Homework solutions: stable theories

Introduction to Model Theory

Due March 10, 2022

1. $(\mathbb{C}, +, \cdot)$ is an algebraically closed field. Show that the algebraic set $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$ is reducible, i.e., not a variety.

   Solution. Let $V = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$. Let $i = \sqrt{-1}$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, and so $x^2 + y^2 = 0$ iff $x + iy = 0$ or $x - iy = 0$. Therefore

   $$V = \{(x, y) \in \mathbb{C}^2 : x + iy = 0\} \cup \{(x, y) \in \mathbb{C}^2 : x - iy = 0\} = W_+ \cup W_-,$$

   where $W_\pm = \{(x, y) \in \mathbb{C}^2 : x \pm iy = 0\}$. The two sets W_+ and W_- are algebraic sets, and each is strictly smaller than V: $(1, -i) \in V \setminus W_+$ and $(1, i) \in V \setminus W_-$. Therefore V is reducible. $\square$

2. Consider the theory of dense linear orders (DLO). Let $\varphi(x, y)$ be the formula $x < y$. One can show that $\varphi(x, y)$ has the dichotomy property. Show by giving an example that D_3 is consistent.

   Solution. We need to find $a_{000}, a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111}, b, b_0, b_1, b_{00}, b_{01}, b_{10}, b_{11}$ such that

   $$\begin{gathered}
   a_{000} < b,\ a_{000} < b_0,\ a_{000} < b_{00} \\
   a_{001} < b,\ a_{001} < b_0,\ a_{001} \not< b_{00} \\
   a_{010} < b,\ a_{010} \not< b_0,\ a_{010} < b_{01} \\
   a_{011} < b,\ a_{011} \not< b_0,\ a_{011} \not< b_{01} \\
   a_{100} \not< b,\ a_{100} < b_1,\ a_{100} < b_{10} \\
   a_{101} \not< b,\ a_{101} < b_1,\ a_{101} \not< b_{10} \\
   a_{110} \not< b,\ a_{110} \not< b_1,\ a_{110} < b_{11} \\
   a_{111} \not< b,\ a_{111} \not< b_1,\ a_{111} \not< b_{11}.
   \end{gathered}$$

   Take $a_{000} = 0$, $a_{001} = 2$, $a_{010} = 4$, $a_{011} = 6$, $a_{100} = 8$, $a_{101} = 10$, $a_{110} = 12$, $a_{111} = 14$, $b = 7$, $b_0 = 3$, $b_1 = 11$, $b_{00} = 1$, $b_{01} = 5$, $b_{10} = 9$, and $b_{11} = 13$. $\square$

3. In the structure $M = (\mathbb{R}, +, \cdot, 0, 1, \leq)$, let $\varphi(\bar{x}; \bar{y})$ be the formula $x_1y_1 + x_2y_2 = 1$. Thus $\varphi(\mathbb{R}^2; \bar{b})$ is a line for most $\bar{b} \in \mathbb{R}^2$. It turns out that the formula φ does not have the dichotomy property. Find the largest n such that D_n is consistent.

   Solution. The largest is $n = 2$. First, we show D_2 is consistent. Take $\bar{b} = (1, 1)$ and $\bar{b}_0 = \bar{b}_1 = (1, -1)$. Then $\varphi(M^2, \bar{b}) = \ell$ and $\varphi(M^2, \bar{b}_0) = \varphi(M^2, \bar{b}_1) = \ell'$ for two lines ℓ, ℓ' that are not parallel. Take $\bar{a}_{00}$ to be the point in $\ell \cap \ell'$. Then $\varphi(\bar{a}_{00}, \bar{b})$ and $\varphi(\bar{a}_{00}, \bar{b}_0)$ hold. Take $\bar{a}_{01}$ to be a point on ℓ but not ℓ'. Then $\varphi(\bar{a}_{01}, \bar{b}) \wedge \neg\varphi(\bar{a}_{01}, \bar{b}_0)$ holds. Take $\bar{a}_{10}$ to be a point on ℓ' but not ℓ. Then $\neg\varphi(\bar{a}_{10}, \bar{b}) \wedge \varphi(\bar{a}_{10}, \bar{b}_1)$ holds. Finally, take $\bar{a}_{11}$ to be a point on neither line. Then $\neg\varphi(\bar{a}_{11}, \bar{b}) \wedge \neg\varphi(\bar{a}_{11}, \bar{b}_1)$ holds.

   Next, we show that D_3 is inconsistent. Otherwise, take $\bar{a}_\sigma, \bar{b}_\tau$ for $\sigma \in 2^3$ and $\tau \in 2^{<3}$ as in the definition of D_3. For each $\tau \in 2^{<3}$, we have $M \models \varphi(\bar{a}_{\tau 0}, \bar{b}_\tau)$, and so $\varphi(M^2, \bar{b}_\tau)$ is non-empty (it contains $\bar{a}_{\tau 0}$). Therefore $\varphi(M^2, \bar{b}_\tau)$ is a line ℓ_τ, rather than the empty set. We have

$$\begin{aligned} M &\models \varphi(\bar{a}_{010}, \bar{b}) \wedge \neg\varphi(\bar{a}_{010}, \bar{b}_0) \\ M &\models \varphi(\bar{a}_{001}, \bar{b}) \wedge \varphi(\bar{a}_{001}, \bar{b}_0) \wedge \neg\varphi(\bar{a}_{001}, \bar{b}_{00}) \\ M &\models \varphi(\bar{a}_{000}, \bar{b}) \wedge \varphi(\bar{a}_{000}, \bar{b}_0) \wedge \varphi(\bar{a}_{000}, \bar{b}_{00}) \end{aligned}$$

   The first equation says $\bar{a}_{010} \in \ell$ but $\bar{a}_{010} \notin \ell_0$. Therefore the two lines ℓ and ℓ_0 are not equal. The next two equations imply that $\bar{a}_{001}$ and $\bar{a}_{000}$ are both on ℓ and both on ℓ_0. Therefore, $\{\bar{a}_{001}, \bar{a}_{000}\} \subseteq \ell \cap \ell_0$. As the two lines ℓ, ℓ_0 are not equal, their intersection is a point, and then $\bar{a}_{001} = \bar{a}_{000}$. But the second and third equations show that $\bar{a}_{001} \neq \bar{a}_{000}$, because $\bar{a}_{000} \in \ell_{00}$ and $\bar{a}_{001} \notin \ell_{00}$. So we have a contradiction, and D_3 is inconsistent. □

4. Let T be the complete theory of the structure $(\mathbb{Z}, +, -, 0)$. Show that T is not $\aleph_0$-stable.

   Solution. We will construct $2^{\aleph_0}$-many 1-types over $(\mathbb{Z}, +, -, 0)$, contradicting $\aleph_0$-stability.

   For any finite string $s \in 2^k$, let $D_s \subseteq \mathbb{Z}$ be the set of integers congruent to "s" modulo 10^k. For example, D_{0110} is the set of integers congruent to 110 modulo 10000. That is

$$D_{0110} = \{x \in \mathbb{Z} : x - 110 \text{ is a multiple of } 10000\}.$$

   Then each set D_s is definable (with parameters). For example, D_{0110} is defined by the formula $\varphi(x) \equiv$

$$\exists y \; \underbrace{y + \cdots + y}_{10000 \text{ times}} = x - 110.$$

   Let φ_s be an $L(\mathbb{Z})$-formula defining D_s. Also, note that each D_s is non-empty. For example, the number "s" is in D_s. Also note that if s, s' have the same length and $s \neq s'$, then $D_s \cap D_{s'} = \varnothing$. For example, no number is congruent to both 110 and 101 modulo 1000.

For any $s \in 2^\omega$, let $\Sigma_s(x) = \{\varphi_{rev(s \restriction n)}(x) : n < \omega\}$, where $s \restriction n$ is the restriction of s to the first n bits, and $rev(w)$ denotes the reverse of a finite string w. For example, if $s = 1001011001101001011\ldots$, then

$$\Sigma_s(x) = \{\varphi(x), \varphi_1(x), \varphi_{01}(x), \varphi_{001}(x), \varphi_{1001}(x), \varphi_{01001}(x), \varphi_{101001}(x), \ldots\}$$

This type is finitely satisfiable. Indeed, $\{\varphi_{rev(s \restriction n)}(x) : n < N\}$ is realized by any element of $D_{rev(s \restriction N)}$. For example, any element of D_{101001} (such as 101001) realizes

$$\{\varphi(x), \varphi_1(x), \varphi_{01}(x), \varphi_{001}(x), \varphi_{1001}(x), \varphi_{01001}(x), \varphi_{101001}(x)\}.$$

For each $s \in 2^\omega$, take a complete type $p_s(x) \in S_1(M)$ extending $\Sigma_s(x)$. We claim that $s \mapsto p_s$ is injective. Suppose $s \neq s'$. Then there is n such that $s \restriction n \neq s' \restriction n$. The type p_s extends Σ_s so it contains the formula $\varphi_{rev(s \restriction n)}$. Similarly, $\varphi_{rev(s' \restriction n)}$ is in $p_{s'}$. These two formulas are contradictory, because $D_{rev(s \restriction n)} \cap D_{rev(s' \restriction n)} = \varnothing$. Therefore $p_s \neq p_{s'}$, or else p_s would be contradictory (not finitely satisfiable in $\mathbb{Z}$).

So $s \mapsto p_s$ is an injection, and the number of 1-types over $\mathbb{Z}$ is at least $|2^\omega| = 2^{\aleph_0} > \aleph_0$. $\square$

Alternate solution. We will construct $2^{\aleph_0}$-many 1-types over $\mathbb{Z}$, contradicting $\aleph_0$-stability.

Let $p_0 < p_1 < p_2 < p_3 < \cdots$ be an enumeration of the prime numbers $(2, 3, 5, 7, 11, 13, \ldots)$. Let $\varphi_i(x)$ be the formula saying that x is a multiple of p_i. That is, $\varphi_i(x)$ is the formula

$$\exists y \ (\underbrace{y + \cdots + y}_{p_i \text{ times}} = x).$$

For $S \subseteq \omega$, let Σ_S be the partial type

$$\{\varphi_i(x) : i \in S\} \cup \{\neg\varphi_i(x) : i \notin S\}.$$

We claim that $\Sigma_S(x)$ is finitely satisfiable in $\mathbb{Z}$. Indeed, if S_0, S_1 are two disjoint finite subsets of ω, then

$$\{\varphi_i(x) : i \in S_0\} \cup \{\neg\varphi_i(x) : i \in S_1\}$$

is consistent, because it is realized by $n = \prod_{i \in S_0} p_i$.

So each $\Sigma_S(x)$ is finitely satisfiable. Take a completion $p_S(x) \in S_1(M)$. Then $S \mapsto p_S$ is injective: if $S \neq S'$, without loss of generality there is $i \in S \setminus S'$, and then realizations of p_S satisfy $\varphi_i(x)$ but realizations of $p_{S'}$ satisfy $\neg\varphi_i(x)$.

Therefore $S \mapsto p_S$ is injective, which implies $|S_1(\mathbb{Z})| \geq 2^{\aleph_0}$. $\square$