Chapter 9

Topology

Valuable examples of Boolean algebras, and in particular of complete Boolean algebras, can be constructed using topological spaces. The purpose of this chapter is to go over some of the basic topological notions that will be needed in the construction of these algebras.

A topology is a very general kind of geometry that is suitable for studying continuous functions. The basic notion is that of an open set, an abstraction and generalization of the notion of an open interval of real numbers. A *topological space* is a set X, together with a class $\mathcal{T}$ of subsets of X that satisfies three conditions: first, $\varnothing$ and X are in $\mathcal{T}$; second, $\mathcal{T}$ is closed under finite intersections in the sense that the intersection of any finite family of sets from $\mathcal{T}$ is again in $\mathcal{T}$; third, $\mathcal{T}$ is closed under arbitrary unions in the sense that the union of an arbitrary family of sets from $\mathcal{T}$ is again in $\mathcal{T}$. The elements of X are called *points*, and the members of $\mathcal{T}$ are called *open sets*. An open set containing a point x is called a *neighborhood* of x. The set X is often called the *space*, and the class $\mathcal{T}$ the *topology* of the space. The three conditions on $\mathcal{T}$ say that $\varnothing$ and X are open sets, that the intersection of finitely many open sets is open, and that the union of an arbitrary family of open sets is open.

The classical example of a topological space is the n-dimensional *Euclidean space* $\mathbb{R}^n$. Its points are the n-termed sequences of real numbers. The topology of the space is defined in terms of the notion of distance. The *distance* between two points

$$x = (x_1, \dots, x_n) \qquad \text{and} \qquad y = (y_1, \dots, y_n)$$

is the length of the line segment between them:

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$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

For a given positive real number ϵ and a point x, the (*open*) *ball* with *radius* ϵ and *center* x is defined to be the set of all points whose distance from x is less than ϵ:

$$\{y \in \mathbb{R}^n : d(x, y) < \epsilon\}.$$

In two-dimensional Euclidean space, this is just the interior of the circle of radius ϵ centered at x, and in three-dimensional Euclidean space it is the interior of the sphere of radius ϵ centered at x. A subset P of $\mathbb{R}^n$ is defined to be *open* if for every point x in P, some open ball centered at x is included in P.

The empty set vacuously satisfies the condition for being open. The whole space $\mathbb{R}^n$ satisfies the condition for being open because it includes every open ball. It is easy to check that the union of an arbitrary family of open sets in $\mathbb{R}^n$ is open, as is the intersection of a finite family of open sets. For the proof, consider such a family $\{P_i\}$ of open sets. If a point x is in the union of the family, then x is in one of the sets P_i, by the definition of union. Consequently, some open ball centered at x is included in P_i (because P_i is open) and that same open ball must be included in the union of the family $\{P_i\}$. It follows that this union is an open set. If the family is finite, and if a point x is in the intersection of the family, then x belongs to each set P_i. Consequently, for each index i there is a positive real number ϵ_i such that the open ball of radius ϵ_i centered at x is included in P_i. Let ϵ be the minimum of the (finitely many) radii ϵ_i. The open ball of radius ϵ centered at x is included in each set P_i, so it is included in the intersection of the family $\{P_i\}$. It follows that this intersection is an open set. Conclusion: the class of open subsets of $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$.

In the special case of dimension one, the points of the Euclidean space are identified with real numbers. The open balls are the *open intervals*

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}, \qquad (a, \infty) = \{x \in \mathbb{R} : x > a\},$$
$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \quad (-\infty, \infty) = \mathbb{R},$$

where a and b are real numbers with $a < b$. It is not difficult to see that a set is open in this space if and only if it can be written as a countable union of mutually disjoint open intervals. Examples of open sets in $\mathbb{R}^2$ are also easy to manufacture: the entire space with finitely many points removed is an example of an open set, and so is the set obtained from $\mathbb{R}^2$ by removing any straight line (the set is the union of two open half-planes).

To generalize the notion of a Euclidean space, we must introduce the notion of a metric. A *metric* on a set X is a real-valued function d of two arguments such that for all x, y, and z in X,

$d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$ (*strict positivity*),

$d(x,y) = d(y,x)$ (*symmetry*),

$d(x,z) \leq d(x,y) + d(y,z)$ (*triangle inequality*).

A *metric space* is a set X together with a metric d on X. (The prototypical examples are the Euclidean spaces $\mathbb{R}^n$ with their distance functions.) For each positive real number ϵ, and each point x in X, the *open ball* of radius ϵ centered at x is defined to be the set of points

$$\{y \in X : d(x,y) < \epsilon\}.$$

A subset P of X is defined to be open if for every point x in P, some open ball centered at x is included in P. The resulting class of open sets constitutes a topology called the *metric topology* (induced by d) on X. The proof that the conditions for being a topology really are satisfied by this class of sets is virtually the same as in the case of Euclidean spaces.

Some types of topologies can be defined on arbitrary sets X. One example is the *discrete topology*: every subset of X is declared to be open. Under this topology, X is called a *discrete space*. Another example is the *indiscrete* or *trivial topology*: the only sets declared to be open are, by definition, $\varnothing$ and X. A third example is the *cofinite topology*: a subset of X is defined to be open if it is empty or the complement of a finite set. (It is a simple matter to check that finite intersections and arbitrary unions of cofinite sets are cofinite.) The discrete and cofinite topologies on X coincide when X is finite, but they are obviously different when X is infinite.

A subset Y of a topological space X may be endowed with the *inherited topology* by declaring a subset P of Y to be open ife it can be written in the form

$$P = Y \cap Q$$

for some open subset Q of X. It is easy to check that under this definition the empty set and Y are both open, and that finite intersections and arbitrary unions of open sets are open. (The proof that arbitrary unions of open sets are open uses the infinite distributive law (8.2).) The resulting space Y is said to be a *subspace* of X.

For a concrete example, let X be the space of real numbers and Y the closed interval $[0,1]$. The open intervals of Y, under the inherited topology, are the subintervals

$$[0,a), \qquad (a,b), \qquad (a,1], \qquad \text{and} \qquad [0,1]$$

with $0 \le a < b \le 1$. The open subsets of Y turn out to be the countable unions of pairwise disjoint open intervals. Notice that some of these sets are not open in the topology of X.

The dual of the notion of an open set is that of a closed set. A set of points in a topological space is said to be *closed* if it is the complement of an open set. The intersection of an arbitrary family of closed sets is closed. Indeed, if $\{Q_i\}$ is a family of closed sets, then the complements Q_i' are open, by definition, and therefore the union of the family of complements is open. It follows that the complement of this union is a closed set; since that complement is just the intersection of the family $\{Q_i\}$,

$$\bigcap_i Q_i = \Bigl(\bigcup_i Q_i'\Bigr)',$$

the intersection of the family is closed. An analogous argument shows that a finite union of closed sets is closed. It is an elementary theorem of analysis that a subset P of a metric space (but not a subset of an arbitrary topological space) is closed just in case the limit of a convergent sequence of points from P always belongs to P. More precisely, if $\{x_n\}$ is an infinite sequence of points in P that converges to a limit x in the metric space, then x belongs to P.

As in the case of open sets, it is helpful to gain a sense of what closed sets may look like. In $\mathbb{R}$, the intervals

$$\begin{aligned} (-\infty,a] &= \{x\in\mathbb{R} : x \le a\}, & [a,\infty) &= \{x\in\mathbb{R} : x \ge a\},\\ [a,b] &= \{x\in\mathbb{R} : a \le x \le b\}, & (-\infty,\infty) &= \mathbb{R} \end{aligned}$$

are closed, as is any finite union of them (where a and b are real numbers with $a \le b$). In particular, the sets $[a,a]$ — which are just the singletons $\{a\}$ — are closed. Consequently, any finite set of real numbers is closed (every finite set is the union of a finite class of singletons). The set of positive integers is closed, but the set of the reciprocals of positive integers is not closed; it becomes closed when the integer 0 is adjoined. In $\mathbb{R}^2$, the line segment $\{(x,0) : 0 < x < 1\}$ is neither open nor closed, but the segment $\{(x,0) : 0 \le x \le 1\}$ is closed. Every subset of a discrete space is of course closed. A subset of a space with the cofinite topology is closed if and only if it is finite or the whole space.

Sets that are both open and closed are called *clopen*. The whole space and the empty set are always clopen. Every subset of a discrete space is clopen.

In what follows, let P be an arbitrary subset of a topological space. The *interior* of P is defined to be the union of the open sets that are included in P. This interior is clearly an open set, and in fact it is the largest open set that is included in P. We shall denote it by P°. Examples: in $\mathbb{R}$, the interior of the closed interval $[0, 1]$ is the open interval $(0, 1)$ and the interior of the set of rational numbers is the empty set; in $\mathbb{R}^2$, the interior of the unit square is the unit square with its perimeter removed, while the interior of the line segment $\{(x, 0) : 0 \le x \le 1\}$ is the empty set. In a discrete space every subset coincides with its own interior. In a space with the cofinite topology, the interior of a cofinite subset is itself, and the interior of every other subset is empty.

The dual of the notion of an interior is that of a closure: the *closure* of P is defined to be the intersection of all closed sets that include P. This closure is of course a closed set, and in fact it is the smallest closed set that includes P. It is denoted by P^- (not $\overline{P}$) for typographic reasons. The closure of P can be characterized as the set of points x such that every neighborhood of x has a non-empty intersection with P. For the proof in one direction, suppose some neighborhood of x, say Q, is disjoint from P. The complement Q' is a closed set that includes P, so it belongs to the family of closed sets whose intersection is P^-. Since the point x is not in Q', it cannot belong to this intersection, and therefore it cannot belong to P^-. To establish the reverse implication, suppose every neighborhood of x has a non-empty intersection P. If Q is a closed set that includes P, then Q' is an open set that is disjoint from P, so Q' cannot contain x. It follows that x is in Q. In other words, x belongs to every closed set that includes P; therefore x must belong to the intersection of all such closed sets, and this intersection is just P^-. The closure of set P in a metric space (but not in a arbitrary topological space) can be characterized as the set of all limit points of convergent infinite sequences of points from P.

Here are some examples of closures. In $\mathbb{R}$, the closure of the set of rational points is the entire space, and the closure of the set

$$P = \{1/n : n \text{ is a positive integer}\}$$

is the set $P \cup \{0\}$. In $\mathbb{R}^2$, the closure of an open disk is the open disk together with its perimeter, and the closure of the set of points

$$Q = \{(x, y) : y = \sin(1/x) \text{ and } 0 < x \le 1\}$$

is the union of Q with the set

$$\{(0,y) : -1 \leq y \leq 1\}.$$

In a discrete space, every subset is its own closure. In a space with the cofinite topology, the closure of a finite subset is itself, and the closure of an infinite subset is the whole space.

It is not difficult to see that the closure of a set P is the complement of the interior of the complement of P. In other words,

$$P^- = P^{\prime\circ\prime}. \tag{1}$$

Indeed, P'° is an open set that is included in P', by the definition of the interior of a set. Form the complements of both sets to conclude that P'' — which is just P — is included in the closed set $P^{\prime\circ\prime}$. Since P^- is the smallest closed set that includes P, it follows that P^- is a subset of $P^{\prime\circ\prime}$. To establish the reverse inclusion, consider any closed set Q that includes P. The complement Q' is an open set that is included in P'. Since P'° is the largest open set that is included in P', it follows that Q' is included in P'°. Form the complements of both sets to conclude that Q'' — which is just Q — includes $P^{\prime\circ\prime}$. This argument shows that every closed set that includes P also includes $P^{\prime\circ\prime}$. In particular, the closure P^- includes $P^{\prime\circ\prime}$.

Replace P by P' in (1), and form the complement of both sides to conclude that the interior of P is equal to the complement of the closure of the complement of P, that is,

$$P^\circ = P^{\prime-\prime}. \tag{2}$$

We shall need two properties about the closure operator. First, it preserves inclusion:

$$P \subseteq Q \qquad \text{implies} \qquad P^- \subseteq Q^-. \tag{3}$$

This is immediately evident from the definition of closure. Second, it preserves union:

$$(P \cup Q)^- = P^- \cup Q^-. \tag{4}$$

The inclusion from right to left follows from (3): the sets P and Q are both included in $P \cup Q$, so their closures are included in $(P \cup Q)^-$. The reverse inclusion follows from the simple observation that $P^- \cup Q^-$ is a closed set and it includes $P \cup Q$, so it must include the smallest closed set that includes $P \cup Q$, namely $(P \cup Q)^-$.

A set P is said to be *dense* if its closure is the entire space. This means that every non-empty open set contains points of P. More generally, a set P is said to be *dense in an open set* Q if the closure of P includes Q. For instance, the set of points in $\mathbb{R}^2$ with rational coordinates is dense, while the set of points with positive rational coordinates is dense in the first quadrant of $\mathbb{R}^2$, but not in all of $\mathbb{R}^2$. In a discrete space, no proper subset of the space is dense. In an infinite space with cofinite topology, every infinite subset of the space is dense.

The opposite of being dense is being nowhere dense. A set P is defined to be *nowhere dense* if it is not dense in any non-empty open set. This means that the interior of the closure of P is empty, or what amounts to the same thing, no non-empty open set is included in the closure of P. To say that no non-empty open set is included in P^- is equivalent to saying that every non-empty open set has a non-empty intersection with $P^{-\prime}$. Thus, P is nowhere dense if and only if $P^{-\prime}$ is dense.

Examples of nowhere dense sets are not hard to manufacture. Every finite set of points is nowhere dense in $\mathbb{R}^n$. The integers are nowhere dense in $\mathbb{R}$, and any straight line is nowhere dense in $\mathbb{R}^2$. In a discrete space only the empty set is nowhere dense. In an infinite space with the cofinite topology, a set is nowhere dense just in case it is finite.

A finite union of nowhere dense sets is again nowhere dense, but a countable union of such sets may in fact be dense. For example, the set of points with rational coordinates is dense in $\mathbb{R}^n$, and yet it is a countable union of nowhere dense sets, namely the singletons of points with rational coordinates. A set is said to be *meager* if it is the countable union of nowhere dense sets. (In classically clumsy nomenclature, meager sets are also called *sets of the first category.*) An important result in analysis known as the Baire category theorem says that the interior of a meager set of points in $\mathbb{R}^n$ is always empty. (See Theorem 28 for one version of this theorem.) In a discrete space only the empty set is meager. In an infinite space with the cofinite topology, a set is meager just in case it is countable. In particular, if the space is countably infinite, then every subset is meager.

A point x is a *boundary point* of a set P if every neighborhood of x contains points of P and points of P'. In other words, x is a boundary point of P just in case it belongs to the closure of both P and P'. The *boundary* of P is the set of its boundary points, that is to say, it is the set $P^- \cap P'^-$.

The boundary of a set is always closed, because it is the intersection of two closed sets. In a metric space (but not in an arbitrary topological space), a point x is in the boundary of a set P just in case there is a sequence of points in P that converges to x and a sequence of points in P' that converges to x. If a set P is open, then its complement P' is closed, and therefore $P'^- = P'$; in this case, the boundary of P is just the set-theoretic difference $P^- - P$.

Here are some examples. The boundary of an open ball in $\mathbb{R}^3$ of radius ϵ and center x is the set of points whose distance from x is exactly ϵ. The boundary of the set of points

$$P = \{(1/n, 1/n) : n \text{ is a positive integer}\}$$

in $\mathbb{R}^2$ is the set $P \cup \{(0,0)\}$. More generally, the boundary of any nowhere dense set P in a topological space is just the closure P^-. Indeed, in this case the set $P^{-\prime}$ is dense; since $P^{-\prime} \subseteq P'$, the set P' must also be dense, so that P'^- is the whole space, and therefore $P^- \cap P'^- = P^-$. In a discrete space every set of points has an empty boundary. In an infinite space with the cofinite topology, the boundary of a finite set is itself, the boundary of a cofinite set is its complement, and the boundary of an infinite set with an infinite complement is the whole space.

The closure of a set P ought to be, and is, the union of P with its boundary. To prove this assertion, write $Q = P^- \cap P^{-\prime}$ for the boundary. Obviously, P and Q are both included in P^-, and therefore so is $P \cup Q$. To establish the reverse inclusion, consider a point x in P^-. If x is in P, then certainly x is in the union $P \cup Q$. If x is not in P, then it is in P'. In this case, every neighborhood of x contains a point in P', namely x, and also a point in P, since x is in the closure of P. It follows that x belongs to the boundary Q, by definition, and therefore it belongs to $P \cup Q$.

An open set is said to be *regular* if it coincides with the interior of its own closure. In other words, P is regular if and only if

$$P = P^{-\prime-\prime},$$

by (2). It is convenient, in this connection, to write $P^\perp = P^{-\prime}$; in these terms, P is regular if and only if

$$P = P^{\perp\perp}$$

(where $P^{\perp\perp}$ denotes $(P^\perp)^\perp$). Every open ball in $\mathbb{R}^n$ is regular. So is the open unit square in $\mathbb{R}^2$.

To construct an example of an open set that is not regular, start with a non-empty, nowhere dense set P in a topological space X, and form $P^\perp$. Certainly $P^\perp$ is open, for it is the complement of a closed set. Since P^- is not empty, its complement $P^\perp$ cannot be the whole space X. To prove that $P^\perp$ is not regular, it therefore suffices to show that

$$(P^{\perp})^{\perp\perp} = X. \tag{5}$$

The assumption that P is nowhere dense implies that the complement of its closure, which is just the set $P^{\perp}$, is dense. Consequently, the closure of $P^{\perp}$ is X, and this directly implies (5):

$$(P^{\perp})^{\perp\perp} = (P^{\perp})^{-\prime-\prime} = (P^{\perp-})^{\prime-\prime} = X^{\prime-\prime} = \varnothing^{-\prime} = \varnothing^{\prime} = X.$$

This example reveals the intuition behind regular open sets: they are the open sets without “cracks”. Concrete examples of open sets that are not regular can be obtained by taking for P in the preceding construction any finite set of points or any straight line in $\mathbb{R}^n$ $(n \geq 2)$; the closure P^- then coincides with P, so that $P^{\perp}$ is the space $\mathbb{R}^2$ with finitely many points, or with a straight line, removed. In a discrete space, every set of points is a regular open set. In an infinite space with the cofinite topology, there are only two regular open sets: the empty set and the whole space.

Note incidentally that a set P is open (nothing is said about regularity here) if and only if it has the form $Q^{\perp}$ for some set Q. Indeed, if $P = Q^{\perp}$, then P is the complement of the closed set Q^-, and so it must be open. Conversely, if P is open, and if Q is the complement of P, then Q is closed and therefore

$$Q^{\perp} = Q^{-\prime} = Q^{\prime} = P^{\prime\prime} = P.$$

Exercises

1. Prove that a subset P of a topological space is open just in case every point in P belongs to an open set that is included in P.

2. Prove, using the definition of an open set in the space $\mathbb{R}^n$, that if finitely many points are removed from $\mathbb{R}^n$, the resulting set is open.

3. Prove, using the definition of an open set in the space $\mathbb{R}^n$, that if all the points on some straight line are removed from $\mathbb{R}^n$, the resulting set is open.

4. Show that in a topological space, a finite union of closed sets is always closed.

5. Prove that the class of clopen sets in a topological space is a field of sets.

6. Show that the inherited topology on a subset Y of a topological space X is in fact a topology. In other words, show that the class of sets of the form $Y \cap U$, where U ranges over the open subsets of X, satisfies the three defining conditions for a topology. Show further that a subset P of Y is closed in the inherited topology if and only if there is a closed subset Q of X such that $P = Y \cap Q$.

7. Suppose Y is an open subset of a topological space X, and P an arbitrary subset of Y. Prove that P is open in the inherited topology on Y (Exercise 6) just in case it is open in X.

8. Formulate and prove a version of Exercise 7 for closed sets.

9. Suppose Y is a subspace of a topological space X. Show that if P is a subset of Y, then the closure of P in Y is equal to the intersection with Y of the closure of P in X.

10. (Harder.) Show that the distance function defined on $\mathbb{R}^n$ satisfies the three conditions for being a metric.

11. Show that the class of open sets in a metric space satisfies the conditions for being a topology.

12. Show that a subset of a metric space is open if and only if it is a union of open balls.

13. Let P and Q be subsets of a topological space. Prove the following assertions.

    (a) If $P \subseteq Q$, then $P^\circ \subseteq Q^\circ$.
    (b) $(P \cap Q)^\circ = P^\circ \cap Q^\circ$.
    (c) $(P \cup Q)^\perp = P^\perp \cap Q^\perp$.
    (d) $P \cap Q^- \subseteq (P \cap Q)^-$ whenever P is open.
    (e) $P \cap Q^- = (P \cap Q)^-$ whenever P is clopen.

14. Give a direct proof of equation (2), without using (1).

15. If P and Q are open sets, is the equation

$$(P \cap Q)^- = P^- \cap Q^-$$

    true?

16. (Harder.) If P and Q are open sets, is the equation
$$(P \cap Q)^{-\circ} = (P^- \cap Q^-)^\circ$$
true?

17. Prove that every subset of a discrete space has an empty boundary.

18. In an infinite space with the cofinite topology, prove that the boundary of a finite set is the set itself, the boundary of a cofinite set is the complement of the set, and the boundary of an infinite set with an infinite complement is the whole space.

19. Prove that a set of points and its complement always have the same boundary.

20. Prove that the boundary of a nowhere dense set is just the closure of the set.

21. Prove that the boundary of the union of two sets of points is included in the union of the boundaries of the two sets.

22. Prove that the complement of the boundary of an open set P is equal to $P \cup P^{\perp}$.

23. Prove that in a topological space, the class of sets with countable boundaries is a field of sets.

24. Prove that a finite union of nowhere dense sets is nowhere dense.

25. Prove that a subset of a nowhere dense set is nowhere dense.

26. Prove that in a topological space, the class of sets with nowhere dense boundaries is a field of sets. (This example of a field of sets is due to Stone [67].)

27. Prove that a subset of a meager set is meager.

28. Prove that the union of a countable sequence of meager sets is meager.

29. Prove that in a topological space, the sets with meager boundaries form a field.

30. Prove that every clopen set in a topological space is a regular open set. Conclude that every subset of a discrete space is a regular open set.

31. Prove that in an infinite space with the cofinite topology only the empty set and the whole space are regular open sets.

32. Let X be a set of uncountable cardinality. Define a subset of X to be open if it is empty or the complement of a countable set.

    (a) Prove that the class of open sets so defined satisfies the three conditions for being a topology on X. (It is called the *cocountable topology.*)
    (b) Describe the closed sets.
    (c) Describe the interior of each set.
    (d) Describe the closure of each set.
    (e) Describe the nowhere dense sets.
    (f) Describe the meager sets.
    (g) Describe the boundary of each set.
    (h) Describe the regular open sets.

33. A *linear order* (also called a *total order*) on a set X is a partial order $\leq$ on X (Chapter 7) such that any two elements x and y in X are *comparable*: either $x \leq y$ or $y \leq x$. (The set X itself is said to be *linearly ordered* or *totally ordered.*) Write $x < y$ to mean that $x \leq y$ and $x \neq y$.

    Given a linear order $\leq$ on a set X, define the open intervals of X to be the subsets

    $$(-\infty, a) = \{x \in X : x < a\}, \qquad (a, \infty) = \{x \in X : x > a\},$$
    $$(a, b) = \{x \in X : a < x < b\}, \quad (-\infty, \infty) = X,$$

    for a, b in X, and define the open sets of X to be the unions of arbitrary families of open intervals. Prove that the class of sets so defined is a topology for X. (It is called the *order topology* on X.)

34. Prove that a subset of the (Euclidean) space of real numbers is open if and only if it can be written as a countable union of mutually disjoint open intervals.

35. (Harder.) Prove that there are $2^{\aleph_0}$ open sets in the space of real numbers.

36. Let Q be an open ball in $\mathbb{R}^n$ and P a non-empty, nowhere dense subset of Q. Prove that the set $Q \cap P^{\perp} = Q - P^{-}$ is open, but not regular.

37. Can every open set in $\mathbb{R}^n$ be written as the union of a family of regular open sets?

38. Can every open set in an arbitrary topological space be written as the union of a family of regular open sets?

39. For every subset P of a topological space, prove that $P^{\perp}$ is the largest open set that is disjoint from P. Conclude that P is a regular open set if and only if it is the largest open set that is disjoint from the largest open set that is disjoint from P.

40. (Harder.) What is the largest number of distinct sets obtainable from a subset of $\mathbb{R}^n$ by repeated applications of closure and complementation? Construct an example for which this largest number is attained. (The question and its answer are both due to Kuratowski [36].)

Chapter 10

Regular Open Sets

The purpose of this chapter is to discuss one more example of a Boolean algebra. This example, the most intricate of all the ones so far, is one in which the elements of the Boolean algebra are subsets of a set. However, the operations are not the usual set-theoretic ones, so the Boolean algebra is not a field of sets. Artificial examples of this kind are not hard to manufacture; the example that follows arises rather naturally and plays an important role in the general theory of Boolean algebras.

Recall (Chapter 9) that an open set in a topological space X is regular if it coincides with the interior of its own closure. The next theorem (due to MacNeille [43] and Tarski [75]) asserts that the regular open sets constitute a complete Boolean algebra of sets, the *regular open algebra of* X.

Theorem 1. *The class of all regular open sets of a topological space X is a complete Boolean algebra with respect to the distinguished Boolean elements and operations defined by*

$$0 = \varnothing, \tag{1}$$

$$1 = X, \tag{2}$$

$$P \wedge Q = P \cap Q, \tag{3}$$

$$P \vee Q = (P \cup Q)^{\perp\perp}, \tag{4}$$

$$P' = P^{\perp}. \tag{5}$$

The infimum and the supremum of a family $\{P_i\}$ of regular open sets are, respectively,

$$\Big(\bigcap_i P_i\Big)^{\perp\perp} \qquad \textit{and} \qquad \Big(\bigcup_i P_i\Big)^{\perp\perp}.$$

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The proof of the theorem depends on several small lemmas of some independent interest. The first thing to prove is that the right sides of (1)–(5) are regular open sets. For (1) and (2) this is obvious, but for (3), for instance, it is not. To say that the intersection of two regular open sets is regular may sound plausible (this is what is involved in (3)), and it is true. It is, however, just as plausible to say that the union of two regular open sets is regular, but that is false. Example: let P and Q be disjoint open half-planes in $\mathbb{R}^2$ separated by a line (a nowhere dense set), say P consists of the points to the right of the y-axis, and Q consists of the points to the left; then $P \cup Q$ is open, but not regular, since

$$(P \cup Q)^{\perp\perp} = \mathbb{R}^2.$$

In intuitive terms, an open set is regular if there are no cracks in it; the trouble with the union of two regular open sets is that there might be a crack between them. This example helps to explain the necessity for the possibly surprising definition (4). It is obvious that something unusual, such as (5) for instance, is needed in the definition of complementation; the set-theoretic complement of an open set (regular or not) is quite unlikely to be open.

Lemma 1. *If $P \subseteq Q$, then $Q^{\perp} \subseteq P^{\perp}$.*

Proof. Closure preserves inclusions and complementation reverses them.

Lemma 2. *If P is open, then $P \subseteq P^{\perp\perp}$.*

Proof. Since $P \subseteq P^{-}$, it follows, by complementation, that $P^{\perp} \subseteq P'$. Now apply closure: since P' is closed, it follows that $P^{\perp-} \subseteq P'$, and this is the complemented version of what is wanted.

Lemma 3. *If P is open, then $P^{\perp} = P^{\perp\perp\perp}$.*

Proof. Apply Lemma 1 to the conclusion of Lemma 2 to get $P^{\perp\perp\perp} \subseteq P^{\perp}$, and apply Lemma 2 to the open set $P^{\perp}$ (in place of P) to get the reverse inclusion.

It is an immediate consequence of Lemma 3 that if P is open, and all the more if P is regular, then $P^{\perp}$ is regular; this proves that the right side of (5) belongs to the class of regular open sets. Since $(P \cup Q)^{\perp}$ is always open, the same thing is true for (4). To settle (3), one more argument is needed.

Lemma 4. *If P and Q are open, then $(P \cap Q)^{\perp\perp} = P^{\perp\perp} \cap Q^{\perp\perp}$.*

Proof. The set $P \cap Q$ is included in P and in Q, so the set $(P \cap Q)^{\perp\perp}$ is included in $P^{\perp\perp}$ and in $Q^{\perp\perp}$, by two applications of Lemma 1. Consequently,

$$(P \cap Q)^{\perp\perp} \subseteq P^{\perp\perp} \cap Q^{\perp\perp}.$$

The reverse inclusion depends on the general topological fact that if P is open, then

$$P \cap Q^{-} \subseteq (P \cap Q)^{-}.$$

(It must be checked that every neighborhood U of a point x in $P \cap Q^{-}$ has a non-empty intersection with $P \cap Q$. The point x is in Q^{-}, by assumption, and $U \cap P$ is a neighborhood of x, so $U \cap P$ must intersect Q in some point. Of course, U meets $P \cap Q$ in the same point.) Complementing this relation, we get

$$(P \cap Q)^{\perp} \subseteq P' \cup Q^{\perp}.$$

Apply the operations of closure and complement to arrive at

$$(P' \cup Q^{\perp})^{-\prime} \subseteq (P \cap Q)^{\perp\perp}.$$

Closure distributes over unions, and P' is closed (whence $P'^{-\prime} = P'' = P$), so the preceding inclusion may be written in the form

$$P \cap Q^{\perp\perp} \subseteq (P \cap Q)^{\perp\perp}. \tag{6}$$

An application of (6) with $P^{\perp\perp}$ in place of P, followed by an application of (6) with the roles of P and Q interchanged, yields

$$P^{\perp\perp} \cap Q^{\perp\perp} \subseteq (P^{\perp\perp} \cap Q)^{\perp\perp} \subseteq (P \cap Q)^{\perp\perp\perp\perp}.$$

The desired conclusion follows from Lemma 3.

Lemma 4 implies immediately that the intersection of two regular open sets is regular, and hence that the right side of (3) belongs to the class of regular open sets.

So far it has been shown that the class of regular open sets of a topological space X is closed under the operations defined by (1)–(5). To complete the proof of the first assertion of Theorem 1, it must now be shown that these operations satisfy some system of axioms for Boolean algebras. It is less trouble to verify every one of the conditions (2.11)–(2.20) than to prove that some small subset of them is sufficient to imply the rest. In the verifications of (2.11), (2.12), (2.13), (2.15), (2.16), (2.17), (2.18), and (2.19), nothing is needed beyond the definitions and some straightforward computations involving Lemma 3 and the equation

$$(P \cup Q)^{\perp} = P^{\perp} \cap Q^{\perp} \tag{7}$$

(valid for any two sets P and Q). The proof of (7) is quite easy. Closure distributes over union, by (9.3), so

$$(P \cup Q)^{-} = P^{-} \cup Q^{-}.$$

Form the complement of both sides of this equation to arrive at (7).

The validity of the distributive axioms (2.20) in the algebra of regular open sets depends on Lemma 4. Here is the verification of the first of these axioms:

$$\begin{aligned} P \wedge (Q \vee R) &= P \cap (Q \cup R)^{\perp\perp} = P^{\perp\perp} \cap (Q \cup R)^{\perp\perp} \\ &= (P \cap (Q \cup R))^{\perp\perp} = ((P \cap Q) \cup (P \cap R))^{\perp\perp} \\ &= (P \wedge Q) \vee (P \wedge R). \end{aligned}$$

The first and last equalities follow from definitions (3) and (4), the second from the assumed regularity of P, the third from Lemma 4, and the fourth from the distributive law (2.10) for intersection over union.

It remains to verify the complement laws (2.14); this amounts to showing that

$$P \cap P^{\perp} = \varnothing \qquad \text{and} \qquad (P \cup P^{\perp})^{\perp\perp} = X.$$

The first identity is obvious, since $P^{\perp} \subseteq P'$. The second one is not; one way to proceed is by means of a little topological lemma that has other applications also.

Lemma 5. *The boundary of an open set is a nowhere dense closed set.*

Proof. The boundary of an open set P is the set $P^{-} \cap P'$ (see p. 60). If the boundary of P included a non-empty open set, then that open set would have a non-empty intersection (namely itself) with P^{-}, and, at the same time, it would be disjoint from P (because it is included in P'). This contradicts the fundamental property of closure (often used as the definition — see p. 57).

Lemma 5 implies that if P is open, and all the more if it is regular, then the complement of the boundary of P, that is, $P \cup P^{\perp}$, is a dense open set. It follows that $(P \cup P^{\perp})^{\perp} = \varnothing$ and hence that $(P \cup P^{\perp})^{\perp\perp} = X$. This completes the proof of the first assertion of Theorem 1. The second assertion of the theorem follows from the next lemma and its dual.

Lemma 6. *The supremum of a family $\{P_i\}$ of regular open sets is $(\bigcup_i P_i)^{\perp\perp}$.*

Proof. Write $P = (\bigcup_i P_i)^{\perp\perp}$. Each of the sets P_i is included in their union, so Lemma 2 implies that $P_i \subseteq P$ for every i. (Since the meet of two regular open sets is the same as their intersection, the Boolean order relation for regular open sets is the same as ordinary set-theoretic inclusion.) To prove that the upper bound P is the least possible one, suppose Q is a regular open set such that $P_i \subseteq Q$ for every i. The proof that then $P \subseteq Q$ is quite easy: just observe that $\bigcup_i P_i \subseteq Q$ and apply Lemma 1 twice to obtain $P \subseteq Q^{\perp\perp} = Q$.

It is worth pointing out again that the Boolean algebra of regular open sets is not a field of sets, much less a complete field of sets. The example preceding Lemma 1 shows that the join of two regular open sets may be different from their union.

It is also worth mentioning that, in general, this algebra fails to be completely distributive. Consider, for instance, the regular open algebra of the open unit interval $(0, 1)$. (Warning to the would-be expert. Compactness, or its absence, has nothing to do with this example; the endpoints were omitted for notational convenience only.) Let I be the set of non-negative integers and let J be the set consisting of the two numbers $+1$ and -1. To define $P(i, j)$, split the interval into 2^i open intervals of length 2^{-i}; let $P(i, -1)$ be the union of the open left halves of these intervals and let $P(i, +1)$ be the union of their open right halves.

For example, when $i = 1$, the interval $(0, 1)$ is split into the two open intervals $(0, 1/2)$ and $(1/2, 1)$ of length $1/2$. The open left halves of these intervals are $(0, 1/4)$ and $(1/2, 3/4)$, and the open right halves are $(1/4, 1/2)$ and $(3/4, 1)$. The regular open sets $P(1, -1)$ and $P(1, +1)$ are defined by

$$P(1, -1) = (0, 1/4) \cup (1/2, 3/4) \quad \text{and} \quad P(1, +1) = (1/4, 1/2) \cup (3/4, 1).$$

The union of these two sets is the entire space $(0, 1)$ with the points $1/4$, $1/2$, and $3/4$ removed. Consequently, the join of the two sets is the entire space.

In general, the union of the two sets $P(i, -1)$ and $P(i, +1)$ is the entire space with the points of the form $k/2^{i+1}$ removed for $0 < k < 2^{i+1}$. These latter points form a nowhere dense subset of the space, so the join

$$P(i, -1) \vee P(i, +1)$$

is equal to the entire space $(0, 1)$ for each i. It follows that the left side of equation (8.3) is the unit element of the algebra under consideration.

For each function a in J^I, the intersection $\bigcap_i P(i, a(i))$ coincides with the intersection of a nested sequence of open intervals whose lengths go to zero; consequently, the intersection contains at most one point, whatever the function a may be. In fact, the only point that can be in the intersection is the real number whose binary representation has 0 or 1 in the $(i+1)$th place according as $a(i)$ is -1 or $+1$. It follows that the infimum $\bigwedge_i P(i, a(i))$ is the zero element of our algebra for every function a; hence, so is the right side of equation (8.3).

This last argument can be clarified with an example. Suppose a is a function from I to J whose first four values are

$$a(0) = -1, \quad a(1) = +1, \quad a(2) = +1, \quad \text{and} \quad a(3) = -1.$$

Then,

$$\begin{aligned}
P(0, a(0)) &= (0, 1/2),\\
P(1, a(1)) &= (1/4, 2/4) \cup (3/4, 4/4),\\
P(2, a(2)) &= (1/8, 2/8) \cup (3/8, 4/8) \cup (5/8, 6/8) \cup (7/8, 8/8),\\
P(3, a(3)) &= (0/16, 1/16) \cup (2/16, 3/16) \cup \cdots \cup (14/16, 15/16),\\
&\vdots
\end{aligned}$$

The intersection of these sets coincides with the intersection of the open intervals

$$(0, 1/2), \qquad (1/4, 2/4), \qquad (3/8, 4/8), \qquad (6/16, 7/16), \qquad \ldots .$$

If there is a point in this intersection, it can only be the real number whose binary representation begins with $.0110\ldots$. Consequently, the infimum of the family $\{P(i, a(i))\}$ is the empty set.

Exercises

1. Prove that a subset Q of a topological space is regular and open if and only if $Q = P^{\perp\perp}$ for some set P.

2. Prove, for an arbitrary subset P of a topological space, that

$$P^{\perp -} = P^{\perp\perp\perp -}.$$

3. Show that Boolean axioms (2.11), (2.12), (2.13), (2.15), (2.16), and (2.18) are valid in the algebra of regular open sets.

4. Show that the De Morgan laws (2.17) are valid in the algebra of regular open sets.

5. Show that the associative laws (2.19) are valid in the algebra of regular open sets.

6. Show that the distributive law for join over meet in (2.20) is valid in the algebra of regular open sets.

7. Describe the Boolean algebra of regular open subsets of a discrete space.

8. Describe the Boolean algebra of regular open subsets of an infinite space with the cofinite topology.

9. (Harder.) A closed subset P of a topological space X is called *regular* if it is equal to the closure of its interior: $P = P^{\circ -}$. Define operations of join, meet, and complement on the class of regular closed subsets of X, and prove that the resulting algebra is a complete Boolean algebra. (This dual formulation of the regular open algebra is due to Tarski [75].)

10. (Harder.) Prove, using the last assertion of Theorem 1 and the infinite version of the De Morgan laws (Lemma 8.1), that if $\{P_i\}$ is a family of regular open sets, then

$$\Big(\bigcap_i P_i\Big)^{-\prime-\prime} = \Big(\bigcap_i P_i^-\Big)^{\prime-\prime}.$$

Show that this is not necessarily true for arbitrary open sets, and give a direct topological proof for regular open sets.

11. This exercise refers to the notation introduced in the final example of the chapter. Consider the function a from the set of non-negative integers into the set $\{-1, +1\}$ defined by

$$a(i) = \begin{cases} +1 & \text{if } \ i \text{ is even,} \\ -1 & \text{if } \ i \text{ is odd.} \end{cases}$$

   (a) Write out explicitly the sets $P(0, a(0))$, $P(1, a(1))$, $P(2, a(2))$, and $P(3, a(3))$.
   (b) The intersection of the family $\{P(i, a(i))\}$ coincides with the intersection of which family of open intervals?
   (c) What is the binary representation of the only real number that can be in this intersection?
   (d) Is that real number in the intersection?

12. Repeat the preceding exercise for the function a defined by

$$a(i) = \begin{cases} +1 & \text{if} \quad i > 0, \\ -1 & \text{if} \quad i = 0. \end{cases}$$