On f-Generic Types in Presburger Arithmetic

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Link

1.1 Definable groups and f-generics

Presburger arithmetic: the complete first-order theory of the ordered group of integers $(\mathbb{Z}, +, <, 0)$.

Let T be a complete theory, with a monster model M. We also work with a larger monster model M^* in which we can take realizations of global types over M.

Suppose $G = G(M)$ is a definbale group in T, let $S_G(M)$ denote the space of global types containing the formula defining G. Given $p \in S_G(M)$ and $g \in G$, we let gp denote the translate $\{\varphi(g^{-1}x) : \varphi(x) \in p\}$ of p.

Definition 1.1. Let $p \in S_G(M)$ be a global G-type.

1. p is **definable (over G)** if, for any formula $\varphi(\bar{x}, \bar{y})$ there is a formula $d_p[\varphi](\bar{y})$ over G s.t., for any $\bar{b} \in G$, $\varphi(\bar{x}, \bar{b}) \in p$ iff $G \models d_p[\varphi](\bar{b})$
2. p is **f-generic** if, for every formula $\phi(x) \in p$ there is a small model M_0 s.t. no translate $\phi(gx)$ of $\phi(x)$ forks over M_0
3. p is **strongly f-generic** if there is a small model M_0 s.t. no translate gp of p forks over M_0
4. p is **definably f-generic** if there is a small model M_0 s.t. every translate gp is definable over M_0

1.2 End extensions of discrete orders

Assume $\mathcal{L}$ contains a symbol $<$ and T extends the theory of linear orders. We say that T is **definably complete** if any nonempty definable subset of M, with an upper bound in M, has a least upper bound in M, and similarly for lower bounds. Note that this does not depend on the model M.

If T is definably complete, and we further assume that M is discretely ordered by $<$, then it follows that definable subsets of M contain their least upper bound and greatest lower bound. We will say T is **discretely ordered** to indicate that the ordering $<$ on M is discrete.

In a totally ordered structure, algebraic closure and definable closure coincide.

Given a tuple $\bar{a} \in (M^*)^n$, we let $M(\bar{a}) = \mathrm{dcl}(M\bar{a})$.

Definition 1.2. Given subsets $A \subseteq B$ of M^*, we say B is an **end extension** of A if, for all $b \in B \setminus A$, either $b < a$ for all $a \in A$ or $b > a$ for all $a \in A$.

Lemma 1.3. *Suppose T is discretely ordered and definably complete. Fix a non-isolated type $p \in S_n(M)$ and a realization $\bar{a}$ in M^*. If $M(\bar{a})$ is not an end extension of M then*

1. *p is not definable*

2. *p has at least two distinct coheirs to M^**

Proof. Since $M(\bar{a})$ is not an end extension of M, we may fix an M-definable function $f : (M^*)^n \to M^*$, and $m_1, m_2 \in M$ s.t. $f(\bar{a}) \notin M$ and $m_1 < f(\bar{a}) < m_2$. Define the upwards closed set

$$X = \{m \in M : p \models f(\bar{a}) < m\}$$

Then m_1 and m_2 witness that X is nonempty and not all of M. If X has a minimal element m_0 and m_0^- is the immediate predecessor of m_0 in M, then we must have $m_0^- \leq f(\bar{a}) < m_0$ and so $f(\bar{a}) = m_0^- \in M$, which is a contradiction. So X has no minimal element, and therefore cannot be M-definable. This proves part 1.

Now define

$$C = \{c \in M^* : m < c < m' \text{ for all } m \in M \setminus X \text{ and } m' \in X\}$$

Then $f(\bar{a}) \in C$, and so $C \neq \emptyset$. We define the following partial types over M^*:

$$q_1 = p \cup \{m < f(\bar{x}) < c : m \in M \setminus X, c \in C\}$$
$$q_2 = p \cup \{c < f(\bar{x}) < m : c \in C, m \in X\}$$

Note that q_1 and q_2 are distinct since $C \neq \emptyset$. If we can show that they are each finitely satisfiable in M, then they will extend to distinct coheirs of p, which proves part 2. So we show q_1 is finitely satisfiable in M.

Fix a formula $\varphi(\bar{x}) \in p$ and some $m \in M \setminus X$ (which exists since X is not all of M) . Set

$$A = \{m' \in f(\varphi(M^n)) : m < m'\}$$

Then A is an M-definable subset of M, which is nonempty since $\bar{a} \in A(M^*)$. Since A is bounded below by m, we may fix a minimal element $m_0 \in A$. By elementarity, m_0 is the minimal element of $A(M^*)$. In particular, $m_0 < f(\bar{a})$, and so $m_0 \in M \setminus X$. In particular, $m_0 < f(\bar{a})$, and so $m_0 \in M \setminus X$. By definition of A, $m_0 = f(\bar{a}')$ for some $\bar{a}' \in M^n$ s.t. $M \models \varphi(\bar{a}')$. Altogether, we have $M \models \varphi(\bar{a}')$ and $m < f(\bar{a}') < c$ for any $c \in C$. $\square$

Suppose T is discretely ordered and definably complete. If, moreover, $\mathrm{dcl}(\emptyset)$ is nonempty, then T has definable Skolem functions by picking out either the maximal element of a definable set or the least element greater than some $\emptyset$-definable constant. It follows that $M(\bar{a})$ is the unique prime model over $M\bar{a}$.

1.3 Presburger arithmetic

Let $T = \mathrm{Th}(\mathbb{Z}, +, <, 0)$. Let G denote a sufficiently saturated model of T, and let G^* denote a larger elementary extension of G, which is sufficiently saturated w.r.t. G. We treat types over G as *global types*, but use G^* as an even larger monster model in which we can realize such types.

Note that T satisfies the properties discussed above: it is discretely ordered and definably complete, with $\mathrm{dcl}(\emptyset)$ nonempty. Therefore, for $\bar{a} \in G^*$, $G(\bar{a})$ is the prime model over $G\bar{a}$. Recall that T has quantifier elimination in the expanded language $\mathcal{L}^* = \{+, <, 0, 1, (D_n)_{n<\omega}\}$ where D_n is a unary predicate interpreted as $n\mathbb{Z}$. Consequently, given $\bar{a} \in G^*$, $G(\bar{a})$ is the divisible hull of the subgroup of G^* generated by $G\bar{a}$.

Given $a \in G^*$ and $n > 0$, let $[a]_n \in \{0, 1, \dots, n-1\}$ be the unique remainder of a modulo n. Given $\bar{k} \in \mathbb{Z}^n$, we let $s_{\bar{k}}(\bar{x})$ denote the definable function $\bar{x} \mapsto k_1 x_1 + \dots + k_n x_n$

Proposition 1.4. *1. Let $G_0 \prec G$ be a small model, and fix $a, b \in G$*

(a) If $G_0 < a < b$ then there is some $c \in G$ s.t. $b < c$ and $a \equiv_{G_0} c$

(b) If $a < b < G_0$ then there is some $c \in G$ s.t. $c < a$ and $b \equiv_{G_0} c$

2. For any $p \in S_n(G)$ and $\bar{a} \models p$, if $G(\bar{a})$ is not an end extension of G then there are $h_1, h_2 \in G$ and $\bar{k} \in \mathbb{Z}^n$ s.t. $h_1 < s_{\bar{k}}(\bar{a}) < h_2$ and $s_{\bar{k}}(\bar{a}) \notin G$.

Proof. 1. By quantifier elimination and saturation of G it is enough to fix an integer $N > 0$ and find $c \in G$ s.t. $b < c$ and $[c]_n = [a]_n$ for all $0 < n \leq N$. To find such an element, simply note that $\bigcap_{0<n\leq N} nG + [a]_n$ is nonempty as it contains a and is therefore a single coset $mG + r$ for some $m, r \in \mathbb{Z}$(chinese remainder theorem). So we may choose $c = b - [b]_m + m + r$

2. By assumption, there is $b \in \mathrm{dcl}(G\bar{a}) \setminus G$ and $h_1', h_2' \in G$ s.t. $h_1' < b < h_2'$. By the description of definable closure in Presburger arithmetic, there are integers $r \in \mathbb{Z}^+$, $\bar{k} \in \mathbb{Z}^n$ and some $h_0 \in G$ s.t. $rb = s_{\bar{k}}(\bar{a}) + h_0$. Now let $h_i = rh_i' - h_0$.

□

1.4 Definable types in Presburger arithmetic

Consider the situation where G is the monster model M, and the definable group is $G^n = \mathbb{Z}^n(G)$, for a fixed $n > 0$, under coordinate addition. In particular.

Definition 1.5. A type $p \in S_n(G)$ is **algebraically independent** if for all (some) $\bar{a} \vDash p$, $a_i \notin G(\bar{a}_{\neq i})$ for all $1 \leq i \leq n$.

Lemma 1.6. *Suppose $p \in S_n(G)$ is algebraically independent and for all (some) $\bar{a} \vDash p$, $G(\bar{a})$ is an end extension of G. Then p is definable over $\emptyset$.*

Proof. Let $\mathbb{Z}^n_*$ denote $\mathbb{Z}^n \setminus \{0\}$. By quantifier elimination, it suffices to give definitions for atomic formulas of the following forms:

- $\varphi_1(\bar{x}, \bar{y}) := (s_{\bar{k}}(\bar{x}) = t(\bar{y}))$, where $\bar{k} \in \mathbb{Z}^n_*$ and $t(\bar{y})$ is a term in variables $\bar{y}$.
- $\varphi_2(\bar{x}, \bar{y}) := (s_{\bar{k}}(\bar{x}) > t(\bar{y}))$, where $\bar{k} \in \mathbb{Z}^n_*$ and $t(\bar{y})$ is a term in variables $\bar{y}$
- $\varphi_3(\bar{x}, \bar{y}) := ([s_{\bar{k}}(\bar{x}) + t(\bar{y})]_m = 0)$, where $\bar{k} \in \mathbb{Z}^n_*$, $m \in \mathbb{Z}^+$, and $t(\bar{y})$ is a term in variables $\bar{y}$.

Fix $\bar{a} \vDash p$ and fix $\bar{k} \in \mathbb{Z}^n_*$. Since p is algebraically independent, it follows that $s_{\bar{k}}(\bar{a}) \notin G$. Since $G(\bar{a})$ is an end extension of G, we may partition $\mathbb{Z}^n_* = S^+ \cup S^-$ where

$$S^+ = \{\bar{k} : s_{\bar{k}}(\bar{a}) > G\} \quad \text{and} \quad S^- = \{\bar{k} : s_{\bar{k}}(\bar{a}) < G\}$$

Note that S^+ and S^- depends only on p, and not choice of realization $\bar{a}$. Moreover, for any $\bar{k} \in \mathbb{Z}^n$ and $m > 0$, the integer $[s_{\bar{k}}(\bar{a})]_m \in \{0, \dots, m-1\}$ depends only on p. We now give the following definitions for p (note that they are formulas over $\emptyset$):

$$\begin{aligned}
d_p[\varphi_1](\bar{y}) &:= (y_1 \neq y_1) \\
d_p[\varphi_2](\bar{y}) &:= \begin{cases} y_1 = y_1 & \bar{k} \in S^+ \\ y_1 \neq y_1 & \bar{k} \in S^- \end{cases} \\
d_p[\varphi_3](\bar{y}) &:= ([t(\bar{y}) + [s_{\bar{k}}(\bar{a})]_m]_m = 0)
\end{aligned}$$

$\square$

Theorem 1.7. *Given $p \in S_n(G)$, TFAE*

1. *p is definable over G*
2. *p has a unique coheir to G^**
3. *For any (some) $\bar{a} \vDash p$, $G(\bar{a})$ is an end extension of G*

Proof. $1 \Rightarrow 2$: True for any NIP theory

$2 \Rightarrow 3$: 1.3

$3 \Rightarrow 1$: We may assume p is non-isolated. We proceed by induction on n. If $n = 1$ then p is algebraically independent since it is non-isolated, and so we apply Lemma 1.6. Assume the result for $n' < n$ and fix $p \in S_n(G)$. If p is algebraically independent then we apply Lemma 1.6. So assume, W.L.O.G., that we have $\bar{a} \models p$ with $a_n \in G(\bar{a}_{<n})$. Let $q = \text{tp}(\bar{a}_{<n}/G) \in S_{n-1}(G)$. By assumption, $G(\bar{a}_{<n}) = G(\bar{a})$ is an end extension of G, and so q is definable by induction. Fix a G-definable function $f : (G^*)^{n-1} \to G^*$ s.t. $f(\bar{a}< n) = a_n$. Fix a formula $\varphi(\bar{x}, \bar{y})$ and define

$$\psi(\bar{x}_{<n}, \bar{y}) := \varphi(\bar{x}_{<n}, f(\bar{x}_{<n}), \bar{y})$$

Let $d_q[\psi](\bar{y})$ be an $\mathcal{L}_G$-formula s.t., for any $\bar{b} \in G$, $\psi(\bar{x}_{<n}, \bar{b}) \in q$ iff $G \models d_q[\psi](\bar{b})$. Then for any $\bar{b} \in G$, we have

$$\varphi(\bar{x}, barb) \in p \Leftrightarrow G^* \models \varphi(\bar{a}, \bar{b}) \Leftrightarrow G^* \models \psi(\bar{a}_{<n}, \bar{b}) \Leftrightarrow G \models d_q[\psi](\bar{b})$$

□

1.5 f-generics in Presburger arithmetic

Proposition 1.8. *Any f-generic $p \in S_n(G)$ is algebraically independent*

Proof. Suppose p is not algebraically independent. W.L.O.G., fix $\bar{a} \models p$ with $a_n \in G(\bar{a}_{<n})$. Then there are $r, k_1, \ldots, k_{n-1} \in \mathbb{Z}$ and $b \in G$ s.t. $ra_n = b + k_1a_1 + \cdots + k_{n-1}a_{n-1}$. Consider the formula $\phi(\bar{x}; b) := rx_n = b + k_1x_1 + \cdots + k_{n-1}x_{n-1}$, and note that $\phi(\bar{x}; b) \in p$. We fix a small model $G_0 \prec G$, and find a translate of $\phi(\bar{x}; b)$ that forks over G_0.

Pick $c \in rG$ s.t. $b - c \notin G_0$, and set $g = \frac{c}{r}$. Let $\bar{g} = (0, \ldots, 0, g)$ and set $\psi(\bar{x}; b, \bar{g}) := \phi(\bar{x} + \bar{g}; b)$. By construction, we may find automorphism $\sigma_i \in \text{Aut}(G/G_0)$ s.t. $\sigma_i(b-c) \neq \sigma_j(b-c)$ for all $i \neq j$. ($b-c$ is not almost G_0-definable, therefore it has infinite orbits) Setting $b_i = \sigma_i(b)$ and $\bar{g}_i = \sigma_i(\bar{g})$, we have that $\{\psi(\bar{x}; b_i, \bar{g}_i) : i < \omega\}$ is 2-inconsistent. So $\psi(\bar{x}; b, \bar{g})$ forks over G_0 □

Theorem 1.9. *If $p \in S_n(G)$ is algebraically independent, TFAE*

1. *p is f-generic*

2. *p is strongly f-generic*

3. *p is definable f-generic*

4. *p is definable over G*

5. *p is definable over $\emptyset$*

6. *For any (some) $\bar{a} \vDash p$, $G(\bar{a})$ is an end extension of G*

Proof. $4 \Leftrightarrow 6$: 1.7

$6 \Rightarrow 5$: 1.6

$5 \Rightarrow 4$: trivial

$1 \Rightarrow 6$: Suppose $G(\bar{a})$ is not an end extension of G, and fix $\bar{k} \in \mathbb{Z}^n$ and $h_1, h_2 \in G$ s.t. $s_{\bar{k}}(\bar{a}) \notin G$ and $h_1 < s_{\bar{k}}(\bar{a}) < h_2$. Consider the formula $\phi(\bar{x}; h_1, h_2) := h_1 < s_{\bar{k}}(\bar{x}) < h_2$, and note that $\phi(\bar{x}; h_1, h_2) \in p$. We fix a small model $G_0 \prec G$, and find a translate of $\phi(\bar{x}; h_1, h_2)$ that forks over G_0. W.L.O.G., assume $b > 0$ and also $h_1 > 0$. Let k_i be a nonzero element of the tuple $\bar{k}$. By saturation of G, we may find $g \in G$ s.t. $k_i g > c$ for all $c \in G_0$. Let $\bar{g} \in G^n$ be s.t. $g_j = 0$ for all $j \neq i$ and $g_i = g$. For $t \in \{1, 2\}$, set $c_t = h_t + k_i g \in G$. Then $\phi(\bar{x} - \bar{g}; h_1, h_2)$ is equivalent to $c_1 < s_{\bar{k}}(\bar{x}) < c_2$. Since $c < c_1$ for all $c \in G_0$, by Proposition 1.4, that $\phi(\bar{x} - \bar{g}; h_1, h_2)$ forks over G_0, as desired. (By increase g, we can show that $\phi(\bar{x}; h_1, h_2; g_i))$ is 2-inconsistent or something. So p is not f-generic.

$6 \Rightarrow 3$: Suppose $G(\bar{a})$ is an end extension of G. For any $\bar{g} \in G^n$, we have $G(\bar{a}) = G(\bar{g} + \bar{a})$, and $\bar{g}p$ is still algebraically independent. Therefore, for any $\bar{g} \in G^n$, we use Lemma 1.6 to conclude that $\bar{g}p$ is definable over $\emptyset$. $\square$

2 Introduction and Preliminaries

2.1 Introduction

Marcin Petrykowski gave a nice description of f-generic types in groups $(R, +) \times (R, +)$ with $(R, <, +, \cdot)$ with $(R, <, +, \cdot)$ an o-minimal expansion of real closed field. An analogs question is: What are the f-generic types of G^n, the product of n copies of ordered additive groups $(\mathbb{Z}, +, <)$ of integers.

Let M be an elementary extension of $(\mathbb{Z}, +, <, 0)$, $\mathbb{M} \succ M$ a monster model. G denotes the additive group $(\mathbb{M}, +)$, $S_G(M)$ the space of complete types over M extending the formula '$x \in G$'. G^0 is the definable connected component of G. Namely, G^0 is the intersection of all definable subgroups of G with finite index.

Let L_n denote the space of homogeneous n-ary $\mathbb{Q}$-linear functions. For $f, g \in L_n$ and $\alpha, \beta \in \mathbb{M}^n$ s.t. $\alpha \in \text{dom}(f)$ and $\beta \in \text{dom}(g)$, by $f(\alpha) \ll_M g(\beta)$ we mean that for all $a, b \in M$ and $k, l \in \mathbb{N}^+$, $kf(\alpha) + a < lg(\beta) + b$. By $f(\alpha) \sim_M g(\beta)$ we mean that neither v$f(\alpha) \ll_M g(\beta)$ nor $g(\beta) \ll_M f(\alpha)$. Let

$f_0, \dots, f_m \in L_n$, we say $0 \ll_M f_1(\alpha) \ll_M \cdots \ll_M f_m(\alpha)$ is a maximal positive chain of α over M if for any $g \in L_n$ with $g(\alpha) > 0$, neither $f_m(\alpha) \ll_M g(\alpha)$ nor $g(\alpha) \ll_M f_1(\alpha)$

Theorem 2.1. *Let $M \succ \mathbb{Z}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in (G^n)^0$. Then there exists a finite subset $\{f_0, \dots, f_m\} \subset L_n$ s.t. $f_0(\alpha) = 0 \ll_M f_1(\alpha) \ll_M \cdots \ll_M f_m(\alpha)$ is the maximal positive chain of α over M. If α realizes an f-generic type $p \in S_{G^n}(M)$ then for every $\beta \in G^0$, $p = \mathrm{tp}(\alpha, \beta/M) \in S_{G^{n+1}}(M)$ is an f-generic type iff one of the following holds:*

1. *$f_m(\alpha) \ll_M \beta$ or $\beta \ll_M -f_m(\beta)$*
2. *there is i with $0 \le i < m$ and $g \in L_n$ s.t. $f_i(\alpha) \ll_M \epsilon(\beta - g(\alpha)) \ll_M f_{i+1}(\alpha)$ where $\epsilon = \pm 1$*
3. *there is i with $1 \le i \le m$ and $g \in L_n$ s.t. for all $h \in L_n$ with $h(\alpha) \sim_M f_i(\alpha)$ there is an irrational number $r_h \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $q_1 h(\alpha) < \beta - g(\alpha) < q_2 h(\alpha)$ for all $q_1, q_2 \in \mathbb{Q}$ with $q_1 < r_h < q_2$*

2.2 Preliminaries

Definition 2.2. 1. A definable subset $X \subseteq G$ is f**-generic** if for some/any model M over which X is defined and any $g \in G$, gX does not divide over M. Namely, for any M-indiscernible sequence $(g_i : i < \omega)$ with $g = g_0$, $\{g_i X : i < \omega\}$ is consistent.

Remark. The class of all non-weakly generic formulas forms an ideal. So any weakly generic type $p \in S_G(M)$ has a global extension $\bar{p} \in S_G(\mathbb{M})$ which is weakly generic.

T is said to be (or have) NIP if for any indiscernible sequence $(b_i : i < \omega)$ formula $\psi(x, y)$ and $a \in \mathbb{M}$, there is an eventual truth value of $\psi(a, b_i)$ as $i \to \infty$.

A type definable over A subgroup $H \le G$ has bounded index if $|G/H| < 2^{|T|+|A|}$. For groups definbale in NIP structures, the smallest type-definable subgroup G^{00} exists. Namely, the intersection of all type-definable subgroup of bounded index still has bounded index. We call G^{00} the **type-definable connected component** of G. Another model theoretic invariant is G^0, called the definably-connected component of G, which is the intersection of all definable subgroup of G of finite index.

The Keisler measure over M on X, with X a definable set over M, is a finitely additive measure on the Boolean algebra of definable subsets of X over M.

A definable group G is **definably amenable** if it admits a global (left) G-invariant probability Keisler measure

Fact 2.3. *Assuming NIP, a nip group G is definably amenable iff it admits a global type $p \in S_G(\mathbb{M})$ with bounded G-orbit.*

Fact 2.4. *For a definable amenable NIP group G, we have*

- *weakly generic definable subsets, formulas and types coincide with f-generic definable subsets, formulas, and types, respectively*
- *$p \in S_G(\mathbb{M})$ is f-generic iff it has bounded G-orbit*
- *$p \in S_G(\mathbb{M})$ is f-generic iff it is G^{00}-invariant*
- *A type-definable subgroup H fixing a global f-generic type is exactly G^{00}*

Remark. Assuming that G is definable amenable NIP group. By Remark 2.2, we see that any f-generic $p \in S_G(M)$ has an f-generic global extension $\bar{p} \in S_G(\mathbb{M})$

Assume that $T = \mathrm{Th}(\mathbb{Z}, +, \{D_n\}_{n\in\mathbb{N}^+}, <, 0)$ is the first order theory of integers in Presburger language $L_{Pres} = (+, \{D_n\}_{n\in\mathbb{N}^+}, <, 0)$ where each D_n is a unary predicate symbol for the set of elements divisible by n. $\mathbb{M}$ is the monster model of T.

T has quantifier elimination and cell decomposition.

Definition 2.5. We call a function $f : X \subseteq M^m \to M$ **linear** if there is a constant $\gamma \in M$ and integers a_i, $0 \le c_i < n_i$ for $i = 1, \dots, m$ s.t. $D_{n_i}(x_i - c_i)$ and

$$f(x) = \sum_{1\le i\le m} a_i(\frac{x_i - c_i}{n_i}) + \gamma$$

for all $x = (x_1, \dots, x_m) \in X$. We call f **piecewise linear** if there is a finite partition $\mathcal{P}$ of X s.t. all restrictions $f|_A$, $A \in \mathcal{P}$ are linear.

Note that $x \in \mathrm{dom}(f)$ iff $D_{n_i}(x_i - c_i)$ for each i.

Definition 2.6.

- A (0)-cell is a point $\{a\} \subset M$.
- An (1)-cell is a set with infinite cardinality of the form

$$\{x \in M | a \square_1 x \square_2 b, D_n(x - c)\}$$

with $a, b \in M$, integers $0 \le c < n$ and $\square_i$ either $\le$ or no condition.

- Let $i_j \in \{0,1\}$ for $j = 1, \dots, m$ and $x = (x_1, \dots, x_m)$. A $(i_1, \dots, i_m, 1)$-cell is a set A of the form

$$\{(x,t) \in M^{m+1} \mid x \in D, f(x)\square_1 t \square_2 g(x), D_n(t-c)\}$$

with $D = \pi_m(A)$ an $(i_1, \dots, i_m)$-cell. $f, g : D \to M$ linear functions, $\square_i$ either $\leq$ or no condition and integers $0 \leq c < n$ s.t. the cardinality of the fibers $A_x = \{t \in M \mid (x,t) \in A\}$ can not be bounded uniformly in $x \in D$ by an integers.

- An $(i_1, \dots, i_m, 0)$-cell is a set A of the form

$$\{(x,t) \in M^{m+1} \mid x \in D, t = g(x)\}$$

with $g : D \to M$ a linear function and $D \in M^m$ an $(i_1, \dots, i_m)$-cell

Fact 2.7 ([**?**]Cell Decomposition Theorem)**.** *Let $X \subset M^m$ and $f : X \to G$ be definable. Then there exists a finite partition $\mathcal{P}$ of X into cells, s.t. the restriction $f|_A : A \to M$ is linear for each cell $A \in \mathcal{P}$. Moreover, if X and f are S-definable, then the parts A can be taken S-definable.*

By the Cell Decomposition Theorem, we conclude that every definable subset of M^n is a finite union of cells. So every definable subset $X \subseteq M$ is a finite union of points and intervals mod some $n \in \mathbb{N}$. This implies that T has NIP.

From now on, we assume that $G = (\mathbb{M}, +)$ is the additive group of the Presburger arithmetic. Namely, G is defined by the formula "$x = x$", $G = \mathbb{M}$ as a set, and $G(M) = M$ for any $M \prec \mathbb{M}$. For any n-tuple $x = (x_1, \dots, x_n)$, by $D_m(x)$ we mean $\bigwedge_{1 \leq i \leq n} D_m(x_i)$. For any $\alpha \in \mathbb{M}$, and $A \subseteq \mathbb{M}$, by $\alpha > A$ we mean $\alpha > a$ for all $a \in \mathrm{acl}(A)$.

$\mathrm{dcl}(A) = \mathrm{acl}(A)$ since $\mathbb{M}$ is a linear order If $a \in \mathrm{acl}(A)$, then suppose $\varphi(\mathbb{M})$ is finite, then $\varphi(\mathbb{M})$ lies in some finite interval in A

Fact 2.8. *For every $n \in \mathbb{N}$*

- *G^n is definably amenable;*
- *the type-definable connected component of G^n is $\bigcap_{m \in \mathbb{N}^+} D_m(\mathbb{M}^n)$*

Proof. Let $x = (x_1, \dots, x_n)$ be an n-tuple. Let $\Pi(x)$ be the partial type of form

$$\{x_1 > \mathbb{M}\} \wedge \{x_2 > \mathrm{dcl}(\mathbb{M}, x_1)\} \wedge \dots \\ \wedge \{x_n > \mathrm{dcl}(\mathbb{M}, x_1, \dots, x_{n-1})\} \wedge \{D_m(x) : m \in \mathbb{N}^+\}$$

By the cell decomposition theorem, and induction on n, it is easy to see that Π determines a unique type $p \in S_{G^n}(\mathbb{M})$. Moreover, Π is invariant under $\bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$.

Since $D_m(\mathbb{M}^n)$ is a definable subgroup of G^n of finite index, $G^{00} \leq \bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$. Thus p is G^{00}-invariant and hence has a bounded orbit.

By Fact 2.3 G^n is definably amenable and $G^{n00} = \bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$ □

Corollary 2.9. *$G^{n0} = G^{n00}$ for all $n \in \mathbb{N}^+$.*

Remark.

- G^0 is a densely linear ordered divisible abelian group, hence is isomorphic to an ordered vector space over $\mathbb{Q}$.
- For every $n \in \mathbb{N}^+$, $(G^0)^n = (G^n)^0$

Proof. divisibility and abelian is trivial. For any $a, b \in G^0$, $\frac{a+b}{2} \in G^0$. □

Fact 2.10. *Suppose that f is an M-definable function from $X \subseteq \mathbb{M}^n$ to $Y \subseteq \mathbb{M}$. Then for any $\alpha \in (G^0)^n$ there are $q_1, \dots, q_n \in \mathbb{Q}$ and $a \in M$ s.t. $f(\alpha) = q_1\alpha_1 + \cdots + q_n\alpha_n + a$*

Proof. By Cell Decomposition we may assume f is linear. Then apply remark 2.2, $\alpha \in (G^n)^0$, therefore $\alpha_i \in G^0$ and we don't need the c_i. □

Definition 2.11. We call the function f of the form $q_1x_1 + \cdots + q_nx_n + a$ with $q_1, \dots, q_n \in \mathbb{Q}$ and $a \in M$ an **n-ary $\mathbb{Q}$-linear function** over M. If $a = 0$, we call f a **homogeneous** n-ary $\mathbb{Q}$-linear function. By $L_n(M)$ we mean the space of all n-ary $\mathbb{Q}$-linear functions over M, and L_n the space of all homogeneous n-ary $\mathbb{Q}$-linear functions.

It is easy to see that any $f \in L_n(M)$ is M-definable, and there is a natural number m s.t. $D_m(\mathbb{M}^n) \subseteq \mathrm{dom}(f)$ (common factor). In particular, $(G^0)^n \subseteq \mathrm{dom}(f)$. By Fact 2.7 and Fact 2.10 we conclude that:

Corollary 2.12. *If $\alpha = (\alpha_1, \dots, \alpha_n) \in (G^0)^n$, then for any $\phi(x_1, \dots, x_n) \in \mathrm{tp}(\alpha/M)$ there is a formula $\psi(x_1, \dots, x_n) \in \mathrm{tp}(\alpha/M)$ of the form*

$$\theta(x_1, \dots, x_{n-1}) \wedge D_m(x_n) \wedge (f_1(x_1, \dots, x_{n-1}) \square_1 x_n \square_2 f_2(x_1, \dots, x_{n-1}))$$

with $m \in \mathbb{N}$, $\theta(M)$ a cell, $f_i \in L_{n-1}(M)$, and $\square_i$ either $\leq$ or no condition, s.t. $M \models \forall x(\psi(x) \to \phi(x))$.

Remark. There are only 2 f-generic types contained in every coset of G^0. More precisely, for any model M,

$$p^+(x) = \{D_n(x) \mid n \in \mathbb{N}^+\} \cup \{x > a \mid a \in M\}$$
$$p^-(x) = \{D_n(x) \mid n \in \mathbb{N}^-\} \cup \{x < a \mid a \in M\}$$

Then every f-generic type over M is one of $G(M)$-translates of p^+ or p^-.

3 Main results

3.1 The f-generics of G^2

Let $\mathbb{M}$ be the saturated model of $\mathrm{Th}(\mathbb{Z}, +, D_n, <, 0, 1)_{n\in\mathbb{N}+}$, T the theory of Presburger Arithmetic.

Proposition 3.1. *For any $M \succ \mathbb{Z}$, the f-generic type* $\mathrm{tp}(\alpha, \beta/M) \in S_{G^2}(M)$*, with $\alpha, \beta \in G^0$, has one of the following forms:*

- $\beta > \mathrm{dcl}(M, \alpha)$ ($+\infty$-*type*)
- $\beta < \mathrm{dcl}(M, \alpha)$ ($-\infty$-*type*)
- *there is some $q \in \mathbb{Q}$ s.t. $q\alpha + m < \beta < (q + \frac{1}{n})\alpha$ for all $m \in M$ and $n \in \mathbb{N}$* (q^+-*type*)
- *there is some $q \in \mathbb{Q}$ s.t. $(q - \frac{1}{n})\alpha < \beta < q\alpha + m$ for all $m \in M$ and $n \in \mathbb{N}$* (q^--*type*)
- *there is some $r \in \mathbb{R}$ s.t. $q_1\alpha < \beta < q_2\alpha$ for all $q_1, q_2 \in \mathbb{Q}$ with $q_1 < r < q_2$* (r^0-*type*)

Proof. Let $p = \mathrm{tp}(\alpha, \beta/M)$ be a f-generic type which contained in $(G^2)^0$. By the cell decomposition, we may assume that every formula $\phi(x, y)$ in p is of the form

$$D_n(x) \wedge (a < x) \wedge D_n(y) \wedge (f_1(x) \square_1 y \square_2 f_2(x))$$

with $n \in \mathbb{N}$, $a \in M$, $f_i : D_n(M) \to M$ linear, and $\square_i$ either $\leq$ or no condition.

If every formula in p contains a cell of the form $D_n(x) \wedge D_n(y) \wedge f_1(x) \leq y$, it's then a $+\infty$-type

Similar for $-\infty$-type.

Otherwise there are linear functions $f_1(x) = q_1x + b_1$ and $f_2(x) = q_2x + b_2$, with $q_1, q_2 \in \mathbb{Q}$ and $b_1, b_2 \in M$ s.t. the cell

$$D_n(x) \wedge (a < x) \wedge D_n(y) \wedge (f_1(x) \leq y \leq f_2(x))$$

is contained in p, where both nq_1 and nq_2 are some integers. We call the above cell a (n, a, q_1, q_2)-cell.

Let

$$Q_1 = \{t \in \mathbb{Q} : \text{there is an } (n, a, t, q_2)\text{-cell which is contained in } p(x, y)\}$$
$$Q_2 = \{t \in \mathbb{Q} : \text{there is an } (n, a, q_1, t)\text{-cell which is contained in } p(x, y)\}$$

Then both Q_1 and Q_2 are nonempty.
Claim: (Q_1, Q_2) is a cut of $\mathbb{Q}$

Proof. Clearly $q_1 \leq q_2$whenever $q_1 \in Q_1$ and $q_2 \in Q_2$. Otherwise p is inconsistent.

By Remark ref:1.5, let $\bar{p} \in S_{G^2}(\mathbb{M})$ be any global f-generic type containing p. Now $\bar{p}$ is G^{2^0}-invariant. If there are $q_1 \in Q_1$ and $q_2 \in Q_2$ s.t. $q_1 = q_2$, take $g \in G^{2^0}$ s.t. $g > M$, we see that the partial type $(gp) \cup p$ is inconsistent, but $(gp) \cup p \subseteq \bar{p}$, a contradiction. So $q_1 < q_2$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$. «Problem5»

Suppose that there is $q \in \mathbb{Q}$ s.t. $q_1 < q$ for all $q_1 \in Q_1$. Then for some $n \in \mathbb{N}$ and any $a \in M$, any (n, a, q_1, q)-cell is consistent with p and hence contained in p. So $q \in Q_2$. Similarly, if $q < q_2$ for all $q_2 \in Q_2$, then $q \in Q_1$. So (Q_1, Q_2) is a cut of $\mathbb{Q}$. □

Let $r \in \mathbb{R}$ be the real number determined by the cut (Q_1, Q_2). By the G^{2^0}-invariance of $\bar{p}$, we have

- If $r = q \in Q_1$, then p is a q^+-type
- If $r = q \in Q_2$, then p is a q^--type
- If $r \notin \mathbb{Q}$, then p is a r-type

□

Definition 3.2. $\alpha \in \mathbb{M}$ is **bounded** over M if there are $a, b \in M$ s.t. $a < \alpha < b$, and unbounded otherwise

Remark. By the above argument, it is easy to conclude that for any $\alpha, \beta \in G^0$, if both tp(α/M) and tp(β/M) are f-generic. Then either tp$(\alpha, \beta/M)$ is f-generic, or there is $q_1, q_2 \in \mathbb{Q}$ s.t. $q_1\alpha + q_2\beta$ is bounded over M

Corollary 3.3. *Let* tp$(\alpha, \beta/M)$ *be a f-generic type which contained in G^{2^0}. Then* tp$(q_1\alpha, q_2\beta/M)$ *is f-generic for all $q_1, q_2 \in \mathbb{Q} \setminus \{0\}$.*

Corollary 3.4. *Let $\alpha, \beta \in G^0$. Then $\mathrm{tp}(\alpha, \beta/M)$ is an f-generic type iff for all $q_1, q_2 \in \mathbb{Q}$, $q_1\alpha + q_2\beta$ is unbounded over M whenever $q_1^2 + q_2^2 \neq 0$. In particular, both α and β are unbounded over M, and $\{\alpha, \beta\}$ is algebraic independent over M.*

Remark. By Remark 2.2, every f-generic type of G^2 over M is one of $G^2(M)$-translate of some f-generic type contained in G^{2^0}. So it suffices to study the f-generic types contained in G^{2^0}

Corollary 3.5. *Every global f-generic type of G^2 contained in G^{2^0} is $\emptyset$-definable.*

Proof. Let $\phi(x, y, z)$ be a formula. Then we may assume that ϕ is finitely many union of the following cells:

$$\begin{aligned} C_i(x,y,z) = {} & D_{n_{1i}}(z - c_{1i}) \wedge D_{n_{2i}}(x - c_{2i}) \wedge D_{n_{3i}}(y - c_{3i}) \wedge \\ & (a_{1i} \square_{1i} z \square_{2i} a_{2i}) \wedge (h_{1i}(z) \square_{3i} x \square_{4i} h_{2i}(z)) \wedge (f_{1i}(x,z) \square_{5i} y \square_{6i} f_{2i}(x,z)) \end{aligned}$$

where $i = 1, \dots, m$, $c_{1i}, c_{2i}, c_{3i}, a_{1i}, a_{2i} \in \mathbb{Z}$, $\square_{1i}, \dots, \square_{6i}$ either $\leq$ or no condition, $h_{li}(x) = b_{li}(\frac{z-c_{1i}}{n_{1i}}) + \gamma_{li}$ and $f_{li}(x,z) = d_{li}(\frac{x-c_{2i}}{n_{2i}}) + e_{li}(\frac{z-c_{1i}}{n_{1i}}) + \xi_{li}$ for $l = 1, 2$ and $b_{li}, d_{li}, \gamma_{li}, \xi_{li} \in \mathbb{Z}$

Let $p = \mathrm{tp}(\alpha, \beta/\mathbb{M})$ be a global f-generic type of G^2 contained in G^{2^0}. We assume that, for example, $\alpha > \mathbb{M}$ and p is a q^+-type for some $q \in \mathbb{Q}$. Then $\phi(x, y, b) \in p$ iff there is some $i \leq m$ s.t.

1. $\mathbb{M} \models D_{n_{2i}}(c_{2i}) \wedge D_{n_{2i}}(c_{3i})$
2. $\square_{4i}$ is no condition.

$\square$

3.2 An equivalence relation on homogeneous linear functions

Let $L_n = \{q_1x_1 + \dots + q_nx_n \mid q_1, \dots, q_n \in \mathbb{Q}\}$ is the space of all homogeneous n-ary $\mathbb{Q}$-linear functions, and $L_n(M) = \{f + a \mid f \in L_n, a \in M\}$ for any $M \prec \mathbb{M}$. For each $f \in L_n(M)$, there is $m \in \mathbb{N}^+$ s.t. f is $\emptyset$-definable from $D_m(G^n)$ to G

Definition 3.6. $M \prec \mathbb{M}$, $f, g \in L_n(M)$, $\alpha \in (G^n)^0$

- $f(\alpha) \ll_M g(\alpha)$ if
$$nf(\alpha) + a < mg(\alpha) + b$$
for all $n, m \in \mathbb{N}^+$, and $a, b \in M$
- $f \sim_{M\alpha} g$ if neither $f(\alpha) \ll_M g(\alpha)$ nor $g(\alpha) \ll_M f(\alpha)$

For any $f \in L_n(M)$, there is $g \in L_n$ s.t. $f \in [g]_{M\alpha}$.

Remark. If both $f(\alpha)$ and $g(\alpha)$ are positive (or negative), then $f(\alpha) \ll_M g(\alpha)$ iff $\mathrm{dcl}(M, f(\alpha)) < g(\alpha)$ (or $f(\alpha) < \mathrm{dcl}(M, g(\alpha))$)

Lemma 3.7. *Suppose $\alpha_1, \alpha_2 \in G^0$. Then $\{|f| \mid f \in L_2(M)\}$ has at most 5 elements*

Proof. Let $p = \mathrm{tp}(\alpha_1, \alpha_2/M)$. Suppose p is not f-generic. Then by Corollary 3.4, $q_1\alpha_1 + q_2\alpha_2$ is bounded over M for some $q_1, q_2 \in \mathbb{Q}$. If $q_1 \neq 0$, then for each $f \in L_2$ there is $g \in L_1(M)$ s.t. $f(\alpha_1, \alpha_2) \sim_M g(\alpha_2)$. Assume that $\alpha_2 > 0$. Then

$$\{|g|_{M\alpha_2} \mid g \in L_1(M)\} = \{[0]_{M\alpha_2}\}$$

if α_2 is bounded over M, and

$$\{|g|_{M\alpha_2} \mid g \in L_1(M)\} = \{[-x_2]_{M\alpha_2}, [0]_{M\alpha_2}, [x_2]_{M_{\alpha_2}}\}$$

Now suppose that p is an f-generic type. W.L.O.G., we assume that $\alpha_1 > 0$.

- Suppose that p is a q-TYPE with $q \in \mathbb{Q}$, say a q^+-type.

  Let $h(x_1, x_2) = ax_1 + bx_2 \in L_2$ and $g(x_1, x_2) = a'x_1 + b'x_2 \in L_2$ s.t. $h(\alpha_1, \alpha_2) > 0$, $g(\alpha_1, \alpha_2) > 0$ and $h(\alpha) \gg_M g(\alpha)$. Then we have

  $$a\alpha_1 + b\alpha_2 > n(a'\alpha_1 + b'\alpha_2)$$

  for all $n \in \mathbb{N}^+$.

  If $b' = 0$, we conclude that either $\alpha_2 < \mathrm{dcl}(M, \alpha_1)$ or $\alpha_2 > \mathrm{dcl}(M, \alpha_1)$, and hence p should be an ∞-TYPE, a contradiction.

  If $b' < 0$, then $a' > -b'q$ as $a'\alpha_1 + b'\alpha_2 > 0$. For any sufficiently large $n \in \mathbb{N}^+$ we have

  $$(a - na')\alpha_1 + (b - nb')\alpha_2 > 0$$

  We now assume that $b - nb' > 0$. Since $\alpha_2 < (q + \frac{1}{m})\alpha_1$ for all $m \in \mathbb{N}^+$, we have

  $$(a - na')\alpha_1 + (b - nb')(q + \frac{1}{m})\alpha > 0$$

  which implies that for all sufficiently large $m, n \in \mathbb{N}^+$,

  $$(a + b(q + \frac{1}{m})) - n(a' + b'(q + \frac{1}{m})) > 0$$

So $a' + b'(q + \frac{1}{m}) \leq 0$ for all $0 < m \in \mathbb{N}$. But $a' > -b'q$, so for sufficiently large m, $a' > -b'(q + \frac{1}{m})$. A contradiction.

We conclude that $b' > 0$. For sufficiently large n, $(b - nb') < 0$ and hence $(b - nb')q\alpha_1 > (b - nb')\alpha_2$. So we have

$$(a - na')\alpha_1 + (b - nb')q\alpha_1 > 0$$

which implies that

$$(a + bq) - n(a' + b'q) > 0$$

for all sufficiently large $n \in \mathbb{N}$, and hence $a' + b'q \leq 0$. Since $a'\alpha_1 + b'\alpha_2 > 0$, we have $a' + b'q \geq 0$. So $a' + b'q = 0$. For any $h'(x_1, x_2) = a''x_1 + b''x_2$ with $b'' > 0$ and $a'' + b''q = 0$, there is some $n \in \mathbb{N}$ s.t. $h' = nh$ or $h = nh'$. So in this case

$$\{[f]_{M\alpha} \mid f \in L_2\} = \{[-h]_{M\alpha}, [-g]_{M\alpha}, [0]_{M\alpha}, [g]_{M\alpha}, [h]_{M\alpha}\}$$

- Suppose that p is an ∞-TYPE. p is an ∞-TYPE iff $\text{tp}(\alpha_2, \alpha_1/M)$ is a 0-TYPE

- Suppose p is an r-TYPE with $r \in \mathbb{R} \setminus \mathbb{Q}$, and let $h(x_1, x_2) = ax_1 + bx_2$ and $g(x_1, x_2) = a'x_1 + b'x_2$ as above.

  If $b' < 0$, then $a' > -b'r$. Let $r < q \in \mathbb{Q}$ s.t. $a' > -b'q$. For all sufficiently large $n \in \mathbb{N}$, we have

  $$(a - na')\alpha_1 + (b - nb')q\alpha_1 > (a - na')\alpha_1 + (b - nb')\alpha_2 > 0$$

  which implies that

  $$(a + bq) - n(a' + b'q) > 0$$

  This is a contradiction as $a' + b'q > 0$.

  If $b' > 0$, then $a'\alpha_1 + b'\alpha - 2 > 0$ implies that there is some $q \in \mathbb{Q}$ s.t. $a' + b'q > 0$ and $q < r$.

  For all sufficiently large $n \in \mathbb{N}$, we have

  $$(a - na')\alpha_1 + (b - nb')q\alpha_1 > (a - na')\alpha_1 + (b - nb')\alpha_2 > 0$$

  which implies that

  $$(a + bq) - n(a' + b'q) > 0$$

  This is a contradiction since $a' + b'q > 0$. So $[h]_{M\alpha} = [g]_{M\alpha}$ whenever $h(\alpha) > 0$, $g(\alpha) > 0$, and $f, g \in L_2$, and hence

  $$\{[f]_{M\alpha} \mid f \in L_2(M)\} = \{[-h]_{M\alpha}, [0]_{M\alpha}, [h]_{M\alpha}\}$$

□

Corollary 3.8. *Suppose that* $p = \text{tp}(\alpha_1, \alpha_2/M)$ *is* f*-generic with* $\alpha_1, \alpha_2 \in G^0$*. Let* $f_1(x_1, x_2) = ax_1 + bx_2$ *and* $f_2(x_1, x_2) = a'x_1 + b'x_2$ *be lienar functions s.t.* $f_i(\alpha_1, \alpha_2) > 0$*. If* $f_1(\alpha_1, \alpha_2) \ll_M f_2(\alpha_1, \alpha_2)$ *then* p *is a* q*-TYPE with* $q \in \mathbb{Q}$ *and* $a + bq = 0$

Lemma 3.9. *For any* $\alpha = (\alpha_1, \dots, \alpha_n) \in G^{0^n}$ *and* $\beta \in G^0$*,* $\{[f]_{M\alpha\beta} \mid f \in L_{n+1}\}$ *is finite*

Proof. By IH, there are finitely many n-ary linear functions $h_1, \dots, h_k \in L_n$ s.t. $0 \ll_M h_1(\alpha) \ll_M \cdots \ll_M h_k(\alpha)$ and

$$\{[h]_{M\alpha} \mid 0 < h(\alpha) \in L_n\} = \{[h_1]_{M\alpha}, \dots, [h_k]_{M\alpha}\}$$

Claim: For each $\epsilon \in \{1, \dots, k\}$, there do not exist $u_i \in [h_\epsilon]_{M\alpha}$, $c_i \in \mathbb{Q}$, and $\gamma \in G^0$, with $i \in \mathbb{N}^+$ s.t.

$$u_1(\alpha_1, \dots, \alpha_n) + c_1\gamma \ll_M u_2(\alpha_1, \dots, \alpha_n) + c_2\gamma \ll_M \dots$$

is an infinite chain
Claim: If there are $\epsilon \in \{1, \dots, k\}$, $u_i \in [h_\epsilon]_{M\alpha}$, $c_i \in \mathbb{Q}$, and $\gamma \in G^0$ with $i \in \mathbb{N}^+$ s.t.

$$u_1(\alpha_1, \dots, \alpha_n) + c_1\gamma \ll_M u_2(\alpha_1, \dots, \alpha_n) + c_2\gamma \ll_M \dots$$

is an infinite chain. Then $\text{tp}(u_i(\alpha)/\gamma/M)$ is a q_i-TYPE with $q_i \in \mathbb{Q} \setminus \{0\}$ for all $i \in \mathbb{N}^+$

Proof. If there are $j \in \mathbb{N}^+$, $d_1, d_2 \in \mathbb{Q}$ s.t. $d_1 u_j(\alpha_1, \dots, \alpha_n) + d_2\gamma$ is bounded over M, then

$$-\frac{d_1}{d_2}u_j(\alpha_1, \dots, \alpha_n) + a < \gamma < -\frac{d_1}{d_2}u_j(\alpha_1, \dots, \alpha_n) + b$$

for some $a, b \in M$. So we conclude that

$$(u_1(\alpha_1, \dots, \alpha_n) + c_i\gamma) \sim_M (u_i(\alpha_1, \dots, \alpha_n) - c_i\frac{d_1}{d_2}u_j(\alpha_1, \dots, \alpha_n))$$

Let

$$v_i(\alpha_1, \dots, \alpha_n) = u_i(\alpha_1, \dots, \alpha_n) - c_i\frac{d_1}{d_2}u_j(\alpha_1, \dots, \alpha_n)$$

then we have an infinite chain of

$$v_1(\alpha_1, \dots, \alpha_n) \ll_M v_2(\alpha_1, \dots, \alpha_n) \ll_M \dots$$

which contradicts IH.

We now assume that $d_1 u_i(\alpha_1, \dots, \alpha_n) + d_2\gamma$ is unbounded over M for all $i \in \mathbb{N}^+$ and $d_1, d_2 \in \mathbb{Q}$ s.t. $d_1^2 + d_2^2 \neq 0$. Therefore tp$(u_i(\alpha), \gamma/M)$ is f-generic for each $i \in \mathbb{N}^+$. As $u_{i+1} \sim_{M\alpha} u_i$, there exists $q \in \mathbb{Q}$ s.t. for all $m \in \mathbb{N}^+$,

$$qu_i(\alpha) + c_{i+1}\gamma > u_{i+1}(\alpha) + c_{i+1}\gamma > m(u_i(\alpha) + c_i\gamma)$$

By Corollary ref:2.14, tp$(u_i(\alpha), \gamma/M)$ is either non-f-generic, or a $-c_i^{-1}$-TYPE. □

We now turn to Claim 1.

Proof. For a contradiction, let $1 \leq t \leq k$ be the least number s.t. there exist $u_i \in [h_t]_{M\alpha}$, $c_i \in \mathbb{Q}$ and $\gamma \in G^0$ with $i \in \mathbb{N}^+$ s.t.

$$u_1(\alpha) + c_1\gamma \ll_M u_2(\alpha) + c_2\gamma \ll_M \dots$$

□

□

4 Problem

2.2
2.2
1.2
1.3

3.2