Cantor-Bendixson rank and Morley rank

Advanced model theory

May 5–7, 2022

Reference in the book: Section 17.3. (But the first two sections are copied from last class, and are based on Section 15.2.)

1 Independent sequences

Definition 1.1. A family $(A_i : i \in I)$ is *independent* over B if $A_i \mathop{\smile\!\!\!\!\!\!|}_B A_{\neq i}$ for each $i \in I$, where $A_{\neq i} = \{A_j : j \neq i\}$.

Example 1.2. A_1, A_2 are independent over B if $A_1 \mathop{\smile\!\!\!\!\!\!|}_B A_2$ and $A_2 \mathop{\smile\!\!\!\!\!\!|}_B A_1$. By symmetry, this just means $A_1 \mathop{\smile\!\!\!\!\!\!|}_B A_2$.

Fact 1.3. *In the theory of $\mathbb{R}$-vector spaces, if $v_1, \ldots, v_n$ are non-zero vectors, then $v_1, \ldots, v_n$ is independent over $\varnothing$ iff $v_1, \ldots, v_n$ are linearly independent (in the sense of linear algebra), meaning that*

$$x_1 v_1 + \cdots + x_n v_n \neq 0.$$

for non-zero $\bar{x} \in \mathbb{R}^n$.

Proposition 1.4. *$(A_i : i \in I)$ is independent over B iff $(A_i : i \in I_0)$ is independent over B for every finite $I_0 \subseteq I$.*

Proof. Monotonicity and finite character. □

Lemma 1.5. *Let $(A_i : i \leq \alpha)$ be a sequence. Suppose $(A_i : i < \alpha)$ is independent over B and $A_\alpha \mathop{\smile\!\!\!\!\!\!|}_B A_{<\alpha}$ where $A_{<\alpha} = \{A_i : i < \alpha\}$. Then $(A_i : i \leq \alpha)$ is independent over B.*

Proof. We must show $A_i \mathop{\smile\!\!\!\!\!\!|}_B \{A_j : j \leq \alpha,\ j \neq i\}$ for each $i \leq \alpha$. When $i = \alpha$, this is assumed. Suppose $i < \alpha$. Let $C_i = \{A_j : j < \alpha,\ j \neq i\}$. We want to show $A_i \mathop{\smile\!\!\!\!\!\!|}_B C_i A_\alpha$. We know $A_i \mathop{\smile\!\!\!\!\!\!|}_B C_i$ because $(A_i : i < \alpha)$ is independent. We know $A_\alpha \mathop{\smile\!\!\!\!\!\!|}_B A_i C_i$ by assumption. By base monotonicity, $A_\alpha \mathop{\smile\!\!\!\!\!\!|}_{BC_i} A_i$. By symmetry, $A_i \mathop{\smile\!\!\!\!\!\!|}_{BC_i} A_\alpha$. By transitivity, the known facts $A_i \mathop{\smile\!\!\!\!\!\!|}_B C_i$ and $A_i \mathop{\smile\!\!\!\!\!\!|}_{BC_i} A_\alpha$ imply $A_i \mathop{\smile\!\!\!\!\!\!|}_B C_i A_\alpha$, as desired. □

Proposition 1.6. *If $(A_i : i < \alpha)$ is a sequence and $A_i \mathop{\smile\!\!\!\!\!\!|}_B A_{<i}$ for each $i < \alpha$, where $A_{<i} = \{A_j : j < i\}$, then $(A_i : i < \alpha)$ is independent over B.*

Proof. By induction on α using Lemma 1.5. $\square$

Example 1.7. If $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots$ is a Morley sequence over B, then $\bar{a}_1, \bar{a}_2, \ldots$ is independent over B, because $\mathrm{tp}(\bar{a}_i/B\bar{a}_{<i})$ has a B-definable global extension, implying it doesn't fork over B, implying $\bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_B \bar{a}_{<i}$.

More generally, if $p_1, \ldots, p_n$ are B-invariant types and $(\bar{a}_1, \ldots, \bar{a}_n)$ realizes $(p_1 \otimes \cdots \otimes p_n) \upharpoonright B$, then $\bar{a}_1, \ldots, \bar{a}_n$ is independent over B.

2 Bases in strongly minimal theories

Suppose T is strongly minimal for this section. If $A \subseteq \mathbb{M}$, then there is a unique type $p \in S_1(A)$ such that $b \models p \iff b \notin \mathrm{acl}(A)$. If such a type did not exist, there would be two $b, b' \notin \mathrm{acl}(A)$ with $\mathrm{tp}(b/A) \neq \mathrm{tp}(b'/A)$. Then there is an A-definable set $D \subseteq \mathbb{M}$ with $b \in D$, $b' \in \mathbb{M} \setminus D$. By strong minimality, D or $\mathbb{M} \setminus D$ is finite. Then b or b' is algebraic.

This type is called the *transcendental type* over A. If $p \in S_1(\mathbb{M})$ is the global transcendental type, then $p \upharpoonright A$ is the transcendental type over A, because if $N \succeq \mathbb{M}$ and $b \in N \setminus \mathrm{acl}(\mathbb{M})$, then $b \notin \mathrm{acl}(A)$ and so $\mathrm{tp}(b/\mathbb{M}) \upharpoonright A = \mathrm{tp}(b/A)$ is the transcendental type over A.

The transcendental type over A has a global $\varnothing$-definable extension, so it is stationary (Lemma 6.1 in the April 21–28 notes). Say that $b \in \mathbb{M}$ is *transcendental* if $b \notin \mathrm{acl}(\varnothing)$.

Lemma 2.1. *If b is transcendental, then $b \mathop{\smile\!\!\!\!\!\mid}_\varnothing C$ iff $b \notin \mathrm{acl}(C)$.*

Proof. Let p be the global transcendental type. By Lemma 6.3 in the April 21–28 notes, $\mathrm{tp}(b/C) \sqsupseteq \mathrm{tp}(b/\varnothing)$ holds iff $\mathrm{tp}(b/C) = p \upharpoonright C$, which just means that $b \notin \mathrm{acl}(C)$. $\square$

Proposition 2.2. *A sequence of transcendentals $(b_i : i < \alpha)$ is independent over $\varnothing$ iff $b_i \notin \mathrm{acl}(\{b_j : j < i\})$ for each i.*

Proof. Proposition 1.6. $\square$

Remark 2.3. Let p be the global transcendental type. A finite sequence of transcendentals $(b_1, \ldots, b_n)$ is independent iff $b_i \notin \mathrm{acl}(b_1, \ldots, b_{i-1})$ for each i, iff $b_i \models p \upharpoonright b_1 b_2 \cdots b_{i-1}$ iff $(b_1, \ldots, b_n) \models p^{\otimes n} \upharpoonright \varnothing$. Independent sequences of transcendentals are just Morley sequences of p.

Lemma 2.4. *Let I_1, I_2 be two independent sets. Let $f : I_1 \to I_2$ be a bijection. Then f is a partial elementary map.*

Proof. Suppose $b_1, \ldots, b_n \in I_1$ map to $c_1, \ldots, c_n \in I_2$. Then $\mathrm{tp}(b_1, \ldots, b_n) = p^{\otimes n} \upharpoonright \varnothing = \mathrm{tp}(c_1, \ldots, c_n)$. $\square$

Definition 2.5. Suppose $M \preceq \mathbb{M}$. A *basis* of M is a maximal independent set of transcendentals in M.

Every $M \preceq \mathbb{M}$ has a basis by Zorn's lemma and Proposition 1.4.

Proposition 2.6. *Let B be a basis of $M \preceq \mathbb{M}$. Then $M = \operatorname{acl}(B)$.*

Proof. Otherwise, take $c \in M \setminus \operatorname{acl}(B)$. Then $c \mathop{\smile\!\!\!\!|}_{\varnothing} B$ by Lemma 2.1, so Lemma 1.5 shows $B \cup \{c\}$ is independent, contradicting maximality. $\square$

Theorem 2.7. *The strongly minimal theory T is κ-categorical for all $\kappa > |L|$.*

Proof. Suppose $M_1, M_2 \preceq \mathbb{M}$ have $|M_1| = |M_2| = \kappa > |L|$. Take a basis $B_i \subseteq M_i$ for $i = 1, 2$. Then $|B_i| \leq |M_i| = \kappa$. If $|B_i| < \kappa$, then $|M_i| = |\operatorname{acl}(B_i)| \leq |B_i| + |L| < \kappa$, a contradiction. Therefore $|B_1| = |B_2| = \kappa$. Take a bijection $f : B_1 \to B_2$. By Lemma 2.4, f is a partial elementary map, and so it extends to an automorphism $\sigma \in \operatorname{Aut}(\mathbb{M})$. Then $\sigma(M_1) = \sigma(\operatorname{acl}(B_1)) = \operatorname{acl}(\sigma(B_1)) = \operatorname{acl}(B_2) = M_2$. Therefore $M_1 \cong M_2$. $\square$

Later, we will use *ranks* to see the following:

- If B, B' are bases of a model M, then $|B| = |B'|$. The *dimension* of M is defined to be $|B|$ for any basis M.
- M is determined up to isomorphism by its dimension.

(The argument above handles the case where the dimension is $> |L|$.)

3 More topology on type spaces

Note: (This section builds off `notes-13.pdf` and `homework-12.pdf` from *last semester's* lesson on the omitting types theorem, around December 16, 2021.)

Fix a set of parameters A and consider the type space $S_n(A)$. If $\varphi(x_1, \ldots, x_n) \in L(A)$, then $[\varphi] := \{p \in S_n(A) : p(\bar{x}) \vdash \varphi(\bar{x})\}$. If $D \subseteq \mathbb{M}^n$ is A-definable, then $[D] = [\varphi]$ for φ defining D. Sets of the form $[\varphi]$ are called *clopen sets*. Note

$$[D_1 \cap D_2] = [D_1] \cap [D_2]$$
$$[D_1 \cup D_2] = [D_1] \cup [D_2]$$
$$[\mathbb{M}^n \setminus D] = S_n(A) \setminus [D_1],$$

so $D \mapsto [D]$ is an isomorphism of boolean algebras from A-definable subsets of $\mathbb{M}^n$ to clopen sets in $S_n(A)$.

Fact 3.1 (Compactness)**.** *Let $\mathcal{F}$ be a collection of clopen sets in $S_n(A)$. If $\mathcal{F}$ has the finite intersection property, then $\bigcap \mathcal{F} \neq \varnothing$.*

Fact 3.2 (Compactness, dual form)**.** *Let $\mathcal{F}$ be a collection of clopen sets in $S_n(A)$. If $\bigcup \mathcal{F} = S_n(A)$, then there is finite $\mathcal{F}_0 \subseteq \mathcal{F}$ with $\bigcup \mathcal{F}_0 = S_n(A)$.*

Facts 3.1, 3.2 are Lemmas 4, 5 in last semester's `notes-13.pdf`.

Definition 3.3. $X \subseteq S_n(A)$ is *open* (resp. *closed*) if X is a possibly-infinite union (resp. intersection) of clopen sets.

Remark 3.4. If $\Sigma(\bar{x})$ is a set of $L(A)$-formulas in the variables $\bar{x}$, then

$$\bigcap_{\varphi\in\Sigma}[\varphi] = \{p \in S_n(A) : \forall\varphi \in \Sigma\ (p(\bar{x}) \vdash \varphi(\bar{x}))\} = \{p \in S_n(A) : p \supseteq \Sigma\} = \{\mathrm{tp}(\bar{b}/A) : \bar{b} \models \Sigma\}.$$

Thus, closed sets in $S_n(A)$ correspond to partial types over A.

The following three facts are (essentially[1]) Problems 1–3 on last semester's `homework-12.pdf`.

Fact 3.5. *Singletons $\{p\}$ are closed.*

Fact 3.6. *X is clopen iff X is closed and open.*

Fact 3.7. *Let $\mathcal{F}$ be a collection of closed sets in $S_n(A)$. If $\mathcal{F}$ has the FIP, then $\bigcap\mathcal{F} \neq \varnothing$.*

Corollary 3.8. *Let $\mathcal{F}$ be a non-empty collection of non-empty closed sets in $S_n(A)$. Suppose one of the following holds:*

1. *$\mathcal{F}$ is linearly ordered: $\forall X, Y \in \mathcal{F}$, $X \subseteq Y$ or $Y \subseteq X$.*
2. *$\mathcal{F}$ is filtered: $\forall X, Y \in \mathcal{F}\ \exists Z \in \mathcal{F}$ such that $Z \subseteq X \cap Y$.*

Then $\mathcal{F}$ has the FIP, so $\bigcap\mathcal{F} \neq \varnothing$.

Proposition 3.9 (Total separation)**.** *If $p, q \in S_n(A)$ and $p \neq q$, then there is a clopen set $U \subseteq S_n(A)$ with $p \in U$ and $q \notin U$.*

Proof. Take $\varphi \in p$ with $\varphi \notin q$. Then $p \in [\varphi]$ but $q \notin [\varphi]$. □

Proposition 3.10. *Suppose $C \subseteq S_n(A)$ is closed and $p \in C$. Then one of two things happens:*

1. *There is a clopen set $U \ni p$ such that $U \cap C = \{p\}$.*
2. *For every clopen set $U \ni p$, the intersection $U \cap C$ is infinite.*

Proof. Suppose $U \cap C$ is finite for some clopen set U. Let $\{p, q_1, \ldots, q_m\}$ enumerate $U \cap C$. By total separation, there are clopen sets V_i with $p \in V_i$, $q_i \notin V_i$. Let U' be the clopen set $U \cap \bigcap_{i=1}^n V_i$. Then $p \in U'$ but $q_i \notin U'$, so $U' \cap C = \{p\}$. □

Definition 3.11. Suppose C is closed and $p \in C$.

1. p is an *isolated point* of C if case (1) holds in Proposition 3.10.
2. p is an *accumulation point* of C if case (2) holds in Proposition 3.10.

[1]Fact 3.7 is dual to Problem 3 on `homework-12.pdf`. To prove Fact 3.7 from Problem 3, consider the family of complements $\mathcal{F}' = \{S_n(A) \setminus C : C \in \mathcal{F}\}$. If $\bigcap\mathcal{F} = \varnothing$, then $\mathcal{F}'$ is a cover of $S_n(A)$, and Problem 3 gives a finite subcover, which implies that $\mathcal{F}$ *doesn't* have the FIP.

3. The *derived set* C' is the set of accumulation points.

Proposition 3.12. *If C is a closed set, then C' is a closed set.*

Proof. For each isolated point $p \in C$, take a clopen set U_p with $U_p \cap C = \{p\}$. Then C' is the complement of the open set $(S_n(A) \setminus C) \cup \bigcup_{p \in C \setminus C'} U_p$. $\square$

Proposition 3.13. *If C is an infinite closed set, then $C' \neq \varnothing$.*

Proof. Otherwise, for any $p \in S_n(A)$ there is a clopen set $U_p \ni p$ such that $U_p \cap C'$ is finite. (If $p \notin C$, then there is clopen U_p with $p \in U_p \subseteq S_n(A) \setminus C$, meaning $U_p \cap C = \varnothing$. If $p \in C$, then U_p exists by definition of "isolated point.") By Fact 3.2, $S_n(A)$ is a finite union $\bigcup_{i=1}^m U_{p_i}$ for some $p_1, \ldots, p_m \in S_n(A)$. Then $C = C \cap S_n(A) = \bigcup_{i=1}^m (C \cap U_{p_i})$ is finite. $\square$

Definition 3.14. A *perfect set* is a closed set $C \subseteq S_n(A)$ with $C = C'$. $S_n(A)$ is *scattered* if there are no non-empty perfect subsets of $S_n(A)$.

4 Cantor-Bendixson rank

Consider $S_n(A)$ for some $n < \omega$, $A \subseteq \mathbb{M}$. Define a descending chain of closed sets $E_\alpha \subseteq S_n(A)$ for ordinals α as follows:

- $E_0 = S_n(A)$.
- $E_{\alpha+1}$ is the derived set E'_α.
- If α is a limit ordinal, $E_\alpha = \bigcap_{\beta < \alpha} E_\beta$.

The chain can only decrease $2^{|S_n(A)|}$ times, so it stops decreasing at some point. Let E_∞ be E_α for sufficiently large α. Note E_∞ is perfect (but possibly empty).

Remark 4.1. "∞" is a formal symbol greater than all ordinals. Be careful: $\alpha < \infty$ means "α is an ordinal", not "α is finite".

Definition 4.2. The *Cantor-Bendixson rank* $\mathrm{R}(p)$ of $p \in S_n(A)$ is the maximum α such that $p \in E_\alpha$. If $C \subseteq S_n(A)$ is closed set, then the Cantor-Bendixson rank $\mathrm{R}(C)$ is the maximum α such that $C \cap E_\alpha \neq \varnothing$.[2]

The maximum exists because if α is a limit ordinal and $E_\beta \cap C \neq \varnothing$ for all $\beta < \alpha$, then $E_\alpha \cap C = \bigcap_{\beta < \alpha}(E_\beta \cap C) \neq \varnothing$ by Corollary 3.8. Note $E_\alpha = \{p \in S_n(A) : \mathrm{R}(p) \geq \alpha\}$.

Remark 4.3. $\mathrm{R}(p)$ is characterized by the fact that

$\mathrm{R}(p) \geq \alpha + 1$ iff p is an accumulation point of types with rank $\geq \alpha$.

Remark 4.4. $\mathrm{R}(C) = \max\{\mathrm{R}(p) : p \in C\}$.

[2] If $C = \varnothing$, we set $\mathrm{R}(C) = -\infty$, where $-\infty$ is a formal symbol less than all ordinals.

Definition 4.5. Then Cantor-Bendixson rank of a formula or partial type over A is the Cantor-Bendixson rank of the associated closed set in $S_n(A)$. The Cantor-Bendixson rank of an A-definable or A-type-definable set is the Cantor-Bendixson rank of the formula or partial type that defines it.

Proposition 4.6. *Suppose $C_1, C_2 \subseteq S_n(A)$ are closed.*

1. *If $C_1 \subseteq C_2$, then* $\mathrm{R}(C_1) \leq \mathrm{R}(C_2)$.
2. $\mathrm{R}(C_1 \cup C_2) = \max(\mathrm{R}(C_1), \mathrm{R}(C_2))$.
3. *If $\Sigma(\bar{x}), \Phi(\bar{x})$ are partial types and $\Sigma(\bar{x}) \vdash \Phi(\bar{x})$, then* $\mathrm{R}(\Sigma) \leq \mathrm{R}(\Phi)$.
4. $\mathrm{R}(\varphi \vee \psi) = \max(\mathrm{R}(\varphi), \mathrm{R}(\psi))$.

Proof. (1) and (2) are clear from Remark 4.4. (3) and (4) are direct consequences of (1) and (2). ☐

Proposition 4.7. 1. *If $C \subseteq S_n(A)$ is closed, then* $\mathrm{R}(C) = \min\{\mathrm{R}(U) : U \supseteq C,\ U \text{ is clopen}\}$.

2. *If $\Sigma(\bar{x})$ is a set of $L(A)$-formulas, then* $\mathrm{R}(\Sigma) = \min\{\mathrm{R}(\varphi) : \Sigma(\bar{x}) \vdash \varphi(\bar{x})\}$.

Proof. Note (1) $\iff$ (2). We prove (1). Certainly $\mathrm{R}(C)$ is $\leq$ the minimum. We must show that $\mathrm{R}(C)$ is $\geq$ the minimum. If $\mathrm{R}(C) = \infty$ there is nothing to show. Suppose $\mathrm{R}(C) = \alpha < \infty$. Then $C \cap E_{\alpha+1} = \varnothing$. As C is a filtered intersection $\bigcap\{U : U \supseteq C,\ U \text{ is clopen}\}$, we have

$$\varnothing = C \cap E_{\alpha+1} = \bigcap\{U \cap E_{\alpha+1} : U \supseteq C,\ U \text{ is clopen}\}$$

By Corollary 3.8, some $U \cap E_{\alpha+1} = \varnothing$. Then $\alpha+1 > \mathrm{R}(U) \geq \mathrm{R}(C) = \alpha$, so $\mathrm{R}(U) = \mathrm{R}(C)$. ☐

Proposition 4.8. *If $C \subseteq S_n(A)$ is closed and $-\infty < \mathrm{R}(C) < \infty$, then there are finitely many $p \in C$ with $\mathrm{R}(p) = \mathrm{R}(C)$.*

Proof. Otherwise, $\{p \in C : \mathrm{R}(p) = \mathrm{R}(C)\} = C \cap E_{\mathrm{R}(C)}$ would have an accumulation point q by Proposition 3.13, and then $q \in C$, $q \in E_{\mathrm{R}(C)+1}$, and $\mathrm{R}(C) \geq \mathrm{R}(q) \geq \mathrm{R}(C)+1$, absurd. ☐

Lemma 4.9. *Suppose $U \subseteq S_n(A)$ is clopen and $\mathrm{R}(U) \geq \alpha + 1$. Then U is a disjoint union $U_1 \sqcup U_2$ of two clopen sets U_1, U_2 such that $\mathrm{R}(U_1) \geq \alpha + 1$ and $\mathrm{R}(U_2) \geq \alpha$.*

Proof. Take $p \in U \cap E_{\alpha+1}$. Then p is an accumulation point of E_α so $U \cap E_\alpha$ is infinite. Take $q \in U \cap E_\alpha$ with $q \neq p$. By Proposition 3.9 there is a clopen set V with $p \in V$, $q \notin V$. Let $U_1 = U \cap V$ and $U_2 = U \setminus V$. Then $p \in U_1$ and $q \in U_2$, so $\mathrm{R}(U_1) \geq \mathrm{R}(p) \geq \alpha + 1$ and $\mathrm{R}(U_2) \geq \mathrm{R}(q) \geq \alpha$. ☐

If X, Y are sets, let $X \Delta Y$ denote $(X \setminus Y) \cup (Y \setminus X)$.

Proposition 4.10. *If $U \subseteq S_n(A)$ is clopen and α is an ordinal, the following are equivalent:*

1. $\mathrm{R}(U) \geq \alpha + 1$.

2. *There are pairwise disjoint clopen sets* $U_1, U_2, \ldots \subseteq U$ *with* $\mathrm{R}(U_i) \geq \alpha$ *for each* α.

3. *There are clopen sets* $U_1, U_2, \ldots \subseteq U$ *with* $\mathrm{R}(U_i \Delta U_j) \geq \alpha$ *for* $i \neq j$.

Proof. (1) $\implies$ (2): by Lemma 4.9 we can find a clopen set $U_1 \subseteq U$ with $\mathrm{R}(U_1) \geq \alpha$ and $\mathrm{R}(U \setminus U_1) \geq \alpha + 1$. Applying Lemma 4.9 to $U \setminus U_1$ we can find a clopen set $U_2 \subseteq U \setminus U_1$ with $\mathrm{R}(U_2) \geq \alpha$ and $\mathrm{R}(U \setminus (U_1 \cup U_2)) \geq \alpha + 1$, etc.

(2) $\implies$ (3): if the U_i are disjoint, then $\mathrm{R}(U_i \Delta U_j) = \mathrm{R}(U_i \cup U_j) \geq \mathrm{R}(U_i) \geq \alpha$.

(3) $\implies$ (1): suppose (3) holds but $\mathrm{R}(U) \leq \alpha$. Then $U \cap E_\alpha$ is finite, possibly empty. The map

$$\begin{aligned} \omega &\to Pow(U \cap E_\alpha) \\ i &\mapsto U_i \cap (U \cap E_\alpha) = U_i \cap E_\alpha \end{aligned}$$

is not injective (since the range is finite), so there are $i \neq j$ with

$$U_i \cap E_\alpha = U_j \cap E_\alpha$$

or equivalently, $(U_i \Delta U_j) \cap E_\alpha = \varnothing$. Then $\mathrm{R}(U_i \Delta U_j) < \alpha$, a contradiction. □

Proposition 4.10(2) gives an alternate definition of $\mathrm{R}(U)$ for clopen U, and then Proposition 4.7 determines $\mathrm{R}(C)$ for closed C (including points). Rephrasing in terms of A-definable sets, we get the following definition. Suppose D is an A-definable set.

- $\mathrm{R}(D) \geq 0$ iff D is non-empty.
- If α is a limit ordinal, then $\mathrm{R}(D) \geq \alpha$ iff $\mathrm{R}(D) \geq \beta$ for all $\beta \leq \alpha$.
- $\mathrm{R}(D) \geq \alpha + 1$ iff there are pairwise disjoint A-definable subsets $D_1, D_2, \ldots \subseteq D$ with $\mathrm{R}(D_i) \geq \alpha$ for all i.

If φ is an $L(A)$-formula, then $\mathrm{R}(\varphi) = \mathrm{R}(\varphi(\mathbb{M}^n))$, and if Σ is a type over A then $\mathrm{R}(\Sigma) = \min\{\mathrm{R}(\varphi) : \Sigma \vdash \varphi\}$ as in Proposition 4.7.

Definition 4.11. A family $\mathcal{F}$ of sets is *k-inconsistent* if $\bigcap_{i=1}^k D_i = \varnothing$ for any distinct $D_1, \ldots, D_k \in \mathcal{F}$.

Lemma 4.12. *Let D be A-definable. Let $D_1, D_2, \ldots$ be distinct A-definable subsets of D, with $\mathrm{R}(D_i) \geq \alpha$. Suppose $\{D_1, D_2, \ldots\}$ is k-inconsistent. Then $\mathrm{R}(D) \geq \alpha + 1$.*

Proof. Suppose not. Let $U_i \subseteq S_n(A)$ correspond to D_i. As in the proof of Proposition 4.10, the map

$$\begin{aligned} \omega &\to Pow(U \cap E_\alpha) \\ i &\mapsto U_i \cap (U \cap E_\alpha) = U_i \cap E_\alpha \end{aligned}$$

is non-injective, and in fact some fiber is infinite. Passing to a subsequence, we may assume $U_i \cap E_\alpha = U_j \cap E_\alpha$ for all i, j. Then $\bigcap_{i=1}^k U_i \supseteq \bigcap_{i=1}^k (U_i \cap E_\alpha) = U_1 \cap E_\alpha \neq \varnothing$, as $\mathrm{R}(U_1) \geq \alpha$. So the family is not actually k-inconsistent. □

5 Whether $S_n(A)$ is scattered

Proposition 5.1. $\mathrm{R}(S_n(A)) < \infty$ *iff* $S_n(A)$ *is scattered (i.e., has no non-empty perfect set).*

Proof. $\mathrm{R}(S_n(A)) < \infty$ means $E_\infty = \varnothing$. If $E_\infty \neq \varnothing$ then E_∞ is a non-empty perfect set. Conversely, suppose $K \subseteq S_n(A)$ is non-empty and perfect. Then $K \subseteq E_\alpha$ for all α by induction on α, so $K \subseteq E_\infty$ and $E_\infty \neq \varnothing$. □

Lemma 5.2. *If* $U \subseteq S_n(A)$ *is clopen and* $\mathrm{R}(U) = \infty$*, then* U *is a disjoint union* $U = U_1 \sqcup U_2$ *of two clopen sets* U_1, U_2 *with* $\mathrm{R}(U_1) = \mathrm{R}(U_2) = \infty$.

Proof. Like Lemma 4.9. □

Lemma 5.3. *1.* $\mathrm{R}(S_n(A)) = \infty$ *if and only if there is a tree of non-empty clopen sets* $(U_\sigma : \sigma \in 2^{<\omega})$ *such that* U_σ *is a disjoint union* $U_{\sigma 0} \sqcup U_{\sigma 1}$ *for each* $\sigma \in 2^{<\omega}$.

2. If $\mathrm{R}(S_n(A)) = \infty$*, then* $|S_n(A)| \geq 2^{\aleph_0}$.

3. If $\mathrm{R}(S_n(A)) < \infty$*, then* $|S_n(A)| \leq |L(A)|$.

Proof. 1. Suppose $\mathrm{R}(S_n(A)) = \infty$. Take $U = S_n(A)$. By Lemma 5.2, $U = U_0 \sqcup U_1$ where U_0, U_1 are clopen sets with $\mathrm{R}(U_0) = \mathrm{R}(U_1) = \infty$. By Lemma 5.2 we can split U_0 into two clopen sets U_{00}, U_{01}, we can split U_1 into two clopen sets U_{10}, U_{11}, etc.

Conversely, suppose a tree (U_σ) exists. We claim by induction on α that $\mathrm{R}(U_\sigma) \geq \alpha$ for all σ. The base case $\alpha = 0$ holds as the $U_\sigma \neq \varnothing$. The limit ordinal case is immediate. For successor ordinals $\alpha + 1$, suppose $\mathrm{R}(U_\sigma) \geq \alpha$ for all σ. Then $U_{\sigma 1}, U_{\sigma 01}, U_{\sigma 001}, \ldots$ are disjoint clopen subsets of U_σ with rank $\geq \alpha$, so $\mathrm{R}(U_\sigma) \geq \alpha + 1$.

Therefore $\mathrm{R}(U_\sigma) = \infty$ for all σ, which implies $\mathrm{R}(S_n(A)) = \infty$.

2. Take a tree $(U_\sigma : \sigma \in 2^{<\omega}\}$ as in (1). For $\tau \in 2^\omega$ let $C_\tau = \bigcap_{n=0}^\infty U_{\tau \restriction n}$. Then C_τ is a non-empty closed set by Corollary 3.8. Take $p_\tau \in C_\tau$. The p_τ are distinct, since if $\tau \neq \tau'$ then $\tau \restriction n \neq \tau' \restriction n$ for some $n < \omega$, and then $p_\tau \in U_{\tau \restriction n}$, $p_{\tau'} \in U_{\tau' \restriction n}$, $U_{\tau \restriction n} \cap U_{\tau' \restriction n} = \varnothing$. So we get at least $2^{\aleph_0}$-many points.

3. Suppose $\mathrm{R}(S_n(A)) < \infty$. For each $p \in S_n(A)$, let $\alpha_p = \mathrm{R}(p)$. Then p is isolated in E_{α_p}, so there is a clopen set $U_p \ni p$ with $U_p \cap E_{\alpha_p} = \{p\}$. Then p is the unique type in U_p of maximal rank. Therefore $p \mapsto U_p$ is injective. But $|\{U_p : p \in S_n(A)\}| \leq |L(A)|$ since each U_p is defined by an $L(A)$-formula. □

Remark 5.4. The tree in Lemma 5.3(1) corresponds to a tree of non-empty A-definable sets $(D_\sigma : \sigma \in 2^{<\omega})$ with $D_\sigma \subseteq \mathbb{M}^n$, such that D_σ is a disjoint union $D_{\sigma 0} \sqcup D_{\sigma 1}$.

Theorem 5.5. *If* L *is countable and* T *is stable, then* $\lambda_0(T)$ *(the smallest* λ *such that* T *is* λ*-stable) is either* $\aleph_0$ *or* $2^{\aleph_0}$.

Proof. By the proof of Lemma 1 in the March 10 notes, T is λ-stable for $\lambda = 2^{|L|} = 2^{\aleph_0}$. So $\lambda_0(T) \leq 2^{\aleph_0}$. Suppose $\aleph_0 < \lambda_0 < 2^{\aleph_0}$. Then T is not $\aleph_0$-stable. Take countable $A \subseteq \mathbb{M}$ with $|S_1(A)| > \aleph_0$. By Lemma 5.3(2,3), either $|S_1(A)| \leq |L(A)| = \aleph_0$ (no) or $|S_1(A)| \geq 2^{\aleph_0}$ (yes). Then $|A| \leq \aleph_0 \leq \lambda_0$ but $|S_1(A)| \geq 2^{\aleph_0} > \lambda_0$, so λ_0-stability fails, a contradiction. □

6 Morley rank

Definition 6.1. *Morley rank* $\mathrm{RM}(-)$ means Cantor-Bendixson rank in $S_n(\mathbb{M})$.

The choice of the monster model doesn't matter:

Lemma 6.2. *Let $M \preceq \mathbb{M}$ be $\aleph_0$-saturated. Let $\Sigma(\bar{x})$ be a partial type over M. Then $\mathrm{RM}(\Sigma(\bar{x}))$ agrees with the Cantor-Bendixson rank of $\Sigma(\bar{x})$ in $S_n(M)$.*

Proof. Let $\mathrm{R}(-)$ denote Cantor-Bendixson rank in $S_n(M)$.
Claim. If D is M-definable, and α is an ordinal, then $\mathrm{R}(D) \geq \alpha \iff \mathrm{RM}(D) \geq \alpha$.

Proof. By induction on α. When $\alpha = 0$, both sides say $D \neq \varnothing$. The case of limit ordinals is easy. Suppose $\alpha = \beta + 1$. By Proposition 4.10(2), the left and right sides say

1. There are disjoint M-definable sets $D_1, D_2, \ldots \subseteq D$ with $\mathrm{R}(D_i) \geq \beta$.

2. There are disjoint $\mathbb{M}$-definable sets $D_1, D_2, \ldots \subseteq D$ with $\mathrm{RM}(D_i) \geq \beta$.

Then (1) $\implies$ (2) by induction. Suppose (2) holds. Let $D = \psi(\mathbb{M}, \bar{b})$, with $\bar{b} \in M$, and let $D_i = \varphi_i(\mathbb{M}, \bar{c}_i)$ for $i < \omega$, with $\bar{c}_i \in \mathbb{M}$. By $\aleph_0$-saturation we can realize $\mathrm{tp}(\bar{c}_0, \bar{c}_1, \ldots / \bar{b})$ in M. Moving the $\bar{c}_i$ and D_i by $\sigma \in \mathrm{Aut}(\mathbb{M}/\bar{b})$, we may assume the $\bar{c}_i$ are in M. Then the D_i are M-definable, and (2) $\implies$ (1) by induction. $\square_{\text{Claim}}$

The Claim shows $\mathrm{R}(D) = \mathrm{RM}(D)$ for M-definable D, i.e., $\mathrm{R}(\varphi) = \mathrm{RM}(\varphi)$ for $\varphi(\bar{x}) \in L(M)$. By Proposition 4.7(2), we are done. $\square$

Example 6.3. Let D be definable.

1. $\mathrm{RM}(D) \geq 0$ iff $D \neq \varnothing$.

2. $\mathrm{RM}(D) \geq 1$ iff there are disjoint non-empty definable sets $D_1, D_2, \ldots \subseteq D$. So $\mathrm{RM}(D) \geq 1$ iff D is infinite.

3. $\mathrm{RM}(D) \geq 2$ iff there are disjoint infinite definable sets $D_1, D_2, \ldots \subseteq D$.

4. If T is strongly minimal, then $\mathrm{RM}(\mathbb{M}) = 1$, because there are not disjoint infinite definable sets $D_1, D_2 \subseteq \mathbb{M}$.

Proposition 6.4. *Let $f : X \to Y$ be a definable function.*

1. *If f is surjective, then* $\mathrm{RM}(X) \geq \mathrm{RM}(Y)$.

2. *If f is a bijection, then* $\mathrm{RM}(X) = \mathrm{RM}(Y)$.

3. *If f is an injection, then* $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.

Proof. 1. We prove by induction on α that if $f : X \to Y$ is a surjection and $\mathrm{RM}(Y) \geq \alpha$, then $\mathrm{RM}(X) \geq \alpha$. When $\alpha = 0$, this says $Y \neq \varnothing \implies X \neq \varnothing$, which is true by surjectivity. Limit ordinals are trivial. Suppose $\mathrm{RM}(Y) \geq \alpha+1$. Take disjoint definable $D_1, D_2, \ldots \subseteq Y$ with $\mathrm{RM}(D_i) \geq \alpha$. Let $D_i' = f^{-1}(D_i) \subseteq X$. Then $f : D_i' \to D_i$ is surjective, so $\mathrm{RM}(D_i') \geq \alpha$ by induction. The D_i' are disjoint, so $\mathrm{RM}(X) \geq \alpha+1$. This proves the claim, then shows $\mathrm{RM}(X) \geq \mathrm{RM}(Y)$ (taking $\alpha = \mathrm{RM}(Y)$).

2. Apply (1) to f and f^{-1}.

3. f is a bijection from X to the image $f(X)$, so $\mathrm{RM}(X) = \mathrm{RM}(f(X)) \leq \mathrm{RM}(Y)$. □

Example 6.5. Suppose $\mathbb{M}$ is infinite. Take $a_1, a_2, \ldots \in \mathbb{M}$ distinct. By Proposition 6.4, $\mathrm{RM}(\mathbb{M}^n) = \mathrm{RM}(\mathbb{M}^n \times \{a_i\})$ because of the definable bijection. The sets $\mathbb{M}^n \times \{a_1\}, \mathbb{M}^n \times \{a_2\}, \ldots$ are disjoint definable subsets of $\mathbb{M}^{n+1}$. Therefore $\mathrm{RM}(\mathbb{M}^{n+1}) \geq \mathrm{RM}(\mathbb{M}^n) + 1$. By induction on n, $\mathrm{RM}(\mathbb{M}^n) \geq n$.

Lemma 6.6. *Let $f : X \to Y$ be a definable function with finite fibers. Then* $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.

Proof. We prove $\alpha \leq \mathrm{RM}(X) \implies \alpha \leq \mathrm{RM}(Y)$ by induction on α. The zero and limit cases are easy. Suppose $\mathrm{RM}(X) \geq \alpha + 1$. Take disjoint definable sets $D_1, D_2, \ldots \subseteq X$ with $\mathrm{RM}(D_i) \geq \alpha$. The map $D_i \to f(D_i)$ has finite fibers. By induction, $\mathrm{RM}(D_i) \geq \alpha \implies \mathrm{RM}(f(D_i)) \geq \alpha$. Let k bound the fibers of f. Then $\{f(D_1), f(D_2), \ldots\}$ is $(k+1)$-inconsistent. By Lemma 4.12, $\mathrm{RM}(Y) \geq \alpha + 1$. □

Example 6.7. In ACF, the set $D = \{(x, y) : x^2 + y^2 = 1\}$ has Morley rank 1, because it is infinite and the map

$$\begin{aligned} D &\to \mathbb{M}^1 \\ (x, y) &\mapsto x \end{aligned}$$

has finite fibers.

Definition 6.8. For $\bar{a} \in \mathbb{M}^n$ and $B \subseteq \mathbb{M}$, $\mathrm{RM}(\bar{a}/B) = \mathrm{RM}(\mathrm{tp}(\bar{a}/B))$.

Proposition 6.9. *Fix $B \subseteq \mathbb{M}$.*

1. *If $\bar{a} \in \mathbb{M}^n$, then* $\mathrm{RM}(\bar{a}/B) = \min\{\mathrm{RM}(X) : \bar{a} \in X,\ X \text{ is } B\text{-definable}\}$.

2. *If X is B-definable, then* $\mathrm{RM}(X) = \max\{\mathrm{RM}(\bar{a}/B) : \bar{a} \in X\}$.

Proof. 1. Proposition 4.7.[3]

[3] Actually we're using a slightly stronger version of Proposition 4.7: if $C \subseteq S_n(A)$ is a filtered intersection $\bigcup_{i \in I} U_i$ of clopen sets U_i, then $\mathrm{R}(C) = \min_{i \in I} \mathrm{R}(U_i)$.

2. If $\bar{a} \in X$ then $[\mathrm{tp}(\bar{a}/B)] \subseteq [X] \subseteq S_n(\mathbb{M})$ so $\mathrm{RM}(\bar{a}/B) \leq \mathrm{RM}(X)$. Take $p \in [X] \subseteq S_n(\mathbb{M})$ with $\mathrm{RM}(p)$ maximal, so $\mathrm{RM}(p) = \mathrm{RM}(X)$. Let $q = p \restriction B$. Let $\bar{a}$ realize q. Then $\bar{a} \in X$, and $\mathrm{RM}(p) \leq \mathrm{RM}(q) \leq \mathrm{RM}(X) = \mathrm{RM}(p)$, so $\mathrm{RM}(\bar{a}/B) = \mathrm{RM}(q) = \mathrm{RM}(X)$. □

Lemma 6.10. *If $\bar{b} \in \mathrm{acl}(\bar{a}C)$, then $\mathrm{RM}(\bar{b}/C) \leq \mathrm{RM}(\bar{a}/C)$.*

Proof. Let π_1, π_2 be the coordinate projections $\pi_1(\bar{x}, \bar{y}) = \bar{x}$, $\pi_2(\bar{x}, \bar{y}) = \bar{y}$. If D is a small enough C-definable set containing $(\bar{a}, \bar{b})$, then…

- $\mathrm{RM}(\pi_1(D)) = \mathrm{RM}(\bar{a}/C)$, by Proposition 6.9.
- $\mathrm{RM}(\pi_2(D)) = \mathrm{RM}(\bar{b}/C)$, by Proposition 6.9.
- The fibers of $D \to \pi_1(D)$ are finite[4], because $\bar{b} \in \mathrm{acl}(\bar{a}C)$.

By Proposition 6.4 and Lemma 6.6,

$$\mathrm{RM}(\bar{b}/C) = \mathrm{RM}(\pi_2(D)) \leq \mathrm{RM}(D) \leq \mathrm{RM}(\pi_1(D)) = \mathrm{RM}(\bar{a}/C).$$ □

Theorem 6.11. *Suppose T is strongly minimal. Let p be the global transcendental type. Let $p^{\otimes n} = \underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}$. Let C be a set of parameters and $\bar{a}$ be an n-tuple.*

1. *If $\bar{a}$ doesn't realize $p^{\otimes n} \restriction C$, then $\mathrm{RM}(\bar{a}/C) < n$.*
2. *$\mathrm{RM}(\mathbb{M}^n) = n$.*
3. *If $\bar{a}$ realizes $p^{\otimes n} \restriction C$, then $\mathrm{RM}(\bar{a}/C) = n$.*

Proof. We prove (1)–(3) together, by induction on n.

1. It can't happen that $a_i \notin \mathrm{acl}(Ca_1, \ldots, a_{i-1})$ for all $i \leq n$, as this would mean $\bar{a}$ realizes $p^{\otimes n}$. Therefore there is some i such that $a_i \in \mathrm{acl}(Ca_1, \ldots, a_{i-1})$. Then $\bar{a} \in \mathrm{acl}(Ca_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. By Lemma 6.10 and induction,
$$\mathrm{RM}(\bar{a}/C) \leq \mathrm{RM}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n/C) \leq \mathrm{RM}(\mathbb{M}^{n-1}) \leq n - 1.$$
2. By Example 6.5, we know $\mathrm{RM}(\mathbb{M}^n) \geq n$. Take $\aleph_0$-saturated $M \preceq \mathbb{M}$. For types over M, Morley rank agrees with Cantor-Bendixson rank in $S_n(M)$, by Lemma 6.2. Applying part (1) to $C = M$, we see that all types in $S_n(M)$ have rank $< n$, with at most one exception. Then the set $E_n \subseteq S_n(M)$ is finite or empty, so $E_{n+1} = \varnothing$, and $\mathrm{R}(S_n(M)) \leq n$, which means $\mathrm{RM}(\mathbb{M}^n) \leq n$.
3. By Proposition 6.9, $\mathrm{RM}(\mathbb{M}^n) = \max\{\mathrm{RM}(p) : p \in S_n(C)\}$. By part (2), there is *some* $q \in S_n(C)$ with $\mathrm{RM}(q) = n$. By part (1), $q = p^{\otimes n} \restriction C$, because all the other possibilities have rank $< n$. □

[4] In other words, for any $\bar{a}'$, there are only finitely many $\bar{b}'$ with $(\bar{a}', \bar{b}') \in D$

Recall that if T is strongly minimal and $M \models T$, a *basis* for M is a maximal independent set $B \subseteq M$. M has at least one basis, and if B is a basis, then $M = \operatorname{acl}(B)$. (See Section 2 above.)

Theorem 6.12. *Suppose T is strongly minimal and $M \models T$. Let B_1, B_2 be two bases. Then $|B_1| = |B_2|$.*

Proof. Suppose not. Without loss of generality, $|B_1| < |B_2|$. There are two cases:

- B_2 is infinite. For each $a \in B_1 \subseteq M = \operatorname{acl}(B_2)$, take a finite set $S_a \subseteq B_2$ with $a \in \operatorname{acl}(S_a)$. This is possible because the formula showing $a \in \operatorname{acl}(B_2)$ uses only finitely many parameters from B_2. Let $I = \bigcup_{a \in S_a} \subseteq B_2$. Then $|I| < |B_2|$: if B_1 is infinite then $|I| \le |B_1|$, and if B_1 is finite then I is finite. Note $B_2 \subseteq \operatorname{acl}(I)$, so $M = \operatorname{acl}(B_2) \subseteq \operatorname{acl}(\operatorname{acl}(I)) = \operatorname{acl}(I) \subseteq M$. Therefore $M = \operatorname{acl}(I)$. As $|I| < |B_2|$ there is $e \in B_2 \setminus I$. Then $e \in M = \operatorname{acl}(I)$. Also $I \subseteq B_2 \setminus \{e\}$. Therefore $e \in M = \operatorname{acl}(I) \subseteq \operatorname{acl}(B_2 \setminus \{e\})$, and B_2 is *not* independent.

- B_2 is finite, and so B_1 is finite as well. Let $B_1 = \{a_1, \ldots, a_n\}$ and $B_2 = \{c_1, \ldots, c_m\}$ where $n = |B_1| < m = |B_2|$. Let p be the global transcendental type. Recall that independent sets are Morley sequences of p over $\varnothing$. By Theorem 6.11(3), $\operatorname{RM}(\bar{a}/\varnothing) = n$ and $\operatorname{RM}(\bar{c}/\varnothing) = m$. In particular, $\operatorname{RM}(\bar{a}/\varnothing) < \operatorname{RM}(\bar{c}/\varnothing)$. But $\bar{c} \in \operatorname{acl}(\bar{a})$, so this contradicts Lemma 6.10. □

The *dimension* of a strongly minimal model M is the cardinality of any basis of M. By the proof of Theorem 2.7 plus Theorem 6.12, the dimension is a complete isomorphism invariant of models of T—two models are isomorphic iff they have the same dimension.

7 Totally transcendental theories

If $L_0 \subseteq L$, let $S_n^{L_0}(A)$ denote the set of n-types over A in the reduct $\mathbb{M} \restriction L_0$. Recall $S_n(A)$ is scattered iff $\operatorname{R}(S_n(A)) < \infty$ (Proposition 5.1).

Lemma 7.1. *For fixed $n < \infty$, the following are equivalent:*

1. *$S_n(\mathbb{M})$ is scattered.*
2. *$S_n(A)$ is scattered for all $A \subseteq \mathbb{M}$.*
3. *$S_n(A)$ is scattered for all countable $A \subseteq \mathbb{M}$.*
4. *$S_n^{L_0}(A)$ is scattered for countable $L_0 \subseteq L$ and countable $A \subseteq \mathbb{M}$.*

Proof. By Lemma 5.3, all these conditions say that there's no tree $(D_\sigma : \sigma \in 2^{<\omega})$ with $D_\sigma \subseteq \mathbb{M}^n$ definable and $D_\sigma = D_{\sigma 0} \sqcup D_{\sigma 1}$. (If such a tree exists, it's defined using only countably many parameters and countably many symbols in the language.) □

Theorem 7.2. *If L is countable, the following are equivalent:*

1. $\mathrm{RM}(\Sigma(\bar{x})) < \infty$ *for any partial type* $\Sigma(x_1, \ldots, x_n)$.

2. $\mathrm{RM}(x = x) < \infty$. *(That is,* $\mathrm{RM}(\mathbb{M}^1) < \infty$.*)*

3. *T is ω-stable.*

Proof. (1) $\implies$ (2): clear

(2) $\implies$ (3): (2) says $\mathrm{R}(S_1(\mathbb{M})) < \infty$, i.e., $S_1(\mathbb{M})$ is scattered. By Lemma 7.1, $S_1(A)$ is scattered for countable A. By Lemma 5.3(3), $S_1(A)$ is countable for countable A, which is ω-stability.

(3) $\implies$ (1): By ω-stability, $S_n(A)$ is countable for countable A. By Lemma 5.3(2), $S_n(A)$ is scattered for countable A. By Lemma 7.1, $S_n(\mathbb{M})$ is scattered, which implies $\mathrm{R}(C) < \infty$ for any closed set $C \subseteq S_n(\mathbb{M})$. This is (1). $\square$

Theorem 7.3. *The following are equivalent for any theory T:*

1. $\mathrm{RM}(\Sigma(\bar{x})) < \infty$ *for any partial type* $\Sigma(x_1, \ldots, x_n)$.

2. $\mathrm{RM}(x = x) < \infty$.

3. *For every countable $L_0 \subseteq L$, the reduct $T \restriction L_0$ is ω-stable.*

Proof. Similar to Theorem 7.2, but using the *fourth* condition of Lemma 7.1. $\square$

Definition 7.4. T is *totally transcendental* if the equivalent conditions hold:

- $\mathrm{RM}(x = x) < \infty$.
- $\mathrm{RM}(\Sigma(\bar{x})) < \infty$ for any partial type $\Sigma(\bar{x})$.

In countable languages, “totally transcendental” is equivalent to “ω-stable.”

Example 7.5. Strongly minimal theories are λ-stable for $\lambda \geq |L|$. Strongly minimal theories in countable languages are ω-stable, hence totally transcendental. If T is strongly minimal, so is any reduct. Therefore, strongly minimal theories are totally transcendental. (Or see Theorem 6.11.)

Theorem 7.6. *If T is totally transcendental, then T is λ-stable for any $\lambda \geq |L|$. In particular, T is superstable.*

Proof. Since $\mathrm{R}(S_n(\mathbb{M})) < \infty$, $S_n(\mathbb{M})$ is scattered, and so $S_n(A)$ is scattered for any A, by Lemma 7.1. If $|A|, |L| \leq \lambda$, then

$$|S_n(A)| \leq |L(A)| \leq \lambda$$

by Lemma 5.3(3). $\square$

8 Morley rank and forking

Suppose T is totally transcendental.

Lemma 8.1. *Suppose $p \in S_n(A)$.*

1. *If $q \in S_n(\mathbb{M})$ extends p, then $\mathrm{RM}(q) \leq \mathrm{RM}(p)$.*

2. *There is $r \in S_n(\mathbb{M})$ with $r \supseteq p$, $\mathrm{RM}(r) = \mathrm{RM}(p)$.*

3. *If $q \in S_n(\mathbb{M})$ extends p and $\mathrm{RM}(q) = \mathrm{RM}(p)$, then $q \sqsupseteq p$.*

4. *If $q \in S_n(\mathbb{M})$ extends p and $q \sqsupseteq p$, then $\mathrm{RM}(q) = \mathrm{RM}(p)$.*

Proof. The first two points hold because $\mathrm{RM}(p)$ is defined to be $\{\max \mathrm{RM}(q) : q \in S_n(\mathbb{M}),\ q \supseteq p\}$.

3. By Proposition 4.8, there are finitely many such q. Then $\{\sigma(q) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite, so q is almost A-definable and $q \sqsupseteq p$ by Proposition 5.6 in the April 21 notes.

4. Take r as in (2). By (3), $r \sqsupseteq p$. Then r, q are non-forking extensions of p. By Proposition 4.5 in the April 21, notes, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(r) = p$. Then $\mathrm{RM}(q) = \mathrm{RM}(r) = \mathrm{RM}(p)$. □

Proposition 8.2. *If $p \in S_n(A)$ and $q \in S_n(B)$ is an extension.*

1. $\mathrm{RM}(q) \leq \mathrm{RM}(p)$.

2. $\mathrm{RM}(q) = \mathrm{RM}(p)$ *iff* $q \sqsupseteq p$.

Proof. 1. Clear.

2. Take r a global non-forking extension of q. Then $\mathrm{RM}(r) = \mathrm{RM}(q)$. By full transitivity of $\sqsupseteq$ and by Lemma 8.1(3,4), we see $q \sqsupseteq p \iff r \sqsupseteq p \iff \mathrm{RM}(r) = \mathrm{RM}(p) \iff \mathrm{RM}(q) = \mathrm{RM}(p)$. □

Note the parallel between $\mathrm{RM}(p)$ and the bound $\mathrm{bd}(p)$:

- If q is a forking extension of p, then $\mathrm{RM}(q) < \mathrm{RM}(p)$ and $\mathrm{bd}(q) < \mathrm{bd}(p)$.

- If q is a nonforking extension of p, then $\mathrm{RM}(q) = \mathrm{RM}(p)$ and $\mathrm{bd}(q) = \mathrm{bd}(p)$.

This gives another proof of superstability: if $\beta_1 > \beta_2 > \cdots$ is a descending chain in the fundamental order, we can get a forking chain

$$\begin{gathered} p_1 \subseteq p_2 \subseteq p_3 \subseteq \cdots \\ p_1 \not\sqsubseteq p_2 \not\sqsubseteq p_3 \not\sqsubseteq \cdots \\ \mathrm{bd}(p_i) = \beta_i, \end{gathered}$$

as in Proposition 12.1 in the April 28 notes. Then

$$\mathrm{RM}(p_1) > \mathrm{RM}(p_2) > \mathrm{RM}(p_3) > \cdots$$

which is impossible—the ordinals are well-ordered and there are no descending chains.

9 Density of isolated types

Recall from last semester that $p \in S_n(A)$ is *isolated* if the following equivalent conditions hold:

- p is an isolated point in the closed set $S_n(A) \subseteq S_n(A)$.
- There is a clopen set U such that $U \cap S_n(A) = \{p\}$.
- $\{p\}$ is clopen.
- There is an $L(A)$-formula $\varphi(\bar{x})$ generating $p(\bar{x})$.
- There is an $L(A)$-formula $\varphi(\bar{x})$ such that for any $\bar{b} \in \mathbb{M}^n$,

$$\operatorname{tp}(\bar{b}/A) = p \iff \bar{b} \models p \iff \mathbb{M} \models \varphi(\bar{b}).$$

We say that $\varphi(\bar{x})$ "isolates" p if these conditions hold.

Lemma 9.1. *Suppose $S_n(A)$ is scattered. If $U \subseteq S_n(A)$ is clopen and non-empty, then there is an isolated type $p \in S_n(A)$ with $p \in U$.*

Proof. Since $S_n(A)$ is scattered, the non-empty closed set U is not perfect. Take $p \in U \setminus U'$. Then p is an isolated point of U. Take a clopen set V such that $V \cap U = \{p\}$. Then the clopen set $V \cap U$ isolates p in $S_n(A)$. $\square$

Theorem 9.2. *Suppose T is totally transcendental. If $A \subseteq \mathbb{M}$ is small and $D \subseteq \mathbb{M}^n$ is A-definable and non-empty, then there is $\bar{b} \in D$ such that $\operatorname{tp}(\bar{b}/A)$ is isolated.*

Proof. By definition of totally transcendental, $\mathrm{R}(S_n(\mathbb{M})) < \infty$, i.e., $S_n(\mathbb{M})$ is scattered (Proposition 5.1). By Lemma 7.1, this implies that $S_n(A)$ is scattered. By Lemma 9.1, there is an isolated type $p \in S_n(A)$ with $p \in [D]$. Take $\bar{b} \in \mathbb{M}^n$ realizing p. Then $\operatorname{tp}(\bar{b}/A) = p$ is isolated and $\bar{b} \in D$ as $p \in [D]$. $\square$