Seminar on Topological Dynamics of Definable Group Actions

Introduction

Definition. A G-flow is a compact (Hausdorff!) topological space X together with a group G acting on X by homeomorphisms.

- A subflow is a closed subset also closed under the action of G.
- A flow is minimal if it doesn't have proper subflow.
- A G-set is homogeneous if the action is transitive.
- A flow is point transitive if it contains a dense G-orbit.
- If $cl(o(x))$ is minimal, then we call the point x almost periodic.

Remark 1. Topological dynamics is concerned with the orbits of the actions of G on various G-flows, and particularly with minimal flows.

Proposition 1. Assume X is a G-flow, $o(x)$ denotes the orbit of $x \in X$, $cl(A)$ denotes the closure of $A \subset X$, $\mathcal{O}$ denotes an orbit.

1. $cl(o(x))$ is a point transitive subflow of X.
2. If $x \in cl(\mathcal{O})$, $cl(o(x)) \subseteq cl(\mathcal{O})$.
3. Any orbit $\mathcal{O}$ contains a minimal subflow.(By Zorn's lemma)
4. Any minimal flow is point transitive.
5. Any point in a minimal flow is almost periodic.
6. The intersection of subflows is still a subflow.
7. Any two distinct minimal flows are disjoint.

Proof. 1. We only need to check $cl(o(x))$ is closed under group action. Note $G \cdot o(x) = o(x)$, if y is a accumulative point of $o(x)$, gy is a accumulative point of $G \cdot o(x)$ by homeomorphism.

2. Trivial.
3. Use compactness and Zorn's lemma.
4. By (1).
5. By (1).
6. Easy to check.
7. By (6).

$\square$

There is a natural notion of a morphism of G-flows(called G-mapping), just a combinition of G-equivariant(G-set) and continuous function(topological space). So G-flows becomes a category. Point transitive G-flows are a subcategory.

There is a unique largest universal point transitive G-flow, namely $X = \beta G$, the space of ultrafilters of G, where the action is left translation by G. The orbit consisting of the principal ultrafilters is dense in βG. For every point transitive G-flow X there is a surjective G-mapping $\beta G \to X$ and every minimal flow in X is an image of a minmal flow in βG under the map.

Remark 2. Here is a more detailed explaination. Assume X is a topological space, $f : A \to X$ is function, $\mathcal{U}$ is a ultrafilter, then $b = \varinjlim_{\mathcal{U}} f(x)$ is an ultralimit if for any neighborhood $N \ni b$, $\{a \in A : f(a) = b\} \in \mathcal{U}$. X is Hausdorff iff for any A and f, there is at most one ultralimit. X is compact iff for any A and f, there is at least one ultralimit. With the help of these facts, $\beta(-) : \mathbf{Set} \to \mathbf{Compactum}$ (the category of compact Hausdorff topological space) is actually the left adjoint of $I : \mathbf{Compactum} \to \mathbf{Set}$ where I is the forgetful functor. Let $p \in X$ such that $G \cdot p$ is dense in X, a map $g \mapsto g \cdot p$ corresponds to a map $\beta G \to X$. The image is closed and contains $G \cdot p$, so it's a surjection.

Definition. Let $T = Th(M)$ be a complete theory, M be a model, G be a definable group. A G-set V is definable if the underlying set V and the group action are both definable.

The largest homogeneous G-set is $V = G$, with the action by left translation. Definable G-sets don't carry topology, but We have various(?) topological spaces related to a G-set V in model theory.

Let $\mathfrak{C}$ be a monster model (κ-saturated and strongly κ-homogeneous for some sufficiently large κ), $M \preceq N$ are small models. By V^N we mean the set $\varphi(N)$ where φ defines V in M. If V is definable G-set, then G^N acts on V^N and $G^{\mathfrak{C}}$ acts on $V^{\mathfrak{C}}$ and also G acts on $V^{\mathfrak{C}}$.

Assume E an equivalence relation on $V^{\mathfrak{C}}$ type definable over M. For any formula $\varphi(x,y) \in E(x,y)$, we may assume $\varphi \vdash V(x) \wedge V(y)$. $\varphi(x,y)$ is reflexive, and we can require it to be symmetric since $\varphi(x,y) \wedge \varphi(y,x)$ is always symmetric. Although we can't require φ to be transitive (as a relation), we have the following lemma.

Lemma 2. Let E be a equivalence relation on $V^{\mathfrak{C}}$ type definable over M, for any symmetric $\psi(x,y) \in E$, there is a symmetric $\varphi(x,y)$ such that $\varphi(x,y) \wedge \varphi(y,z) \vdash \psi(x,z)$. Note $\varphi(x,y) \vdash \psi(x,y)$ as $z = y$.

Proof. $\{\varphi(x,y) \wedge \varphi(x,z) : \varphi \in E\} \vdash \psi(x,z)$ for any $\psi \in E$ because E is an equivalence relation and $\psi \supseteq E$. By compactness, we get a desired $\varphi(x,y)$. $\square$

Proposition 3. Assume E is an equivalence relation on $V^{\mathfrak{C}}$, type definable over M. The following conditions are equivalent.

1. $|V^{\mathfrak{C}}/E| < \kappa$.
2. $|V^{\mathfrak{C}}/E| \leq 2^{|T|+|M|}$.
3. For any symmetric $\varphi(x,y)$, there is a number $n_\varphi < \omega$ and n_φ elements $a_1, ..., a_{n_\varphi} \in V^{\mathfrak{C}}$ satisfying that for any $i < j$, $\neg\varphi(a_i, a_j)$, and for any $c \in V^{\mathfrak{C}}$, $\varphi(c, a_i)$ for some $i \leq n_\varphi$.
4. For any symmetric $\varphi(x,y)$ and $a \in V^{\mathfrak{C}}$, there is $c \in M$ with $\varphi(a,c)$. So we can require the elements $a_1, ..., a_{n_\varphi}$ in (3) to live in M.

If E satisify the equivalent condition, we say E has bounded number of classes, or briefly, E is bounded.

Proof. (2) $\implies$ (1): Trivial.
(3) $\implies$ (2): $|V^{\mathfrak{C}}/E| \leq {\aleph_0}^{|T|+|M|} = 2^{|T|+|M|}$.
(1) $\implies$ (3): If not, we can build a κ-sequence by induction. Assume we have a_i for $i < \alpha$, then we choose a_α realizing $\{\neg\varphi(x, a_i) \wedge V(x) : i < \alpha\}$, which is finitely satisfiable. Since $\neg\varphi(a_i, a_j) \implies \neg E(a_i, a_j)$, $(a_i)_{i<\kappa}$ are from different equivalence class of E.

(3) $\implies$ (4): Assume $a_1, ..., a_k$ are the representatives, $a_1, ..., a_l \in M$, $a_{l+1}, ..., a_k \notin M$, and for any $c \in M$ and $i > l$, there are $\neg\varphi(c, a_i)$. Let $\psi \equiv \bigwedge_{i=1}^{l} \neg\varphi(a_i, x)$, then $\psi(M) = \emptyset$ while $\{a_{l+1}, ..., a_k\} \subseteq \psi(\mathfrak{C})$, contradicting with Tarski-Vaught test.
(4) $\implies$ (2): $|M|^{|T|+|M|} = 2^{|T|+|M|}$.

□

Assume E is an equivalence relation on $V^{\mathfrak{C}}$, type definable over M, with bounded number of classes(shortly: btde-relation). On $V^{\mathfrak{C}}$ we have a natrual topology where the closed set are the type definable set ($\mathfrak{C}$ is the parameter set). This topology is discrete.

On the quotient set $V^{\mathfrak{C}}/E$ (we also denote it by V_E), there is a natural topology, with the closed set Z where $\pi^{-1}(Z) \subseteq V^{\mathfrak{C}}$ is a type definable set. Here $\pi : V^{\mathfrak{C}} \to V^{\mathfrak{C}}/E$ is the quotient map. This topology is called the logic topology or the *Kim-Pillay* Topology on V_E.

Proposition 4. Assume E is a equivalence relation type definable by M, we equip V_E with logic topology.

1. $Y \subseteq V_E$ is closed iff for some type definable set A, $Y = \{a_E : a \in A\}$. This is the definition of closed set in the paper.

2. A basis of open sets is the collection of all $U_{a\varphi}$. Here $U_{a\varphi} = \{b_E : \varphi(a', b') \text{ for all } E(a, a'), E(b, b')\}$, $\varphi \in E$.

3. V_E is Hausdorff.

4. V_E is compact iff E is bounded.

Proof. 1. $\Rightarrow$: Trivial. $\Leftarrow$: $\pi^{-1}(Y) = \{a : a_E \in Y\}$ is type defined by $\exists y(E(x, y) \wedge A(y))$.

2. $U_{a\varphi}$ is open because $\{b : b_E \notin U_{a\varphi}\}$ is type definable by $\exists y \exists z(E(a, y) \wedge E(x, z) \wedge \neg\varphi(y, z))$.

   Assume U is open and $a_E \in U$, $V_E \backslash U$ is closed and type defined by a partial type $\Sigma(x)$ where $\Sigma(V^{\mathfrak{C}}) = \pi^{-1}(V_E \backslash U)$. Choose $\psi \in \Sigma$ such that $\neg\psi(a)$. Since $E(x, y) \wedge \Sigma(x) \vdash \Sigma(y)$, we have $\varphi(x, y) \wedge \Sigma(y) \vdash \psi(y)$ for some $\varphi \in E(x, y)$ by compactness. Then $a_e \in U_{a\varphi} \subseteq U$.

3. Suppose $a_E \neq b_E$, then there is $\varphi \in E$ such that $\neg\varphi(a, b)$. Let $\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z)$, then $U_{a\varphi'} \ni a$ and $U_{b\varphi'} \ni b$ don't intersect.

4. Assume E is bounded, $(F_i)_{i\in I}$ is a family of closed sets with finite intersection property. The number of E-classes is bounded, so the number of closed set is also bounded by at most a power. $\bigcap_{i\in I} F_i$ is realized by some $a \in V^{\mathfrak{C}}$.

   Assume V_E is compact, for any $\varphi \in E$, let $(a_i)_{i\in I}$ be representatives. Now $V_E = \bigcup_{i\in I} U_{a_i\varphi}$, so there is finite $I_0 \subseteq I$ such that $V_E = \bigcup_{i\in I_0} U_{a_i\varphi}$ by compactness. These are finite representatives.

□

A btde-relation E on $V^{\mathfrak{C}}$ is G-invariant if for any $x, y \in V^{\mathfrak{C}}$, $xEy \implies$ (or $\iff$, equivalently) $gxEgy$ for any $g \in G$. In this case, a group action of G on V (is a homeomorphism) induces a homeomorphism on V_E. Hence V_E becomes a G-flow. We call any G-flow of this kind a definable G-flow.

Proposition 5. If V is a homogeneous G-set, then the flow V_E is point transitive.

Proof. We claim the orbit consisting a_E for $a \in V$ is dense in V_E. Suppose $[b]_E \cap W = \emptyset$ where W is a type definable set of V_E. We need to prove $[a]_E \cap W = \emptyset$ for some $a \in V$. Let symmetric φ, φ' be with $\varphi(\mathfrak{C}, b) \cap W = \emptyset$, $\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z)$. Since E is bounded, there is $a \in \varphi'(\mathfrak{C}, b) \cap M \subseteq V$. Let $c \in \varphi'(\mathfrak{C}, a)$, then $c \in \varphi(\mathfrak{C}, b)$, so $\varphi'(\mathfrak{C}, a) \cap W = \emptyset$, and hence $[a]_E \cap W = \emptyset$. □

Remark 3. Transitivity of group action is a first order property, so V_E is a homogeneous $G^{\mathfrak{C}}$-set.

Proposition 6. Assume E is a btde-relation, $tp(a/M) = tp(b/M)$, then aEb.

Proof. Assume symmtric φ' satisfying $\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z)$. Because E is bounded, there is $c \in M$ such that $\varphi'(a, c)$. Since $tp(a/M) = tp(b/M)$, we have $\varphi'(b, c)$ and then $\varphi(a, b)$. So for any φ we have $\varphi(a, b)$, this is $E(a, b)$.

□

There is a finest btde-relation, namely $\equiv_M$, given by $x \equiv_M y \iff tp(x/M) = tp(y/M)$. $\equiv_M$ is G-invariant. Hence $V_{\equiv_M}$ is the largest definable G-flow, where the group action is by left translation on type.

Our interest is $G_{\equiv_M}$(also denoted by $S_G(M)$). Any point transitive definable G-flow V_E is isomorphic to $G_{E'}$ for some E' coarser than $\equiv_M$.

Remark 4. Generic type is of the central notions in stable theory. The paper is to generalize generic type to a broader, unstable context, introducing weak generic types. In this paper, we relate notions to the basic ideas of Topological dynamics as a good language to set up.

If $M \preceq N$ are small models, then $S_G(N)$ is a point transitive G^N-flow, and also a (point transitive?) G-flow. The natural restriction $r : S_G(N) \to S_G(M)$ is a morphism of G-flows. Investigation of the relationship between the topological dynamics of $S_G(N)$ and $S_G(M)$ has a new, specifically model theoretical flavour.

In the stable context, generic types on $S_G(M)$ are thought of as "large" types, and then it is natural that the restriction of a generic type in $S_G(N)$ to M is still a generic type. Moreover, a type $q \in S_G(M)$ is generic iff $q|M$ is generic in $S_G(M)$ and the extension $q \supseteq q|M$ is non-forking. So in the stable context the notion of a generic type is closedly related to forking independence.

Inside weak generic types, we distinguish an even smaller subset of almost periodic types. We will see which notion is a better counterpart of the notion of generic type by investigating their extension and restriction properties. A complicated example in Section 3 show that restriction of weak generic type is still a weak generic but not true for almost periodic type.