Midterm review

Introduction to Model Theory

November 1, 2021

Here is the important information to review for the midterm exam. Nothing from the third hour will appear on the exam. For now, nothing related to the back-and-forth method of Fraïssé's theorem will appear on the exam.

- Know what languages, structures, sentences, formulas, and terms are (Chapter 3 of the textbook.) Some things to note:
  - A term is an element-valued expression (like $x + y$), a formula is a truth-valued expression (like $x > y$).
  - A sentence is a formula with no free variables.
- If M is a structure and σ is a sentence, then $M \models \sigma$ means that M satisfies σ, i.e., σ is true in M.
  - This relation is defined precisely in the textbook in Section 2.1 (for relations), in Section 3.2 (for general structures).
  - Also—and this is important—if $\phi(x_1, \ldots, x_n)$ is a formula (with free variables), we can substitute in elements of M. So if $a_1, \ldots, a_n \in M$, it makes sense to talk about whether M satisfies $\phi(a_1, \ldots, a_n)$, written $M \models \phi(a_1, \ldots, a_n)$ (or $M \models \phi(\bar{a})$ as an abbreviation). For example, if M is the structure $(\mathbb{Z}, +, \cdot)$ then we might write something like $\mathbb{Z} \models 2 \cdot 2 + 3 = 7$, even though $2, 3, 7$ aren't symbols in the language. The idea is that we are writing $\mathbb{Z} \models \phi(2, 3, 7)$, where $\phi(x, y, z)$ is the formula $x \cdot x + y = z$.
- A *theory* is a set of sentences. If M is a structure and T is a theory, then $M \models T$ means that for every $\sigma \in T$, we have $M \models \sigma$. We say that M is a *model* of T if $M \models T$. We say that M *satisfies* T if $M \models T$.
- The difference between a "structure" and a "model." For any language L, there is a class of L-*structures*. Then if T is an L-theory (a set of L-sentences), there is a class of *models of* T. So usually, a "structure" means any L-structure, and a "model" means a structure that is a model of T, where T and L are the theory and language we are working with. But sometimes we model theorists slip up and say "model" when

we mean "structure," because the name of the subject is model theory not structure theory.

- Two structures M_1, M_2 are *elementarily equivalent*, written $M_1 \equiv M_2$, if for every sentence σ,
$$M_1 \models \sigma \iff M_2 \models \sigma.$$
In other words, they satisfy exactly the same sentences.

- Understand what it means for two structures to be *isomorphic*. The definition can be found on pages 1, 32, and 35 of the textbook. If M_1 and M_2 are isomorphic, then they are elementarily equivalent.

- Elementary substructure, elementary extension, and elementary embeddings:
  - $M \preceq N$ means that $M \subseteq N$ and for every formula $\phi(x_1, \ldots, x_n)$ and every tuple $(a_1, \ldots, a_n) \in M^n$, we have
  $$M \models \phi(\bar{a}) \iff N \models \phi(\bar{a}).$$
  - This has to hold for all n, including $n = 0$. When $n = 0$, ϕ is a sentence and this says that $M \equiv N$. So $M \preceq N$ implies $M \equiv N$. But $M \preceq N$ is a much stronger condition, since it says that M and N believe the same things about elements of M. For example, if $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$, then $(2\mathbb{Z}, \leq) \equiv (\mathbb{Z}, \leq)$, since they are isomorphic. But $(2\mathbb{Z}, \leq) \not\preceq (\mathbb{Z}, \leq)$, since if $\phi(x, y)$ is the formula $\exists z : x < z \ \wedge \ z < y$, then
  $$(2\mathbb{Z}, \leq) \not\models \phi(0, 2)$$
  $$(\mathbb{Z}, \leq) \models \phi(0, 2).$$
  - If $M \preceq N$, we say that M is an *elementary substructure* of N. Also, we say that N is an *elementary extension* of M.
  - An *elementary embedding* from M to N is a function $f : M \to N$ such that for every formula $\phi(x_1, \ldots, x_n)$ and every tuple $(a_1, \ldots, a_n)$, we have
  $$M \models \phi(a_1, \ldots, a_n) \iff N \models \phi(f(a_1), \ldots, f(a_n)).$$
  - If $f : M \to N$ is an elementary embedding, then f is an injection (take $\phi(x, y)$ to be the formula $x = y$).
  - $M \preceq N$ means exactly that $M \subseteq N$ and the embedding $i : M \to N$ is an elementary embedding.
  - An elementary embedding from M to N is the same thing as an isomorphism from M to an elementary substructure of N.

- If we have an elementary embedding $f : M \to N$, it turns out we can always "move N by an isomorphism" to make it be an elementary *extension* of M. Specifically, there is at least one isomorphism $j : N \to N'$ such that $M \preceq N'$ and the composition $(j \circ f) : M \to N'$ is the inclusion.

- The compactness theorem: let T be a theory (a set of sentences). Suppose that every finite subset $T_0 \subseteq T$ has a model. Then T has a model. This is proven in the textbook, but I find the proof hard to follow there (because it assumes too much algebra and topology), and so I wrote up a separate set of notes. They were the class notes for October 14 and 21.

- Be familiar with the technique of adding constant symbols, especially the language $L(A)$ obtained by adding a constant symbol for each element of a set $A \subseteq M$, and the theory $T(A)$ consisting of all the $L(A)$-sentences true in M.
  - For examples of this technique, see Lemma 4.11 to Theorem 4.14 in the textbook, as well as Section 5.1.
  - In particular, in the case $A = M$, the theory $T(M)$ is called the *elementary diagram* of M, and models of $T(M)$ are (almost) the same thing as elementary extensions of M.
  - More precisely, a model of $T(M)$ is the same thing as an L-structure N plus an elementary embedding $M \to N$.

- If L is a language, be aware of what $|L|$ means. See Section 3.3 of the textbook, where Poizat calls it $card(L)$. At any rate
  - $|L|$ is the number of symbols in the language (constants symbols, function symbols, and relation symbols). Except that if there are finitely many, then we set $|L| = \aleph_0$. ($\aleph_0$ is the size of $\mathbb{N}$. It's also called ω because set theorists code cardinal numbers as ordinal numbers in a specific way and the cardinal number $\aleph_0$ is identified with the ordinal number ω. If all this is new to you, it might be a good idea to review Chapter 8 of Poizat, which doesn't depend on any of the previous chapters, doesn't contain any model theory, and is a brief introduction to cardinal and ordinal numbers.)
  - The end result is that $|L|$ is the number of L-sentences.

- If M is a structure, then $|M|$ is the size of M.

- The Löwenheim-Skolem theorem. Actually, there are three of these theorems:
  1. If M is an L-structure and $|L| \leq \kappa \leq |M|$, then there is an elementary substructure $N \preceq M$ with $|N| = \kappa$. More generally, suppose $A \subseteq M$. If $\max(|A|, |L|) \leq \kappa \leq |M|$, then there is an elementary substructure $N \preceq M$ with $A \subseteq N$ and $|N| =$

κ. This is called the *Downward Löwenheim-Skolem theorem* or *Löwenheim's theorem*, and is proved in the textbook (Theorem 2.5 for relations, Theorem 3.1 for structures).

2. The *Löwenheim-Skolem theorem*: let T be an L-theory. Suppose T has an infinite model. Then for every cardinal $\kappa \geq |L|$, there is a model $M \models T$ with $|M| = \kappa$. This is Theorem 4.10 in the textbook.
   - Also, you can weaken the assumption. Instead of assuming that T has an infinite model, you can assume that for each natural number n, there is a model of T that has size at least n.
3. The *upward Löwenheim-Skolem theorem*. (But some people call this the Löwenheim-Skolem theorem.) Let M be an infinite L-structure. Let κ be a cardinal with $\kappa \geq \max(|L|, |M|)$. Then there is an elementary extension $N \succeq M$ with $|N| = \kappa$. This isn't in the textbook (as far as I know). It's a homework problem on the homework due this Thursday.

- The Tarski-Vaught criterion/test for elementary substructures. This is Theorem 2.4 in the textbook, where it is called Tarski's test. Poizat states the Tarski-Vaught test for relations, but generalizes it to structures on page 36. Anyway, here is what it says: let M be a structure. Let A be a subset. Then A is an elementary substructure of M iff the following condition is true:

  > Let $\phi(\bar{x}, y)$ be a formula. Let $\bar{a}$ be a tuple in the smaller set A. Suppose the big structure M satisfies $M \models \exists y\ \phi(\bar{a}, y)$. Then there is $b \in A$ such that $M \models \phi(\bar{a}, b)$.

  (Later, we might talk about *definable sets*. A set $X \subseteq M^n$ is *definable* with parameters from A if it has the form $\{\bar{b} \in M^n : M \models \phi(\bar{a}, \bar{b})\}$ for some formula $\phi(x_1, \ldots, x_m; y_1, \ldots, y_n)$ and some tuple $\bar{a} \in A^m$. The Tarski-Vaught test says that A is an elementary substructure of M if and only if for any A-definable non-empty set $S \subseteq M$, we have $S \cap A \neq \varnothing$. This might be an easier form to remember.)

- Ultrafilters. (See the special notes on compactness for the October 14 and October 21 lessons.)
  - An *ultrafilter* on a set I is a family $\mathcal{U} \subseteq \mathcal{P}(I)$ such that
    1. $I \in \mathcal{U}$ and $\varnothing \notin \mathcal{U}$.
    2. If $X, Y \in \mathcal{U}$, then $X \cap Y \in \mathcal{U}$.
    3. If $X \subseteq Y \subseteq I$ and $X \in \mathcal{U}$, then $Y \in \mathcal{U}$.
    4. For any $X \subseteq I$, either $X \in \mathcal{U}$ or $(I \setminus X) \in \mathcal{U}$.

    If only the first three conditions hold, then we call $\mathcal{U}$ a *filter*.
  - The intuition for ultrafilters is that if $X \subseteq I$, then either X is "big" or "small." $\mathcal{U}$ is the set of big sets. $\mathcal{P}(I) \setminus \mathcal{U}$ is the set of "small" sets. A set is big if and only

if its complement is small. The union of two small sets is small; the intersection of two big sets is big. Any subset of a small set is small. Any superset of a big set is big. $\varnothing$ is small. I is big.

  - A family of sets $\mathcal{S} \subseteq \mathcal{P}(I)$ has the *finite intersection property* (FIP) if for any $n \geq 0$ and $X_1, \ldots, X_n \in \mathcal{S}$, the intersection $\bigcap_{i=1}^n X_i$ is non-empty.
  - Fact: if $\mathcal{S}$ has the FIP, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{S}$. (This is Lemmas 13+15 in the special notes.)

- Ultraproducts. (See the special notes on compactness for the October 14 and October 21 lessons.)
  - Let M_i be a set for each $i \in I$. The *product* $\prod_{i \in I} M_i$ is the set of functions $f : I \to \bigcup_{i \in I} M_i$ such that for every $i \in I$, we have $f(i) \in M_i$.
    * The intuition is that if I is a finite set $\{i_1, i_2, \ldots, i_n\}$, then there is a bijection between $\prod_{i \in I} M_i$ and the cartesian product $M_{i_1} \times M_{i_2} \times \cdots \times M_{i_n}$ sending a function f to the tuple $(f(i_1), f(i_2), f(i_3), \ldots, f(i_n))$.
    * In the case where $I = \mathbb{N}$, we often think of $\prod_{i \in \mathbb{N}} M_i$ as a set of "infinite tuples" $(a_1, a_2, a_3, \ldots)$ where $a_1 \in M_1$, $a_2 \in M_2$, $a_3 \in M_3$, and so on.
  - Now suppose each M_i is **non-empty** and there is some ultrafilter $\mathcal{U}$ on I. Then the *ultraproduct* $\prod_{i \in I} M_i/\mathcal{U}$ is the set of equivalence classes of the equivalence relation on $\prod_{i \in I} M_i$ defined by setting $f \sim g$ if and only if

$$\{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

    The intuition is that f and g are equivalent if $f(i) = g(i)$ for "most" i.
    * I wrote the equivalence class of f as f^*, in order to tie to the proof of compactness via Henkin's method. Some other people write it as $[f]$. Iin Poizat's textbook, he talks about ultraproducts in section 4.1 and doesn't introduce a notation for the class of f.
  - Now suppose each M_i is not just a set, but an L-structure. Then we can make the ultraproduct $M = \prod_{i \in I} M_i/\mathcal{U}$ into an L-structure as well, as follows:
    * If c is a constant symbol in L, define $a \in \prod_{i \in I} M_i$ by $a(i) = c^{M_i}$. (c^{M_i} means "the interpretation of the symbol c in the L-structure M_i.".) Then define a^M to be the class of f.
    * If f is a function symbol... let's suppose that f is binary (2-ary) for simplicity. Take two elements of the ultraproduct $M = \prod_{i \in I} M_i/\mathcal{U}$. We can write them as a^* and b^* for some $a, b \in \prod_{i \in I} M_i$. Define $c \in \prod_{i \in I} M_i$ by $c(i) = f^{M_i}(a(i), b(i))$ for all i. Then $f^M(a^*, b^*)$ is defined to be c^*. (It takes some work to check that this is well-defined.)
    * If R is, for example, a 2-ary relation symbol in the language, then we define R^M as follows. Take two elements of the ultraproduct. We can write them as

a^* and b^* for some $a, b \in \prod_{i \in I} M_i$. Then define $R^M(a^*, b^*)$ to be true if and only if

$$\{i \in I : M_i \models R(a(i), b(i))\} \in \mathcal{U}.$$

In other words, $M \models R(a^*, b^*)$ if and only if $M_i \models R(a(i), b(i))$ for "most" $i \in I$. Again, there is work to check that this is well-defined.

- At the end of the day, we get an L-structure called the *ultraproduct*.
- The key property of ultraproducts is *Łoś's theorem*. (Łoś is pronounced "wosh". The ł is pronounced like w and the ś is pronounced like x in pinyin.) Łoś's theorem says that if we have an ultraproduct $M = \prod_{i \in I} M_i / \mathcal{U}$ and we have an L-formula $\phi(x_1, \ldots, x_n)$, and some elements $a_1, a_2, \ldots, a_n \in \prod_{i \in I} M_i$, then

$$M \models \phi(a_1^*, \ldots, a_n^*) \iff \{i \in I : M_i \models \phi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}.$$

  The idea is that M satisfies $\phi(a_1^*, \ldots, a_n^*)$ if and only if M_i satisfies $\phi(a_1^*, \ldots, a_n^*)$ for "most" i, where we interpret a^* as $a(i)$.
- Specializing Łoś's theorem to the case where ϕ is an L-sentence (no free variables), we see that the ultraproduct $M = \prod_{i \in I} M_i / \mathcal{U}$ satisfies the following:

$$M \models \phi \iff \{i \in I : M_i \models \phi\} \in \mathcal{U}.$$

  In other words, M satisfies a sentence ϕ if and only if "most" of the M_i satisfy ϕ. This uniquely determines the complete theory of M, and can be used to give a proof of the compactness theorem.
- It may be helpful to notice that we can read off the structure/definition of ultraproducts from Łoś's theorem.
  * If we take the formula $\phi(x, y)$ to be $x = y$, then Łoś's theorem says

$$M \models a^* = b^* \iff \{i \in I : M_i \models a(i) = b(i)\} \in \mathcal{U},$$

    or equivalently

$$a^* = b^* \iff \{i \in I : a(i) = b(i)\} \in \mathcal{U}$$

    which is the definition we used above.
  * If R is a 2-ary relation symbol and we take $\phi(x, y) = R(x, y)$, then Łoś says

$$M \models R(a^*, b^*) \iff \{i \in I : M_i \models R(a(i), b(i))\} \in \mathcal{U},$$

    which is how we defined R^M above.
  * If R is a 2-ary function symbol and we take $\phi(x, y, z)$ to be the formula $(f(x, y) = z)$, then Łoś says

$$M \models f(a^*, b^*) = c^* \iff \{i \in I : M_i \models f(a(i), b(i)) = c(i)\} \in \mathcal{U}.$$

    Well, then, if we decide to take $c(i)$ to be $f(a(i), b(i))$ for all i, then $M_i \models f(a(i), b(i)) = c(i)$ will hold for *all* i, so M must also think that $f(a^*, b^*) = c^*$. This was how we defined f^M above.

This was the approach used in the class notes. Rather than going through the standard explicit definition of ultraproducts (given above, and on page 40 of the textbook), we used Łoś's theorem as the "definition" of ultraproducts. The reason this worked was because Łoś's theorem specifies what the theory of the ultraproduct should be, the theory happens to have the witness property, and we happened to have just built the machinery to buil a model from a theory with the witness property.

- From Łoś's theorem, you can see that if all the M_i are models of some theory T, then the ultraproduct is also a model of T.
- Suppose N is an ultraproduct $\prod_{i \in I} M/\mathcal{U}$, i.e., the ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$ where $M_i = M$ for all i. Then we call N the *ultrapower*, and we sometimes write it as $M^{\mathcal{U}}$ or $M^I/\mathcal{U}$. From Łoś's theorem you can see that $M \equiv M^{\mathcal{U}}$, and in fact there is an elementary embedding $M \to M^{\mathcal{U}}$. This was a homework problem.
- If $M = (\mathbb{R}, +, \cdot, \leq, 0, 1)$ and if $I = \mathbb{N}$, and we take a *non-principal ultrafilter*, then the ultrapower $M^{\mathcal{U}}$ is called "the" set of *hyperreal numbers*. There is a more concrete description of this ultrapower given in the notes on compactness and ultrafilters. The elementary embedding from $\mathbb{R}$ to $\mathbb{R}^{\mathcal{U}}$ is the map sending x to the class of the infinite tuple $(x, x, x, x, \ldots)$. (This map is often called the *diagonal embedding* because it's similar to the map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ sending x to (x, x), whose image is a diagonal line in the plane.)

• Types (section 5.1 of the textbook, and the class notes for October 28).

  - If $A \subseteq M$ and $\bar{b}$ is an n-tuple in M (i.e., $\bar{b} \in M^n$), then the *type* of $\bar{b}$ over A is the set of formulas with parameters from A satisfied by $\bar{b}$. So it is $\{\phi(x_1, \ldots, x_n) \in L(A) : M \models \phi(b_1, \ldots, b_n)\}$. Equivalently, it is $\{\phi(\bar{x}, \bar{a}) : \phi(\bar{x}, y_1, \ldots, y_m) \in L,\ \bar{a} \in A^m,\ M \models \phi(\bar{x}, \bar{a})\}$. The type of $\bar{b}$ over A is written $\operatorname{tp}(\bar{b}/A)$. When A is empty we often just write this as $\operatorname{tp}(\bar{b})$.
  - An *n-type* over A is something of the form $\operatorname{tp}(\bar{b}/A)$, where $\bar{b}$ is an n-tuple in M or in an elementary extension. The set of n-types over A is written $S_n(A)$.
  - If $\Sigma(x_1, \ldots, x_n)$ is a set of $L(A)$-formulas in the variables $\bar{x}$, then $\bar{b}$ *satisfies* or *realizes* $\Sigma(\bar{x})$ if $M \models \phi(\bar{b})$ for every $\phi \in \Sigma$. We also say that $\bar{b}$ is a *realization* of $\Sigma(\bar{x})$. When $\Sigma(\bar{x})$ is a type over A, a realization of Σ is the same thing as a tuple $\bar{b}$ such that $\operatorname{tp}(\bar{b}/A) = \Sigma(\bar{x})$. (This is one of the homework problems.)
  - It turns out (see the notes for October 28) that if $\Sigma(\bar{x})$ is a set of $L(A)$-formulas, then...
    * $\Sigma(\bar{x})$ can be extended to a type over A if and only if $\Sigma(\bar{x})$ is finitely satisfiable in M. This means that every finite subset of $\Sigma(\bar{x})$ is satisfied by a tuple from M. (But $\Sigma(\bar{x})$ itself might only be satisfied in an elementary extension of M.) A *partial type* over A is a set of $L(A)$-formulas with that property.
    * A partial type $\Sigma(\bar{x})$ is a type if and only if it is a maximal partial type.

* Sometimes "types" and "partial types" are called "complete types" and "types," respectively.

- Combining this machinery, we see that if $\Sigma(\bar{x})$ is a set of $L(A)$-formulas that is finitely satisfiable in M, then there is a complete type $p(\bar{x})$ extending $\Sigma(\bar{x})$, and there is an elementary extension $N \succeq M$ and a tuple $\bar{b}$ in N with $p = \mathrm{tp}(\bar{b}/A)$. Then $\bar{b}$ realizes p, so it satisfies $\Sigma(\bar{x})$. And so, any set of $L(A)$-formulas which is finitely satisfiable in M is satisfied in an elementary extension of M.