Homework 7

Introduction to Model Theory

Due 2021-11-25

1. Let M be a substructure of N. Let $\bar{a} \in M^n$ be a tuple and $\varphi(x_1, \ldots, x_n)$ be a quantifier-free formula. Show that $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.

2. Let M be an ω-saturated elementary extension of $(\mathbb{R}, +, \cdot, -, 0, 1, \leq)$. Suppose that $a \in M$. Show that there is $b \in M$ such that $b > a^n$ for all positive integers n.

3. Let K be a field, and $x, y \in K$ be elements. Show that $xy = 0 \iff (x = 0 \lor y = 0)$. *Hint:* Both directions are non-trivial. The $\Leftarrow$ direction only uses the ring axioms, while the $\Rightarrow$ direction needs some of the field axioms. The main difficulty is proving the "zero law" $0x = x$. The proof of Theorem 13 in the notes contains a useful technique to prove this. *Note:* This exercise is part of Fact 2 in the notes, which you may not cite.

4. Let a, b be positive integers. Let g be the greatest common divisor of a and b (the biggest positive integer which divides a and b). Show that $g = ax + by$ for some $x, y \in \mathbb{Z}$. *Hint:* show that the set $\{ax + by : x, y \in \mathbb{Z}\}$ is an ideal in $\mathbb{Z}$. Figure out which ideal it is. You will probably want to use Theorem 15 in the notes.

5. If $x, y, n \in \mathbb{Z}$ and $n > 0$, then $x \equiv y \pmod{n}$ means $x - y \in n\mathbb{Z}$. Show that $\equiv$ is an equivalence relation.

6. Suppose that $x \equiv x' \pmod{n}$ and $y \equiv y' \pmod{n}$. Show that $xy \equiv x'y' \pmod{n}$.

7. Suppose that p is a prime and $x \not\equiv 0 \pmod{p}$. Show that there is y such that $xy \equiv 1 \pmod{p}$. *Hint:* apply Exercise 4 to x and p.

Remark. Exercise 7 is basically how one constructs fields of characteristic $p > 0$.