Homework 10: independent sequences and Cantor-Bendixson rank

Advanced Model Theory

Due May 12, 2022

There are 5 problems.

1. Work in a stable theory. Suppose the sequence $(a_1, \ldots, a_n)$ (of length n) is independent over $\varnothing$. Suppose the sequence $(b_1, \ldots, b_m)$ is independent over $\varnothing$. Suppose

$$\{a_1, \ldots, a_n\} \underset{\varnothing}{\perp} \{b_1, \ldots, b_m\}.$$

   Show that the concatenated sequence $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is independent over $\varnothing$.

The remaining problems are about Cantor-Bendixson rank. For fixed $A \subseteq M$ and $n < \omega$, the Cantor-Bendixson rank of an A-definable set $D \subseteq M^n$ is an ordinal or $\pm\infty$ characterized as follows:

- $\mathrm{R}(D) \geq 0$ iff $D \neq \varnothing$.
- $\mathrm{R}(D) \geq \alpha + 1$ iff there are pairwise disjoint A-definable subsets $D_1, D_2, \ldots \subseteq D$ such that $\mathrm{R}(D_i) \geq \alpha$.
- If β is a limit ordinal, then $\mathrm{R}(D) \geq \beta$ iff $\mathrm{R}(D) \geq \alpha$ for all $\alpha < \beta$.

It may help to look ahead at Section 6 in the May 5–7 notes for examples of how to use this characterization.

2. If $T = \mathrm{Th}(\mathbb{R}, \leq)$ and $A = \mathbb{R}$ and $n = 1$, show that $\mathrm{R}(\mathbb{R}) \geq 3$, i.e., $\mathrm{R}(S_1(\mathbb{R})) \geq 3$. *Hint:* you can show that $\mathrm{R}(U) \geq \alpha$ for any open interval $U = (a, b) \subseteq \mathbb{R}$ and any ordinal α.

3. If $T = \mathrm{Th}(\mathbb{Z}, +)$ and $A = \varnothing$ and $n = 1$, show that the definable set $\mathbb{Z}$ has Cantor-Bendixson rank ∞, or equivalently, that $\mathrm{R}(S_1(\varnothing)) = \infty$. *Hint:* unlike the other problems, this problem is most easily done using a fact we proved in class, rather than the direct definition of Cantor-Bendixson rank given above.

4. If $T = ACF_0 = \mathrm{Th}(\mathbb{C}, +, \cdot)$, and $A = \mathbb{C}$ and $n = 3$, show that the definable set

$$D = \{(x, y, z) \in \mathbb{C}^3 : x + y + z = 0\}$$

has Cantor-Bendixson rank at least 2. *Hint:* this can be proven using the definition of $\mathrm{R}(D)$ given above.

5. Let $(M, \approx)$ be a set M with an equivalence relation $\approx$, such that each equivalence class is infinite and there are infinitely many equivalence classes. Show that $S_1(M)$ has Cantor-Bendixson rank at least 2. *Hint:* this can be proven using the definition of $\mathrm{R}(D)$ given above.