Homework 10

Introduction to Model Theory

Due 2021-12-9

Definition. Let M be a structure. If $\varphi(x_1, \ldots, x_n)$ is an $L(M)$-formula, then $\varphi(M^n)$ means $\{\bar{a} \in M^n : M \models \varphi(\bar{a})\}$.

A *definable set* is a set of the form $\varphi(M^n)$ for some $L(M)$-formula $\varphi(\bar{x})$. If $A \subseteq M$, an A-definable set is a set defined by an $L(A)$-formula, i.e., a formula with parameters from A. Note that "definable" means "M-definable."

1. Let M be a κ-saturated structure for some $\kappa > \aleph_0$. Let X_i be a definable subset of M^n for $i = 0, 1, 2, \ldots$. Suppose that $X_0 \subseteq \bigcup_{i=1}^{\infty} X_i$. Show that there is an n such that $X_0 \subseteq \bigcup_{i=1}^{n} X_i$. *Hint:* this is related to the thing called κ-compactness in the notes. More generally, if a definable set X is covered by a "small" family of definable sets, then there is a finite subcover, where "small" means "less than κ."

2. Consider the structure $(\mathbb{C}, +, \cdot)$ (the field of complex numbers). Let $\mathcal{F}$ be a family of definable subsets of $\mathbb{C}^1$ (i.e., $\mathbb{C}^n$ with $n = 1$). Suppose $\mathcal{F}$ has the finite intersection property—any finite intersection of sets in $\mathcal{F}$ is non-empty. If $|\mathcal{F}| < |\mathbb{C}|$, show that $\bigcap \mathcal{F} \neq \varnothing$. *Hint:* Use the following fact: if $X \subseteq \mathbb{C}$ is definable, then X or $\mathbb{C} \setminus X$ is finite (this is part of Lemma 33 in the November 18–25 notes).

3. Show that $\mathbb{C}$ is $|\mathbb{C}|$-saturated. *Hint:* reverse the proof of Theorem 10 in the notes.

4. Show that $\mathbb{R}$ is not $|\mathbb{R}|$-saturated. *Hint:* $\aleph_1$-compactness already fails.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be the complex conjugation map

$$f(x + iy) = x - iy \text{ for } x, y \in \mathbb{R}.$$

   Show that the structure $(\mathbb{C}, +, \cdot, f)$ is *not* $|\mathbb{C}|$-saturated. *Hint:* define $\mathbb{R}$.

6. Let M be an infinite structure. Show that M is not $|M|^+$-saturated. *Hint:* use the formulas $x \neq a$ for $a \in M$.