Many-sorted logic

Advanced model theory

Supplementary notes for April 7

Reference: page 28 of Marker's model theory textbook (posted on eLearning).

So far, we have considered *single-sorted* logic. *Many-sorted* logic is a slight generalization of single-sorted logic. Many-sorted logic allows us to consider structures with multiple "sorts" of elements.[1]

1 Motivating examples

Here are a couple examples of "many-sorted" structures.

Definition 1. A *projective plane* consists of a set P of *points*, a set L of *lines*, and a binary relation $R \subseteq P \times L$ called *incidence* satisfying the following axioms:

1. If x, y are distinct points, then there is a unique line ℓ incident to x and y.
2. If ℓ, ℓ' are distinct lines, then there is a unique point x incident to ℓ and ℓ'.
3. There is at least one point and at least one line.
4. Every line is incident to at least three points.
5. Every point is incident to at least three lines.

In this case, the two "sorts" of elements are P (points) and L (lines). See Wikipedia for more about projective planes, including examples.

Definition 2. An *ultrametric space*[2] consists of a set X and a set I, a function $d : X \times X \to I$, a relation $\leq$ on I, and an element $0 \in I$ such that the following axioms hold:

1. $\leq$ is a linear order on I, and 0 is the least element.
2. $d(x, y) = d(y, x)$ for $x, y \in X$.

[1] Sorts are like "types" in computer science and category theory. In model theory we have to use a different name because "type" already means something else.

[2] More properly, this should be called an "I-valued ultrametric space." Normally, an ultrametric space means an $\mathbb{R}_{\geq 0}$-valued ultrametric space.

3. $d(x, y) = 0 \iff x = y$, for $x, y \in X$.

4. $d(x, z) \leq \max(d(x, y), d(y, z))$ for $x, y, z \in X$.

The function $d(x, y)$ is called the *distance* from x to y.

In this case, the two "sorts" are X and I. See Section 6.4 of Poizat's textbook for more about ultrametric spaces. Here is an example of an ultrametric space:

Example 3. Let X be the set of functions $f : \mathbb{N} \to \mathbb{N}$. Let I be $[0, +\infty) \subseteq \mathbb{R}$ with its usual order. If $f, g \in X$, let

$$d(f, g) = \begin{cases} 2^{-\min\{i : f(i) \neq g(i)\}} & \text{if } f \neq g \\ 0 & \text{if } f = g. \end{cases}$$

Then $(X, I, d, \leq, 0)$ is an ultrametric space.

Here are some other natural examples of many-sorted structures arising in mathematics:

1. Categories. The two sorts are the objects and the morphisms.

2. Pairs (K, V) where K is a field and V is a K-vector space. The two sorts are the scalars K and the vectors V.

3. Group actions. The two sorts are the group G and the set X that G acts on.

2 First approximation: many-sorted structures

This section is a first attempt at formalizing many-sorted structures. For comparison, recall the "definition" of single-sorted structures:

Definition 4. A *(single-sorted) structure* consists of a set M and a collection of *functions*, *relations*, and *constants*. Each function is a function $f : M^{n_f} \to M$ for some number n_f called the *arity* of f. Each relation is a relation $R \subseteq M^{n_R}$ for some number n_R called the *arity* of R. Each constant is an element of M.

Many-sorted structures are then "defined" as follows:

Definition 5. A *many-sorted structure* consists of a collection of *sorts*, *functions*, *relations*, and *constants*. Each sort is a set. Each function is a function $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ where $X_1, X_2, \ldots, X_n, Y$ are sorts. Each relation is a relation $R \subseteq X_1 \times X_2 \times \cdots \times X_n$ where $X_1, \ldots, X_n$ are sorts. Each constant is an element of a sort.

For example:

1. The single-sorted structure $(\mathbb{R}, +, \times, 0, 1, -, \leq)$ consists of the set $\mathbb{R}$, the binary function $+ : \mathbb{R}^2 \to \mathbb{R}$, the binary function $\times : \mathbb{R}^2 \to \mathbb{R}$, the constant $0 \in \mathbb{R}$, the constant $1 \in \mathbb{R}$, the unary function $- : \mathbb{R} \to \mathbb{R}$, and the binary relation $(\leq) \subseteq \mathbb{R}^2$.

2. Let (X, I) be an ultrametric space. The many-sorted structure has two sorts, X, and I. There is one binary function $d : X \times X \to I$. There is one constant $0 \in I$. There is one binary relation $(\leq) \subseteq I \times I$.

This approach works if we only need to consider definable sets and formulas within a fixed structure. If we want to talk about theories or elementary equivalence, we need to define *many-sorted languages* before we can properly define many-sorted structures.

3 Many-sorted languages

Definition 6. A *many-sorted language* consists of the following data:

1. A set $\mathcal{S}$ of *sorts*
2. A set $\mathcal{F}$ of *function symbols.*
   - For each $f \in \mathcal{F}$, a finite non-empty list of sorts $(X_1, X_2, \ldots, X_n, Y)$, called the *signature* of f.
3. A set $\mathcal{R}$ of *relation symbols.*
   - For each $R \in \mathcal{R}$, a finite list of sorts $(X_1, \ldots, X_n)$, called the *signature* of R.
4. A set $\mathcal{C}$ of *constant symbols.*
   - For each $c \in \mathcal{C}$, a sort X, called the *signature* of c.

Contrast this with the definition of single-sorted languages, such as Definition 1.1.1 in Marker's model theory textbook (posted on eLearning).

Example 7. The *language of ultrametric spaces* consists of

1. Two sorts, X and I.
2. One function symbol d, with signature (X, X, I).
3. One relation symbol $\leq$, with signature (I, I).
4. One constant symbol 0, with signature I.

The *language of projective planes* consists of

1. Two sorts, P and L.
2. One relation symbol “incidence”, with signature (P, L).

Now we can properly define many-sorted structures:

Definition 8. Let $L = (\mathcal{S}, \mathcal{F}, \mathcal{R}, \mathcal{C})$ be a many-sorted language. An *L-structure* M consists of the following data:

1. For each sort $X \in \mathcal{S}$, a set X^M.

2. For each function symbol $f \in \mathcal{F}$ with signature $(X_1, \ldots, X_n, Y)$, a function $f^M : X_1^M \times X_2^M \times \cdots \times X_n^M \to Y^M$.

3. For each relation symbol $R \in \mathcal{R}$ with signature $(X_1, \ldots, X_n)$, a relation $R^M \subseteq X_1^M \times \cdots \times X_n^M$.

4. For each constant symbol $c \in \mathcal{C}$ with signature X, an element $c^M \in X^M$.

For example, if L_{ultra} is the language of ultrametric spaces, then an L_{ultra}-structure M consists of the following: a set X^M, a set I^M, a function $d^M : X^M \times X^M \to I^M$, a relation $(\leq^M) \subseteq I^M \times I^M$, and an element $0^M \in I^M$. (But the axioms of ultrametric spaces needn't hold.)

4 Many-sorted model theory

All of model theory generalizes easily to many-sorted languages and structures. Here are some notes on what changes:

1. In formulas, the variables need to be assigned to sorts. For example, one of the axioms of ultrametric spaces says

$$\forall x, y, z\ (d(x, z) \leq d(x, y)\ \vee\ d(x, z) \leq d(y, z))$$

where x, y, z are variables in the X sort. Another axiom says

$$\forall s, t\ ((s \leq t\ \wedge\ t \leq s)\ \rightarrow\ s = t)$$

where s, t are variables in the I sort.

- The right way to say this is that *each sort comes with its own bank of variable symbols*. Above, the variables for the sort X are $x, y, z, \ldots$, and the variables for the sort I are $s, t, \ldots$.

2. In formulas, instances of the function/relation/constant symbols (as well as $=$) should respect the sorts. For example, "$d(0, x) = s$" isn't an L_{ultra}-formula, because the symbol 0 is in the sort I, and both arguments of d should come from the sort X.

3. A definable set D lives in a product of sorts $X_1 \times \cdots \times X_n$. A union of two sorts $X_1 \cup X_2$ isn't really a definable set, since no formula defines it.[3]

[3] In practice, we can identify $X_1 \cup X_2$ with a definable set, such as the definable set $(X_1 \times \{d\} \times \{d\}) \cup (\{c\} \times X_2 \times \{e\}) \subseteq X_1 \times X_2 \times X_2$ for some $c \in X_1$ and distinct $d, e \in X_2$.

4. An embedding from an L-structure M to an L-structure N consists of an injection X^M to X^N for each sort $X \in L$, respecting the function/constant/relation symbols. An automorphism of an L-structure M consists of a bijection $X^M \to X^M$ for each sort $X \in L$, preserving the symbols. An automorphism of M isn't allowed to permute the sorts.

5. Rather than talking about spaces of n-types, we talk about $\bar{x}$-types, where $\bar{x}$ is a tuple of variables. (Each variable is assigned a sort.) We write $S_{\bar{x}}(A)$ for the space of $\bar{x}$-types over A. For example, in L_{ultra}, if x is a variable in the sort X, and s is a variable in the sort I, then $S_{(x,s)}(A)$ is the set of types $p(x, s)$ over A where x is in the sort X and s is in the sort I. Similarly, $S_x(A)$ is the set of 1-types over A in the X sort, and $S_s(A)$ is the set of 1-types over A in the I sort.

6. When taking ultraproducts, construct each sort separately. If N is an ultraproduct $\prod_{i\in I}^{/\mathcal{U}} M_i$ of some L-structures M_i, and X is a sort in L, then N^X is the ultraproduct of sets $\prod_{i\in I}^{/\mathcal{U}}(M_i^X)$. In other words, when forming the ultraproduct $\prod_{i\in I}^{/\mathcal{U}} M_i$, you only consider tuples $(x_i : i \in I)$ where *all* the x_i come from one sort.

5 Coding many-sorted logic in single-sorted logic

Let M be a many-sorted structure with finitely many sorts $X_1, \ldots, X_n$. Then we can convert M into a single-sorted structure whose underlying set (alias "universe", "domain") is the disjoint union $X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$. A function symbol $f : X_{i_1} \times \cdots \times X_{i_m} \to X_j$ in the old structure is converted to an $(m + 1)$-ary relation symbol in the new structure. Relation symbols and constant symbols carry over without modification. We add a unary relation symbol for each X_i. The resulting single-sorted structure carries the same information as the original many-sorted structure. This provides a way to convert a many-sorted theory into a single-sorted theory.

Example 9. Compare our many-sorted definition of ultrametric spaces (Definition 2) with the single-sorted definition in Section 6.4 of Poizat's textbook.

Remark 10. This approach only works in contexts with finitely many sorts. In class this week, we will need a many-sorted theory with infinitely many sorts.