Homework 12

Introduction to Model Theory

Due 2021-12-23

In problems (1)–(3), T is a complete theory in a countable language and $S_n(T)$ is the space of n-types. A *clopen* set is a set of the form

$$[\varphi] = \{p \in S_n(T) : \varphi \in p\}$$

An *open* set is a union (possibly an infinite union) of clopen sets.

1. Show that if $p \in S_n(T)$ then $S_n(T) \setminus \{p\}$ is open (a union of clopen sets).

2. Suppose $X \subseteq S_n(T)$ is open and the complement $S_n(T) \setminus X$ is also open. Show that X is clopen. *Hint*: use Lemma 5 in the notes.

3. Suppose I is a set and $U_i \subseteq S_n(T)$ is open for each $i \in I$. Suppose $\bigcup_{i \in I} U_i = S_n(T)$. Show that there is a finite set $I_0 \subseteq I$ such that $\bigcup_{i \in I_0} U_i = S_n(T)$. *Hint:* this looks like Lemma 5 in the notes, but now the sets in the cover are open rather than clopen. It may help to consider the family

   $$\{X \subseteq S_n(T) : X \text{ is clopen and there is } i \in I \text{ such that } X \subseteq U_i\}.$$

   (There are other ways to approach this problem.)

4. Let $S_3(\mathrm{DLO})$ be the space of 3-types in DLO (the theory of dense linear endpoints). What is the cardinality of $S_3(\mathrm{DLO})$?[1] *Hint:* use quantifier-elimination, or the characterization of ω-isomorphisms, in DLO.

5. Let K be an infinite field and $t \in K$ be a non-zero element. Suppose the type-space $S_1(\{t\})$ is finite. Show that there is a positive integer n such that $t^n = 1$. *Hint:* otherwise, the set $\{t, t^2, t^3, t^4, \ldots\}$ is infinite.

6. Let T be a complete theory of infinite fields. Show that T is not ω-categorical. *Hint:* use the previous problem, the Ryll-Nardzewski theorem, Lemma 13 in the notes, and compactness to show that models of T must be finite.

[1] We proved that DLO is ω-categorical on the first day of class. By the Ryll-Nardzewski Theorem, $S_n(\mathrm{DLO})$ is finite for all n.