Stability

Qi'ao Chen

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A combination of various notes from [Pillay(2018)] [Chernikov(2019)] [Tent and Ziegler(2012)]

A monster model $\mathfrak{C}$

2 Preliminaries

2.1 Indiscernibles

Definition 2.1. Let I be a linear order and $\mathfrak{A}$ an L-structure. A family $(a_i)_{i\in I}$ of elements of A is called a **sequence of indiscernibles** if for all L-formulas $\varphi(x_1,\dots,x_n)$ and all $i_1<\dots<i_n$ and $j_1<\dots<j_n$ from I

$$\mathfrak{A}\vDash\varphi(a_{i_1},\dots,a_{i_n})\leftrightarrow\varphi(a_{j_1},\dots,a_{j_n})$$

or

$$\operatorname{tp}(a_{i_1}, \dots, a_{i_n}) = \operatorname{tp}(a_{j_1}, \dots, a_{j_n})$$

Theorem 2.2. *Compactness let us "stretch" indiscernibles. Let $(a_i : i \in \omega)$ be indiscernibles in $\mathfrak{C}$, and $(I, <)$ an ordering. Then there exists an indiscernible $(b_i : i \in I)$ in $\mathfrak{C}$ s.t. $\forall i_1 < \dots < i_n \in I$*

$$\operatorname{tp}(a_1, \dots, a_n) = \operatorname{tp}(b_{i_1}, \dots, b_{i_n})$$

Indiscernible sequence are a fundamental tool of model theory, and there are many ways to obtain them.

Theorem 2.3 (Ramsey, extended)**.** *Let $n_1, \dots, n_r < \omega$. For each $i = 1, \dots, r$, let $X_{i,1}$, $X_{i,2}$ be a partition of $[\omega]^{n_i}$. Then there is an infinite subset $Y \subseteq \omega$ which is homogeneous, i.e., $\forall i = 1, \dots, r$, either $[Y]^{n_i} \subseteq X_{i,1}$ or $[Y]^{n_i} \subseteq Y_{i,2}$*

Proposition 2.4. *For each $n \in \omega$, let $\Sigma_n(x_1, \dots, x_n)$ be a collection of L-formulas in variables $x_1, \dots, x_n$. Suppose that there are $a_1, a_2, \dots \in \mathfrak{C}$ s.t.*

$$\vDash \Sigma_n(a_{i_1}, \dots, a_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

Then there is an indiscernible $(b_i : i \in \omega)$ in $\mathfrak{C}$ s.t.

$$\vDash \Sigma_n(b_{i_1}, \dots, b_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

we can expand $\bigcup_{n\in\omega} \Sigma_n$ and obtain the Ehrenfeucht-Mostowski type $\mathrm{EM}((a_i)_{i\in\omega})$. This is just the Standard Lemma in Tent

Example 2.1. Suppose $\Sigma_2 = \{x_1 \neq x_2\}$. Then the proposition yields the existence of infinite indiscernible sequences

Proof. Consider

$$\begin{aligned}\Gamma(x_1, x_2, \dots) = \{\varphi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \varphi(x_{j_1}, \dots, x_{j_n}) :\\ i_1 < \dots < i_n, j_1 < \dots < j_n \in \omega, \varphi \in L\}\\ \cup \bigcup_n \Sigma(x_1, \dots, x_n)\end{aligned}$$

Let $\Gamma'(x_1, \dots, x_n) \subseteq_f \Gamma$. Let $\varphi_1, \dots, \varphi_r$ be the L-formulas appearing in Γ'. For $i = 1, \dots, r$, let

$$X_{i,1} = \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \vDash \varphi_i(a_{j_1}, \dots, a_{j_n})\}$$
$$X_{i,2} = \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \vDash \neg\varphi_i(a_{j_1}, \dots, a_{j_n})\}$$

By Ramsey's theorem, there exists an infinite $Y \subseteq \mathbb{N}$ s.t. $\forall i = 1, \dots, r$, $[Y]^{n_i}$ is either contained in $X_{i,1}$ or in $X_{i,2}$. Write $Y = \{k_1 < k_2 < \dots\}$. Interpret each x_i as a_{k_i} to satisfy Γ' $\square$

Definition 2.5. Let $M \prec N \prec \mathfrak{C}$ be models, and $p(\bar{x}) \in S_{\bar{x}}(N)$. We say p is finitely satisfiable in M, or $p(\bar{x})$ is a **coheir** of $p \restriction M \in S_{\bar{x}}(M)$, if every $\varphi(\bar{x}) \in p(\bar{x})$ is satisfied by some $\bar{a} \in M$

Remark. $p(\bar{x}) \in S_n(N)$ is finitely satisfiable (f.s.) in M iff $p(\bar{x})$ is in the topological closure of $\{\mathrm{tp}(\bar{a}/N) : \bar{a} \in M\} \subseteq S_n(N)$

Lemma 2.6. *Suppose $p(\bar{x}) \in S_{\bar{x}}(M)$ and $M \prec N$, then there is $p'(\bar{x}) \in S_{\bar{x}}(N)$ s.t. $p \subseteq p'$ and p' is f.s. in M*

Proof. Consider $\Gamma(\bar{x}) = p(\bar{x}) \cup \{\neg\varphi(\bar{x}) : \varphi(\bar{x}) \in L_N$ and not realized in $M\}$. Let $\Gamma \supseteq_f \Gamma' = \{\Psi(\bar{x}), \neg\varphi_1(\bar{x}), \dots, \neg\varphi_r(\bar{x})\} \in p$. Then any solution $\bar{a}$ of Ψ in M satisfies Γ' as $M \vDash \forall\bar{x}(\neg\varphi_i(\bar{x}))$ □

Remark. Let $i_M : M^{\bar{x}} \to S_{\bar{x}}(M)$ s.t. $m \mapsto \mathrm{tp}(m/M)$. Define $i_N : M^{\bar{x}} \to S_{\bar{x}}(N)$ similarly. Let $r : S_{\bar{x}}(N) \to S_{\bar{x}}(M)$. Note that $r \circ i_N = i_M$ and the set of types in $S_{\bar{x}}(N)$ that are f.s. in M is exactly the closure of $i_N(M^{\bar{x}})$ in $S_{\bar{x}}(N)$. Hence its image under r is closed. However the image must contain $i_M(M^{\bar{x}})$ which is dense in $S_{\bar{x}}(M)$. Therefore it must be onto, which proves the desired result

r is continuous and $r(\overline{i_N(M^n)}) \supseteq i_M(M^n)$ is closed. $\overline{i_M(M^n)} = S_n(M)$. Then r is onto? Then its preimage of p is what we want

Proposition 2.7. *Let $p(\bar{x}) \in S_{\bar{x}}(M)$, $N \succ M$ be $|M|^+$-saturated, and $p'(\bar{x}) \in S_{\bar{x}}(N)$ a coheir of p. Let $\bar{a}_1, \bar{a}_2, \dots \in N$ be defined as follows*

$$\begin{aligned}
&\bar{a}_1 \textit{ realises } p(\bar{x}) \\
&\bar{a}_2 \textit{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1) \\
&\bar{a}_3 \textit{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1, \bar{a}_2) \\
&\dots
\end{aligned}$$

Then $(\bar{a}_i : i \in \omega)$ is indiscernible over M

Proof. We prove by induction on k that for any $n \leq k$ and $i_1 < \dots < i_n \leq k$ and $j_1 < \dots < j_n \leq k$, we have

$$\mathrm{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/M) = \mathrm{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/M)$$

Assume this is true for k and consider $k+1$. Let $i_1 < \dots < i_n \leq k$, $j_1 < \dots < j_n \leq k$. We need to show that

$$\mathrm{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}_{k+1}/M) = \mathrm{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}_{k+1}/M)$$

Consider a formula $\varphi(\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1}) \in L_M$. Assume by contradiction that

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}_{k+1}) \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}_{k+1})$$

But $\mathrm{tp}(\bar{a}_{k+1}/M, \bar{a}_1, \dots, \bar{a}_k)$ is f.s. in M, so there is $\bar{a}' \in M$ s.t.

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}') \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}')$$

contradicting IH $\square$

2.2 Definability and Generalizations

Definition 2.8. $X \subseteq \mathfrak{C}^n$ is **definable almost over** A if there is an A-definable equivalence relation E on $\mathfrak{C}^n$ with finitely many classes and X is a union of some E-classes

Lemma 2.9. *Let $\mathbb{D}$ be a definable class and A a set of parameters. T.F.A.E.*

1. *$\mathbb{D}$ is definable over A*
2. *$\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix A pointwise*

$S \subseteq K^{\mathrm{alg}} \Rightarrow M \setminus S \subseteq K^{\mathrm{alg}}$

Proof. $\Rightarrow$ is easy as for any $F \in \mathrm{Aut}(\mathfrak{C}/A)$ and $\mathbb{D} = \varphi(\mathfrak{C}, \bar{a})$, $\mathfrak{C} \vDash \varphi(\bar{s}, \bar{a})$ iff $\mathfrak{C} \vDash \varphi(F(\bar{s}), \bar{a})$. StackExchange

$x \in \mathbb{D} \Leftrightarrow \vDash \varphi(x, \bar{a}) \Leftrightarrow \varphi(F(x), F(\bar{a})) \leftrightarrow \varphi(F(x), \bar{a}) \Leftrightarrow F(x) \in \mathbb{D}$

$\Leftarrow$. Another proof from Chernikov. Assume that $\mathbb{D} = \varphi(\mathfrak{C}, b)$ where $b \in \mathfrak{C}$, and let $p(y) = \mathrm{tp}(b/A)$

Claim 1. $p(y) \vdash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$, which says that for any realisations b', $\varphi(\mathfrak{C}, b) = \varphi(\mathfrak{C}, b')$

Indeed, let $b' \vDash p(y)$ be arbitrary. Then $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$ so there is some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ with $\sigma(b) = b'$. Then $\sigma(X) = \varphi(\mathfrak{C}, b')$ and by assumption $\sigma(X) = X$, thus $\varphi(\mathfrak{C}, b) = X = \varphi(\mathfrak{C}, b')$.

There is some $\psi(y) \in p$ (there is a finite subset of $p(y)$ that does the job and we take the conjunction) s.t.

$$\psi(y) \vDash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$$

Let $\theta(x)$ be the formula $\exists y(\psi(y) \wedge \varphi(x, y))$. Note that $\theta(x)$ is an $L(A)$-formula, as $\psi(y)$ is

Claim 2. $X = \theta(\mathfrak{C})$

If $a \in X$, then $\vDash \varphi(a,b)$, and as $\psi(y) \in \text{tp}(b/A)$ we have $\vDash \theta(a)$. Conversely, if $\vDash \theta(a)$, let b' be s.t. $\vDash \psi(b') \wedge \varphi(a,b')$. But by the choice of ψ this implies that $\vDash \varphi(a,b)$

$\Leftarrow$ Let $\mathbb{D}$ be defined by φ, defined over $B \supset A$. Consider the maps

$$\mathfrak{C} \xrightarrow{\tau} S(B) \xrightarrow{\pi} S(A)$$

where $\tau(c) = \text{tp}(c/B)$ and π is the restriction map. Let Y be the image of $\mathbb{D}$ in $S(A)$. Since $Y = \pi[\varphi]$. Y is closed. Note that $\tau(\mathbb{D}) = [\varphi]$. $\tau(\mathbb{D}) = \{\text{tp}(c/B) : \mathfrak{C} \vDash \varphi(c)\} \subseteq [\varphi]$. For any $q(x) \in [\varphi]$, as $\mathfrak{C}$ is saturated, $\mathfrak{C} \vDash q(d)$ and $d \in \mathbb{D}$. Thus $q \in \tau(\mathbb{D})$. π is continuous

Assume that $\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix A pointwise. Since elements which have the same type over A are conjugate by an automorphism of $\mathfrak{C}$, this means that $\mathbb{D}$-membership depends only on the type over A, i.e., $\mathbb{D} = (\pi\tau)^{-1}(Y)$. For any $\text{tp}(c/A) = \text{tp}(d/A)$ and $c \in \mathbb{D}$, as c and d are conjugate, $d \in \mathbb{D}$.

For any $c \notin \mathbb{D}$, $\pi\tau(c) \in Y$ iff $\text{tp}(c/A) \in \pi[\varphi]$ iff there is $d \in \mathbb{D}$ s.t. $\text{tp}(c/A) = \text{tp}(d/A)$ but then $c \in \mathbb{D}$.

This implies that $[\varphi] = \pi^{-1}(Y)$ $\tau(\mathbb{D}) = [\varphi] = \tau(\tau^{-1}\pi^{-1})(Y) = \pi^{-1}(Y)$, or $S(A) \setminus Y = \pi[\neg\varphi]$; hence $S(A) \setminus Y$ is also closed and we conclude that Y is clopen. By Lemma **??** $Y = [\psi]$ for some $L(A)$-formula ψ. This ψ defines $\mathbb{D}$. For any $d \in \mathfrak{C}$

$$\vDash \psi(d) \Leftrightarrow \text{tp}(d/A) \Leftrightarrow d \in \mathbb{D}$$

□

A slight generalization of the previous lemma

Lemma 2.10. *Let $X \subseteq \mathfrak{C}^n$ be definable. TFAE*

1. *X is almost A-definable, i.e., there is an A-definable equivalence relation E on $\mathfrak{C}^n$ with finitely many classes, s.t. X is a union of E-classes*

2. *The set $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ is finite*

3. *The set $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ is small*

Proof. $1 \to 2$. Let $\varphi(x_1, x_2) \in L(A)$ be the A-definable equivalence relation E, and let $b_1, \ldots, b_n \in M$ be representatives in each equivalence class so that each class can be written as $[b_i] = \varphi(\mathfrak{C}, b_i)$. Given $\sigma \in \text{Aut}(\mathfrak{C}/A)$, since $\varphi(x_1, x_2) \leftrightarrow \varphi(\sigma(x_1), \sigma(x_2))$, the image of each $[b_i]$ under σ will be

$$\sigma([b_i]) = \{\sigma(x) : \varphi(x, b_i)\} = \{x' : \varphi(x', \sigma(b_i))\} = \{x : \varphi(x, b_{j_i})\} = [b_{j_1}]$$

for some $j_i \leq n$. Now X is a disjoint union of some $[b_i]$'s, so $\sigma(X)$ is a disjoint union of some $[b_j]$'s. Since there are only finitely many equivalence classes, there can only be finitely many possibilities for disjoint unions of these classes

$2 \to 1$. Let $X = \varphi(\mathfrak{C}, b)$ and $p(y) = \text{tp}(b/A)$. Given $\sigma \in \text{Aut}(\mathfrak{C}/A)$, we have $\sigma(X) = \varphi(\mathfrak{C}, \sigma(b))$. Then from assumption, there must be distinct $b_1, \dots, b_n$ s.t.

$$\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\} = \{\varphi(\mathfrak{C}, b_i) : i \leq n\}$$

Now if $\text{tp}(b'/A) = \text{tp}(b/A)$, then strong homogeneity yields some $\sigma \in \text{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Then the above argument again shows that $\varphi(x, b')$ defines $\sigma(X)$ for some $\sigma \in \text{Aut}(\mathfrak{C}/A)$. Thus $\sigma(X) = \varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i \leq k$. Therefore $p(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. By compactness there is some $\psi(y) \in p$ s.t. $\psi(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. Now define $E(x_1, x_2)$ as

$$\forall y(\psi(y) \to (\phi(x_1, y) \leftrightarrow \phi(x_2, y)))$$

so it is A-definable. It is easy to check that E is an equivalence relation with finitely many classes, and that X is a union of E-classes ($a_1 E a_2$ iff they agree on $\phi(x, b_i)$ for all $i \leq k$, and so $X = \phi(\mathfrak{C}, b_0)$ is given by the union of all possible combinations intersected with it)

$3 \to 1$ Assume for contradiction that

$$|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| = \lambda \geq \omega$$

we can find λ-many elements $(b_i : i < \lambda) \subset \mathfrak{C}$ to represent the distinct images under automorphisms. Then the set

$$q(y) = p(y) \cup \{\neg \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b_i)) : i < \lambda\}$$

will be finitely satisfiable. Thus $q(y)$ is realised by some b'. But such b' has the same type as b over A and so strong homogeneity yields some $\sigma \in \text{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Applying such σ on X gives the image $\varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i < \lambda$, a contradiction $\square$

Proposition 2.11. *We can identify definable sets with continuous functions in a certain settings*

1. *Formulas $\varphi(\bar{x}), \psi(\bar{x}) \in L_A$ are equivalent iff $[\varphi(\bar{x})] = [\psi(\bar{x})]$*
2. *The clopen subsets of $S_{\bar{x}}(A)$ are precisely the basic clopen sets*

3. *Clopen subsets X of $S_{\bar{x}}(A)$ correspond exactly to continuous functions f : $S_{\bar{x}}(A) \to 2$ (with discrete topology) where $f(p(\bar{x})) = 1$ if $p(\bar{x}) \in X$ and 0 otherwise*

4. *The definable subsets of $\mathfrak{C}^c$ are in one-to-one correspondence with continuous functions from $S_{\bar{x}}(A)$ to 2*

Proof. 3. If X is clopen, then $f^{-1}(2) = S_{\bar{x}}(A)$, $f^{-1}(0) = \emptyset$, $f^{-1}(\{1\}) = X$, $f^{-1}(\{0\}) = X^c$

4. By 1, definable sets are in one-to-one correspondence with basic clopen subsets. By 2, basic clopen sets are exactly all of the clopen subsets, so definable sets are in one-to-one correspondence with clopen sets. By 3, clopen sets are in one-to-one correspondence with continuous functions $f : S_{\bar{x}}(A) \to 2$

□

2.3 Imaginaries and T^{eq}

Definition 2.12. The **definable closure** $\text{dcl}(A)$ of A is the set of elements c for which there is an $L(A)$-formula $\varphi(x)$ s.t. c is the unique element satisfying φ. Elements or tuples a and b are said to be **interdefinable** if $a \in \text{dcl}(b)$ and $b \in \text{dcl}(a)$.

Lemma 2.13. *Assume $A \subseteq \mathfrak{C}$ and $\bar{b} \in \mathfrak{C}$*

1. *$\bar{b} \in \text{acl}(A)$ iff $\{f(\bar{b}) : f \in Aut(\mathfrak{C}/A)\}$ is finite*

2. *$\bar{b} \in \text{dcl}(A)$ iff $f(\bar{b}) = \bar{b}$ for all $f \in Aut(\mathfrak{C}/A)$*

Proof. 1. Suppose $\bar{b} \in \text{acl}(A)$ with witness $\exists^{\leq k}\varphi(\bar{x})$. Then $\varphi(\mathfrak{C})$ is A-definable and hence is $\text{Aut}(\mathfrak{C}/A)$-invariant by Lemma 2.9

Suppose the finiteness. Since the composition of automorphisms is an automorphism, this set is $\text{Aut}(\mathfrak{C}/A)$-invariant and therefore A-definable by some $\varphi(\bar{x})$.

2. $\{\bar{b}\}$ is $\text{Aut}(\mathfrak{C}/A)$-invariant

□

The first motivation to develop T^{eq} is dealing with quotient objects, without leaving the context of first order logic. That is, if E is some definable equivalence relation on some definable set X, we want to view X/E as a definable set

We work in the setting of multi-sorted languages. Let L be a 1-sorted language and let T be a (complete) L-theory. We shall build a many-sorted language L^{eq}-theory T^{eq}. We will ensure that in natural sense, L^{eq} contains L and T^{eq} contains T

First we define L^{eq}. Consider the set L-formula $\varphi(x, y)$, up to equivalence, such that T models that φ is an equivalence relation. For each φ, define s_φ to be a new sort in L^{eq}. Of particular importance is $s_=$, the sort given by the formula "$x = y$". = is an equivalence relation This sort $s_=$ will yield, in each model of T^{eq}, a model of T

Also define f_φ to be a function symbol with domain sort $s_=^n$ (where φ has n free variables) and codomain sort s_φ

For each m-place relation symbol $R \in L$, make R^{eq} an m-place relation symbol in L^{eq} on $s_=^m$. Likewise for all constant and function symbols in L. Finally, for the sake of formality, we put a unique equality symbol $=_\varphi$ on each sort

Remark. Let N be an L^{eq} structure. Then N has interpretations $s_\varphi(N)$ of each sort s_φ and $f_\varphi(N) : s_=(N)^{n_{f_\varphi}} \to s_\varphi(N)$ of each function symbol f_φ. Additionally, N will contain an L-structure consisting of $s_=$ and interpretations of the symbols of L inside of $s_=$

Definition 2.14. T^{eq} is the L^{eq}-theory which is axiomatised by the following

1. T, where the quantifiers in the formulas of T now range over the sort $s_=$
2. For each suitable L-formula $\varphi(x, y)$, the axiom $\forall_{s_=}\bar{x}\forall_{s_=}\bar{y}(\varphi(x, y) \leftrightarrow f_\varphi(\bar{x}) = f_\varphi(\bar{y}))$
3. For each L-formula φ, the axiom $\forall_{s_\varphi} y \exists_{s_=}\bar{x}(f_\varphi(\bar{x}) = y)$

Axioms 2 and 3 simply state that f_φ is the quotient function for the equivalence relation given by φ

Definition 2.15. Let $M \vDash T$. Then M^{eq} is the L^{eq} structure s.t. $s_=(M^{\mathrm{eq}}) = M$ and for each suitable L-formula $\varphi(x, y)$ of n variables, the sort $s_\varphi(M^{\mathrm{eq}})$ is equal to $M^{n_{f_\varphi}}/E$ where E is the equivalence relation defined by $\varphi(x, y)$ and $f_\varphi(M^{\mathrm{eq}})(b) = b/E$

Example 2.2 (Projective planes)**.** From Hodges.

Suppose A is a three-dimensional vector space over a finite field, and let L be the first-order language of A. Then we can write a formula $\theta(x, y)$ of L which expresses 'vectors x and y are non-zero and are linearly dependent on each other'. The formula θ is an equivalence formula of A, and the sort s_θ is the set of points of the projective plane P associated with A

Now $M^{\text{eq}} \vDash T^{\text{eq}}$. Moreover, passing from T to T^{eq} is a canonical operation, in the following sense

Lemma 2.16. 1. *For any $N \vDash T^{\text{eq}}$, there is an $M \vDash T$ s.t. $N \cong M^{\text{eq}}$*

2. *Suppose $M, N \vDash T$ are isomorphic, and let $h : M \cong N$. Then h extends uniquely to $h^{\text{eq}} : M^{\text{eq}} \cong N^{\text{eq}}$*

3. *T^{eq} is a complete L^{eq}-theory*

4. *Suppose $M, N \vDash T$ and let $\bar{a} \in M$, $\bar{b} \in N$ with $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$. Then $\text{tp}_{M^{\text{eq}}}(\bar{a}) = \text{tp}_{N^{\text{eq}}}(\bar{b})$*

Proof. 1. Take $M = s_=(N)$

2. Let $h^{\text{eq}} : M^{\text{eq}} \to N^{\text{eq}}$ be defined as $h^{\text{eq}}(f_\varphi(M^{\text{eq}})(b)) = f_\varphi(N^{\text{eq}})(h(b))$ for each $\varphi \in L$. This defines a function on M^{eq}, because $f_\varphi(M^{\text{eq}})$ is surjective by the T^{eq} axioms. Moreover h^{eq} is well-defined. Suppose $f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b')$, then $\varphi(b, b')$ and hence $\varphi(h(b), h(b'))$, therefore $f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b'))$. Injectivity is the same since $\varphi(b, b') \leftrightarrow \varphi(h(b), h(b'))$.

$$\begin{aligned} f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b')) &\Leftrightarrow h(b)/E_\varphi = h(b')/E_\varphi \\ &\Leftrightarrow \varphi(h(b), h(b')) \\ &\Leftrightarrow \varphi(b, b') \\ &\Leftrightarrow f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b') \end{aligned}$$

3. Let $M, N \vDash T^{\text{eq}}$, we want to show that they are elementary equivalent. Assume the generalized continuum hypothesis. By GCH, there are $M', N' \vDash T^{\text{eq}}$ which are λ saturated of size λ, for some large λ (strongly inaccessible), which $M \preceq M'$ and $N \preceq N'$. Since we want to show elementary equivalence, we can replace M, N with M' and N'. By 1, we have $M = M_0^{\text{eq}}$, $N = N_0^{\text{eq}}$ for some $M_0, N_0 \vDash T$. Furthermore, M_0, N_0 are λ-saturated of size λ. By assumption, T is complete, so $M_0 \equiv N_0$, and therefore $M_0 \cong N_0$. By 2, $M \cong N$, and therefore $M \equiv N$

We could simply prove that there is a back and forth system between M and N, using such a system between $M \supset M_0 \vDash T$ and $N \supset N_0 \vDash T$ $M_0 \equiv N_0$ iff $M_0 \sim_\omega N_0$. We want to show that $M \sim_\omega N$. For any $p \in \omega$,

- given $a \in s_=(M)$, choose according to M
- given $a \in s_\varphi(M)$, then there is $\bar{b}\bar{c} \in s_=(M)$ s.t. $f_\varphi(M^{\mathrm{eq}})(\bar{b}\bar{c}) = a$ and $\varphi(\bar{b}, \bar{c})$. If $\bar{b} \in s_=(M^{\mathrm{eq}})^n$, then there is a local isomorphism $\bar{b} \mapsto \bar{d}$ as $M \sim_\omega N$. Take $b = \bar{d}/E_\varphi$.

4. Let $M, N \vDash T$, they are elementary submodels of $\mathfrak{C}$. Since $\mathrm{tp}_M(\bar{a}) = \mathrm{tp}_N(\bar{b})$, there exists an $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ with $\sigma(\bar{a}) = \bar{b}$. By 2, this automorphism extends to $\sigma^{\mathrm{eq}} : \mathfrak{C}^{\mathrm{eq}} \to \mathfrak{C}^{\mathrm{eq}}$ with $\sigma^{\mathrm{eq}}(a) = b$, hence $\mathrm{tp}_{M^{\mathrm{eq}}}(a) = \mathrm{tp}_{\mathfrak{C}^{\mathrm{eq}}}(a) = \mathrm{tp}_{\mathfrak{C}^{\mathrm{eq}}}(b) = \mathrm{tp}_{N^{\mathrm{eq}}}(b)$

□

Corollary 2.17. *Consider the Strong space $S_{(s_=)^n}(T^{\mathrm{eq}})$. The forgetful map $\pi : S_{(s_=)^n}(T^{\mathrm{eq}}) \to S_n(T)$ is a homeomorphism*

Proof. Observe that it is continuous and surjective. By 4 of the previous lemma it is injective. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism □

Proposition 2.18. *Let $\varphi(x_1, \dots, x_k)$ be an L^{eq} formula, where x_i is of sort S_{E_i}. There is an L-formula $\psi(\bar{y}_1, \dots, \bar{y}_k)$ s.t.*

$$T^{\mathrm{eq}} \vDash \forall \bar{y}_1, \dots, \bar{y}_k(\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$$

Proof. Let n be the length of $\bar{y}_1, \dots, \bar{y}_k$. Consider the set $\pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$, it is a clopen subset of $S_n(T)$ by the previous lemma, hence equal to $\psi(\bar{y}_1, \dots, \bar{y}_k)$ for some formula ψ.

Guess the intuition is $[\varphi] = [\psi]$ iff $\vDash \varphi \leftrightarrow \psi$. Consider $\pi[\psi(\bar{y}_1, \dots, \bar{y}_k)] = \pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ and as π is homeomorphism, $[\psi(\bar{y}_1, \dots, \bar{y}_k)] = [\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ □

This proposition also shows that T^{eq} is complete since f_{E_i} is surjective

Corollary 2.19.
1. *Let $M, N \vDash T$, and let $h : M \to N$ be an elementary embedding. Then $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to N^{\mathrm{eq}}$ is also an elementary embedding*
2. *$\mathfrak{C}^{\mathrm{eq}}$ is also κ-saturated*

Proof.
1. $h : M \to \mathrm{im}(h)$ is an isomorphism and can extend to $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to (\mathrm{im}(h))^{\mathrm{eq}}$, and $(\mathrm{im}(h))^{\mathrm{eq}} \subseteq N^{\mathrm{eq}}$
2. By Proposition 2.18

□

Remark. For $M \vDash T$, a definable set $X \subseteq M^n$ can be viewed as an element of M^{eq}. Suppose X is defined in M by $\varphi(\bar{x}, \bar{a})$ where $\bar{a} \in M$. Consider the equivalence relation E_ψ defined by $\psi(\bar{y}_1, \bar{y}_2) = \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$ $\bar{y}_1 \sim \bar{y}_2$ iff this $\varphi(M, \bar{y}_1) = \varphi(M, \bar{y}_2)$, and consider $c = \bar{a}/E_\psi = f_\psi(\bar{a}) \in M^{\text{eq}}$. Then X is defined in M^{eq} by $\chi(\bar{x}, c) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f_\psi(\bar{y}) = c)$. Moreover, if $c' \in S_\psi(M^{\text{eq}})$ and $\forall \bar{x}(\chi(\bar{x}, c) \leftrightarrow \chi(\bar{x}, c'))$, then $c = c'$. To see this, let $c' = f_\psi(\bar{a}')$, and let X' be defined in M by $\varphi(\bar{x}, \bar{a}')$. Then X' is defined in M^{eq} by $\chi(\bar{x}, c')$, so we have that $X = X'$ (in M^{eq}). And then $X = X'$ (in M) so $c = f_\psi(\bar{a}) = f_{\psi'}(\bar{a}') = c'$

Definition 2.20. With the above considerations in mind, given $M \vDash T$ and a definable set $X \subseteq M^n$, we call such a $c \in M^{\text{eq}}$ a **code** for X

Remark. Any automorphism of $\mathfrak{C}^{\text{eq}}$ fixes a definable set X set-wise iff it fixes a code for X. However, the choice of a code for X will depend on the formula φ used to define it

$$\begin{aligned}\sigma(X) = X &\Leftrightarrow \sigma(X) = \{\sigma(x) : \varphi(x, b)\} = \{x : \varphi(x, \sigma(b))\} = \{x : \varphi(x, b)\} = X \\ &\Leftrightarrow \forall x(\varphi(x, b) \leftrightarrow \varphi(x, \sigma(b))) \\ &\Leftrightarrow \psi(b, \sigma(b)) \Leftrightarrow f_\psi(b) = f_\psi(\sigma(b))\end{aligned}$$

We can think of $\mathfrak{C}^{\text{eq}}$ as adjoining codes for all definable equivalence relations (as c/E' codes $E'(x, c)$ for an arbitrary equivalence relation E)

Definition 2.21. Let $A \subseteq M \vDash T$. Then $\text{acl}^{\text{eq}}(A) = \{c \in M^{\text{eq}} : c \in \text{acl}_{M^{\text{eq}}}(A)\}$ and $\text{dcl}^{\text{eq}}(A)$ is defined similarly

Remark. Suppose $A \subseteq M \prec N$, then $\text{acl}_{N^{\text{eq}}}(A), \text{dcl}_{N^{\text{eq}}}(A) \subseteq M^{\text{eq}}$, so this notation is unambiguous

Lemma 2.22. *Let $M \vDash T$, a definable subset X of M^n, and $A \subseteq M$. Then X is almost A-definable iff X is definable in M^{eq} by a formula with parameters in* $\text{acl}^{\text{eq}}(A)$

Proof. We can work in $\mathfrak{C}$, since $M \prec \mathfrak{C}$. Let c be a code for X. From 2.10 X is almost A-definable iff $|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| < \omega$ iff $|\{\sigma(c) : \sigma \in \text{Aut}(\mathfrak{C}^{\text{eq}}/A)\}| < \omega$ (note that σ extends uniquely in $\mathfrak{C}^{\text{eq}}$), that is, $c \in \text{acl}^{\text{eq}}(A)$.

$$\begin{aligned}\sigma(b)/E = \sigma'(b)/E &\Leftrightarrow \forall x(\varphi(x, \sigma(b)) \leftrightarrow \varphi(x, \sigma'(b))) \\ &\Leftrightarrow \sigma(X) = \sigma'(X)\end{aligned}$$

$\square$

Definition 2.23. Let $\bar{a}, \bar{b} \in \mathfrak{C}$ have length n. Let $\bar{a}, \bar{b}$ have the same strong type over A (written as $\mathrm{stp}_{\mathfrak{C}}(\bar{a}/A) = \mathrm{stp}_{\mathfrak{C}}(\bar{a}/A)$) if $E(\bar{a}, \bar{b})$ for any finite equivalence relation (finitely many classes) defined over A

Remark. If $\varphi(\bar{x})$ is a formula over A, then it defines an equivalence with two classes $E(\bar{x}_1, \bar{x}_2)$ iff $(\varphi(\bar{x}_1) \wedge \varphi(\bar{x}_2)) \vee (\neg\varphi(\bar{x}_1) \wedge \neg\varphi(\bar{x}_2))$. Hence strong types are a refinement of types

Hence for any formula if $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/B)$, at least we have $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$

Lemma 2.24. *If $A = M \prec \mathfrak{C}$, then* $\mathrm{tp}_{\mathfrak{C}}(a/M) \vDash \mathrm{stp}_{\mathfrak{C}}(a/M)$

$$\mathrm{tp}_{\mathfrak{C}}(a/M) = \mathrm{tp}_{\mathfrak{C}}(b/M) \Rightarrow \mathrm{stp}_{\mathfrak{C}}(a/M) = \mathrm{stp}_{\mathfrak{C}}(b/M)$$

Proof. Let E be an equivalence relation with finitely many classes, defined over M, and $\bar{b}$ another realization of $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/M)$, we want to show $E(a, b)$. Since E has only finitely many classes, and M is a model, there are representants $e_1, \ldots, e_n$ of each E-class in M. Hence we must have $E(a, e_i)$ for some i, and therefore $E(b, e_i)$, which yields $E(a, b)$ □

Lemma 2.25. *Let $A \subseteq M \vDash T$, and let $\bar{a}, \bar{b} \in M$. TFAE*

1. $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$
2. *$\bar{a}, \bar{b}$ satisfy the same formulas almost A-definable*
3. $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}_{\mathfrak{C}}(\bar{b}/\operatorname{acl}^{\mathrm{eq}}(A))$

Proof. $3 \to 2$. 2.22. Suppose $X = \varphi(\mathfrak{C}, \bar{d})$ is almost A-definable, then $\bar{a}, \bar{b} \in \varphi(\mathfrak{C}, \bar{d})$ iff $\bar{a}, \bar{b} \in \theta(\mathfrak{C}) := \exists \bar{y}(\varphi(\mathfrak{C}, \bar{y}) \wedge \bar{y}/E_\psi = \bar{c})$ where $\bar{c} = \bar{d}/E_\psi \in \operatorname{acl}^{\mathrm{eq}}(A)$.

$2 \to 3$

$1 \to 2$. Let X be almost definable over A. We want to show that $\bar{a} \in X$ iff $\bar{b} \in X$.

Since X is almost definable over A, there is an A-definable equivalence relation E with finitely many classes, and $\bar{c}_1, \ldots, \bar{c}_n$ s.t. for all $\bar{x} \in M$, we have $\bar{x} \in X$ iff $M \vDash E(\bar{x}, \bar{c}_1) \vee \cdots \vee E(\bar{x}, \bar{c}_n)$. Hence $E(\bar{a}, \bar{c}_i)$ for some i, so by assumption $E(\bar{b}, \bar{c}_i)$.

$2 \to 1$. Let E be an A-definable equivalence relation with finitely many classes, we want to show that $E(\bar{a}, \bar{b})$. The set $X = \{\bar{x} \in M : E(\bar{x}, \bar{a})\}$ is definable almost over A. But $\bar{a} \in X$, so $\bar{b} \in X$, hence $E(\bar{a}, \bar{b})$ □

Here is a note from scanlon

Definition 2.26. An **imaginary element** of $\mathfrak{A}$ is a class a/E where $a \in A^n$ and E is a definable equivalence relation on A^n

Definition 2.27. $\mathfrak{A}$ **eliminates imaginaries** if, for every definable equivalence relation E on A^n there exists definable function $f : A^n \to A^m$ s.t. for $x, y \in A^n$ we have

$$xEy \Leftrightarrow f(x) = f(y)$$

Remark. The definition give above is what Hodges calls **uniform elimination of imaginaries**

Remark. If $\mathfrak{A}$ eliminates imaginaries, then for any definable set X and definable equivalence relation E on X, there is a definable set Y and a definable bijection $f : X/E \to Y$. Of course this is not literally true, we should rather say that there is a definable map $f' : X \to Y$ s.t. f' is invariant on the equivalence classes defined by E

So elimination of imaginaries is saying that quotients exists in the category of definable sets

Remark. If $\mathfrak{A}$ eliminates imaginaries then for any imaginaries element $a/E = \tilde{a}$ there is some tuple $\hat{a} \in A^m$ s.t. $\tilde{a}$ and $\hat{a}$ are **interdefinable**, i.e. there is a formula $\varphi(x, y)$ s.t.

- $\mathfrak{A} \vDash \varphi(a, \tilde{a})$
- If $a'Ea$ then $\mathfrak{A} \vDash \varphi(a', \hat{a})$
- If $\varphi(b, \hat{a})$ then bEa
- If $\varphi(a, c)$ then $c = \hat{a}$

To get the formula φ we use the function f given by the definition of elimination of imaginaries and let $\varphi(x, y) := f(x) = y$

Almost conversely, if for every $\mathfrak{A}' \equiv \mathfrak{A}$ every imaginary in $\mathfrak{A}'$ is interdefinable with a **real** (non-imaginary) tuple then $\mathfrak{A}$ eliminates imaginaries

$$\{\forall xy(xEy \leftrightarrow f_E(x) = f_E(y)) \mid \forall E\}$$

Example 2.3. For any structure $\mathfrak{A}$, every imaginary in $\mathfrak{A}_A$ is interdefinable with a sequence of real elements

Example 2.4. Let $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$. Then $\mathfrak{A}$ eliminates imaginaries. For example, to eliminate the "odd/even" equivalence relation, E, we can define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(x) = y \Leftrightarrow xEy \wedge \forall z[xEz \to y < z \vee y = z]$$

Definition 2.28. $\mathfrak{A}$ has **definable choice functions** if for any formula $\theta(\bar{x}, \bar{y})$ there is a definable function $f(\bar{y})$ s.t.

$$\forall \bar{y} \exists \bar{x} [\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(f(\bar{y}), \bar{y})]$$

(i.e., f is a skolem function for θ) and s.t.

$$\forall \bar{y} \forall \bar{z} [\forall \bar{x} (\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(\bar{x}, \bar{z})) \rightarrow f(\bar{y}) = f(\bar{z})]$$

Proof. If $\mathfrak{A}$ has definable choice functions then $\mathfrak{A}$ eliminates imaginaries □

Proof. Given a definable equivalence relation E on A^n let f be a definable choice function for $E(\bar{x}, \bar{y})$. Since E is an equivalence relation we have $\forall \bar{y} E(f(\bar{y}), \bar{y})$ and

$$\forall \bar{y} \bar{z} [\bar{y}/E = \bar{z}/E \rightarrow f(\bar{y}) = f(\bar{z})]$$

Thus $f(\bar{y}) = f(\bar{z}) \Leftrightarrow \bar{y} E \bar{z}$ □

Example 2.5. We now see that $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$ eliminates imaginaries. Basically since $\mathfrak{A}$ is well ordered, we can find a least element to witness membership of definable sets, hence we have definable functions

Example 2.6. $\mathfrak{A} = (\mathbb{N}, \equiv \mod 2)$ does not eliminate imaginaries

First note that the only definable subsets of $\mathbb{N}$ are $\emptyset, \mathbb{N}, 2\mathbb{N}, (2n+1)\mathbb{N}$. This is because $\mathfrak{A}$ has an automorphisms which switches $(2n+1)\mathbb{N}$ and $2\mathbb{N}$

Now suppose $f : \mathbb{N} \to \mathbb{N}^m$ eliminates the equivalence relation $\equiv \mod 2$, i.e.,

$$f(x) = f(y) \Leftrightarrow y \equiv 2 \mod 2$$

The $\mathrm{im}(f)$ is definable and has cardinality 2. Since there are no definable subsets of $\mathbb{N}$ of cardinality 2, we must have $m > 1$. Now let $\pi : \mathbb{N}^m \to \mathbb{N}$ be a projection. Then $\pi(\mathrm{im}(f))$ is a finite nonempty definable subset of $\mathbb{N}$. But no such set exists

Proposition 2.29. *If $\mathfrak{A}$ eliminates imaginaries, then $\mathfrak{A}_A$ eliminates imaginaries*

Proof. The idea is that an equivalence relation with parameters can be obtained as a fiber of an equivalence relation in more variables. Let $E \subseteq A^n$ be an equivalence relation definable in $\mathfrak{A}_A$. Let $\varphi(x, y; z) \in L$ and $a \in A^l$ be s.t.

$$xEy \Leftrightarrow \mathfrak{A} \vDash \varphi(x, y; a)$$

We now define

$$\psi(x,u,y,v) = \begin{cases} u = v \wedge \text{"}\varphi \text{ defines an equivalence relation"} & \text{or} \\ u \neq v & \text{or} \\ \text{"}\varphi(x,y,v) \text{ does not define an equivalence relation"} \end{cases}$$

Now ψ defines an equivalence relation on A^{n+l}. Let $f : A^{n+l} \to A^m$ eliminate ψ, then $f(-,a)$ eliminates E □

Back to [Pillay(2018)]

Definition 2.30. 1. T has elimination of imaginaries (EI) if for any model $M \vDash T$ and $e \in M^{\text{eq}}$, there is a $\bar{c} \in M$ s.t. $e \in \text{dcl}_{M^{\text{eq}}}(\bar{c})$ and $\bar{c} \in \text{dcl}_{M^{\text{eq}}}(e)$

2. T has weak elimination of imaginaries if, as above, except $\bar{c} \in \text{acl}_{M^{\text{eq}}}(e)$

3. T has geometric elimination of imaginaries if, as above, except $e \in \text{acl}_{M^{\text{eq}(\bar{c})}}$ and $\bar{c} \in \text{acl}_{M^{\text{eq}}}(e)$

Note that in particular, elimination of imaginaries imply the existence of codes for definable sets

Proposition 2.31. *TFAE*

1. *T has EI*

2. *For some model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation E, there is a partition of M^n into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

3. *For any model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation E, there is a partition of M^n into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

4. *For any model $M \vDash T$, and any definable $X \subseteq M^n$ there is an L-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in M$ s.t. X is defined by $\varphi(x, \bar{b})$ and for all $\bar{b}' \in M$ if X is defined by $\varphi(\bar{x}, \bar{b}')$ then $\bar{b} = \bar{b}'$. We call such a $\bar{b}$ a code for X*

1#+BEGIN$_{\text{proof}}$ $2 \Leftrightarrow 3$. Since we concern only $\emptyset$-definable relations and functions, if it is true in some model, then it is true in any model

$1 \to 2$. Let $\pi_E : S^n_= \to S_E$ the canonical definable quotient map. Let $e \in S_E$. By assumption, there is $k \in \mathbb{N}$ and $\bar{c} \in \mathfrak{C}^k$ s.t. e and $\bar{c}$ are interdefinable. In other words, there is a formula $\varphi_e(x, \bar{y})$ over $\emptyset$ s.t. $\varphi_e(e, \bar{c})$. Moreover, $|\varphi_e(\mathfrak{C}, \bar{c})| = |\varphi_e(e, \mathfrak{C})| = 1$

Let

$$
\begin{aligned}
X_e = \{\bar{x} \in \mathfrak{C}, \vDash \exists! \bar{y} \varphi_e(\pi_E(\bar{x}), \bar{y}) \\
\wedge \forall z (E(\bar{x}, \bar{z}) \leftrightarrow \\
(\forall y (\varphi_e(\pi_E(\bar{x}), \bar{y}) \leftrightarrow \varphi_e(\pi_E(\bar{z}), \bar{y}))) \}
\end{aligned}
$$

This means that φ_e defines a function on X_e, and that this function separates E-classes. We want to define $X_e = \pi^{-1}(e)$, but $\pi^{-1}(e)$ is not $\emptyset$-definable and so we want to simulate it.

$|\varphi_e(e, \mathfrak{C})| = 1$ is showed by $\exists! \bar{y} \varphi_e(\pi_E(\bar{x}), \bar{y})$.

Now $E(\bar{x}, \bar{z})$ iff $\pi_E(\bar{x})$

$\varphi_e(\pi_E(\bar{x}), \mathfrak{C}) = \varphi_e(\pi_E(\bar{z}),)$

Then $\pi^{-1}(\{a\}) \subset X_e$. Indeed, let $\bar{a} \in \pi^{-1}(\{a\})$, then $\bar{c}$ is the realization of $\varphi_e(\pi_E(\bar{a}), \bar{y})$, hence the first half of the conjunction is true

Conversely, suppose that $\vDash \forall \bar{y} (\varphi_e(\pi_E(\bar{a}), \bar{y}) \leftrightarrow \varphi_e(\pi_E(\bar{b}), \bar{y}))$ for some $\bar{b}$. By assumption we have $\varphi_e(\pi_E(\bar{a}), \bar{c})$, hence $\varphi_e(\pi_E(\bar{b}), \bar{c})$. But by definition of φ_e, this implies that $e = \pi_E(\bar{b})$ and by definition of π_E, this yields $E(\bar{a}, \bar{b})$

Since each X_e contains $\pi^{-1}(\{a\})$, we get $\mathfrak{C}^n = \bigcup_{e \in \pi_E(\mathfrak{C}^n)} X_e$, and by compactness, there are $e_1, \dots, e_l$ s.t. $\mathfrak{C}^n = \bigcup_{i=1}^{l} X_{e_i}$. Consider $f_i = \varphi_{e_i} \circ \pi_E$ on each X_{e_i}, #+END$_{\text{proof}}$

3 TODO Problems

2.1 2.3

4 Index

This is a functional link that will open a buffer of clickable index entries:

5 References

References

[Pillay(2018)] Anand Pillay. Topics in Stability Theory. https://www3.nd.edu/~apillay/Topics_in_Stability_Theory__No_Intro_.pdf, 2018.

[Chernikov(2019)] Artem Chernikov. Lecture notes on stability theory. https://www.ams.org/open-math-notes/omn-view-listing?listingId=110792, 2019.

[Tent and Ziegler(2012)] Katrin Tent and Martin Ziegler. *A course in model theory*. Number 40. Cambridge University Press, 2012.