Heirs and definable types

Advanced Model Theory

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Reference in the book: Sections 11.1–2 (beware typos!)

Motivation. Recall that a theory T is *κ-categorical* if T has a unique model of size κ, up to isomorphism. Our goal for the course is to prove *Morley's theorem*:

> Let T be a theory in a countable language. If T is κ-categorical for one uncountable cardinal κ, then T is κ-categorical for all uncountable cardinals κ.

Theories that are κ-categorical for one (hence all) $\kappa > \aleph_0$ are called *uncountably categorical theories*.

The statement of Morley's theorem is not as interesting as the proof, which will be a tour of **stability theory**.

1 Definable types

Suppose $p(\bar{x}) \in S_n(M)$.

Definition 1. $p(\bar{x})$ is a *definable type* if for every formula $\varphi(\bar{x}; \bar{y})$, the set

$$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is definable, defined by some $L(M)$-formula $d\varphi(\bar{y})$.

Example (Constant types)**.** If $p = \mathrm{tp}(\bar{a}/M)$ for $\bar{a} \in M$, then

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff M \models \varphi(\bar{a}, \bar{b}),$$

so p is definable with $d\varphi(\bar{y}) = \varphi(\bar{a}, \bar{y})$.

Definition 2. T is *strongly minimal* if for any model $M \models T$, M is infinite and for any definable $X \subseteq M^1$, either X or $M \setminus X$ is finite.

ACF, the theory of *algebraically closed fields* like $(\mathbb{C}, +, \cdot)$, is strongly minimal.

Proposition 3. *If T is strongly minimal and $M \models T$, there is a 1-type $p(x) \in S_1(M)$ such that*

$$\varphi(x, \bar{b}) \in p(x) \iff \exists^\infty a \in M : M \models \varphi(a, \bar{b}).$$

Moreover, $p = \mathrm{tp}(c/M)$ for any $N \succeq M$ and $c \in N \setminus M$.

Proof. Take $N \succ M$ and $c \in N \setminus M$; let $p(x) = \mathrm{tp}(c/M)$. We must show

$$N \models \varphi(c, \bar{b}) \iff \exists^\infty a \in M : M \models \varphi(a, \bar{b}).$$

Suppose $(\Rightarrow)$ fails, so $N \models \varphi(c, \bar{b})$ but $\varphi(M, \bar{b})$ is a finite set $\{e_1, \ldots, e_n\} \subseteq M$. Then $\forall x\ (\varphi(x, \bar{b}) \to \bigvee_{i=1}^n (x = e_i))$ holds in M and therefore N, and so $c \in \{e_1, \ldots, e_n\} \subseteq M$, a contradiction. Thus $(\Rightarrow)$ holds.

Conversely, suppose $N \models \neg\varphi(c, \bar{b})$. By $(\Rightarrow)$, the set $\neg\varphi(M, \bar{b})$ is infinite. By strong minimality, the complement $\varphi(M, \bar{b})$ is finite. This proves $(\Leftarrow)$. □

$p(x)$ is called the *transcendental 1-type*

Proposition 4. 1. *T eliminates the $\exists^\infty$ quantifier: for any formula $\varphi(x, \bar{y})$ there is a formula $\psi(\bar{y})$ equivalent to $\exists^\infty x\ \varphi(x, \bar{y})$.*

2. *If $M \models T$, the transcendental 1-type $p \in S_1(M)$ is definable.*

Proof. 1. Fix $\varphi(x, \bar{y})$. We claim there is $n_\varphi < \omega$ such that for every $M \models T$ and $\bar{b} \in M$,

$$|\varphi(M, \bar{b})| < n_\varphi \text{ or } |\neg\varphi(M, \bar{b})| < n_\varphi.$$

Otherwise, by compactness we can find $M \models T$ and $\bar{b} \in M$ such that $\varphi(M, \bar{b})$ and its complement have size $> n$ for all finite n, contradicting strong minimality. Thus n_φ exists, and then

$$\underbrace{\exists^\infty x\ \varphi(x, \bar{b})}_{\text{not first-order}} \iff \underbrace{\exists^{\geq n_\varphi} x\ \varphi(x, \bar{b})}_{\text{first-order}}.$$

2. For each $\varphi(x, \bar{y})$, $d\varphi(\bar{y})$ is the formula (equivalent to) $\exists^\infty x\ \varphi(x, \bar{y})$. □

Corollary 5. *If $p \in S_1(M)$ and M is strongly minimal, then p is definable.*

Definition 6. A theory T is *stable* if all n-types over models are definable.

Motivation. Later we will see the following:

- Strongly minimal theories are stable.
- Uncountably categorical theories are stable.
- Strongly minimal complete theories are uncountably categorical, and uncountably categorical theories are very close to being strongly minimal.

2 Heirs and strong heirs

Suppose $M \preceq N$ and $p \in S_n(M)$. An *extension* or *son* of p is $q \in S_n(N)$ with $q \supseteq p$, i.e., $p = q \restriction M$.

Definition 7 (Heirs)**.** $q \in S_n(N)$ is an *heir* of p, written $p \sqsubseteq q$, if for any $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$ with $\bar{b} \in M$ and $\bar{c} \in N$, there is $\bar{c}' \in M$ with $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p(\bar{x})$.

We will see soon that (1) definable types have unique heirs, (2) this property characterizes definable types, and (3) the unique heir is a definable type with the same definition as the original type.

The notation $\sqsubseteq$ and $\sqsupseteq$ isn't very standard, except in the special case of stable theories.

Lemma 8. *Suppose $M_1 \preceq M_2 \preceq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$, with $p_1 \subseteq p_2 \subseteq p_3$.*

1. *If $p_1 \sqsubseteq p_2 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_3$.*

2. *If $p_1 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_2$.*

Proof. 1. Suppose $\varphi(\bar{x}, \bar{b}, \bar{c}) \in p_3(\bar{x})$ with $\bar{b} \in M_1$ and $\bar{c} \in M_3$. Then there is $\bar{c}' \in M_2$ with $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p_2(\bar{x})$. Then there is $\bar{c}'' \in M_1$ with $\varphi(\bar{x}, \bar{b}, \bar{c}'') \in p_1(\bar{x})$.

2. Suppose $\varphi(\bar{x}, \bar{b}, \bar{c}) \in p_2(\bar{x})$ with $\bar{b} \in M_1$ and $\bar{c} \in M_2$. Then $\varphi(\bar{x}, \bar{b}, \bar{c}) \in p_3(\bar{x})$, so there is $\bar{c}' \in M_1$ with $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p_1(\bar{x})$. $\square$

Definition 9. If $p \in S_n(M)$, then (M, dp) is the expansion of M by relation symbols $d\varphi(\bar{y})$ for each $\varphi(\bar{x}, \bar{y})$, interpreted as follows:

$$(M, dp) \models d\varphi(\bar{b}) \iff \varphi(\bar{x}, \bar{b}) \in p(\bar{x}).$$

Remark. p is definable if and only if the new relations in (M, dp) are definable in the old structure M.

Remark 10. The class of structures of the form (M, dp) with $M \models T$ and $p \in S_n(M)$ is an elementary class, axiomatized by T plus the following:

$$\forall \bar{y}_1, \ldots, \bar{y}_m \left(\bigwedge_{i=1}^{m} d\varphi_i(\bar{y}_i) \to \exists \bar{x} \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, \bar{y}_i) \right) \text{ for formulas } \varphi_1(\bar{x}, \bar{y}_1), \ldots, \varphi_n(\bar{x}, \bar{y}_n)$$

$$\forall \bar{y}\ (d\varphi(\bar{y}) \vee d\neg\varphi(\bar{y})) \qquad \text{for each formula } \varphi(\bar{x}, \bar{y})$$

The first axiom schema expresses finite satisfiability of p, and the second expresses completeness of p.

Lemma 11. *If $(M, dp) \preceq (N, dq)$, then $M \preceq N$ and q is an heir of p.*

Proof. The fact that $(N, dq) \succeq (M, dp)$ implies $N \succeq M$. Then:

- $q \supseteq p$: if $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ (with $\bar{b} \in M$), then $(M, dp) \models d\varphi(\bar{b})$, so $(N, dq) \models d\varphi(\bar{b})$, and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$.

- $q \sqsupseteq p$: suppose $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$, with $\bar{b} \in M$ and $\bar{c} \in N$. Then $(N, dq) \models d\varphi(\bar{b}, \bar{c})$, and $(N, dq) \models \exists \bar{z}\ d\varphi(\bar{b}, \bar{z})$. Then $(M, dp) \models \exists \bar{z}\ d\varphi(\bar{b}, \bar{z})$, so there is $\bar{c}' \in M$ with $M' \models d\varphi(\bar{b}, \bar{c}')$ and $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p(\bar{x})$. □

Corollary 12. *If $p \in S_n(M)$, then there is $M_0 \preceq M$ with $|M_0| \leq |T|$, such that $p \sqsupseteq (p \restriction M_0)$.*

Proof. Apply downward Löwenheim-Skolem to (M, dp) to find $(M_0, dq) \preceq (M, dp)$ with $|M_0| \leq |T|$. Then $q = p \restriction M_0$ and $p \sqsupseteq q$. □

Definition 13. If $M \preceq N$ and $p \in S_n(M)$ and $q \in S_n(N)$, then q is a *strong heir* of p if $(N, dq) \succeq (M, dp)$.

Proposition 14 (Types have heirs)**.** *Suppose $M \preceq N$ and $p \in S_n(M)$.*

1. *There is $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p.*

2. *There is $q \in S_n(N)$ an heir of p.*

Proof.
1. Let $\bar{c}$ be an infinite tuple enumerating N. Then $\mathrm{tp}^L(\bar{c}/M)$ is finitely satisfiable in M, hence finitely satisfiable in the expansion (M, dp). Therefore it is satisfied in some $(N', dq) \succeq (M, dp)$. So there is $\bar{e}$ in N' with $\mathrm{tp}^L(\bar{e}/M) = \mathrm{tp}^L(\bar{c}/M)$. Then the map $f(c_i) = e_i$ is an L-elementary embedding of N into N' extending $\mathrm{id}_M : M \to M$. Moving N' by an isomorphism, we may assume $N' \succeq N$.

2. Take $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p. Let $q = q' \restriction N$. Then $q' \supseteq q \supseteq p$ and $q' \sqsupseteq p$, so $q \sqsupseteq p$. □

Next week, we will see that every heir of p arises this way, as a restriction of a strong heir of p.

3 Heirs and definable types

Proposition 15. *Let $p \in S_n(M)$ be definable and $N \succeq M$.*

1. *p has a unique heir $q \in S_n(N)$.*

2. *For $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N$,*

$$\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \iff N \models d_p\varphi(\bar{b}) \tag{$*$}$$

3. *In particular, q is definable with $d_q\varphi = d_p\varphi$ for all φ.*

Proof.

Claim. If $q \in S_n(N)$ and $q \sqsupseteq p$, then q satisfies $(*)$.

Proof. Take $\bar{a} \in N' \succeq N$ realizing q (and p). If $(*)$ fails then

$$\begin{gathered}(\varphi(\bar{x},\bar{b}) \in q(\bar{x}) \iff\!\!\!\!\!\!\!/\;\; N \models d_p\varphi(\bar{b})) \\ N' \models \neg(\varphi(\bar{a},\bar{b}) \leftrightarrow d_p\varphi(\bar{b})) \\ \neg(\varphi(\bar{x},\bar{b}) \leftrightarrow d_p\varphi(\bar{b})) \in q(\bar{x}).\end{gathered}$$

As $q \sqsupseteq p$, there is $b' \in M$ such that

$$\begin{gathered}\neg(\varphi(\bar{x},\bar{b}') \leftrightarrow d_p\varphi(\bar{b}')) \in p(\bar{x}) \\ N' \models \neg(\varphi(\bar{a},\bar{b}') \leftrightarrow d_p\varphi(\bar{b}')) \\ (\varphi(\bar{x},\bar{b}') \in p(\bar{x}) \iff\!\!\!\!\!\!\!/\;\; M \models d_p\varphi(\bar{b}')),\end{gathered}$$

contradicting the definition of $d_p\varphi$. $\square_{\text{Claim}}$

There is at least one heir, and at most one $q \in S_n(N)$ satisfying $(*)$, so therefore there is a unique heir q and it satisfies $(*)$. Point (3) is clear. $\square$

Example. Suppose T is strongly minimal and $M \preceq N$ are models of T. Let p and q be the transcendental 1-types over M and N. For any $\varphi(x,\bar{y})$,

$$d_p\varphi(\bar{y}) \equiv (\exists^\infty x\ \varphi(x,\bar{y})) \equiv d_q\varphi(\bar{y}),$$

so q is the unique heir of p.

Proposition 16. *The following are equivalent for $p \in S_n(M)$.*

1. *p is definable.*
2. *For every $N \succeq M$, p has a unique heir over N.*

Proof. (1) $\implies$ (2) is above. Conversely, suppose p has unique heirs. Then for any $N \succeq M$, p has at most one strong heir over N. Therefore, there is at most one way to expand N to an elementary extension of (M, dp). Then the elementary diagram (M, dp) explicitly defines the relations $d\varphi$. By Beth's implicit definability theorem, (M, dp) is a expansion of M by definable relations, meaning p is definable. $\square$

Proposition 17. *Suppose $M_1 \preceq M_2 \preceq M_3$, and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$, with $p_1 \subseteq p_2 \subseteq p_3$. Suppose p_1 is definable. Then $p_1 \sqsubseteq p_2 \sqsubseteq p_3$ if and only if $p_1 \sqsubseteq p_3$.*

Proof. We only need to show the implication $p_1 \sqsubseteq p_3 \implies p_2 \sqsubseteq p_3$. Suppose $p_1 \sqsubseteq p_3$. Take $p_2' \sqsupseteq p_1$ and $p_3' \sqsupseteq p_2'$. By the uniqueness of heirs of definable types, $p_2' = p_2$, and p_2 is definable. Then $p_3' = p_3$, and $p_3 = p_3' \sqsupseteq p_2' = p_2$. $\square$

4 Types in ACF

Recall ACF is the theory of algebraically closed fields like $\mathbb{C}$ or $\mathbb{Q}^{alg}$. If $K \models \text{ACF}$, then $K[x_1, \ldots, x_n]$ denotes the set of polynomials $P(x_1, \ldots, x_n)$ in n variables.

A *positive quantifier free formula* is a quantifier-free formula that doesn't use $\neg$.

Fix a model $M \models \text{ACF}$.

Definition 18. A set $V \subseteq M^n$ is an *algebraic set* if

$$V = \varphi(M^n; \bar{b}) = \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}$$

where φ is positive quantifier free. In other words, an algebraic set is a set defined by a positive quantifier free $L(M)$-formula.

Remark 19. V is an algebraic set iff V is defined by finitely may polynomial equations

$$V = \{\bar{a} \in M^n : P_1(\bar{a}) = \cdots = P_m(\bar{a}) = 0\}.$$

Indeed, such a set is defined by the positive quantifier-free formula $\bigwedge_{i=1}^m (P_i(\bar{x}) = 0)$. Conversely, any positive quantifier-free formula can be put into conjunctive normal form

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{\ell_i} (P_j(\bar{x}) = R_j(x))$$

which is equivalent to

$$\bigwedge_{i=1}^{m} \left(\underbrace{\prod_{j=1}^{\ell_i} (P_j(\bar{x}) - R_j(\bar{x}))}_{\text{a polynomial}} = 0 \right).$$

Example (Twisted cubic)**.** $\{(t, t^2, t^3) : t \in M\}$ is an algebraic set in M^3. It is defined by the system of equations

$$y = x^2$$
$$z = x^3$$

i.e., by the two polynomials $x^2 - y$ and $x^3 - z$.

Lemma 20. *1. M^n and $\varnothing$ are algebraic sets.*

2. If $V, W \subseteq M^n$ are algebraic sets, then $V \cap W$ and $V \cup W$ are algebraic sets.

3. Any finite subset of M^n is an algebraic set.

Proof. (1) and (2) are easy, and then (3) reduces to showing that a point $\{\bar{a}\}$ is an algebraic set, which is easy, as it's defined by the system of polynomial equations $\{x_i = a_i : 1 \leq i \leq n\}$. $\square$

Fact (Quantifier elimination). *Every definable set $D \subseteq M^n$ is a finite boolean combination of algebraic sets.*

Fact (Consequence of Hilbert's basis theorem). *The class of algebraic sets has the descending chain condition (DCC): there is no infinite descending chain of algebraic sets $V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \cdots$.*

Proof sketch (if you know ring theory). For V an algebraic set, let $I(V)$ be the set of polynomials in $K[\bar{x}]$ which vanish on V. Note that $I(V)$ is an ideal in $K[\bar{x}]$.

Claim. If V, W are algebraic sets and $V \supsetneq W$, then $I(V) \subsetneq I(W)$.

Proof. $I(V) \subseteq I(W)$ is easy. Take $\bar{a} \in V \setminus W$. Then $\bar{a} \notin W$ implies that $P(\bar{a}) \neq 0$ for one of the polynomials P defining W. Then P vanishes on W but not V, so $P \in I(W) \setminus I(V)$, and $I(W) \supsetneq I(V)$. $\square_{\text{Claim}}$

Then a descending chain of algebraic sets would give an ascending chain of ideals in $K[\bar{x}]$, contradicting Hilbert's basis theorem, which says $K[\bar{x}]$ is Noetherian. $\square$

Remark. In what follows, ACF can be replaced by any theory T with quantifier elimination, such that the class of sets defined by positive quantifier-free formulas satisfies the descending chain condition. One other theory with this property is the theory DCF of *differentially closed fields*, discussed in Section 6.2 of the textbook.

Corollary 21. *If $\mathcal{S}$ is a non-empty collection of algebraic sets, then $\mathcal{S}$ contains at least one minimal element.*

Corollary 22. *An infinite intersection $\bigcap_{i \in I} V_i$ of algebraic sets is an algebraic set.*

Proof. Apply Corollary 21 to the set of finite intersections $\{\bigcap_{i \in I_0} V_i : I_0 \subseteq_f I\}$. (Here, $A \subseteq_f B$ means A is a finite subset of B.) $\square$

Corollary 23. *If $S \subseteq K[\bar{x}]$ is any set of polynomials, possibly infinite, then the subset of M^n defined by S is an algebraic set. All algebraic sets arise this way. (This is the usual definition of "algebraic set.")*

Corollary 24 (Noetherian induction). *Let $\mathcal{S}$ be a class of algebraic sets. Suppose the following holds:*

> *If X is an algebraic set, and every algebraic set $Y \subsetneq X$ is in $\mathcal{S}$, then $X \in \mathcal{S}$.*

Then every algebraic set is in $\mathcal{S}$.

Proof. Otherwise, take a minimal algebraic set not in $\mathcal{S}$ and get a contradiction. $\square$

Definition 25. An algebraic set V is *reducible* if $V = W_1 \cup W_2$ for algebraic sets $W_1, W_2 \subsetneq V$. A *variety* is a non-empty irreducible algebraic set.

Remark 26. If V is an algebraic variety, then the set of algebraic proper subsets of V is closed under finite unions.

Proposition 27. *If V is an algebraic set, then V is a finite union of varieties.*

Proof.

- $V = \varnothing$: V is a union of zero varieties.
- V is irreducible: V is a union of one variety (itself).
- V is reducible: $V = X \cup Y$ where $X, Y \subsetneq V$. By Noetherian induction, X and Y are finite unions of varieties, and so V is too. □

Example. In M^1, the algebraic sets are

- Finite sets
- M.

Thus, the varieties are

- Points
- M.

Fact. *In M^2, the varieties are exactly the following:*

- *Points*
- M^2.
- $\{(x, y) \in M^2 : P(x, y) = 0\}$ *where $P(x, y)$ is a prime polynomial in $M[x, y]$.*

In M^3, one gets similar things, plus an additional class of "1-dimensional varieties," one of which is the twisted cubic.

Let $V \subseteq M^n$ be a variety.

Definition 28. The *generic type* of V is the type generated by the following formulas:

- $x \in V$.
- $x \notin W$ for each algebraic proper subset $W \subsetneq V$.

We wil write this type as $p_V(\bar{x})$.

Proposition 29. *Let V be a variety.*

1. *$p_V(\bar{x})$ is a consistent complete type.*
2. *If W is an algebraic set, then $p_V(\bar{x}) \vdash \bar{x} \in W$ if and only if $W \supseteq V$.*

Proof. Finite satisfiability: given finitely many proper algebraic subsets $W_1, \ldots, W_m \subsetneq V$, we have $V \supsetneq \bigcup_{i=1}^m W_i$ by Remark 26, so there is $\bar{a} \in M^n$ with $\bar{a} \in V$ and $\bar{a} \notin W_i$ for $1 \le i \le m$.

(2): If $W \supseteq V$, then $p_V(\bar{x}) \vdash \bar{x} \in V \vdash \bar{x} \in W$. If $W \not\supseteq V$, then $(W \cap V) \subsetneq V$, so $p_V(\bar{x}) \vdash \bar{x} \notin W \cap V$. But $p_V(\bar{x}) \vdash \bar{x} \in V$, so $p_V(\bar{x}) \vdash \bar{x} \notin W$.

Completeness: by (2), for any positive quantifier-free formula $\varphi(\bar{x})$,

$$p_V(\bar{x}) \vdash \varphi(\bar{x}) \text{ or } p_V(\bar{x}) \vdash \neg\varphi(\bar{x}).$$

Then the same holds for quantifier-free formulas (which are boolean combinations of positive quantifier-free formulas), and then the same holds for all formulas (which are equivalent to quantifier-free formulas). □

Example. The generic type of the variety M^1 is the transcendental type of Proposition 3.

Theorem 30. *The map $V \mapsto p_V$ is a bijection from the set of varieties $V \subseteq M^n$ to $S_n(M)$.*

Proof. Injectivity: suppose V, W are varieties and $V \neq W$. Without loss of generality, $V \not\subseteq W$. Then $p_W(\bar{x}) \vdash \bar{x} \in W$, but $p_V(\bar{x}) \not\vdash \bar{x} \in W$, so $p_W \neq p_V$.

Surjectivity: fix $p \in S_n(M)$. Take V a minimal algebraic set such that $p(\bar{x}) \vdash \bar{x} \in V$. (There's at least one such V, namely $V = M^n$.) V is non-empty because p is consistent. If V is reducible as $V = X \cup Y$ for smaller algebraic sets X, Y, then $p(\bar{x}) \vdash \bar{x} \in X$ or $p(\bar{x}) \vdash \bar{x} \in Y$ by completeness, contradicting the choice of V. Thus V is a variety. By choice of V, $p(\bar{x}) \vdash \bar{x} \in V$. If $W \subsetneq V$, then $p(\bar{x}) \not\vdash \bar{x} \in W$ (by choice of V), so $p(\bar{x}) \vdash \bar{x} \notin W$ (by completeness of p). Then $p(\bar{x}) \vdash p_V(\bar{x})$, so $p = p_V$, as both are complete types. □

Proposition 31. *Let $V \subseteq M^n$ be a variety, defined by a formula φ.*

1. *φ defines a variety $V_N \subseteq N^n$.*
2. *V_N depends only on V, not on the choice of φ.*

Proof. Take ψ a positive quantifier-free formula defining V. Then $\forall \bar{x}\ (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is satisfied by M, and therefore by N. Let $V_N = \varphi(N^n) = \psi(N^n)$. This depends only on ψ, not on φ, so the choice of φ doesn't matter. As ψ is positive quantifier free, V_N is an algebraic set. As $M \models \exists \bar{x}\ \varphi(\bar{x})$, the same holds in N, so V_N is non-empty. Suppose for the sake of contradiction that V_N is reducible. Then $V_N = W_1 \cup W_2$, where for $i = 1, 2$, W_i is an algebraic proper subset of V_N defined by $\theta_i(\bar{x}, \bar{b}_i)$ for some positive quantifier-free L-formula θ_i and tuple of parameters $\bar{b}_i \in N$. Then

$$N \models \exists \bar{y}_1, \bar{y}_2 \left(\forall \bar{x} \ \left(\psi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{2} \theta_i(\bar{x}, \bar{y}_i) \right) \wedge \bigwedge_{i=1}^{2} \exists \bar{x}\ (\psi(\bar{x}) \wedge \neg\theta_i(\bar{x}, \bar{y}_i)) \right)$$

namely $(\bar{y}_1, \bar{y}_2) = (\bar{b}_1, \bar{b}_2)$. The same holds in $M \preceq N$, which means that $V = \bigcup_{i=1}^2 \theta_i(M^n, \bar{b}_i')$ for some $\bar{b}_i' \in N$, contradicting irreducibility of V. □

Theorem 32. *Let $M \preceq N$ be models of ACF. Let $V \subseteq M^n$ be a variety, and let $V_N \subseteq N^n$ be its extension. Then $p_{V_N} \in S_n(N)$ is the unique heir of $p_V \in S_n(M)$.*

Proof. Let $q \in S_n(N)$ be an heir of p_V. Let φ be an $L(M)$-formula defining V and V_N. Then $\varphi(\bar{x}) \in p_V(\bar{x}) \subseteq q(\bar{x})$, so $q(\bar{x}) \vdash \bar{x} \in V_N$. Suppose $q(\bar{x}) \not\vdash \bar{x} \notin W$ for some algebraic proper subset $W \subsetneq V_N$. By completeness of q, $q(\bar{x}) \vdash \bar{x} \in W$. Let $\psi(\bar{x}, \bar{b})$ be a positive quantifier-free formula defining W. Let $\theta(\bar{b})$ be the $L(M)$-formula

$$\forall \bar{x}\ (\psi(\bar{x}, \bar{b}) \to \varphi(\bar{x}))\ \wedge\ \exists \bar{x}\ (\varphi(\bar{x})\ \wedge\ \neg\psi(\bar{x}, \bar{b})).$$

(This says $\psi(M^n, \bar{b}) \subsetneq \varphi(M^n)$.) In particular, $N \models \theta(\bar{b})$, because $W \subsetneq V$. Then $q(\bar{x}) \vdash \psi(\bar{x}, \bar{b}) \wedge \theta(\bar{b})$, because $\theta(\bar{b})$ is true in N. Because $q \sqsupseteq p_V$, there is $\bar{b}' \in M$ such that

$$p_V(\bar{x}) \vdash \psi(\bar{x}, \bar{b}')\ \wedge\ \theta(\bar{b}').$$

In particular, $M \models \theta(\bar{b}')$ (or else $p_V(\bar{x})$ is inconsistent). If $W_0 = \psi(\bar{x}, \bar{b}')$, then W_0 is an algebraic proper subset of V, as $M \models \theta(\bar{b}')$. Then $p_V(\bar{x}) \vdash \psi(\bar{x}, \bar{b}')$ means $p_V(\bar{x}) \vdash \bar{x} \in W_0$, contradicting the definition of p_V.

So we see that $q(\bar{x}) \vdash \bar{x} \in V_N$, and $q(\bar{x}) \vdash \bar{x} \notin W$ for any proper algebraic subset $W \subseteq V_N$. Thus $q = p_{V_N}$, and p_{V_N} is the only heir of p. $\square$

Corollary 33. *In ACF, all types over models are definable. ACF is stable.*

Motivation. On some level, all stable theories look like ACF, especially the strongly minimal and uncountably categorical theories. We will see examples of this throughout the course.

5 1-types in DLO

DLO is the theory of *dense linear orders* without endpoints, things like $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$. It is complete and has quantifier elimination.

Suppose $M \models$ DLO and $p(x) \in S_1(M)$. By quantifier elimination, p is determined by the two sets:

$$X^- = \{b \in M : p(x) \vdash b \leq x\}$$
$$X^+ = \{b \in M : p(x) \vdash b \geq x\}.$$

There are six possibilites:

X^-	X^+	Explanation	Example
$(-\infty, b]$	$[b, +\infty)$	Constant type at b	$\mathrm{tp}(2/\mathbb{Q})$
$(-\infty, b]$	$(b, +\infty)$	Type at b^+	$\mathrm{tp}((2+\epsilon)/\mathbb{Q})$
$(-\infty, b)$	$[b, +\infty)$	Type at b^-	$\mathrm{tp}((2-\epsilon)/\mathbb{Q})$
M	$\varnothing$	Type at $+\infty$	$\mathrm{tp}(10/(-1,1))$
$\varnothing$	M	Type at $-\infty$	$\mathrm{tp}(-10/(-1,1))$
(has no maximum)	*(has no minimum)*	Irrational type	$\mathrm{tp}(\pi/\mathbb{Q})$

Lemma 34. *Suppose $M \preceq N \models \mathrm{DLO}$ and $p \in S_1(M)$ is the type at b^+. If $q \in S_1(N)$ is an heir, then q is the type at b^+.*

Proof. $p(x) \vdash x > b$, so $q(x) \vdash x > b$. If q isn't the type at b^+, then there is $c > b$ in N such that $q(x) \vdash x > c$. Then $q(x) \vdash x > c \ \wedge \ c > b$. Because $q \sqsupseteq p$, there is $c' \in M$ such that $p(x) \vdash x > c' \ \wedge c' > b$. Then $c' > b$ (as p is consistent), and then c' shows p is *not* the type at b^+, a contradiction. $\square$

So if $p \in S_1(M)$ is the type at b^+, then p has a unique heir over any $N \succeq M$. Therefore p is definable.

Similarly...

1. The unique heir of the type at b^- is the type at b^-.

2. The unique heir of the type at $\pm\infty$ is the type at $\pm\infty$.

3. The unique heir (*or extension*) of the constant type at b is the constant type at b.

Then all these types are definable. On the other hand...

Fact (see pages 228–229 the textbook)**.** *If $p \in S_1(M)$ is an irrational type and $N \succeq M$, then every non-constant extension of p to N is an heir of p. There is more than one such extension.*

Consequently, irrational types are not definable. Here is an example:

Example. $\mathrm{tp}(\pi/\mathbb{Q})$ is not definable because $\{b \in \mathbb{Q} : p(x) \vdash x < b\} = (\pi, +\infty) \cap \mathbb{Q}$, which is not a definable subset of $(\mathbb{Q}, \leq)$.