Homework 5 solutions: Ramsey's theorem and indiscernible sequences

Advanced Model Theory

Due March 31, 2022

Let $\mathbb{M}$ be a monster model elementary extension of $(\mathbb{R}, +, \cdot, -, 0, 1, \leq)$. Let $a_1, a_2, a_3, \ldots$ be a non-constant indiscernible sequence of singletons/elements in $\mathbb{M}$.

1. Show that $a_1, a_2, \ldots$ is not totally indiscernible.

   Solution. If the sequence is totally indiscernible, then $\mathrm{tp}(a_1, a_2) = \mathrm{tp}(a_2, a_1)$, so $a_1 \leq a_2 \iff a_2 \leq a_1$. This is impossible unless $a_1 = a_2$, in which case the sequence is constant, a contradiction. □

2. Show that $a_1 a_2 > 0$.

   Solution. Break into cases depending on the sign of a_1:

   - If $a_1 = 0$, then $a_i = 0$ for all i by indiscernibility, and the sequence is constant, a contradiction.
   - If $a_1 > 0$, then $a_i > 0$ for all i by indiscernibility. Therefore $a_i a_j > 0$ for any i, j.
   - If $a_1 < 0$, then $a_i < 0$ for all i by indiscernibility. Therefore $a_i a_j > 0$ for any i, j.

   □

3. Suppose $a_2 - a_1 \geq 1$. Show that $a_2 - a_1 \geq 7$.

   Solution. If $a_2 - a_1 \geq 1$, then $a_i - a_j \geq 1$ for all $i > j$, by indiscernibility. In particular, $a_{i+1} - a_i \geq 1$ for any i. Then

   $$a_8 - a_1 = \sum_{i=1}^{7} (a_{i+1} - a_i) \geq \sum_{i=1}^{7} 1 = 7.$$

   Therefore (a_1, a_8) satisfies the formula

   $$\varphi(x, y) \iff (x_2 - x_1 \geq 1 + 1 + 1 + 1 + 1 + 1 + 1).$$

   By indiscernibility, (a_1, a_2) satisfies this formula, so $a_2 - a_1 \geq 7$. □

4. Show that at least one of the following is true: $a_2 < (1.01) \cdot a_1$ or $a_2 > 200 \cdot a_1$.

   Solution. Note if q is a rational number, then there is a formula $\psi_q(x, y)$ *without parameters* such that
   $$\psi_q(x, y) \iff y \geq q \cdot x.$$
   For example, if $q = n/m$ where n, m are positive integers, we can define $\psi_q(x, y)$ as
   $$y \cdot (\underbrace{1 + \cdots + 1}_{m \text{ times}}) \geq x \cdot (\underbrace{1 + \cdots + 1}_{n \text{ times}}).$$
   Let q be a positive rational number. By indiscernibility,
   $$\mathbb{M} \models \psi_q(a_1, a_2) \iff \mathbb{M} \models \psi_q(a_2, a_3) \iff \mathbb{M} \models \psi_q(a_1, a_3).$$
   In other words,
   $$a_2 \geq q \cdot a_1 \iff a_3 \geq q \cdot a_2 \iff a_3 \geq q \cdot a_1.$$
   Suppose $a_2 \geq q \cdot a_1$. Then
   $$\begin{gathered} qa_2 \geq q^2 a_1 \text{ (since } q > 0) \\ a_3 \geq qa_2 \text{ (by indiscernibility)} \\ a_3 \geq q^2 a_1 \text{ (by transitivity)} \\ a_2 \geq q^2 a_1 \text{ (by indiscernibility).} \end{gathered}$$
   So we just showed
   $$a_2 \geq qa_1 \implies a_2 \geq q^2 a_1 \tag{1}$$
   for any positive rational number q. Now suppose for the sake of contradiction that $1.01a_1 \leq a_2 \leq 200a_1$. This implies $1.01a_1 \leq 200a_1$, and so $a_1 \geq 0$. By question 2, $a_1 \neq 0$, and so $a_1 > 0$.

   By Equation (1), the fact that $1.01a_1 \leq a_2$ implies
   $$(1.01)^2 a_1 \leq a_2.$$
   By Equation (1), this in turn implies $(1.01)^4 a_1 \leq a_2$. Continuing on in this way, we see $(1.01)^{2^n} a_1 \leq a_2$ for any n, by induction on n. For sufficiently large n, $(1.01)^{2^n} > 200$. Then
   $$a_2 \geq (1.01)^{2^n} a_1 > 200a_1$$
   because $a_1 > 0$. This contradicts the assumption that $a_2 \leq 200a_1$. □

5. Show that $a_i + a_j \neq a_k$ for any i, j, k (possibly equal).

   Solution. Suppose for the sake of contradiction that $a_i + a_j = a_k$ for some i, j, k. It suffices to prove that the sequence is constant. Without loss of generality, $i \leq j$. Then there are eight cases:

- $i = j = k$. Then $a_i + a_i = a_i$, and $a_i = 0$. By indiscernibility, $a_1 = a_2 = 0$, and the sequence is constant
- $k < i = j$. By indiscernibility, $a_1 = a_3 + a_3$ and $a_2 = a_3 + a_3$. Then $a_1 = a_2$, and the sequence is constant.
- $i = j < k$. By indiscernibility, $a_1 + a_1 = a_2$ and $a_1 + a_1 = a_3$. Then $a_2 = a_3$ and the sequence is constant.
- $k < i < j$. By indiscernibility, $a_1 = a_3 + a_4$ and $a_2 = a_3 + a_4$. Then $a_1 = a_2$ and the sequence is constant.
- $i = k < j$. Then $a_i + a_j = a_k = a_i$, so $a_j = 0$. By indiscernibility, $a_1 = a_2 = 0$, and the sequence is constant.
- $i < k < j$. By indiscernibility, $a_1 + a_4 = a_2$ and $a_1 + a_4 = a_3$. Then $a_2 = a_3$ and the sequence is constant.
- $i < j = k$. Then $a_i + a_j = a_k = a_j$, so $a_i = 0$. By indiscernibility, $a_1 = a_2 = 0$, and the sequence is constant.
- $i < j < k$. By indiscernibility, $a_1 + a_2 = a_3$ and $a_1 + a_2 = a_4$. Then $a_3 = a_4$, and the sequence is constant. $\square$

Alternate solution. By indiscernibility, a_i, a_j, a_k have the same sign (positive or negative). Then $a_i + a_j = a_k \implies |a_i| + |a_j| = |a_k|$. Since no element of the sequence is zero, we see $|a_i| < |a_k|$ and $|a_j| < |a_k|$. Therefore $k \neq i$ and $k \neq j$. By indiscernibility, $a_{2i} + a_{2j} = a_{2k}$, since (i, j, k) has the same order type as $(2i, 2j, 2k)$. As $2k \notin \{2i, 2j\}$, we also see that $(2i, 2j, 2k + 1)$ has the same order type. (For example, $2k < 2i \iff 2k + 1 < 2i$.) So

$$a_{2i} + a_{2j} = a_{2k}$$
$$a_{2i} + a_{2j} = a_{2k+1}.$$

Then $a_{2k} = a_{2k+1}$, and the sequence is constant by indiscernibility. $\square$

6. Show that there is an indiscernible sequence $b_1, b_2, b_3, \ldots$ such that $b_2 > 200 \cdot b_1$. *Hint:* extract an indiscernible sequence from a well-chosen sequence in $\mathbb{R}$.

   Solution. Let $c_n = 1000^n$ for each n. Let $b_1, b_2, b_3, \ldots$ be an indiscernible sequence in $\mathbb{M}$ extracted from $c_1, c_2, c_3, \ldots$. The formula $x_2 > 200x_1$ is part of the EM type of $\{c_n\}$, because $1000^j \geq 1000^{i+1} > 200 \cdot 1000^i$ for any $j > i$. Therefore the formula $x_2 > 200x_1$ is also part of the EM type of $b_1, b_2, \cdots$, and so $b_2 > 200 \cdot b_1$. $\square$