Ramsey's theorem and indiscernible sequences

Advanced model theory

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Reference in the book: Ramsey's theorem is in Section 12.9, Ehrenfeucht-Mostowski models are 18.5, and the rest of the material is scattered across Chapter 12.

1 Ramsey's theorem

If X is a set and κ is a cardinal, then $[X]^\kappa$ denotes the collection of subsets of X of size κ. If C is a set of "colors," we think of a function $f : [X]^\kappa \to C$ as a "coloring" of the κ-element subsets of X. A subset $Y \subseteq X$ is *homogeneous* (for f) if f is constant on $[Y]^\kappa$.

Definition 1. If N, m, n, k are cardinals, then the notation

$$N \to (m)^n_k$$

means that whenever $|X| = N$ and $|C| = k$ and $f : [X]^n \to C$ is a function, there is a homogeneous subset $Y \subseteq X$ with $|Y| = m$.

Fact 2. *If $|X| = 6$ and $|C| = 2$ and $f : [X]^2 \to C$ is a function, then there is a homogeneous $Y \subseteq X$ with $|Y| = 3$. In symbols,*

$$6 \to (3)^2_2$$

This is often phrased as follows: if there are 6 people at a party, you can find 3 people who are all friends or 3 people who are all strangers. X is the set of 6 people. $[X]^2$ is the set of pairs. $C = \{0, 1\}$. $f : [X]^2 \to C$ is

$$f(\{a, b\}) = \begin{cases} 0 & \text{if } a \text{ knows } b \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3 (Finite Ramsey's theorem)**.** *For any finite n, m, k, there is finite N such that $N \to (m)^n_k$.*

Proof. Let L be the language with k distinct n-ary relation symbols $R_1, \ldots, R_k$. Let T be the L-theory saying:

- If $R_j(x_1, \ldots, x_n)$ holds, then the x_i are distinct.

- If $x_1, \ldots, x_n$ are distinct, then $R_j(\bar{x})$ holds for exactly one j.
- $R_j(x_1, \ldots, x_n) \leftrightarrow R_j(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any permutation σ.

A model of T is a set M with a coloring of the n-element subsets of M using k colors. There is an L-formula φ such that $M \models \varphi$ iff M has a homogeneous subset of size m. If $N \not\to (m)^n_k$, then there is $M \models T \cup \{\neg\varphi\}$ with $|M| = N$. If $N \not\to (m)^n_k$ for all $N < \omega$, then by compactness we can find an infinite model of $T \cup \{\neg\varphi\}$. By Theorem 17 last week, we can find a non-constant indiscernible sequence $a_1, a_2, a_3, \ldots$ in a model $M \models T \cup \{\neg\varphi\}$. By indiscernibility,

$$R_\ell(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \leftrightarrow R_\ell(a_{j_1}, \ldots, a_{j_n})$$

for $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$. This means that $\{a_1, a_2, a_3, \ldots\}$ is a homogeneous set. In particular $\{a_1, \ldots, a_m\}$ is a homogeneous set, contradicting $M \models \neg\varphi$. $\square$

Fact 4 (Infinite Ramsey's theorem). *If $n, k < \omega$, then $\aleph_0 \to (\aleph_0)^n_k$. In other words, for any coloring $f : [\mathbb{N}]^n \to k$ there is an infinite homogeneous subset $S \subseteq \mathbb{N}$.*

See Theorem 12.40 in the textbook for a proof.

2 Extracting indiscernible sequences

Work in a monster model $\mathbb{M}$. Let $(I, \leq)$ be an infinite linear order (often ω) and let $(\bar{a}_i : i \in I)$ be a sequence. Let B be a set of parameters.

Definition 5. The *Ehrenfeucht-Mostowski type* of a sequence $(\bar{a}_i : i \in I)$ over a set of parameters $B \subseteq \mathbb{M}$ is the set of $L(B)$-formula $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$ such that

$$\forall i_1 < \cdots < i_n \in I : \mathbb{M} \models \varphi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}).$$

The EM type is written $\mathrm{tp}^{EM}(\bar{a}/B)$.

Remark 6. The EM type is really a sequence $(\Sigma_1, \Sigma_2, \Sigma_3, \ldots)$ where Σ_n is a partial n-type over B containing the n-ary formulas $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$ in the EM-type.

Example. In $(\mathbb{R}, \leq)$, the EM type of the sequence $1, 1, 2, 2, 3, 3, 4, 4, \ldots$ over $\varnothing$ contains the formula $x_1 \leq x_2$ but not the formula $x_1 < x_2$.

Remark 7. If $(\bar{a}_i : i \in I)$ is a sequence and $(\bar{a}_i : i \in I_0)$ is a subsequence (meaning $I_0 \subseteq I$), then $\mathrm{tp}^{EM}((\bar{a}_i : i \in I)/B) \subseteq \mathrm{tp}^{EM}((\bar{a}_i : i \in I_0)/B)$.

Definition 8. If $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$ is a formula, a sequence $(\bar{a}_i : i \in I)$ is "φ-indiscernible" if for any $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$,

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \leftrightarrow \varphi(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}).$$

If Δ is a set of formulas, then $(\bar{a}_i : i \in I)$ is "Δ-indiscernible" if it is φ-indiscernible for any $\varphi \in \Delta$.

Lemma 9. *Let $(\bar{a}_i : i \in I)$ be an infinite sequence.*

1. *For any $m < \omega$ and finite Δ, there is a subsequence of length m that is Δ-indiscernible.*

2. *For any linear order J and any set of formulas Δ, there is a Δ-indiscernible sequence $(\bar{b}_j : j \in J)$ with $\mathrm{tp}^{EM}(\bar{b}/\varnothing) \supseteq \mathrm{tp}^{EM}(\bar{a}/\varnothing)$.*

Proof. 1. By induction on $|\Delta|$. If $\Delta = \varnothing$ then any subsequence is Δ-indiscernible. Otherwise write $\Delta = \Delta_0 \cup \{\varphi\}$ for some formula $\varphi(x_1, \ldots, x_n)$. Take N such that $N \to (m)^n_2$. By induction there is a subsequence $(\bar{b}_i : i < N)$ that is Δ_0-indiscernible. Define $f : [N]^n \to 2$ by

$$f(\{i_1, \ldots, i_n\}) = \begin{cases} 1 & \text{if } \mathbb{M} \models \varphi(\bar{b}_{i_1}, \ldots, \bar{b}_{i_n}) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 0 \leq i_1 < i_2 < \cdots < i_n < N.$$

As $N \to (m)^n_2$, there is a subsequence $(\bar{c}_i : i < m)$ that is homogeneous, and therefore φ-indiscernible.

2. By compactness we reduce to the case where Δ, J are finite. Then we can take $(\bar{b}_j : j \in J)$ to be a Δ-indiscernible subsequence of $(\bar{a}_i : i \in I)$, of length $|J|$. □

Theorem 10 (Extracting indiscernibles)**.** *Let $(\bar{a}_i : i \in I)$ be an infinite sequence in $\mathbb{M}$. Let C be a set of parameters. Let J be an infinite linear order. Then there is a C-indiscernible sequence $(\bar{b}_j : j \in J)$ with $\mathrm{tp}^{EM}(\bar{b}/C) \supseteq \mathrm{tp}^{EM}(\bar{a}/C)$.*

Proof. Naming parameters (i.e., replacing L with $L(C)$), we may assume $C = \varnothing$. Let Δ be the set of all formulas and apply Lemma 9(2). □

We call $(\bar{b}_j : j \in J)$ a *C-indiscernible sequence extracted from* $(\bar{b}_i : i \in I)$.

Example. Let $(a_i : i < \omega)$ be a sequence of distinct elements. Let $(b_i : i < \omega)$ be an indiscernible sequence extracted from $\bar{a}$. The formula $x_1 \neq x_2$ is in the EM-type of $\bar{a}$, so it's in the EM-type of $\bar{b}$, meaning that $b_i \neq b_j$ for $i < j$. Then $(\bar{b}_i : i < \omega)$ is a non-constant indiscernible sequence. This recovers Theorem 17 last week.

Example. Take the ordered field structure on $\mathbb{R}$ and take a monster model $\mathbb{M} \succeq \mathbb{R}$. Let $b_1, b_2, b_3, \ldots$ be an indiscernible sequence extracted from $1, 2, 3, \ldots$. The formula $x_1 > 0$ is in the EM type, so every $b_i > 0$. The formula $x_2 - x_1 \geq 1$ is in the EM-type, so $b_j - b_i \geq 1$ for $j > i$.

Remark 11. $(\bar{a}_i : i \in I)$ is B-indiscernible iff $\mathrm{tp}^{EM}(\bar{a}/B)$ is "complete," in the sense that for any $L(B)$-formula $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$, either φ or $\neg\varphi$ is in $\mathrm{tp}^{EM}(\bar{a}/B)$.

Therefore, if we extract an indiscernible sequence from an indiscernible sequence, the new sequence has the same EM-type as the old one:

Theorem 12 (Stretching indiscernible sequences)**.** *Let $(\bar{a}_i : i \in I)$ be a C-indiscernible sequence. Let J be an infinite linear order. Then there is a C-indiscernible sequence $(\bar{b}_j : j \in J)$ with $\mathrm{tp}^{EM}(\bar{b}/C) = \mathrm{tp}^{EM}(\bar{a}/C)$.*

Remark 13. If $(\bar{a}_i : i \in I)$ is B-indiscernible, then its type over B is determined by $\mathrm{tp}^{EM}(\bar{a}/B)$ and the order type of I. In particular, if $(\bar{c}_i : i \in I)$ is another B-indiscernible sequence with the same EM type over B, then the two sequences have the same type over B, and there is $\sigma \in \mathrm{Aut}(\mathbb{M}/B)$ with $\sigma(\bar{a}_i) = \bar{c}_i$ for all i.

Theorem 14 (Extending indiscernible sequences)**.** *Let $(\bar{a}_i : i \in I)$ be a C-indiscernible sequence. Let J be a linear order extending I. Then there are $\bar{a}_j$ for $j \in J \setminus I$ such that $(\bar{a}_j : j \in J)$ is a C-indiscernible sequence.*

Proof. Let $(\bar{b}_j : j \in J)$ be a C-indiscernible sequence extracted from $\bar{a}$. The subsequence $(\bar{b}_i : i \in I)$ has the same EM-type as $(\bar{a}_i : i \in I)$, so there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(\bar{b}_i) = \bar{a}_i$. Define $\bar{a}_j = \sigma(\bar{b}_j)$ for all $j \in J$. □

Example.
- An indiscernible sequence indexed by ω can be extended to one indexed by $\omega + \omega$ (adding another ω terms on the end).
- An indiscernible sequence indexed by $\mathbb{Z}$ can be extended to one indexed by $\mathbb{R}$.

3 The order property and indiscernible sequences

Theorem 15. *The following are equivalent for a formula $\varphi(\bar{x}; \bar{y})$.*

1. *φ has the order property, i.e., there are $\bar{a}_i, \bar{b}_i$ for $i \in \mathbb{Z}$ such that*
$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$
2. *Same as (1) but $((\bar{a}_i, \bar{b}_i) : i < \omega)$ is indiscernible.*
3. *There is an indiscernible sequence $(\bar{a}_i : i < \omega)$ and an element $\bar{b}$ such that*
$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}) \iff i < 0.$$

Proof. (1) $\implies$ (2): extract an indiscernible sequence from $((\bar{a}_i, \bar{b}_i) : i \in \mathbb{Z})$.

(2) $\implies$ (3): take $\bar{b} = \bar{b}_0$.

(3) $\implies$ (1): For any $j \in \mathbb{Z}$, the shifted tuple $(\bar{a}_{i+j} : i \in \mathbb{Z})$ has the same type as $(\bar{a}_i : i \in \mathbb{Z})$ by indiscernibility, so there is $\sigma_j \in \mathrm{Aut}(\mathbb{M})$ such that $\sigma_j(\bar{a}_i) = \bar{a}_{i+j}$ for any i. Let $\bar{b}_j = \sigma_j(\bar{b})$. Then $(\bar{a}_i, \bar{b}_j) = \sigma_j(\bar{a}_{i-j}, \bar{b})$, so
$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff \mathbb{M} \models \varphi(\bar{a}_{i-j}, \bar{b}) \iff i - j < 0,$$
showing (1). □

Corollary 16. *T is unstable iff there is an indiscernible sequence $(a_i : i \in \mathbb{Z})$ and a definable set $D \subseteq \mathbb{M}^1$ such that $a_i \in D \iff i < 0$.*

4 Total indiscernibility

Definition 17. Let $(I, \leq)$ be an infinite set. Let $(\bar{b}_i : i \in I)$ be a sequence. Then $(\bar{b}_i : i \in I)$ is *totally A-indiscernible* if for any distinct $i_1, \ldots, i_n \in I$ and any distinct $j_1, \ldots, j_n \in I$,

$$\bar{b}_{i_1} \cdots \bar{b}_{i_n} \equiv_A \bar{b}_{j_1} \cdots \bar{b}_{j_n}$$

When A isn't specified, "indiscernible" and "totally indiscernible" mean (totally) indiscernible over $\varnothing$.

Example. In DLO, the sequence $1, 2, 3, 4, \ldots$ is indiscernible but not totally indiscernible, because $(1, 2) \not\equiv (2, 1)$, as $1 < 2$ but $2 \not< 1$.

More generally,

Proposition 18. *If T is unstable, there is an indiscernible sequence that is not totally indiscernible.*

Proof. Some formula $\varphi(\bar{x}, \bar{y})$ has the order property. By Theorem 15 there is an indiscernible sequence $((\bar{a}_i, \bar{b}_i) : i < \omega)$ such that $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j$. Then

$$\mathbb{M} \models \varphi(\bar{a}_1, \bar{b}_2)$$
$$\mathbb{M} \models \neg\varphi(\bar{a}_2, \bar{b}_1).$$

so the sequence isn't totally indiscernible. □

Definition 19. $\mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_n/B)$ is "symmetric" if for any permutation σ of $\{1, \ldots, n\}$,

$$\bar{a}_1, \ldots, \bar{a}_n \equiv_B \bar{a}_{\sigma(1)}, \ldots, \bar{a}_{\sigma(n)}. \tag{$*$}$$

Remark 20. Let σ_i be the permutation exchanging i and $i + 1$. If $(*)$ holds for $\sigma = \sigma_i$ for all i, then $\mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_n/B)$ is symmetric, because any permutation σ can be written as a composition of σ_i's:

$$\sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_m}$$

for some $m, i_1, \ldots, i_m$.

Remark 21. Let $(\bar{a}_i : i \in I)$ be a B-indiscernible sequence. Let p_n be $\mathrm{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/B)$ for any $i_1 < \cdots < i_n$. Then $\bar{a}$ is totally B-indiscernible iff each p_n is symmetric.

Proof. Total indiscernibility implies symmetry easily.

Conversely, suppose each p_n is symmetric. Let $i_1, \ldots, i_n$ be distinct and let $j_1, \ldots, j_n$ be distinct. There are permutations σ, τ of $\{1, \ldots, n\}$ such that

$$i_{\sigma(1)} < \cdots < i_{\sigma(n)} \text{ and } j_{\tau(1)} < \cdots < j_{\tau(n)}.$$

Then

$$\bar{a}_{i_1}, \ldots, \bar{a}_{i_n} \equiv_A \bar{a}_{i_{\sigma(1)}}, \ldots, \bar{a}_{i_{\sigma(n)}} \equiv_A \bar{a}_{j_{\sigma(1)}}, \ldots, \bar{a}_{j_{\sigma(n)}} \equiv_A \bar{a}_{j_1}, \ldots, \bar{a}_{j_n}.$$ □

Remark 22. In particular, whether a B-indiscernible sequence $(\bar{a}_i : i \in I)$ is totally B-indiscernible depends only on $\mathrm{tp}^{EM}(\bar{a}/B)$. (The EM type is the sequence of types $p_1, p_2, p_3, \ldots$)

Lemma 23. *Let $(\bar{a}_i : i \in \mathbb{Z})$ be B-indiscernible. Let $C = \{\bar{a}_i : i \notin \{0,1\}\}$. Suppose $\bar{a}_0, \bar{a}_1 \equiv_{BC} \bar{a}_1, \bar{a}_0$. Then the sequence is totally B-indiscernible.*

Proof. By assumption there is $\alpha_0 \in \mathrm{Aut}(\mathbb{M}/B)$ such that α swaps $\bar{a}_0, \bar{a}_1$ and fixes $\bar{a}_i$ for $i \neq 0, 1$. By indiscernibility, for any $i \in \mathbb{Z}$ there is $\alpha_i \in \mathrm{Aut}(\mathbb{M}/B)$ which swaps $\bar{a}_i, \bar{a}_{i+1}$ and fixes $\bar{a}_j$ for $j \neq i, i+1$. With σ_i as in Remark 20, this implies

$$\bar{a}_1, \ldots, \bar{a}_n \equiv_B \bar{a}_{\sigma_i(1)}, \ldots, \bar{a}_{\sigma_i(n)}.$$

Then $\mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_n/B)$ is symmetric, and $(\bar{a}_i : i \in \mathbb{Z})$ is totally B-indiscernible. ☐

Proposition 24. *If $\mathbb{M}$ is stable and A is a small set of parameters, then $\mathbb{M}$ is stable as an $L(A)$-structure.*

Proof. Otherwise there is an $L(A)$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b}_i, \bar{c}_i$ such that

$$\mathbb{M} \models \varphi(\bar{b}_i, \bar{c}_j) \iff i < j.$$

Write $\varphi(\bar{x}, \bar{y})$ as $\psi(\bar{x}, \bar{y}, \bar{a})$ for some L-formula $\psi(\bar{x}, \bar{y}, \bar{z})$ and some $\bar{a} \in A$. Then

$$\mathbb{M} \models \psi(\bar{b}_i; \bar{c}_j, \bar{a}) \iff i < j$$

so $\psi(\bar{x}; \bar{y}, \bar{z})$ has the order property. ☐

Theorem 25. *The following are equivalent:*

1. *T is stable.*
2. *Every indiscernible sequence $(\bar{a}_i : i \in I)$ is totally indiscernible.*
3. *Every B-indiscernible sequence $(\bar{a}_i : i \in I)$ is totally B-indiscernible.*

Proof. (3) $\Longrightarrow$ (2) is trivial, and (2) $\Longrightarrow$ (1) is Proposition 18.

(1) $\Longrightarrow$ (3): suppose for the sake of contradiction that T is stable and $(\bar{a}_i : i \in I)$ is B-indiscernible but not totally B-indiscernible. By Theorem 12 and Remark 22, we may assume I is $\mathbb{Z}$. Let $C = \{\bar{a}_i : i \in \mathbb{Z} \setminus \{0,1\}\}$. By Lemma 23, $\bar{a}_0, \bar{a}_1 \not\equiv_{BC} \bar{a}_1, \bar{a}_0$. Take an $L(BC)$-formula $\varphi(\bar{x}, \bar{y})$ such that

$$\mathbb{M} \models \varphi(\bar{a}_0, \bar{a}_1) \ \wedge \ \neg\varphi(\bar{a}_1, \bar{a}_0).$$

By Theorem 14, we can extend $\{\bar{a}_i : i \in \mathbb{Z}\}$ to a B-indiscernible sequence $\{\bar{a}_i : i \in \mathbb{R}\}$. Now $\{\bar{a}_i : i \in \mathbb{R}, \ -1 < i < 2\}$ is BC-indiscernible (see point 5 in the Appendix). Therefore

$$-1 < i < j < 2 \implies \bar{a}_i \bar{a}_j \equiv_{BC} \bar{a}_0 \bar{a}_1$$
$$-1 < i < j < 2 \implies \mathbb{M} \models \varphi(\bar{a}_i, \bar{a}_j) \ \wedge \ \neg\varphi(\bar{a}_j, \bar{a}_i).$$

Therefore φ has the order property, witnessed by $\bar{a}_i, \bar{a}_j$ for i, j in the interval $[-0.9, 1.9]$.[1] By Proposition 24, T is unstable. $\square$

Remark 26. There are unstable theories in which any B-indiscernible sequence of singletons is totally B-indiscernible.[2]

Corollary 27. *Suppose T is stable. Let $(\bar{a}_i : i \in I)$ be an indiscernible sequence and D be a definable set. Then $\{i \in I : \bar{a}_i \in D\}$ is finite or cofinite in I.*

Proof. Suppose not. By Theorem 25, the sequence is totally indiscernible. Take distinct $\ldots, i_{-2}, i_{-1}, i_0, i_1, i_2, \ldots$ such that $\bar{a}_{i_j} \in D$ for $j < 0$ and $\bar{a}_{i_j} \notin D$ for $j \geq 0$. Then $(\bar{a}_{i_j} : j \in \mathbb{Z})$ is (totally) indiscernible and contradicts Corollary 16. $\square$

Therefore, in a stable theory, an indiscernible sequence acts a little bit like a strongly minimal set.

5 Ehrenfeucht-Mostowski models

In this section, we assume all languages are countable.

Definition 28. A structure M has *Skolem functions* if for any formula $\varphi(\bar{x}; y)$ there is a function symbol $f_\varphi(\bar{x})$ such that

$$M \models \forall \bar{x}\ (\exists y\ \varphi(\bar{x}, y) \to \varphi(\bar{x}, f_\varphi(\bar{x}))).$$

In other words, $\varphi(\bar{a}, M) \neq \varnothing \implies f_\varphi(\bar{a}) \in \varphi(\bar{a}, M)$.

Note that the complete theory of M determines whether M has Skolem functions.

Proposition 29. *Suppose M has Skolem functions and A is a substructure. Then $A \preceq M$.*

Proof. Use the Tarski-Vaught criterion. Suppose $M \models \exists y\ \varphi(\bar{a}, y)$ for some formula $\varphi(\bar{x}, y)$ and $\bar{a} \in A$. We must show there is $b \in A$ such that $M \models \varphi(\bar{a}, b)$. Take $b = f_\varphi(\bar{a})$. $\square$

Proposition 30. *If M is non-empty, then some expansion of M has Skolem functions.*

Proof. For each L-formula $\varphi(\bar{x}, y)$, define a function $f_\varphi(\bar{x})$ as follows:

$$f_\varphi(\bar{a}) = \begin{cases} \text{any element of } \varphi(\bar{a}, M) & \text{if } \varphi(\bar{a}, M) \neq \varnothing \\ \text{anything} & \text{if } \varphi(\bar{a}, M) = \varnothing. \end{cases}$$

Expand M with new function symbols for all of the f_φ. Repeat ω more times (since each step adds new formulas φ to worry about). $\square$

[1] Here we are secretly using the following fact. Let $\varphi(\bar{x}; \bar{y})$ be a formula. Let $\bar{a}_i, \bar{b}_i$ be tuples in $\mathbb{M}$ for $i \in \mathbb{Z}$ such that $i < j \implies \mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j)$ and $i > j \implies \mathbb{M} \models \neg\varphi(\bar{a}_i, \bar{b}_j)$ (but the behavior of $\varphi(\bar{a}_i, \bar{b}_j)$ is unspecified when $i = j$). Then φ has the order property. Why? Because $\mathbb{M} \models \varphi(\bar{a}_{2i+1}, \bar{b}_{2j}) \iff i < j$ for $i, j \in \mathbb{Z}$.

[2] If you know what it is, the theory of the random graph has this property.

Recall the notation $\langle A\rangle$ for the substructure generated by A. The elements of $\langle A\rangle$ are exactly the things of the form $t(a_1,\ldots,a_n)$ where t is a term and $a_1,\ldots,a_n \in A$. The size of $\langle A\rangle$ is at most the size of A plus $\aleph_0$, because the language is countable.

Theorem 31. *Let T be a complete theory and κ be an infinite cardinal. There is a model $M \models T$ of size κ with the following property: for any countable $A \subseteq M$, only countably many types over A are realized in M:*

$$|A| \le \aleph_0 \implies |\{\mathrm{tp}(c/A) : c \in M\}| \le \aleph_0. \tag{$*$}$$

Proof. By Proposition 30 we can replace T with an expansion T' that has Skolem functions. (Property $(*)$ is preserved in reducts, so we can go back from T' to T at the end.) Assume T has Skolem functions.

Take a non-constant indiscernible sequence $(b_i : i < \kappa)$ in the monster model $\mathbb{M}$ and let $M = \langle\{b_i : i < \kappa\}\rangle$. Then $M \preceq \mathbb{M}$ by Proposition 29, so $M \models T$. Also $|M| = \kappa$.

Fix countable $A \subseteq M$. For each $c \in A$ there is finite $I_c \subseteq \kappa$ such that $c \in \langle\{b_i : i \in I_c\}\rangle$. Let $I_0 = \bigcup_{c\in A} I_c$ and let $A_0 = \{b_i : i \in I_0\}$. Then I_0 and A_0 are countable, and $A \subseteq \langle A_0\rangle$.

Because $A \subseteq \langle A_0\rangle$, every $L(A)$-formula is equivalent to an $L(A_0)$-formula. Therefore $\mathrm{tp}(c/A_0)$ determines $\mathrm{tp}(c/A)$. It suffices to show that countably many types over A_0 are realized in M. Replacing A with A_0, we may assume $A = A_0 = \{b_i : i \in I_0\}$. To review, I_0 is a countable subset of κ.

Suppose $c \in M$. Then $c = t(b_{i_1},\ldots,b_{i_n})$ for some term t and some $i_1,\ldots,i_n \in \kappa$. By indiscernibility, $\mathrm{tp}(c/A)$ is determined by the following data:

- The term $t(x_1,\ldots,x_n)$.
- $\mathrm{tp}(b_{i_1},\ldots,b_{i_n}/A)$, which is in turn determined by the following data (see point 4 in the Appendix):
  - The relative order of $i_1,\ldots,i_n$.[3]
  - The cut in I_0 defined by each i_j, i.e., the two sets

    $$\{i \in I_0 : i \ge i_j\} \text{ and } \{i \in I_0 : i > i_j\},$$

    both of which are determined by their minimum members.

There are only countably many possibilities, so $\{\mathrm{tp}(c/A) : c \in M\}$ is countable. □

Theorem 32. *Let T be a complete theory (in a countable language). If T is κ-categorical for some uncountable κ, then T is $\aleph_0$-stable.*

Proof. Suppose not. Take a monster model $\mathbb{M} \models T$ and a countable $A \subseteq \mathbb{M}$ with $|S_1(A)| > \aleph_0$. Take $(b_i : i < \aleph_1)$ realizing distinct types over A. Let N be a model of size $\kappa \ge \aleph_1$ containing $A \cup \{b_i : i < \aleph_1\}$. Then N has a countable subset A over which uncountably many types are realized. Therefore N is not isomorphic to the model of size κ from Theorem 31. □

[3] In class we arranged $i_1 < \cdots < i_n$, and then this one doesn't matter.

A Appendix: a remark on tuples from indiscernible sequences

(I wrote this section after class, in order to clarify some of the proofs.)

Suppose $(\bar{a}_i : i \in I)$ is an indiscernible sequence.

1. If $i_1, \ldots, i_n \in I$, then $\mathrm{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n})$ is determined by the relative order of the elements $i_1, \ldots, i_n$, i.e., the quantifier free type of $(i_1, \ldots, i_n)$ in the structure $(I, \leq)$.

2. In fact, for any L-formula $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$, there is a quantifier-free formula $\psi(\bar{x}_1, \ldots, \bar{x}_n)$ in the language of orders such that

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \iff (I, \leq) \models \psi(i_1, \ldots, i_n).$$

   This follows because there are only finitely many possibilities for $\mathrm{qftp}(i_1, \ldots, i_n)$, so any condition on $\mathrm{qftp}(i_1, \ldots, i_n)$ can be expressed by a formula.

3. Let I_0 be a subset of I. Let $A = \{\bar{a}_i : i \in I_0\}$. Then $\mathrm{tp}(\bar{a}_j/A)$ is determined by the relative order of j and the elements of I_0. In particular, if we hold I_0 fixed and let j vary, $\mathrm{tp}(\bar{a}_j/A)$ is determined by the cut of j in I_0, i.e., the two sets

$$\{i \in I_0 : i < j\} \text{ and } \{i \in I_0 : i > j\}.$$

   Another way of saying this is that $\mathrm{tp}(\bar{a}_j/A)$ is determined by the quantifier-free type of j over I_0 in the structure $(I, \leq)$.

4. More generally, if $j_1, \ldots, j_n \in I$ and $A = \{\bar{a}_i : i \in I_0\}$, then $\mathrm{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}/A)$ is determined by the relative order of $j_1, \ldots, j_n$ amongst themselves, plus the cuts in I_0 defined by each j_k. That is, $\mathrm{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}/A)$ is determined by $\mathrm{qftp}(j_1, \ldots, j_n/I_0)$ in the structure $(I, \leq)$. This was used in the proof of Theorem 31.

5. Now suppose that J_0, J_1, J_2 are subsets of I such that $J_0 < J_1 < J_2$ (in the sense that every element of J_0 is less than every element of J_1, etc.). If $i \in J_1$, then the cut of i in $J_0 \cup J_2$ is always (J_0, J_2); it doesn't depend on i. Therefore, if $i_1, \ldots, i_n \in J_1$, then $\mathrm{qftp}(i_1, \ldots, i_n/(J_0 \cup J_2))$ is determined by $\mathrm{qftp}(i_1, \ldots, i_n)$. Returning to the indiscernible sequence, suppose $C = \{\bar{a}_i : i \in J_0 \cup J_2\}$. Then the subsequence $(\bar{a}_i : i \in J_1)$ is C-indiscernible. This was used in the proof of Theorem 25, with

$$\begin{aligned} I &= \mathbb{R} \\ J_0 &= \{\ldots, -4, -3, -2, -1\} \\ J_1 &= (-1, 2) = \{x \in \mathbb{R} : -1 < x < 2\} \\ J_2 &= \{2, 3, 4, 5, \ldots\}. \end{aligned}$$