Extra notes: stable formulas

Advanced Model Theory

March 23, 2022

In class, we have given several equivalent definitions of stability:

1. No formula has the order property.
2. All types over models are definable.
3. T is λ-stable for some λ.
4. No formula has the dichotomy property.

It turns out that these equivalences hold formula-by-formula:

Theorem A. *Let $\varphi(\bar{x};\bar{y})$ be a formula. The following are equivalent:*

1. *$\varphi(\bar{x};\bar{y})$ does not have the order property.*
2. *For any model M, any φ-type over M is definable.*
3. *For any set A, the number of φ-types over A is at most $|A| + \aleph_0$.*
4. *There is an infinite cardinal λ such that if A has size at most λ, then the number of φ-types over A is at most λ.*
5. *For any set A, any φ-type over A is definable.*
6. *$\varphi(\bar{x};\bar{y})$ does not have the dichotomy property.*

(See §1 below for the definition of φ-types and definability of φ-types.) A formula is said to be **stable** if it satisfies the equivalent conditions listed above. A theory is stable iff all formulas are stable.

We prove the equivalence of the first four conditions in §2, and the last two in §3. Section 3 builds off the earlier set of notes `dichotomy-and-definable-types.pdf`, which otherwise isn't used here. In the process, we will resolve some loose ends from class:

- Fact 15 from the March 17 notes said that the dichotomy property and order property are equivalent. This follows from Theorem A.

- Proposition 10 in the March 3 notes said that if some formula has the dichotomy property, then the theory isn't λ-stable for any λ. We give a new proof of this fact here, *not* using D_α and the suspicious application of compactness in Proposition 5 of the March 3 notes.[1]

1 φ-types

(This section is entirely copied from the file `dichotomy-and-definable-types.pdf`. If you've read those notes, you can skip this section.)

Fix a theory T, a monster model $\mathbb{M}$, and a formula $\varphi(x_1, \ldots, x_n; \bar{y})$.

Definition 1. If $B \subseteq \mathbb{M}$ is a set and $\bar{a} \in \mathbb{M}^n$, then $\text{tp}^\varphi(\bar{a}/B)$ is the partial type

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B,\ \mathbb{M} \models \varphi(\bar{a}, \bar{b})\} \ \cup\ \{\neg\varphi(\bar{x}; \bar{b}) : \bar{b} \in B,\ \mathbb{M} \models \neg\varphi(\bar{a}, \bar{b})\}.$$

In other words, $\text{tp}^\varphi(\bar{a}/B)$ is the set of formulas in $\text{tp}(\bar{a}/B)$ of the form φ or $\neg\varphi$.

Remark 2. Suppose $\bar{a}, \bar{b} \in \mathbb{M}^n$.

1. $\text{tp}^\varphi(\bar{a}/C) = \text{tp}^\varphi(\bar{b}/C)$ if and only if

$$\forall \bar{c} \in C : \mathbb{M} \models (\varphi(\bar{a}, \bar{c}) \leftrightarrow \varphi(\bar{b}, \bar{c}))$$

   So $\bar{a}$ and $\bar{b}$ have the same φ-type over C iff they satisfy exactly the same formulas of the form $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$.

2. $\bar{b} \in \mathbb{M}^n$ realizes $\text{tp}^\varphi(\bar{a}/C)$ if and only if $\text{tp}^\varphi(\bar{a}/C) = \text{tp}^\varphi(\bar{b}/C)$.

Warning 3. Some authors like Pillay in his book *Geometric Stability Theory* use "φ-type" to mean something slightly different. But the definition here is more common in contemporary research.

Definition 4. $S_\varphi(B)$ is the set $\{\text{tp}^\varphi(\bar{a}/B) : \bar{a} \in \mathbb{M}^n\}$.

Definition 5. A φ-type $p \in S_\varphi(B)$ is *definable* if there is an $L(B)$-formula $\psi(\bar{y})$ such that

$$\text{For any } \bar{b} \in B,\ \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \mathbb{M} \models \psi(\bar{b}).$$

[1] Proposition 5 in the March 3 notes said that if φ has the dichotomy property, then D_α is consistent for any ordinal α. Supposedly this follows from compactness. I'm sure this is correct and one can write down a valid proof. But it seems to me that the proof would be very messy, which is why I skipped it in class.

2 Stable formulas

Theorem 6. *The following are equivalent for a formula $\varphi(\bar{x};\bar{y})$.*

1. *φ does not have the order property.*
2. *For any model $M \preceq \mathbb{M}$, every φ-type over M is definable.*
3. *For any set $A \subseteq \mathbb{M}$, we have $|S_\varphi(A)| \leq |A| + \aleph_0$.*
4. *There is a cardinal λ such that if $A \subseteq \mathbb{M}$ and $|A| \leq \lambda$, then $|S_\varphi(A)| \leq \lambda$.*

Proof. We will prove the following implications:

$$\begin{tikzcd}
(1) \arrow[r] \arrow[dr, dashed] & (2) \arrow[d, dashed] \arrow[dl] \\
(4) \arrow[u] & (3) \arrow[l]
\end{tikzcd}$$

The solid arrows work in general. We will prove (2) $\implies$ (3) when the language is countable, and then prove (1) $\implies$ (3) for the general case.

- (1) $\implies$ (2): Proposition 12 in the March 17 notes.
- (2) $\implies$ (4): Take any $\lambda \geq |L|$. Given $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$, there is a small model $M \preceq \mathbb{M}$ containing A with $|M| \leq \lambda$, by the Löwenheim-Skolem theorem. Every φ-type over A extends to a φ-type over M, and so $|S_\varphi(A)| \leq |S_\varphi(M)|$. By (2), every φ-type over M is definable. A φ-type $p \in S_\varphi(M)$ is determined by the definable set $\{\bar{b} \in M : \varphi(\bar{x},\bar{b}) \in p(\bar{x})\}$. The number of such sets is at most $|L(M)| \leq \lambda$. Therefore $|S_\varphi(A)| \leq |S_\varphi(M)| \leq \lambda$.
- (2) $\implies$ (3) <u>when L is countable</u>: Condition (3) says that condition (4) holds for *any* $\lambda \geq \aleph_0$. When L is countable, the proof of (2) $\implies$ (4) gives this stronger statement.
- (3) $\implies$ (4): Clear.
- (4) $\implies$ (1): Similar to the proof of Theorem 8 in the March 17 notes. For completeness, here are the details. Suppose for the sake of contradiction that φ has the order property. Fix λ as in (4). By Lemma 7 in the March 17 notes, there is a linear order I with a dense subset $S \subseteq I$ such that $|I| > \lambda$ and $|S| \leq \lambda$. Take $\bar{a}_i, \bar{b}_i$ for $i \in I$ such that

  $$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

  Let $C = \{\bar{b}_j : j \in S\}$. As in the proof of Theorem 8 in the March 17 notes, there is an injection

  $$\begin{aligned} I \setminus S &\to S_\varphi(C) \\ i &\mapsto \mathrm{tp}^\varphi(\bar{a}_i/C). \end{aligned}$$

  Then $|C| \leq |S| \leq \lambda$ but $|S_\varphi(C)| \geq |I \setminus S| > \lambda$, contradicting (4).

This completes the proof in the case when L is countable. In particular, (1)–(4) are equivalent in any structure over a countable language.

It remains to show that (1) $\implies$ (3). If $L_0 \subseteq L$ and φ is an L_0-formula, then the truth of (1) and (3) doesn't change when we pass to the reduct $\mathbb{M} \restriction L_0$. Passing to a reduct, we may assume the language is countable. Then (1) $\implies$ (3) was proved above. $\square$

Definition 7. A formula φ is *stable* if it satisfies the equivalent conditions of Theorem 6, that is, φ doesn't have the order property.

A theory is stable iff all formulas are stable. By Corollary 14 in the March 17 notes, it suffices to check stability of formulas of the form $\varphi(x; y_1, \ldots, y_m)$ (with x being a single variable rather than a tuple of variables).

3 Stability and the dichotomy property

Here are two more equivalent definitions of stability of φ:

Theorem 8. *The following are equivalent for a formula φ:*

1. *φ does not have the dichotomy property.*
2. *For any set $A \subseteq \mathbb{M}$, any φ-type over A is definable (in the sense of Definition 5).*
3. *φ is stable.*

Proof. (1) $\implies$ (2): Theorem 10 in the document `dichotomy-and-definable-types.pdf`.

(2) $\implies$ (3): If all φ-types over arbitrary sets are definable, then all φ-types over models are definable, which is condition (2) in Theorem 6.

(3) $\implies$ (1): Suppose φ has the dichotomy property. Then "D_ω" is consistent, which means there are $\bar{a}_\sigma$ for $\sigma \in 2^\omega$ and $\bar{b}_\tau$ for $\tau \in 2^{<\omega}$ such that if σ extends $\tau 0$, then $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds, and if σ extends $\tau 1$, then $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds. Let $A = \{\bar{b}_\tau : \tau \in 2^{<\omega}\}$. Then A is countable. As in the proof of Proposition 10 in the March 3 notes, the elements $\bar{a}_\sigma$ all have distinct φ-types over A, and so $|S_\varphi(A)| = 2^{\aleph_0} > |A| + \aleph_0 = \aleph_0$, contradicting condition (3) in Theorem 6.

(If one doesn't care about condition (2) and only wants to show (1) $\iff$ (3), the direction (1) $\implies$ (3) can also be proved more directly using Remark 12 below.) $\square$

Corollary 9. *φ has the dichotomy property iff φ has the order property.*

This was stated as Fact 15 in the March 17 notes.

Corollary 10. *T is stable iff no formula of the form $\varphi(x; \bar{y})$ has the dichotomy property.*

This was condition (4) in Theorem 2 in the March 10 notes.

Corollary 11. *If φ has the dichotomy property, then for any λ there is a set A with $|A| \leq \lambda$ and $|S_\varphi(A)| > \lambda$. (This is the negation of condition (4) in Theorem 6.) Every φ-type over A extends to a complete n-type, so $|S_n(A)| > \lambda$ as well.*

This recovers Proposition 10 in the March 3 notes, *without using dubious Proposition 5 in the March 3 notes*, and without any use of D_α for $\alpha > \omega$.

Remark 12. The fact that the dichotomy property and order property are equivalent is a combinatorial statement, and there are combinatorial proofs. One direction is easy: Suppose $\varphi(\bar{x}, \bar{y})$ has the order property. Consider the order on $I = 2^\omega \cup 2^{<\omega}$ used in Lemma 7 in the March 17 notes. Because φ has the order property, there are $\bar{a}_i, \bar{b}_i$ for $i \in I$ such that $\varphi(\bar{a}_i, \bar{b}_j) \iff i < j$. Now one can check that $(\bar{a}_\sigma : \sigma \in 2^\omega)$ and $(\bar{b}_\tau : \tau \in 2^{<\omega})$ witness the dichotomy property.

The other direction is harder. If I remember correctly, there is a proof in the final chapter of Hodges' *Shorter Model Theory*.