The bigger picture of Henkin's method and Gödel's completeness theorem

Introductory Model Theory

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In class, we used Henkin's method to prove the compactness theorem. Henkin's method and the compactness theorem are part of a bigger picture, touching on the other branches of mathematical logic (proof theory, computability theory, and set theory). This document is a sketch of the bigger picture.

1 Provability and proof theory

Let $\Gamma(\bar{x})$ be a set of formulas. Let $\phi(\bar{x})$ be a formula. The notation

$$\Gamma(\bar{x}) \vdash \phi(\bar{x})$$

means

> The formula $\phi(\bar{x})$ can be proven from the formulas in $\Gamma(\bar{x})$.

The relation $\vdash$ is called *provability*.

For example, $x = y,\ y = z \vdash \{x = z\}$ is true, because we can prove $x = z$ from $x = y$ and $y = z$. Similarly, if we are working in a language with two constant symbols c and d, then

$$\forall x\ x = c \vdash d = c.$$

There are several different ways to precisely define the provability relation $\vdash$, such as

- Natural deduction systems
- Hilbert-style systems
- Gentzen-style sequent calculi.

We give an example of a Hilbert-style system below in Section 1.1. All these different proof systems are equivalent, so the choice does not impact the definition of $\vdash$.

The study of proof systems and provability is called *proof theory*.[1] Here are some typical results of proof theory:

[1] **Warning:** I know very little proof theory, so my description here could be completely wrong.

- The Curry-Howard isomorphism, connecting intuitionistic logic to typed lambda calculus.
- Cut elimination in sequent calculus. Roughly speaking, this is a way of simplifying or normalizing proofs.
- Gentzen's proof of the consistency of Peano Arithmetic using ϵ_0-induction. This is somehow a consequence of cut elimination.
- Craig's interpolation theorem. This says that if σ is an L-sentence and σ' is an L'-sentence and $\sigma \vdash \sigma'$, then there is an $(L \cap L')$-sentence τ such that $\sigma \vdash \tau$ and $\tau \vdash \sigma'$.

Among the branches of mathematical logic, proof theory is probably the closest to "logic" in the colloquial sense, since it studies the laws of logic—the rules of deduction.

1.1 One way to define $\vdash$

Fix some signature L. All formulas will be L-formulas. Before defining $\vdash$, we need to discuss some stylistic differences between model theory and proof theory.

In model theory, we usually take $\vee, \wedge, \neg, \bot, \top, \forall, \exists$ as logical primitives, and define $\rightarrow$, $\leftarrow$, and $\leftrightarrow$ from them. For example, $A \rightarrow B$ is defined to mean $(\neg A) \vee B$.

In contrast, proof theorists apparently prefer to take $\vee, \wedge, \rightarrow, \bot, \forall, \exists$ as logical primitives, and then define

$$\begin{aligned} A \leftarrow B &:= B \rightarrow A \\ A \leftrightarrow B &:= (A \leftarrow B) \wedge (A \rightarrow B) \\ \neg A &:= A \rightarrow \bot \\ \top &:= \neg\bot. \end{aligned}$$

Also, model theorists prefer to view $=$ as a logical symbol, something that is present in any theory and doesn't need to be included in the signature. In contrast, proof theorists apparently prefer to view $=$ as a relation symbol in the signature, which comes with axioms like

$$\forall x : x = x$$
$$\forall \bar{x}, y, z : (y = z \wedge \phi(\bar{x}, y) \rightarrow \phi(\bar{x}, z)).$$

We can now define the provability relation $\vdash$. The following is adapted from Definition 2.4.1 in *Basic Proof Theory*, 2nd edition, by Troelstra and Schwichtenberg. Any errors are my fault, not theirs.

Definition 1. The relation $\vdash$ is generated by the following properties:

- Assumptions: If $A \in \Gamma$, then $\Gamma \vdash A$.
- Modus Ponens: If $\Gamma \vdash A \rightarrow B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$.

- Generalization rule: If $\Gamma \vdash A(x, \bar{y})$, then $\Gamma \vdash \forall x\ A(x, \bar{y})$.
- Axioms: If D is one of the following logical axioms, then $\Gamma \vdash D$:

$$
\begin{gathered}
A \to (B \to A), \qquad (A \to (B \to C)) \to ((A \to B) \to (A \to C)), \\
A \to A \vee B, \qquad B \to A \vee B, \\
(A \to C) \to ((B \to C) \to (A \vee B \to C)), \\
A \wedge B \to A, \qquad A \wedge B \to B, \qquad A \to (B \to (A \wedge B)), \\
(\forall x\ A(x, \bar{y})) \to A(t(\bar{y}), \bar{y}), \qquad A(t(\bar{y}), \bar{y}) \to \exists x\ A(x, \bar{y}), \\
(\forall x\ (B(\bar{z}) \to A(x, \bar{z}))) \to (B(\bar{z}) \to \forall y\ A(y, \bar{z})) \\
(\forall x\ (B(x, \bar{z}) \to A(\bar{z}))) \to ((\exists y\ B(y, \bar{z})) \to A(\bar{z})), \\
\bot \to A, \qquad \neg\neg A \to A,
\end{gathered}
$$

  where A, B, C are formulas and t is a term.

More concretely, $\Gamma \vdash A$ holds if there is a sequence $B_1, \ldots, B_n$ such that

- For each i, one of the following holds:
  - $B_i \in \Gamma$.
  - B_i is obtained from B_j and B_k by Modus Ponens, where $j, k < i$.
  - B_i is obtained from B_j by the Generalization rule, where $j < i$.
  - B_i is one of the logical axioms listed above.
- $B_n = A$.

Definition 1 is a "Hilbert-style system." Among the main types of proof systems, Hilbert-style systems are the most straightforward, but also the most clumsy, and the least interesting to proof-theorists.

2 The soundness and completeness theorems

Recall that if M is a structure and σ is a sentence, then $M \models \sigma$ means that M satisfies σ. This was defined in class, and you can also find a definition on page 19 of Poizat's model theory textbook.

There are two other uses of the symbol $\models$. If T is a set of sentences, then $M \models T$ means that $M \models \sigma$ for all $\sigma \in T$. We say that M is a *model* of T if $M \models T$.

Lastly, if T is a set of sentences and σ is a sentence, then $T \models \sigma$ means that every model of T satisfies σ, i.e.,

$$M \models T \implies M \models \sigma.$$

2.1 The soundness theorem

Theorem 2 (Soundness)**.** *If $T \vdash \sigma$, then $T \models \sigma$.*

By definition of $\models$, we can rephrase this as follows:

If $T \vdash \sigma$ and $M \models T$, then $M \models \sigma$.

This says that the laws of proofs are logically sound[2]: if M satisfies the sentences in T and if σ can be proven from T, then M satisfies σ. This is intuitively unsurprising. In fact, the proof of the soundness theorem is a straightforward proof by induction on the length of the proof of σ from T.

2.2 The completeness theorem

The converse to the soundness theorem is the *completeness theorem.*

Theorem 3 (Completeness)**.** *If $T \models \sigma$, then $T \vdash \sigma$.*

The completeness theorem is due to Gödel, and is one of the central theorems of mathematical logic[3] The intuition of the completeness theorem is that if σ *should* be provable from T, then σ is *actually* provable. Therefore, the laws of proofs are complete.

For example, Definition 1 has a long and random-looking list of logical axioms. One might naturally wonder whether any axioms are missing. The completeness theorem ensures that our list is complete, and no further laws of logic are required.

In fact, the completeness theorem says that any further additions to the laws of logic will be either redundant or unsound:

Proposition 4. *Let $\vdash'$ be a relation between theories and sentences. Suppose that $T \vdash \sigma$ implies $T \vdash' \sigma$. Then one of two things happens:*

- *$T \vdash \sigma$ is equivalent to $T \vdash' \sigma$ for all T, σ.*
- *$\vdash'$ is not sound: there is $M \models T$ and $T \vdash' \sigma$ with $M \not\models \sigma$.*

Proof. If $\vdash$ is not equivalent to $\vdash'$, then there are T and σ such that $T \vdash' \sigma$ but $T \not\vdash \sigma$. By the completeness theorem, $T \not\models \sigma$. Therefore there is $M \models T$ such that $M \not\models \sigma$. $\square$

2.3 The completeness theorem, the compactness theorem, and Henkin's method

Recall that a theory T is *consistent* if it does not prove a contradiction ($T \not\vdash \bot$). With a little work, the completeness theorem can be reformulated as follows:

[2]As an adjective, "sound" means "valid, not defective, legal, secure, solid, healthy, . . . "

[3]The completeness theorem should not be confused with the *incompleteness* theorem. See Section 4 for a comparison of the two theorems.

Theorem 5 (Completeness, alternate form)**.** *If T is consistent, then T has a model.*

This looks similar to the compactness theorem:

Theorem 6 (Compactness theorem)**.** *If T is finitely satisfiable, then T has a model.*

In fact, the compactness theorem is an easy consequence of the completeness theorem:

Proof (of Theorem 6 from Theorem 5). Suppose T is finitely satisfiable. We claim T has a model. By Theorem 5 it suffices to show that T is consistent. Otherwise, $T \vdash \perp$. The proof of $\perp$ from T uses only finitely many assumptions from T. So there is a finite $T_0 \subseteq T$ such that $T_0 \vdash \perp$. Then T_0 is inconsistent, so it has no model by the soundness theorem[4]. Therefore T_0 is not finitely satisfiable, a contradiction. □

In class, we proved the compactness theorem using Henkin's method[5] It turns out that Henkin's method can also be used to prove the completeness theorem. Roughly speaking, you can replace "finitely satisfiable" with "consistent" in the proof, and everything works.

Historically, Henkin's method was first developed to prove the completeness theorem, and the compactness theorem was deduced as a corollary of the completeness theorem. For more about this history, see pages 48 and 53 of Poizat's textbook.

2.4 Soundness and completeness as a bridge between model theory and proof theory

The soundness and completeness theorems together say

$$T \vdash \sigma \iff T \models \sigma.$$

This is an equivalence between a syntactic proof-theoretic concept ($\vdash$) and a semantic model-theoretic concept ($\models$). This can be used to transfer statements between proof theory and model theory. For example, consider Craig's interpolation theorem:

Theorem 7. *Let L, L' be two languages. Let ϕ be an L-sentence and ψ be an L'-sentence. If $\phi \vdash \psi$, then there is an $(L \cap L')$-sentence θ such that $\phi \vdash \theta$ and $\theta \vdash \psi$.*

Craig's interpolation is a purely proof-theoretic statement. However, it can be given a model-theoretic proof using saturated models or elementary chains (see Theorem 5.5.3 of Hodges' *Shorter Model Theory*). The completeness theorem allows one to convert the model-theoretic statement into a statement about provability.

In the other direction, there is a purely proof-theoretic proof of the interpolation theorem, using cut-elimination in sequent calculus. Using the completeness theorem, we can convert this into a model-theoretic statement, and obtain useful consequences like the following:

[4]If $M \models T_0$, then $M \models \perp$. But $M \models \perp$ is false by definition.

[5]Henkin's method is the method where you start with a finitely satisfiable theory T, build a larger theory $T' \supseteq T$ that is finitely satisfiable, complete, and has the witness property, and "read off" a model of T' from the atomic sentences in T'. Next week, we will see another method to prove compactness using ultraproducts.

Theorem 8 (Robinson's joint consistency lemma)**.** *Let M be an L-structure and M' be an L'-structure. Suppose that the reducts $M|(L \cap L')$ and $M'|(L \cap L')$ are elementarily equivalent. Then there is an $(L \cup L')$-structure N such that $N|L \equiv M$ and $N|L' \equiv M'$.*

In summary, the completeness theorem yields applications of model theory to proof theory and applications of proof theory to model theory, by serving as a bridge between the two subjects.

3 Complete theories

An L-theory T is *complete* if for any L-sentence σ, either $T \vdash \sigma$ or $T \vdash \neg\sigma$.[6]

Theorem 9. *If $M \models T$ and T is a complete theory, then*

$$\{\sigma : M \models \sigma\} = \{\sigma : T \vdash \sigma\}$$

In other words, T is a complete axiomatization of M: every sentence σ true in M is provable from T.

Proof. We claim that for any sentence σ, the following are equivalent:

1. $T \vdash \sigma$.
2. $M \models \sigma$.
3. $M \not\models \neg\sigma$.
4. $T \nvdash \neg\sigma$.

Indeed, (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) by the soundness theorem. (2) $\Longleftrightarrow$ (3) by definition of $\models$. Finally, (4) $\Longrightarrow$ (1) because T is complete. $\square$

The completeness theorem gives a model-theoretic criterion for a theory to be complete:

Theorem 10. *T is complete if and only if any two models of T are elementarily equivalent.*

Proof. First suppose T is complete. Let M, N be two models. By Theorem 9,

$$\{\sigma : M \models \sigma\} = \{\sigma : T \vdash \sigma\} = \{\sigma : N \models \sigma\},$$

and so $M \equiv N$. Conversely, suppose T is *not* complete. Then there is a sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$. By the completeness theorem, there are models $M, N \models T$ such that $M \models \neg\sigma$ and $T \models \sigma$. Then $M \not\equiv N$. $\square$

[6] This definition is slightly different from the definition in class, where $\sigma \in T$ or $\neg\sigma \in T$, for any sentence σ. If $\overline{T}$ denotes the closure of T under logical consequence, i.e., $\overline{T} = \{\sigma : T \vdash \sigma\}$, then T is complete in the sense here iff $\overline{T}$ is complete in the sense we used in class. Note that the two theories T and $\overline{T}$ are logically equivalent. We usually ignore these distinctions in model theory.

Example. Recall the theory DLO of non-empty dense linear orders without endpoints. In Chapters 1–2 of Poizat's textbook, we saw that any two models of DLO are elementarily equivalent, using the back-and-forth method. By Theorem 10, DLO is complete.

We will see similar examples later in the course. Theorem 10 is often the most practical way to show that a theory is complete.

3.1 Completeness and foundations

The Zermelo-Fraenkel axioms of set theory (ZFC) are conventionally taken as the foundational axioms of mathematics. The theorems of mathematics are exactly the sentences σ such that $\mathrm{ZFC} \vdash \sigma$.

Pretend for the moment that ZFC is a consistent and complete theory. If σ is any sentence, then $\mathrm{ZFC} \vdash \sigma$ or $\mathrm{ZFC} \vdash \neg\sigma$. In other words, either σ or its negation $\neg\sigma$ is a theorem of mathematics—we can either prove or disprove σ. This would mean that ZFC yields the answer to all mathematical questions. In fact, it would imply that there is an algorithm to tell whether a given statement is a theorem of mathematics; see Theorem 11 below.

Unfortunately, ZFC is *not* a consistent and complete theory. The problem is not specific to ZFC; *any* reasonable foundation for mathematics will be incomplete, for reasons related to Gödel's incompleteness theorem. We will say more about this in Section 4 below.

In fact, there are specific set-theoretic statements which are neither provable nor disprovable from ZFC. The most famous is the *continuum hypothesis*. In one form, this says

> Let S be an infinite subset of $\mathbb{R}$. Then S has the same cardinality as $\mathbb{N}$ or $\mathbb{R}$. (In other words, there is a bijection from S to $\mathbb{N}$ or from S to $\mathbb{R}$.)

By work of Cohen and Gödel, the continuum hypothesis (CH) can be neither proven nor disproven from ZFC.

Let σ be a sentence that is neither proven nor disproven from the ZFC axioms

$$\mathrm{ZFC} \nvdash \sigma \text{ and } \mathrm{ZFC} \nvdash \neg\sigma.$$

By the completeness theorem, the only way this can happen is if there are models M, N of ZFC such that $M \models \sigma$ and $N \models \neg\sigma$. And indeed, this is how Cohen and Gödel proved the independence of CH. Gödel constructed a model of ZFC called the *constructible universe* L, and showed that L satisfied the continuum hypothesis. Then Cohen used *forcing* to construct a model $V[G]$ in which the continuum hypothesis is false.[7]

[7]Technically, we can't prove that ZFC is consistent, i.e., we can't prove that any models of ZFC exist. What Gödel really did was show that for any model $M \models \mathrm{ZFC}$, one can construct another model $L^M \models \mathrm{ZFC}$ such that L^M satisfies the continuum hypothesis. Therefore, *if* ZFC is consistent, then $\mathrm{ZFC} \nvdash \neg\mathrm{CH}$. What Cohen did is similar.

3.2 Complete theories and computability

Complete theories can be used to show that certain sets are *computable* in the sense of computability theory.[8]

Theorem 11. *Let T be a consistent and complete theory. Suppose that T is finite, or more generally, computable, or more generally, computably enumerable. Then the following set is computable:*

$$\{\sigma : T \vdash \sigma\}.$$

In other words, there is an algorithm which takes a sentence σ as input, and correctly determines whether σ is provable from T.

Proof. By the proof of Theorem 9, we know the following two sets are complementary:

$$\{\sigma : T \vdash \sigma\}$$
$$\{\sigma : T \vdash \neg\sigma\}.$$

It's easy to see that both sets are computably enumerable, essentially because the set of proofs is computable. Therefore, both sets are computable. □

In practice, most theories one encounters are computably enumerable. (The exception is things like "the true theory of arithmetic"—the set of all sentences satisfied by the structure $(\mathbb{N}, +, \cdot, 0, 1, \leq)$.)

Corollary 12. *Let M be a model of a complete theory T. Suppose that T is finite, or more generally, computable, or more generally, computably enumerable. Then the following set is computable:*

$$\{\sigma : M \models \sigma\}.$$

In other words, there is an algorithm which takes a sentence σ as input, and correctly determines whether σ is true or false in M.

Example. The structure $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ is a model of a certain computable theory RCF (*real closed fields*). Using back-and-forth equivalence, one can show that any two models of RCF are elementarily equivalent. By Theorem 10, RCF is complete. By Corollary 12, there is an algorithm which calculates whether a given sentence is true in $\mathbb{R}$.

4 The incompleteness theorem

Recall that ZFC is the set of Zermelo-Fraenkel axioms of set theory, which are conventionally taken as a foundation for mathematics. Another theory of interest is *Peano Arithmetic* (PA), a set of axioms for the structure $(\mathbb{N}, +, \cdot, 0, 1, \leq)$. While PA is much weaker than ZFC, PA can prove most theorems of number theory, and can arguably serve as a foundational system for "finite mathematics".

Here are two versions of Gödel's famous *incompleteness theorem*:

[8] Computability theory is also called recursion theory. Computable sets are also called recursive sets. Computably enumerable sets are also called recursively enumerable sets.

Theorem 13 (Incompleteness theorem for Peano Arithmetic)**.** *PA is incomplete.*

Theorem 14 (Incompleteness theorem for ZFC)**.** *If ZFC is consistent, then ZFC is incomplete.*

Remark 15. We should make a few remarks on the statements of the theorems.

1. The original theorem by Gödel was not for Peano Arithmetic or ZFC, but instead for the ramified type theory developed in Russell and Whitehead's *Principia Mathematica*. But the same argument works in any of these three contexts.

2. We don't need to assume that PA is consistent in Theorem 13, because we already know it has a model $(\mathbb{N}, +, \cdot, 0, 1, \leq)$.[9]

3. Gödel's original argument required a stronger assumption (ω-consistency rather than consistency), but one can weaken the assumption to consistency using Rosser's trick. For more information, see Wikipedia.

Gödel's incompleteness theorem tells us that ZFC cannot provide the answers to all mathematical questions. More precisely, the set

$$\{\sigma : \text{ZFC} \vdash \sigma\}$$

is *not* the set of true sentences about the set-theoretic universe. In theory, this could hold for two different reasons: a problem with ZFC or a problem with $\vdash$. In the first case, we are missing some laws of set theory. In the second case, we are missing some laws of proofs. However, Gödel's *completeness* theorem rules out the second possibility. So we are forced to concede that ZFC is missing some of the properties of sets.

In summary,

- Gödel's *completeness* theorem says that the laws of logic are complete—none are missing.

- Gödel's *incompleteness* theorem says that the axioms of set theory are not complete—they fail to completely describe the set-theoretic universe.

We might hope to "fix" ZFC by adding in the missing properties of sets. Unfortunately, this doesn't work, as Gödel's incompleteness theorem applies to any computable consistent theory extending ZFC. In fact, it applies to any reasonable foundation for mathematics. So incompleteness is an unavoidable problem.

[9] This doesn't work for ZFC, because the set-theoretic universe $(V, \in)$ is not technically a model. A model is a *set* with some functions and relations, but V is a *proper class*—it's too big to be a set. For technical reasons, model theory doesn't work with class-sized models. In fact one cannot define the relation $M \models \sigma$ if M is class-sized.

Remark 16. In the early work in symbolic logic by Frege and others, set theory was viewed as a part of logic. The axioms of set theory were viewed as the same type of thing as the rules of deduction in first-order logic. However, Gödel's completeness and incompleteness theorems helped clarify the important distinction between these two things. The laws of first-order logic are complete; we know we aren't missing any. On the other hand, the laws of set-theory are incomplete; we know we are definitely missing some. For this reason and others, the axioms of set theory are no longer seen as purely "logical" laws.

4.1 Second-order vs first-order logic

Second-order logic is similar to first-order logic, except that we can quantify over subsets of the structure, in addition to quantifying over elements. For example, in a linear order $(M, \leq)$, we could write an axiom like

$$\forall S \subseteq M \left[(\exists x \in M : x \in S) \to (\exists x_0 \in M : x_0 \in S \ \wedge \ (\forall y : y \in S \to x_0 \leq y))\right].$$

(This says that $(M, \leq)$ is well-ordered.) Second-order logic is much more expressive than first-order logic. Unfortunately, the technique of Gödel's incompleteness theorem can be used to show that second order logic is "incomplete," in the sense that we cannot define a sensible notion of provability $\vdash$ and get the completeness theorem to hold. For this reason, mathematical foundations (like ZFC) are usually layered on top of first-order logic, even if we are interested in sets, which would naturally appear in second-order logic.

Second-order logic also fails to satisfy the compactness theorem, which is the central tool of model theory. For this reason, model theory focuses almost exclusively on first-order logic.

On the other hand, first-order logic is much less expressive than second-order logic. Above, we gave a second-order sentence which says "$(M, \leq)$ is well-ordered." This condition cannot be expressed in first-order logic, however. Since well-orderedness is a very important property in mathematics, this is a little disappointing.

Model theorists see this as a feature, not a bug. The weakness of first-order logic allows things like the compactness theorem and Löwenheim-Skolem theorem to be true. Modern model theory is mostly the art of using these two theorems to deduce interesting mathematical consequences.