Seminar on Topological Dynamics of Definable Group Actions

Section 2,3(part)

March 25, 2022

1 Extensions of types

Assume $M \preceq N \preceq \mathfrak{C}$ where M, N are both small. Assume G, V are definable in M, G acts transtively on V. Then G^N acts transtively on V^N because it's a first order property. There is a natural restriction function $r : S_V(N) \to S_V(M)$ (it's a G^M-mapping). We can consider weakly generic types and almost periodic types both in $S_V(N)$ and $S_V(M)$.

We have a useful lemma about weakly generic sets and types.

Lemma 1. *Let X be a flow.*

1. *If U_1, U_2 are not weakly generic, then $U_1 \cup U_2$ is not weakly generic.*
2. *Every open set U containing $WGen$ is generic.*
3. *Every partial weakly generic type extends to a complete weakly generic type.*

Proof.

1. Suppose V is non-generic, then $U_2 \cup V$ is non-generic since U_2 is not weakly generic, and $U_1 \cup (U_2 \cup V)$ is non-generic since U_1 is not weakly generic. Hence $U_1 \cup U_2$ is not weakly generic.
2. Since no point in $X \backslash U$ is weakly generic, by compactness, we can find non-weakly-generic open sets $V_1, ..., V_k$ such that $U \cup V_1 \cup ... \cup V_k = X$. By (1), $V_1 \cup ... \cup V_k$ is non-weakly-generic, so U is generic.
3. By 1. $\square$

Lemma 2. *Assume $\varphi(x)$ is formula over M.*

1. *φ is generic in V^M iff it is generic in V^N.*
2. *If φ is weakly generic in V^M, then it's weakly generic in V^N. If M is ω-saturated, then the converse holds.*

3. *If $p \in S_V(M)$ is weakly generic, then there is a weakly generic type $q \in S_V(N)$ extending p, that is $WGen_V(M) \subseteq r(WGen_V(N))$.*

4. *If M is ω-saturated, then for every weakly generic type $q \in S_V(N)$, $q|M$ is weakly generic in $S_V(M)$. Together with 3, that is, $WGen_V(M) = r(WGen_V(N))$.*

Proof. We may assume $\varphi(x) \vdash V(x)$.

1. The formula $\exists A \subseteq G(|A| = k)(\forall x \in V \bigvee_{a\in A} \varphi(a^{-1}x))$ describes the generic set.

2. If φ is weakly generic in V^M, there is a non-generic formula over M ψ such that $\varphi \vee \psi$ is generic in V^M, then we apply 1.

   If φ is weakly generic in V^N, then there is a non-generic formula over N ψ such that $\varphi \vee \psi$ is generic in V^N. We regard $\psi = \psi'(x; \bar{b})$ where ψ' is a $L(M)$-formula and $\bar{b}$ are parameters from N. There is a partial type of y over finite parameters appearing in φ and ψ' forcing $\varphi(x) \vee \psi'(x; \bar{y})$ be generic in V^M (one formula) and $\psi'(x; \bar{y})$ non-generic in V^M (infinite fomulas). If M is ω-saturated, the realization $\bar{a}$ makes $\psi'(x; \bar{a})$ a non-generic formula in V^M and $\varphi(x) \vee \psi'(x; \bar{a})$ generic in V^M.

3. It follows 1 and lemma 1.3.

4. It follows 2. □

Recall that a type $q \in S(N)$ is an heir of $p \in S(M)$ if $p \subseteq q$ and for every $\varphi(x; \bar{c}) \in q$ where $\bar{c}$ comes from N, there is $\varphi(x; \bar{b}) \in p$ where $\bar{b}$ comes from M. A type $q \in S(N)$ is a coheir of $p \in S(M)$ if $p \subseteq q$ and q is finitely satisfiable in M.

Lemma 3. *Assume $p \in S_V(M)$, $q \in S_V(N)$, $p \subseteq q$.*

1. *If q is an heir of p, then $r(cl(G^N \cdot q)) \subseteq cl(G^M \cdot p)$.*

2. *$r(cl(G^N \cdot q))$ is G^M-invariant in $S_V(M)$.*

Proof.

1. Suppose not, there is a formula φ (we may assume $\varphi(x) \vdash V(x)$) over M such that $[\varphi] \cap cl(G^M \cdot p) = \emptyset$ and $\varphi \in g \cdot q$ for some $g \in G^N$. So $\varphi(g^{-1} \cdot x) \in q$. Since q is an heir of p, we have $g' \in G^M$ such that $\varphi(g'^{-1} \cdot x) \in p$, so $\varphi \in g' \cdot p$, a contradiction.

2. $cl(G^N \cdot q)$ is G^N-invariant, so it's G^N-invariant because G^M is a subgroup of G^N. Then $r(cl(G^N \cdot q))$ is G^M-invariant because r is a G^M-mapping. □

Proposition 4. *Assume $p \in S_V(M)$ is almost periodic, then there is an almost periodic $q \in S_V(N)$ extending p.*

Proof. For any $q \in S_V(M)$ extending p, $r(cl(G^N \cdot q)) \subseteq cl(G^M \cdot p)$ and it's G^M-invariant, hence $r(cl(G^N \cdot q)) = cl(G^M \cdot p)$ by minimality.

Let $q_0 \in S_V(M)$ be an heir of p, so $r(cl(G^N \cdot q_0)) = cl(G^M \cdot p)$. Let $q_1 \in cl(G^N \cdot q_0)$ be almost periodic, then $cl(G^N \cdot q_1) \subseteq cl(G^N \cdot q_0)$ and $r(cl(G^N \cdot q_1)) \subseteq r(cl(G^N \cdot q_0))$, hence $r(cl(G^N \cdot q_1)) = cl(G^M \cdot p)$ by minimality (of $cl(G^M \cdot p)$).

Now every $q \in cl(G^N \cdot q_1)$ with $r(q) = p$ is an almost periodic extension of p. □

There is a question if every weakly generic type in $S_V(M)$ has a weakly generic heir extension in $S_V(N)$. Lemma 2.4. partly answer it.

Corollary 5. *Assume G is a 0-definable group in an ω-saturated model N and $p \in S_G(N)$. Then the following are equivalent.*

1. *p is weakly generic.*
2. *For every ω-saturated $M \preceq N$, $p|M$ is weakly generic in $S_G(M)$.*
3. *For any finite $A \subseteq N$, there is an ω-saturated $M \preceq N$ containing A, such that $p|M$ is weakly generic in $S_G(M)$.*

Proof. It's just lemma 2.4. □

Assume $M \preceq N$ are ω-saturated, a non-weakly-generic type in $S_V(M)$ can't extend to a weakly generic type in $S_V(N)$ by lemma 2.4. However, it can happen that a weakly generic but not almost periodic type extend to an almost periodic type. We will see it in the second example of section 3.

2 An example

Consider the group $(\mathbb{Z}, +)$ acting on $2^{\mathbb{Z}}$ by right shift. Namely, let $k \in \mathbb{Z}$,$f \in 2^{\mathbb{Z}}$, then $(k \cdot f)(n) = f(n-k)$. The topology of $2^{\mathbb{Z}}$ is just the countable product of discrete topology $2 = \{0, 1\}$. There is an $\eta \in 2^{\mathbb{Z}}$ whose orbit $\mathbb{Z} \cdot \eta$ is dense(just concatenate all finite string to construct that). So $2^{\mathbb{Z}}$ is point transitive. If $f \in 2^{\mathbb{Z}}$ is periodic, $|o(f)|$ is finite, hence f is almost periodic.

Periodic functions are dense, so $WGen(2^{\mathbb{Z}}) = 2^{\mathbb{Z}}$. There are functions not almost periodic, such as η. So the notion of weakly generic point is weaker than almost periodic point in topological dynamics.

Now we are going to transfer the example into model-theoretic setting, interprete $2^{\mathbb{Z}}$ as a space of types over some model. If M is a model, M embeds into $S(M)$ as constant types.(What are algebraic types?) The image of M is discrete and $S(M)$ is a compactification. We can identify Z with the dense orbit $Z \cdot \eta$, then $2^{\mathbb{Z}}$ is also a compactification of $\mathbb{Z}$. We need to expand $(\mathbb{Z}, +)$ to make $2^{\mathbb{Z}}$ a space of types over $\mathbb{Z}$.

For a set A, let $2^{\subseteq A}$ $[2^{<A}]$ denote the set of all [finite] functions $\sigma \subseteq A \times 2$. $\mathbb{Z}$ also acts by right shift on $2^{\subseteq A}$ and $2^{<A}$. The sets $[\sigma] = \{f \in 2^{\mathbb{Z}} : \sigma \subseteq f\}$ form a basis of topology on $2^{\mathbb{Z}}$.

We expand $(\mathbb{Z}, +)$ by unary predicates $P_\sigma(x)$, $\sigma \in 2^{<\mathbb{Z}}$, defined by $P_\sigma(n) \iff n \cdot \eta \in [\sigma]$. Let $M = (\mathbb{Z}, +, P_\sigma)_{\sigma \in 2^{<\mathbb{Z}}}$, $T = Th(M)$.

Proposition 6. *For all $n \in \mathbb{Z}$ and $P_\sigma(x)$, the following hold.*

1. *$P_\sigma(x - n)$ is equivalent in M to $P_{n \cdot \sigma}(x)$.*
2. *$\neg P_\sigma(n)$ is equivalent to the disjunction $\bigvee_{v \in 2^X \setminus \{\sigma\}} P_v(x)$ where $X = dom(\sigma)$*

3. *If $\sigma_1, \sigma_2 \in 2^{<\mathbb{Z}}$ are compatible, then $P_{\sigma_1} \wedge P_{\sigma_2}$ is consistent, otherwise, it is inconsistent.*

Proof. Easy to check. $\square$

Proposition 7. *Let $\Delta = \{P_\sigma(x) : \sigma \in 2^{<\mathbb{Z}}\}$, consider the Δ-type space.*

1. *Each Δ-type over M is determined by its formulas without parameters. $S_\Delta(\emptyset) = S_\Delta(M)$.*

2. *Each function $f \in 2^{\mathbb{Z}}$ determines a complete Δ-type $p_f(x)$, generated by $\{P_{f\restriction X}(x) : X \subseteq \mathbb{Z}$ is finite$\}$. This is a bijection from $2^{\mathbb{Z}}$ to $S_\Delta(\emptyset)$. Moreover, $S_\Delta(\emptyset)$ is isomorphic to $2^{\mathbb{Z}}$ via $p_f \mapsto f$ as $\mathbb{Z}$-flow, where $\mathbb{Z}$ acts on $S_\Delta(\emptyset)$ by translation.*

3. *$tp_\Delta(0) = p_\eta$.*

Proof.

1. By Proposition 6.1.

2. Each p_f is complete by the previous proposition, it's consistent since the orbit of η is dense. If $f(n) = 0$, $g(n) = 1$, let $\sigma = \{(n, 0)\}$, then $P_\sigma(x) \in p_f$ and $P_\sigma(x) \notin p_g$, hence the map is injective. It's surjective because any complete Δ-type also determines a function in $2^{\mathbb{Z}}$ as $f(n) = 0 \iff P_{\{(n,0)\}}(x) \in p$. An basis of $S(\emptyset)$ maps to a basis of $2^{\mathbb{Z}}$ and vise versa.

3. Easy to check. $\square$

Remark 8. Let's take a review about the relationship of all the spaces($\mathbb{Z}$-flows). $2^{\mathbb{Z}}$ is a compactification of $\mathbb{Z}$, and $2^{\mathbb{Z}} \cong S_\Delta(\emptyset) \cong S_\Delta(M)$ via $f \mapsto p_f$ as $\mathbb{Z}$-flows. $S(M)$ is also a compactification of $\mathbb{Z}$, also $S_\Delta(M)$ is a quotient space of $S(M)$, and the equivalence relation is $\mathbb{Z}$-invariant.

By the isomorphism and discussion at the start of the section, every type in $S_\Delta(M)$ is weakly generic but not every type is almost periodic. Actually, it also holds for $S(M)$.

Proposition 9. *In $S(M)$ there are weakly generic types not almost periodic.*

Proof. Let $P_f = \{p \in WGen(M) : p_f \subseteq p\}$ for $f \in 2^{\mathbb{Z}}$. It's closed because $P_f = WGen(M) \cap [p_f]$, and it's non-empty because $\pi(WGen(M)) = WGen(S_\Delta(M)) = S_\Delta(M)$ where π is the natural surjective morphism from $S(M)$ to $S_\Delta(M)$ by Lemma 1.4 of the paper.

We claim that no type in P_η is almost generic. Assume $r \in P_\eta$, $\mathbb{Z}r \subseteq WGen(M)$ because $r \in WGen(M)$, then $cl(\mathbb{Z}r) \subseteq WGen(M)$. For every $\sigma \in 2^{<\mathbb{Z}}$, $\mathbb{Z}r$ meets $[P_\sigma]$, hence for any $f \in 2^{\mathbb{Z}}$, we have $cl(\mathbb{Z}r) \cap P_f \neq \emptyset$.

For periodic f with period k, let $\bar{P}_f = \bigcup_{i<k} P_{i \cdot f}$, the set is closed and $\mathbb{Z}$-invariant, and $cl(\mathbb{Z}r) \cap \bar{P}_f \subsetneq \bar{P}_f$. So $cl(\mathbb{Z}r)$ is not minimal. $\square$

Remark 10. Is $tp(0/M)$ is weakly generic? If so, we can also prove the proposition by choosing $tp(0/M)$.