{eq}}$  on  $s_=^m$ . Likewise for all constant and function symbols in  $L$ . Finally, for the sake of formality, we put a unique equality symbol  $=_\varphi$  on each sort

*Remark.* Let  $N$  be an  $L^{\text{eq}}$  structure. Then  $N$  has interpretations  $s_\varphi(N)$  of each sort  $s_\varphi$  and  $f_\varphi(N) : s_\varphi^n(N) \rightarrow s_\varphi(N)$  of each function symbol  $f_\varphi$ . Additionally,  $N$  will contain an  $L$ -structure consisting of  $s_=$  and interpretations of the symbols of  $L$  inside of  $s_=$

**Definition 2.14.**  $T^{\text{eq}}$  is the  $L^{\text{eq}}$ -theory which is axiomatised by the following

1.  $T$ , where the quantifiers in the formulas of  $T$  now range over the sort  $s_=$
2. For each suitable  $L$ -formula  $\varphi(x, y)$ , the axiom  $\forall_{s_=} \bar{x} \forall_{s_=} \bar{y} (\varphi(x, y) \leftrightarrow f_\varphi(\bar{x}) = f_\varphi(\bar{y}))$
3. For each  $L$ -formula  $\varphi$ , the axiom  $\forall_{s_\varphi} y \exists_{s_=} \bar{x} (f_\varphi(\bar{x}) = y)$

Axioms 2 and 3 simply state that  $f_\varphi$  is the quotient function for the equivalence relation given by  $\varphi$

**Definition 2.15.** Let  $M \models T$ . Then  $M^{\text{eq}}$  is the  $L^{\text{eq}}$  structure s.t.  $s_=(M^{\text{eq}}) = M$  and for each suitable  $L$ -formula  $\varphi(x, y)$  of  $n$  variables, the sort  $s_\varphi(M^{\text{eq}})$  is



equal to  $M^{n_{f_\varphi}}/E$  where  $E$  is the equivalence relation defined by  $\varphi(x, y)$  and  $f_\varphi(M^{\text{eq}})(b) = b/E$

**Example 2.2** (Projective planes). From Hodges.

Suppose  $A$  is a three-dimensional vector space over a finite field, and let  $L$  be the first-order language of  $A$ . Then we can write a formula  $\theta(x, y)$  of  $L$  which expresses ‘vectors  $x$  and  $y$  are non-zero and are linearly dependent on each other’. The formula  $\theta$  is an equivalence formula of  $A$ , and the sort  $s_\theta$  is the set of points of the projective plane  $P$  associated with  $A$

Now  $M^{\text{eq}} \models T^{\text{eq}}$ . Moreover, passing from  $T$  to  $T^{\text{eq}}$  is a canonical operation, in the following sense

- Lemma 2.16.**    1. For any  $N \models T^{\text{eq}}$ , there is an  $M \models T$  s.t.  $N \cong M^{\text{eq}}$
2. Suppose  $M, N \models T$  are isomorphic, and let  $h : M \cong N$ . Then  $h$  extends uniquely to  $h^{\text{eq}} : M^{\text{eq}} \cong N^{\text{eq}}$
3.  $T^{\text{eq}}$  is a complete  $L^{\text{eq}}$ -theory
4. Suppose  $M, N \models T$  and let  $\bar{a} \in M, \bar{b} \in N$  with  $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$ . Then  $\text{tp}_{M^{\text{eq}}}(\bar{a}) = \text{tp}_{N^{\text{eq}}}(\bar{b})$

*Proof.*    1. Take  $M = s_=(N)$

2. Let  $h^{\text{eq}} : M^{\text{eq}} \rightarrow N^{\text{eq}}$  be defined as  $h^{\text{eq}}(f_\varphi(M^{\text{eq}})(b)) = f_\varphi(N^{\text{eq}})(h(b))$  for each  $\varphi \in L$ . This defines a function on  $M^{\text{eq}}$ , because  $f_\varphi(M^{\text{eq}})$  is surjective by the  $T^{\text{eq}}$  axioms. Moreover  $h^{\text{eq}}$  is well-defined. Suppose  $f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b')$ , then  $\varphi(b, b')$  and hence  $\varphi(h(b), h(b'))$ , therefore  $f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b'))$ . Injectivity is the same since  $\varphi(b, b') \leftrightarrow \varphi(h(b), h(b'))$ .

$$\begin{aligned} f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b')) &\Leftrightarrow h(b)/E_\varphi = h(b')/E_\varphi \\ &\Leftrightarrow \varphi(h(b), h(b')) \\ &\Leftrightarrow \varphi(b, b') \\ &\Leftrightarrow f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b') \end{aligned}$$

3. Let  $M, N \models T^{\text{eq}}$ , we want to show that they are elementary equivalent. Assume the generalized continuum hypothesis. By GCH, there are  $M', N' \models T^{\text{eq}}$  which are  $\lambda$  saturated of size  $\lambda$ , for some large  $\lambda$  (strongly inaccessible), which  $M \leq M'$  and  $N \leq N'$ . Since we want to show elementary equivalence, we can replace  $M, N$  with  $M'$  and

$N'$ . By 1, we have  $M = M_0^{\text{eq}}, N = N_0^{\text{eq}}$  for some  $M_0, N_0 \models T$ . Furthermore,  $M_0, N_0$  are  $\lambda$ -saturated of size  $\lambda$ . By assumption,  $T$  is complete, so  $M_0 \equiv N_0$ , and therefore  $M_0 \cong N_0$ . By 2,  $M \cong N$ , and therefore  $M \equiv N$ .

We could simply prove that there is a back and forth system between  $M$  and  $N$ , using such a system between  $M \supset M_0 \models T$  and  $N \supset N_0 \models T$ .  $M_0 \equiv N_0$  iff  $M_0 \sim_\omega N_0$ . We want to show that  $M \sim_\omega N$ . For any  $p \in \omega$ ,

- given  $a \in s_=(M)$ , choose according to  $M$
- given  $a \in s_\varphi(M)$ , then there is  $\bar{b}\bar{c} \in s_=(M)$  s.t.  $f_\varphi(M^{\text{eq}})(\bar{b}\bar{c}) = a$  and  $\varphi(\bar{b}, \bar{c})$ . If  $\bar{b} \in s_=(M^{\text{eq}})^n$ , then there is a local isomorphism  $\bar{b} \mapsto \bar{d}$  as  $M \sim_\omega N$ . Take  $b = \bar{d}/E_\varphi$ .

4. Let  $M, N \models T$ , they are elementary submodels of  $\mathfrak{C}$ . Since  $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$ , there exists an  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  with  $\sigma(\bar{a}) = \bar{b}$ . By 2, this automorphism extends to  $\sigma^{\text{eq}} : \mathfrak{C}^{\text{eq}} \rightarrow \mathfrak{C}^{\text{eq}}$  with  $\sigma^{\text{eq}}(a) = b$ , hence  $\text{tp}_{M^{\text{eq}}}(a) = \text{tp}_{\mathfrak{C}^{\text{eq}}}(a) = \text{tp}_{N^{\text{eq}}}(b)$ .

□

**Corollary 2.17.** Consider the Strong space  $S_{(s_=)^n}(T^{\text{eq}})$ . The forgetful map  $\pi : S_{(s_=)^n}(T^{\text{eq}}) \rightarrow S_n(T)$  is a homeomorphism

*Proof.* Observe that it is continuous and surjective. By 4 of the previous lemma it is injective. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

**Proposition 2.18.** Let  $\varphi(x_1, \dots, x_k)$  be an  $L^{\text{eq}}$  formula, where  $x_i$  is of sort  $S_{E_i}$ . There is an  $L$ -formula  $\psi(\bar{y}_1, \dots, \bar{y}_k)$  s.t.

$$T^{\text{eq}} \models \forall \bar{y}_1, \dots, \bar{y}_k (\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$$

*Proof.* Let  $n$  be the length of  $\bar{y}_1, \dots, \bar{y}_k$ . Consider the set  $\pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ , it is a clopen subset of  $S_n(T)$  by the previous lemma, hence equal to  $\psi(\bar{y}_1, \dots, \bar{y}_k)$  for some formula  $\psi$ .

Guess the intuition is  $[\varphi] = [\psi]$  iff  $\models \varphi \leftrightarrow \psi$ . Consider  $\pi[\psi(\bar{y}_1, \dots, \bar{y}_k)] = \pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$  and as  $\pi$  is homeomorphism,  $[\psi(\bar{y}_1, \dots, \bar{y}_k)] = [\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ . □

This proposition also shows that  $T^{\text{eq}}$  is complete since  $f_{E_i}$  is surjective. Also, for any  $\bar{c} \in \mathfrak{C}$ ,  $\bar{c} \in \text{dcl}^{\text{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \text{dcl}(\emptyset)$ ,  $\bar{c} \in \text{acl}^{\text{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \text{acl}(\emptyset)$ .

**Corollary 2.19.** 1. Let  $M, N \models T$ , and let  $h : M \rightarrow N$  be an elementary embedding. Then  $h^{\text{eq}} : M^{\text{eq}} \rightarrow N^{\text{eq}}$  is also an elementary embedding

2.  $\mathfrak{C}^{\text{eq}}$  is also  $\kappa$ -saturated

*Proof.* 1.  $h : M \rightarrow \text{im}(h)$  is an isomorphism and can extend to  $h^{\text{eq}} : M^{\text{eq}} \rightarrow (\text{im}(h))^{\text{eq}}$ , and  $(\text{im}(h))^{\text{eq}} \subseteq N^{\text{eq}}$

2. By Proposition 2.18

□

*Remark.* For  $M \models T$ , a definable set  $X \subseteq M^n$  can be viewed as an element of  $M^{\text{eq}}$ . Suppose  $X$  is defined in  $M$  by  $\varphi(\bar{x}, \bar{a})$  where  $\bar{a} \in M$ . Consider the equivalence relation  $E_\psi$  defined by  $\psi(\bar{y}_1, \bar{y}_2) = \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$   $\bar{y}_1 \sim \bar{y}_2$  iff this  $\varphi(M, \bar{y}_1) = \varphi(M, \bar{y}_2)$ , and consider  $c = \bar{a}/E_\psi = f_\psi(\bar{a}) \in M^{\text{eq}}$ . Then  $X$  is defined in  $M^{\text{eq}}$  by  $\chi(\bar{x}, c) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f_\psi(\bar{y}) = c)$ . Moreover, if  $c' \in S_\psi(M^{\text{eq}})$  and  $\forall \bar{x}(\chi(\bar{x}, c) \leftrightarrow \chi(\bar{x}, c'))$ , then  $c = c'$ . To see this, let  $c' = f_\psi(\bar{a}')$ , and let  $X'$  be defined in  $M$  by  $\varphi(\bar{x}, \bar{a}')$ . Then  $X'$  is defined in  $M^{\text{eq}}$  by  $\chi(\bar{x}, c')$ , so we have that  $X = X'$  (in  $M^{\text{eq}}$ ). And then  $X = X'$  (in  $M$ ) so  $c = f_\psi(\bar{a}) = f_\psi(\bar{a}') = c'$

**Definition 2.20.** With the above considerations in mind, given  $M \models T$  and a definable set  $X \subseteq M^n$ , we call such a  $c \in M^{\text{eq}}$  a **code** for  $X$

*Remark.* Any automorphism of  $\mathfrak{C}^{\text{eq}}$  fixes a definable set  $X$  set-wise iff it fixes a code for  $X$ . However, the choice of a code for  $X$  will depend on the formula  $\varphi$  used to define it

$$\begin{aligned} \sigma(X) = X &\Leftrightarrow \sigma(X) = \{\sigma(x) : \varphi(x, b)\} = \{x : \varphi(x, \sigma(b))\} = \{x : \varphi(x, b)\} = X \\ &\Leftrightarrow \forall x(\varphi(x, b) \leftrightarrow \varphi(x, \sigma(b))) \\ &\Leftrightarrow \psi(b, \sigma(b)) \Leftrightarrow f_\psi(b) = f_\psi(\sigma(b)) \end{aligned}$$

We can think of  $\mathfrak{C}^{\text{eq}}$  as adjoining codes for all definable equivalence relations (as  $c/E'$  codes  $E'(x, c)$  for an arbitrary equivalence relation  $E$ )

**Definition 2.21.** Let  $A \subseteq M \models T$ . Then  $\text{acl}^{\text{eq}}(A) = \{c \in M^{\text{eq}} : c \in \text{acl}_{M^{\text{eq}}}(A)\}$  and  $\text{dcl}^{\text{eq}}(A)$  is defined similarly

*Remark.* Suppose  $A \subseteq M < N$ , then  $\text{acl}_{N^{\text{eq}}}(A), \text{dcl}_{N^{\text{eq}}}(A) \subseteq M^{\text{eq}}$ , so this notation is unambiguous

**Lemma 2.22.** Let  $M \models T$ , a definable subset  $X$  of  $M^n$ , and  $A \subseteq M$ . Then  $X$  is almost  $A$ -definable iff  $X$  is definable in  $M^{\text{eq}}$  by a formula with parameters in  $\text{acl}^{\text{eq}}(A)$

*Proof.* We can work in  $\mathfrak{C}$ , since  $M < \mathfrak{C}$ . Let  $c$  be a code for  $X$ . From 2.10  $X$  is almost  $A$ -definable iff  $|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| < \omega$  iff  $|\{\sigma(c) : \sigma \in \text{Aut}(\mathfrak{C}^{\text{eq}}/A)\}| < \omega$  (note that  $\sigma$  extends uniquely in  $\mathfrak{C}^{\text{eq}}$ , that is,  $c \in \text{acl}^{\text{eq}}(A)$ ).

$$\begin{aligned} \sigma(b)/E = \sigma'(b)/E &\Leftrightarrow \forall x(\varphi(x, \sigma(b)) \leftrightarrow \varphi(x, \sigma'(b))) \\ &\Leftrightarrow \sigma(X) = \sigma'(X) \end{aligned}$$

□

**Definition 2.23.** Let  $\bar{a}, \bar{b} \in \mathfrak{C}$  have length  $n$ . Let  $\bar{a}, \bar{b}$  have the same strong type over  $A$  (written as  $\text{stp}_{\mathfrak{C}}(\bar{a}/A) = \text{stp}_{\mathfrak{C}}(\bar{b}/A)$ ) if  $E(\bar{a}, \bar{b})$  for any finite equivalence relation (finitely many classes) defined over  $A$

*Remark.* If  $\varphi(\bar{x})$  is a formula over  $A$ , then it defines an equivalence with two classes  $E(\bar{x}_1, \bar{x}_2)$  iff  $(\varphi(\bar{x}_1) \wedge \varphi(\bar{x}_2)) \vee (\neg\varphi(\bar{x}_1) \wedge \neg\varphi(\bar{x}_2))$ . Hence strong types are a refinement of types

Hence for any formula if  $\text{stp}(\bar{a}/A) = \text{stp}(\bar{b}/B)$ , at least we have  $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$

**Lemma 2.24.** If  $A = M < \mathfrak{C}$ , then  $\text{tp}_{\mathfrak{C}}(a/M) \models \text{stp}_{\mathfrak{C}}(a/M)$

$$\text{tp}_{\mathfrak{C}}(a/M) = \text{tp}_{\mathfrak{C}}(b/M) \Rightarrow \text{stp}_{\mathfrak{C}}(a/M) = \text{stp}_{\mathfrak{C}}(b/M)$$

*Proof.* Let  $E$  be an equivalence relation with finitely many classes, defined over  $M$ , and  $\bar{b}$  another realization of  $\text{tp}_{\mathfrak{C}}(\bar{a}/M)$ , we want to show  $E(a, b)$ . Since  $E$  has only finitely many classes, and  $M$  is a model, there are representants  $e_1, \dots, e_n$  of each  $E$ -class in  $M$ . Hence we must have  $E(a, e_i)$  for some  $i$ , and therefore  $E(b, e_i)$ , which yields  $E(a, b)$  □

**Lemma 2.25.** Let  $A \subseteq M \models T$ , and let  $\bar{a}, \bar{b} \in M$ . TFAE

1.  $\text{stp}(\bar{a}/A) = \text{stp}(\bar{b}/A)$
2.  $\bar{a}, \bar{b}$  satisfy the same formulas almost  $A$ -definable
3.  $\text{tp}_{\mathfrak{C}}(\bar{a}/\text{acl}^{\text{eq}}(A)) = \text{tp}_{\mathfrak{C}}(\bar{b}/\text{acl}^{\text{eq}}(A))$

*Proof.*  $3 \rightarrow 2$ . 2.22. Suppose  $X = \varphi(\mathfrak{C}, \bar{d})$  is almost  $A$ -definable, then  $\bar{a}, \bar{b} \in X$  iff  $\bar{a}, \bar{b} \in \theta(\mathfrak{C}) := \exists \bar{y}(\varphi(\mathfrak{C}, \bar{y}) \wedge \bar{y}/E_{\psi} = \bar{c})$  where  $\bar{c} = \bar{d}/E_{\psi} \in \text{acl}^{\text{eq}}(A)$ .

$2 \rightarrow 3$

$1 \rightarrow 2$ . Let  $X$  be almost definable over  $A$ . We want to show that  $\bar{a} \in X$  iff  $\bar{b} \in X$ .

Since  $X$  is almost definable over  $A$ , there is an  $A$ -definable equivalence relation  $E$  with finitely many classes, and  $\bar{c}_1, \dots, \bar{c}_n$  s.t. for all  $\bar{x} \in M$ , we have  $\bar{x} \in X$  iff  $M \models E(\bar{x}, \bar{c}_1) \vee \dots \vee E(\bar{x}, \bar{c}_n)$ . Hence  $E(\bar{a}, \bar{c}_i)$  for some  $i$ , so by assumption  $E(\bar{b}, \bar{c}_i)$ .

$2 \rightarrow 1$ . Let  $E$  be an  $A$ -definable equivalence relation with finitely many classes, we want to show that  $E(\bar{a}, \bar{b})$ . The set  $X = \{\bar{x} \in M : E(\bar{x}, \bar{a})\}$  is definable almost over  $A$ . But  $\bar{a} \in X$ , so  $\bar{b} \in X$ , hence  $E(\bar{a}, \bar{b})$   $\square$

Here is a note from scanlon

**Definition 2.26.** An **imaginary element** of  $\mathfrak{A}$  is a class  $a/E$  where  $a \in A^n$  and  $E$  is a definable equivalence relation on  $A^n$

**Definition 2.27.**  $\mathfrak{A}$  **eliminates imaginaries** if, for every definable equivalence relation  $E$  on  $A^n$  there exists definable function  $f : A^n \rightarrow A^m$  s.t. for  $x, y \in A^n$  we have

$$xEy \Leftrightarrow f(x) = f(y)$$

*Remark.* The definition give above is what Hodges calls **uniform elimination of imaginaries**

*Remark.* If  $\mathfrak{A}$  eliminates imaginaries, then for any definable set  $X$  and definable equivalence relation  $E$  on  $X$ , there is a definable set  $Y$  and a definable bijection  $f : X/E \rightarrow Y$ . Of course this is not literally true, we should rather say that there is a definable map  $f' : X \rightarrow Y$  s.t.  $f'$  is invariant on the equivalence classes defined by  $E$

So elimination of imaginaries is saying that quotients exists in the category of definable sets

*Remark.* If  $\mathfrak{A}$  eliminates imaginaries then for any imaginaries element  $a/E = \tilde{a}$  there is some tuple  $\hat{a} \in A^m$  s.t.  $\tilde{a}$  and  $\hat{a}$  are **interdefinable**, i.e. there is a formula  $\varphi(x, y)$  s.t.

- $\mathfrak{A} \models \varphi(a, \tilde{a})$
- If  $a' Ea$  then  $\mathfrak{A} \models \varphi(a', \hat{a})$
- If  $\varphi(b, \hat{a})$  then  $bEa$
- If  $\varphi(a, c)$  then  $c = \hat{a}$

To get the formula  $\varphi$  we use the function  $f$  given by the definition of elimination of imaginaries and let  $\varphi(x, y) := f(x) = y$

Almost conversely, if for every  $\mathfrak{A}' \equiv \mathfrak{A}$  every imaginary in  $\mathfrak{A}'$  is interdefinable with a **real** (non-imaginary) tuple then  $\mathfrak{A}$  eliminates imaginaries

$$\{\forall xy(xEy \leftrightarrow f_E(x) = f_E(y)) \mid \forall E\}$$

**Example 2.3.** For any structure  $\mathfrak{A}$ , every imaginary in  $\mathfrak{A}_A$  is interdefinable with a sequence of real elements

**Example 2.4.** Let  $\mathfrak{A} = (\mathbb{N}, <, \equiv \pmod{2})$ . Then  $\mathfrak{A}$  eliminates imaginaries. For example, to eliminate the “odd/even” equivalence relation,  $E$ , we can define  $f : \mathbb{N} \rightarrow \mathbb{N}$  by

$$f(x) = y \Leftrightarrow xEy \wedge \forall z[xEz \rightarrow y < z \vee y = z]$$

**Definition 2.28.**  $\mathfrak{A}$  has **definable choice functions** if for any formula  $\theta(\bar{x}, \bar{y})$  there is a definable function  $f(\bar{y})$  s.t.

$$\forall \bar{y} \exists \bar{x} [\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(f(\bar{y}), \bar{y})]$$

(i.e.,  $f$  is a skolem function for  $\theta$ ) and s.t.

$$\forall \bar{y} \forall \bar{z} [\forall \bar{x} (\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(\bar{x}, \bar{z})) \rightarrow f(\bar{y}) = f(\bar{z})]$$

*Proof.* If  $\mathfrak{A}$  has definable choice functions then  $\mathfrak{A}$  eliminates imaginaries  $\square$

*Proof.* Given a definable equivalence relation  $E$  on  $A^n$  let  $f$  be a definable choice function for  $E(\bar{x}, \bar{y})$ . Since  $E$  is an equivalence relation we have  $\forall \bar{y} E(f(\bar{y}), \bar{y})$  and

$$\forall \bar{y} \bar{z} [\bar{y}/E = \bar{z}/E \rightarrow f(\bar{y}) = f(\bar{z})]$$

Thus  $f(\bar{y}) = f(\bar{z}) \Leftrightarrow \bar{y}E\bar{z}$   $\square$

**Example 2.5.** We now see that  $\mathfrak{A} = (\mathbb{N}, <, \equiv \pmod{2})$  eliminates imaginaries. Basically since  $\mathfrak{A}$  is well ordered, we can find a least element to witness membership of definable sets, hence we have definable functions

**Example 2.6.**  $\mathfrak{A} = (\mathbb{N}, \equiv \pmod{2})$  does not eliminate imaginaries

First note that the only definable subsets of  $\mathbb{N}$  are  $\emptyset, \mathbb{N}, 2\mathbb{N}, (2n+1)\mathbb{N}$ . This is because  $\mathfrak{A}$  has an automorphisms which switches  $(2n+1)\mathbb{N}$  and  $2\mathbb{N}$

Now suppose  $f : \mathbb{N} \rightarrow \mathbb{N}^m$  eliminates the equivalence relation  $\equiv \pmod{2}$ , i.e.,

$$f(x) = f(y) \Leftrightarrow x \equiv y \pmod{2}$$

The  $\text{im}(f)$  is definable and has cardinality 2. Since there are no definable subsets of  $\mathbb{N}$  of cardinality 2, we must have  $m > 1$ . Now let  $\pi : \mathbb{N}^m \rightarrow \mathbb{N}$  be a projection. Then  $\pi(\text{im}(f))$  is a finite nonempty definable subset of  $\mathbb{N}$ . But no such set exists

**Proposition 2.29.** *If  $\mathfrak{A}$  eliminates imaginaries, then  $\mathfrak{A}_A$  eliminates imaginaries*

*Proof.* The idea is that an equivalence relation with parameters can be obtained as a fiber of an equivalence relation in more variables. Let  $E \subseteq A^n$  be an equivalence relation definable in  $\mathfrak{A}_A$ . Let  $\varphi(x, y; z) \in L$  and  $a \in A^l$  be s.t.

$$xEy \Leftrightarrow \mathfrak{A} \models \varphi(x, y; a)$$

We now define

$$\psi(x, u, y, v) = \begin{cases} u = v \wedge \text{"}\varphi \text{ defines an equivalence relation"} & \text{or} \\ u \neq v & \text{or} \\ \text{"}\varphi(x, y, v) \text{ does not define an equivalence relation"} \end{cases}$$

Now  $\psi$  defines an equivalence relation on  $A^{n+l}$ . Let  $f : A^{n+l} \rightarrow A^m$  eliminate  $\psi$ , then  $f(-, a)$  eliminates  $E$   $\square$

Back to [Pillay(2018)]

- Definition 2.30.**
1.  $T$  has elimination of imaginaries (EI) if for any model  $M \models T$  and  $e \in M^{\text{eq}}$ , there is a  $\bar{c} \in M$  s.t.  $e \in \text{dcl}_{M^{\text{eq}}}(\bar{c})$  and  $\bar{c} \in \text{dcl}_{M^{\text{eq}}}(e)$
  2.  $T$  has weak elimination of imaginaries if, as above, except  $\bar{c} \in \text{acl}_{M^{\text{eq}}}(e)$  (that is,  $e \in \text{dcl}_{M^{\text{eq}}}(\bar{c})$  and  $\bar{c} \in \text{acl}_{M^{\text{eq}}}(e)$ )
  3.  $T$  has geometric elimination of imaginaries if, as above, except  $e \in \text{acl}_{M^{\text{eq}}}(\bar{c})$  and  $\bar{c} \in \text{acl}_{M^{\text{eq}}}(e)$

Note that in particular, elimination of imaginaries imply the existence of codes for definable sets

**Proposition 2.31.** *TFAE*

1.  $T$  has EI
2. For some model  $M \models T$ , we have that for any  $\emptyset$ -definable equivalence relation  $E$ , there is a partition of  $M^n$  into  $\emptyset$ -definable sets  $Y_1, \dots, Y_r$  and for each  $i = 1, \dots, r$  a  $\emptyset$ -definable  $f_i : Y_i \rightarrow M^{k_i}$  where  $k_i \geq 1$  s.t. for each  $i = 1, \dots, r$ , for all  $\bar{b}_1, \bar{b}_2 \in Y_i$ , we have  $E(\bar{b}_1, \bar{b}_2)$  iff  $f_i(\bar{b}_1) = f_i(\bar{b}_2)$
3. For any model  $M \models T$ , we have that for any  $\emptyset$ -definable equivalence relation  $E$ , there is a partition of  $M^n$  into  $\emptyset$ -definable sets  $Y_1, \dots, Y_r$  and for each  $i = 1, \dots, r$  a  $\emptyset$ -definable  $f_i : Y_i \rightarrow M^{k_i}$  where  $k_i \geq 1$  s.t. for each  $i = 1, \dots, r$ , for all  $\bar{b}_1, \bar{b}_2 \in Y_i$ , we have  $E(\bar{b}_1, \bar{b}_2)$  iff  $f_i(\bar{b}_1) = f_i(\bar{b}_2)$

4. For any model  $M \models T$ , and any definable  $X \subseteq M^n$  there is an  $L$ -formula  $\varphi(\bar{x}, \bar{y})$  and  $\bar{b} \in M$  s.t.  $X$  is defined by  $\varphi(\bar{x}, \bar{b})$  and for all  $\bar{b}' \in M$  if  $X$  is defined by  $\varphi(\bar{x}, \bar{b}')$  then  $\bar{b} = \bar{b}'$ . We call such a  $\bar{b}$  a code for  $X$ .

most typos i've ever seen in a proof

*Proof.*  $2 \Leftrightarrow 3$ . Since we concern only  $\emptyset$ -definable relations and functions, if it is true in some model, then it is true in any model

$1 \rightarrow 2$ . Let  $\pi_E : S^n \rightarrow S_E$  the canonical definable quotient map. Let  $e \in S_E$ . By assumption, there is  $k \in \mathbb{N}$  and  $\bar{c} \in \mathfrak{C}^k$  s.t.  $e$  and  $\bar{c}$  are interdefinable. In other words, there is a formula  $\varphi_e(x, \bar{y})$  over  $\emptyset$  s.t.  $\varphi_e(e, \bar{c})$ . Moreover,  $|\varphi_e(\mathfrak{C}, \bar{c})| = |\varphi_e(e, \mathfrak{C})| = 1$

Let

$$\begin{aligned} X_e = \{ \bar{x} \in \mathfrak{C}, \models \exists! \bar{y} \varphi_e(\pi_E(\bar{x}), \bar{y}) \\ \wedge \forall z (E(\bar{x}, \bar{z}) \leftrightarrow \\ (\forall y (\varphi_e(\pi_E(\bar{x}), \bar{y}) \leftrightarrow \varphi_e(\pi_E(\bar{z}), \bar{y}))) \} \end{aligned}$$

This means that  $\varphi_e$  defines a function on  $X_e$ , and that this function separates  $E$ -classes.

Then  $\pi^{-1}(\{e\}) \subset X_e$ .

Since each  $X_e$  contains  $\pi^{-1}(\{e\})$ , we get  $\mathfrak{C}^n = \bigcup_{e \in \pi_E(\mathfrak{C}^n)} X_e$ , and by compactness, there are  $e_1, \dots, e_l$  s.t.  $\mathfrak{C}^n = \bigcup_{i=1}^l X_{e_i}$ . **As each  $X_e$  is  $\emptyset$ -definable. Let  $\bar{x} \in X_e \Leftrightarrow \theta_e(\bar{x})$ . Then  $\mathfrak{C}^n = \bigcup_e X_e$  means that  $\models \forall \bar{x} \bigvee_e \theta_e(\bar{x})$  and thus  $[\bigvee_e \theta_e] = S_n(\emptyset) = \bigcup_e [\theta_e]$ . Then by compactness...  $\models \forall \bar{x} (\bar{x} = \bar{x} \leftrightarrow \bigvee_e \theta_e(\bar{x}))$**  Naively, we can pick  $f_i = \varphi_{e_i} \circ \pi_E$ , but  $X_{e_i}$  are not disjoint

However we can consider  $Y_1, \dots, Y_r$  to be the atoms of the boolean algebra generated by the  $X_i$ . These are disjoint, and we can pick, for each  $Y_j$ , appropriate  $f_j$ , to get the result

$3 \rightarrow 4$ . Let  $X = \varphi(\mathfrak{C}, \bar{a})$ . Consider the  $\emptyset$ -definable equivalence relation  $E(\bar{y}, \bar{z}) \Leftrightarrow \forall x (\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{z}))$ . Let  $Y_i$  and  $f_i$  be as in 3 and say  $\bar{a} \in Y_1$ , and let  $\bar{b} = f_1(\bar{a})$ . Then  $\exists \bar{y} (f_1(\bar{y}) = \bar{b} \wedge \varphi(\bar{x}, \bar{y}))$  defines  $X$ , call this formula  $\psi$

We have to show that  $\bar{b}$  is unique. Let  $\bar{b}'$  be s.t.  $\exists \bar{y} (f_1(\bar{y}) = \bar{b}' \wedge \varphi(\bar{x}, \bar{y}))$  also defines  $X$ , and let  $\bar{a}_0$  be as the  $\bar{y}$  in the formula. Then  $\varphi(x, \bar{a}_0)$  defines  $X$ , hence  $\bar{a}_0 E \bar{a}$ , which implies  $\bar{b}' = f_1(\bar{a}_0) = f_1(\bar{a}) = \bar{b}$

$4 \rightarrow 1$ . Let  $e \in \mathfrak{C}^{\text{eq}}$ , then  $e = \pi_E(\bar{a})$  for some  $\bar{a} \in \mathfrak{C}^n$  and some  $\emptyset$ -definable equivalence relation  $E$

The set  $X = \{ \bar{x} \in \mathfrak{C}^n \mid \models E(\bar{x}, \bar{a}) \}$  has a code  $\bar{b} \in \mathfrak{C}^k$ , so that  $X = \psi(\mathfrak{C}^n, \bar{b})$ . We are going to prove interdefinability of  $e$  and  $\bar{b}$  using automorphisms of  $\mathfrak{C}$



First suppose that  $\sigma \in \text{Aut}(\mathfrak{C})$ , and fixes  $e$ . We have  $\mathfrak{C}^{\text{eq}} \models \forall \bar{x} (\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b}))$ . Applying  $\sigma$ , we get  $\mathfrak{C}^{\text{eq}} \models \forall \bar{x} (\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \sigma(\bar{b})))$ . But  $\bar{b}$  is a code for  $X$ , hence  $\bar{b} = \sigma(\bar{b})$ . This implies  $\bar{b} \in \text{dcl}(e)$

Now suppose  $\sigma \in \text{Aut}(\mathfrak{C})$  and fixes  $\bar{b}$ . Again  $\mathfrak{C}^{\text{eq}} \models \forall \bar{x} (\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b}))$  and  $\mathfrak{C}^{\text{eq}} \models \forall \bar{x} (\pi_E(\bar{x}) = \pi_E(\sigma(\bar{a})) \leftrightarrow \psi(\bar{x}, \bar{b}))$ . But  $\psi(\bar{a}, \bar{b}), e = \pi_E(\bar{a}) = \pi_E(\sigma(\bar{a})) = \sigma(e)$ . Hence  $e \in \text{dcl}(\bar{b})$   $\square$

Note that condition 2 is somewhat unsatisfying, as we would like to have a quotient function for  $E$ , that is,  $r = 1$

**Proposition 2.32.** *Suppose  $T$  eliminates imaginaries. We get  $r = 1$  in condition 2 iff  $\text{dcl}(\emptyset)$  has at least two elements*

*Proof.* First, suppose that  $r = 1$ . Consider the equivalence on  $\mathfrak{C}^2$  given by  $E((x, y), (x', y'))$  iff  $x = y \leftrightarrow x' = y'$ . In other words, the  $E$  classes are the diagonal and its complement (only two). Then  $\pi_E(\mathfrak{C}^2)$  has two elements, and they belong to  $\text{dcl}^{\text{eq}}(\emptyset)$ . But because  $T$  eliminates imaginaries, this implies that there is also two elements in  $\text{dcl}(\emptyset)$  by Proposition 2.18

Second, suppose that  $\text{dcl}(\emptyset)$  contains two constants  $a$  and  $b$ . Let  $Y_i, f_i$  be as in condition 2. Using  $a$  and  $b$ , we can find some number  $k$  and functions  $g_i : \mathfrak{C}^{k_i} \rightarrow \mathfrak{C}^k$  s.t.  $g_i(\mathfrak{C}^{k_i})$  are pairwise disjoint. We can check that the  $\emptyset$ -definable function  $f : \mathfrak{C}^n \rightarrow \mathfrak{C}^k$  sending  $y \in Y_i$  to  $g_i(f_i(y))$  has all the required properties  $\square$

*Remark.* Elimination of imaginaries also makes sense for many sorted theories

**Proposition 2.33** (Assume  $T$  1-sorted).  *$T^{\text{eq}}$  has elimination of imaginaries*

*Proof.* Prove a strong version of 2 in Proposition 2.31 **that is, we don't need to distinguish  $Y_1, \dots, Y_r$  and  $f_1, \dots, f_r$ .** Let  $E'$  be a  $\emptyset$ -definable equivalence relation on a sort  $s_E$  in some model  $M^{\text{eq}}$  of  $T^{\text{eq}}$ . By Proposition 2.18 there is an  $L$ -formula  $\psi(\bar{y}_1, \bar{y}_2)$  ( $\bar{y}_i$  the appropriate length) s.t. for all  $\bar{a}_1, \bar{a}_2 \in M$ ,  $M \models \psi(\bar{a}_1, \bar{a}_2)$  iff  $M^{\text{eq}} \models E'(f_E(\bar{a}_1), f_E(\bar{a}_2))$ . So  $\psi(\bar{y}_1, \bar{y}_2)$  is an  $L$ -formula defining an equivalence relation on  $M^k$  for the suitable length  $k$ . Consider the map  $h$ , taking  $e \in S_E$  to  $f_\psi(\bar{a})$  for any  $\bar{a} \in M^k$  s.t.  $f_E(\bar{a}) = e$  for any  $\bar{a} \in M^k$  s.t.  $f_E(\bar{a}) = e$ . Suppose  $f_E(\bar{a}) = e = f_E(\bar{a}')$ , we easily see that  $f_\psi(\bar{a}) = f_\psi(\bar{a}')$ , hence the map  $h$  is well-defined, and satisfies 2 of 2.31  $\square$

## 2.4 Examples and counterexamples

**Example 2.7.** The theory of an infinite set has weak elimination of imaginaries but not full elimination of imaginaries

*Proof.* First, we show that  $T$  has weak elimination of imaginaries. Let  $M$  be an infinite set and let  $e \in M^{\text{eq}}$  be an imaginary element. Suppose that. Let  $A \subset M$  be a finite set over which  $X$  is definable ?? . Consider the set

$$\hat{A} := \bigcap_{\substack{\sigma \in \text{Aut}(M) \\ \sigma(X)=X}} \sigma(A)$$

Since  $A$  is finite, there are  $\sigma_1, \dots, \sigma_n$  s.t.  $\hat{A} = \bigcap_i \sigma_i(A)$

To see that  $T$  does not have full elimination of imaginaries, observe that there is never a code for any finite set. Indeed, if  $M$  is an infinite set,  $X \subset_f M$ , and  $\bar{a} \in M$ , we can find a permutation of  $M$  which fixes  $X$  as a set but does not fix  $\bar{a}$ , meaning  $\bar{a}$  could not be a code for  $X$   $\square$

**Example 2.8.** Let  $T = \text{Th}(M, <, \dots)$  where  $<$  is a total well-ordering. Then  $T$  has elimination of imaginaries

*Proof.* Every definable set has a least element. We verify (2) in 2.31. Let  $E$  be a  $\emptyset$ -definable equivalence relation on  $M^n$ . Let  $f : M^n \rightarrow M^n$  s.t. for any  $\bar{a}$ ,  $f(\bar{a})$  is the least element of the  $E$ -class of  $\bar{a}$ . Notice that  $f$  is  $\emptyset$ -definable, and for all  $\bar{a}, \bar{b}$ ,  $f(\bar{a}) = f(\bar{b})$  iff  $E(\bar{a}, \bar{b})$   $\square$

**Lemma 2.34.** Let  $T$  be strongly minimal and  $\text{acl}(\emptyset)$  be infinite (in some, any model). Then  $T$  has weak elimination of imaginaries

*Proof.* Fix a model  $M$ . Let  $e \in M^{\text{eq}}$  **Ok, now i think the convention for pillay is that  $e \in M^{\text{eq}}$  is automatically imaginary,** so  $e = \bar{a}/E$  for some  $\bar{a}$  and  $E$  some  $\emptyset$ -definable equivalence relation. Let  $A = \text{acl}_{M^{\text{eq}}}(e) \cap M$ .  $A$  is infinite as it contains  $\text{acl}(\emptyset)$ .  $A$  is infinite as it contains  $\text{acl}(\emptyset)$ .

We first prove that there exists some  $\bar{b} \subset A$  s.t.  $E(\bar{a}, \bar{b})$ . Let  $X_1 = \{y_1 \in M : M \models \exists y_2 \dots y_n (\bar{y} E \bar{a})\}$ . It is definable over  $e$ . If  $X_1$  is finite, any  $b_1 \in X_1$  then belongs to  $A$ . Otherwise,  $X_1$  is cofinite, hence meets the infinite set  $A$ . Either way,  $X_1 \cap A \neq \emptyset$  and we have  $b_1 \in X_1 \cap A$

Now let  $X_2 = \{y_2 \in M : M \models \exists y_3 \dots y_n (b_1 \bar{y} E \bar{a})\}$ . We remark  $X_2 \neq \emptyset$  since  $b_1 \in X_1$ . Now  $X_2$  is either finite or cofinite since  $T$  is strongly minimal. By the same argument above, we may find  $b_2 \in X_2 \cap A$ . Then repeating this process, we may find  $\bar{b} \subset A$ . Therefore  $\bar{b} \in \text{acl}_{M^{\text{eq}}}(e)$ .

Finally notice that  $e \in \text{dcl}_{M^{\text{eq}}}(\bar{b})$  since  $\bar{a}/E = \bar{b}/E = e$   $\square$

**Example 2.9.** The theory  $\text{ACF}_p$  has elimination of imaginaries, for any  $p$

*Proof.* By Lemma 2.34,  $\text{ACF}_p$  has weak elimination of imaginaries. Therefore it suffices to show that every finite set can be coded. Let  $K$  be an algebraically closed field and let  $X = \{c_1, \dots, c_n\} \subseteq K$ . Consider the polynomial

$$\begin{aligned} P(x) &= \prod_{i=1}^n (x - c_i) \\ &= x^n + e_{n-1}x^{n-1} + \dots + e_1x + e_0 \end{aligned}$$

Then we may take the tuple  $\bar{e} = (e_n, \dots, e_0)$  to be our code for  $X$ .  $\square$

### 3 Stability

Throughout this chapter we will fix a complete theory  $T$  in some language  $L$ . Moreover, we will have no problem in working in  $T^{\text{eq}}$  (that is to say, to assume  $T = T^{\text{eq}}$ )

For a given theory  $T$ , the spectrum functions is given as

$$\begin{aligned} I(T, -) &: \text{Card} \rightarrow \text{Card} \\ I(T, \lambda) &= \# \text{ of models of } T \text{ or cardinality } \lambda \text{ (up to isomorphism)} \end{aligned}$$

#### 3.1 Local Stability

- Definition 3.1.** 1. Let  $M \models T$ . We say  $\varphi(\bar{x}, \bar{y})$  is **stable in**  $M$  if it is **not** the case that there are  $\bar{a}_i, \bar{b}_i \in M$ , for  $i < \omega$ , s.t. **either** for all  $i, j < \omega$ ,  $M \models \varphi(\bar{a}_i, \bar{b}_j)$  iff  $i \leq j$ , **or**, for all  $i, j < \omega$ ,  $M \models \neg\varphi(\bar{a}_i, \bar{b}_j)$  iff  $i \leq j$
2.  $\varphi(\bar{x}, \bar{y})$  is **stable** (for  $T$ ) if it is stable in  $M$  for all  $M \models T$
3.  $T$  is **stable** if every  $L$ -formula  $\varphi(\bar{x}, \bar{y})$  is stable (for  $T$ )

*Remark.* A formula  $\varphi(\bar{x}, \bar{y})$  is stable for  $T$  iff it is not the case that there are  $\bar{a}_i, \bar{b}_i \in \mathfrak{C}$ , for  $i < \omega$ , s.t.  $\mathfrak{C} \models \varphi(\bar{a}_i, \bar{b}_j)$  iff  $i \leq j$  for all  $i, j < \omega$

For simplicity, from now onwards we will write tuples simply as  $x$  and  $a$ .

**Lemma 3.2.** 1. Suppose  $\varphi(x, y)$  and  $\psi(x, z)$  are stable for  $T$ . Then  $\neg\varphi(x, y)$ ,  $(\varphi \vee \psi)(x, yz)$  and  $(\varphi \wedge \psi)(x, yz)$  are also stable

*Proof.* 1. Enrich the language  $L$  with the tuples of constant symbols  $\bar{c}_i, \bar{d}_i$ ,  $i \in \omega$ , and consider  $\Sigma = T \cup \{\neg\varphi(\bar{c}_i, \bar{d}_j) : i \leq j\} \cup \{\varphi(\bar{c}_i, \bar{d}_j) : i > j\}$  of formulas in the new language. For  $\Sigma' \subset_f \Sigma$ , let  $n$  be the largest index.  $\square$

## 4 TODO Problems

2.1 2.3 2.4

## 5 Index

This is a functional link that will open a buffer of clickable index entries:

## 6 References

### References

- [Pillay(2018)] Anand Pillay. Topics in Stability Theory. [https://www3.nd.edu/~apillay/Topics\\_in\\_Stability\\_Theory\\_\\_No\\_Intro\\_.pdf](https://www3.nd.edu/~apillay/Topics_in_Stability_Theory__No_Intro_.pdf), 2018.
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- [Tent and Ziegler(2012)] Katrin Tent and Martin Ziegler. *A course in model theory*. Number 40. Cambridge University Press, 2012.