Homework 11

Introduction to Model Theory

Due 2021-12-16

1. Show that the field $\mathbb{C}$ is strongly $|\mathbb{C}|$-homogeneous. You may assume the statements on last week's homework assignment.

2. Show that the field $\mathbb{R}$ is strongly κ-homogeneous for any cardinal κ. *Hint:* $\mathbb{R}$ doesn't even have automorphisms.

3. Let $S = \{0, 1\} \times \mathbb{Z}$ and let $\leq$ be the lexicographic order on S:

$$
\begin{aligned}
(0, x) &< (1, y) \\
(0, x) \leq (0, y) &\iff x \leq y \\
(1, x) \leq (1, y) &\iff x \leq y.
\end{aligned}
$$

   Show that $(S, \leq)$ is not strongly ω-homogeneous, but some expansion of $(S, \leq)$ is strongly ω-homogeneous.

In the following problems, let T be the complete theory of the structure $(\mathbb{R}, +, \cdot, 0)$. So a model of T is a structure $(M, +, \cdot, 0)$ that is elementarily equivalent to $\mathbb{R}$.

4. Suppose $M \models T$. Show that there is at most one linear order $\leq$ on M such that the following hold:

   - If $x \leq y$, then $x + z \leq y + z$.
   - If $x \leq y$, and $0 \leq z$, then $xz \leq yz$.

   (The idea is that these properties are the implicit definition of $\leq$.)

   Hint: if you get stuck; do problem 5 first and use the solution as a guide.

5. Write down an explicit definition of $\leq$ in $(\mathbb{R}, +, \cdot)$, that is, a formula $\phi(x, y)$ in the language $\{+, \cdot, 0\}$ such that $\mathbb{R} \models \phi(a, b)$ iff $a \leq b$.