Homework 7

Introduction to Model Theory

Due 2021-11-18

If the homework is turned in n days late, the grade will be scaled by a factor of $(1 - n/5)$. If you have questions about the homework, please ask them in office hours or in the class WeChat group.

1. Let M and N be L-structures. Let T be the set of all L-sentences satisfied by M. Show that $M \equiv N$ if and only if $N \models T$.

Let $(M, \leq)$ and $(N, \leq)$ be linear orders. An *embedding* from $(M, \leq)$ to $(N, \leq)$ is a function $i : M \to N$ such that for any $a, b \in M$, $M \models a \leq b \iff N \models i(a) \leq i(b)$.

2. Show that if $(M, \leq)$ is a countable linear order, then there is an embedding $(M, \leq) \to (\mathbb{Q}, \leq)$. *Hint:* the proof is a little like the proof that any two countable dense linear orders are isomorphic.

Suppose L is a first-order language and L' is a bigger first-order language (with new symbols). If M is an L'-structure, then $M \restriction L$ denotes the *reduct*—the L-structure obtained from M by forgetting the newly added symbols. For example, if $L' = \{+, \cdot, \leq\}$ and $L = \{+\}$ and $M = (\mathbb{R}, +, \cdot, \leq)$, then $M \restriction L = (\mathbb{R}, +)$.

3. Let L be a language and L' be a bigger language. Let M_1 be an L-structure and M_2 be an L'-structure. Suppose that $M_1 \equiv M_2 \restriction L$. (In other words, for every L-sentence φ, $M_1 \models \varphi \iff M_2 \models \varphi$.) Show that there is an L'-structure M_3 with an L'-elementary embedding $i_2 : M_2 \to M_3$ and an L-elementary embedding $i_1 : M_1 \to (M_3 \restriction L)$.

In the following problems, let L be the language $\{\leq, P\}$, where $\leq$ is a binary relation and P is a unary relation (written $P(x)$). Let $(M, \leq, P)$ be some L-structure with the following properties:

- $(M, \leq)$ is a linear order.
- There are infinitely many $a \in M$ satisfying $P(-)$.

For example, $(M, \leq, P)$ could be $(\mathbb{R}, \leq, \mathbb{Z})$ (so $P = \mathbb{Z}$, i.e., $P(a)$ is true if and only if $a \in \mathbb{Z}$).

4. Show that there is a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$. *Hint:* add some new constant symbols to the language and use compactness.

5. Show that there is a *countable* structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$.

6. Show that there is a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $f : (N, \leq) \to (P^N, \leq)$.

7. Show that there is an elementary extension $(N, \leq, P^N) \succeq (M, \leq, P)$ and an embedding $f : (N, \leq) \to (P^N, \leq)$. *Hint:* add f to the language as a function symbol.

Note: In Problem n, you may assume the conclusions of Problems $1, \ldots, n-1$.