Homework10

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Exercise 1. *Work in a stable theory. Suppose the sequence $(a_1, \dots, a_n)$ is independent over $\emptyset$. Suppose the sequence $(b_1, \dots, b_m)$ is independent over $\emptyset$. Suppose*

$$\{a_1, \dots, a_n\} \mathop{\downarrow}_{\emptyset} \{b_1, \dots, b_m\}$$

Show that $(a_1, \dots, a_n, b_1, \dots, b_m)$ is independent over $\emptyset$

Proof. Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_m)$, for any $1 \leq i \leq n$, since $\bar{a}$ is independent over $\emptyset$, $a_i \downarrow_{\emptyset} a_{\neq i}$. Also as $\bar{a} \downarrow_{\emptyset} \bar{b}$, we have $a_i a_{\neq i} \downarrow_{\emptyset} \bar{b}$, by base monotonicity, $a_i \downarrow_{a_{\neq i}} \bar{b}$. Then with $a_i \downarrow_{\emptyset} a_{\neq i}$, monotonicity gives $a_i \downarrow_{\emptyset} a_{\neq i}\bar{b}$. Similarly we can prove $b_i \downarrow_{\emptyset} b_{\neq i}\bar{a}$. Therefore $\bar{a}\bar{b}$ is independent over $\emptyset$ $\square$

Exercise 2. *If $T = \mathrm{Th}(\mathbb{R}, \leq)$ and $A = \mathbb{R}$ and $n = 1$, show that $R(\mathbb{R}) \geq 3$, i.e., $R(S_1(\mathbb{R})) \geq 3$*

Proof. First we prove that $R(U) \geq \alpha$ for any open interval $U = (a, b) \subseteq \mathbb{R}$ and any ordinal α by induction.

1. Apparently for any open interval U, $U \neq \emptyset$ and therefore $R(U) \geq 0$.
2. For any open interval $U = (a, b) \subseteq \mathbb{R}$, we can take $U_i = (a + \frac{b-a}{i+2}, a + \frac{b-a}{i+1})$ for all $i \in \omega$, then $U_0, U_1, \dots$ are pairwise disjoint $\mathbb{R}$-definable subsets of U and $R(U_i) \geq \alpha$ for each α, therefore $R(U) \geq \alpha + 1$
3. Limit ordinal case is obvious.

$R(\mathbb{R}) \geq 3$ if and only if there are pairwise disjoint $\mathbb{R}$-definable subseteq $D_1, D_2, \dots \subseteq \mathbb{R}$ such that $R(D_i) \geq 2$, and we can take $D_i = (i, i+1)$ for $i = 1, 2, 3, \dots$. For each D_i, $R(D_i) \geq 2$, therefore $R(\mathbb{R}) \geq 3$ $\square$

Exercise 3. *If $T = \mathrm{Th}(\mathbb{Z}, +)$ and $A = \emptyset$ and $n = 1$, show that the definable set $\mathbb{Z}$ has Cantor-Bendixon rank ∞.*

Proof. First let $\{0\}$ is defined by $\varphi(y) := \forall x(x + y = x)$. Now we prove that $R(n\mathbb{Z} \setminus \{0\}) \geq \alpha$ for any ordinal α and any $n \in \mathbb{N} \setminus \{0\}$

1. As they are all nonempty, $R(n\mathbb{Z} \setminus \{0\}) \geq 0$ for any positive integer n
2. For any $n \in \mathbb{N} \setminus \{0\}$, then $n\mathbb{Z} \setminus \{0\}$ has disjoint definable subsets $(n \cdot p_1)\mathbb{Z} \setminus \{0\}, (n \cdot p_2)\mathbb{Z} \setminus \{0\}, \ldots$ where $p_1, p_2, \ldots$ are strictly increasing primes. As for each i, $R((n \cdot p_1)\mathbb{Z} \setminus \{0\}) \geq \alpha$, then $R(n\mathbb{Z} \setminus \{0\}) \geq \alpha + 1$
3. Limit ordinal case is immediate

Therefore $R(\mathbb{Z} \setminus \{0\}) = \infty$ and since $R(\mathbb{Z}) \geq R(\mathbb{Z} \setminus \{0\})$, $R(\mathbb{Z}) = \infty$ □

Exercise 4. *If $T = ACF_0 = \mathrm{Th}(\mathbb{C}, +, \cdot)$ and $A = \mathbb{C}$ and $n = 3$. Show that the definable set*

$$D = \{(x, y, z) \in \mathbb{C}^3 : x + y + z = 0\}$$

has Cantor-Bendixon rank at least 2

Proof. For any $i \in \omega$, let $D_i = \{(x, y, z) \in \mathbb{C}^3 : x + y = i\}$, then $D_0, D_1, D_2, \ldots$ are disjoint $\mathbb{C}$-definable subsets of D. Now for any $i \in \omega$, for any $j \in \omega$, let $D_{ij} = \{(x, y, z) \in \mathbb{C}^3 : x + y = i \wedge y = j\}$. When fixing i, each D_{ij} is disjoint nonempty set, therefore $R(D_{ij}) \geq 0$ and $R(D_i) \geq 1$. Thus $R(D) \geq 2$. □

Exercise 5. *Let $(M, \approx)$ be a set M with an equivalence relation $\approx$, s.t. each equivalence class is infinite and there are infinite many equivalence classes. Show that $S_1(M)$ has Cantor-Bendixson rank at least 2*

Proof. As there are infinitely many equivalence class, we can take $a_1, a_2, \ldots$ such that their equivalence class is different. Then $[x \approx a_1], [x \approx a_2], \ldots$ are disjoint clopen subsets of $S_1(M)$. Since each equivalence class is infinite, in $[a_i]$,we can take infinitely many different $a_{i1}, a_{i2}, \cdots \in [a_i]$. Thus for each $[x \approx a_i]$, there are pairwise disjoint M-definable subsets $[x = a_{i1}], [x = a_{i2}], \ldots$ of it and therefore $R([x \approx a_i]) \geq 1$ and so $R(S_1(M)) \geq 2$ □