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Cook Theorem

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Outline

wef

Theorem (Cook-Levin Theorem)

1. *$\mathtt{SAT}$ is **NP**-complete*
2. *$\mathtt{3SAT}$ is **NP**-complete*

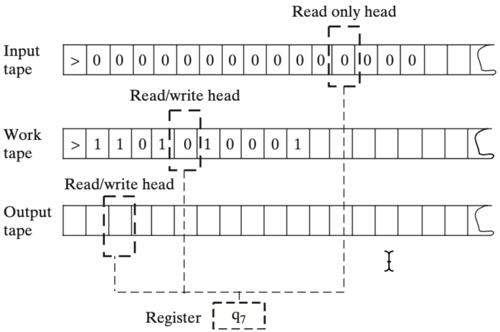
Turing machine

Definition

A TM M is described by a tuple (Γ, Q, δ) containing

- A finite set Γ of the symbols that M's tapes can contain. We assume that Γ contains a designated "blank" symbol, denoted $\square$; a designated "start" symbol, denoted $\triangleright$; and the numbers 0 and 1. We call Γ the alphabet of M
- A finite set Q of possible states M' register can be in. We assume that Q contains a designated start state, denoted q_{start}, and a designated halting state, denoted q_{halt}
- A function $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{\mathsf{L,S,R}\}^k$, where $k \geq 2$, describing the rules M use in performing each step. This function is called the transition function of M

Turing machine



Efficiency and running time

Definition (Computing a function and running time)

Let $f : \{0,1\}^* \to \{0,1\}^*$ and let $T : \mathbb{N} \to \mathbb{N}$ be some functions, and let M be a Turing machine. We say that M computes f if for every $x \in \{0,1\}^*$ whenever M is initialized to the start configuration on input x, then it halts with $f(x)$ written on its output tape. We say M computes f in $T(n)$-time if its computation on every input x requires at most $T(|x|)$ steps

The class **P**

A complexity class is a set of functions that can be computed within given resource bounds. We say that a machine decides a language $L \subseteq \{0,1\}^*$ if it computes the function $f_L : \{0,1\}^* \to \{0,1\}$ where $f_L(x) = 1 \Leftrightarrow x \in L$

Definition

Let $T : \mathbb{N} \to \mathbb{N}$ be some function. A language L is in **DTIME**$(T(n))$ iff there is a deterministic Turing machine that runs in time $c \cdot T(n)$ for some constant $c > 0$ and decides L

Definition

P $= \bigcup_{c \geq 1}$ **DTIME**(n^c)

The class **NP**

Definition

A language $L \subseteq \{0,1\}^*$ is in **NP** if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M (called the verifier for L) such that for every $x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x,u) = 1$, then we call u a certificate for x w.r.t. L and M

The class **NP**

Definition

For every function $T : \mathbb{N} \to \mathbb{N}$ and $L \subseteq \{0,1\}^*$, we say that $L \in \textbf{NTIME}(T(n))$ if there is a constant $c > 0$ and a $c \cdot T(n)$-time NDTM M s.t. for every $x \in \{0,1\}^*$, $x \in L \Leftrightarrow M(x) = 1$

Theorem

$\textbf{\textit{NP}} = \bigcup_{c \in \mathbb{N}} \textbf{\textit{NTIME}}(n^c)$

Proof.

The main idea is that the sequence of nondeterministic choices made by an accepting computation of an NDTM can be viewed as a certificate that the input is in the language, and vice versa $\square$

The class **NP**

Theorem

$\textbf{\textit{NP}} = \bigcup_{c \in \mathbb{N}} \textbf{\textit{NTIME}}(n^c)$

Proof.

Suppose $p : \mathbb{N} \to \mathbb{N}$ is a polynomial and L is decided by a NDTM N that runs in time $p(n)$. For every $x \in L$, there is a sequence of nondeterministic choices that makes N reach q_{accept} on input x. We can use this sequence as a certificate for x. □

Reducibility

Definition

A language $L \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to a language $L' \subseteq \{0,1\}^*$ (sometimes shortened to just "polynomial-time reducible"), denoted by $L \leq_p L'$ if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ s.t. for every $x \in \{0,1\}^*$, $x \in L$ iff $f(x) \in L'$

We say that L' is **NP**-hard if $L \leq_p L'$ for every $L \in$ **NP**. We say that L' is **NP**-complete if L' is **NP**-hard and $L' \in$ **NP**

Goal

Theorem (Cook-Levin Theorem)

1. *SAT is **NP**-complete*
2. *$3SAT$ is **NP**-complete*

Oblivious Turing machine

Definition

Define a TM M to be oblivious if its head movements do not depend on the input but only on the input length. That is, M is oblivious if for every input $x \in \{0,1\}^*$ and $i \in \mathbb{N}$, the location of each of M's heads at the ith step of execution on input x is only a function of $|x|$ and i.

Theorem

For any Turing machine M that decides a language in time $T(n)$, there exists an oblivious Turing machine that decides the same language in $T(n)^2$

A lemma

Lemma

For every Boolean function $f : \{0,1\}^l \to \{0,1\}$, there is an l-variable CNF formula φ of size $l2^l$ s.t. $\varphi(u) = f(u)$ for every $u \in \{0,1\}^l$, where the size of a CNF formula is defined to be the number of $\wedge / \vee$ symbols it contains

Proof.

For every $v \in \{0,1\}^l$, there exists a clause $C_v(z_1, \dots, z_l)$ s.t. $C_v(v) = 0$ and $C_v(u) = 1$ for every $u \neq v$.
We let φ be the AND of all the clauses C_v for v s.t. $f(v) = 0$

$$\varphi = \bigwedge_{v : f(v) = 0} C_v(z_1, \dots, z_l)$$

Note that φ has size at most $l2^l$. $\square$

Main lemma

Lemma

*$\mathtt{SAT}$ is **NP**-hard*

Proof.

Let L be an **NP** language. By definition, there is a polynomial time TM M s.t. for every $x \in \{0,1\}^*$, $x \in L \Leftrightarrow M(x,u) = 1$ for some $u \in \{0,1\}^{p(|x|)}$, where $p : \mathbb{N} \to \mathbb{N}$ is some polynomial. We show L is polynomial-time Karp reducible to $\mathtt{SAT}$ by describing a polynomial-time transformation $x \to \varphi_x$ from strings to CNF formulae s.t. $x \in L$ iff φ_x is satisfiable. Equivalently

$$\varphi_x \in \mathtt{SAT} \quad \text{iff} \quad \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x \circ u) = 1$$

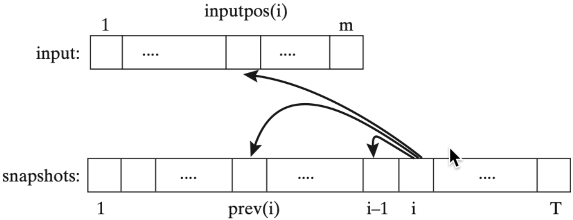
where $\circ$ denotes concatenation $\square$

Assumption

Assume

1. M only has two tapes - an input tape and a work/output tape
2. M is an oblivious TM in the sense that its head movement does not depend on the contents of its tapes. That is, M's computation takes the same time for all inputs of size n, and for every i the location of M's head at the ith step depends only on i and the length of the input

Proof

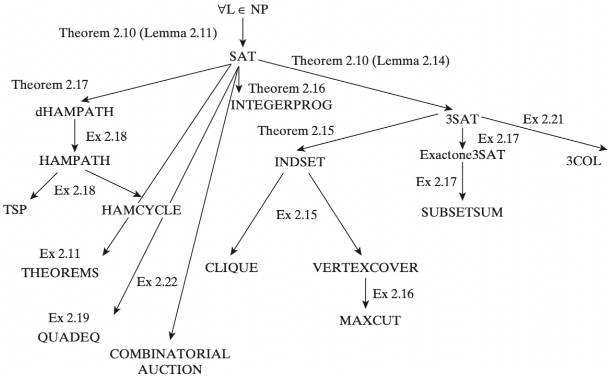


Proof

The previous discussion shows this latter condition occurs iff there exists a string $y \in \{0,1\}^{n+p(n)}$ and a sequence of strings $z_1, \ldots, z_{T(n)} \in \{0,1\}^c$ (where $T(n)$ is the number of steps M takes on inputs of length $n + p(n)$) satisfying the following four conditions

1. The first n bits of y are equal to x
2. The string z_1 encodes the initial snapshot of M. That is, z_1 encodes the triple $\langle \triangleright, \square, q_{\mathsf{start}} \rangle$.
3. For every $i \in \{2, \ldots, T(n)\}$, $z_i = F(z_{i-1}, z_{\mathsf{prev}(i)}, y_{\mathsf{inputpos}(i)})$.
4. The last string $z_{T(n)}$ encodes a snapshot where the machine halts and outputs 1

The web of reductions



nondeterministic turing machines

Maybe some examples