Exercise

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Exercise 1. Suppose G is infinite planar

Proof. Let $\mathcal{L} = \{E, R, W, B, Y\}$,

$$\begin{aligned}\sigma = \forall x((&R(x) \wedge \neg W(x) \wedge \neg B(x) \wedge \neg Y(x))\vee \\ &(\neg R(x) \wedge W(x) \wedge \neg B(x) \wedge \neg Y(x))\vee \\ &(\neg R(x) \wedge \neg W(x) \wedge B(x) \wedge \neg Y(x))\vee \\ &\neg(R(x) \wedge \neg W(x) \wedge \neg B(x) \wedge Y(x)))\end{aligned}$$

$\sigma_R : \forall x, y(E(x,y) \to \neg(R(x) \wedge R(y)))$ and $\sigma_W, \sigma_B, \sigma_Y$ similarly.

$\mathrm{Diag}_{\mathrm{el}}(G) = \{\phi(a_1, \dots, a_n) \mid G \vDash \phi(a_1, \dots, a_n), a_i \in V, \phi \in L\}$

Let $L_V = L \cup V$

Let $\Sigma = \mathrm{Diag}(G) \cup \{\sigma, \sigma_R, \sigma_W, \sigma_B, \sigma_Y\}$. Σ is finitely satisfiable. For any finite $\Delta \subset \mathrm{Diag}(G)$, assume $a_1, \dots, a_m$ occurs in Δ, then the subgraph T of G with vertices $a_1, \dots, a_m$ s.t. Ea_ia_j in T iff Ea_ia_j in G is a model of Δ. As we can color T in 4 colors, Δ is satisfiable and thus Σ is satisfiable.

Thus Σ has a model G' with $f : G \xrightarrow{\prec} G'$ an elementary map. Prove

Let $f(a) = a^{G'}$. For any $a_1, a_2 \in G$

1. If a_1, a_2 are distinct elements of G, then $a_1 \neq a_2 \in \mathrm{Diag}_{\mathrm{el}}(G)$. Hence $f(a_1) \neq f(a_2)$
2. For any relation R, if $\bar{a} \in R^G$, then $R(\bar{a}) \in \mathrm{Diag}_{\mathrm{el}}(G)$, hence $f(\bar{a}) \in R^{G'}$

   If $\bar{a} \notin R^G$, then $\neg R(\bar{a}) \in \mathrm{Diag}_{\mathrm{el}}(G)$, hence $f(\bar{a}) \notin R^{G'}$

As G' has 4 color, so does G. $\square$