Tame Topology And O-minimal Structures

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Contents

1 Some Elementary Results

Definition 1.1. A **structure** on a nonempty set R is a sequence $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{N}}$ s.t. for each $m \geq 0$

1. $\mathcal{S}_m$ is a boolean algebra of subsets of R^m
2. if $A \in \mathcal{S}_m$, then $R \times A$ and $A \times R$ belong to $\mathcal{S}_{m+1}$ $(\forall)$

3. $\{(x_1, \dots, x_m) \in R^m : x_1 = x_m\} \in \mathcal{S}_m$
4. if $A \in S_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ where $\pi : R^{m+1} \to R^m$ is the projection map on the first m coordinates ($\exists$)
5. $\{a\} \in \mathcal{S}_1$ for $a \in R$

Fact 1.2. *If $(R, \dots)$ is a model-theoretic structure and $\mathcal{S}_n = \{D \subseteq R^n : D$ is definable$\}$, then $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a structure on R*

Definition 1.3. $X \subseteq \mathbb{C}^n$ is **constructible** if $X = \bigcup_{i=1}^m Y_i$ where each Y_i has the form

$$\{\bar{x} \in \mathbb{C}^n : P_1(\bar{x}) = 0, \dots, P_n(\bar{x}) = 0, Q_1(\bar{x}) \neq 0, \dots, Q_n(\bar{x}) \neq 0\}$$

Fact 1.4. *If $S_m = \{D \subseteq \mathbb{C}^m : D$ constructible$\}$, then $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a structure on $\mathbb{C}$*

Theorem 1.5 (Chevalley's Theorem, Quantifier elimination in $\mathbb{C}$)**.** *Projections works*

Definition 1.6. $X \subseteq \mathbb{R}^n$ is **semialgebraic** if X is a finite union of sets of the form

$$\{\bar{x} \in \mathbb{R}^n : P_1(\bar{x}) = 0, \dots, P_n(\bar{x}) = 0, Q_1(\bar{x}) > 0, \dots, Q_m(\bar{x}) > 0\}$$

Semialgebraic sets are closed under intersection, union, complement, cartesian product, projection

Fact 1.7 (Tarski-Seidenberg)**.** *Semialgebraic sets are a structure on $\mathbb{R}$ (projection)*

Fact 1.8. *If $f : X \to Y$ is definable*

1. *if f^{-1} exists then f^{-1} is definable*
2. *if $g : Y \to Z$ is definable, then $g \circ f$ is definable*
3. *if $A \subseteq X$ is definable, then $f(A)$ is definable*
4. *if $A \subseteq Y$ is definable, then $f^{-1}(A)$ is definable*
5. *If $A \subseteq X$ is definable, then so is $f \restriction A$*

Given functions $f, g : X \to R_\infty$ on a set $X \subseteq R^m$ we put

$$(f, g) := \{(x, r) \in X \times R : f(x) < r < g(x)\}$$
$$[f, g] := \{(x, r) \in X \times R_\infty : f(x) \le r \le g(x)\}$$

We consider (f, g) as a subset of R^{m+1}; also $[f, g] \subseteq R^{m+1}$ if f and g are R-valued

Definition 1.9. Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints. An **o-minimal structure** on $(R, <)$ is by definition a structure $\mathcal{S}$ on R s.t.

1. $\{(x, y) \in R^2 : x < y\} \in \mathcal{S}_2$
2. the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points

In $\mathbb{R}$, "definable" = "semialgebraic", in $\mathbb{Q}$, "definable" = "semilinear"

Fact 1.10. *Semialgebraic sets are an o-minimal structure on* $\mathbb{R}$

context

- $(R, \le)$ dense linear order with no endpoints
- for each n, there's $\mathcal{S}_n$

Fix an o-minimal structure $\mathcal{S}$ on $(R, <)$
Why o-minimality?

1. results results for definable sets
2. a bunch of o-minimal structures exist

Fact 1.11 (Wilkie)**.** *There is an o-minimal structure on* $\mathbb{R}$ *where* $\exp(-)$, $\log(-)$ *are definable*

$\sin(x)$ cannot be definable in o-minimal structure on $\mathbb{R}$

Lemma 1.12. *Let $A \subseteq R$ be definable. Then*

1. $\inf(A)$ *and* $\sup(A)$ *exist in* R_∞ (*dedekind completeness for definable sets*)
2. *the boundary* $bd(A) := \{x \in R :$ *each interval containing x intersects both A and $R - A$*$\}$ *is finite, and if $a_1 < \cdots < a_k$ are the points of $bd(A)$ in order, then each interval (a_i, a_{i+1}), where $a_0 = -\infty$ and $a_{k+1} = +\infty$ is either part of A or disjoint from A*

3. *If $|X| = \infty$ then $X \supseteq I$ for some I*

4. *If X is dense in I, then $|X| = \infty$, $X \supseteq J$ (not true in $\mathbb{Q}/\mathbb{R}$) ($X \subseteq I$ is **dense** in I if $\forall J \subseteq I (J \cap X \neq \emptyset)$)*

5. *If $p \in R$, then $\exists b > a$ s.t. $(a, b) \subseteq X$ or $(a, b) \cap X = \emptyset$. Locally,*

Proof. 2. $bd(X \cup Y) \subseteq bd(X) \cup bd(Y)$

3. X is a union of interval and points

□

Lemma 1.13. *1. If $A \subseteq R^m$ is definable, so are $cl(A)$ and $int(A)$*

2. *If $A \subseteq B \subseteq R^m$ are definable sets, and A is open in B, then there is a definable open $U \subseteq R^m$ with $U \cap B = A$*

Proof.

$$(x_1, \dots, x_m) \in \mathrm{cl}(A)$$
$$\Leftrightarrow$$
$$(\forall y_1, \dots, y_m \forall z_1, \dots, z_m [y_1 < x < z_1 \wedge \dots \wedge y_m < x_m < z_m) \rightarrow$$
$$\exists a_1, \dots, a_m (y_1 < a < z_1) \wedge \dots \wedge y_m < a_m < z_m \wedge (a_1, \dots, a_m) \in A]$$

take U is as the union of all boxes in R^m whose intersection with B is contained in A □

Definition 1.14. A set $X \subseteq R^m$ is **definably connected** if X is definable and X is not the union of two disjoint nonempty definable open subsets of X

Lemma 1.15. *1. the definably connected subsets of R are the following: the empty set, the intervals, the sets $[a, b)$ with $-\infty < a < b \leq +\infty$, the sets $(a, b]$ with $-\infty \leq a < b < +\infty$ and the sets $[a, b)$ with $-\infty < a \leq b < +\infty$, and the sets $[a, b]$ with $-\infty < a \leq b < +\infty$*

2. *the image of a definably connected set $X \subseteq R^m$ under a definable continuous map $f : X \rightarrow R^n$ is definably connected*

3. *if X and Y are definable subsets of R^m, $X \subseteq Y \subseteq cl(X)$, and X is definably connected, then Y is definably connected*

4. *if X and Y are definably connected subsets of R^m and $X \cap Y \neq \emptyset$, then $X \cup Y$ is definably connected*

Proof. 3. suppose $Y = U_1 \cup U_2$ where U_1, U_2 are definably open, then $X \subseteq U_1$ or $X \subseteq U_2$ $\square$

note the following special case of (2):

If the function $f : [a, b] \to R$ is definable and continuous, then f assumes all values between $f(a)$ and $f(b)$

Lemma 1.16. *If $I, J \subseteq R$ intervals, $X \subseteq R$ definable, $I < J$, $|I \setminus X| = \infty = |J \cap X|$, then there is a s.t. $I < a < J$, and there is $c < a < b$ s.t. $(c, a) \cap X = \emptyset$, $(a, b) \subseteq X$*

Proof. take $a = \inf X \setminus bd(X)$ $\square$

1.1 O-minimal ordered groups and rings

Order group is a group equipped with a linear order that is invariant under left and right multiplication:

$$x < y \Rightarrow zx < zy \wedge xz < yz$$

Lemma 1.17. *The only definable subsets of R that are also subgroups are $\{1\}$ and R*

Proof. Given a definable subgroup G we first show that G is convex: if not, then there are $g \in G$, $r \in R - G$ with $1 < r < g$. This gives a sequence

$$1 < r < g < rg < g^2 < rg^2 < g^3 < ...$$

whose terms alternate in being in and out of the definable set G.

So G is convex, hence assuming $G \neq \{1\}$ we have $s := \sup(G) > 1$ with $(1, s) \subseteq G$. If $G = +\infty$, then clearly $R = G$. If $s < +\infty$, then we take any $g \in (1, s)$ and obtain $s = gg^{-1}s \in G$, since $g^{-1}s \in (1, s)$ hence $s < gs \in G$ $\square$

Proposition 1.18. *Suppose $(R, <, \mathcal{S})$ is an o-minimal structure and $\mathcal{S}$ contains a binary operation $\cdot$ on R, s.t. $(R, <, \cdot)$ is an ordered group. Then the group $(R, \cdot)$ is abelian, divisible and torsion-free*

Proof. for each $r \in R$ the centralizer $C_r := \{x \in R : rx = xr\}$ is a definable subgroup containing r, so $C_r = R$ by the lemma. Hence R is abelian. For each $n > 0$ the subgroup $\{x^n : x \in R\}$ is definable, hence equal to R. Every ordered group is torsion free $\square$

Remark. Let $(R, <, +)$ be an ordered abelian group, $R \neq \{0\}$, so $(R, <)$ has no endpoints. Assume also that the linearly ordered set $(R, <)$ is dense. Then the addition operation $+ : R^2 \to R$ and the additive inverse operation $- : R \to R$ are continuous w.r.t. the interval topology, that is, $(R, +)$ is a topological group w.r.t. the interval topology

An **ordered ring** is a ring (associative with 1) equipped with a linear order < s.t.

1. $0 < 1$
2. < is translation invariant
3. < is invariant under multiplication by positive elements

Note that then the additive group of the ring is an ordered group, that the ring has no zero divisors, that $x^2 \geq 0$ for all x, and that $k \mapsto k \cdot 1 : \mathbb{Z} \to$ ring is a strictly increasing ring embedding

suppose our ordered ring is moreover a **division ring**: for each $x \neq 0$ there is y with $x \cdot y = 1$. It is easy to check that such a y is unique, and satisfies $y \cdot x = 1$ and that $x > 0$ implies $y > 0$. It is easy to see that the additive group is divisible, the underlying ordered set is dense without endpoints, and the maps $(x, y) \to xy$ and $x \mapsto x^{-1}$ are continuous w.r.t. the interval topology

An **ordered field** is an ordered division ring with commutative multiplication. Examples: field of reals, field of rational numbers. Define **real closed field** to be an ordered field s.t. if $f(X)$ is a one-variable polynomial with coefficients in the field and $a < b$ are elements in the field with $f(a) < 0 < f(b)$, then there is $c \in (a, b)$ in the field with $f(c) = 0$

Proposition 1.19. *Suppose $(R, <, \mathcal{S})$ is an o-minimal structure and $\mathcal{S}$ contains binary operations $+ : R^2 \to R$ and $\cdot : R^2 \to R$ s.t. $(R, <, +, \cdot)$. Then $(R, <, +, \cdot)$ is a real closed field*

Proof. For each $r \in R$ we have a definable additive subgroup rR of $(R, +)$, hence $rR = R$ if $r \neq 0$. This shows that $(R, <, +, \cdot)$ is an ordered division ring. Let $Pos(R) = \{r \in R : r > 0\}$. Clearly $Pos(R)$ is an ordered multiplicative group. By restricting $\mathcal{S}$ to $Pos(R)$ it follows from the previous proposition that multiplication is commutative on $Pos(R)$, hence on all of R. So $(R, <, +, \cdot)$ is an ordered field. Each one-variable polynomial $f(X) \in R[X]$ gives rise to a definable continuous function $x \mapsto f(x) : R \to R$. Now apply 1.15 $\square$

1.2 Model-theoretic structures

Definition 1.20. A model-theoretic structure $\mathcal{R} = (R, <, \dots)$ where $<$ is a dense linear order without endpoints on R, is called **o-minimal** if $\mathrm{Def}(\mathcal{R}_R)$ is an o-minimal structure on $(R, <)$, in other words, every set $S \subseteq R$ that is definable in $\mathcal{R}$ using constants is a union of finitely many intervals and points

$(\mathbb{R}, +, \cdot, \leq)$, $(\mathbb{R}, +, \leq)$, $(\mathbb{Q}, +, \leq)$, $(\mathbb{R}, \leq)$, $(\mathbb{Q}, \leq)$
Wilkie's theorem: $(\mathbb{R}, +, \cdot, \leq, \exp)$,

1.3 The simplest o-minimal structures

Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints

We prove below that the model theoretic structure $(R, <)$ is o-minimal

Let $1 \leq i \leq m$. The function $(x_1, \dots, x_m) \mapsto x_i : R^m \to R$ will be denoted by x_i. The **simple** functions on R^m are by definition these coordinate functions $x_1, \dots, x_m$ and the constant functions $R^m \to R$

Let $f_1, \dots, f_N$ be simple functions on R^m, and let $\epsilon : \{1, \dots, N\}^2 \to \{-1, 0, 1\}$ be given. Then we put

$$\begin{aligned} \epsilon(f_1, \dots, f_N) := \{x \in R^m : &\forall (i,j) \in \{1, \dots, N\}^2 \\ &f_i(x) < f_j(x) \text{ if } \epsilon(i,j) = -1 \\ &f_i(x) = f_j(x) \text{ if } \epsilon(i,j) = 0 \\ &f_i(x) > f_j(x) \text{ if } \epsilon(i,j) = 1\} \end{aligned}$$

If ξ and η are the restrictions of f_i and f_j to $\epsilon(f_1, \dots, f_N)$, then either $\xi < \eta$ or $\xi = \eta$ or $\xi > \eta$. Let $\xi_1 < \cdots < \xi_k$ be the restrictions of $f_1, \dots, f_N$ to $\epsilon(f_1, \dots, f_N)$ arranged in increasing order. One checks easily that the sets $\Gamma(\xi_j)$ $(1 \leq j \leq k)$ and the sets (ξ_j, ξ_{j+1}) $(0 \leq j \leq k$, where $\xi_0 = -\infty$ and $\xi_{k+1} = +\infty$ by convention) are exactly the nonempty subsets of R^{m+1} of the form $\epsilon'(f_1, \dots, f_N, x_{m+1})$ where

$$\epsilon' : \{1, \dots, N, N+1\}^2 \to \{-1, 0, 1\}$$

is an extension of ϵ. suppose $x_{m+1}(x) = y$, we only need to know the relation among $f_1(x), \dots, f_N(x), y$. And $\bigcup \Gamma(\xi_j) \cup \bigcup (\xi_j, \xi_{j+1}) = \epsilon(f_1, \dots, f_N) \times R$

Define a **simple set** in R^m to be the subset of R^m of the form $\epsilon(f_1, \dots, f_N)$ with $f_1, \dots, f_N$ simple functions on R^m and $\epsilon : \{1, \dots, N\}^2 \to \{-1, 0, 1\}$. We have just proved that if $S \subseteq R^{m+1}$ is simple, then its image under the projection map

$$(x_1, \dots, x_m, x_{m+1}) \mapsto (x_1, \dots, x_m) : R^{m+1} \to R^m$$

is simple in R^m

Proposition 1.21. *The subsets of R^m that are definable in $(R, <)$ using constants are exactly the finite unions of simple sets in R^m*

Proof. Let $\mathcal{S}_m$ be the collection of finite unions of simple sets in R^m. Clearly $\mathcal{S}_m$ is a boolean algebra of subsets of R^m, and each set in $\mathcal{S}_m$ is definable in $(R, <)$ using constants. Texts above show that $\mathcal{S} := (\mathcal{S}_m)_{m\in\mathbb{N}}$ is a structure on the set R, hence the sets in $\mathcal{S}_m$ are exactly the subsets of R^m definable in $(R, <)$ using constants $\square$

Corollary 1.22. *The model-theoretic structure $(R, <)$ is o-minimal*

1.4 Semilinear sets

In this section we show that the sets definable using constants in an ordered vector space over an ordered field are exactly the semilinear sets.

definition

2 Semialgebriac sets

2.1 Thom's lemma and continuity of roots

Lemma 2.1. *Let $\alpha \in \mathbb{C}$ be a zero of the monic polynomial*

$$a_0 + a_1T + \cdots + a_{d-1}T^{d-1} + T^d \in \mathbb{C}[T], d \geq 1$$

Then $|\alpha| \leq 1 + \max\{|a_i| : i = 0, \ldots, d-1\}$

Proof. Put $M := \max\{|a_i| : i = 0, \ldots, d-1\}$ and suppose $\alpha > 1 + M$. Then $\left|a_0 + a_1\alpha + \cdots + a^{d-1}\alpha^{d-1}\right| \leq M(1 + abs\alpha + \cdots + |\alpha|^{d-1}) = M(|\alpha|^d - 1)/(|\alpha| - 1) < |\alpha|^d$, contradicting $0 = |f(\alpha)|$ $\square$

Lemma 2.2 (Thom)**.** *Let $f_1, \ldots, f_k \in \mathbb{R}[T]$ be nonzero polynomials s.t. if $f_i' \neq 0$, then $f_i' \in \{f_1, \ldots, f_k\}$. Let $\epsilon : \{1, \ldots, k\} \to \{-1, 0, 1\}$, and put*

$$A_\epsilon := \{t \in \mathbb{R} : \text{sgn}(f_i(t)) = \epsilon(i), i = 1, \ldots, k\} \subseteq \mathbb{R}$$

Then A_ϵ is empty, a point, or an interval. If $A_\epsilon \neq \emptyset$, then its closure is given by

$$cl(A_\epsilon) = \{t \in \mathbb{R} : \text{sgn}(f_i(t)) \in \{\epsilon(i), 0\}, i = 1, \ldots, k\}$$

If $A_\epsilon = \emptyset$, then $\{t \in \mathbb{R} : \text{sgn}(f_i(t)) \in \{\epsilon(i), 0\}, i = 1, \ldots, k\}$ is empty or a point

We call ϵ a **sign condition** for $f_1, \dots, f_k$. The 3^k possible sign conditions ϵ determine 3^K disjoint sets A_ϵ, which together cover the real line $\mathbb{R}$. The second statement of the lemma says that for nonempty A_ϵ its closure can be obtained by relaxing all strict inequalities to weak inequalities

Proof. By induction on k. The lemma holds trivially for $k = 0$. Let $f_1, \dots, f_k, f_{k+1} \in \mathbb{R}[T] - \{0\}$ be polynomials s.t. if $f_i' \neq 0$, then $f_i' \in \{f_1, \dots, f_{k+1}\}$. We may assume that $\deg(f_{k+1}) = \max\{\deg(f_i) : 1 \leq i \leq k+1\}$. Let $\epsilon' : \{1, \dots, k+1\} \to \{-1, 0, 1\}$, and let ϵ be the restriction of ϵ' to $\{1, \dots, k\}$. By the inductive hypothesis, A_ϵ is empty, a point or an interval. It A_ϵ is empty or a point, so is $A_{\epsilon'} = A_\epsilon \cap \{t \in \mathbb{R} : \mathrm{sgn}(f_{k+1}(t)) = \epsilon'(k+1)\}$, and the other properties to be checked in this case follow easily from the inductive hypothesis on A_ϵ

Suppose A_ϵ is an interval. Since f_{k+1}' has a constant sign on A_ϵ, the function f_{k+1} is either strictly monotone on A_ϵ, or constant. In both cases, it is routine to check that $A_{\epsilon'} = A_\epsilon \cap \{t \in \mathbb{R} : \mathrm{sgn}(f_{k+1}(t)) = \epsilon'(k+1)\}$ has the required properties $\square$

Lemma 2.3 (Continuity of roots)**.** *Let $f(T) = a_0 + a_1 + \dots + a_d T^d \in \mathbb{C}[T]$ be a polynomial that has no zero on the boundary circle $|z - c| = r$ of a given open disc $|z - c| < r$ in the complex plane ($c \in \mathbb{C}, r > 0$). Then there is $\epsilon > 0$ s.t. if $|a_i - b_i| \leq \epsilon$ for $i = 0, \dots, d$ then $g(T) := b_0 + b_1 T + \dots + b_d T_d \in \mathbb{C}[T]$ also has no zero on the circle, and f and g have the same number of zeros in the disc*

3 Cell Decomposition

Fix an arbitrary o-minimal structure $(R, <, \mathcal{S})$. Instead of saying that a set $A \subseteq R^m$ belongs to $\mathcal{S}$, we say that A is definable

3.1 The monotonicity theorem and the finiteness lemma

Theorem 3.1 (Monotonicity theorem)**.** *Let $f : (a, b) \to R$ be a definable function on the interval (a, b). Then there are points $a_1 < \dots < a_k$ in (a, b) s.t. on each subinterval (a_j, a_{j+1}) with $a_0 = a$, $a_{k+1} = b$, the function is either constant, or strictly monotone and continuous*

We derive this from the threes below. In these lemmas we consider a definable function $f : I \to R$ on an interval I

Lemma 3.2. *There is a subinterval of I on which f is constant or injective*

Lemma 3.3. *If f is injective, then f is strictly monotone on a subinterval of I*

Lemma 3.4. *If f is strictly monotone, then f is continuous on a subinterval of I*

These lemmas imply the monotonicity theorem as follows:
Let

$$X := \{x \in (a,b) : \text{on some subinterval of } (a,b) \text{ containing } x \text{ the function } f \\ \text{is either constant, or strictly monotonicity and continuous}\}$$

Now $(a,b)-X$ must be finite, since otherwise it would contain an interval I; applying successively lemmas 3.2, 3.3, 3.4 we can make I so small that f is either constant, or strictly monotone and continuous on I. But then $I \subseteq X$, a contradiction

Since $(a,b)-X$ is finite, we can reduce the proof of the theorem to the case that $(a,b) = X$, by replacing (a,b) by each of the finitely many intervals of which the open set X consists. In particular, we may assume that f is continuous. By splitting up (a,b) further we can reduce to one of the following three cases

Case 1. For all $x \in (a,b)$, f is constant on some neighborhood of x

Case 2. For all $x \in (a,b)$, f is strictly increasing on some neighborhood of x

Case 3. For all $x \in (a,b)$, f is strictly decreasing on some neighborhood of x

Case 1. Take $x_0 \in (a,b)$ and put

$$s := \sup\{x : x_0 < x < b, f \text{ is constant on } [x_0, x)\}$$

Then $s = b$, since $s < b$ implies that f is constant on some neighborhood of s, contradiction. From $s = b$ it follows that f is constant on $[x_0, b)$. Similarly we prove that f is constant on $(a, x_0]$, therefore f is constant on (a,b)

Case 2. Take $x_0 \in (a,b)$ and put

$$s := \sup\{x : x_0 < x < b, f \text{ is strictly increasing on } [x_0, x)\}$$

Then $s = b$, since $s < b$ leads to a contradiction

We now prove the lemmas

Proof of Lemma 3.2. If some $y \in R$ had infinite preimage $f^{-1}(y)$, then this preimage would contain a subinterval of I and f would take the constant value y on that subinterval. So we may assume that each $y \in R$ has finite preimage. Then $f(I)$ is infinite, and so contains an interval J. Define an "inverse" $g : J \to I$ by

$$g(y) := \min\{x \in I : f(x) = y\}$$

Since g is injective by definition, $g(J)$ is infinite, and hence $g(J)$ contains a subinterval of I, and f is necessarily injective on this subinterval

If $x_1, x_2 \in J' \subseteq g(J)$, $x_i = g(y_i)$, $f(x_1) = f(x_2) \Rightarrow y_1 = y_2 \Rightarrow x_1 = x_2$ and f is injective $\square$

Fix $f : I \to R$, $a \in I$, $\Phi_{-+}(a)$ means $\exists \epsilon$ s.t. if $x \in (a-\epsilon, a)$ then $f(x) < f(a)$, and if $x \in (a, a+\epsilon)$ then $f(x) > f(a)$. "locally increasing"

$\Phi_{+-}(a)$, $\Phi_{++}(a)$, Φ_{--} is similar

$\Phi_{00}(a)$, $\exists \epsilon$, $x \in (a-\epsilon, a+\epsilon) \Rightarrow f$ is increasing

Definition 3.5. $a \in slbd(D)$ if $(a-\epsilon, a) \cap D = \emptyset$, $(a, a+\epsilon) \subseteq D$, strong left boundary

Fact 3.6. *If $X, Y \subseteq R$, $|X| = |Y| = \infty$, $X < Y$, if $D \subseteq R$, $X \cap D = \emptyset$, $Y \subseteq D$ then $\exists a \in slbd(D)$, $X \leq a \leq Y$*

Lemma 3.7. *If $\Phi_{-+}(a)$, $\forall a \in I$, then f is increasing*

Proof. suppose $a, b \in I$, $a < b$, $f(a) \geq f(b)$. there is ϵ s.t. if $x \in (a, a+\epsilon)$ then $f(x) > f(a)$, and if $x \in (b-\epsilon, b)$, $f(x) < f(b) \leq f(a)$.

$D = \{x : f(x) \leq f(a)\}$, $(a, a+\epsilon) \cap D = \emptyset$, $(b-\epsilon, b) \subseteq D$, then $\exists c \in slbd(D)$, $c - \delta, c \cap D = \emptyset$ and $(c, c+\delta) \subseteq D$, so $\Phi_{-+}(c)$ is false $\square$

Lemma 3.8.

1. *If $\forall a \in I$, $\Phi_{+-}(a)$, then f is decreasing*
2. *If $\forall a \in I$, $\Phi_{00}(a)$, then f is constant*

Lemma 3.9. *If $f : I \to R$ injective, $a \in I$, then $\Phi_{++}(a)$ or $\Phi_{+-}(a)$ or $\Phi_{-+}(a)$ or $\Phi_{--}(a)$*

if f is not injective, then there may be 9 cases

Fact 3.10. *If $D \subseteq R$ definable, $a \in R$, then there is ϵ s.t. $(a, a+\epsilon) \subseteq D$ or $(a, a+\epsilon) \cap D = \emptyset$ and $(a-\epsilon, a) \subseteq D$ or $(a-\epsilon, a) \cap D = \emptyset$*

Proof. Let $D = \{x \in I : f(x) > f(a)\}$, then the fact gives 4 cases $\square$

Lemma 3.11. *If $f : I \to R$ is definable*

1. *It can't be that: $\forall a \in I$, $\Phi_{++}(a)$*
2. *It can't be that: $\forall a \in I$, $\Phi_{--}(a)$*

Proof. 1. Assume $\forall x \Phi_{++}(x)$

$\Psi_{+-}(a) \Leftrightarrow \exists y, \epsilon$, if $x \in (a-\epsilon, a)$, then $f(x) > y$, $x \in (a, a+\epsilon)$, $f(x) < y$

Let $I = (a, b)$, $S = \{x \in I \mid \exists x' \in I, x' > x, f(x') < f(x)\}$

Case 1: $(\exists \epsilon)(b-\epsilon, b) \cap S = \emptyset$. Then on the interval $(b-\epsilon, b)$, f is increasing, $\Phi_{++}(x)$ doesn't hold

Case 2: $(\exists \epsilon)(b-\epsilon, b) \subseteq S$

Take $x_0 \in (b-\epsilon, b)$, $x_0 \in S$, and we could get a decreasing sequence

Let $D = \{x \in I : f(x) > f(x_0)\}$. So there are infinitely many points $< x_0$ in D, and infinitely many points $> x_0$ not in D

$\exists c$ s.t. $(c-\epsilon, c) \subseteq D$, $(c, c+\epsilon) \cap D = \emptyset$. So $\Psi_{+-}(c)$ is true

□

Lemma 3.12. $\exists J \subseteq I$, $\forall x \in J$, $\Psi_{+-}(x)$,

Proof. $S = \{x \in I : \Psi_{+-}(x)\}$. If S is finite, replace I with $I' \subseteq I \setminus S$, replace f with $f|_{I'}$, apply previous lemma, get $c \in I'$, $\Psi_{+-}(c)$, a contradiction □

Similarly, $\exists J \subseteq I$, $\forall x \in J$, $\Psi_{-+}(x)$

Combine these, get $I \supseteq I' \supseteq I''$, $\forall x \in I'$, $\Psi_{+-}(x)$, and $\forall x \in I''$, $\Psi_{+-}(x)$, a contradiction

Lemma 3.13. *If $f : I \to R$, $\exists a \in I$, $\Phi_{-+}(a)$ or $\Phi_{+-}(a)$ or $\Phi_{00}(a)$*

Proof. By Lemma 3.2, there is $J \subseteq I$, if $f|_J$ is constant, then we are done.

If $f|_J$ is injective, let $S_{+-} = \{a \in J, \Phi_{+-}(a)\}$ and other sets similarly. $J = S_{+-} \cup S_{++} \cup S_{-+} \cup S_{--}$. If $|S_{++}| = \infty$, there is $I' \subseteq S_{++}$, a contradiction. Therefore S_{--} and S_{++} are finite. But $|J|$ is infinite, so S_{+-} or S_{-+} is nonempty □

Lemma 3.14. *$f : I \to R$, $\exists c_0 < c_1 < \cdots < c_n$, $I = (c_0, c_n)$, $f|_{(c_i, c_{i+1})}$ is constant or decreasing or increasing*

Proof. Let $E = I \setminus (S_{+-} \cup S_{-+} \cup S_{00})$. If $|E| = \infty$, then $J \subseteq E$ and $f|_E$ contradicts 3.13. Take $\{c_0, \dots, c_n\} \supseteq E \cup bd(I) \cup bd(S_{+-}) \cup bd(S_{-+}) \cup bd(S_{00})$.

So all the sets respect the partition

$$(c_0, c_1), \{c_1\}, (c_1, c_2), \dots, (c_{n-1}, c_n)$$

□

Lemma 3.15. *If $f : I \to R$ definable and $S = \{x \in I : f \text{ is not continuous at } x\}$, then S is finite*

Proof. S is definable. If $|S| = \infty$, take $J \subseteq S$, replace f with $f|_J$, we may assume f is nowhere continuous. By Lemma 3.14, there is $J \subseteq I$, $f|_J$ is constant or monotone. Replace f with $f|_J$, now f is monotone (constant is continuous). Assume f is increasing, then f is injective, $|f(I)| = \infty$, take $J \subseteq f(I)$, $[c, d] \subseteq f(I)$, $c = f(a)$, $d = f(b)$, $x \in (a, b) \Rightarrow f(x) \in (c, d)$. f is strictly increasing. if $y \in (c, d) \subseteq f(I)$, so $\exists x \in I$, $y = f(x)$, therefore f is surjective. Also f is order-preserving, thus f is continuous on (a, b) (since we are using order to define the topology). But f is continuous at nowhere, so a contradiction ☐

Then the monotonicity theorem follows from the proof of Lemma 3.14 (modify the boundary to include the discontinuous points)

Corollary 3.16. *If $f : (a, b) \to R$ definable, $\lim_{x\to a^+} f(x)$ exists in R_∞*

Proof.

1. Take ϵ, $f|_{a,a+\epsilon}$ is continuous and monotone. Then $\lim_{x\to a^+} f(x)$ is $\sup\{f(x) : x \in (a, a + \epsilon)\}$ or $\inf\{f(x) : x \in (a, a + \epsilon)\}$

☐

Corollary 3.17. *If $f : [a, b] \to R$ is definable and continuous, then $\max_{x\in[a,b]f(x)}$ and $\min_{x\in[a,b]} f(x)$ exist*

Proof. Take maximum for each piece and combine ☐

Uniform Finiteness

Suppose $D \subseteq R^n \times R$, for $\bar{a} \in R^n$, $D_{\bar{a}} = \{y \in R : (a, y\} \in D$

Theorem 3.18 (Uniform Finitness)**.** *Suppose $\forall \bar{a}$, $|D_{\bar{a}} < \infty|$. Then $\exists N < \infty \forall \bar{a} |D_{\bar{a}}| < N$*

For now, consider $n = 1$.

Fix $D \subseteq R^2$ definable, $|D_a| < \infty$ for all $a \in R$

Definition 3.19. $(a, b) \subseteq R \times R_\infty$ is **normal** if either

- $(a, b) \notin \mathrm{cl}(D)$, $(\exists\epsilon)(a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \cap D = \emptyset$
- $(a, b) \in D$ and $(\exists\epsilon, \delta)D \cap (a - \epsilon, a + \epsilon) \times (b - \delta, b + \delta)$ is $\Gamma(f)$ for some continuous function f

Otherwise (a, b) is abnormal

Remark. $\{(x, y)$ normal$\}$ is open, $\{(x, y)$ abnormal$\}$ is closed.

Definition 3.20. $a \in R$ is **good** if $\forall b \in R_\infty$, (a, b) is normal, is **bad** if $\exists b \in R_\infty$, (a, b) is abnormal

This is a definable definition

Lemma 3.21. *$\{x \in R : x$ is bad$\}$ is finite*

Proof. Otherwise, take $I \subseteq B$, $\forall x \in I$, $\{y \in R_\infty : (x, y)$ abnormal$\}$ is closed, nonempty.

Let $f(x) = \min\{y \in R_\infty : (x, y)$ abnormal$\}$, $f : I \to R_\infty$ definable.

$\forall x$, break into cases based on these questions

- $f(x) = -\infty$ vs $f(x) \in R$ vs $f(x) = +\infty$
- $(x, f(x)) \in D$ vs not
- whether $\exists y > f(x)$, $(x, y) \in D$
- whether $\exists y < f(x)$, $(x, y) \in D$

So 24 pieces

Shrink I to make all the answers constant

Assume $\forall x \in I$, $f(x) \in R$, $(x, f(x)) \in D$, $(\exists y < f(x))(x, y) \in D$, $(\exists z > f(x))(x, z) \in D$

Let $g(x) = \max\{y : y < f(x), (x, y) \in D\}$, $h(x) = \min\{y : y > f(x), (x, y) \in D\}$

D_x is finite and we can take the min and max

For each $x \in I$, $(x, f(x)) \in D$, $(x, f(x))$ is abnormal, if $f(x) < y < h(x)$, then $(x, y) \notin D$

Idea: apply monotonicity theorem, get f, g, h continuous, then $(x, f(x))$ is normal

Use monotonicity theorem to get $J \subseteq I$, $f|_J, g|_J, h|_J$ are continuous

Take $a \in J$, $(a, f(a)) \in D$, $(a, f(a))$ is normal. Take ϵ s.t. $g(a) + \epsilon < f(a) - \epsilon$, $f(a) + \epsilon < h(a) - \epsilon$. Take δ s.t. if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$, same for g, h

If $x \in (a - \epsilon, a + \epsilon)$, then $(x, f(x)) \in D$, $(a - \delta, a + \delta) \times (f(a) - \epsilon, f(a) + \epsilon) \cap D$ is $\Gamma(f|_{(a-\delta, a+\delta)})$

if $(x, y) \in D$ and $x \in (a - \delta, a + \delta)$, $y \in (f(a) - \epsilon, f(a) + \epsilon)$ then $y = f(x)$ or else $y \neq f(x)$, then $y \in D_x$, so either $y \geq h(x)$ or $y \leq g(x)$. But $|x - a| < \delta$, $|h(x) - h(a)| < \epsilon$ $\square$

Lemma 3.22. *If $I \subseteq R$ and $\forall x \in I$, x is good, then $\exists n < \infty$, $\forall a \in I$, $|D_a| = n$*

Proof. Take $a_0 \in I$, let $n = \left|D_{a_0}\right| < \infty$.

Let $S = \{a \in I : |D_a| = n\}$.

Goal: S and $I \setminus S$ are open.

This is sufficient because o-minimality shows intervals are definably connected. $I \neq X \cup Y$ where X, Y are disjoint, nonempty, definable and open. So if $I = X \cup Y$, $X \cap Y = \emptyset$, X, Y definable open, then one of them is empty

Let $S_n = \{a \in I : |D_a|\} = n$, $S = S_n$

Goal: each S_n is open. Then $I \setminus S_n$ and S_n are open

Idea: $I \to \mathbb{N}$, $a \mapsto |D_a|$ is locally constant

Fix $a \in I$, let $n = |D_a|$, $D_a = \{b_1, \dots, b_n\}$, $b_1 < \dots < b_n$. Since a is good, so (a, b_i) is normal. Take ϵ, δ small enough, then $(a-\delta, a+\delta) \times (b_i-\epsilon, b_i+\epsilon) \cap D$ is a graph of a continuous function $f_i : (a-\delta, a+\delta) \to R$. Take ϵ small enough s.t. $b_i + \epsilon < b_{i+1} - \epsilon$. If $x \in (a-\delta, a+\delta)$, then $(f_1(a), \dots, f_n(a)) \subseteq D_x$, then $|D_x| \geq n$.

Suppose $|D_x| \geq n$, if $(a-\delta, a+\delta) \subseteq S_n$, then we are done. Otherwise, by o-minimality, $(a, a+\delta') \cap S_n =$ or $(a-\delta', a) \cap S_n = \emptyset$.

Shrinking δ', δ we can assume $\delta = \delta'$. WLOG, $a < x < a + \delta$, $x \notin S_n$.

Let $g(x) = \min D_x \setminus \{f_1(x), \dots, f_n(x)\}$. By monotonicity theorem, $\lim_{x \to a^+} g(x) =: c \in R_\infty$

(a, c) is normal because a is good.

$(a, c) = \lim_{x \to a^+}(x, g(|x|))$, so $(a, c) \in \mathrm{cl}(D)$. Because (a, c) is normal, $(a, c) \in D$ and D looks like $\Gamma(h)$ around (a, c). $\square$

Proof of uniform boundedness. Let $b_1, \dots, b_n$ be the bad points.

$$(-\infty, b_1), (b_1, b_2), \dots, (b_{n-1}, b_n), (b_n, +\infty)$$

and $|D_a|$ is constant on each intervals $\square$

3.2 The cell decomposition theorem

for each definable set X in R^m we put

$$C(X) := \{f : X \to R : f \text{ definable and continuous}\}$$
$$C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$$

where we regard $-\infty$ and $+\infty$ as constant functions on X

For $f, g \in C_\infty(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\}$$

So $(f, g)_X$ is a definable subset of R^{m+1}

Definition 3.23. Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones of length m. An $(i_1, \dots, i_m)$**-cell** is a definable subset of R^m obtained by induction on m as follows:

1. a (0)-cell is a one-element set $\{r\} \subseteq R$, a (1)-cell is an interval $(a, b) \subseteq R$
2. suppose $(i_1, \dots, i_m)$-cells are already defined, then an $(i_1, \dots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where X is an $(i_1, \dots, i_m)$-cell; further, an $(i_1, \dots, i_m, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, \dots, i_m)$-cell and $f, g \in C_\infty(X)$, $f < g$

So a $(0, 0)$-cell is a "point" $\{(r, s)\} \subseteq R^2$, a $(0, 1)$-cell is an "interval" on a vertical line $\{a\} \times R$, and a $(1, 0)$-cell is the graph of a continuous definable function defined on an interval.

Definition 3.24. A **cell in** R^m is an $(i_1, \dots, i_m)$-cell for some (necessarily unique) sequence $(i_1, \dots, i_m)$. Since the $(1, \dots, 1)$-cells are exactly the cells which are open in their ambient space R^m, we call these **open cells**

The non-open cells are "thin":
The union of finitely many non-open cells in R^m has empty interior

Proposition 3.25. *Each cell is locally closed, i.e., open in its closure*

Proof. Let $C \subseteq R^{m+1}$ be a cell. Put $B := \pi(C) \subseteq R^m$ and assume inductively that the cell B is open in its closure $\mathrm{cl}(B)$, so that $\mathrm{cl}(B) - B$ is a closed set. If $C = \Gamma(f)$ with $f : B \to R$ a definable continuous function, then $\mathrm{cl}(C) - C$ is contained in $(\mathrm{cl}(B) - B) \times R$, hence C is open in the closed set $C \cup ((\mathrm{cl}(B) - B) \times R)$

If $C = (f, g)$ with $f, g : B \to R$ definable continuous functions on B, $f < g$, then one verifies that $\mathrm{cl}(C) - C \subseteq \Gamma(f) \cup \Gamma(g) \cup ((\mathrm{cl}(B) - B) \times R)$ and that C is open in the closed set $C \cup \Gamma(f) \cup \Gamma(g) \cup ((\mathrm{cl}(B) - B) \times R)$ □

we consider the point-space R^0 as a cell, or ()-cell, where () is the sequence of length 0

Each cell is homeomorphic under a coordinate projection to an open cell. We now make this explicit. Let $i = (i_1, \dots, i_m)$ be a sequence of zeros and ones

Define $p_i : R^m \to R^k$ as follows: let $\lambda(1) < \dots < \lambda(k)$ be the indices $\lambda \in \{1, \dots, m\}$ for which $i_\lambda = 1$, so that $k = i_1 + \dots + i_m$; then

$$p_i(x_1, \dots, x_m) := (x_{\lambda(1), \dots, x_{\lambda(k)}})$$

It is easy to show by induction on m that p_i maps each i-cell A homeomorphically onto an open cell $p_i(A)$ in R^k. We denote $p_i(A)$ also by $p(A)$ and the homeomorphism $p_i|A : A \to p(A)$ by p_A. Clearly $p_A = \mathrm{id}_A$ if A is an open cell

If A is a cell in R^{m+1} then $\pi(A)$ is a cell in R^m, where $\pi : R^{m+1} \to R^m$ is the projection on the first m coordinates. Here is a simple application of this fact

Proposition 3.26. *Each cell is definably connected*

Proof. For intervals and points this is stated in 1.15

If A is a cell in R^{m+1}, then we assume inductively that the cell $\pi(A)$ in R^m is definably connected and use the fact that each fiber $\pi^{-1}(x) \cap A$ is definably connected $\square$

Theorem 3.27 (Cell decomposition)**.** *If $X \subseteq R^m$ is definable, then $\exists C_1, \dots, C_n$ cells, $X = \bigcup_{i=1}^n C_i$, $C_i \cap C_j = \emptyset$ for $i \neq j$.*

Example 3.1. In $(\mathbb{R}, +, 0, \leq, 0, 1)$, $X = \{(x, y) : x^2 + y^2 \leq 1\}$ is a $(1, 1)$-cell

Theorem 3.28. *For any $m \in \mathbb{N}$,*

1. *$(Cell_m)$: any definable $A \subseteq R^m$ has a cell decomposition. $A = \bigcup_{i=1}^n C_i$*

2. *(Con_m): if $f : A \to R$ definable, then $\exists$ cell decomposition $A = \bigcup_{i=1}^n C_i$ s.t. $f|_{C_i}$ is continuous for all i*

3. *(Fin_m): if $A \subseteq R^m \times R$ definable, and if $A_{\bar{x}} = \{y \in R : (\bar{x}, y) \in A\}$ is finite $\forall \bar{x} \in R^m$, then $\exists N \in \mathbb{N}$, $\forall \bar{x} \in R^m$, $|A_{\bar{x}}| \leq N$.*

Proof stragegy: Cell_1, Con_1, Fin_1, Cell_2, Con_2, Fin_2, and so on.
(Cell_1) is the definition of o-minimality.
(Con_1) is the monotonicity theorem.
(Fin_1) is the uniform finiteness part 1.
Suppose $m > 1$, take $m = 2$ for simplicity.

If $D \subseteq R$ definable, and $\mathrm{bd}(D) = \{x_1, \dots, x_n\}$, $x_1 < \cdots < x_n$, then D is the union of some of

$$c_0 := (-\infty, x_1), c_1 := \{x_1\}, c_2 := (x_1, x_2), \dots, c_{2n-1} := \{x_n\}, c_{2n} := (x_n, +\infty)$$

The "shape" of D is the string $(j_0, \dots, j_{2n})$ where $j_i = 1$ if $C_i \subseteq D$ and 0 if $C_i \cap D = \emptyset$

(Cell$_2$): Fix definable $A \subseteq R^2$, $A_x = \{y \in R : (x, y) \in A\}$, bd$(A_x)$ is finite for all $x \in R$. By (Fin$_1$), $\exists N \in \mathbb{N}$ s.t. $\forall x$, $|\text{bd}(A_x)| \leq N$, then $\{shape(A_x) : x \in R\}$ is finite. $\{x \in R : shape(A_x) = 0011001\}$ is definable.

By (Cell$_1$), can partition R into cells $C_1, \ldots, C_n$ s.t. $shape(A_x)$ is a constant for $x \in C_i$.

Let $f_i(x)$ be the ith smallest element of bd(A_x), then f is definable. We can further partition and WMA $f_1, \ldots, f_N$ is continuous on $C_1, \ldots, C_n$ using (Con$_1$).

(Con$_2$):

Lemma 3.29. *If $B \subseteq R^2$ is a box and $f : B \to R$ is continuous in each variable separately and monotone in each variable separately, then f is continuous.*

Proof. WLOG, f is increasing in each variable.

Fix $(x, y) \in B$, $\epsilon > 0$, $\exists \delta_1 > 0$ s.t. $f(x + \delta_1, y) < f(x, y) + \epsilon$. $\exists \delta_2 > 0$ s.t. $f(x, y + \delta_2) < f(x, y) + \epsilon$.

$\exists \delta_3, \delta_4 > 0$, $f(x - \delta_3, y - \delta_4) > f(x, y) - \epsilon$.

If $x' \in (x - \delta_3, x + \delta_4)$ and $y' \in (y - \delta_4, y + \delta_2)$, then

$$f(x, y) - \epsilon < \cdots < f(x', y') < f(x + \delta_1, y + \delta_2) < f(x, y) + \epsilon$$

$\square$

For $x \in A$ ask questions:

- is f continuous at x
- is f continuous in first variable at x
- is f continuous in second variable at x
- is f increasing in first variable
- is f decreasing in first variable
- increasing in second
- decreasing in second

By (Cell$_2$), $A = \bigcup_{i=1}^n C_i$, answers are constant on each C_i. Fix $C = C_i$

- if f is continuous $\forall x \in C$, then $f|_C$ is continuous

- if C is not a $(1,1)$-cell, then there is a coordinate projection $\pi : R^2 \to R$ s.t. $C \to \pi(C)$ is a bijection, $\pi(C)$ is a cell. $C \to \pi(C)$ is a homeomorphism, subcells of C corresponds to subcells of $\pi(C)$

  (Con_1) on $\pi(C)$ implies (Con_2) on C by homeomorphism.

- C is a $(1,1)$-cell, f is continuous nowhere on C. Take $(a,b) \in C$, look at $f(x,b)$ for $x \in (a-\epsilon, a+\epsilon)$. Apply monotonicity theorem, get a' s.t. f at (a',b) is continuous and monotone in x, then f is continuous and monotone in 1st coordinate everywhere in C. Similary, f is cts & monotone in 2nd coordinate. By the lemma, f is continuous,

Fix $A \subseteq R^2 \times R$, $\left|A_{(x,y)} < \infty\right|$ for any $x, y \in R$

Definition 3.30. $(x,y,z) \in R^2 \times R_\infty$ is **normal** if

- locally, A is $\emptyset$, $(x,y,z) \notin \text{cl}(A)$, or
- locally, A is $\Gamma(f)$, f is continuous

abnormal otherwise
(x,y) is **good** if $\forall z \in R_\infty$, (x,y,z) is normal. Bad otherwise

Lemma 3.31. *If $B \subseteq R^2$ is a box, then there is $(x,y) \in B$ s.t. (x,y) is good.*

Proof. Similar, use Con_2 □

Lemma 3.32. *If $B \subseteq R^2$ is a box and $\forall (x,y) \in B$, (x,y) is good, then $\left|A_{(x,y)}\right|$ is constant for $(x,y) \in B$*

Proof. Suppose $B = I \times J$, fix $a_0 \in I$, let $A' = \{(y,z) : (a_0, y, z) \in A\}$, $A'_y = A_{(a_0,y)}$. A'_y is finite for each y.
Check A' is good on J. Every $y \in J$ is good w.r.t. A'.
By the $m = 1$ version of lemma, $\left|A'_y\right|$ is constant for $y \in J$
$f(x,y) = \left|A_{(x,y)}\right|$ doesn't depend on 2nd coordinate.
Similary, $f(x,y)$ is constant on 1st coordinate
So f is constant on B. □

Lemma 3.33. *If C is an open cell, $(1,1)$-cell, and $\forall (x,y) \in C$ is good, then $\left|A_{(x,y)}\right|$ is constant for $(x,y) \in C$*

Proof. Take $(x_0, y_0) \in C$, let $n = \left|A_{(x_0,y_0)}\right|$, let $D = \{(x,y) \in C : \left|A_{(x,y)} = n\right|\}$. Using **??**, can show that D is open, $C \setminus D$ open
But C is definably connected

$$\chi_D(x) \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases}$$

want χ_D constant in the cell.

1. $\chi_D(x, y)$ doesn't depend on y, so $\chi_D(x, y) = f(x)$
2. $f(x)$ doesn't depend on x

$\square$

Proof of Fin_2: Use (Cell_2) to split R^2 into $R^2 = \bigcup_{i=1}^{n} C_i$, where for i, (normal is definable)

1. $\forall (x, y) \in C_i$, (x, y) is good
2. C_i is bad.

We only need uniform finiteness above each C_i

Fix C_i

Case 1: C_i is a $(1, 1)$-cell, all the points in C_i is good.

Case 2: C_i is not a $(1, 1)$-cell. Take a projection π s.t. $\pi(C_i)$ is an cell. Uniform finiteness holds over $\pi(C_i)$, then uniform finiteness holds over C_i, transfering the definable bijection π

R is o-minimal.

Lemma 3.34. *If $\varphi(\bar{x}, y)$ is a formula and $|\varphi(\bar{a}, M)| < \infty$ for all $\bar{a} \in R^m$, then $\exists N < \infty$ s.t. $\forall \bar{a} \in R^m$, $|\varphi(\bar{a}, M)| < N$*

Remark. If $\varphi(\bar{x}, y)$ a formula and we let $\varphi'(\bar{x}, y)$ be

$$\forall z, w[z < y < w \to (\exists s : z < s < w \wedge \varphi(\bar{x}, s)) \wedge (\exists s : z < s < w \wedge \neg\varphi(\bar{x}, s))]$$

Then $\varphi'(\bar{a}, R)$ is $\text{bd}(\varphi(\bar{a}, R))$

Theorem 3.35. *If $R \equiv R'$ and R is o-minimal, then R' is o-minimal*

Proof. If $D \subseteq R'$ definable, (want D be the union of finite intervals)

$D = \varphi(\bar{a}, R')$ for some $\bar{a} \in R'$. Let $\varphi'(\bar{x}, y)$ be the formula from the remark.

$\varphi'(\bar{b}, R') = \text{bd}(\varphi(\bar{b}, R'))$ for $\bar{b} \in (R')^m$, so $\varphi'(\bar{b}, R) = \text{bd}(\varphi(\bar{b}, R))$ for all $\bar{b} \in R^m$.

O-minimality $\Rightarrow \left|\varphi'(\bar{b}, R)\right| < \infty$ for all $\bar{b} \in R^m$. Uniform finiteness gives N s.t. $\forall \bar{b} \in R^m$, $\left|\varphi'(\bar{b}, R)\right| < N$, therefore $\varphi'(\bar{b}, R')$ is finite for all $\bar{b} \in (R')^m$. Take $\bar{b} = \bar{a}$, therefore $\mathrm{bd}(D)$ is finite.

Claim: In R', If $y < z$ and $[y, z] \cap \mathrm{bd}(D)$, then $y \in D \Leftrightarrow z \in D$

Proof: True in R by o-minimality.

$$R \vDash \forall \bar{x}, y, z(y < z \wedge \neg\exists w(y \leq w \leq z \wedge \varphi'(\bar{x}, w)) \rightarrow [\varphi(\bar{x}, y) \leftrightarrow \varphi(\bar{x}, z)])$$

If $\mathrm{bd}(D) = \{c_1, \dots, c_m\}$, $c_1 < \dots < c_m$, the claim shows that D respects the partition of R'.

So D is a union of some points and intervals $\square$

Definition 3.36. T is o-minimal is every model of T is o-minimal

Example 3.2. DLO, RCF, ODAG

Theorem 3.37. *If $\mathcal{M}$ is o-minimal, then* $\mathrm{Th}(\mathcal{M})$ *is o-minimal*

Definition 3.38. $\mathcal{M}$ is minimal if $\forall$ definable $D \subseteq M$, D is either finite or cofinite.

Definition 3.39. $\mathcal{M}$ is strongly minimal if

1. $\mathcal{M} \vDash T$ where T is strongly minimal, or
2. $\forall \mathcal{N} \equiv \mathcal{M}$, $\mathcal{N}$ is minimal.

Fact 3.40. $(\mathbb{C}, +, \cdot)$ *is strongly minimal,* $(\mathbb{N}, \leq)$ *is minimal but not strongly minimal*

Idea: strong o-minimality = o-minimality

Definition 3.41. A **decomposition** of R^m is a special kind of partition of R^m into finitely many cells. The definition is by induction on m

1. a decomposition of $R^1 = R$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

   where $a_1 < \dots < a_k$ are points

2. a decomposition of R^{m+1} is a finite partition of R^{m+1} into cells A s.t. the set of projections $\pi(A)$ is a decomposition of R^m

Let $\mathcal{D} = \{A(1), \dots, A(k)\}$ be a decomposition of R^m, $A(i) \neq A(j)$ if $i \neq j$, and let for each $i \in \{1, \dots, k\}$ functions $f_{i1} < \dots < f_{in(i)}$ in $C(A_i)$ be given

Then

$$\mathcal{D}_i := \{(-\infty, f_{i1}), (f_{i1}, f_{i2}), \dots, (f_{in(i)}, +\infty), \Gamma(f_{i1}), \dots, \Gamma(f_{in(i)})\}$$

is a partition of $A(i) \times R$ and one easily checks that $\mathcal{D}^* := \mathcal{D}_1 \cup \dots \cup \mathcal{D}_k$ is a decomposition of R^{m+1}, and that every decomposition of R^{m+1} arises in this way from a decomposition $\mathcal{D}$ of R^m. We write $\mathcal{D} = \pi(\mathcal{D}^*)$

A decomposition $\mathcal{D}$ of R^m is said to be **partition** a set $S \subseteq R^m$ if each cell in $\mathcal{D}$ is either part of S or disjoint from S, in other words, if S is a union of cells in $\mathcal{D}$.

Theorem 3.42 (Cell Decomposition Theorem)**.** *1. (I_m)Given any definable sets $A_1, \dots, A_k \subseteq R^m$ there is a decomposition of R^m partitioning each of $A_1, \dots, A_k$*

2. (II_m)For each definable function $f : A \to R$, $A \subseteq R^m$, there is a decomposition $\mathcal{D}$ of R^m partitioning A s.t. the restriction $f|B : B \to R$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous

(I_1) holds by o-minimality, and that (II_1) follows from the monotonicity theorem

We now assume that $\mathrm{I}_1, \dots, \mathrm{I}_m$ and $\mathrm{II}_1, \dots, \mathrm{II}_m$ hold

The proof is lengthy. The first step is to generalize the finiteness lemma of the previous section. Call a set $Y \subseteq R^{m+1}$ **finite over** R^m if for each $x \in R^m$ the fiber $Y_x := \{r \in R : (x, r\} \in Y$ is finite; call Y **uniformly finite over** R^m if there is $N \in \mathbb{N}$ s.t. $|Y_x| \leq N$ for all $x \in R^m$

Lemma 3.43 (Uniform Finitness Property)**.** *Suppose the definable subset Y of R^{m+1} is finite over R^m, then Y is uniformly finite over R^m*

Proof. ☐

Lemma 3.44. *Let X be a topological space, $(R_1, <)$, $(R_2, <)$ dense linear orderings without endpoints and $f : X \times R_1 \to R_2$ a function s.t. for each $(x, r) \in X \times R_1$*

1. *$f(x, \cdot) : R_1 \to R_2$ is continuous*

2. *$f(\cdot, r) : X \to R_2$ is continuous*

Then f is continuous

Proof. Let $(x, r) \in X \times R_1$ and $f(x, r) \in J$, where J is an interval in R_2. We shall find a neighborhood U of x and an interval I around r s.t. $f(U \times I) \subseteq J$. By (1) there are r_-, r_+ in R_1 s.t. $r_- < r < r_+$ and $f(x, r_-), f(x, r_+) \in J$. Now use (2) to get a neighborhood U of x s.t. $f(U \times \{r_-\}) \subseteq J$ and $f(U \times \{r_+\}) \subseteq J$. We claim that then $f(U \times I) \subseteq J$ for $I = (r_-, r_+)$

Let $x' \in U$ and $r_- < r' < r_+$. Assume $f(x', \cdot)$ is increasing, then $f(x', r_-) \leq f(x', r') \leq f(x', r_+)$ and $f(x', r_-)$, $f(x', r_+)$ are both in J, hence $f(x', r')$ is in J $\square$

A **definably connected component** of a nonempty definable set $X \subseteq R^m$ is by definition a maximal definably connected subset of X

Proposition 3.45. *Let $X \subseteq R^m$ be a nonempty definable set. Then X has only finitely many definably connected components. They are open and closed in X and form a finite partition of X*

Proof. Let $\{C_1, \dots, C_k\}$ be a partition of X into k disjoint cells. For each nonempty set of indices $I \subseteq \{1, \dots, k\}$, put $C_I := \bigcup_{i \in I} C_i$. Among the $2^k - 1$ sets C_I, let C' be maximal w.r.t. being definably connected.

Claim: If a set $Y \subseteq X$ is definably connected and $C' \cap Y \neq \emptyset$, then $Y \subseteq C'$

Put $C_Y := \bigcup\{C_i : C_i \cap Y \neq \emptyset\}$. Since the C_i's cover X we have $Y \subseteq C_Y$, so C_Y is the union of Y with certain cells that intersect Y. Hence C_Y is definably connected . By maximality of C' it follows that $C' \cup C_Y = C'$. Hence $Y \subseteq C_Y \subseteq C'$, which proves the claim.

It follows in particular that C' is a definably connected component of X. Further the claim shows that the sets C' are the only definable connected components of X. Note that because the closure in X of a definably connected subset of X is also definably connected, the definably connected components of X are closed in X. Hence they are open in X $\square$

3.3 Definable families

Let $S \subseteq R^{m+n} = R^m \times R^n$ be definable. For each $a \in R^m$ we put

$$S_a := \{x \in R^n : (a, x) \in S\} \subset R^n$$

We view S as describing the family of sets $(S_a)_{a \in R^m}$. Such a family is called a **definable family** (of subsets of R^n, with parameter space R^m). The sets S_a are also called the **fibers** of the family

Example 3.3. Let $\mathcal{R} := (\mathbb{R}, <, +, \cdot)$ and consider the formula

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

This defines a relation $S \subseteq \mathbb{R}^6 \times \mathbb{R}^2$. For each point $(a, b, c, d, e, f) \in \mathbb{R}^6$ the subset $S_{(a,b,c,d,e,f)} \in \mathbb{R}^2$ consists of the points (x, y) satisfying the equation

In the following $\pi : R^{m+n} \to R^m$ denotes the projection on the first m coordinates

Proposition 3.46. *1. Let C be a cell in R^{m+n} and $a \in \pi(C)$. Then C_a is a cell in R^n*

2. Let $\mathcal{D}$ be a decomposition of R^{m+n} and $a \in R^m$. Then the collection

$$\mathcal{D}_a := \{C_a : C \in \mathcal{D}, a \in \pi(C)\}$$

is a decomposition of R^n

Proof. For $n = 1$ this is immediate from the definitions

Suppose the proposition holds for a certain n, and let C be a cell in $R^{m+(n+1)}$. Let $\pi_1 : R^{m+(n+1)} \to R^{m+n}$ be the obvious projection map, so that $\pi \circ \pi_1 : R^{m+(n+1)} \to R^m$ is the projection on the first m coordinates

If $C = \Gamma(f)$, then $C_a = \Gamma(f_a)$, where $f_a : (\pi_1 C) \to R$ is defined by $f_a(x) = f(a, x)$

If $C = (f, g)_D$ with $D = \pi_1 C$, then $C_a = (f_a, g_a)_E$ where $E = D_a$

In both cases C_a is a cell in R^{n+1} □

Corollary 3.47. *Let $S \subseteq R^m \times R^n$ be definable. Then there is a number $M_S \in \mathbb{N}$ s.t. for each $a \in R^m$ the set $S_a \subseteq R^n$ has a partition into at most M_S cells. In particular, each fiber S_a has at most M_S definably connected components*

Proof. Take a decomposition $\mathcal{D}$ of R^{m+n} partitioning S. Then for each $a \in R^m$ the decomposition $\mathcal{D}_a = \{C_a : C \in \mathcal{D}, a \in \pi C\}$ of R^m consists of at most $|\mathcal{D}|$ cells and partitions S_a. So we can take $M_S = |\mathcal{D}|$ □

Corollary 3.48. *Let $S \subseteq R^m \times R^n$ be definable. Then there is a natural number M_S s.t. for each $a \in R^m$ the set $S_a \subseteq R^n$ has at most M_S isolated points. In particular, each finite fiber S_a has cardinality at most M_S*

4 Definable invariants: dimension and euler characteristic

4.1 Dimension

We define the **dimension** of a nonempty definable set $X \subseteq R^m$ by

$$\dim(X) := \max\{i_1 + \dots + i_m : X \text{ contains an } (i_1, \dots, i_m)\text{-cell}\}$$

To the empty set we assign the dimension $-\infty$

Goal:

Theorem 4.1. 1. *well-defined*

2. *if $f : A \to B$ is a definable bijection*

3. $\dim(A) > 0 \Leftrightarrow |A| = \infty$

4. $\dim(A \times B) = \dim(A) + \dim(B)$

5. $\dim(R^n) = n$

6. *If $f : A \to B$ is a definable surjection, if $\dim(f^{-1}(b)) = k$ for all $b \in B$, then $\dim(A) = k + \dim(B)$*

7. *If $\varphi(\bar{x}, \bar{y})$ is a formula, then*

$$S(k) = \{\bar{b} \in R^m : \dim(\varphi(R^n, \bar{b})) = k\}$$

is definable.

8. $\dim(cl(A)) = \dim(A)$, $\partial A := cl(A) \setminus A$, $\dim(\partial A) < \dim(A)$

Definition 4.2. If $A = \sqcup_{i=1}^n C_i$, $\chi(A) = \sum_{i=1}^n (-1)^{\dim(C_i)}$

Theorem 4.3. • *well-defined*

- $\chi(A \times B) = \chi(A) \cdot \chi(B)$
- $\chi(A \cup B) = \chi(A) \cdot \chi(B) - \chi(A \cap B)$
- *If $|A| < \infty$, $\chi(A) = |A|$*
- (7) *in above and replace* dim *with* χ
- *If $f : A \to B$*

Lemma 4.4. *If $f : A_1 \to A_2$ a definable injection, then $m \leq \dim(A_1) \Rightarrow m \leq \dim(A_2)$*

Theorem 4.5. 1. *If $f : A_1 \to A_2$ is a definable injection, then $\dim(A_1) \leq \dim(A_2)$.*

2. *If $f : A_1 \to A_2$ is a definable bijection, then $\dim(A_1) = \dim(A_2)$.*

3. *If $A_1 \subseteq A_2$, then $\dim(A_1) \leq \dim(A_2)$*

$D \subseteq R^n$ definable, $a \in \text{int}(D) \Leftrightarrow \exists \text{Box } B \ni a, B \subseteq D$.
$\text{int}(D) \neq \emptyset \Leftrightarrow \exists \text{Box } B, B \subseteq D$

Remark. If $C \subseteq R^n$ is an $(i_1, \dots, i_n)$-cell, then

- if $i_1 = i_2 = \dots = i_n = 1$, then C is open
- otherwise, $\text{int}(C) = \emptyset$ and C is nowhere dense.

Definition 4.6. D is **nowhere dense** if $\forall$ box B, $\exists$ box $B' \subseteq B$, $B' \cap D = \emptyset$.

Lemma 4.7. *If D_1, D_2 are nowhere dense, then $D_1 \cup D_2$ is nowhere dense.*

Proof. Given a box B_1, $\exists B_2 \subseteq B_1$, $B_2 \cap D_1 = \emptyset$, $\exists B_3 \subseteq B_2$, $B_3 \cap D_2 = \emptyset$, $B_3 \cap (D_1 \cup D_2) = \emptyset$. □

If $A \subseteq R^m$ definable, $A = \bigcup_{i=1}^n C_i$, C_i are cells, then either

- some C_i is open, then $\text{int}(A) \neq \emptyset$, or
- all C_i are nowhere dense, so A is nowhere

Corollary 4.8. *If $D_1, \dots, D_n \subseteq R^m$ are definable and $int(D_i) = \emptyset$, then $int(\bigcup_{i=1}^m D_i) = \emptyset$.*

$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, $\text{int}(\mathbb{Q}) = \text{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, $\text{int}(\mathbb{R}) = \mathbb{R}$

Proof. $\text{int}(D_i) = \emptyset \Rightarrow D_i$ is nowhere dense. □

Theorem 4.9. *If $D_1, D_2 \subseteq R^n$ definable, then $\dim(D_1 \cup D_2) = \max(\dim(D_1), \dim(D_2))$*

Proof. $\max(\dim(D_1), \dim(D_2)) \leq \dim(D_1 \cup D_2)$.
Claim: If $m \leq \dim(D_1 \cup D_2)$, then $m \leq \max(\dim(D_1), \dim(D_2))$

Take $B \subseteq R^m$, definable injection $f : B \hookrightarrow D_1 \cup D_2$. If $x \in B$, $f(x) \in D_1$ or $f(x) \in D_2$. So $B = f^{-1}(D_1) \cup f^{-1}(D_2)$. $\text{int}(B) = B \neq \emptyset$, so $\text{int}(f^{-1}(D_1)) \neq \emptyset$ or $\text{int}(f^{-1}(D_2)) \neq \emptyset$.

$\exists$ box $B' \subseteq f^{-1}(D_1)$ and $f|_{B'} : B' \hookrightarrow D_1$, so $m \leq \dim(D_i) \leq \max(\dim(D_1), \dim(D_2))$ □

Theorem 4.10. *If $D \subseteq R^n$ definable and int$(D) \neq \emptyset$*

1. (I_n): *$\nexists$ definable injection $f : D \to R^{n-1}$*
2. (II_n): *If $f : D \to R^n$ a definable injection, then int$(f(D)) \neq \emptyset$.*

Proof. $n = 1$, II_1, $|D| = \infty$, $|f(D)| = \infty$, then o-minimality says $f(D)$ contains an interval

I_n: By Cell decomposition, can shrink B, get $f|_B$ continuous. And at least one of cell is open, take box B from it.

$B = B_0 \times (a, b)$, $B_0 \subseteq R^{n-1}$, take $c \in (a, b)$, let $g : B_0 \to R^{n-1}$, $g(x) = f(x, c)$. By II_{n-1}, $g(B_0)$ has interior, take $p = g(x_0) \in \text{int}(g(B_0))$, f is continuous, $\exists c' \in (a, b)$ s.t. $c' \neq c$, $f(x_0, c') \in \text{int}(g(B_0))$, $f(x_0, c') = g(x_1) = f(x_1, c)$ for some $x_1 \in B_0$, but f is injective.

II_n: $f : D \to R^n$ injection, int$(D) \neq \emptyset$. Assume int$(f(D)) = \emptyset$.

$f(D) = \sqcup_{i=1}^n C_i$ cells, no C_i is open. $D = \bigcup_{i=1}^n f^{-1}(C_i)$, int$(D) \neq \emptyset$. There is i s.t. int$(f^{-1}(C_i)) \neq \emptyset$, there is box $B \subseteq f^{-1}(C_i)$, therefore there is definable injection $B \to C_i \to \pi(C_i) \subseteq R^{n-1}$, contradicts I_n $\square$

Corollary 4.11. $\dim(R^n) = n$

If $D \subseteq R^n$, int$(D) \neq \emptyset$, then $\dim(D) = n$

int$(D) \neq \emptyset \Leftrightarrow \dim(D) = n$

dimension theory rules out things like space-filling curves. No definable surjection

$$[0, 1] \to [0, 1]^2$$

No Hilbert curves,

Lemma 4.12. *If $A \subseteq R^m$ is an open cell and $f : A \to R^m$ an injective definable map, then $f(A)$ contains an open cell*

Proof. Clearly for $m = 1$. Let $m > 1$ and assume inductively the lemma holds for lower values of m. Taking a decomposition of R^m that partitions $f(A)$ we have

$$f(A) = C_1 \cup \cdots \cup C_k \text{ for cells } C_i \text{ in } R^m$$

Then

$$A = f^{-1}(C_1) \cup \cdots \cup f^{-1}(C_k)$$

so at least one of the $f^{-1}(C_i)$, say $f^{-1}(C_1)$, contains a box B , and by taking B suitably small we may assume that $f|B$ is continuous. We now claim that C_1 is open.

If not, then by composing $f|B : B \to C_1$ with a definable homeomorphism of C_1 with a cell in R^{m-1} we obtain a definable continuous injective map $g : B \to R^{m-1}$. Write $B = B' \times (a, b)$

Take c with $a < c < b$ and consider the map $h : B' \to R^{m-1}$ given by $h(x) = f(x, c)$. By the inductive assumption applied to h we get $h(B') \supseteq D$ for some box D in R^{m-1}. Let y be a point in D and take x in B' with $h(x) = y$

If $c' \neq c$ is sufficiently close to c, then $g(x, c')$ will be in D, so $g(x, c') = h(x') = g(x', c)$ for some $x' \in B'$. This contradicts the injectivity of g $\square$

Box is a cell

Proposition 4.13. *1. If $X \subseteq Y \subseteq R^m$ and X, Y are definable, then* $\dim X \leq \dim Y \leq m$

2. *If $X \subseteq R^m$ and $Y \subseteq R^n$ are definable and there is a definable bijection between X and Y, then* $\dim X = \dim Y$

3. *If $X, Y \subseteq R^m$ are definable, then* $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$

Proof. 2. Let $f : X \to Y$ be a definable bijection and $d = \dim X$, $e = \dim Y$. It is enough to show $d \leq e$.

Let A be an $(i_1, \dots, i_m)$-cell contained in X, with $d = i_1 + \dots + i_m$. Then $f \circ (p_A^{-1}) : p(A) \to Y$ is an injective map and $p(A)$ an open cell. Replacing X by $p(A)$, Y by $f(A)$ and f by $f \circ (p_A^{-1})$ we may as well assume that $d = m$ and that X is an open cell in R^d. Let $Y = C_1 \cup \dots \cup C_k$ be a partition of $Y = f(X)$ into cells. Then $X = f^{-1}(C_1) \cup \dots \cup f^{-1}(C_k)$, so by the cell decomposition theorem $f^{-1}(C_i)$ contains an open cell B since X is open, for some i. Fix such i and B

Let $C_i = C \subseteq R^n$ be a $(j_1, \dots, j_n)$-cell. We shall prove that $d \leq j_1 + \dots + j_n$.

Suppose $d > j_1 + \dots + j_n$, the composition

$$B \xrightarrow{f|B} C \xrightarrow{p_C} p(C) \subseteq R^{j_1 + \dots + j_n}$$

is an injective map. Identify $R^{j_1 + \dots + j_n}$ with a non-open cell $(R^{j_1 + \dots + j_n}) \times \{p\}$ in R^d, where $p \in R^{d-(j_1 + \dots + j_n)}$, we obtain a contradiction with lemma 3.3

3. Let $d = \dim(X \cup Y)$, and let A be an $(i_1, \dots, i_m)$-cell contained in $X \cup Y$ with $d = i_1 + \dots + i_m$. The open cell $pA \subseteq R^d$ is the union of $p_A(A \cap X)$ and $p_A(A \cap Y)$, so by the cell decomposition theorem, one of these sets,

say $p_A(A \cap X)$, contains a box B in R^d. Then $p_A^{-1}(B)$ is an $(i_1, \dots, i_m)$-cell contained in X, so that

$$\dim X \geq d \geq \dim X$$

$\square$

Theorem 4.14. *If C is an $(i_1, \dots, i_n)$-cell, then $\dim(C) = \sum_{j=1}^n i_j$.*

Proof. there is $\pi : R^n \to R^m$ s.t. $C \to \pi(C)$ is a homeomorphism, $\pi(C)$ is open. $\square$

Lemma 4.15. *If $C \subseteq R^n$ is an $(i_1, \dots, i_n)$-cell, $C' \subseteq R^m$ is a $(j_1, \dots, j_m)$-cell, then $C \times C'$ is an $(i_1, \dots, i_n, j_1, \dots, j_m)$-cell*

Theorem 4.16. *If $A \subseteq R^n$, $B \subseteq R^m$ definable, then*

$$\dim(A \times B) = \dim(A) + \dim(B)$$

Proof. Do cell decomposition on each one $\square$

Theorem 4.17. *If $A \subseteq R^{n+m}$, if $A_x = \{y \in R^m : (x, y\} \in A$ for all $x \in R^n$, if $S(k) = \{x : \dim(A_x) = k\}$, then $S(k)$ is definable and $\dim(A) = \max_k(k + \dim(S(k)))$*

Lemma 4.18. *If $D \subseteq R^n \times R^m$, if $\pi(\bar{x}, \bar{y}) = \bar{x}$, $\dim(D) \geq \dim(\pi(D))$*

Proof. If $C \subseteq R^{n+m}$ is a cell, then $\pi(C)$ is an cell $\square$

Theorem 4.19. *If $f : A \to B$ is a definable surjection, then $\dim(A) \geq \dim(B)$*

Proof. $\square$

Theorem 4.20. *If $D \subseteq R^n$, $m \leq n$, then $\dim(D) \geq m$ iff there is some coordinate projection $\pi : R^n \to R^m$ s.t. int$(\pi(D)) \neq \emptyset$.*

Proof. If $\exists \pi : R^n \to R^m$ and $\text{int}(\pi(D)) \neq \emptyset$, then $\dim(D) \geq \dim(\pi(D)) = m$

If $\dim(D) \geq m$, then $\exists$ cell $C \subseteq D$ s.t. C is an $(i_1, \dots, i_n)$-cell and $l = i_1 + \dots + i_n \geq m$.

There is $\pi_0 : R^n \to R^l$ s.t. $\pi_0(C)$ is an open cell.

Then take $\pi_1 : R^l \to R^m$ $\square$

$\partial D = \text{cl}(D) \setminus D$

Theorem 4.21. *If $A \subseteq R^n$ definable, $A \neq \emptyset$, then $\dim(\partial A) < \dim(A)$*

Proof. □

The next result says among other things that the dimension of a set from a definable family depends "definably" on its parameters

Proposition 4.22. *Let $S \subseteq R^m \times R^n$ be definable. For $d \in \{-\infty, 0, 1, \dots, n\}$ put*

$$S(d) := \{a \in R^m : \dim S_a = d\}$$

Then $S(d)$ is definable and the part of S above $S(d)$ has dimension given by

$$\dim\left(\bigcup_{a \in S(d)} \{a\} \times S_a\right) = \dim(S(d)) + d$$

Proof. Let $\mathcal{D}$ be a decomposition of R^{m+n} partitioning S □

$D = \bigsqcup_{i=1}^n C_i$, $\chi(D) = \sum_{i=1}^n (-1)^{\dim(C_i)}$

If K is a definable filed in o-minimal R, then K is not algebraically closed with $char(K) = p$

5 Tame topology

$(R, +, 0, 1, \leq, \dots)$ o-minimal ordered abelian group.

Note if $n \in \mathbb{N}$, $n > 0$, then $f(x) = nx$ is onto because bd(nR) is empty: take $c \in$ bd(nR), $(c - \epsilon, c) \subseteq nR$, $(c, c + \epsilon) \cap nR = \emptyset$ (maybe). Take δ so small that $n\delta < \epsilon$. Take $c - n\delta \in nR$ and $c + n\delta \notin nR$, impossible since $c - n\delta \in nR$ and $2n\delta \in nR$

$(R, +, 0, \leq) =\equiv (\mathbb{Q}, +, 0, \leq) \equiv (\mathbb{R}, +, 0, \leq)$

Definition 5.1. If $D \subseteq R^n$ definable, $D \neq \emptyset$, $\gamma(D) \in D$ is defined as follows:

1. if $D \subseteq R^1$,
   (a) if $D = \{a\}$, take $\gamma(D) = a$
   (b) if $D = (a, b)$, take $\gamma(D) = \frac{a+b}{2}$
   (c) if $(D = (-\infty, a))$, take $\gamma(D) = a - 1$
   (d) if $D = (a, +\infty)$, take $a + 1$
   (e) if $D = (-\infty, \infty)$, take 0
   (f) if D is arbitrary, suppose bd$(D) = \{c_1, \dots, c_m\}$, $c_1 < \dots < c_m$, and D is a union of some of $S_0 = (-\infty, c_1)$, $S_1 = \{c_1\}$, $S_2 = (c_1, c_2)$,..., $S_{2m+1} = (c_m, +\infty)$, then take minimal i s.t. $S_i \subseteq D$ and let $\gamma(D) = \gamma(S_i)$.

2. if $D \subseteq R^{n+1}$, $n \geq 1$, let $\pi(D) \subseteq R^n$, $\pi(\bar{x}, y) = \bar{x}$, let $\bar{a} = \gamma(\pi(D))$, let $D_{\bar{a}} = \{y : (\bar{a}, y\} \in D\} \neq \emptyset$, let $b = \gamma(D_{\bar{a}})$

Theorem 5.2. *$\gamma(D) \in D$, if $D \neq \emptyset$ and if $\{D_{\bar{a}}\}_{\bar{a} \in Y}$ is definable, then $\bar{a} \mapsto \gamma(D_{\bar{a}})$ is definable*

Theorem 5.3. *if D is definable, E is a definable equivalence relation on D, then $\exists f : D \to D'$ surjective definable s.t. $f(x) = f(y) \Leftrightarrow xEy$. (so f induces a bijection $D/E \to D'$)*

Proof. let $f(x) = \gamma([x]_E)$, if xEy, $f(x) = f(y)$. if $f(x) = f(y)$, then xEy. □

Theorem 5.4. *If $\bar{c} \in cl(D)$, $D \subseteq R^n$, then $\bar{c} = \lim_{x \to 0} f(x)$ where $f : (0, 1) \to D$.*

Proof. Let $D_\epsilon = \{\bar{x} \in D : \|\bar{x} - \bar{c}\| < \epsilon\}$, $D_\epsilon \neq \emptyset$ as $\bar{c} \in \text{cl}(D)$.

Take $f(\epsilon) = \gamma(D_\epsilon)$, $f : R_{>0} \to D$, we can make f continuous by scale it and move it □

Assume R is ordered field.

Definition 5.5. $D \subseteq R^n$ is definably compact if $\forall$ continuous $f : (0, 1) \to D$, $\lim_{x \to 0} f(x) \in D$

Theorem 5.6. *D is definably compact iff D is closed and bounded*

Proof. If $\text{cl}(D) \supsetneq D$, take $\bar{c} \in \text{cl}(D) \setminus D$, so D is not definably compact

if D is not bounded, take $A_N = \{\bar{x} \in D : \|\bar{x}\| > N\}$, $A_N \neq \emptyset$, $\forall N > 0$, so let $f(x) = \gamma(A_{1/x})$, $\lim f(x)$ doesn't exist.

If D is closed and bounded, and $f : (0, 1) \to D$ is continuous, then $\lim_{x \to 0} f(x)$ (monotonicity theorem) exists in R^n, D is closed and therefore $\bar{c} \in D$ □

Theorem 5.7. *If $f : D \to R^n$ continuous definable, D is definably compact, then $f(D)$ is definably compact*

Proof. If not, take $g : (0, 1) \to f(D)$ continuous and $\lim_{x \to 0} g(x) \notin f(D)$

let $h(x) = \gamma(\{y \in D : f(y) = g(x)\})$ definable, $f(h(x)) = g(x)$. h is continuous on $(0, \epsilon)$, definable compactness implies that $\lim_{x \to 0} h(x) \in D$, $f(\lim_{x \to 0} h(x)) = \lim_{x \to 0} f(h(x)) \in D$ □

Corollary 5.8. *If $f : D \to R$ continuous, D definably compact, $D \neq \emptyset$, then $\max(f(D))$, $\min(f(D))$ exist.*

Theorem 5.9. *If D is definably compact and $f : D \to R^m$ is definable, continuous, injective, then $D \to f(D)$ is a homeomorphism*

Definition 5.10. D is **definably path connected** if $\forall a, b \in D\ \exists$ continuous definable $f : [0, 1] \to D$, $f(0) = a$, $f(1) = b$.

D is **definably connected** if if $f : D \to \{0, 1\}$ is continuous, then f is constant

Remark. If D is definably path connected, then D is definably connected
cells are path connected, definably homeomorphic to a box

Lemma 5.11. *If $D_1 \cap D_2 = \emptyset$ and D_1 and D_2 are path-connected, cl$(D_1) \cap D_2 \neq \emptyset$, then $D_1 \cup D_2$ is definably path connected.*

Proof. take $\bar{c} \in \mathrm{cl}(D_1) \cap D_2$, there is path from D_1 to $\bar{c}$ ☐

Lemma 5.12. *If D is definable and $D_1, \dots, D_n$ are the path connected components, each D_i is definable, clopen in D*

Corollary 5.13. *D is path connected $\Leftrightarrow$ D is connected.*

over $\mathbb{R}$
Fix o-minimal $(R, +, \cdot, \leq, \dots)$ where $(R, +, \cdot, \leq)$ is an o-minimal field
If $f : I \to R$ is definable

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$
$$f^+(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x}$$
$$f^-(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x}$$

$f^+(x), f^-(x)$ always exist in $R_\infty = R \cup \{\pm\infty\}$
f is differentiable at x if $f^+(x) = f^-(x) \notin \{\pm\infty\}$

Lemma 5.14. *If f has a local minimum at x, then $f^-(x) \leq 0 \leq f^+(x)$*
similar for local maximum.
If $f'(x)$ exists then $f'(x) = 0$ in either case

Lemma 5.15. *If $f : I \to R$ is definable, then $\exists a \in I$ s.t. $f^+(a) \leq f^-(a)$ and $f^+(x) \neq +\infty$ and $f^-(x) \neq -\infty$*

Proof. Shrink I so that f is continuous. Take $[d, e] \subseteq I$, choose $a, b, c \in R$ so that if $g(x) = f(x) + ax^2 + bx + c$ then $g(d) = 0 < g(\frac{d+e}{c}) > g(e) = 0$. Now g has a local maximum at $x \in (d, e)$. $g^+(x) \leq 0 \leq g^-(x)$, $f^+(x) + 2ax + b \leq 0 \leq f^-(x) + 2ax + b$ ☐

Theorem 5.16. *If $f : I \to R$ definable*

1. *$\exists S \subseteq_f I$, $\forall x \in I \setminus S$, $f^+(x) \leq f^-(x)$ and $f^+(x) \neq \pm\infty$, $f^-(x) \neq \infty$*

Theorem 5.17. *If $f : [a, b] \to R$ is definable and continuous and differentiable on (a, b), then*

1. *(Rolle's theorem) If $f(a) = f(b)$ then $\exists c \in (a, b)$, $f'(c) = 0$*
2. *(Mean value theorem) $\exists c \in (a, b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$*

Theorem 5.18. *If $f : R^n \to R$ is definable, then $\exists S \subseteq R^n$,* $\dim(S) < n$ *s.t. f is differentiable on $R^n \setminus S$*

6 Triangulation

An **affine subspace** of R^n is a set like $a + V = \{a + x : x \in V\}$ where $a \in R^n$ and V is a linear subspace

$a_0, \dots, a_k \in R^n$ are **affine independent** if there is no affine subspace $A \subset R^n$ s.t. $\dim(A) < k$ and $A \supseteq \{a_0, \dots, a_k\}$

Definition 6.1. A **k-simplex** in R^n is a set like

$$(a_0, \dots, a_k) := \{t_0 a_0 + \dots + t_k a_k : t_0, \dots, t_k \in R_{>0}, t_0 + \dots + t_k = 1\}$$

where $a_0, \dots, a_k$ is affine independent.

A **face** of a k-simplex $(a_0, \dots, a_k)$ is one of the form $(b_0, \dots, b_l)$ where $\{b_0, \dots, b_l\} \subseteq \{a_0, \dots, a_k\}$

A **closed simplicial complex** is a finite set K of simplices in R^n s.t.

1. pairwise disjoint
2. if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$

A **subcomplex** is a subset $K' \subseteq K$

A **simplicial complex** is a subcomplex of a closed simplecial complex

$|K| = \bigcup K$

$|K|$ is definably compact iff K is a closed simplicial complex

Theorem 6.2 (Triangulation theorem)**.** *If $D \subseteq R^n$ is definable, then $\exists$ simplicial complex K and a definable homeomorphism $\Phi : D \to |K|$*

If $D_1, \dots, D_m \subseteq D$ definable, can choose K, Φ so that $\exists K_1, \dots, K_n \subseteq K$ and $\Phi(D_i) = K_i$

Small number of homeomorphism classes
$A \cong B$ is there is definable homeomorphism $A \to B$
$[A] = \{B : B \cong A\}$.

Theorem 6.3. *$\{[A] : A \subseteq R^n$ definable, $n < \infty\}$ is countable*

Proof. WMA A is $|K|$ for some simplicial complex K. If $a_0, \dots, a_n$ are the 0-simplices in K then $[|K|]$ is determined by the "abstract simplicial complex"

$$(\{a_0, \dots, a_n\}, \{\{b_0, \dots, b_m\} \subseteq \{a_0, \dots, a_n\} : (b_0, \dots, b_m) \in K\})$$

□

Fact 6.4. *If $\{D_a\}_{a \in X}$ is a definable family, then $\{[D_a] : a \in X\}$ is finite.*

Theorem 6.5. *If D is definable, then D is locally definably connected, locally contractible*
If D is definably compact, then $D \cong |K|$, K is a closed simplicial complex
If $R = \mathbb{R}$ then D is a finite CW-complex

Theorem 6.6 (Extension)**.** *If $D \subseteq R^n$ definable and $A \subseteq D$ definable and closed in D and $f : A \to R$ is definable and continuous, then $\exists g : D \to R$, $g \supseteq f$, g is definable and continuous*

Proof. Trivial. □

7 Problems

3.2 3.2