Final review

Introduction to Model Theory

December 23, 2021

Section 1

Languages, structures, formulas, satisfaction

Languages

Definition

A *language* (or *signature*) consists of

1. A set of *function symbols*
2. A set of *relation symbols*
3. A set of *constant symbols*
4. A map assigning to each function symbol or relation symbol X a nonnegative integer called the *arity* of X.

A k*-ary relation symbol* or k*-ary function symbol* is a relation symbol or function symbol of arity k.

- *Unary* and *binary* mean 1-ary and 2-ary.
- The *language of orders* has one binary relation symbol $\leq$.
- The *language of rings* has binary function symbols $+, \times$, a unary function symbol $-$, and two constant symbols 0 and 1.

Structures

Fix a language L.

Definition

An L*-structure* M consists of

- A set M, sometimes called the *domain* of the structure, or the set of *elements*.
- For each constant symbol c in L, an element $c^M \in M$.
- For each k-ary relation symbol R in L, a subset $R^M \subseteq M^k$.
- For each k-ary function symbol f in L, a function $f^M : M^k \to M$.

If X is a symbol, then X^M is called the *interpretation* of X in M.

Structures can be empty.

Structures

Suppose L_{or} is the language of ordered rings $\{+, -, \times, 0, 1, \leq\}$. An L_{or}-structure is the following:

- A set M.
- Functions

$$+^M : M^2 \to M$$
$$-^M : M \to M$$
$$\times^M : M^2 \to M$$

- Elements $0^M \in M$ and $1^M \in M$.
- A relation $(\leq^M) \subseteq M^2$.

Usually we write $+^M, -^M, \times^M, 0^M, 1^M, \leq^M$ as $+, -, \times, 0, 1, \leq$.
An L_{or}-structure needn't be an ordered ring.

Terms

Fix a language L and a set of *variables*. The set of L-terms is generated by the following:

- Any variable is a term.
- Any constant symbol is a term.
- If f is a k-ary function symbol and $s_1, \ldots, s_k$ are terms, then $f(s_1, \ldots, s_k)$ are terms.

Examples:

- In the language of rings, $+(x, \times(y, 1))$ is a term, and we usually write it as $x + (y \times 1)$ or just $x + y \cdot 1$.
- In the language of orders, the only terms are variables.

Terms: notation

- If we say "$t(x_1, \ldots, x_n)$ is a term," we mean that $t(x_1, \ldots, x_n)$ is a term in the variables $\{x_1, \ldots, x_n\}$
  - That is, the only variables appearing in $t(x_1, \ldots, x_n)$ are $\{x_1, \ldots, x_n\}$.
- We can do substitutions: we can replace one or more of the x_i with other terms.
  - In the language of rings, if $t(x, y)$ is $x + y \cdot y$, then $t(x. y + z)$ is $x + (y + z) \cdot (y + z)$.
- We often abbreviate a tuple of variables $(x_1, \ldots, x_n)$ as $\bar{x}$.
  - The length n should be clear from context.

Formulas

The set of L-*formulas* is generated by the following:

1. If t, s are terms, then "$t = s$" is a formula.
2. If R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula.
3. If φ is a formula, then $\neg\varphi$ is a formula.
4. If φ, ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are formulas.
5. $\bot, \top$ **are formulas**.
6. If φ is a formula and x is a variable, then $\exists x\ \varphi$ and $\forall x\ \varphi$ are formulas.

Semantics of terms

Let M be an L-structure, $t(x_1, \ldots, x_n)$ be a term, and $\bar{a} \in M^n$. We define $t(\bar{a})^M$ recursively as follows:

- If $t(\bar{x}) = x_i$, then $t(\bar{a})^M = a_i$.
- If $t(\bar{x})$ is a constant symbol c, then $t(\bar{a})^M = c^M$.
- If $t(\bar{x})$ is $f(s_1(\bar{x}), \ldots, s_k(\bar{x}))$ for some k-ary function symbol f, then $t(\bar{a})^M = f^M(s_1(\bar{a})^M, \ldots, s_k(\bar{a})^M)$.

Idea

t^M is obtained by replacing the symbols in t with their interpretations in M.

In an L_{or}-structure M, $(0 + 1 \cdot a)^M = 0^M +^M 1^M \cdot^M a$.

Semantics of formulas ($\models$)

Let M be an L-structure, $\varphi(x_1, \ldots, x_n)$ be a term, and $\bar{a} \in M^n$. We define $M \models \varphi(\bar{a})$ recursively:

- $M \models \top$
- $M \not\models \bot$.
- $M \models t(\bar{a}) = s(\bar{a}) \iff t(\bar{a})^M = s(\bar{a})^M$
- $M \models R(t_1(\bar{a}), \ldots, t_n(\bar{a})) \iff (t_1(\bar{a})^M, \ldots, t_n(\bar{a})^M) \in R^M$
- $M \models \varphi(\bar{a}) \wedge \psi(\bar{a}) \iff (M \models \varphi(\bar{a})$ and $M \models \psi(\bar{a}))$
- $M \models \varphi(\bar{a}) \vee \psi(\bar{a}) \iff (M \models \varphi(\bar{a})$ or $M \models \psi(\bar{a}))$
- $M \models \neg\varphi(\bar{a}) \iff M \not\models \varphi(\bar{a})$
- $M \models \exists x\ \varphi(x, \bar{a}) \iff \exists b \in M\ (M \models \varphi(b, \bar{a}))$
- $M \models \forall x\ \varphi(x, \bar{a}) \iff \forall b \in M\ (M \models \varphi(b, \bar{a}))$.

Semantics of formulas ($\models$)

Idea

"$M \models \varphi$" is φ with the following changes:

- Each symbol in L is replaced with its interpretation in M.
- $\forall x$ becomes $\forall x \in M$
- $\exists x$ becomes $\exists x \in M$

Example

An L_{or}-structure satisfies $\exists x\ (x \cdot x \leq x)$ iff

$$\exists a \in M\ (a \cdot^M a \leq^M a).$$

Idea

$M \models \varphi(a_1, \ldots, a_n)$ means that $\varphi(a_1, \ldots, a_n)$ is "true inside M."

$L(M)$

Suppose L is a language and M is an L-structure.

- $L(M)$ is the language obtained by adding a new constant symbol for each element of M.
- We can regard M as an $L(M)$-structure by interpreting the each new symbol c as the corresponding element of M.

Idea

A formula or term in $L(M)$ is a formula or term with parameters from M.

Remark

The map $(-)^M$ that evaluates terms is really a map from $L(M)$-terms (with no variables) to M.
The relation $M \models \varphi$ is really a relation between structures M and $L(M)$-sentences φ.

Constants and 0-ary functions

Constant symbols are equivalent to 0-ary function symbols.

- We can think of a constant symbol like 1 as a function 1() which takes no inputs, and output the value 1.
- In general, a k-ary function is $M^k \to M$.
- When $k = 0$, M^0 is a singleton $\{()\}$, where $()$ is the tuple of length 0.
- A 0-ary (or "nullary") function is $\{()\} \to M$, which amounts to an element of M.
- So we don't *really* need constant symbols.

Section 2

Theories and models

Theories and models

Let L be a language.

- An L-*theory* is a set of L-sentences.
- If M is an L-structure and T is an L-theory, then

$$M \models T$$

  means that $M \models \varphi$ for every $\varphi \in T$.
- A *model of* T is an L-structure M such that $M \models T$.

Elementary equivalence

Definition

Two L-structures M_1, M_2 are *elementarily equivalent* if

$$M_1 \models \varphi \iff M_2 \models \varphi$$

for any L-sentence φ.

This implies that $M_1 \models T \iff M_2 \models T$, for any theory T.

Logical implication

If T is an L-theory and φ is an L-sentence, then

$$T \vdash \varphi$$

means that every model of T satisfies φ:

$$M \models T \implies M \models \varphi.$$

Some authors write $T \models \varphi$ rather than $T \vdash \varphi$.

Fact (Gödel's completeness theorem)

$T \vdash \varphi$ iff φ is provable from T.

Consistent theories

A theory T is *inconsistent* if the following equivalent conditions hold:

- T has no models.
- $T \vdash \bot$.
- There is a sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Otherwise, T is *consistent*.

Complete theories

A consistent theory T is *complete* if the following equivalent conditions hold:

- If $M_1, M_2 \models T$, then $M_1 \equiv M_2$.
- For any sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

(Some authors use the stronger sense of "complete" where $\varphi \in T$ or $\neg\varphi \in T$.)

Remark

If T is complete and $M \models T$, then for any sentence φ,

$$T \vdash \varphi \iff M \models \varphi.$$

Logically equivalent theories

Two L-theories T_1, T_2 are *logically equivalent* if the following equivalent conditions hold:

- T_1 and T_2 have the same models.
- $T_1 \vdash \varphi \iff T_2 \vdash \varphi$ for any φ.
- $\varphi \in T_1 \implies T_2 \vdash \varphi$ and $\varphi \in T_2 \implies T_1 \vdash \varphi$.

The complete theory of a structure

Let M be an L-structure.

- The *complete theory of* M, written $\mathrm{Th}(M)$, is the set of L-sentences φ such that $M \models \varphi$.
- $N \models \mathrm{Th}(M) \iff N \equiv M$.
- If $M \models T$ and T is complete, then T is logically equivalent to $\mathrm{Th}(M)$.

Elementary classes

An *elementary class* is a class of structures of the form

$$\{M : M \text{ is a model of } T\}$$

for some theory T.

Warning

Some authors require T to be finite, but this is unusual in modern model theory.

Section 3

More about formulas

Atomic formulas

An *atomic formula* is a formula of one of the forms

- $t(\bar{x}) = s(\bar{x})$
- $R(t_1(\bar{x}), \ldots, t_n(\bar{x}))$.

i.e., a formula built without the logical connectives $\forall, \exists, \neg, \vee, \wedge, \top, \bot$.

Quantifier-free formulas

A *quantifier-free formula* is a formula without quantifiers $(\forall, \exists)$.

- $(x + y \cdot y \leq -z) \vee \bot$ is a quantifier-free formula.
- $\exists x \ (x \leq x)$ is not quantifier-free.

Conjunctions

- The AND operation $(\wedge)$ is called "conjunction."
- If $\varphi_1, \ldots, \varphi_n$ are formulas, their *conjunction* is

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n.$$

- When $n = 0$, the conjunction is $\top$.
- The conjunction is often written $\bigwedge_{i=1}^n \varphi_i$.

Remark

$M \models \bigwedge_{i=1}^n \varphi_i$ iff for every $i \in \{1, \ldots, n\}$, $M \models \varphi_i$.
So $\bigwedge_{i=1}^n$ works a little like "$\forall i$."

Disjunction

- The OR operation $(\vee)$ is called "disjunction."
- If $\varphi_1, \ldots, \varphi_n$ are formulas, their *disjunction* is

$$\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n.$$

- When $n = 0$, the disjunction is $\bot$.
- The disjunction is often written $\bigvee_{i=1}^n \varphi_i$.

Remark

$M \models \bigvee_{i=1}^n \varphi_i$ iff there is $i \in \{1, \ldots, n\}$ such that $M \models \varphi_i$
So $\bigvee_{i=1}^n$ works a little like "$\exists i$."

Boolean combinations

Let S be a set of formulas.

1. Let S_1 be the smallest set of formulas containing S and closed under disjunction and conjunction (including $\top, \bot$). Formulas in S_1 are called *positive boolean combinations* of formulas in S.
2. Let S_2 be the smallest set of formulas containing S and closed under disjunction, conjunction, and negation $(\neg)$. Formulas in S_2 are called *boolean combinations* of formulas in S.

Quantifier-free formulas = boolean combinations of atomic formulas.

More logical symbols

- $\varphi \rightarrow \psi$ means "φ implies ψ," i.e.,

$$\neg\varphi \vee \psi.$$

- $\varphi \leftarrow \psi$ means $\psi \rightarrow \varphi$.
- $\varphi \leftrightarrow \psi$ means "φ iff ψ"

$$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

"Every man is mortal" becomes $\forall x \ (Man(x) \rightarrow Mortal(x))$.

More logical symbols

- $\exists! x \ \varphi(x)$ means "there is a unique x such that $\varphi(x)$ holds," i.e.,

$$(\exists x \ \varphi(x)) \\ \wedge \\ \forall x, y \ \ (\varphi(x) \wedge \varphi(y) \rightarrow x = y))$$

Fewer logical symbols

Every formula is logically equivalent to a formula built from atomic formulas using

$$\neg, \exists, \wedge$$

i.e., not using $\forall, \vee, \top, \bot$.
Why?

$$\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$$
$$\forall x\ \varphi(x) \equiv \neg\exists x\ \neg\varphi(x)$$
$$\top \equiv \forall x\ (x = x)$$
$$\bot \equiv \neg\top.$$

Definable sets

If $\varphi(x_1, \ldots, x_n)$ is an L-formula and M is an L-structure, then

$$\varphi(M^n) := \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}.$$

- Sometimes $\varphi(M^n)$ is written $\varphi(M)$.
- Sets of the form $\varphi(M^n)$ are called *$\varnothing$-definable* or *0-definable* sets.

In $(\mathbb{R}, +, \cdot)$, the formula

$$\varphi(x, y) = \exists z\ (x + z \cdot z = y)$$

defines the relation $\leq$.

Definable sets

If $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an L-formula and $\bar{b} \in M^m$, then

$$\varphi(M^n, \bar{b}) = \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}.$$

- Sets of the form $\varphi(M^n, \bar{b})$ are called *M-definable sets*.
- If $\bar{b} \in A^n$ for some $A \subseteq M$, we say the set is *A-definable*.
- "definable" by itself means M-definable or $\varnothing$-definable, depending on the author.

Section 4

Examples of theories

Equivalence relations

An *equivalence relation* is a model of the theory

$$\forall x\ (x \sim x)$$
$$\forall x, y\ (x \sim y \to y \sim x)$$
$$\forall x, y, z\ (x \sim y\ \wedge\ y \sim z \to x \sim z).$$

Partial orders

A *partial order* is a model of the theory

$$\forall x\ (x \leq x)$$
$$\forall x, y\ (x \leq y\ \wedge\ y \leq x \to x = y)$$
$$\forall x, y, z\ (x \leq y\ \wedge\ y \leq z \to x \leq z).$$

Example: the powerset $(P(X), \subseteq)$.

Linear orders

A *linear order* is a partial order satisfying

$$\forall x, y\ (x \leq y\ \vee\ y \leq z).$$

Example: $(\mathbb{R}, \leq)$.

Dense linear orders (DLO)

A *dense linear order* (without endpoints) is a linear order satisfying

$$\begin{gathered} \exists x\ (\top) \\ \forall x, y\ (x < y \to \exists z\ (x < z\ \wedge\ z < y)) \\ \forall x\ \exists y\ x < y \\ \forall x\ \exists y\ y < x, \end{gathered}$$

where $x < y$ means $x \leq y\ \wedge\ x \neq y$.

- Examples: $(\mathbb{R}, \leq)$, $(\mathbb{Q}, \leq)$.
- Non-examples: $(\mathbb{Z}, \leq)$, $([0,1], \leq)$.

The theory of dense linear orders is usually denoted DLO.

Rings

A *ring* is a model of the theory

$$\begin{aligned} \forall x, y, z\ \Big(& x + y = y + x\ \wedge\ x \cdot y = y \cdot x\ \wedge\ x \cdot 1 = x\ \wedge\ x + 0 = x \\ & \wedge\ x + (-x) = 0\ \wedge\ x + (y + z) = (x + y) + z \\ & \wedge\ x \cdot (y \cdot z) = (x \cdot y) \cdot z\ \wedge\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)\Big) \end{aligned}$$

Examples: $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}$, or the ring of polynomials $R[x]$ for any ring R.

Fields

A *field* is a ring satisfying

$$0 \neq 1\ \wedge\ \forall x\ (x \neq 0 \to \exists y\ (x \cdot y = 1)).$$

Examples: $\mathbb{R}, \mathbb{Q}, \mathbb{C}$, but not $\mathbb{Z}$.

Algebraically closed fields (ACF)

An *algebraically closed field* is a field satisfying the axioms

$$\forall y_0, y_1, \dots, y_n\ (y_n \neq 0 \to \exists x\ (y_n x^n + \cdots + y_2 x^2 + y_1 x + y_0 = 0)$$

for each $n = 1, 2, 3, \dots$.

- The theory of algebraically closed fields is denoted ACF.
- $\mathbb{C} \models \mathrm{ACF}$, but $\mathbb{Q}$ and $\mathbb{R}$ are not models.

ACF_0

ACF_0 is ACF plus the axiom schema

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} \neq 0$$

for $n = 1, 2, 3, \dots$.

- $\mathbb{C}$ is a model.
- Models of ACF_0 are called algebraically closed fields *of characteristic 0*.
- ACF_0 is a complete theory.

The algebraic numbers

A complex number $z \in \mathbb{C}$ is *algebraic* if there are rational numbers $a_0, \ldots, a_n$ with $a_n \neq 0$ and

$$a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0 = 0.$$

- The set of algebraic numbers is denoted $\mathbb{Q}^{alg}$.
- $\mathbb{Q}^{alg}$ is a field.
- $\mathbb{Q}^{alg} \models \mathrm{ACF}_0$, and $\mathbb{Q}^{alg} \equiv \mathbb{C}$. In fact, $\mathbb{Q}^{alg} \preceq \mathbb{C}$.

ACF_p

If $p = 2, 3, 5, 7, \ldots$, then ACF_p is ACF plus the axiom

$$\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0.$$

- A model of ACF_p is called an algebraically closed field *of characteristic* p.
- ACF_p is consistent.
- ACF_p is a complete theory.
- The completions of ACF are exactly ACF_0, ACF_2, ACF_3, ACF_5, …

Ordered fields

The theory of *ordered fields* is the theory of fields plus the theory of linear orders plus the axioms

$$\forall x, y, z\ (x \leq y \rightarrow x + z \leq y + z)$$
$$\forall x, y, z\ (x \leq y\ \wedge\ 0 \leq z \rightarrow xz \leq yz).$$

$\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.

Real closed fields (RCF)

RCF is the theory of ordered fields plus the axiom schema

$$\begin{aligned} &\forall w_0, \ldots, w_n, x, y : \\ &\qquad \Big((w_n x^n + \cdots + w_2 x^2 + w_1 x + w_0 < 0) \\ &\qquad \wedge\ (w_n y^n + \cdots + w_2 y^2 + w_1 y + w_0 > 0) \\ &\qquad \wedge\ x < y \Big) \\ &\qquad \rightarrow \exists z\ (x < z\ \wedge\ z < y\ \wedge\ w_n z^n + \cdots + w_1 z + w_0 = 0) \end{aligned}$$

for each n.

- This is the intermediate value theorem for polynomials.
- $\mathbb{R} \models \mathrm{RCF}$.
- RCF is complete.
- Models of RCF are called *real closed fields*.

Section 5

Partial elementary maps and embeddings

Partial elementary maps

Let M, N be L-structures. A *partial elementary map* is a partial function f, where

- $\mathrm{dom}(f) \subseteq M$.
- $\mathrm{im}(f) \subseteq N$.
- If $a_1, \ldots, a_n \in \mathrm{dom}(f)$ and $\varphi(x_1, \ldots, x_n)$ is an L-formula, then

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n)).$$

Example

Take $\varphi(x, y) = (x = y)$. Then

$$a = b \iff M \models a = b \iff N \models f(a) = f(b) \iff f(a) = f(b).$$

So partial elementary maps are injective, hence bijections from $\mathrm{dom}(f)$ to $\mathrm{im}(f)$.

Partial elementary maps and elementary equivalence

The following are equivalent for L-structures M and N:

1. $M \equiv N$.
2. $\varnothing$ is a partial elementary map from M to N
3. There is a partial elementary map f from M to N.

Isomorphisms

Let M, N be L-structures.

- An *isomorphism* from M to N is a bijection $f : M \to N$ such that
  - If c is a constant symbol, then $g(c^M) = c^N$.
  - If g is an n-ary function symbol and $a_1, \ldots, a_n \in M$, then
  $$f(g^M(a_1, \ldots, a_n)) = g^N(f(a_1), \ldots, f(a_n)).$$
  - If R is an n-ary relation symbol and $a_1, \ldots, a_n \in M$, then
  $$(a_1, \ldots, a_n) \in R^M \iff (f(a_1), \ldots, f(a_n)) \in R^N.$$

Isomorphism

M is isomorphic to N ($M \cong N$) if there is an isomorphism from M to N. This is an equivalence relation:

- $M \cong M$.
- If $M \cong N$ then $N \cong M$.
- If $M_1 \cong M_2$ and $M_2 \cong M_3$, then $M_1 \cong M_3$.

Isomorphisms

Let f be a bijection from M to N. Then the following are equivalent:

- f is an isomorphism.
- f is a partial elementary map.

Consequently, $M \cong N \implies M \equiv N$.

Substructures

Let M be an L-structure.

Definition

A set $A \subseteq M$ is a *substructure* if

- For every constant symbol $c \in L$, we have $c^M \in A$.
- For every function symbol $f \in L$, the set A is closed under f^M.

Then we can make A be an L-structure by defining

$$c^A = c^M$$
$$f^A(x_1, \ldots, x_n) = f^M(x_1, \ldots, x_n)$$
$$R^A(x_1, \ldots, x_n) \iff R^M(x_1, \ldots, x_n).$$

So we can regard substructures as structures, not just sets.

Substructures

If M is an L-structure and $A \subseteq M$, then the substructure *generated by* A is

$$\langle A \rangle_M = \{t(a_1, \ldots, a_n) : t(x_1, \ldots, x_n) \text{ is an } L\text{-term and } a_1, \ldots, a_n \in A\}.$$

- This is the smallest substructure of M containing A.
- We say M is *finitely generated* if $M = \langle A \rangle_M$ for some finite set $A \subseteq M$.

If L has no function symbols or constant symbols, then "finitely generated" = "finite."

Embeddings

An *embedding* from M to N is a function $f : M \to N$ such that...

- If c is a constant symbol, then $g(c^M) = c^N$.
- If g is an n-ary function symbol and $a_1, \ldots, a_n \in M$, then

$$f(g^M(a_1, \ldots, a_n)) = g^N(f(a_1), \ldots, f(a_n)).$$

- If R is an n-ary relation symbol and $a_1, \ldots, a_n \in M$, then

$$(a_1, \ldots, a_n) \in R^M \iff (f(a_1), \ldots, f(a_n)) \in R^N.$$

Remark

An isomorphism is the same thing as a bijective embedding.

Embeddings and substructures

- If A is a substructure of M, then the inclusion $A \hookrightarrow M$ is an embedding.
- If $f : M \to N$ is an embedding, then $f(M)$ is a substructure of N, and M is isomorphic to $f(M)$.
- An embedding from M to N is the same thing as a substructure $A \subseteq N$ and an isomorphism $M \to A$.

Elementary substructures

A substructure $N \subseteq M$ is an *elementary substructure* if for any $\varphi(x_1, \ldots, x_n)$ and $\bar{a} \in N^n$,

$$N \models \varphi(\bar{a}) \iff M \models \varphi(\bar{a}).$$

Equivalently, this means that the inclusion $N \hookrightarrow M$ is a partial elementary map.

Remark

You can think of this as a technical tool which shows up throughout model theory.

Elementary extensions

- $M \preceq N$ means that M is an elementary substructure of N.
- $M \succeq N$ means $N \preceq M$.
- We say M is an *elementary extension* of N if N is an elementary substructure of M.
- More generally, M is an *extension* of N if N is a substructure of M.

Elementary embeddings

An *elementary embedding* from M to N is a function $f : M \to N$ such that for any $\varphi(x_1, \ldots, x_n)$ and $\bar{a} \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n)).$$

Equivalently, an elementary embedding from M to N is a partial elementary map f with $\mathrm{dom}(f) = M$.

Elementary embeddings

- A substructure $M \subseteq N$ is an elementary substructure iff the inclusion $M \hookrightarrow N$ is an elementary embedding.
- If $f : M \to N$ is an elementary embedding, then $\mathrm{im}(f) = f(M)$ is an elementary substructure of N, and $M \cong f(M)$.
- An elementary embedding from M to N amounts to an elementary substructure $A \preceq N$ and an isomorphism $M \cong A$.

A technicality

Suppose $f : M \to N$ is an elementary embedding.

- There is an isomorphism $N \to N'$ such that $N' \succeq M$ and the composition $M \to N \to N'$ is the inclusion $M \hookrightarrow N'$.

Idea

If $M \to N$ is an elementary embedding, then, up to isomorphism, N is an elementary extension of M.

The same holds if we delete the word "elementary" everywhere.

$T(M)$

If M is an L-structure, then $T(M)$ (Poizat's notation) is the set of all $L(M)$-sentences true in M.

- This is usually called the *elementary diagram* of M.
- If $N \succeq M$, then $N \models T(M)$ by definition of $\succeq$.
- If $N \models T(M)$, then there is an elementary embedding $M \to N$ given by sending $c \in M$ to c^N.
  - Morally: if $N \models T(M)$, then $N \succeq M$ (up to isomorphism).

Tarski-Vaught test

Let M be an L-structure and $A \subseteq M$ be a subset.

Fact

$A \preceq M$ iff the following is true: for any formula $\varphi(x, y_1, \ldots, y_n)$ and any $a_1, \ldots, a_n \in A$, if $M \models \exists x\, \varphi(x, a_1, \ldots, a_n)$, then there is $b \in A$ such that $M \models \varphi(b, a_1, \ldots, a_n)$.

- In other words, if $\varphi(M, \bar{a})$ is non-empty, then $\varphi(M, \bar{a}) \cap A \neq \varnothing$.
- In other words, if $X \subseteq M$ is A-definable and non-empty, then $X \cap A \neq \varnothing$.
- A intersects every non-empty A-definable subset of M.

The other theorem of Tarski and Vaught

Definition

An *elementary chain* is a family $\{M_i\}_{i \in I}$, where $(I, \leq)$ is a linear order, each M_i is a structure, and

$$i \leq j \implies M_i \preceq M_j.$$

Theorem (Tarski-Vaught)

Given an elementary chain $\{M_i\}_{i \in I}$, let $N = \bigcup_{i \in I} M_i$. Then $N \succeq M_i$ for any i.

In particular, we can add N to the end of the chain, and it's still an elementary chain.

Section 6

Compactness and ultraproducts

The compactness theorem

- T is *satisfiable* if it has a model.
- T is *finitely satisfiable* if every finite subset $T_0 \subseteq T$ is satisfiable.

Theorem (Compactness)

If T is finitely satisfiable, then T is satisfiable.

Elementary amalgamation

Theorem

If $M_1 \equiv M_2$, then there is a structure N and elementary embeddings

$$M_1 \to N$$
$$M_2 \to N$$

Equivalently, there is N with elementary substructures isomorphic to M_1 and M_2.

- Proof idea: use compactness to find a model of $T(M_1) \cup T(M_2)$.

Löwenheim-Skolem

Theorem

Let M be an infinite L-structure. Suppose $\kappa \geq |L|$. Then there is $N \equiv M$ with $|N| = \kappa$.

Corollary

If an L-theory T has an infinite model, then T has models of size κ for all $\kappa \geq |L|$.

Downward Löwenheim-Skolem

Theorem

Let M be an infinite L-structure. Suppose $|L| \leq \kappa \leq |M|$. Then there is an elementary substructure $N \preceq M$ with $|N| = \kappa$.

In fact,

Theorem

Let M be an infinite L-structure. Let A be a subset. There is an elementary substructure $N \preceq M$ with $N \supseteq A$ and $|N| = \max(|A|, |L|)$.

Upward Löwenheim-Skolem

Theorem

Let M be an infinite L-structure. Suppose $\kappa \geq \max(|L|, |M|)$. Then there is an elementary extension $N \succeq M$ with $|N| = \kappa$.

κ-categoricity

Let κ be an infinite cardinal.

Definition

T is *κ-categorical* if there is a unique model of size κ, up to isomorphism.

Theorem (Łoś-Vaught test, aka Vaught's criterion)

Suppose T is κ-categorical and $\kappa \geq |L|$.

- *Any two infinite models of T are elementarily equivalent.*
- *If all models of T are infinite, then T is complete.*

DLO is $\aleph_0$-categorical. ACF_0 is $\aleph_1$-categorical.

The witness property

An L-theory T has the *witness property* (or is *Henkinized*) if the following holds:

- For any formula $\varphi(x)$, if $\exists x\ \varphi(x)$ is in T, then there is a constant symbol $c \in L$ such that $\varphi(c) \in T$.

Canonical models

Suppose

- T has the witness property.
- T is finitely satisfiable.
- T is complete in the strong sense that $\varphi \in T$ or $\neg\varphi \in T$ for any φ.

Then T has a "canonical model" M where every element of M is named by a constant symbol.

Remark

In fact, T is essentially $T(M)$.

Compactness via Henkin's method

Theorem

Let T be a finitely satisfiable L-theory. Then there is a larger language $L' \supseteq L$ and a larger theory $T' \supseteq T$ such that

- *T' has the witness property.*
- *T' is finitely satisfiable.*
- *For any φ, either $\varphi \in T'$ or $\neg\varphi \in T'$.*

Then T' has a model M (the canonical model), and the reduct $M \restriction L$ is a model of the original theory T.

Ultrafilters

Let I be a set.

Definition

A (proper) *filter* on I is a set $\mathcal{F} \subseteq P(I)$ such that...

- $I \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$.
- If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.
- If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.

Ultrafilters

An *ultrafilter* on I is a filter $\mathcal{F} \subseteq P(I)$ satisfying the equivalent conditions:

- $\mathcal{F}$ is a maximal filter.
- For any $X \subseteq I$, either $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$.

Finite intersection property (FIP)

Definition

A family of sets $\mathcal{F} \subseteq P(I)$ has the *finite intersection property* (FIP) if for any $X_1, \ldots, X_n \in \mathcal{F}$, $\bigcap_{i=1}^n X_i \neq \varnothing$.

We let $n = 0$, in which case $\bigcap_{i=1}^n X_i = I$.

Fact

$\mathcal{F}$ has the FIP iff $\mathcal{F}$ is contained in an ultrafilter.

Principal and non-principal ultrafilters

If $a \in I$, there is a *principal ultrafilter*

$$\{X \subseteq I : a \in X\}$$

Other filters are called *non-principal ultrafilters*.

Ultraproducts

Let I be a set and M_i be an L-structure for each $i \in I$.

- The *product* $\prod_{i \in I} M_i$ is the set of functions $f : I \to \bigcup_{i \in I} M_i$ such that $f(i) \in M_i$ for all $i \in I$.
- If $I \in \{0, 1, 2, \ldots, \omega\}$, we can identify $\prod_{i \in I} M_i$ with the set of tuples of length I, where the jth element of the tuple comes from M_j.

Ultraproducts: the set

Let I be a set and M_i be an L-structure for each $i \in I$. Let $\mathcal{U}$ be an ultrafilter on I.

- The *ultraproduct* $\prod_{i \in I} M_i$ is the quotient of $\prod_{i \in I} M_i$ by the equivalence relation where

$$a \sim b \iff \{i \in I : a(i) = b(i)\} \in \mathcal{U}.$$

We write the equivalence class of a as $[a]$.

Ultraproducts: the structure

- If $c \in L$ is a constant symbol, we interpret c in the ultraproduct as $[a]$, where $a(i) = c^{M_i}$ for all i.
- If $f \in L$ is an n-ary function symbol, we interpret f in the ultraproduct by

$$f([a_1], \ldots, [a_n]) = b,$$

where $b(i) = f^{M_i}(a_1(i), \ldots, a_n(i))$.
  - Idea: f is evaluated coordinate-by-coordinate.
- If $R \in L$ is an n-ary relation symbol, we interpret R in the ultraproduct by

$$R([a_1], \ldots, [a_n]) \iff \{i \in I : R^{M_i}(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}.$$

Łoś's theorem

Theorem (Łoś)

Let N be an ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$. Let $\varphi(x_1, \ldots, x_n)$ be a formula. Then

$$N \models \varphi([a_1], \ldots, [a_n]) \iff \{i \in I : M_i \models \varphi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}.$$

Corollary

If φ is a sentence, then

$$N \models \varphi \iff \{i \in I : M_i \models \varphi\} \in \mathcal{U}.$$

This can be used to give a proof of compactness.

Ultrapowers

An *ultrapower* is an ultraproduct of the form $\prod_{i \in I} M/\mathcal{U}$, i.e., with all the structures M_i being the same structure M. The ultrapower is also written $M^I/\mathcal{U}$ or $M^{\mathcal{U}}$.

Fact

There is an elementary embedding $M \to M^I/\mathcal{U}$ given by sending $a \in M$ to $[f]$, where $f(i) = a$ for all i.

If the ultrapower is non-principal, this is usually a proper elementary extension.

Section 7

Types

Partial types

Let M be an L-structure, A be a subset, and $x_1, \dots, x_n$ be variables.

Definition

A *partial n-type over A* is a set Σ of $L(A)$-formulas in the variables $x_1, \dots, x_n$ that is finitely satisfiable in M: for any $\psi_1(\bar{x}), \dots, \psi_n(\bar{x}) \in \Sigma$, there is $\bar{a} \in M$ such that $M \models \bigwedge_{i=1}^{n} \psi(\bar{a})$.

- We often write $\Sigma(x_1, \dots, x_n)$ to indicate that Σ is a type in the variables $x_1, \dots, x_n$.

Realizations of partial types

Theorem

Let $\Sigma(\bar{x})$ be a partial n-type over $A \subseteq M$. Then there is an elementary extension $N \succeq M$ and a tuple $\bar{a} \in N^n$ which realizes $\Sigma(\bar{x})$, in the sense that

$$\psi(\bar{x}) \in \Sigma(\bar{x}) \implies N \models \psi(\bar{a}).$$

We sometimes write this as $N \models \Sigma(\bar{a})$, or $\bar{a} \models \Sigma$.

The type of a tuple

Suppose $B \subseteq M \preceq N$ and $\bar{a} \in N^n$. Then

$$\mathrm{tp}(\bar{a}/B) = \{\varphi(x_1, \dots, x_n) : \varphi \text{ is an } L(B)\text{-formula and } N \models \varphi(\bar{a})\}.$$

Then $\mathrm{tp}(\bar{a}/B)$ is a partial n-type over B.

(Complete) types

Let $p(\bar{x})$ be an n-type over $A \subseteq M$. Then p is a *complete type* if the following equivalent conditions hold:

- $p = \mathrm{tp}(\bar{b}/A)$ for some n-tuple $\bar{b}$ in an elementary extension $N \succeq M$.
- p is a maximal partial type.
- For any $L(A)$-formula $\varphi(x_1, \dots, x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$.

Complete types are also called *types*.

Remark

This is analogous to how if T is a consistent L-theory, then following are equivalent:

- $T = \mathrm{Th}(M)$ for some L-structure M.
- T is a maximal consistent theory.
- For any sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$.

The space of n-types

$S_n(A)$ is the space of (complete) n-types over A.

- $S_n(A) = \{\mathrm{tp}(\bar{b}/A) : \bar{b} \in N^n, \ N \succeq M\}$.

Remark

$S_n(A)$ has the structure of a topological space, but we didn't discuss this much in class.

How to think of types over $\varnothing$

If $\bar{a}$ is an n-tuple in N_1 and $\bar{b}$ is an n-tuple in N_2, then the following are equivalent:

- $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$
- For any formula $\varphi(x_1, \dots, x_n)$,

$$N_1 \models \varphi(\bar{a}) \iff N_2 \models \varphi(\bar{b}).$$

- $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a partial elementary map.

Similarly, $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$ iff $\bar{a}$ and $\bar{b}$ satisfy the same $L(C)$-formulas.

Section 8

κ-saturated models

κ-saturation

Definition

A structure M is *κ-saturated* if the following holds: for any $A \subseteq M$ with $|A| < \kappa$ and any $p \in S_1(A)$, the type p is realized in M.

Theorem

If M is κ-saturated and $A \subseteq M$ and $|A| < \kappa$ and $p \in S_n(A)$, then p is realized in M.

Consequences of κ-saturation

Suppose M is κ-saturated.

Theorem (κ-universality)

If $N \equiv M$ and $|N| \leq \kappa$, then there is an elementary embedding $N \to M$. Equivalently, there is $N' \preceq M$ with $N \cong N'$.

Theorem (κ-compactness)

Let $\mathcal{F} \subseteq P(M^n)$ be a family of definable sets with the FIP. If $|\mathcal{F}| < \kappa$, then $\bigcap \mathcal{F} \neq \varnothing$.
Equivalently, if $\Sigma(\bar{x})$ is a partial type and $|\Sigma| \leq \kappa$, then Σ is realized in M.

Strong κ-homogeneity

Definition

A structure M is *strongly κ-homogeneous* if the following holds: for any partial elementary map f from M to M with $|\mathrm{dom}(f)| = |\mathrm{im}(f)| < \kappa$, there is an automorphism $\sigma \in \mathrm{Aut}(M)$ extending f.

Consequences of strong κ-homogeneity

Definition

$\mathrm{Aut}(M/A)$ is the set of automorphisms $\sigma \in \mathrm{Aut}(M)$ which fix A pointwise, in the sense that $\sigma(x) = x$ for all $x \in A$.

Suppose M is strongly κ-homogeneous.

Theorem

Suppose $A \subseteq M$ and $|A| < \kappa$. Let $\bar{b}, \bar{c}$ be n-tuples in M. Then the following are equivalent:

- $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$.
- *There is $\sigma \in \mathrm{Aut}(M/A)$ such that $\sigma(\bar{b}) = \bar{c}$.*

Idea (If you know group theory...)

If M is κ-saturated and strongly κ-homogeneous, and $A \subseteq M$ has $|A| < \kappa$, then $S_n(A)$ can be identified with the space of orbits of $\mathrm{Aut}(M/A)$ acting on M^n.

Existence

Theorem

Given M and κ, there is an elementary extension $N \succeq M$ that is κ-saturated and strongly κ-homogeneous.

Theorem

If M is an infinite structure, then M is not κ-saturated for any $\kappa > |M|$.

Saturated models

Definition

Let M be an infinite structure of size κ.

- M is *saturated* if it is κ-saturated.
- M is *strongly homogeneous* if it is κ-saturated.

Theorem

If M is saturated, then M is strongly homogeneous.

Theorem

If T is a complete theory and κ is a cardinal, then T has at most one saturated model of size κ.

Saturated models

If T is a complete theory, then $S_n(T)$ denotes $S_n^M(\varnothing)$ for any $M \models T$. Equivalently,

$$S_n(T) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in M^n,\ M \models T\}.$$

Definition

A complete theory T is *small* if $S_n(T)$ is countable for all n.

Theorem

T has a countable saturated model iff T is small.

Beth's implicit definability theorem

Theorem

Let T be an L-theory. Let $L(P)$ be L plus a new n-ary relation symbol P. Let T' be an $L(P)$-theory. Suppose that for every $M \models T$, there is a unique $P \subseteq M^n$ such that $(M, P) \models T'$. Then there is an L-formula $\varphi(x_1, \ldots, x_n)$ such that

$$M \models T \implies (M, \varphi(M^n)) \models T'.$$

Idea

T' is an "implicit definition" of P, and φ is an "explicit definition" of P.

The proof uses κ-saturated strongly κ-homogeneous models.

Section 9

Back-and-forth equivalence

Local isomorphisms

Let M, N be L-structures. A *local isomorphism* or *0-isomorphism* is a partial map f where

- $\mathrm{dom}(f)$ is a finitely-generated substructure of M.
- $\mathrm{im}(f)$ is a finitely-generated substructure of N.
- f is an isomorphism from $\mathrm{dom}(f)$ to $\mathrm{im}(f)$.

If L has only relation symbols, then "finitely-generated substructure" means "finite subset."

Karpian families

A *Karpian family* between M and N is a set $\mathcal{K}$ of local isomorphisms such that...

- If $f \in \mathcal{K}$ and $a \in M$, there is $g \in \mathcal{K}$ with $g \supseteq f$ and $a \in \mathrm{dom}(g)$.
- If $f \in \mathcal{K}$ and $b \in N$, there is $g \in \mathcal{K}$ with $g \supseteq f$ and $b \in \mathrm{im}(g)$.

Karpian families are usually called "back-and-forth systems."

Karpian families

Fact

If $\mathcal{K}$ is a Karpian family and $f \in \mathcal{K}$, then f is a partial elementary map.

Fact

If $M, N \models \mathrm{DLO}$, then the set of all local isomorphisms is a Karpian family.

Fact

If M, N are ω-saturated, then the set of local isomorphisms that are partial elementary maps is a Karpian family.

∞-equivalences

We didn't discuss it in class but...

- An ∞-*equivalence* from M to N is a local isomorphism belonging to some Karpian family.
- The set of all ∞-equivalences is a Karpian family.
  - (The union of all Karpian families is a Karpian family.)
- Two structures M, N are said to be ∞-equivalent if there is an ∞-equivalence between them (or equivalently, $\langle\varnothing\rangle_M \to \langle\varnothing\rangle_N$ exists and is an ∞-equivalence).

Graded back-and-forth systems

A *graded back-and-forth system* between M and N is a sequence of classes $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$ of local isomorphisms such that

- If $f \in \mathcal{S}_{p+1}$ and $a \in M$, there is $g \in \mathcal{S}_p$ with $g \supseteq f$ and $a \in \mathrm{dom}(g)$.
- If $f \in \mathcal{S}_{p+1}$ and $b \in N$, there is $g \in \mathcal{S}_p$ with $g \supseteq f$ and $b \in \mathrm{im}(g)$.

ω-isomorphisms

Definition

A local isomorphism f is a *p-isomorphism* if $f \in \mathcal{S}_p$ for some graded back-and-forth system $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots)$.

Definition

A local isomorphism f is an *ω-isomorphism* if f is a p-isomorphism for all $p < \omega$.

This is Poizat's non-standard terminology; don't use it.

Fraïssé's theorem

Theorem

If f is an ω-isomorphism, then f is a partial elementary map.

Now, **suppose L contains finitely many relation symbols and constant symbols, and no function symbols.**

Theorem

Let f be a local isomorphism. Then the following are equivalent:

- *f is an ω-isomorphism.*
- *f is a partial elementary map.*

Back-and-forth in ω-saturated models

Fact

If M, N are ω-saturated, then the set of local isomorphisms that are partial elementary maps is a Karpian family.

If f is a local isomorphism, then then the following are equivalent:

1. f is a partial elementary map
2. f is an ∞-isomorphism
3. f is an ω-isomorphism.

(1) $\implies$ (2) is the Fact. (2) $\implies$ (3) holds in general. (3) $\implies$ (1) is the direction of Fraïssé's theorem that always holds.

Section 10

Quantifier-elimination

Quantifier-free types

- $\mathrm{qftp}(\bar{a}/B)$ is the set of *quantifier-free* $L(B)$-formulas satisfied by $\bar{a}$.
- Usually we're interested in the case $B = \varnothing$:
  - $\mathrm{qftp}(\bar{a})$ is the set of quantifier-free L-formulas satisfied by $\bar{a}$.

Quantifier-free types

If $\bar{a} \in M^n$ and $\bar{b} \in N^n$, then the following are equivalent:

- $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$
- For any quantifier-free L-formula $\varphi(x_1, \ldots, x_n)$,
$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b}).$$
- For any atomic L-formula $\varphi(x_1, \ldots, x_n)$,
$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b}).$$
- There is an isomorphism from $\langle \bar{a} \rangle_M$ to $\langle \bar{b} \rangle_N$ sending $\bar{a}$ to $\bar{b}$.
- There is a local isomorphism from M to N extending $\{(a_1, b_1), \ldots, (a_n, b_n)\}$.

Quantifier elimination

A theory T has *quantifier elimination* if for any formula $\varphi(\bar{x})$, there is a quantifier-free formula $\psi(\bar{x})$ such that
$$T \vdash \forall \bar{x}\ (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

In other words, every formula is equivalent to a quantifier-free formula.

Quantifier elimination

Theorem

Let T be a theory. The following are equivalent:

1. *T has quantifier elimination.*
2. *For any models $M, N \models T$ and n-tuples $\bar{a} \in M^n$ and $\bar{b} \in N^n$,*
$$\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b}) \implies \mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b}).$$

(1) $\Longrightarrow$ (2) is trivial; (2) $\Longrightarrow$ (1) is non-trivial and uses compactness.

Idea

If $\mathrm{qftp}(\bar{a})$ determines $\mathrm{tp}(\bar{a})$, then the theory has quantifier elimination.

Quantifier elimination

The following are equivalent:

- T has quantifier-elimination.
- $\mathrm{qftp}(\bar{a})$ determines $\mathrm{tp}(\bar{a})$ in models of T.
- $\mathrm{qftp}(\bar{a})$ determines $\mathrm{tp}(\bar{a})$ in ω-saturated models of T.
- If M, N are ω-saturated models of T and f is a local isomorphism, then f is a partial elementary map.
- If M, N are ω-saturated models of T and f is a local isomorphism, then f is an ∞-isomorphism.
- If M, N are ω-saturated models of T, then the set of all local isomorphisms is a Karpian family.

Quantifier elimination criterion

Theorem

T has quantifier elimination iff the following holds: if $M, N \models T$ and $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$ and $\alpha \in M$, then there is $\beta \in N$ such that $\mathrm{qftp}^M(\bar{a}, \alpha) = \mathrm{qftp}^N(\bar{b}, \beta)$.

This criterion is useful in combination with

Fact

$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$ *iff there is an isomorphism* $\langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ *sending* $\bar{a}$ *to* $\bar{b}$.

Important theories with QE

These theories have quantifier elimination:

- Algebraically closed fields (ACF), in the language of rings.
- Real closed fields (RCF), in the language of ordered rings.
- Dense linear orders (DLO), in the language of orders.

Discrete linear orders

Let T be the theory of discrete linear orders without endpoints, like $\mathbb{Z}$.

- T doesn't have quantifier elimination.
- Two n-tuples $\bar{a}$ and $\bar{b}$ have the same type iff $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$ and
$$\forall i, j \leq n : d(a_i, a_j) = d(b_i, b_j).$$
- Therefore T has quantifier elimination if we expand the language with binary relations $R_n(x, y)$ for $n < \omega$, where $M \models R_n(a, b)$ iff $d(a, b) = n$.

Consequences of quantifier elimination

Suppose T has quantifier elimination. If M, N are models of T, then the following are equivalent:

1. $M \equiv N$.
2. $\mathrm{tp}^M() = \mathrm{tp}^N()$.
3. $\mathrm{qftp}^M() = \mathrm{qftp}^N()$.
4. $\langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$.

Example

Two algebraically closed fields are elementarily equivalent iff they have the same characteristic. ACF_0 is a complete theory.

Similarly, DLO, RCF, and the theory of discrete linear orders are complete.

Consequences of quantifier elimination

Suppose T has quantifier elimination. Then every definable set is quantifier-free definable.

- Every definable set has the form $\varphi(M^n)$ for some quantifier-free φ.
- Every definable set is a boolean combination of sets defined by atomic formulas.

Example

If M is an algebraically closed field and $X \subseteq M$ is definable, then X is a finite set or the complement of a finite set.
(We only need to check sets defined by atomic formulas like $P(x) = Q(x)$ for some polynomials P, Q over M. If $P = Q$ this set is M; otherwise it's the finite set of roots of $P - Q$.)

Section 11

ω-categoricity

Assumptions

- L is a countable language
- T is a complete L-theory
- The models of T are infinite

Isolated types

Work in a model M.
A type $p \in S_n(A)$ is *isolated* if there is an $L(A)$-formula $\varphi(x)$ such that

$$\text{tp}^N(b/A) = p \iff N \models \varphi(b),$$

for $N \succeq M$.

Remark
This means that $\{p\}$ is open in the topology on $S_n(A)$.

Omitted and realized types

- A type $p \in S_n(A)$ is *realized* if there is $b \in M^n$ with $p = \text{tp}(b/A)$.
- Otherwise, p is *omitted* (in M). We say that M *omits* the type p.

Theorem
If $p \in S_n(A)$ is isolated, then p is realized.

Omitting types theorem

Theorem
Let Π be a countable set of non-isolated types in $\bigcup_n S_n(T)$. Then there is a countable model M omitting every type in Π.

ω-categoricity

Definition
T is ω-categorical if T has a unique model of size ω.

Fix a complete theory T in a countable language, such that the models of T are infinite.

Theorem (Ryll-Nardzewski)
T is ω-categorical iff $S_n(T)$ is finite for all $n < \omega$.

Proof sketch

(1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6) $\implies$ (7) $\implies$ (8) $\implies$ (9)

1. $S_n(T)$ is finite for all $n < \omega$.
2. $S_n(A)$ is finite if $n < \omega$ and A is finite.
3. Every type in $S_n(A)$ is isolated.
4. Every type in $S_n(A)$ is realized.
5. Every model is ω-saturated.
6. Every countable model is saturated.
7. Any two countable models are isomorphic (ω-categoricity).
8. No countable model omits any type in $S_n(T)$
9. Every type in $S_n(T)$ is isolated.

From the proof...

If T is ω-categorical, then

- Every model is ω-saturated.
- Every countable model is saturated.
- Every countable model is strongly homogeneous.

If T eliminates quantifiers, then

- Every local isomorphism is an ∞-isomorphism.
- Every local automorphism on a countable model extends to an automorphism.