

# Introduction To Model Theory

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## 1 Back-and-forth Equivalence I

Convention: Relations and functions are sets of pairs  $(x, y)$

**Definition 1.1.** A **binary relation** is a pair  $(E, R)$  where  $E$  is a set and  $R \subseteq E^2$ . We call  $E$  the **universe** of the relation. For  $a, b \in E$ , write  $aEb$  if  $(a, b) \in R$

We abbreviate  $(E, R)$  as  $R$  or  $E$ , if  $E$  or  $R$  is clear

**Example 1.1.**  $(\mathbb{R}, <)$ ,  $(\mathbb{R}, =)$ ,  $(\mathbb{R}, \geq)$ ,  $(\mathbb{Z}, <)$

**Definition 1.2.** A binary relation  $R$  is said to be

- **reflexive** if  $aRa$  ( $\forall a \in E$ )
- **symmetric** if  $aRb \Rightarrow bRa$  ( $\forall a, b \in E$ )
- **transitive** if  $aRb \wedge bRc \Rightarrow aRc$  ( $\forall a, b, c \in E$ )
- **antisymmetric** if  $aRb \wedge bRa \Rightarrow a = b$  ( $\forall a, b \in E$ )
- **total** if  $aRb \vee bRa$  ( $\forall a, b \in E$ )
- an **equivalence relation** if it's reflexive, symmetric and transitive
- a **partial order** if it's reflexive, antisymmetric and transitive
- a **linear order** if it's a total partial order

**Example 1.2.**  $=$  is an equivalence relation

$\subseteq$  is a partial order

$\leq$  is a linear order

**Definition 1.3.** An **isomorphism** from  $(E, R)$  to  $(E', R')$  is a bijection  $f : E \rightarrow E'$  s.t. for any  $a, b \in E$ ,  $aRb \Leftrightarrow f(a)R'f(b)$ . Two binary relations  $(E, R)$  and  $(E', R')$  are **isomorphic** ( $\cong$ ) if there is an isomorphism between them

**Example 1.3.**  $f : (\mathbb{Z}, <) \rightarrow (2\mathbb{Z}, >)$  and  $f(x) = -2x$  is an isomorphism.  
 $x < y \Leftrightarrow -2x > -2y$

$\cong$  is an equivalence relation

**Definition 1.4.** A **local isomorphism** from  $R$  to  $R'$  is an isomorphism from a finite restriction of  $R$  to a finite restriction of  $R'$ . The set of local isomorphisms from  $R$  to  $R'$  is denoted  $S_0(R, R')$ . For  $f \in S_0(R, R')$ ,  $\text{dom}(f)$  and  $\text{im}(f)$  denote the domain and range of  $f$

**Example 1.4.**  $(\mathbb{Z}, <)$  is a restriction of  $(\mathbb{R}, <)$

**Example 1.5.** Suppose  $R = R' = (\mathbb{Z}, <)$ , there is  $f \in S_0(R, R')$  given by  $\text{dom}(f) = \{1, 2, 3\}$  and  $\text{im}(f) = \{10, 20, 30\}$  and  $f(1) = 10, f(2) = 20, f(3) = 30$

**Definition 1.5.** Let  $f, g$  be local isomorphisms from  $R$  to  $R'$ . Then  $f$  is a **restriction** of  $g$  if  $f \subseteq g$  and  $f$  is an **extension** of  $g$  if  $f \supseteq g$ .

**Example 1.6.**  $g : \{0, 1, 2, 3\} \rightarrow \{5, 10, 20, 30\}$ ,  $g$  extends  $f$  in the previous example

**Definition 1.6.** Let  $R, R'$  be binary relations with universe  $E, E'$ . A **Karpian family** for  $(R, R')$  is a set  $K \subseteq S_0(R, R')$  satisfying the following two conditions for any  $f \in K$

1. (**forth**) if  $a \in E$  then there is  $g \in K$  with  $g \supseteq f$  and  $a \in \text{dom}(g)$
2. (**back**) if  $b \in E'$  then there is  $g \in K$  with  $g \supseteq f$  and  $b \in \text{im}(g)$

$R$  and  $R'$  are  $\infty$ -**equivalent**, write  $R \sim_\infty R'$ , if there is a non-empty Karpian family

**Proposition 1.7.** If  $f : (E, R) \rightarrow (E', R')$  an isomorphism and  $K = \{g \subseteq f : g \text{ is finite}\}$ , then  $K$  is Karpian and  $R \sim_\infty R'$

*Proof.* Suppose  $g \in K$

- (forth) Suppose  $a \in E$ , take  $b = f(a)$  and let  $h = g \cup \{(a, b)\}$ . Then  $h \subseteq f$ , so  $h \in K$ ,  $h \supseteq g$ ,  $a \in \text{dom}(h)$
- (back) similarly

□

**Proposition 1.8.** If  $(E, R)$  and  $(E', R')$  are countable and  $R \sim_\infty R'$ , then  $R \cong R'$

*Proof.* Let  $K \subseteq S_0(R, R')$  be Karpian,  $K \neq \emptyset$ ,  $E = \{e_1, e_2, e_3, \dots\}$ ,  $E' = \{e'_1, e'_2, e'_3, \dots\}$

Recursively build  $f_1 \subseteq f_2 \subseteq \dots, f_i \in K$

Let  $f_1$  be anything in  $K$  as  $K$  is non-empty.

$f_{2i}$  some extension of  $f_{2i-1}$  with  $e_i \in \text{dom}(f_{2i})$

$f_{2i+1}$  some extension of  $f_{2i}$  with  $e'_i \in \text{im}(f_{2i+1})$

Now let  $g = \bigcup_{i=1}^\infty f_i$ , then  $g$  is an isomorphism

□

**Definition 1.9.** A **dense linear order without endpoints** (DLO) is a linear order  $(C, \leq)$  satisfying

1.  $C \neq \emptyset$
2.  $\forall x, y \in C, x < y \Rightarrow \exists z \in C \ x < z < y$
3.  $\forall x \in C, \exists y, z \in C \ y < x < z$

**Example 1.7.**  $(\mathbb{Q}, \leq), (\mathbb{R}, \leq)$   
 non-example:  $(\mathbb{Z}, \leq), ([0, 1], \leq)$

**Proposition 1.10.** Let  $(C, \leq)$  and  $(C', \leq)$  be DLO's. Then  $S_0(C, C')$  is Karpian.  
 So  $C \sim_\infty C'$

*Proof.* Let  $f \in S_0(C, C')$ ,  $\text{dom}(f) = \{a_1, \dots, a_n\}$ ,  $a_1 < \dots < a_n$  and  $\text{im}(f) = \{b_1, \dots, b_n\}$ ,  $b_1 < \dots < b_n$ . Since  $f$  is a local isomorphism,  $f(a_i) = b_i$

- (forth) Suppose  $a \in C$ . We want  $b \in C'$  s.t.  $f \cup \{(a, b)\} \in S_0(C, C')$ .
  - if  $a_i < a < a_{i+1}$ . We take  $b \in C'$  s.t.  $b_i < b < b_{i+1}$  since dense
  - if  $a < a_1$ . We take  $b \in C'$  s.t.  $b < b_1$  since no endpoints
  - if  $a > a_n$ , take  $b \in C'$  s.t.  $b > b_n$
  - if  $a = a_i$ , take  $b = b_i$
- (back) similar

□

**Proposition 1.11.** If  $(C, \leq)$  and  $(C', \leq)$  are countable DLOs, then  $C \sim_\infty C'$ , so  $C \cong C'$

Hence

$$\begin{aligned}
 (\mathbb{Q}, \leq) &\cong (\mathbb{Q} \setminus \{0\}, \leq) \\
 &\cong (\mathbb{Q} \cup \{\sqrt{2}\}, \leq) \\
 &\cong (\mathbb{Q} \cap (0, 1), \leq)
 \end{aligned}$$

**Definition 1.12.** Let  $R, R'$  be binary relations with universe  $E, E'$

- A **0-isomorphism** from  $R$  to  $R'$  is a local isomorphism from  $R$  to  $R'$
- For  $p > 0$ , a  **$p$ -isomorphism** from  $R$  to  $R'$  is a local isomorphism  $f$  from  $R$  to  $R'$  satisfying the following two conditions
  1. (**forth**) For any  $a \in E$ , there is a  $(p-1)$ -isomorphism  $g \supseteq f$  with  $a \in \text{dom}(g)$

2. (**back**) For any  $b \in E'$ , there is a  $(p-1)$ -isomorphism  $g \supseteq f$  with  $b \in \text{im}(g)$

- An  $\omega$ -**isomorphism** from  $R$  to  $R'$  is a local isomorphism  $f$  from  $R$  to  $R'$  s.t.  $f$  is a  $p$ -isomorphism for all  $p < \omega$

The set of  $p$ -isomorphisms from  $R$  to  $R'$  is denoted  $S_p(R, R')$

**Example 1.8.** Suppose  $R = R' = (\mathbb{Z}, <)$ ,  $f : \{2, 4\} \rightarrow \{1, 2\}$  is a local isomorphism with  $f(2) = 1$  and  $f(4) = 2$ . Then  $f \notin S_1(\mathbb{Z}, \mathbb{Z})$  (forth) fails. For  $a = 3$ , there is no  $b$  s.t.  $1 < b < 2$

$g : \{2, 4\} \rightarrow \{1, 5\}$  is a 1-isomorphism but not a 2-isomorphism

**Proposition 1.13.** If  $f \in S_p(R, R')$  and  $g \subseteq f$ , then  $g \in S_p(R, R')$

*Proof.* if  $p = 0$  easy

if  $p > 0$  (forward),  $\forall a \in E, \exists h \in S_{p-1}(R, R')$  has  $a \in \text{dom}(h)$  and  $h \supseteq f \supseteq g$   $\square$

**Proposition 1.14.**  $S_p(R, R') \neq \emptyset$  iff  $\emptyset \in S_p(R, R')$

*Proof.*  $\Leftarrow$  immediate

$\Rightarrow$ . Suppose  $f \in S_p(R, R')$ . Then  $\emptyset \subseteq f$ . Hence  $\emptyset \in S_p(R, R')$ .  $\square$

**Definition 1.15.**  $R$  and  $R'$  are  $p$ -**equivalent**, written  $R \sim_p R'$ , if there is a  $p$ -isomorphism from  $R \rightarrow R'$

$R$  and  $R'$  are  $\omega$ -**equivalent** or **elementarily equivalent**, written  $R \sim_\omega R'$  or  $R \equiv R'$ , if there is an  $\omega$ -isomorphism from  $R$  to  $R'$

Note:  $R \sim_\omega R'$  iff  $S_\omega(R, R') \neq \emptyset$  iff  $\emptyset \in S_\omega(R, R')$  iff  $\forall p \emptyset \in S_p(R, R')$  iff  $\forall p R \sim_p R'$

**Definition 1.16.** Let  $R, R'$  be binary relations with universe  $E, E'$ . The Ehrenfeucht-Fraïssé game of length  $n$ , denoted  $\text{EF}_n(R, R')$  is played as follows

- There are two players, the Duplicator and Spoiler
- There are  $n$  rounds
- In the  $i$ th round, the Spoiler chooses either an  $a_i \in E$  or a  $b_i \in E'$
- The Duplicator responds with a  $b_i \in E'$  or an  $a_i \in E$  respectively

- At the ends of the game, the Duplicator wins

$$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

is a local isomorphism from  $R$  to  $R'$

- Otherwise, the Spoiler wins

**Example 1.9.** For  $\text{EF}_3(\mathbb{Q}, \mathbb{R})$

$\mathbb{Q}$	$\mathbb{R}$
S: $a_1 = 7$	D: $b_1 = 7$
D: $a_2 = 1.4$	S: $b_2 = \sqrt{2}$
D: $a_3 = -10$	S: $b_3 = 1.41$

So  $D$  wins

**Example 1.10.**  $\text{EF}_3(\mathbb{R}, \mathbb{Z})$

$\mathbb{R}$	$\mathbb{Z}$
D: $a_1 = 1$	S: $b_1 = 1$
D: $a_2 = 1.1$	S: $b_2 = 2$
S: $a_3 = 1.01$	

$D$  fails

**Proposition 1.17.**  $\text{EF}_n(R, R')$  is a win for Duplicator iff  $R \sim_n R'$

**Proposition 1.18.** In  $\text{EF}_n(R, R')$  if moves so far are  $a_1, b_1, \dots, a_i, b_i$ ,  $p = n - 1$ ,  $f = \{(a_1, b_1), \dots, (a_i, b_i)\}$ . Then Duplicator wins iff  $f \in S_p(R, R')$

## 2 Back-and-forth Equivalence II

**Definition 2.1.** Let  $(M, R), (M', R')$  be binary relations.. The Ehrenfeucht-Fraïssé game of length  $n$ , denoted  $\text{EF}_n(M, M')$  is played as follows

- There are two players, the Duplicator and Spoiler
- There are  $n$  rounds
- In the  $i$ th round, the Spoiler chooses either an  $a_i \in M$  or a  $b_i \in M'$
- The Duplicator responds with a  $b_i \in M'$  or an  $a_i \in M$  respectively



- At the ends of the game, the Duplicator wins

$$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

is a local isomorphism from  $R$  to  $R'$

- Otherwise, the Spoiler wins

**Lemma 2.2.** *Suppose we are playing  $EF_n(M, M')$  and there have been  $q$  rounds so far, with  $p = n - q$  rounds remaining. Suppose the moves so far are  $(a_1, b_1), \dots, (a_n, b_n)$ . Let  $f = \{(a_1, b_1), \dots, (a_q, b_q)\}$ . Then the following are equivalent*

- Duplicator has a winning strategy
- $f$  is a  $p$ -isomorphism

*Proof.* By induction on  $p$ .

if  $p = 0$ , then the game is over, so Duplicator wins iff  $f \in S_0(M, M')$

$p > 0$ . If  $f$  isn't a local isomorphism, then Duplicator will definitely lose, and  $f$  isn't a  $p$ -isomorphism. So we may assume  $f \in S_0(M, M')$ . Then the following are equivalent

- Duplicator wins
- For any  $a_{q+1} \in M$ , there is a  $b_{q+1} \in M'$  s.t. Duplicator wins in the position  $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$ , AND for any  $b_{q+1} \in M'$ , there is a  $a_{q+1} \in M$  s.t. Duplicator wins in the position  $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$ ,
- For any  $a_{q+1} \in M$  there is a  $b_{q+1} \in M'$  s.t.  $f \cup \{(a_{q+1}, b_{q+1})\} \in S_{p-1}(M, M')$  (by induction), AND ...
- For any  $a_{q+1} \in M$ , there is  $g \in S_{p-1}(M, M')$  s.t.  $g \supseteq f$  and  $a_{q+1} \in \text{dom}(g)$ , AND ....
- $f \in S_p(M, M')$

□

**Theorem 2.3.** *If  $M$  is  $p$ -equivalent to  $M'$ , then  $EF_p(M, M')$  is a win for the Duplicator. Otherwise it is a win for the Spoiler*

*Proof.* We need to prove  $\emptyset \in EF_p(M, M')$

□

**Theorem 2.4.** *Every  $(p + 1)$ -isomorphism is a  $p$ -isomorphism*

*Proof.* By induction on  $p$ .

$p = 0$ : every 1-isomorphism is a 0-isomorphism.  $\square$

So  $S_0(M, M') \supseteq S_1(M, M') \supseteq S_2(M, M') \supseteq \dots$  In terms of the Ehrenfeucht-Fraïssé game

**Theorem 2.5.** Suppose  $s \in S_p(M, M')$  and  $t \in S_p(M', M'')$  and  $\text{dom}(t) = \text{im}(s)$ . Then  $u := t \circ s \in S_p(M, M'')$

**Corollary 2.6.** If  $M \sim_p M'$  and  $M' \sim_p M''$ , then  $M \sim_p M''$

*Proof.*  $\emptyset \in S_p(M, M')$  and  $\emptyset \in S_p(M', M'')$ , hence  $\emptyset \in S_p(M, M'')$   $\square$

**Theorem 2.7.** Suppose  $s \in S_p(M, M')$ . Then  $s^{-1} \in S_p(M, M')$

*Proof.* Since  $s \in S_p(M, M')$ ,  $s$  is a local isomorphism from  $M$  onto  $M'$ . As  $s$  is an bijection,  $s^{-1}$  is also a bijection.  $\square$

**Corollary 2.8.** If  $M \sim_p M'$ , then  $M' \sim_p M$

$\sim_p$  is an equivalence relation

**Theorem 2.9.** Let  $K$  be a Karpian family for  $(M, R)$  and  $(M', R')$ . Then  $K \subseteq S_p(M, M')$  for all  $p$ . (also for all  $\alpha$ )

**Corollary 2.10.** If  $M, M'$  are DLOs, then  $S_0(M, M') = S_p(M, M')$  for all  $p$ .  $M \sim_\omega M'$

**Corollary 2.11.**  $A \cong B \implies A \sim_\infty B \implies A \sim_\omega B \implies A \sim_p B$

**Corollary 2.12.**  $\sim_p$  and  $\sim_\omega$  are equivalence relations

**Theorem 2.13.** Suppose  $(\mathbb{Q}, \leq) \sim_\omega (C, R)$ . Then  $(C, R)$  is a DLO

*Proof.* Suppose  $(C, R)$  is not a DLO and break into cases

- $R$  is not reflexive. As  $\emptyset \in S_1(\mathbb{Q}, C)$ . Spoiler chooses  $b_1 \in C$  s.t.  $(b_1, b_1) \notin R$ . Then duplicator must choose  $a_1 \in \mathbb{Q}$  s.t.  $a_1 \not\leq a_1$ , impossible
- $R$  is antisymmetric.  $\emptyset \in S_2(\mathbb{Q}, C)$ . Let  $b_1, b_2 \in C$  s.t.  $b_1 R b_2$  and  $b_2 R b_1$ . We want to show that  $b_1 = b_2$ . Since  $\emptyset \in S_2(\mathbb{Q}, C)$ , we have a local isomorphism  $\{(a_1, b_1), (a_2, b_2)\} \in S_0(\mathbb{Q}, C)$ . Hence  $a_1 \leq a_2$  and  $a_2 \leq a_1$ . As so  $a_1 = a_2$ . As this is a bijection,  $b_1 = b_2$ .
- $R$  is transitive.  $\emptyset \in S_3(\mathbb{Q}, C)$ . Let  $b_1, b_2, b_3 \in C$  s.t.  $b_1 R b_2$  and  $b_2 R b_3$ .  $\square \square \square \square a_1, a_2, a_3 \in \mathbb{Q}$  s.t.  $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \in S_0(\mathbb{Q}, C)$ .

- $R$  is total.  $\square\square\square S_2(\mathbb{Q}, C)$ .
- $(C, R)$  has no maximum.  $\forall b_1 \in C$
- $(C, R)$  has no minimum
- $(C, R)$  is dense. For any  $b_1 \neq b_2 \in C$  s.t.  $b_1 R b_2$ .  $S_3(\mathbb{Q}, C)$

□

**Corollary 2.14.** *The class of DLOs is the  $\sim_\omega$ -equivalence class of  $(\mathbb{Q}, \leq)$*

**Definition 2.15.** A linear order  $(C, \leq)$  is **discrete** without endpoints if  $C \neq \emptyset$  and

$$\forall a \exists b : a \triangleleft b$$

$$\forall b \exists a : a \triangleleft b$$

where  $a \triangleleft b$  means  $a < b$  and not  $\exists c : a < c < b$

**Example 2.1.**  $(\mathbb{Z}, \leq)$ . So is  $(C, \leq)$ , where

$$\begin{aligned} C = & \{ \dots, -3, -2, -1 \} \cup \\ & \{ -1/2, -1/3, -1/4, -1/5, \dots \} \cup \\ & \{ \dots, 1/5, 1/4, 1/3, 1/2 \} \cup \\ & \{ 1, 2, 3, \dots \} \end{aligned}$$

**Definition 2.16.** Let  $(C, <)$  be discrete. If  $a \leq b \in C$ , then  $d(a, b)$  is the size of  $[a, b) = \{x \in C : a \leq x < b\}$  or  $\infty$  if infinite. If  $a > b$ , then  $d(a, b) = d(b, a)$  (definition)

$$d(a, b) = 0 \Leftrightarrow a = b$$

**Lemma 2.17.** *Let  $(C, <)$  and  $(C', <)$  be discrete linear orders without endpoints. Suppose  $a_1 < \dots < a_n$  in  $C$  and  $b_1 < \dots < b_n$  in  $C'$ . Let  $f$  be the local isomorphism  $f(a_i) = b_i$ . Suppose that for every  $1 \leq i < n$ , we have*

$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or } d(a_i, a_{i+1}) \geq 2^p \leq d(b_i, b_{i+1})$$

*Then  $f$  is a  $p$ -isomorphism*

IDEA: a 0-isomorphism needs to respect the order. A 1-isomorphism needs to respect the order plus the relation  $d(x, y) = 1$  (to make sure we can find the point). A 2-isomorphism needs to respect the order plus the relation  $d(x, y) = i$  for  $i = 1, 2, 3$ . A 3-isomorphism needs to respect the order plus the relations  $d(x, y) = i$  for  $i = 1, 2, 3, \dots, 7$

this is like binary search algorithm:D

*Proof.*     •  $a_i < a < a_{i+1}$   
               – if  $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$   
                   which means they are finite

□

**Theorem 2.18.** *Let  $(C, \leq)$  and  $(C', \leq')$  be discrete linear orders without points. Then  $\emptyset$  is a  $p$ -equivalence from  $(C, \leq)$  to  $(C', \leq)$  for all  $p$ . Therefore  $(C, \leq) \sim \omega(C', \leq)$ .*

*Remark.* If  $(\mathbb{Z}, \leq) \sim_\omega (C, R)$ , then  $(C, R)$  is a dense linear order

**Definition 2.19.** Let  $(M, R), (M', R')$  be binary relations.. The **infinite Ehrenfeucht-Fraïssé game**, denoted  $EF_\infty(M, M')$  is played as follows

- There are two players, the Duplicator and Spoiler
- There are infinitely many rounds (indexed by  $\omega$ )
- In the  $i$ th round, the Spoiler chooses either an  $a_i \in M$  or a  $b_i \in M'$
- The Duplicator responds with a  $b_i \in M'$  or an  $a_i \in M$  respectively
- if  $\{(a_1, b_1), \dots, (a_n, b_n)\}$  is not a local isomorphism, then the Spoiler immediately wins
- The Duplicator wins if the Spoiler has not won by the end of the game

**Theorem 2.20.** *TFAE*

1.  $R \sim_\infty R'$ , i.e., there is a non-empty Karpian family  $K$
2. Duplicator has a winning strategy for  $EF_\infty(M, M')$
3. Spoiler does not have a winning strategy for  $EF_\infty(M, M')$

*Proof.*  $1 \rightarrow 2$ . Karpian family is the winning strategy

□

### 3 Connections to Back-and-Forth Technique

**Theorem 3.1** (Fraïssé's Theorem). *Let  $(M, R)$  and  $(N, S)$  be  $m$ -ary relations, let  $\bar{a} \in M^n$  and  $\bar{b} \in N^n$ . Then  $\bar{a}$  and  $\bar{b}$  are  $p$ -equivalent iff*

$$(M, R) \models f(\bar{a}) \iff (N, S) \models f(\bar{b})$$

*for any formula  $f(\bar{x})$  with quantifier rank at most  $p$*

*Proof.*  $\Rightarrow$ . Induction on  $p$ . If  $\bar{a} \sim_0 \bar{b}$ , then by definition, they satisfy the same atomic formulas. Therefore they satisfy the same quantifier-free formulas.

Suppose that  $\bar{a} \sim_{p+1} \bar{b}$ . The formula  $f := (\exists y)g(\bar{x}, y)$  has quantifier rank at most  $p + 1$ . So  $g(\bar{x}, y)$  is a formula of quantifier rank at most  $p$ .  $(M, R) \models f(\bar{a})$  iff there is a  $c \in M$  s.t.  $(M, R) \models g(\bar{a}, c)$ . Then there is a  $d \in N$  s.t.  $\bar{a}c \sim_p \bar{b}d$ . By IH,  $(N, S) \models g(\bar{b}, d)$  and thus  $(N, S) \models (\exists y)g(\bar{b}, y)$ . Another direction is similar  $\square$

To prove the converse we need the following lemma

**Lemma 3.2.** *If the arity  $m$  of a relation, and the integers  $n$  and  $p$  are fixed, there is only finite number  $C(n, p)$  of  $p$ -equivalence classes of  $n$ -tuples*

$(M, R_1, \bar{a}_1), \dots, (M, R_n, \bar{a}_n)$ . For any  $(M, R)$  and  $\bar{a} \in M$ ,  $\exists 1 \leq i \leq n$  s.t.  $\bar{a} \sim_p \bar{a}_i$

*Proof.* Induction on  $p$ . If  $p = 0$ , then consider a set of symbols  $X = \{x_1, \dots, x_n\}$ . There are at most finitely many  $m$ -ary relations defined on  $X$ . Also there are at most finitely many ways to interpret the relation “=” on  $X$ . Let  $(M, R)$  and  $(N, S)$  be  $m$ -ary relations,  $\bar{a} \in M^n$  and  $\bar{b} \in N^n$ . Let  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_n\}$ . Let  $R_A = R \cap A^m$  and  $S_B = S \cap B^m$ . If  $p = 0$ ,  $\bar{a} \sim_0 \bar{b}$  iff  $R_A$  is isomorphic to  $S_B$  via  $a_i \mapsto b_i, i = 1, \dots, n$ . So there are at most finitely many 0-equivalence classes of  $n$ -tuples

By IH, there exists relations  $\{(M_k, R_k) \mid k \leq C(n + 1, p)\}$  and  $\{\bar{d}_k \in M_k^{n+1} \mid k \leq C(n + 1, p)\}$  s.t. each  $n + 1$ -tuple is  $p$ -equivalent to some  $\bar{d}_k$ . Now consider an arbitrary relation  $(M, R)$  and an  $n$ -tuple  $\bar{a}$ , we define  $[\bar{a}] = \{k \mid \exists c \in M(\bar{a}c \sim_p \bar{d}_k)\}$ . For any relation  $(N, S)$  and  $\bar{b} \in N^n$ ,  $\bar{a} \sim_{p+1} \bar{b} \Leftrightarrow [\bar{a}] = [\bar{b}]$   $\square$

*Proof (continued).* We now show that if  $\bar{a}$  and  $\bar{b}$  satisfy the same formulas of QR at most  $p$ , then  $\bar{a} \sim_p \bar{b}$ .

Claim: For each  $p$ -equivalence class  $C$ , there is a formula  $f_C$  of QR  $p$  s.t. the tuples in  $C$  are exactly those satisfy  $f_C$ .  $(M, R, \bar{a}) \in C \Leftrightarrow R \models f_C(\bar{a})$ .

Induction on  $p$ . If  $p = 0$ , given an  $n$ -tuple  $\bar{a}$ , there are finitely many atomic formulas with variables  $x_1, \dots, x_n$ .  $n^2 + n^m$ .  $\{x_i = x_j \mid i, j \leq n\}$  and  $\{r(x_{i_1}, \dots, x_{i_m}) \mid i_j \leq n\}$ .

Let  $f_C$  be the conjunction of those satisfied by  $\bar{a}$  and negation of the others. Then  $f_C$  characterizes the 0-equivalence class of  $\bar{a}$ . (characterizes  $R|_{\{a_1, \dots, a_n\}}$ )

Now prove  $p + 1$ . Let  $\bar{a}$  be an  $n$ -tuple of  $(M, R)$ . Let  $f_1(\bar{x}, y), \dots, f_k(\bar{x}, y)$  characterize all the  $p$ -equivalence classes  $C_1, \dots, C_k$  on  $n+1$ -tuples. Let  $\langle \bar{a} \rangle = \{i \leq k \mid (M, R) \models (\exists y)f_i(\bar{a}, y)\}$ .  $\langle \bar{a} \rangle = [\bar{a}]$

Let  $f_C(\bar{x}) = \bigwedge_{i \in \langle \bar{a} \rangle} (\exists y) f_i(\bar{x}, y) \wedge \bigwedge_{i \notin \langle \bar{a} \rangle} \neg(\exists y) f_i(\bar{x}, y)$ .  $\bar{b} \sim_{p+1} \bar{a}$  iff  $[\bar{a}] = [\bar{b}]$   
iff  $\langle \bar{a} \rangle = \langle \bar{b} \rangle$  iff  $f_C(\bar{b})$  holds □

bracket system

## 4 Compactness

### 4.1 Ultraproducts

If  $I$  is a nonempty set, a **filter** is a set  $F$  of subsets of  $I$  s.t.

- $I \in F, \emptyset \notin F$
- if  $X, Y \in F$ , then  $X \cap Y \in F$
- if  $X \in F$  and  $X \subset Y$ , then  $Y \in F$

A **filter prebase**  $B$  is a set of subsets of  $I$  contained in a filter; this means that the intersection of a finite number of elements of  $B$  is never empty. The filter  $F_B$  consisting of subsets of  $I$  containing a finite intersection of elements of  $B$  is the smallest filter containing  $B$ ; we call it the **filter generated** by  $B$ . If, in addition, the intersection of two elements of  $B$  is always in  $B$ , we call  $B$  a **filter base**

**Example 4.1.** Let  $J$  be a set and  $I$  the set of finite subsets of  $J$ ; for every  $i \in I$ , let  $I_i = \{j : j \in I, j \supset i\}$ , and let  $B$  be the set of all the  $I_i$ . Then  $I_i \cap I_j = I_{i \cup j}$ ;  $B$  is closed under finite intersections and does contain  $\emptyset$ ; It is therefore a filter base.

**Theorem 4.1.** A filter  $F$  of subsets of  $I$  is an **ultrafilter** iff for every subset  $A$  of  $I$ , either  $A$  or its complement  $I - A$  is in  $F$

**Theorem 4.2.** Let  $U$  be an ultrafilter of subsets of  $I$ . If  $I$  is covered by finitely many subsets  $A_1, \dots, A_n$ , then one of the  $A_i$  is in  $U$ ; moreover, if the  $A_i$  are pairwise disjoint, exactly one of the  $A_i$  is in  $U$

Ultrafilter and Compactness

A topological space  $X$  is compact if and only if every ultrafilter in  $X$  is convergent

## 4.2 Applications of Compactness

**Lemma 4.3.** *If  $M$  and  $N$  are elementarily equivalent structures, then  $M$  can be embedded into an ultraproduct of  $N$*

*Proof.* Let  $I$  be the set of injections from finite subset of  $M$  to  $N$ . If  $f(\bar{a})$  is a formula with parameters  $\bar{a}$  in  $M$ ,  $M \models f(\bar{a})$ , let  $I_{f(\bar{a})}$  denote the set of such injections  $s$  whose universe contains  $\bar{a}$  and s.t.  $N \models f(s(\bar{a}))$ . The set  $I_{f(\bar{a})}$  is never empty, as  $M \models f(\bar{a})$ , so  $M \models \exists \bar{x}(f(\bar{x}) \wedge D(\bar{x}))$ , where  $D$  is the conjunction of the formulas  $x_i = x_j$  if  $a_i = a_j$ , and  $x_i \neq x_j$  otherwise, and  $N$  also satisfies this formula. On the other hand,  $I_{f(\bar{a})} \cap I_{g(\bar{b})} = I_{f(\bar{a}) \wedge g(\bar{b})}$ , so the  $I_{f(\bar{a})}$  form a filter base, which can be extended to an ultrafilter

Define a function  $S$  from  $M$  to  $N^U$  as follows: If  $a \in M$ , the  $i$ th coordinate of  $Sa$  is  $ia$  if  $i$  is defined at  $a$ , and any element of  $N$  otherwise (We are excluding the case of empty universes, which is trivial.) Note that  $\{i : i \text{ is defined at } a\} = I_{a=a}$ , and that changing the coordinates outside of  $I_{a=a}$  will not change  $Sa$  modulo  $U$ , so  $S$  is well-defined. **If  $a = b$ , then  $S(a) = S(b)$  iff  $\{i : N \models i(a) = i(b)\} = I_{a=b} \in U$ . If  $a \neq b$ , then  $I_{a \neq b} \in U$ , hence  $S$  is an injection.**

$N^U \models \phi(S(\bar{a}))$  iff  $\{i : N \models \phi(i(\bar{a}))\} \in U$ . If  $M \models \phi(\bar{a})$ , then  $\{i : N \models \phi(i(\bar{a}))\} = I_{\phi(\bar{a})}$ .  $\square$

## 5 Quantifier elimination

**Theorem 5.1.** *If two structures  $M$  and  $N$  are elementarily equivalent and  $\omega$ -saturated, they are  $\infty$ -equivalent: More precisely, two tuples of the same type (over  $\emptyset$ ), one in  $M$  and the other in  $N$ , can be matched up by an infinite back-and-forth construction*

If  $M$  is  $\omega$ -saturated, then for every  $\bar{a}$  of  $M$  and every  $p$  of  $S_n(\bar{a})$ ,  $p$  is realised in  $M$

An  $\omega$ -saturated model therefore realises all absolute  $n$ -types for all  $n$ . This condition, however, is not sufficient for a model to be  $\omega$ -saturated. Example: let  $T$  be the theory of discrete order without endpoints;  $M$  is  $\omega$ -saturated iff it has the form  $\mathbb{Z} \times \mathbb{C}$  where  $\mathbb{C}$  is a dense chain without endpoints, while it realizes all pure  $n$ -types iff it has the form  $\mathbb{Z} \times \mathbb{C}$  where  $\mathbb{C}$  is an infinite chain

If  $T$  is a complete theory and  $M$  is an  $\omega$ -saturated model of  $T$ , then every denumerable model  $N$  of  $T$  can be elementarily embedded in  $M$ . In fact, if  $N = \{a_0, a_1, \dots, a_n, \dots\}$ , we can successively realize, in  $M$ , the type of  $a_0$ , then the type of  $a_1$  over  $a_0$ , ..., the type of  $a_{n+1}$  over  $(a_0, \dots, a_n)$ , ...

As two denumerable, elementarily equivalent,  $\omega$ -saturated structures are isomorphic. Under what conditions does a complete theory  $T$  have a (unique)  $\omega$ -saturated denumerable model? That happens iff for every  $n$ ,  $S_n(T)$  is (finite or) denumerable. (Here, we do not assume that  $T$  is denumerable)

In fact, this condition further implies that for every  $\bar{a} \in M$ ,  $S_1(\bar{a})$  is denumerable (because to say that  $b$  and  $c$  have the same type over  $\bar{a}$  is to say that  $\bar{a}b$  and  $\bar{a}c$  have the same type over  $\emptyset$ ). It is clearly necessary, because a denumerable model can realize only denumerable many  $n$ -types. To see that it is sufficient: Let  $A_1$  be a denumerable subset of  $M$  that realizes all 1-types over  $\emptyset$ ; then let  $A_2$  be a denumerable subset of  $M$  that realises all 1-types over finite subsets of  $A_1$ ; etc. Let  $A = \bigcup A_n$ .  $A$  satisfies Tarski's test so it is an elementary submodel of  $M$

**Theorem 5.2.** *Let  $T$  be a theory, not necessarily complete, and let  $F$  be a nonempty set of formulas  $f(\bar{x})$  in the language  $L$  of  $T$ , having for free variables only  $\bar{x} = (x_1, \dots, x_n)$ , s.t. two  $n$ -tuples from models of  $T$  have the same type whenever they satisfy the same formulas of  $F$ . Then for every formula  $g(\bar{x})$  of  $L$  in these variables, there is some  $f(\bar{x})$  that is a Boolean combination of elements of  $F$  s.t.  $T \models \forall \bar{x} (f(\bar{x}) \leftrightarrow g(\bar{x}))$*

*Proof.* Consider the clopen set  $[g(\bar{x})]$  in  $S_n(T)$ . If  $[g] = \emptyset$ , then  $[g] = [f \wedge \neg f]$ , and if  $[g] = S_n(T)$ , then  $[g] = [f \vee \neg f]$ , where  $f$  is an arbitrary element of  $F$ , which is nonempty. Consider  $p \in [g]$  and  $q \notin [g]$ . There is  $f_{p,q} \in F$  s.t.  $p \models f_{p,q}(\bar{x})$  and  $q \models \neg f_{p,q}(\bar{x})$ . **If  $p$  and  $q$  are different, then they are realised by two tuples satisfying different formulas of  $F$ . Here we consider the model amalgamated by the model realising  $p$  and the model realising  $q$ . Thus such  $f_{p,q}$  exists**

Keeping  $p$  fixed and varying  $q$ , all the  $[f_{p,q}]$  and  $\neg[g]$  form a family of closed sets whose intersection is empty;  $\bigcup [\neg f_{p,q}] \supset [\neg g]$ . by compactness, one of its finite subfamilies must have empty intersection, meaning that for some  $h_p = f_{p,q} \wedge \dots \wedge f_{p,q_n} \in [h_p] \subset [g]$

Now when we vary  $p$ ,  $[g]$  is a compact set that is covered by the open sets  $[h_p]$ , so a finite number of them are enough to cover it; the disjunction of these  $h_p$ , module  $T$ , is equivalent to  $g$   $\square$

Note that if we want that every sentence be equivalent module  $T$  to a quantifier-free sentence; that requires, naturally, that the set of sentences without quantifiers be nonempty, meaning that the language **involves** constant symbols, or else nullary relation symbols.



A theory  $T$  is **model complete** if it has the following property: If  $M, N \models T$  and if  $N \subseteq M$ , then  $N \leq M$

Two theories  $T_1$  and  $T_2$  in the same language  $L$ , are **companions** if every model of one can be embedded into a model of the other

**Theorem 5.3.** *Two theories are companions of each other iff they have the same universal consequences (a sentence being called **universal** if it is of the form  $\forall x_1, \dots, x_n f(x_1, \dots, x_n)$  with  $f$  quantifier-free)*

*Proof.* A universal sentence  $f$  that is true in a structure is always true in its substructure; if  $T_1 \models f$  and if there is a model of  $T_2$  that doesn't satisfy  $f$ , it cannot be extended to a model of  $T_1$

Conversely, suppose that  $T_1$  and  $T_2$  have the same universal consequences, and let  $M_1 \models T_1$ . We name each element of  $M_1$  by a new constant, and let  $D(M_1)$  be the set of all *quantifier-free* sentences in the new language that are true in  $M_1$ . If  $D(M_1) \models f(a_1, \dots, a_n)$ , then  $M \models \exists \bar{x} f(\bar{x})$ , so  $\forall \bar{x} \neg f(\bar{x})$  is not a consequence of  $T_1$ , and therefore not of  $T_2$ . There is therefore some model  $M_2 \models T_2$  with  $\bar{b} \in M_2$  s.t.  $M_2 \models f(\bar{b})$ . By compactness, this means that  $D(M_1) \cup T_2$  is consistent, in other words, that  $M_1$  embeds into a model of  $T_2$   $\square$

A theory  $T$  therefore has a minimal companion, which we shall denote by  $T_\forall$ , which is axiomatized by the universal consequences of  $T$ .

A theory  $T'$  is a **model companion** of  $T$  if it is a companion of  $T$  that is model complete

**Theorem 5.4.** *A theory has at most one model companion*

*Proof.* Let  $T_1$  and  $T_2$  be model companions of  $T$ . Therefore  $T_1$  and  $T_2$  are companions. Let  $M_1 \models T_1$ ; it embeds into a  $N_1 \models T_2$ , which embeds into a  $M_2 \models T_1$ . We get a chain  $M_1 \subset N_1 \subset M_2 \subset N_2 \subset \dots \subset M_n \subset N_n \subset \dots$ , whose limit we call  $P$ . As  $T_1$  is model complete, the chain of  $M_n$  is elementary, and  $P$  is an elementary extension of  $M_1$ ; similarly  $N_1 \leq P$ . Therefore  $M_1$  is also a model of  $T_2$ ; by symmetry  $T_1$  and  $T_2$  have the same models, meaning  $T_1 = T_2$   $\square$

We say that  $T'$  is a **model completion** of  $T$  if it is a model companion of  $T$  and also the following condition is satisfied: if  $M \models T$ , embeds into a model  $M_1 \models T'$  and into a model  $M_2 \models T'$ , then a tuple  $\bar{a}$  of  $M$  satisfies the same formulas in  $M_1$  and in  $M_2$

Naturally a model complete theory is its own model completion, and it is clear that a theory that admits quantifier elimination is the model completion of every one of its companions. A theory is the model completion

of every one of its companions iff it is the model completion of the weakest of them all,  $T_\forall$

In the particular case where for every  $n > 0$  we can take for  $F$  the quantifier-free formulas, we say that the theory  $T$  **eliminates quantifiers** or **admits quantifier elimination**.

**Theorem 5.5.** *The model completion of a universal theory (i.e., one that is axiomatized by universal sentences) admits quantifier elimination*

*Proof.* Let  $\bar{a}$  and  $\bar{b}$  satisfying the same quantifier-free formulas, be in two models  $M_1$  and  $M_2$  of this theory  $T'$ , and let  $N_1 \subseteq M_1$ ,  $N_2 \subseteq M_2$  generated by  $\bar{a}$  and  $\bar{b}$  respectively.  $\square$

DLO has quantifier elimination

Facts. In DLO, any 0-isomorphism is an  $\omega$ -isomorphism.

Suppose  $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$ , want  $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$

$\exists f : \langle \bar{a} \rangle_{\mathfrak{M}} \rightarrow \langle \bar{b} \rangle_{\mathfrak{N}}$  an isomorphism by Theorem 6,  $f \in S_0(\mathfrak{M}, \mathfrak{N}) = S_\omega(\mathfrak{M}, \mathfrak{N})$ . Then by Fraïssé's theorem,  $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$

$M \equiv N \Leftrightarrow \langle \emptyset \rangle_M \cong \langle \emptyset \rangle_N \Leftrightarrow \text{char}(M) = \text{char}(N)$

same characteristic determine same minimal subring

$M^n / \text{Aut}(M/A) \cong S_n(A)$

Algebraically closed fields are axiomatized by the field axioms plus the axiom schema

$$\forall y_0, \dots, y_n \left( y_n \neq 0 \rightarrow \exists x \sum_{i=0}^n y_i x^i = 0 \right)$$

**Lemma 5.6.** *If  $K \models \text{ACF}$ , then  $K$  is infinite*

*Proof.* If  $K = \{a_1, \dots, a_n\}$ , then  $P(x) = 1 + \prod_{i=1}^n (x - a_i)$  has no root in  $K$   $\square$

If  $M \models \text{ACF}$  and  $K$  is a subfield, then  $K^{\text{alg}}$  denotes the set of  $a \in M$  algebraic over  $K$

**Lemma 5.7.** *Given uncountable  $M, N \models \text{ACF}$ , suppose  $\bar{a} \in M^n$  and  $\bar{b} \in N^n$  and  $\text{qftp}^M(\bar{a}) = \text{qftp}^N(\bar{b})$ . Suppose  $\alpha \in M$ . Then there is  $\beta \in N$  s.t.  $\text{qftp}^M(\bar{a}, \alpha) = \text{qftp}^N(\bar{b}, \beta)$*

*Proof.* Let  $A = \langle \bar{a} \rangle_M$  and  $B = \langle \bar{b} \rangle_N$ . There is an isomorphism  $f : A \rightarrow B$  and we can extend  $f$  to an isomorphism  $f : \text{Frac}(A) \rightarrow \text{Frac}(B)$  (Note that  $A$  and  $B$  are subrings since they are only closed under multiplication and addition). Moving  $N$  by an isomorphism we may assume  $\text{Frac}(A) =$

$\text{Frac}(B)$  and  $f = id_{\text{Frac}(A)}$ . (In particular,  $\bar{a} = \bar{b}$ ). let  $K = \text{Frac}(A)$ . Let  $K = \text{Frac}(A)$

**Claim.** There is  $\beta \in N$  with  $I(\alpha) = I(\beta)$  in  $K$

Suppose  $\alpha$  is algebraic over  $K$  with minimal polynomial  $P(x)$ . Take  $\beta \in N$  with  $P(\beta) = 0$ . Let  $Q(x)$  be the minimal polynomial over  $\beta$  over  $K$ . Then  $P(x) \in Q(x) \cdot K[x]$ . But  $P(x)$  is irreducible, so  $P(x) = Q(x)$ . Then  $I(\alpha) = I(\beta)$

suppose  $\alpha$  is transcendental, since there are only countable many solutions, there is transcendental  $\beta \in N$ . Then  $I(\alpha) = I(\beta) = 0$

Take such  $\beta$ , let  $I = I(\alpha) = I(\beta)$

- If  $P(x) \in K[x]$ ,  $P(\alpha) = 0 \Leftrightarrow P(x) \in I \Leftrightarrow P(\beta) = 0$
- If  $P(x), Q(x) \in K[x]$ , then  $P(\alpha) = Q(\alpha) \Leftrightarrow (P - Q)(\alpha) = 0 \Leftrightarrow (P - Q)(\beta) = 0 \Leftrightarrow P(\beta) = Q(\beta)$
- Hence if  $\varphi(x)$  is an atomic  $\mathcal{L}(K)$ -formula, then  $M \models \varphi(\alpha) \Leftrightarrow N \models \varphi(\beta)$
- so is quantifier-free  $\varphi(x) \in \mathcal{L}(K)$

□

**Lemma 5.8.** Lemma 5.7 holds if we replace “uncountable” with “ $\omega$ -saturated”

*Proof.* Take uncountable  $M' \geq M$  and  $N' \geq N$ , this is possible since models of ACF are infinite. By Lemma 5.7, there is  $\beta_0 \in N'$  s.t.  $\text{qftp}(\bar{a}, \alpha) = \text{qftp}(\bar{b}, \beta_0)$ . By  $\omega$ -saturation, we can find  $\beta \in N$  s.t.  $\text{tp}(\beta/\bar{b}) = \text{tp}(\beta_0/\bar{b})$ . Then  $\text{tp}(\bar{b}, \beta) = \text{tp}(\bar{b}, \beta_0)$  □

**Theorem 5.9.** ACF has quantifier elimination

**Theorem 5.10.** Suppose  $M, N \models \text{ACF}$ , then  $M \equiv N \Leftrightarrow \text{char}(M) = \text{char}(N)$

*Proof.* TFAE

- $M \equiv N$
- for every sentence  $\varphi$ ,  $M \models \varphi \Leftrightarrow N \models \varphi$
- for every quantifier-free sentence  $\varphi$ ,  $M \models \varphi \Leftrightarrow N \models \varphi$
- for every atomic sentence  $\varphi$ ,  $M \models \varphi \Leftrightarrow N \models \varphi$
- for any terms  $t_1, t_2$ ,  $M \models t_1 = t_2 \Leftrightarrow N \models t_1 = t_2$
- for any term  $t$ ,  $M \models t = 0 \Leftrightarrow N \models t = 0$

- for any  $n \in \mathbb{Z}$ ,  $M \models n = 0 \Leftrightarrow N \models n = 0$
- $\{n \in \mathbb{Z} : n^M = 0\} = \{n \in \mathbb{Z} : n^N = 0\}$
- $\text{char}(M) = \text{char}(N)$

□

**Corollary 5.11.**  $ACF_p$  is complete for each  $p$

**Corollary 5.12.**  $\mathbb{C}$  is completely axiomatized by  $ACF_0$

**Lemma 5.13.** Let  $M$  be algebraically closed. Let  $K$  be a field. Let  $\varphi(x)$  be an  $\mathcal{L}(K)$ -formula in one variable. Let  $D = \varphi(M)$ . Then there is a finite subset  $S \subseteq K^{\text{alg}}$  s.t.  $D = S$  or  $D = M \setminus S$ , that is, either  $D \subseteq K^{\text{alg}}$  or  $M \setminus K \subseteq K^{\text{alg}}$

*Proof.* By Q.E., we may assume  $\varphi$  is quantifier-free. Then  $\varphi$  is a boolean combination of atomic formulas

Let  $\mathcal{F} = \{S : S \subseteq_f K^{\text{alg}}\} \cup \{M \setminus S : S \subseteq_f K^{\text{alg}}\}$ . Note that  $\mathcal{F}$  is closed under boolean combinations. So we may assume  $\varphi$  is an atomic formula

Then  $\varphi(x)$  is  $(P(x) = 0)$  for some  $P(x) \in K[x]$ . If  $P(x) \equiv 0$ , then  $\varphi(M) = M \in \mathcal{F}$ . Otherwise  $\varphi(M) \subseteq_f K^{\text{alg}}$ , so  $\varphi(M) \in \mathcal{F}$  □

**Lemma 5.14.** Suppose  $M \leq N \models ACF$  and  $K$  is a subfield of  $M$ . Suppose  $c \in N$  is algebraic over  $K$ . Then  $c \in M$

*Proof.* Let  $P(x)$  be the minimal polynomial of  $c$  over  $K$ . Let  $b_1, \dots, b_n$  be the roots of  $P(x)$  in  $M$ . Then

$$M \models \forall x \left( P(x) = 0 \rightarrow \bigvee_{i=1}^n x = b_i \right)$$

so the same holds in  $N$ . Then  $P(c) = 0 \Rightarrow c \in \{b_1, \dots, b_n\} \subseteq M$  □

**Theorem 5.15.** If  $M \models ACF$  and  $K$  is a subfield, then  $K^{\text{alg}}$  is a subfield of  $M$  and  $(K^{\text{alg}})^{\text{alg}} = K^{\text{alg}}$

*Proof.* Suppose  $a, b \in K^{\text{alg}}$ . We claim  $a + b \in K^{\text{alg}}$ . Let  $P(x)$  and  $Q(y)$  be the minimal polynomials of  $a, b$  over  $K$ . Let  $\varphi(z)$  be the  $\mathcal{L}(K)$ -formula

$$\exists x, y (P(x) = 0 \wedge Q(y) = 0 \wedge x + y = z)$$

Then  $M \models \varphi(a + b)$  and  $\varphi(M) = \{x + y : P(x) = 0 = Q(y)\}$  is finite. Thus  $a + b \in \varphi(M) \subseteq K^{\text{alg}}$

A similar argument shows  $K^{\text{alg}}$  is closed under the field operations, so  $K^{\text{alg}}$  is a subfield of  $M$  □

**Theorem 5.16.** Suppose  $M \models ACF$  and  $K$  is a subfield. TFAE

1.  $K = K^{alg}$
2.  $K \models ACF$
3.  $K \leq M$

*Proof.*  $1 \rightarrow 2$ : suppose  $P(x) \in K[x]$  has degree  $> 0$ . Then there is  $c \in M$  s.t.  $P(c) = 0$ . By definition,  $c \in K^{alg} = K$

$2 \rightarrow 3$ : quantifier elimination

$3 \rightarrow 1$ . 5.14 □

**Corollary 5.17.** If  $M \models ACF$  and  $K$  is a subfield, then  $K^{alg} \models ACF$

$K^{alg}$  is called the **algebraic closure** of  $K$ . It is independent of  $M$ :

**Theorem 5.18.** Let  $M, N$  be two algebraically closed fields extending  $K$ . Let  $(K^{alg})_M$  and  $(K^{alg})_N$  be  $K^{alg}$  in  $M$  and  $N$ , respectively. Then  $(K^{alg})_M \cong (K^{alg})_N$

## 6 Saturated Models

**Lemma 6.1.** Let  $S_0 \subseteq S_1 \subseteq \dots \subseteq S_\alpha \subseteq \dots$  be an increasing chain of sets indexed by  $\alpha < \kappa$  for some regular cardinal  $\kappa$ . If  $A \subseteq \bigcup_{\alpha < \kappa} S_\alpha$  and  $|A| < \kappa$ , then  $A \subseteq S_\alpha$  for some  $\alpha < \kappa$

*Proof.* define  $f : A \rightarrow \kappa$  by  $f(x) = \min\{\alpha : x \in S_\alpha\}$ . Then  $|f(A)| \leq |A| < \kappa$ , so  $\alpha := \sup f(A) < \kappa$ . For any  $x \in A$ , we have  $f(x) \leq \alpha$  and so  $x \in S_{f(x)} \subseteq S_\alpha$  □

**Theorem 6.2.** If  $M$  is a structure and  $\kappa$  is a cardinal, there is a  $\kappa$ -saturated  $N \geq M$

*Proof.* Build an elementary chain

$$M_0 \leq M_1 \leq \dots \leq M_\alpha \leq \dots$$

of length  $\kappa^+$ , where

1.  $M_0 = M$
2.  $M_{\alpha+1}$  is an elementary extension of  $M_\alpha$  realizing every type in  $S_1(M_\alpha)$
3. If  $\alpha$  is a limit ordinal, then  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$

Let  $N = \bigcup_{\alpha < \kappa^+} M_\alpha$ . If  $A \subseteq N$  and  $|A| < \kappa$ , then  $A \subseteq M_\alpha$  for some  $\alpha < \kappa^+$   $\square$

**Theorem 6.3.** Suppose  $M$  is  $\kappa$ -saturated. If  $A \subseteq M$  and  $|A| < \kappa$ , then every  $p \in S_n(A)$  is realized in  $M$

*Proof.* Take  $N \geq M$  containing a realization  $\bar{a}$  of  $p$ . We can extend the partial elementary map  $\text{id}_A : A \rightarrow A$  to  $f : A \cup \{a_1, \dots, a_n\} \rightarrow B$  where  $B \subseteq M$ . Then  $\text{tp}^M(f(\bar{a})/A) = \text{tp}^N(\bar{a}/A) = p$ , so  $f(\bar{a})$  realizes  $p$  in  $M$   $\square$

**Lemma 6.4.** For any  $M$  there is an elementary extension  $N \geq M$  with the following properties:

- Every type over  $M$  is realized in  $N$
- If  $A, B \subseteq M$  and  $f : A \rightarrow B$  is a partial elementary map, then there is  $\sigma \in \text{Aut}(N)$  with  $\sigma \supseteq f$

*Proof.* Build an elementary chain

$$M = M_0 \leq M_1 \leq \dots$$

of length  $\omega$ , where  $M_{i+1}$  is  $|M_i|^+$ -saturated. Every  $p \in S_n(M)$  is realized in  $M_1$

For the second point, let  $f : A \rightarrow B$  be given. Recursively build an increasing chain of partial elementary maps  $f_n$  with  $\text{dom}(f_n), \text{im}(f_n) \subseteq M_n$  as follows:

- $f_0 = f$
- If  $n > 0$  is odd, then  $f_n$  is a partial elementary map extending  $f_{n-1}$  with  $\text{dom}(f_n) = M_{n-1}$  and  $\text{im}(f_n) \subseteq M_n$
- If  $n > 0$  is even, then  $f_n$  is a partial elementary map extending  $f_{n-1}$  with  $\text{dom}(f_n) \subseteq M_n$  and  $\text{im}(f_n) = M_{n-1}$

$\square$

**Theorem 6.5.** If  $M$  is a structure and  $\kappa$  is a cardinal, there is a strongly  $\kappa$ -homogeneous  $\kappa$ -saturated  $N \geq M$

*Proof.* Build an elementary chain

$$M_0 \leq M_1 \leq \dots \leq M_\alpha \leq \dots$$

of length  $\kappa^+$ .  $\square$

**Lemma 6.6.** *Let  $M$  be a  $\kappa$ -saturated  $L$ -structure. For  $L_0 \subseteq L$ , the reduct  $M \upharpoonright L_0$  is  $\kappa$ -saturated*

**Lemma 6.7.** *Let  $M$  be an  $L$ -structure and  $\kappa$  be a cardinal. There is an  $L$ -structure  $N \succeq M$  s.t. for every  $L_0 \subseteq L$ , the reduct  $N \upharpoonright L_0$  is  $\kappa$ -saturated and  $\kappa$ -strongly homogeneous*

**Definition 6.8.** Let  $T$  be an  $L(R)$ -theory

1.  $R$  is **implicitly defined** in  $T$  if for every  $L$ -structure  $M$ , there is at most one  $R \subseteq M^n$  s.t.  $(M, R) \models T$
2.  $R$  is **explicitly defined** in  $T$  if there is an  $L$ -formula  $\phi(x_1, \dots, x_n)$  s.t.  $T \vdash \forall \bar{x} (R(\bar{x}) \leftrightarrow \phi(\bar{x}))$

**Lemma 6.9.** *Suppose  $R$  is not explicitly defined in  $T$ . Then there are  $M, N \models T$  and  $\bar{a} \in M^n, \bar{b} \in N^n$  s.t.*

- $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$
- $M \models R(\bar{a})$  and  $N \models \neg R(\bar{b})$

*Proof.* Suppose not. Let  $S = \{\text{tp}^L(\bar{a}) : M \models T, \bar{a} \in M^n\}$ . For  $p \in S$ , one of two things happens

1. Every realization of  $p$  satisfies  $R$
2. Every realization of  $p$  satisfies  $\neg R$

Otherwise we can find a realization  $\bar{a}$  satisfying  $R$  and a realization  $\bar{b}$  satisfying  $\neg R$ , as desired.

By compactness, for each  $p \in S$  there is an  $L$ -formula  $\phi_p(\bar{x}) \in p(\bar{x})$  s.t. one of two things happens

1.  $T \cup \{\phi_p(\bar{x})\} \vdash R(\bar{x})$
2.  $T \cup \{\phi_p(\bar{x})\} \vdash \neg R(\bar{x})$

Let  $\Sigma(\bar{x}) = T \cup \{\neg \phi_p(\bar{x}) : p \in S\}$ . If  $\Sigma(\bar{x})$  is consistent, there is  $M \models T$  and  $\bar{a} \in M^n$  satisfying  $\Sigma(\bar{x})$ . Let  $p = \text{tp}^L(\bar{a})$ , so it satisfies  $\phi_p$  but it also satisfies  $\neg \phi_p$ , a contradiction

Therefore  $\Sigma(\bar{x})$  is inconsistent. By compactness there are  $p_1, \dots, p_n, q_1, \dots, q_m \in S$  s.t.

$$\begin{aligned} T &\vdash \bigvee_{i=1}^n \phi_{p_i}(\bar{x}) \vee \bigvee_{i=1}^m \phi_{q_i}(\bar{x}) \\ T \cup \{\phi_{p_i}(\bar{x})\} &\vdash R(\bar{x}) \quad \text{for } i = 1, \dots, n \\ T \cup \{\phi_{q_i}(\bar{x})\} &\vdash \neg R(\bar{x}) \quad \text{for } i = 1, \dots, m \end{aligned}$$

Then  $T \vdash \forall \bar{x} (R(\bar{x}) \leftrightarrow \bigvee_{i=1}^n \phi_{p_i}(\bar{x}))$ . The  $\leftarrow$  is by the choice of the  $\phi_{p_i}$ . The  $\rightarrow$  is because if none of the  $\phi_{p_i}$  hold, then one of the  $\phi_{q_i}$  holds, and then  $\neg R$  must hold.

Finally  $\bigvee_{i=1}^n \phi_{p_i}(\bar{x})$  is an explicit definition of  $R$

If  $m = 0$ , then  $T \vdash R(\bar{x})$ , if  $n = 0$ , then  $T \vdash \neg R(\bar{x})$   $\square$

**Theorem 6.10** (beth). *If  $R$  is implicitly defined in  $T$ , then  $R$  is explicitly defined in  $T$*

*Proof.* **Case 1:**  $T$  is complete.

If  $R$  is not explicitly defined, we obtain  $M, N \models T$  and  $\bar{a} \in M^n, \bar{b} \in N^n$  with  $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$  but  $M \models R(\bar{a})$  and  $N \models \neg R(\bar{a})$ . Since  $T$  is complete, we have  $M \equiv N$ . By elementary amalgamation, we may find elementary embeddings  $M \rightarrow N', N \rightarrow N'$ . Replacing  $M$  and  $N$  by  $N'$  and  $N'$ , we may choose  $M = N$ . By Lemma 6.7, we may replace  $M$  with an elementary extension and assume  $M$  and  $M \upharpoonright L$  are  $\aleph_0$ -saturated and  $\aleph_0$ -strongly homogeneous. The fact that  $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$  implies that there is an automorphism  $\sigma \in \text{Aut}(M \upharpoonright L)$  with  $\sigma(\bar{a}) = \bar{b}$ . Let  $R' = \sigma(R)$ . Let  $M' = (M \upharpoonright L, R')$ . Then  $\sigma$  is an isomorphism from  $M$  to  $M'$ , so  $M' \models T$ . But  $M' \upharpoonright L = M \upharpoonright L$ . Because  $R$  is implicitly defined,  $R = R'$ . But then

$$\bar{a} \in R \Leftrightarrow \sigma(\bar{a}) \in \sigma(R) \Leftrightarrow \bar{b} \in R' \Leftrightarrow \bar{b} \in R$$

contradicting the fact that  $M \models R(\bar{a})$  and  $M \models \neg R(\bar{b})$

**Case 2:**  $T$  is not complete. Any completion of  $T$  implicitly defines  $R$ . By Case 1, any completion of  $T$  explicitly defines  $R$ . So in any model  $M \models T$ , there is an  $L$ -formula  $\phi_M$  s.t.  $M \models \forall \bar{x} (R(\bar{x}) \leftrightarrow \phi_M(\bar{x}))$

Assume  $R$  is not explicitly defined, there are  $M, N \models T$  and  $\bar{a} \in M^n, \bar{b} \in N^n$ , with  $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$  and  $M \models R(\bar{a})$  and  $N \models \neg R(\bar{a})$ . Let  $T'$  be the  $L$ -theory obtained from  $T$  by replacing every  $R$  with  $\phi_M$ . Then  $M \models T'$ . The type  $\text{tp}^L(\bar{a})$  contains the following



- $\phi_M(\bar{x})$
- sentences in  $T'$

So  $N \models \phi_M(\bar{b})$  and  $N \models T'$ .

Let  $R' = \{\bar{c} \in N^n : N \models \phi_M(\bar{c})\}$ . Then  $(N \upharpoonright L, R') \models T$  because  $N \models T'$ . Therefore  $R' = R$  because  $R$  is implicitly defined. But  $N \models \phi_M(\bar{b})$  and  $N \models \neg R(\bar{b})$ , a contradiction  $\square$

**Theorem 6.11.** *Let  $T$  be a complete theory. Then  $T$  has a countable  $\omega$ -saturated model iff  $T$  is small*

*Proof.*  $\Rightarrow$ : trivial

$\Leftarrow$ : Suppose  $S_n(T)$  is countable for any  $n$ . Take some  $\omega$ -saturated model  $M^+$ . For each finite set  $A \subseteq M^+$  and type  $p \in S_1(A)$ , take some element  $c_{A,p} \in M$  realizing  $p$ . Define an increasing chain of countable subsets  $A_0 \subseteq A_1 \subseteq \dots \subseteq M^+$  as follows

- $A_0 = \emptyset$
- $A_{i+1} = A_i \cup \{c_{A,p} : A \subseteq_f A_i, p \in S_1(A)\}$

each  $A_i$  is countable, and define  $M = \bigcup_{i=0}^{\infty} A_i$ , which is countable

Now we only need to prove that  $M$  is  $\omega$ -saturated and  $M \leq M^+$   $\square$

## 7 Prime models

### 7.1 Omitting types theorem

**Theorem 7.1** (Baire Category Theorem for  $S_n(A)$ ). *Let  $U_1, U_2, \dots$  be dense open sets. Then  $\bigcap_{i=1}^{\infty} U_i$  is dense*

**Lemma 7.2.**  *$S_n(A)$  is finite iff all types in  $S_n(A)$  are isolated*

*Proof.* If each  $p \in S_n(A)$  is isolated. The family  $\{\{p\} : p \in S_n(A)\}$  covers  $S_n(A)$ , so there is a finite cover. This is impossible unless  $S_n(A)$  is finite  $\square$

**Definition 7.3.** A set  $X \subseteq S_n(A)$  is **comeager** if  $X \supseteq \bigcap_{i=1}^{\infty} U_i$  for some dense open sets  $U_i$

Work in  $S_{\omega}(T)$ .

**Lemma 7.4.** *If  $X_1, X_2, \dots$  are comeager, then  $\bigcap_{i=1}^{\infty} X_i$  is comeager*

**Lemma 7.5.** *For any formula  $\phi(x_0, \dots, x_n, y)$ , there is a dense open set  $Z_\phi$  s.t. if  $M \models T$ ,  $\bar{c} \in M^\omega$ ,  $\text{tp}^M(\bar{c}) \in Z_\phi$  and  $M \models \exists y \phi(c_0, \dots, c_n, y)$ , then there is  $i < \omega$  s.t.  $M \models \phi(c_0, \dots, c_n, c_i)$*

*Proof.* Take  $A = [\neg \exists y \phi(x_0, \dots, x_n, y)]$  and  $B_i = [\phi(x_0, \dots, x_n, x_i)]$  for  $i < \omega$ . Let  $Z_\phi = A \cup \bigcup_{i=0}^\infty B_i$ , which is open. If  $p = \text{tp}^M(\bar{c}) \in Z_\phi$  and  $M \models \exists y \phi(c_0, \dots, c_n, y)$  then  $p \notin A$ , so there is  $i < \omega$  s.t.  $p \in B_i$  meaning  $M \models \phi(c_0, \dots, c_n, c_i)$

It remains to show that  $Z_\phi$  is dense. Take non-empty  $[\psi] \subseteq S_\omega(T)$ ; we claim  $Z_\phi \cap [\psi] \neq \emptyset$ . Take  $p = \text{tp}^M(\bar{e}) \in [\psi]$ . We may assume  $p \notin Z_\phi$ , or we are done. Then  $p \notin A$ , so  $M \models \exists y \phi(e_0, \dots, e_n, y)$ . Take  $b \in M$  s.t.  $M \models \phi(e_0, \dots, e_n, b)$ . Take  $i > n$  large enough that  $x_i$  doesn't appear in  $\phi$ . Let  $\bar{c} = (e_0, \dots, e_{i-1}, b, e_{i+1}, e_{i+2}, \dots)$ . We have  $M \models \psi(\bar{e})$  because  $\text{tp}(\bar{e}) \in [\psi]$  and therefore  $M \models \psi(\bar{c})$ , so  $\text{tp}(\bar{c}) \in [\psi]$ . Also  $M \models \phi(c_0, \dots, c_n, c_i)$   $\square$

**Proposition 7.6.** *There is a comeager set  $W \subseteq S_\omega(T)$  s.t. if  $\text{tp}^M(\bar{c}) \in W$ , then  $\{c_i : i < \omega\} \leq M$*

*Proof.* Let  $W = \bigcap_\phi Z_\phi$ . Suppose  $\text{tp}^M(\bar{c}) \in W$ . Then for any  $\phi(x_0, \dots, x_n, y)$ , if  $M \models \exists y \phi(c_0, \dots, c_n, y)$ , then there is  $i < \omega$  s.t.  $M \models \phi(c_0, \dots, c_n, c_i)$ . By Tarski-Vaught,  $\{c_i : i < \omega\} \leq M$ .  $\square$

**Lemma 7.7.** *Let  $p \in S_n(T)$  be non-isolated. For any  $(j_1, \dots, j_n) \in \mathbb{N}^n$ , there is a dense open set  $V_{p, \vec{j}} \subseteq S_\omega(T)$  s.t.  $\text{tp}^M(\bar{c}) \in V_{p, \vec{j}} \Leftrightarrow \text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$*

*Proof.* Let  $V_{p, \vec{j}} = V = \bigcup_{\phi \in p} [\neg \phi(x_{j_1}, \dots, x_{j_n})]$ . If  $\text{tp}^M(\bar{c}) \in V$ , then there is some  $\phi \in p$  s.t.  $M \models \neg \phi(c_{j_1}, \dots, c_{j_n})$ , and so  $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$ . Conversely, if  $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$ , there is  $\phi \in p$  s.t.  $M \models \neg \phi(c_{j_1}, \dots, c_{j_n})$ , and then  $\text{tp}^M(\bar{c}) \in V$

It remains to show that  $V$  is dense. Suppose  $[\psi] \subseteq S_\omega(T)$  is non-empty. Take  $q = \text{tp}^M(\bar{e}) \in [\psi]$ . We may assume  $q \notin V$ . By choice of  $V$ ,  $\text{tp}^M(e_{j_1}, \dots, e_{j_n}) = p$ . Take  $m$  large enough so that  $m \geq \max(j_1, \dots, j_n)$  and  $\psi$  is a formula in  $x_0, \dots, x_m$ . Let  $\phi(y_1, \dots, y_n)$  be

$$\exists x_0, \dots, x_m \psi(x_0, \dots, x_m) \wedge \bigwedge_{i=1}^n (y_i = x_{j_i})$$

Then  $(e_{j_1}, \dots, e_{j_n})$  satisfies  $\phi$ , and so  $\phi \in p$ . As  $p$  is non isolated, there is  $N \models \phi(d_1, \dots, d_n)$  with  $\text{tp}^N(d_1, \dots, d_n) \neq p$ . By definition of  $\phi$  there are  $c_0, \dots, c_m \in N$  with  $N \models \psi(c_0, \dots, c_m)$  and  $(d_1, \dots, d_n) = (c_{j_1}, \dots, c_{j_n})$ . Choose  $c_{m+1}, c_{m+2}, \dots \in N$  arbitrarily. Then  $\bar{c} = (c_i : i < \omega) \in N^\omega$  and

$\text{tp}(\bar{c}) \in [\psi]$ , and  $\text{tp}(c_{j_1}, \dots, c_{j_n}) = \text{tp}(d_1, \dots, d_n) \neq p$ , so  $\text{tp}(\bar{c}) \in V$ , showing  $V \cap [\psi] \neq \emptyset$   $\square$

**Proposition 7.8.** *Let  $p \in S_n(T)$  be non-isolated. There is a comeager set  $V_p \subseteq S_\omega(T)$  s.t. if  $\text{tp}^M(\bar{c}) \in V_p$ , then  $p$  is not realized by a tuple in  $\{c_i : i < \omega\}$*

*Proof.* Let  $V_p = \bigcap_{\bar{j} \in \mathbb{N}^n} V_{p, \bar{j}}$ . If  $\text{tp}^M(\bar{c}) \in V_p$ , then for any  $j_1, \dots, j_n \in \mathbb{N}$

$$\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$$

$\square$

**Theorem 7.9** (Omitting types theorem). *Let  $\Pi$  be a countable set of pairs  $(p, n)$ , where  $n < \omega$  and  $p$  is a non-isolated type in  $S_n(T)$ . There is a countable model  $M \models T$  omitting  $p$  for every  $(p, n) \in \Pi$*

*Proof.* The set  $Q = W \cap \bigcap_{(p, n) \in \Pi} V_p$  is comeager, hence non-empty. Take  $\text{tp}^N(\bar{c}) \in Q$ . Then  $M := \{c_i : i < \omega\} \preceq N$  because  $\text{tp}^N(\bar{c}) \in W$ . For  $(p, n) \in \Pi$ ,  $M$  omits  $p$  because  $\text{tp}(\bar{c}) \in V_p$   $\square$

**Theorem 7.10** (Ryll-Nardzewski). *Let  $T$  be a complete theory in a countable language. Then  $T$  is  $\omega$ -categorical iff  $S_n(T)$  is finite for every  $n < \omega$*

*Proof.* Suppose  $S_n(T)$  is infinite for some  $n$ . By 7.2 there is a non-isolated  $p \in S_n(T)$ . By 7.9 there is a countable model  $M_0 \models T$  omitting  $p$ . Take an elementary extension  $M_1 \geq M_0$  where  $p$  is realized by  $\bar{a} \in M_1^n$ . By Löwenheim–Skolem Theorem we may assume  $M_1$  is countable. Then  $M_1 \not\equiv M_0$   $\square$

## 8 Heirs and definable types

### 8.1 Definable types

**Definition 8.1.**  $p(\bar{x})$  is a **definable type** if for every formula  $\varphi(\bar{x}; \bar{y})$  the set

$$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is definable, defined by some  $L(M)$ -formula  $d\varphi(\bar{y})$

**Proposition 8.2.** *If  $T$  is strongly minimal and  $M \models T$ , there is a 1-type  $p(x) \in S_1(M)$  s.t.*

$$\varphi(x, \bar{b}) \in p(x) \Leftrightarrow \exists^\infty a \in M : M \models \varphi(a, \bar{b})$$

*Moreover,  $p = \text{tp}(c/M)$  for any  $N \geq M$  and  $c \in N \setminus M$*

*Proof.* Take  $N \succ M$  and  $c \in N \setminus M$ ; let  $p(x) = \text{tp}(c/M)$ . We must show that

$$N \models \varphi(c, \bar{b}) \Leftrightarrow \exists^\infty a \in M : M \models \varphi(a, \bar{b})$$

$\Rightarrow$ : if not,  $N \models \varphi(c, \bar{b})$  but  $\varphi(M, \bar{b})$  is finite, then  $c \in M$

$\Leftarrow$ : if  $N \models \neg\varphi(c, \bar{b})$ , then  $\neg\varphi(M, \bar{b})$  is infinite and so  $\varphi(M, \bar{b})$  is finite  $\square$

$p(x)$  is called the **transcendental 1-type**

**Proposition 8.3.** *If  $T$  is strongly minimal*

1.  *$T$  eliminates the  $\exists^\infty$  quantifier*
2. *If  $M \models T$ , the transcendental 1-type  $p \in S_1(M)$  is definable*

*Proof.* 1. For any  $\varphi(x, y)$ , there is  $n_\varphi < \omega$  s.t. for every  $M \models T$  and  $\bar{b} \in M$

$$|\varphi(M, \bar{b})| < n_\varphi \text{ or } |\neg\varphi(M, \bar{b})| < n_\varphi$$

2. For each  $\varphi(x, \bar{y})$ ,  $d\varphi(\bar{y})$  is the formula  $\exists^\infty x \varphi(x, \bar{y})$

$\square$

**Corollary 8.4.** *If  $p \in S_1(M)$  and  $M$  is strongly minimal, then  $p$  is definable*

**Definition 8.5.** A theory  $T$  is **stable** if all  $n$ -types over models are definable

## 8.2 Heirs and strong heirs

Suppose  $M \leq N$  and  $p \in S_n(M)$ . An **extension** or **son** of  $p$  is  $q \in S_n(N)$  with  $q \supseteq p$ , i.e.,  $p = q \upharpoonright M$

**Definition 8.6** (Heirs).  $q \in S_n(N)$  is an **heir** of  $p$ , written  $p \sqsubseteq q$ , if for any  $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$  with  $\bar{b} \in M$  and  $\bar{c} \in N$ , there is  $\bar{c}' \in M$  with  $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p(\bar{x})$

**Lemma 8.7.** *Suppose  $M_1 \leq M_2 \leq M_3$  and  $p_i \in S_n(M_i)$  for  $i = 1, 2, 3$ , with  $p_1 \subseteq p_2 \subseteq p_3$*

1. *If  $p_1 \sqsubseteq p_2 \sqsubseteq p_3$ , then  $p_1 \sqsubseteq p_3$*
2. *If  $p_1 \sqsubseteq p_3$ , then  $p_1 \sqsubseteq p_2$*

**Definition 8.8.** If  $p \in S_n(M)$ , then  $(M, dp)$  is the expansion of  $M$  by relation symbols  $d\varphi(\bar{y})$  for each  $\varphi(\bar{x}, \bar{y})$ , interpreted as follows:

$$(M, dp) \models d\varphi(\bar{b}) \Leftrightarrow \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$$

*Remark.*  $p$  is definable iff the new relations in  $(M, dp)$  are definable in the old structure  $M$

*Remark.* The class of structures of the form  $(M, dp)$  with  $M \models T$  and  $p \in S_n(M)$  is an elementary class, axiomatized by  $T$  plus the following:

$$\forall \bar{y}_1 \dots \bar{y}_m \left( \bigwedge_{i=1}^m d\varphi_i(\bar{y}) \rightarrow \exists \bar{x} \bigwedge_{i=1}^m \varphi_i(\bar{x}, \bar{y}_i) \right) \text{ for formulas } \varphi_1(\bar{x}, \bar{y}_1), \dots, \varphi_n(\bar{x}, \bar{y}_n) \\ \forall \bar{y} (d\varphi(\bar{y}) \vee d\neg\varphi(\bar{y})) \text{ for each formula } \varphi(\bar{x}, \bar{y})$$

Any model of such theory has an underlying  $p$

**Lemma 8.9.** *If  $(M, dp) \leq (N, dq)$ , then  $M \leq N$  and  $p \sqsubseteq q$*

*Proof.*  $(N, dq) \geq (M, dp)$  implies  $N \geq M$ . Then:

- $q \supseteq p$ : if  $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$  (with  $\bar{b} \in M$ ), then  $(M, dp) \models d\varphi(\bar{b})$ , so  $(N, dq) \models d\varphi(\bar{b})$ , and  $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$
- $q \sqsupseteq p$ : suppose  $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$ , with  $\bar{b} \in M$  and  $\bar{c} \in N$ . Then  $(N, dq) \models d\varphi(\bar{b}, \bar{c})$ , and  $(N, dq) \models \exists \bar{z} d\varphi(\bar{b}, \bar{z})$ . Then  $(M, dp) \models \exists \bar{z} d\varphi(\bar{b}, \bar{z})$

□

**Corollary 8.10.** *If  $p \in S_n(M)$ , then there is  $M_0 \leq M$  with  $|M_0| \leq |T|$ , s.t.  $p \sqsupseteq (p \upharpoonright M_0)$*

*Proof.* Apply downward Löwenheim–Skolem theorem to  $(M, dp)$  to find  $(M_0, dq) \leq (M, dp)$  with  $|M_0| \leq |T|$ . Then  $q = p \upharpoonright M_0$  and  $p \sqsupseteq q$  □

**Definition 8.11.** If  $M \leq N$  and  $p \in S_n(M)$  and  $q \in S_n(N)$ , then  $q$  is a **strong heir** of  $p$  if  $(N, dq) \geq (M, dp)$

**Proposition 8.12** (Types have heirs). *Suppose  $M \leq N$  and  $p \in S_n(M)$*

1. *There is  $N' \geq N$  and  $q' \in S_n(N')$  a strong heir of  $p$*
2. *There is  $q \in S_n(N)$  an heir of  $p$*

*Proof.* 1. Let  $\bar{c}$  be an infinite tuple enumerating  $N$ . Then  $\text{tp}^L(\bar{c}/M)$  is finitely satisfiable in  $M$ , hence finitely satisfiable in the expansion  $(M, dp)$ . Therefore it is satisfied in some  $(N', dq) \geq (M, dp)$ . So there is  $\bar{e}$  in  $N'$  with  $\text{tp}^L(\bar{e}/M) = \text{tp}^L(\bar{c}/M)$ . Then the map  $f(c_i) = e_i$  is an  $L$ -elementary embeddings of  $N$  into  $N'$  extending  $\text{id}_M : M \rightarrow M$ . Moving  $N'$  by an isomorphism, we may assume  $N' \geq N$

2. Take  $N' \geq N$  and  $q' \in S_n(N')$  a strong heir of  $p$ . Let  $q = q' \upharpoonright N$ . Then  $q' \supseteq q \supseteq p$  and  $q' \sqsupseteq p$ , so  $q \sqsupseteq p$ .

□

### 8.3 Heirs and definable types

**Proposition 8.13.** *Let  $p \in S_n(M)$  be definable and  $N \geq M$*

1.  *$p$  has a unique heir  $q \in S_n(N)$*
2. *For  $\varphi(\bar{x}, \bar{y})$  and  $\bar{b} \in N$*

$$\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Leftrightarrow N \models d_p \varphi(\bar{b}) \quad (*)$$

3. *In particular,  $q$  is definable with  $d_q \varphi = d_p \varphi$  for all  $\varphi$*

*Proof. Claim.* If  $q \in S_n(N)$  and  $q \sqsupseteq p$ , then  $q$  satisfies (\*)

Take  $\bar{a} \in N' \geq N$  realizing  $q$ . If (\*) fails then

$$\begin{aligned} (\varphi(\bar{x}, \bar{b})) \in q(\bar{x}) &\not\Leftrightarrow N \models d_p \varphi(\bar{b}) \\ N' \models \neg(\varphi(\bar{a}, \bar{b}) \leftrightarrow d_p \varphi(\bar{b})) \\ \neg(\varphi(\bar{x}, \bar{b}) \leftrightarrow d_p \varphi(\bar{b})) &\in q(\bar{x}) \end{aligned}$$

As  $q \sqsupseteq p$ , there is  $b' \in M$  s.t.

$$\begin{aligned} \neg(\varphi(\bar{x}, \bar{b}') \leftrightarrow d_p \varphi(\bar{b}')) &\in p(\bar{x}) \\ N' \models \neg(\varphi(\bar{a}, \bar{b}') \leftrightarrow d_p \varphi(\bar{b}')) \\ \varphi(\bar{x}, \bar{b}') \in p(\bar{x}) &\not\Leftrightarrow M \models d_p \varphi(\bar{b}') \end{aligned}$$

a contradiction

There is at least one heir, and at most one heir satisfying (\*) □

**Example 8.1.** Suppose  $T$  is strongly minimal and  $M \leq N$  are models of  $T$ . Let  $p$  and  $q$  be the transcendental 1-types over  $M$  and  $N$ . For any  $\varphi(x, \bar{y})$

$$d_p \varphi(\bar{y}) \equiv (\exists^\infty x \varphi(x, \bar{y})) \equiv d_q \varphi(\bar{y})$$

so  $q$  is the unique heir of  $p$

**Proposition 8.14.** *TFAE for  $p \in S_n(M)$*

1.  *$p$  is definable*
2. *For every  $N \geq M$ ,  $p$  has a unique heir over  $N$*

*Proof.* Suppose  $p$  has unique heirs. Then for any  $N \geq M$ ,  $p$  has at most one strong heir over  $N$ . Therefore there is at most one way to expand  $N$  to an elementary extension of  $(M, dp)$ . Then the elementary diagram  $(M, dp)$  implicitly defines the relations  $d\varphi$ . By Beth's implicit definability theorem,  $(M, dp)$  is an expansion of  $M$  by definable relations, meaning  $p$  is definable  $\square$

**Proposition 8.15.** Suppose  $M_1 \leq M_2 \leq M_3$  and  $p_i \in S_n(M_i)$  for  $i = 1, 2, 3$  with  $p_1 \subseteq p_2 \subseteq p_3$ . Suppose  $p_1$  is definable. Then  $p_1 \sqsubseteq p_2 \sqsubseteq p_3$  iff  $p_1 \sqsubseteq p_3$

*Proof.* We only need to show the implication  $p_1 \sqsubseteq p_3 \Rightarrow p_2 \sqsubseteq p_3$ . Suppose  $p_1 \sqsubseteq p_3$ . Take  $p'_2 \sqsupseteq p_1$  and  $p'_3 \sqsupseteq p'_2$ . By the uniqueness of heirs of definable types,  $p'_2 = p_2$  and  $p_2$  is definable. Then  $p'_3 = p_3$   $\square$

## 8.4 Types in ACF

A **positive quantifier free formula** is a quantifier-free formula that doesn't use  $\neg$

Fix a model  $M \models \text{ACF}$

**Definition 8.16.** A set  $V \subseteq M^n$  is an **algebraic set** if

$$V = \varphi(M^n; \bar{b}) = \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}$$

where  $\varphi$  is positive quantifier free.

*Remark.*  $V$  is an algebraic set iff  $V$  is defined by finitely many polynomial equations

$$V = \{\bar{a} \in M^n : P_1(\bar{a}) = \dots = P_m(\bar{a}) = 0\}$$

**Lemma 8.17.** 1.  $M^n$  and  $\emptyset$  are algebraic sets

2. If  $V, W \subseteq M^n$  are algebraic sets, then  $V \cap W$  and  $V \cup W$  are algebraic sets

3. Any finite subset of  $M^n$  is an algebraic set

**Fact 8.18** (Quantifier elimination). Every definable set  $D \subseteq M^n$  is a finite boolean combination of algebraic sets

**Fact 8.19** (Consequence of Hilbert's basis theorem). The class of algebraic sets has the descending chain condition (DCC): there is no infinite chain of algebraic sets  $V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$

**Corollary 8.20.** If  $\mathcal{S}$  is a non-empty collection of algebraic sets, then  $\mathcal{S}$  contains at least one minimal element

**Corollary 8.21.** *An infinite intersection  $\bigcap_{i \in I} V_i$  of algebraic sets is an algebraic set*

**Corollary 8.22.** *If  $S \subseteq K[\bar{x}]$  is any set of polynomials, possibly infinite, then the subset of  $M^n$  defined by  $S$  is an algebraic set. All algebraic sets arise this way*

**Corollary 8.23** (Noetherian induction). *Let  $\mathcal{S}$  be a class of algebraic sets. Suppose the following holds*

*If  $X$  is an algebraic set, and every algebraic set  $Y \subsetneq X$  is in  $\mathcal{S}$ , then  $X \in \mathcal{S}$*

*Then every algebraic set is in  $\mathcal{S}$*

**Definition 8.24.** An algebraic set  $V$  is **reducible** if  $V = W_1 \cup W_2$  for algebraic sets  $W_1, W_2 \subsetneq V$ . A **variety** is a non-empty irreducible algebraic set

*Remark.* If  $V$  is an algebraic variety, then the set of algebraic proper subsets of  $V$  is closed under finite unions

**Proposition 8.25.** *If  $V$  is an algebraic set, then  $V$  is a finite union of varieties*

*Proof.* •  $V = \emptyset$ :  $V$  is a union of zero varieties

•  $V$  is irreducible:  $V$  is a union of one variety

•  $V$  is reducible:  $V = X \cup Y$  where  $X, Y \subsetneq V$ . By Noetherian induction!

□

**Definition 8.26.** The **generic type** of  $V$  is the type generated by the following formulas

1.  $x \in V$
2.  $x \notin W$  for each algebraic proper subset  $W \subsetneq V$

We will write this type as  $p_V(\bar{x})$

Note that  $x \in V$  and  $x \notin W$  is all definable

**Proposition 8.27.** *Let  $V$  be a variety*

1.  $p_V(\bar{x})$  is a consistent complete type
2. If  $W$  is an algebraic set, then  $p_V(\bar{x}) \vdash \bar{x} \in W \Leftrightarrow W \supseteq V$

*Proof.* Finite satisfiability: given finitely many proper algebraic subsets  $W_1, \dots, W_m \subsetneq V$ , we have  $V \not\supseteq \bigcup_{i=1}^m W_i$ , so there is  $\bar{a} \in V$  and  $\bar{a} \notin W_i$  for  $1 \leq i \leq m$



1. If  $W \supseteq V$ , then  $p_V(\bar{x}) \vdash \bar{x} \in V \vdash \bar{x} \in W$ . If  $W \not\supseteq V$ , then  $(W \cap V) \subsetneq V$ , so  $p_V(\bar{x}) \vdash \bar{x} \notin W \cap V$ . But  $p_V(\bar{x}) \vdash \bar{x} \in V$  so  $p_V(\bar{x}) \vdash \bar{x} \notin W$

Completeness: by 2, for any positive quantifier-free formula  $\varphi(\bar{x})$

$$p_V(\bar{x}) \vdash \varphi(\bar{x}) \text{ or } p_V(\bar{x}) \vdash \neg\varphi(\bar{x})$$

□

**Theorem 8.28.** *The map  $V \mapsto p_V$  is a bijection from the set of varieties  $V \subseteq M^n$  to  $S_n(M)$*

*Proof.* Injectivity: suppose  $V, W$  are varieties and  $V \neq W$ . WLOG,  $V \not\subseteq W$ . Then  $p_W(\bar{x}) \vdash \bar{x} \in W$  but  $p_V(\bar{x}) \not\vdash \bar{x} \in W$ , so  $p_V \neq p_W$

Surjectivity: fix  $p \in S_n(M)$ . Take  $V$  a minimal algebraic set s.t.  $p(\bar{x}) \vdash \bar{x} \in V$ . (There is at least one such  $V$ , namely  $M^n$ ).  $V$  is non-empty because  $p$  is consistent. If  $V$  is reducible as  $V = X \cup Y$  for smaller algebraic sets  $X, Y$ , then  $p(\bar{x}) \vdash \bar{x} \in X$  or  $p(\bar{x}) \vdash \bar{x} \in Y$  by completeness, contradicting the choice of  $V$ . Thus  $V$  is a variety. By choice of  $V$ ,  $p(\bar{x}) \vdash \bar{x} \in V$ . □

**Proposition 8.29.**  *$N \geq M$ , let  $V \subseteq M^n$  be a variety, defined by a formula  $\varphi$*

1.  $\varphi$  defines a variety  $V_N \subseteq N^n$
2.  $V_N$  depends only on  $V$ , not on the choice of  $\varphi$

*Proof.* Take  $\psi$  a positive quantifier-free formula defining  $V$ . Then  $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$  is satisfied by  $M$ , and therefore by  $N$ . Let  $V_N = \psi(N)$ . As  $\psi$  is positive quantifier free,  $V_N$  is an algebraic set. As  $M \models \exists \bar{x}\psi(\bar{x})$ ,  $V_N$  is non-empty. If  $V_N = W_1 \cup W_2$  where  $W_1, W_2$  are algebraic proper subsets of  $V_N$  defined by  $\theta_i(\bar{x}, \bar{b}_i)$  for some positive quantifier-free  $L$ -formula  $\theta_i$  and tuple of parameters  $\bar{b}_i \in N$ . Then

$$N \models \exists \bar{y}_1 \bar{y}_2 \left( \forall \bar{x} \left( \psi(\bar{x}) \leftrightarrow \bigvee_{i=1}^2 \theta_i(\bar{x}, \bar{y}_i) \right) \wedge \bigwedge_{i=1}^2 \exists \bar{x}(\psi(\bar{x}) \wedge \neg \theta_i(\bar{x}, \bar{y}_i)) \right)$$

which implies  $V$  is reducible □

**Theorem 8.30.** *Let  $M \leq N$  be models of ACF. Let  $V \subseteq M^n$  be a variety, and let  $V_N \subseteq N^n$  be its extension. Then  $p_{V_N} \in S_n(N)$  is the unique heir of  $p_V \in S_n(M)$*

*Proof.* Let  $q \in S_n(N)$  be an heir of  $p_V$ . Let  $\varphi$  be an  $L(M)$ -formula defining  $V$  and  $V_N$ . Then  $\varphi(\bar{x}) \in p_V(\bar{x}) \subseteq q(\bar{x})$ , so  $q(\bar{x}) \vdash \bar{x} \in V_N$ . Suppose  $q(\bar{x}) \not\vdash \bar{x} \notin W$  for some algebraic  $W \subsetneq V_N$ ,  $q(\bar{x}) \vdash \bar{x} \in W$ . Let  $\psi(\bar{x}, \bar{b})$  be a positive quantifier-free formula defining  $W$ . Let  $\theta(\bar{b})$  be the  $L(M)$ -formula

$$\forall \bar{x}(\psi(\bar{x}, \bar{b}) \rightarrow \varphi(\bar{x})) \wedge \exists \bar{x}(\varphi(\bar{x}) \wedge \neg \psi(\bar{x}, \bar{b}))$$

which says  $\psi(M^n, \bar{b}) \subsetneq \varphi(M^n)$ .  $N \models \theta(\bar{b})$  since  $W \subsetneq V$ . Then  $q(\bar{x}) \vdash \psi(\bar{x}, \bar{b}) \wedge \theta(\bar{b})$ . Because  $q \supseteq p_V$ , there is  $\bar{b}' \in M$  s.t.

$$p_V(\bar{x}) \vdash \psi(\bar{x}, \bar{b}') \wedge \theta(\bar{b}')$$

Thus we find an algebraic proper subset of  $V$  □

General fact: If  $q \supseteq p$ , suppose  $\forall \bar{b}(\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in p(\bar{x}))$ , then  $\forall \bar{b} \in N$ ,  $\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in q(\bar{x})$

## 8.5 1-types in DLO

# 9 Stable Theories

## 9.1 Strong heirs from ultrapowers

**Definition 9.1.** If  $p \in S_n(M)$ ,  $I$  set,  $\mathcal{U}$  ultrafilter on  $I$ ,  $M^\mathcal{U} = M^I/\mathcal{U}$ . The **ultrapower type**  $p^\mathcal{U} \in S_n(M^\mathcal{U})$  is the strong heir of  $p$  s.t.  $(M^\mathcal{U}, dp^\mathcal{U}) = (M, dp)^\mathcal{U}$

$p^\mathcal{U}$  is a strong heir of  $p$

If  $\varphi(\bar{x}, \bar{y}) \in L$ ,  $\bar{b} \in M^\mathcal{U}$  represented by  $(\bar{b}_i : i \in I) \in M^I$ ,

$$\varphi(\bar{x}, \bar{b}) \in p^\mathcal{U} \Leftrightarrow (M, dp)^\mathcal{U} \models d\varphi(\bar{b}) \Leftrightarrow \{i \in I \mid (M, dp) \models d\varphi(\bar{b}_i)\} \in \mathcal{U} \Leftrightarrow \{i \in I \mid \varphi(\bar{x}, \bar{b}_i) \in p(\bar{x})\} \in \mathcal{U}$$

**Proposition 9.2.** Suppose  $M \leq N$ ,  $p \in S_n(M)$ ,  $q \in S_n(N)$ ,  $q \supseteq p$ . Then there is  $I$ , ultrafilter  $\mathcal{U}$  on  $I$  s.t. (for some copy of  $M^\mathcal{U}$ , moved by isomorphism),  $M \leq N \leq M^\mathcal{U}$ ,  $p \subseteq q \subseteq p^\mathcal{U}$

*Proof.* Let  $I = \{f : N \rightarrow M \mid f \supseteq \text{id}_M\}$ .

Note that if  $\phi(\bar{x}, \bar{b}) \in q(\bar{x})$ ,  $\bar{b} \in N$ , there is  $f \in I$ ,  $\phi(\bar{x}, f(\bar{b})) \in p(\bar{x})$ . (has some duplicate variable problem, if  $b_1 = b_2$ , but  $c_1 \neq c_2$ , but maybe we could take some equivalent formulas)

For each  $\phi(\bar{x}, \bar{b})$ ,  $\bar{b} \in N$ , let  $S_{\phi, \bar{b}} = \{f \in I \mid \phi(\bar{x}, f(\bar{b})) \in p(\bar{x})\}$ . Let  $\mathcal{F} = \{S_{\phi, \bar{b}} \mid \phi(\bar{x}, \bar{b}) \in q(\bar{x})\}$

**Claim**  $\mathcal{F}$  has F.I.P

Suppose  $\phi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$ ,  $1 \leq i \leq m$ . So  $\bigwedge_{i=1}^m \phi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$ , then there is  $f \in I$  s.t.  $\bigwedge_{i=1}^m \phi_i(\bar{x}, f(\bar{b}_i)) \in p(\bar{x})$ . Then  $f \in S_{\phi_i, \bar{b}_i}$ , so  $\bigcap_{i=1}^n S_{\phi_i, \bar{b}_i} \neq \emptyset$

Thus there is  $\mathcal{U} \supseteq \mathcal{F}$ . Form  $M^{\mathcal{U}}, p^{\mathcal{U}}$ . Let  $g : N \rightarrow M^{\mathcal{U}}$  as follows. If  $c \in N$ ,  $g(c) = [(f(c) : f \in I)]$ . Note if  $c \in M$ , then  $f(c) = c$  for all  $f$ , and so  $g \upharpoonright M = \text{id}_M$

For any  $\phi(\bar{x}, \bar{y})$ ,  $\bar{b} \in N$ ,  $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow S_{\phi, \bar{b}} \in \mathcal{F} \Rightarrow S_{\phi, \bar{b}} \in \mathcal{U} \Rightarrow \{f \in I \mid \phi(\bar{x}, f(\bar{b})) \in p(\bar{x})\} \in \mathcal{U} \Leftrightarrow \phi(\bar{x}, g(\bar{b})) \in p^{\mathcal{U}}$

So  $g : N \rightarrow M^{\mathcal{U}}$ ,  $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \phi(\bar{x}, g(\bar{b})) \in p^{\mathcal{U}}$ .  $N \models \phi(\bar{b}) \Rightarrow M^{\mathcal{U}} \models \phi(g(\bar{b}))$ . WLOG,  $N \leq M^{\mathcal{U}}$  and  $g \supseteq \text{id}_N$ .  $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \phi(\bar{x}, \bar{b}) \in p^{\mathcal{U}}$ .  $q \subseteq p^{\mathcal{U}}$ .  $\square$

Since we can prove compactness by ultrapower. Everything we get from compactness can be got by some ultrapower

**Corollary 9.3.** *Every heir of  $p$  extends to a strong heir of  $p$*

## 9.2 Stability

**Definition 9.4.** If  $\alpha$  is an ordinal, then  $2^\alpha =$  strings of length  $\alpha$  in alphabet  $\{0, 1\}$

**Definition 9.5.**  $\varphi(\bar{x}, \bar{y})$  be a formula. For  $\alpha$  an ordinal, take variables  $\bar{x}_\sigma$  for  $\sigma \in 2^\alpha$ ,  $\bar{y}_\tau$  for  $\tau \in 2^{<\alpha}$ .

$D_\alpha = \{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 0\} \cup \{\neg\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 1\}$   
 $\varphi(\bar{x}, \bar{y})$  has the **dichotomy property** if

1.  $D_\omega$  is consistent
2.  $D_n$  is consistent for all  $n \in \omega$
3.  $D_\alpha$  is consistent for all  $\alpha$

1-3 are equivalent

**Example 9.1.**  $D_2$  is  $\varphi(x_{00}, y), \varphi(x_{00}, y_0), \varphi(x_{01}, y), \neg\varphi(x_{01}, y_0)$  and so on

$y \wedge y_0 \vee y_1 \wedge x_{00} \wedge x_{01} \wedge x_{10} \wedge x_{11}$

**Proposition 9.6.** Fix  $T, \mathbb{M}$ , and an integer  $n < \omega$ . Suppose there is a small model  $M \preceq \mathbb{M}$  and a type  $p \in S_n(M)$  that is not definable, then some formula  $\varphi(x_1, \dots, x_n, \bar{y})$  has the dichotomy property

*Proof.* Because  $p$  is not definable, there is an  $N \geq M$ ,  $q_1, q_2 \in S_n(N)$ ,  $q_1, q_2 \sqsupseteq p$  and  $q_1 \neq q_2$ . There is  $\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x}) \setminus q_2(\bar{x})$ ,  $\bar{b} \in N$ .

**Claim** If  $M' \geq N$ ,  $p' \in S_n(M')$ ,  $p' \sqsupseteq p$ , then there is some  $N' \geq M'$ ,  $q'_1, q'_2 \in S_n(N')$ ,  $q'_1, q'_2 \sqsupseteq p'$ ,  $q'_1, q'_2 \sqsupseteq p$ . and there is  $\bar{b}' \in N'$ ,  $\varphi(\bar{x}, \bar{b}') \in q'_1$ ,  $\neg\varphi(\bar{x}, \bar{b}') \in q'_2$

There is  $M^\mathcal{U}$  s.t.  $M \leq M' \leq M^\mathcal{U}$ ,  $p \subseteq p' \subseteq p^\mathcal{U}$ . Then  $M' \leq M^\mathcal{U} \leq N^\mathcal{U}$  and  $p \subseteq p^\mathcal{U} \subseteq q_i^\mathcal{U}$  for  $i = 1, 2$ . Take  $N' = N^\mathcal{U}$ ,  $q'_i = q_i^\mathcal{U}$ , and  $\bar{b}'$  to be the image of  $\bar{b}$  under the elementary embedding  $N \rightarrow N^\mathcal{U}$

Recursively build a tree of  $(M, p) / \setminus (M_0, p_0) (M_1, p_1)$

build  $(M_\tau, p_\tau, \varphi(x, b_\tau))$  for  $\tau \in 2^{<\omega}$

$M_{\langle \rangle} = M$ ,  $p_\tau \sqsupseteq p$ .  $M_{\tau 0} = M_{\tau 1} \geq M_\tau$ ,  $\bar{b}_\tau \in M_{\tau 0}$ ,  $\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 0}(\bar{x})$ ,  $\neg\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 1}(\bar{x})$ .

Then  $\varphi$  has dichotomy □

working in  $\mathbb{M}$

**Proposition 9.7.** *If some  $\varphi(x_1, \dots, x_n, \bar{y})$  has dichotomy property, then for every cardinal  $\lambda \geq \aleph_0$ , there is  $A \subseteq \mathbb{M}$ ,  $|A| \leq \lambda$ ,  $|S_n(A)| > \lambda$*

*Proof.* take smallest cardinal  $\mu$  s.t.  $2^\mu > \lambda$ ,  $\mu \leq \lambda$ . note that  $|2^{<\mu}| = |\bigcup_{\alpha < \mu} 2^\alpha| \leq \lambda$ .

$\varphi$  has dichotomy proposition, so  $D_\mu$  is consistent. In the monster, there are  $\bar{a}_\sigma$  for  $\sigma \in 2^\mu$ ,  $\bar{b}_\tau$  for  $\tau \in 2^{<\mu}$  s.t. if  $\sigma$  extends  $\tau 0$  then  $\mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau)$  and if  $\sigma$  extends  $\tau 1$  then  $\mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ . Let  $A = \{\bar{b}_\tau : \tau \in 2^{<\mu}\}$ . Then  $|A| \leq \lambda$  but  $\text{tp}(a_\sigma/A) \neq \text{tp}(a_{\sigma'}/A)$  for  $\sigma \neq \sigma'$ . Thus  $|S_n(A)| \geq 2^\mu > \lambda$ . □

**Lemma 9.8.** *for  $\lambda$  infinite, TFAE*

1.  $\forall A \subseteq \mathbb{M}$ , if  $|A| \leq \lambda$ , then  $\forall n$ ,  $|S_n(A)| \leq \lambda$
2.  $\forall A \subseteq \mathbb{M}$ , if  $|A| \leq \lambda$ , then  $|S_1(A)| \leq \lambda$

*Proof.*  $2 \rightarrow 1$ : By induction on  $n$ ,  $|S_{n-1}(A)| \leq \lambda$ . Then we can find  $\bar{b}_\alpha \in \mathbb{M}^{n-1}$  for  $\alpha < \lambda$  s.t.

$$S_{n-1}(A) = \{\text{tp}(\bar{b}_\alpha/A) : \alpha < \lambda\}$$

For each  $\alpha$ ,  $|A\bar{b}_\alpha| \leq \lambda \Rightarrow |S_1(A\bar{b}_\alpha)| \leq \lambda$ . So we can find  $c_{\alpha, \beta} \in \mathbb{M}$  for  $\beta < \lambda$  s.t.

$$S_1(A\bar{b}_\alpha) = \{\text{tp}(c_{\alpha, \beta}/A\bar{b}_\alpha) : \beta < \lambda\} \text{ (for } \alpha < \lambda)$$

**Claim:** if  $p \in S_n(A)$  then  $p = \text{tp}(\bar{b}_\alpha c_{\alpha, \beta}/A)$  for some  $\alpha, \beta < \lambda$

Take  $(\bar{b}', c') \in \mathbb{M}^n$  realizing  $p$ . Then  $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}_\alpha/A)$  for some  $\alpha < \lambda$ . Moving  $(\bar{b}', c')$  by an automorphism in  $\text{Aut}(\mathbb{M}/A)$ , we may assume  $\bar{b}' =$

$\bar{b}_\alpha$ . Then  $\text{tp}(c/A\bar{b}_\alpha) = \text{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha)$  for some  $\beta < \lambda$ . Moving  $c'$  by an automorphism in  $\text{Aut}(\mathbb{M}/A\bar{b}_\alpha)$ , we may assume  $c' = c_{\alpha,\beta}$

By the claim,  $|S_n(A)| \leq \lambda^2 = \lambda$  □

**Definition 9.9.**  $T$  is  $\lambda$ -stable if  $|A| \leq \lambda \Rightarrow |S_1(A)| \leq \lambda$

**Proposition 9.10.** If  $\lambda \geq |L|$ , TFAE

1.  $\forall A \subseteq \mathbb{M}$ , if  $|A| \leq \lambda$ , then  $\forall n$ ,  $|S_n(A)| \leq \lambda$
2.  $\forall A \subseteq \mathbb{M}$ , if  $|A| \leq \lambda$ , then  $|S_1(A)| \leq \lambda$
3. If  $M \preceq \mathbb{M}$ ,  $|M| \leq \lambda \Rightarrow |S_1(M)| \leq \lambda$
4. If  $M \preceq \mathbb{M}$ ,  $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$

*Proof.* 3  $\rightarrow$  1: Let  $A \subseteq \mathbb{M}$ ,  $|A| \leq \lambda$ , using downward Löwenheim–Skolem Theorem to get a model  $A \subseteq M \preceq \mathbb{M}$  and  $|A| + |L| = |M|$

4  $\rightarrow$  2: similar □

**Example 9.2.** strongly minimal theory is  $\lambda$ -stable for  $\lambda \geq |L|$

Given  $A \subseteq \mathbb{M}$ ,  $\exists M \preceq \mathbb{M}$ ,  $|M| \leq \lambda$ .  $S_1(M)$  = const types + transcendental types, so  $|S_1(M)| = |M| + 1$

$\lambda$ -stable  $\Rightarrow$  no  $\varphi$  has D.P  $\Rightarrow$  all types are definable

**Lemma 9.11.** Suppose  $\forall M \preceq \mathbb{M}$ ,  $\forall p \in S_1(M)$  is definable. Then  $T$  is  $\lambda$ -stable for some  $\lambda$

*Proof.* Take  $\lambda = 2^{|L|} > |L|$ . Suppose  $M \preceq \mathbb{M}$  and  $|M| \leq \lambda$ .  $p \in S_1(M)$  is determined by  $\varphi \in L \mapsto d_p \varphi \in L(M)$ ,  $|S_1(M)| \leq |L(M)|^{|L|} \leq \lambda^{|L|} = 2^{|L|}$  □

**Theorem 9.12.** TFAE

1.  $T$  is  $\lambda$ -stable for some  $\lambda$
2. no formula  $\varphi(\bar{x}, \bar{y})$  has D.P.
3. no  $\varphi(x, \bar{y})$  has D.P.
4.  $M \models T$ ,  $p \in S_1(M) \Rightarrow p$  is definable
5.  $M \models T$ ,  $p \in S_n(M) \Rightarrow p$  is definable

*Proof.*  $5 \rightarrow 1$ : Let  $\lambda = 2^{|L|}$ . Note that  $\lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|} = \lambda$ . Take  $A \subseteq \mathbb{M}$  with  $|A| \leq \lambda$ . By downward Löwenheim–Skolem Theorem, there is a small model  $M \preceq \mathbb{M}$  with  $A \subseteq M$  and  $|M| \leq \lambda$ . Every  $n$ -type over  $A$  extends to an  $n$ -type over  $M$ , so  $|S_n(A)| \leq |S_n(M)|$ . It remains to show  $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$ . (That is, we may assume  $A$  is a small model  $M$ ). By (5), every  $n$ -type over  $M$  is definable. A definable type is determined by the map  $\varphi \mapsto d\varphi$ , which is a function from  $L$ -formulas to  $L(M)$ -formulas. So the number of (definable) types over  $M$  is at most  $|L(M)|^{|L|} \leq \lambda^{|L|} = \lambda$   $\square$

### 9.3 The dichotomy property and definability of types

We will prove

If no formula has the dichotomy property, then all types over *arbitrary sets* are definable

#### 9.3.1 The dichotomy property

Fix a complete theory  $T$  and monster model  $\mathbb{M}$ . Fix a formula  $\varphi(\bar{x}; \bar{y})$

**Definition 9.13.** “ $D_\alpha$  is consistent” if there are  $(\bar{a}_\sigma : \sigma \in 2^\alpha)$  and  $(\bar{b}_\tau : \tau \in 2^{<\alpha})$  s.t.

$$\begin{aligned} \sigma \sqsupseteq \tau 0 &\Rightarrow \mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau) \\ \sigma \sqsupseteq \tau 1 &\Rightarrow \mathbb{M} \models \neg \varphi(\bar{a}_\sigma, \bar{b}_\tau) \end{aligned}$$

*Remark.* “ $D_\alpha$ ” is the name for the set of formulas  $\{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 0\} \cup \{\neg \varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 1\}$

**Lemma 9.14.** If  $D_n$  is consistent for all  $n < \omega$ , then  $D_\omega$  is consistent

*Proof.* Let  $F$  be a finite fragment of  $D_\omega$ . By compactness, it suffices to show that  $F$  is consistent. Take  $n$  bigger than the length of  $\tau$  for any  $\bar{y}_\tau$  appearing in  $F$ . Because  $D_n$  is consistent, there are  $(\bar{a}^0_\sigma : \sigma \in 2^n)$  and  $(\bar{b}^0_\tau : \tau \in 2^{<n})$ . Define  $\square$

#### 9.3.2 $\varphi$ -types

Continue to fix  $T, \mathbb{M}, \varphi(\bar{x}; \bar{y})$ . Let  $n$  be the length of the variable tuple  $\bar{x}$

**Definition 9.15.** If  $B \subseteq \mathbb{M}$  is a set and  $\bar{a} \in \mathbb{M}^n$ , then  $\text{tp}^\varphi(\bar{a}/B)$  is the partial type

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \varphi(\bar{a}, \bar{b})\} \cup \{\neg \varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \neg \varphi(\bar{a}, \bar{b})\}$$

**Definition 9.16.**  $S_\varphi(B) = \{\text{tp}^\varphi(\bar{a}/B) : \bar{a} \in \mathbb{M}^n\}$

**Definition 9.17.** A  $\varphi$ -type  $p \in S_\varphi(B)$  is **definable** if there is an  $L(B)$ -formula  $\psi(\bar{y})$  s.t.

$$\forall \bar{b} \in B, \quad \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

**Theorem 9.18.** Suppose  $\varphi$  does not have the dichotomy property. Then every  $\varphi$ -type over any set is definable

### 9.3.3 Proof of Theorem 9.18

**Definition 9.19.** Let  $\Sigma(\bar{x})$  be a small set of  $L(\mathbb{M})$ -formulas. Define “ $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ ” by recursion on  $n$ :

- $R_{\varphi,2}(\Sigma(\bar{x})) \geq 0$  iff  $\Sigma(\bar{x})$  is consistent
- $R_{\varphi,2}(\Sigma(\bar{x})) \geq n + 1$  iff there is  $\bar{b} \in \mathbb{M}$  s.t.

$$\begin{aligned} R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) &\geq n \\ R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) &\geq n \end{aligned}$$

**Lemma 9.20.**  $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$  iff there are  $(\bar{a}_\sigma : \sigma \in 2^n)$  and  $(\bar{b}_\tau : \tau \in 2^{<n})$  s.t.

- If  $\sigma$  extends  $\tau 0$ , then  $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$  holds
- If  $\sigma$  extends  $\tau 1$ , then  $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$  holds
- Each  $\bar{a}_\sigma$  satisfies  $\Sigma(\bar{x})$

**Definition 9.21.**  $R_{\varphi,2}(\Sigma(\bar{x}))$  is the largest  $n$  s.t.  $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ , or  $-\infty$  if there is no such  $n$ , or  $+\infty$  if  $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$  for all  $n$

$R_{\varphi,2}(\Sigma(\bar{x}))$  is called the “ $\varphi$ -2-rank” or “ $R_{\varphi,2}$ -rank” of  $\Sigma$

*Remark (Monotonicity).* If  $\Sigma(x) \vdash \Sigma'(x)$ , then  $R_{\varphi,2}(\Sigma(\bar{x})) \leq R_{\varphi,2}(\Sigma'(\bar{x}))$

*Remark.* From Lemma 9.20 we see that  $R_{\varphi,2}(\{\bar{x} = \bar{x}\}) \geq n$  iff “ $D_n$ ” is consistent. By Lemma 9.14,

$$R_{\varphi,2}(\{\bar{x} = \bar{x}\}) = +\infty \text{ iff } \varphi \text{ has the dichotomy property}$$

In particular, if  $\varphi$  does not have the dichotomy property, then  $R_{\varphi,2}(\{\bar{x} = \bar{x}\})$  is finite. By Monotonicity,  $R_{\varphi,2}(\Sigma(\bar{x}))$  is finite for any  $\Sigma(\bar{x})$

*Remark (Definability).* Suppose  $\Sigma(\bar{x})$  is a finite partial type over  $A \subseteq \mathbb{M}$  and suppose  $n < \omega$ . Then the set

$$\{\bar{b} \in \mathbb{M} : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n\}$$

is  $A$ -definable

Now we can prove Theorem 9.18. Suppose  $p \in S_{\varphi}(B)$ . Take a finite subtype  $\Sigma(\bar{x}) \subseteq_f p(\bar{x})$  minimizing  $R_{\varphi,2}(\Sigma(\bar{x}))$ . Then  $\Sigma(\bar{x})$  is a partial type over  $B$ . Let  $k = R_{\varphi,2}(\Sigma(\bar{x}))$

**Claim.** If  $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ , then

$$\begin{aligned} R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) &= k \\ R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) &< k \end{aligned}$$

*Proof.* Monotonicity gives

$$\begin{aligned} R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) &\leq k \\ R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) &\leq k \end{aligned}$$

If the first inequality is sharp, then it contradicts the minimality of  $\Sigma(\bar{x})$ . If the second inequality is not sharp, then

$$R_{\varphi,2}(\Sigma(\bar{x})) \geq k + 1$$

□

**Claim.** If  $\neg\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ , then

$$\begin{aligned} R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) &< k \\ R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) &= k \end{aligned}$$

Combining the two claims, we see that the set

$$\{\bar{b} \in B : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is exactly

$$\{\bar{b} \in B : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k\}$$

By Definability, there is an  $L(B)$ -formula  $\psi(\bar{x})$  s.t.

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

Therefore  $p$  is a definable  $\varphi$ -type



### 9.3.4 Remarks on the proof

1. In the proof of theorem 9.18, the finite subtype  $\Sigma(\bar{x}) \subseteq p(\bar{x})$  chosen to minimize  $R_{\varphi,2}(\Sigma(\bar{x}))$  actually has

$$R_{\varphi,2}(\Sigma(\bar{x})) = R_{\varphi,2}(p(\bar{x}))$$

This comes from the following

- Fact 9.22.** (a) If  $n < \omega$  and if  $R_{\varphi,2}(\Sigma_0(\bar{x})) \geq n$  for every finite subtype  $\Sigma_0(\bar{x})$ , then  $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$
- (b)  $R_{\varphi,2}(\Sigma(\bar{x}))$  is the minimum of  $R_{\varphi,2}(\Sigma_0(\bar{x}))$  as  $\Sigma_0$  ranges over finite subtypes of  $\Sigma$ .

2. We have discussed  $R_{\varphi,2}$  for partial types, but we can also define it for definable sets. If  $D$  is a definable set, defined by a finite type  $\Sigma(\bar{x})$ , then  $R_{\varphi,2}(D) := R_{\varphi,2}(\Sigma(\bar{x}))$ . The

### 9.3.5 Consequences of Theorem 9.18

**Theorem 9.23.** Suppose that no formula  $\varphi(x_1, \dots, x_n; \bar{y})$  has the dichotomy property. For any model  $M \models T$  and any  $p \in S_n(M)$ ,  $p$  is definable

*Proof.* Take  $\bar{a} \in \mathbb{M}$  realizing  $p$ . Let  $\varphi(\bar{x}; \bar{y})$  be a formula. By Theorem 9.18,  $\text{tp}^\varphi(\bar{a}/M)$  is definable, therefore  $p$  is definable  $\square$

*Remark.* When  $A \subseteq \mathbb{M}$  is arbitrary, one says that  $p \in S_n(A)$  is **definable** if for any  $\varphi(\bar{x}; \bar{y})$  there is an  $L(A)$ -formula  $\psi(\bar{y})$  s.t. for  $\bar{b} \in A$

$$\varphi(\bar{x}; \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

The proof of Theorem 9.23 shows more generally that

If no formula has the dichotomy property, then any  $p \in S_n(A)$  is definable for any  $A$

Therefore in a stable theory, any type over *any* set is definable

**Warning 9.24.** Definable types over arbitrary sets are not as well-behaved as definable types over models. For example

1. A definable type over  $A \subseteq \mathbb{M}$  can have more than one  $A$ -definable extension to  $\mathbb{M}$ : In DLO, there is only one 1-type over  $\emptyset$ . It has two different  $\emptyset$ -definable extensions to  $\mathbb{M}$ : the types at  $+\infty$  and  $-\infty$

2. A definable type over  $A \subseteq \mathbb{M}$  can have no  $A$ -definable extensions to  $\mathbb{M}$ : In ACF,  $\text{tp}(\sqrt{2}/\mathbb{Q})$  is definable (because ACF is stable), but there is no  $\mathbb{Q}$ -definable extension to the monster model  $\mathbb{M}$ . Indeed, there are exactly two extensions to  $\mathbb{M}$ , namely  $\text{tp}(\sqrt{2}/\mathbb{M})$  and  $\text{tp}(-\sqrt{2}/\mathbb{M})$ . These are exchanged by some automorphisms in  $\text{Aut}(\mathbb{M}/\mathbb{Q})$  so neither one can be  $\mathbb{Q}$ -definable

**Theorem 9.25.** Assume  $T$  is stable and  $M \models T$ . Let  $D \subseteq M^n$  be  $\emptyset$ -definable. If  $X \subseteq D$  is definable (with parameters from  $M$ ), then  $X$  is definable over parameters from  $D$

*Proof.* Let  $\psi(\bar{y})$  be a formula defining  $D$  and let  $\varphi(\bar{a}; \bar{y})$  be a formula defining  $X$ . Then  $\text{tp}^\varphi(\bar{a}/D)$  is definable by Theorem 9.18. Therefore there is a formula  $\theta(\bar{y})$  with parameters from  $D$  s.t. if  $\bar{b} \in D$ , then

$$(M \models \varphi(\bar{a}; \bar{b})) \Leftrightarrow \varphi(\bar{a}; \bar{b}) \in \text{tp}^\varphi(\bar{a}/D) \Leftrightarrow (M \models \theta(\bar{b}))$$

In other words, if  $\psi(\bar{y})$  holds, then  $\varphi(\bar{a}; \bar{y})$  is equivalent to  $\theta(\bar{y})$ . Therefore  $X$  is defined by  $\psi(\bar{y}) \wedge \theta(\bar{y})$ , a formula with parameters from  $D$   $\square$

**Fact 9.26.** Let  $M$  be a stable structure and  $D \subseteq M^n$  be  $A$ -definable. Let  $N$  be the structure whose domain is  $D$ , with an  $m$ -ary relation symbol for each  $A$ -definable  $X \subseteq D^m$ . Then the definable subsets of  $D^m$  in  $M$  agree with the definable subsets of  $D^m$  in  $N$

## 9.4 Coheirs

**Definition 9.27.** If  $M \leq N$ , if  $p \in S_n(M)$ , if  $q \in S_n(N)$ , then  $q$  is a **coheir** of  $p$  if  $q \supseteq p$  and  $q$  is finitely satisfiable in  $M$  (for any  $\phi(x) \in q(x)$ , there is  $a \in M$  s.t.  $N \models \phi(a)$ )

**Example 9.3.**  $\mathbb{Q}^{\text{alg}} \leq \mathbb{C}$ ,  $q = \text{tp}(\pi/\mathbb{C})$ ,  $p = \text{tp}(\pi/\mathbb{Q}^{\text{alg}})$ .  $q \supseteq p$ , but  $q$  isn't a coheir since  $x = \pi \in q(x)$

**Example 9.4.** If  $M \leq N$  strongly minimal,  $q(x) \in S_1(N)$ ,  $p(x) \in S_1(M)$  is the transcendental 1-type,  $p \subseteq q$ , then  $q$  is a coheir of  $p$ ,

If  $\varphi(x) \in q(x)$ , then  $\varphi(N)$  is cofinite and  $M$  is infinite, so  $\varphi(N) \cap M \neq \emptyset$

**Lemma 9.28.** If  $M \leq N$ ,  $\Sigma(\bar{x})$  partial type over  $N$ ,  $\Sigma(\bar{x})$  is f.sat. in  $M$ , then  $\exists q(\bar{x}) \in S_n(N)$ ,  $q(\bar{x})$  is f.sat. in  $M$

*Proof.* Let  $\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(N) : \forall \bar{a} \in M, N \models \psi(\bar{a})\}$

If  $\bar{a} \in M$ , then  $\bar{a}$  satisfies  $\Psi$

**Claim**  $\Sigma(\bar{x})$  fsat in  $M \Rightarrow \Sigma \cup \Psi$  is fsat  $\Rightarrow q \in S_n(N)$ ,  $q \supseteq \Sigma \cup \Psi$

If  $q$  isn't fast. in  $M$  then  $\varphi(\bar{x}) \in q(\bar{x})$ ,  $\varphi(\bar{x})$  not sat. in  $M$   $\square$

**Theorem 9.29.** *If  $p \in S_n(M)$ ,  $N \geq M$ , then  $\exists q \in S_n(N)$ ,  $q$  is a coheir of  $p$*

**Theorem 9.30.** *Suppose  $M_1 \leq M_2 \leq M_3$ ,  $p_1 \in S_n(M_1)$ ,  $p_2 \in S_n(M_2)$ ,  $p_2$  is a coheir of  $p_1$ . Then  $\exists p_3 \in S_n(M_3)$ ,  $p_3$  is a coheir of  $p_1$  and  $p_2$*

## 9.5 Coheir Independence

### 9.5.1 Coheir independence

**Definition 9.31.** Let  $M$  be a small model,  $\bar{a}, \bar{b}$  small tuples (possibly infinite). Then  $\bar{a}$  is **coheir independent** from  $\bar{b}$  over  $M$ , written

$$\bar{a} \downarrow_M^u \bar{b}$$

if  $\text{tp}(\bar{a}/M\bar{b})$  is finitely satisfiable in  $M$

*Remark.* The relation  $A \downarrow_M^u B$  is finitary w.r.t. the arguments  $A$  and  $B$ , in the following sense.  $A \downarrow_M^u B$  holds iff the following does:

For any finite tuple  $\bar{a} \in A$  and any finite tuple  $\bar{b} \in B$ , we have  $\bar{a} \downarrow_M^u \bar{b}$

Since a formula  $\varphi(\bar{x}, \bar{y})$  can only refer to finitely many variables

*Remark.* The relation  $\downarrow^u$  can be used to define heirs and coheirs, as follows. Suppose  $M, N$  are small models with  $M \leq N$ . Suppose  $p \in S_n(M)$  and  $q \in S_n(N)$  with  $q \supseteq p$ . Take  $\bar{a} \in \mathbb{M}^n$  realizing  $q$

1.  $q = \text{tp}(\bar{a}/N)$  is a coheir of  $p = \text{tp}(\bar{a}/M)$  iff  $\bar{a} \downarrow_M^u N$
2.  $q = \text{tp}(\bar{a}/N)$  is an heir of  $p = \text{tp}(\bar{a}/M)$  iff  $N \downarrow_M^u \bar{a}$

### 9.5.2 Existence

**Lemma 9.32.** *Let  $M$  be a small model and  $\bar{a}, \bar{b}$  be tuples, possibly infinite*

1. *There is  $\sigma \in \text{Aut}(\mathbb{M}/M)$  s.t.  $\sigma(\bar{a}) \downarrow_M^u \bar{b}$*
2. *There is  $\sigma \in \text{Aut}(\mathbb{M}/M)$  s.t.  $\bar{a} \downarrow_M^u \sigma(\bar{b})$*

*Proof.* 1. Let  $\alpha$  be the length of  $\bar{a}$  and  $\bar{x}$  be an  $\alpha$ -tuple of variables. Let

$$\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(M\bar{b}) : \psi(\bar{x}) \text{ is satisfied by every } \bar{a}' \in M^\alpha\}$$

If  $\varphi(\bar{x}) \in \text{tp}(\bar{a}/M)$ , then there is  $\bar{a}' \in M^\alpha$  satisfying  $\varphi(\bar{x})$  because  $\text{tp}(\bar{a}/M)$  is finitely satisfiable in  $M$ . Then  $\bar{a}'$  satisfies  $\{\varphi(\bar{x})\} \cup \Psi(\bar{x})$ .

This shows  $\text{tp}(\bar{a}/M) \cup \Psi(\bar{x})$  is finitely satisfiable, hence realized by some  $\bar{a}' \in \mathbb{M}^\alpha$

Then  $\bar{a}'$  realizes  $\text{tp}(\bar{a}/M)$ , so  $\text{tp}(\bar{a}'/M) = \text{tp}(\bar{a}/M)$ , and there is  $\sigma \in \text{Aut}(\mathbb{M}/M)$  s.t.  $\sigma(\bar{a}) = \bar{a}'$ . Finally  $\bar{a}' \downarrow_M^u \bar{b}$  by choice of  $\Psi(\bar{x})$ : if  $\varphi(\bar{x}) \in \text{tp}(\bar{a}'/M\bar{b})$  and  $\varphi(\bar{x})$  isn't satisfiable in  $M$ , then  $M \models \neg \exists \bar{x} \varphi(\bar{x})$  and  $M \models \forall \bar{x} \neg \varphi(\bar{x})$ , hence  $\neg \varphi(\bar{x}) \in \Psi(\bar{x})$  and  $\bar{a}$  doesn't satisfy  $\varphi(\bar{x})$ , a contradiction

2. By 1, there is  $\tau \in \text{Aut}(\mathbb{M}/M)$  s.t.  $\tau(\bar{a}) \downarrow_M^u \bar{b}$ . Let  $\sigma = \tau^{-1}$ . Then  $\sigma(\tau(\bar{a})) \downarrow_{\sigma(M)}^u \sigma(\bar{b})$ , or equivalently,  $\bar{a} \downarrow_M^u \sigma(\bar{b})$

□

**Corollary 9.33.** Suppose  $p \in S_n(M)$  and  $N \geq M$

1. There is  $q \in S_n(M)$  s.t.  $q$  is a coheir of  $p$
2. There is  $q \in S_n(M)$  s.t.  $q$  is an heir of  $p$

*Proof.* 1. Take  $\bar{a} \in \mathbb{M}^n$  realizing  $p$ . Let  $\bar{b}$  enumerate  $N$ . By Lemma, there is  $\sigma \in \text{Aut}(\mathbb{M}/M)$  s.t.  $\sigma(\bar{a}) \downarrow_M^u \bar{b}$ , i.e.,  $\sigma(\bar{a}) \downarrow_M^u N$ . Thus  $\text{tp}(\sigma(\bar{a})/N)$  is a coheir of  $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M) = p$

2. Similarly we have  $N \downarrow_M^u \sigma(\bar{a})$ , and thus  $\text{tp}(\sigma(\bar{a})/N)$  is an heir of  $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M)$

□

### 9.5.3 “u” for “ultrafilter”

**Proposition 9.34.** Let  $\bar{a}$  be an  $\alpha$ -tuple in  $\mathbb{M}$ . Let  $M$  be a small model and  $B$  a small set. TFAE

1.  $\bar{a} \downarrow_M^u B$
2. There is an ultrafilter  $\mathcal{U}$  on the set  $M^\alpha$  s.t. for any  $L(MB)$ -formula  $\varphi(\bar{x})$

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Leftrightarrow \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}$$

*Proof.*  $\Rightarrow$ : For  $\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)$ , let  $I = M^\alpha$  and  $\mathcal{F} = \{\varphi(M^\alpha) : \varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)\}$ . We claim that  $\mathcal{F}$  has FIP. Let  $\mathcal{U}$  be an ultrafilter on  $M^\alpha$  extending  $\mathcal{F}$ . Then for any  $L(MB)$ -formula

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \in \mathcal{F} \Rightarrow \varphi(M^\alpha) \in \mathcal{U} \Leftrightarrow \{\bar{a}' \in M : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}$$

Then

$$\varphi(\bar{x}) \notin \text{tp}(\bar{a}/MB) \Rightarrow \neg\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \notin \mathcal{U}$$

$\Leftarrow$ :

□

**Proposition 9.35.** Suppose  $p \in S_n(M)$  and  $N \succeq M$

1. If  $q \in S_n(N)$  is a coheir of  $p$ , then there is an ultrafilter  $\mathcal{U}$  on  $M^n$  s.t.

$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\} \quad (\star)$$

2. Conversely, if  $\mathcal{U}$  is an ultrafilter on  $M^n$  and we define  $q(\bar{x})$  according to  $(\star)$ , then  $q(\bar{x}) \in S_n(N)$  and  $q$  is a coheir of  $p$

*Proof.* 1. Take  $\bar{a}$  realizing  $q$  and  $p$ , then  $\bar{a} \downarrow_M^u N$ . Apply proposition 9.34

2. It suffices to show that  $q$  is finitely satisfiable in  $M$  and complete

□

**Corollary 9.36** (Coheirs extend). Suppose  $M \preceq N \preceq N'$  and  $p \in S_n(M)$  and  $q \in S_n(N)$  is a coheir of  $p$ , then is  $q' \in S_n(N')$  with  $q' \supseteq q$  and  $q'$  is a coheir of  $p$

*Proof.* By proposition 9.35 there is an ultrafilter  $\mathcal{U}$  on  $M^n$  s.t.

$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\}$$

Take  $q'(\bar{x}) = \{\varphi(\bar{x}) \in L(N') : \varphi(M^n) \in \mathcal{U}\}$

□

*Remark.* Suppose  $q \in S_n(N)$  is an heir of  $p \in S_n(M)$ . Then  $N \downarrow_M^u \bar{a}$  for a realization  $\bar{a}$ . Proposition 9.34 gives an ultrafilter  $\mathcal{U}$  and tells us something, ultimate conclusion is

There is an ultrapower  $M^\mathcal{U} \succeq N$  s.t.  $p^\mathcal{U} \supseteq q$

#### 9.5.4 Symmetry

Suppose  $q \in S_n(N)$  is an extension of  $p \in S_n(M)$ .

In stable theory, coheir and heir are the same thing, so for any  $q \in S_n(N)$  and  $p \in S_n(M)$ ,  $M \preceq N$

$$\bar{a} \downarrow_M^u N \Leftrightarrow N \downarrow_M^u \bar{a}$$

**Theorem 9.37.** If  $T$  is stable, then

$$\bar{a} \downarrow_M^u \bar{b} \Leftrightarrow \bar{b} \downarrow_M^u \bar{a}$$

*Proof.* It suffices to prove  $\Rightarrow$ . Let  $\alpha$  be the length of  $\bar{a}$ . Take a small model  $N$  containing  $M$  and  $\bar{b}$ . By the method of 9.36, one can find a type  $q \in S_\alpha(N)$  extending  $\text{tp}(\bar{a}/M\bar{b})$  finitely satisfiable in  $M$ . Take  $\bar{a}'$  realizing  $q$ . Then  $\bar{a}' \downarrow_M^u N$ . Also  $\text{tp}(\bar{a}'/M\bar{b}) = q \upharpoonright (M\bar{b}) = \text{tp}(\bar{a}/M\bar{b})$ , so there is  $\sigma \in \text{Aut}(\mathbb{M}/M\bar{b})$  s.t.  $\sigma(\bar{a}') = \bar{a}$ . Then

$$\bar{a}' \downarrow_M^u N \Rightarrow \sigma(\bar{a}') \downarrow_{\sigma(M)}^u \sigma(N) \Leftrightarrow \bar{a} \downarrow_M^u \sigma(N)$$

Replacing  $N$  with  $\sigma(N)$ , we may assume  $\bar{a} \downarrow_M^u N$ . Therefore we have  $N \downarrow_M^u \bar{a}$ . As  $\bar{b} \in N$ , this implies  $\bar{b} \downarrow_M^u \bar{a}$   $\square$

### 9.5.5 Finitely satisfiable types commute with definable types

Recall that if  $M \leq N \leq \mathbb{M}$ , then

$$N \downarrow_M^u \bar{a} \Leftrightarrow \text{tp}(\bar{a}/N) \supseteq \text{tp}(\bar{a}/M)$$

Therefore the following lemma generalizes the fact that definable types have unique types

**Lemma 9.38.** *Let  $M$  be a small model. Suppose  $\text{tp}(\bar{a}/M)$  is definable and  $\bar{b} \downarrow_M^u \bar{a}$ . Then  $\text{tp}(\bar{a}/M\bar{b})$  is  $p \upharpoonright M\bar{b}$ , where  $p$  is the  $M$ -definable global type extending  $\text{tp}(\bar{a}/M)$*

*Proof.* We must show that for any  $L$ -formula  $\varphi(\bar{x}, \bar{y}, \bar{z})$  and any  $\bar{c} \in M$ ,

$$\varphi(\bar{x}, \bar{b}, \bar{c}) \in \text{tp}(\bar{a}/M\bar{b}) \Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})$$

Otherwise, these things are true

$$\begin{aligned} \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c}) \\ \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c}) \\ (\varphi(\bar{a}, \bar{y}, \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{y}, \bar{c})) \in \text{tp}(\bar{b}/M\bar{a}) \end{aligned}$$

As  $\bar{b} \downarrow_M^u$ , there is  $\bar{b}' \in M$  s.t.

$$\begin{aligned} \mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \\ \mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \\ \varphi(\bar{x}, \bar{b}', \bar{c}) \in \text{tp}(\bar{a}/M) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \end{aligned}$$

A contradiction  $\square$

**Lemma 9.39.** *Let  $p \in S_n(\mathbb{M})$  be finitely satisfiable in a small model  $M$ . If  $\bar{a} \models p \upharpoonright M\bar{b}$ , then  $\bar{a} \downarrow_M^u \bar{b}$*

**Theorem 9.40.** *Let  $p, q$  be global types. Suppose  $p$  is definable over some small set  $A$ . ( $p$  is  $A$ -invariant) Suppose  $q$  is finitely satisfiable in some small set  $B$  ( $q$  is  $B$ -invariant by 9.49). Then  $p$  and  $q$  commute*

*Proof.* Otherwise, there is an  $L(\mathbb{M})$ -formula  $\varphi(\bar{x}, \bar{y})$  s.t.

$$\begin{aligned} (p \otimes q)(\bar{x}, \bar{y}) &\vdash \varphi(\bar{x}, \bar{y}) \\ (q \otimes p)(\bar{y}, \bar{x}) &\vdash \neg \varphi(\bar{x}, \bar{y}) \end{aligned}$$

The formula  $\varphi$  uses only finitely many parameters  $\bar{c}$  from  $\mathbb{M}$ . By Löwenheim–Skolem Theorem there is a small model  $M$  containing  $AB\bar{c}$ . Then  $\varphi(\bar{x}, \bar{y})$  is an  $L(M)$ -formula. Also,  $p$  is  $M$ -definable and  $q$  is finitely satisfiable in  $M$ . Note that  $p, q$  and  $p \otimes q, q \otimes p$  are  $M$ -invariant types. Take  $(\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright M$  and  $\bar{a} \models p \upharpoonright M, \bar{b} \models q \upharpoonright M\bar{a}$ . By Lemma 9.39,  $\bar{b} \downarrow_M^u \bar{a}$

Now  $\text{tp}(\bar{a}/M)$  is the definable type  $p \upharpoonright M$ , so by Lemma 9.39

$$\bar{a} \models p \upharpoonright M\bar{b}$$

Thus  $(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$

It follows that  $(q \otimes p)(\bar{y}, \bar{x})$  and  $(p \otimes q)(\bar{x}, \bar{y})$  have the same restriction to  $M$ . Then  $\varphi$  leads to a contradiction  $\square$

### 9.5.6 Types commute in stable theories

Assume the theory  $T$  is stable

**Proposition 9.41** (Assuming stability). *Let  $p \in S_n(\mathbb{M})$  be a global type and  $M$  be a small model. TFAE*

1.  $p$  is finitely satisfiable in  $M$
2.  $p$  is  $M$ -invariant
3.  $p$  is  $M$ -definable

*Proof.*  $1 \rightarrow 2$ : 9.49

$2 \rightarrow 3$ : 9.51  $\square$

**Theorem 9.42** (Assuming stability). *Let  $p(\bar{x}), q(\bar{y})$  be two invariant global types. Then  $p$  and  $q$  commute*

*Proof.* The types  $p$  and  $q$  are invariant over small sets  $A$  and  $B$  respectively. Take a small model  $M$  containing  $A \cup B$ . Then  $p$  and  $q$  are  $M$ -invariant. By Proposition 9.41,  $p$  is  $M$ -definable and  $p$  is finitely satisfiable in  $M$ . Therefore  $p$  and  $q$  commute by Theorem 9.40  $\square$

### 9.5.7 Morley products and $\downarrow^u$

Let  $M$  be a small model. If  $p$  and  $q$  are  $M$ -definable types, then the Morley product  $p \otimes q$  is also  $M$ -definable by 9.63. Since  $M$ -definable global types corresponds to  $(M)$ -definable types over  $M$  (Proposition 9.48), we can regard  $\otimes$  as an operation on definable types over  $M$

If  $T$  is stable, then all types over  $M$  are definable, and we get an operation

$$\begin{aligned} S_n(M) \times S_n(M) &\rightarrow S_{m+n}(M) \\ (p, q) &\mapsto p \otimes q \end{aligned}$$

The following theorem shows that, at least in stable theories, there is a very close connection between the Morley product  $p \otimes q$  and the coheir independence relation  $\bar{a} \downarrow_M^u \bar{b}$

**Theorem 9.43.** *Assume  $T$  is stable. Let  $M \preceq \mathbb{M}$  be a small model and  $\bar{a}, \bar{b}$  be tuples in  $\mathbb{M}$ . Then*

$$\bar{a} \downarrow_M^u \bar{b} \Leftrightarrow \text{tp}(\bar{b}, \bar{a}/M) = \text{tp}(\bar{b}/M) \otimes \text{tp}(\bar{a}/M)$$

*Proof.* First suppose  $\bar{a} \downarrow_M^u \bar{b}$ . Then  $\text{tp}(\bar{a}/M\bar{b})$  is finitely satisfiable in  $M$ . By Lemma 9.28, there is a global type  $p$  which is finitely satisfiable in  $M$  and extends  $\text{tp}(\bar{a}/M\bar{b})$ . By Proposition 9.41,  $p$  is  $M$ -definable. Then  $p$  is the unique  $M$ -definable global extension of the definable type  $\text{tp}(\bar{a}/M)$ . Let  $q$  be the unique  $M$ -definable global extension of the definable type  $\text{tp}(\bar{b}/M)$ . Then

$$\bar{b} \models q \upharpoonright M \quad \text{and} \quad \bar{a} \models p \upharpoonright M\bar{b}$$

because  $p$  extends  $\text{tp}(\bar{a}/M\bar{b})$ . Therefore

$$(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$$

or equivalently,  $\text{tp}(\bar{b}, \bar{a}/M) = (q \otimes p) \upharpoonright M$ .

Conversely, suppose  $\text{tp}(\bar{b}, \bar{a}/M) = \text{tp}(\bar{b}/M) \otimes \text{tp}(\bar{a}/M)$ . Let  $q$  be the unique  $M$ -definable global extension of the definable type  $\text{tp}(\bar{b}/M)$  and let  $p$  be the unique  $M$ -definable global extension of the definable type  $\text{tp}(\bar{a}/M)$  by 9.48. Then

$$(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$$



or equivalently

$$\bar{b} \models q \upharpoonright M \quad \text{and} \quad \bar{a} \models p \upharpoonright M\bar{b}$$

By Proposition 9.41  $p$  is finitely satisfiable in  $M$ , and so

$$\bar{a} \models p \upharpoonright M\bar{b} \Rightarrow \bar{a} \underset{M}{\overset{u}{\downarrow}} \bar{b}$$

by Lemma 9.39 □

## 9.6 Invariant types

**Lemma 9.44.** *If  $X \subseteq \mathbb{M}^n$ , TFAE*

1.  $\sigma(X) = X$  if  $\sigma \in \text{Aut}(\mathbb{M}/A)$
2. If  $\bar{a}, \bar{b} \in \mathbb{M}^n$ ,  $\bar{a} \equiv_A \bar{b} \Rightarrow (\bar{a} \in X \Leftrightarrow \bar{b} \in X)$
3. There is  $f : S_n(A) \rightarrow \{0, 1\}$  s.t.  $\bar{a} \in X \Leftrightarrow f(\text{tp}(\bar{a}/A)) = 1$

*Proof.* rewrite (2) as

- If  $\bar{a}, \bar{b} \in \mathbb{M}^n$ ,  $\sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(\bar{a}) = \sigma(\bar{b})$ , then  $\bar{a} \in X \Leftrightarrow \bar{b} \in X$
  - If  $\bar{a} \in M$ ,  $\sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\bar{a} \in X \Leftrightarrow \sigma(\bar{a}) \in X$
- 

**Definition 9.45.**  $X \subseteq \mathbb{M}^n$  is  **$A$ -invariant** if  $\forall \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(X) = X$

**Example 9.5.** If  $X$  is  $A$ -definable, then  $X$  is  $A$ -invariant

**Lemma 9.46.** *If  $D \subseteq \mathbb{M}^n$  is definable and  $A$ -invariant, then  $D$  is  $A$ -definable*

*Proof.* Step 1: If  $\bar{b} \in D$  then  $\text{tp}(\bar{b}/A) \vdash \bar{x} \in D$ , by compactness, there is  $\varphi(\bar{x}) \in \text{tp}(\bar{b}/A)$  s.t.  $\varphi(\bar{x}) \vdash \bar{x} \in D$ ,  $\varphi(\mathbb{M}^n) \subseteq D$

Step 2: So then  $D$  is covered by  $A$ -definable subsets of  $D$ . By compactness,  $\bar{D}$  is covered by finitely many of them, which implies  $D$  is  $A$ -definable

Suppose  $D = \psi$ , then  $[\psi] = \bigcup [\varphi_i]$  □

**Definition 9.47.**  $p$  is  **$A$ -definable** if  $\forall \varphi$ ,  $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$  is  $A$ -definable

*Remark.* 1.  $p$  is  $A$ -definable  $\Rightarrow p$  is  $A$ -invariant

2. If  $p$  is definable, then  $p$  is  $A$ -invariant  $\Leftrightarrow p$  is  $A$ -definable

3. If  $p$  is definable then  $p$  is  $A$ -definable for some small  $A$

Each  $d_p\varphi$  uses only finitely many parameters

**Proposition 9.48.** *Suppose  $M \leq \mathbb{M}$ , small*

1. *If  $p \in S_n(M)$  definable and  $p^{\mathbb{M}}$  is its heir over  $\mathbb{M}$ , then  $p^{\mathbb{M}} \in S_n(\mathbb{M})$  is  $M$ -definable*
2.  *$p \mapsto p^{\mathbb{M}}$  is a bijection from definable types over  $M$  to  $M$ -definable types over  $\mathbb{M}$*

*Proof.* 1.  $p^{\mathbb{M}}$  has the same definition as  $p$ , so it's  $M$ -definable

2.  $q \mapsto q \upharpoonright M$  is an inverse to  $p \mapsto p^{\mathbb{M}}$

□

Warning: an  $M$ -invariant type  $p$  is not determined by  $p \upharpoonright M$ . If  $A \subseteq \mathbb{M}$ ,  $A$ -definable type  $p$  is not determined by  $p \upharpoonright A$ . Only works for models  
CHECK

**Theorem 9.49.** *Suppose  $M \leq \mathbb{M}$  and  $p \in S_n(M)$*

1. *If  $q \in S_n(\mathbb{M})$  and  $q$  is a coheir of  $p$ , then  $q$  is  $M$ -invariant*
2.  *$\exists q \in S_n(\mathbb{M}), p \subseteq q$  is  $M$ -invariant*

*Proof.* If  $q$  is a coheir of  $p$ , but  $q$  is not  $M$ -invariant, then  $\exists \bar{b}, \bar{c}, \bar{b} \equiv_M \bar{c}$ ,  $\varphi(\bar{x}, \bar{b}) \in q$ ,  $\varphi(\bar{x}, \bar{c}) \notin q$ . Then  $\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$ . Because  $q$  is fsat. in  $M$ ,  $\exists \bar{a} \in M$ ,  $M \models \varphi(\bar{a}, \bar{b}) \wedge \neg\varphi(\bar{a}, \bar{c})$ , so  $\bar{b} \not\equiv_M \bar{c}$  □

In stable theories:

**Lemma 9.50.** *If  $T$  is stable and  $p$  is  $A$ -invariant, then  $p$  is  $A$ -definable*

**Theorem 9.51.** *Suppose  $T$  stable,  $M \leq \mathbb{M}$  small,  $p \in S_n(M)$ . Let  $p^{\mathbb{M}}$  the global heir.*

1.  *$p^{\mathbb{M}}$  is the only  $M$ -invariant global type extending  $p$*
2.  *$p^{\mathbb{M}}$  is the only global coheir of  $p$*
3. *If  $M \leq N \leq \mathbb{M}$  and  $q$  is the heir of  $p$  over  $N$ , then  $q$  is the unique coheir of  $p$  over  $N$*

*Proof.* 1.  $M$ -invariant  $\Leftrightarrow M$ -definable

2. there is some coheir of  $p$ . Any coheir is  $M$ -invariant, so  $p^M$  is the only coheir

□

**Corollary 9.52.** *In a stable theory, coheirs are unique and coheir=heir*

**Corollary 9.53.** *In a stable theory, “coheir” is transitive*

## 9.7 Morley sequences and the order property

### 9.7.1 Morley sequence

**Lemma 9.54.** *If  $p, q$  are  $A$ -invariant global types,  $p \in S_n(\mathbb{M})$ ,  $q \in S_m(\mathbb{M})$ , then there is  $r \in S_{n+m}(A)$  s.t.  $(\bar{b}, \bar{c}) \models r$  iff*

$$\bar{b} \models p \upharpoonright A \quad \text{and} \quad \bar{c} \models q \upharpoonright (A\bar{b}) \quad (\star)$$

*Proof.* Let  $X = \{(\bar{b}, \bar{c}) : \bar{b} \models p \upharpoonright A \text{ and } \bar{c} \models q \upharpoonright A\bar{b}\}$ . If  $(\bar{b}, \bar{c}) \in X$  and  $\sigma \in \text{Aut}(\mathbb{M}/A)$ , then  $\sigma(\bar{b}) \models \sigma(p \upharpoonright A) = p \upharpoonright A$  and  $\sigma(\bar{c}) \models q \upharpoonright A\sigma(\bar{b})$ . So  $\sigma(\bar{b}, \bar{c}) \in X$ ,  $X$  is  $A$ -invariant

Fix  $\bar{b}_0 \models p \upharpoonright A$ ,  $\bar{c}_0 \models q \upharpoonright A\bar{b}_0$ , so  $(\bar{b}_0, \bar{c}_0) \in X$ . Let  $r = \text{tp}(\bar{b}_0, \bar{c}_0/A)$ . If  $(\bar{b}, \bar{c}) \models r$ , then  $(\bar{b}, \bar{c}) \in X$

Conversely, if  $(\bar{b}, \bar{c}) \in X$ , want  $(\bar{b}, \bar{c}) \models r$ , i.e.,  $(\bar{b}, \bar{c}) \equiv_A (\bar{b}_0, \bar{c}_0)$

$\bar{b} \models p \upharpoonright A = \text{tp}(\bar{b}_0/A)$  so  $\bar{b} \equiv_A \bar{b}_0$ ,  $\exists \sigma \in \text{Aut}(A)$ ,  $\sigma(\bar{b}) = \bar{b}_0$ . Replace  $(\bar{b}, \bar{c})$  with  $(\sigma(\bar{b}), \sigma(\bar{c})) = (\bar{b}_0, \sigma(\bar{c}))$ .

WMA  $\bar{b} = \bar{b}_0$ . Then  $\bar{c}$  and  $\bar{c}_0$  both satisfy  $q \upharpoonright A\bar{b}_0$ . Move  $\bar{c}$  by  $\tau \in \text{Aut}(\mathbb{M}/A\bar{b}_0)$ , we may assume  $\bar{c} = \bar{c}_0$ . Then  $\bar{c} \equiv_{A\bar{b}_0} \bar{c}_0 \Rightarrow \bar{b}\bar{c} \equiv_A \bar{b}_0\bar{c}_0$  □

**Proposition 9.55.** *If  $p \in S_n(\mathbb{M})$ ,  $q \in S_m(\mathbb{M})$  and both are  $A$ -invariant, then there is  $A$ -invariant  $p \otimes q \in S_{n+m}(\mathbb{M})$  s.t. for any small  $A' \supseteq A$ ,*

$$(\bar{b}, \bar{c}) \models (p \otimes q) \upharpoonright A' \Leftrightarrow \bar{b} \models p \upharpoonright A' \text{ and } \bar{c} \models q \upharpoonright A'\bar{b}$$

*Proof.* Note  $p, q$  are  $A'$ -invariant for any  $A'$ -invariant, so lemma gives  $r_{A'} \in S_{n+m}(A')$  for each  $A' \supseteq A$  s.t.  $(\bar{b}, \bar{c}) \models r_{A'} \Leftrightarrow$  the condition

If  $A'' \supseteq A' \supseteq A$ , if  $(\bar{b}, \bar{c}) \models r_{A''}$  then  $(\bar{b}, \bar{c}) \models r_{A'}$  so  $r_{A'} \models r_{A''} \upharpoonright A'$ .

Let  $p \otimes q = \bigcup_{A'} r_{A'}$ , then  $p \otimes q \in S_{n+m}(\mathbb{M})$  and  $r_{A'} = p \otimes q \upharpoonright A'$  □

If  $\sigma \in \text{Aut}(\mathbb{M}/A)$ , then  $\sigma(p \otimes q) = \sigma(p) \otimes \sigma(q) = p \otimes q$ , so  $p \otimes q$  is  $A$ -invariant

**Fact 9.56.** *If  $p \in S_n(M)$   $A$ -invariant where  $M$  is  $|A|^+$ -saturated and  $N \geq M$ , then  $p$  has a unique  $A$ -invariant extension over  $N$*

**Fact 9.57.** If  $p, q \in S_{n+m}(\mathbb{M})$   $A$ -invariant, take  $\bar{b} \models p, \bar{b} \in \mathbb{M}_1 \geq \mathbb{M}$ , take  $\bar{c} \models q \upharpoonright \mathbb{M}_1$  then  $\text{tp}(\bar{b}, \bar{c}/\mathbb{M}) = p \otimes q$

**Definition 9.58.** The **(Morley) product** of invariant types  $p, q$  is  $p \otimes q$

If  $p, q$  are  $A$ -invariant, then  $(\bar{b}, \bar{c}) \models (p \otimes q) \upharpoonright A \Leftrightarrow \bar{b} \models p \upharpoonright A$  and  $\bar{c} \models q \upharpoonright A\bar{b}$

**Definition 9.59.**  $\text{acl}(A) = \bigcup \{ \varphi(\mathbb{M}) : \varphi(x) \in L(A), |\varphi(\mathbb{M})| < \infty \}$

**Fact 9.60.** In ACF, if  $K$  a subfield of  $\mathbb{M}$ , then  $\text{acl}(K)$  is  $K^{\text{alg}}$

**Fact 9.61.** In any theory  $T$ ,  $\text{acl}(-)$  is a finitary closure operation

**Example 9.6.** If  $T$  is strongly minimal and  $p \in S_1(\mathbb{M})$  transcendental 1-type, what is  $p \otimes p$

$b \models p \upharpoonright A \Leftrightarrow b \notin \text{acl}(A)$

Therefore  $(b, c) \models (p \otimes p) \upharpoonright A$  iff  $b \models p \upharpoonright A$  and  $c \models p \upharpoonright A\bar{b}$  iff  $b \notin \text{acl}(A)$  and  $c \notin \text{acl}(A\bar{b})$

idea:  $b, c$  are algebraically independent over  $A$

In stable theories,  $(p \otimes q)(x, y)$  is the “most free” completion of  $p(\bar{x}) \cup q(\bar{y})$

**Example 9.7.** Suppose  $\mathbb{M} \models \text{ACF}$ . let  $p_V$  denote generic type of a variety  $V \subseteq \mathbb{M} \{x \in V\} \cup \{x \notin W : W \subsetneq V, W \text{ algebraic}\}$

If  $V \subseteq \mathbb{M}^n, W \subseteq \mathbb{M}^m$  varieties, then  $V \times W$  is a variety, and  $p_V \otimes p_W = p_{V \times W}$

*Proof.*  $p_V \otimes p_W = p_Z$  for some variety  $Z \subseteq \mathbb{M}^{n+m}$ . Take small  $M \preceq \mathbb{M}$  s.t.  $V, W, Z$  are  $M$ -definable. Take  $\bar{a} \models p_V \upharpoonright M$ , take small  $N \preceq \mathbb{M}, N \supseteq M\bar{a}$ . Take  $\bar{b} \models p_W \upharpoonright N$ , so  $(\bar{a}, \bar{b}) \models p_V \otimes p_W \upharpoonright M = p_Z \upharpoonright M$ .

“ $x \in V \in p_V \upharpoonright M$ ”,  $\bar{a} \in V, \bar{b} \in W$ , so  $(\bar{a}, \bar{b}) \in V \times W$ .

**Fact:**  $p_Z(\bar{x}) \upharpoonright M \vdash \bar{x} \in U \Leftrightarrow Z \subseteq U$  for  $U$  algebraic

So  $(\bar{a}, \bar{b}) \in V \otimes W \Leftrightarrow Z \subseteq V \times W$

Suppose  $Z \subsetneq V \times W$ . Take  $(\bar{a}_0, \bar{b}_0) \in V \times W \setminus Z$ . Let  $Z_{\bar{a}} = \{\bar{y} \in M : (\bar{a}, \bar{y}) \in Z\}$ , then  $Z_{\bar{a}}$  is an algebraic set over  $N \supseteq M_{\bar{a}}$

L

□

**Definition 9.62.** invariant types  $p, q$  “commute” if  $p \otimes q(\bar{x}, \bar{y}) = q \otimes p(\bar{y}, \bar{x})$

**Example 9.8.** In ACF, any two types commutes

$p_V \otimes p_W = p_{V \times W} = p_W \otimes p_V$

If  $p$  is a definable type and  $\varphi(\bar{x}, \bar{y})$  is a formula, then  $(d_p \bar{x})\varphi(\bar{x}, \bar{y})$  means  $d\varphi(\bar{y})$ , the formula defining  $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$

$d_p \bar{x}$  works like quantifier, free variables in  $(d_p \bar{x})\varphi(\bar{x}, \bar{y})$  are  $\bar{y}$

**Example 9.9.** Suppose  $\mathbb{M} \models T$  strongly minimal, let  $p =$  transcendental 1-type,  $\varphi()$

**Proposition 9.63.** *If  $p, q$  are  $A$ -definable global types, then  $p \otimes q$  is  $A$ -definable and  $(d_{p \otimes q}(\bar{x}, \bar{y}))\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{z})$*

*Proof.* Fix  $\bar{c} \in \mathbb{M}$ , take  $M \preceq \mathbb{M}$  s.t.  $\bar{c} \in M$  and  $M \supseteq A$ , so  $p, q$  are  $M$ -definable. Take  $\bar{a} \models p \upharpoonright M$  and  $\bar{b} \models q \upharpoonright M\bar{a}$ , so  $(\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright M$ . So

$$\begin{aligned} \varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q &\Leftrightarrow \varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q \upharpoonright M \\ &\Leftrightarrow \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \\ &\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \upharpoonright M\bar{a} \\ &\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \\ &\Leftrightarrow \mathbb{M} \models (d_q \bar{y})\varphi(\bar{a}, \bar{y}, \bar{c}) \\ &\Leftrightarrow (d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \in p(\bar{x}) \\ &\Leftrightarrow (d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \end{aligned}$$

□

**Example 9.10.** in a strongly minimal theory, if  $p \in S_1(\mathbb{M})$  is transcendental and  $q = p \otimes p$  then  $(d_q(x, y))\varphi(x, y, \bar{z})$  is  $\exists^\infty x \exists^\infty y \varphi(x, y, \bar{z})$

Two definable types  $p, q$  commute iff  $(d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_q \bar{y})(d_p \bar{x})\varphi(\bar{x}, \bar{y}, \bar{z})$   
Let  $A$ -invariant  $p \in S_n(\mathbb{M})$

**Definition 9.64.** A **Morley sequence** of  $p$  over  $A$  is a sequence  $\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots \in \mathbb{M}^n$  s.t.

$$\bar{b}_1 \models p \upharpoonright A, \bar{b}_2 \models p \upharpoonright A\bar{b}_1, \dots, \bar{b}_i \models p \upharpoonright A\bar{b}_1 \dots \bar{b}_{i-1} \dots$$

$$\text{So } (\bar{b}_1, \dots, \bar{b}_n) \models \underbrace{p \otimes \dots \otimes p}_{n \text{ times}}$$

**Example 9.11.** If  $T$  is strongly minimal,  $p$  is transcendental 1-type, a Morley sequence over  $A$  is  $b_1, b_2, \dots$  s.t.  $b_1 \notin \text{acl}(A), b_2 \notin \text{acl}(Ab_1), \dots$

**Example 9.12.** In DLO, in  $(\mathbb{R}, \leq)$ ,  $1, 2, 3, 4, \dots$  is indiscernible

An increasing sequence is indiscernible in DLO

**Theorem 9.65.** *If  $p \in S_n(\mathbb{M})$   $A$ -invariant and  $(\bar{b}_i : i < \omega)$  is a Morley sequence of  $p$  over  $A$ , then it is  $A$ -indiscernible*

## 9.7.2 Order Property

*Remark.* If  $\varphi$  has O.P., then  $\neg \varphi$

### 9.7.3 Instability from the order property

**Lemma 9.66.** For any infinite  $\lambda \geq \aleph_0$  there is a linear order  $(I, \leq)$  and  $S \subseteq I$  s.t.  $|I| > \lambda$ ,  $|S| \leq \lambda$ ,  $S$  is dense in  $I$

*Proof.* there is  $\mu$  s.t.  $|2^\mu| > \lambda$  and  $|2^{<\mu}| \leq \lambda$ .

Let  $I = 2^\mu \cup 2^{<\mu}$  and  $S = 2^{<\mu}$  □

**Theorem 9.67.** If  $\varphi(\bar{x}, \bar{y})$  has O.P., then  $T$  is not  $\lambda$ -stable for any  $\lambda$

*Proof.* Take  $I \supseteq S$  s.t.  $S$  dense in  $I$ ,  $|S| \leq \lambda$ ,  $|I| > \lambda$

$\bar{a}_i, \bar{b}_j, i, j \in \mathbb{Z}$ ,  $\varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$ . By compactness, we can take any linear order. There is  $\bar{a}_i, \bar{b}_j$  for  $i, j \in I$  s.t.  $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$

Let  $C = \{\bar{b}_j : j \in S\}$ ,  $|C| \leq \lambda$ .

**Claim**  $I \setminus S \rightarrow S_n(C)$ ,  $i \mapsto \text{tp}(\bar{a}_i/C)$  is an injection

If  $i_1 < i_2$ , then there is  $j \in S$ ,  $i_1 < j < i_2$  then  $\varphi(\bar{a}_{i_1}, \bar{b}_j) \wedge \neg \varphi(\bar{a}_{i_2}, \bar{b}_j)$ ,  $\bar{b}_j \in C$ , so  $\bar{a}_{i_1} \not\equiv_C \bar{a}_{i_2}$   
 $|S_n(C)| \geq |I \setminus S| > \lambda$  □

### 9.7.4 The order property from instability

**Lemma 9.68.** Suppose  $\varphi(\bar{x}, \bar{y})$  doesn't have O.P. Let  $n_\varphi$  be from Lemma 9. Let  $\bar{b}_1, \bar{b}_2, \dots$  be indiscernible (over  $\emptyset$ ). Then there is no  $\bar{a}$  s.t.  $\mathbb{M} \models \varphi(\bar{a}, \bar{b}_i)$  for  $0 \leq i < n_\varphi$  s.t.

*Proof.*  $n = n_\varphi$ . Suppose  $\bar{a}$  exists, for  $0 \leq$  □

**Lemma 9.69.** Suppose  $\varphi(x_1, \dots, x_n; \bar{y})$  doesn't have O.P.. Take  $N > \max(n_\varphi, n_{\neg\varphi})$ . let  $p$  be an  $A$ -invariant type over  $\mathbb{M}$ . Let  $a_1, a_2, \dots$  be a Morley sequence of  $p$  over  $A$

1. If  $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ , then  $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b})$  for most of  $i < 2N$
2. If  $\varphi(\bar{x}, \bar{b}) \notin p(\bar{x})$ , then  $\mathbb{M} \models \neg \varphi(\bar{a}_i, \bar{b})$  for most of  $i < 2N$

**Example 9.13.** If  $T$  is strongly minimal then  $T$  is stable if  $\varphi(x, \bar{y})$  has the O.P., then there is  $a_i, \bar{b}_i \in \mathbb{M}$   $\mathbb{M} \models \varphi(a_i, \bar{b}_j) \Leftrightarrow i < j$  for  $i, j \in \mathbb{Z}$

So  $\varphi(\mathbb{M}, \bar{b}_0)$  is neither finite or cofinite

### 9.7.5 Commuting types

**Theorem 9.70.** If  $T$  is stable and  $p$  and  $q$  are global types (all types are definable and hence invariant for some  $A$ ), then  $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$

*Proof.* Suppose not. Take  $\varphi(\bar{x}, \bar{y}) \in L(\mathbb{M})$ .  $\varphi(\bar{x}, \bar{y}) \in (p \otimes q)(\bar{x}, \bar{y})$ ,  $\varphi(\bar{x}, \bar{y}) \notin (q \otimes p)(\bar{y}, \bar{x})$ .

Take  $A$  s.t.  $p, q$  are  $A$ -definable and  $\varphi(\bar{x}, \bar{y}) \in L(A)$

Take  $p \otimes q \otimes p \otimes q \otimes \dots$

$((b_i, c_i) : i \in \omega)$  a Morley sequence of  $p \otimes q$  over  $A$

If  $i \leq j$ ,  $(b_i, c_j) \models p \otimes q \upharpoonright A$ ,  $\mathbb{M} \models \varphi(b_i, c_j)$

If  $i > j$ ,  $(c_j, b_i) \models q \otimes p \upharpoonright A$ ,  $\mathbb{M} \models \neg \varphi(b_i, c_j)$  □

## 9.8 Ramsey's theorem and indiscernible sequences

**Definition 9.71.**  $X$  set,  $C$  a set of "colors", then  $f : [X]^\kappa \rightarrow C$  is a coloring of  $\kappa$ -elements subsets of  $X$

**Definition 9.72.**  $Y \subseteq X$  is **homogeneous** if  $f \upharpoonright [Y]^\kappa$  is constant

**Definition 9.73.** If  $N, m, n, k$  are cardinals,  $N \rightarrow (m)_k^n$  means that if  $|X| = N$ ,  $|C| = k$ ,  $f : [X]^n \rightarrow C$ , then there is  $Y \subseteq X$ ,  $Y$  is homogeneous and has size  $m$

**Fact 9.74** (Friends and strangers theorem).  $|X| = 6$ ,  $|C| = 2$  and  $f : [X]^2 \rightarrow C$ , then there is  $Y \subseteq X$  homogeneous and size 3

**Theorem 9.75** (Finite Ramsey's theorem). If  $n, m, k \in \omega$  then there is  $N < \omega$  s.t.  $N \rightarrow (m)_k^n$

*Proof.* Let  $L = \{R_1, \dots, R_k\}$ ,  $R_i$  is an  $n$ -ary predicate (relation) symbol.  $T$  is the  $L$ -theory that says:

- If  $R_i(\bar{x})$  then  $\bar{x}$  is distinct
- If  $\bar{x}$  is distinct then  $R_i(\bar{x})$  holds for exactly one  $i$
- If  $\bar{y}$  is a permutation of  $\bar{x}$ ,  $R_i(\bar{x}) \leftrightarrow R_i(\bar{y})$

A model of  $T$  is a set  $M$  and a coloring of  $[M]^n$

Let  $\varphi$  be the formula s.t.  $M \models \varphi \Leftrightarrow$  there is a homogeneous  $Y \subseteq M$ ,  $|Y| = m$

$$\exists y_1, \dots, y_m \bigwedge_{1 \leq i_1 < \dots < i_n \leq m} \bigwedge_{1 \leq j_1 < \dots < j_n \leq m} \text{same color}$$

Suppose  $N \not\rightarrow (m)_k^n$ , then  $\exists M \models T$   $|M| = N$  and  $M \not\models \varphi$ . Suppose  $N \not\rightarrow (m)_k^n$  for any  $N < \omega$ , then by compactness,  $T \cup \{\neg \varphi\}$  has infinite models. By theorem 17 last week, there is  $M \models T \cup \{\neg \varphi\}$ , indiscernible sequence  $a_1, a_2, \dots \in M$  not constant, but indiscernibility  $\Rightarrow \{a_1, a_2, \dots\}$  is homogeneous,  $\{a_1, \dots, a_m\}$  is homogeneous □

**Fact 9.76** (Infinite Ramsey's theorem).  $\aleph_0 \rightarrow (\aleph_0)_k^n$  for  $n, k \in \omega$

extracting indiscernibles

Working  $\mathbb{M} \models T$ . If  $(I, \leq)$  is a linear order and  $(\bar{a}_i : i \in I)$  is a sequence in  $\mathbb{M}$  and if  $B \subseteq \mathbb{M}$

**Definition 9.77.**  $\text{tp}^{\text{EM}}(\bar{a}/B) = \{\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B) : \forall i_1 < \dots < i_n \in I, \mathbb{M} \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n})\}$ , the **Ehrenfeucht-Mostowski type** over  $B$

*Remark.*  $\text{tp}^{\text{EM}}$  is really a sequence of partial types over  $B, \Sigma_1, \Sigma_2, \dots$

**Example 9.14.** In  $(\mathbb{R}, \leq)$ ,  $1, 1, 2, 2, 3, 3, 4, 4, \dots$

$(x_1 \leq x_2) \in \text{tp}^{\text{EM}}(\dots)$

$x_1 < x_2 \notin \text{tp}^{\text{EM}}$

*Remark.* If  $(\bar{a}_i : i \in I)$  is a sequence,  $I_0 \subseteq I$ , then  $\text{tp}^{\text{EM}}((\bar{a}_i : i \in I)/B) \subseteq \text{tp}^{\text{EM}}((\bar{a}_i : i \in I_0)/B)$

**Definition 9.78.** If  $\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B)$ ,  $(\bar{a}_i : i \in I)$  is “ $\varphi$ -indiscernible” if  $\forall i_1 < \dots < i_n, \forall j_1 < \dots < j_n$ ,

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) \leftrightarrow \varphi(\bar{a}_{j_1}, \dots, \bar{a}_{j_n})$$

*Remark.*  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible iff it is  $\varphi$ -indiscernible for all  $\varphi \in L(B)$

**Definition 9.79.** If  $\Delta$  is a set of formulas,  $\bar{a}$  is  $\Delta$ -indiscernible if it is  $\varphi$ -indiscernible for all  $\varphi \in \Delta$

**Lemma 9.80.** Let  $(\bar{a}_i : i \in I)$  be infinite

1. If  $m < \omega$ ,  $\Delta$  is a finite set of  $L$ -formulas, then there is  $\Delta$ -indiscernible subsequence of length  $m$
2. If  $(J, \leq)$  is a linear order,  $\Delta$  a set of formulas, then there is  $(\bar{b}_j : j \in J) \in \mathbb{M}$  s.t.  $\bar{b}$  is  $\Delta$ -indiscernible and  $\text{tp}^{\text{EM}}(\bar{b}) \supseteq \text{tp}^{\text{EM}}(\bar{a})$

*Proof.* 1. By induction on  $|\Delta|$ .

$|\Delta| = 0$ , take any subsequence of length  $m$

$|\Delta| > 0$ ,  $\Delta = \Delta_0 \cup \{\varphi\}$ ,  $\varphi(x_1, \dots, x_n)$ . Ramsey: there is  $N \rightarrow (m)_2^n$ , by induction there is subsequence  $(\bar{b}_i : i < N)$   $\Delta_0$ -indiscernible. Define  $f : [N]^n \rightarrow \{0, 1\}$  by

$$f(\{i_1, \dots, i_n\}) = \begin{cases} 1 & \mathbb{M} \models \varphi(b_{i_1}, \dots, b_{i_n}) \\ 0 & \text{otherwise} \end{cases}$$

there is subsequence  $(\bar{c}_i : i < m)$  that is homogeneous,  $\varphi$ -indiscernible



2. By compactness, we may assume  $J$  is finite,  $\Delta$  is finite. By part 1

□

**Theorem 9.81.** *If  $(\bar{a}_i : i \in I)$  an infinite sequence,  $B$  is a set of parameters,  $(J, \leq)$  infinite linear order, then there is  $B$ -indiscernible sequence  $(\bar{b}_j : j \in J)$  with  $\text{tp}^{\text{EM}}(\bar{b}/B) \supseteq \text{tp}^{\text{EM}}(\bar{a}/B)$*

*Proof.* Apply Lemma 9.80 with  $\Delta = \{\text{all the } L(B)\text{-formulas}\}$

□

“Extracting indiscernible sequences”

**Example 9.15** (=Theorem 17 last week). If  $|\mathbb{M}| = \infty$ , take distinct  $a_0, a_1, a_2, \dots \in \mathbb{M}$ ,  $x_1 \neq x_2 \in \text{tp}^{\text{EM}}(\bar{a})$ . Take  $b_0, b_1, \dots$  indiscernible, extracted from  $\bar{a}$ , then  $(x_1 \neq x_2) \in \text{tp}^{\text{EM}}(\bar{a}) \subseteq \text{tp}^{\text{EM}}(\bar{b})$ , so  $b_i \neq b_j$  for  $i < j$ . So  $\bar{b}$  is a non-constant indiscernible sequence

**Example 9.16.** Suppose  $\mathbb{M} \geq (\mathbb{R}, +, \cdot, \leq, 0, 1, -)$ . Suppose  $b_1, b_2, b_3, \dots$  is indiscernible, extracted from  $1, 2, 3, \dots$

$$\begin{aligned} x_1 > 0 &\in \text{tp}^{\text{EM}}(\bar{a}) \subseteq \text{tp}^{\text{EM}}(\bar{b}) \\ x_2 - x_1 &\geq 1 \in \text{tp}^{\text{EM}}(\bar{b}) \end{aligned}$$

*Remark.*  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible iff  $\text{tp}^{\text{EM}}(\bar{a}/B)$  is “complete”, i.e.,  $\forall \varphi(x_1, \dots, x_n) \in L(B)$ ,  $\varphi \in \text{tp}^{\text{EM}}$  or  $\neg \varphi \in \text{tp}^{\text{EM}}$

**Theorem 9.82.** *If  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible, if  $(J, \leq)$  is a linear order, then there is  $B$ -indiscernible  $(\bar{b}_j : j \in J)$  with  $\text{tp}^{\text{EM}}(\bar{b}/B) = \text{tp}^{\text{EM}}(\bar{a}/B)$*

*Remark.* If  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible, then  $\text{tp}(\bar{a}/B)$  is determined by  $\text{tp}^{\text{EM}}(\bar{a}/B)$  and  $(I, \leq)$

$$\mathbb{M} \models \varphi(a_{i_1}, \dots, a_{i_n}) \Leftrightarrow \varphi \in \text{tp}^{\text{EM}}(\bar{a}/B)$$

So if  $(\bar{a}_i : i \in I)$ ,  $(\bar{b}_i : i \in I)$  both  $B$ -indiscernible and  $\text{tp}^{\text{EM}}(\bar{a}/B) = \text{tp}^{\text{EM}}(\bar{b}/B)$ , then  $\text{tp}(\bar{a}/B) = \text{tp}(\bar{b}/B)$

**Theorem 9.83** (extending indiscernibles). *If  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible, if  $(J, \leq)$  extends  $(I, \leq)$ , then  $\exists \bar{a}_j$  for  $j \in J \setminus I$  s.t.  $(\bar{a}_j : j \in J)$  is  $B$ -indiscernible*

*Proof.* extract  $B$ -indiscernible  $(\bar{c}_j : j \in J)$  from  $(\bar{a}_i : i \in I)$ ,  $\text{tp}^{\text{EM}}(\bar{c}/B) = \text{tp}^{\text{EM}}(\bar{a}/B)$

the subsequence  $(\bar{c}_i : i \in I)$  has same EM-type as

there is  $\sigma \in \text{Aut}(\mathbb{M}/B)$  s.t.  $\sigma(\bar{c}_i) = \bar{a}_i$  for  $i \in I$ . Define  $\bar{a}_j := \sigma(\bar{c}_j)$  for  $j \in J \setminus I$

□

**Theorem 9.84.** *If  $\varphi(\bar{x}, \bar{y}) \in L$ , TFAE*

1.  $\varphi$  has O.P.,  $\bar{a}_i, \bar{b}_i, i \in \mathbb{Z}, \mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$
2. same as (1) but  $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$  is indiscernible
3. There is an indiscernible  $(\bar{a}_i : i \in \mathbb{Z})$  some  $\bar{b}$  s.t.  $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$

*Proof.* 1  $\rightarrow$  2: extract an indiscernible sequence from

2  $\rightarrow$  3: take  $\bar{b} = \bar{b}_0$

3  $\rightarrow$  1: For any  $j \in \mathbb{Z}, (\bar{a}_i : i \in \mathbb{Z}) \equiv_B (\bar{a}_{i+j} : i \in \mathbb{Z})$ , there is  $\sigma_j \in \text{Aut}(\mathbb{M})$ ,  $\sigma_j(\bar{a}_i) = \bar{a}_{i+j}$ . Let  $\bar{b}_j = \sigma_j(\bar{b})$ . Then  $\bar{a}_i \bar{b}_j = \sigma(\bar{a}_{i-j} \bar{b})$   
 $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow \mathbb{M} \models \varphi(\bar{a}_{i-j}, \bar{b}) \Leftrightarrow i - j < 0 \Leftrightarrow i < j$   $\square$

**Corollary 9.85.**  $T$  is unstable  $\Leftrightarrow$  there is  $\varphi(\bar{x}, \bar{y})$  with O.P.  $\Leftrightarrow (\bar{a}_i : i \in \mathbb{Z})$ ,  $\varphi(\bar{x}, \bar{y}), \bar{b}$  s.t.  $\varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$

Total indiscernibility

**Example 9.17.** In DLO, 1,2,3,4,... is indiscernible but not totally indiscernible

In a totally

**Proposition 9.86.** If  $T$  is unstable, then  $\exists$  indiscernible sequence that isn't totally indiscernible

*Proof.* Take  $\varphi$  with O.P., take  $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$  witnessing O.P., then  $\varphi(a_1, b_2) \wedge \neg \varphi(a_2, b_1)$ , so  $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$  isn't totally indiscernible  $\square$

**Definition 9.87.**  $\text{tp}(a_1, \dots, a_n/B)$  is **symmetric** if  $\forall$  permutation  $\sigma \in S(n)$   $\bar{a}_1, \dots, \bar{a}_n \equiv_B \bar{a}_{\sigma(1)}, \dots, \bar{a}_{\sigma(n)}$

*Remark.* Let  $\sigma_i$  be the permutation swapping  $i$  and  $i+1$  and fixing everything else.

$\text{tp}(\bar{a}_1, \dots, \bar{b}_n/B)$  is symmetric iff it holds for each  $\sigma_i$

*Remark.* Let  $(\bar{a}_i : i \in I)$  be  $B$ -indiscernible. Let  $p_n = \text{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/B)$  for any  $i_1 < \dots < i_n$ . Then  $(\bar{a}_i : i \in I)$  is totally  $B$ -indiscernible iff each  $p_n$  is symmetric

*Remark.* If  $(\bar{a}_i : i \in I)$  is  $B$ -indiscernible, then  $\text{tp}^{\text{EM}}(\bar{a}/B)$  determines whether  $\bar{a}$  is totally  $B$ -indiscernible

$\text{tp}^{\text{EM}}$  is  $p_1, p_2, \dots$

**Lemma 9.88.** Let  $(\bar{a}_i : i \in \mathbb{Z})$  be  $B$ -indiscernible. Let  $C = \{\bar{a}_i : i \notin \{0, 1\}\}$ . If  $\bar{a}_0 \bar{a}_1 \equiv_{BC} \bar{a}_1 \bar{a}_0$ . Then  $(\bar{a}_i : i \in \mathbb{Z})$  is totally  $B$ -indiscernible

*Proof.* there is  $\sigma_0 \in \text{Aut}(\mathbb{M}/BC)$ ,  $\sigma_0(\bar{a}_0) = \bar{a}_1$ ,  $\sigma(\bar{a}_1) = \bar{b}_0$

By indiscernibility, there is  $\alpha_i \in \text{Aut}(\mathbb{M}/B)$  s.t.  $\alpha_i$  swaps  $\bar{a}_i, \bar{a}_{i+1}$  fixes  $\bar{a}_j$  for  $j \notin \{i, i+1\}$ . This means  $\bar{a}_1 \dots \bar{a}_n \equiv_B \bar{a}_{\sigma_i(1)} \dots \bar{a}_{\sigma_i(n)}$  so  $\text{tp}(\bar{a}_1, \dots, \bar{a}_n/B)$  is symmetric  $\square$

**Proposition 9.89.** *If  $\mathbb{M}$  is stable and  $A \subseteq \mathbb{M}$  small, then  $\mathbb{M}$  is stable as an  $L(A)$ -structure*

*Proof.* Otherwise, there is  $L(A)$ -formula  $\varphi(\bar{x}, \bar{y})$  with the O.P.  $\varphi(\bar{x}, \bar{y}, \bar{c})$  for some  $\bar{c} \in A$ ,  $\bar{b}_i \bar{c}$  is the new  $\bar{b}$   $\square$

**Theorem 9.90.** *TFAE*

1.  $T$  is stable
2. every indiscernible sequence is totally indiscernible
3.  $B$ -indiscernible  $\Rightarrow$  totally  $B$ -indiscernible

*Proof.*  $3 \rightarrow 2$ : trivial

$1 \rightarrow 3$ : Suppose  $T$  stable but  $(\bar{a}_i : i \in I)$   $B$ -indiscernible not totally  $B$ -indiscernible

Extract  $(\bar{a}'_i : i \in I)$  from  $(\bar{a}_i : i \in I)$  some  $\square$

**Corollary 9.91.** *If  $T$  is stable, if  $(\bar{a}_i : i \in I)$  is indiscernible, if  $D$  is definable,  $\{i \in I : \bar{a}_i \in D\}$  is finite or cofinite in  $I$*

*Proof.* Suppose not. Take  $i_1, i_2, \dots \in I$  s.t.  $a_{i_1}, a_{i_2}, \dots \notin D$ ,  $\square$

## 9.9 Invariant types clarification

### 9.9.1 Morley products via fixed monster model

*Remark.* One can think of a global type as a compatible family of types over small sets.

More precisely, one can think of a global type  $p \in S_n(\mathbb{M})$  as a map that assigns to each small set  $A \subseteq \mathbb{M}$  a complete  $n$ -type  $p_A \in S_n(A)$ , namely  $p_A = p \upharpoonright A$

**Warning 9.92.** *We only consider  $A$ -invariant types  $p$  when  $A$  is small and  $p$  is a global type (strongly homogeneous) In particular, we shouldn't talk about  $A$ -invariant types over small sets (we can extend partial elementary map to small sets). We shouldn't talk about  $\mathbb{M}$ -invariant types (strongly homogeneity only works for small sets)*

### 9.9.2 Morley products by varying the monster

**Definition 9.93.** Suppose  $M$  is a model and  $A \subseteq M$ . Suppose every  $n$ -type over  $A$  is realized in  $M$  for every  $n < \omega$ . We say that  $p \in S_n(M)$  is  **$A$ -invariant** if the following property holds:

- If  $\bar{b}, \bar{c} \in M$ ,  $\bar{b} \equiv_A \bar{c}$  and  $\varphi(\bar{x}, \bar{y})$  is a formula, then

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow \varphi(\bar{x}, \bar{c}) \in p(\bar{x})$$

We only want to consider  $A$ -invariant types over models  $M$  s.t.  $M$  realizes every type over  $A$ .

*Remark.* Suppose  $M$  is  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous,  $A \subseteq M$ ,  $|A| < \kappa$  and  $p \in S_n(M)$ . Then  $p$  is  $A$ -invariant iff

$$\forall \sigma \in \text{Aut}(M/A) \sigma(p) = p$$

**Lemma 9.94.** Suppose  $p \in S_n(M)$  is  $A$ -invariant. Let  $N \succeq M$  be an elementary extension. Then there is a unique  $q \in S_n(N)$  extending  $p$  s.t.  $q$  is  $A$ -invariant

*Proof.* Given  $\varphi(\bar{x}, \bar{y})$  and  $\bar{c} \in N$ . Let  $\bar{b}$  be a realization of  $\text{tp}(\bar{c}/A)$  inside  $M$ . Define  $\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$  to hold iff  $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ .

Now we have a well-defined set of  $L(N)$ -formulas  $q(\bar{x})$  s.t. if  $\bar{c} \in N$  and  $\bar{b} \in M$  and  $\bar{c} \equiv_A \bar{b}$

$$\varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \Leftrightarrow \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \quad (\star)$$

First we check  $q(\bar{x})$  is finitely satisfiable. Otherwise, take a contradictory set of formulas  $\{\varphi_i(\bar{x}, \bar{c}_i) : 1 \leq i \leq m\}$  in  $q(\bar{x})$ . Take  $(\bar{b}_1, \dots, \bar{b}_m) \in M$  realizing  $\text{tp}(\bar{c}_1, \dots, \bar{c}_m/A)$ . Then  $\{\varphi_i(\bar{x}, \bar{b}_i) : 1 \leq i \leq m\} \subseteq p(\bar{x})$ . As  $p(\bar{x})$  is consistent,

$$M \models \exists \bar{x} \bigwedge_{i=1}^m \varphi_i(\bar{x}, \bar{b}_i)$$

which implies

$$N \models \exists \bar{x} \bigwedge_{i=1}^m \varphi_i(\bar{x}, \bar{c}_i)$$

Then  $q(\bar{x})$  is finitely satisfiable, i.e., a partial type over  $N$ . It is also complete: if  $\varphi(\bar{x}, \bar{c})$  is an  $L(N)$ -formula, take  $\bar{b} \in M$  realizing  $\text{tp}(\bar{c}/A)$ . Then

$$\begin{aligned} \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) &\text{ or } \neg \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \\ \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) &\Rightarrow \varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \\ \neg \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) &\Rightarrow \neg \varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \end{aligned}$$

Also  $q(\bar{x})$  extends  $p(\bar{x})$

Finally  $q(\bar{x})$  is  $A$ -invariant by construction

Uniqueness: any  $A$ -invariant  $q \in S_n(N)$  extending  $p$  must satisfy  
( $\star$ )  $\square$

**Example 9.18.** If  $p \in S_n(M)$  is  $A$ -definable, then the unique  $A$ -invariant extension to  $N$  is the heir of  $p$ ,

**Warning 9.95.** In Definition 9.93 and Lemma 9.94, it's really necessary to assume that  $M$  realizes every  $n$ -types over  $A$ .

Embed  $M = (\mathbb{R}, \leq)$  into a monster model  $N \models DLO$ . Let  $A = \mathbb{Q} \subseteq M$ . No two elements of  $\mathbb{R}$  have the same type over  $\mathbb{Q}$ . Therefore any type  $p \in S_n(\mathbb{R})$  satisfies the definition of " $\mathbb{Q}$ -invariance" if we ignore the fact that  $\mathbb{R}$  doesn't realize all types over  $\mathbb{Q}$ . Then we can prove the following:

1. If  $\alpha \in \mathbb{R}$  is irrational, then the constant type  $\text{tp}(\alpha/\mathbb{R})$  has no  $\mathbb{Q}$ -invariant extension to  $N$  (The only extension is  $\text{tp}(\alpha/N)$ , which is not  $\mathbb{Q}$ -invariant)
2. Suppose  $\beta \in N$  is greater than every element of  $\mathbb{R}$ . Then  $\text{tp}(\beta/\mathbb{R})$  is the type at  $+\infty$  over  $\mathbb{R}$ . It turns out that  $\text{tp}(\beta/\mathbb{R})$  has two distinct  $\mathbb{Q}$ -invariant extensions to  $N$  (One is the coheir and the other is the heir)

Fix a model  $M$ , a subset  $A \subseteq M$  s.t. every  $n$ -type over  $A$  is realized in  $M$  for all finite  $n$ , and fix an  $A$ -invariance type  $p$ . For any elementary extension  $N \geq M$ , let  $p|N$  denote the  $A$ -invariant extension of  $N$  as in Lemma 9.94. More generally, if  $B \subseteq N$ , let  $p|B$  denote the restriction of  $p|N$  to  $B$

*Remark.* we can think of  $p|B$  as the type over  $B$  characterized by the fact that

$$\begin{aligned} \varphi(\bar{x}, \bar{c}) \in p|B &\Leftrightarrow \varphi(\bar{x}, \bar{c}') \in p \\ &\text{for } \bar{c}' \in M \text{ realizing } \text{tp}(\bar{c}/A) \end{aligned}$$

**Definition 9.96.** Let  $p, q$  be  $A$ -invariant types in  $M$ . Take  $\bar{a}$  realizing  $p|M$  and  $\bar{b}$  realizing  $q|M\bar{a}$ . Define  $p \otimes q$  to be  $\text{tp}(\bar{a}, \bar{b}/M)$

**Lemma 9.97.** This is well-defined, independent of the choice of  $\bar{a}, \bar{b}$ . Moreover  $p \otimes q$  is  $A$ -invariant

*Proof.* Take  $\bar{a}, \bar{b}' \models p|M$   $\bar{b} \models q|M\bar{a}$ ,  $\bar{b}' \models q|M\bar{a}'$ . WLOG these realizations live in a monster model  $\mathbb{M} \geq M$

**Claim:**  $\text{tp}(\bar{a}, \bar{b}/M) = \text{tp}(\bar{a}', \bar{b}'/M)$

Otherwise, there is  $\bar{c} \in M$  and a formula  $\varphi(\bar{x}, \bar{y}, \bar{c})$  s.t.

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \wedge \neg \varphi(\bar{a}', \bar{b}', \bar{c})$$

This implies

$$\begin{aligned}\varphi(\bar{a}, \bar{y}, \bar{c}) &\in \text{tp}(\bar{b}/M\bar{a}) = q|M\bar{a} \subseteq q|\mathbb{M} \\ \neg\varphi(\bar{a}', \bar{y}, \bar{c}) &\in \text{tp}(\bar{b}'/M\bar{a}) = q|M\bar{a}' \subseteq q|\mathbb{M}\end{aligned}$$

Note  $\bar{a}, \bar{a}'$  both realize  $p$ , so they have the same type over  $M$ , this implies  $\bar{a} \equiv_{A\bar{c}} \bar{a}'$ , which is equivalent to  $\bar{a}\bar{c} \equiv_A \bar{a}'\bar{c}$ . As  $q|\mathbb{M}$  is  $A$ -invariant, we see

$$\varphi(\bar{a}, \bar{y}, \bar{c}) \in q|\mathbb{M} \Leftrightarrow \varphi(\bar{a}', \bar{y}, \bar{c}) \in q|\mathbb{M}$$

This shows that  $p \otimes q$  is well-defined. Next we claim that  $p \otimes q$  is  $A$ -invariant. Otherwise,

$$p \otimes q \vdash \varphi(\bar{x}, \bar{y}, \bar{c}) \wedge \neg\varphi(\bar{x}, \bar{y}, \bar{c}')$$

for two  $\bar{c}, \bar{c}' \in M$  with  $\bar{c} \equiv_A \bar{c}'$ . By definition of  $p \otimes q$ , this means

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \wedge \neg\varphi(\bar{a}, \bar{b}, \bar{c}')$$

□

## 10 Fundamental Order and Forking

### 10.1 The fundamental order

Fix  $n < \omega$

**Definition 10.1.** If  $M \preceq \mathbb{M}$ ,  $p \in S_n(M)$ ,  $\varphi(x_1, \dots, x_n; \bar{y})$ .  $p$  **represents**  $\varphi$  if  $\exists \bar{b} \in M$   $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ .  $p$  **omits**  $\varphi$  otherwise

The **class** of  $p$  is  $[p] = \{\varphi : p \text{ represents } \varphi\}$

$[p] \leq [q]$  if  $[p] \supseteq [q]$

The **fundamental order** is  $\{[p] : M \preceq \mathbb{M}, p \in S_n(M)\}$ , with  $\leq$  (depends on  $n$ ).  $p \leq q$  means  $[p] \leq [q]$

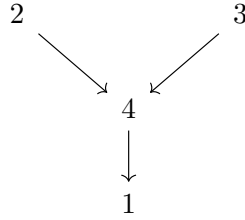
*Remark.*  $\leq$  is a partial order on the fundamental order but a preorder on the class  $\{p : M \models T, p \in S_n(M)\}$

$[p]$  is not a standard notation

**Example 10.1.**  $n = 1$ ,  $\varphi(x, y) := x = y$ .  $p \in S_1(M)$  represents  $p$  iff  $\exists b \in M$ ,  $x = b \in p(x)$  iff  $p$  is a constant type

**Example 10.2.**  $n = 1$ ,  $T = \text{DLO}$ , there are four classes:

1. constant types
2. types at  $+\infty$
3. types at  $-\infty$
4. others



$x = y$  is represented in 1

$x < y$  is represented in 1,3,4  $\text{tp}(2/\mathbb{R})$  has  $x < 3$ ,  $\text{tp}(-\infty/\mathbb{R})$  has  $x < 0$ ,  $\text{tp}(\sqrt{2}/\mathbb{Q})$  has  $x < 2$ ,  $\text{tp}(+\infty/\mathbb{R})$  doesn't have  $x < b$

$x > y$  is represented in 1,2,4

$\text{tp}(\sqrt{2}/\mathbb{Q})$  and  $\text{tp}(0^+/\mathbb{R})$  have the same class

Goal: in a stable theory: if  $q$  is an extension of  $p$ , then if  $q \sqsupseteq p$ , then  $[q] = [p]$ , if  $q \sqsupset p$ , then  $[q] < [p]$

**Proposition 10.2.** Suppose  $M \preceq N$ ,  $p \in S_n(M)$ ,  $q \in S_n(N)$ ,  $p \subseteq q$

1.  $[q] \leq [p]$
2.  $[q] = [p]$  iff for any  $L$ -formula  $\varphi(\bar{x}, \bar{y})$ , if  $\bar{b} \in N$  and  $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$ , then  $\exists \bar{b}' \in M \varphi(\bar{x}, \bar{b}') \in p$
3. if  $q \sqsupset p$ , then  $[q] = [p]$

*Proof.* 1. every formula  $\varphi$  represented by  $p$  is represented by  $q$

2.  $[q] = [p] \Leftrightarrow [q] \geq [p] \Leftrightarrow [q] \subseteq [p] \Leftrightarrow$  this condition

3.

□

*Remark.* Suppose  $M \preceq N$ ,  $p \in S_n(M)$ ,  $q \in S_n(N)$ ,  $p \subseteq q$

1.  $[q] = [p]$  means that  $\forall \varphi(\bar{x}, \bar{y}) \in L, \exists \bar{b} \in N, \varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \exists \bar{b}' \in M \varphi(\bar{x}, \bar{b}') \in p(\bar{x})$

2. but  $q \sqsupseteq p$  considers  $L(M)$ -formulas

$q \sqsupseteq p$  iff  $[q] = [p]$  in  $L(M)$

**Proposition 10.3.**  $M, N \leq \mathbb{M}$ ,  $p \in S_n(M)$ ,  $q \in S_n(N)$ , then  $[p] \geq [q]$  iff  $\exists$  ultrafilter  $\mathcal{U}$  and elementary embedding  $M \rightarrow N^{\mathcal{U}}$  making  $q^{\mathcal{U}} \sqsupseteq p$

*Proof.*  $\Rightarrow$  similar to 9.2

$\Leftarrow$ :  $[q^{\mathcal{U}}] = [q]$  because  $q^{\mathcal{U}} \sqsupseteq q$ ,  $[q^{\mathcal{U}}] \leq [p]$  because  $q^{\mathcal{U}} \sqsupseteq p$   $\square$

## 10.2 The fundamental order in stable theory

Assume  $T$  is stable

**Lemma 10.4.** Suppose  $M \leq N \leq \mathbb{M}$ ,  $p \in S_n(M)$ ,  $q_1, q_2 \in S_n(N)$ ,  $q_1, q_2 \sqsupseteq p$  and  $[q_1] = [p] = [q_2]$ . Then  $q_1 = q_2$ .

*In other words, there is at most one extension of  $p$  to  $N$  with the same class as  $p$*

*Proof.* similar to 9.6

Suppose  $q_1 \neq q_2$ ,  $\exists \varphi(\bar{x}, \bar{b})$  s.t.  $\varphi \in q_1$ ,  $\neg \varphi \in q_2$

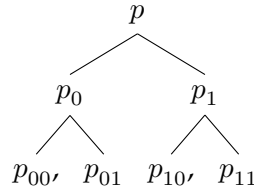
Let  $\beta = [p]$

**Claim:** If  $M' \leq \mathbb{M}$ ,  $p' \in S_n(M')$ ,  $[p'] = \beta$ , then  $\exists N' \geq M'$ ,  $\exists q'_1, q'_2 \in S_n(N')$ ,  $q'_1, q'_2 \sqsupseteq p'$ ,  $[q'_1] = [q'_2] = \beta$  and  $\exists \bar{b}' \in N'$ ,  $\varphi(\bar{x}, \bar{b}') \in q'_1$  and  $\neg \varphi \in q'_2$ .  
 $[p'] \geq [p]$ , so there  $\mathcal{U}$ , elementary embedding  $M' \rightarrow M^{\mathcal{U}}$  s.t.  $p^{\mathcal{U}} \sqsupseteq p'$ .

Then we have  $M' \rightarrow M^{\mathcal{U}} \rightarrow N^{\mathcal{U}}$

$[q_1^{\mathcal{U}}] = [q_1] = \beta = [q_2] = [q_2^{\mathcal{U}}]$ . Let  $q'_i = q_i^{\mathcal{U}}$ ,  $N' = N^{\mathcal{U}}$

Using the claim, we can build a tree of types



where  $p_{\sigma 0}$  and  $p_{\sigma 1}$  are extensions of  $p_{\sigma}$  differing by a formula  $\varphi(\bar{x}, \bar{b}_{\sigma})$ . Then  $\varphi$  has the dichotomy property  $\square$

**Proposition 10.5.** If  $M \leq N$ ,  $p \in S_n(M)$ ,  $q \in S_n(N)$ ,  $q \sqsupseteq p$

1.  $q \sqsupseteq p \Leftrightarrow [q] = [p]$

2.  $q \not\sqsupseteq p \Leftrightarrow [q] < [p]$

*Proof.* Let  $q'$  be the heir of  $p$ ,  $q' \in S_n(N)$

If  $q \sqsupseteq p$ , then  $q = q'$

If  $[q] = [p]$ , then  $[q] = [q'] = [p]$  so Lemma 10.4 shows  $q = q'$   $\square$



### 10.3 bounds

$T$  is stable

Fix  $A \subseteq \mathbb{M}, p \in S_n(A)$

**Definition 10.6.** If  $M \leq \mathbb{M}, M \supseteq A$ , then  $\text{Ex}_M(p) = \{[q] : q \in S_n(M), q \supseteq p\}$

**Lemma 10.7.** Every chain in  $\text{Ex}_M(p)$  has an upper bound

*Proof.* Let  $F = \{q \in S_n(M) : q \supseteq p\}$ . Suppose  $\{[q_i] : i \in I\}$  is a chain,  $q_i \in F$ ,  $(I, \leq)$  a linear order,  $[q_i] \leq [q_j]$  for  $i \leq j$

If  $i \leq j$ ,  $q_i$  omits  $\varphi$ , then  $q_j$  omits  $\varphi$

Let  $\Sigma(\bar{x}) = \{\neg\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \text{ omitted by some } q, \bar{b} \in M\}$

**Claim:**  $p(\bar{x}) \cup \Sigma(\bar{x})$  is consistent

suppose  $\varphi_1, \dots, \varphi_m$ ,  $\varphi_j$  is omitted by  $q_{i_j}$ ,  $i_j \in I$ ,  $\bar{b}_1, \dots, \bar{b}_m \in M$ . Want  $p \cup \{\neg\varphi_j(\bar{x}, \bar{b}_j) : 1 \leq j \leq m\}$  consistent. WLOG,  $i_1 \leq \dots \leq i_m$ , then  $\varphi_j(\bar{x}, \bar{y})$  is omitted by  $q_{i_m}$  for all  $j$ . Then  $q_{i_m}(\bar{x})$  extends  $p(\bar{x}) \cup \{\neg\varphi_j(\bar{x}, \bar{b}_j) : 1 \leq j \leq m\}$

Take  $q(\bar{x}) \in S_n(M)$  a completion of  $p(\bar{x}) \cup \Sigma(\bar{x})$ . Then  $q \in F$ , so  $[q] \in \text{Ex}_M(p)$ .  $\square$

**Definition 10.8.**  $\text{Bd}_M(p) = \{\text{maximal } \beta \in \text{Ex}_M(p)\}$

Elements of  $\text{Bd}_M(p)$  are called **bounds** of  $p$

**Corollary 10.9.**  $\forall \beta \in \text{Ex}_M(p), \exists \beta' \in \text{Bd}_M(p), \beta' \geq \beta$ , and  $\text{Bd}_M(p)$  is not empty

**Example 10.3.** Suppose  $A \leq \mathbb{M}, p \in S_n(A)$ ,  $A$  is a model

**Claim:**  $[p] = \max \text{Ex}_M(p)$ , so  $\text{Bd}_M(p) = \{[p]\}$

Take  $q \in S_n(M), q \supseteq p$ , then  $[q] = [p]$ ,  $[q] \in \text{Ex}_M(p)$ . If  $r \in S_n(M), r \supseteq p$ , then  $[r] \leq [p]$ , so if  $p \in \text{Ex}_M(p)$  then  $\beta \leq [p]$

**Lemma 10.10.** Suppose  $M, N \leq \mathbb{M}, M, N \supseteq A, p \in S_n(A)$

1.  $\forall \beta \in \text{Ex}_M(p), \exists \beta' \in \text{Ex}_N(p), \beta' \geq \beta$

2.  $\text{Bd}_M(p) = \text{Bd}_N(p)$

*Proof.* 1. Take  $M' \leq \mathbb{M}, M' \supseteq M \cup N, \beta \in \text{Ex}_M(p)$  means  $\exists q \in S_n(M), q \supseteq p, [q] = \beta$

Let  $q' \in S_n(M')$  be  $q' \supseteq q$

Let  $r = q' \upharpoonright N$ . Then  $r \supseteq p$ , so  $[r] \in \text{Ex}_N(p)$ .  $[r] \geq [q'] = [q] = \beta$

2. suppose  $\beta \in \text{Bd}_M(p)$

- by 1, there is  $\beta' \in \text{Ex}_N(p)$  with  $\beta \leq \beta'$
- by Corollary 10.9, there is  $\beta'' \in \text{Bd}_N(p)$  with  $\beta' \leq \beta''$
- By 1, there is  $\beta''' \in \text{Ex}_M(p)$  with  $\beta'' \leq \beta'''$

Then  $\beta \leq \beta' \leq \beta'' \leq \beta''' \in \text{Ex}_M(p)$ . Therefore

$$\beta = \beta' = \beta'' = \beta'''$$

This shows  $\text{Bd}_M(p) \subseteq \text{Bd}_N(p)$

□

Since  $\text{Bd}_M(p)$  doesn't depend on  $M$ , we write it as  $\text{Bd}(p)$

## 10.4 Theorem of the bound

$T$  is stable

**Definition 10.11.**  $p \in S_n(\mathbb{M})$  is **Lascar  $A$ -invariant** if  $p$  is  $M$ -invariant for every  $A \subseteq M \leq \mathbb{M}$

weaker than being  $A$ -invariant in stable theory

**Lemma 10.12.** If  $A \subseteq M \leq \mathbb{M}$ ,  $p \in S_n(A)$ ,  $q \in S_n(M)$ ,  $q \supseteq p$ ,  $[q] \in \text{Bd}(p)$ . Let  $q^{\mathbb{M}}$  be the global heir of  $q$ . Then  $q^{\mathbb{M}}$  is Lascar  $A$ -invariant

*Proof.* By 10.2,  $[q^{\mathbb{M}}] = [q] \in \text{Bd}(p)$ . If  $q^{\mathbb{M}}$  isn't Lascar  $A$ -invariant, there is small  $N \supseteq A$   $q^{\mathbb{M}}$  isn't  $N$ -invariant, not  $N$ -definable. Then  $q^{\mathbb{M}} \not\sqsubseteq q^{\mathbb{M}} \upharpoonright N$  (or else  $q^{\mathbb{M}}$  would be  $N$ -definable 9.48). By Proposition 10.5,  $[q^{\mathbb{M}} \upharpoonright N] > [q^{\mathbb{M}}] = [q]$

Let  $r = q^{\mathbb{M}} \upharpoonright N$ ,  $r \supseteq p$ , so  $[r] \in \text{Ex}_N(p)$ ,  $[q] \in \text{Bd}(p) = \text{Bd}_N(p)$  is maximal in  $\text{Ex}_N(p)$ , but  $[r] > [q]$ ,  $[r] \in \text{Ex}_N(p)$  □

**Lemma 10.13.** Fix  $\bar{b}$  and  $A$ , then  $\exists M \supseteq A$ ,  $M \leq \mathbb{M}$ , the global heir of  $\text{tp}(\bar{b}/M)$  is Lascar  $A$ -invariant. Also given  $\beta \in \text{Bd}(\text{tp}(\bar{b}/A))$ , can make  $\text{tp}(\bar{b}/M)$  and it's heir have class  $\beta$

*Proof.* Take  $\beta \in \text{Bd}(p)$ ,  $p = \text{tp}(\bar{b}/A)$ . Take  $M \supseteq A$   $M \leq \mathbb{M}$ . Take  $q \in S_n(M)$ ,  $[q] = \beta$ . Take  $\bar{b}_0 \models q$ ,  $\text{tp}(\bar{b}_0/A) = \text{tp}(\bar{b}/A)$ . There is  $\sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(\bar{b}_0) = \bar{b}$ .

Move  $M, q, \bar{b}_0$  by  $\sigma$ . We may assume  $\bar{b}_0 = \bar{b}$ , so  $\text{tp}(\bar{b}/M) = q$ ,  $[q] = \beta$ .

By 10.12,  $q^{\mathbb{M}}$  is Lascar  $A$ -invariant □

**Lemma 10.14.** Fix  $\bar{b}$ ,  $A$ . Suppose  $M_1, M_2 \leq \mathbb{M}$ ,  $M_1, M_2 \supseteq A$ . Let  $p_i \in S_n(\mathbb{M})$  be the heir of  $\text{tp}(\bar{b}/M_i)$ . Suppose  $p_1, p_2$  are Lascar  $A$ -invariant, then  $p_1 = p_2$

*Proof.* Suppose  $p_1 \neq p_2$ . Take  $\varphi(\bar{x}, \bar{c}) \in p_1(\bar{x})$ ,  $\neg\varphi(\bar{x}, \bar{c}) \in p_2$ .

Lemma 10.13 shows there is  $M_3 \leq \mathbb{M}$ ,  $M_3 \supseteq A$  s.t.  $\text{tp}(\bar{c}/M_3) \sqsubseteq r \in S_n(\mathbb{M})$  and  $r$  is Lascar  $A$ -invariant.

Take  $\bar{e} \models r \upharpoonright M_1 M_2 M_3 \bar{b}$ . Note  $\bar{b} \models p_1 \upharpoonright M_1$  and  $\bar{e} \models r \upharpoonright M_1 \bar{b}$ . Then  $(\bar{b}, \bar{e}) \models (p_1 \otimes r) \upharpoonright M_1$  since  $p_1, r$  are  $M_1$ -invariant. In stable theory, product commutes. Therefore  $(\bar{e}, \bar{b}) \models (r \otimes p_1) \upharpoonright M_1$ . Then  $\bar{b} \models p_1 \upharpoonright M_1 \bar{e}$ .

$\bar{e} \models r \upharpoonright M_3 = \text{tp}(\bar{c}/M_3)$ ,  $\bar{e} \equiv_{M_3} \bar{c}$ ,  $p_1$  is  $M_3$ -invariant. Hence  $\varphi(\bar{x}, \bar{e}) \in p_1$ . So  $\mathbb{M} \models \varphi(\bar{c}, \bar{e})$

Same argument with  $p_2$ , get  $\mathbb{M} \models \neg\varphi(\bar{c}, \bar{e})$ , a contradiction  $\square$

**Theorem 10.15.** If  $p \in S_n(A)$ ,  $|\text{Bd}(p)| = 1$

*Proof.* Take  $\bar{b} \models p$ ,  $\beta_1, \beta_2 \in \text{Bd}(p)$ . Lemma 10.13, there is  $A \subseteq M_1, M_2 \leq \mathbb{M}$  s.t.  $[\text{tp}(\bar{b}/M_i)] = \beta$  if  $p_i = \text{tp}(\bar{b}/M_i)$ ,  $p_i^{\mathbb{M}}$  is Lascar  $A$ -invariant.

Lemma 10.14  $p_1^{\mathbb{M}} = p_2^{\mathbb{M}}$   $\square$

**Definition 10.16.**  $\text{bd}(p)$  = the bound of  $p$

example

## 10.5 Non-forking extensions

Assume stability

**Proposition 10.17.** If  $A \subseteq B$ ,  $p \in S_n(A)$ ,  $q \in S_n(B)$ ,  $p \subseteq q$ , then  $\text{bd}(q) \leq \text{bd}(p)$

*Proof.* Take  $M \supseteq B$ ,  $M \leq \mathbb{M}$ ,  $r \in S_n(M)$  extending  $q$  with  $[r] = \text{bd}(q)$ . Then  $r$  extends  $p$ , so  $[r] \in \text{Ex}_M(p)$ . As  $\text{bd}(p)$  is the maximum of  $\text{Ex}_M(p)$  we must have  $[r] \leq \text{bd}(p)$   $\square$

**Definition 10.18.** If  $A \subseteq B$ ,  $p \in S_n(A)$ ,  $q \in S_n(B)$ ,  $q \supseteq p$ ,  $q$  is a **nonforking extension** of  $p$  iff  $\text{bd}(q) = \text{bd}(p)$

**Proposition 10.19.** If  $M \leq N$  and  $q \in S_n(N)$  extends  $p \in S_n(M)$ , then  $q$  is a non-forking extension of  $p$  iff  $q$  is an heir of  $p$

Proposition 10.19 ensures the notation  $q \sqsupseteq p$  is unambiguous

*Proof.*  $\text{bd}(p) = [p]$  and  $\text{bd}(q) = [q]$   $\square$

**Proposition 10.20** (Full transitivity). Suppose  $A_1 \subseteq A_2 \subseteq A_3$  and  $p_i \in S_n(A_i)$  for  $i = 1, 2, 3$  with  $p_1 \subseteq p_2 \subseteq p_3$ . Then  $p_1 \sqsubseteq p_3$  iff  $p_1 \sqsubseteq p_2$  and  $p_2 \sqsubseteq p_3$

**Proposition 10.21** (Extension). *If  $p \in S_n(A)$  and  $B \supseteq A$ , then there is at least one  $q \in S_n(B)$  with  $q \supseteq p$*

*Proof.* Take a small model  $M \supseteq B$ . Then  $\text{bd}(p) \in \text{Bd}(p) \subseteq \text{Ex}_M(p)$ , so there is  $r \in S_n(M)$  extending  $p$  with  $[r] = \text{bd}(p)$ . Let  $q = r \upharpoonright B$ . Then  $\text{bd}(r) = \text{bd}(p)$ , so  $r \supseteq p$ . By full transitivity,  $q \supseteq p$   $\square$

## 10.6 Forking formulas and Lascar invariance

**Lemma 10.22.** *If  $A \subseteq M \preceq \mathbb{M}$  and if the global heir of  $\text{tp}(\bar{b}/M)$  is Lascar  $A$ -invariant, then  $\text{tp}(\bar{b}/M) \supseteq \text{tp}(\bar{b}/A)$*

*Proof.* Let  $\beta$  be the bound of  $\text{tp}(\bar{b}/A)$ . By Lemma 10.13 there is a small model  $M' \supseteq A$  s.t. the global heir of  $\text{tp}(\bar{b}/M')$  is Lascar  $A$ -invariant and has class  $\beta$ . By Lemma 10.14  $\text{tp}(\bar{b}/M')$  and  $\text{tp}(\bar{b}/M)$  have the same global heir. By Proposition 10.2 they have the same class. Then the class of  $\text{tp}(\bar{b}/M)$  is  $\beta = \text{bd}(\text{tp}(\bar{b}/A))$ , implying  $\text{tp}(\bar{b}/M) \supseteq \text{tp}(\bar{b}/A)$   $\square$

**Proposition 10.23** (Forking and Lascar  $A$ -invariance). *If  $p$  is a global type and  $A \subseteq \mathbb{M}$ , then  $p \supseteq (p \upharpoonright A)$  iff  $p$  is Lascar  $A$ -invariant*

*Proof.* First suppose  $p \supseteq (p \upharpoonright A)$ . For any small model  $M \supseteq A$ , we have  $p \supseteq (p \upharpoonright M)$  by Full transitivity, which then means  $p$  is the heir of  $p \upharpoonright M$  by Proposition 10.19. Then  $p$  is  $M$ -definable, so  $p$  is Lascar  $A$ -invariant

Conversely, suppose  $p$  is Lascar  $A$ -invariant. Take a small model  $M \supseteq A$  and take  $\bar{b} \models p \upharpoonright M$ . Then  $p$  is  $M$ -definable, so  $p$  is the global heir of  $p \upharpoonright M = \text{tp}(\bar{b}/M)$ . By Lemma 10.22,  $\text{tp}(\bar{b}/M) \supseteq \text{tp}(\bar{b}/A) = p \upharpoonright A$ . But  $p$  is the heir of  $\text{tp}(\bar{b}/M)$ , so  $p \supseteq \text{tp}(\bar{b}/M) \supseteq p \upharpoonright A$ . By transitivity we have  $p \supseteq (p \upharpoonright A)$   $\square$

**Corollary 10.24.** *If  $A \subseteq B$  and  $q \in S_n(B)$  extends  $p \in S_n(A)$ , then  $q \supseteq p$  iff some global extension of  $q$  is Lascar  $A$ -invariant*

**Definition 10.25.** An  $L(\mathbb{M})$ -formula  $\varphi(\bar{x})$  **forks over**  $A$  if every global type containing it fails to be Lascar  $A$ -invariant

**Proposition 10.26** (Finite Character). *If  $A \subseteq B$  and  $q \in S_n(B)$  extends  $p \in S_n(A)$ , then  $q \not\supseteq p$  ( $q$  is a forking extension of  $p$ ) iff some formula in  $q$  forks over  $A$*

*Proof.* For any model  $M$ , let  $\Sigma_M(\bar{x})$  be the global partial type

$$\{\varphi(\bar{x}; \bar{b}) \leftrightarrow \varphi(\bar{x}; \bar{c}) : \varphi \in L, \bar{b} \equiv_M \bar{c}\}$$

A global type  $p \in S_n(\mathbb{M})$  extends  $\Sigma_M$  iff it is  $M$ -invariant, iff it is  $M$ -definable. Define  $\Sigma_A(\bar{x})$  to be the union of  $\Sigma_M(\bar{x})$  for  $M$  ranging over small models

containing  $A$ . Then  $p \in S_n(\mathbb{M})$  extends  $\Sigma_A(\bar{x})$  iff it is Lascar  $A$ -invariant. Therefore an  $L(\mathbb{M})$ -formula  $\psi(\bar{x})$  forks over  $A$  iff  $\Sigma_A(\bar{x}) \cup \{\psi(\bar{x})\}$  is inconsistent. By Corollary 10.24,  $q \not\sqsubseteq p$  iff  $\Sigma_A(\bar{x}) \cup q(\bar{x})$  is inconsistent. Then the result follows by compactness  $\square$

**Intuition** if  $\varphi$  forks over  $A$ , then  $\varphi(\mathbb{M})$  is “small”, and  $\{\varphi(\mathbb{M}) : \varphi \text{ forks over } A\}$  is an ideal

## 10.7 The dichotomy property and the fundamental order

**Lemma 10.27.** *Assume stability. Suppose  $M \leq N \leq \mathbb{M}$ ,  $p \in S_n(M)$ , and  $q_1, q_2 \in S_n(N)$  are exteqqeQ*

# 11 Algebraic closure and imaginaries

## 11.1 Many-sorted logic

### 11.1.1 First approximation: many-sorted structures

**Definition 11.1.** A **(single sorted) structure** consists of a set  $M$  and a collection of **functions**, **relations** and **constants**. Each function is a function  $f : M^{n_f} \rightarrow M$  for some number  $n_f$  called the **arity** of  $f$ . Each relation is a relation  $R \subseteq M^{n_R}$  for some  $n_R$  called the **arity** of  $R$ . Each constant is an element of  $M$

**Definition 11.2.** A **many-sorted structure** consists of a collection of **sorts**, **functions**, **relations** and **constants**. Each sort is a set. Each function is a function  $f : X_1 \times X_2 \times \cdots \times X_n \rightarrow Y$  where  $X_1, X_2, \dots, X_n, Y$  are sorts. Each relation is a relation  $R \subseteq X_1 \times \cdots \times X_n$  where  $X_1, \dots, X_n$  are sorts. Each constant is an element of a sort

This approach works if we only need to consider definable sets and formulas within a fixed structure. If we want to talk about theories or elementary equivalence, we need to define many-sorted languages before we can properly define many-sorted structures.

### 11.1.2 Many-sorted languages

**Definition 11.3.** A **many-sorted language** consists of the following data:

1. A set  $\mathcal{S}$  of **sorts**

2. A set  $\mathcal{F}$  of **function symbols**: for each  $f \in \mathcal{F}$ , a finite non-empty list of sorts  $(X_1, \dots, X_n, Y)$ , called the **signature** of  $f$
3. A set  $\mathcal{R}$  of **relation symbols**: for each  $R \in \mathcal{R}$ , a finite list of sorts  $(X_1, \dots, X_n)$ , called the **signature** of  $R$
4. A set  $\mathcal{C}$  of **constant symbols**: for each  $c \in \mathcal{C}$ , a sort  $X$ , called the **signature** of  $c$

## 11.2 Definable closure

Work in a monster model  $\mathbb{M}$

**Fact 11.4.** *If  $\mathcal{F}$  is a small family of definable  $D \subseteq \mathbb{M}^n$ , suppose  $X \subseteq \mathbb{M}^n$  definable. Suppose  $X$  is “infinite boolean combination” of  $\mathcal{F}$ , i.e., if  $\bar{a}, \bar{b} \in \mathbb{M}^n$  and  $\forall D \in \mathcal{F}$ ,  $\bar{a} \in D \Leftrightarrow \bar{b} \in D$ , then  $\bar{a} \in X \Leftrightarrow \bar{b} \in X$ . Then  $X$  is a (finite) boolean combination of sets in  $\mathcal{F}$*

*Proof.* WLOG,  $\mathcal{F}$  is closed under finite boolean combination

- If  $\bar{a} \in X$ ,  $\exists D \in \mathcal{F}$ ,  $\bar{a} \in D$
- $X$  is a finite union of things in  $\mathcal{F}$

□

**Fact 11.5** (9.46). *If  $D \subseteq \mathbb{M}^n$  definable,  $A \subseteq \mathbb{M}$  small, then  $D$  is  $A$ -definable iff  $D$  is  $A$ -invariant*

**Definition 11.6.** If  $D_1, \dots, D_{n+1}$  are  $A$ -definable, and  $f : D_1 \times \dots \times D_n \rightarrow D_{n+1}$ , then  $f$  is  **$A$ -definable** if  $\Gamma(f) = \{(\bar{a}, b) : b = f(\bar{a})\}$  is definable

**Definition 11.7.** If  $A \subseteq \mathbb{M}$ ,  $\text{dcl}(A) = \{b \in \mathbb{M} : \{b\} \text{ is } A\text{-definable}\}$

**Example 11.1.** In a field,  $a \div b \in \text{dcl}(\{a, b\})$  because  $\{a \div b\}$  is  $\varphi(\mathbb{M})$ ,  $\varphi(x) := bx = a$

Note: if  $\bar{b} \in \mathbb{M}^n$ ,  $\bar{b} \in \text{dcl}(A)^n \Leftrightarrow \{\bar{b}\} \subseteq \mathbb{M}^n$  is  $A$ -definable

**Proposition 11.8.** *If  $\bar{b} \in \mathbb{M}^n$ ,  $A \subseteq \mathbb{M}$ , TFAE*

1.  $\bar{b} \in \text{dcl}(A)$ , i.e.,  $\{\bar{b}\}$  is  $A$ -definable
2.  $\forall \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(\bar{b}) = \bar{b}$ , i.e.  $\{\bar{b}\}$  is  $A$ -invariant

3.  $\bar{b}$  is the only realization of  $\text{tp}(\bar{b}/A)$

*Proof.*  $1 \Leftrightarrow 2$  by fact since  $\{\bar{b}\}$  is definable

$2 \Leftrightarrow 3$ : let  $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ . □

**Proposition 11.9.** 1.  $A \subseteq \text{dcl}(A)$

2.  $A \subseteq B \Rightarrow \text{dcl}(A) \subseteq \text{dcl}(B)$

3.  $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$

4.  $D$  is  $A$ -definable  $\Leftrightarrow D$  is  $\text{dcl}(A)$ -definable

Conditions 1-3 say that  $\text{dcl}(-)$  is an abstract “closure operator”

*Proof.* 4. If  $D$  is  $\text{dcl}(A)$ -definable,  $\sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $b \in \text{dcl}(A)$ ,  $\sigma(b) = b$  by Proposition 11.8,  $\sigma \in \text{Aut}(\mathbb{M}/\text{dcl}(A))$ ,  $\sigma(D) = D$ , so  $D$  is  $A$ -invariant

3. take  $b \in \text{dcl}(\text{dcl}(A))$ ,  $\{b\}$  is  $\text{dcl}(A)$ -definable,  $\{b\}$  is  $A$ -definable,  $b \in \text{dcl}(A)$  □

**Definition 11.10.**  $A$  is **definably closed** if  $\text{dcl}(A) = A$

**Proposition 11.11.**  $\text{dcl}(A)$  is the smallest definably closed set containing  $A$

*Proof.*  $\text{dcl}(A)$  is definably closed

$\text{dcl}(A) \supseteq A$

Now suppose  $B = \text{dcl}(B)$  and  $B \supseteq A$ , by “monotonicity”,  $\text{dcl}(B) \supseteq \text{dcl}(A)$ , so  $B \supseteq \text{dcl}(A)$  □

**Fact 11.12.** if  $M \models \text{ACF}_0$ , if  $A \subseteq M$ , then  $A = \text{dcl}(A) \Leftrightarrow A$  is a subfield of  $M$

$\Rightarrow$ : easy  $\Leftarrow$ : harder

It fails in  $\text{ACF}_p$ , in  $\text{ACF}_p$ ,  $p > 0$ ,  $K \subseteq M$  is definably closed  $\Leftrightarrow \forall x \in K$ ,  $\sqrt[p]{x} \in K$

**Definition 11.13.**  $\bar{a}, \bar{b}$  are **interdefinable** iff  $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b})$  iff  $\bar{a} \in \text{dcl}(\bar{b})$  and  $\bar{b} \in \text{dcl}(\bar{a})$

**Lemma 11.14.**  $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b}) \Leftrightarrow \text{Aut}(\mathbb{M}/\bar{a}) = \text{Aut}(\mathbb{M}/\bar{b})$

*Proof.* By Proposition 11.8,  $\text{dcl}(\bar{a}) \subseteq \text{dcl}(\bar{b}) \Leftrightarrow \bar{a} \in \text{dcl}(\bar{b}) \Leftrightarrow \text{Aut}(\mathbb{M}/\bar{b}) \subseteq \text{Aut}(\mathbb{M}/\bar{a})$  □

**Lemma 11.15.** If  $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b})$ , then  $\exists \emptyset$ -definable bijection  $f : X \rightarrow Y$  s.t.  $f(\bar{a}) = \bar{b}$

*Proof.*  $\bar{a} \in \text{dcl}(\bar{b})$ , so there is  $L$ -formula  $\varphi_1(\bar{x}, \bar{y})$  s.t.  $\{\bar{a}\} = \varphi_1(\mathbb{M}, \bar{b})$ , also there is  $\{\bar{b}\} = \varphi_2(\bar{a}, \mathbb{M})$

Let  $\varphi = \varphi_1 \wedge \varphi_2$ , can replace  $\varphi_1, \varphi_2$  with  $\varphi$ .

Let  $\psi(\bar{x}, \bar{y})$  be  $\varphi(\bar{x}, \bar{y}) \wedge \exists! \bar{w} \varphi(\bar{w}, \bar{y}) \wedge \exists! \bar{z} \varphi(\bar{x}, \bar{z})$ . Then  $\psi(\bar{x}, \bar{y})$  defines a bijection and  $f(\bar{a}) = \bar{b}$

We can extract the definition of  $f(\bar{a}) = \bar{b}$  and fill  $f$  with garbage  $\square$

### 11.3 Algebraic closure

**Definition 11.16.**  $\text{acl}(A) = \bigcup \{D \subseteq \mathbb{M}^1 : D \text{ is } A\text{-definable}, |D| < \infty\}$

**Example 11.2.** In fields,  $\sqrt{a} \in \text{acl}(a)$  because  $\{\sqrt{a}, -\sqrt{a}\}$  is  $\{a\}$ -definable

Note:  $\bar{b} \in \text{acl}(A) \Leftrightarrow$  there is  $A$ -definable  $D \subseteq \mathbb{M}^n, \bar{b} \in D, |D| < \infty$

If  $\bar{b} \in D \subseteq \mathbb{M}^n$ , then let  $D_i = \pi_i(D), \pi_i(\bar{x}) = x_i$ ,

**Proposition 11.17.** If  $\bar{b} \in \mathbb{M}^n, A \subseteq \mathbb{M}$  small, let  $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$

1. If  $\bar{b} \in \text{acl}(A)$ , then  $S$  is finite and  $A$ -definable

2. If  $\bar{b} \notin \text{acl}(A)$ , then  $S$  is large ( $S$  is not small)

*Proof.* 1.  $\bar{b} \in \text{acl}(A)$ , there is  $D$  finite,  $A$ -definable,  $\bar{b} \in D$ .  $\sigma(\bar{b}) \in \sigma(D) = D$  for  $\sigma \in \text{Aut}(\mathbb{M}/A)$ , so  $S \subseteq D$ ,  $S$  is finite, then  $S$  is definable. Also  $S$  is  $A$ -invariant

for each  $a \in D \setminus S$ , we have a  $L(A)$ -formula  $\varphi_a$  s.t.  $\mathbb{M} \models \neg \varphi_a(a) \wedge \varphi_a(b)$ . Then  $\bigwedge_{a \in D \setminus S} \varphi_a$  extract  $S$  from  $D$

2. If  $\bar{b} \notin \text{acl}(A)$  but  $S$  is small. Let  $\Sigma(\bar{x}) = \text{tp}(\bar{b}/A) \cup \{\bar{x} \neq \bar{c} : \bar{c} \in S\}$ .  $\Sigma(\bar{x})$  is inconsistent. By Compactness, there is  $\psi(\bar{x}) \in \text{tp}(\bar{b}/A)$ ,  $\bar{c}_1, \dots, \bar{c}_m \in S$ ,  $\{\psi(\bar{x}), \bar{x} \neq \bar{c}_1, \dots, \bar{x} \neq \bar{c}_m\}$  is inconsistent. Thus  $D = \psi(\mathbb{M}) \subseteq \{\bar{c}_1, \dots, \bar{c}_m\}$ .  $D$  is  $A$ -definable,  $\bar{b} \in D$ ,  $D$  is finite, so  $\bar{b} \in \text{acl}(A)$   $\square$

**Proposition 11.18.** 1.  $A \subseteq \text{acl}(A)$

2.  $A \subseteq B \Rightarrow \text{acl}(A) \subseteq \text{acl}(B)$

3.  $\text{acl}(\text{acl}(A)) = \text{acl}(A)$



*Proof.* 3. Take  $b \in \text{acl}(\text{acl}(A))$ , then  $b \in \varphi(\mathbb{M}, \bar{c}) = D$ ,  $D$  is finite,  $D$  is  $\text{acl}(A)$ -definable,  $\bar{c} \in \text{acl}(A)$

$\mathcal{F} = \{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\} = \{\varphi(\mathbb{M}, \sigma(\bar{c})) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  is finite by Proposition 11.17,  $\sigma(D)$  is finite as  $D$  is finite. Therefore  $D' = \bigcup \mathcal{F}$  is finite,  $A$ -invariant,  $b \in D'$  as  $b \in D \in \mathcal{F}$ . Therefore  $b \in \text{acl}(A)$   $\square$

**Definition 11.19.**  $A$  is **algebraically closed** if  $A = \text{acl}(A)$

**Proposition 11.20.**  $\text{acl}(A)$  is the smallest algebraically closed set containing  $A$

**Proposition 11.21.** If  $M \leq \mathbb{M}$ , then  $\text{acl}(M) = M$

*Proof.* Otherwise,  $\text{acl}(M) \supsetneq M$ , take  $b \in \text{acl}(M) \setminus M$ .  $S = \{\sigma(b) : \sigma \in \text{Aut}(\mathbb{M}/M)\}$ . By Proposition 11.17,  $S$  is finite,  $M$ -definable.  $M$  is  $M$ -invariant so  $S \cap M = \emptyset$ , contradicting to Tarski-Vaught criterion  $\square$

**Proposition 11.22.** If  $M \models \text{ACF}$  and  $K$  is a subfield. TFAE

1.  $K = \text{acl}(K)$
2.  $K \models \text{ACF}$
3.  $K \leq M$

Idea: in ACF, field theoretic algebraic closure = model theoretic algebraic closure

*Proof.* 1  $\rightarrow$  2: Take  $P(x)$ ,  $P \neq 0$ ,  $M \models \text{ACF}$ ,  $P(x) = c \cdot (x - r_1) \dots (x - r_n)$ ,  $c \in K$ ,  $r_1, \dots, r_n \in M$ .  $D = \{r_1, \dots, r_n\}$  is  $K$ -definable.  $K = \text{acl}(K)$  implies  $D \subseteq \text{acl}(K) = K$

2  $\rightarrow$  3: q.e.  $\square$

**Fact 11.23.** If  $T$  is strongly minimal,  $A \subseteq M \models T$

$$A \leq M \Leftrightarrow |A| = \infty \text{ and } A = \text{acl}(A)$$

## 11.4 Imaginaries

**Definition 11.24.** An  $A$ -**interpretable set** is  $X/E$  where  $X$  is  $A$ -definable and  $E \subseteq X^2$  is  $A$ -definable equivalence relation on  $X$

Interpretable =  $\mathbb{M}$ -interpretable. 0-interpretable =  $\emptyset$ -interpretable

**Definition 11.25.** If  $M$  is a structure,  $M^{\text{eq}}$  is the expansion of  $M$  by

- A new sort for every 0-interpretable  $D/E$
- Relation symbols for  $D \rightarrow D/E$  (i.e., if  $D \subseteq M^n$ , add  $R \subseteq M^n \times (D/E)$  where  $R(a_1, \dots, a_n, b) \Leftrightarrow \bar{a} \in D, [a]_E = b$ ) If  $D \subsetneq M^n$  can't add a function symbol

$M$  is called the **home sort** (when  $M$  is one-sorted)

$M^{\text{eq}}$  is the expansion of  $M$  obtained by adding each 0-interpretable set as a new sort, with enough data to connect the new sorts to the old sorts

If  $\mathbb{M}$  is a monster model, then  $\mathbb{M}^{\text{eq}}$  is a monster model (7) with the “same” automorphism group (6) and the “same” small models (5). If we restrict our attention to the original sorts from  $M$ , then  $\mathbb{M}^{\text{eq}}$  and  $\mathbb{M}$  have the same definable sets (1) (4) and the same partial elementary maps (2). However,  $\mathbb{M}^{\text{eq}}$  have some new elements, and the definable sets in  $\mathbb{M}^{\text{eq}}$  correspond exactly to the interpretable sets in the original structure  $\mathbb{M}$  (8) (9). On the other hand, the new elements of  $\mathbb{M}^{\text{eq}}$  are definable from the old elements (3). So  $\mathbb{M}^{\text{eq}}$  is a **way of covering interdefinable sets into definable sets while preserving most other things**

**Fact 11.26.** 1. If  $X \subseteq M^n$ ,  $X$  is 0-definable in  $M \Leftrightarrow X$  is 0-definable in  $M^{\text{eq}}$ .

*In other words,  $M^{\text{eq}}$  doesn't define any new sets on the original sorts of  $M$*

2. If  $A, B \subset M$ ,  $f : A \rightarrow B$  bijection, then  $f$  is a partial elementary map in  $M \Leftrightarrow f$  is a partial elementary map in  $M^{\text{eq}}$

3. In  $M^{\text{eq}}$ ,  $\text{dcl}(M) = M^{\text{eq}}$

4. Consequently, any  $M^{\text{eq}}$ -definable set  $X \subseteq M^n$  is  $M$ -definable in  $M^{\text{eq}}$ , and therefore  $M$ -definable in  $M$

5. If  $N \leq M$  then  $N^{\text{eq}} \leq M^{\text{eq}}$  (more precisely,  $N^{\text{eq}} \cong \text{dcl}_{M^{\text{eq}}}(N) \leq M^{\text{eq}}$ ) This gives an  $\leq$ -preserving bijection.

*Moreover, all elementary substructures of  $M$  arise this way. This yields an order-preserving bijection between the elementary substructures of  $M$  and the elementary substructures of  $M^{\text{eq}}$ . In particular, all elementary substructures of  $M^{\text{eq}}$  arise this way*

6. If  $\sigma \in \text{Aut}(M)$ ,  $\sigma$  acts on  $M^{\text{eq}}$  in a natural way.  $\sigma$  induces  $\hat{\sigma} \in \text{Aut}(M^{\text{eq}})$ . This gives an isomorphism  $\text{Aut}(M) \cong \text{Aut}(M^{\text{eq}})$

5 and 6 come from equivalence of categories  $\text{Mod } T \rightarrow \text{Mod } T^{\text{eq}}$

7.  $M$  is  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous  $\Leftrightarrow M^{\text{eq}}$  is  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous

8. If  $D/E$  is 0-interpretable in  $M$ , then  $D/E$  is 0-definable in  $M^{\text{eq}}$
9. If  $X$  is 0-definable in  $M^{\text{eq}}$ , then there is 0-interpretable  $D/E$  in  $M$  and a 0-definable bijection  $f : X \rightarrow D/E$  in  $M^{\text{eq}}$

*Ideas (8-9): interpretable in  $M \Leftrightarrow$  definable in  $M^{\text{eq}}$*

*Proof.* Only some remarks

1. More generally, one can show that  $X \subseteq M^n \times \prod_{j=1}^m (D_j/E_j)$  is 0-definable in  $M^{\text{eq}}$  iff  $\tilde{X} \subseteq M^n \times \prod_{j=1}^m D_j$  is 0-definable in  $M$  where  $\tilde{X} = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n, [b_1]_E, \dots, [b_m]_{E_M}) \in X\}$ .
3. using the definable functions  $D \rightarrow D/E$
4. Note (1) means that if  $\bar{x}$  is a tuple of variables in the old sorts of  $M$ , then any  $L^{\text{eq}}$ -formula  $\phi(x)$  is equivalent to an  $L$ -formula
5. Behind the scenes, there is a theory  $T^{\text{eq}}$  and  $M \models T \Rightarrow M^{\text{eq}} \models T^{\text{eq}}$ . Moreover, all models of  $T^{\text{eq}}$  have the form  $M^{\text{eq}}$  up to isomorphism. Finally, elementary embeddings  $M \rightarrow N$  correspond bijectively to elementary embeddings  $M^{\text{eq}} \rightarrow N^{\text{eq}}$
7.  $\Leftarrow$  is easier by (2). If you just want a monster model  $\mathbb{M}$  s.t.  $\mathbb{M}^{\text{eq}}$  is a monster model, you can do the following: take  $M \models T$ , construct  $M^{\text{eq}}$ , take some monster elementary extension  $U \geq M^{\text{eq}}$ , then check that  $U$  is  $\mathbb{M}^{\text{eq}}$  for some  $\mathbb{M} \geq M$

If  $M$  is  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous.  $\kappa$ -saturation is not hard in terms of a compactness-like property: if  $|A| < \kappa$  and a collection of  $A$ -definable sets has FIP, then it has non-empty intersection. To transfer this from  $M$  to  $M^{\text{eq}}$ , one takes the  $A$ -definable sets in  $M^{\text{eq}}$  and lifts them to  $A$ -definable sets in  $M$  using the maps  $D \rightarrow D/E$

9. This comes down to the following things. First, if  $D/E$  and  $D'/E'$  are two interpretable sets, then  $(D/E) \times (D'/E')$  "is" an interpretable set, namely  $(D \times D')/E''$  where  $(a, b)E''(c, d) \Leftrightarrow aEc \wedge bE'd$ . Secondly if  $X$  is a definable subset of  $D/E$ , then  $X$  "is" an interpretable set  $D'/E'$ , where  $D' = \{a \in D : [a]_E \in X\}$  and  $E'$  is the restriction of  $E$  to  $D'$

□

From now on, we use the word "interpretable" to mean "definable in  $\mathbb{M}^{\text{eq}}$ " and "definable" to mean "definable" in  $\mathbb{M}$

An **imaginary** is an element of  $\mathbb{M}^{\text{eq}}$

## 11.5 Elimination of imaginaries

**Definition 11.27.**  $T$  has **elimination of imaginaries** (EI) if  $\forall a \in \mathbb{M}^{\text{eq}}, \exists \bar{b} \in \mathbb{M}, \text{dcl}(a) = \text{dcl}(\bar{b})$

**Definition 11.28.**  $T$  has **uniform EI** if

1. If  $D/E$  is 0-interpretable, then there is 0-definable  $Y$ , there is bijection  $f : D/E \rightarrow Y$ , 0-interpretable (= 0-definable in  $M^{\text{eq}}$ )
2. If  $D/E$  is 0-interpretable, then there is  $Y$  0-definable, 0-definable surjection  $g : D \rightarrow Y$  s.t.  $g(x) = g(y)$  iff  $E(x, y)$

$$1 \Leftrightarrow 2$$

Note: uniform EI implies EI

If  $e \in D/E$ , there is  $Y$  0-definable, bijection  $f : D/E \rightarrow Y$ ,  $e$  interpretable with  $f(e) \in \mathbb{M}^n$

**Lemma 11.29.** If  $T$  has EI, if  $D/E$  is 0-interpretable, then there is 0-interpretable  $X_i \subseteq D/E$ ,  $X_i \cap X_j = \emptyset$ ,  $D/E = \bigcup_{i=1}^n X_i$ , each  $X_i$  has 0-interpretable bijection to a 0-definable set  $f_i : X_i \rightarrow Y_i$

*Proof.* Say 0-interpretable  $X \subseteq D/E$  is “good” if there is 0-definable  $Y$ , 0-interpretable bijection  $f : X \rightarrow Y$ .

If  $X' \subseteq X$ ,  $X$  is “good”,  $X'$  is 0-interpretable, then  $X'$  is good

**Claim**  $D/E$  is covered by good sets

If  $e \in D/E$ , E.I. implies there is  $\bar{b} \in \mathbb{M}^m$ ,  $\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(\bar{b})$ . Lemma 11.15 implies there is 0-interpretable bijection  $f : X \rightarrow Y$ ,  $f(e) = \bar{b}$ ,  $X$  is good

There are at most  $|L|$ -many good sets. By saturation,  $D/E = \bigcup_{i=1}^n X_i$  (class of good sets is small)

Replace  $X_i$  with  $X_i \setminus (X_1 \cup \dots \cup X_{i-1})$ , we may assume the  $X_i$  are pairwise disjoint □

**Theorem 11.30.** Suppose  $T$  has one-sort and  $|\text{dcl}(\emptyset)| \geq 2$ , then  $T$  has E.I.  $\Leftrightarrow T$  has uniform E.I.

*Proof.*  $\Rightarrow$ : Take  $D/E$  0-interpretable. Lemma 11.29 gives  $D/E = \coprod_{i=1}^n X_i$ ,  $f_i : X_i \rightarrow Y_i$ ,  $Y_i$  0-definable

Fix  $a, b \in \text{dcl}(\emptyset)$

By replacing  $y_i$  with  $y_i \times \{(a, a, \dots, a)\}$ . WMA there is  $m$  s.t.  $Y_i \subseteq \mathbb{M}^m$   $\forall i$ . Take  $N \gg 0$ ,  $2^N > n$ , take distinct  $\bar{c}_1, \dots, \bar{c}_n \in \{a, b\}^N$

Replacing  $Y_i$  with  $Y_i \times \{\bar{c}_i\}$ , now  $Y_i$ s are disjoint □

**Example 11.3.** DLO has E.I., doesn't have uniform E.I.

$D = M^2$ ,  $E$  two class  $\{(x, y) : x = y\}$  and  $\{(x, y) : x \neq y\}$

uniform E.I. would imply  $D/E \leftrightarrow Y$ ,  $Y$  is 0-definable. But there is no 0-definable  $Y$  with two elements

*Remark.*  $\mathbb{M}^{\text{eq}}$  has uniform E.I.

If  $D/E$  is 0-interpretable and  $E' \subseteq (D/E) \times (D/E)$  is a 0-interpretable equivalence relation on  $D/E$ , then  $(D/E)/E'$  is also 0-interpretable. In fact, it's  $D/E''$  where

$$E''(\bar{a}, \bar{b}) \Leftrightarrow E'([\bar{a}]_E, [\bar{b}]_E)$$

Therefore  $\mathbb{M}^{\text{eq}} \approx (\mathbb{M}^{\text{eq}})^{\text{eq}}$

**Example 11.4.** DLO is an example of a theory with elimination of imaginaries but not uniform elimination of imaginaries

Let  $E$  be the equivalence relation on  $\mathbb{M}^2$  with two classes, one of which is the line  $y = x$  and the other is its complement. If there was a 0-interpretable bijection from  $\mathbb{M}^2/E$  to  $Y \subseteq \mathbb{M}^n$ , then  $Y$  would contain two elements, both of which are in  $\text{dcl}(\emptyset)$ . But  $\text{dcl}(\emptyset) = \emptyset$ , so  $Y$  cannot have any elements unless  $n = 0$  and when  $n = 0$  the set  $Y$  can only have one element.

## 11.6 Codes

**Definition 11.31.** A real tuple or imaginary  $e$  is a **code** for  $D$  ( $e$  **codes**  $D$ ) if  $\{\sigma \in \text{Aut}(\mathbb{M}) : \sigma(D) = D\} = \text{Aut}(\mathbb{M}^{\text{eq}}/e) = \text{Aut}(\mathbb{M}/e)$

*Remark.* If  $e, e'$  code  $D$ , then  $\text{Aut}(\mathbb{M}/e) = \text{Aut}(\mathbb{M}/e')$ , so  $\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(e')$  by Lemma 11.14

*Remark.* If  $e$  codes  $D$ , then  $D$  is  $A$ -definable  $\Leftrightarrow e \in \text{dcl}^{\text{eq}}(A)$

*Proof.* TFAE

- $D$  is  $A$ -definable
- $D$  is  $A$ -invariant
- $\forall \sigma \in \text{Aut}(\mathbb{M}/A), \sigma(D) = D$
- $\forall \sigma \in \text{Aut}(\mathbb{M}/A), \sigma(e) = e$
- $e \in \text{dcl}^{\text{eq}}(A)$

□

**Example 11.5.** Suppose  $T = \text{ACF}$  and  $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$ . Let  $P(x) = \prod_{i=1}^n (x - r_i)$ . Write  $P(x)$  as  $x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$ . Then  $(c_0, \dots, c_{n-1})$  is a code for  $S$ . Indeed

$$\begin{aligned} \sigma(\bar{c}) = \bar{c} &\Leftrightarrow \sigma(P(x)) \equiv P(x) \\ &\Leftrightarrow \prod_{i=1}^n (x - \sigma(r_i)) \equiv \prod_{i=1}^n (x - r_i) \\ &\Leftrightarrow \{\sigma(r_1), \dots, \sigma(r_n)\} = \{r_1, \dots, r_n\} \\ &\Leftrightarrow \sigma(S) = S \end{aligned}$$

**Example 11.6.** If  $D/E$  is 0-interpretable and  $e \in D/E$ , then  $e$  is an  $E$ -equivalence class  $X = E(\mathbb{M}, \bar{a})$ , and  $\sigma(e) = \sigma(X)$  for all  $\sigma$ . Therefore  $\sigma$  codes  $X$

**Lemma 11.32.** Let  $\varphi(\bar{x}, \bar{y})$  be a formula. Let  $f(\bar{y})$  be a 0-definable function s.t.

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f(\bar{b}) = f(\bar{c})$$

Then  $f(\bar{b})$  is a code for  $\varphi(\mathbb{M}, \bar{b})$ , for each  $\bar{b}$

**Proposition 11.33.** TFAE

1.  $T$  has uniform elimination of imaginaries
2. For any formula  $\varphi(\bar{x}; \bar{y})$ , there is a 0-definable function  $f_\varphi(\bar{y})$  s.t.

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f_\varphi(\bar{b}) = f_\varphi(\bar{c})$$

*Proof.* 1  $\rightarrow$  2: apply uniform E.I. to  $\mathbb{M}^n/E$ , where  $E(\bar{b}, \bar{c}) \Leftrightarrow (\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}))$

2  $\rightarrow$  1: given a 0-interpretable set  $D/E$ ,

$$E(\bar{b}, \bar{c}) \Leftrightarrow E(\mathbb{M}, \bar{b}) = E(\mathbb{M}, \bar{c}) \Leftrightarrow f_E(\bar{b}) = f_E(\bar{c})$$

for  $\bar{b}, \bar{c} \in D$ . So we have a 0-definable function on  $D$  satisfying condition 2 of Definition □

**Corollary 11.34.** If  $T$  has uniform elimination of imaginaries, then every definable set has a code in  $\mathbb{M}$

**Corollary 11.35.** Every definable set has a code in  $\mathbb{M}^{\text{eq}}$

**Proposition 11.36.** TFAE

1.  $T$  has elimination of imaginaries
2. Every definable  $D \subseteq \mathbb{M}^n$  has a code in  $\mathbb{M}$

*Proof.*  $1 \rightarrow 2$ : given  $D$  take a code  $e \in \mathbb{M}^{\text{eq}}$ , then take  $\bar{b} \in \mathbb{M}^m$  interdefinable with  $e$ . Then

$$\text{Aut}(\mathbb{M}/\bar{b}) = \text{Aut}(\mathbb{M}/e) = \{\sigma \in \text{Aut}(\mathbb{M}) : \sigma(D) = D\}$$

so  $\bar{b}$  is a code for  $D$

$2 \rightarrow 1$ : if  $e \in D/E \subseteq \mathbb{M}^{\text{eq}}$ , then  $e$  codes a definable set  $X$  □

**Corollary 11.37.**  $\text{dcl}^{\text{eq}}(e)$  is the smallest definably closed set defining  $D$

## 11.7 Elimination of imaginaries and naming parameters

**Proposition 11.38.** Uniform elimination of imaginaries is preserved by naming parameters

*Proof.* Fix  $D/E$ ,  $D \subseteq \mathbb{M}^n$  □

## 12 Forking and stability spectra

### 12.1 EI in PA and ACF

$\mathbb{Q}$  is 0-definable

**Theorem 12.1.** If complete  $T \supseteq \text{PA}$  (e.g.,  $T = \text{Th}(\mathbb{N})$ ), then  $T$  has uniform E.I.

*Proof.* Fix interpretable  $D/E$ . Want  $D/E \rightarrow Y$  definable bijection, or  $D/E \rightarrow \mathbb{M}^n$  definable injection

Take  $f : D/E \rightarrow \mathbb{M}^n$ ,  $f(X) = \min(X)$ ,  $\min$  is w.r.t. lexicographic order on  $\mathbb{M}^n$ .  $\text{PA} \Rightarrow \mathbb{M}, \mathbb{M}^n$  are definably well-ordered □

Consider  $T = \text{ACF}_0$

**Fact 12.2.** If  $S \subseteq_f \mathbb{M}^n$ , then  $\exists^\top S^\top \in \mathbb{M}$

*Proof.*  $n = 1$ , if  $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$  form  $P(x) = \prod_{i=1}^m (x - r_i)$ . Then  $P(x) = x^m + c_{m-1}x^{m-1} + \dots + c_1x + c_0$ ,  $\bar{c}$  is a code for  $S$

$n = 2$ , for  $q \in \mathbb{Q}$ , let  $\pi_q : \mathbb{M}^2 \rightarrow \mathbb{M}$ ,  $(x, y) \mapsto y - qx$ . Let  $A = \{\ulcorner \pi_q(S)^\top : q \in \mathbb{Q} \urcorner\} \subseteq \mathbb{M}$

**Claim:** If  $\sigma \in \text{Aut}(\mathbb{M})$ ,  $\sigma(S) = S \Leftrightarrow \sigma \in \text{Aut}(\mathbb{M}/A)$

$\Rightarrow: \forall q \in \mathbb{Q}, \sigma(\pi_q(S)) = \pi_q(\sigma(S)) = \pi_q(S)$  ( $\pi_q$  is 0-definable since  $\mathbb{Q} \subseteq \text{dcl}(\emptyset)$ ), then  $\sigma(\ulcorner \pi_q(S) \urcorner) = \ulcorner \pi_q(S) \urcorner, \sigma \in \text{Aut}(\mathbb{M}/A)$   
 $\Leftarrow: \text{Suppose } \sigma \in \text{Aut}(\mathbb{M}/A) \text{ but } S' = \sigma(S) \neq S, \text{ then } |S \cup S'| > |S| = |S'|. \exists q \in \mathbb{Q} \text{ s.t. } S \cup S' \rightarrow \pi_q(S \cup S') \text{ is injective (take any } q \notin \{\frac{y_1 - y_0}{x_1 - x_0} : (x_0, y_0), (x_1, y_1) \in S \cup S'\}. \text{ Then } |\pi_q(S) \cup \pi_q(S')| = |\pi_q(S \cup S')| = |S \cup S'| > |S| \geq |\pi_q(S)|. \text{ Therefore } \pi_q(S) \neq \pi_q(S') = \sigma(\pi_q(S)). \text{ Then } \sigma(\ulcorner \pi_q(S) \urcorner) \neq \ulcorner \pi_q(S) \urcorner, \sigma \notin \text{Aut}(\mathbb{M}/A)$

$S$  is  $A$ -invariant, so  $S$  is  $A$ -definable, so  $\exists \bar{b} \in A, S$  is  $\bar{b}$ -definable

If  $\sigma \in \text{Aut}(\mathbb{M})$ , then

$$\sigma(\bar{b}) = \bar{b} \Rightarrow \sigma(S) = S \Rightarrow \sigma \in \text{Aut}(\mathbb{M}/A) \Rightarrow \sigma(\bar{b}) = \bar{b}$$

□

**Lemma 12.3.** *If  $A \subseteq \mathbb{M}^{\text{eq}}$ , if  $D \subseteq \mathbb{M}^n$  is  $A$ -definable and  $D \neq \emptyset$ , then  $\exists \bar{b} \in D, \bar{b} \in \text{acl}^{\text{eq}}(A)$*

*Proof.* Need  $|\text{acl}(\emptyset)| = \infty$  and strongly minimality

Induction on  $n$

$n = 1$ : if  $|D| < \infty, D \subseteq \text{acl}^{\text{eq}}(A)$ , take any  $b \in D$

If  $|\mathbb{M} \setminus D| < \infty, D \cap \mathbb{Q} \neq \emptyset$ , take  $b \in D \cap \mathbb{Q}$

$n > 1$ :  $D' = \{\bar{b} \in \mathbb{M}^{n-1} : \exists c \in \mathbb{M}, (\bar{b}, c) \in D\}, D' \neq \emptyset, D'$  is  $A$ -definable, there is  $\bar{b} \in D', \bar{b} \in \text{acl}^{\text{eq}}(A)$ .  $D'' = \{c \in \mathbb{M} : (\bar{b}, c) \in D\}, D'' \neq \emptyset, D''$  is  $A\bar{b}$ -definable, there is  $c \in D'', c \in \text{acl}^{\text{eq}}(A\bar{b}) \subseteq \text{acl}^{\text{eq}}(\text{acl}^{\text{eq}}(A)) = \text{acl}^{\text{eq}}(A)$  □

**Theorem 12.4.**  *$\text{ACF}_0$  has uniform E.I.*

*Proof.*  $\text{ACF}_0 \vdash 0 \neq 1$  so uniform EI  $\Leftrightarrow$  EI. Just need EI (11.30)

Take  $e \in D/E, e = \ulcorner X \urcorner$  for some  $E$ -equivalence class  $X$

By Lemma 12.3, there is  $\bar{a} \in X, \bar{a} \in \text{acl}^{\text{eq}}(e)$ . Let  $S = \{\sigma(\bar{a}) : \sigma \in \text{Aut}(\mathbb{M}/e)\} = \text{realizations of } \text{tp}(\bar{a}/e)$ . By Proposition 11.17,  $|S| < \infty, S$  is  $e$ -definable. By the fact, take  $\ulcorner S \urcorner \in \mathbb{M}^m$ .  $\ulcorner S \urcorner \in \text{dcl}^{\text{eq}}(e)$  because  $S$  is  $e$ -definable. Given  $S, X$  is the unique  $E$ -equivalence class containing  $S$ . So  $e = \ulcorner X \urcorner \in \text{dcl}^{\text{eq}}(\ulcorner S \urcorner)$

$$\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(\ulcorner S \urcorner)$$

If  $\sigma \in \text{Aut}(\mathbb{M})$  and  $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner$ , then  $\sigma(S) = S$ . As  $D, E$  are 0-definable,  $\sigma(X)$  is some  $E$ -equivalence class. But  $S \subseteq X \Rightarrow \sigma(S) \subseteq \sigma(X)$ , so  $S \subseteq \sigma(X)$ . Therefore  $\sigma(X)$  must be the same  $E$ -equivalence class as  $X$ , then  $\sigma(X) = X$ . This argument shows  $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner \Rightarrow \sigma(X) = X$ , which means  $X$  is  $\ulcorner S \urcorner$ -definable □

**Fact 12.5.**  *$\text{ACF}_p$  has uniform E.I.*



## 12.2 Stability and $\mathbb{M}^{\text{eq}}$

**Theorem 12.6.**  $\mathbb{M}$  is  $\lambda$ -stable  $\Leftrightarrow \mathbb{M}^{\text{eq}}$  is  $\lambda$ -stable

*Proof.*  $\Leftarrow$ : is easy.

$\Rightarrow$ : Suppose  $A \subseteq \mathbb{M}^{\text{eq}}$ ,  $|A| \leq \lambda$ , want  $|S_{\bar{x}}(A)| \leq \lambda$

Take  $B \subseteq \mathbb{M}$ ,  $A \subseteq \text{dcl}^{\text{eq}}(B)$ ,  $|B| \leq \lambda$ . If  $e \in D/E$ ,  $e = [\bar{a}]_E$ , then  $e \in \text{dcl}^{\text{eq}}(\bar{a})$ . Then  $\text{tp}(e/B)$  determines  $\text{tp}(e/A)$ , so  $|S_{\bar{x}}(B)| \geq |S_{\bar{x}}(A)|$ , suffice to show  $|S_{\bar{x}}(B)| \leq \lambda$ . If  $\bar{x} \in D/E$ , let  $\bar{y}$  live in  $D$ , let  $\pi : D \rightarrow D/E$ . If  $\bar{c} \in D$ ,  $\text{tp}(\bar{c}/B)$  determines  $\text{tp}(\pi(\bar{c})/B)$ , so  $|S_D(B)| \geq |S_{D/E}(B)|$ , but  $|S_D(B)| \leq \lambda$   $\square$

## 12.3 Almost $A$ -definability

**Proposition 12.7.** If  $A \subseteq \mathbb{M}$ ,  $\text{acl}(A) = \bigcap_{M \leq \mathbb{M}, A \subseteq M} M$

*Proof.* If  $M \supseteq A$ ,  $M \leq \mathbb{M}$ ,  $\text{acl}(A) \subseteq \text{acl}(M) = M$  11.21

If  $b \notin \text{acl}(A)$ , then  $\{\sigma(b) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  is large. Take some  $M_0 \supseteq A$ ,  $M_0 \leq \mathbb{M}$ . Take  $\sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(b) \notin M_0$  since that is large,  $b \notin \sigma^{-1}(M_0) := M$ ,  $M \supseteq A$ ,  $M \leq \mathbb{M}$ ,  $b \notin \bigcap_{M \supseteq A} M$   $\square$

**Proposition 12.8.** If  $D \subseteq \mathbb{M}^n$  is definable,  $A \subseteq \mathbb{M}^{\text{eq}}$  small. TFAE

1.  $D$  is  $\text{acl}^{\text{eq}}(A)$ -definable
2.  $\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  finite
3.  $\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  is small
4.  $D$  is  $M$ -definable  $\forall M \leq \mathbb{M}$ ,  $M \supseteq A$  (forking)

*Proof.* By remark 11.6, the proposition is equivalent to

1.  $\ulcorner D \urcorner \in \text{acl}^{\text{eq}}(A)$
2.  $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  finite
3.  $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  small
4.  $\ulcorner D \urcorner \in M^{\text{eq}}$ ,  $\forall M \supseteq A$

1  $\leftrightarrow$  2  $\leftrightarrow$  3 by Proposition 11.17

1  $\leftrightarrow$  4 Proposition 12.7  $\square$

$D$  is **almost  $A$ -definable** if 1-4 hold

**Proposition 12.9.** *If  $p \in S_n(\mathbb{M})$  is definable and  $A \subseteq \mathbb{M}^{\text{eq}}$ , TFAE*

1.  $p$  is  $\text{acl}^{\text{eq}}(A)$ -definable
2.  $\{\sigma(p) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$  is small
3.  $p$  is  $M$ -definable for every  $M \supseteq A$ ,  $M \preceq \mathbb{M}$  (Lascar  $A$ -invariant)

*Proof.* Apply Proposition 12.8 to

$$D_\varphi = \{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

□

## 12.4 Theorem of the bound

Assume  $T$  is stable. Recall 10.5, 10.9, 10.10

**Fact 12.10.** *If  $p \in S_n(M)$ ,  $q \in S_n(N)$ ,  $p \subseteq q$ , then  $[q] \leq [p]$  and  $[q] = [p]$  iff  $q \supseteq p$*

If  $p \in S_n(A)$ ,  $A \subseteq M$ ,  $\text{Ex}_M(p) = \{[q] : q \in S_n(M), q \supseteq p\}$ ,  $\text{Bd}_M(p)$  = maximal elements of  $\text{Ex}_M(p)$ . Zorn's lemma  $\Rightarrow$  if  $\beta \in \text{Ex}_M(p) \exists \beta' \in \text{Bd}_M(p), \beta' \supseteq \beta$

**Fact 12.11.** *If  $A \subseteq M, N$ , then  $\text{Bd}_M(p) = \text{Bd}_N(p)$ , and we can just write  $\text{Bd}(p)$*

A “bound” of  $p$  is an element of  $\text{Bd}(p)$

**Lemma 12.12.** *If  $p \in S_n(A)$ ,  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ , if  $[q] \in \text{Bd}(p)$ , then*

1. *If  $A \subseteq M \preceq \mathbb{M}$ , then  $q$  is  $M$ -definable*
2.  *$q$  is  $\text{acl}^{\text{eq}}(A)$ -definable*

*Proof.* 1. Let  $r = q \upharpoonright M$ . If  $r \sqsubseteq q$  then  $q$  is  $M$ -definable. If  $r \not\sqsubseteq q$ , then  $[q] < [r]$ , contradicting  $[q] \in \text{Bd}(p)$

2. Proposition 12.9

□

**Corollary 12.13.** *If  $p \in S_n(A)$ ,  $\exists q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ ,  $q$  is  $\text{acl}^{\text{eq}}(A)$ -definable*

*Proof.* Take  $\beta \in \text{Bd}(p) = \text{Bd}_{\mathbb{M}}(p) \subseteq \text{Ex}_{\mathbb{M}}(p)$ ,  $\beta = [q]$  for some  $q \in S_n(\mathbb{M})$ , then by 12.12 □

**Proposition 12.14.** *If  $A = \text{acl}^{\text{eq}}(A)$ , if  $p \in S_n(A)$ , if  $q_1, q_2 \in S_n(\mathbb{M})$ ,  $q_1, q_2 \supseteq p$ ,  $q_1, q_2$   $A$ -definable, then  $q_1 = q_2$*

*Proof.* If not, take  $\varphi(\bar{x}, \bar{b}), q_1(\bar{x}) \vdash \varphi(\bar{x}, \bar{b}), q_2(\bar{x}) \vdash \neg\varphi(\bar{x}, \bar{b})$ .  $\text{tp}(\bar{b}/A)$  has a global  $A$ -definable extension  $r$  by Corollary 12.13. Take  $\bar{c} \models q_1 \upharpoonright A\bar{b}$ , then  $\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x}) \upharpoonright A\bar{b}$ .  $\bar{b} \models r \upharpoonright A$  and  $\bar{c} \models q_1 \upharpoonright A\bar{b}$ .  $(\bar{b}, \bar{c}) \models (r \otimes q_1) \upharpoonright A$ ,  $(\bar{c}, \bar{b}) \models (q_1 \otimes r) \upharpoonright A$ ,  $\bar{c} \models q_1 \upharpoonright A$  and  $\bar{b} \models r \upharpoonright A\bar{c}$ , that is,  $\bar{c} \models q_2 \upharpoonright A$ , therefore  $(\bar{c}, \bar{b}) \models (q_2 \otimes r) \upharpoonright A$ ,  $(\bar{b}, \bar{c}) \models (r \otimes q_2) \upharpoonright A$ ,  $\bar{c} \models q_2 \upharpoonright A\bar{b}$ , a contradiction  $\square$

**Proposition 12.15.** *If  $A = \text{acl}^{\text{eq}}(A)$ ,  $p \in S_n(A)$*

1.  $\text{Bd}(p) = \{\beta\}$
2. *If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$  then  $[q] = \beta \Leftrightarrow q$  is  $A$ -definable*
3.  *$\exists! q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ ,  $[q] = \beta$ ,  $q$  is  $A$ -definable*

*Proof.* 1. If  $\beta_1, \beta_2 \in \text{Bd}(p)$ , take  $q_1, q_2 \in S_n(\mathbb{M})$ ,  $q_1, q_2 \supseteq p$ ,  $[q_i] = \beta_i$ . Lemma 12.12 shows that  $q_1, q_2$  is  $A$ -definable, Proposition 12.14 says  $q_1 = q_2$ , then  $\beta_1 = \beta_2$

2. If  $[q] = \beta$ , then  $q$  is  $A$ -definable by Lemma 12.12.

If  $q$  is  $A$ -definable, take some  $q' \in S_n(\mathbb{M})$ ,  $q' \supseteq p$ ,  $[q'] = \beta$ . Then Lemma 12.12 shows that  $q'$  is  $A$ -definable. By Proposition 12.14,  $q = q'$

3. Existence by 12.13, uniqueness by 12.14

$\square$

**Theorem 12.16** (Theorem of the bound). *If  $A \subseteq \mathbb{M}^{\text{eq}}$ , (small),  $p \in S_n(A)$ , then*

1.  $\text{Bd}(p) = \{\beta\}$
2. *If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ , then  $q$  is  $\text{acl}^{\text{eq}}(A)$ -definable  $\Leftrightarrow [q] = \beta$*
3. *If  $X = \{q \in S_n(\mathbb{M}) : q \supseteq p, q \text{ is } \text{acl}^{\text{eq}}(A)\text{-definable}\}$ ,  $Y = \{q \in S_n(\text{acl}^{\text{eq}}(A)) : q \supseteq p\}$ , there is a bijection  $X \rightarrow Y$ ,  $q \mapsto q \upharpoonright \text{acl}^{\text{eq}}(A)$*
4. *If  $q_1, q_2 \in S_n(\mathbb{M})$ ,  $q_1, q_2 \supseteq p$ ,  $q_1, q_2$   $\text{acl}^{\text{eq}}(A)$ -definable, then  $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(q_1) = q_2$*

*Proof.* 3. Proposition 12.15 (3)

4. Take  $c_i \models q_i \upharpoonright \text{acl}^{\text{eq}}(A)$  for  $i = 1, 2$ ,  $\text{tp}(c_1/A) = p = \text{tp}(c_2/A)$ .  $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(c_1) = c_2$ .  $\sigma(\text{tp}(c_1/\text{acl}^{\text{eq}}(A))) = \text{tp}(c_2/\text{acl}^{\text{eq}}(A))$ , (if  $a \in \text{acl}^{\text{eq}}(A)$ , then  $\sigma(a) \in \text{acl}^{\text{eq}}(A)$ )  $\sigma(q_1 \upharpoonright \text{acl}^{\text{eq}}(A)) = q_2 \upharpoonright \text{acl}^{\text{eq}}(A)$ ,  $\sigma(q_1) \upharpoonright \text{acl}^{\text{eq}}(A) = q_2 \upharpoonright \text{acl}^{\text{eq}}(A)$ ,  $\sigma(q_1), q_2$  are both almost  $A$ -definable, both  $\supseteq p$ , (3) shows that  $\sigma(q_1) = q_2$

1. If  $\beta_1, \beta_2 \in \text{Bd}(p) \subseteq \text{Ex}_{\mathbb{M}}(p)$ , take  $q_i \in S_n(\mathbb{M})$ ,  $q_i \supseteq p$ ,  $[q_i] = \beta_i$  for  $i = 1, 2$ . Lemma 12.12 shows that  $q_1, q_2$  are  $\text{acl}^{\text{eq}}(A)$ -definable, (4) shows that  $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(q_1) = q_2$ ,  $\beta_1 = [q_1] = [\sigma(q_1)] = [q_2] = \beta_2$
2. If  $[q] = \beta \in \text{Bd}(p)$ , then  $q$  is  $\text{acl}^{\text{eq}}(A)$ -definable by Lemma 12.12. Take  $q' \in S_n(\mathbb{M})$ ,  $q' \supseteq p$ ,  $[q'] = \beta$ , Lemma 12.12  $\Rightarrow$   $q'$  is  $\text{acl}^{\text{eq}}(A)$ -definable. (4)  $\Rightarrow \exists \sigma \in \text{Aut}(\mathbb{M}/A)$ ,  $\sigma(q) = q'$ ,  $[q] = [\sigma(q)] = \beta$

□

*Remark.*  $\text{stp}(\bar{b}/A) := \text{tp}(\bar{b}/\text{acl}^{\text{eq}}(A))$ , the **strong type** of  $\bar{b}$  over  $A$ . A set  $X$  is  $\text{acl}^{\text{eq}}(A)$ -invariant iff

$$\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A) \Rightarrow (\bar{b} \in X \Leftrightarrow \bar{c} \in X)$$

Proposition ?? (3) says that there is a bijection between strong types over  $A$  and  $\text{acl}^{\text{eq}}(A)$ -invariant global types

One can define strong types without using  $\mathbb{M}^{\text{eq}}$ . It turns out that  $\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A)$  iff  $\bar{b}E\bar{c}$  for every  $A$ -definable “finite” equivalence relation  $E$  on  $\mathbb{M}^n$   
 $\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A) \Leftrightarrow \forall X$  almost  $A$ -definable,  $\bar{b} \in X \Leftrightarrow \bar{c} \in X$

## 12.5 Forking

Let  $\text{Bd}(p) = \{\text{bd}(p)\}$ . If  $p \in S_n(M)$ ,  $\text{bd}(p) = \max \text{Ex}_M(p) = [p]$

*Remark.* If  $p \in S_n(A)$ ,  $q \in S_n(B)$ ,  $q \supseteq p$ , then  $\text{Ex}_{\mathbb{M}}(q) \subseteq \text{Ex}_{\mathbb{M}}(p)$ ,  $\text{bd}(q) \leq \text{bd}(p)$

**Definition 12.17.**  $q \in S_n(B)$ ,  $p \in S_n(A)$ , if  $q \supseteq p$ ,  $q$  is a **nonforking extension**, written  $q \sqsupseteq p$  if  $\text{bd}(q) = \text{bd}(p)$ ,

*Remark.* If  $p \in S_n(M)$  and  $M \leq \mathbb{M}$  then  $\text{bd}(p) = \max \text{Ex}_M(p) = \max\{[p]\} = [p]$

**Proposition 12.18** (Full transitivity). *If  $p_1 \subseteq p_2 \subseteq p_3$ , then*

$$p_1 \sqsubseteq p_2 \wedge p_2 \sqsubseteq p_3 \Leftrightarrow p_1 \sqsubseteq p_3$$

**Proposition 12.19** (Extension). *If  $p \in S_n(A)$ ,  $B \supseteq A$ , then there is  $q \in S_n(B)$ ,  $q \sqsupseteq p$*

*Proof.* Take  $M \supseteq B$ ,  $M \leq \mathbb{M}$ , take  $r \in S_n(M)$ ,  $r \supseteq p$  with  $[r] = \text{bd}(p) \in \text{Ex}_M(p)$ , then  $\text{bd}(r) = \text{bd}(p)$ ,  $r \sqsupseteq p$ . □

**Proposition 12.20.** *If  $p \in S_n(A)$ ,  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ , then  $q \sqsupseteq p \Leftrightarrow [q] = \text{bd}(p) \Leftrightarrow q$  is  $\text{acl}^{\text{eq}}(A)$ -definable*

In light of Proposition 12.20, Proposition 12.16 is really a statement about global non-forking extensions. In particular, global non-forking extensions of  $p$  correspond to extensions of  $p$  to  $\text{acl}^{\text{eq}}(A)$ , and any two extensions are conjugate over  $A$

*Proof.*  $q$  is almost  $A$ -definable by Proposition 12.16  $\square$

**Proposition 12.21.** *If  $q \in S_n(B)$ ,  $p \in S_n(A)$ ,  $q \supseteq p$ , then  $q \sqsupseteq p \Leftrightarrow \exists r \in S_n(\mathbb{M})$ ,  $r \supseteq q$ ,  $r$  is  $\text{acl}^{\text{eq}}(A)$ -definable*

*Proof.* If  $q \sqsupseteq p$ , by extension there is  $r \in S_n(\mathbb{M})$  with  $r \sqsupseteq q$  and then  $r \sqsupseteq p$  by transitivity and  $r$  is  $\text{acl}^{\text{eq}}(A)$ -definable by 12.20

If  $r \supseteq q$  and  $r$  is  $\text{acl}^{\text{eq}}(A)$ -definable, then  $r \sqsupseteq p$  by 12.20, and so  $q \sqsupseteq p$  by transitivity  $\square$

**Proposition 12.22.** *If  $p \in S_n(A)$  if  $q \in S_n(\text{acl}^{\text{eq}}(A))$ ,  $q \supseteq p$ , then  $q \sqsupseteq p$*

*Proof.* 12.13,  $q$  has a global  $\text{acl}^{\text{eq}}(A)$ -definable extension  $\square$

**Definition 12.23.**  $\varphi(x) \in L(\mathbb{M})$  **forks** over  $A$  if there is no  $p \in S_n(\mathbb{M})$  s.t.  $p(\bar{x}) \models \varphi(\bar{x})$  and  $p(\bar{x})$  is  $\text{acl}^{\text{eq}}(A)$ -definable

**Proposition 12.24.** *If  $q \supseteq p \in S_n(A)$ , then  $q \not\sqsupseteq p$  iff  $\exists \varphi(x) \in q(x)$ ,  $\varphi$  forks over  $A$*

*Proof.* Let  $\Sigma_A(\bar{x}) = \{\varphi(\bar{x}, \bar{b}) \leftrightarrow \varphi(\bar{x}, \bar{c}) : \bar{b} \equiv_{\text{acl}^{\text{eq}}(A)} \bar{c}\}$ . If  $r(\bar{x}) \in S_n(\mathbb{M})$ , then  $r \supseteq \Sigma_A \Leftrightarrow r$  is  $\text{acl}^{\text{eq}}(A)$ -invariant  $\Leftrightarrow r$  is  $\text{acl}^{\text{eq}}(A)$ -definable. Therefore

1. An  $L(\mathbb{M})$ -formula  $\varphi(\bar{x})$  forks over  $A$  iff  $\Sigma_A(\bar{x}) \cup \{\varphi(\bar{x})\}$  is inconsistent
2.  $q \sqsupseteq p$  iff  $q(\bar{x}) \cup \Sigma_A(\bar{x})$  is consistent by 12.21

Then  $\varphi(\bar{x})$  forks over  $A \Leftrightarrow \{\varphi(\bar{x})\} \cup \Sigma_A(\bar{x}) \vdash \perp$   $\square$

We say that  $q \in S_n(B)$  **forks over**  $A$  if  $q$  contains a formula which forks over  $A$

## 12.6 Stationary types

**Lemma 12.25.** *TFAE for  $p \in S_n(A)$*

1.  $p$  has a unique non-forking extension over  $\mathbb{M}$
2.  $p$  has a unique non-forking extension over any  $B \supseteq A$

3.  $p$  has a unique extension to  $\text{acl}^{\text{eq}}(A)$
4.  $p$  has an  $A$ -definable extensions over  $\mathbb{M}$

*Proof.*  $1 \rightarrow 2$ : obvious

$2 \rightarrow 3$ : take  $B = \text{acl}^{\text{eq}}(A)$ , use Proposition 12.22

$3 \leftrightarrow 1$ : Proposition 12.16

$1 \rightarrow 4$ : If  $q$  is the unique non-forking extension, then  $\text{Aut}(\mathbb{M}/A)$  fixes  $q$  by symmetry, so  $q$  is  $A$ -invariant and  $A$ -definable

$4 \rightarrow 1$ : Let  $q$  be the  $A$ -definable extension. Then  $q$  is  $\text{acl}^{\text{eq}}(A)$ -definable (= non-forking). If  $q'$  is any other non-forking extension, then  $q' = \sigma(q) = q$  for some  $\sigma \in \text{Aut}(\mathbb{M}/A)$  by Proposition 12.16  $\square$

**Example 12.1.** 1. If  $p \in S_n(M)$  and  $M \leq \mathbb{M}$ , then  $p$  is stationary

2. If  $A = \text{acl}^{\text{eq}}(A)$  and  $p \in S_n(A)$ , then  $p$  is stationary. (It has an  $A$ -definable extension by Corollary 12.13) In the language of Remark 12.4, strong types are stationary
3. If  $p \in S_n(A)$  is stationary and  $q \in S_n(B)$  is a non-forking extension, then  $q$  is stationary (By transitivity, every global non-forking extension of  $q$  is a global non-forking extension of  $p$ . So  $q$  has no more global non-forking extension than  $p$ )
4. In a strongly minimal theory, the transcendental type over  $A$  is stationary (The global transcendental type is  $\emptyset$ -definable and extends it)
5. In  $\mathbb{C} \models \text{ACF}$ ,  $\text{tp}(i/\mathbb{Q}) = \text{tp}(-i/\mathbb{Q})$  is *not* stationary, because it has two extensions to  $\text{acl}(\mathbb{Q}) = \mathbb{Q}^{\text{alg}}$ , namely  $\text{tp}(i/\mathbb{Q}^{\text{alg}})$  and  $\text{tp}(-i/\mathbb{Q}^{\text{alg}})$

**Lemma 12.26.** Let  $p \in S_n(A)$  be stationary, let  $q \in S_n(B)$  be an extension of  $p$ , and let  $p^{\mathbb{M}} \in S_n(\mathbb{M})$  be the unique  $A$ -definable global extension of  $p$ . Then  $q \sqsupseteq p$  iff  $q = p^{\mathbb{M}} \upharpoonright B$

*Proof.* Note  $p^{\mathbb{M}} \sqsupseteq p$ . If  $q = p^{\mathbb{M}} \upharpoonright B$ , then  $q \sqsupseteq p$  by full transitivity. Conversely if  $q \sqsupseteq p$ , take some  $r' \in S_n(\mathbb{M})$  with  $r' \sqsupseteq q \sqsupseteq p$ . Then  $r' = p^{\mathbb{M}}$  by stationary, so  $q \subseteq r' = p^{\mathbb{M}}$  and  $q = p^{\mathbb{M}} \upharpoonright B$   $\square$

## 12.7 Local Character

**Definition 12.27.**  $\kappa_n(T)$  is the smallest infinite cardinal s.t. there is no descending chain  $(\beta_\alpha : \alpha < \kappa)$  of length  $\kappa$  in the fundamental order for  $n$ -types.

*Remark.*  $\kappa_n(T) \leq |L|^+$ . Otherwise, there is a descending chain  $(\beta_\alpha : \alpha < |L|^+)$ . If  $\beta_\alpha = [p_\alpha]$ , then  $[p_\alpha] \subsetneq [p_{\alpha+1}]$ , and we can take  $\varphi_\alpha \in [p_{\alpha+1}] \setminus [p_\alpha]$ . Then  $\varphi_\alpha$  are pairwise distinct

**Proposition 12.28** (Local character). *If  $p \in S_n(A)$ , then there is  $B \subseteq A$  with  $|B| < \kappa_n(T)$  s.t.  $p \supseteq (p \upharpoonright B)$*

*Proof.* Suppose not ( $p \not\supseteq p \upharpoonright B$  for  $B \subseteq A$  with  $|B| < \kappa_n(T)$ )

Let  $\kappa = \kappa_n(T)$ . Build  $(\bar{b}_\alpha : \alpha < \kappa)$  in  $A$  as follows:

Step  $\alpha$ :  $p \not\supseteq p \upharpoonright \{\bar{b}_\gamma : \gamma < \alpha\}$ ,  $B_\alpha = \{\bar{b}_\gamma : \gamma < \alpha\}$ ,  $\exists \varphi(x, \bar{b}_\alpha) \in p(x)$ ,  $\varphi(x, \bar{b}_\alpha)$  forks over  $B_\alpha$

$(B_\alpha : \alpha < \kappa)$ . For  $\alpha < \gamma$ ,  $p \upharpoonright B_\gamma \supseteq p \upharpoonright B_\alpha$ ,  $\text{bd}(p \upharpoonright B_\alpha) \leq \text{bd}(p \upharpoonright B_\gamma)$ ,  $p \upharpoonright B_{\alpha+1} \ni \varphi(x, \bar{b}_\alpha)$  forks over  $B_\alpha$ , so  $p \upharpoonright B_{\alpha+1} \not\supseteq p \upharpoonright B_\alpha$ , then  $\text{bd}(p \upharpoonright B_\gamma) < \text{bd}(p \upharpoonright B_\alpha)$   $\square$

## 12.8 Stability spectra

$\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$

**Lemma 12.29.** *If  $M \models T$ ,  $\lambda$  cardinal,  $\beta, \beta'$  in fundamental order,  $\beta > \beta'$ , if  $N \geq M$ ,  $N$  very saturated, strongly homogeneous, if  $p \in S_n(M)$ ,  $[p] = \beta$ , then*

$$|\{q \in S_n(N) : q \supseteq p, [q] = \beta'\}| > \lambda$$

*Proof.* Since  $\beta' \leq [p]$ ,  $\exists N_0 \geq M$ ,  $q_0 \in S_n(N_0)$ ,  $q_0 \supseteq p$ ,  $[q_0] = \beta'$  (10.3). WLOG,  $|N_0| \leq |M| + |L|$ . Embed  $N_0 \hookrightarrow N$ . WLOG,  $M \leq N_0 \leq N$ . Take  $q \in S_n(N)$  the heir of  $q_0$ ,  $[q] = \beta' < \beta = [p]$ ,  $q \supseteq p$ ,  $q \not\supseteq p$ . (Treat  $N$  as monster,  $q$  is not  $\text{acl}^{\text{eq}}(M)$ -definable). Then  $\{\sigma(q) : \sigma \in \text{Aut}(N/M)\}$  is big  $\square$

**Proposition 12.30.** *If  $\lambda^\mu > \lambda$  for some  $\mu < \kappa(T)$ , then  $T$  is not  $\lambda$ -stable ( $\lambda \geq \aleph_0$ )*

*Proof.* Take  $\mu$  minimal,  $\lambda^{<\mu} \leq \lambda$

Take  $(\beta_\alpha : \alpha < \mu)$  in descending fundamental order

Take  $p \in S_1(M)$ ,  $[p] = \beta_0$ , build  $M_0 \leq M_1 \leq \dots$  of length  $\mu$  s.t.  $M_{\alpha+1}$  is  $(|M_\alpha| + \lambda)^+$ -saturated and -strongly homogeneous

Lemma 12.29 gives distinct  $p_i$ ,  $i < \lambda$ ,  $p_i \in S_1(M_1)$ ,  $p_i \supseteq p$ ,  $[p_i] = \beta_1$

$\forall i < \lambda$ , Lemma 12.29 gives distinct  $p_{ij} \in S_1(M_2)$ ,  $[p_{ij}] = \beta_2$ ,  $p_{ij} \supseteq p_i$ , etc.

Get  $p_\sigma$  for  $\sigma \in \lambda^{<\mu}$ , if  $\sigma \in \lambda^\alpha$ ,  $p_\sigma \in S_1(M_\alpha)$ . If  $\tau$  extends  $\sigma$ , then  $p_\tau \supseteq p_\sigma$ ,  $[p_\alpha] = \beta_\alpha$ ,  $p_{\sigma i} \neq p_{\sigma j}$  for  $i \neq j$ .

For each  $\sigma$ ,  $i, j$ ,  $i \neq j$ ,  $i, j < \lambda$ , take  $\varphi(x, b) \in p_{\sigma i}$ ,  $\varphi(x, b) \notin p_{\sigma j}$ . Collect all the  $b$  in a set  $B \subseteq \bigcup_{\alpha < \mu} M_\alpha$ ,  $|B| = \lambda^{<\mu} \cdot \lambda \cdot \lambda \leq \lambda$

**Claim:**  $\lambda^\mu \rightarrow S_1(B), \sigma \mapsto p_\sigma \upharpoonright B$  is injective

Since we have chosen the parameters to distinguish all these types

If  $\sigma, \tau \in \lambda^\mu, \sigma \neq \tau$ , there is  $\alpha < \mu, \sigma \upharpoonright \alpha = \tau \upharpoonright \alpha, \sigma \upharpoonright (\alpha + 1) \neq \tau \upharpoonright (\alpha + 1)$   $\square$

**Corollary 12.31.** *If  $T$  is  $\lambda$ -stable, then  $\lambda \geq \kappa(T)$*

*Proof.* Take  $\mu = \lambda$   $\square$

**Lemma 12.32.** *If  $T$  is  $\lambda$ -stable,  $A \subseteq \mathbb{M}, |A| \leq \lambda$ , then  $S_1(\text{acl}^{\text{eq}}(A)) \leq \lambda$*

*Proof.* Take  $A = A_0 \subseteq A_1 \subseteq \dots$  of length  $\omega, A_{i+1} = A_i \cup \{\text{a realization of each type over } A_i\}$ . If  $|A_i| \leq \lambda$ , then  $|S_1(A_1)| \leq \lambda$ , so  $|A_i| \leq \lambda$ . Let  $M = \bigcup_{i=0}^{\infty} A_i, |M| \leq \lambda$

**Claim**  $M \preceq \mathbb{M}$

Use Tarski-Vaught criterion. If  $D \subseteq \mathbb{M}, D$  is  $M$ -definable,  $D \neq \emptyset$ , then we want  $D \cap M \neq \emptyset$ .  $D$  is  $A_i$ -definable,  $i < \omega$ , take  $b_0 \in D, \text{tp}(b_0/A_i)$  is realized by  $b \in A_{i+1}$ , therefore  $b \in D, D \cap M \neq \emptyset$

$\text{dcl}^{\text{eq}}(M) = M^{\text{eq}} \supseteq \text{acl}^{\text{eq}}(A), \lambda \geq |S_1(M)|$   $\square$

**Definition 12.33.**  $\lambda_0(T) = \min\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$

By Corollary 12.31,  $\lambda_0(T) \geq \kappa(T)$

**Theorem 12.34.**  $T$  is  $\lambda$ -stable  $\Leftrightarrow \lambda \geq \lambda_0(T)$  and  $\forall \mu < \kappa(T), \lambda^\mu \leq \lambda$

*Proof.*  $\Leftarrow$ : Fix  $A \subseteq \mathbb{M}, |A| \leq \lambda$ .

Goal:  $|S_1(A)| \leq \lambda$

If  $p \in S_1(A)$ , by local character, there is  $B \subseteq A, p \sqsupseteq p \upharpoonright B$  and  $\mu = |B| < \kappa(T)$ . Number of choices of  $B$  is  $\leq |A|^\mu \leq \lambda$

If  $c \models p, p = \text{tp}(c/A)$ , determined by  $\text{tp}(c/\text{acl}^{\text{eq}}(A))$ , determined by  $\text{tp}(c/\text{acl}^{\text{eq}}(B))$ , and the number of choices for  $\text{tp}(c/\text{acl}^{\text{eq}}(B))$  is  $\leq \lambda$  by lemma 12.32  $\square$

*Remark.* Theorem 12.34 shows that  $\kappa(T)$  and  $\lambda_0(T)$  determine the **stability spectrum** of  $T$ , the set  $S = \{\lambda : T \text{ is } \lambda\text{-stable}\}$

## 12.9 Superstability

**Definition 12.35.**  $T$  is **superstable** if  $T$  is stable and  $\kappa(T) = \aleph_0$  (fundamental order has DCC)

If  $T$  is superstable and  $p \in S_n(A)$ , then  $\exists A_0 \subseteq_f A, p \sqsupseteq (p \upharpoonright A_0)$  by local character



**Proposition 12.36.** *T is superstable iff T is  $\lambda$ -stable for all sufficiently large  $\lambda$*

*Proof.* If  $\kappa(T) = \aleph_0$ . Then  $T$  is  $\lambda$ -stable  $\Leftrightarrow \lambda \geq \lambda_0(T)$  and  $\forall \mu < \aleph_0$  for all  $\mu < \aleph_0$

If  $\kappa > \aleph_0$ ,  $\lambda = \aleph_{\alpha+\omega}$  for any ordinal  $\alpha$ , then  $\text{cf}(\lambda) = \omega$ .  $\lambda^{\aleph_0} > \lambda$  by König's lemma (if  $\forall i \in I, \lambda_i > \kappa_i$ , then  $\prod_{i \in I} \lambda_i > \sum_{i \in I} \kappa_i$ ),  $\lambda^{\aleph_0} = \prod_{i \in \aleph_0} \lambda > \sum_{i \in \aleph_0} \aleph_{\alpha+i} = \lambda$

Let  $\mu = \aleph_0$ ,  $\mu < \kappa(T)$ ,  $\lambda^\mu > \lambda$  so not  $\lambda$ -stable

If  $T$  is not superstable, then  $\forall \alpha$ ,  $T$  is not  $\aleph_{\alpha+\omega}$ -stable □

Now suppose  $|L| = \aleph_0$ ,  $\aleph_0 \leq \lambda_0(T)$ ,  $\aleph \leq \kappa(T)$

Corollary 12.31:  $\lambda_0(T) \geq \kappa(T)$

?:  $\kappa(T) \leq |L|^+ = \aleph_1$

$T$  is  $2^{\aleph_0}$ -stable

So  $\lambda_0 \leq 2^{\aleph_0}$

**Fact 12.37.** *If  $\lambda_0 > \aleph_0$  then  $\lambda_0 \geq 2^{\aleph_0}$*

So there are three possibilities

name	$\kappa(T)$	$\lambda_0(T)$	Spectrum
$\omega$ -stable	$\aleph_0$	$\aleph_0$	$\{\lambda : \lambda \geq \aleph_0\}$
Superstable but not $\omega$ -stable	$\aleph_0$	$2^{\aleph_0}$	$\{\lambda : \lambda \geq 2^{\aleph_0}\}$
Stable but not superstable	$\aleph_1$	$2^{\aleph_0}$	$\{\lambda : \lambda^{\aleph_0} = \lambda\} = \{\lambda^{\aleph_0} : \lambda\}$

**Fact 12.38.** • *Strongly minimal theory is  $\omega$ -stable*

- $(\mathbb{Z}, +)$  is superstable but not  $\omega$ -stable
- separably closed fields (other than ACF), free groups (other than  $(\mathbb{Z}, +)$ ) are stable, but not superstable

## 12.10 Forking calculus

**Definition 12.39.**  $\bar{a} \downarrow_C \bar{b}$  if  $\text{tp}(\bar{a}/C\bar{b}) \sqsubseteq \text{tp}(\bar{a}/C)$

$\bar{a}$  and  $\bar{b}$  are **independent over C**

**Lemma 12.40.** *Suppose  $C = \text{acl}^{\text{eq}}(C)$ ,  $\bar{a}, \bar{b}$  are tuples. Let  $p, q$  be the unique global  $C$ -definable types (they are stationary) extend  $\text{tp}(\bar{a}/C)$ ,  $\text{tp}(\bar{b}/C)$ . Then*

1.  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C$
2.  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{b} \downarrow_C \bar{a})$

*Proof.*

$$\begin{aligned}
& (\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C \\
& \Leftrightarrow \bar{b} \models q \upharpoonright C \text{ and } \bar{a} \models p \upharpoonright C\bar{b} \\
& \Leftrightarrow \bar{a} \models p \upharpoonright C\bar{b} \\
& \Leftrightarrow \text{tp}(\bar{a}/C\bar{b}) \subseteq p \\
& \Leftrightarrow \text{tp}(\bar{a}/C\bar{b}) \supseteq \text{tp}(\bar{a}/C) \quad (\text{Lemma 12.26}) \\
& \Leftrightarrow \bar{a} \downarrow_C \bar{b}
\end{aligned}$$

$\text{tp}(\bar{a}/C)$  and  $\text{tp}(\bar{b}/C)$  are stationary by 12.25 (4)

Types commute in stable theory (9.70) □

**Lemma 12.41.**  $\forall C, \bar{a}, \bar{b}$

1.  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a} \downarrow_{\text{acl}^{\text{eq}}(C)} \bar{b}$
2.  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{b} \downarrow_C \bar{a}$

*Proof.*  $1 \rightarrow 2$  by 12.40

$$\begin{array}{ccc}
& \text{tp}(\bar{a}/C\bar{b}) & \sqsubseteq & \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C\bar{b})) \\
& \subseteq & & \sqsubseteq \sqcup \\
\text{tp}(\bar{a}/c) & & & \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C)\bar{b}) \\
& \sqsubseteq & & \subseteq \\
& \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(\bar{c})) & & 
\end{array}$$

Therefore  $\text{tp}(\bar{a}/C) \sqsubseteq \text{tp}(\bar{a}/C\bar{b}) \Leftrightarrow \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(\bar{c})) \sqsubseteq \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C)\bar{b})$  □

**Lemma 12.42.** If  $\bar{a}, \bar{a}'$  enumerate  $A$ ,  $\bar{b}, \bar{b}'$  enumerate  $B$ , then  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a}' \downarrow_C \bar{b}'$

*Proof.*  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \text{tp}(\bar{a}/CB) \supseteq \text{tp}(\bar{a}/C) \Leftrightarrow \bar{a} \downarrow_C \bar{b}' \Leftrightarrow \bar{b}' \downarrow_C \bar{a} \Leftrightarrow \bar{b}' \downarrow_C \bar{a}'$  □

**Definition 12.43.**  $A \downarrow_C B$  if  $\bar{a} \downarrow_C \bar{b}$  for tuples  $\bar{a}, \bar{b}$  enumerating  $A, B$

**Proposition 12.44.** 1. (Symmetry)  $A \downarrow_C B \Leftrightarrow B \downarrow_C A$

2. (Monotonicity) If  $A' \subseteq A, B' \subseteq B$  then  $A \downarrow_C B \Rightarrow A' \downarrow_C B'$

3. (*right Transitivity*)  $A \downarrow_C B, A \downarrow_{CB} B' \Rightarrow A \downarrow_C BB'$
4. (*left Transitivity*)  $A \downarrow_C B, A' \downarrow_{CA} B \Rightarrow AA' \downarrow_C B$
5. (*Base monotonicity*)  $A \downarrow_C BB' \Rightarrow A \downarrow_{CB} B'$
6. (*Normality*)  $A \downarrow_C B \Rightarrow A \downarrow_C BC$
7. (*Invariance*) If  $\sigma \in \text{Aut}(\mathbb{M})$  then  $A \downarrow_C B \Rightarrow \sigma(A) \downarrow_{\sigma(C)} \sigma(B)$
8. (*Extension*) Given  $A, B, C, \exists A' \equiv_C A$  with  $A' \downarrow_C B$
9. (*Finite character*) If  $A_0 \downarrow_C B_0, \forall A_0 \subseteq_f A, B_0 \subseteq_f B$ , then  $A \downarrow_C B$
10. if  $C_1 \subseteq C_2 \subseteq C_3, A \downarrow_{C_1} C_3 \Leftrightarrow (A \downarrow_{C_1} C_2 \text{ and } A \downarrow_{C_2} C_3)$

*Proof.*  $C \subseteq CB \subseteq CBB'$ , so  $\text{tp}(A/CBB') \supseteq \text{tp}(A/C) \Leftrightarrow \text{tp}(A/CBB') \supseteq \text{tp}(A/CB)$  and  $\text{tp}(A/CB) \supseteq \text{tp}(A/C)$  this gives transitivity, base monotonicity and monotonicity

Extension: Proposition ??  $\text{tp}(A/C)$  has a non-forking extension  $q$  to  $BC$ , take  $A' \models q$ , then  $A \equiv_C A'$

Finite character: only need to show  $\forall B_0 \subseteq_f B (A \downarrow_C B_0) \Rightarrow A \downarrow_C B$

If  $\text{tp}(A/BC) \not\supseteq \text{tp}(A/C)$ , then  $\varphi \in \text{tp}(A/BC)$ ,  $\varphi$  forks over  $C$ ,  $\varphi \in L(B_0C), A \not\downarrow_C B_0$   $\square$

## 12.11 Examples

**Proposition 12.45.** If  $p, q$  are  $C$ -definable types and  $\bar{a} \models p \upharpoonright C, \bar{b} \models q \upharpoonright C$ , thne

$$\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright C$$

*Proof.*  $\text{tp}(\bar{a}/C), \text{tp}(\bar{b}/C)$  stationary, like Lemma 12.40  $\square$

In ACF,  $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a} \downarrow_{\text{acl}(C)} \bar{b}$ ,  $\text{acl}(M)$  is a model,  $\text{tp}(\bar{a}/M) = p_V, \text{tp}(\bar{b}/M) = p_W$ , then  $\bar{a} \downarrow_M \bar{b} \Leftrightarrow (\bar{a}, \bar{b})$  is generic on  $V \times W$

**Proposition 12.46.**  $\bar{a} \downarrow_B \bar{a} \Leftrightarrow \bar{a} \in \text{acl}^{\text{eq}}(B)$

*Proof.*  $\Rightarrow$ :  $S_n(\mathbb{M}) \ni p \supseteq \text{tp}(\bar{a}/B\bar{a}) \supseteq \text{tp}(\bar{a}/B), p = \text{tp}(\bar{a}/\mathbb{M}), p$  is  $\text{acl}^{\text{eq}}(B)$ -definable,  $\bar{a} \in \text{acl}^{\text{eq}}(B)$

$\Leftarrow$ : By “extension”,  $\exists \sigma \in \text{Aut}(\mathbb{M}/B), \sigma(\text{acl}^{\text{eq}}(B)) \downarrow_B \text{acl}^{\text{eq}}(B), \sigma(\text{acl}^{\text{eq}}(B)) = \text{acl}^{\text{eq}}(\sigma(B)) = \text{acl}^{\text{eq}}(B)$

Actually  $\text{acl}^{\text{eq}}(B) \downarrow_B \text{acl}^{\text{eq}}(B)$ , by monotonicity,  $\bar{a} \downarrow_B \bar{a}$   $\square$

**Proposition 12.47.**  $A \downarrow_C B \Rightarrow \text{acl}^{\text{eq}}(AC) \cap \text{acl}^{\text{eq}}(BC) = \text{acl}^{\text{eq}}(C)$

if  $e \in \text{acl}(AC)$  and  $e \in \text{acl}(BC)$  then  $e \in \text{acl}(C)$

**Proposition 12.48.** Let  $T = \text{theory of } \mathbb{R}\text{-vector spaces}$ . If  $A \subseteq M$

1.  $T$  is complete, has q.e.
2.  $T$  is strongly minimal, stable
3.  $A \downarrow_{\emptyset} B \Leftarrow \text{span}(A) \cap \text{span}(B) = \{0\}$
4.  $A \downarrow_C B \Leftrightarrow \text{span}(AC) \cap \text{span}(BC) = \text{span}(C)$

**Proposition 12.49.**  $\kappa_n(T)$  doesn't depend on  $n$

*Proof. Claim:*  $\kappa_n(T)$  is smallest  $\kappa \geq \aleph_0$  s.t.  $\nexists \bar{a} \in \mathbb{M}^n$ , increasing  $(C_\alpha : \alpha < \kappa)$ ,  $C_\alpha \subseteq \mathbb{M}$  s.t.  $\text{tp}(\bar{a}/C_{\alpha+1}) \supseteq \text{tp}(\bar{a}/C_\alpha)$  for all  $\alpha$

Given a forking chain,  $\text{bd}(\text{tp}(\bar{a}/C_{\alpha+1})) < \text{bd}(\text{tp}(\bar{a}/C_\alpha))$  so descending chain of length  $\kappa$  in fundamental order

Conversely, given a descending chain □

## 12.12 Independence sequence

**Definition 12.50.**  $(A_i : i \in I)$  is **independent** over  $B$  if  $A_i \downarrow_B A_{\neq i}$  where  $A_{\neq i} = \{A_j : j \neq i\}$

**Example 12.2.**  $(A_1, A_2)$  is indepdent over  $B \Leftrightarrow A_1 \downarrow_B A_2$

**Fact 12.51.**  $T = \mathbb{R}\text{-vector spaces}$ , if  $v_1, \dots, v_n \in \mathbb{N} \setminus \{0\}$ , then  $(v_1, \dots, v_n)$  is independent over  $\emptyset$  iff they are indepdent in vector spaces

**Proposition 12.52.**  $(A_i : i \in I)$  is independent over  $B \Leftrightarrow \forall I_0 \subseteq_f I$   $I$  is independent over  $B$

*Proof.* Finite character & monotonicity □

**Lemma 12.53.** Let  $(A_i : i \leq \alpha)$  be a sequence. Let  $A_{<\alpha} = (A_i : i < \alpha)$ . Suppose  $A_{<\alpha}$  is independent over  $B$  and  $A_\alpha \downarrow_B A_{<\alpha}$ . Then  $A_{\leq \alpha}$  is independent over  $B$

*Proof.* Want to show if  $i \leq \alpha$ , then  $A_i \downarrow_B \{A_j : j \leq \alpha, j \neq i\}$

Suppose  $i < \alpha$ ,  $C_i = A_{<\alpha} \setminus \{A_i\}$ . Want  $A_i \downarrow_B A_\alpha C_i$ . We know  $A_i \downarrow_B C_i$ , also  $A_\alpha \downarrow_B A_i C_i$ . By base monotonicity,  $A_\alpha \downarrow_{BC_i} A_i$ , by symmetry,  $A_i \downarrow_{BC_i} A_\alpha$ , by transitivity,  $A_i \downarrow_B A_\alpha C_i$  □

**Proposition 12.54.** If  $(A_i : i < \alpha)$  is a sequence and  $A_i \perp_B \{A_j : j < i\}$  for each  $i$ , then  $(A_i : i < \alpha)$  is independent

**Example 12.3.**  $(A_1, A_2, A_3, A_4)$  is indepdent over  $B \Leftrightarrow A_2 \perp_B A_1$  and  $A_3 \perp_B A_1 A_2$  and  $A_4 \perp_B A_1 A_2 A_3$

**Example 12.4.** If  $\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots$  is a Morley sequence over  $B$ , then  $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots)$  is independent over  $B$

$\bar{a}_i \models p \upharpoonright B\bar{a}_1\bar{a}_2 \dots \bar{a}_{i-1}$ ,  $\text{tp}(\bar{a}_i/B\bar{a}_1 \dots \bar{a}_{i-1}) \subseteq p$ ,  $\text{tp}(\bar{a}_i/B\bar{a}_1 \dots \bar{a}_{i-1})$  has a  $B$ -definable global extension,  $\text{tp}(\bar{a}_i/B\bar{a}_1 \dots \bar{a}_{i-1}) \supseteq \text{tp}(\bar{a}_i/B)$ ,

More generally if  $p_1, \dots, p_n$  are  $B$ -definable and  $(\bar{a}_1, \dots, \bar{a}_n) \models (p_1 \otimes \dots \otimes p_n) \upharpoonright B$  then  $(\bar{a}_1, \dots, \bar{a}_n)$  is independent over  $B$

Related to the fact that if  $\text{tp}$

### 12.13 Bases in strongly minimal theories

$T$  is strongly minimal,  $A \subseteq M$ ,  $b, b' \in M \setminus \text{acl}(A)$ , then  $\text{tp}(b/A) = \text{tp}(b'/A)$ . otherwise there is  $A$ -definable  $D \subseteq M$  with  $b \in D$  and  $b' \notin D$ . If  $D$  is finite, balala, if  $D$  is cofinite, balabala

**Definition 12.55.** The **transcendental type** over  $A$  is  $\text{tp}(b/A)$  for any  $b \notin \text{acl}(A)$

Let  $p \in S_1(M)$  be transcendental, then  $p \upharpoonright A$  is  $\text{tp}(b/A)$  where  $b \in N \geq M$ ,  $b \notin \text{acl}(M) = M \supseteq \text{acl}(A)$ , so  $\text{tp}(b/A)$  is transcendental, so  $p \upharpoonright A$  is the transcendental over  $A$

The transcendental type over  $A$  has a global  $\emptyset$ -definable extension, so it is stationary by 12.25

$p$  is  $\emptyset$ -invariant,  $\emptyset$ -definable

**Definition 12.56.**  $b$  is **transcendental** if  $b \notin \text{acl}(\emptyset)$

**Lemma 12.57.** If  $b$  is transcendental, then  $b \perp_{\emptyset} C$  iff  $b \notin \text{acl}(C)$

*Proof.* Let  $p$  be the global transcendental type.  $\text{tp}(b/\emptyset)$  is stationary so  $\text{tp}(b/C) \supseteq \text{tp}(b/\emptyset) \Leftrightarrow \text{tp}(b/C) = p \upharpoonright C \Leftrightarrow b \notin \text{acl}(C)$  12.26  $\square$

**Proposition 12.58.** A sequence  $(b_i : i < \alpha)$  of transcendentals is independent over  $\emptyset$  iff  $b_i \notin \text{acl}(\{b_j : j < i\})$  for each  $i$

*Remark.* If  $p \in S_1(M)$  transcendental, if  $b_1, \dots, b_n \in M \setminus \text{acl}(\emptyset)$ , then  $(b_1, \dots, b_n)$  is independent over  $\emptyset$  iff  $(b_1, \dots, b_n) \models p^{\otimes n} \upharpoonright \emptyset$

**Lemma 12.59.** *If  $I_1, I_2 \subseteq \mathbb{M} \setminus \text{acl}(\emptyset)$  are independent sets and if  $f : I_1 \rightarrow I_2$  a bijection, then  $f$  is a partial elementary map*

*Proof.* If  $b_1, \dots, b_n \in I_1$  then  $b_1, \dots, b_n \models p^{\otimes n} \upharpoonright \emptyset$  and  $f(\bar{b}) \models p^{\otimes n} \upharpoonright \emptyset$ , so  $\bar{b} \equiv_{\emptyset} f(\bar{b})$   $\square$

**Definition 12.60.** If  $M \preceq \mathbb{M}$ , a **basis** of  $M$  is a maximal independent set  $I \subseteq M \setminus \text{acl}(\emptyset)$

Exists by Zorn's lemma

**Proposition 12.61.** *If  $B$  is a basis of  $M$ , then  $\text{acl}(B) = M$*

*Proof.* Otherwise,  $\text{acl}(B) \subsetneq M$ , take  $c \in M \setminus \text{acl}(B)$ ,  $c \perp_{\emptyset} B$  by Lemma 12.57,  $B \cup \{c\}$  is independent by Lemma ??  $\square$

**Theorem 12.62.**  *$T$  is  $\lambda$ -categorical for  $\lambda > |L|$*

*Proof.* Suppose  $M_1, M_2 \preceq \mathbb{M}$ ,  $|M_i| = \lambda$  for  $i = 1, 2$ . Take basis  $B_i \subseteq M_i$ ,  $|B_i| \leq |M_i| = \lambda$

$|M_i| = \text{acl}(B_i) \leq |B_i| + |L|$ . So  $|B_i| = \lambda$ . Take bijection  $f : B_1 \rightarrow B_2$ , then  $f$  is partial elementary, and there is  $\sigma \in \text{Aut}(\mathbb{M})$ ,  $\sigma \supseteq f$ ,  $\sigma(M_1) = \sigma(\text{acl}(B_1)) = \text{acl}(\sigma(B_1)) = \text{acl}(B_2) = M_2$   $\square$

**Definition 12.63.**  $\dim(M) = |B|$  for any basis  $B \subseteq M$

**Fact 12.64.** *This is well-defined.  $\dim(M_1) = \dim(M_2) \Rightarrow M_1 \cong M_2$*

## 13 Cantor-Bendixson rank and Morley rank

### 13.1 More topology on type spaces

Fix  $A \subseteq \mathbb{M}$ ,  $n$ ,

Iso of boolean algebras

$$\{A\text{-definable sets } D \subseteq \mathbb{M}^n\} \xrightarrow{\cong} \{\text{clopen sets in } S_n(A)\}$$

*Remark.* If  $\Sigma(\bar{x})$  is a set of  $L(A)$ -formulas, then  $\bigcap_{\varphi \in \Sigma} [\varphi] = \{p \in S_n(A) : \forall \varphi \in \Sigma : p \vdash \varphi\} = \{p \in S_n(A) : p \vdash \Sigma\}$ , the completions of  $\Sigma$

Closed sets in  $S_n(A)$  corresponds to partial type over  $A$

**Fact 13.1.**  $\{p\}$  is closed

**Fact 13.2.**  $X$  is clopen  $\Leftrightarrow X$  is closed and open

**Fact 13.3.** If  $\mathcal{F}$  is a collection of closed sets and  $\mathcal{F}$  has FIP, then  $\bigcap \mathcal{F} \neq \emptyset$

**Corollary 13.4.** If  $\mathcal{F}$  is a nonempty collection of nonempty closed sets in  $S_n(A)$

1. If  $\mathcal{F}$  is a chain
  2. Or if  $\mathcal{F}$  is filtered ( $\forall X, Y \in \mathcal{F} \exists Z \in \mathcal{F}, Z \subseteq X \cap Y$ )
- then  $\mathcal{F}$  has FIP  $\Rightarrow \bigcap \mathcal{F} \neq \emptyset$

**Proposition 13.5** (Total separation). If  $p, q \in S_n(A), p \neq q$ , then  $\exists U \in S_n(A)$  clopen,  $p \in U, q \notin U$

*Proof.* Take  $\varphi$  s.t.  $p \vdash \varphi, q \vdash \neg\varphi$  □

**Proposition 13.6.** If  $C \subseteq S_n(A)$  closed, and  $p \in C$ , then 1 or 2 holds

1.  $\exists$  clopen  $U, p \in U, U \cap C = \{p\}$
2.  $\forall$  clopen  $U \ni p, |U \cap C| = \infty$

*Proof.* If 2 is false, then there is clopen  $U$   $U \cap C = \{p, q_1, \dots, q_n\}$ , 13.5 to get a nice  $U$  □

**Definition 13.7.** Suppose  $C$  is closed and  $p \in C$

1.  $p$  is an **isolated point** of  $C$  if case 1 holds
2.  $p$  is an **accumulation point** of  $C$  if case 2 holds
3. The **derived set**  $C'$  of  $C = \{p \in C : p \text{ is an accumulation point}\}$

**Proposition 13.8.** If  $C$  is closed, then  $C'$  is closed

*Proof.*  $\forall p \in C$  isolated, take  $U_p$  clopen with  $U_p \cap C = \{p\}$ ,  $S_n(A) \setminus C' = (S_n(A) \setminus C) \cup \bigcup_{p \in C \setminus C'} U_p$  open □

**Proposition 13.9.** If  $C \subseteq S_n(A)$  closed and  $|C| = \infty$ , then  $C'$  is not empty

*Proof.* Otherwise, every  $p \in C$  is isolated,  $\forall p \in S_n(A)$ , there is  $U_p$  clopen,  $U_p \cap C = \emptyset$  or  $\{p\}$

If  $p \in S_n(A) \setminus C$ , there is clopen  $U_p$  with  $p \in U_p \subseteq S_n(A) \setminus C, U_p \cap C = \emptyset$   
 $\bigcup_{p \in S_n(A)} U_p = S_n(A)$ , by compactness, there is  $p_1, \dots, p_m, S_n(A) = \bigcup_{i=1}^m U_{p_i}$ ,  
 $C = C \cap S_n(A) = \bigcup_{i=1}^m (C \cap U_{p_i})$  which is finite □

**Definition 13.10.**  $K \subseteq S_n(A)$  is **perfect** if  $K$  is closed and  $K' = K$   
 $S_n(A)$  is **scattered** if there is no  $K \subseteq S_n(A)$  s.t.  $K \neq \emptyset, K' = K$

### 13.2 Cantor-Bendixson rank

Continue to fix  $S_n(A)$

**Definition 13.11.** For ordinal  $\alpha$ , define  $E_\alpha \subseteq S_n(A)$  closed by recursion

$$\begin{aligned} E_0 &= S_n(A) \\ E_{\alpha+1} &= E'_\alpha \\ E_\beta &= \bigcap_{\alpha < \beta} E_\alpha \text{ for limit ordinal } \beta \end{aligned}$$

Then chain  $(E_\alpha : \alpha \in \text{On})$  can only decrease  $|S_n(A)|$  times

So  $E_\alpha = E_{\alpha+1}$  for some  $\alpha$ ,

Let  $E_\infty = E_\alpha$  for all sufficiently large  $\alpha$ , and  $E_\infty$  is perfect, but possibly empty

*Remark.*  $\infty$  is a formal symbol with  $\infty > \alpha$  for all  $\alpha \in \text{On}$ , therefore  $\alpha < \infty$  means  $\alpha$  is an ordinal

**Definition 13.12.**  $R(p) = \max\{\alpha \mid p \in E_\alpha\}$  (possibly  $\infty$ ) **Cantor-Bendixson rank**

If  $C \subseteq S_n(A)$  closed,  $R(C) = \max\{\alpha \mid E_\alpha \cap C \neq \emptyset\}$

This is well-defined if  $\beta$  is a limit ordinal and  $E_\alpha \cap C \neq \emptyset, \forall \alpha < \beta$ , then

$\bigcap_{\alpha < \beta} E_\alpha \cap C \neq \emptyset$  by compactness

So  $\{\alpha \mid E_\alpha \cap C \neq \emptyset\}$  is “closed”

Note:  $p \in E_\alpha \Leftrightarrow \alpha \leq R(p)$ , so  $E_\alpha = \{p \in S_n(A) : R(p) \geq \alpha\}$

*Remark.*  $R(p)$  is characterized by

$$R(p) \geq \alpha + 1 \Leftrightarrow p \text{ is an accumulation point of } \{q \in S_n(A) : R(q) \geq \alpha\}$$

$$R(p) = 0 \Leftrightarrow p \text{ is isolated}$$

*Remark.*  $R(C) = \max\{p \in C : R(p)\}$

$$\begin{aligned} \alpha \leq R(C) &\Leftrightarrow E_\alpha \cap C \neq \emptyset \Leftrightarrow \exists p \in C : p \in E_\alpha \\ &\Leftrightarrow \exists p \in C : R(p) \geq \alpha \\ &\Leftrightarrow \alpha \leq \max_{p \in C} R(p) \end{aligned}$$

**Definition 13.13.** If  $\varphi \in L(A)$ ,  $R(\varphi)$  is  $R([\varphi])$ . If  $\Sigma(\bar{x}) \in L(A)$ ,  $R(\Sigma) = R(\bigcap_{\varphi \in \Sigma} [\varphi])$ . If  $D \subseteq \mathbb{M}^n$   $A$ -definable by  $\varphi$ ,  $R(D) = R([\varphi])$

All these depend on  $A$



**Proposition 13.14.** Suppose  $C_1, C_2 \subseteq S_n(A)$  closed

1. If  $C_1 \subseteq C_2$  then  $R(C_1) \leq R(C_2)$
2.  $R(C_1 \cup C_2) = \max(R(C_1), R(C_2))$

If  $\Sigma, \Psi$  are partial types over  $A$

3. If  $\Sigma \vdash \Psi$ , then  $R(\Sigma) \leq R(\Psi)$

If  $\varphi, \psi \in L(A)$

4.  $R(\varphi \vee \psi) = \max(R(\varphi), R(\psi))$

**Proposition 13.15.** 1. If  $C \subseteq S_n(A)$  closed, then

$$R(C) = \min\{R(U) : U \supseteq C, U \text{ clopen}\}$$

2. If  $\Sigma(\bar{x}) \subseteq L(A)$ , then  $R(\Sigma) = \min\{R(\varphi) : \Sigma \vdash \varphi\}$

*Proof.* Note  $1 \leftrightarrow 2$ . Certainly  $LHS \leq RHS$ . If  $R(C) = \infty$ , there is nothing to show. Suppose  $R(C) = \alpha < \infty$ , then  $C \cap E_{\alpha+1} = \emptyset$ . As  $C$  is a filtered intersection  $\bigcap\{U : U \supseteq C, U \text{ clopen}\}$ , we have

$$\emptyset = C \cap E_{\alpha+1} = \bigcap\{U \cap E_{\alpha+1} : U \supseteq C, U \text{ clopen}\}$$

then there is some  $U \cap E_{\alpha+1} = \emptyset$ . Then  $\alpha + 1 > R(U) \geq R(C) = \alpha$ , so  $R(U) = R(C)$   $\square$

**Proposition 13.16.** If  $C \subseteq S_n(A)$  closed and  $-\infty < R(C) < \infty$  then  $\{p \in C : R(p) = R(C)\}$  is finite

*Proof.* Let  $\alpha = R(C)$ . If  $E_\alpha \cap C$  is infinite, then there is  $q \in (E_\alpha \cap C)'$  by ??,  $q \in (E_\alpha \cap C)' \subseteq E'_\alpha = E_{\alpha+1}$ , so  $R(C) \geq \alpha + 1$   $\square$

**Lemma 13.17.** If  $U \subseteq S_n(A)$  clopen and  $R(U) \geq \alpha + 1$ , then there is  $U_1, U_2$  clopen,  $U = U_1 \sqcup U_2$  and  $R(U_1) \geq \alpha + 1$ ,  $R(U_2) \geq \alpha$

*Proof.* Take  $p \in E_{\alpha+1} \cap U$ ,  $p \in E'_\alpha$ ,  $U \cap E_\alpha$  is infinite, then there is  $q \in U \cap E_\alpha$ ,  $q \neq p$ . By ??, there is clopen  $V$ ,  $p \in V$ ,  $q \notin V$ , let  $U_1 = U \cap V$ ,  $U_2 = U \setminus V$   $\square$

**Proposition 13.18.** If  $U \subseteq S_n(A)$  clopen, TFAE

1.  $R(U) \geq \alpha + 1$
2. there is clopen  $U_1, U_2, \dots \subseteq U$ ,  $U_i \cap U_j = \emptyset$ ,  $i \neq j$ , and  $R(U_i) \geq \alpha$

3. there is clopen  $U_1, U_2, \dots \subseteq U$  s.t.  $R(U_i \triangle U_j) \geq \alpha$  (idea: modulo the ideal  $\{U : R(U) < \alpha\}$ , there are infinitely many subsets of  $U$ )

*Proof.*  $2 \rightarrow 3$ : If  $U_i \cap U_j = \emptyset$ , then  $U_i \triangle U_j = U_i \cup U_j$ ,  $R(U_i \triangle U_j) = R(U_i \cup U_j) \geq \alpha$

$3 \rightarrow 1$ :  $U_1 \triangle U_2 \subseteq U$ ,  $R(U) \geq R(U_1 \triangle U_2) \geq \alpha$ . Suppose  $R(U) < \alpha + 1$ , then  $R(U) = \alpha$ . By Proposition 13.16,  $U \cap E_\alpha = \{q_1, \dots, q_m\}$ ,  $\omega \rightarrow \text{Pow}(U \cap E_\alpha)$ ,  $i \mapsto U_i \cap E_\alpha$ , cannot be infinitive, so there is  $i < j$ ,  $U_i \cap E_\alpha = U_j \cap E_\alpha$ ,  $(U_i \triangle U_j) \cap E_\alpha = \emptyset$ ,  $R(U_i \triangle U_j) < \alpha$   $\square$

If  $D \subseteq \mathbb{M}^n$  is  $A$ -definable, then

$$R(D) \geq 0 \Leftrightarrow D \neq \emptyset$$

$$R(D) \geq \alpha + 1 \Leftrightarrow \exists A\text{-definable } D_1, D_2, \dots \subseteq D, D_i \cap D_j = \emptyset, R(D_i) \geq \alpha$$

$$R(D) \geq \beta \Leftrightarrow R(D) \geq \alpha, \forall \alpha < \beta$$

Morley rank is Cantor-Bendixson rank when  $A = \mathbb{M}$

**Definition 13.19.** A family  $\mathcal{F}$  of definable sets  $k$ -inconsistent ( $k \in \omega$ ) if  $\forall$  distinct  $D_1, \dots, D_k \in \mathcal{F}$ ,  $\bigcap_{i=1}^k D_i = \emptyset$

**Lemma 13.20.** If  $D$  is  $A$ -definable, if  $D_1, D_2, \dots \subseteq D$  are  $A$ -definable subsets,  $R(D_i) \geq \alpha$ , and  $\{D_1, D_2, \dots\}$  is  $k$ -inconsistent, then  $R(D) \geq \alpha + 1$

$k = 2$  is the definition

*Proof.*  $D \supseteq D_1$  so  $R(D) \geq R(D_1) \geq \alpha$ . If  $R(D) \not\geq \alpha + 1$ , then  $R(D) = \alpha$ . Let  $E_\alpha = \{p \in S_n(A) : R(p) = \alpha\}$ ,  $[D] \cap E_\alpha \neq \emptyset$ ,  $|[D] \cap E_\alpha| < \infty$ , otherwise if  $q \in (D \cap E_\alpha)'$ , then  $q \in [D] \cap E_{\alpha+1}$

Consider  $\omega \rightarrow \text{Pow}([D] \cap E_\alpha)$ ,  $i \mapsto [D_i] \cap E_\alpha$ . Some fiber is infinite, pass to a subsequence, WMA  $[D_i] \cap E_\alpha = [D_1] \cap E_\alpha$

$\bigcap_{i=1}^k D_i = \emptyset$ ,  $\bigcap_{i=1}^k [D_i] = \emptyset$ ,  $\bigcap_{i=1}^k ([D_i] \cap E_\alpha) = \emptyset = [D_1] \cap E_\alpha$ , then  $R(D_1) < \alpha$   $\square$

### 13.3 When is $S_n(A)$ scattered

**Proposition 13.21.**  $R(S_n(A)) < \infty \Leftrightarrow S_n(A)$  is scattered (no non-empty perfect sets)

*Proof.* If  $R(S_n(A)) = \infty$ , then  $E_\infty \neq \emptyset$ ,  $E_\infty$  is a nonempty perfect space

Suppose  $S_n(A)$  is not scattered, there is  $\emptyset \neq K \subseteq S_n(A)$ ,  $K$  perfect, then  $K \subseteq E_\alpha$  by induction on  $\alpha$ .  $K \subseteq E_\alpha \Rightarrow K = K' \subseteq E'_\alpha = E_{\alpha+1}$   $\square$

**Lemma 13.22.** *If  $U \subseteq S_n(A)$  clopen and  $R(U) = \infty$ , then there is clopen  $U_1, U_2$ ,  $U = U_1 \sqcup U_2$ ,  $R(U_1) = \infty = R(U_2)$*

*Proof.* like 13.17 □

**Lemma 13.23.** 1.  $R(S_n(A)) = \infty \Leftrightarrow$  there is  $(U_\sigma : \sigma \in 2^{<\omega})$  s.t.  $U_\sigma = U_{\sigma 0} \sqcup U_{\sigma 1} \forall \sigma, U_\sigma \neq \emptyset, U_\sigma$  clopen

2. If  $R(S_n(A)) = \infty$ , then  $|S_n(A)| \geq 2^{\aleph_0}$

3. If  $R(S_n(A)) < \infty$ , then  $|S_n(A)| \leq |L(A)| = |L| + |A|$

*Proof.* 1. If  $R(S_n(A)) = \infty$ , take  $U = S_n(A)$ . By 13.22,  $U = U_0 \sqcup U_1$  where  $U_0, U_1$  are clopen sets with  $R(U_0) = R(U_1) = \infty$ . Also we can split  $U_0$  into two clopen sets  $U_{00}$  and  $U_{01}$ , split  $U_1$  into two clopen sets  $U_{10}$  and  $U_{11}$

Conversely, suppose a tree  $(U_\sigma)$  exists. We claim by induction on  $\alpha$  that  $R(U_\sigma) \geq \alpha$  for all  $\sigma$ . The base case  $\alpha = 0$  holds as the  $U_\sigma \neq \emptyset$ . For successor ordinals  $\alpha + 1$ , suppose  $R(U_\sigma) \geq \alpha$  for all  $\sigma$ , then  $U_{\sigma 1}, U_{\sigma 01}, U_{\sigma 001}, \dots$  are disjoint clopen subsets of  $U_\sigma$  with rank  $\geq \alpha$ , so  $R(U_\sigma) \geq \alpha + 1$

2. Take a tree  $(U_\sigma : \sigma \in 2^{<\omega})$  as in 1. For  $\tau \in 2^\omega$  let  $C_\tau = \bigcap_{n=0}^\infty U_{\tau \upharpoonright n}$ . Then  $C_\tau$  is nonempty closed set. Take  $p_\tau \in C_\tau$ . The  $p_\tau$  is distinct. So we get at least  $2^{\aleph_0}$ -many points

3. Suppose  $R(S_n(A)) < \infty$ , for each  $p \in S_n(A)$  let  $\alpha_p = R(p)$ . Then  $p$  is isolated in  $E_{\alpha_p}$ , so there is a clopen set  $U_p \ni p$  with  $U_p \cap E_{\alpha_p} = \{p\}$ . Then  $p$  is the unique type in  $U_p$  of maximal rank  $(U_p)$  □

**Theorem 13.24.** *If  $L$  is countable and  $T$  is stable, then  $\lambda_0(T) = \aleph_0$  or  $2^{\aleph_0}$*

### 13.4 Morley rank

**Definition 13.25.** Morley rank RM is Cantor-Bendixson rank on monster model

If  $p \in S_n(\mathbb{M})$ ,

1.  $\text{RM}(p) \geq 0$  always

2.  $\text{RM}(p) \geq \alpha + 1 \Leftrightarrow p \in (\{q \in S_n(\mathbb{M}) : \text{RM}(q) \geq \alpha\})'$

If  $\Sigma(\bar{x})$  is a partial type over  $\mathbb{M}$

$$[\Sigma] = \{p \in S_n(\mathbb{M}) : p \supseteq \Sigma\} \subseteq S_n(\mathbb{M})$$

$$\text{RM}(\Sigma) = \max_{p \in S_n(\mathbb{M}), p \supseteq \Sigma} \text{RM}(p)$$

If  $\varphi(x_1, \dots, x_n) \in L(M)$ ,  $\text{RM}(\varphi) = \text{RM}(\{\varphi\}) = \text{RM}([\varphi])$

If  $D$  is definable,  $D = \varphi(\mathbb{M}^n)$ , then  $\text{RM}(D) = \text{RM}(\varphi)$

$$\text{RM}(\Sigma) = \min_{\Sigma_0 \subseteq_f \Sigma} \text{RM}(\Sigma_0) = \min_{\Sigma \vdash \varphi} \text{RM}(\varphi)$$

$$\text{RM}(D) \geq \alpha + 1 \Leftrightarrow \text{there is definable } D_1, D_2, D_3, \dots \subseteq D, D_i \cap D_j = \emptyset, \text{RM}(D_i) \geq \alpha$$

**Lemma 13.26.** *If  $M \preceq \mathbb{M}$  is  $\aleph_0$ -saturated, if  $\Sigma(\bar{x})$  is a partial type over  $M$ , then  $\text{RM}(\Sigma) = R(\Sigma)$  over  $M$*

*Proof.* By Proposition ??, we may assume  $\Sigma$  is  $\{\varphi\}$ ,  $D \subseteq \mathbb{M}^n$ ,  $D$  is  $M$ -definable, want  $\text{RM}(D) = R(D)$

**Claim:** if  $\alpha \in \text{On}$ ,  $D$  is  $M$ -definable, then

$$\text{RM}(D) \geq \alpha \Leftrightarrow R(D) \geq \alpha$$

Induction on  $\alpha$ . If  $\alpha = 0$ , both side says  $D$  is nonempty  
limit ordinal case

$\alpha + 1$ :  $\text{RM}(D) \geq \alpha + 1$  means there are definable  $D_1, D_2, \dots \subseteq D$ , disjoint,  $\text{RM}(D_i) \geq \alpha$ .  $R(D) \geq \alpha + 1$  means there are  $M$ -definable definable  $D_1, D_2, \dots \subseteq D$ , disjoint,  $R(D_i) \geq \alpha$ , by induction,  $\text{RM}(D) \geq \alpha$ . So  $R(D) \geq \alpha + 1 \Rightarrow \text{RM}(D) \geq \alpha + 1$

If  $\text{RM}(D) \geq \alpha + 1$ , let  $D = \psi(\mathbb{M}^n, \bar{b})$ ,  $D_i = \varphi_i(\mathbb{M}^n, \bar{c}_i)$ . By  $\aleph_0$ -saturation, we can find  $\bar{c}'_i \in M$  s.t.  $\bar{c}_1 \bar{c}_2 \dots \equiv_{\bar{b}} \bar{c}'_1 \bar{c}'_2 \dots$ . Let  $D'_i = \varphi_i(\mathbb{M}^n, \bar{c}'_i)$ ,  $D'_i$  are  $M$ -definable, there is  $\sigma \in \text{Aut}(\mathbb{M}/\bar{b})$ ,  $\sigma(\bar{c}_i) = \bar{c}'_i$ ,  $\sigma(D_i) = D'_i$ ,  $\sigma(D) = D$ .  $\text{RM}(D'_i) = \text{RM}(D_i) \geq \alpha$ ,  $D'_i \cap D'_j = \emptyset$  for  $i \neq j$ ,  $D'_i \subseteq D$   $\square$

**Example 13.1.**  $|\mathbb{M}| = \infty$ ,  $S_1(A) = \{p\}$ ,  $R(S_1(A)) = 0$ , but  $\text{RM}(\mathbb{M}) > 0$

**Example 13.2.** If  $D$  is definable

1.  $\text{RM}(D) \geq 0 \Leftrightarrow D \neq \emptyset$
2.  $\text{RM}(D) \geq 1 \Leftrightarrow$  there are disjoint nonempty definable subsets  $D_1, D_2, \dots \subseteq D \Leftrightarrow |D| = \infty$
3.  $\text{RM}(D) \geq 2 \Leftrightarrow$  there are disjoint infinite definable subsets  $D_1, D_2, \dots \subseteq D$

If  $T$  is strongly minimal, then  $\text{RM}(\mathbb{M}) = 1$

Later  $\text{RM}(\mathbb{M}^n) = n$

**Proposition 13.27.** *If  $f : X \rightarrow Y$  is definable ( $X, Y, \Gamma(f)$  are definable)*

1. *If  $f$  is surjective, then  $\text{RM}(X) \geq \text{RM}(Y)$*
2. *If  $f$  is bijective, then  $\text{RM}(X) = \text{RM}(Y)$*
3. *If  $f$  is injective, then  $\text{RM}(X) \leq \text{RM}(Y)$*

*Proof.* 1. **Claim:** if  $\alpha \in \text{On}$ ,  $f : X \rightarrow Y$  definable surjection,  $\text{RM}(Y) \geq \alpha$ , then  $\text{RM}(X) \geq \alpha$

$\alpha + 1$ : if  $\text{RM}(Y) \geq \alpha + 1$ , there is definable  $D_1, D_2, \subseteq Y$  disjoint,  $\text{RM}(D_i) \geq \alpha$ .  $f^{-1}(D_i) \subseteq X$ , as  $f^{-1}(D_i) \rightarrow D_i$  is surjective,  $\text{RM}(D_i) \geq \alpha \Rightarrow \text{RM}(f^{-1}(D_i)) \geq \alpha$

2.  $f^{-1}$  and  $f$  are surjection

3. if  $f : X \rightarrow Y$  injective,  $f : X \rightarrow f(X)$  is bijection

□

**Example 13.3.** If  $|\mathbb{M}| = \infty$ , then  $\text{RM}(\mathbb{M}^n) \geq n$

*Proof.*  $n + 1$ : take  $a_1, a_2, \dots \in \mathbb{M}$  distinct, then  $D_i = \mathbb{M}^n \times \{a_i\}$  and there is definable  $\pi : D_i \rightarrow \mathbb{M}^n$  and  $\text{RM}(D_i) = \text{RM}(\mathbb{M}^n) \geq n$  by induction □

**Lemma 13.28.** *Let  $f : X \rightarrow Y$  be definable, and has finite fibers, then  $\text{RM}(X) \leq \text{RM}(Y)$*

*Proof.* If  $\alpha \in \text{On}$ , prove  $\alpha \leq \text{RM}(X) \Rightarrow \alpha \leq \text{RM}(Y)$  by induction on  $\alpha$

$\alpha = 0$ ,  $X \neq \emptyset \Rightarrow Y \neq \emptyset$ , as else  $f : X \rightarrow Y$  can't exist

$\alpha + 1$ : If  $\text{RM}(X) \geq \alpha + 1$ , take  $D_1, D_2, \dots \subseteq X$  disjoint,  $\text{RM}(D_i) \geq \alpha$ ,  $f(D_i)$  no longer disjoint. By compactness, there is  $k$  s.t.  $|f^{-1}(\bar{b})| \leq k$  for all  $\bar{b} \in Y \setminus \{f(D_i)\}_{i \in \omega}$  is  $(k + 1)$ -inconsistent

If  $\bar{b} \in \bigcap_{i=1}^{k+1} f(D_i)$ , then there is  $\bar{a}_i \in D_i$ ,  $f(\bar{a}_i) = \bar{b}$ , then  $|f^{-1}(\bar{b})| \geq k + 1$

By Lemma 13.20, and  $\text{RM}(f(D_i)) \geq \alpha$  by induction □

**Example 13.4.** In  $\text{ACF}_0$ , the set  $D = \{(x, y) : x^2 + y^2 = 1\}$  has  $\text{RM}(D) = 1$  because  $D$  is infinite and we have  $D \rightarrow \mathbb{M}$ ,  $(x, y) \mapsto x$  has finite fibres so  $\text{RM}(D) \leq \text{RM}(\mathbb{M}) = 1$  as it's strongly minimal

**Definition 13.29.** Fix  $\bar{a} \in \mathbb{M}^n$ ,  $B \subseteq \mathbb{M}$ ,  $\text{RM}(\bar{a}/B) = \text{RM}(\text{tp}(\bar{a}/B))$

**Proposition 13.30.** Fix  $B \subseteq \mathbb{M}$

1. If  $\bar{a} \in \mathbb{M}^n$ , then  $\text{RM}(\bar{a}/B) = \min\{\text{RM}(X) : X \subseteq \mathbb{M}^n, X \text{ is } B\text{-definable}, \bar{a} \in X\}$
2. If  $X$  is  $B$ -definable, then  $\text{RM}(X) = \max_{\bar{a} \in X} \text{RM}(\bar{a}/B)$

*Proof.* 1. Proposition ??

$$\begin{aligned} \text{RM}(\bar{a}/B) &= \text{RM}(\text{tp}(\bar{a}/B)) = \min_{\Sigma_0 \subseteq_f \text{tp}(\bar{a}/B)} \text{RM}(\Sigma_0) \\ &= \min_{\varphi \in \text{tp}(\bar{a}/B)} \text{RM}(\varphi) \end{aligned}$$

2. If  $\bar{a} \in X$ ,  $[\text{tp}(\bar{a}/B)] \subseteq [X] \subseteq S_n(M)$ , if  $X = \varphi(\mathbb{M})$ ,  $\varphi \in L(B)$ , then  $\varphi \in \text{tp}(\bar{a}/B)$ , so  $[\text{tp}(\bar{a}/B)] \subseteq [\varphi]$ . So  $\text{RM}(\text{tp}(\bar{a}/B)) \leq \text{RM}(X)$ , so  $\text{RM}(X) \geq \max_{\bar{a} \in X} \text{RM}(\bar{a}/B)$

Take  $p \in S_n(\mathbb{M})$ ,  $p \in [X]$  with  $\text{RM}(p) = \text{RM}(X)$ , let  $q = p \upharpoonright B$ ,  $p \vdash x \in X$ , then  $p \vdash q \vdash \varphi$  and  $\text{RM}(p) \leq \text{RM}(q) \leq \text{RM}(\varphi) = \text{RM}(X) = \text{RM}(p)$ , so  $\text{RM}(q) = \text{RM}(X)$ . Take  $a \models q$ ,  $\mathbb{M} \vdash \varphi(a)$ , so  $a \in X$ ,  $\text{RM}(a/B) = \text{RM}(q) = \text{RM}(X)$

□

**Example 13.5.** If  $T$  is strongly minimal,  $a \in \mathbb{M}$ ,  $B \subseteq \mathbb{M}$ ,

$$\text{RM}(a/B) = \begin{cases} 0 & a \in \text{acl}(B) \\ 1 & a \notin \text{acl}(B) \end{cases}$$

$$\text{RM}(a/B) = \min\{\text{RM}(X) : X \text{ } B\text{-definable}, X \ni a\},$$

**Lemma 13.31.** If  $\bar{b} \in \text{acl}(\bar{a}C)$ , then  $\text{RM}(\bar{b}/C) \leq \text{RM}(\bar{a}/C)$

*Proof.* Let  $\pi_1(\bar{x}, \bar{y}) = \bar{x}$ ,  $\pi_2(\bar{x}, \bar{y}) = \bar{y}$ ,  $|\bar{x}| = |\bar{a}|$ ,  $|\bar{y}| = |\bar{b}|$

If  $D$  is  $C$ -definable and  $D \ni (\bar{a}, \bar{b})$  and  $D$  is small enough, then

1.  $\text{RM}(\pi_1(D)) = \text{RM}(\bar{a}/C)$
2.  $\text{RM}(\pi_2(D)) = \text{RM}(\bar{b}/C)$
3.  $D \rightarrow \pi_1(D)$  has finite fibers because  $\bar{b} \in \text{acl}(C\bar{a})$

$$\text{Then } \text{RM}(\bar{b}/C) = \text{RM}(\pi_2(D)) \leq \text{RM}(D) \leq \text{RM}(\pi_1(D)) = \text{RM}(\bar{a}/C) \quad \square$$

**Theorem 13.32.** If  $T$  is strongly minimal, let  $p \in S_1(\mathbb{M})$  be transcendental, let  $C \subseteq \mathbb{M}$ ,  $\bar{a} \in \mathbb{M}^n$

1. If  $\bar{a} \not\models p^{\otimes n} \upharpoonright C$ , then  $\text{RM}(\bar{a}/C) < n$
2.  $\text{RM}(\mathbb{M}^n) = n$
3. If  $\bar{a} \models p^{\otimes n} \upharpoonright C$ , then  $\text{RM}(\bar{a}/C) = n$

*Proof.* Prove 1-3 jointly by induction on  $n$

1. If  $a_i \notin \text{acl}(Ca_1 \dots a_{i-1})$  for all  $i$ , then  $a_i \models p \upharpoonright Ca_1 \dots a_{i-1}$ ,  $\bar{a} \models p^{\otimes n} \upharpoonright C$ , a contradiction  
then there is  $i$  s.t.  $a_i \in \text{acl}(Ca_1 \dots a_{i-1})$ ,
2. take  $\aleph_0$ -saturated  $M \leq \mathbb{M}$ , so CB-rank=Morley rank. If  $q \in S_n(M)$  and  $q \neq p^{\otimes n} \upharpoonright M$ , then  $\text{RM}(q) < n$  by part 1. If  $\alpha = n$ , then  $E_\alpha = \{q \in S_n(M) : R(q) \geq n\} \subseteq \{p^{\otimes n} \upharpoonright M\}$ , as  $E_n \neq \emptyset$ , so  $R(p^{\otimes n} \upharpoonright M) = n$ . So  $E'_n = \emptyset$
3. Proposition ?? gives  $n = \text{RM}(\mathbb{M}^n) = \max_{\bar{a} \in \mathbb{M}^n} \text{RM}(\bar{a}/C)$ , and part 1 gives the answer

□

**Theorem 13.33.** If  $T$  is strongly minimal,  $M \models T$ , if  $B_1, B_2$  are two different bases, then  $|B_1| = |B_2|$

*Proof.* WMA  $|B_1| \leq |B_2|$ .

1.  $|B_2| < \infty$ , let  $B_1 = \{a_1, \dots, a_n\}$ ,  $B_2 = \{b_1, \dots, b_m\}$
2.  $|B_2| = \kappa \geq \aleph_0$ , suppose  $|B_1| < |B_2|$ ,  
If  $a \in B_1$ ,  $a \in M = \text{acl}(B_2)$ , there is  $S_a \subseteq_f B_2$ ,  $a \in \text{acl}(S_a)$ , let  $S = \bigcup_{a \in B_1} S_a$ , then  $|S| < |B_2|$ ,  $B_1 \subseteq \text{acl}(S)$ ,  $M = \text{acl}(B_1) \subseteq \text{acl}(\text{acl}(S)) = \text{acl}(S)$ . Then  $B_2$  is not independent

□

### 13.5 Totally transcendental theory

If  $L_0 \subseteq L$ , let  $S_n^{L_0}(A) = n$ -types over  $A$  in  $\mathbb{M} \upharpoonright L_0$

Recall  $S_n(A)$  is scatters iff  $R(S_n(A)) < \infty$ , that is  $R(S_n(A))$  is an ordinal

**Lemma 13.34.** Fix  $n < \infty$ , TFAE

1.  $R(S_n(\mathbb{M})) < \infty$

2.  $R(S_n(A)) < \infty$  for all  $A \subseteq \mathbb{M}$
3.  $R(S_n(A)) < \infty$  for all  $A \subseteq \mathbb{M}$  with  $|A| \leq \aleph_0$
4.  $R(S_n^{L_0}(A)) < \infty$  for all countable  $A \subseteq \mathbb{M}$  and countable  $L_0 \subseteq L$

*Proof.* By Lemma ??,  $\neg 1$  says there is  $\{D_\sigma\}_{\sigma \in 2^{<\omega}}$ ,  $D_\sigma \subseteq \mathbb{M}^n$  definable,  $D_\sigma \neq \emptyset$ ,  $D_\sigma = D_{\sigma 0} \sqcup D_{\sigma 1}$ ,  
 $\neg 3$  says  $D_\sigma$  is  $A$ -definable  $\square$

**Theorem 13.35.** Suppose  $|L| = \aleph_0$ , TFAE

1.  $\text{RM}(\Sigma(\bar{x})) < \infty$  for any  $\Sigma(x_1, \dots, x_n)$  partial type 2
2.  $\text{RM}(x = x) < \infty$
3.  $T$  is  $\omega$ -stable

*Proof.*  $1 \rightarrow 2$ : clear

$2 \rightarrow 3$ : 2 says that  $R(S_1(\mathbb{M})) < \infty$ , Lemma 13.34 and ?? gives stability

$3 \rightarrow 1$ : ??  $\square$

**Theorem 13.36.** For any  $T$ , TFAE

1.  $\text{RM}(\Sigma(x_1, \dots, x_n)) < \infty$
2.  $\text{RM}(\mathbb{M}') = \text{RM}(x = x) < \infty$
3.  $T \upharpoonright L_0$  is  $\omega$ -stable for any countable  $L_0 \subseteq L$

**Definition 13.37.**  $T$  is **totally transcendental** (t.t.) if 1-3 hold

**Example 13.6.** strongly minimal is t.t.

**Theorem 13.38.** If  $T$  is t.t., then  $T$  is  $\lambda$ -stable for all  $\lambda \geq |L|$

*Proof.* If  $\lambda \geq |L|$ ,  $A \subseteq \mathbb{M}$ ,  $|A| \leq \lambda$ , then  $S_1(A)$  is scattered,  $|S_1(A)| \leq |L| + |A| \leq \lambda$   $\square$

**Corollary 13.39.** t.t.  $\Rightarrow$  superstable



### 13.6 Morley rank and forking

Assume  $T$  is t.t.

**Lemma 13.40.** Suppose  $p \in S_n(A)$

1. If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ , then  $\text{RM}(q) \leq \text{RM}(p)$
2. there is  $r \in S_n(\mathbb{M})$ ,  $r \supseteq p$  with  $\text{RM}(r) = \text{RM}(p)$
3. If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$  and  $\text{RM}(q) = \text{RM}(p)$ , then  $q \sqsupseteq p$
4. If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$  and  $q \sqsupseteq p$ , then  $\text{RM}(q) = \text{RM}(p)$

*Proof.* 1 - 2: by definition

3.

□

If  $q \in S_n(\mathbb{M})$ ,  $q \supseteq p$ , then  $\text{RM}(q) = \text{RM}(p) \Leftrightarrow q \sqsupseteq p$

**Proposition 13.41.** If  $p \in S_n(A)$ ,  $A \subseteq B$ ,  $q \in S_n(B)$ ,  $q \supseteq p$ , then

1.  $\text{RM}(q) \leq \text{RM}(p)$
2.  $\text{RM}(q) = \text{RM}(p) \Leftrightarrow q \sqsupseteq p$

*Proof.* 1.

2. Take  $r \in S_n(\mathbb{M})$ ,  $r \sqsupseteq q$ ,  $\text{RM}(r) = \text{RM}(q)$  by Lemma ??, so  $q \sqsupseteq p \Leftrightarrow r \sqsupseteq p$

□

### 13.7 Density

Recall  $p \in S_n(A)$  is **isolated** if  $\{p\}$  is clopen,  $\{p\} = [\varphi]$ ,  $\varphi \in L(A)$ ,  $a \models p \Leftrightarrow \varphi(a)$

**Lemma 13.42.** If  $S_n(A)$  is scattered, if  $U \subseteq S_n(A)$  clopen, nonempty, then there is  $p \in S_n(A)$  isolated with  $p \in U$

*Proof.*  $U$  is not perfect, take  $p \in U$  isolated in  $U$ , then there is clopen □

**Theorem 13.43.** If  $T$  is t.t., if  $D \subseteq \mathbb{M}^n$ ,  $A$ -definable,  $D \neq \emptyset$ , then there is  $\bar{b} \in D$  s.t.  $\text{tp}(\bar{b}/A)$  is isolated in  $S_n(A)$

*Proof.*  $T$  is t.t.  $\Rightarrow R(S_n(\mathbb{M})) < \infty$ ,  $R(S_n(A)) < \infty$ ,  $S_n(A)$  scattered, there is  $p \in S_n(A)$   $p$  is isolated and  $p \in [D]$ , take  $\bar{b} \models p$ ,  $\text{tp}(\bar{b}/A) = p$  isolated,  $\bar{b} \in D$  because  $p \in [D]$  □

## A Metric Spaces

$\mathbb{R}_{\geq 0}$  denotes  $[0, +\infty] = \{x \in \mathbb{R} : x \geq 0\}$

**Definition A.1.** A **metric** on a set  $M$  is a function  $d : M \times M \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties

1.  $d(x, y) = 0 \Leftrightarrow x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z)$

**Example A.1.**  $M = \mathbb{R}^2$ ,  $d(x, y)$  = (the distance from  $x$  to  $y$ )

$$d(x_1, x_2; y_1, y_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

**Example A.2.** The **Manhattan metric** on  $\mathbb{R}^2$  is given by

$$d(x_1, x_2; y_1, y_2) = |x_1 - y_1| + |x_2 - y_2|$$

measure distances in a city grid

**Example A.3.** Let  $M$  be the set of strings. The **edit distance** from  $x$  to  $y$  is the minimum number of intersections, deletions, and substitutions to go from  $x$  to  $y$

$$\begin{aligned} d(\text{drip}, \text{rope}) &= 3 \\ \text{drip} &\mapsto \text{drop} \mapsto \text{rop} \mapsto \text{rope} \end{aligned}$$

Edit distance is a metric on  $M$

**Definition A.2.** A **metric space** is a pair  $(M, d)$  where  $M$  is a set and  $d$  is a metric space

- $(\mathbb{R}^n, d_{\text{Euclidean}})$  where  $d_{\text{Euclidean}}$  is the usual Euclidean distance
- $(\mathbb{R}^2, d_{\text{Manhattan}})$  where  $d_{\text{Manhattan}}$  is the Manhattan distance

Often we abbreviate  $(M, d)$  as  $M$ , when  $d$  is clear

Fix a metric space  $(M, d)$

**Definition A.3.** If  $p \in M$  and  $\epsilon > 0$ , then

$$B_\epsilon(p) = \{x \in M : d(x, p) < \epsilon\}$$

$$\bar{B}_\epsilon(p) = \{x \in M : d(x, p) \leq \epsilon\}$$

$B_\epsilon(p)$  and  $\bar{B}_\epsilon(p)$  are called the **open** and **closed** balls of radius  $\epsilon$  around  $p$

**Example A.4.** In  $\mathbb{R}^2$  with the Euclidean metric, the open ball of radius 2 around  $(0, 0)$  the open disk

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2^2\}$$

**Example A.5.** In  $\mathbb{R}^2$  with the Manhattan metric, the open ball of radius 1 around  $(0, 0)$  the open disk

$$\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$$

Suppose  $p \in M$  and  $X \subseteq M$

**Definition A.4.**  $p$  is an **interior point** of  $X$  if  $X$  contains an open ball of positive radius around  $p$

In particular,  $p$  must be an element of  $X$

**Example A.6.** If  $X = [-1, 1] \times [-1, 1]$ , then  $(0, 0)$  is an interior point of  $X$ , but  $(1, 0)$  and  $(0, 2)$  are not

**Definition A.5.** The **interior**  $\text{int}(X)$  is the set of interior points

Warning: There are metric spaces where the interior of  $\bar{B}_\epsilon(p)$  isn't  $B_\epsilon(p)$

**Definition A.6.** A set  $X \subseteq M$  is **open** if  $X = \text{int}(X)$ , i.e., every point of  $X$  is an interior point of  $X$

**Example A.7** (in  $\mathbb{R}$ ). The set  $(-1, 2)$  is open. The sets  $[-1, 2]$  and  $[-1, 2)$  are not; they have interior  $(-1, 2)$

Fact: the interior  $\text{int}(X)$  is the unique largest open set contained in  $X$

Let  $a_1, a_2, \dots$  be a sequence in a metric space  $(M, d)$  and let  $p$  be a point

**Definition A.7.** " $\lim_{i \rightarrow \infty} a_i = p$ " if for every  $\epsilon > 0$ , there is  $n$  s.t.

$$\{a_n, a_{n+1}, a_{n+2}, \dots\} \subseteq B_\epsilon(p)$$

**Example A.8.** Work in  $\mathbb{R}$  with the usual distance. Let  $a_n = 1/n$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$  but  $\lim_{n \rightarrow \infty} a_n \neq 1$

Fact: For any sequence  $a_1, a_2, a_3, \dots$  in  $(M, d)$ , there is at most one point  $p$  s.t.  $\lim_{i \rightarrow \infty} a_i = p$

If such a  $p$  exists, it is called the **limit**, and written  $\lim_{i \rightarrow \infty} a_i$

let  $X$  be a set and  $p$  be a point in a metric space  $(M, d)$

**Definition A.8.**  $p$  is an **accumulation point** of  $X$  if  $p = \lim_{n \rightarrow \infty} a_n$  for some sequence  $a_n$  in  $X$

Equivalently

**Definition A.9.**  $p$  is an accumulation point of  $X$  if for every  $\epsilon > 0$ , we have  $B_\epsilon(p) \cap X \neq \emptyset$

**Definition A.10.** The **closure** of  $X$ , written  $\text{cl}(X)$  or  $\bar{X}$ , is the set of accumulation points

**Definition A.11.** A set  $X \subseteq M$  is **closed** if  $X = \text{cl}(X)$

Fact: The closure  $\text{cl}(X)$  is the unique smallest closed set containing  $X$

**Example A.9.** Work in  $\mathbb{R}$  with the distance  $d(x, y) = |x - y|$

$\mathbb{Q}$  is neither closed nor open

$\mathbb{R}$  is both closed and open, so is *emptyset*

Let  $X^c$  denote the complement  $M \setminus X$

Fact:  $X$  is closed iff  $X^c$  is open

Fact:  $\text{int}(X) = \text{cl}(X^c)^c$  and  $\text{cl}(X) = \text{int}(X^c)^c$

Let  $(M, d)$  and  $(M', d)$  be metric spaces. Let  $f : M \rightarrow M'$  be a function

**Definition A.12.**  $f$  is **continuous** if

$$\lim_{n \rightarrow \infty} a_n = p \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(p)$$

for  $a_1, a_2, a_3, \dots, p \in M$

idea:  $f$  is continuous iff  $f$  preserves limits

**Example A.10.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Then  $\lim_{n \rightarrow \infty} 1/n = 0$ , but

$$\lim_{n \rightarrow \infty} f(1/n) = \lim_{n \rightarrow \infty} 1 = 1 \neq -1 = f(0)$$

**Proposition A.13.** Fix  $f : (M, d) \rightarrow (M', d)$ . The following are equivalent

1.  $f$  is continuous
2. For every open set  $U \subseteq M'$ , the preimage  $f^{-1}(U)$  is open
3. For every  $p \in M$ , for every  $\epsilon > 0$ , there is  $\delta > 0$  s.t. for every  $x \in M$ ,

$$d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$$

Fact: The functions  $\sin$ ,  $\cos$ ,  $\exp$ ,  $\sqrt[3]{\cdot}$  and polynomials are continuous

**Proposition A.14.** If  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, then  $f + g, f \cdot g, f - g, f \circ g$  are continuous

**Proposition A.15.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f(x) \neq 0$  for all  $x$ , then  $1/f(x)$  is continuous. If  $f(x) \geq 0$  for all  $x$ , then  $\sqrt{f(x)}$  is continuous

**Example A.11.** This function is continuous

$$h(x) = \exp\left(\frac{1}{1+x^2}\right) - \frac{1}{17 + \sin(\sqrt[3]{x})}$$

**Definition A.16.** A function  $f : M \rightarrow M'$  is **Lipschitz continuous** if there is  $c \in \mathbb{R}$  s.t. for any  $x, y \in M$

$$d(f(x), f(y)) \leq c \cdot d(x, y)$$

**Example A.12** (In  $\mathbb{R}$ ). The function  $f(x) = |x| + |x - 1|$  is Lipschitz continuous with  $c = 2$

**Proposition A.17.** If  $f$  is Lipschitz continuous, then  $f$  is continuous

**Example A.13.** The function  $f(x) = x^2$  is continuous but not Lipschitz continuous

**Definition A.18.** Let  $(M, d)$  be a metric space and  $S \subseteq M$  be a set. Then  $(S, d')$  is a metric space, where  $d'(x, y) = d(x, y)$  for  $x, y \in S$

- $d'$  is the restriction of  $d$  to  $S \times S$
- We say that  $(S, d')$  is a **subspace** of  $(M, d)$

Let  $(M, d), (M', d)$  be metric spaces,  $S \subseteq M$  and  $f : S \rightarrow M'$  be a function

**Definition A.19.**  $f$  is **continuous** if  $f$  is continuous as a map from the subspace  $(S, d')$  to  $(M', d)$

**Example A.14** (in  $\mathbb{R}$ ). Let  $f : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$  be given by  $f(x) = 1/x$ . Then  $f$  is continuous

**Definition A.20.** An **isometry** or **isomorphism** from  $(M, d)$  to  $(M', d')$  is a bijection  $f : M \rightarrow M'$  s.t. for any  $x, y \in M$

$$d(x, y) = d'(f(x), f(y))$$

**Example A.15** (in  $\mathbb{R}^2$ ). The map  $(x, y) \mapsto (x + 1, y - 7)$  is an isometry

So is the map  $(x, y) \mapsto (3/5x + 4/5y, -4/5x + 3/5y)$

These two metric spaces are isometric via the isometry  $x \mapsto (x, 0)$

- $\mathbb{R}$  with the usual distance
- The subspace  $\mathbb{R} \times \{0\}$  inside  $\mathbb{R}^2$  with the usual distance

**Proposition A.21.** *The isometries of  $\mathbb{R}^2$  are exactly the rotations, translations, reflections and glide reflections*

Let  $X$  be a non-empty set in a metric space

**Definition A.22.** The **diameter** of  $X$ , written  $\text{diam}(X)$ , is

$$\sup\{d(p, q) : p, q \in X\}$$

(Possibly  $\text{diam}(X) = +\infty$ )

**Example A.16.** In  $\mathbb{R}^2$  with the usual metric, the diameter of  $B_r(p)$  is  $2r$

Work in a metric space  $M$

**Definition A.23.** A **Cauchy sequence** is a sequence  $a_1, a_2, a_3, \dots$  s.t.

$$\lim_{n \rightarrow \infty} \text{diam}(\{a_n, a_{n+1}, a_{n+2}, \dots\}) = 0$$

**Proposition A.24.** *Every sequence which converges to a point in  $M$  is a Cauchy sequence*

**Proposition A.25.** *Let  $a_1, a_2, a_3, \dots$  be a sequence in a metric space  $(M, d)$ . The following are equivalent*

- *The sequence is a Cauchy sequence*

- There is some metric space  $M'$  s.t.  $M$  is a subspace of  $M'$ , and  $\lim_{n \rightarrow \infty} a_n$  converges in  $M'$

**Proposition A.26.** In  $\mathbb{R}$ , every Cauchy sequence converges

This fails in the subspace  $\mathbb{Q}$

**Definition A.27.** A metric space  $(M, d)$  is **complete** if every Cauchy sequence in  $M$  converges (to a point in  $M$ )

**Example A.17.**  $\mathbb{R}$  is complete. The subspace  $\mathbb{Q}$  and  $(-1, 1)$  are not complete

Let  $(M, d)$  be a metric space

**Definition A.28.** The **completion** of  $M$  is a new metric space  $\overline{M}$ . Objects of  $\overline{M}$  are equivalence classes of Cauchy sequences in  $M$ . Two Cauchy sequences  $(a_i)_{i \in \mathbb{N}}$  and  $(b_i)_{i \in \mathbb{N}}$  are equivalent if  $\lim_{i \rightarrow \infty} d(a_i, b_i) = 0$ . The distance in  $\overline{M}$  between two Cauchy sequences  $(a_i)_{i \in \mathbb{N}}$  and  $(b_i)_{i \in \mathbb{N}}$  is  $\lim_{i \rightarrow \infty} d(a_i, b_i)$

**Proposition A.29.** This is well-defined, and  $\overline{M}$  is complete

**Proposition A.30.** If we identify  $c \in M$  with the constant sequence  $c, c, c, c, \dots$  then  $M$  is a dense subspace of  $\overline{M}$ . If  $M$  is complete, then  $\overline{M} = M$

**Example A.18.**  $\mathbb{R}$  is the completion of  $\mathbb{Q}$  w.r.t. its usual metric

**Example A.19.** The  **$p$ -adic norm** on  $\mathbb{Q}$  is defined by

$$|0|_p = 0$$

$$|p^k a/b|_p = p^{-k} \text{ if } a, b \text{ are integers not divisible by } p$$

For example,  $|1.3|_5 = |5^{-1} \cdot 13/2|_5 = 5^1$

The  **$p$ -adic metric** on  $\mathbb{Q}$  is given by  $d(x, y) = |x - y|_p$ . This is an incomplete metric. The completion is called  $\mathbb{Q}_p$ , the set of  **$p$ -adic numbers**

**Definition A.31.**  $C([0, 1])$  is the space of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$

**Proposition A.32.** There is a metric on  $C([0, 1])$  where  $d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}$ . This makes  $C([0, 1])$  into a complete metric space.

**Definition A.33.** A metric space  $(M, d)$  is **connected** if the only clopen sets are  $M$  and  $\emptyset$ . Otherwise  $M$  is disconnected

**Definition A.34.** A set  $X \subseteq M$  is **connected** (resp. **disconnected**) if the subspace  $(X, d)$  is connected or disconnected as a metric space.

**Proposition A.35.**  *$X$  is disconnected iff there is a non-constant continuous function  $f : X \rightarrow \{0, 1\}$*

**Example A.20.** The set  $[-10, -1] \cup [1, 10]$  is disconnected, as witnessed by

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

**Example A.21.** The set  $[-10, 10] \setminus \{0\}$  is disconnected

**Example A.22.** The set  $\mathbb{Q}$  is disconnected, witnessed by

$$f(x) = \begin{cases} 0 & x < \sqrt{2} \\ 1 & x > \sqrt{2} \end{cases}$$

The set  $\mathbb{R} \setminus \mathbb{Q}$  is disconnected by a similar argument

**Proposition A.36.** *If  $X \subseteq \mathbb{R}$  is non-empty, then the following are equivalent*

- $X$  is connected
- $X$  is convex: if  $a, b \in X$ , then  $[a, b] \subseteq X$
- $X$  is an interval, a set of the form

$$\begin{aligned} & [a, b], (a, b), (a, b], [a, b) \\ & (-\infty, a), (-\infty, a], [a, +\infty), (a, +\infty), (-\infty, \infty) \end{aligned}$$

**Proposition A.37.** *Let  $f : M \rightarrow M'$  be continuous. If  $X \subseteq M$  is connected, then  $f(X) \subseteq M'$  is connected*

**Corollary A.38** (Intermediate Value Theorem). *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) < y < f(b)$ , then there is  $x \in [a, b]$  with  $f(x) = y$*

*Proof.*  $f([a, b])$  is connected, hence convex, so it contains  $y \in [f(a), f(b)]$ . Therefore there is  $x \in [a, b]$  with  $f(x) = y$  □

There are discontinuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the IVT  
classify infinite set with only 1 unary predicate



## **B Midterm review [5/7]**

- ☒ 2.24 : definable types, heirs
- ☒ 3.3 : dichotomy property,  $\lambda$ -stability, stability
- ☒ 3.10 : stability, coheirs, invariant types, Morley products
- ☒ 3.17 : Morley sequences, the order property, indiscernible sequences
- ☐ 3.24 : Ramsey's theorem, indiscernible sequence, EM-types, total indiscernibility
- ☒ 4.7 : definable closure, algebraic closure, elimination of imaginaries
- ☐ Homework
  - ☐ 1
  - ☐ 2
  - ☐ 3
  - ☒ 4
  - ☐ 5
  - ☐ 6
  - ☐ 7

## **C Problems want to ask**

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