Homework 5

Introduction to Model Theory

Due 2021-10-28

If the homework is turned in n days late, the grade will be scaled by a factor of $(1 - n/5)$. If you have questions about the homework, please ask them in office hours or in the class WeChat group.

1. Let M be a structure and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Let $M^{\mathcal{U}}$ be the ultrapower $\prod_{i \in \mathbb{N}} M/\mathcal{U}$. Let $f : M \to M^{\mathcal{U}}$ be the map sending $a \in M$ to the class of $(a, a, a, \ldots)$, i.e., the class of the constant function $g_a(x) = a$. Show that f is an elementary embedding. *Hint:* use Łoś's theorem.

2. Say that a set $S \subseteq \mathbb{N}$ is *cofinite* if $\mathbb{N} \setminus S$ is finite. Show that there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing every cofinite set (and possibly other sets as well).

3. For $i \in \mathbb{N}$, let $\mathcal{U}_i$ be the set $\{S \subseteq \mathbb{N} : i \in S\}$. Show that $\mathcal{U}_i$ is an ultrafilter on $\mathbb{N}$. Such ultrafilters are called *principal ultrafilters*.

4. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Show that $\mathcal{U}$ contains every cofinite set.

5. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing every cofinite set. Let $\mathbb{R}^{\mathcal{U}}$ be the ultrapower of the structure $(\mathbb{R}, +, \cdot, 0, 1)$. Show that $\mathbb{R}^{\mathcal{U}}$ is not isomorphic to $\mathbb{R}$. *Hint:* let $a \in \mathbb{R}^{\mathcal{U}}$ be the class of the tuple $(0, 1, 2, 3, 4, 5, \ldots)$. Show that a satisfies all the formulas of the form
$$\exists y : 1 + 1 + \cdots + 1 + y \cdot y = x.$$