Model Theory for Dummies: An Introduction

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1 Structures and Theories

1.1 Languages and Structures

Definition 1.1. A language $\mathcal{L}$ is given by specifying the following data

1. A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$
3. a set of constant symbols $\mathcal{C}$

Definition 1.2. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data

1. a nonempty set M called the **universe**, **domain** or **underlying set** of $\mathcal{M}$
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols f, R and c. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

Definition 1.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An **$\mathcal{L}$-embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \dots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \dots, a_{n_f} \in M$
2. $(a_1, \dots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \dots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \dots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective $\mathcal{L}$-embedding is called an **$\mathcal{L}$-isomorphism**. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either $\mathcal{M}$ is a **substrcture** of $\mathcal{N}$ or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$

The **cardinality** of $\mathcal{M}$ is $|M|$, the cardinality of the universe of $\mathcal{M}$

Definition 1.4. The set of **$\mathcal{L}$-terms** is the smallest set $\mathcal{T}$ s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$

2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$

3. if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \ldots, n_{n_f}) \in \mathcal{T}$

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \to M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If s is a constant symbol c, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$

2. If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$

3. If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

Definition 1.5. ϕ is an **atomic $\mathcal{L}$-formula** if ϕ is either

1. $t_1 = t_2$ where t_1 and t_2 are terms

2. $R(t_1, \ldots, t_{n_R})$

The set of **$\mathcal{L}$-formulas** is the smallest set $\mathcal{W}$ containing the atomic formulas s.t.

1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$

2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$

3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable v **occurs freely** in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \ldots, v_n)$ to make explicit the free variables in ϕ

Definition 1.6. Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$ and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \vDash \phi\bar{a}$ as follows

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$

2. If ϕ is $R(t_1, \ldots, t_{m_R})$ then $\mathcal{M} \vDash \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$

3. If ϕ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If ϕ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\mathcal{M}$ **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in $\mathcal{M}$

Proposition 1.7. *Suppose that $\mathcal{M}$ is a substrcture of $\mathcal{N}$, $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$*

Proof. **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. ☐

Definition 1.8. We say that two $\mathcal{L}$-strctures $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences ϕ

We let $\mathrm{Th}(\mathcal{M})$, the **full theory** of $\mathcal{M}$ be the set of $\mathcal{L}$-sentences ϕ s.t. $\mathcal{M} \models \phi$

Theorem 1.9. *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$*

Proof. Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \dots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$ for all formulas ϕ ☐

1.2 Theories

Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-theory** T is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a **model** of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory T s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

Example 1.1 (Linear Orders)**.** Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. The class of linear order is axiomatized by the $\mathcal{L}$-sentences

$$\begin{aligned}
&\forall x\, \neg(x < x) \\
&\forall x \forall y \forall z\, ((x < y \wedge y < z) \to x < z) \\
&\forall x \forall y\, (x < y \vee x = y \vee y < x)
\end{aligned}$$

Example 1.2 (Groups). Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary function symbol and e is a constant symbol. The class of groups is axiomatized by

$$\forall x\ e \cdot x = x \cdot e = x$$
$$\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x \exists y\ x \cdot y = y \cdot x = e$$

Example 1.3 (Ordered Abelian Groups). Let $\mathcal{L} = \{+, <, 0\}$, where + is a binary function, < is a binary relation symbol, and 0 is a constant symbol. The axioms for order groups are

1. the axioms for additive groups
2. the axioms for linear orders
3. $\forall x \forall y \forall z (x < y \to x + z < y + z)$

Example 1.4 (Left R-modules). Let R be a ring with multiplicative identity 1. Let $\mathcal{L} = \{+, 0\} \cup \{r : r \in R\}$ where $+$ is a binary function symbol, 0 is a constant, and r is a unary function symbol for $r \in R$. In an R-module, we will interpret r as scalar multiplication by R. The axioms for R-modules are

$$\forall x\ \ r(x + y) = r(x) + r(y) \text{ for each } r \in R$$
$$\forall x\ \ (r + s)(x) = r(x) + s(x) \text{ for each } r, s \in R$$
$$\forall x\ \ r(s(x)) = rs(x) \text{ for } r, s \in R$$
$$\forall x\ \ 1(x) = x$$

Example 1.5 (Rings and Fields). Let $\mathcal{L}_r$ be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants. The axioms for rings are given by

$$\forall x \forall y \forall z\ (x - y = z \leftrightarrow x = y + z)$$
$$\forall x\ x \cdot 0 = 0$$
$$\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x\ x \cdot 1 = 1 \cdot x = x$$
$$\forall x \forall y \forall z\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
$$\forall x \forall y \forall z\ (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

We axiomatize the class of fields by adding

$$\forall x \forall y\ x \cdot y = y \cdot x$$
$$\forall x\ (x \neq 0 \to \exists y\ x \cdot y = 1)$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \dots \forall a_{n-1} \exists x \; x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \dots$. Let ACF be the axioms for algebraically closed fields.

Let ψ_p be the $\mathcal{L}_r$-sentence $\forall x \; \underbrace{x + \dots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic p. For $p > 0$ a prime, let $\mathsf{ACF}_p = \mathsf{ACF} \cup \{\psi_p\}$ and $\mathsf{ACF}_0 = \mathsf{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic p and zero respectively

Definition 1.10. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a **logical consequence** of T and write $T \vDash \phi$ if $\mathcal{M} \vDash \phi$ whenever $\mathcal{M} \vDash T$

Proposition 1.11. *1. Let $\mathcal{L} = \{+, <, 0\}$ and let T be the theory of ordered abelian groups. Then $\forall x (x \neq 0 \to x + x \neq 0)$ is a logical consequence of T*

2. Let T be the theory of groups where every element has order 2. Then $T \nvDash \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \land x_2 \neq x_3 \land x_1 \neq x_3)$

Proof. 1. $\mathbb{Z}/2\mathbb{Z} \vDash T \land \neg\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \land x_2 \neq x_3 \land x_1 \neq x_3)$ ☐

1.3 Definable Sets and Interpretability

Definition 1.12. Let $\mathcal{M} = (M, \dots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^b$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \vDash \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** X. We say that X is A**-definable** or **definable over** A if there is a formula $\psi(\bar{v}, w_1, \dots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines X

A number of examples using $\mathcal{L}_r$, the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^{m} a_i X^i$. Let $\phi(v, w_0, \dots, w_n)$ be the formula

$$w_n \cdot \underbrace{v \cdots v}_{n\text{-times}} + \dots + w_1 \cdot v + w_0 = 0$$

Then $\phi(v, a_0, \dots, a_n)$ defines Y. Indeed, Y is A-definable for any $A \supseteq \{a_0, \dots, a_n\}$

- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula
$$\exists z(z \neq 0 \wedge y = x + z^2)$$
Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is $\emptyset$-definable

- Consider the natural numbers $\mathbb{N}$ as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an $\mathcal{L}$-formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by e halts on input x in at most s steops. Thus the Turing machine with program e halts on input x if and only if

$\mathbb{N} \models \exists s\, T(e, x, s)$. So the halting computations is definable

Proposition 1.13. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.*

1. *$M^n \in D_n$*
2. *for all n-ary function symbols f of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in D_{n+1}*
3. *for all n-ary relation symbols R of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$*
4. *for all $i, j \leq n$, $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\} \in D_n$*
5. *if $X \in D_n$, then $M \times X \in D_{n+1}$*
6. *each D_n is cloed under complement, union and intersection*
7. *if $X \in D_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$, then $\pi(X) \in D_n$*
8. *if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$*

Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$

Proposition 1.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subset M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise(that is, if σ is an automorphism of M and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

Proof.
$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$
In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ $\square$

Definition 1.15. A subset S of a field L is **algebraically independent** over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K

Corollary 1.16. *The set of real numbers is not definable in the field of complex numbers*

Proof. If $\mathbb{R}$ where definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of $\mathbb{C}$ s.t. $\sigma|A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and $\mathbb{R}$ is not definable over A $\square$

We say that an $\mathcal{L}_0$-structure $\mathcal{N}$ is **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on X so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$

For example, let K be a field and G be $\mathrm{GL}_2(K)$, the group of invertible 2×2 matrices over K. Let $X = \{(a, b, c, d) \in K^4 : ad - bc \neq 0\}$. Let $f : X^2 \to X$ by

$$f((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) = \\ (a_1a_2 + b_1c_2, a_1b_2 + b_1d_2, c_1a_2 + d_1c_2, c_1b_2 + d_1d_2)$$

X and f are definable in $(K, +, \cdot)$, and the set X with operation f is isomorphic to $\mathrm{GL}_2(K)$, where the identity element of X is $(1, 0, 0, 1)$

Clearly, $(\mathrm{GL}_n(K), \cdot, e)$ is definably interpreted in $(K, +, \cdot, 0, 1)$. A **linear algebraic group** over K is a subgroup of $\mathrm{GL}_n(K)$ defined by polynomial equations over K. Any linear algebraic group over K is definably interpreted in K

Let F be an infinite field and let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in F, a \neq 0$. This group is isomorphic to the group of affine transformations $x \mapsto ax + b$, where $a, b \in F$ and $a \neq 0$

We will show that F is definably interpreted in the group G. Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tau \neq 0$. Let

$$A = \{g \in G : g\alpha = \alpha g\} = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\}$$

$$B = \{g \in G : g\beta = \beta g\} = \{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0\}$$

Clearly A, B are definable using parameters α and β

B acts on A by conjugation

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}$$

We can define the map $i : A \backslash \{1\} \to B$ by $i(a) = b$ if and only if $b^{-1}ab = \alpha$, that is

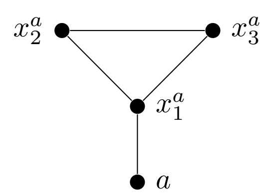
$$i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define an operation $*$ on A by

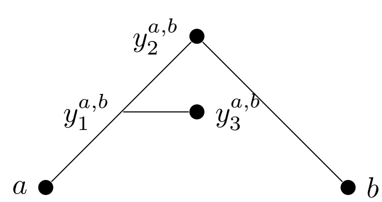
$$a * b = \begin{cases} i(b)a(i(b))^{-1} & \text{if } b \neq I \\ 1 & \text{if } b = I \end{cases}$$

where I is the identity matrix. Now $(F, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$

Very complicated structures can often be interpreted in seemingly simpler ones. For example, any structure in a countable language can be interpreted in a graph. Let $(A, <)$ be a linear order. For each $a \in A$, G_A will have vertices a, x_1^a, x_2^a, x_3^a and contain the subgraph

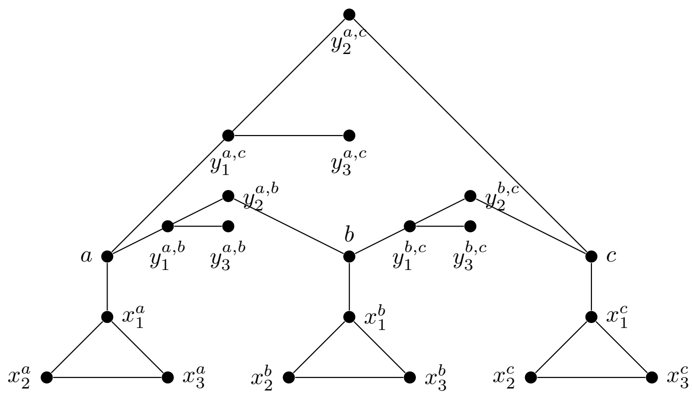


If $a < b$, then G_A will have vertices $y_1^{a,b}, y_2^{a,b}, y_3^{a,b}$ and contain the subgraph



Let $V_A = A \cup \{x_1^a, x_2^a, x_3^a : a \in A\} \cup \{y_1^{a,b}, y_2^{a,b}, y_3^{a,b} : a, b \in A$ and $a < b\}$, and let R_A be the smallest symmetric relation containing all edges drawn above.

For example, if A is the three-element linear order $a < b < c$, then G_A is the graph



Let $\mathcal{L} = \{R\}$ where R is a binary relation. Let $\phi(x, u, v, w)$ be the formula asserting that x, u, v, w are distinct, there are edges $(x, u), (u, v), (v, w), (u, w)$ and these are the only edges involving u, v, w. $G_A \vDash \phi(a, x_1^a, x_2^a, x_3^a)$ for all $a \in A$.

$\psi(x, y, u, v, w)$ asserts that x, y, u, v, w are distinct. $(x, u), (u, v), (u, w), (v, y)$

Define $\theta_i(z)$ as follows:

$$\begin{aligned}
\theta_0(z) &:= \exists u \exists v \exists w \ \phi(z, u, v, w) \\
\theta_1(z) &:= \exists x \exists v \exists w \ \phi(x, z, v, w) \\
\theta_2(z) &:= \exists u \exists u \exists w \ \phi(x, u, z, w) \\
\theta_3(z) &:= \exists x \exists y \exists v \exists w \ \psi(x, y, z, v, w) \\
\theta_4(z) &:= \exists x \exists y \exists u \exists w \ \psi(x, y, u, z, w) \\
\theta_5(z) &:= \exists x \exists y \exists u \exists v \ \psi(x, y, u, v, z)
\end{aligned}$$

If $a, b \in A$ and $a < b$, then

$$G_A \vDash \theta_0(a) \wedge \theta_1(x_1^a) \wedge \theta_2(x_2^a) \wedge \theta_2(x_3^a)$$

and

$$G_A \vDash \theta_3(y_1^{a,b}) \wedge \theta_4(y_2^{a,b}) \wedge \theta_5(y_3^{a,b})$$

Lemma 1.17. *If $(A, <)$ is a linear order, then for all vertices x in G, there is a unique $i \leq 5$ s.t. $G_A \vDash \theta_i(x)$*

Let T be the $\mathcal{L}$-theory with the following axioms

1. R is symmetric and irreflexive
2. for all x, exactly one θ_i holds
3. if $\theta_0(x)$ and $\theta_0(y)$ then $\neg R(x, y)$
4. if $\exists u \exists v \exists w \ \psi(x, y, u, v, w)$
   then $\forall u_1 \forall v_1 \forall w_1 \neg \psi(y, x, u_1, v_1, w_1)$
5. if $\exists u \exists v \exists w \ \psi(x, y, u, v, w)$ and $\exists u \exists v \exists w \ \psi(y, z, u, v, w)$ then
   $\exists u \exists v \exists w \ \psi(x, z, u, v, w)$
6. if $\theta_0(x)$ and $\theta_0(y)$, then either $x = y$ or $\exists u \exists v \exists w \ \psi(x, y, u, v, w)$ or $\exists u \exists v \exists w \ \psi(y, x, u, v, w)$
7. if $\phi(x, u, v, w) \wedge \phi(x, u', v', w')$, then $u = u', v = v', w = w'$
8. if $\psi(x, y, u, v, w) \wedge \psi(x, y, u', v', w')$, then $u' = u, v = v', w = w'$

If $(A, <)$ is a linear order, then $G_A \vDash T$
Suppose $G \vDash T$. Let $X_G = \{x \in G : G \vDash \theta_0(x)\}$

Lemma 1.18. *If $(A, <)$ is a linear order, then $(X_{G_A}, <_{G_A}) \cong (A, <)$. Moreover, $G_{X_G} \cong G$ for all $G \vDash T$*

Definition 1.19. An $\mathcal{L}_0$-structure $\mathcal{N}$ is **interpretable** in an $\mathcal{L}$-structure M if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X, and for each symbol of $\mathcal{L}_0$ we can find definable E-invariant sets on X s.t. X/E with the induced structure is isomorphic to $\mathcal{N}$

1.4 Answers to Exercises

Exercise 1.4.1. 1. transform ψ to CNF

2. prenex normal form

Exercise 1.4.2. 1.
$$\begin{array}{cc} s & rs \\ \bullet & \bullet \\ e & r \\ \bullet & \bullet \end{array}$$

2. enumerate $\mathcal{M}$'s functions, relations and constants

Exercise 1.4.3. [1] Note that every $\mathcal{L}$-structure $\mathcal{M}$ of size κ is isomorphic to an $\mathcal{L}$-structure with domain κ. For each relation symbols, we have 2^κ options. If the language has size λ, this is at most $(2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\lambda,\kappa)}$

Exercise 1.4.4.

$$\begin{aligned} T \vDash \phi &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \vDash T \to \mathcal{M} \vDash \phi \\ &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \vDash T' \to \mathcal{M} \vDash \phi \\ &\Leftrightarrow T' \vDash \phi \end{aligned}$$

Exercise 1.4.5. Follow the definition

Exercise 1.4.6. Since there is no model $\mathcal{M}$ s.t. $\mathcal{M} \vDash T$. It's true that $T \vDash \phi$

Exercise 1.4.7.

1. Suppose $\mathcal{M} \vDash \phi$, then $E^{\mathcal{M}}$ is an equivalent relation and each equivalence class's cardinality is 2
2. follows from number theory
3. [DJMM12]

Exercise 1.4.8. TBD

Exercise 1.4.9. $G(f) = \{(\bar{x},\bar{y}) \in M^{n+m} \mid \phi(\bar{x},\bar{y})\}$ and $G(g) = \{(\bar{y},\bar{z}) \in M^{m+l} \mid \psi(\bar{y},\bar{z})\}$. Hence $G(g\circ f) = \{(\bar{x},\bar{z}) \in M^{n+l} \mid \phi(\bar{x},\bar{y}) \wedge \psi(\bar{y},\bar{z})\}$

Exercise 1.4.10. $\phi(\bar{a}, b)$ really defines a function and since $\phi(\bar{a}, y) \to y = b$

2 Basic Techniques

2.1 The Compactness Theorem

Some points of proofs

- Proofs are finite
- (Soundness) If $T \vdash \phi$, then $T \vDash \phi$
- If T is a finite set of sentences, then there is an algorithm that, when given a sequence of $\mathcal{L}$-formulas σ and an $\mathcal{L}$-sentence ϕ, will decide whether σ is a proof of ϕ from T

A language $\mathcal{L}$ is **recursive** if there is an algorithm that decides whether a sequence of symbols is an $\mathcal{L}$-formula. An $\mathcal{L}$-theory T is **recursive** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input, decides whether $\phi \in T$

[1] stackexchange

Proposition 2.1. *If $\mathcal{L}$ is a recursive language and T is a recursive $\mathcal{L}$-theory, then $\{\phi : T \vdash \phi\}$ is recursively enumerable; that is, there is an algorithm that when given ϕ as input will halt accepting if $T \vdash \phi$ and not halt if $T \nvdash \phi$*

Proof. There is $\sigma_0, \sigma_1, \ldots$ a computable listing of all finite sequence of $\mathcal{L}$-formulas. At stage i, we check to see whether σ_i is a proof of ψ from T. If it is, then halt. □

Theorem 2.2 (Gödel's Completeness Theorem)**.** *Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then $T \vDash \phi$ if and only if $T \vdash \phi$*

We say that an $\mathcal{L}$-theory T is **inconsistent** if $T \vdash (\phi \wedge \neg\phi)$ for some sentence ϕ.

Corollary 2.3. *T is consistent if and only if T is satisfiable*

Proof. Supose that T is not satisfiable, then every model of T is a model of $\phi \wedge \neg\phi$. Thus by the Completeness theorem $T \vdash (\phi \wedge \neg\phi)$ □

Theorem 2.4 (Compactness Theorem)**.** *T is satisfiable if and only if every finite subset of T is satisfiable*

Proof. If T is not satisfiable, then T is inconsistent. Let σ be a proof of a contradiction from T. Because σ is finite, only finitely many assumptions from T are used in the proof. Thus there is a finite $T_0 \subseteq T$ s.t. σ is a proof of a contradiction from T_0 □

2.1.1 Henkin Constructions

A theory T is **finitely satisfiable** if every finite subset of T is satisfiable. We will show that every finitely satisfiable theory T is satisfiable.

Definition 2.5. We say that an $\mathcal{L}$-theory T has the **witness property** if whenever $\phi(v)$ is an $\mathcal{L}$-formula with one free variable v, then there is a constant symbol $c \in \mathcal{L}$ s.t. $T \vdash (\exists v \phi(v)) \to \phi(c) \in T$

An $\mathcal{L}$-theory T is **maximal** if for all ϕ either $\phi \in T$ or $\neg\phi \in T$

Lemma 2.6. *Suppose T is a maximal and finitely satisfiable $\mathcal{L}$-theory. If $\Delta \subseteq T$ is finite and $\Delta \vDash \psi$, then $\psi \in T$*

Proof. If $\psi \notin T$, then $\neg\psi \in T$ but $\Delta \cup \{\psi\}$ is unsatisfiable □

Lemma 2.7. *Suppose that T is a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property. Then T has a model. In fact, if κ is a cardinal and $\mathcal{L}$ has at most κ constant symbols, then there is $\mathcal{M} \vDash T$ with $|\mathcal{M}| \leq \kappa$*

Proof. Let $\mathcal{C}$ be the set of constant symbols of $\mathcal{L}$. For $c, d \in \mathcal{C}$, we say $c \sim d$ if $c = d \in T$

Claim 1 $\sim$ is an equivalence relation.

The universe of our model will be $M = \mathcal{C}/\sim$. Clearly $|M| \leq \kappa$. We let c^* denote the equivalence class of c and interprete c as its equivalence class, that is, $c^{\mathcal{M}} = c^*$

Suppose that R is an n-ary relation symbol of $\mathcal{L}$

Claim 2 Suppose that $c_1, \dots, c_n, d_1, \dots, d_n \in \mathcal{C}$ and $c_i \sim d_i$ for $i = 1, \dots, n$, then $R(\bar{c})$ if and only if $R(\bar{d})$

By Lemma 2.6, if one of $R(\bar{c})$ and $R(\bar{d})$ is in T, then both are in T

$$R^{\mathcal{M}} = \{(c_1^*, \dots, c_n^*) : R(c_1, \dots, c_n) \in T\}$$

Suppose that f is an n-ary function symbol of $\mathcal{L}$ and $c_1, \dots, c_n \in \mathcal{C}$. Because $\underline{\emptyset \models \exists v f(c_1, \dots, c_n) = v}$, and T has the witness property, then there is $c_{n+1} \in \mathcal{C}$ s.t. $f(c_1, \dots, c_n) = c_{n+1} \in T$. As above, if $d_i \sim c_i$ for $i = 1, \dots, n+1$, then $f(d_1, \dots, d_n) = d_{n+1} \in T$. Thus we get a well-defined function $f^{\mathcal{M}} : M^n \to M$ by

$$f^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^* \text{ if and only if } f(c_1, \dots, c_n) = d \in T$$

Claim 3 Suppose that t is a term using free variables from $v_1, \dots, v_n$. If $c_1, \dots, c_n, d \in \mathcal{C}$, then $t(c_1, \dots, c_n) = d \in T$ if and only if $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$

($\Rightarrow$) If t is a constant symbol, then $c = d \in T$ and $c^{\mathcal{M}} = c^* = d^*$

If t is the variable v_i, then $c_i = d \in T$ and $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = c_i^* = d^*$

Suppose that the claim is true for $t_1, \dots, t_m$ and t is $f(t_1, \dots, t_m)$. Using the witness property and Lemma 2.6, we can find $d, d_1, \dots, d_n \in \mathcal{C}$ s.t. $t_i(c_1, \dots, c_n) = d_i \in T$ for $i \leq m$ and $f(d_1, \dots, d_m) = d \in T$. By our induction hypothesis, $t_i^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d_i^*$ and $f^{\mathcal{M}}(d_1^*, \dots, d_m^*) = d^*$. Thus $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$

($\Leftarrow$) Suppose $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. By the witness property, there is a $e \in \mathcal{C}$ s.t. $t(c_1, \dots, c_n) = e \in T$. Using the ($\Rightarrow$) direction of the proof, $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$. Thus $e^* = d^*$ and $e = d \in T$

Claim 4 For all $\mathcal{L}$-formulas $\phi(v_1, \dots, v_n)$ and $c_1, \dots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ if and only if $\phi(\bar{c}) \in T$

Suppose that ϕ is $t_1 = t_2$. By Lemma 2.6 and the witness property, we can find d_1 and d_2 s.t. $t_1(\bar{c}) = d_1, t_2(\bar{c}) = d_2 \in T$. By Claim 3, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$. Then

$$\begin{aligned}
\mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow d_1^* = d_2^* \\
&\Leftrightarrow d_1 = d_2 \in T \\
&\Leftrightarrow t_1(\bar{c}) = t_2(\bar{c}) \in T
\end{aligned}$$

Suppose that ϕ is $R(t_1, \dots, t_m)$. There are $d_1, \dots, d_m \in \mathcal{C}$ s.t. $t_i(\bar{c}) = d_i \in T$. Thus

$$\begin{aligned} \mathcal{M} \vDash \phi(\bar{c}^*) &\Leftrightarrow \bar{d}^* \in R^{\mathcal{M}} \\ &\Leftrightarrow R(\bar{d}) \in T \\ &\Leftrightarrow \phi(\bar{c}) \in T \end{aligned}$$

Suppose that the claim is true for ϕ. If $\mathcal{M} \vDash \neg\phi(\bar{c}^*)$, then $\mathcal{M} \nvDash \phi(\bar{c}^*)$. By the inductive hypothesis, $\phi(\bar{c}) \notin T$. Thus by maximality, $\neg\phi(\bar{c}) \in T$. On the other hand, if $\neg\phi(\bar{c}) \in T$, then because T is finitely satisfiable, $\phi(\bar{c}) \notin T$. Thus, by induction, $\mathcal{M} \nvDash \phi(\bar{c}^*)$. □

Lemma 2.8. *Let T be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory s.t. any $\mathcal{L}^*$-theory extending T^* has the witness property. We can choose $\mathcal{L}^*$ s.t. $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$*

Proof. We first show that there is a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}_1$-theory $\mathcal{L}_1 \supseteq T$ s.t. for any $\mathcal{L}$-formula $\phi(v)$ there is an $\mathcal{L}_1$-constant symbol c s.t. $T_1 \vDash (\exists v \phi(v)) \to \phi(c)$. For each $\mathcal{L}$-formula $\phi(v)$, let c_ϕ be a new constant symbol and let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$. For each $\mathcal{L}$-formula $\phi(v)$, let Θ_ϕ be the $\mathcal{L}_1$-sentence $(\exists v \phi(v)) \to \phi(c_\phi)$. Let $T_1 = T \cup \{\Theta_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$

Claim T_1 is finitely satisfiable

Suppose that Δ is a finite subset of T_1. Then $\Delta = \Delta_0 \cup \{\Theta_{\phi_1}, \dots, \Theta_{\phi_n}\}$ where Δ_0 is a finite subset of T and there is $\mathcal{M} \vDash \Delta_0$. We will make $\mathcal{M}$ into an $\mathcal{L} \cup \{c_{\phi_1}, \dots, c_{\phi_n}\}$-structure $\mathcal{M}'$. If $\mathcal{M} \vDash \exists v \phi(v)$, choose a_i some element of M s.t. $\mathcal{M} \vDash \phi(a_i)$ and let $c_{\phi_i}^{\mathcal{M}'} = a_i$. Otherwise, let $c_{\phi_i}^{\mathcal{M}'}$ be any element of $\mathcal{M}$. Clearly $\mathcal{M}' \vDash \Theta_{\phi_i}$ for $i \leq n$. Thus T_1 is finitely satisfiable.

We now iterate the construction above to build a sequence of languages $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$ and a sequence of finitely satisfiable $\mathcal{L}_i$-theories $T \subseteq T_1 \subseteq T_2 \subseteq \dots$ s.t. if $\phi(v)$ is an $\mathcal{L}_i$-formula then there is a constant symbol $c \in \mathcal{L}_{i+1}$ s.t. $T_{i+1} \vDash (\exists v \phi(v)) \to \phi(c)$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $|\mathcal{L}_i|$ is the number of relation, function and constant symbols in $\mathcal{L}_i$, then there are at most $|\mathcal{L}_i| + \aleph_0$ formulas in $\mathcal{L}_i$. Thus by induction, $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$ □

Lemma 2.9. *Suppose that T is a finitely satisfiable $\mathcal{L}$-theory and ϕ is an $\mathcal{L}$-sentence, then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable*

Corollary 2.10. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal finitely satisfiable $\mathcal{L}$-theory $T' \supseteq T$*

Proof. Let I be the set of all finitely satisfiable $\mathcal{L}$-theory containing T. We partially order I by inclusion. If $C \subseteq I$ is a chain, let $T_C = \bigcup\{\Sigma : \Sigma \in C\}$. If Δ is a finite subset of T_C, then there is a $\Sigma \in C$ s.t. $\Delta \subseteq \Sigma$, so T_C is finitely satisfiable and $T_C \supseteq \Sigma$ for all $\Sigma \in C$. Thus every chain in I has an upper bound, and we can apply Zorn's lemma to find a $T' \in I$ maximal w.r.t. the partial order. $\square$

Theorem 2.11 (strengthening of Compactness Theorem)**.** *If T is a finitely satisfiable $\mathcal{L}$-theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ*

Proof. By Lemma 2.8, we can find $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory s.t. any $\mathcal{L}^*$-theory extending T^* has the witness property and the cardinality of $\mathcal{L}^*$ is at most κ. By Corollary 2.10, we can find a maximal finitely satisfiable $\mathcal{L}^*$-theory $T' \supseteq T^*$. Because T' has the witness property, Lemma 2.7 ensures that there is $\mathcal{M} \vDash T$ with $|M| \leq \kappa$ $\square$

Proposition 2.12. *Let $\mathcal{L} = \{\cdot, +, <, 0, 1\}$ and let* Th($\mathbb{N}$) *be the full $\mathcal{L}$-theory of the natural numbers. There is $\mathcal{M} \vDash$* Th($\mathbb{N}$) *and $a \in M$ s.t. a is larger than every natural number*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant symbol and let

$$T = \mathrm{Th}(\mathbb{N}) \cup \{\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} < c : \text{for } n = 1, 2, \dots\}$$

If Δ is a finite subset of T we can make $\mathbb{N}$ a model of Δ by interpreting c as a suitably large natural number. Thus T is finitely satisfiable and there is $\mathcal{M} \vDash T$. $\square$

Lemma 2.13. *If $T \vDash \phi$, then $\Delta \vDash T$ for some finite $\Delta \subseteq T$*

Proof. Suppose not. Let $\Delta \subseteq T$ be finite. Because $\Delta \nvDash \phi$, $\Delta \cup \{\neg\phi\}$ is satisfiable. Thus $T \cup \{\neg\phi\}$ is finitely satisfiable and by the compactness theorem, $T \nvDash \phi$ $\square$

2.2 Complete Theories

Definition 2.14. An $\mathcal{L}$-theory T is called **complete** if for any $\mathcal{L}$-sentence ϕ either $T \vDash \phi$ or $T \vDash \neg\phi$

For $\mathcal{M}$ an $\mathcal{L}$-structure, then the full theory

$$\mathrm{Th}(\mathcal{M}) = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \vDash \phi\}$$

is a complete theory.

Proposition 2.15. *Let T be an $\mathcal{L}$-theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where each c_α is new constant symbol, and let T^* be the $\mathcal{L}^*$-theory $T \cup \{c_\alpha \neq c_\beta : \alpha, \beta < \kappa, \alpha \neq \beta\}$. Clearly if $\mathcal{M} \vDash T^*$, then $\mathcal{M}$ is a model of T of cardinality at least κ. Thus by Theorem 2.11, it suffices to show that T^* is finitely satisfiable. But if $\Delta \subseteq T^*$ is finite, then $\Delta \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta, \alpha, \beta \in I\}$, where I is a finite subset of κ. Let $\mathcal{M}$ be an infinite model of T. We can interpret the symbols $\{c_\alpha : \alpha \in I\}$ as $|I|$ distinct elements of M. Because $\mathcal{M} \vDash \Delta$, T^* is finitely satisfiable. $\square$

Definition 2.16. Let κ be an infinite cardinal and let T be a theory with models of size κ. We say that T is **κ-categorical** if any two models of T of cardinality κ are isomorphic.

Let $\mathcal{L} = \{+, 0\}$ be the language of additive groups and let T be the $\mathcal{L}$-theory of torsion-free divisible Abelian groups. The axioms of T are the axioms for Abelian groups together with the axioms

$$\forall x(x \neq 0 \to \underbrace{x + \cdots + x}_{n\text{-times}} \neq 0)$$

$$\forall y \exists x \underbrace{x + \cdots + x}_{n\text{-times}} = y$$

for $n = 1, 2, \ldots$

Proposition 2.17. *The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$*

Proof. We first argue that models of T are essentially vector spaces over the field of rational numbers $\mathbb{Q}$. If V is any vector space over $\mathbb{Q}$, then the underlying additive group V is a model of T. Check StackExchange. On the other hand, if $G \vDash T$, $g \in G$ and $n \in \mathbb{N}$ with $g > 0$, we can find $h \in G$ s.t. $nh = g$. If $nk = g$, then $n(h - k) = 0$. Because G is torsion-free there is a unique $h \in G$ s.t. $nh = g$. We call this element g/n. We can view G as a $\mathbb{Q}$-vector space under the action $\frac{m}{n} g = m(g/n)$

Two $\mathbb{Q}$-vector spaces are isomorphic if and only if they have the same dimension. Thus the model of T are determined up to isomorphism by their dimension. If G has dimension λ, then $|G| = \lambda + \aleph_0$. If κ is uncountable and G has cardinality κ, then G has dimension κ. Thus for $\kappa > \aleph_0$ any two models of T of cardinality κ are isomorphic $\square$

Lemma 2.18. *Field of uncountable cardinality κ has transcendence degree κ* [2]

Proof. We prove the theorem for fields with characteristic $p = 0$.

Since each characteristic 0 field contains a copy of $\mathbb{Q}$ as its prime field, we can view F as a field extension over $\mathbb{Q}$. We will show that F has a subset of cardinality κ which is algebraically independent over $\mathbb{Q}$.

We build the claimed subset of F by transfinite induction and implicit use of the axiom of choice.

Let $S_0 = \emptyset$

Let S_1 be a singleton containing some element of F which is not algebraic over $\mathbb{Q}$. This is possible since algebraic numbers are countable

Define $S_{\alpha+1}$ to be S_α together with an element of F which is not a root of any non-trivial polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$ since there are only $|\mathbb{Q} \cup S_\alpha| = \aleph_0 + |\alpha| < \kappa$ polynomials

Define $S_\beta = \bigcup\limits_{\alpha<\beta} S_\alpha$

Let $P(x_1, \dots, x_n)$ be a non-trivial polynomial with coefficients in $\mathbb{Q}$ and elements $a_1, \dots, a_n$ in F. W.L.O.G., it is assumed that a_n was added at an ordinal $\alpha + 1$ later than the other elements. Then $P(a_1, \dots, a_{n-1}, x_n)$ is a polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$. Hence $P(a_1, \dots, a_n) \neq 0$. $\square$

Proposition 2.19. *ACF_p is κ-categorical for all uncountable cardinals κ*

Proof. Two algebraically closed fields are isomorphic if and only if they have the same characteristic and transcendence degree. See `AdvancedModernAlgebra.org`. By Lemma 2.18, an algebraically closed field of transcendence degree λ has cardinality $\lambda + \aleph_0$. $\square$

Theorem 2.20 (Vaught's Test)**.** *Let T be a satisfiable theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete*

Proof. Suppose T is not complete. Then there is a sentence ϕ s.t. $T \not\vDash \phi$ and $T \not\vDash \neg\phi$. Because $T \not\vDash \psi$ if and only if $T \cup \{\neg\psi\}$ is satisfiable, the theories $T_0 = T \cup \{\phi\}$ and $T_1 = T \cup \{\neg\phi\}$ are satisfiable. Because T has no finite models, both T_0 and T_1 have infinite models. By Proposition 2.15 we can find $\mathcal{M}_0$ and $\mathcal{M}_1$ of cardinality κ with $\mathcal{M}_i \vDash T_i$. Because $\mathcal{M}_0$ and $\mathcal{M}_1$ disagree about ϕ, they are not elementarily equivalent, and hence by Theorem 1.9, nonisomorphic. $\square$

Definition 2.21. We say that an $\mathcal{L}$-theory T is **decidable** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input decides whether $T \vDash \phi$

[2] proofwiki

Lemma 2.22. *Let T be a recursive complete satisfiable theory in a recursive language $\mathcal{L}$. Then T is decidable*

Proof. Because T is satisfiable $A = \{\phi : T \models \phi\}$ and $B = \{\phi : T \models \neg\phi\}$ are disjoint. Because T is consistent $A \cup B$ is the set of all $\mathcal{L}$-sentences. By the Completeness Theorem, $A = \{\phi : T \vdash \phi\}$ and $B = \{\phi : T \vdash \neg\phi\}$. By Proposition 2.1 A and B are recursively enumerable. But any recursively enumerable set with a recursively enumerable complement is recursive. $\square$

Corollary 2.23. *For $p = 0$ or p prime, ACF_p is decidable. In particular,* $\mathrm{Th}(\mathbb{C})$, *the first-order theory of the field of complex numbers, is decidable*

Corollary 2.24. *Let ϕ be a sentence in the language of rings. The following are equivalent*

1. *ϕ is true in the complex number*
2. *ϕ is true in every algebraically closed field of characteristic zero*
3. *ϕ is true in some algebraically closed field of characteristic zero*
4. *There are arbitrarily large primes p s.t. ϕ is true in some algebraically closed field of characteristic p*
5. *There is an m s.t. for all $p > m$, ϕ is true in all algebraically closed fields of characteristic p*

Proof. By Proposition 2.19 and Vaught's Test, ACF_p is complete.

$(2) \to (5)$. Suppose that $\mathsf{ACF}_0 \models \phi$. By Lemma 2.13, there is a finite $\Delta \subseteq \mathsf{ACF}_0$ s.t. $\Delta \models \phi$. Thus if we choose p large enough, then $\mathsf{ACF}_p \models \Delta$.

$(4) \to (2)$. Suppose $\mathsf{ACF}_0 \not\models \phi$. Because ACF_0 is complete, $\mathsf{ACF}_0 \models \neg\phi$. $\square$

2.3 Up and Down

Definition 2.25. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, then an $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called an **elementary embedding** if

$$\mathcal{M} \models \phi(a_1, \dots, a_n) \leftrightarrow \mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$$

for all $\mathcal{L}$-formulas $\phi(v_1, \dots, v_n)$ and all $a_1, \dots, a_n \in M$

If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that it is an **elementary substructure** and write $\mathcal{M} \prec \mathcal{N}$ if the inclusion map is elementary. $\mathcal{N}$ is an **elementary extension** of $\mathcal{M}$

Definition 2.26. $\mathcal{M}$ is an $\mathcal{L}$-structure. Let $\mathcal{L}_M$ be the language where we add to $\mathcal{L}$ constant symbols m for each element of M. The **atomic diagram** of $\mathcal{M}$ is $\{\phi(m_1, \dots, m_n) : \phi$ is either an atomic $\mathcal{L}$-formula or the negation of an atomic $\mathcal{L}$-formula and $\mathcal{M} \models \phi(m_1, \dots .m_n)\}$. The **elementary diagram** of $\mathcal{M}$ is

$$\{\phi(m_1, \dots, m_n) : \mathcal{M} \models \phi(m_1, \dots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\}$$

We let $\mathrm{Diag}(\mathcal{M})$ and $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ denote the atomic and elementary diagrams of $\mathcal{M}$

Lemma 2.27. *1. Suppose that $\mathcal{N}$ is an $\mathcal{L}_M$-structure and $\mathcal{N} \models \mathrm{Diag}(\mathcal{M})$, then viewing $\mathcal{N}$ as an $\mathcal{L}$-structure, there is an $\mathcal{L}$-embedding of $\mathcal{M}$ into $\mathcal{N}$*

2. If $\mathcal{N} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, then there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$

Proof. 1. Let $j : M \to N$ by $j(m) = m^{\mathcal{N}}$. If $m_1 \neq m_2 \in \mathrm{Diag}(\mathcal{M})$; thus $j(m_1) \neq j(m_2)$ so j is an embedding. If f is a function symbols of $\mathcal{L}$ and $f^{\mathcal{M}}(m_1, \dots, m_n) = m_{n+1}$, then $f(m_1, \dots, m_n) = m_{n+1}$ is a formula in $\mathrm{Diag}(\mathcal{M})$ and $f^{\mathcal{N}}(j(m_1), \dots, j(m_n)) = j(m_{n+1})$. If R is a relation symbol and $\bar{m} \in R^{\mathcal{M}}$, then $R(m_1, \dots, m_n) \in \mathrm{Diag}(\mathcal{M})$ and $(j(m_1), \dots, j(m_n)) \in R^{\mathcal{N}}$. Hence j is an $\mathcal{L}$-embedding

2. j is elementary.

□

Theorem 2.28 (Upward Löwenheim–Skolem Theorem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and κ be an infinite cardinal $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then, there is $\mathcal{N}$ an $\mathcal{L}$-structure of cardinality κ and $j : \mathcal{M} \to \mathcal{N}$ is elementary*

Proof. Because $\mathcal{M} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable. By Theorem 2.11, there is $\mathcal{N} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ of cardinality κ. By Lemma 2.27, there is an elementary $j : \mathcal{M} \to \mathcal{N}$ □

Proposition 2.29 (Tarski-Vaught Test)**.** *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \bar{w})$ and $\bar{a} \in M$, if there is $b \in N$ s.t. $\mathcal{N} \models \phi(b, \bar{a})$, then there is $c \in M$ s.t. $\mathcal{N} \models \phi(c, \bar{a})$*

Proof. We need to show that for all $\bar{a} \in M$ and all $\mathcal{L}$-formulas $\psi(\bar{v})$

$$\mathcal{M} \models \psi(\bar{a}) \Leftrightarrow \mathcal{N} \models \psi(\bar{a})$$

In Proposition 1.7, we showed that if $\phi(\bar{v})$ is quantifier free then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\phi(\bar{a})$ □

We say that an $\mathcal{L}$-theory T has **built-in Skolem functions** if for all $\mathcal{L}$-formulas $\phi(v, w_1, \dots, w_n)$ there is a function symbol f s.t. $T \models \forall \bar{w}((\exists v \phi(v, \bar{w})) \to \phi(f(\bar{w}), \bar{w}))$. In other words, there are enough function symbols in the language to witness all existential statements.

Lemma 2.30. *Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-theory s.t. T^* has built-in Skolem functions, and if $\mathcal{M} \models T$, then we can expand $\mathcal{M}$ to $\mathcal{M}^* \models T^*$. We can choose $\mathcal{L}^*$ s.t. $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

*We call T^* a **skolemization** of T*

Proof. We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$ and $\mathcal{L}_i$-theories T_i s.t. $T = T_0 \subseteq T_1 \subseteq \dots$

Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L} \cup \{f_\phi : \phi(v, w_1, \dots, w_n) \text{ an } \mathcal{L}_i\text{-formula}, n = 1, 2, \dots\}$, where f_ϕ is an n-ary function symbol. For $\phi(v, \bar{w})$ an $\mathcal{L}_i$-formula, let Ψ_ϕ be the sentence

$$\forall \bar{w}((\exists v \phi(v, \bar{w})) \to \phi(f_\phi(\bar{w}), \bar{w}))$$

and let $T_{i+1} = T_i \cup \{\Psi_\phi : \phi \text{ an } \mathcal{L}_i\text{-formula}\}$

Claim If $\mathcal{M} \models T_i$, then we can interpret the function symbols of $\mathcal{L}_{i+1} \backslash \mathcal{L}_i$ so that $\mathcal{M} \models T_{i+1}$

Let c be some fixed element of M. If $\phi(v, w_1, \dots, w_n)$ is an $\mathcal{L}_i$-formula, we find a function $g : M^n \to M$ s.t. $\bar{a} \in M^n$ and $X_{\bar{a}} = \{b \in M : \mathcal{M} \models \phi(b, \bar{a})\}$ is nonempty, then $g(\bar{a}) \in X_{\bar{a}}$, and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$. Thus if $\mathcal{M} \models \exists v \phi(v, \bar{a})$, then $\mathcal{M} \models \phi(g(\bar{a}), \bar{a})$. If we interpret f_ϕ as g, then $\mathcal{M} \models \Psi_\phi$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $\phi(v, \bar{w})$ is an $\mathcal{L}^*$-formula, then $\phi \in \mathcal{L}_i$ for some i and $\Psi_\phi \in T_{i+1} \subseteq T^*$, so T^* has built in Skolem functions. By iterating the claim, we see that for any $\mathcal{M} \models T$ we can interpret the symbols of $\mathcal{L}^* \backslash \mathcal{L}$ to make $\mathcal{M} \models T^*$

$|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ ◻

Theorem 2.31 (Löwenheim–Skolem Theorem)**.** *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq M$, there is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ s.t. $X \subseteq N$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$*

Proof. By Lemma 2.30, we may assume that $\mathrm{Th}(\mathcal{M})$ has built in Skolem functions (otherwise we may extend $\mathcal{L}$ to some $\mathcal{L}^*$). Let $X_0 = X$. Given X_i, let $X_{i+1} = X_i \cup \{f^{\mathcal{M}}(\bar{a}) : f \text{ an } n\text{-ary function symbol}, \bar{a} \in X_i^n, n = 1, 2, \dots\}$. Let $N = \bigcup X_i$, then $|N| \leq |X| + |\mathcal{L}| + \aleph_0$ If f is an n-ary function symbol of $\mathcal{L}$ and $\bar{a} \in N^n$, then $\bar{a} \in X_i^n$ for some i and $f^{\mathcal{M}}(\bar{a}) \in X_{i+1} \subseteq N$. Thus $f^{\mathcal{M}} | N : N^n \to N$. Thus we can interpret f as $f^{\mathcal{N}} = f^{\mathcal{M}} | N^n$. If R is an n-ary relation symbol, let $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$. If c is a constant symbol of $\mathcal{L}$, there is a Skolem function $f \in \mathcal{L}$ s.t. $f(x) = c^{\mathcal{M}}$ for all $x \in M$ (for example, f is the Skolem function for the formula $v = c$) . Thus $c^{\mathcal{N}} \in N$

If $\phi(v, \bar{w})$ is any $\mathcal{L}$-formula, $\bar{a}, b \in M$ and $\mathcal{M} \models \phi(b, \bar{a})$, then $\mathcal{M} \models \phi(f(\bar{a}), \bar{a})$ for some function symbol f of $\mathcal{L}$. By construction, $f^{\mathcal{M}}(\bar{a}) \in N$. Thus by Proposition 2.29 $\mathcal{N} \prec \mathcal{M}$ □

Definition 2.32. A **universal sentence** is one of the form $\forall \bar{v} \phi(\bar{v})$, where ϕ is quantifier-free. We say that an $\mathcal{L}$-theory T has a **universal axiomatization** if there is a set of universal $\mathcal{L}$-sentences Γ s.t. $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$ for all $\mathcal{L}$-structures $\mathcal{M}$

Theorem 2.33. *An $\mathcal{L}$-theory T has a universal axiomatization if and only if whenever $\mathcal{M} \models T$ and $\mathcal{N}$ is a substructure of $\mathcal{M}$, then $\mathcal{N} \models T$. In other words, a theory is preserved under substructure if and only if it has a universal axiomatization*

Proof. Suppose that $\mathcal{N} \subseteq \mathcal{M}$. By Proposition 1.7, if $\phi(\bar{v})$ is a quantifier-free formula and $\bar{a} \in N$, then $\mathcal{N} \models \phi(\bar{a})$ if and only if $\phi(\bar{a})$. Thus if $\mathcal{M} \models \forall \bar{v} \phi(\bar{v})$, then so does $\mathcal{N}$

Suppose that T is preserved under substructures. Let $\Gamma = \{\phi : \phi$ is universal and $T \models \phi\}$. Clearly, if $\mathcal{N} \models T$, then $\mathcal{N} \models \Gamma$. For the other direction, suppose that $\mathcal{N} \models \Gamma$. We claim that $\mathcal{N} \models T$

Claim $T \cup \mathrm{Diag}(\mathcal{N})$ is satisfiable

Suppose not. Then, by the Compactness Theorem, there is a finite $\Delta \subseteq \mathrm{Diag}(\mathcal{N})$ s.t. $T \cup \Delta$ is not satisfiable. Let $\Delta = \{\psi_1, \ldots, \psi_n\}$. Let $\bar{c}$ be the new constant symbols from N used in $\psi_1, \ldots, \psi_n$ and say $\psi_i = \phi_i(\bar{c})$, where ϕ_i is a quantifier-free $\mathcal{L}$-formula. Because the constants in $\bar{c}$ do not occur in T, if there is a model of $T \cup \{\exists \bar{v} \bigwedge \phi_i(\bar{v})\}$, then by interpreting $\bar{c}$ as witness to the existential formula, $T \cup \Delta$ would be satisfiable. Thus $T \models \forall \bar{v} \bigvee \neg \phi_i(\bar{v})$. As the latter formula is universal, $\forall \bar{v} \bigvee \neg \phi_i(\bar{v}) \in \Gamma$, contradicting $\mathcal{N} \models \Gamma$.

By Lemma 2.27, there is $\mathcal{M} \models T$ with $\mathcal{M} \supseteq \mathcal{N}$. Because T is preserved under substructure, $\mathcal{N} \models T$ and Γ is a universal axiomatization □

Definition 2.34. Suppose that $(I, <)$ is a linear order. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for $i \in I$. We say that $(\mathcal{M}_i : i \in I)$ is a chain of $\mathcal{L}$-strctures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i < j$. If $\mathcal{M}_i \prec \mathcal{M}_j$ for $i < j$, we call $(\mathcal{M}_i : i \in I)$ an **elementary chain**

If $(\mathcal{M}_i : i \in I)$ is a nonempty chain of structures, then we can define $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$, the union of the chain, as follows. $M = \bigcup_{i \in I} M_i$. if c is a constant in the language, then $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$ for all $i, j \in I$. Let $c^{\mathcal{M}} = c^{\mathcal{M}_i}$.

Suppose that $\bar{a} \in M$. Because I is linearly ordered, we can find $i \in I$ s.t. $\bar{a} \in M_i$. If f is a function symbol of $\mathcal{L}$ and $i < j$, then $f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_j}(\bar{a})$. Thus $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$ is a well-defined function. Similarly, $R^{\mathcal{M}} = \bigcup_{i \in I} R^{\mathcal{M}_i}$

Proposition 2.35. *Suppose that $(I, <)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain. Then $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each $\mathcal{M}_i$*

Proof. We prove by induction on formulas that

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M}_i \models \phi(\bar{a})$$

for all $i \in I$, all formulas $\phi(\bar{v})$, and all $\bar{a} \in M_i^n$

Because $\mathcal{M}_i$ is a substructure of $\mathcal{M}$, by Proposition 1.7 this is true for all atomic ϕ. $\neg\phi$ and $\phi \vee \psi$ is easy.

Suppose that ϕ is $\exists v \psi(v, \bar{w})$ and the chain holds for ψ. If $\mathcal{M}_i \models \psi(b, \bar{a})$, then so does $\mathcal{M}$. Thus if $\mathcal{M}_i \models \phi(\bar{a})$, then so does $\mathcal{M}$. On the other hand, if $\mathcal{M} \models \psi(b, \bar{a})$, there is $j \geq i$ s.t. $b \in M_j$. By induction, $\mathcal{M}_j \models \psi(b, \bar{a})$, so $\mathcal{M}_j \models \phi(\bar{a})$. Because $\mathcal{M}_i \prec \mathcal{M}_j$, $\mathcal{M}_i \models \phi(\bar{a})$ □

2.4 Back and Forth

2.4.1 Dense Linear Orders

Let $\mathcal{L} = \{<\}$ and let DLO be the theory of dense linear orders without endpoints. DLO is axiomatized by the axioms for linear orders plus the axioms

$$\forall x \forall y \; (x < y \rightarrow \exists z \; x < z < y)$$
$$\forall x \exists y \exists z \; y < x < z$$

Theorem 2.36. *The theory DLO is $\aleph_0$-categorical and complete*

Proof. Let $(A, <)$ and $(B, <)$ be two countable models of DLO. Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be one-to-one enumerations of A and B. We will build a sequence of partial bijections $f_i : A_i \rightarrow B_i$ where $A_i \subset A$ and $B_i \subset B$ are finite s.t. $f_0 \subseteq f_1 \subseteq \ldots$ and if $x, y \in A_i$ and $x < y$, then $f_i(x) < f_i(y)$. We call f_i a **partial embedding**. We will build these sequences s.t. $A = \bigcup A_i$ and $B = \bigcup B_i$. In this case, $f = \bigcup f_i$ is the desired isomorphism from $(A, <)$ to $(B, <)$

At odd stages of the construction we will ensure that $\bigcup A_i = A$, and at even stages we will ensure that $\bigcup B_i = B$

<u>stage 0</u>: Let $A_0 = B_0 = f_0 = \emptyset$

stage $n+1=2m+1$: We will ensure that $a_m \in A_{n+1}$.

If $a_m \in A_n$, then let $A_{n+1} = A_n$, $B_{n+1} = B_n$ and $f_{n+1} = f_n$. Suppose that $a_m \notin A_n$. To add a_m to the domain of our partial embedding, we must find $b \in B \backslash B_n$ s.t.

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$. In other words, we must find $b \in B$, which is the image under f_n of the cut of a_m in A_n. Exactly one of the following holds:

1. a_m is greater than every element of A_n, or

2. a_m is than than every element of A_n, or

3. there are α and $\beta \in A_n$ s.t. $\alpha < \beta$,$\gamma \leq \alpha$ or $\gamma \geq \beta$ for all $\gamma \in A_n$ and $\alpha < a_m < \beta$

In case 1 because B_n is finite and $B \models \mathsf{DLO}$,we can find $b \in B$ greater than every element of B_n. Similar for case 2. In case 3, because f_n is a partial embedding, $f_n(\alpha) < f_n(\beta)$ and we can choose $b \in B_n$ s.t. $f_n(\alpha) < b < f_n(\beta)$. Note that

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$

stage $n+1=2m+2$: We will ensure $b_m \in B_{n+1}$

Again, if b_m is already in B_n, then we make no changes. Otherwise, we must find $a \in A$ s.t. the image of the cut of a in A_n is the cut of b_m in B_n. This is done in odd case.

Clearly, at odd stages we have ensured that $\bigcup A_n = A$ and at even stages we have ensured that $\bigcup B_n = B$. Because each f_n is a partial embedding, $f = \bigcup f_n$ is an isomorphism from A onto B

But there are no finite dense linear orders, Vaught's test implies that DLO is complete $\square$

2.4.2 The Random Graph

Let $\mathcal{L} = \{R\}$, where R is a binary relation symbol. We will consider an $\mathcal{L}$-theory containing the graph axioms $\forall x \ \neg R(x,x)$ and $\forall x \forall y \ R(x,y) \rightarrow R(y,x)$. Let ψ_n be the "extension axiom"

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n \ \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} x_1 \neq y_j \rightarrow \exists z \ \bigwedge_{i=1}^{n} (R(x_i, z) \wedge \neg R(y_i, z)) \right)$$

We let T be the theory of graphs where we add $\{\exists x \exists y\ x \neq y\} \cup \{\psi_n : n = 1, 2, \dots\}$ to the graph axioms. A model of T is a graph where for any finite disjoint sets X and Y we can find a vertex with edges going to every vertex in X and no vertex in Y

Theorem 2.37. *T is satisfiable and $\aleph_0$-categorical. In particular, T is complete and decidable*

Proof. We first build a countable model of T. Let G_0 be any countable graph

Claim There is a graph $G_1 \supseteq G_0$ s.t. G_1 is countable and if X and \$Y\$are disjoint finite subsets of G_0 then there is $z \in G_1$ s.t. $R(x, z)$ for $x \in X$ and $\neg R(y, z)$ for $y \in Y$

Let the vertices of G_1 be the vertices of G_0 plus new vertices z_X for each $X \subseteq G_0$. The edges of G_1 are the edges of G together with new edges between x and z_X whenever $X \subseteq G_0$ is finite and $x \in X$.

We iterate this construction to build a sequence of countable graphs $G_0 \subset G_1 \subset \dots$ s.t. if X and Y are disjoint finite subsets of G_i, then there is $z \in G_{i+1}$ s.t. $R(x, z)$ for $x \in X$ and $\neg R(y, z)$ for $y \in Y$. Thus $G = \bigcup G_n$ is a countable model of T

Next we show that T is $\aleph_0$-categorical. Let G_1 and G_2 be countable models of T. Let $a_0, a_1, \dots$ list G_1, and let $b_0, b_1, \dots$ list G_2. We will build a sequence of finite partial one-to-one maps $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ s.t. for all x, y in the doamin of f_s,

$$G_1 \vDash R(x, y) \Leftrightarrow G_2 \vDash R(f_s(x), f_s(y))$$

Let $f_0 = \emptyset$ <u>stage $s + 1 = 2i + 1$</u>: We make sure that a_i is in the domain

If a_i is in the domain of f_s, let $f_{s+1} = f_s$. If not, let $\alpha_1, \dots, \alpha_m$ list the domain of f_s and let $X = \{j \leq m : R(\alpha_j, a_i)\}$ and let $Y = \{j \leq m : \neg R(\alpha_j, a_i)\}$. Because $G_2 \vDash T$, we can find $b \in G_2$ s.t. $G_2 \vDash R(f_s(\alpha_j), b)$ for $j \in X$ and $G_2 \vDash \neg R(f_s(\alpha_j), b)$ for $j \in Y$. Let $f_{s+1} = f_s \cup \{(a_i, b)\}$.

<u>stage $s + 1 = 2i + 2$</u>: Similar $\square$

Let $\mathcal{G}_N$ be the set of all graphs with vertices $\{1, 2, \dots, N\}$. We consider a probability measure on $\mathcal{G}_N$ where we make all graphs equally likely. This is the same as constructing a random graph where we independently decide whether there is an edge between i and j with probability $\frac{1}{2}$. For any $\mathcal{L}$-sentence ϕ,

$$p_N(\phi) = \frac{|\{G \in \mathcal{G}_N : G \vDash \phi\}|}{|\mathcal{G}_N|}$$

is the probability that a random element of $\mathcal{G}_N$ satisfies ϕ

Lemma 2.38. $\lim_{N\to\infty} p_N(\psi_n) = 1$

Proof. Fix n. Let G be a random graph in $\mathcal{G}_N$ where $N > 2n$. Fix $x_1, \dots, x_n, y_1, \dots, y_n, z \in G$ distinct. Let q be the probability that

$$\neg\left(\bigwedge_{i=1}^{n}(R(x_i, z)) \wedge \neg R(y_i, z)\right)$$

Then $q = 1 - 2^{-2n}$. Because these probabilities are independent, the probability that

$$G \models \neg\exists z \neg\left(\bigwedge_{i=1}^{n}(R(x_i, z)) \wedge \neg R(y_i, z)\right)$$

is q^{N-2n}. Let M be the number of pairs of disjoint subsets of G of size n. Thus

$$p_N(\neg\psi_n) \leq Mq^{N-2n} < N^{2n}q^{N-2n}$$

Because $q < 1$

$$\lim_{N\to\infty} p_N(\neg\psi_n) = \lim_{N\to\infty} N^{2n}q^N = 0$$

$\square$

Theorem 2.39 (Zero-One Law for Graphs)**.** *For any $\mathcal{L}$-sentence ϕ either* $\lim_{N\to\infty} p_N(\phi) = 0$ *or* $\lim_{N\to\infty} p_N(\phi) = 1$. *Moreover, T axiomatizes* $\{\phi : \lim_{N\to\infty} p_N(\phi) = 1\}$, *the* ***almost sure theory graphs****. The almost sure theory of graphs is decidable and complete*

Proof. If $T \models \phi$, then there is n s.t. if G is a graph and $G \models \psi_n$, then $G \models \phi$. Thus, $p_N(\phi) \geq \phi_N(\psi_n)$ and by Lemma 2.38, $\lim_{N\to\infty} p_N(\phi) = 1$. $\square$

2.4.3 Ehrenfeucht-Fraïssé Games

Let $\mathcal{L}$ be a language and $\mathcal{M} = (M, \dots)$ and $\mathcal{N} = (N, \dots)$ be two $\mathcal{L}$-structures with $M \cap N = \emptyset$. If $A \subseteq M$, $B \subseteq N$ and $f : A \to B$, we wsay that f is a **partial embedding** if $f \cup \{(c^{\mathcal{M}}, c^{\mathcal{N}}) : c$ a constant of $\mathcal{L}\}$is a bijection preserving all relations and functions of $\mathcal{L}$

We will define an infinite two-player game $G_\omega(\mathcal{M}, \mathcal{N})$. We will call the two players player I and player II; together they will build a partial embedding f from M to N. A play of the game will consist of ω stages. At the ith-stage, player I moves first and either plays $m_i \in M$, challenging player II to put m_i into the domain of f, or $n_i \in N$, challenging player II to put n_i into the range. If player I plays $m_i \in M$, then player II must play $n_i \in N$, whereas if player I plays $n_i \in M$, then player II must play $m_i \in M$. Player II wins the play of the game if $f = \{(m_i, n_i) : i = 1, 2, ...\}$ is the graph of a partial embedding.

A **strategy** for player II in $G_\omega(\mathcal{M}, \mathcal{N})$ is a function τ s.t. if player I's first n moves are $c_1, \ldots, c_n$, then player II's nth move will be $\tau(c_1, \ldots, c_n)$. We say that player II uses the strategy τ in the play of the game if the play looks like

Player I	Player II
c_1	
	$\tau(c_1)$
c_2	
	$\tau(c_1, c_2)$
c_3	
	$\tau(c_1, c_2, c_3)$
$\vdots$	$\vdots$

We say that τ is a **winning strategy** for player II, if for any sequence of plays $c_1, ...$ player I makes, player II will win by following τ. We define strategies for player I analogously

For example, suppose that $\mathcal{M}, \mathcal{N} \models \mathsf{DLO}$. Then player II has a winning strategy. Suppose that up to stage n they have built a partial embedding $g : A \to B$. If player I plays $a \in M$, then player II plays $b \in N$ s.t. the cub b makes in B is the image of the cut of a in A under g. Similar for player I's $b \in N$

Proposition 2.40. *If $\mathcal{M}$ and $\mathcal{N}$ is countable, then the second player has a wining strategy in G_ω if and only if $\mathcal{M} \cong \mathcal{N}$*

Proof. If $\mathcal{M} \cong \mathcal{N}$, player II can win by playing according to the isomorphism

Suppose that player II has a winning strategy. Let $m_0, m_1, ...$ list M and $n_0, n_1, ...$ list N. Consider a play of the game where the second player uses the winning strategy and the first player plays $m_0, n_0, m_1, n_1, m_2, n_2, ...$ If f is the partial embedding build during this play of the game then the domain of f is M and the range of f is N. Thus f is an isomorphism $\square$

Fix $\mathcal{L}$ a finite language with no function symbols, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We define a game $G_n(\mathcal{M}, \mathcal{N})$ for $n = 1, 2, ...$ The game will have n rounds similar to ω rounds . Player II wins if $\{(a_i, b_i) : i = 1, \ldots, n\}$ is the graph of a partial embedding from $\mathcal{M}$ into $\mathcal{N}$. We call $G_n(\mathcal{M}, \mathcal{N})$ an **Ehrenfeucht-Fraïssé Games**

Theorem 2.41. *Let $\mathcal{L}$ be a finite language without function symbols and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if the second player has a wining strategy in $G_n(\mathcal{M}, \mathcal{N})$ for all n*

We need several lemmas.

Lemma 2.42. *One of the players has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$*

Proof. Suppose that player II does not have a winning strategy. Then there is some move player I can make in round one so that player II has no move available to force a win. Player I makes that move. Now, whatever player II does, there is still a move that if made by player I means that player II cannot force a win. $\square$

We inductively define depth(ϕ), the **quantifier depth** of an $\mathcal{L}$-formula ϕ, as follows

depth(ϕ) $= 0$ if and only if ϕ is quantifier-free

depth($\neg\phi$) $=$ depth(ϕ)

depth($\phi \wedge \psi$) $=$ depth($\phi \vee \psi$) $= \max\{$depth(ϕ), depth(ψ)$\}$

depth($\exists v\phi$) $=$ depth(ϕ) $+ 1$

We say that $\mathcal{M} \equiv_n \mathcal{N}$ if $\mathcal{M} \vDash \phi \Leftrightarrow \mathcal{N} \vDash \phi$ for all sentences of depth at most n. We will show player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$

Lemma 2.43. *For each n and l, there is a finite list of formulas $\phi_1, \ldots, \phi_k$ of depth at most n in free variables $x_1, \ldots, x_l$ s.t. every formula of depth at most n in free variables $x_1, \ldots, x_l$ is equivalent to some ϕ_i*

Proof. We first prove this for quantifier-free formulas. Because $\mathcal{L}$ is finite and has no function symbols, there are only finitely many atomic $\mathcal{L}$-formulas in free variables $x_1, \ldots, x_l$. Let $\sigma_1, \ldots, \sigma_s$ list all such formulas.

If ϕ is a Boolean combination of formulas $\tau_1, \ldots, \tau_s$, then there is S a collection of subsets of $\{1, \ldots, s\}$ s.t.

$$\vDash \phi \leftrightarrow \bigvee_{X \in S} \left(\bigwedge_{i \in X} \tau_i \wedge \bigwedge_{i \notin X} \neg\tau_i \right)$$

This gives a list of 2^{2^s} formulas s.t. every Boolean combination of $\tau_1, \ldots, \tau_s$ is equivalent to a formula in this list. In particular, because quantifier free formulas are Boolean combinations of atomic formulas, there is a finite list of depth-zero formulas s.t. every depth-zero formula is equivalent to one in the list.

Because formulas of depth $n + 1$ are Boolean combinations of $\exists v\phi$ and $\forall v\phi$ where ϕ has depth at most n $\square$

Lemma 2.44. *Let $\mathcal{L}$ be a finite language without function symbols and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. The second player has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$*

Proof. Induction on n

Suppose that $\mathcal{M} \equiv_n \mathcal{N}$. Consider a play of the game where in round one player I plays $a \in M$. We claim that there is $b \in \mathcal{N}$ s.t. $\mathcal{M} \vDash \phi(a) \Leftrightarrow \mathcal{N} \vDash \phi(b)$ whenever depth$(\phi) < n$. Let $\phi_0(v), \ldots, \phi_m(v)$ list, up to equivalance, all formulas of depth less than n. Let $X = \{i \leq m : \mathcal{M} \vDash \phi_i(a)\}$, and let $\Phi(v)$ be the formula

$$\bigwedge_{i \in X} \phi_i(v) \wedge \bigwedge_{i \notin X} \neg\phi_i(v)$$

Then, depth$(\exists v \Phi(v)) \leq n$ and $\mathcal{M} \vDash \Phi(a)$; thus there is $b \in N$ s.t. $\mathcal{N} \vDash \Phi(b)$. Player II plays b in round one

If $n = 1$, the game has now concluded and $a \mapsto b$ is a partial embedding so player II wins. Suppose that $n > 1$

Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. View $\mathcal{M}$ and $\mathcal{N}$ as $\mathcal{L}^*$-structures $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ where we interpret the new constant as a and b respectively. Because

$$\mathcal{M} \vDash \phi(a) \Leftrightarrow \mathcal{N} \vDash \phi(b)$$

for $\phi(v)$ an $\mathcal{L}$-formula with depth$(\phi) < n$, $(\mathcal{M}, a) \equiv_{n-1} (\mathcal{N}, b)$. By induction, player II has a winning strategy in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))$. If player's second play is d, player II responds as if d was player I's first play in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))'$ and continues playing using this strategy, that is, in round i player I has plays $a, d_2, \ldots, d_i$, then player II plays $\tau(d_2, \ldots, d_i)$, where τ is his winning strategy in $G((\mathcal{M}, a), (\mathcal{N}, b))$. $\square$

2.4.4 Scott-Karp Analysis

Definition 2.45. Let $\mathcal{L}$ be a language and κ an infinite cardinal. The formulas of the infinitary logic $\mathcal{L}_{\kappa,\omega}$ are defined inductively as follows:

1. Every atomic $\mathcal{L}$-formula is a formula of $\mathcal{L}_{\kappa,\omega}$
2. If X is a set of formulas of $\mathcal{L}_{\kappa,\omega}$ s.t. all of the free variables come from a fixed finite set and $|X| < \kappa$, then

   $$\bigwedge_{\phi \in X} \phi \quad \text{and} \quad \bigvee_{\phi \in X} \phi$$

   are formulas of $\mathcal{L}_{\kappa,\omega}$

3. If ϕ s a formula of $\mathcal{L}_{\kappa,\omega}$, then so are $\neg\phi$, $\forall v\ \phi$ and $\exists v\ \phi$

We say that ϕ is a formula of $\mathcal{L}_{\infty,\omega}$ if it is an $\mathcal{L}_{\kappa,\omega}$-formula for some infinite cardinal κ.

2.5 Exercises

Exercise 2.5.1. We say that an ordered group $(G, +, <)$ is **Archimedean** if for all $x, y \in G$ with $x, y > 0$ there is an integer m s.t. $|x| < m|y|$. Show that there are non-Archimedean fields elementarily equivalent to the field of real numbers

Exercise 2.5.2. Let T be an $\mathcal{L}$-theory and $T_\forall$ be all of the universal sentences ϕ s.t. $T \vDash \phi$. Show that $\mathcal{A} \vDash T_\forall$ if and only if there is $\mathcal{M} \vDash T$ with $\mathcal{A} \subseteq \mathcal{M}$

Proof. Comes from Quantifier Elimination Tests and Examples

Consider the theory $T' = T \cup \text{Diag}(\mathcal{A})$ in the language $\mathcal{L}_A$. We will show by contradiction that T' is satisfiable.

Suppose that T' is not satisfiable. Then by the Compactness Theorem, already some finite subset $\Delta \subseteq T'$ is not satisfiable. By forming conjunctions we may assume that the part of Δ coming from $\text{Diag}(\mathcal{A})$ consists only of one formula $\phi(\bar{a})$ for some $\bar{a} \in\ A$, where $\phi(\bar{a})$ is a conjunction of atomic formulas and the negation of atomic formulas. Thus we will assume that $T \cup \{\phi(\bar{a})\}$ is not satisfiable.

On the other hand, viewing T as an $\mathcal{L}_{\bar{a}}$-theory, and because $T \cup \{\phi(\bar{a})\}$ is not satisfiable, we obtain $T \vDash \neg\phi(\bar{a})$. We will show that this implies $T \vDash \forall\bar{v}\neg\phi(\bar{v})$: Let $\mathcal{C}$ be an $\mathcal{L}$-structure with $\mathcal{C} \vDash T$. Let n be the number of components in $\bar{a}$ and $c_1, \dots, c_n \in C$. Let C' be the $\mathcal{L}_{\bar{a}}$-structure which expands $\mathcal{C}$ by the constant symbols that we interpret as $c_1, \dots, c_n$ respectively. Then $\mathcal{C}' \vDash T$ and hence $\mathcal{C}' \vDash \neg\phi(\bar{c})$. As this follows for any tuple in C, we get $\mathcal{C} \vDash \forall\bar{v}\neg\phi(\bar{v})$

Since $T_\forall$ consists exactly of the universal formulas which hold in all models of T, we obtain $T_\forall \vDash \forall x\neg\phi(x)$. Hence also $\mathcal{A} \vDash \forall x\neg\phi(x)$, a contradiction

Therefore T' is indeed satisfiable ∎

3 Algebraic Examples

3.1 Quantifier Elimination

Let $\phi(a, b, c)$ be the formula

$$\exists x \; ax^2 + bx + c = 0$$

By the quadratic formula,

$$\mathbb{R} \vDash \phi(a, b, c) \leftrightarrow [(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))]$$

whereas in the complex numbers

$$\mathbb{C} \vDash \phi(a, b, c) \leftrightarrow (a \neq 0 \vee b \neq 0 \vee c = 0)$$

Definition 3.1. We say that a theory T has **quantifier elimination** if for every formula ϕ there is a quantifier-free formula ψ s.t.

$$T \vDash \phi \leftrightarrow \psi$$

Lemma 3.2. *Let $(A, <)$ and $(B, <)$ be countable dense linear orders, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$, s.t. $a_1 < \dots < a_n$ and $b_1, \dots < b_n$. Then there is an isomorphism $f : A \to B$ s.t. $f(a_i) = b_i$ for all $i = 1, \dots, n$*

Proof. Modify the proof of Theorem 2.36 starting with $A_0 = \{a_1, \dots, a_n\}$, $B_0 = \{b_1, \dots, b_n\}$, and the partial isomorphism $f_0 : A_0 \to B_0$, where $f_0(a_i) = b_i$. ☐

Theorem 3.3. *DLO has quantifier elimination*

Proof. First, suppose that ϕ is a sentence. If $\mathbb{Q} \vDash \phi$, then because DLO is complete, DLO $\vDash \phi$, and

$$\mathsf{DLO} \vDash \phi \leftrightarrow x_1 = x_1$$

whereas if $\mathbb{Q} \vDash \neg\phi$

$$\mathsf{DLO} \vDash \phi \leftrightarrow x_1 \neq x_1$$

Now suppose that ϕ is a formula with free variables $x_1, \dots, x_n$ where $n \geq 1$. We will show that there is a quantifier-free formula ψ with free variables from among $x_1, \dots, x_n$ s.t.

$$\mathbb{Q} \vDash \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

Because DLO is complete,

$$\mathsf{DLO} \vDash \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

so this will suffices.

For $\sigma : \{(i,j) : 1 \le i < j \le n\} \to 3$, let $\chi_\sigma(x_1, \ldots, x_n)$ be the formula

$$\bigwedge_{\sigma(i,j)=0} x_i = x_j \wedge \bigwedge_{\sigma(i,j)=1} x_i < x_j \wedge \bigwedge_{\sigma(i,j)=2} x_i > x_j$$

We call χ_σ a **sign condition**.

Let $\mathcal{L}$ be the language of linear orders and ϕ be an $\mathcal{L}$-formula with $n \ge 1$ free variables. Let Λ_ϕ be the set of sign conditions s.t. there is $\bar{a} \in \mathbb{Q}$ s.t. $\mathbb{Q} \vDash \chi_\sigma(\bar{a}) \wedge \phi(\bar{a})$

case 1: $\Lambda_\phi = \emptyset$

Then $\mathbb{Q} \vDash \forall \bar{x} \neg \phi(\bar{x})$ and $\mathbb{Q} \vDash \phi(\bar{x}) \leftrightarrow x_1 \neq x_1$

case 2: $\Lambda_\phi \neq \emptyset$

Let

$$\psi_\phi(\bar{x}) = \bigwedge_{\sigma \in \Lambda_\phi} \chi_\sigma(\bar{x})$$

By choice of Λ_ϕ,

$$\mathbb{Q} \vDash \phi(\bar{x}) \to \psi_\phi(\bar{x})$$

On the other hand, suppose that $\bar{b} \in \mathbb{Q}$ and $\mathbb{Q} \vDash \psi_\phi(\bar{b})$. Let $\sigma \in \Lambda_\phi$ s.t. $\mathbb{Q} \vDash \chi_\sigma(\bar{b})$. There is $\bar{a} \in \mathbb{Q}$ s.t. $\mathbb{Q} \vDash \phi(\bar{a}) \wedge \chi_\sigma(\bar{a})$. By Theorem 2.36, there is f, an automorphism of $(\mathbb{Q}, <)$ s.t. $f(\bar{a}) = \bar{b}$. By Theorem 1.9, $\mathbb{Q} \vDash \phi(\bar{b})$. Thus $\phi(\bar{b}) \leftrightarrow \psi_\phi(\bar{b})$ □

Theorem 3.4. *Suppose that $\mathcal{L}$ contains a constant symbol c, T is an $\mathcal{L}$-theory, and $\phi(\bar{v})$ is an $\mathcal{L}$-formula. The following are equivalent:*

1. *There is a quantifier-free $\mathcal{L}$-formula $\psi(\bar{v})$ s.t. $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$*
2. *If $\mathcal{M}$ and $\mathcal{N}$ are models of T, $\mathcal{A}$ is an $\mathcal{L}$-structure, $\mathcal{A} \subseteq \mathcal{M}$, and $\mathcal{A} \subseteq \mathcal{N}$, then $\mathcal{M} \vDash \phi(\bar{a})$ if and only if $\mathcal{N} \vDash \phi(\bar{a})$ for all $\bar{a} \in \mathcal{A}$*

Proof. (1) $\to$ (2). Suppose that $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$, where ψ is quantifier-free. Let $\bar{a} \in \mathcal{A}$, where $\mathcal{A}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$ and the latter structures are models of T. In Proposition 1.7, we saw that quantifer-free formulas are preserved under substructure and extension. Thus

$$\begin{aligned} \mathcal{M} \vDash \phi(\bar{a}) &\Leftrightarrow \mathcal{M} \vDash \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{A} \vDash \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{N} \vDash \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{N} \vDash \phi(\bar{a}) \end{aligned}$$

$(2) \to (1)$. First, if $T \vDash \forall \bar{v} \phi(\bar{v})$, then $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow c = c)$.Second, if $T \vDash \forall \bar{v} \neg \phi(\bar{v})$, then $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow c \neq c)$.

Thus, we may assume that both $T \cup \{\phi(\bar{v})\}$ and $T \cup \{\neg\phi(\bar{v})\}$ are satisfiable

Let $\Gamma(\bar{v}) = \{\psi(\bar{v}) : \psi$ is quantifier free and $T \vDash \forall \bar{v}(\phi(\bar{v}) \to \psi(\bar{v}))\}$. Let $d_1, \dots, d_m$ be new constant symbols. We will show that $T \cup \Gamma(\bar{d}) \vDash \phi(\bar{d})$. Then, by compactness, there are $\psi_1, \dots, \psi_n \in \Gamma$ s.t. Let $p(\bar{v}) = \{\psi(\bar{v}) \to \phi(\bar{v}) \mid \psi(\bar{v}) \in \Gamma(\bar{v})\}$. Then $T \vDash p(\bar{d})$ and we apply the compactness.

$$T \vDash \forall \bar{v} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{v}) \to \phi(\bar{v}) \right)$$

Thus

$$T \vDash \forall \bar{v} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{v}) \leftrightarrow \phi(\bar{v}) \right)$$

and $\bigwedge_{i=1}^{n} \psi_i(\bar{v})$ is quantifier-free

Claim $T \cup \Gamma(\bar{d}) \vDash \phi(\bar{d})$

Suppose not. Let $\mathcal{M} \vDash T \cup \Gamma(\bar{d}) \cup \{\neg\phi(\bar{d})\}$. Let $\mathcal{A}$ be the substructure of $\mathcal{M}$ generated by $\bar{d}$

Let $\Sigma = T \cup \mathrm{Diag}(\mathcal{A}) \cup \phi(\bar{d})$. If Σ is unsatisfiable, then there are quantifier-free formulas $\psi_1(\bar{d}), \dots, \psi_n(\bar{d}) \in \mathrm{Diag}(\mathcal{A})$ s.t.

$$T \vDash \forall \bar{v} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{v}) \to \neg\phi(\bar{v}) \right)$$

as $T \cup \mathrm{Diag}(\mathcal{A})$ is consistent. The only evildoer is $\phi(\bar{d})$. Then we have $T \cup \mathrm{Diag}(\mathcal{A}) \vDash \phi(\bar{d})$ and again by compactness. But then

$$T \vDash \forall \bar{v} \left(\phi(\bar{v}) \to \bigvee_{i=1}^{n} \neg\psi_i(\bar{v}) \right)$$

so $\bigvee_{i=1}^{n} \neg\psi_i(\bar{v}) \in \Gamma$ and $\mathcal{A} \vDash \bigvee_{i=1}^{n} \neg\psi_i(\bar{d})$, a contradiction. Thus, Σ is satisfiable

Let $\mathcal{N} \vDash \Sigma$. Then $\mathcal{N} \vDash \phi(\bar{d})$. Because $\Sigma \supseteq \mathrm{Diag}(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{N}$, by Lemma 2.27. But $\mathcal{M} \vDash \neg\phi(\bar{d})$; thus $\mathcal{N} \vDash \neg\phi(\bar{d})$, a contradiction $\square$

if $\mathcal{L}$ doesn't contain a constant symbol, there are no quantifier-free sentences, but for each sentence we can find a quantifier-free formula $\psi(v_1)$ s.t. $T \vDash \phi \leftrightarrow \psi(v_1)$

Lemma 3.5. *Let T be an $\mathcal{L}$-theory. Suppose that for every quantifier-free $\mathcal{L}$-formula $\theta(\bar{v}, w)$ there is a quantifier-free formula $\psi(\bar{v})$ s.t. $T \vDash \exists w\theta(\bar{v}, w) \leftrightarrow \psi(\bar{v})$. Then T has quantifier elimination*

Proof. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula. We wish to show to show that $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ for some quantifier-free formula $\psi(\bar{v})$

If ϕ is quantifier-free, there is nothing to prove. Suppose that for $i = 0, 1$, $T \vDash \forall \bar{v}(\theta_i(\bar{v}) \leftrightarrow \psi_i(\bar{v}))$, where ψ_i is quantifier-free.

If $\phi(\bar{v}) = \neg\theta_0(\bar{v})$, then $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \neg\psi_0(\bar{v}))$

Suppose that $T \vDash \forall \bar{v}(\theta(\bar{v}, w) \leftrightarrow \psi_0(\bar{v}, w))$, where ψ_0 is quantifier-free and $\phi(\bar{v}) = \exists w\theta(\bar{v}, w)$. Then $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \exists w\psi_0(\bar{v}, w))$. By our assumptions, there is a quantifier-free $\psi(\bar{v})$ s.t. $T \vDash \forall \bar{v}(\exists w\psi_0(\bar{v}, w) \leftrightarrow \psi(\bar{v}))$. But then $T \vDash \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ □

Combining Theorem 3.4 and Lemma 3.5 gives us the following test for quantifier elimination (Restrict the form of ϕ)

Corollary 3.6. *Let T be an $\mathcal{L}$-theory. Suppose that for all quantifier-free formulas $\phi(\bar{v}, w)$, if $\mathcal{M}, \mathcal{N} \vDash T$, $\mathcal{A}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$, $\bar{a} \in A$, and there is $b \in M$ s.t. $\mathcal{M} \vDash \phi(\bar{a}, b)$, then there is $c \in N$ s.t. $\mathcal{N} \vDash \phi(\bar{a}, c)$. Then T has a quantifier elimination*

Proof. Check this notes Quantifier Elimination Tests and Examples

We need to show that $T \vDash \forall \bar{v}(\exists w\phi(\bar{v}, w) \leftrightarrow \psi(\bar{v}))$. Suppose that $\mathcal{M} \vDash \exists w\phi(\bar{v}, w)$, then $\mathcal{N} \vDash \exists w\phi(\bar{v}, w)$. Note that $\mathcal{M}$ and $\mathcal{N}$ are interchangeable. □

3.1.1 Divisible Abelian Groups

In Proposition 2.17 we showed that the theory of nontrivial torsion-free divisible Abelian groups is κ-categorical for uncountable cardinals and hence complete by Vaught's test.

Work with the language $\mathcal{L} = \{+, -, 0\}$ because its convenient

Let DAG be the $\mathcal{L}$-theory of nontrivial torsion-free divisible Abelian groups

Lemma 3.7. *Suppose G and H are nontrivial torsion free divisible Abelian groups, $G \subseteq H$, $\psi(\bar{v}, w)$ is quantifier-free, $\bar{a} \in G$, $b \in H$, and $H \vDash \phi(\bar{a}, b)$. Then there is $c \in G$ s.t. $G \vDash \phi(\bar{a}, c)$*

Proof. We first note that ψ can be put in disjunctive normal form, namely there are atomic or negated atomic formulas $\theta_{i,j}(\bar{v}, w)$ s.t.

$$\psi(\bar{v}, w) \leftrightarrow \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \theta_{i,j}(\bar{v}, w)$$

Because $H \vDash \psi(\bar{a}, b)$, $H \vDash \bigwedge_{j=1}^{m} \theta_{i,j}(\bar{a}, b)$ for some i. Thus, without loss of generality, we may assume that ψ is a conjunction of atomic and negated atomic formulas. If $\theta(v_1, \ldots, v_m, w)$ is an atomic formula, then for some integers $n_1, \ldots, n_m, m$, $\theta(\bar{v}, w)$ is $\sum n_i v_i + mw = 0$

Thus we may assume that

$$\psi(\bar{a}, w) = \bigwedge_{i=1}^{s} \sum_{j=1}^{m} n_{i,j} a_j + m_i w = 0 \wedge \bigwedge_{i=1}^{s} \sum_{j=1}^{m} n'_{i,j} a_j + m'_i w \neq 0$$

Let $g_i = \sum n_{i,j} a_j$ and $h_i = \sum n'_{i,j} a'_j$. Then $g_i, h_i \in G$ and

$$\psi(\bar{a}, w) \leftrightarrow \bigwedge g_i + m_i w = 0 \wedge \bigwedge h_i + m'_i w \neq 0$$

If any $m_i \neq 0$, then $b = -g_i/m_i \in G$ and $G \vDash \theta(\bar{a}, b)$, so suppose that $\psi(\bar{a}, w) = \bigwedge h_i + m'_i w \neq 0$. Thus $\psi(\bar{a}, w)$ is satisfied by any element of H that is not equal to any one of $\frac{-h_1}{m'_1}, \ldots, \frac{-h_s}{m'_s}$. Because G is infinite, there is an element of G satisfying $\psi(\bar{a}, w)$ $\square$

Lemma 3.8. *Suppose that G is a torsion-free Abelian group. Then there is a torsion-free divisible Abelian group H, called the **divisible hull** of G, and an embedding $i : G \to H$ s.t. if $j : G \to H'$ is an embedding of G into a torsion-free divisible Abelian group, then there is $h : H \to H'$ s.t. $j = h \circ i$*

Proof. If G is the trivial group, then we take $H = \mathbb{Q}$ since every torsion free divisible Abelian group can be viewed as a vector space over $\mathbb{Q}$. So suppose that G is non-trivial

Let $X = \{(g, n) : g \in G, n \in \mathbb{N}, n > 0\}$. We think of (g, n) as g/n

We define an equivalence relation $\sim$ on X by $(g, n) \sim (h, m)$ if and only if $mg = nh$. Let $H = X/\sim$. For $(g, n) \in X$, let $[(g, n)]$ denote the $\sim$-class of (g, n). We define $+$ on H by $[(g, n)] + [(h, m)] = [(mg + nh, mn)]$. We must show that $+$ is well defined

Suppose that $(g_0, n_0) \sim (g, n)$. We claim that $(mg_0 + n_0 h, mn_0) \sim (mg + nh, mn)$.

Similarly we can define $-$ by $[(g, n)] - [(h, m)] = [(mg - nh, mn)]$. It is easy to show that $(H, +)$ is an Abelian group

If $[(g, m)] \in H$ and $n > 0$, then $n[(g, m)] = [(ng, m)]$. If $(ng, m) \sim (0, k)$, then $kng = 0$. Because $k, n > 0$ and G is torsion free, $g = 0$. Then $[(g, m)] = [(0, 1)]$. Thus H is torsion free.

Suppose that $[(g, m)] \in H$ and $n > 0$, then $n[(g, mn)] = [(g, m)]$. Thus H is divisible.

We can embed G into H by the map $i(g) = [(g, 1)]$

Suppose that H' is a divisible torsion-free Abelian group and $j : G \to H'$ is an embedding. Let $h : H \to H'$ by $h([g, n]) = j(g)/n$ $\square$

Theorem 3.9. *DAG has quantifier elimination*

Proof. Suppose that G_0 and G_1 are torsion-free divisible Abelian groups, G is a common subgroup of G_0 and G_1, $\bar{g} \in G$, $h \in G_0$ and $G_0 \vDash \phi(\bar{g}, h)$, where ϕ is quantifier-free. Let H be the divisible hull of G. Because we can embed H into G_0, by Lemma 3.7, $H \vDash \exists w \phi(\bar{g}, w)$. Because we can embed H into G_1, there is $h' \in G_1$ s.t. $G_1 \vDash \phi(\bar{g}, h')$. By Corollary 3.6, DAG has quantifier elimination $\square$

Quantifier elimination gives us a good picture of the definable sets in a model of DAG. Suppose that $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ is an atomic formula. Then there are integers $k_1, \dots, k_n$ and $l_1, \dots, l_m$ s.t. $\phi(\bar{v}, \bar{w}) \leftrightarrow \sum k_i x_i + \sum l_i y_i = 0$. If $G \vDash$ DAG and $a_1, \dots, a_m \in G$, $\phi(\bar{v}, \bar{a})$ defines $\{\bar{g} \in G^n : \sum k_i g_i + \sum l_i a_i = 0\}$, a hyperplane in G^n. Because any $\mathcal{L}$-formula $\phi(\bar{v}, \bar{w})$ is equivalent in DAG to a Boolean combination of atomic $\mathcal{L}$-formulas, every definable subset of G^n is a Boolean combination of hyperplanes

In particular, suppose that $\bar{a} \in G^m$ and $\phi(v, \bar{a})$ defines a subset of G. The "hyperplanes" in G are just single points. Thus, $\{g \in G : G \vDash \phi(g, \bar{a})\}$ is either finite or cofinite. Thus every definable subset of G was definable already in the language of equality

Definition 3.10. We say that an $\mathcal{L}$-theory T is **strongly minimal** if for any $\mathcal{M} \vDash T$ every definable subset of M is either finite or cofinite

Corollary 3.11. *DAG is strongly minimal*

If T is a theory then $T_\forall$ is the set of all universal consequences of T. In Exercise 2.5.2 we saw that $\mathcal{A} \vDash T_\forall$ if and only if there is $\mathcal{M} \vDash T$ with $\mathcal{A} \subseteq \mathcal{M}$. One consequence of Lemma 3.8 is that every torsion-free Abelian group is a substructure of a nontrivial divisible Abelian group. Because the axioms for torsion-free Abelian groups are universal, *DAG$_\forall$ is exactly the theory of torsion-free Abelian groups.*

We say that a theory T has **algebraically prime models** if for any $\mathcal{A} \vDash T_\forall$ there is $\mathcal{M} \vDash T$ and an embedding $i : \mathcal{A} \to \mathcal{M}$ s.t. for all $\mathcal{N} \vDash T$ and embeddings $j : \mathcal{A} \to \mathcal{N}$ there is $h : \mathcal{M} \to \mathcal{N}$ s.t. $j = h \circ i$.

$$\begin{tikzcd}
\mathcal{A} \vDash T_\forall \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{M} \vDash T \arrow[d, "h"] \\
& \mathcal{N} \vDash T
\end{tikzcd}$$

If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, we say that $\mathcal{M}$ is **simply closed** in $\mathcal{N}$ and write $\mathcal{M} \prec_s \mathcal{N}$ if for any quantifier free formula $\phi(\bar{v}, w)$ and any $\bar{a} \in M$, if $\mathcal{N} \models \exists w\ \phi(\bar{a}, w)$ then so does $\mathcal{M}$. Lemma 3.7 says that if G and H are models of DAG and $G \subseteq H$, then $G \prec_s H$

Corollary 3.12. *Suppose that T is an $\mathcal{L}$-theory s.t.*

1. *T has algebraically prime models and*
2. *$\mathcal{M} \prec_s \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ are models of T*

Then T has quantifier elimination

Proof. Suppose $\mathcal{A} \models T_\forall$, then □

Definition 3.13. An $\mathcal{L}$-theory T is **model-complete** $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}, \mathcal{N} \models T$

Proposition 3.14. *If T has quantifier elimination, then T is model-complete*

Proof. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are models of T. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula, and let $\bar{a} \in M$. There is a quantifier-free formula $\psi(\bar{v})$ s.t. $\mathcal{M} \models \forall \bar{v}(\phi(\bar{v} \leftrightarrow \psi(\bar{v})))$. Because quantifier-free formulas are preserved under substructures and extensions, $\mathcal{M} \models \psi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$. Thus $\mathcal{M} \prec \mathcal{N}$ □

Proposition 3.15. *Let T be a model-complete theory. Suppose that there is $\mathcal{M}_0 \models T$ s.t. $\mathcal{M}_0$ embeds into every model of T. Then T is complete*

Proof. If $\mathcal{M} \models T$, then $\mathcal{M}_0 \prec \mathcal{M}$. In particular $\mathcal{M}_0 \equiv \mathcal{M}$. □

Because $(\mathbb{Q}, +, 0)$ embeds in every model of DAG, this gives another proof of the completeness of DAG

3.1.2 Ordered Divisible Abelian Groups

Let $\mathcal{L} = \{+, 0, <, 0\}$ and let ODAG be the theory of nontrivial divisible ordered Abelian groups. The axioms for ordered Abelian groups are universal and hence contained in $\mathsf{ODAG}_\forall$.

We start by trying to identify $\mathsf{ODAG}_\forall$. Axioms for ordered Abelian groups are universal and hence contained in $\mathsf{ODAG}_\forall$. We claim that these axioms suffice. We must show that every ordered Abelian group embeds in an ordered divisible Abelian group. Because ordered groups are torsion-free, it suffices to show that the ordering of the group extends to an ordering of the divisible hull.

Lemma 3.16. *Let G be an ordered Abelian group and H be the divisible hull of G. We can order H s.t. $i : G \to H$ is order-preserving, $(H, +, <) \models$ ODAG and if $H' \models$ ODAG and $j : G \to H'$ is an embedding, then there is an embedding $h : H \to H'$ s.t. $j = h \circ i$*

Proof. We let $\frac{g}{n}$ denote $[(g, n)]$. We can order H by $\frac{g}{n} < \frac{h}{m}$ if and only if $mg < nh$. If $g < h$, then $\frac{g}{1} < \frac{h}{1}$ so this extends the ordering of G. If $\frac{g_1}{n_1} < \frac{g_2}{n_2}$ and $\frac{h_1}{m_1} \leq \frac{h_2}{m_2}$, then $n_2 g_1 < n_1 g_2$ and $m_2 h_1 \leq m_1 h_2$. Then,

$$m_1 m_2 n_2 g_1 + n_1 n_2 m_2 h_1 < m_1 m_2 n_1 g_2 + n_1 n_2 m_1 h_2$$

and

$$\frac{m_1 g_1 + n_1 h_1}{m_1 n_1} < \frac{m_2 g_2 + n_2 h_2}{m_2 n_2}$$

Thus, < makes H an ordered group

If H' is another ordered disivible Abelian group and $j : G \to H'$ is an embedding, let h be as in Lemma 3.8 □

To prove quantifier elimination, we must how that if G and H are ordered divisible Abelian groups and $G \subseteq H$, then $G \prec_s H$

Suppose that $\phi(v, \bar{w})$ is a quantifier-free formula, $\bar{a} \in G$, and for some $b \in H$, $H \models \phi(b, \bar{a})$. As above, it suffices to consider the case where ϕ is a conjunction of atomic and negated atomic formulas. If $\theta(v, \bar{w})$ is atomic, then θ is equivalent to either $\sum n_i w_i + mv = 0$ or $\sum n_i w_i + mv > 0$ for some $n_i, m \in \mathbb{Z}$. In particular, there is an element $g \in G$ s.t. $\theta(v, \bar{a})$ is of the form $mv = g$ or $mv > g$. Also not that for any formula $mv \neq g$ is equivalent to $mv > g$ or $-mv > g$. Thus we may assume that

$$\phi(v, \bar{a}) \leftrightarrow \bigwedge m_i v = g_i \bigwedge n_i v > h_i$$

where $g_i, h_i \in G$ and $m_i, n_i \in \mathbb{Z}$

If there is actually a conjunct $m_i v = g_i$, then we must have $b = \frac{g_i}{m_i} \in G$; otherwise $\phi(v, \bar{a}) = \bigwedge m_i v > h_i$. Let $k_0 = \min\{\frac{h_i}{m_i} : m_i < 0\}$ and $k_i = \max\{\frac{h_i}{m_i} : m_i > 0\}$. Then $c \in H$ satisfies $\phi(v, \bar{a})$ if and only if $k_0 < v < k_1$. Because b satisfies ϕ, we must have $k_0 < k_1$. But any ordered divisible Abelian group is densely ordered because if $g < h$ then $g < \frac{g+h}{2} < h$, so there is $d \in G$ s.t. $k_0 < d < k_1$. Thus $G \prec_s H$

Corollary 3.17. *ODAG is a complete decidable theory with quantifier elimination. In particular, every ordered divisible Abelian group is elementarily equivalent to $\mathbb{Q}, +, <$*

Proof. By Lemma 3.16, $\mathsf{ODAG}_\forall$ is the theory of ordered Abelian groups and ODAG has algebraically prime models. From Corollary 3.12 we see that ODAG has quantifier elimination. The ordered group of rational embeds into every ordered divisible Abelian group; thus by Proposition 3.15, ODAG is complete. Because ODAG has a recursive axiomatization, it is decidable by Lemma 2.22 $\square$

ODAG is not strongly minimal. For example, $\{a \in \mathbb{Q} : a < 0\}$ is infinite and coinfinite. On the other hand, *definable subsets are quite well-behaved*. Suppose that G is an ordered divisible Abelian group and $X \subseteq G$ definable. By quantifier elimination, X is a Boolean combination of sets defined by atomic formulas. If $\phi(v, w_1, \dots, w_n)$ is atomic, then there are integers $k_0, \dots, k_n$ s.t. ϕ is equivalent to either

$$k_0 v + \sum k_i w_i = 0$$

or

$$k_0 v + \sum k_i w_i > 0$$

If $\bar{a} \in G^n$, in the first case $\phi(v, \bar{a})$ defines a finite set whereas in the second case it defines an interval. It follows that X is a finite union of points and intervals with endpoints in $G \cup \{\pm\infty\}$

Definition 3.18. We say the an ordered structure $(M, <, \dots)$ is **o-minimal** if for any definable $X \subseteq M$ there are finitely many intervals $I_1, \dots, I_m$ with endpoints in $M \cup \{\pm\infty\}$ and a finite set X_0 s.t. $X = X_0 \cup I_1 \cup \dots \cup I_m$

3.1.3 Presburger Arithmetic

Let $\mathcal{L} = \{+, -, <, 0, 1\}$ and consider the $\mathcal{L}$-theory of the ordered group of integers. In fact this theory will not have quantifer elimination in the language $\mathcal{L}$. Let $\psi_n(v)$ be the formula

$$\exists y \; v = \underbrace{y + \dots + y}_{n\text{-times}}$$

It turns out that this is the only obstruction to quantifer elimination. Let $\mathcal{L}^* = \mathcal{L} \cup \{P_n : n = 2, 3, \dots\}$ whre P_n is a unary predicate which we will interpret as the elements divisible by n

For any language $\mathcal{L}$ and $\mathcal{L}$-theory T, there is a language $\mathcal{L}' \supseteq \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supseteq T$ s.t. for any $\mathcal{M} \models T$ we can interpret the new symbols of $\mathcal{L}'$ to make $\mathcal{M}' \models T'$ s.t. for any subset of M^k definable using $\mathcal{L}'$ is already definable using $\mathcal{L}$, and any $\mathcal{L}'$-formula is equivalent to an atomic $\mathcal{L}'$-formula

Let $\mathcal{L}' = \mathcal{L} \cup \{R_\phi : \phi \text{ an } \mathcal{L}\text{-formula}\}$, where if ϕ is a formula in n free variables, R_ϕ is an n-ary predicate symbol. Let T' be the theory obtained by adding to T the sentences

$$\forall \bar{v}(\phi(\bar{v}) \leftrightarrow R_\phi(\bar{v}))$$

Consider the $\mathcal{L}^*$-theory, which we call Pr for **Presburger arithmetic**, with axioms:

1. axioms for ordered Abelian groups
2. $0 < 1$
3. $\forall x(x \leq 0 \vee x \geq 1)$
4. $\forall x(P_n(x) \leftrightarrow \exists y\ x = \underbrace{y + \cdots + y}_{n\text{-times}})$, for $n = 2, 3, \ldots$
5. $\forall x \bigvee_{i=0}^{n-1} [P_n(x + \underbrace{1 + \cdots + 1}_{i \text{ times}}) \wedge \bigwedge_{j \neq i} \neg P_n(x + \underbrace{1 + \cdots + 1}_{j\text{times}})]$ for $n = 2, 3, \ldots$

Suppose that $(G, +, -, <, 0, 1)$ is a model of Pr. For each n, axiom (4) asserts that $P_n^G = nG$. Axiom (5) asserts that $\frac{G}{nG} \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$

3.2 Algebraically Closed Fields

Lemma 3.19. *Check this.*

ACF$_\forall$ is the theory of integral domains

ACF$_\forall$ axiomatize the theory of integral domains. Actually this is what we want as we consider integral domains later and prove a stronger version

Consider a different version. Let $\mathcal{L} = \{0, 1, +, -, *\}$ be the language of rings and T is the theory of fields, then $T_\forall$ is the theory of integral domains. For if R is an integral domain, then it is a subring of its field of fractions K, and $K \vDash T$ and hence $R \vDash T_\forall$ by Exercise 2.5.2. So any integral domain models $T_\forall$. Conversely, if S is a ring and $S \vDash T_\forall$ then we need to check that S is an integral domain, so we need to check $0 \neq 1$, that $xy = yx$ and that $xy = 0 \Rightarrow x = 0 \vee y = 0$.

Proof. The axioms for integral domains are universal consequences of ACF. If D is an integral domain, then the algebraic closure of the fraction field of D is a model of ACF. Because every integral domain is a subring of an algebraically closed field, $\mathsf{ACF}_\forall$ is the theory of integral domains by Exercise 2.5.2 $\square$

Theorem 3.20. *ACF has quantifier elimination*

Proof. We will apply Corollary 3.12. If D is an integral domain, then the algebraic closure of the fraction field of D embeds into any algebraically closed field containing D. Thus ACF has algebraically prime models

To prove quantifier elimination, we need only show that if K and F are algebraically closed fields, $F \subseteq K$, $\phi(x, \bar{y})$ is quantifier-free, $\bar{a} \in F$, and $K \vDash \phi(b, \bar{a})$ for some $b \in K$, then $F \vDash \exists v\ \phi(v, \bar{a})$

As in Lemma 3.7, we may assume that $\phi(x, \bar{y})$ is a conjunction of atomic and negated atomic formulas. In the language of rings, atomic formulas $\phi(v_1, \dots, v_n)$ are of the form $p(\bar{v}) = 0$, where $p \in \mathbb{Z}[x_1, \dots, x_n]$. If $p(X, \bar{Y}) \in \mathbb{Z}[X, \bar{Y}]$, we can view $p(X, \bar{a})$ as a polynomial in $F[X]$. Thus there are polynomials $p_1, \dots, p_n, q_1, \dots, q_m \in F[X]$ s.t. $\phi(v, \bar{a})$ is equivalent to

$$\bigwedge_{i=1}^{n} p_i(v) = 0 \wedge \bigwedge_{i=1}^{m} q_i(v) \neq 0$$

If any of the polynomials p_i are nonzero, then b is algebraic over F. In this case, $b \in F$ because F is algebraically closed. Thus we may assume that $\phi(v, \bar{a})$ is equivalent to

$$\bigwedge_{i=1}^{m} q_i(v) \neq 0$$

But $q_i(X) = 0$ has only finitely many solutions for each $i \leq m$. Thus there are only finitely many elements of F that do not satisfy F. Because algebraically closed fields are infinite, there is a $c \in F$ s.t.

$$F \vDash \phi(c, \bar{a})$$

$\square$

Corollary 3.21. *ACF is model-complete and ACF_p is complete where $p = 0$ or p is prime*

Proof. Suppose that $K, L \vDash \mathsf{ACF}_p$. Let ϕ be any sentence in the language of rings. By quantifer elimination, there is a quantifer-free sentence ψ s.t.

$$\mathsf{ACF} \vDash \phi \leftrightarrow \psi$$

Because quantifer-free sentences are preserved under extension and substructure,

$$K \vDash \psi \Leftrightarrow \mathbb{F}_p \vDash \psi \Leftrightarrow L \vDash \psi$$

Thus $K \equiv L$ and ACF_p is complete $\square$

3.2.1 Zariski Closed and Constructible Sets

Let K be a field. If $S \subseteq K[X_1, \dots, X_n]$, let $V(S) = \{a \in K^n : p(a) = 0 \text{ for all } p \in S\}$. If $Y \subseteq K^n$, we let $I(Y) = \{f \in K[X_1, \dots, X_n] : f(\bar{a}) = 0 \text{ for all } \bar{a} \in Y\}$. We say $X \subseteq K^n$ is **Zariski closed** if $X = V(S)$ for some $S \subseteq K[X_1, \dots, X_n]$

The **radical** of an ideal I in a commutative ring R , denoted by $\sqrt{I}$, is defined as

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$$

I is a radical ideal iff $I = \sqrt{I}$

Lemma 3.22. *Let K be a field*

1. *If $X \subseteq K^n$, then $I(X)$ is a radical ideal*
2. *If X is Zariski closed, then $X = V(I(X))$*
3. *If X and Y are Zariski closed and $X \subseteq Y \subseteq K^n$, then $I(Y) \subseteq I(X)$*
4. *If $X, Y \subseteq K^n$ are Zariski closed, then $X \cup Y = V(I(X) \cap I(Y))$ and $X \cap Y = V(I(X) + I(Y))$*

Proof.

1. Suppose that $p, q \in I(X)$ and $f \in K[X_1, \dots, X_n]$. If $a \in X$, then $p(a)+q(a) = f(a)p(a) = 0$. Thus $p+q, fp \in I(X)$ and $I(X)$ is an ideal. If $f^n \in I(X)$ and $a \in X$, then $f^n(a) = 0$ so $f(a) = 0$. Thus $f \in I(X)$ and $I(X)$ is a radical ideal
2. If $a \in X$ and $p \in I(X)$, then $p(a) = 0$. Thus $X \subseteq V(I(X))$. If $a \in V(I(X)) \setminus X$, then there is $p \in I(X)$ s.t. $p(a) \neq 0$, a contradiction
3. If $p \in I(Y)$ and $a \in X$, then $p(a) = 0$ and $I(Y) \subseteq I(X)$. By (2), if $I(X) = I(Y)$, then $X = Y$
4. If $p \in I(X) \cap I(Y)$, then $p(a) = 0$ for $a \in X$ or $a \in Y$. Thus $X \cup Y \subseteq V(I(X) \cap I(Y))$. If $a \notin X \cup Y$, there are $p \in I(X)$ and $q \in I(Y)$ s.t. $p(a) \neq 0$ and $q(a) \neq 0$. But then $p(a)q(a) \neq 0$. Because $pq \in I(X) \cap I(Y)$, $a \notin V(I(X) \cap I(Y))$

If $a \in X \cap Y$, $p \in I(X)$, $q \in I(Y)$, then $p(a) + q(a) = 0$. Thus $X \cap Y \subseteq V(I(X) + I(Y))$. If $a \notin X$, then there is $p \in I(X) \subseteq I(X) + I(Y)$ s.t. $p(a) \neq 0$. Thus $a \notin V(I(X) + I(Y))$. Similarly, if $a \notin Y$, then $a \notin V(I(X) + I(Y))$

□

Theorem 3.23 (Hilbert's Basis Theorem). *If K is a field, then the polynomial ring $K[X_1, \dots, X_n]$ is a Noetherian ring, (i.e., there are no infinite ascending chains of ideals). In particular, every ideal is finitely generated*

Corollary 3.24.

1. *There are no infinite descending sequences of Zariski closed sets*
2. *If X_i is Zariski closed for $i \in I$, then there is a finite $I_0 \subseteq I$ s.t.*

$$\bigcap_{i \in I} X_i = \bigcap_{i \in I_0} X_i$$

In particular, an arbitrary intersection of Zariski closed sets is Zariski closed

4 Realizing and Omitting Types

4.1 Types

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\mathcal{L}_A$ be the language obtained by adding to $\mathcal{L}$ constant symbols for each $a \in A$. We can naturally view $\mathcal{M}$ as an $\mathcal{L}_A$-structure by interpreting the new symbols in the obvious way. Let $\mathrm{Th}_A(\mathcal{M})$ be the set of all $\mathcal{L}_A$-sentences true in $\mathcal{M}$. Note that $\mathrm{Th}_A(\mathcal{M}) \subseteq \mathrm{Diag}_{el}(\mathcal{M})$

Definition 4.1. Let p be the set of $\mathcal{L}_A$-formulas in free variables $v_1, \dots, v_n$. We call p an n**-type** if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that p is a **complete** n**-type** if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables from $v_1, \dots, v_n$. We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types.

Remark. Wu's remark: guess here $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable means that there is a model $\mathfrak{N} \vDash \mathrm{Th}_A(\mathcal{M})$ that realizes p, which is slightly different from "there is an elementary extension of $\mathfrak{M}$ that realizes p"

Consider $\mathcal{M} = (\mathbb{Q}, <)$and $A = \mathbb{N}$, let $q(v) = \{\phi(v) \in \mathcal{L}_A : \mathcal{M} \vDash \phi(\frac{1}{2})\}$. $q(v)$ is a complete 1-type

We sometimes refer to incomplete types as **partial types**

By the compactness theorem, we could replace "satisfiable" by "finitely satisfiable"

If $\mathcal{M}$ is any $\mathcal{L}$-structure, $A \subset M$, and $\bar{a} = (a_1, \dots, a_n) \in M^n$, let $\text{tp}^{\mathcal{M}}(\bar{a}/A) = \{\phi(v_1, \dots, v_n) \in \mathcal{L}_A : \mathcal{M} \vDash \phi(a_1, \dots, a_n)\}$. Then $\text{tp}^{\mathcal{M}}(\bar{a}/A)$ is a complete n-type. We write $\text{tp}^{\mathcal{M}}(\bar{a})$ for $\text{tp}^{\mathcal{M}}(\bar{a}/\emptyset)$

Definition 4.2. If p is an n-type over A, we say that $\bar{a} \in M^n$ **realizes** p if $\mathcal{M} \vDash \phi(\bar{a})$ for all $\phi \in p$. If p is not realized in $\mathcal{M}$ we say that $\mathcal{M}$ **omits** p.

$1/2$ realizes $q(v)$. And there are many realizations of $q(v)$ in $\mathcal{M}$. Suppose that $r \in \mathbb{Q}$ and $0 < r < 1$. We can construct an automorphism σ of $\mathcal{M}$ that fixes every natural number but $\sigma(1/2) = r$. Because σ fixes all elements of A, σ is also an $\mathcal{L}_A$-automorphism. By Theorem 1.9

$$\mathcal{M} \vDash \phi(1/2) \iff \mathcal{M} \vDash \phi(r)$$

In fact, the elements of $\mathbb{Q}$ that realize $q(v)$ are exactly the rational number s s.t. $0 < s < 1$

Proposition 4.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and p an n-type over A. There is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ s.t. p is realized in $\mathcal{N}$.*

Proof. Let $\Gamma = p \cup \text{Diag}_{\text{el}}(\mathcal{M})$. We claim that Γ is satisfiable

Suppose that Δ is a finite subset of Γ. W.L.O.G., Δ is the single formula

$$\phi(v_1, \dots, v_n, a_1, \dots, a_m) \wedge \psi(a_1, \dots, a_m, b_1, \dots, b_l)$$

where $a_1, \dots, a_m \in A$, $b_1, \dots, b_l \in M \setminus A$, $\phi(\bar{v}, \bar{a}) \in p$ and $\mathcal{M} \vDash \psi(\bar{a}, \bar{b})$. Let $\mathcal{N}_0$ be a model of the satisfiable set of sentences $p \cup \text{Th}_A(\mathcal{M})$. Because $\exists \overline{w} \psi(\bar{a}, \overline{w}) \in \text{Th}_A(\mathcal{M})$,

$$\mathcal{N}_0 \vDash \phi(\bar{v}, \bar{a}) \wedge \exists \overline{w} \psi(\bar{a}, \overline{w})$$

By interpreting $b_1, \dots, b_l$ as witnesses to $\exists \overline{w} \psi(a_1, \dots, a_m, \overline{w})$, we make $\mathcal{N}_0 \vDash \Delta$. Thus Δ is satisfiable.

By the Compactness Theorem, Γ is satisfiable. Let $\mathcal{N} \vDash \Gamma$. Because $\mathcal{N} \vDash \text{Diag}_{\text{el}}(\mathcal{M})$, the map that sends $m \in M$ to the interpretation of the constant symbol m in $\mathcal{N}$ is an elementary embedding. Let $c_i \in N$ be the interpretation of v_i. Then $(c_1, \dots, c_n)$ is a realization of p. $\square$

If $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, then $\text{Th}_A(\mathcal{M}) = \text{Th}_A(\mathcal{N})$. Thus $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$

Corollary 4.4. *$p \in S_n^{\mathcal{M}}(A)$ iff there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ s.t. $p = \text{tp}^{\mathcal{N}}(\bar{a}/A)$*

Proof. If $\bar{a} \in N^n$, then $\text{tp}^{\mathcal{N}}(\bar{a}/A) \in S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

On the other hand if $p \in S_n^{\mathcal{M}}(A)$, then by Proposition 4.3 there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in \mathcal{M}$ realizing p. Because p is complete, if $\phi(\bar{v}) \in \mathcal{L}_A$, then exactly one of $\phi(\bar{v})$ and $\neg\phi(\bar{v})$ is in p. Thus $\phi(\bar{v}) \in \text{tp}^{\mathcal{N}}(\bar{a}/A)$ iff $\phi(\bar{v}) \in p$ and $p = \text{tp}^{\mathcal{N}}(\bar{a}/A)$ □

Proposition 4.5. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\bar{a}, \bar{b} \in M^n$ s.t. $\text{tp}^{\mathcal{M}}(\bar{a}/A) = \text{tp}^{\mathcal{M}}(\bar{b}/A)$. Then there is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ and σ an automorphism of $\mathcal{N}$ fixing all elements of A s.t. $\sigma(\bar{a}) = \bar{b}$.*

If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $B \subseteq M$, we say that $f : B \to N$ is a **partial elementary map** iff

$$\mathcal{M} \models \phi(\bar{b}) \iff \mathcal{N} \models \phi(f(\bar{b}))$$

for all $\mathcal{L}$-formulas ϕ and all finite sequences $\bar{b} \in B$

Lemma 4.6. *Let $\mathcal{M}, \mathcal{N}, B$ be as above and let $f : B \to N$ be partial elementary. If $b \in M$, there is an elementary extension $\mathcal{N}_1$ of $\mathcal{N}$ and $g : B \cup \{b\} \to \mathcal{N}_1$ a partial elementary map extending f.*

Proof. Let $\Gamma = \{\phi(v, f(a_1), \dots, f(a_n)) : \mathcal{M} \models \phi(b, a_1, \dots, a_n), a_1, \dots, a_n \in B\} \cup \text{Diag}_{\text{el}}(\mathcal{N})$. Note that here we have the range of f and therefore the range of $\phi(f(\bar{b}))$

Suppose that we find a structure $\mathcal{N}_1$ and an element $c \in N_1$ satisfying all of the formulas in Γ, then we are done.

Thus it suffices to show that Γ is satisfiable. By the Compactness Theorem it suffices to show that every finite subset of Γ is satisfiable in $\mathcal{N}$. Taking conjunctions, it is enough to show that if $\mathcal{M} \models \phi(, a_1, \dots, a_n)$, then $\mathcal{N} \models \exists v \phi(v, f(a_1), \dots, f(a_n))$ but this is clear because $\mathcal{M} \models \exists v \phi(v, a_1, \dots, a_n)$ and f is partial elementary □

Corollary 4.7. *If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, $B \subseteq M$ and $f : B \to N$ is a partial elementary map, then there is $\mathcal{N}'$ an elementary extension of $\mathcal{N}$ and $g : \mathcal{M} \to \mathcal{N}'$ an elementary embedding*

Proof. Let $\kappa = |M|$, and let $\{a_\alpha : \alpha < \kappa\}$ be an enumeration of M. Let $\mathcal{N}_0 = \mathcal{N}$, $B_0 = B$, and $g_0 = f$. Let $B_\alpha = B \cup \{a_\beta : \beta < \alpha\}$. We inductively build an elementary chain $(N_\alpha : \alpha < \kappa)$ and $g_\alpha : B_\alpha \to N_\alpha$ partial elementary s.t. $g_\beta \subseteq g_\alpha$ for $\beta < \alpha$

If $\alpha = \beta + 1$ and $g_\beta : B_\beta \to N_\beta$ is partial elementary, then by Lemma 4.6 we can find $N_\beta \prec N_\alpha$ and $g_\alpha : B_\alpha \to N_\alpha$

If α is a limit ordinal, let $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ and $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. By Proposition 2.35 $\mathcal{N}_\alpha$ is an elementary extension of N_β for $\beta < \alpha$ and f_α is a partial elementary map.

Let $\mathcal{N}' = \bigcup_{\alpha<\kappa} \mathcal{N}_\alpha$ and $g = \bigcup_{\alpha<\kappa} g_\alpha$. Again by Proposition 2.35 $\mathcal{N} \prec \mathcal{N}'$ and g is partial elementary. But $\mathrm{dom}(g) = M$, so g is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}'$ $\square$

Proof of 4.5. Let $f : A \cup \{a\} \to A \cup \{b\}$ s.t. $f|A$ is the identity and $f(a) = b$. Because $\mathrm{tp}^{\mathcal{M}}(a/A) = \mathrm{tp}^{\mathcal{M}}(b/A)$, f is a partial elementary map. By Corollary 4.7 there is $\mathcal{N}_0$ an elementary extension of $\mathcal{M}$ and $f_0 : \mathcal{M} \to \mathcal{N}_0$ an elementary embedding extending f. We will build a sequence of elementary extensions

$$\mathcal{M} = \mathcal{M}_0 \prec \mathcal{N}_0 \prec \mathcal{M}_1 \prec \mathcal{N}_1 \prec \mathcal{M}_2 \prec \mathcal{N}_2 \prec \ldots$$

and elementary embeddings $f_i : \mathcal{M}_i \to \mathcal{N}_i$ s.t. $f_0 \subseteq f_1 \subseteq f_2 \ldots$ and N_i is contained in the image of f_{i+1}. Having done this, let

$$\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i = \bigcup_{i<\omega} \mathcal{M}_i$$

and $\sigma = \bigcup f_i$. By Proposition 2.35 $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ and $\sigma : \mathcal{N} \to \mathcal{N}$ is an elementary map s.t. $\sigma|A$ is the identity and $\sigma(a) = b$. By construction σ is surjective. Thus σ is the desired automorphism.

Given $f_i : \mathcal{M}_i \to \mathcal{N}_i$ we can view f_i^{-1} as a partial elementary map from the image of f_i into $\mathcal{M}_i \prec \mathcal{N}_i$. By Corollary 4.7 we can find $\mathcal{M}_{i+1}$ an elementary extension of $\mathcal{N}_i$ and extend f_i^{-1} to an elementary embedding $g_i : \mathcal{N}_i \to \mathcal{M}_{i+1}$ $\square$

4.1.1 Stone Spaces

For ϕ an $\mathcal{L}_A$-formula with free variables from $v_1, \ldots, v_n$, let

$$[\phi] = \{p \in S^{\mathcal{M}}(A) : \phi \in p\}$$

If p is a complete type and $\phi \vee \psi \in p$, then $\phi \in p$ or $\psi \in p$. Thus $[\phi \vee \psi] = [\phi] \cup [\psi]$

The **Stone topology** on $S_n^{\mathcal{M}}(A)$ is the topology by taking the sets $[\phi]$ as basic open sets.

Lemma 4.8. *1. $S_n^{\mathcal{M}}(A)$ is compact*

2. *if $S_n^{\mathcal{M}}(A)$ is totally disconnected, that is if $p, q \in S_n^{\mathcal{M}}(A)$ and $p \neq q$, then there is a clopen set X s.t. $p \in X$ and $q \notin X$*

Proof. 1. It suffices to show that every cover of $S_n^{\mathcal{M}}(A)$ by basic open sets has a finite
subcover. Suppose not. Let $C = \{[\phi_i(\bar{v})] : i \in I\}$ be a cover of $S_n^{\mathcal{M}}(A)$ by basic open sets with no finite subcover. Let

$$\Gamma = \{\neg\phi_i(\bar{v}) : i \in I\}$$

We claim that $\Gamma \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. If I_0 is a finite subset of I, then because there is no finite subcover of C, there is a type p s.t.

$$p \notin \bigcup_{i \in I_0} [\phi_i]$$

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ containing a realization $\bar{a}$ of p. Then

$$\mathcal{N} \models \mathrm{Th}_A(\mathcal{M}) \cup \bigwedge_{i \in I_0} \neg\phi_i(\bar{a})$$

Hence Γ is satisfiable

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$, and let $\bar{a} \in \mathcal{N}$ realize Γ. Then

$$\mathrm{tp}^{\mathcal{N}}(\bar{a}/A) \in S_n^{\mathcal{M}}(A) \setminus \bigcup_{i \in I} [\phi_i(\bar{v})]$$

a contradiction

2. if $p \neq q$, there is a formula ϕ s.t. $\phi \in p$ and $\neg\phi \in q$. Thus $[\phi]$ is a basic clopen set separating p and q.

$\square$

Lemma 4.9. 1. *If $A \subseteq B \subset M$ and $p \in S_n^{\mathcal{M}}(B)$, let $p|A$ be the set of $\mathcal{L}_A$-formulas in p. Then $p|A \in S_n^{\mathcal{M}}(A)$ and $p \mapsto p|A$ is a continuous map from $S_n^{\mathcal{M}}(B)$ onto $S_n^{\mathcal{M}}(A)$*

2. *if $f : \mathcal{M} \to \mathcal{N}$ is an elementary embedding and $p \in S_n^{\mathcal{M}}(A)$, let*

$$f(p) = \{\phi(\bar{v}, f(\bar{a})) : \phi(\bar{v}, \bar{a}) \in p\}$$

Then $f(p) \in S_n^{\mathcal{N}}(f(A))$ and $p \mapsto f(p)$ is continuous

3. *if $f : A \to \mathcal{N}$ is partial elementary, then $S_n^{\mathcal{M}}(A)$ is homeomorphic to $S_n^{\mathcal{N}}(f(A))$*

Proof. 1. Because $p|A \cup \mathrm{Th}_A(\mathcal{M}) \subseteq p \cup \mathrm{Th}_B(\mathcal{M})$, $p|A \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. Because $p|A$ is the set of all $\mathcal{L}_A$-formulas in p, $p|A$ is complete. If ϕ is an $\mathcal{L}_A$-formula, then

$$\{p \in S_n^{\mathcal{M}}(B) : \phi \in p\} = [\phi]$$

Thus the map is continuous. Here we consider the basic open sets.

if $q \in S_n^{\mathcal{M}}(A)$, there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N$ realizing q. Then $p = \mathrm{tp}^{\mathcal{N}}(\bar{a}/B) \in S_n^{\mathcal{M}}(B)$ and $p|A = q$. Thus the restriction map is surjective

2. Suppose Δ is a finite subset of $f(p)$. Say

$$\Delta = \{\phi_1(\bar{v}, f(\bar{a}), \dots, \phi_m(\bar{v}, f(\bar{a})))\}$$

where $\phi_1(\bar{v}, \bar{a}), \dots, \phi_m(\bar{v}, \bar{a}) \in p$. Because $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable,

$$\mathcal{M} \vDash \exists \bar{v} \bigwedge_{i=1}^{m} \phi_i(\bar{v}, \bar{a})$$

Because f is elementary

$$\mathcal{N} \vDash \exists \bar{v} \bigwedge_{i=1}^{m} \phi_i(\bar{v}, \bar{a})$$

and $f(p) \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is satisfiable. $f(p)$ is complete since $\mathfrak{M} \equiv \mathfrak{N}$.

Because

$$\{p \in S_n^{\mathcal{M}}(A) : \phi(\bar{v}, f(\bar{a})) \in f(p)\} = [\phi(\bar{v}, \bar{a})]$$

$p \mapsto f(p)$ is continuous

3. since we map onto $f(A)$.

□

Definition 4.10. We say that $p \in S_n^{\mathcal{M}}(A)$ is **isolated** if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$

Proposition 4.11. *Let $p \in S_n^{\mathcal{M}}(A)$. The following are equivalent*

1. *p is isolated*

2. $\{p\} = [\phi(\bar{v})]$ *for some* $\mathcal{L}_A$*-formula* $\phi(\bar{v})$*. We say that* $\phi(\bar{v})$ *isolates* p

3. *There is an* $\mathcal{L}_A$*-formula* $\phi(\bar{v}) \in p$ *s.t. for all* $\mathcal{L}_A$*-formulas* $\psi(\bar{v})$*,* $\psi(\bar{v}) \in p$ *iff*

$$\mathrm{Th}_A(\mathcal{M}) \models \phi(\bar{v}) \to \psi(\bar{v})$$

Proof. $1 \to 2$. If X is open, then

$$X = \bigcup_{i \in I} [\phi_i]$$

for some collection of formulas $\{\phi_i : i \in I\}$. If $\{p\}$ is open, then $\{p\} = [\phi]$ for some formula ϕ

$2 \to 3$. $\square$

4.1.2 Examples

Dense Linear Order.

Let $\mathcal{L} = \{<\}$. Let $\mathcal{M} = (M, <)$ be a dense linear order without endpoints and let $A \subseteq M$. Let $p \in S_1^{\mathcal{M}}(A)$. If $a \in A$, then because p is a complete type, exactly one of the formulas $v = a$, $v < a$, or $v > a$ is in p.

case 1: p is realized in A

$v = a \in p$ for some $a \in A$. In this case, $p = \{\psi(v) : \mathcal{M} \models \psi(a)\}$ and p is isolated by the formula $v = a$.

case 2: Otherwise

Let $L_p = \{a \in A : a < v \in p\}$ and $U_p = \{a \in A : v < a \in p\}$. If $a < v, v < b \in p$, because $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable, $a < b$. Thus, $a < b$ for $a \in L_p$ and $b \in U_p$ and L_p and U_p determine a cut in the ordering $(A, <)$

Also note that if A is the disjoint union of L and U where $a < b$ for $a \in L$ and $b \in U$, then $\mathrm{Th}_A(\mathcal{M}) \cup \{a < v : a \in L\} \cup \{v < b : b \in U\}$ is satisfiable. Thus, there is a type p with $L_p = L$ and $U_p = U$.

We claim that the cut completely determines p; that is,

$$\{p\} = \bigcap_{a \in L_p} [a < v] \cap \bigcap_{a \in U_p} [v < b]$$

Suppose that $q \neq p$, $L_p = L_q$ and $U_p = U_q$. Because the only atomic formulas are $u = v$ and $u < v$, p and q determine the same cut in A, and they contain the same atomic formulas. Because quantifier-free formulas are Boolean combinations of atomic formulas, p and q contain the same quantifier-free formulas. Because every formula is equivalent to a quantifier-free formula, $p = q$

Using the identification between types and cuts, we can give a complete description of all types in $S_1^{\mathbb{Q}}(\mathbb{Q})$

For $a \in \mathbb{Q}$, let p_a be the unique type containing $v = a$.

Let $p_{+\infty}$ be the unique type p with $L_p = \infty$ and $U_p = \emptyset$, and let $p_{-\infty}$ be the unique type p with $L_p = \emptyset$ and $U_p = \mathbb{Q}$. For $r \in \mathbb{R} \setminus \mathbb{Q}$, let p_r be the unique type p with $L_p = \{a \in \mathbb{Q} : a < r\}$ and $U_p = \{b \in \mathbb{Q} : r < b\}$. For $c \in \mathbb{Q}$, let p_{c^+} be the unique type p with $L_p = \{a \in \mathbb{Q} : a \leq c\}$ and $U_p = \{b \in \mathbb{Q} : c < b\}$ and p_{c^-} be the unique type p with $L_p = \{a \in \mathbb{Q} : a < c\}$ and $U_p = \{b \in \mathbb{Q} : c \leq b\}$. These are all possible types. Note in particular that $\left|S_1^{\mathbb{Q}}(\mathbb{Q})\right| = 2^{\aleph_0}$

We return to the general case where $\mathcal{M} \models \mathsf{DLO}$ and $A \subseteq M$ is nonempty. Aside from the types realized by elements of A, what types in $S_1^{\mathcal{M}}$ are isolated? Suppose that L_p has a largest element a and U_p has a smallest element b. Then $p \in [a < v < b]$. Moreover, $\mathrm{Th}_A(\mathcal{M}) \models a < v < b \to c < v < d$ for all $c \in L_p$ and $d \in U_p$. Thus $a < v < b$ isolates p. Similarly, if $U_p = \emptyset$ and L_p has a greatest element a, then $a < v$ isolates p, and if U_p has a smallest element b and $L_p = \emptyset$, then $v < b$ isolates p.

We claim that these are the only possibilities. For example, suppose that $U_p \neq \emptyset$ and has no least element. Suppose that $\phi(v)$ isolates p. Because U_p and L_p determine p,

$$\mathrm{Th}_A(\mathcal{M}) \cup \{a < v : a \in L_p\} \cup \{v < b : v \in U_p\} \models \phi(v)$$

Thus we can find $a \in L_p \cup \{-\infty\}$ and $b \in U_p$ s.t.

$$\mathrm{Th}_A(\mathcal{M}) \models \{a < v < b\} \to \phi(v)$$

There is $c \in U_p$ s.t. $c < b$. Because $a < c < b$, $\mathcal{M} \models \phi(c)$. But then the type containing $v = c$ is in $[\phi(v)]$ contradicting the fact that $[\phi(v)]$ isolates p.

Proposition 4.12. *Let $\mathcal{M} \models$ DLO and let $A \subseteq M$ be nonempty. Types in $S_1^{\mathcal{M}}(A)$ not realized by elements of A correspond to cuts in the ordering of A. A nonrealized type p is nonisolated if either $U_p \neq \emptyset$ has no least element or $L_p \neq \emptyset$ has no greatest element*

Algebraically Closed Fields.

Let $K \models \mathsf{ACF}$, and let $A \subseteq K$. We first argue that, W.L.O.G., we may assume that A is a field. Let k be a subfield of K generated by A. If $p \in S_n^K(k)$, then $p|A \in S_n^K(A)$. We claim that the restriction map is a bijection.

By Lemma 4.9, we know it is surjective. Suppose that $q \in S_n^K(A)$. For $b_1, \dots, b_l \in k$, there are $a_1, \dots, a_m \in A$ s.t. for each i there is $q_i(\overline{X}) \in \mathbb{Z}[X_1, \dots, X_l, \overline{Y}]$ s.t. $b_i = q_i(\bar{a})$.

4.2 Omitting Types and Prime Models

For T an $\mathcal{L}$-theory, we let $S_n(T)$ be the set of all complete n-types p s.t. $p \cup T$ is satisfiable. If T is complete and $\mathcal{M} \models T$, then $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$ since $\mathcal{M} \models \phi$ iff $T \models \phi$. Also, $S_n^{\mathcal{M}}(A) = S_n(\mathrm{Th}_A(\mathcal{M}))$

In particular, $S_n(T)$ is a totally disconnected compact topological space with basic open sets

$$[\phi] = \{p : \phi \in p\}$$

For p a complete type, p is isolated in $S_n(T)$ iff $\{p\} = [\phi]$ for some ϕ

Definition 4.13. Let $\phi(v_1, \dots, v_n)$ be an $\mathcal{L}$-formula s.t. $T \cup \{\phi(\bar{v})\}$ is satisfiable, and let p be an n-type. We say that ϕ **isolates** p if

$$T \models \forall \bar{v}(\phi(\bar{v}) \to \psi(\bar{v}))$$

for all $\psi \in p$.

Proposition 4.14. *If $\phi(\bar{v})$ isolates p, then p is realized in any model of $T \cup \{\exists \bar{v}\ \phi(\bar{v})\}$. In particular, if T is complete, then every isolated type is realized.*

Theorem 4.15 (Omitting Types Theorems)**.** *Let $\mathcal{L}$ be countable language, T an $\mathcal{L}$-theory, and p a (possibly incomplete) nonisolated n-types over $\emptyset$. Then there is a countable $\mathcal{M} \models T$ omitting p.*

Proof. Let $C = \{c_0, c_1, \dots\}$ be countably many new constant symbols, and let $\mathcal{L}^* = \mathcal{L} \cup C$. As in the proof of the Compactness Theorem, we will build $T^* \supseteq T$, a complete $\mathcal{L}^*$-theory with the witness property and build $\mathcal{M} \models T^*$ as in Lemma 2.7. We will arrange the construction s.t. for all $d_1, \dots, d_n \in C$, there is a formula $\phi(\bar{v}) \in p$ s.t. $T^* \models \neg\phi(d_1, \dots, d_n)$. This will ensure that $d_1^{\mathcal{M}}, \dots, d_n^{\mathcal{M}}$ does not realize p. Because every element of M is the interpretation of a constant symbol in C, $\mathcal{M}$ omits p.

We will construct a sequence $\theta_0, \theta_1, \theta_2, \dots$ of $\mathcal{L}^*$-sentences s.t.

$$\models \theta_t \to \theta_s$$

for $t > s$ and $T^* = T \cup \{\theta_i : i = 0, 1, \dots\}$ is a satisfiable extension of T

Let $\phi_0, \phi_1, \phi_2, \dots$ list all $\mathcal{L}^*$-sentences. To ensure that T^* is complete, we will either have

$$\models \theta_{3i+1} \to \phi_i$$

or

$$\models \theta_{3i+1} \to \neg\phi_i$$

If ϕ_i is $\exists v\ \psi(v)$ and $\models \theta_{3i+1} \to \phi_i$, then

$$\models \theta_{3i+2} \to \psi(c)$$

for some $c \in C$. This will ensure that T^* has the witness property. Let $\bar{d}_0, \bar{d}_1, \ldots$ list all n-tuples from C. We will choose θ_{3i+3} to ensure that $\bar{d}_i^{\mathcal{M}}$ does not realize p in the canonical model of T^*

stage 0: Let θ_0 be $\forall x\ x = x$

Suppose that we have constructed θ_s s.t. $T \cup \theta_s$ is satisfiable. There are three cases to consider

stage $s+1 = 3i+1$: (Completeness) If $T \cup \{\theta_s, \phi_i\}$ is satisfiable then θ_{s+1} is $\theta_s \wedge \phi_i$; otherwise θ_{s+1} is $\theta_s \wedge \neg\phi_i$. In either case $T \cup \theta_{s+1}$ is satisfiable. Note that if $\theta_s \wedge \phi_i$ is the case, then $\neg(\theta_s \wedge \neg\phi_i) \equiv \theta_s \to \phi_i$

stage $s+1 = 3i+2$: (witness property) Suppose that ϕ_i is $\exists v\ \psi(v)$ for some formula ψ and $T \models \theta_s \to \phi_i$. In this case we want to find a witness for ψ. Let $c \in C$ be a constant that does not occur in $T \cup \{\theta_s\}$. Because only finitely many constants from C have been used so far, we can always find such a c. Let $\theta_{s+1} = \theta_s \wedge \psi(c)$. If $\mathcal{N} \models T \cup \{\theta_s\}$, then there is $a \in N$ s.t. $\mathcal{N} \models \psi(a)$. By letting $c^{\mathcal{N}} = a$, we have $\mathcal{N} \models \theta_{s+1}$. Thus in this case $T \cup \{\theta_{s+1}\}$ is satisfiable.

If ϕ_i is not of the correct form or $T \not\models \theta_s \to \phi_i$, then let θ_{s+1} be θ_s

stage $s+1 = 3i+3$: (omitting p) Let $\bar{d}_i = (e_1, \ldots, e_n)$. let $\psi(v_1, \ldots, v_n)$ be the $\mathcal{L}$-formula obtained from θ_s by replacing each occurrence of e_i by v_i and then replacing every other constant symbol $c \in C \setminus \{e_0, \ldots, e_n\}$ occuring in θ_s by the variable v_c and putting a $\exists v_c$ quantifier in front. In particular, we get rid of all of the constants in θ_s from C either by replacing them by variables or by quantifying over them. For example, if θ_s is

$$\forall x \exists y cx + e_1 e_2 = y^2 + de_2$$

where c, d, e_1, e_2 are distinct constants in C, then $\psi(v_1, v_2)$ would be

$$\exists v_c \exists v_d \forall x \exists y\ v_c x + v_1 v_2 = y^2 + de_2$$

Because p is nonisolated, there is a formula $\phi(\bar{v}) \in p$ s.t.

$$T \not\models \forall \bar{v}(\psi(\bar{v}) \to \phi(\bar{v})) \tag{$\star$}$$

Let θ_{s+1} be $\theta_s \wedge \neg\phi(\bar{d}_i)$. We must argue that $T \cup \theta_{s+1}$ is satisfiable. By ($\star$) there is $\mathcal{N} \models T$ with $\bar{a} \in N$ s.t.

$$\mathcal{N} \models \psi(\bar{a}) \wedge \neg\phi(\bar{a})$$

We can make $\mathcal{N}$ into a model of θ_{s+1} by interpreting the constants $c \in C \setminus \{e_1, \dots, e_n\}$ as the witnesses to v_c and e_i as a_i

This completes the construction. Let $T^* = T \cup \{\theta_0, \theta_1, \dots\}$. Because $T \cup \{\theta_s\}$ is satisfiable for each s, T^* is satisfiable. If ϕ is any $\mathcal{L}$-sentence, then $\phi = \phi_i$ for some i, and at stage $3i+1$ we ensure that $T^* \models \phi$ or $T^* \models \neg\phi$. Thus, T^* is complete

Also, T^* has the witness property.

If $\mathcal{M}$ is the canonical model of T^* constructed as in Lemma 2.7 we claim that $\mathcal{M}$ omits p. $\square$

The proof can be generalized to omit countably many types at once

Theorem 4.16. *Let $\mathcal{L}$ be a countable language, and let T be an $\mathcal{L}$-theory. Let X be a countable collection of nonisolated types over $\emptyset$. There is a countable $\mathcal{M} \models T$ that omits all of the types $p \in X$*

The assumption of countability of $\mathcal{L}$ is necessary in the Omitting Types Theorem. Suppose that $\mathcal{L}$ is the language with two disjoint sets of constant symbols C and D, where C is uncountable and $|D| = \aleph_0$. Let T be the theory $\{a \neq b : a, b \in C\}$ and p be the type $\{v \neq d : d \in D\}$. Because every model of T is uncountable, there is always an element that is not the interpretation of a constant in D. Thus, every model of T realizes p. On the other hand, if $\phi(v)$ is any $\mathcal{L}$-formula, then, because only countably many constants from D occur in $T \cup \{\phi(v)\}$, there is $d \in D$ s.t. $T \cup \{\phi(d)\}$ is satisfiable. Thus, p is nonisolated

Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ and let PA be the axioms for Peano arithmetic PA. Suppose that $\mathcal{M}, \mathcal{N} \models \mathsf{PA}$. We say that $\mathcal{N}$ is an **end extension** of $\mathcal{M}$ if $N \supset M$ and $a < b$ for all $a \in M$ and $b \in N \setminus M$

Theorem 4.17. *If $\mathcal{M}$ is a countable model of PA, then there is $\mathcal{M} \prec \mathcal{N}$ s.t. $\mathcal{N}$ is a proper end extension of $\mathcal{M}$*

Proof. Consider the language $\mathcal{L}^*$ where we have constant symbols for all elements of M and a new constant symbol c. Let $T = \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c > m : m \in M\}$ and for $a \in M \setminus \mathbb{N}$ let p_a be the type $\{v < a, v \neq m : m \in M\}$. if $\mathcal{N}$ omits each p_a, then $\mathcal{N}$ is an end extension of $\mathcal{M}$. By Theorem 4.16, it suffices to show that each p_a is nonisolated

Suppose that $\phi(v)$ is an $\mathcal{L}^*$ formula isolating p_a. Let $\phi(v) = \theta(v, c)$, where θ is an $\mathcal{L}_M$-formula. Then

$$T \cup \theta(v, c) \models v < a$$

Because $T \cup \{\theta(v, c)\}$ is satisfiable (definition),

$$\mathcal{M} \models \forall x \exists y > x \exists v < a\ \theta(v, y)$$

The Pigeonhole Principle is provable in Peano arithmetic. Thus

$$\mathcal{M} \models [\forall x \exists y > x \exists v < a \theta(v, y)] \to \exists v < a \forall x \exists y > x \theta(v, y) \qquad (\star)$$

Thus there is $m < a$ s.t.

$$\mathcal{M} \models \forall x \exists y > x \theta(m, y)$$

We claim that $T \cup \{\theta(m, c)\}$ is satisfiable. If not, there is $n \in M$ s.t.

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) + c > n \models \neg\theta(m, c)$$

contradicting $(\star)$. Thus $\phi(v)$ does not isolate p_a, a contradiction $\square$

4.2.1 Prime and Atomic Models

We use the Omitting Types Theorem to study small models of a complete theory. For the remainder of this section, we will assume that $\mathcal{L}$ is a countable language and T is a complete $\mathcal{L}$-theory with infinite models

Definition 4.18. We say that $\mathcal{M} \models T$ is a **prime model** of T if whenever $\mathcal{N} \models T$ there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$

Let $T = \mathsf{ACF}_0$. If $K \models \mathsf{ACF}_0$ and F is the algebraic closure of $\mathbb{Q}$, then there is an embedding of F into K. Because ACF_0 is model complete this embedding is elementary. Thus F is a prime model of ACF_0

Consider $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ and let T be $\mathrm{Th}(\mathbb{N})$. If $\mathcal{M} \models T$, then we can view $\mathbb{N}$ as an initial segment of $\mathcal{M}$. We claim that this embedding is elementary. We use the Tarski-Vaught test (Proposition 2.29). Let $\phi(v, w_1, \dots, w_m)$ be an $\mathcal{L}$-formula and let $n_1, \dots, n_m \in \mathbb{N}$ s.t. $\mathcal{M} \models \exists v\ \phi(v, \bar{n})$. Let ψ be the $\mathcal{L}$-sentence

$$\exists v\ \phi(v, \underbrace{1 + \cdots + 1}_{n_1\text{-times}}, \dots, \underbrace{1 + \cdots + 1}_{n_m\text{-times}})$$

Then $\mathcal{M} \models \psi$ and $\mathbb{N} \models \psi$ because $\mathcal{M} \equiv \mathbb{N}$. But then, for some $s \in \mathbb{N}$

$$\mathbb{N} \models \phi(s, \underbrace{1 + \cdots + 1}_{n_1\text{-times}}, \dots, \underbrace{1 + \cdots + 1}_{n_m\text{-times}})$$

and

$$\mathbb{N} \models \phi(\underbrace{1 + \cdots + 1}_{s\text{-times}}, \underbrace{1 + \cdots + 1}_{n_1\text{-times}}, \dots, \underbrace{1 + \cdots + 1}_{n_m\text{-times}})$$

Because the latter statement is an $\mathcal{L}$-sentence,

$$\mathcal{M} \models \phi(\underbrace{1 + \cdots + 1}_{s\text{-times}}, \underbrace{1 + \cdots + 1}_{n_1\text{-times}}, \ldots, \underbrace{1 + \cdots + 1}_{n_m\text{-times}})$$

and $\mathcal{M} \models \phi(s, n_1, \ldots, n_m)$. By the Tarski-Vaught test, $\mathcal{N} \prec \mathcal{M}$. Thus $\mathbb{N}$ is a prime model of T

Suppose $\mathcal{M}$ is a prime model of T. Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. If $\bar{a} \in M^n$ realizes $p \in S_n(T)$, then so does $j(\bar{a})$ (definition). If $p \in S_n(T)$ is nonisolated, there is $\mathcal{N}$ s.t. $\mathcal{N}$ omits p. If $\mathcal{M}$ realizes p, then we can elementarily embed $\mathcal{M}$ into $\mathcal{N}$; thus $\mathcal{M}$ must also omit p. In particular, if $\bar{a} \in M^n$, then $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ must be isolated. This leads us to the following definition

Definition 4.19. $\mathcal{M} \models T$ is **atomic** if $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ is isolated for all $\bar{a} \in M^n$

Prime models are atomic

Theorem 4.20. *Let $\mathcal{L}$ be a countable language and let T be a complete $\mathcal{L}$-theory with infinite models. Then $\mathcal{M} \models T$ is prime iff it is countable and atomic*

Proof. $\Rightarrow$. Because $\mathcal{L}$ is countable, T has a countable model. Thus, the prime model must be countable

$\Leftarrow$. Let $\mathcal{M}$ be countable and atomic. Let $\mathcal{N} \models T$. We must construct an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Let $m_0, m_1, \ldots, m_n, \ldots$ be an enumeration of M. For each i, let $\theta_i(v_0, \ldots, v_i)$ isolate the type of $(m_0, \ldots, m_i)$. We will build $f_0 \subseteq f_1 \subseteq \ldots$ a sequence of partial elementary maps from $\mathcal{M}$ into $\mathcal{N}$ where the domain of f_i is $\{m_0, \ldots, m_{i-1}\}$. Then $f = \bigcup_{i=0}^{\infty} f_i$ is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$

Let $f_0 = \emptyset$. Because $\mathcal{M} \equiv \mathcal{N}$, f_0 is partial elementary

Given f_s, let $n_i = f(m_i)$ for $i < s$. Because $\theta_s(m_0, \ldots, m_s)$ and f_s is partial elementary

$$\mathcal{N} \models \exists v \theta_s(n_0, \ldots, n_{s-1}, v)$$

Let $n_s \in N$ s.t. $\mathcal{N} \models \theta_s(n_0, \ldots, n_s)$. Because θ_s isolates $\mathrm{tp}^{\mathcal{M}}(m_0, \ldots, m_s)$

$$\mathrm{tp}^{\mathcal{M}}(m_0, \ldots, m_s) = \mathrm{tp}^{\mathcal{N}}(n_0, \ldots, n_s)$$

Thus $f_{s+1} = f_s \cup \{(m_s, n_s)\}$ is a partial elementary map $\square$

Lemma 4.21. *Suppose that $(\bar{a}, \bar{b}) \in M^{m+n}$ realizes an isolated type in $S_{m+n}(T)$. Then $\bar{a}$ realizes an isolated type in $S_m(T)$. Indeed if $A \subseteq M$ and $(\bar{a}, \bar{b}) \in M^{m+n}$ realizes an isolated type in $S^{\mathcal{M}}_{m+n}(A)$, then $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A)$ is isolated*

Proof. Let $\phi(\bar{v}, \overline{w})$ isolate $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$. We claim that $\exists \overline{w} \ \ \phi(\bar{v}, \overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/A)$. Let $\psi(\bar{v})$ be any $\mathcal{L}_A$-formula s.t. $\mathcal{M} \vDash \psi(\bar{a})$. We must show that

$$\text{Th}_A(\mathcal{M}) \vDash \forall \bar{v}(\exists \overline{w} \phi(\bar{v}, \overline{w}) \to \psi(\bar{v}))$$

Suppose no, then there is $\bar{c} \in M^m$ s.t.

$$\mathcal{M} \vDash \exists \overline{w} \phi(\bar{c}, \overline{w}) \wedge \neg \psi(\bar{c})$$

Let $\bar{d} \in M^n$ s.t. $\mathcal{M} \vDash \phi(\bar{c}, \bar{d}) \wedge \neg \psi(\bar{c})$. Because $\phi(\bar{v}, \overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{d}/A)$

$$\text{Th}_A(\mathcal{M}) \vDash \forall \bar{v} \forall \overline{w}(\phi(\bar{v}, \overline{w}) \to \psi(\bar{v}))$$

This is a contradiction because

$$\psi(\bar{v}) \in \text{tp}^{\mathcal{M}}(\bar{a}/A) \subset \text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$$

$\square$

Definition 4.22. The isolated types are **dense** in T if every consistent L-formula $\psi(x_1, \dots, x_n)$ belongs to an isolated type $p(x_1, \dots, x_n) \in S_n(T)$

Theorem 4.23. *Let $\mathcal{L}$ be a countable language and let T be a complete $\mathcal{L}$-theory with infinite models. Then, the following are equivalent*

1. *T has a prime model*
2. *T has an atomic model $\mathcal{M}$*
3. *the isolated types in $S_n(T)$ are dense for all n (in the sense of topology)*

Proof. $2 \to 3$. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula s.t. $[\phi(\bar{v})]$ is a nonempty open set in $S_n(T)$. We must show that $[\phi(\bar{v})]$ contains an isolated type

Let $\mathcal{M} \vDash T$ be atomic. Because T is complete and $T \cup \{\phi(\bar{v})\}$ is satisfiable, $T \vDash \exists \bar{v} \phi(\bar{v})$. Thus there is $\bar{a} \in M^n$ s.t. $\mathcal{M} \vDash \phi(\bar{a})$. Then $\text{tp}^{\mathcal{M}}(\bar{a}) \in [\phi]$ and because $\mathcal{M}$ is atomic, $\text{tp}^{\mathcal{M}}(\bar{a})$ is isolated.

$3 \to 2$. (From tent) A structure $\mathfrak{M}_0$ is atomic iff for all n the set

$$\Sigma_n(x_1, \dots, x_n) = \{\neg \varphi(x_1, \dots, x_n) \mid \varphi(x_1, \dots, x_n) \text{ complete}\}$$

is not realised in $\mathfrak{M}_0$. By Theorem 4.16 it is enough to show that the $\Sigma_n(x_1, \dots, x_n)$ are not isolated in T. This is the case iff for every consistent $\psi(x_1, \dots, x_n)$ there is a complete formula $\varphi(x_1, \dots, x_n)$ with $T \nvDash \forall \bar{x}(\psi(\bar{x}) \to \neg \varphi(\bar{x}))$. We conclude that Σ_n is not isolated iff the isolated n-types are dense $\square$

Theorem 4.24. *Suppose that T is a complete theory in a countable language and $A \subseteq M \models T$ is countable. If $\left|S_n^{\mathcal{M}}(A)\right| < 2^{\aleph_0}$, then*

1. *the isolated types in $S_n^{\mathcal{M}}(A)$ are dense*
2. $\left|S_n^{\mathcal{M}}(A)\right| \leq \aleph_0$

In particular, if $|S_n(T)| < 2^{\aleph_0}$, then T has a prime model

Proof. 1. We first prove that the isolated types are dense. Suppose that there is a formula ϕ s.t. $[\phi]$ contains no isolated types. Because ϕ does not isolate a type, we can find ψ s.t. $[\phi \wedge \psi] \neq \emptyset$ and $[\phi \wedge \neg\psi] \neq \emptyset$. Because $[\phi]$ does not contain an isolated type, neither does $[\phi \wedge \pm\psi]$

We build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ s.t.

(a) each $[\phi_\sigma]$ is nonempty but contains no isolated types

(b) if $\sigma \subset \tau$, then $\phi_\tau \models \phi_\sigma$

(c) $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$

Let $\phi_\emptyset = \phi$ for some formula ϕ where $[\phi]$ contains no isolated types. Suppose that $[\phi_\sigma]$ is nonempty but contains no isolated types. As above, we can find ψ s.t. $[\phi_\sigma \wedge \psi]$ and $[\phi_\sigma \wedge \neg\psi]$ are both nonempty and neither contains an isolated type. Let $\phi_{\sigma,0} = \phi \wedge \psi$ and $\phi_{\sigma,1} = \phi \wedge \neg\psi$

Let $f : \omega \to 2$. Because

$$[\phi_{f|0}] \supseteq [\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \cdots$$

and $S_n^{\mathcal{M}}(A)$ is compact, there is

$$p_f \in \bigcap_{n=0}^{\infty} [\phi_{f|n}]$$

If $g \neq f$, we can find m s.t. $f|m = g|m$ but $f(m) \neq g(m)$. By construction, $\phi_{f|m+1} \models \neg\phi_{g|m+1}$; thus $p_f \neq p_g$. Because $f \mapsto p_f$ is a one-to-one function from 2^ω into $S_n^{\mathcal{M}}(A)$, $\left|S_n^{\mathcal{M}}(A)\right| = 2^{\aleph_0}$

2. Suppose that $\left|S_n^{\mathcal{M}}(A)\right| > \aleph_0$. We claim that $\left|S_n^{\mathcal{M}}(A)\right| = 2^{\aleph_0}$. Because $\left|S_n^{\mathcal{M}}(A)\right| > \aleph_0$, and there are only countably many $\mathcal{L}_A$-formulas, there is a formula ϕ s.t. $|[\phi]| > \aleph_0$ as $S_n^{\mathcal{M}}(A) = \bigcup[\phi]$.

Claim if $|[\phi]| > \aleph_0$, there is an $\mathcal{L}_A$-formula ψ s.t. $|[\phi \wedge \psi]| > \aleph_0$ and $|[\phi \wedge \neg\psi]| > \aleph_0$

Suppose not. Let $p = \{\psi(\bar{v}) : |[\phi \wedge \psi]| > \aleph_0\}$. p is complete since $[\phi] = [\phi \wedge \psi] \cup [\phi \wedge \neg\psi]$. We claim that p is satisfiable. Suppose $\psi_1, \dots, \psi_m \in p$. Either $\psi_1 \wedge \cdots \wedge \psi_m \in p$, in which case $\{\psi_1, \dots, \psi_m\} \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable, or $\neg\psi_1 \vee \cdots \vee \neg\psi_m \in p$. Because

$$[\neg\psi_1 \vee \cdots \vee \neg\psi_m] = [\neg\psi_1] \cup \cdots \cup [\neg\psi_m]$$

We must have $|[\phi \wedge \neg\psi_i]| > \aleph_0$ for some i, a contradiction. Thus $p \in S_n^{\mathcal{M}}(A)$. Moreover, if $\psi \notin p$, then $[\phi \wedge \psi] \leq \aleph_0$. But

$$[\phi] = \bigcup_{\psi \notin p} [\phi \wedge \psi] \cup \{p\}$$

(Consider $[\phi] \setminus \bigcup_{\psi \notin p} [\phi \wedge \psi]$, which is exactly p) Because $[\phi]$ is the union of at most $\aleph_0$ sets each of size at most $\aleph_0$, we have $|[\phi]| \leq \aleph_0$, a contradiction.

We build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ s.t.

(a) if $\sigma \subset \tau$, then $\phi_\tau \vDash \phi_\sigma$

(b) $\phi_{\sigma,i} \vDash \neg\phi_{\sigma,1-i}$

(c) $|[\phi_\sigma]| > \aleph_0$

Let $\phi_\emptyset = \phi$ for some ϕ with $|[\phi]| > \aleph_0$. Given ϕ_σ where $|[\phi_\sigma]| > \aleph_0$, by the chain we can find ψ s.t. $|[\phi_\sigma \wedge \psi]| > \aleph_0$ and $|[\phi_\sigma \wedge \neg\psi]| > \aleph_0$. Let $\phi_{\sigma,0} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma,1} = \phi_\sigma \wedge \neg\psi$

As in 1, for each $f \in 2^\omega$ there is a

$$p_f \in \bigcap_{m=0}^{\infty} [\phi_{f|m}]$$

and if $f \neq g$, then $p_f \neq p_g$. Thus $\left|S_n^{\mathcal{M}}(A)\right| = 2^{\aleph_0}$

□

4.2.2 Countable Homogeneous Models

Our next goal is to show that prime models are unique up to isomorphism

Definition 4.25. Let κ be an infinite cardinal. We say that $\mathcal{M} \vDash T$ is **κ-homogeneous** if whenever $A \subset M$ with $|A| < \kappa$, $f : A \to M$ is a partial elementary map, and $a \in M$, there is $f^* \supseteq f$ s.t. $f^* : A \cup \{a\} \to M$ is partial elementary

We say that $\mathcal{M}$ is **homogeneous** if it is $|M|$-homogeneous

In homogeneous models, partial elementary maps are just restrictions of automorphisms

Proposition 4.26. *Suppose that $\mathcal{M}$ is homogeneous, $A \subset M$, $|A| < |M|$, and $f : A \to M$ is a partial elementary map. Then there is an automorphism σ of $\mathcal{M}$ with $\sigma \supseteq f$.*

In particular, if $\mathcal{M}$ is homogeneous and $\bar{a}, \bar{b} \in M^n$ realize the same n-type, then there is an automorphism σ of $\mathcal{M}$ with $\sigma(\bar{a}) = \bar{b}$

Proof. Let $|M| = \kappa$ and let $(a_\alpha : \alpha < \kappa)$ be an enumeration of M. We build a sequence of partial elementary maps $(f_\alpha : \alpha < \kappa)$ extending f with $f_\alpha \subseteq f_\beta$ for $\alpha < \beta$ s.t. a_α is in the domain and image of $f_{\alpha+1}$ and $|f_{\alpha+1}| \leq |f_\alpha| + 2 < \kappa$. Then $\sigma = \bigcup_{\alpha<\kappa} f_\alpha$ is the desired automorphism. Let $f_0 = f$.

If α is a limit ordinal and f_β is partial elementary with

$$|f_\beta| \leq |A| + |\beta| + \aleph_0 < \kappa$$

for all $\beta < \alpha$, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then f_α is partial elementary and

$$|f_\alpha| \leq |\alpha|(|A| + |\alpha| + \aleph_0) \leq |A| + |\alpha| + \aleph_0 < \kappa$$

Given f_α with $|f_\alpha| < \kappa$, because $\mathcal{M}$ is homogeneous, there is $b \in M$ s.t. if $g_\alpha = f_\alpha \cup \{(a_\alpha, b)\}$, then g_α is partial elementary. Note that g_α^{-1} is also partial elementary. Thus there is $c \in M$ s.t. $g_\alpha^{-1} \cup \{(a_\alpha, c)\}$ is partial elementary. Thus $f_{\alpha+1} = g_\alpha \cup \{(c, a_\alpha)\}$ is partial elementary, $|f_{\alpha+1}| \leq |f_\alpha| + 2 \leq |A| + |\alpha| + \aleph_0$ and a_α is in the domain and range of $f_{\alpha+1}$ $\square$

If $\mathcal{M}$ is homogeneous and $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$, then $\bar{a} \mapsto \bar{b}$ is a partial elementary map that must extend to an automorphism

Lemma 4.27. *If $\mathcal{M}$ is atomic, then $\mathcal{M}$ is $\aleph_0$-homogeneous. In particular, countable atomic models are homogeneous*

Proof. Suppose that $\bar{a} \mapsto \bar{b}$ is elementary and $c \in M$. Let $\phi(\bar{v}, w)$ isolate $\mathrm{tp}^{\mathcal{M}}(\bar{a}, c)$. Because $\mathcal{M} \models \exists w\, \phi(\bar{a}, w)$ and $\bar{a} \mapsto \bar{b}$ is elementary, $\mathcal{M} \models \exists w\, \phi(\bar{b}, w)$. Suppose that $\mathcal{M} \models \phi(\bar{b}, d)$. Because $\phi(\bar{v}, w)$ isolates a type, $\mathrm{tp}^{\mathcal{M}}(\bar{a}, c) = \mathrm{tp}^{\mathcal{M}}(\bar{b}, d)$. Thus $\bar{a}, c \mapsto \bar{b}, d$ is elementary $\square$

Theorem 4.28. *Let T be a complete theory in a countable language. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable homogeneous models of T and $\mathcal{M}$ and $\mathcal{N}$ realize the same types in $S_n(T)$ for $n \geq 1$. Then $\mathcal{M} \cong \mathcal{N}$.*

Proof. We build an isomorphism $f : \mathcal{M} \to \mathcal{N}$ by a back-and-forth argument. We will build $f_0 \subset f_1 \subset \cdots$ a sequence of partial elementary maps with finite domain, and let $f = \bigcup_{i=0}^{\infty} f_i$. Let $a_0, a_1, \ldots$ enumerate M and $b_0, b_1, \ldots$ enumerate N. We will ensure that $a_i \in \mathrm{dom}(f_{2i+1})$ and $b_i \in \mathrm{im}(f_{2i+2})$.

stage 0: Let $f_0 = \emptyset$. Because T is complete f_0 is partial elementary

We inductively assume that f_s is partial elementary. Let $\bar{a}$ be the domain of f_s and $\bar{b} = f_s(\bar{a})$

stage $s+1 = 2i+1$: Let $p = \mathrm{tp}^{\mathcal{M}}(\bar{a}, a_i)$. Because $\mathcal{M}$ and $\mathcal{N}$ realize the same types, we can find $\bar{c}, d \in N$ s.t. $\mathrm{tp}^{\mathcal{N}}(\bar{c}, d) = p$. Note that $\mathrm{tp}^{\mathcal{N}}(\bar{c}) = \mathrm{tp}^{\mathcal{M}}(\bar{a})$ by choice of $\bar{c}$ and $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{N}}(\bar{b})$ because f_s is partial elementary. Thus $\mathrm{tp}^{\mathcal{N}}(\bar{c}) = \mathrm{tp}^{\mathcal{N}}(\bar{b})$. Because $\mathcal{N}$ is homogeneous, there is $e \in N$ s.t. $\mathrm{tp}^{\mathcal{N}}(\bar{b}, e) = \mathrm{tp}^{\mathcal{N}}(\bar{c}, d) = p$. Thus $f_{s+1} = f_s \cup \{(a_i, e)\}$ is partial elementary with a_i in the domain

stage $s+1 = 2i+2$: the same □

Corollary 4.29. *Let T be a complete theory in a countable language. If $\mathcal{M}$ and $\mathcal{N}$ are prime models of T, then $\mathcal{M} \cong \mathcal{N}$*

Proof. By Theorem 4.20, $\mathcal{M}$ and $\mathcal{N}$ are atomic. Because the types in $S_n(T)$ realized in an atomic model are exactly the isolated types, $\mathcal{M}$ and $\mathcal{N}$ realize the same types. By Lemma 4.27, countable atomic models are homogeneous. Thus by Theorem 4.28 $\mathcal{M} \cong \mathcal{N}$ □

4.2.3 Prime Model Extensions of ω-Stable Theories

Suppose that $\mathcal{M} \models T$ and $A \subseteq M$. We say that $\mathcal{M}$ is **prime over** A if whenever $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary, there is an elementary $f^* : \mathcal{M} \to \mathcal{N}$ extending f

Let L be any linear order. We build $L^* \models \mathsf{DLO}$ prime over L as follows. If L has a least element a, add a copy of $\mathbb{Q}$ below a. If L has a greatest element b, add a copy of $\mathbb{Q}$ above b. If $c, d \in L$ with $c < d$ but there are no element of L between c and d, add a copy of $\mathbb{Q}$ between c and d. We add no new elements. Then $L^* \models \mathsf{DLO}$ and that if $f : L \to \mathcal{M} \models \mathsf{DLO}$, then f extends to $f^* : L^* \to \mathcal{M}$. Because DLO has quantifier elimination, it is model-complete and $f^*$is elementary

For ACF, if R is any integral domain and F is the algebraic closure of the fraction field of R, then F is prime over R and any embedding of R into an algebraically closed field K extends to F. Because ACF is model-complete, this map is elementary

Definition 4.30. Let T be a complete theory in a countable language, and let κ be an infinite cardinal. We say that T is **κ-stable** if whenever $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \kappa$, then $\left|S_n^{\mathcal{M}}(A)\right| = \kappa$

We say that $\mathcal{M}$ is **κ-stable** if $\mathrm{Th}(\mathcal{M})$ is κ-stable

By Corollary **??** ACF is ω-stable. On the other hand, $\left|S_1^{\mathbb{Q}}(\mathbb{Q})\right| = 2^{\aleph_0}$ so DLO is not ω-stable

Theorem 4.31. *Let T be a complete theory in a countable language. If T is ω-stable, then T is κ-stable for all infinite cardinals κ*

Proof. Suppose that $\mathcal{M} \models T$, $A \subseteq M$, $|A| = \kappa$ and $\left|S_n^{\mathcal{M}}(A)\right| > \kappa$. Because there are only κ formulas with parameters from A, there is some $\mathcal{L}_A$-formula $\phi_\emptyset(\bar{v})$ s.t. $|[\phi_\emptyset]| > \kappa$ as $S_n^{\mathcal{M}}(A) = \bigcup[\phi]$ where ϕ is consistent with $\mathrm{Th}_A(\mathcal{M})$. The argument from Theorem 4.24 (2) can be extended to show that if $|[\phi]| > \kappa$ there is an $\mathcal{L}_A$-formula ψ s.t. $|[\phi \wedge \psi]| > \kappa$ and $|[\phi \wedge \neg\psi]| > \kappa$

Then we build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ s.t.

1. if $\sigma \subset \tau$ then $\phi_\tau \models \phi_\sigma$
2. $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$
3. $|[\phi_\sigma]| > \kappa$

Let A_0 be set of all parameters from A occurring in any formula ϕ_σ. A_0 is countable. Arguing as in Theorem 4.24 (2) $\left|S_n^{\mathcal{M}}(A_0)\right| = 2^{\aleph_0}$, contradicting the ω-stability of T □

Remark. As long as we make some property P coherent for ϕ, $\phi \wedge \neg\psi$ and $\phi \wedge \psi$, we can use the technique to make a ω-length tree and get something $\left|S_n^{\mathcal{M}}(A)\right| = 2^{\aleph_0}$

Proposition 4.32. *Let T be a complete theory in a countable language. If T is ω-stable, then for all $\mathcal{M} \models T$ and $A \subseteq M$, the isolated types in $S_n^{\mathcal{M}}(T)$ are dense*

Proof. Suppose not. We can build a binary tree of formulas as in Theorem 4.24 (1). As in Theorem 4.31, we can find a countable $A_0 \subseteq A$ s.t. all parameters come from A_0. But then $\left|S_n^{\mathcal{M}}(A_0)\right| = 2^{\aleph_0}$, contradicting the ω-stability □

Theorem 4.33. *Suppose that T is ω-stable. Let $\mathcal{M} \models T$ and $A \subseteq M$. There is $\mathcal{M}_0 \prec \mathcal{M}$, a prime model extension of A. Moreover, we can choose $\mathcal{M}_0$ s.t. every element of $\mathcal{M}_0$ realizes an isolated type over A*

Proof. We will find an ordinal δ and build a sequence of sets $(A_\alpha : \alpha \leq \delta)$ where $A_\alpha \subseteq M$ and

1. $A_0 = A$
2. if α is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$
3. if no element of $M \setminus A_\alpha$ realizes an isolated type over A_α, we stop and let $\delta = \alpha$; otherwise, pick a_α realizing an isolated type over A_α, and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with universe A_δ

Claim 1 $\mathcal{M}_0 \prec \mathcal{M}$.

We apply the Tarski-Vaught test. Suppose that $\mathcal{M} \models \exists v\phi(v, \bar{a})$, where $\bar{a} \in A_\delta$. By Proposition 4.32, the isolated types in $S^{\mathcal{M}}(A_\delta)$ are dense. Thus, there is $b \in M$ s.t. $\mathcal{M} \models \phi(b, \bar{a})$ and $\text{tp}^{\mathcal{M}}(b/A_\delta)$ is isolated. By Choice of δ, $b \in A_\delta$. Thus by Proposition 2.29, $\mathcal{M}_0 \prec \mathcal{M}$

Claim 2 $\mathcal{M}_0$ is a prime model extension of A

Suppose that $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary. We show by induction that there are $f = f_0 \subset \cdots \subset f_\alpha \subset \cdots \subset f_\delta$ where $f_\alpha : A_\alpha \to \mathcal{N}$ is elementary

if α is a limit ordinal, we let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$

Given $f_\alpha : A_\alpha \to \mathcal{N}$ partial elementary, $\phi(v, \bar{a})$ isolate $\text{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)$. □

Lemma 4.34. *Suppose that $A \subseteq B \subseteq \mathcal{M} \models T$ and every $\bar{b} \in B^m$ realizes an isolated type in $S^{\mathcal{M}}_m(A)$. Suppose that $\bar{a} \in M^n$ realizes an isolated type in $S^{\mathcal{M}}_n(B)$. Then $\bar{a}$ realizes an isolated type in $S^{\mathcal{M}}_n(A)$*

Proof. Let $\phi(\bar{v}, \overline{w})$ be an $\mathcal{L}$-formula and $\bar{b} \in B^m$ s.t. $\phi(\bar{v}, \bar{b})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$. Let $\theta(\overline{w})$ be an $\mathcal{L}_A$-formula isolating $\text{tp}^{\mathcal{M}}(\bar{b}/A)$. We first claim that $\phi(\bar{v}, \overline{w}) \wedge \theta(\overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$

Suppose that $\mathcal{M} \models \psi(\bar{a}, \bar{b})$. Because $\phi(\bar{v}, \bar{b})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$

$$\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}, \bar{b}) \to \psi(\bar{v}, \bar{b}))$$

Thus, because $\theta(\overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{b}/A)$

$$\text{Th}_A(\mathcal{M}) \models \forall \overline{w}(\theta(\overline{w}) \to \forall \bar{v}(\phi(\bar{v}, \overline{w}) \to \psi(\bar{v}, \overline{w})))$$

and

$$\text{Th}_A(\mathcal{M}) \models \forall \overline{w} \forall \bar{v}(\theta(\overline{w}) \wedge \phi(\bar{v}, \overline{w}) \to \psi(\bar{v}, \overline{w}))$$

as desired

Because $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$ is isolated, so is $\text{tp}^{\mathcal{M}}(\bar{a}/A)$ by Lemma 4.21 □

For ω-stable theories (indeed, for theories that are κ-stable for some κ), prime model extensions are unique

Theorem 4.35. *Let T be ω-stable. Suppose that $\mathcal{M} \models T$ and $\mathcal{N} \models T$ are prime model extensions of A and $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$. Then there is $f : \mathcal{M} \to \mathcal{N}$, an isomorphism fixing A.*

4.3 Saturated and Homogeneous Models

Assume that T is a complete theory with infinite models in a countable language $\mathcal{L}$

Definition 4.36. Let κ be an infinite cardinal. $\mathcal{M} \models T$ is **κ-saturated** if for all $A \subseteq M$, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$

$\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated

Proposition 4.37. *Let $\kappa \geq \aleph_0$. TFAE:*

1. *$\mathcal{M}$ is κ-saturated*
2. *if $A \subseteq M$ with $|A| < \kappa$ and p is a (possibly incomplete) n-type over A, then p is realized in $\mathcal{M}$*
3. *if $A \subseteq M$ with $|A| < \kappa$ and $p \in S_1^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$*

Proof. $3 \to 1$. Induction on n. Let $p \in S_n^{\mathcal{M}}(A)$. Let $q \in S_{n-1}^{\mathcal{M}}(A)$ be the type $\{\phi(v_1, \dots, v_{n-1}) : \phi \in p\}$. By induction, $\mathcal{M} \models q(\bar{a})$ for some $\bar{a}$. Let $r \in S_1^{\mathcal{M}}(A \cup \{a_1, \dots, a_{n-1}\})$ be the type $\{\psi(\bar{a}, w) : \psi(v_1, \dots, v_n) \in p\}$. Hence we can realize r by some b. Then $(\bar{a}, b)$ realizes p □

Homogeneity is a weak form of saturation

Proposition 4.38. *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is κ-homogeneous*

Proof. Suppose that $A \subseteq M$, $|A| < \kappa$ and $f : A \to M$ is partial elementary. Let $b \in M \setminus A$. Let

$$\Gamma = \{\phi(v, f(\bar{a})) : \bar{a} \in A^m \wedge \mathcal{M} \models \phi(b, \bar{a})\}$$

If $\phi(v, f(\bar{a})) \in \Gamma$, then $\mathcal{M} \models \exists v\ \phi(v, \bar{a})$ and hence, because f is partial elementary, $\mathcal{M} \models \exists v\ \phi(v, f(\bar{a}))$. Thus because Γ is closed under conjunctions, Γ is satisfiable (guess by compactness). Because $\mathcal{M}$ is saturated, there is $c \in M$ realizing Γ. Thus $f \cup \{(b, c)\}$ is elementary and $\mathcal{M}$ is κ-homogeneous □

4.3.1 Countably Saturated Models

If $\mathcal{M}$ is $\aleph_0$-saturated, then $\mathcal{M}$ realizes every type in $S_n(T)$

Proposition 4.39. *If $\mathcal{M} \models T$, then $\mathcal{M}$ is $\aleph_0$-saturated iff $\mathcal{M}$ is $\aleph_0$-homogeneous and $\mathcal{M}$ realizes all types in $S_n(T)$*

Proof. Since T is complete, $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$

$\Leftarrow$. Let $\bar{a} \in M^m$ and let $p \in S_n^{\mathcal{M}}(\bar{a})$. Let $q \in S_{n+m}(T)$ be the type $\{\phi(\bar{v}, \overline{w}) : \phi(\bar{v}, \bar{a}) \in p\}$. By assumption, there is $(\bar{b}, \bar{c}) \in M^{n+m}$ realizing q. Because $\mathrm{tp}^{\mathcal{M}}(\bar{c}) = \mathrm{tp}^{\mathcal{M}}(\bar{a})$ as they realize the same type and $\mathcal{M}$ is $\aleph_0$-homogeneous, there is $\bar{d} \in \mathcal{M}$ s.t. $\mathrm{tp}^{\mathcal{M}}(\bar{a}, \bar{d}) = \mathrm{tp}^{\mathcal{M}}(\bar{c}, \bar{b})$. Hence $\bar{d}$ realizes p and $\mathcal{M}$ is $\aleph_0$-saturated $\square$

Corollary 4.40. *If $\mathcal{M}, \mathcal{N} \models T$ are countable saturated models, then $\mathcal{M} \cong \mathcal{N}$*

Proof. Because $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_0$-homogeneous and both realize all types in $S_n(T)$ for all $n < \omega$, by Theorem 4.28 $\mathcal{M} \cong \mathcal{N}$ $\square$

We can extend models to $\aleph_0$-homogeneous models without increasing the cardinality

Proposition 4.41. *Let $\mathcal{M} \models T$. There is $\mathcal{M} \prec \mathcal{N}$ s.t. $\mathcal{N}$ is $\aleph_0$-homogeneous and $|N| = |M|$*

Proof. We first argue that we can find $\mathcal{M} \prec \mathcal{N}_1$ s.t. $|M| = |N_1|$, and if $\bar{a}, \bar{b}, c \in M$ and $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$, then there is $d \in N_1$ s.t. $\mathrm{tp}^{\mathcal{N}_1}(\bar{a}, c) = \mathrm{tp}^{\mathcal{N}_1}(\bar{b}, d)$

Let $(\bar{a}_\alpha, \bar{b}_\alpha, c_\alpha) : \alpha < |M|$ list all tuples $(\bar{a}, \bar{b}, c)$ where $\bar{a}, \bar{b}, c \in M$ and $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$. We build an elementary chain $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_\alpha \prec \cdots$ for $\alpha < |M|$

Let $\mathcal{M}_0 = \mathcal{M}$

If α is a limit ordinal, let $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$

Given $\mathcal{M}_\alpha$, let $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ with $|M_\alpha| = |M_{\alpha+1}|$ s.t. there is $d \in \mathcal{M}_\alpha$ with $\mathrm{tp}^{\mathcal{M}_{\alpha+1}}(\bar{b}, d) = \mathrm{tp}^{\mathcal{M}_{\alpha+1}}(\bar{a}, c)$. First we should note that $\mathrm{tp}^{\mathcal{M}_{\alpha+1}}(\bar{a}, c) = \mathrm{tp}^{\mathcal{M}_\beta}(\bar{a}, c)$ for any $\beta < \alpha + 1$. Let $p(x) = \{\phi(\bar{b}, x) : \phi(\bar{v}, w) \in \mathrm{tp}^{\mathcal{M}}(\bar{a}, c)\}$. $p \cup \mathrm{Th}(\mathcal{M})$ is finitely satisfiable since for each $\phi(\bar{b}, x) \in p(x)$, $\mathcal{M} \models \exists x \phi(\bar{b}, x)$ and p is closed under conjunction. Thus $p(x)$ is a type, thus is realized in an elementary extension of $\mathcal{M}_{\alpha+1}$.

Let $\mathcal{N}_1 = \bigcup_{\alpha<|M|} \mathcal{M}_\alpha$. Because $\mathcal{N}_1$ is a union of $|M|$ models of size $|M|$, $|N_1| = |M|$

We now build $\mathcal{N}_0 \prec \mathcal{N}_1 \prec \cdots$ s.t. $|N_i| = |M|$ and if $\bar{a}, \bar{b}, c \in N$ and $\mathrm{tp}^{\mathcal{N}_i}(\bar{a}) = \mathrm{tp}^{\mathcal{N}_i}(\bar{b})$, then there is $d \in N_{i+1}$ s.t. $\mathrm{tp}^{\mathcal{N}_{i+1}}(\bar{a}, c) = \mathrm{tp}^{\mathcal{N}_{i+1}}(\bar{b}, d)$

Let $\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i$. Clearly $|N| = |M|$ and $\mathcal{N}$ is $\aleph_0$-homogeneous $\square$

Propositions 4.40 and 4.41 allows us to characterize theories with countable saturated models

Theorem 4.42. *T has a countable saturated model iff $|S_n(T)| \leq \aleph_0$ for all n*

Proof. $\Rightarrow$. If T has a countable saturated model $\mathcal{M}$, by Proposition 4.39 $\mathcal{M}$ realizes all types in $S_n(T)$, but there are only countably many possibilities.

$\Leftarrow$. Let $p_0, p_1, \dots$ list all elements of $\bigcup_{n\in\omega} S_n(T)$. Let $\mathcal{M}_0 \vDash T$. Iterating Proposition 4.3, we build $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots$ s.t. $\mathcal{M}_i$ is countable and $\mathcal{M}_{i+1}$ realizes p_i. Thus $\mathcal{M} = \bigcup_{i\in\omega} \mathcal{M}_i$ is countable and contains realizations of all types in $S_n(T)$ for $n < \omega$. By proposition 4.41, there is $\mathcal{M} \prec \mathcal{N}$ s.t. $\mathcal{N}$ is countable and $\aleph_0$-homogeneous. By Corollary 4.40 $\mathcal{N}$ is $\aleph_0$-saturated □

Corollary 4.43.
1. *If T has a countable saturated model, then T has a prime model*
2. *If T has fewer than $2^{\aleph_0}$ countable models, then T has a countable saturated model and a prime model*

Proof.
1. if T has a countable saturated model, then $|S_n(T)|$ is countable for all n. By Theorem 4.24, the isolated types are dense in $S_n(T)$ for all n. Thus, by Theorem 4.23 T has a prime model
2. It suffices to show that $S_n(T)$ is countable for all $n < \omega$. Suppose not. By Theorem 4.24, if $|S_n(T)| > \aleph_0$, then $|S_n(T)| = 2^{\aleph_0}$. Each n-type must be realized in some countable model. Because each countable model realizes only countably many n-types, if there are $2^{\aleph_0}$ n-types, then there must be $2^{\aleph_0}$ nonisomorphic countable models

□

Example 4.1 (Dense Linear Orders)**.** We will show that $(\mathbb{Q}, <)$ is saturated. Suppose $A \subset \mathbb{Q}$ is finite. Suppose that $A = \{a_1, \dots, a_n\}$ where $a_1 < \cdots < a_n$. By the analysis of types in DLO, there are exactly $2m + 1$ types in $S_1(A)$. Each type is isolated by one of the formulas $v = a_i$, $v < a_0$, $a_i < v < a_{i+1}$, or $a_m < v$. Clearly all of these types are realized in $\mathbb{Q}$. Note that in this case $\mathbb{Q}$ is both saturated, atomic and prime.

4.3.2 Existence of Saturated Models

Theorem 4.44. *For all $\mathcal{M}$, there is a κ^+-saturated $\mathcal{M} \prec \mathcal{N}$ with $|N| \leq |M|^\kappa$*

We need $|M|^\kappa$ since $\kappa^+ \leq 2^\kappa$.

Proof. **Claim** For any $\mathcal{M}$ there is $\mathcal{M} \prec \mathcal{M}'$ s.t. $|M'| \leq |M|^\kappa$, and if $A \subseteq M$, $|A| \leq \kappa$ and $p \in S_1^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}'$

We first note that

$$|\{A \subseteq M : |A| \leq \kappa\}| \leq |M|^\kappa$$

because for each such A there is f mapping κ onto A. Also for each such A, $\left|S_1^{\mathcal{M}}(A)\right| \leq 2^\kappa$. Let $(p_\alpha : \alpha < |M|^\kappa)$ list all types in $S_1^{\mathcal{M}}(A)$ for $n < \omega$, $A \subseteq M$ with $|A| \leq \kappa$. We build an elementary chain $(\mathcal{M}_\alpha : \alpha < |M|^\kappa)$ as follows

1. $\mathcal{M}_0 = \mathcal{M}$
2. $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ for α a limit ordinal
3. $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ with $|M_{\alpha+1}| = |M_\alpha|$, $\mathcal{M}_{\alpha+1}$ realizes p_α

By induction we see that $|M_\alpha| \leq |M|^\kappa$. Let $\mathcal{M}' = \bigcup_{\alpha<|M|^\kappa} \mathcal{M}_\alpha$. Then $|M'| \leq |M|^\kappa$ and $\mathcal{M}'$ is the desired model.

We build an elementary chain $(\mathcal{N}_\alpha : \alpha < \kappa^+)$ s.t. each $|N_\alpha| \leq |M|^\kappa$ and

1. $\mathcal{N}_0 = \mathcal{M}$
2. $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$ for α a limit ordinal
3. $\mathcal{N}_\alpha \prec \mathcal{N}_{\alpha+1}$, $|\mathcal{N}_\alpha| \leq |M|^\kappa$, and if $A \subseteq N_\alpha$ with $|A| \leq \kappa$ and $p \in S_n^{\mathcal{N}_\alpha}(A)$, then p is realized in $\mathcal{N}_{\alpha+1}$. This is possible because, by induction
$$|N_\alpha|^\kappa \leq (|M|^\kappa)^\kappa = |M|^\kappa$$

Let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. Because $\kappa^+ \leq |M|^\kappa$, N is the union of at most $|M|^\kappa$ sets of size $|M|^\kappa$ so $|N| \leq |M|^\kappa$. Suppose that $|A| \subseteq N$, $|A| \leq \kappa$, and $p \in S_n^{\mathcal{N}}(A)$. Because κ^+ is a regular cardinal, there is $\alpha < \kappa^+$ s.t. $A \subset N_\alpha$ and p is realized in $\mathcal{N}_{\alpha+1} \prec \mathcal{N}$. Thus $\mathcal{N}$ is κ^+-saturated $\square$

Corollary 4.45. *Suppose that $2^\kappa = \kappa^+$. Then there is a saturated model of T of size κ^+. In particular, if the Generalized Continuum Hypothesis is true, there are saturated models of size κ^+ for all κ*

If $|S_n(T)| = 2^{\aleph_0}$, then any $\aleph_0$-saturated model has size $2^{\aleph_0}$. If $\aleph_1 < 2^{\aleph_0}$, then there is no saturated mode of size $\aleph_1$

Corollary 4.46. *Suppose that $\kappa \geq \aleph_1$ is regular and $2^\lambda \leq \kappa$ for $\lambda < \kappa$. Then there is a saturated model of size κ. In particular, if $\kappa \geq \aleph_1$ is strongly inaccessible, then there is a saturated model of size κ*

Proof. Let $\mathcal{M} \models T$ with $|M| = \kappa$. If $\kappa = \lambda^+$ for $\lambda < \kappa$, then the corollary follows from Corollary 4.45. Thus we may assume that κ is a limit cardinal. We build an elementary chain $(\mathcal{M}_\lambda : \lambda < \kappa, \lambda$ a cardinal). Each $\mathcal{M}_\lambda$ will have cardinality κ. Let $\mathcal{M}_0 = \mathcal{M}$

Let $\mathcal{M}_\lambda = \bigcup_{\mu<\lambda} \mathcal{M}_\mu$ for λ a limit cardinal. Because $\mathcal{M}_\alpha$ is the union of fewer than κ models of size κ, $|M_\lambda| = \kappa$.

Given $\mathcal{M}_\lambda$, by Theorem 4.44 there is $\mathcal{M}_\lambda \prec \mathcal{M}_{\lambda^+}$ s.t. $\mathcal{M}$ is λ^+-saturated and $|M_{\lambda^+}| \leq \kappa^\lambda = \kappa$

Let $\mathcal{N} = \bigcup \mathcal{M}_\lambda$. Because κ is a regular limit cardinal, $\kappa = \aleph_\kappa$. Thus because κ is regular, if $A \subset N$ and $|A| < \kappa$, then there is $\lambda < \kappa$ s.t. $A \subset M_\lambda$. Thus, if $p \in S_n^{\mathcal{N}}(A)$, then p is realized in $\mathcal{M}_{\lambda^+} \prec \mathcal{N}$ ◻

4.3.3 Homogeneous and Universal Models

Definition 4.47. $\mathcal{M} \models T$ is κ**-universal** if for all $\mathcal{N} \models T$ with $|N| < \kappa$ there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$

We say that $\mathcal{M}$ is **universal** if it is $|M|^+$-universal

Lemma 4.48. *Let $\kappa \geq \aleph_0$. If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is κ^+-universal*

Proof. Here, T at least should contain $\mathrm{Th}(\mathcal{M})$

Let $\mathcal{N} \models T$ with $|N| \leq \kappa$. Let $(n_\alpha : \alpha < \kappa)$ enumerate N. Let $A_\alpha = \{n_\beta : \beta < \alpha\}$. We build a sequence of partial elementary maps $f_0 \subset f_1 \subset \cdots \subset f_\alpha \subset \cdots$ for $\alpha < \kappa$ with $f_\alpha : A_\alpha \to \mathcal{M}$

Let $f_0 = \emptyset$. (Justification)

And if α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$

Given $f_\alpha : A_\alpha \to \mathcal{M}$ partial elementary, let

$$\Gamma(v) = \{\phi(v, f_\alpha(\bar{a})) : \mathcal{M} \models \phi(n_\alpha, \bar{a})\}$$

Because f_α is partial elementary and $|A_\alpha| < \kappa$, Γ is satisfiable and, by κ-saturation, realized in some b in $\mathcal{M}$. The $f_{\alpha+1} = f_\alpha \cup \{(n_\alpha, b)\}$ is the desired partial elementary map ◻

Theorem 4.49. *Let $\kappa \geq \aleph_0$. The following are equivalent*

1. *$\mathcal{M}$ is κ-saturated*

2. *$\mathcal{M}$ is κ-homogeneous and κ^+-universal*

   If $\kappa \geq \aleph_1$, 1 and 2 also equivalent to

3. *$\mathcal{M}$ is κ-homogeneous and κ-universal*

Proof. By Proposition 4.38 and Lemma 4.48, $1 \to 2$. Clearly $2 \to 3$

$2 \to 1$. Let $A \subseteq M$ with $|A| < \kappa$, and let $p \in S_1^{\mathcal{M}}(A)$. We can find $\mathcal{N} \vDash \mathrm{Th}_A(\mathcal{M})$ s.t. $A \subseteq N$ and there is $a \in N$ realizing p. If $\kappa = \aleph_0$, then we can choose $\mathcal{N}$ with $|N| = \aleph_0$. If $\kappa \geq \aleph_1$, then we choose $\mathcal{N}$ with $|N| < \kappa$. By assumption, there is an elementary embedding $f : \mathcal{N} \to \mathcal{M}$. Because $f|A$ is partial elementary, by κ-homogeneity, there is $b \in M$ s.t.

$$\mathrm{tp}^{\mathcal{M}}(b/A) = \mathrm{tp}^{\mathcal{M}}(f(a)/f(A)) = \mathrm{tp}^{\mathcal{N}}(a/A) = p$$

Thus $\mathcal{M}$ is κ-saturated

Note that $\mathcal{N}$ is built on $\mathcal{L}_A$. □

Corollary 4.50. *$\mathcal{M}$ is saturated iff it is homogeneous and universal*

Theorem 4.51. *If $\mathcal{M}$ and $\mathcal{N}$ are saturated models of T of cardinality κ, then $\mathcal{M} \cong \mathcal{N}$*

Proof. By Corollary 4.40 we may assume that $\kappa \geq \aleph_1$. Let $(m_\alpha : \alpha < \kappa)$ enumerate $\mathcal{M}$ and $(n_\alpha : \alpha < \kappa)$ enumerate $\mathcal{N}$. We build a sequence of partial embeddings $f_0 \subset \dots \subset f_\alpha \cdots$ for $\alpha < \kappa$ s.t. $m_\alpha \in \mathrm{dom}(f_{\alpha+1})$ and $n_\alpha \in \mathrm{im}(f_{\alpha+1})$. Let A_α denote the domain of f_α. We will have $|A_\alpha| \leq |\alpha| + \aleph_0 < \kappa$ for all α

Let $f_0 = \emptyset$ WTF? □

4.3.4 Vaught's Two-Cardinal Theorem

Tent's is better on this topic□

If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(v_1, \dots, v_n)$ is an $\mathcal{L}$-formula, we let $\phi(\mathcal{M}) = \{\bar{x} \in M^n : \mathcal{M} \vDash \phi(\bar{x})\}$

Definition 4.52. Let $\kappa > \lambda \geq \aleph_0$. We say that an $\mathcal{L}$-theory T has a **(κ,λ)-model** if there is $\mathcal{M} \vDash T$ and $\phi(\bar{v})$ an $\mathcal{L}$-formula s.t. $|M| = \kappa$ and $|\phi(\mathcal{M})| = \lambda$

(κ, λ)-models are an obstruction to κ-categoricity. If T is a theory in a countable language with infinite models, then an easy compactness argument shows that there is $\mathcal{M} \vDash T$ of cardinality κ where every $\emptyset$-definable subsets of $\mathcal{M}$ has cardinality κ. Guess add κ constants, and partition constants into ω groups of κ constants. This is finitely satisfiable. If T also has a (κ, λ)-model, then T is not κ-categorical. Out main goal is the following theorem of Vaught

Theorem 4.53. *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model*

Definition 4.54. We say that $(\mathcal{N}, \mathcal{M})$ is a **Vaughtian pair** of models of T if $\mathcal{M} \prec \mathcal{N}$, $M \neq N$ and there is an $\mathcal{L}_M$-formula ϕ s.t. $\phi(\mathcal{M})$ is infinite and $\phi(\mathcal{M}) = \phi(\mathcal{N})$

For example, if $\mathcal{M}$ and $\mathcal{N}$ are nonstandard models of Peano arithmetic and $\mathcal{N}$ is a proper elementary end extension of $\mathcal{M}$, then $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair. If a is any infinite element of $\mathcal{M}$, then the formula $v < a$ defines an infinite set containing no elements of $N \setminus M$

Lemma 4.55. *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then there is $(\mathcal{N}, \mathcal{M})$ a Vaughtian pair of models of T*

Proof. Let $\mathcal{N}$ be a (κ, λ)-model. Suppose that $X = \phi(\mathcal{N})$ has cardinality λ. By the Löwenheim–Skolem Theorem, there is $\mathcal{M} \prec \mathcal{N}$ s.t. $X \subseteq M$ and $|M| = \lambda$. $\square$

We would like to show that if there is a Vaughtian pair, then there is a Vaughtian pair of countable models.

Let $\mathcal{L}^* = \mathcal{L} \cup \{U\}$, where U is a unary predicate symbol. If $\mathcal{M} \subseteq \mathcal{N}$ are $\mathcal{L}$-structures, we consider the pair $(\mathcal{N}, \mathcal{M})$ as an $\mathcal{L}^*$-structure by interpreting U as M

If $\phi(v_1, \dots, v_n)$ is an $\mathcal{L}$-formula, we define $\phi^U(\bar{v})$, the restriction of ϕ to U, inductively as follows:

1. if ϕ is atomic, then ϕ^U is $U(v_1) \wedge \cdots \wedge U(v_n) \wedge \phi$
2. if ϕ is $\neg\psi$, then ϕ^U is $\neg\psi^U$
3. if ϕ is $\psi \wedge \theta$, then ϕ^U is $\psi^U \wedge \theta^U$
4. if ϕ is $\exists v\ \psi$, then ϕ^U is $\exists v\ U(v) \wedge \psi^U$

If $\mathcal{M} \subset \mathcal{N}$, $\bar{a} \in M^k$ and we view $(\mathcal{N}, \mathcal{M})$ as an $\mathcal{L}^*$-structure, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $(\mathcal{N}, \mathcal{M}) \vDash \phi^U(\bar{a})$

guess $U(v)$ iff $v \in M$.

Lemma 4.56. *If $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair for T, then there is a Vaughtian pair $(\mathcal{N}_0, \mathcal{M}_0)$ where $\mathcal{N}_0$ is countable*

Proof. Let ϕ be an $\mathcal{L}_M$-formula s.t. $\phi(\mathcal{M})$ is infinite and $\phi(\mathcal{M}) = \phi(\mathcal{N})$. Let $\overline{m}_0$ be the parameters from M occuring in ϕ. By the Löwenheim–Skolem Theorem, there is $(\mathcal{N}_0, \mathcal{M}_0)$ a countable $\mathcal{L}^*$-structure s.t. $\overline{m} \in M_0$ and $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$. Because $\mathcal{M} \prec \mathcal{N}$, for any formula $\psi(v_1, \dots, v_k)$

$$(\mathcal{N}, \mathcal{M}) \vDash \forall \bar{v} \left(\left(\bigwedge_{i=1}^{k} U(v_i) \wedge \psi(\bar{v}) \right) \to \psi^U(\bar{v}) \right)$$

Because $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$, these sentences are also true in $(\mathcal{N}_0, \mathcal{M}_0)$, so $\mathcal{M}_0 \prec \mathcal{N}_0$ (Tarski's test)

Let $\phi(\bar{v})$ be an $\mathcal{L}_M$-formula with infinitely (maybe some restrictions) many realizations in $\mathcal{M}$ and none in $\mathcal{N} \setminus \mathcal{M}$., witnessing that $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair. For each k, the sentences

$$\exists \bar{v}_1 \dots \exists \bar{v}_k \left(\bigwedge_{i<j} \bar{v}_i \neq \bar{v}_j \wedge \bigwedge_{i=1}^{k} \phi(\bar{v}_i) \right)$$

hold in $(\mathcal{N}, \mathcal{M})$, as do the sentences $\exists x \neg U(x)$ and

$$\forall \bar{v} \left(\phi(\bar{v}) \to \bigwedge U(v_i)\right)$$

Because these sentences also hold in $(\mathcal{N}_0, \mathcal{M}_0)$, this structure is also a Vaughtian pair. $\square$

Lemma 4.57. *Suppose that $\mathcal{M}_0 \prec \mathcal{N}_0$ are countable models of T. We can find $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$ s.t. $\mathcal{N}$ and $\mathcal{M}$ are countable, homogeneous, and realize the same types in $S_n(T)$. By Theorem 4.28 $\mathcal{M} \cong \mathcal{N}$*

Proof. **Claim 1** If $\bar{a} \in M_0$ and $p \in S_n(\bar{a})$, then there is $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}', \mathcal{M}')$ s.t. p is realized in $\mathcal{M}'$ (as $\mathcal{M}_0 \prec \mathcal{N}_0$, so $S_n^{\mathcal{M}_0}(\bar{a}) = S_n^{\mathcal{N}_0}(\bar{a})$)

Let $\Gamma(\bar{v}) = \{\phi^U(\bar{v}, \bar{a}) : \phi(\bar{v}, \bar{a}) \in p\} \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{N}_0, \mathcal{M}_0)$ (ensures $\mathcal{M}_0 \prec \mathcal{N}_0$). If $\phi_1, \dots, \phi_m \in p$, then $\mathcal{N}_0 \vDash \exists \bar{v} \bigwedge \phi_i(\bar{v}, \bar{a})$ as $\exists \bar{v} \bigwedge \phi_i \in p$, thus $\mathcal{M}_0 \vDash \exists \bar{v} \bigwedge \phi_i(\bar{v}, \bar{a})$ and $(\mathcal{N}_0, \mathcal{M}_0) \vDash \exists \bar{v} \bigwedge \phi_i^U(\bar{v}, \bar{a})$. Thus $\Gamma(\bar{v})$ is satisfiable. Let $(\mathcal{N}', \mathcal{M}')$ be a countable elementary extension realizing Γ.

By iterating Claim 1, we can find $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}^*, \mathcal{M}^*)$ countable s.t. if $\bar{a} \in M_0$ and $p \in S_n(\bar{a})$, then p is realized in $\mathcal{M}^*$

Claim 2 If $\bar{b} \in N_0$ and $p \in S_n(\bar{b})$, then there is $(\mathcal{M}_0, \mathcal{N}_0) \prec (\mathcal{N}', \mathcal{M}')$ s.t. p is realized in $\mathcal{N}'$

Let $\Gamma(\bar{v}) = p \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{N}_0, \mathcal{M}_0)$. If $\phi_1, \dots, \phi_m \in p$, then $\mathcal{N}_0 \vDash \exists \bar{v} \bigwedge_i (\bar{v}, \bar{b})$; thus we can find a countable elementary extension of $(\mathcal{N}_0, \mathcal{M}_0)$ realizing p.

We build an elementary chain of countable models

$$(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}_1, \mathcal{M}_1) \prec \cdots$$

s.t.

1. if $p \in S_n(T)$ is realized in $\mathcal{N}_{3i}$, then p is realized in $\mathcal{M}_{3i+1}$
2. if $\bar{a}, \bar{b}, c \in \mathcal{M}_{3i+1}$ and $\mathrm{tp}^{\mathcal{M}_{3i+1}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}_{3i+1}}(\bar{b})$, then there is $d \in M_{3i+2}$ s.t. $\mathrm{tp}^{\mathcal{M}_{3i+2}}(\bar{a}, c) = \mathrm{tp}^{\mathcal{M}_{3i+2}}(\bar{b}, d)$

3. if $\bar{a}, \bar{b}, c \in \mathcal{N}_{3i+2}$ and $\text{tp}^{\mathcal{N}_{3i+2}}(\bar{a}) = \text{tp}^{\mathcal{N}_{3i+2}}(\bar{b})$, then there is $d \in N_{3i+3}$ s.t. $\text{tp}^{\mathcal{N}_{3i+3}}(\bar{a}, c) = \text{tp}^{\mathcal{N}_{3i+3}}(\bar{b}, d)$

1 and 2 are done by using the first claim, 3 is done by the second claim.

Let $(\mathcal{N}, \mathcal{M}) = \bigcup_{i<\omega}(\mathcal{N}_i, \mathcal{M}_i)$. Then $(\mathcal{N}, \mathcal{M})$ is a countable Vaughtian pair. By 1, $\mathcal{M}$ and $\mathcal{N}$ realize the sames types. By 2 and 3, $\mathcal{M}$ and $\mathcal{N}$ are homogeneous □

Proof of 4.53. Suppose that T has a (κ, λ)-model. By the lemmas above, we can find $(\mathcal{N}, \mathcal{M})$ a countable Vaughtian pair s.t. $\mathcal{M}$ and $\mathcal{N}$ are homogeneous models realizing the same types. Let $\phi(\bar{v})$ be an $\mathcal{L}_M$-formula with infinitely many realizations in M and none in $N \setminus M$.

We build an elementary chain $(\mathcal{N}_\alpha : \alpha < \omega_1)$, each $\mathcal{N}_\alpha$ is isomorphic to $\mathcal{N}$ and $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$. In particular, $N_{\alpha+1} \setminus N_\alpha$ contains no elements satisfying ϕ

Let $\mathcal{N}_0 = \mathcal{N}$. For α a limit ordinal, let $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$. Because $\mathcal{N}_\alpha$ is a union of models isomorphic to $\mathcal{N}$, $\mathcal{N}_\alpha$ is homogeneous and realizes the same types as $\mathcal{N}$ so $\mathcal{N}_\alpha \cong \mathcal{N}$ by Theorem 4.28 (interesting)

Given $\mathcal{N}_\alpha \cong \mathcal{N}$, because $\mathcal{N} \cong \mathcal{M}$ there is $\mathcal{N}_{\alpha+1}$ an elementary extension of $\mathcal{N}_\alpha$ s.t. $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha)$. Clearly $\mathcal{N}_{\alpha+1} \cong \mathcal{N}_\alpha$. Extend $f : \mathcal{M} \cong \mathcal{N}_\alpha$ to $f' : \mathcal{N} \cong \mathcal{N}_{\alpha+1}$

Let $\mathcal{N}^* = \bigcup_{\alpha<\omega_1} \mathcal{N}_\alpha$. Then $|N^*| = \aleph_1$ and if $\mathcal{N}^* \vDash \phi(\bar{a})$, then $\bar{a} \in M$; thus $\mathcal{M}^*$ is an $(\aleph_1, \aleph_0)$-model □

Corollary 4.58. *If T is $\aleph_1$-categorical, then T has no Vaughtian pairs and hence no (κ, λ) models for $\kappa > \lambda \geq \aleph_0$.*

If T is ω-stable, we can prove a partial converse to Vaught's Theorem

Lemma 4.59. *Suppose that T is ω-stable, $\mathcal{M} \vDash T$, and $|M| \geq \aleph_1$. There is a proper elementary extension $\mathcal{N}$ of $\mathcal{M}$ s.t. if $\Gamma(\overline{w})$ is a countable type over M realized in $\mathcal{N}$, then $\Gamma(\overline{w})$ is realized in $\mathcal{M}$*

Theorem 4.60. *Suppose that T is ω-stable and there is an $(\aleph_1, \aleph_0)$-model of T. If $\kappa > \aleph_1$, then there is a $(\kappa, \aleph_0)$-model of T.*

4.4 The Number of Countable Models

T a complete theory in a countable language with infinite models

For any infinite cardinal κ, we let $I(T, \kappa)$ be the number of nonisomorphic model of T of cardinality κ

$I(\mathsf{DLO}, \aleph_0) = 1$

4.4.1 $\aleph_0$-categorical Theories

Theorem 4.61. *The following are equivalent*

1. *T is $\aleph_0$-categorical*
2. *Every type in $S_n(T)$ is isolated for $n < \omega$*
3. *$|S_n(T)| < \aleph_0$ for all $n < \omega$*
4. *For each $n < \omega$, there is a finite list of formulas*

$$\phi_1(v_1, \dots, v_n), \dots, \phi_m(v_1, \dots, v_n)$$

   s.t. for every formula $\psi(v_1, \dots, v_n)$

$$T \vDash \phi_i(\bar{v}) \leftrightarrow \psi(\bar{v})$$

   for some $i \leq m$

Proof. $1 \to 2$. If $p \in S_n(T)$ is nonisolated, then there is a countable $\mathcal{M} \vDash T$ omitting p. There is also a countable $\mathcal{N} \vDash T$ realizing p: as $p \cup T$ is satisfiable, let $\mathcal{N} \vDash p(n) \cup T$ and consider language $\mathcal{L}(c)$. Then by LST, we can get a countable elementary substructure $\mathcal{N}'$ of $\mathcal{N}$. Clearly $\mathcal{M} \ncong \mathcal{N}$

$2 \to 3$. Suppose that $S_n(T)$ is infinite. For each $p \in S_n(T)$, let ϕ_p isolates p. Because $\bigcup_{p \in S_n(T)} [\phi_p] = S_n(T)$ and $S_n(T)$ is compact, there are $p_1, \dots, p_m$ s.t $[\phi_{p_1}] \cup \dots \cup [\phi_{p_m}] = S_n(T)$. Because $[\phi_p] = \{p\}$, $S_n(T)$ is finite

$3 \to 4$. for each i, we can find a formula θ_i s.t. $\theta_i \in p_i$ and $\neg\theta_i \in p_j$ for $i \neq j$ (e.g. $\neg\psi_1 \wedge \dots \wedge \neg\psi_{i-1} \wedge \psi_i \wedge \neg\psi_{i+1} \wedge \dots \wedge \psi_n$). Then θ_i isolates p_i. For any formula $\psi(v_1, \dots, v_n)$

$$T \vDash \psi(\bar{v}) \leftrightarrow \bigvee_{\psi \in p_i} \theta_i$$

Thus each ψ with free variables $v_1, \dots, v_n$ is equivalent to $\bigvee_{i \in S} \theta_i$ for some $S \subseteq \{1, \dots, m\}$. There are at most 2^m such formulas.

$4 \to 1$. Let $\mathcal{M}$ be a countable model of T. If $\bar{a} \in M^n$, let $S_{\bar{a}} = \{i \leq m : \mathcal{M} \vDash \phi_i(\bar{a})\}$. Then $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ is isolated by

$$\bigwedge_{i \in S_{\bar{a}}} \phi_i(\bar{v}) \wedge \bigwedge_{i \notin S_{\bar{a}}} \neg\phi(\bar{v})$$

Thus $\mathcal{M}$ is atomic and hence by Theorem 4.20 prime. Because there is a unique prime model, T is $\aleph_0$-categorical $\square$

b is **algebraic over** A if there is a formula $\phi(v, \overline{w})$ and $\bar{a} \in A$ s.t. $\mathcal{M} \models \phi(b, \bar{a})$ and $\{x \in M : \mathcal{M} \models \phi(x, \bar{a})\}$ is finite. Also, $\text{acl}(A) = \{b \in A : b \text{ is algebraic over } A\}$

Corollary 4.62. *Suppose that T is $\aleph_0$-categorical. There is a function $f : \mathbb{N} \to \mathbb{N}$ s.t. if $\mathcal{M} \models T$, $A \subset M$ and $|A| \leq n$, then $|acl(A)| \leq f(n)$*

Proof. By Theorem 4.61, $|S_{n+1}(T)|$ is finite. Let $q_1, \dots, q_k$ list all $n+1$-types. Let

$$X = \Big\{ i : q_i \text{ contains a formula } \phi(v, \overline{w}) \text{ s.t. for some } N \quad \mathcal{M} \models \forall v_0, \dots, v_N \left(\bigwedge_{i=0}^{N} \phi(v_i, \overline{w}) \to \bigvee_{i<j\leq N} v_i = v_j \right) \Big\}$$

For $i \in X$, let N_i be the least N s.t. for some formula ϕ

$$\forall v_0, \dots, v_N \left(\bigwedge_{i=0}^{N} \phi(v_i, \overline{w}) \to \bigvee_{i<j} v_i = v_j \right)$$

is in q_i.

If $a, b_1, \dots, b_n \in M$ and a is algebraic over $\bar{b}$, then $(a, \bar{b})$ realizes some $q_i \in X$ and $\left|\{x : (x, \bar{b}) \text{ realizes } q_i\}\right| \leq N_i$. Thus

$$|\text{acl}(b_1, \dots, b_n)| \leq \sum_{i \in X} N_i$$

Let

$$f(n) = \sum_{i \in X} N_i$$

□

Corollary 4.63. *If F is an infinite field, then the theory of F is not $\aleph_0$-categorical*

Proof. By compactness, we can find an elementary extension K of F s.t. K contains a transcendental element t. Because $t, t^2, t^3, \dots$ are distinct, $\text{acl}(t)$ is infinite. Thus, by Corollary 4.62 $\text{Th}(F)$ is not $\aleph_0$-categorical □

A group G is **locally finite** if for any finite $X \subseteq G$, the subgroup generated by X is finite

Corollary 4.64. *Let G be an infinite group*

1. *if* $\mathrm{Th}(G)$ *is* $\aleph_0$*-categorical, then* G *is locally finite. Moreover, there is a number* b *s.t. if* $g \in G$*, then* $g^n = 1$ *for some* $n \leq b$ *(we say that* G *has* ***bounded exponent****)*

2. *if* G *is an infinite Abelian group of bounded exponent, then* $\mathrm{Th}(G)$ *is* $\aleph_0$*-categorical*

Proof. 1. By Corollary 4.62 there is a function $f : \mathbb{N} \to \mathbb{N}$ s.t. if $|X| \leq n$, the group generated by X has size at most $f(n)$. We need to justify that the group generated by X is the $\mathrm{acl}(X)$. But in X, we can only TALK about the elements generated by X. So $|(X)| \leq |\mathrm{acl}X|$

In particular, if $g \in G$, then $g^n = 1$ for some $n \leq f(1)$.

$\square$

Lemma 4.65. *Let* $\kappa \geq \aleph_0$*. Let* $A \subset M$ *with* $|A| < \kappa$*. Let* $\mathcal{M}_A$ *be the* $\mathcal{L}_A$*-structure obtained from* $\mathcal{M}$ *by interpreting the new constant symbols in the natural way. If* $\mathcal{M}$ *is* κ*-saturated, then so is* $\mathcal{M}_A$

Proof. For all $B \subseteq M$, if $|B| < \kappa$ and $p \in S_n^{\mathcal{M}_A}(B)$, note that $S_n^{\mathcal{M}}(A, B) = S_n^{\mathcal{M}_A}(B)$. For p is actually $p(\bar{x}, \bar{b}, \bar{a})$ and $p(\bar{x}, \bar{b}, \bar{a}) \cup \mathrm{Th}(\mathcal{M}_A)$ means $p(\bar{x}, \bar{b}, \bar{a}) \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable. Hence $p \in S_n^{\mathcal{M}}(A, B)$ $\square$

Theorem 4.66. $I(T, \aleph_0) \neq 2$

Proof. Suppose that $I(T, \aleph_0) = 2$. By Corollary 4.43 (2), there is $\mathcal{N}$ a prime model of T and $\mathcal{M}$ a countable saturated model of T. Because T is not $\aleph_0$-categorical, by Theorem 4.61, there is a nonisolated type $p \in S_n(T)$ for some n. The type is realized in $\mathcal{M}$ and omitted in $\mathcal{N}$. Let $\bar{a} \in M$ realize p. Let T^* be the $\mathcal{L}_{\bar{a}}$-theory of $\mathcal{M}_{\bar{a}}$

By Theorem 4.61, there are infinitely many T-inequivalent formulas in the free variables $v_1, \dots, v_n$. As they are still T^*-inequivalent, T^* is not $\aleph_0$-categorical. By Lemma 4.65, $\mathcal{M}_{\bar{a}}$ is a saturated $\mathcal{L}_{\bar{a}}$-structure. Thus by Corollary 4.43 (1), T^* has a countable atomic model $\mathcal{A}$. Let $\mathcal{B}$ denote the $\mathcal{L}$-reduct of $\mathcal{B}$. Because $\mathcal{A} \models T^*$, $\mathcal{B}$ contains a realization of p, thus $\mathcal{B} \ncong \mathcal{N}$. Because T^* is not $\aleph_0$-categorical, there is a nonisolated $\mathcal{L}_{\bar{a}}$-type. This type is not realized in $\mathcal{A}$. Thus $\mathcal{A}$ is not saturated. If $\mathcal{B}$ were saturated, then by Lemma 4.65 $\mathcal{A}$ would be saturated. Thus $\mathcal{B} \ncong \mathcal{M}$ and $I(T, \aleph_0) \geq 3$. $\square$

4.4.2 Morley's Analysis of Countable Models

Next we prove Morley's theorem that if $I(T, \aleph_0) > \aleph_1$, then $I(T, \aleph_0) = 2^{\aleph_0}$

Definition 4.67. A **fragment** of $\mathcal{L}_{\omega_1,\omega}$ is a set of $\mathcal{L}_{\omega_1,\omega}$-formulas containing all first-order formulas and closed under subformulas, finite Boolean combinations, quantification, and change of free variables

If F is a fragment of $\mathcal{L}_{\omega_1,\omega}$, we say that $\mathcal{M} \equiv_F \mathcal{N}$ if

$$\mathcal{M} \vDash \phi \quad \text{iff} \quad \mathcal{N} \vDash \phi$$

for all sentences $\phi \in F$

If F is a fragment of $\mathcal{L}_{\omega_1,\omega}$, we say that $p \subset F$ is an F**-type** if there is countable $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \dots, a_n \in M$ s.t. $p = \{\phi(v_1, \dots, v_n) \in F : \mathcal{M} \vDash \phi(\bar{a})\}$. Let $S_n(F, T)$ be the set of all F-types realized by some n-tuples in some countable model of T

4.5 Exercise

Exercise 4.5.1.

1. Let $\mathcal{M} = (X, <)$ be a dense linear order, let $A \subset M$ and $\bar{b}, \bar{c} \in M^n$ with $b_1 < \dots < b_n$ and $c_1 < \dots < c_n$. Show that $\text{tp}^{\mathcal{M}}(\bar{a}/A) = \text{tp}^{\mathcal{M}}(\bar{b}/A)$ iff $b_i < a \Leftrightarrow c_i < a$ and $b_i > a \Leftrightarrow c_i > a$ for all $i = 1, \dots, n$ and $a \in A$. In particular, show that any two elements of X realize the same 1-type over $\emptyset$

Proof.

1. Since DLO has

□

Exercise 4.5.2. Suppose that $A \subseteq B$, $\theta(\bar{v})$ is a formula with parameters from A, and θ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$. Then θ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/A)$

Proof. TRIVIAL as $A \subseteq B$ □

Exercise 4.5.3. Suppose that $A \subset M$, $\bar{a}, \bar{b} \in M$ s.t. $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$ is isolated. Show that $\text{tp}^{\mathcal{M}}(\bar{a}/A, \bar{b})$ is isolated

Proof. Suppose $\theta(\bar{v}, \bar{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$. Then for each $\phi(\bar{v}, \bar{w}) \in \text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$

$$\text{Th}(\mathcal{M}_A) \vDash \forall \bar{v} \forall \bar{w} (\theta(\bar{v}, \bar{w}) \to \phi(\bar{v}, \bar{w}))$$

For each $\psi(\bar{v}) \in \text{tp}^{\mathcal{M}}(\bar{a}/A, \bar{b})$ □

Exercise 4.5.4. Suppose that $\mathcal{M}$ is κ-saturated, and $(\phi_i(\bar{v}) : i \in I)$ and $(\theta_j(\bar{v}) : j \in J)$ are sequences of $\mathcal{L}_M$-formulas s.t. $|I|, |J| < \kappa$ and

$$\mathcal{M} \vDash \bigvee_{i \in I} \phi_i(\bar{v}) \leftrightarrow \neg \left(\bigvee_{j \in J} \theta_j(\bar{v}) \right)$$

Show that there are finite sets $I_0 \subseteq I$ and $J_0 \subseteq J$ s.t.

$$\mathcal{M} \models \bigvee_{i \in I} \phi_i(\bar{v}) \leftrightarrow \bigvee_{i \in I_0} \phi_i(\bar{v})$$

Proof. □

5 Indiscernibles

5.1 Partition Theorems

For X a set and κ,λ (possibly finite) cardinals, we let $[X]^\kappa$ be the collection of all subsets of X of size κ. We call $f : [X]^\kappa \to \lambda$ a **partition** of $[X]^\kappa$. We say that $Y \subseteq X$ is **homogeneous** for the partition f if there is $\alpha < \lambda$ s.t. $f(A) = \alpha$ for all $A \in [Y]^\kappa$ (i.e. f is a constant on $[Y]^\kappa$). Finally, for cardinals κ,η,μ, and λ, we write $\kappa \to (\eta)^\mu_\lambda$ if whenever $|X| > \kappa$ and $f : [X]^\mu \to \lambda$, then there is $Y \subseteq X$ s.t. $|Y| \geq \eta$ and Y is homogeneous for f

Theorem 5.1 (Ramsey's Theorem)**.** *If $k, n < \omega$, then $\aleph_0 \to (\aleph_0)^n_k$*

Some applications:

Any sequence of real numbers $(r_0, r_1, \dots)$ has a monotonic subsequence. Let $f : [\mathbb{N}]^2 \to 3$ by

$$f(\{i,j\}) = \begin{cases} 0 & i < j \text{ and } r_i < r_j \\ 1 & i < j \text{ and } r_i = r_j \\ 2 & i < j \text{ and } r_i > r_j \end{cases}$$

By Ramsey's Theorem, there is $Y \subseteq N$ an infinite homogeneous set for f. Let $j_0 < j_1 < \dots$ list Y. There is $c < 3$ s.t. $f(\{j_m, j_n\}) = c$ for $m < n$.

Suppose G is an infinite graph. Let $f : [G]^2 \to 2$ by

$$f(\{a,b\}) = \begin{cases} 1 & (a,b) \text{ is an edge of } G \\ 0 & (a,b) \text{ is not an edge of } G \end{cases}$$

By Ramsey's Theorem, there is an infinite $H \subseteq G$ homogeneous for f. If f is constantly 1 on $[H]^2$, then H is a complete subgraph, and if f is constantly 0, there are no edges.

Proof. Induction on n. For $n = 1$ Ramsey's Theorem asserts that if X is infinite, $k < \omega$, and $f : X \to k$, then $f^{-1}(i)$ is infinite for some $i < k$. This is just the Pigeonhole Principle.

Suppose that we have proved that if $i < n$, $k < \omega$, X is infinite, and $f : [X]^i \to k$, then there is an infinite $Y \subseteq X$ homogeneous for f.

We could always replace X by a countable subset of X; thus, W.L.O.G., we may assume that $X = \mathbb{N}$.

Let $f : [\mathbb{N}]^n \to k$. For $a \in \mathbb{N}$, let $f_a : [\mathbb{N} \setminus \{a\}]^{n-1} \to k$ by $f_a(A) = f(A \cup \{a\})$. We build a sequence $0 = a_0 < a_1 < \dots$ in $\mathbb{N}$ and $\mathbb{N} = X_0 \supset X_1 \supset \dots$ a sequence of infinite sets as follows. Given a_i and X_i, let $X_{i+1} \subset X_i \setminus \{0, 1, \dots, a_i\}$ be homogeneous for f_{a_i}. Let a_{i+1} be the least element of X_{i+1}

Let $c_i < k$ be s.t. $f_{a_i}(A) = c_i$ for all $A \in [X_{i+1}]^{n-1}$. By the Pigeonhole Principle, there is $c < k$ s.t. $\{i : c_i = c\}$ is infinite. Let $X = \{a_i : c_i = c\}$. We claim that X is homogeneous for f. Let $x_1 < \dots < x_n$ where each $x_i \in X$, there is an i s.t. $x_1 = a_i$ and $x_2, \dots, x_n \in X_i$. Thus

$$f(\{x_1, \dots, x_n\}) = f_{x_1}(\{x_2, \dots, x_n\}) = c_i = c$$

and X is homogeneous for f. □

Theorem 5.2 (Finite Ramsey Theorem)**.** *For all $k, n, m < \omega$, there is $l < \omega$ s.t. $l \to (m)^n_k$*

Proof. Suppose that there is no l s.t. $l \to (m)^n_k$. For each $l < \omega$, let

$$\begin{aligned} T_l = \{f : [\{0, \dots, l-1\}]^n \to k :\ & \text{there is no } X \subseteq \{0, \dots, l-1\} \\ & \text{of size at least } m,\ \text{homogeneous for } f \end{aligned}$$

Clearly each T_l is finite since n and k are finite. if $f \in T_{l+1}$ there is a unique $g \in T_l$ s.t. $g \subset f$. Thus if we order $T = \bigcup T_l$ by inclusion, we get a finite branching tree. Each T_l is not empty, so T is an infinite finite branching tree. By Kőnig's Lemma (Lemma A.5) we can find $f_0 \subset f_1 \subset f_2 \dots$ with $f_i \in T_i$

Let $f = \bigcup f_i$. Then $f : [\mathbb{N}]^n \to k$. By Ramsey's Theorem, there is an infinite $X \subseteq \mathbb{N}$ homogeneous for f. Let $x_1, \dots, x_m$ be the first m elements of X and let $s > x_m$. Then $\{x_1, \dots, x_m\}$ is homogeneous for f_s, a contradiction □

Proposition 5.3. $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$

Proof. We define $F : [2^\omega]^2 \to \omega$ by $F(\{f, g\})$ is the least n s.t. $f(n) = g(n)$. Clearly, we cannot find $\{f, g, h\}$ s.t. $f(n) \neq g(n)$, $g(n) \neq h(n)$ and $f(n) \neq h(n)$ □

On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2_{\aleph_0}$. This is the special case of an important generalization of Ramsey's Theorem. For κ an infinite cardinal and α an ordinal, we inductively define $\beth_\alpha(\kappa)$ by $\beth_0(\kappa) = \kappa$ and

$$\beth_\alpha(\kappa) = \sup_{\beta<\alpha} 2^{\beth_\beta(\kappa)}$$

In particular, $\beth_1(\kappa) = 2^\kappa$. We let $\beth_\alpha = \beth_\alpha(\aleph_0)$. Under the Generalized Continuum Hypothesis, $\beth_\alpha = \aleph_\alpha$

Theorem 5.4 (Erdős–Rado theorem)**.** $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$

Proof. Induction on n. For $n = 0$, $\kappa^+ \to (\kappa^+)^{n+1}_\kappa$ is just the Pigeonhole Principle

Suppose that we have proved the theorem for $n-1$. Let $\lambda = \beth_n(\kappa)^+$, and let $f : [\lambda]^{n+1} \to \kappa$. For $\alpha < \lambda$, let $f_\alpha : [\lambda \setminus \{\alpha\}]^n \to \kappa$ by $f_\alpha(A) = f(A \cup \{\alpha\})$.

We build $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\alpha \subseteq \ldots$ for $\alpha < \beth_{n-1}(\kappa)^+$ s.t. $X_\alpha \subseteq \beth_n(\kappa)^+$ and each X_α has cardinality at most $\beth_n(\kappa)$. Let $X_0 = \beth_n(\kappa)$. If α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$

Suppose we have X_α with $|X_\alpha| = \beth_n(\kappa)$. Because

$$\beth_n(\kappa)^{\beth_{n-1}(\kappa)} = (2^{\beth_{n-1}(\kappa)})^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa)$$

there are $\beth_n(\kappa)$ subsets of X_α of cardinality $\beth_{n-1}(\kappa)$. Also note that if $Y \subset X_\alpha$ and $|Y| = \beth_{n-1}(\kappa)$, then there are $\beth_n(\kappa)$ functions $g : [Y]^n \to \kappa$ because

$$\kappa^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa)$$

Thus we can find $X_{\alpha+1} \subseteq X_\alpha$ s.t. $|X_{\alpha+1}| = \beth_n(\kappa)$ and if $Y \subset X_\alpha$ with $|Y| = \beth_{n-1}(\kappa)$ and $\beta \in \lambda \setminus Y$, then there is $\gamma \in X_{\alpha+1} \setminus Y$ s.t. $f_\beta | [Y]^n = f_\gamma | [Y]^n$ $\square$

5.2 Order Indiscernibles

Let $\mathcal{M}$ be an $\mathcal{L}$-structure

Definition 5.5. Let I be an infinite set and suppose that $X = \{x_i : i \in I\}$ is a set of distinct elements of $\mathcal{M}$. We say that X is an **indiscernible set** if whenever $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are two sequences of m distinct elements of I, then $\mathcal{M} \models \phi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_m})$

For example, suppose that F is an algebraically closed field of infinite transcendence degree and $x_1, x_2, \ldots$ is an infinite algebraically independent set. For any two sequence $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$, there is an automorphism σ of F with $\sigma(x_{i_k}) = x_{j_k}$ for $k = 1, \ldots, m$. it follows that $x_1, x_2, \ldots$ is an infinite set of indiscernibles.

If $(A, <)$ is an infinite linear order, then because we cannot have $a < b$ and $b < a$ there is no set of indiscernibles of size 2.

Definition 5.6. Let $(I, <)$ be an ordered set, and let $(x_i : i \in I)$ be a sequence of distinct elements of M, we say that $(x_i : i \in I)$ is a sequence of **order indiscernibles** if whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < \cdots < j_m$ are two increasing sequences from I, then $\mathcal{M} \models \phi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_m})$

For example, in $(\mathbb{Q}, <)$, by quantifier elimination, if $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$, then $\mathbb{Q} \models \phi(\bar{x}) \leftrightarrow \phi(\bar{y})$ for all ϕ. Thus $\mathbb{Q}$, itself, is a sequence of order indiscernibles

Theorem 5.7. *Let T be a theory with infinite models. For any infinite linear order $(I, <)$, there is $\mathcal{M} \models T$ containing $(x_i : i \in I)$, a sequence of order indiscernibles*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i \in I\}$. Let Γ be the union of

- T
- $c_i \neq c_j$ for $i, j \in I$ with $i \neq j$
- $\phi(c_{i_1}, \ldots, c_{i_m}) \to \phi(c_{j_1}, \ldots, c_{j_m})$ for all $\mathcal{L}$-formulas $\phi(\bar{v})$, where $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ are increasing sequences from I

If $\mathcal{M} \models \Gamma$, then $(c_i^{\mathcal{M}} : i \in I)$ is an infinite sequence of order indiscernibles. It suffices to show that Γ is satisfiable. Suppose that $\Delta \subset \Gamma$ is finite. Let I_0 be the finite subset of I s.t. if c_i occurs in Δ, then $i \in I_0$. Let $\phi_1, \ldots, \phi_m$ be the formulas s.t. Δ asserts indiscenibility w.r.t. the formula ϕ_i, $i \leq m$. Let $v_1, \ldots, v_n$ be the free variables from $\phi_1, \ldots, \phi_m$, $i \leq m$.

Let $\mathcal{M}$ be an infinite model of T. Fix < any linear order of $\mathcal{M}$. We will define a partition $F : [M]^n \to \mathcal{P}(\{1, \ldots, m\})$. If $A = \{a_1, \ldots, a_n\}$ where $a_1 < \cdots < a_n$, then

$$F(A) = \{i : \mathcal{M} \models \phi_i(a_1, \ldots, a_n)\}$$

Because F partitions $[M]^n$ into at most 2^m sets, we can find an infinite $X \subseteq M$ homogeneous for F. Let $\eta \subseteq \{1, \ldots, m\}$ s.t. $F(A) = \eta$ for $A \in [X]^n$.

Suppose that I_0 is a finite subset of I. Choose $(x_i : i \in I_0)$ s.t. each $x_i \in X$ and s.t. $x_i < x_j$ if $i < j$. If $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ then

$$\mathcal{M} \models \phi_k(x_{i_1}, \ldots, x_{i_n}) \iff k \in \eta \iff \mathcal{M} \models \phi_k(x_{j_1}, \ldots, x_{j_n})$$

If we interpret c_i as x_i for $i \in I_0$, then we make $\mathcal{M}$ a model of Δ. Note that here $x_i \in M$ -.- $\square$

if $(x_i : i \in I)$ is any sequence of order indiscernibles in M, we can order $X = \{x_i : i \in I\}$ by $x_i < x_j$ if $i < j$. In this way, we frequently identify X and I

Suppose that $\psi(x, y)$ is a formula in the language s.t. in some $\mathcal{M} \models T$, ψ linearly orders an infinite set Y. When we did the construction above, we could add the condition that $\psi(c_i, c_j)$ for $i < j$. We would then restrict the partition to $[Y]^m$ and let the ordering < be the ordering determined by ψ. In this way, we would get an infinite sequence of indiscernibles $(x_i : i \in I)$ s.t. $\psi(x_i, x_j)$ iff $i < j$

5.2.1 Ehrenfeucht-Mostowski Models

Suppose that our theory has built-in Skolem functions. Then when we have a model containing an infinite sequence of order indiscernibles, we can form the elementary submodel generated by the indiscernibles.

Let T be an $\mathcal{L}$-theory. By Lemma 2.30 we can find $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ and $\mathcal{L}^*$-theory with built-in Skolem functions, s.t. if $\mathcal{M}$ is any model of T, we can interpret the symbols of $\mathcal{L}^*$ s.t. $\mathcal{M} \models T^*$. Note that if I is a sequence of order indiscernibles for $\mathcal{L}^*$, then I is also a sequence of order indiscernibles for $\mathcal{L}$.

If $\mathcal{M} \models T^*$ and $X \subseteq M$, let $\mathcal{H}(X)$ be the $\mathcal{L}^*$-substructure of $\mathcal{M}$ generated by X. We call $\mathcal{H}(X)$ the **Skolem hull** of X. Because X has built in Skolem functions, $\mathcal{H}(X) \prec \mathcal{M}$. Models built as Skolem hulls of sequences of order indiscernibles are called **Ehrenfeucht-Mostowski models**

If I is an infinite set of order indiscernibles, then order-preserving premutations of I induce automorphisms of $\mathcal{H}(I)$.

Lemma 5.8. *Suppose that T^* is an $\mathcal{L}^*$-theory with built-in Skolem functions. Let $\mathcal{M} \models T^*$. Let $I \subseteq M$ be an infinite sequence of order indiscernibles. Suppose that $\tau : I \to I$ is an order-preserving permutation. Then there is an automorphism $\sigma : \mathcal{H}(I) \to \mathcal{H}(I)$ extending τ*

Proof. For each element $a \in \mathcal{H}(I)$, there is a Skolem term t and $x_1 < x_2 < \dots < x_n \in I$ s.t. $a = t(x_1, \dots, x_n)$. Let $\sigma(a) = t(\tau(x_1), \dots, \tau(x_n))$.

We first show that σ is well-defined. Suppose that there is a second Skolem term s s.t. $a = s(x_1, \dots, x_n)$. Because

$$\mathcal{M} \models t(x_1, \dots, x_n) = s(x_1, \dots, x_n)$$

and τ is order-preserving,

$$\mathcal{M} \models t(\tau(x_1), \dots, \tau(x_n)) = s(\tau(x_1), \dots, \tau(x_n))$$

Thus σ is well defined.

We must show that σ is an automorphism. If $a = t(\bar{x})$ and $t(\tau^{-1}(\bar{x}))$, then $\sigma(b) = a$ so σ is surjective

Let $\phi(v_1, \dots, v_m)$ be any $\mathcal{L}^*$-formula, and let $a_1, \dots, a_m \in \mathcal{H}(I)$. There are terms $t_1, \dots, t_m$ and $\bar{x} \in I$ s.t. $a_i = t_i(\bar{x})$. By indiscenibility

$$\begin{aligned} \mathcal{M} \models \phi(a_1, \dots, a_m) \quad &\Leftrightarrow \quad \mathcal{M} \models \phi(t_1(\bar{x}), \dots, t_m(\bar{x})) \\ &\Leftrightarrow \quad \mathcal{M} \models \phi(t_1(\tau(\bar{x})), \dots, t_m(\tau(\bar{x}))) \\ &\Leftrightarrow \quad \mathcal{M} \models \phi(\sigma(a_1), \dots, \sigma(a_m)) \end{aligned}$$

Thus σ is an automorphism $\square$

Lemma 5.8 shows that it would be useful to find order indiscernibles where there are many order-preserving permutations. Indeed, once we have an infinite sequence of order indiscernibles, we can find them of any given order type.

Let $X = (x_i : i \in I)$ be a sequence of order indiscernibles in $\mathcal{M}$. Let

$$\text{tp}(I) = \{\phi(v_1, \dots, v_n) : \mathcal{M} \models \phi(x_{i_1}, \dots, x_{i_n}), i_1 < \dots < i_n \in I, n \in \omega\}$$

We call $\text{tp}(X)$ the **type of the indiscernibles**. Note that $\text{tp}(X)$ is maximal as it's a sequence of order indiscernibles.

Lemma 5.9. *Let T^* be an $\mathcal{L}^*$-theory with built-in Skolem functions. Suppose that $X = (x_i : i \in I)$ is an infinite sequence of order indiscernibles in $\mathcal{M} \models T^*$. If $(J, <)$ is any infinite ordered set, we can find $\mathcal{N} \models T^*$ containing a sequence of order indiscernibles $Y = (y_j : j \in J)$ and $\text{tp}(X) = \text{tp}(Y)$.*

Proof. Add to $\mathcal{L}^*$ constant symbols c_j for $j \in J$ and let

$$\Gamma = T^* \cup \{c_i \neq c_j : i, j \in J, i \neq j\} \cup \{\phi(c_{i_1}, \dots, c_{i_m}) : i_1 < \dots < i_m \in J \text{ and } \phi \in \text{tp}(X)\}$$

If Δ is a finite subset of Γ, then by choosing elements of X we can make $\mathcal{M}$ a model of Δ

If $\mathcal{N} \models \Gamma$, then the interpretation of the $(c_j : j \in J)$ is the desired indiscernible sequence. $\square$

Lemma 5.10. *Suppose that T^* is an $\mathcal{L}^*$-theory with built-in Skolem functions. If I is a sequence of order indiscernibles in $\mathcal{M} \models T^*$ and J is a sequence of order indiscernibles in $\mathcal{N} \models T^*$ with $\text{tp}(I) = \text{tp}(J)$, then any order-preserving map $\tau : I \to J$ extends to an elementary embedding $\sigma : \mathcal{H}(I) \to \mathcal{H}(J)$*

Proof. If $a = t(x_1, \dots, x_n)$ for t a term and $x_1, \dots, x_n \in I$ we let $\sigma(a) = t(\tau(x_1), \dots, \tau(x_n))$. We then argue as in Lemma 5.8 that this map is well-defined and elementary $\square$

We give several applications of this method

Corollary 5.11. *Let T be an T-theory with infinite models. For any $\kappa \geq |\mathcal{L}| + \aleph_0$, there is $\mathcal{N} \models T$ of cardinality κ with 2^κ automorphisms*

Proof. Let $\mathcal{L}^*$ and T^* be as above. We can find $\mathcal{M} \models T^*$ containing an infinite sequence of order indiscernibles I.

Claim. There is a linear order $(X, <)$ of size κ with 2^κ order-preserving permutations

Let $X = \kappa \times \mathbb{Q}$ with the lexicographic ordering $(\alpha, q) < (\beta, r)$ if $\alpha < \beta$ or $\alpha = \beta$ and $q < r$. For each $A \subseteq \kappa$ let σ_A be the order-preserving permutation

$$\sigma_A((\alpha, q)) = \begin{cases} (\alpha, q) & \alpha \in A \\ (\alpha, q+1) & \alpha \notin A \end{cases}$$

Clearly $\sigma_A = \sigma_B$ iff $A = B$. Thus there are 2^κ order-preserving permutations of X.

By Lemma 5.9 we can find $\mathcal{N} \models T^*$ containing J a sequence of order indiscernibles of order type $(X, <)$. By lemma 5.8 each order preserving permutation of the indiscernibles induces an automorphisms of $\mathcal{H}(J)$. Thus $\mathcal{H}(J)$ has 2^κ automorphisms and $|\mathcal{H}(J)| = \kappa$. When viewed as an $\mathcal{L}$-structure, $\mathcal{N}$ still has 2^κ automorphisms $\square$

Corollary 5.12. *Suppose that T^* is an $\mathcal{L}^*$-theory with built-in Skolem functions, $\mathcal{M} \models T^*$, $\mathcal{M}$ omits p (a type over $\emptyset$), and $\mathcal{M}$ contains an infinite sequence of order indiscernibles I. There are arbitrarily large models of T^* omitting p.*

Proof. Let $\kappa \geq \aleph_0$. By Lemma 5.9, we can find $\mathcal{N} \models T^*$ containing a sequence or order indiscernibles J with $|J| \geq \kappa$, and $\text{tp}(I) = \text{tp}(J)$. Then $|\mathcal{H}(J)| \geq \kappa$. Suppose that $(a_1, \dots, a_n) \in \mathcal{H}(J)$ realizes p. Let $a_i = t_i(x_1, \dots, x_m)$, where t_i is a Skolem term, $x_1 < \dots < x_m$, and each $x_i \in J$. If $y_1 < \dots < y_m$ is an increasing sequence in I, then, because $\text{tp}(I) = \text{tp}(J)$

$$\mathcal{M} \models \phi(t_1(\bar{y}), \dots, t_n(\bar{y})) \Leftrightarrow \mathcal{N} \models \phi(a_1, \dots, a_n)$$

Thus $(t_1(\bar{y}), \dots, t_n(\bar{y}))$ realizes $p \in \mathcal{M}$, a contradiction. $\square$

Theorem 5.13. *Let $\mathcal{L}$ be countable and T be an $\mathcal{L}$-theory with infinite models. For all $\kappa \geq \aleph_0$, there is $\mathcal{M} \models T^*$ with $|M| = \kappa$ s.t. if $A \subseteq M$, then $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types in $S_n^{\mathcal{M}}(A)$.*

Proof. We assume that $n = 1$. Let $\mathcal{L}^*$ and T^* be as above. Let $\mathcal{M} \models T$ be the Skolem hull of a sequence of order indiscernibles I or order type $(\kappa, <)$. Then $|M| = \kappa$.

Let $A \subseteq M$. For each $a \in A$, there is a term t_a and $\bar{x}_a$ a sequence from I s.t. $a = t_a(\bar{x}_a)$. Let $X = \{x \in I : x \text{ occurs in some } \bar{x}_a\}$. Then $|X| \leq |A| + \aleph_0$

If $y_1 < \cdots < y_n$ and $z_1 < \cdots < z_n$, we say that $\bar{y} \sim_X \bar{z}$ if for all $x \in X$, $y_i < x$ iff $z_i < x$ and $y_i = x$ iff $z_i = x$ for $i = 1, \ldots, n$

Claim If $\bar{y} \sim_X \bar{z}$ and t is a Skolem term, then $t(\bar{y})$ and $t(\bar{z})$ realize the same type in $S_1^{\mathcal{M}}(A)$.

Let $a_1, \ldots, a_m \in A$. Because $\bar{y}$ and $\bar{z}$ are in the same position in the ordering w.r.t. X, by indiscenibility

$$\begin{aligned} \mathcal{M} \models \phi(t(\bar{y}), a_1, \ldots, a_m) \quad &\leftrightarrow \quad \mathcal{M} \models \phi(t(\bar{y}), t_{a_1}(\bar{x}_{a_1}), \ldots, t_{a_m}(\bar{x}_{a_m})) \\ &\Leftrightarrow \quad \mathcal{M} \models \phi(t(\bar{z}), t_{a_1}(\bar{x}_{a_1}), \ldots, t_{a_m}(\bar{x}_{a_m})) \\ &\Leftrightarrow \quad \mathcal{M} \models \phi(t(\bar{z}), a_1, \ldots, a_m) \end{aligned}$$

It suffices to show that $|I^n / \sim_X| \leq |A| + \aleph_0$.For $y \in I \setminus X$, let $C_y = \{x \in X : x < y\}$. Then $\bar{y} \sim_X \bar{z}$ iff for each i:

1. if $y_i \in X$, then $y_i = z_i$ and
2. if $y_i \notin X$, then $z_i \notin X$ and $C_{y_i} = C_{z_i}$

Because I is well-ordered, $C_y = C_z$ iff $C_y = C_z = \emptyset$ or

$$\inf\{i \in I : i > C_y\} = \inf\{i \in I : i > C_z\}$$

In particular, there are at most $|X| + 1$ possible cuts C_y. It follows that $|I^n / \sim_X| \leq |A| + \aleph_0$ and $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types over A $\square$

Corollary 5.14. *Let T be a complete theory in a countable language with infinite models, and let $\kappa \geq \aleph_1$. If T is κ-categorical, then T is ω-stable*

Proof. If T is not ω-stable, then we can find a countable $\mathcal{M} \models T$ with $A \subseteq M$ s.t. $\left|S_n^{\mathcal{M}}(A)\right| > \aleph_0$. By Compactness, we can find $\mathcal{M} \prec \mathcal{N}_0$ of cardinality κ (as long as $\kappa > \aleph_0$) realizing uncountably many types in $S_n^{\mathcal{M}}(A)$. By Theorem 5.13 we can find $\mathcal{N}_1 \models T$ of cardinality κ s.t. for all $B \subset M$ if $|B| = \aleph_0$, then $\mathcal{N}_1$ realizes at most $\aleph_0$ types over B. Then $\mathcal{N}_0 \ncong \mathcal{N}_1$, contradicting κ-categoricity $\square$

Combining Corollary 5.14 with Theorem 4.60

Corollary 5.15. *Let T be a complete theory in a countable language with infinite models. If $\kappa \geq \aleph_1$ and T is κ-categorical, then T has no Vaughtian pairs and hence no (κ, λ)-models for $\kappa > \lambda \geq \aleph_0$*

Proof. Because T is κ-categorical, T is ω-stable. If there is a Vaughtian pair, then by Theorem 4.53 there is an $(\aleph_1, \aleph_0)$-model, and by Theorem 4.60, a $(\kappa, \aleph_0)$-model. Because we can find a model of T of cardinality κ where every infinite definable set has cardinality κ, this is a contradiction. □

5.2.2 Indiscernibles in Stable Theories

We have seen that, although it is always possible to find infinite sequences of order indiscernibles, for some theories we cannot find infinite indiscernibles sets.

Lemma 5.16. *For any infinite cardinal κ, there is a dense linear order $(A, <)$ with $B \subset A$ s.t. B is dense in A and $|B| \leq \kappa < A$*

Proof. Let $\lambda \leq \kappa$ be least s.t. $2^\lambda > \kappa$. Let A be the set of all functions from λ to $\mathbb{Q}$. If we order A by $f < g$ iff $f(\alpha) < g(\alpha)$, where α is least s.t. $f(\alpha) \neq g(\alpha)$(lexicographic order□), then $(A, <) \vDash \mathsf{DLO}$.

Let B be the set of sequences in A that are eventually 0 (Maybe, that is, there is a $\gamma < \lambda$ s.t. $f(\beta) = 0$ for all $\gamma < \beta < \lambda$) Then

$$|B| = \sup\{\mu < \lambda : 2^\mu\} \leq \kappa$$

and, for all $f, g \in X$, if $f < g$, there is $h \in Y$ s.t. $f < h < g$ □

Theorem 5.17. *Suppose that $\mathcal{L}$ is a countable language, κ is an infinite cardinal and T is a κ-stable $\mathcal{L}$-theory. If $\mathcal{M} \vDash T$ and $X \subseteq M$ is an infinite sequence of order indiscernibles, then X is a set of indiscernibles*

Proof. Let $\phi(v_1, \dots, v_n)$ be an $\mathcal{L}$-formula and $x_1, \dots, x_n$ be an increasing sequence from I s.t. $\mathcal{M} \vDash \phi(x_1, \dots, x_n)$. Let S_n be the group of all permutations of $\{1, \dots, n\}$. Let $\Gamma_\phi = \{\sigma \in S_n : \mathcal{M} \vDash \phi(x_{\sigma(1)}, \dots, x_{\sigma(n)})\}$. To show that X is a set of indiscernibles, we must show that $\Gamma_\phi = S_n$

Claim $\Gamma_\phi = S_n$.

Suppose not. Because every permutation is a product of transpositions we can find $\sigma \in \Gamma_\phi$ and $\tau \in S_n \setminus \Gamma_\phi$ s.t. $\tau = \sigma \circ \mu$ for some transpositions μ (If we can't find such σ, then $S_n \setminus \Gamma_\phi = S_n$). Say $\mu(y_1, \dots, y_n) = (y_1, \dots, y_{m-1}, y_{m+1}, y_m, y_{m+2}, \dots, y_n)$.

Let $\psi(v_1, \dots, v_n)$ be the formula $\phi(v_{\sigma(1)}, \dots, v_{\sigma(n)})$. Then

$$\mathcal{M} \models \psi(x_1, \dots, x_n)$$

but

$$\mathcal{M} \models \neg\psi(x_1, \dots, x_{m-1}, x_{m+1}, x_m, x_{m+2}, \dots, x_n)$$

Let $(A, <)$ and B be as in Lemma 5.16, we can find $\mathcal{N} \models T$ containing a sequence of order indiscernibles Y of order type $(A, <)$ with tp$(Y) =$ tp$(X)$by Lemma 5.9. Let Y_0 be the subset of Y corresponding to B. If $y_1 < \cdots < y_n$ are in Y, then

$$\mathcal{N} \models \psi(y_1, \dots, y_n)$$

and

$$\mathcal{N} \models \neg\psi(y_1, \dots, y_{m-1}, y_{m+1}, y_m, y_{m+2}, \dots, y_n)$$

If $x, y \in Y$ and $x < y$ we can find $z_1, \dots, z_{n-1}$ in Y_0 s.t. $z_1 < \cdots < z_{k-1} < x < z_k < y < z_{k+1} < \cdots < z_{n-1}$. Then

$$\mathcal{N} \models \psi(z_1, \dots, z_{k-1}, x, z_k, \dots, z_{n-1})$$

but

$$\mathcal{N} \models \neg\psi(z_1, \dots, z_{k-1}, y, z_k, \dots, z_{n-1})$$

Thus any two elements of Y realize distinct 1-types over Y_0. Because $|Y_0| \leq \kappa < |Y|$, T is not κ-stable, a contradiction $\square$

5.2.3 Applications of Erdős–Rado

😄

5.3 A Many-Models Theorem

Let T be a complete theory in a countable language with infinite models.

Definition 5.18. We say that T is **stable** if it. is λ-stable for some $\lambda \geq \aleph_0$; otherwise we say T is **unstable**

Shelah proved that if T is unstable, then T has 2^κ nonisomorphic models of cardinality κ for all $\kappa \geq \aleph_0$. Indeed, he showed that unless T is κ-stable for all $\kappa \geq 2^{\aleph_0}$, then T has the maximal number of nonisomorphic models for each uncountable cardinal κ.

Assumptions.

- T is a complete theory in a countable language $\mathcal{L}$ with infinite models

- there is a binary relation symbol $<$ in the language
- there is $\mathcal{M} \models T$ containing an infinite set linearly ordered by $<$

Theorem 5.19. *If $\kappa \geq \aleph_1$, then there are 2^κ nonisomorphic models of T*

We will prove this only for regular $\kappa > \aleph_1$.

6 ω-Stable Theories

6.1 Uncountably Categorical Theories

Throughout this chapter, T will be a complete theory in a countable language with infinite models

We say that T is **uncountably categorical** if it is κ-categorical for some uncountable κ

Theorem 6.1 (Categoricity Theorem)**.** *If T is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ*

In Theorem 5.14 and 5.15, we proved two important facts about uncountably categorical theories

- if $\kappa \geq \aleph_1$ and T is κ-categorical, then T is ω-stable
- if $\kappa \geq \aleph_1$ and T is κ-categorical, then T has no Vaughtian pairs

6.1.1 Strongly Minimal Sets

If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(\bar{v})$ is an $\mathcal{L}_M$-formula, we will let $\phi(\mathcal{M})$ denote the elements of M that satisfy ϕ

Definition 6.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $D \subseteq M^n$ be an infinite definable set. We say that D is **minimal** in $\mathcal{M}$ if for any definable $Y \subseteq D$ either Y is finite or $D \setminus Y$ is finite. If $\phi(\bar{v}, \bar{a})$ is the formula that defines D, then we also say that $\phi(\bar{v}, \bar{a})$ is minimal

We say that D and ϕ are **strongly minimal** if ϕ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$

We say that a theory T is **strongly minimal** if the formula $v = v$ is strongly minimal

Let $\mathcal{L} = \{E\}$ and consider the $\mathcal{L}$-structure $\mathcal{M}$, where E is an equivalence relation with one class of size n for $n = 1, 2, \ldots$ and no infinite classes. In this structure, $v = v$ is a minimal formula, but suppose that $\mathcal{M} \prec \mathcal{N}$ and $a \in N$ s.t. the equivalence class of a is infinite. Then the formula vEa defines an infinite-coinfinite subset of the universe. Thus the formula $v = v$ is not strongly minimal

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $D \subseteq M$ be strongly minimal. We will consider acl_D, the algebraic closure relation restricted to D. Recall that b is algebraic over A if there is a formula $\phi(x, \bar{a})$ with $\bar{a} \in A$ s.t. $\phi(\mathcal{M}, \bar{a})$ is finite and $\phi(b, \bar{a})$. For $A \subseteq D$, we let $\mathrm{acl}_D(A) = \{b \in D : b \text{ is algebraic over } A\}$

If K is an algebraically closed field and $A \subseteq K$, then $\mathrm{acl}(A)$ is the algebraic closure of the subfield generated by A (Proposition **??**). If G is a torsion-free divisible Abelian group, then $\mathrm{acl}(A)$ is the $\mathbb{Q}$-vector space span of A (Exercise **??**)

Lemma 6.3. *1. $acl(acl(A)) = acl(A) \supseteq A$*

2. If $A \subseteq B$, then $acl(A) \subseteq acl(B)$

3. If $a \in acl(A)$, then $a \in acl(A_0)$ for some finite $A_0 \subseteq A$

Proof. 1. For any $a \in A$, just consider $x = a$. Hence $A \subseteq \mathrm{acl}(A)$. Let $B = \mathrm{acl}(A)$. For any $c \in \mathrm{acl}(\mathrm{acl}(A))$, there is a formula ϕ s.t. $\phi(\mathcal{M}, \bar{b})$ is finite and $\phi(c, \bar{b})$. For each b, the is a formula ψ_b s.t. $\psi_b(\mathcal{M}, \bar{a})$ is finite and $\psi_b(b, \bar{a})$. Then we could use $x \in \psi_b(\mathcal{M}, \bar{a})$ to represent b.

3. $\phi(\mathcal{M}, \bar{a})$ is finite and $\phi(a, \bar{a})$.

□

Lemma 6.4 (Exchange Principle)**.** *Suppose that $D \subset M$ is storngly minimal, $A \subseteq D$, and $a, b \in D$. If $a \in acl(A \cup \{b\}) \setminus acl(A)$, then $b \in acl(A \cup \{a\})$.*

Abuse of symbols, acl here is actually acl_D

Proof. We write $\mathrm{acl}(A, b)$ for $\mathrm{acl}(A \cup \{b\})$.

Suppose that $a \in \mathrm{acl}(A, b) \setminus \mathrm{acl}(A)$. Suppose that $\mathcal{M} \models \phi(a, b)$, where ϕ is a formula with parameters from A and $|\{x \in D : \phi(x, b)\}| = n$. Let $\psi(w)$ be the formula asserting that $|\{x \in D : \phi(x, w)\}| = n$. If $\psi(w)$ defines a finite subset of D, ψ is an $\mathcal{L}_A$-formula as ϕ and $\psi(D)$, so $b \in \mathrm{acl}_D(A)$ then $b \in \mathrm{acl}(A)$ and $a \in \mathrm{acl}(A)$, a contradiction □

A Set Theory

A.1 Cardinals

$\kappa > \aleph_0$ is **inaccessible** if κ is a regular limit cardinal

A.2 Cardinal Arithmetic

Corollary A.1. *1. If $|I| = \kappa$ and $|A_i| \leq \kappa$ for all $i \in I$, then $|\bigcup A_i| \leq \kappa$*

2. If κ is regular, $|I| < \kappa$ and $|A_i| < \kappa$ for all $i \in I$, then $|\bigcup A_i| < \kappa$

3. Let κ be an infinite cardinal. Let X be a set and $\mathcal{F}$ a set of functions $f : X^{n_f} \to X$. Suppose that $|\mathcal{F}| \leq \kappa$ and $A \subseteq X$ with $|A| \leq \kappa$. Let $cl(A)$ be the smallest subset of X containing A closed under the functions in $\mathcal{F}$. Then $|cl(A)| \leq \kappa$

Lemma A.2. *Let κ, λ, and μ be cardinals*

1. $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$
2. *if $\lambda \geq \aleph_0$ and $2 \leq \kappa < \lambda$, then $2^\lambda = \kappa^\lambda = \lambda^\lambda$*
3. *if κ is regular and $\lambda < \kappa$, then $\kappa^\lambda = \sup\{\kappa, \mu^\lambda : \mu < \kappa\}$*

Proof. 3. If $f : \lambda \to \kappa$, because κ is regular, then is $\alpha < \kappa$ s.t. $f : \lambda \to \alpha$. Thus $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$. The right-hand side is the union of κ sets each of size μ^λ for some $\mu < \kappa$

□

An inaccessible cardinal κ is **storngly inaccessible** if $2^\lambda < \kappa$ for all $\lambda < \kappa$

Corollary A.3. *If κ is strongly inaccessible and $\lambda < \kappa$, then $\kappa^\lambda = \kappa$.*

A.3 Finite Branching Trees

Definition A.4. A **finite branching tree** is a partial order $(T, <)$ s.t.

1. there is $r \in T$ s.t. $r \leq x$ for all $x \in T$
2. if $x \in T$, then $\{y : y < x\}$ is finite and linearly ordered by <
3. if $x \in T$, then there is a finite (possibly empty) set $\{y_1, \dots, y_m\}$ of incomparable elements s.t. each $y_i > x$ and if $z > x$, then $z \geq y_i$ for some i

A **path** through T is a function $f : \omega \to T$ s.t. $f(n) < f(n+1)$ for all n

Lemma A.5 (Kőnig's Lemma)**.** *If T is an infinite finite branching tree, then there is a path through T*

Proof. Let $S(x) = \{y : y \geq x\}$ for $x \in T$. We inductively define $f(n)$ s.t. $S(f(n))$ is infinite for all n. Let r be the minimal element of T, then $S(r)$ is infinite. Let $f(0) = r$. Given $f(n)$, let $\{y_1, \dots, y_m\}$ be the immediate successors of $f(n)$. Because $S(f(n)) = S(y_1) \cup \dots \cup S(y_n)$, $S(y_i)$ is infinite for some i. Let $f(n+1) = y_i$. □

B Reference

References

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