1. Back-and-forth

Introductory Model Theory

September 16, 2021

Important things to know about the class:

- The undergrad class is two hours; the grad class is three hours; the third hour is a high-level overview of algebra and topology for the graduate students.
- I will be gone for part of October.
- There is a WeChat group.
- We're following Poizat's textbook *A Course in Model Theory*; you can find an electronic copy on the Fudan Library website, or on eLearning for this class.
- Poizat has an unusual approach in the first two chapters, which we will follow. If you have seen model theory or mathematical logic before, things will make sense in a couple weeks.

The recommended reading for Today is Chapter 1 of Poizat. You're encouraged to read the introduction and preface as well.

1 Binary relations

Convention. We use the standard conventions that relations and functions are sets of tuples:

- If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then $f = \{(x, x^2) : x \in \mathbb{R}\}$.
- $\geq$ is the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq y\}$.

Definition 1. A *binary relation* is a pair (E, R) where E is a set and $R \subseteq E^2$. We call E the *universe* of the relation. For $a, b \in E$, we write aRb if $(a, b) \in R$.

We abbreviate (E, R) as R or E, if E or R is clear from context.

Example. These are binary relations: $(\mathbb{R}, <)$, $(\mathbb{R}, =)$, $(\mathbb{R}, \geq)$, $(\mathbb{Z}, <)$.

Definition. A binary relation (E, R) is

- *Reflexive* if xRx for $x \in E$

- *Symmetric* if $xRy \implies yRx$ for $x, y \in E$
- *Transitive* if xRy and $yRz \implies xRz$ for $x, y, z \in E$
- *Anti-symmetric* if xRy and $yRx \implies x = y$ for $x, y \in E$
- *Total* if xRy or yRx for $x, y \in E$.
- An *equivalence relation* if it is reflexive, symmetric, and transitive.
- A *partial order* if it is reflexive, anti-symmetric, and transitive.
- A *linear order* if it is reflexive, anti-symmetric, transitive, and total.

Example. $\leq$ is a linear order. $=$ is an equivalence relation. $\subseteq$ is a partial order.

2 Isomorphisms

Definition 2. An *isomorphism* from (E, R) to (E', R') is a bijection $f : E \to E'$ such that for any $a, b \in E$, $aRb \iff f(a)R'f(b)$. Two binary relations (E, R) and (E', R') are *isomorphic* ($\cong$) if there is an isomorphism between them.

Example. Let $2\mathbb{Z} = \{2x : x \in \mathbb{Z}\}$. There is an isomorphism

$$\begin{gathered}(\mathbb{Z}, <) \to (2\mathbb{Z}, >) \\ x \mapsto -2x\end{gathered}$$

because $x < y \iff (-2x) > (-2y)$.

Isomorphism is an equivalence relation:

$$\begin{gathered}R \cong R \\ R \cong S \implies S \cong R \\ R \cong S \text{ and } S \cong T \implies R \cong T\end{gathered}$$

3 Restrictions and local isomorphisms

Definition 3. (E, R) is a *restriction* of (E', R') if $E \subseteq E'$ and $R = R' \cap E^2$. (E, R) is an *extension* of (E', R') if (E', R') is a restriction of (E, R).

Example. $(\mathbb{Z}, <)$ is a restriction of $(\mathbb{R}, <)$. $(\mathbb{R}, <)$ is an extension of $(\mathbb{Z}, <)$.

Definition 4. A relation R with universe E is *finite* if E is finite, and *countable* if E is countable.

Definition 5. A *local isomorphism* from R to R' is an isomorphism from a finite restriction of R to a finite restriction of R'. The set of local isomorphisms from R to R' is denoted $S_0(R, R')$. For $f \in S_0(R, R')$, $\mathrm{dom}(f)$ and $\mathrm{im}(f)$ denote the domain and range of f.

Example. There is a local isomorphism s from $(\mathbb{Z}, <)$ to $(\mathbb{Z}, <)$ given by

$$\mathrm{dom}(s) = \{1, 2, 3\}$$
$$\mathrm{im}(s) = \{10, 20, 30\}$$
$$s(1) = 10 \qquad s(2) = 20 \qquad s(3) = 30.$$

Definition 6. Let f, g be local isomorphisms from R to R'. Then f is a *restriction* of g if $f \subseteq g$, and f is an *extension* of g if $f \supseteq g$.

Example. Let $t \in S_0(\mathbb{Z}, \mathbb{Z})$ be $\{0, 1, 2, 3\} \to \{0, 10, 20, 30\}$. Then t extends s from before.

4 Karpian families

Definition 7. Let R, R' be binary relations with universes E, E'. A *Karpian family* for (R, R') is a set $K \subseteq S_0(R, R')$ satisfying the following two conditions for any $f \in K$:

1. **(forth)** If $a \in E$ then there is $g \in K$ with $g \supseteq f$ and $a \in \mathrm{dom}(g)$.
2. **(back)** If $b \in E'$ then there is $g \in K$ with $g \supseteq f$ and $b \in \mathrm{im}(g)$.

R and R' are ∞-*equivalent*, written $R \sim_\infty R'$, if there is a non-empty Karpian family.

Proposition. *Let f be an isomorphism from (E, R) to (E', R'). Let K be the family of finite restrictions of f. Then K is a Karpian family.*

Proof. Suppose s is a finite restriction of f.

(forth) Given $a \in E$, let $b = f(a)$ and $t = s \cup \{(a, b)\}$. Then $t \supseteq s$, $a \in \mathrm{dom}(t)$, and $t \in K$.

(back) Similar. $\square$

Proposition. *Let $(E, R), (E', R')$ be countable. Suppose $R \sim_\infty R'$. Then $R \cong R'$.*

Proof. Let

$$E = \{e_1, e_2, \ldots\}$$
$$E' = \{e'_1, e'_2, \ldots\}.$$

Recursively build an ascending chain

$$f_1 \subseteq f_2 \subseteq \cdots$$

in K, by choosing

- $f_1 \in K$ arbitrary (K is non-empty)
- $f_{2i} \in K$ is an extension of f_{2i-1} such that $e_i \in \mathrm{dom}(f_{2i})$ (forth condition).
- $f_{2i+1} \in K$ is an extension of f_{2i} such that $e'_i \in \mathrm{im}(f_{2i+1})$ (back condition).

Let $f = \bigcup_{n=1}^{\infty} f_n$. Then $e_i \in \mathrm{dom}(f_{2i}) \subseteq \mathrm{dom}(f)$ and $e'_i \in \mathrm{im}(f_{2i+1}) \subseteq \mathrm{im}(f)$ so $\mathrm{dom}(f) = E$ and $\mathrm{im}(f) = E'$, and f is an isomorphism from R to R'. □

Definition. A *dense linear order without endpoints* (DLO) is a linear order $(C, \leq)$ satisfying the following conditions:

- $C \neq \varnothing$
- $\forall x \in C\ \exists y, z \in C : y < x < z$. (There are no greatest or least elements.)
- $\forall x, y \in C : x < y \to \exists z \in C\ (x < z < y)$. (The order is "dense".)

Example. $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$ are DLOs. $(\mathbb{Z}, \leq)$ and $([0,1], \leq)$ are not.

Proposition. *Let $(C, <)$ and $(C', <)$ be DLOs. Then $S_0(C, C')$ is a Karpian family.*

Proof. Let s be a local isomorphism from C to C'. Enumerate

$$\begin{gathered} \mathrm{dom}(s) = \{a_1, \ldots, a_n\} \qquad a_1 < \cdots < a_n \\ \mathrm{im}(s) = \{b_1, \ldots, b_n\} \qquad b_1 < \cdots < b_n \\ s(a_i) = b_i. \end{gathered}$$

(forth) Given $a \in C$, we want $t \in S_0(R, R')$ with $t \supseteq s$ and $a \in \mathrm{dom}(t)$. Break into cases:

- $a = a_i$ for some i. Take $t = s$.
- $a < a_1$. Take some $b < b_1$ (no endpoints) and set $t = s \cup \{(a, b)\}$.
- $a_i < a < a_{i+1}$ for some i. Take some $b \in C'$ with $b_i < b < b_{i+1}$ (density) and set $t = s \cup \{(a, b)\}$.
- $a > a_n$. Take some $b > b_n$ (no endpoints) and set $t = s \cup \{(a, b)\}$.

(back) Similar. □

Corollary. *Any two countable DLOs are isomorphic.*

5 p-isomorphisms

Definition 8. Let R, R' be binary relations with universes E, E'.

- A *0-isomorphism* from R to R' is a local isomorphism from R to R'.
- For $p > 0$, a *p-isomorphism* from R to R' is a local isomorphism f from R to R' satisfying the following two conditions:
  1. **(forth)** For any $a \in E$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $a \in \mathrm{dom}(g)$.
  2. **(back)** For any $b \in E'$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $b \in \mathrm{im}(g)$.
- An *ω-isomorphism* from R to R' is a local isomorphism f from R to R' such that f is a p-isomorphism for all $p < \omega$.

The set of p-isomorphisms from R to R' is denoted $S_p(R, R')$.

Example. If $R = R' = (\mathbb{Z}, <)$, then $f : \{2, 4\} \to \{1, 2\}$ isn't in $S_1(\mathbb{Z}, \mathbb{Z})$ because forth fails for $a = 3$. But $g : \{2, 4\} \to \{1, 5\}$ is a 1-isomorphism.

Proposition. *If $g \in S_p(R, R')$ and $f \subseteq g$, then $f \in S_p(R, R')$. (Restrictions of p-isomorphisms are p-isomorphisms.)*

Proof. If $p = 0$, then $f \in S_0(R, R')$ easily. Suppose $p > 0$.

(forth) Given $a \in E$, there is $h \in S_{p-1}(R, R')$ with $a \in \mathrm{dom}(h)$ and $h \supseteq g \supseteq f$.

(back) Given $b \in E'$, there is $h \in S_{p-1}(R, R')$ with $b \in \mathrm{im}(t)$ and $h \supseteq g \supseteq f$. □

Proposition. *The following are equivalent:*

1. $S_p(R, R') \neq \varnothing$
2. $\varnothing \in S_p(R, R')$.

Proof. (2) $\Longrightarrow$ (1) trivial.
(1) $\Longrightarrow$ (2): take $f \in S_p(R, R')$. Then $\varnothing \subseteq f$. □

Definition 9. R and R' are *p-equivalent*, written $R \sim_p R'$, if there is a p-isomorphism from R to R'.

R and R' are *ω-equivalent* or ***elementarily equivalent***, written $R \sim_\omega R'$ or $R \equiv R'$, if there is an ω-isomorphism from R to R'.

Note $R \sim_\omega R' \iff S_\omega(R, R') \neq \varnothing \iff \varnothing \in S_\omega(R, R') \iff \forall p\ (\varnothing \in S_p(R, R')) \iff \forall p\ (R \sim_p R')$.

6 Ehrenfeucht-Fraïssé games

Definition 10. Let R, R' be binary relations with universes E, E'. The Ehfrenfeucht-Fraïssé game of length n, denoted $\mathrm{EF}_n(R, R')$, is played as follows.

- There are two players, the Duplicator and Spoiler.
- There are n rounds.
- In the ith round, the Spoiler chooses either an $a_i \in E$ or a $b_i \in E'$.
- The Duplicator responds with a $b_i \in E'$ or an $a_i \in E$, respectively.
- At the end of the game, the Duplicator wins if

  $$\{(a_1, b_1), \ldots, (a_n, b_n)\}$$

  is a local isomorphism from R to R'.
- Otherwise, the Spoiler wins.

Example. Here is a game of $\mathrm{EF}_3(\mathbb{Q}, \mathbb{R})$:

- Spoiler chooses $a_1 = 7$. Duplicator chooses $b_1 = 7$.
- Spoiler chooses $b_2 = \sqrt{2}$. Duplicator chooses $a_2 = 1.4$.
- Spoiler chooses $b_3 = 1.41$. Duplicator chooses $a_3 = -10$.
- Duplicator wins.

Example. Here is a game of $\mathrm{EF}_3(\mathbb{R}, \mathbb{Z})$:

- Spoiler chooses $b_1 = 1$. Duplicator chooses $a_1 = 1$.
- Spoiler chooses $b_2 = 2$. Duplicator chooses $a_2 = 1.1$.
- Spoiler chooses $a_3 = 1.01$. Duplicator chooses $b_3 = 0$ and loses.

Example. Try playing $\mathrm{EF}_4(\mathbb{Z}, \mathbb{N})$. Who wins?
Let $C = \{1, -1, 1/2, -1/2, 1/3, -1/3, \ldots\}$. Try playing $\mathrm{EF}_4(C, \mathbb{Z})$. Who wins?

Lemma. *In $\mathrm{EF}_n(R, R')$, suppose there have been q rounds, and $a_1, b_1, \ldots, a_q, b_q$ have been chosen. Let $p = n - q$, the number of remaining rounds. Then Duplicator has a winning strategy if and only if $s = \{(a_1, b_1), \ldots, (a_q, b_q)\}$ is an p-isomorphism.*

Corollary. *If R is p-equivalent to R', then $\mathrm{EF}_p(R, R')$ is a win for the Duplicator. Otherwise it is a win for the Spoiler.*

7 More about p-isomorphisms

Proposition. *Every $(p+1)$-isomorphism is a p-isomorphism.*

Proof. By induction on p.

$p = 0$: every 1-isomorphism is a 0-isomorphism. True by definition.

$p > 0$. Suppose s is a $p+1$-isomorphism. We claim s is a p-isomorphism.

(forth) Given $a \in E$, there is $t \in S_p(R, R')$ such that $t \supseteq s$ and $a \in \mathrm{dom}(t)$. By induction, $t \in S_{p-1}(R, R')$.

(back) Given $b \in E'$, there is $t \in S_p(R, R')$ such that $t \supseteq s$ and $b \in \mathrm{im}(t)$. By induction, $t \in S_{p-1}(R, R')$. □

So $S_0(R, R') \supseteq S_1(R, R') \supseteq S_2(R, R') \supseteq \cdots$. In terms of the Ehfrenfeuch-Fraïssé game, if we reduce the number of remaining rounds, it can only help the Duplicator.

Proposition. *Suppose $s \in S_p(R, R')$ and $t \in S_p(R', R'')$ and $\mathrm{dom}(t) = \mathrm{im}(s)$. Then $u := t \circ s \in S_p(R, R'')$.*

Proof. By induction on p. For $p = 0$, this says we can compose (local) isomorphisms; this is easy.

Suppose $p > 0$. Let E, E', E'' be the universes of R, R', R''.

(forth) Given $a \in E$, there is $b \in E'$ such that $s' := s \cup \{(a, b)\} \in S_{p-1}(R, R')$. There is $c \in E''$ such that $t' := t \cup \{(b, c)\} \in S_{p-1}(R', R'')$. By induction, $u' := t' \circ s' = u \cup \{(a, c)\} \in S_{p-1}(R, R'')$.

(back) Similar. □

Corollary. *If $R \sim_p R'$ and $R' \sim_p R''$, then $R \sim_p R''$.*

Proposition. *Suppose $s \in S_p(R, R')$. Then $s^{-1} \in S_p(R', R)$.*

Proof. Exercise. □

Corollary. *If $R \sim_p R'$, then $R' \sim_p R$.*

Proposition. *Let K be a Karpian family for (E, R) and (E', R'). If $s \in K$, then s is a p-isomorphism for all p.*

Proof. By induction on p. For $p = 0$, we have $s \in K \subseteq S_0(R, R')$ by definition.

Suppose $p > 0$:

(forth) For any $a \in E$ there is $t \in K$ with $t \supseteq s$ and $a \in \mathrm{dom}(t)$. By induction $t \in S_{p-1}(R, R')$.

(back) Similar. □

Corollary. *If E, E' are DLOs, then $S_0(E, E') = S_p(E, E')$ for all p. $E \sim_\omega E'$.*

Corollary. *Suppose $f : (E, R) \to (E', R')$ is an isomorphism. If s is a finite restriction of f, then s is a p-isomorphism (for any p).*

Corollary. $A \cong B \implies A \sim_\infty B \implies A \sim_\omega B \implies A \sim_p B$.

Corollary. *$\sim_p$ and $\sim_\omega$ are equivalence relations.*

8 More dense linear orders

Proposition. *Suppose $(\mathbb{Q}, \leq) \sim_\omega (C, R)$. Then (C, R) is a DLO.*

Proof. Suppose (C, R) is not a DLO and break into cases:

- R not reflexive. Spoiler chooses $b_1 \in C$ such that $(b_1, b_1) \notin R$. Then Duplicator must choose $a_1 \in \mathbb{Q}$ such that $a_1 \not\leq a_1$, impossible.

- R not antisymmetric. Spoiler chooses $b_1, b_2 \in C$ such that $b_1 \neq b_2$ but $b_1 R b_2$ and $b_2 R b_1$. Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ such that $a_1 \neq a_2$ and $a_1 \leq a_2$ and $a_2 \leq a_1$, impossible.

- R not transitive. Spoiler chooses $b_1, b_2, b_3 \in C$ such that $b_1 R b_2$ and $b_2 R b_3$ but $b_1 \not R b_3$. Duplicator must choose a_1, a_2, a_3 with $a_1 \leq a_2 \leq a_3$ and $a_1 \not\leq_3$, impossible.

- R not total. Spoiler chooses $b_1, b_2 \in C$ with $b_1 \not R b_2$ and $b_2 \not R b_1$. Again, Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ with $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, impossible.

- (C, R) has a maximum. Spoiler chooses $b_1 = \max(C)$. Duplicator chooses some $a_1 \in \mathbb{Q}$. Spoiler chooses $a_2 \in \mathbb{Q}$ greater than a_1. Spoiler must choose $b_2 \in C$ with $b_2 > b_1$, impossible.

- (C, R) has a minimum. Similar.

- (C, R) is not dense. Spoiler chooses $b_1, b_2 \in C$ with $b_1 < b_2$ with nothing between them. Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ with $A_1 < a_2$. Spoiler then chooses $a_3 = (a_1 + a_2)/2$, so that $a_1 < a_3 < a_2$. Duplicator must choose $b_3 \in C$ with $b_1 < b_3 < b_2$, impossible. $\square$

Corollary. *The class of DLOs is the $\sim_\omega$-equivalence class of* $(\mathbb{Q}, \leq)$

9 Discrete linear orders

Definition. A linear order $(C, \leq)$ is *discrete* without endpoints if $C \neq \varnothing$ and

$$\forall a \exists b : a \lhd b$$
$$\forall b \exists a : a \lhd b,$$

where $a \lhd b$ means $a < b$ and not $\exists c : a < c < b$.

Example. $(\mathbb{Z}, \leq)$ and $\{1, -1, 1/2, -1/2, 1/3, -1/3, \ldots\}$ are discrete without endpoints.

Definition. Let $(C, <)$ be discrete. If $a \leq b \in C$, then $d(a, b)$ is the size of $[a, b) = \{x \in C : a \leq x < b\}$, or ∞ if infinite. If $a > b$, then $d(a, b) = d(b, a)$.

Lemma. *Let $(C, <)$ and $(C', <)$ be discrete linear orders without endpoints. Suppose $a_1 < \cdots < a_n$ in C and $b_1 < \cdots < b_n$ in C'. Let s be the local isomorphism $s(a_i) = b_i$. Suppose that for every $1 \leq i < n$, we have*

$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or } d(a_i, a_{i+1}) \geq 2^p \leq d(b_i, b_{i+1}).$$

Then s is a p-isomorphism.

Proof. By induction on p. $p = 0$ is trivial.

Suppose $p > 0$. We verify the forth condition (back is similar). Let $a \in C$ be given. We must find $b \in C'$ such that $s \cup \{(a, b)\}$ is a $(p-1)$-isomorphism. Break into cases:

- If $a < a_1$ and $d(a, a_1) = q < \infty$, take $b < b_1$ such that $d(b, b_1) = q$.
- If $a < a_1$ and $d(a, a_1) = \infty$, take $b < b_1$ such that $d(b, b_1) = 2^{p-1}$.
- If $a > a_n$, do something similar.
- If $a_i < a < a_{i+1} \ldots$
  - If $d(a_i, a_{i+1}) < 2^p$, then $d(b_i, b_{i+1}) = d(a_i, a_{i+1})$. Take b with $b_i < b < b_{i+1}$ and $d(b_i, b) = d(a_i, a)$.
  - If $d(a_i, a_{i+1}) \geq 2^p$, then $d(b_i, b_{i+1}) \geq 2^p$. There are three cases:
    * If $d(a_i, a) = q < 2^{p-1}$, then take $b > b_i$ with $d(b_i, b) = q$.
    * If $d(a, a_{i+1}) = q < 2^{p-1}$, take $b < b_{i+1}$ with $d(b, b_{i+1}) = q$.
    * If $d(a_i, a) \geq 2^{p-1} \leq d(a_{i+1}, a)$, take $b > b_i$ with $d(b_i, b) = 2^{p-1}$.
- If $a = a_i$, take $b = b_i$. □

In fact, Theorem 1.8 in Poizat shows that this condition exactly characterizes p-isomorphisms.

Proposition. *Let $(C, \leq)$ and $(C', \leq)$ be discrete linear orders without endpoints. Then $\varnothing$ is a p-equivalence from $(C, \leq)$ to $(C', \leq)$ for all p. Therefore $(C, \leq) \sim_\omega (C', \leq)$.*

Remark. Again, one can show that if $(\mathbb{Z}, \leq) \sim_\omega (C, R)$, then (C, R) is a discrete linear order without endpoints. So the discrete linear orders without endpoints are exactly the $\sim_\omega$-equivalence class of $(\mathbb{Z}, \leq)$.

Remark. Let $C = \{1, -1, 1/2, -1/2, 1/3, -1/3, \ldots\}$. Then C is a discrete linear order without endpoints, so $(C, \leq) \sim_\omega (\mathbb{Z}, \leq)$. But $(C, \leq) \not\sim_\infty (\mathbb{Z}, \leq)$, since $(C, \leq) \not\cong (\mathbb{Z}, \leq)$. So $\sim_\infty$ is stronger than $\sim_\omega$.

10 The infinite Ehrenfeucht-Fraïssé game

Definition 11. Let R, R' be binary relations with universes E, E'. The *infinite Ehrenfeucht-Fraïssé game*, denoted $\mathrm{EF}_\infty(R, R')$, is played as follows:

- There are two players, the Duplicator and Spoiler.
- There are infinitely many rounds (indexed by ω).
- In the nth round, the Spoiler chooses either an $a_n \in E$ or a $b_n \in E'$.
- The Duplicator responds with a $b_n \in E'$ or an $a_n \in E$, respectively.
- If $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ is *not* a local isomorphism, then the Spoiler immediately wins.
- The Duplicator wins if the Spoiler has not won by the end of the game.

Proposition. *The following are equivalent:*

1. *$R \sim_\infty R'$, i.e., there is a non-empty Karpian family K.*
2. *Duplicator has a winning strategy for $\mathrm{EF}_\infty(R, R')$.*
3. *Spoiler does not have a winning strategy for $\mathrm{EF}_\infty(R, R')$.*