# Homework 3 solutions: coheirs and invariant types

Advanced Model Theory

Due March 17, 2022

Let DLO be the theory of dense linear orders[1], like $(\mathbb{Q}, \leq)$ or $(\mathbb{R}, \leq)$. Fix a model $M \models \mathrm{DLO}$.

1. Show that the collection of formulas $x > a$ for $a \in M$ generates a complete type $\tau_M(x) \in S_1(M)$. In other words, show that the partial type $\{(x > a) : a \in M\}$ has a unique completion.

   *Solution.* Let $\Gamma(x) = \{(x > a) : a \in M\}$. Then $\Gamma(x)$ is finitely satisfiable, so it can be extended to a complete type. It remains to show that the extension is unique. Otherwise, take two distinct completions $p_1, p_2$. Take an $|M|^+$-saturated elementary extension $N \succeq M$. Then $p_1$ and $p_2$ are realized by $b_1, b_2 \in N$. So $b_1, b_2$ both satisfy $\Gamma(x)$, but have distinct types over $M$. The fact that $b_1, b_2$ satisfy $\Gamma$ means that $b_1 > M$ and $b_2 > M$. Let $f = \mathrm{id}_M \cup \{(b_1, b_2)\}$. Then $f$ is an order-preserving bijection from $M \cup \{b_1\}$ to $M \cup \{b_2\}$. By quantifier elimination in DLO, $f$ is a partial elementary map. Therefore $p_1 = \mathrm{tp}(b_1/M) = \mathrm{tp}(b_2/M) = p_2$, a contradiction. □

The type $\tau_M$ of Exercise 1 is called the *type at* $+\infty$.

2. Show that $\tau_M$ is definable.

   *Solution.* See below. □

3. Suppose $N \succeq M$. Show that $\tau_N$ is an heir of $\tau_M$.

   *Solution for problems 2 and 3.* Suppose $N \succeq M$ and $p \in S_1(N)$ is an heir of $\tau_M$. We claim that $p = \tau_N$. If $p(x) \vdash x > a$ for every $a \in N$, then $p = \tau_N$ because $p$ contains all the formulas $x > a$ generating $\tau_N$. Otherwise, $p(x) \vdash x \leq a$ for some $a \in N$. As $p$ is an heir of $\tau_M$, there must be $a' \in M$ such that $\tau_M(x) \vdash x \leq a'$, which contradicts the definition of $\tau_M$.

   This proves the claim. Letting $N$ vary, we see that $\tau_M$ has a unique heir over any elementary extension $N \succeq M$ (because there is always at least one heir). Therefore $\tau_M$ is a definable type (problem 2), and its unique heir over $N \succeq M$ is $\tau_N$ (problem 3). □

[1] Non-empty, without endpoints.

4. Suppose $N \succeq M$ and $N$ is $|M|^+$-saturated. Show that $\tau_N$ is *not* a coheir of $\tau_M$.

   *Solution.* As $N$ is $|M|^+$-saturated, the type $\tau_M$ is realized by some $c \in N$. Then $c > M$. The formula $x > c$ is part of $\tau_N(x)$, but is not satisfied by any element of $M$. Therefore $\tau_N$ is not finitely satisfiable in $M$. $\square$

5. If $N \succeq M$, show that $\tau_M$ has a unique coheir over $N$.

   *Solution.* Let $U = \{a \in N : a > M\}$. So $U$ is the set of realizations of $\tau_M$ in $N$. Let $p \in S_1(N)$ be a coheir of $\tau_M$.

   - If $a \in U$, then the formula $x \geq a$ is not finitely satisfiable in $M$, so it cannot be in $p(x)$. Then $p(x) \vdash x < a$ by completeness of $p$.
   - If $a \notin U$, then there is $b \in M$ with $a \leq b$. Then $p(x)$ implies $\tau_M(x)$ which implies $x > b$ which implies $x > a$.

   So we see

   $$\begin{aligned} a \in U &\implies p(x) \vdash x < a \\ a \notin U &\implies p(x) \vdash x > a. \end{aligned}$$

   By quantifier elimination in DLO, this generates a complete type. $\square$

6. Give an example of models $M \preceq N$ of DLO where $\tau_N$ is a coheir of $\tau_M$ (and $M \neq N$).

   *Solution.* Take $M = (\mathbb{Q}, \leq)$ and $N = (\mathbb{R}, \leq)$. The embedding $M \preceq N$ is order-preserving, so it is an elementary embedding by quantifier elimination. Therefore $M \preceq N$ (and $M \neq N$). Let $p \in S_1(N)$ be the unique coheir of $\tau_M$. By the analysis in the solution to problem 5,

   $$\begin{aligned} a \in U &\implies p(x) \vdash x < a \\ a \notin U &\implies p(x) \vdash x > a. \end{aligned}$$

   where $U = \{a \in N : a > M\}$. By choice of $M, N$, the set $U$ is empty, and so $p(x) \vdash x > a$ for all $a$. This implies $p(x) = \tau_N(x)$. Therefore $\tau_N$ is the coheir of $\tau_M$. $\square$