Chapter 2
Stability

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1 Historic remarks and motivations

- Can a first order theory T determine its models;
- Any theory T with an infinite model has models of arbitrary infinite cardinalities (L-S-T);
- For a fixed infinite cardinal κ, how many models of T has cardinal κ?;
- Consider the function $I_T(-) : \kappa \mapsto \#\{\text{models of } T \text{ of card. } \kappa\}$.
- Then $1 \leq I_T(\kappa) \leq 2^\kappa$;
- $\#\{L\text{-structures of card. } \kappa\} \leq 2^\kappa$;

Fact 1.1. *[Morley's Theorem] Let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable cardinal κ, then $I_T(\kappa) = 1$ for all uncountable cardinal κ. (Categoricity)*

Example 1.2. .

- The Theory of infinite sets;
- The theory of vector space over a fixed countable field;
- The theory of algebraicly closed fields with fixed char;
- The theory of $(\mathbb{Z}, S, 0)$.

Shelah's stability theory intended to generalize Morley's Theory and classify the complete first order theories.

Conjecture 1.3. *[Morley] Let T be countable, then the function $I_T(\kappa)$ is non-decreasing on uncountable cardinals.*

Fact 1.4. *[Shelah's Main gap theorem] Let T be a countable first order complete theory T. then one of these situations holds:*

- $\forall \alpha,\ I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$
- $\forall \alpha,\ I_T(\aleph_\alpha) < \beth_{\aleph_1}(|\alpha|)$.

Here, $\beth_0(\kappa) = \kappa$, $\beth_\alpha(\kappa) = 2^{\beth_{\alpha+1}(\kappa)}$, and $\beth_\nu(\kappa) = \sup\{\beth_\alpha(\kappa) |\ \alpha < \nu\}$ for limit ordinals ν.

Remark 1.5. .

- The name "Main Gap" refers to the gap between $\beth_{\aleph_1}(|\alpha|)$ and $2^{\aleph_\alpha}$ $(\alpha \geq \omega)$
- Depending on α this may be no gap at all;
- But in general $\beth_{\aleph_1}(|\alpha|)$ goes only moderately compared to $2^{\aleph_\alpha}$;
- The case $I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$ is called the "non-structure case", we have a kind of chaos.
- The second case, namely, the case where there are relatively few non-isomorphic models, is called the "structure case";
- In this case every model can be characterized up to isomorphism in terms of certain invariants;
- The most important "dividing lines" on the space of first-order theories is "stability";
- Main gap theorem says that: "If T is a first-order theory and is stable and . . . , then the class of models looks like . . . . Otherwise, there's no hope".

2 Counting types and stability

Definition 2.1. For a complete first order theory T, let $f_T : \text{Card} \to \text{Card}$ be defined by

$$f_T(\kappa) = \sup\{\ |S_1M|\ :\ M \models T,\ |M| = \kappa\},$$

for κ an infinite cardinal.

It is esay to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$.

Fact 2.2. *Let T be an arbitrary complete theory in a first order language. The $f_T(\kappa)$ is one of the following functions*

$$\kappa, \kappa + 2^{\aleph_0},\ \operatorname{ded} k,\ (\operatorname{ded} k)^{\aleph_0},\ 2^\kappa$$

Here

$$\operatorname{ded}\kappa = \sup\{\ |I| :\ I \textit{ is a linear order with a dense subset of size } \kappa\}$$

$$\operatorname{ded}\kappa = \sup\{\ \lambda :\ \ \textit{There is a linear order of size } \kappa \textit{ with } \lambda \textit{ cuts}\}$$

Lemma 2.3. $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$.

Proof. $\kappa < \operatorname{ded}\kappa$:

- Let μ be minimal such that $2^\mu > \kappa$;
- Consider 2^μ as a set of $0-1$ sequence of length μ;
- then $2^{<\mu}$ is a dense subset of 2^μ;
- $\mu \leq \kappa \implies 2^{<\mu} \leq \kappa$;
- so $\operatorname{ded}\kappa \geq \mu > \kappa$.

$\operatorname{ded}\kappa \leq 2^\kappa$:

- Every cut is determined by the subset of elements in its lower half.

□

Definition 2.4. Let $M \models T$.

- A formula $\phi(x, y)$, with its variables partitioned into two groups x, y, has the k-order property, $k \in \omega$, if there are some $a_i \in M_x$, $b_j \in M_y$ for $i, j < k$ such that

$$M \models \phi(a_i, b_j) \iff i < j$$

- $\phi(x, y)$ has the order property if it has the k-order property for all $k \in \omega$;
- We say that a formula $\phi(x, y) \in L$ is stable if there is some $k \in \omega$ such that it does not have the k-order property.
- A theory is stable if it implies that all formulas are stable (note that this is indeed a property of a theory).

Proposition 2.5. *Assume that T is unstable, then $f_T(\kappa) \geq \operatorname{ded}\kappa$ for all cardinals $\kappa \geq |T|$.*

Proof.

- Fix a cardinal κ. Let $\phi(x, y) \in L$ be a formula has the k-order property for all $k \in \omega$;
- Let $(I, <)$ be a dense linear order order of size κ;
- Let $a_{i\in I}$ and $b_{i\in I}$ be two sequences of new constants;
- Then $\{\phi(a_i, b_j)|\ i < j\} \cup \{\neg\phi(a_i, b_j)|\ i \geq j\}$ is consistent with T;
- By compactness, there is a model $\mathcal{M} \models T$ and $a_{i\in I}$, $b_{i\in I}$ from M such that

$$\mathcal{M} \models \phi(a_i, b_j) \iff i < j$$

- By L-S-T, we may assume that $|M| = \kappa$;
- For any cut $C = (A, B)$ in I

$$\Phi_C(x) = \{\phi(x, b_j)|\ i \in B\} \cup \{\neg\phi(x, b_j)|\ j \in A\}$$

  is a partial type over M;
- It is easy to see that $C_1 \neq C_2 \implies \Phi_{C_1} \cup \Phi_{C_2}$ is inconsistent;
- Let $p_C(x) \in S_x(M)$ be a complete extension of $\Phi_C(x)$;
- Then $|S_x(M)| \geq$ number of cuts in I;
- As I is arbitrary,

$$f_T(\kappa) = \sup\{|S_x(M)|\ M \models T, |M| = \kappa\} \geq \operatorname{ded} \kappa$$

□

Recall

Fact 2.6 (Ramsey Theory). $\aleph_0 \to (\aleph_0)^n_k$ *holds for all* $n, k \in \omega$ *(i.e.for any coloring of subsets of N of size n in k colors, there is some infinite subset I of N such that all n-element subsets of I have the same color).*

Lemma 2.7. *Let* $\phi(x, y)$, $\psi(x, z)$ *be stable formulas (where* y, z *are not necessarily disjoint tuples of variables). Then:*

1. *Let* $\phi^*(y, x) = \phi(x, y)$, *i.e. we switch the roles of the variables. Then* $\phi^*(y, x)$ *is stable.*
2. $\neg\phi(y, x)$ *is stable.*
3. $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ *and* $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ *are stable.*
4. *If* $y = uv$ *and* $c \in M_v$ *then* $\theta(x, u) := \phi(x, uc)$ *is stable.*
5. *If* T *is stable, then every* L^{eq}*-formula is stable as well.*

Proof. .
(1) and (2) are trivial.
(3):

- Suppose that $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ is unstable;
- there are $(a_i, b_i, c_i|\ i \in \mathbb{N})$ such that

$$\phi(a_i, b_j) \vee \psi(a_i, c_j) \iff i < j$$

- Let $f : [\mathbb{N}]^2 \to \{0,1\}$ defined by: for each $i < j \in \mathbb{N}$

$$f(i,j) = 1 \iff \models \psi(a_i, c_j) \text{ and } f(i,j) = 0 \iff \models \neg\psi(a_i, c_j)$$

- By Ramsey, there is a infinite subset $I \subseteq J$ such that
- f is constant on I;
- If $f(I) = 1$, then $\forall i, j \in I(\psi(a_i, b_j) \iff i < j)$
- If $f(I) = 0$, then $\forall i, j \in I(\phi(a_i, b_j) \iff i < j)$
- So either ϕ or ψ is unstable.

(4): Trivial. □

Theorem 2.8 (Erdös-Makkai). *Let B be an infinite set, $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(c_{i<\omega}) \subseteq B$ and $(S_{i<\omega}) \subseteq \mathcal{F}$ such that one of the following holds:*

1. $c_i \in S_j \iff j < i (\forall i, j \in \omega)$,
2. $c_i \in S_j \iff i < j (\forall i, j \in \omega)$.

We need the following lemma:

Lemma 2.9. *Let X be a set and $Y_1, ..., Y_n$ are subsets of X. Define:*

$$E(x,y) := \bigwedge_{i=1}^{n} (x \in X_i \iff y \in X_i).$$

Then E is an equivalence relation on X and $Z \subseteq X$ is a boolean combination of X_i's iff

$$E(x,y) \implies (x \in Z \iff y \in Z)$$

Proof. Exercise. □

We now proof the Theorem 2.8

Proof.
- Choose $\mathcal{F}' \subseteq \mathcal{F}$ such that
  - $|\mathcal{F}'| = |B|$;
  - For any finite $B_0, B_1 \subseteq B$,

$$\exists S \in \mathcal{F}(B_1 \subseteq S \wedge B_2 \subseteq B \backslash S) \implies \exists S' \in \mathcal{F}'(B_1 \subseteq S' \wedge B_2 \subseteq B \backslash S').$$

- $\mathcal{F}'$ exists as there are at most $|B|$-many different pairs of finite subsets of B;

- $|\mathcal{F}| > |\mathcal{F}'| \implies \exists S^* \in \mathcal{F}$ which is not a boolean combination of elements of $\mathcal{F}'$;
- Let $a_0 \in S^*$ and $b_0 \notin S^*$;
- There is $S_0 \in \mathcal{F}'$ s.t. $a_0 \in S_0$ and $b_0 \notin S_0$;
- Since S^* is NOT a boolean combination of $\{S_0\}$, there are a_1, b_1 s.t. :
  - $a_1 \in S_0 \iff b_1 \in S_0$, and ;
  - $a_1 \in S^*$ but $b_1 \notin S^*$.
- Now $\{a_0, a_1\} \subseteq S^*$ and $\{b_0, b_1\} \subseteq B \backslash S^*$;
- By the assumption of $\mathcal{F}'$, $\exists S_1 \in \mathcal{F}'(\{a_0, a_1\} \subseteq S_1 \wedge \{b_0, b_1\} \subseteq B \backslash S_1)$;
- Since S^* is NOT a b. c. of $\{S_0, S_1\}$, there are a_2, b_2 s.t. :
  - $a_2 \in S_i \iff b_2 \in S_i$, for $i < 2$, and ;
  - $a_2 \in S^*$ but $b_2 \notin S^*$.
- ...
- Inductively, we have infinite sequences $(a_{i<\omega}) \subseteq S^*$ and $(b_{i<\omega}) \subseteq B \backslash S^*$ s.t.
  - $a_n \in S_i \iff b_n \in S_i$, for $i < n$;
  - $\{a_0, ..., a_n\} \subseteq S_n$, $\{b_0, ..., b_n\} \subseteq B \backslash S_n$

  By Ramsey, there is an infinite $I \subseteq \omega$ such that
  - either $\forall n > j \in I(a_n \in S_j) \implies \forall i, j \in I(b_i \in S_j \iff i > j)$,
  - or $\forall n > j \in I(a_n \notin S_j) \implies \forall i, j \in I(a_i \in S_j \iff i \leq j)$
- In the first case we set $c_i = b_i$;
- In the second case we set $c_i = a_{i+1}$;

$\square$

Definition 2.10. Let $\phi(x, y)$ be a formula, by a complete ϕ-type over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x, b), \neg\phi(x, b)$ where b ranges over A. Let $S_\phi(A)$ be the space of all complete ϕ-types over A.

Proposition 2.11. *Assume that $|S_\phi(B)| > |B|$ for some infinite set of parameters B. Then $\phi(x, y)$ is unstable.*

Proof. .

- For $a \in \mathbb{M}_x$, $\mathrm{tp}_\phi(a/B)$ is determined by $\phi(a, B) = \{b \in B | \models \phi(a, b)\}$;

- $|S_\phi(B)| > |B| \implies |\{\phi(a,B)|\ a \in \mathbb{M}_x\}| > |B|$;
- By Erdös-Makkai, there are sequences $(a_{i<\omega})$ and $(b_{i<\omega})$ s.t.

$$\text{either } \models \phi(a_i, b_j) \iff i < j, \text{ or } \models \phi(a_i, b_j) \iff j < i.$$

$\square$

3 Local ranks and definability of types

Definition 3.1. We define Shelah's local 2-rank taking values in $\{-\infty\} \cup \omega \cup \{+\infty\}$ by induction on $n \in \omega$. Let Δ be a set of L-formulas, and $\theta(x)$ a partial type over $\mathbb{M}$.

- $R_\Delta(\theta(x)) \geq 0 \iff \theta$ is consistent (and $-\infty$ otherwise);
- $R_\Delta(\theta(x)) \geq n+1$ if $\exists \phi(x,y) \in \Delta$ and $a \in \mathbb{M}_y$ s.t.

$$R_\Delta(\theta(x) \wedge \phi(x,a)) \geq n \text{ and } R_\Delta(\theta(x) \wedge \neg\phi(x,a)) \geq n$$

- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and $R_\Delta(\theta(x)) \not\geq n+1$
- $R_\Delta(\theta(x)) = +\infty$ if $R_\Delta(\theta(x)) \geq n$ for all $n \in \omega$.

If ϕ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$.

Proposition 3.2. *$\phi(x,y)$ is stable iff $R_\phi(x=x)$ is finite.*

Proof. Assume that $\phi(x,y)$ is unstable:

- By compactness, there is a sequence $(a_i b_i | \ i \in [0,1])$ such that

$$\models \phi(a_i, b_j) \iff i < j$$

- Both $\phi(x, b_{\frac{1}{2}})$ and $\neg\phi(x, b_{\frac{1}{2}})$ contain dense subsequences of a_i's.
- Each of these sets can be split again, by $\phi(x, b_{\frac{1}{4}})$ and $\phi(x, b_{\frac{3}{4}})$;
- ...

Conversely, assume that the rank is infinite:

- We can find a infinity tree of parameters

$$B = \{b_\eta | \ \eta \in 2^{<\omega}\}$$

  such that

- for each $\eta \in 2^\omega$, let
$$\Phi_\eta = \{\phi^{\eta(n)}(x, b_{\eta|_n}) | \ n \in \omega\},$$
  where $\phi^1 = \phi$ and $\phi^0 = \neg\phi$;
- Then each Φ_η is consistent;
- Different Φ_η's are inconsistent;
- $|S_\phi(B)| \geq 2^{|B|} \implies \phi(x,y)$ is unstable.

$\square$

Definition 3.3. .

- Let $\phi(x, y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is definable over B if there is some $L(B)$-formula $\psi(y)$ such that for all $a \in A$

$$\phi(x, a) \in p \iff \models \psi(a)$$

- A type $p \in S_x(A)$ is definable over B if $p|_\phi$ is definable over B forall $\phi(x, y) \in L$.
- A type is definable if it is definable over its domain.
- We say that types in T are uniformly definable if for every $\phi(x, y)$ there is some $\psi(y, z)$ such that every type can be defined by an instance of $\psi(y, z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that

$$\phi(x, a) \in p \iff \models \psi(a, b),$$

  forall $a \in A$.

Remark 3.4. .

- Let $A \subseteq \mathbb{M}_x$, and $B \subseteq A$. We say that B is externally definable if there is some $\mathbb{M}$-definable set X such that $B = X \cap A$
- If $X = \phi(\mathbb{M}, c)$. Then $\mathrm{tp}_\phi(c/A)$ is definable iff $B = X \cap A$ is in fact internally definable.
- A set is called stably embedded if for every externally definable subset of it is internally definable.

Example 3.5. Consider $(\mathbb{Q}, <) \models DLO$, and let $p = \mathrm{tp}(\pi/\mathbb{Q})$. Then $x < y \in p(y) \iff x < \pi$. By QE, p is not definable.

Lemma 3.6. .

1. *The set* $\{e \in \mathbb{M}^k | \ R_\phi(\theta(x, e)) \geq n\}$ *is definable for all* $n \in \omega$*;*
2. *If* $R_\phi(\theta(x)) = n$*, then for any* $a \in \mathbb{M}_y$*, at most one of* $\theta(x) \wedge \phi(x, a)$*,* $\theta(x) \wedge \neg\phi(x, a)$ *has* R_ϕ*-rank* n*.*

Proof. (1):

- Induction on n.
- $n = 0:\ R_\phi(\theta(x, e)) \geq 0 \iff \models \exists x(\theta(x, e))$;

- $n = k+1$:

$$R_\phi(\theta(x,e)) \geq k{+}1 \iff \exists y(\Big(R_\phi(\theta(x,e) \wedge \phi(x,y)) \geq k\Big) \wedge \Big(R_\phi(\theta(x,e) \wedge \neg\phi(x,y)) \geq k\Big))$$

(2): Trivial. □

Proposition 3.7. *Let $\phi(x,y)$ be a stable formula. Then all ϕ-types are uniformly definable.*

Proof. .

- Suppose that $R_\phi(x=x)$ is $n \in \omega$;
- Let $p \in S_\phi(A)$;
- Then there is $\chi(x) \in p$ such that $R_\phi(\chi(x)) = \min\{R_\phi(\varphi(x)) |\ \varphi \in p\}$;
- For each $b \in A_y$, either $\phi(x,b) \in p$ or $\neg\phi(x,b) \in p$;
- either $R_\phi(\chi(x) \wedge \phi(x,b)) < n$ or $R_\phi(\chi(x) \wedge \neg\phi(x,b)) < n$;
- $R_\phi(\chi(x))$ is minimal $\Longrightarrow$ $\big(\phi(x,b) \in p \iff R_\phi(\chi(x) \wedge \phi(x,b)) = n\big)$.

□

Theorem 3.8. *The following are equivalent for a formula $\phi(x,y)$.*

1. *$\phi(x,y)$ is stable;*
2. *$R_\phi(x=x) < \omega$;*
3. *All ϕ-types are uniformly definable;*
4. *All ϕ-types over models are uniformly definable;*
5. *$S_\phi(M) \leq \kappa$ for all $\kappa \geq |L|$ and $M \models T$ with $|M| = \kappa$;*
6. *There is some κ such that $|S_\phi(M)| < \operatorname{ded}\kappa$ for all $M \models T$ with $|M| = \kappa$.*

Proof. .

- (1) $\iff$ (2) by Proposition 3.2;
- (2) $\implies$ (3) by Proposition 3.7;
- (3) $\implies$ (4) is obvious;
- (4) $\implies$ (5), each ϕ-type is determined by a $L(M)$-formula, its own definition;
- (5) $\implies$ (6) is obvious;

- (6) $\implies$ (1) is by Proposition 2.5.

$\square$

Global case:

Theorem 3.9. *Let T be a complete theory. Then the following are equivalent.*

1. *T is stable;*
2. *There is NO sequence of tuples $(c_i |\ i \in \omega)$ from $\mathbb{M}$ and formula $\phi(z_1, z_2) \in L(M)$ such that*
$$\models \phi(c_i, c_j) \iff i < j;$$
3. *$f_T(\kappa) \leq \kappa^{|T|}$ for all infinite cardinals κ;*
4. *There is some κ such that $f_T(\kappa) \leq \kappa$;*
5. *There is some κ such that $f_T(\kappa) < \operatorname{ded} \kappa$;*
6. *All formulas of the form $\phi(x, y)$ where x is a singleton variable, are stable;*
7. *All types over models are definable.*

Proof. .

- (1) $\implies$ (2) by definition;
- (2) $\implies$ (1):
  - Let $\psi(x, y)$ be a formula with order property witnessed by sequence
  $$\{(a_i, b_i)|\ i < \omega\};$$
  - Let $\phi(x_1 y_1;\ x_2 y_2) := \psi(x_1, y_2)$ and $c_i := a_i b_i$;
  - Then $\models \phi(c_i, c_j) \iff i < j$.
- (1) $\implies$ (3) :$S_x(M) \to \prod_{\phi \in L} S_\phi(M)$ is injective;
- (3) $\implies$ (4) is obvious;
- (4) $\implies$ (5) is obvious;
- (5) $\implies$ (1) is by Proposition 2.5.
- (6) $\iff$ $(1-5)$: Fix some κ, then $S_1(M) \leq \kappa$ for all M with $|M| = \kappa$ iff $S_n(M) \leq \kappa$ for all M with $|M| = \kappa$;
- (7) $\iff$ $(1-5)$ by Theorem 3.9

□

Example 3.10.

- Stability $\iff$ all types over all models are definable;
- Some unstable theories have certain special models over which all types are definable;
- $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$, all types over $\mathbb{R}$ are uniformly definable;
- $\mathcal{M} = (\mathbb{Q}_p, +, \times, 0, 1)$, all types over $\mathbb{Q}_p$ are uniformly definable.

4 Indiscernible sequences and stability

Definition 4.1. Given a linear order I, a sequence of tuples $(a_i|\ i \in I)$ with $a_i \in \mathbb{M}_x$ is indicernible over a set of parameters A if $a_{i_0}, ..., a_{i_n} \equiv_A a_{j_0}, ..., a_{j_n}$ for all $i_0 < ... < i_n$ and $j_0 < ... < j_n$ from I and all $n \in \omega$.

Example 4.2. .

1. A constant sequence is indiscernible over any set;
2. A subsequence of a A-indiscernible sequence is A-indiscernible;
3. In the theory of equality, any sequence of singletons is indiscernible;
4. Any increasing (or decreasing) sequence of singletons in a dense linear order is indiscernible;
5. Any basis in a vector space is an indiscernible sequence.

Definition 4.3. For any sequence $\bar{a} = (a_i|\ i \in I)$ and a set of parameters B, we define $\mathrm{EM}(\bar{a}/B)$, the Ehrenfeucht-Mostowski type of the sequence $\bar{a}$ over B, as a partial type over B in countably many variables indexed by and given by the following collection of formulas

$$\phi(x_0, ..., x_n) \in L(B)|\ \forall i_0 < ... < i_n,\ \models \phi(a_{i_0}, ..., a_{i_n},\ n \in \omega)$$

Exercise 4.4. For any sequence $\bar{a} = (a_i|\ i \in I)$ and a set of parameters B. If J is an infinite linear order, then there is a sequence $\bar{b} = (b_i|\ i \in b)$ which realizes the EM-type of $\bar{a}$ over A, i.e.

$$\models \phi(b_{i_0}, ..., b_{i_n}) \text{ for all } i_0 < ... < i_n \in I,\ \phi \in \mathrm{EM}(\bar{a}/A)$$

Exercise 4.5. If $\bar{a} = (a_i|\ i \in I)$ is an A-indiscernible sequence. Then $\mathrm{EM}(\bar{a}/A)$ is a complete ω-type over A.

Let $\bar{a} = (a_i|\ i \in I)$ and $\bar{b} = (b_j|\ j \in J)$ be A-indiscernible sequences. We denote $\bar{a} \equiv_{\mathrm{EM},A} \bar{b}$ if $\mathrm{EM}(\bar{a}/A) = \mathrm{EM}(\bar{b}/A)$

Proposition 4.6. *Let $\bar{a} = (a_i|\ i \in J)$be an arbitrary sequence in $\mathbb{M}$, where J is an arbitrary linear order and A is a small set of parameters. Then for any small linear order I we can find (in $\mathbb{M}$) an A-indiscernible sequence $(b_i|\ i \in I)$ realize the* EM*-type of $\bar{a}$ over A.*

Proof.

1. Let $\{c_i|\ i \in I\}$ be a set of new constants;
2. Let $L' = L \cup \{c_i|\ i \in I\}$;
3. Let $T' \supseteq T$ be in L' containing the following axioms:
   - $\phi(c_{i_0}, ..., c_{i_n})$ for all $i_0 < ... < i_n \in I$ and $\phi \in \mathrm{EM}(\bar{a}/A)$;
   - $\psi(c_{i_0}, ..., c_{i_n}) \leftrightarrow \psi(c_{j_0}, ..., c_{j_n})$ for all $i_0 < ... < i_n, j_0 < ... < j_n \in I$ and $\psi \in L(A)$

4. It is enough to show that T' is consistent;
5. By compactness, it is enough to show that every finite $T_0 \subseteq T'$ is consistent;
6. T_0 involves only finitely many formulas $\Delta \subseteq L(A)$ with at most n variables, and new constants $\{c_{k_0}, ..., c_{k_m}\}$;
7. Let $(b_i|\ i \in I) \subseteq \mathbb{M}$ realize the EM-type of $\bar{a}$ over A;
8. By Ramsey, there is an infinite subset $I_0 \subseteq I$ such that:
   - either $\models \phi(b_{i_0}, ..., b_{i_n})$ for all $\phi \in \Delta$ and $i_0 < ... < i_n \in I$;
   - or $\models \neg\phi(b_{i_0}, ..., b_{i_n})$ for all $\phi \in \Delta$ and $i_0 < ... < i_n \in I$.
9. Let $i_0 < ... < i_m \in I_0$ and interpret $c_{k_0}, ..., c_{k_m}$ as $b_{i_0}, ..., b_{i_m}$;
10. Then $\mathbb{M} \models T_0$.

□

Corollary 4.7. *If $\bar{a} = (a_i|\ i \in I)$ is an A-indiscernible sequence and $J \supseteq I$ is an arbitrary linear order, then (in $\mathbb{M}$) there is an A-indiscernible sequence $(b_j|\ j \in J)$ such that $b)i = a_i$ for all $i \in I$ (every thing involved is small).*

Proof. .

- Let $(c_j|\ j \in J) \subseteq \mathbb{M}$ realize the EM-type of $\bar{a}$ over A;
- Then the subsequence $(c_j|\ j \in I)$ realize the type of $(a_i|\ i \in I)$ over A;
- Namely, $(c_j|\ j \in I) \equiv_A (a_i|\ i \in I)$;
- By homogeneity, there is $(b_j|\ j \in J) \supseteq (a_i|\ i \in I)$ such that $(c_j|\ j \in J) \equiv_A (b_j|\ j \in J)$.

□

Lemma 4.8. *If $\bar{a} = (a_i|\ i \in I)$ is an infinite A-indiscernible sequence, then for all $S \subseteq I$ and $a_i \notin \mathrm{acl}(A, a_{j\in S})$*

- $a_i \in \mathrm{acl}(A, a_{j\in S}) \iff \exists S_0 \subseteq_{\text{fin}} S(a_i \in \mathrm{acl}(A, a_{j\in S_0}))$;
- Let $(b_i|\ i < \omega) \equiv_{\mathrm{EM},A} (a_i|\ i \in I)$;
- Then for any $i_0 < ... < i_n \in I$ and $j_0 < ... < j_n \in \mathbb{Q}$

$$a_{i_k} \in acl(A, \{a_{i_s}|\ s \neq k, s \leq n) \iff b_{j_k} \in acl(A, \{b_{j_s}|\ s \neq k, s \leq n)$$

- WLG, we assume that $I = (\mathbb{Q}, <)$;

- Suppose that
$$a_{i_k} \in \mathrm{acl}(A, \{a_{i_s} |\ s \neq k, s \leq n)$$

- Suppose that the formula $\phi(x_0, ..., x_k, ..., x_n) \in L(A)$ witness the property;

- Namely, $\models \phi(a_{i_0}, ..., a_{i_n})$ and $\phi(a_{i_0}, ..., \mathbb{M}, ..., a_{i_n})$ is finite;

- Then for any $q \in \mathbb{Q}$ realizing the same cut of a_{i_k} over $\{a_{i_s} |\ s \neq k, s \leq n)$, we have
$$\models \phi(a_{i_0}, ..., a_q, ..., a_{i_n})$$

- So $\phi(a_{i_0}, ..., \mathbb{M}, ..., a_{i_n})$ is infinite, a contradiction.

Exercise 4.9. Start with the sequence $(1, 2, 3, ...)$ in $(, +, \times, 0, 1) \models ACF_0$. Give an explicit example of an indiscernible sequence based on it.

A more power result is :

Proposition 4.10. *Let A be a set of parameters. If $\kappa \geq |T|+|A|$, $\lambda = \beth_{(2^\kappa)^+}$, and $(a_i|\ i < \lambda)$ is a sequence of tuples a_i of the same length $\leq \kappa$, then there is an A-indiscernible sequence $(b_i|\ i < \omega)$ such that for each $n < \omega$ there are $i_0 < ... < i_n < \lambda$ such that*
$$b_0, ..., b_n \equiv_A a_0, ..., a_n.$$

See Enrique Casanovas's "Simple theories ang hyperimaginaries", Prop. 1.6 for a proof (Using Erdös-Rado theorem: $\beth_n(\kappa)^+ \to \kappa^{+}{}^{n+1}_{\kappa}$, which means if f is a coloring of the $n+1$-element subsets of a set of cardinality $\beth_n(\kappa)^+$, in κ many colors, then there is a homogeneous set of cardinality κ^+, instead of Ramsey).

Definition 4.11. A sequence $(a_i|\ i \in I)$ is totally indiscernible over A if $a_{i_0}...a_{i_n} \equiv_A a_{j_0}...a_{j_n}$ for any $i_0 \neq ... \neq i_n,\ j_0 \neq ... \neq j_n$ from I (so the order of the indices doesn't matter any longer).

Theorem 4.12. *T is stable if and only if every indiscernible sequence is totally indiscernible.*

Proof. $\Rightarrow$

- Suppose that T is stable, $(a_i|\ i \in I)$ is indiscernible over A;

- If $(a_i|\ i \in I)$ is NOT totally indiscernible;

- then there are $i_0 \neq ... \neq i_n,\ j_0 \neq ... \neq j_n$ from I such that $a_{i_0}...a_{i_n} \not\equiv_A a_{j_0}...a_{j_n}$;

- WLG, assume that $I = (\mathbb{Q}, <)$ and $i_0 = 0, ..., i_n = n$;

- there is $\sigma \in S_{n+1}$, the permutation group of $\{0, ..., n\}$, such that
$$a_{\sigma(0)}...a_{\sigma(n)} \equiv_A a_{j_0}...a_{j_n}$$

- $\sigma = \tau_1 ... \tau_m$, a product of a sequence of transpositions of two consecutive elements;
- there is $0 < k < m$ such that $a_{\tau_k(0)} ... a_{\tau_k(n)} \not\equiv_A a_0 ... a_n$;
- Assume that $\tau_k = (s, s+1)$, then there is a $L(A)$-formula $\psi(x_0, ..., x_n)$ such that

$$\models \psi(a_0, ..., a_s, a_{s+1}, ..., s_n) \wedge \neg\psi(a_0, ..., a_{s+1}, a_s, ..., s_n);$$

- Let $\phi(x, y) := \psi(a_0, ..., a_{s-1}, x, y, a_{s+2}, ..., s_n)$;
- Then for all $s < q, r < s+1$, $\models \phi(a_q, a_r) \iff a_q < a_r$.

$\Leftarrow$

- Assume that T is unstable;
- Then $\exists\ \phi(x, y) \in L$ has the order property, witnessed by a sequence $\bar{c} = (c_i |\ i \in \omega)$. Namely

$$\models \phi(c_i, c_j) \iff i < j$$

- Let $\bar{a} = (a_i |\ i \in \omega)$ be an indiscernible sequence based on $\bar{c}$;
- Then

$$\models \phi(a_i, a_j) \iff i < j.$$

  So $\bar{a}$ is NOT totally indiscernible.

□

Proposition 4.13. *For any stable formula $\phi(x, y)$, in an arbitrary theory, there is some $k_\phi \in \omega$ depending just on ϕ such that for any indiscernible sequence $I \subseteq \mathbb{M}_x$ and any $b \in \mathbb{M}_y$, either $|\phi(I, b)| \leq k_\phi$ or $|\neg\phi(I, b)| \leq k_\phi$.*

Proof. .

- Suppose that $|\phi(I, b)| > k$ or $|\neg\phi(I, b)| > k$;
- By compactness, we assume that $I = \omega$;
- either $\phi(I, b)$ or $\neg\phi(I, b)$ is infinite;
- Assume that $\neg\phi(I, b)$ is infinite;
- Then there is subsequence $J = \{n_0 < n_1 < ...\} \subseteq \omega$ such that

$$\models \phi(a_{n_i}, b) \iff i \leq k$$

- Let $c_i = a_{n_i}$, and $b_k = b$, we have $\models \bigwedge_{i \leq k} \phi(c_i, b_k) \wedge \bigwedge_{i=k+1}^{3k} \neg\phi(c_i, b_k)$;

- Since c_i is indiscernible, we have

$$\models \exists y\Big(\bigwedge_{i\leq k}\phi(c_i,y)\wedge\bigwedge_{i=k+1}^{3k}\neg\phi(c_i,y)\Big)\rightarrow\exists y\Big(\bigwedge_{i\leq j}\phi(c_i,y)\wedge\bigwedge_{i=j+1}^{k}\neg\phi(c_i,y)\Big)$$

  for each $j<k$;

- Let b_j realize $\bigwedge_{i\leq j}\phi(c_i,y)\wedge\bigwedge_{i=j+1}^{k}\neg\phi(c_i,y)$;

- Then $\models\phi(c_i,b_j)\iff i\leq j$, so ϕ has k-order property. Since ϕ is stable, k_ϕ exists.

$\square$

Corollary 4.14. *In a stable theory, we can define the average type of an indiscernible sequence $b=(b_i)_{i\in I}$ over a set of parameters A as*

$$\mathrm{Av}(b/A)=\{\phi(x)\in L(A)|\ \models\phi(b_i)\ \textit{for all but finitely many}\ i\in I\}$$

By Proposition 4.13 it is a complete consistent type over A.

5 Stable=NIP$\bigcap$NSOP and the classification picture

The failure of stability can occur in one of the following two "orthogonal" ways.

Definition 5.1. [NSOP]

- A (partitioned) formula $\phi(x, y) \in L$ has the strict order property, or SOP, if there is an infinite sequence $(b_i)_{i \in \omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j \in \omega$;
- A theory T has SOP if some formula does.
- T is NSOP if it does not have the strict order property.

Remark 5.2. .

- If $\phi(x, y)$ has SOP, then by Proposition we can choose an indiscernible sequence $(b_i)_{i \in \omega}$ satisfying the condition above.
- If we have arbitrary long finite sequences $(b_i)_{i<n}$ satisfying the condition above for a fixed formula $\phi(x, y)$, then it has SOP by compactness.
- A typical example of an SOP theory is given by DLO.
- T is NSOP if and only if all formulas with parameters are NSOP (can incorporate the parameters into the sequence of bi's),if and only if all formulas $\phi(x, y)$ with x singleton are NSOP.

Exercise 5.3. T has SOP if and only if there is a definable partial order with infinite chains

Definition 5.4. [NIP]

- A (partitioned) formula $\phi(x, y)$ has the independence property, or IP, if (in $\mathbb{M}$) there are infinite sequences $(b_i)_{i \in \omega}$ and $(a_s)_{s \subseteq \omega}$ such that

$$\models \phi(a_s, b_i) \iff i \in s.$$

- A theory T has IP if some formula does, otherwise T is NIP.

Remark 5.5. .

- If we have arbitrary long finite sequences $(b_i)_{i<n}$ satisfying the condition above for a fixed formula $\phi(x, y)$, then by compactness we can find an infinite sequence satisfying the condition above, hence $\phi(x, y)$ has IP.
- If $\phi(x, y)$ has IP, then by Ramsey and compactness we can choose an indiscernible sequence $(b_i)_{i \in \omega}$ in the definition above.

- A typical example of a theory with IP is given by the theory of the countable random graph, i.e. the theory of a single (symmetric, irreflexive) binary relation $E(x, y)$ axiomatized by the following list of "extension axioms", for each $n \in \omega$

$$\forall x_0 \neq ... \neg x_{n-1} \neq y_0 \neq ... \neq y_{n-1} \exists z \Big(\bigwedge_{i<n} E(x_i, z) \wedge \bigwedge_{i<n} \neg E(y_i, z) \Big)$$

- T is NIP if and only if all formulas with parameters are NIP, if and only if all formulas $\phi(x, y)$ with x singleton are NIP. Also $\phi(x, y)$ is NIP if and only if $\phi^*(y, x) = \phi(x, y)$ is NIP. [see Pierre Simon: "A Guide to NIP Theories"]

Lemma 5.6. *A formula $\phi(x, y)$ has IP if and only if for there is an indiscernible sequence $\bar{b} = (b_n)_{n\in\omega}$ and a parameter c such that*

$$\models \phi(c, b_n) \iff n \textit{ is even.}$$

Proof. .
$\Rightarrow$:

- Suppose that $\phi(x, y)$ has IP;
- There are $\bar{b} = (b_n)_{n\in\omega}$ and $\bar{a} = (a_s)_{s\subseteq\omega}$ such that $\phi(a_s, b_n) \iff n \in s$
- we amy assume that $\bar{b}$ is indiscernible and let $s = \{0, 2, 4, ...\}$.
- Let $c = a_s$, then $\models \phi(c, b_n) \iff n$ is even.

$\Leftarrow$:

- Let $\bar{b} = (b_n)_{n\in\omega}$ be an indiscernible sequence and c a parameter such that

$$\models \phi(a, b_n) \iff n \text{ is even.}$$

- Fix some $n \in \omega$ and $s \subseteq n$
- there is an order-preserving mapping $f : n \to \mathbb{N}$ such that

$$f(s) \subseteq 2\mathbb{N} \ \text{ and } \ f(n\backslash s) \subseteq 2\mathbb{N} + 1$$

- So $\models \exists x\big(\bigwedge_{k\in s} \big(\phi(x, y_{f(k)}) \wedge \bigwedge_{k\notin s} \neg\phi(x, y_{f(k)})\big)$;
- By indicerniblity, $\models \exists x\big(\bigwedge_{k\in s} \big(\phi(x, y_k) \wedge \bigwedge_{k\notin s} \neg\phi(x, y_k)\big)$
- $\implies$ for each $s \subseteq n$ there is a_s such that

$$\models \big(\bigwedge_{k\in s} \big(\phi(a_s, y_k) \wedge \bigwedge_{k\notin s} \neg\phi(a_s, y_k)\big)$$

- By compactness, ϕ has IP.

$\square$

Proposition 5.7. *A formula $\phi(x, y)$ is NIP if and only if for any indiscernible sequence $\bar{b} = (b_i)_{i\in I}$ and a parameter a, the alternation of $\phi(a, y)$ on $\bar{b}$ is finite, bounded by some number $n \in \omega$ depending just on ϕ. That is, there are at most n increasing indices $i_0 < ... < i_{n-1}$ such that*

$$\models \phi(a, b_{i_k}) \leftrightarrow \neg\phi(a, b_{i_{k+1}}) \qquad (\forall k < n - 1)$$

Proof. By Lemma 5.6 and compactness. $\square$

Remark 5.8.

Working in an NIP theory and given an indiscernible sequence $\bar{b} = (b_i)_{i\in I}$ with II an endless order , and A an arbitrary set of parameters, Proposition 5.7 allows us to define a complete consistent type

$$\mathrm{Av}(\bar{b}/A) := \{\phi(x) \in L(A)| \text{ the set } \{i \in I| \models \phi(b_i)\} \text{ is cofinal}\}$$

Theorem 5.9 (Shelah)**.** *T is unstable if and only if it has the independence property or the strict order property.*

Proof. .
$\Leftarrow$:

- strict order property $\implies$ order property, so it is unstable.
- $\phi(x, y)$ has IP $\implies$ $|S_\phi(A)| = 2^{|A|}$, so it is unstable.

$\Rightarrow$:

- Suppose that T is both NSOP and NIP.
- Suppose for a contradiction that T is unstable.
- we will show that unstable+NIP $\implies$ SOP;
- Let $\phi(x, y)$ be a formula and $(a_i)_{i\in\mathbb{Q}}$ and $(b_i)_{i\in\mathbb{Q}}$ be sequences such that

$$\models \phi(a_i, b_j) \iff i < j$$

- We assume that $(b_i)_{i\in\mathbb{Q}}$ is indiscernible;
- Since ϕ has NIP, $\exists\ n \in \omega$, $\exists\ s \subseteq n$ such that

$$\psi_s(x, b_0, ..., b_{n-1}) := \bigwedge_{k\in s} \phi(x, b_k) \wedge \bigwedge_{k<n, k\notin s} \neg\phi(x, b_k)$$

  is inconsistent;

- Assume that $|s| = j$, then

$$\psi_j(x, b_0, ..., b_{n-1}) := \bigwedge_{k<j} \phi(x, b_k) \wedge \bigwedge_{j \leq k < n} \neg\phi(x, b_k)$$

  is consistent;

- there is $\sigma \in S_n$ the permutation group of $\{0, ..., n-1\}$, such that $s = \sigma(\{0, ..., j-1\})$;

- σ is a product of a sequence of transpositions of two consecutive elements;

- There is $s^* \subseteq n$ with $|s^*| = j$ and $k < n-1$ such than

$$\psi_{s^*}(x, b_0, ..., b_k, b_{k+1}, .., b_{n-1}) \;=\; \theta(x, d) \wedge \phi(x, b_k) \wedge \neg\phi(x, b_{k+1})$$

  is consistent and

$$\psi_{s^*}(x, b_0, ..., b_{k+1}, b_k, .., b_{n-1}) \;=\; \theta(x, d) \wedge \neg\phi(x, b_k) \wedge \phi(x, b_{k+1})$$

  is inconsistent, where $d = \{b_0, ..., b_{n-1}\} \backslash \{b_k, b_{k+1}\}$;

- which implies that

$$\theta(x, d) \wedge \phi(x, b_{k+1}) \subsetneq \theta(x, d) \wedge \phi(x, b_k)$$

- For each $k < p < q < k+1$, we have

$$\theta(x, d) \wedge \phi(x, b_q) \subsetneq \theta(x, d) \wedge \phi(x, b_p)$$

- So T has SOP.

$\square$

Exercise 5.10. Show that DLO is NIP, and that the theory of a random graph is indeed NSOP.

Example 5.11. Examples of stable theories.