Homework: heirs and definable types

Introduction to Model Theory

Due March 3, 2022

Please justify your answers.

1. Consider the structure $(\mathbb{Z}, +, \cdot, <)$. Show that there is a complete type $p \in S_1(\mathbb{Z})$ containing the formula $n < x$ for each $n \in \mathbb{Z}$. *Hint:* this only uses things from introductory model theory.

2. Let $p \in S_1(\mathbb{Z})$ be as in the previous problem, meaning that the formula $n < x$ is in $p(x)$ for all $n \in \mathbb{Z}$. Suppose $M \succeq \mathbb{Z}$ and $q \in S_1(M)$ is an heir of p. Show that $q(x)$ contains the formula $n < x$ for each $n \in M$.

3. The structure $(\mathbb{C}, +, \cdot)$ is strongly minimal, so it should eliminate $\exists^\infty$. Find a first-order formula $\varphi(x, y, z)$ equivalent to $\exists^\infty w\ (xw^2 + yw + z = 0)$ in the structure $\mathbb{C}$.

4. Let $M = \mathbb{R} \setminus [0, 2]$ and $N = \mathbb{R} \setminus [0, 1)$, where

$$[0, 2] = \{x \in \mathbb{R} : 0 \le x \le 2\}$$
$$[0, 1) = \{x \in \mathbb{R} : 0 \le x < 1\}.$$

   From quantifier elimination in DLO, one can show that $(M, \le) \preceq (N, \le) \preceq (\mathbb{R}, \le)$. It turns out that $\mathrm{tp}(0/N)$ is an heir of $\mathrm{tp}(0/M)$. Show that $\mathrm{tp}(0/N)$ is *not* a strong heir of $\mathrm{tp}(0/M)$. *Hint:* you will only need $d\varphi(y)$ for $\varphi(x, y) \equiv (x > y)$.