Advanced Set Theory

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Definition 1.1. Fix first order language $\mathcal{L}_1$, $\mathcal{L}_2$ and theory T_1 in $\mathcal{L}_1$, theory T_2 in $\mathcal{L}_2$. We say that T_1 **interpret** T_2 iff there is a function π s.t.

1. $\pi(\forall) = \varphi_\forall(x)$ is a $\mathcal{L}_1$-formula
2. $\pi(\approx) = \varphi_\approx(x, y)$ is a $\mathcal{L}_1$-formula s.t.

   $T_1 \vdash \varphi_\approx(x, y)$defines an equivalent relation on the set defined by $\varphi_\forall(x)$

   e.g., $\forall x \forall y(\varphi_\forall(x) \to \varphi_\forall(y)) \to \varphi_\approx(x, y) \to \varphi_\approx(y, x)$
3. For any n-ary predicate P in $\mathcal{L}_2$, $\pi(P) = \varphi_P(x_1, \dots, x_n)$ and respects the equivalence relation defined by $\varphi_\forall$
4. For constant c in $\mathcal{L}_2$, $\pi(c) = \varphi_c(x)$ s.t. $T_1 \vdash \exists x(\varphi_\forall(x) \wedge \varphi_c(x)) \wedge \forall y, z(\varphi_\forall(y) \to \varphi_\forall(z) \to \varphi_c(y) \to \varphi_c(z) \to \varphi_\approx(y, z))$
5. For n-ary function symbol f in $\mathcal{L}_2$, $\pi(f) = \varphi_f(x_1, \dots, x_n, y)$ s.t. it's a function modulo $\varphi_\approx$

Then we can recursively define the translation of formulas.
For term t we define

$$\varphi_t(x) = \begin{cases} \varphi_\approx(x, t) & t \text{ is a variable(constant) other than } x \\ \exists y_1 \dots y_n((\bigwedge_{i=1}^n \varphi_\forall(y_i) \wedge \varphi_{t_i}(y_i)) \wedge \varphi_f(y_1, \dots, y_n, x)) & t = ft_1 \dots t_n \end{cases}$$

For formulas

1. $(t_1 \approx t_2)^* = \exists x_1 x_2(\varphi_\forall(x_1) \wedge \varphi_\forall(x_2) \wedge \varphi_{t_1}(x_1) \wedge \varphi_{t_2}(x_2) \wedge \varphi_\approx(x_1, x_2))$
2. $(Pt_1 \dots t_n)$
3. $(\forall x \varphi)^* = \forall x(\varphi_\forall(x) \to \varphi^*(x))$

For any $\mathcal{L}_2$-formula φ, $T_2 \vdash \varphi \Rightarrow T_1 \vdash \varphi^*$

Fact 1.2. *If T_1 interprets T_2 then T_1 is consistent implies T_2 is consistent*

Proof. If T_2 is not consistent, then T_1 is not consistent □

Definition 1.3 (Relative consistency)**.** T_2 is **relative consistent** in T_1 iff $\text{Con}(T_1) \to \text{Con}(T_2)$

Usually T_1 and T_2 are recursively enumerable.

Definition 1.4 (Consistency strength)**.** Assume T_1 can interprets Q, T_2 is r.e., we say that the **consistency strength** of T_1 is strictly stronger than T_2 iff $T_1 \vdash \text{Con}(T_2)$

Fact 1.5. *If the consistency strength of T_1 is strictly stronger than T_2 then Con$(T_1) \to$ Con(T_2)*

Proof. If T_2 is not consistent, then $\neg\text{Con}(T_2)$ is a true Σ^0_1-sentence, so $T_1 \vdash \neg\text{Con}(T_2)$ □

ZF,ZF-foundation, ZF-replacement, $V_{\omega+\omega}$ ZF-power set: ZF $\vdash V_{\omega+1} \vDash$ ZF $-Pow$ ZF-Infinite: ZF $\vdash V_\omega \vDash$ ZF $-Inf$
NBG is finitely axiomatizable
class existency axioms

1. $\in \subset V^2$ exists
2. If a class exists, then its complement exists
3. intersection of class exists
4. projection of class exists, $\forall X \exists Y \forall z(z \in Y \leftrightarrow \exists w(z, w) \in X)$

Fact 1.6. *NBG is conservative over ZF, $ZF \vdash \varphi \Leftrightarrow NBG \vdash \varphi^*$*