Overview of stability theory

PHIL630142: Advanced Model Theory

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1 Introduction

This document is a proof-free overview of *stability theory*, the branch of model theory which studies stable theories. A theory is *stable* if it satisfies a certain technical condition (see Fact 2.5). Several important theories from algebra are stable (see Sections 3.1–3.2). Stability is also connected to purely model-theoretic conditions like strong minimality and κ-categoricity (see Section 3.3).

In this overview, we focus on the nicest class of stable theories, the *ω-stable theories* (see Section 2.2). There is a vast collection of tools for working with ω-stable theories.

1. In an ω-stable theory, there is a good notion of "dimension" or "rank" for definable sets, called *Morley rank* (see Section 4).

2. Using Morley rank, one can define the technical notion of a *non-forking extension* of a complete type (see Section 5.2). Intuitively, the non-forking extensions of $p \in S_n(A)$ are the "generic" extensions of p to larger sets of parameters.

3. There is also a notion of *independence* closely connected to non-forking (see Sections 5.3–5.4). These notions of non-forking and independence play an important technical role in stability theory.

4. There is a notion of a "prime model," and more generally, a "prime model over a subset A" (see Section 6). Prime models play a key role in the proof of Morley's theorem, which we sketch in Section 6.2.

The tools of stability theory have been used to great effect in the study of differential fields (see Definition 3.8). Stability theory leads to a rich structure theory for ω-stable fields and groups, which we sketch in Section 7.

Assumption. Throughout, we assume that T is a complete theory <u>in a countable language</u>. We work in a monster model $\mathbb{M} \models T$.

2 Stability

2.1 The definition of stability

Recall that $S_n(A)$ denotes the space of (complete) n-types over $A \subseteq \mathbb{M}$.

Definition 2.1. Let λ be an infinite cardinal. T is *λ-stable* if for any $A \subseteq \mathbb{M}$ and $n < \omega$,

$$|A| \leq \lambda \implies |S_n(A)| \leq \lambda.$$

In other words, there are at most λ complete types over any set of size at most λ.

Definition 2.2. Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula. Then $\varphi(\bar{x}; \bar{y})$ has the *order property* if there are $\bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots \in \mathbb{M}^n$ and $\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots \in \mathbb{M}^m$ such that for any $i, j < \omega$,

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

Example 2.3. In DLO, the formula $\varphi(x, y) = (x \leq y)$ has the order property. For example, in $M = (\mathbb{R}, \leq)$, take $a_i = i + 1$ and $b_j = j$. Then

$$M \models \varphi(a_i, b_j) \iff a_i \leq b_j \iff i + 1 \leq j \iff i < j.$$

Definition 2.4. Let $M \preceq \mathbb{M}$ be a small model. A type $p \in S_n(M)$ is *definable* if for any L-formula $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$, the set

$$\{\bar{b} \in M^m : \varphi(\bar{x}; \bar{b}) \in p(\bar{x})\}$$

is definable in the structure M.

Fact 2.5 ([Poi00, pp. 235, 273])**.** *The following are equivalent:*

- *T is λ-stable for some infinite cardinal λ.*
- *For every small model $M \preceq \mathbb{M}$, every type over M is definable.*
- *No formula has the order property.*

Definition 2.6. T is *stable* if it satisfies the equivalent conditions of Fact 2.5.

2.2 Stability spectra

Definition 2.7. The *stability spectrum* of T is the set of infinite cardinals λ such that T is λ-stable.

Definition 2.8. T is *superstable* if T is λ-stable for all sufficiently large λ.

Fact 2.9 ([Poi00, p. 285])**.** *There are four possible stability spectra for (countable) T:*

1. *T is λ-stable for all infinite λ.*

2. *T is λ-stable for all $\lambda \geq 2^{\aleph_0}$.*

3. *T is λ-stable for all λ such that $\lambda^{\aleph_0} = \lambda$.*

4. *T is not stable for any λ.*

Cases (1)–(3) are stable, and Case (4) is unstable. Cases (1)–(2) are superstable but Cases (3)–(4) are not.[1] Case (1) is $\aleph_0$-stable, but cases (2)–(4) are not. Consequently, the four cases are named as follows:

1. $\aleph_0$-stable (or ω-stable).

2. Strictly superstable.

3. Strictly stable.

4. Unstable.

Remark 2.10. The focus of these notes will be on ω-stable theories, the nicest class of stable theories. Historically, many of the tools of stability theory were first developed within the ω-stable context before being generalized outwards to superstable theories and general stable theories. As one moves from ω-stability to superstability to stability (and beyond), the tools get progressively weaker and more complicated.

3 Examples of stable theories

There are two motivations for the study of stable theories. First, certain theories arising naturally in abstract algebra are stable. Second, certain model-theoretic conditions imply stability.

3.1 Examples from field theory

Recall that a *field* is a structure $(K, +, \cdot)$ satisfying the basic laws of algebra, including the existence of additive and multiplicative inverses (except for 0^{-1}). Some concrete examples of fields are $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

A field K is *algebraically closed* if any polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with coefficients from K can be factored as a product of linear polynomials

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \text{ with } r_1, \ldots, r_n \in K.$$

Fact 3.1 (Fundamental theorem of algebra)**.** *$\mathbb{C}$ is algebraically closed.*

[1] Why isn't Case (3) superstable? König's theorem in cardinal arithmetic [Poi00, p. 179] implies that $\lambda \neq \lambda^{\aleph_0}$ whenever λ has cofinality $\aleph_0$. For example, $\aleph_\omega \neq \aleph_\omega^{\aleph_0}$ because $\aleph_\omega$ has cofinality $\aleph_0$. There are arbitrarily big cardinals with cofinality $\aleph_0$, such as $\aleph_{\aleph_\alpha + \omega}$ for any ordinal α.

In contrast, $\mathbb{Q}$ and $\mathbb{R}$ are not algebraically closed, because the polynomial x^2+1 cannot be factored in those fields.

The theory of algebraically closed fields is denoted ACF.

Fact 3.2 ([Poi00, p. 286]). ACF *is ω-stable, that is, $\aleph_0$-stable.*

By Fact 2.9, this means that ACF is λ-stable for *all* λ.

Remark 3.3. ACF isn't complete. Two algebraically closed fields K, L are elementarily equivalent if and only if they have the same *characteristic*. Consequently, there is one completion of ACF for each possible characteristic. The completions are written

$$\mathrm{ACF}_0, \mathrm{ACF}_2, \mathrm{ACF}_3, \mathrm{ACF}_5, \mathrm{ACF}_7, \mathrm{ACF}_{11}, \ldots$$

Any field K has an algebraically closed extension. There is a minimal such algebraically closed extension called the *algebraic closure*, often written K^{alg} or $\overline{K}$. For example, the algebraic closure of $\mathbb{R}$ is $\mathbb{C}$, and the algebraic closure of $\mathbb{Q}$ is the set of algebraic numbers.

Definition 3.4. A polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is said to be *separable* if its factorization over K^{alg}

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \text{ with } r_1, \ldots, r_n \in K^{alg}$$

has no repeated roots, meaning that $r_i \neq r_j$ for $i \neq j$.

For example, in the field $\mathbb{R}$, the polynomial $x^2 + 1 = (x+i)(x-i)$ is separable but the polynomial $x^2 + 2x + 1 = (x+1)^2$ is not.

Definition 3.5. A field K is *separably closed* if every separable polynomial factors over K into linear polynomials.

This definition looks mysterious, but plays an important role in field theory. Note that any algebraically closed field is separably closed.

Fact 3.6.
1. *For fields of characteristic 0, "separably closed" is equivalent to "algebraically closed."*
2. *For $p = 2, 3, 5, 7, \ldots$, there are separably closed fields of characteristic p that are not algebraically closed.*

The theory of separably closed fields is denoted SCF. The completions of SCF are too complicated to describe here.[2]

Fact 3.7 ([Del98, Theorem 2.3]). *SCF is stable. If M is a separably closed field and M is not algebraically closed, then M is strictly stable (stable but not superstable).*

[2] If K is a separably closed field, the complete theory of K is determined by two invariants—the characteristic and the *Ershov degree*, which measures how far K is from being algebraically closed [Del98, Theorem 2.1].

Here, we say that a structure M is stable, strictly stable, ω-stable, etc., if the complete theory of M is stable, strictly stable, ω-stable, etc.

Another important stable theory comes from differential fields.

Definition 3.8. A *differential field* is a field K with an operation $\partial : K \to K$ satisfying the axioms

$$\begin{aligned}\partial(x+y) &= \partial x + \partial y \\ \partial(xy) &= x\partial y + y\partial x.\end{aligned}$$

The study of differential fields is called *differential algebra*. The simplest examples of differential fields come from fields of functions, with ∂ interpreted as differentiation (taking derivatives). For example, the field $\mathbb{R}(x)$ of rational functions $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ is a differential field if we interpret ∂f as the derivative f'. Conversely, a theorem of Seidenberg shows that any countable differential field of characteristic zero arises from some set of functions in this way [Mar, Theorem 8].

The class of *differentially closed fields* plays a role in differential algebra analogous to the role of algebraically closed fields in field theory. Loosely speaking, a differential field K is differentially closed if any system of differential equations which has a solution in a larger differential field $L \supseteq K$ already has a solution in K.[3] This can be made more precise using the notion of existential closure:

Definition 3.9. Let T be a theory. A model $M \models T$ is *existentially closed* if for any quantifier-free $L(M)$-formula $\varphi(\bar{x})$ and model $N \models T$ extending M,

$$N \models \exists \bar{x}\ \varphi(\bar{x}) \implies M \models \exists \bar{x}\ \varphi(\bar{x}).$$

Fact 3.10 ([Poi00, Theorems 6.4, 6.33])**.** *The existentially closed models of the theory of fields are exactly the algebraically closed fields.*

Fact 3.10 is not entirely trivial, and amounts to the theorem in algebra called *Hilbert's Nullstellensatz* (see [Poi00, Theorem 6.5]).

Definition 3.11. A differential field K is *differentially closed* if it is an existentially closed model of the theory of differential fields.

Fact 3.12 ([Mar00, Theorem 1.1, Corollary 1.2, Corollary 3.1])**.** *There is a theory* DCF_0 *whose models are the differentially closed fields of characteristic 0.* DCF_0 *is complete, has quantifier elimination, and is ω-stable.*

[3] For example, the field of rational functions $\mathbb{C}(x)$ is not differentially closed. Consider the system of equations

$$\begin{aligned}\partial f &= f \\ fg &= 1.\end{aligned}$$

This has a solution $(f, g) = (e^x, e^{-x})$ in the larger field of meromorphic functions on $\mathbb{C}$. But it has no solution in $\mathbb{C}(x)$ because e^x is not a rational function.

The existence, completeness, and quantifier elimination of DCF_0 can be seen from Theorems 6.15, 6.16, and 6.33 in [Poi00]. The ω-stability is proved on p. 286 of [Poi00].

The theory DCF_0 is very rich, and is considered by some to be the "least misleading example" of an ω-stable theory [Mar]. There have been a number of applications of stability theory to DCF_0, yielding new theorems in differential algebra.

3.2 Examples from groups

A *group* is a structure $(G, 1, \cdot, (-)^{-1})$ satisfying the axioms

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x \cdot 1 = 1 \cdot x = x$$
$$x \cdot x^{-1} = x^{-1} \cdot x = 1.$$

A group is *abelian* if it additionally satisfies the axiom

$$x \cdot y = y \cdot x.$$

Abelian groups are sometimes written in additive notation $(G, 0, +, -)$ rather than multiplicative notation $(G, 1, \cdot, (-)^{-1})$.

Fact 3.13 ([Poi00, pp. 286–288])**.**

1. *Any abelian group is stable.*

2. $(\mathbb{Z}, +)$ *is strictly superstable.*

3. $(\mathbb{R}, +)$ *and* $(\mathbb{C}, +)$ *and* $(\mathbb{Q}, +)$ *are* ω*-stable.*

A typical example of a non-abelian group is the group $GL_n(K)$ of invertible $n \times n$ matrices over a field K. A *linear algebraic group* is a subgroup of $GL_n(K)$ defined by algebraic equations. Here are some important linear algebraic groups:

- $GL_n(K)$.

- $SL_n(K)$, the group of matrices $M \in GL_n(K)$ with $\det(M) = 1$.

- $O_n(K)$, the group of orthogonal $n \times n$ matrices, i.e., $M \in GL_n(K)$ such that $M^{-1} = M^{\mathrm{T}}$.

- The group of upper triangular matrices in $GL_n(K)$.

Fact 3.14 ([Poi00, p. 286, Examples 7+15])**.** *If K is algebraically closed, then any linear algebraic group over K is ω-stable.*

Another important class of non-abelian groups is the class of free groups. Let L_{Grp} be the language of groups, including the unary function symbol $(-)^{-1}$ and the constant symbol 1. Let T_{Grp} be the theory of groups. The *free group on n generators*, which we will denote F_n, is the set of L_{Grp}-terms in the variables $x_1, \ldots, x_n$, where we identify two terms $t(\bar{x})$ and $s(\bar{x})$ if $T_{Grp} \vdash \forall \bar{x}\ (t(\bar{x}) = s(\bar{x}))$. For example, we identify the terms $1 \cdot x_1 \cdot x_2^{-1}$ and $(x_2 \cdot x_1^{-1})^{-1}$, because they are equal in any group.

Fact 3.15. *F_1 is isomorphic to $\mathbb{Z}$ via the isomorphism $x_1^n \mapsto n$. In particular, F_1 is abelian and strictly superstable.*

On the other hand, F_n is non-abelian for $n > 1$.

Fact 3.16 ([Sel06, Theorem 3], [Sel13, Theorem 5.1])**.** *Free groups are stable. There is a complete theory T such that $F_n \models T$ for $n > 1$. T is strictly stable.*

Fact 3.16 is a deep theorem of Sela[4] using tools from geometric group theory. The proof is so complicated that nobody I know has managed to read and understand it.

3.3 Stability from model theory

Several model-theoretic conditions imply stability. One important one is strong minimality.

Definition 3.17. A definable set $D \subseteq \mathbb{M}^n$ is *strongly minimal* if D is infinite, but D cannot be written as a disjoint union $X \sqcup Y$ of two infinite definable subsets.

Equivalently, D is strongly minimal if for any definable subset $X \subseteq D$, either X or $D \setminus X$ is finite.

Definition 3.18. A theory T is *strongly minimal* if $\mathbb{M}$ is strongly minimal as a definable set. Equivalently, for any definable set $D \subseteq \mathbb{M}$, either D or $\mathbb{M} \setminus D$ is finite.

A structure M is *strongly minimal* if it is a model of a strongly minimal theory.

Example 3.19. Algebraically closed fields are strongly minimal. This can be proven using quantifier elimination in ACF.

Fact 3.20. *If T is strongly minimal, then T is ω-stable.*

Stability also arises when one tries to count models.

Definition 3.21. Let κ be an infinite cardinal. T is *κ-categorical* if T has a unique model of size κ.

Definition 3.22. T is *uncountably categorical* if T is κ-categorical for some uncountable κ.

Fact 3.23 (Morley's theorem, [Poi00, Corollary 18.25])**.** *If T is uncountably categorical, then T is κ-categorical for* all *uncountable κ.*

[4]Not to be confused with Shelah.

Consequently, uncountably categorical theories are also called $\aleph_1$-*categorical* theories.

Fact 3.24 ([Poi00, Corollary 18.21]). *If T is uncountably categorical, then T is ω-stable.*

In fact, uncountably categorical theories generalize strongly minimal theories:

Fact 3.25 ([Mar02, Corollary 6.1.12]). *Let T be a complete theory. If T is strongly minimal, then T is uncountably categorical. For example,* ACF_p *is uncountably categorical for* $p = 0, 2, 3, 5, \ldots$.

The following is formally similar to Fact 3.24, but the proof is much deeper[5]:

Fact 3.26 (Shelah). *Let T be a complete theory[6] and let κ be an uncountable cardinal. Let λ be the number of models of T of size κ, counted up to isomorphism. If $\lambda \neq 2^\kappa$, then T is superstable.*

The *spectrum* of a theory T is the function which takes an uncountable cardinal κ as input, and outputs the number of models of size κ. Shelah's research program of *classification theory* classifies theories according to their spectra. Fact 3.26 implies that the spectrum of any unstable theory or strictly stable theory is $f(\kappa) = 2^\kappa$. Classification theory can then focus on the remaining cases, which must be superstable.

3.4 Non-examples

Any infinite ordered structure, such as $(\mathbb{Q}, \leq)$ or $(\mathbb{R}, +, \cdot, \leq)$, will be unstable, by the argument from Example 2.3. Consequently, ordered theories like DLO (dense linear orders), RCF (real closed fields), and Presburger arithmetic are never stable. Similarly, any partially ordered set containing an infinite chain will be unstable. For example, most theories of boolean algebras are unstable.

4 Morley rank

Return to the setting of a complete theory T with monster model $\mathbb{M}$.

Definition 4.1. If $D \subseteq \mathbb{M}^n$ is a definable set and α is an ordinal, then the relation $\mathrm{RM}(D) \geq \alpha$ is defined recursively as follows:

- $\mathrm{RM}(D) \geq 0$ iff D is non-empty.
- $\mathrm{RM}(D) \geq \alpha + 1$ iff there are pairwise-disjoint definable subsets $D_0, D_1, D_2, \ldots \subseteq D$ with $\mathrm{RM}(D_i) \geq \alpha$ for all i.
- If β is a limit ordinal, then $\mathrm{RM}(D) \geq \beta$ iff $\mathrm{RM}(D) \geq \alpha$ for all $\alpha < \beta$.

[5] Marker [Mar02, §5.3] gives a proof of the simplest case, which is still very hard.
[6] In a countable language, as always.

If D is non-empty, then $\mathrm{RM}(D)$ is defined to be the largest ordinal α such that $\mathrm{RM}(D) \geq \alpha$, or $+\infty$ if there is no maximum. If $D = \varnothing$, then $\mathrm{RM}(D)$ is defined to be $-\infty$. $\mathrm{RM}(D)$ is called the *Morley rank* of D.

Here, $+\infty$ is a formal symbol greater than all ordinals, and $-\infty$ is a formal symbol less than all ordinals.

Fact 4.2 ([Mar02, Lemma 6.2.7])**.** *Let D be a definable set.*

1. $\mathrm{RM}(D) = -\infty$ *iff D is empty.*
2. $\mathrm{RM}(D) = 0$ *iff D is finite non-empty.*
3. $\mathrm{RM}(D) \geq 1$ *iff D is infinite.*
4. *If $X \subseteq D$, then* $\mathrm{RM}(X) \leq \mathrm{RM}(D)$.
5. *If $D = X \cup Y$, then* $\mathrm{RM}(D) = \max(\mathrm{RM}(X), \mathrm{RM}(Y))$.

Example 4.3. If D is strongly minimal (Definition 3.17), then $\mathrm{RM}(D) = 1$, because D is infinite, but we cannot find two disjoint infinite definable subsets $X, Y \subseteq D$.

The next fact shows that Morley rank is invariant under definable bijections:

Fact 4.4. *Let $f : X \to Y$ be a definable function.*

1. *If f is a bijection, then* $\mathrm{RM}(X) = \mathrm{RM}(Y)$.
2. *If f is an injection, then* $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.
3. *If f is a surjection, then* $\mathrm{RM}(X) \geq \mathrm{RM}(Y)$.

In ACF, Morley rank corresponds to "dimension" or "degrees of freedom." For example, in the field $\mathbb{C}$,

1. The twisted cubic $\{(t, t^2, t^3) : t \in \mathbb{C}\}$ has Morley rank 1.
2. The sphere $\{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\}$ has Morley rank 2.
3. $\mathbb{C}^n$ has Morley rank n.

There is a close connection between Morley rank and ω-stability.

Fact 4.5 ([Mar02, 6.2.14])**.** *The following are equivalent:*

1. *T is ω-stable.*
2. $\mathrm{RM}(\mathbb{M}) < +\infty$.
3. $\mathrm{RM}(D) < +\infty$ *for all definable $D \subseteq \mathbb{M}^n$.*

From this point of view, $+\infty$ is an error value, indicating a failure of ω-stability.

4.1 Morley rank in nice theories

Morley rank is particularly nicely behaved in strongly minimal and uncountably categorical theories[7]. First, Morley rank is finite:

Fact 4.6. *If T is uncountably categorical, then* $\mathrm{RM}(D) < \omega$ *for all definable D.*

Fact 4.7. *If T is strongly minimal, then* $\mathrm{RM}(\mathbb{M}^n) = n$ *for all $n < \omega$, and* $\mathrm{RM}(D) \le n$ *for definable $D \subseteq \mathbb{M}^n$.*

Second, Morley rank satisfies definability and additivity properties:

Fact 4.8. *Suppose T is strongly minimal or uncountably categorical.*

1. *If $\varphi(\bar{x};\bar{y})$ is a formula then the set*

$$\{\bar{b} \in \mathbb{M} : \mathrm{RM}(\varphi(\mathbb{M};\bar{b})) = \alpha\}$$

is definable for each α. (One says that "Morley rank is definable.")

2. *If X, Y are definable sets, then* $\mathrm{RM}(X \times Y) = \mathrm{RM}(X) + \mathrm{RM}(Y)$.

3. *(*"Additivity"*) Let $f : X \to Y$ be a definable surjection. Suppose every fiber $f^{-1}(a)$ has Morley rank k. Then* $\mathrm{RM}(X) = k + \mathrm{RM}(Y)$.

These properties ensure that Morley rank behaves like a good notion of "dimension."

4.2 A less misleading example

In a general ω-stable theory, Morley rank can be infinite. For a representative example, consider the theory DCF_0 of differentially closed fields from Fact 3.12.

Fact 4.9. *Suppose $K \models \mathrm{DCF}_0$.*

1. *If D is definable, then* $\mathrm{RM}(D) = \omega \cdot n + m$ *for some $n, m \in \mathbb{N}$.*

2. $\mathrm{RM}(K) = \omega$.

3. *The "field of constants" $C = \{x \in K : \partial x = 0\}$ has Morley rank 1.*

4. *$C \times K$ has Morley rank $\omega + 1$.*

[7]Facts 4.6–4.8 are well-known, but I had trouble finding a precise reference, due to the current campus lockdown. The strongly minimal case is mostly [Mar02, Theorem 6.2.19, Lemma 6.2.20]. For the uncountably categorical case, see [Poi87, pp. 31–32], which deals with "finite-dimensional theories each of whose dimensions are associated with a type of Morley rank 1." This context generalizes uncountably categorical theories because of the analysis of uncountably categorical theories in [Poi00, Proposition 18.24]. The additivity results hold because Morley rank agrees with Lascar rank in these contexts, and Lascar rank is additive [Poi00, Theorem 19.4].

5. *The equation $\partial x = x$ defines a set of Morley rank 1.*

6. *The equation $\partial^2 x = -x$ defines a set of Morley rank 2.*

Remark 4.10. In calculus, the differential equation $\frac{d^2x}{dt^2} = -x$ has solutions of the form $x = a \sin t + b \cos t$. In particular, there are two degrees of freedom. This is intuitively the reason why $\partial^2 x = -x$ defines a set of rank 2. Similarly, $\partial x = x$ defines a set of rank 1 "because" the generic solution to $\frac{dx}{dt} = x$ is $x = a \cdot e^t$, with one degree of freedom.[8]

5 Non-forking and independence in ω-stable theories

Assume that T is ω-stable. Then every definable set has a well-defined Morley rank less than $+\infty$.

5.1 Morley rank of types

Definition 5.1. If $A \subseteq \mathbb{M}$ is small and $\bar{b} \in \mathbb{M}^n$, then $\mathrm{RM}(\bar{b}/A)$ is the minimum Morley rank of an A-definable set containing $\bar{b}$.

Example 5.2. Suppose T is strongly minimal. If $D \subseteq \mathbb{M}$ is definable, then $\mathrm{RM}(D)$ is 0 or 1 depending on whether D is infinite. Therefore, for $A \subseteq \mathbb{M}$ and $b \in \mathbb{M} = \mathbb{M}^1$,

$$\mathrm{RM}(b/A) = \begin{cases} 0 & \text{if } b \in \mathrm{acl}(A) \\ 1 & \text{if } b \notin \mathrm{acl}(A) \end{cases}$$

where $\mathrm{acl}(A)$ is the *algebraic closure* of A—the union of all finite A-definable sets.

By automorphisms, $\mathrm{RM}(\bar{b}/A)$ depends only on $\mathrm{tp}(\bar{a}/B)$. Therefore, the following definition makes sense:

Definition 5.3. If $A \subseteq \mathbb{M}$ is small and $p \in S_n(A)$, then $\mathrm{RM}(p)$ is $\mathrm{RM}(\bar{b}/A)$ for any $\bar{b}$ realizing p.

Thus $\mathrm{RM}(\bar{b}/A) = \mathrm{RM}(\mathrm{tp}(\bar{b}/A))$.

5.2 Non-forking extensions

Fact 5.4. *Suppose $A \subseteq B \subseteq \mathbb{M}$. Let p be an n-type over A and let $q \in S_n(B)$ be an extension. Then* $\mathrm{RM}(q) \leq \mathrm{RM}(p)$.

Definition 5.5. In the setting of Fact 5.4, we say that q is a *forking extension* of p if $\mathrm{RM}(q) < \mathrm{RM}(p)$. We say that q is a *non-forking extension* of p, written $q \sqsupseteq p$, if $\mathrm{RM}(q) = \mathrm{RM}(p)$.

[8] This intuition isn't 100% reliable—there are differential equations of order n which define sets of Morley rank $m < n$ [Mar00, p. 59].

Non-forking extensions play an important role in stability theory. The intuition of non-forking extensions is that q is a non-forking extension of p iff q is a "generic" extension of p.

Fact 5.6 (Full transitivity, [Poi00, p. 303])**.** *Suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq \mathbb{M}$. Suppose $p_i \in S_n(A_i)$ for $i = 1, 2, 3$, with $p_1 \subseteq p_2 \subseteq p_3$. Then $p_1 \sqsubseteq p_2 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_3$.*

Fact 5.7 (Existence, [Mar02, Theorem 6.3.2(i)])**.** *If $A \subseteq B \subseteq \mathbb{M}$, and $p \in S_n(A)$, then there is at least one non-forking extension $q \in S_n(B)$.*

Over models, non-forking is equivalent to being an heir or coheir:

Fact 5.8 ([Poi00, p. 303] or [Mar02, p. 233])**.** *Let $M \preceq N$ be small models. Let p be an n-type over M and let $q \in S_n(N)$ be an extension. Then the following are equivalent:*

- *q is a non-forking extension of p.*
- *q is the heir of p, i.e., q is the type defined by the same formulas as the definable type p. (Because we are assuming stability, p is definable.)*
- *q is a coheir of p, i.e., q is finitely satisfiable in M.*

In particular, in the context of Fact 5.8, p has a unique non-forking extension to N. This property is called "stationarity."

Definition 5.9. A type $p \in S_n(A)$ is *stationary* if for any $B \supseteq A$, there is a unique non-forking extension of p to B.

Fact 5.10 ([Mar02, Corollary 6.3.12(iii), Corollary 6.3.13])**.**

1. *Any type over a model is stationary.*
2. *If T has elimination of imaginaries[9], any type over $\mathrm{acl}(A)$ is stationary.*

A general type has only finitely many non-forking extensions:

Fact 5.11 ([Mar02, Theorem 6.3.2])**.** *If $A \subseteq B$ and $p \in S_n(A)$, then p has finitely many non-forking extensions to B.*

[9] An *A-interpretable set* is the quotient of an A-definable set by an A-definable equivalence relation. A theory T has *elimination of imaginaries* if for any $\varnothing$-interpretable set X/E there is a $\varnothing$-definable set Y and a $\varnothing$-interpretable bijection f from X/E to Y. It turns out that this technical condition holds in ACF and DCF_0 [Poi00, Theorem 16.21]. Moreover, if T is any theory, there is an "equivalent" (bi-interpretable) theory T^{eq} with elimination of imaginaries, and T^{eq} has the same stability properties as T.

5.3 Ternary independence

Definition 5.12. Let A, B, C be small subsets of $\mathbb{M}$. Then A is *independent* from B over C, written

$$A \mathop{\smile\!\!\!\!\!\!\,|}_C B$$

if for any tuple $\bar{a} \in A$, $\mathrm{tp}(\bar{a}/BC) \sqsupseteq \mathrm{tp}(\bar{a}/C)$.

Remark 5.13. $\mathop{\smile\!\!\!\!\!\!\,|}$ is a ternary (3-ary) relation between subsets of $\mathbb{M}$. It is often called the *(ternary) independence relation*.

Independence is defined from non-forking, and conversely one can recover non-forking from independence:

Fact 5.14. *If $A \subseteq B \subseteq \mathbb{M}$ and $\bar{c} \in \mathbb{M}^n$, then*

$$\bar{c} \mathop{\smile\!\!\!\!\!\!\,|}_A B \iff \mathrm{tp}(\bar{c}/B) \sqsupseteq \mathrm{tp}(\bar{c}/A) \iff \mathrm{RM}(\bar{c}/B) = \mathrm{RM}(\bar{c}/A).$$

Example 5.15. Suppose T is strongly minimal, $C \subseteq \mathbb{M}$, and $a, b \in \mathbb{M} \setminus \mathrm{acl}(C)$. By Example 5.2,

$$a \mathop{\smile\!\!\!\!\!\!\,|}_C b \iff \mathrm{RM}(a/Cb) = \mathrm{RM}(a/C) \iff \mathrm{RM}(a/Cb) = 1 \iff a \notin \mathrm{acl}(Cb).$$

The relation $\mathop{\smile\!\!\!\!\!\!\,|}$ has a mysterious definition, but it satisfies many nice properties. The most important is *symmetry*:

Fact 5.16 (Symmetry, [Poi00, p. 303])**.** $A \mathop{\smile\!\!\!\!\!\!\,|}_C B \iff B \mathop{\smile\!\!\!\!\!\!\,|}_C A$.

We can also write the Full Transitivity property (Fact 5.6) in terms of $\mathop{\smile\!\!\!\!\!\!\,|}$:

Fact 5.17 ([Poi00, p. 303])**.** *If $A_1 \subseteq A_2 \subseteq A_3$, then*

$$B \mathop{\smile\!\!\!\!\!\!\,|}_{A_1} A_3 \iff \left(B \mathop{\smile\!\!\!\!\!\!\,|}_{A_1} A_2 \text{ and } B \mathop{\smile\!\!\!\!\!\!\,|}_{A_2} A_3 \right).$$

For a comprehensive discussion of the axiomatic properties of $\mathop{\smile\!\!\!\!\!\!\,|}$, see [Adl07].

The intuition of $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ is that A and B are independently placed realizations of $\mathrm{tp}(A/C)$ and $\mathrm{tp}(B/C)$. Here are some facts related to this:

Fact 5.18. *Given $C \subseteq \mathbb{M}$ and $p \in S_n(C)$ and $q \in S_m(C)$, there are realizations $\bar{a} \models p$ and $\bar{b} \models q$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_C \bar{b}$. If at least one of p or q is stationary, then $\mathrm{tp}(\bar{a}, \bar{b}/C)$ is uniquely determined by p and q.*

When p and q are both stationary, the unique type $\mathrm{tp}(\bar{a}, \bar{b}/C)$ of independent realizations is called the *Morley product* of p and q, and written $p \otimes q$. It is again stationary.

Fact 5.19. *Suppose T is strongly minimal or uncountably categorical.*[10] *If $\bar{a}, \bar{b}$ are finite tuples, then*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{C} \bar{b} \iff \mathrm{RM}(\bar{a}, \bar{b}/C) = \mathrm{RM}(\bar{a}/C) + \mathrm{RM}(\bar{b}/C)$$
$$\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_{C} \bar{b} \iff \mathrm{RM}(\bar{a}, \bar{b}/C) < \mathrm{RM}(\bar{a}/C) + \mathrm{RM}(\bar{b}/C).$$

Over models, $\mathop{\smile\!\!\!\!\!\!|}$ has a simple definition:

Fact 5.20. *If $M \preceq \mathbb{M}$, then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$ iff $\mathrm{tp}(\bar{a}/M\bar{b})$ is a coheir of $\mathrm{tp}(\bar{a}/M)$, in the sense that $\mathrm{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M.*

5.4 Independent sequences

Definition 5.21. Suppose $B \subseteq \mathbb{M}$, and $A_1, A_2, \ldots, A_n \subseteq M$. Then $A_1, A_2, \ldots, A_n$ are jointly *independent* over B if

$$A_2 \mathop{\smile\!\!\!\!\!\!|}_{B} A_1 \text{ and } A_3 \mathop{\smile\!\!\!\!\!\!|}_{B} A_1 A_2 \text{ and } \cdots \text{ and } A_n \mathop{\smile\!\!\!\!\!\!|}_{B} A_1 A_2 \cdots A_{n-1}.$$

Example 5.22. Two sets A_1 and A_2 are independent over B iff $A_2 \mathop{\smile\!\!\!\!\!\!|}_B A_1$ iff $A_1 \mathop{\smile\!\!\!\!\!\!|}_B A_2$.

Example 5.23. Suppose T is strongly minimal, $C \subseteq \mathbb{M}$, and $a_1, a_2, \ldots, a_n \in \mathbb{M} \setminus \mathrm{acl}(C)$. Generalizing Example 5.15, the sequence $a_1, a_2, \ldots, a_n$ is independent over C iff

$$a_2 \notin \mathrm{acl}(Ca_1) \text{ and } a_3 \notin \mathrm{acl}(Ca_1a_2) \text{ and } \cdots \text{ and } a_n \notin \mathrm{acl}(Ca_1a_2 \cdots a_{n-1}).$$

When we specialize to ACF, this condition means that $a_1, a_2, \ldots, a_n$ are *algebraically independent* over C in the sense of field theory.

Fact 5.24 ([Poi00, Theorem 15.10])**.** *Definition 5.21 is symmetric: if σ is a permutation of $\{1, \ldots, n\}$, then $A_1, \ldots, A_n$ is independent over B if and only if $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$ is independent over B. This generalizes Fact 5.16.*

An independent sequence gives many instances of the ternary independence relation $\mathop{\smile\!\!\!\!\!\!|}$:

Fact 5.25. *Suppose $A_1, \ldots, A_n$ is independent over B. For $S \subseteq \{1, \ldots, n\}$ let $A_S = \bigcup_{i \in S} A_i$. Then $A_{S_1} \mathop{\smile\!\!\!\!\!\!|}_{BA_{S_3}} A_{S_2}$ for any disjoint subsets $S_1, S_2, S_3 \subseteq \{1, \ldots, n\}$.*

For example, if A_1, A_2, A_3 is independent over B, then

$$A_1 \mathop{\smile\!\!\!\!\!\!|}_{B} A_2, \qquad A_1 \mathop{\smile\!\!\!\!\!\!|}_{BA_3} A_2, \qquad A_1A_3 \mathop{\smile\!\!\!\!\!\!|}_{B} A_2, \text{ etc.}$$

Independence is related to the Morley product operation $\otimes$ of Fact 5.18:

[10] In these contexts, Morley rank agrees with *Lascar rank*. Then Fact 5.19 holds by additivity properties of Lascar rank [Poi00, Theorems 19.4, 19.5] together with the fact that Lascar rank governs forking [Poi00, Lemma 17.2].

Fact 5.26. *Suppose $p_1, \ldots, p_n$ are stationary types over B. Let $\bar{a}_i$ be a realization of p_i for $1 \le i \le n$. Then $\bar{a}_1, \ldots, \bar{a}_n$ is independent over B if and only if $(\bar{a}_1, \ldots, \bar{a}_n)$ is a realization of $p_1 \otimes p_2 \otimes \cdots \otimes p_n$.*

The notion of independence can also be extended to infinite sequences:

Definition 5.27. An infinite sequence $(A_i : i \in I)$ is *independent* over B if any finite subsequence is independent over B.

Example 5.28. In ACF, if $(a_i : i \in I)$ is a sequence of transcendental numbers, then $(a_i : i \in I)$ is independent iff it is algebraically independent in the sense of field theory.

6 Prime models and categoricity

Continue to assume that T is ω-stable and $\mathbb{M}$ is a monster model.

Definition 6.1. Let $A \subseteq \mathbb{M}$ be a small set. A *prime model* over A is a small model $M \preceq \mathbb{M}$ containing A, such that if $N \preceq \mathbb{M}$ is any other small model containing A, then there is an elementary embedding $f : M \to N$ fixing the elements of A. (Equivalently, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(M) \subseteq N$.)

Fact 6.2 ([Poi00, Corollaries 17.16, 18.2])**.** *For any small $A \subseteq \mathbb{M}$, there is at least one prime model M over A. Additionally, M is unique up to isomorphism: if M' is another prime model over A, there is an isomorphism $M \to M'$ fixing A.*

In strongly minimal theories, prime models usually come from algebraic closure:

Fact 6.3. *Suppose T is strongly minimal and $A \subseteq \mathbb{M}$ is small. If* $\mathrm{acl}(A)$ *is infinite, then* $\mathrm{acl}(A) \preceq \mathbb{M}$ *and* $\mathrm{acl}(A)$ *is the prime model over A.*

For example, in ACF_0, the prime model over $\varnothing$ is $\mathrm{acl}(\varnothing) = \mathbb{Q}^{alg}$, the set of algebraic numbers.

Example 6.4. Let K be a differential field (Definition 3.8). We can embed K into a monster model $\mathbb{M} \models \mathrm{DCF}_0$ and then form the prime model M over K. The model M is called the *differential closure* of K, and plays a role in differential algebra analogous to the algebraic closure K^{alg} of K in field theory. See [Poi00, §18.3] for more about this. The construction of differential closures was an important application of model theory to differential algebra (see [Poi00, p. 106] for more of the history).

Warning 6.5. In a general ω-stable theory, it can happen that the prime model M over A is not the smallest model containing A. (This happens in DCF_0 according to [Poi00, p. 78].) However, any model smaller than M must contain a copy of M: if $A \subseteq N \prec M$, then there is an isomorphic copy $M' \cong M$ with $A \subseteq M' \preceq N \prec M$.

6.1 Prime models and strongly minimal theories

Prime models and independence play a key role in the fact that strongly minimal theories are uncountably categorical (Fact 3.25), as well as Morley's categoricity theorem (Fact 3.23).

First consider strongly minimal theories. Suppose T is strongly minimal. Say that $a \in \mathbb{M}$ is *transcendental* if $a \notin \operatorname{acl}(\varnothing)$, by analogy with algebraic and transcendental numbers in $\mathbb{C}$. The transcendental elements $a \in \mathbb{M}$ are exactly the realizations of a certain stationary type $p \in S_1(\varnothing)$, called the *transcendental 1-type*. We are interested in sequences of transcendentals which are independent over $\varnothing$. By Fact 5.24, independence is symmetric, so it is really a property of a *set* rather than a *sequence*.

Fact 6.6 ([Mar02, Theorem 6.1.11]). *Assume T is strongly minimal. Let $M \preceq \mathbb{M}$ be a small model.*

1. *There is a maximal independent set B of transcendental elements in M.*
2. $M = \operatorname{acl}(B)$. *In particular, M is the prime model over B by Fact 6.3.*
3. *The cardinality of B is uniquely determined by M. (If B' is another maximal independent subset, then B' has the same cardinality of B.)*
4. *Conversely, M is determined up to isomorphism by the cardinality of B.*

We sketch why each of these things is true. Part (1) is an easy argument using Zorn's lemma plus finitariness of independence (Definition 5.27). Part (2) follows because if $c \in M \setminus \operatorname{acl}(B)$, then $c \mathop{\smile\!\!\!\!\!\!\mid\,}_{\varnothing} B$, and so $B \cup \{c\}$ is independent, contradicting the choice of B. Part (3) is subtle (especially when B is finite), but can be proven using properties of Morley rank. Part (4) works because Fact 5.26 determines the type of B over $\varnothing$ from the cardinality of B. For example, if $B = \{b_1, \ldots, b_n\}$, then $\operatorname{tp}(b_1, \ldots, b_n/\varnothing)$ must be $\underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}$ where $p \in S_1(\varnothing)$ is the transcendental 1-type.

The upshot of Fact 6.6 is that models of the strongly minimal theory T are classified by one invariant, the cardinality of a maximal independent set. Using this, it is not hard to prove that there is at most one model of T of size κ for uncountable κ.[11] This was Fact 3.25.

[11] The situation for countable models is more complicated, because there may be constraints on the cardinality $|B|$. For example, if K is a finite field, then the theory of infinite K-vector spaces is strongly minimal and complete, but $|B|$ must be infinite. (Finite sets have finite algebraic closures in this theory.) Therefore the only countable model is the one with $|B| = \aleph_0$. In particular, T has a unique countable model—it is $\aleph_0$-categorical. On the opposite end of the spectrum, in ACF_0 the invariant $|B|$ can be any of the values $0, 1, 2, 3, \ldots, \aleph_0$, and so we get $\aleph_0$-many distinct countable models. Other intermediate possibilities can occur. For example, there is a theory where $|B|$ can be any of $3, 4, 5, \ldots, \aleph_0$. By considering all the possibilities, one can prove that if T is strongly minimal, then the number of countable models of T is *either* 1 or $\aleph_0$.

6.2 Prime models and uncountably categorical theories

Uncountably categorical theories generalize strongly minimal theories, and there is a similar classification theory for uncountably categorical theories. Fix a complete, uncountably categorical theory T. By Fact 3.24, T is ω-stable. Let $\mathbb{M}$ be a monster model.

Let M_0 be the prime model (over the empty set). Up to isomorphism, every model extends M_0.

Fact 6.7. *There is an M_0-definable strongly minimal set D.*

Let $\varphi(\bar{x})$ be the $L(M_0)$-formula defining D.

Fact 6.8. *There is a unique non-algebraic type $p(\bar{x}) \in S_n(M_0)$ extending $\varphi(\bar{x})$.*

Fact 6.9. *Let κ be a cardinal. Let $(b_i : i < \kappa)$ be an independent sequence of realizations of p. Let M_κ be the prime model over $M_0\bar{b}$.*

1. *M_κ is determined up to isomorphism by κ.*
2. *The size of M_κ is $\kappa + \aleph_0$.*
3. *Up to isomorphism, every model has the form M_κ for some κ.*

See [Poi00, Section 18.6] for proofs of these facts. As in the case of strongly minimal theories, it follows[12] that there is a unique model of size κ for any uncountable κ, which is Morley's Theorem (Fact 3.23).

The situation with countable models is more complicated, as different values of κ can give isomorphic M_κ. But at the end of the day, one can prove that the number of countable models is either 1 or $\aleph_0$. This is part of the *Baldwin-Lachlan Theorem* [Mar02, Theorems 6.1.18, 6.1.22].

7 ω-stable groups and fields

There is a rich theory of stable groups and stable fields. This theory has been applied to DCF_0 to discover non-trivial facts about definable groups in differentially closed fields.[13]

7.1 Stable fields

The theory ACF is strongly minimal, hence ω-stable. In fact, ACF is the only such theory:

Theorem 7.1 (Macintyre, [Poi87, Theorem 3.1])**.** *If a field K is ω-stable, then K is finite or algebraically closed.*

[12] Recall we defined "uncountably categorical" to mean "λ-categorical for *at least one* uncountable λ." So the conclusion is non-trivial.

[13] Surprisingly, this has applications to number theory through work of Hrushovski [Bou98].

This was later generalized by Cherlin and Shelah to superstable fields:

Theorem 7.2 (Cherlin-Shelah, [Poi87, Theorem 6.11])**.** *If a field K is superstable, then K is finite or algebraically closed.*

For the general stable case, we expect something similar:

Conjecture 7.3 (Stable fields conjecture)**.** *If K is a stable field, then K is separably closed (Definition 3.5) or finite.*

There has been little process on Conjecture 7.3 in the past few decades beyond the work of Macintyre, Cherlin, and Shelah.[14]

7.2 ω-stable groups

Let G be a group. Recall that the *index* of a subgroup H is the number of distinct left cosets gH. The index is also the number of distinct right cosets Hg. The index is written $[G : H]$.

Fact 7.4 ([Poi87, p. 16])**.** *Suppose $(G, \cdot)$ is an ω-stable group. There is a minimal definable subgroup $G^0 \subseteq G$ such that G^0 has finite index. G^0 is a normal subgroup. $(G^0, \cdot)$ is itself ω-stable.*

Definition 7.5. Let G be an ω-stable group. The *connected component* of G is the subgroup G^0 of Fact 7.4. G is *connected* if $G = G^0$.

One can show that G^0 is itself connected.

Definition 7.6. Let G be an ω-stable group. A definable set $X \subseteq G$ is *generic* if $\mathrm{RM}(X) = \mathrm{RM}(G)$.

The idea is that generic sets are "big" and non-generic sets are "small".

Fact 7.7. *Let G be an ω-stable group.*

1. *If $X \subseteq G$ is generic, then any left translate gX or right translate Xg is generic.*
2. *$X \subseteq G$ is generic iff G can be written as the union of finitely many left translates $g_1X \cup g_2X \cup \cdots \cup g_nX$.*
3. *Non-generic sets form an ideal:*
    - *If $X, Y \subseteq G$ are non-generic, then $X \cup Y$ is non-generic.*
    - *If $X \subseteq Y \subseteq G$ and Y is non-generic, then X is non-generic.*
    - *$\varnothing$ is non-generic, and G is generic.*

[14] More embarrassingly, the field $\mathbb{C}(x)$ of rational functions $f(x) = \frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}$ is a potential counterexample. (Nobody has been able to prove that $\mathbb{C}(x)$ is unstable.)

4. *If G is connected, then for any definable set $X \subseteq G$, one of the two sets X and $G \setminus X$ is generic and the other is non-generic.*

Most of Fact 7.7 follows easily from properties of Morley rank. Part 2 is [Poi87, Lemma 2.5] and part 4 is [Poi87, Lemma 2.1].

If G has finite Morley rank, then the theory of G is very nice, with formal similarities to strongly minimal and uncountably categorical theories:

Fact 7.8 ([Poi87, pp. 31–32])**.** *Let G be a group of finite Morley rank. Then Morley rank satisfies the properties of Facts 4.6, 4.8, and 5.19.*

Groups of finite Morley rank have a nice structure theory. Here are some of the highlights.

Fact 7.9. *Let G be a connected group of finite Morley rank.*

1. *If G is infinite, then G has an abelian infinite connected definable subgroup [Poi87, Corollary 3.11].*

2. *If $\mathrm{RM}(G) = 1$, then G is abelian [Poi87, Corollary 3.11].*

3. *If $\mathrm{RM}(G) = 2$, then G is solvable [Poi87, Theorem 3.16].*

4. *If G is non-abelian, one of two things happens:*

   - *G is simple, and G is a model of an uncountably categorical theory [Poi87, Proposition 2.12].*

   - *G is definably non-simple: there is a definable normal subgroup $1 \subsetneq H \subsetneq G$ [Poi87, Lemma 2.11]. The normal subgroup H and the quotient group G/H are both groups of finite Morley rank.*

Groups of finite Morley rank decompose into abelian and simple groups of finite Morley rank:

Fact 7.10 ($\approx$ [Poi87, Lemma 2.11])**.** *Let G be a group of finite Morley rank. Then there is a chain of definable subgroups $1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq G$ such that H_i is a normal subgroup of H_{i+1}, and the quotient group H_{i+1}/H_i is simple or abelian. Also, each H_{i+1}/H_i is a group of finite Morley rank.*

Therefore, we would like to understand simple groups of finite Morley rank.

Conjecture 7.11 (Cherlin-Zilber)**.** *If G is a simple group of finite Morley rank, then G is a linear algebraic group over an algebraically closed field.*

There has been a lot of progress on the Cherlin-Zilber conjecture, enough to fill books. Here is one partial result:

Fact 7.12 ([Fré18])**.** *Let G be a simple connected group of finite Morley rank. If $\mathrm{RM}(G) \leq 3$, then G is the projective special linear group $PSL_2(K)$ over an algebraically closed field K.*

8 Further reading

For a more rigorous overview of ω-stable theories, including actual proofs, see Ziegler [Zie98]. For a thorough textbook treatment of ω-stability, see Marker [Mar02, Chapter 6].

For more information on differential fields and DCF, see the surveys by Marker [Mar00] or Wood [Woo98]. Some aspects of DCF are treated rigorously in Poizat's model theory textbook [Poi00, §6.2, 18.3].

A good source of information on stable groups is Poizat's book [Poi87]. There is also a chapter on ω-stable groups in Marker's textbook [Mar02, Chapter 7].

Morley's categoricity theorem (Fact 3.23) is proved in many introductory model theory books, including Poizat [Poi00, Corollary 18.25] and Marker [Mar02, Theorem 6.1.18].

This overview has focused primarily on ω-stable theories. However, many of the tools and theorems can be generalized from the ω-stable case to the general stable case. Morley rank no longer works, though in the superstable case one can instead use Lascar and Shelah ranks (see [Poi00, §17.1–17.2]). Forking and independence fully generalize to stable theories, and play an increasing important role as one moves away from theories of finite Morley rank. Chapters 11–20 of Poizat's textbook [Poi00] treat the general case of stability theory. Another good introduction to stability theory is [Bue17].

In the last couple decades, model theorists have begun to generalize the tools of stability theory to broader contexts beyond stability, such as simple theories and NIP theories. For more information on simple theories, see Grossberg et al's primer [GIL02] or Casanovas's book [Cas11]. For more information on NIP theories, see Simon's book [Sim15].

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