Homework3

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October 13, 2021

Exercise 1. Fix $m, p \in \mathbb{N}$, let $\mathfrak{R}$ be the class of all m-ary relations. Prove that $|\mathfrak{R}/\sim_p|$ is finite

Proof. For any m-ary relations R and R'

$$\begin{aligned} R \sim_p R' &\Leftrightarrow \emptyset \in S_p(R, R') \Leftrightarrow \emptyset \sim_p \emptyset \\ &\Leftrightarrow \text{ for all sentence } \phi \text{ with } QR(\phi) \leq p, R \vDash \phi \text{ iff } R' \vDash \phi \end{aligned}$$

From proofs of Fraïssé's theorem, as up to equivalence, there are only finitely many sentences of quantifier rank q for all $q \in \mathbb{N}$. Then We can list them as $\psi_1, \psi_2, \dots, \psi_n$ for some n whose quantifier rank is at most p such that for any sentence φ with quantifier at most p, φ is equivalent to ψ_a for some $a \in \mathbb{N}$. Since for $i \in \mathbb{N}$, either $R \vDash \psi_i$ or $R \vDash \neg\psi_i$. Take the conjunction of each ψ_i or its negated form and we get a ψ_R such that

$$R \sim_p R' \Leftrightarrow R \vDash \psi_R \wedge R' \vDash \psi_R$$

Thus $R' \in [R]_{\sim_p}$ if and only if $R' \vDash \psi_R$. As there is only 2^n possible ψ's according to $\psi_1, \psi_2, \dots, \psi_n$, we have $|\mathfrak{R}/\sim_p| \leq 2^n$ and thus $|\mathfrak{R}/\sim_p|$ is finite □