Homework6

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Exercise 1. Recall that a linear order $(M, \leq)$ is **well-ordered** if every non-empty subset $S \subseteq M$ has a minimum. Show that the class of well-ordered linear orders is **not** an elementary class. In other words, show that there is no theory T whose models are exactly the well-ordered linear orders.

Proof. The language is $L = \{\leq\}$. Let

$$\varphi_n := \exists x_1 \dots x_n \left(\bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j \wedge x_i \leq x_j) \right)$$
$$\Phi := \{\varphi_n : n \in \mathbb{N}\}$$

If there is such theory T whose models are exactly the well-ordered linear orders. Consider the theory $\Gamma = T \cup \Phi$. For any finite subset Δ of Φ, there is a $\varphi_m \in \Phi$ with biggest m and $\vDash \bigwedge \Delta \leq\leftrightarrow \varphi_m$. Since every ordinal is well-ordered, the ordinal m satisfies both T and φ_m, that is, $(m, \leq) \vDash T \cup \varphi_m$, or equivalently, $(m, \leq) \vDash T \cup \Delta$. Thus by compactness Γ is satisfiable and we have a model $\mathfrak{M} \vDash T \cup \Phi$. Hence $\mathfrak{M}$ is well-ordered and has an infinite descending sequence, a contradiction. $\square$

Exercise 2. Let κ be an infinite cardinal. A theory T is said to be κ-categorical if there is a model $M \vDash T$ of size κ, and any two models of size κ are isomorphic. Prove that: Suppose T is κ-categorical for some infinite $\kappa \geq |L|$. Suppose T has no finite models. Then T is complete: if $M_1, M_2 \vDash T$, then $M_1 \equiv M_2$

Proof. For any model M of T, if $|M| > \kappa$, then by Downward Löwenheim–Skolem Theorem, we can find a model $M' \preceq M$ such that $|M'| = \kappa$.

If $|M| < \kappa$, by Löwenheim–Skolem Theorem, we can find a model $M' \vDash T(M)$ with cardinality κ such that $M \preceq M'$. Furthermore, $M' \equiv M$.

If $|M| = \kappa$, let $M' = M$. Then for any $M_1, M_2 \vDash T$, $M_1' \equiv M_1$ and $M_2' \equiv M_2$. As $|M_1'| = |M_2'| = \kappa$, we have $M_1' \cong M_2'$ and hence $M_1' \equiv M_2'$. Thus $M_1 \equiv M_1' \equiv M_2' \equiv M_2$ □

Exercise 3. Let M be an infinite L-structure. Let κ be a cardinal with $\kappa \geq |M|$ and $\kappa \geq |L|$. Show that there is an elementary extension $N \succeq M$ with $|N| = \kappa$.

Proof. Consider the language $L(M)$ and theory $T(M)$ and M is the infinite model of $T(M)$. Then by Löwenheim–Skolem Theorem, $T(M)$ has a model N of size κ and $M \preceq N$ as $\kappa \geq L(M)$. □

Exercise 4. Let M be an L-structure and A be a subset of M. For $i = 1, 2$, let N_i be an elementary extension of M and let $\bar{b}_i$ be an n-tuple in N_i. Show that the following are equivalent

1. $\bar{b}_1$ realizes tp$(\bar{b}_2/A)$
2. tp$(\bar{b}_1/A) \supseteq$ tp$(\bar{b}_2/A)$
3. tp$(\bar{b}_1/A) =$ tp$(\bar{b}_2/A)$

Proof. $1 \to 2$. For any $\varphi(\bar{x}) \in$ tp$(\bar{b}_2/A)$, $N_1 \vDash \varphi(\bar{b}_1)$ as $\bar{b}_1$ realizes tp$(\bar{b}_2/A)$. Hence tp$(\bar{b}_1/A) \supseteq$ tp$(\bar{b}_2/A)$.

$2 \to 3$. For any $\varphi(\bar{x}) \in$ tp$(\bar{b}_1/A)$. If $N_2 \nvDash \varphi(\bar{b}_2)$, then $\neg\varphi(\bar{x}) \in$ tp$(\bar{b}_2/A) \subseteq$ tp$(\bar{b}_1/A)$. Then both $\neg\varphi(\bar{x}) \in$ tp$(\bar{b}_1/A)$, a contradiction.

$3 \to 1$. As tp$(\bar{b}_1/A) =$ tp$(\bar{b}_2/A)$, for any $\varphi(\bar{x}) \in$ tp$(\bar{b}_2/A)$, $N_1 \vDash \varphi(\bar{b}_1)$ and thus $\bar{b}_1$ realizes tp$(\bar{b}_2/A)$. □