Introduction To Algorithms

CLRS

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1 Graph Algorithms

1.1 Elementary Graph Algorithms

1.1.1 Topological sort

```
procedure TOPOLOGICAL-SORT(G)
    call DFS(G) to compute finishing times v.f for each vertex v
    as each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
end procedure
```

We can perform a topological sort in time $\Theta(V + E)$, since depth-first search takes $\Theta(V + E)$ time and it takes $O(1)$ time to insert each of the $|V|$ vertices onto the front of the linked list

Exercise 1.1.1 (22.4-3). Give an algorithm that determines whether or not a given undirected graph $G = (V, E)$ contains a simple cycle. Your algorithm should run in $O(V)$ time, independent of $|E|$

Proof. If the graph is acylic, then $|E| \leq |V| - 1$ and we can run DFS in $O(|V|)$. If there is a path going back, then at should end in $|V|$th step □

Exercise 1.1.2 (22.4-5). Another way to perform topological sorting on a directed acylic graph $G = (V, E)$ is to repeatedly find a vertex of in-degree 0, output it, and remove it and all of its outgoing edges from the graph. Explain how to implement this idea so that it runs in time $O(V + E)$. What happens to this algorithm if G has cycles?

Proof. □

1.2 Minimum Spanning Trees

1.2.1 Growing a minimum spanning tree

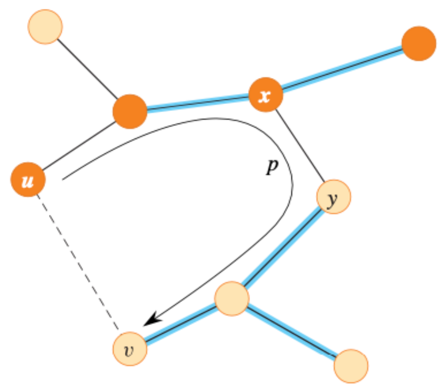
procedure GENERIC-MST(G, w)
    $A = \emptyset$
    while A does not form a spanning tree **do**
        find an edge (u, v) that is safe for A
        $A = A \cup \{(u, v)\}$
    end while
    return A
end procedure

We call an edge **safe** if it can be added safely to A while maintaining the invariant

A **cut** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V. Edge $(u, v) \in E$ **crosses** the cut $(S, V - S)$ if one of its endpoints belongs to S and the other belongs to $V - S$. A cut **respects** a set A of edges if no edge in A crosses the cut. A edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut.

Theorem 1.1. *Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$. Then edge (u, v) is safe for A.*

Proof. Let T be a minimum spanning that includes A, and assume that T does not contain the light edge (u, v),since if it does, we are done. We'll construct another minimum spanning tree T' that includes $A \cup \{(u, v)\}$ by using a cut-and-paste technique, thereby showing that (u, v) is a safe edge for A.



The edge (u, v) forms a cycle with the edges on the single path p from u to v in T. Since u and v are on opposite sides of the cut $(S, V - S)$, at least one edge in T lies on the simple path $p$and also crosses the cut. Let (x, y) be any such edge. The edge (x, y) is not in A, because the cut respects A.Since (x, y) is on the unique simple path from u to v in T, removing (x, y) breaks T into two components. Adding (u, v) reconnected them to form a new spanning tree $T' = (T - \{(x, y)\}) \cup \{(u, v)\}$.

We next show that T' is a minimum spanning tree. Since (u, v) is a light edge crossing $(S, V - S)$ and (x, y) also crosses this cut, $w(u, v) \leq w(x, y)$. Therefore $w(T') \leq w(T)$. □

Corollary 1.2. *Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_C, E_C)$ be a connected component in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A.*

1.2.2 The algorithms of Kruskal and Prim

1. Kruskal's algorithm Kruskal's algorithm finds a safe edge to add to the growing forest by finding, of all the edges that connect any trees in the forest, an edge (u, v) with the lowest weight.

```
procedure MST-Kruskal(G, w)
    A = ∅
    for each vertex v ∈ G.V do
        Make-Set(v)
    end for
    create a single list of the edges in G.E
    sort the list of edges into monotonically increasing order by weight w
    for each edge (u, v) taken from the sorted list in order do
        if Find-Set(u)≠Find-Set(v) then
            A = A ∪ {(u, v)}
            Union(u, v)
        end if
    end for
    Return A
end procedure
```

The running time of Kruskal's algorithm for a graph $G = (V, E)$ depends on the specific implementation of the disjoint-set data structure.

1.3 Single-Source Shortest Paths

```
procedure Initialize-single-source(G, s)
    for v ∈ G.V do
        v.d = ∞
        v.π = nil
    end for
    s.d = 0
end procedure
```

```
procedure Relax(u, v, w)
    if v.d≥u.d+w(u,v) then
        v.d = u.d + w(u, v)
        v.π = u
    end if
end procedure
```

1.3.1 The Bellman-Ford algorithm

```
procedure Initialize-single-source(G, s)
```

for $i = 1$ to $|G, V| - 1$ **do**
  for $(u, v) \in G.E$ **do**
    RELAX(u, v, w)
  end for
end for
for each edge $(u, v) = G.E$ **do**
  if $v.d > u.d + w(u, v)$ **then**
    return False
  end if
end for
end procedure

Lemma 1.3. *Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then after the $|V| - 1$ iterations of the **for** loops, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s*

Proof. Consider any vertex v that is reachable from s, and let $p = \langle v_0, v_1, \dots, v_k \rangle$ where $v_0 = s$ and $v_k = v$ to be any shortest path from s to v. Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \leq |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop relaxes all $|E|$ edges. Among the edges relaxed in the ith iteration, for $i = 1, \dots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$ □

Corollary 1.4. *Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then for each vertex $v \in V$ there is a path from s to v iff BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G*

Theorem 1.5 (Correctness of the Bellman-Ford algorithm)**.** *Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \to \mathbb{R}$. If G contains no negative-weight cycles that are reachable from s, then the algorithm return TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-path tree rooted at s. If G does contain a negative-weight cycle reachable from s, then the algorithm returns FALSE*

Proof. Now suppose that graph G contains a negative-weight cycle that is reachable from the source s; let this cycle be $c = \langle v_0, \dots, v_k \rangle$, where $v_0 = v_k$. Then

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Assume for the purpose of contradiction that the Bellman-Ford algorithm returns TRUE. Thus, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, \dots, k$. Summing the inequalities around cycle c gives us

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k} (v_{i-1}.d + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

But since $\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d$, we have

$$0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

□

Exercise 1.3.1.

1.3.2 Single-source shortest paths in directed acyclic graphs

By relaxing the edges of a weighted dag $G = (V, E)$ according to a topological sort of its vertices, we can compute shortest paths from a single source in $\Theta(V + E)$ time. Shortest paths are always well defined in a dag

procedure DAG-SHORTEST-PATHS(G, w, s)
    topological sort the vertices of G
    INITIALIZE-SINGLE-SOURCE(G, s)
    for each vertex u, taken in topological sorted order **do**
        for each vertex $v \in G.Adj[u]$ **do** RELAX(u, v, w)
        end for
    end for
end procedure

Exercise 1.3.2 (24.2-4). Given an efficient algorithm to count the total number of paths in a directed acylic graph. Analyze your algorithm

1.3.3 Dijkstra's algorithm

Dijkstra's algorithm solves the single-source shortest-paths problem on a weighted, directed graph $G = (V, E)$ for the case in which all edge weights are nonnegative.

```
procedure DIJKSTRA(G, w, s)
    S = ∅
    Q = G.V
    while Q ≠ ∅ do
        u =EXTRACT-MIN(Q)
        S = S ∪ {u}
        for each vertex v ∈ G.Adj[u] do RELAX(u, v, w)
        end for
    end while
end procedure
```

Theorem 1.6 (Correctness of Dijkstra's algorithm)**.** *Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function w and source s, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$*

Proof. Let u be the first vertex for which $u.d \neq \delta(s, u)$ when it is added to set S. Then $u \neq s$ and $\delta(s, u) \neq \infty$. Because there is at least one path, there is a shortest path p from s to u. Prior to adding u to S, path p connects a vertex in S, namely s to a vertex in $V - S$, namely u. Let us consider the first vertex y along p s.t. $y \in V - S$, and let $x \in S$ be y's predecessor along p. We can decompose path p into $s \overset{p_1}{\rightsquigarrow} x \to y \overset{p_2}{\rightsquigarrow} u$

We claim that $y.d = \delta(s, y)$ when u is added to S. But y should be chosen after x $\square$

Exercise 1.3.3.

1.3.4 Proofs of shortest-paths properties

Lemma 1.7 (Triangle inequality)**.** *Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$ and source vertex s. Then for all edges $u, v) \in E$ we have*

$$\delta(s, v) \leq \delta(s, u) + w(u, v)$$

Lemma 1.8 (Upper-bound property)**.** *Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let the graph be initialized by INITIALIZE-SINGLE-SOURCE(G, s). Then $v.d \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once $v.d$ achieves its lower bound $\delta(s, v)$ it never changes*

Proof. By the inductive hypothesis, $x.d \geq \delta(s, x)$ for all $x \in V$ prior to the relaxation. The only d that may change is $v.d$. If it changes, we have

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) \\ &\geq \delta(s, v) \end{aligned}$$

$\square$

Corollary 1.9 (No-path property)**.** *Suppose that in a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, no path connects a source vertex $s \in V$ to a given vertex $v \in V$. Then, after the graph is initialized by INITIALIZE-SINGLE-SOURCE(G, s), we have $v.d = \delta(s, v) = \infty$ and this equality is maintained as an invariant over any sequence of relaxation steps on the edges of G*

Proof. By the upper-bound property, we always have $\infty = \delta(s, v) \leq v.d$ $\square$

Lemma 1.10. *Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $(u, v) \in E$. Then immediately after relaxing edge (u, v) by executing RELAX(u, v, w), we have $v.d \leq u.d + w(u, v)$*

Proof. If prior to relaxing edge (u, v), we have $v.d > u.d + w(u, v)$, then $v.d = u.d + w(u, v)$ afterward. Otherwise $v.d$ doesn't change $\square$

Lemma 1.11 (Convergence property)**.** *Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, let $s \in V$ be a source vertex, and let $s \rightsquigarrow u \to v$ be a shortest path in G for some vertices $u, v \in V$. Suppose G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps that includes the call RELAX(u, v, w) is executed on the edges of G. If $u.d = \delta(s, u)$ at any time prior to the call, then $v.d = \delta(s, v)$ at all times after the call*

Proof. $\square$

Lemma 1.12 (Path-relaxation property)**.** *Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $s \in V$ be a source vertex. Consider any shortest path $p = \langle v_0, \ldots, v_k \rangle$ from $s = v_0$ to v_k. If G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps occurs that includes, in order, relaxing the edges $(v_0, v_1), \ldots, (v_{k-1}, v_k)$ then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times after wards.*

2 Dynamic Programming

2.1 Longest common subsequence

Theorem 2.1 (Optimal substructure of an LCS)**.** *Let $X = \langle x_1, \ldots, x_m \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$ be sequence, and let $Z = \langle z_1, \ldots, z_k \rangle$ be any LCS of X and Y.*

1. *If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.*
2. *If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is an LCS of X_{m-1} and Y.*
3. *If $x_m \neq y_n$ and $z_k \neq y_n$, then Z is an LCS of X and Y_{n-1}*

Proof. 1. If $z_k \neq x_m$, then we could append $x_m = y_n$ to Z

□