Homework 6

Introduction to Model Theory

Due 2021-11-4

If the homework is turned in n days late, the grade will be scaled by a factor of $(1 - n/5)$. If you have questions about the homework, please ask them in office hours or in the class WeChat group.

Some of the problems below make use of *cardinal numbers* from set theory. For an introduction to cardinal and ordinal numbers, see Chapter 8 of the textbook (Poizat's *Course in Model Theory*), especially Sections 8.1 and 8.3.

1. Recall that a linear order $(M, \leq)$ is *well-ordered* if every non-empty subset $S \subseteq M$ has a minimum. Show that the class of well-ordered linear orders is *not* an elementary class. In other words, show that there is no theory T whose models are exactly the well-ordered linear orders. *Hint:* use compactness to get a descending sequence $x_1 > x_2 > x_3 > x_4 > \cdots$.

2. Let κ be an infinite cardinal. A theory T is said to be *κ-categorical* if there is a model $M \models T$ of size κ, and any two models of size κ are isomorphic. For example, DLO is $\aleph_0$-categorical: any two countable dense linear orders are isomorphic, as we proved in the first lecture. Prove the following statement:

   > Suppose T is κ-categorical for some infinite $\kappa \geq |L|$. Suppose T has no finite models. Then T is complete: if $M_1, M_2 \models T$, then $M_1 \equiv M_2$.

   (This is called *Vaught's test* or the *Łoś-Vaught criterion.*) *Hint:* use the Löwenheim-Skolem theorem to change the sizes of the models to κ.

3. Let M be an infinite L-structure. Let κ be a cardinal with $\kappa \geq |M|$ and $\kappa \geq |L|$. Show that there is an elementary extension $N \succeq M$ with $|N| = \kappa$. (This fact is called the *Upward Löwenheim-Skolem Theorem.*) *Hint:* apply the Löwenheim-Skolem theorem to a certain relevant theory.

4. Let M be an L-structure and A be a subset of M. For $i = 1, 2$, let N_i be an elementary extension of M and let $\bar{b}_i$ be an n-tuple in N_i (i.e., $\bar{b}_i \in (N_i)^n$). Show that the following are equivalent:

   (a) $\bar{b}_1$ realizes $\mathrm{tp}(\bar{b}_2/A)$.

(b) $\mathrm{tp}(\bar{b}_1/A) \supseteq \mathrm{tp}(\bar{b}_2/A)$.

(c) $\mathrm{tp}(\bar{b}_1/A) = \mathrm{tp}(\bar{b}_2/A)$.

(As a consequence, this implies that $\bar{b}$ realizes $p \in S_n(A)$ if and only if $p = \mathrm{tp}(\bar{b}/A)$, which is Exercise 10 in the notes.)