Homework 1

Introductory Model Theory

Due 2021-9-23

Recommended reading: *A Course in Model Theory* (Bruno Poizat), Chapter 1.

Definition. An *equivalence relation* is a binary relation $\sim$ with universe E such that

- For any $x \in E$, we have $x \sim x$ ($\sim$ is *reflexive*).
- For any $x, y \in E$, if $x \sim y$ then $y \sim x$ ($\sim$ is *symmetric*).
- For any $x, y, z \in E$, if $x \sim y$ and $y \sim z$, then $x \sim z$ ($\sim$ is *transitive*).

Let $(E, \sim)$ be an equivalence relation. For any $a \in E$, the *equivalence class* of a is the set $[a]_\sim = \{b \in E : a \sim b\}$.

The following exercise is optional, but you should try it if you haven't seen equivalence relations before.

0. Show that $[a]_\sim = [b]_\sim$ if and only if $a \sim b$, and that $[a]_\sim \cap [b]_\sim = \varnothing$ if $a \not\sim b$.

The remaining exercises are required.

1. Suppose a binary relation $(E', \approx)$ is elementarily equivalent to an equivalence relation $(E, \sim)$. Show that $\approx$ is an equivalence relation.[1]

In the following problems, let $\mathcal{K}$ be the collection of equivalence relations $(E, \sim)$ such that for $n = 1, 2, 3, \ldots$, there is exactly one equivalence class of size n.

3. Show that $\mathcal{K}$ is non-empty. *Hint:* $\{\{0\}, \{1, 2\}, \{3, 4, 5\}, \ldots\}$
4. Suppose that $(E, \sim) \in \mathcal{K}$ and $(E', \approx)$ is elementarily equivalent to $(E, \sim)$. Show that $(E', \approx) \in \mathcal{K}$. *Hint:* you can start by applying Problem 1.[2]
5. Suppose that $(E, \sim)$ and $(E', \approx)$ are both in $\mathcal{K}$. Let s be a local isomorphism from $(E, \sim)$ to $(E', \approx)$, and let $p \geq 0$. Suppose that for every $a \in \mathrm{dom}(s)$, the $\sim$-equivalence class of a has the same size as the $\approx$-equivalence class of $s(a)$, or both equivalence classes have size greater than p. Then s is a p-isomorphism from $(E, \sim)$ to $(E', \approx)$. *Hint:* prove this by induction on p.

[1] You may not use Theorem 2.2 in Poizat.

[2] Again, you may not use Theorem 2.2 in Poizat.

6. Suppose that $(E, \sim)$ and $(E', \approx)$ are both in $\mathcal{K}$. Show that $(E, \sim)$ is elementarily equivalent to $(E', \approx)$. *Hint:* apply the previous problem.

7. Construct two equivalence relations $(E, \sim)$ and $(E', \approx)$ such that $(E, \sim) \sim_\omega (E', \approx)$, but $(E, \sim) \not\sim_\infty (E', \approx)$. *Hint:* build two non-isomorphic countable binary relations in $\mathcal{K}$.