Review of ultrapowers and monster models

Advanced Model Theory

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1 Ultrafilters and ultrapowers

If X is a set, let $\mathcal{P}(X)$ denote the power set of X.

Recall an *ultrafilter* $\mathcal{U}$ on a set I is a collection of subsets $\mathcal{U} \subseteq \mathcal{P}(I)$ such that

- $I \in \mathcal{U}$, $\varnothing \notin \mathcal{U}$.
- $X, Y \in \mathcal{U} \implies X \cap Y \in \mathcal{U}$
- $X \in \mathcal{U}$ and $X \subseteq Y \subseteq I$ implies $Y \in \mathcal{U}$.
- For any $X \subseteq I$, either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

This implies:

$$\begin{aligned}(X \cap Y) \in \mathcal{U} &\iff (X \in \mathcal{U} \ \wedge \ Y \in \mathcal{U}) \\ (X \cup Y) \in \mathcal{U} &\iff (X \in \mathcal{U} \ \vee \ Y \in \mathcal{U}) \\ (I \setminus X) \in \mathcal{U} &\iff \neg(X \in \mathcal{U}).\end{aligned}$$

Definition 1. A family of sets $\mathcal{F} \subseteq \mathcal{P}(I)$ has the *finite intersection property* (FIP) if for any $n \geq 0$ and $X_1, \ldots, X_n \in \mathcal{F}$, we have $\bigcap_{i=1}^n X_i \neq \varnothing$.

Fact 2. *If $\mathcal{F} \subseteq \mathcal{P}(I)$ has the FIP, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.*

If M is a structure and $\mathcal{U}$ is an ultrafilter on a set I, there is a structure called the *ultrapower* $M^{\mathcal{U}}$, also written $M^I/\mathcal{U}$. The set M^I is the set of I-tuples in M, i.e., functions from I to M. The underlying set of $M^{\mathcal{U}}$ is M^I modulo the equivalence relation where $f \sim g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}$. Let $[f] \in M^{\mathcal{U}}$ denote the equivalence class of $f \in M^I$. The L-strucutre on $M^{\mathcal{U}}$ is chosen in a certain way that makes Łoś's theorem be true:

Fact 3. *For any L-formula $\varphi(x_1, \ldots, x_n)$ and any $f_1, \ldots, f_n \in M^I$,*

$$M^{\mathcal{U}} \models \varphi([f_1], \ldots, [f_n]) \iff \{i \in I : M \models \varphi(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

For example, when $\varphi(x, y)$ is $(x = y)$, this says that $[f] = [g] \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}$, in agreement with the definition of $\sim$ above.

2 Monster models

Fix a complete L-theory T.

Definition 4. Let κ be a cardinal.

1. A model $M \models T$ is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$, every type over A is realized in M.

2. A model $M \models T$ is *strongly κ-homogeneous* is for every partial elementary map f from M to M with $|\operatorname{dom}(f)| = |\operatorname{im}(f)| < \kappa$, there is an automorphism $\sigma \in \operatorname{Aut}(M)$ extending f.

A *monster model* is a model $\mathbb{M} \models T$ that is κ-saturated and strongly κ-homogeneous, where κ is a cardinal bigger than any cardinals we care about. A set $A \subseteq \mathbb{M}$ is *small* if $|A| < \kappa$. A *small model* is $M \preceq \mathbb{M}$ with $|M| < \kappa$. If $\sigma \in \operatorname{Aut}(\mathbb{M})$ and $A \subseteq \mathbb{M}$, then *σ fixes A pointwise* if $\forall x \in A : \sigma(x) = x$. The notation $\operatorname{Aut}(\mathbb{M}/A)$ denotes the group of automorphisms $\sigma \in \operatorname{Aut}(\mathbb{M})$ which fix A pointwise.

Fact 5. *Let $A \subseteq \mathbb{M}$ be small.*

1. *A is contained in a small model.*

2. *If $p \in S_n(A)$, then p is realized in M.*

3. *If $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$, then there is $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$ with $\sigma(\bar{b}) = \bar{c}$.*

(2) and (3) also hold for α-types instead of n-types, where α is infinite but small.

Also, if $M \preceq \mathbb{M}$ is a small model and $N \succeq M$ with $|N| < \kappa$, then there is an N' such that $M \preceq N' \preceq \mathbb{M}$, and N' is isomorphic to N as an $L(M)$-structure.