Introduction to Model Theory

PHIL130279/PHIL630078

Last updated: September 19, 2021

1 Basic information

Class: Introduction to Model Theory
Location: HGX401.
Weeks: 1–16.
Teacher: Will Johnson, HGW2503, willjohnson@fudan.edu.cn
TA: Sa WANG.
Language: The entire course is in English.
Midterm exam: November 4, 2021, in class, open book.
Final exam: December 30, 2021, in class, open book.
Textbook: *A Course in Model Theory*, by Bruno Poizat. You can access it electronically in the Fudan Library system. We will cover Chapters 1–6 and 9–10.
Office hours: TBA

Other information:

Undergraduate	Graduate
Course code: PHIL130279 **Credits:** 2 **Time:** Thursdays 3–4 (9:55–11:35)	**Course code:** PHIL630078 **Credits:** 3 **Time:** Thursdays 3–5 (9:55–12:30)

Note that the class is one hour longer for graduate students.

2 Prerequisites

This is a mathematics class. A class in mathematical logic is helpful but not strictly required as a prerequisite. You should be familiar with mathematical formalism and reasoning. For example, you should be able to understand something like this:

> Let $f : A \to B$ be a function. Then $f = \{(x, y) \in A \times B : f(x) = y\}$.

3 The third hour

In the third hour (11:45–12:30), I will present slideshows on topics in algebra and topology.

- These topics are important to know if you plan to do research in model theory.
- Graduate students should attend the third hour.
- Undergraduates are welcome to attend the third hour, but do not need to.
- *The material in the third hour will not appear in homework or exams.*

4 Schedule

Week	Date	Main topic	Reading	Third hour
1	9/16	Back-and-forth equivalence I	Chapter 1	Metric spaces
2	9/23	Back-and-forth equivalence II	Chapter 1	Topological spaces I
3	9/30	Formulas, theories, and models	Chapter 2	Topological spaces II
4	10/9	Structures and signatures	Chapter 3	Groups I
5	10/14	Ultraproducts and compactness	Sections 4.1–4.2	Groups II
6	10/21	Henkin's method and applications of compactness	Section 4.3	Commutative rings
7	10/28	Omega-saturated models and quantifier elimination	Chapter 5	Field theory
8	11/4	**MIDTERM EXAM**		
9	11/11	Algebraically closed fields	Section 6.1	Category theory I
10	11/18	Real-closed fields	Section 6.6	Category theory II
11	11/25	Saturated models	Chapter 9	Lattices I
12	12/2	Strongly homogeneous models	Chapter 9	Lattices II
13	12/9	Omitting types	Section 10.1	Modules
14	12/16	Prime models	Section 10.2	Pregeometries (matroids)
15	12/23	The number of countable models	Section 10.3	Universal algebra
16	12/30	**FINAL EXAM**		

The readings are from *A Course in Model Theory*, by Bruno Poizat.

Note: The date of the midterm and final exams are fixed. The rest of the schedule will be adjusted for holidays.

5 Homework and grading

Each week, I will assign homework problems, due 7 days later. Please turn in homework on the duedate, by the end of the day. If the homework is turned in n days late, your grade for the assignment will be multipled by $\left(1 - \frac{n}{5}\right)$. Homework must be in English, preferably in complete sentences.

Undergraduate students and graduate students will have the same homework assignments. Material from the third hour will not appear in homework.

There will be a midterm exam in class on Thursday November 4, and a final exam in class on Thursday December 30. Both exams will be open book.

The final grade will be calculated like so:

- 50% from weekly homework assignments.
- 25% from the midterm exam.
- 25% from the final exam.

After combining these values, the grade will be curved. Undergraduates and graduate students will be curved separately.

6 Attendance policy

Attendance will not affect your grade. Nevertheless, please attend the class when possible.

7 Notes and eLearning

I will not write up notes for the main lectures. The lectures will closely follow the textbook.

The following will be posted on eLearning (`elearning.fudan.edu.cn`):

- Homework assignments and solutions.
- The textbook and other reference material.
- Slides from the third hour.

8 Miscellaneous information

Auditors are welcome to attend the class.

I will be on paternity leave for a few weeks in October. Ningyuan Yao will take over the teaching during this time. During that time, office hours will be canceled and I won't be able to respond to email quickly.

9 About model theory

What is model theory? *Model theory* is a branch of mathematical logic. The main focus of model theory is applying tools from mathematical logic to structures from algebra.

Is model theory related to mathematical modeling? No.

What is mathematical logic? *Mathematical logic* is the mathematical study of logic and reasoning. Aside from model theory, the other major branches of mathematical logic are proof theory, set theory, and computability theory (also called recursion theory).

What is a structure? What is algebra? Roughly speaking, a mathematical *structure* is a set with some operations and relations. The most well-known structure is the set $\mathbb{R}$ of real numbers, which comes with operations of addition ($+$), multiplication ($\times$), subtraction ($-$), and division ($\div$), as well as the order relation ($<$). Some other well-known structures are

1. The set $\mathbb{C}$ of complex numbers, with $+, -, \times, \div$.
2. The set $\mathbb{Z}$ of integers, with $+, -, \times, <$.
3. The set of matrices, with matrix addition, matrix subtraction, and matrix multiplication.
4. The set of polynomials, with addition, subtraction, and multiplication.

These five examples all share certain features in common. For example, all of them satisfy the associative law $x \cdot (y \cdot z)$ and the distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$. A *ring* is a set with three operations called addition, multiplication, and subtraction, satisfying the associative law, distributive law, and several other axioms. The five examples above are all rings. *Ring theory* is the study of rings. It has important applications to number theory, cryptography, information theory, and combinatorics.

Aside from rings, there are many other important classes of mathematical structures, such as groups, fields, categories, and lattices. *Abstract algebra* is the overarching term for the study of these topics.

What are the basic concepts of model theory? A *theory* is a set of axioms. A *model* is a structure satisfying the axioms. Here are two examples:

Theory	Axioms	Models
The theory of rings	The ring axioms (associative law, ...)	Rings
ZFC	The Zermelo-Fraenkel axioms of set theory	Models of set theory

What is an example of an interesting theorem from model theory? Let M be an infinite structure. Then there is another infinite structure N, such that the size of N is bigger than the size of M, but M and N satisfy exactly the same axioms.