# Homework 9

## Introduction to Model Theory

## Due 2021-12-2

1. Show that the following structure is not a field.

| $+$ | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 1 | a | b |
| 1 | 1 | a | b | 0 |
| a | a | b | 0 | 1 |
| b | b | 0 | 1 | a |

| $\times$ | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b |
| a | 0 | a | b | 1 |
| b | 0 | b | 1 | a |

2. Let $\varphi(x)$ be the formula $\exists y\ (x \cdot y + y = 1)$. Find a quantifier-free formula $\psi(x)$ equivalent to $\varphi(x)$ in all algebraically closed fields.

3. Let $\varphi(x, y, z)$ be the formula $\exists w\ (x \cdot w^2 + y \cdot w + z = 0)$. Find a quantifier-free formula $\psi(x, y, z)$ equivalent to $\varphi(x, y, z)$ in all algebraically closed fields.

4. If $M$ is a structure and $\varphi(x)$ is a formula in one variable, then $\varphi(M)$ denotes the set $\{a \in M : M \models \varphi(a)\}$. Show that if $M \preceq N$ and $\varphi(M)$ is finite, then $\varphi(N) = \varphi(M)$. *Warning:* this typically fails when $\varphi(M)$ is infinite.

5. Let $T$ be a theory with quantifier elimination. Let $M$ be a structure and $N$ be an extension (not necessarily an elementary extension). Suppose that $M$ and $N$ are both models of $T$. Let $\varphi(\bar{x})$ be a quantifier-free $L(M)$-formula in several variables. Suppose that $N \models \exists \bar{x}\ \varphi(\bar{x})$. Show that $M \models \exists \bar{x}\ \varphi(\bar{x})$. *Hint:* more generally, you can show that $N$ is an elementary extension of $M$.

6. Let $K$ be an algebraically closed field. Let $L \supseteq K$ be an extension field. Let $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ be polynomials over $K$. Suppose that the system of equations

$$P(x, y, z) = 0$$
$$Q(x, y, z) = 0$$
$$R(x, y, z) = 0$$

has a solution in $L$. Show that it has a solution in $K$. *Hint:* at a minimum you will need quantifier elimination in ACF. You may use the following fact:

**Fact.** *If $F$ is a field, then there is an extension field $M \supseteq F$ such that $M$ is algebraically closed.*