Stability

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Contents

1 Preface

A combination of various notes [Pillay(2018)] [Chernikov(2019)]

2 Preliminaries

2.1 Imaginaries and

The first motivation to develop T^{eq} is dealing with quotient objects, without leaving the context of first order logic. That is, if E is some definable equivalence relation on some definable set X, we want to view X/E as a definable set

We work in the setting of multi-sorted languages. Let L be a 1-sorted language and let T be a (complete) L-theory. We shall build a many-sorted language L^{eq}-theory T^{eq}. We will ensure that in natural sense, L^{eq} contains L and T^{eq} contains T

First we define L^{eq}. Consider the set L-formula $\varphi(x,y)$, up to equivalence, such that T models that φ is an equivalence relation. For each φ, define s_φ to be a new sort in L^{eq}. Of particular importance is $s_=$, the sort given by the formula "$x = y$". This sort $s_=$ will yield, in each model of T^{eq}, a model of T

Also define f_φ to be a function symbol with domain sort $s_=^n$ (where φ has n free variables) and codomain sort s_φ

For each m-place relation symbol $R \in L$, make R^{eq} an m-place relation symbol in L^{eq} on $s_=^m$. Likewise for all constant and function symbols in L. Finally, for the sake of formality, we put a unique equality symbol $=_\varphi$ on each sort

Remark. Let N be an L^{eq} structure. Then N has interpretations $s_\varphi(N)$ of each sort s_φ and $f_\varphi(N) : s_=(N)^{n_{f_\varphi}} \to s_\varphi(N)$ of each function symbol f_φ

Definition 2.1. T^{eq} is the L^{eq}-theory which is axiomatised by the following

1. T, where the quantifiers in the formulas of T now range over the sort $s_=$
2. For each suitable L-formula $\varphi(x, y)$, the axiom $\forall_{s_=} x \forall_{s_=} y(\varphi(x, y) \leftrightarrow f_\varphi(x) = f_\varphi(y))$
3. For each L-formula φ, the axiom $\forall_{s_\varphi} y \exists_{s_=} x(f_\varphi(x) = y)$

Axioms 2 and 3 simply state that f_φ is the quotient function for the equivalence relation given by φ

Definition 2.2. Let $M \vDash T$. Then M^{eq} is the L^{eq} structure s.t. $s_=(M^{\text{eq}}) = M$ and for each suitable L-formula $\varphi(x, y)$ of n variables, the sort $s_\varphi(M^{\text{eq}})$ is equal to $M^{n_{f_\varphi}}/E$ where E is the equivalence relation defined by $\varphi(x, y)$ and $f_\varphi(M^{\text{eq}})(b) = b/E$

Now $M^{\text{eq}} \vDash T^{\text{eq}}$. Moreover, passing from T to T^{eq} is a canonical operation, in the following sense

Lemma 2.3. 1. *For any $N \vDash T^{\text{eq}}$, there is an $M \vDash T$ s.t. $N \cong M^{\text{eq}}$*

2. *Suppose $M, N \vDash T$ are isomorphic, and let $h : M \cong N$. Then h extends uniquely to $h^{\text{eq}} : M^{\text{eq}} \cong N^{\text{eq}}$*
3. *T^{eq} is a complete L^{eq}-theory*
4. *Suppose $M, N \vDash T$ and let $a \in M$, $b \in N$ with $\text{tp}_M(a) = \text{tp}_N(b)$. Then $\text{tp}_{M^{\text{eq}}}(a) = \text{tp}_{N^{\text{eq}}}(b)$*

Proof. 1. Take $M = s_=(N)$

2. Let $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to N^{\mathrm{eq}}$ be defined as $h^{\mathrm{eq}}(f_\varphi(M^{\mathrm{eq}})(b)) = f_\varphi(N^{\mathrm{eq}})(h(b))$ for each $\varphi \in L$. This defines a function on M^{eq}, because $f_\varphi(M^{\mathrm{eq}})$ is surjective by the T^{eq} axioms. Moreover h^{eq} is well-defined. Suppose $f_\varphi(M^{\mathrm{eq}})(b) = f_\varphi(M^{\mathrm{eq}})(b')$, then $\varphi(b, b')$ and hence $\varphi(h(b), h(b'))$, therefore $f_\varphi(N^{\mathrm{eq}})(h(b)) = f_\varphi(N^{\mathrm{eq}})(h(b'))$. Injectivity is the same since $\varphi(b, b') \leftrightarrow \varphi(h(b), h(b'))$.

3. Let $M, N \vDash T^{\mathrm{eq}}$, we want to show that they are elementary equivalent. Assume the generalized continuum hypothesis. By GCH, there are $M', N' \vDash T^{\mathrm{eq}}$ which are λ saturated of size λ, for some large λ (strongly inaccessible), which $M \preceq M'$ and $N \preceq N'$. Since we want to show elementary equivalence, we can replace M, N with M' and N'. By 1, we have $M = M_0^{\mathrm{eq}}$, $N = N_0^{\mathrm{eq}}$ for some $M_0, N_0 \vDash T^{\mathrm{eq}}$. Furthermore, M_0, N_0 are λ-saturated of size λ. By assumption, T is complete, so $M_0 \equiv N_0$, and therefore $M_0 \cong N_0$. By 2, $M \cong N$, and therefore $M \equiv N$

   We could simply prove that there is a back and forth system between M and N, using such a system between $M \supset M_0 \vDash T$ and $N \supset N_0 \vDash T$

4. Let $M, N \vDash T$, they are elementary submodels of $\mathfrak{C}$. Since $\mathrm{tp}_M(a) = \mathrm{tp}_N(b)$, there exists an $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ with $\sigma(a) = b$. By 2, this automorphism extends to $\sigma^{\mathrm{eq}} : \mathfrak{C} \to \mathfrak{C}$ with $\sigma^{\mathrm{eq}}(a) = b$, hence $\mathrm{tp}_{M^{\mathrm{eq}}}(a) = \mathrm{tp}_{\mathfrak{C}^{\mathrm{eq}}}(a) = \mathrm{tp}_{\mathfrak{C}^{\mathrm{eq}}}(b) = \mathrm{tp}_{N^{\mathrm{eq}}}(b)$

□

Corollary 2.4. *Consider the Strong space $S_{(s_=)^n}(T^{\mathrm{eq}})$. The forgetful map $\pi : S_{(s_=)^n}(T^{\mathrm{eq}}) \to S_n(T)$ is a homeomorphism*

Proof. Observe that it is continuous and surjective. By 4 of the previous lemma it is injective. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism □

Proposition 2.5. *Let $\varphi(x_1, \dots, x_k)$ be an L^{eq} formula, where x_i is of sort S_{E_i}. There is an L-formula $\psi(\bar{y}_1, \dots, \bar{y}_k)$ s.t.*

$$T^{\mathrm{eq}} \vDash \forall \bar{y}_1, \dots, \bar{y}_k (\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$$

Proof. Let n be the length of $\bar{y}_1, \dots, \bar{y}_k$. Consider the set $\pi(\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$, it is a clopen subset of $S_n(T)$ by the previous lemma, hence equal to $\psi(\bar{y}_1, \dots, \bar{y}_k)$ for some formula ψ.

For any $\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)) \in p(\bar{y}_1, \dots, \bar{y}_k) \in S_{(s_=)^n}(L^{\mathrm{eq}})$, If $T^{\text{eq}} \vDash \psi(\bar{y}_1, \ldots, \bar{y}_k)$ need further consideration for homeomorphism between two space $\square$

Corollary 2.6. *1. Let $M, N \vDash T$, and let $h : M \to N$ be an elementary embedding. Then $h^{\text{eq}} : M^{\text{eq}} \to N^{\text{eq}}$ is also an elementary embedding*

2. $\mathfrak{C}^{\text{eq}}$ is also κ-saturated

Remark. For $M \vDash T$, a definable set $X \subseteq M^n$ can be viewed as an element of M^{eq}. Suppose X is defined in M by $\varphi(\bar{x}, \bar{a})$ where $\bar{a} \in M$. Consider the equivalence relation E_ψ defined by $\psi(\bar{y}_1, \bar{y}_2) = \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$, and consider $c = \bar{a}/E_\psi = f_\psi(\bar{a}) \in M^{\text{eq}}$. Then X is defined in M^{eq} by $\chi(\bar{x}, c) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f_\psi(\bar{y}) = c)$. Moreover, if $c' \in S_\psi(M^{\text{eq}})$ and $\forall \bar{x}(\chi(\bar{x}, c) \leftrightarrow \chi(\bar{x}, c'))$, then $c = c'$. To see this, let $c' = f_\psi(\bar{a}')$, and let X' be defined in M by $\varphi(\bar{x}, \bar{a}')$. Then X' is defined in M^{eq} by $\chi(\bar{x}, c')$, so we have that $X = X'$ (in M^{eq}). And then $X = X'$ (in M) so $c = f_\psi(\bar{a}) = f_{\psi'}(\bar{a}') = c'$

Definition 2.7. With the above considerations in mind, given $M \vDash T$ and a definable set $X \subseteq M^n$, we call such a $c \in M^{\text{eq}}$ a **code** for X

Remark. Any automorphism of $\mathfrak{C}^{\text{eq}}$ fixes a definable set X set-wise iff it fixes a code for X. However, the choice of a code for X will depend on the formula φ used to define it

$$\begin{aligned} \sigma(X) = X &\Leftrightarrow \sigma(X) = \{\sigma(x) : \varphi(x, b)\} = \{x : \varphi(x, \sigma(b))\} = \{x : \varphi(x, b)\} = X \\ &\Leftrightarrow \forall x(\varphi(x, b) \leftrightarrow \varphi(x, \sigma(b))) \\ &\Leftrightarrow \psi(b, \sigma(b)) \Leftrightarrow f_\psi(b) = f_\psi(\sigma(b)) \end{aligned}$$

We can think of $\mathfrak{C}^{\text{eq}}$ as adjoining codes for all definable equivalence relations (as c/E' codes $E'(x, c)$ for an arbitrary equivalence relation E)

Definition 2.8. Let $A \subseteq M \vDash T$. Then $\text{acl}^{\text{eq}}(A) = \{c \in M^{\text{eq}} : c \in \text{acl}_{M^{\text{eq}}}(A)\}$ and $\text{dcl}^{\text{eq}}(A)$ is defined similarly

Remark. Suppose $A \subseteq M \prec N$, then $\text{acl}_{N^{\text{eq}}}(A), \text{dcl}_{N^{\text{eq}}}(A) \subseteq M^{\text{eq}}$, so this notation is unambiguous

Lemma 2.9. *Let $M \vDash T$, a definable subset X of M^n, and $A \subseteq M$. Then X is almost A-definable iff X is definable in M^{eq} by a formula with parameters in $\text{acl}^{\text{eq}}(A)$*

Proof. We can work in $\mathfrak{C}$, since $M \prec \mathfrak{C}$. Let c be a code for X. From **??** X is almost A-definable iff $|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| < \omega$ iff $|\{\sigma(c) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| < \omega$ (note that σ extends uniquely in $\mathfrak{C}^{\text{eq}}$), that is, $c \in \text{acl}^{\text{eq}}(A)$

$$\sigma(b)/E = \sigma'(b)/E \Leftrightarrow \forall x(\varphi(x, \sigma(b)) \leftrightarrow \varphi(x, \sigma'(b)))$$
$$\Leftrightarrow \sigma(X) = \sigma'(X)$$

□

Definition 2.10. Let $\bar{a}, \bar{b} \in \mathfrak{C}$ have length n. Let $\bar{a}, \bar{b}$ have the same strong type over A (written as $\mathrm{stp}_{\mathfrak{C}}(\bar{a}/A) = \mathrm{stp}_{\mathfrak{C}}(\bar{a}/A)$) if $E(\bar{a}, \bar{b})$ for any finite equivalence relation (finitely many classes) defined over A

Remark. If $\varphi(\bar{x})$ is a formula over A, then it defines an equivalence with two classes $E(\bar{x}_1, \bar{x}_2)$ iff $(\varphi(\bar{x}_1) \wedge \varphi(\bar{x}_2)) \vee (\neg\varphi(\bar{x}_1) \wedge \neg\varphi(\bar{x}_2))$. Hence strong types are a refinement of types

Hence for any formula if $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/B)$, at least we have $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$

Lemma 2.11. *If $A = M \prec \mathfrak{C}$, then $\mathrm{tp}_{\mathfrak{C}}(a/M) \vDash \mathrm{stp}_{\mathfrak{C}}(a/M)$*

Proof. Let E be an equivalence relation with finitely many classes, defined over M, and $\bar{b}$ another realization of $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/M)$, we want to show $E(a, b)$. Since E has only finitely many classes, and M is a model, there are representants $e_1, \dots, e_n$ of each E-class in M. Hence we must have $E(a, e_i)$ for some i, and therefore $E(b, e_i)$, which yields $E(a, b)$ □

Lemma 2.12. *Let $A \subseteq M \vDash T$, and let $\bar{a}, \bar{b} \in M$. TFAE*

1. $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$
2. *$\bar{a}, \bar{b}$ satisfy the same formulas almost A-definable*
3. $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}_{\mathfrak{C}}(\bar{b}/\operatorname{acl}^{\mathrm{eq}}(A))$

Proof. $3 \to 2$. 2.9. Suppose $X = \varphi(\mathfrak{C}, \bar{d})$ is almost A-definable, then $\bar{a}, \bar{b} \in \varphi(\mathfrak{C}, \bar{d})$ iff $\bar{a}, \bar{b} \in \theta(\mathfrak{C}) := \exists \bar{y}(\varphi(\mathfrak{C}, \bar{y}) \wedge \bar{y}/E_\psi = \bar{c})$ where $\bar{c} = \bar{d}/E_\psi \in \operatorname{acl}^{\mathrm{eq}}(A)$.

$2 \to 3$. For any $\varphi(\bar{x}, \bar{d}) \in \mathrm{tp}_{\mathfrak{C}}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(A))$ □

3 Index

This is a functional link that will open a buffer of clickable index entries:

4 References

References

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