Week4

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Exercise 1. Find a set A and a relation $R \subseteq A \times A$ s.t.

$$\exists^\infty x \in A \exists^\infty y \in A : (x,y) \in R$$
$$\neg \exists^\infty y \in A \exists^\infty x \in A : (x,y) \in R$$

Proof. Let $A = \mathbb{N}$ and R be $<$ on $\mathbb{N}$. Then there is infinitely many $r \in \mathbb{N}$ with infinitely many elements of $\mathbb{N}$ greater than it. But there is only finitely many elements which is smaller than it. $\square$

Exercise 2. Consider the structure $(\mathbb{R}, +, -, \cdot, 0, 1, \leq)$. Let $\varphi(x,y)$ be the formula $y - 1 \leq x \wedge x \leq y + 1$. Show that $\varphi(x,y)$ has the order property (in a monster model $\mathbb{M} \succeq \mathbb{R}$)

Proof. Let $b_0 = 2$, $b_i = \frac{1}{i}$ for $i = 1, 2, \ldots.$ Let $a_0 = 0$, $a_i = \frac{1}{i+1} - 1$ for $i = 1, 2, \ldots.$ Then $a_i + 1 = b_{i+1}$ and hence $b_j \in [a_i - 1, a_i + 1]$ for any $i < j$. And for $i \geq j$, $b_j > a_i + 1$. $\square$

Exercise 3. Let $\mathbb{M}$ be a monster model of DLO. Let $\tau \in S_1(\mathbb{M})$ be the type at $+\infty$. Consider the Morley product $\tau \otimes \tau(x,y) \in S_2(\mathbb{M})$. Show that $(\tau \otimes \tau)(x,y)$ is the unique completion of $\tau(x) \cup \tau(y) \cup \{x < y\}$

Proof. Let $\Sigma(x,y) := \tau(x) \cup \tau(y) \cup \{x < y\}$. First we show that Σ has unique extension. Otherwise, take two $p_1, p_2 \in S_2(\mathbb{M})$ containing Σ and suppose p_1, p_2 is realized by x_1y_1, x_2y_2 in $\mathbb{N} \succeq \mathbb{M}$ respectively. Then we have $\mathbb{M} < x_1 < y_1$ and $\mathbb{M} < x_2 < y_2$. Let $f = \mathrm{id}_{\mathbb{M}} \cup \{(x_1, x_2), (y_1, y_2)\}$. Then f is an order-preserving bijection from $\mathbb{M} \cup \{x_1, y_1\}$ to $\mathbb{M} \cup \{x_2, y_2\}$. By quantifier elimination of DLO, f is a partial elementary map and hence $p_1 = \mathrm{tp}(x_1y_1/\mathbb{M}) = \mathrm{tp}(x_2y_2/\mathbb{M}) = p_2$, a contradiction.

Then we show that $\tau \otimes \tau(x,y) \supseteq \Sigma(x,y)$.

First we prove that $\tau(x)$ is $\emptyset$-invariant. Let $\Gamma(x) = \{x > a : a \in \mathbb{M}\}$, then for any $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(\Gamma(x)) = \Gamma(x)$. Then $\sigma(\tau) \supseteq \sigma(\Gamma) = \Gamma$ and by previous exercise, $\tau = \sigma(\tau)$. Then we show that $\tau \otimes \tau(x, y)$ is consistent with $\tau(x) \cup \tau(y) \cup \{x < y\}$. For any finite parameter set $A \subseteq \mathbb{M}$, $a, b \models \tau \otimes \tau(x, y) \restriction_A$ iff $a \models \tau(x) \restriction_A$ and $b \models \tau(y) \restriction_{Aa}$ iff $a, b \models (\tau(x) \cup \tau(y) \cup \{x < y\}) \restriction_A$. Thus $\tau \otimes \tau(x, y) \cup \tau(x) \cup \tau(y) \cup \{x < y\}$ is consistent and hence $\tau \otimes \tau(x, y) \vdash \tau(x) \cup \tau(y) \cup \{x < y\}$ since $\tau \otimes \tau(x, y)$ is complete □

Exercise 4. Let $\mathbb{M}$ be a monster model of a complete theory T. Suppose $\mathbb{M}$ is an expansion of a linear order. Let $p \in S_1(\mathbb{M})$ be a global A-invariant 1-type. Suppose that p commutes with itself. Show that $p = \mathrm{tp}(c/\mathbb{M})$ for some $c \in \mathbb{M}$

Proof. For any $a, b \models (p \otimes p(x, y)) \restriction_A$, $a \models p \restriction_A$ and $b \models p \restriction_{Aa}$. Since p commutes with itself, $a \models p \restriction Ab$. Since $(\mathbb{M}, \leq)$ is a linear order, $x < m \vee x = m \vee x > m \in p(x)$ for any $m \in \mathbb{M}$. If $a = b$, then $x = a \in p(x)$ and p is realized by a. Otherwise, if $a < b$, then $a < x \in p(x)$ as $b \models p \restriction_{Aa}$ and $x < b$ as $a \models p \restriction_{Ab}$. Hence either $a < x \wedge x < b \in p(x)$ or $b < x \wedge x < a \in p(x)$.

I don't know how to continue:(□