Homework 4 solutions: Morley sequences and the order property

Advanced Model Theory

Due March 24, 2022

1. Find a set A and a relation $R \subseteq A \times A$ such that

$$\exists^\infty x \in A \; \exists^\infty y \in A : (x, y) \in R$$
$$\neg\exists^\infty y \in A \; \exists^\infty x \in A : (x, y) \in R.$$

Proof. Let $A = \mathbb{N}$ and let $R(x, y)$ be the relation $x \leq y$. For any $x \in \mathbb{N}$, there are infinitely many $y \in \mathbb{N}$ such that $x \leq y$. In particular, for infinitely many $x \in \mathbb{N}$, there are infinitely many $y \in \mathbb{N}$ such that $x \leq y$.

On the other hand, for any y there are not infinitely many x such that $x \leq y$. So it is not the case that there are infinitely many y such that there are infinitely many x such that $x \leq y$. $\square$

2. Consider the structure $(\mathbb{R}, +, -, \cdot, 0, 1, \leq)$. Let $\varphi(x, y)$ be the formula $y - 1 \leq x \wedge x \leq y + 1$. Show that $\varphi(x, y)$ has the order property (in a monster model $\mathbb{M} \succeq \mathbb{R}$).

Solution. Let $\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ be a strictly increasing sequence of real numbers in the interval $[0, 1]$. For example, it could be the sequence

$$\ldots, 0.80001, 0.8001, 0.801, 0.81, 0.9, 0.99, 0.999, \ldots$$

Let $a_i = b_{i+1} + 1$. Then $a_i \in [1, 2]$ for each i, and so $b_j - 1 \leq 0 \leq 1 \leq a_i$ holds for any $i, j \in \mathbb{Z}$. Therefore,

$$\varphi(a_i, b_j) \iff a_i \leq b_j + 1$$

for any $i, j \in \mathbb{Z}$. Then

$$\varphi(a_i, b_j) \iff a_i \leq b_j + 1 \iff b_{i+1} + 1 \leq b_j + 1 \iff i + 1 \leq j \iff i < j,$$

because the sequence is strictly increasing. So we have a witness of the order property. $\square$

3. Let $\mathbb{M}$ be a monster model of DLO. Let $\tau \in S_1(\mathbb{M})$ be the type at $+\infty$ as in last week's homework. Consider the Morley product $\tau \otimes \tau \in S_2(\mathbb{M})$. Show that $(\tau \otimes \tau)(x, y)$ is the unique completion of $\tau(x) \cup \tau(y) \cup \{x < y\}$.

   Solution. First we prove the following claim: if $M \models \text{DLO}$ and $\tau_M(x) \in S_1(M)$ is the type at infinity, then $\Sigma_M(x, y) := \tau_M(x) \cup \tau_M(y) \cup \{x < y\}$ generates a unique 2-type over M. To see this, take two realizations (a, b) and (a', b') of Σ_M in a further elementary extension $N \succeq M$. Then $M < a < b$ and $M < a' < b'$. Let $f : M \cup \{a, b\} \to M \cup \{a', b'\}$ be the map

$$f(x) = \begin{cases} x & \text{if } x \in M \\ a' & \text{if } x = a \\ b' & \text{if } x = b. \end{cases}$$

   Then f is a local isomorphism, a strictly order-preserving map. By quantifier elimination in DLO, f is a partial elementary map. Therefore $(a, b) \equiv_M (a', b')$. So any two realizations of $\Sigma_M(x, y)$ have the same complete type over M, and $\Sigma_M(x, y)$ generates a complete type.

   This proves the claim. Applying this to the monster model, we see that $\tau(x) \cup \tau(y) \cup \{x < y\}$ generates a complete 2-type over $\mathbb{M}$. Let's call that type $q(x, y)$. Note that $q(x, y)$ is generated by the formulas

$$\{x > a : a \in \mathbb{M}\} \cup \{y > a : a \in \mathbb{M}\} \cup \{x < y\}. \tag{$*$}$$

   It remains to show that $\tau \otimes \tau = q$. It suffices to show that $(\tau \otimes \tau)(x, y)$ implies the generating formulas $(*)$. Suppose $a \in \mathbb{M}$. If $(b, c) \models (\tau \otimes \tau) \restriction \{a\}$, then $b \models \tau \restriction \{a\}$ and $c \models \tau \restriction \{a, b\}$, which means $a < b < c$. Therefore the formula $a < x < y$ must be in $(\tau \otimes \tau)(x, y)$. As this holds for any a, we get that $(\tau \otimes \tau)(x, y)$ contains all the formulas in $(*)$, and therefore $\tau \otimes \tau$ must be q.

   (There are probably other ways to do this problem.) □

4. Let $\mathbb{M}$ be a monster model of a complete theory T. Suppose $\mathbb{M}$ is an expansion of a linear order. (This means that there is a binary relation symbol $\leq$ in the language, and $(\mathbb{M}, \leq)$ is a linear order.) Let $p \in S_1(\mathbb{M})$ be a global A-invariant 1-type. Suppose that p commutes with itself. Show that p is a constant/realized type, meaning that $p = \text{tp}(c/\mathbb{M})$ for some $c \in \mathbb{M}$.

   Solution. Take $(b, c) \models (p \otimes p) \restriction A$. Then

$$b \models p \restriction A \text{ and } c \models p \restriction Ab.$$

   The fact that p commutes with itself means that this condition is symmetric in b and c, and so

$$c \models p \restriction A \text{ and } b \models p \restriction Ac,$$

or equivalently, $(c,b) \models (p \otimes p) \restriction A$. (Or here is a more direct argument. The fact that p commutes with p means that $(p \otimes p)(x,y) = (p \otimes p)(y,x)$. So if $(b,c) \models p \otimes p$, that is, $\mathbb{M} \models (p \otimes p)(b,c)$, then $\mathbb{M} \models (p \otimes p)(c,b)$, that is, $(c,b) \models p \otimes p$.)

The fact that $(b,c) \models (p \otimes p) \restriction A$ means that $(p \otimes p) \restriction A$ is $\mathrm{tp}(b,c/A)$, because $(p \otimes p) \restriction A$ is a complete type over A. So

$$\mathrm{tp}(b,c/A) = (p \otimes p) \restriction A.$$

As (c,b) also realizes this type,

$$\mathrm{tp}(c,b/A) = (p \otimes p) \restriction A.$$

Therefore $\mathrm{tp}(b,c/A) = \mathrm{tp}(c,b/A)$. Now use the order to see

$$b < c \iff c < b.$$

This is impossible unless $b = c$.

Because of how we chose (b,c), we have $c \models p \restriction Ab$, which means $p \restriction Ab = \mathrm{tp}(c/Ab)$. The type $\mathrm{tp}(c/Ab)$ contains the formula $x = b$, because $c = b$. Therefore the larger type $p \supseteq (p \restriction Ab)$ also contains the formula $x = b$. Then p must be $\mathrm{tp}(b/\mathbb{M})$. $\square$