What to review for the final exam

Advanced Model Theory

May 16, 2022

Note that the exam will be open book. You are free to use the textbook and class notes, including electronic copies. You are free to use this document.

These topics might be on the exam:

1. Definable types, invariant types, heirs, coheirs.
2. Stable theories, the order property.
3. Indiscernible sequences.
4. Morley products, Morley sequences.
5. The fundamental order, bounds
6. Algebraic and definable closure. Almost A-definability.
7. Non-forking extensions, the forking calculus, independent sequences.

These topics **won't** be on the exam:

1. Ultrapowers, strong heirs
2. ACF.
3. Strongly minimal theories.
4. The dichotomy property.
5. Stability spectra, $\lambda_0(T)$, $\kappa_n(T)$, superstability.
6. Cantor-Bendixson rank, Morley rank, totally transcendental theories.
7. Ehrenfeucht-Mostowski models, uncountably categorical theories, Morley's theorem.
8. Elimination of imaginaries, multi-sorted logic
9. Material from the "Extra Notes" page on eLearning.

10. Material from the textbook that wasn't discussed in class.

These sections of the notes are relevant:

1. `02-24-notes.pdf`: Sections 1–3 (but not 4–5).
2. `03-03-notes.pdf`: Nothing.
3. `03-10-notes.pdf`: All sections.
4. `03-17-notes.pdf`: All sections.
5. `03-24-notes.pdf`: Sections 1–4 (but not 5–6).
6. `03-31-notes.pdf`: Sections 1–3 (but not 4–6).
7. `04-07-notes.pdf`: Sections 1–2 (but not 3–7).
8. `04-21-28-notes.pdf`: Sections 3–7, 10, 13 (but not 1–2, 8–9, 12, 14).
9. `05-05-07-notes.pdf`: Nothing.
10. `05-12-notes.pdf`: Nothing.

The rest of this document is an incomplete synopsis of the important theorems (but not definitions) from these sections. This hopefully gives a picture of which topics you should review.

1 Synopsis

1. If $p \in S_n(M)$ and $N \succeq M$, then there is at least one $q \in S_n(N)$ that is an heir of p.
2. An heir of an heir is an heir.
3. If $p \in S_n(M)$ is a definable type and $N \succeq M$, then there is a unique heir of p over N. The heir is another definable type $q \in S_n(N)$, and q has the same definition schema as p.
4. A type $p \in S_n(M)$ is definable iff it has a unique heir over every $N \succeq M$.
5. A definable type over M is the same thing as an M-definable type over M, and M-definable types over M correspond bijectively with M-definable types over the monster model $\mathbb{M}$.
6. A global type $p \in S_n(\mathbb{M})$ is A-definable iff it is definable and A-invariant.
7. A global definable type $p \in S_n(\mathbb{M})$ is A-definable for some small A.

8. If a global type $p \in S_n(\mathbb{M})$ is finitely satisfiable in a small set A, then p is A-invariant.

9. If $q \in S_n(N)$ extends $p \in S_n(M)$, then q is a coheir of p iff q is finitely satisfiable in M (this is the definition of "coheir").

10. If $p \in S_n(M)$ and M is a small model in $\mathbb{M}$, then there is a global type $q \in S_n(\mathbb{M})$ extending p such that q is a coheir of p, and therefore q is M-invariant.

11. If $p \in S_n(M)$ and $N \succeq M$, then there is at least one $q \in S_n(N)$ that is a coheir of p.

12. Let λ be an infinite cardinal. The complete theory T is λ-stable iff the following equivalent conditions hold:

    - For any set A in a model of T with $|A| \le \lambda$, we have $|S_1(A)| \le \lambda$.
    - For any $n < \omega$, for any set A in a model of T with $|A| \le \lambda$, we have $|S_n(A)| \le \lambda$.

    When $\lambda \ge |L|$, these are also equivalent to the following things:

    - For any model $M \models T$ with $|M| \le \lambda$, we have $|S_1(M)| \le \lambda$.
    - For any $n < \omega$, for any model $M \models T$ with $|M| \le \lambda$, we have $|S_n(M)| \le \lambda$.

13. The complete theory T is stable iff the following equivalent conditions hold:

    - No formula has the order property.
    - Every type over every model is definable.
    - T is λ-stable for at least one infinite cardinal λ.

14. In a stable theory, q is an heir of p if and only if q is a coheir of p.

15. In a stable theory, if $p \in S_n(M)$ and $N \succeq M$, then there is a unique (co)heir of p over M.

16. In a stable theory, if p_1, p_2, p_3 are types over models with $p_1 \subseteq p_2 \subseteq p_3$, then $p_3 \sqsupseteq p_1 \iff (p_3 \sqsupseteq p_2$ and $p_2 \sqsupseteq p_1)$ where $\sqsupseteq$ means "is the heir of."

17. In any theory, if p, q are global A-invariant types, then there is another global A-invariant type $p \otimes q$ characterized by the fact that for any small set B containing A, a tuple $(\bar{c}, \bar{d})$ realizes $(p \otimes q) \upharpoonright B$ if and only if $\bar{c}$ realizes $p \upharpoonright B$ and $\bar{d}$ realizes $q \upharpoonright B\bar{c}$.

18. In a stable theory, any two global invariant types p and q "commute" in the sense that
    $$(\bar{c}, \bar{d}) \models p \otimes q \iff (\bar{d}, \bar{c}) \models q \otimes p.$$

19. If p is an A-invariant global type, a Morley sequence of p over A is a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots$ where
    $$\bar{a}_i \models p \upharpoonright (A\bar{a}_1\bar{a}_2 \cdots \bar{a}_{i-1}).$$
    So roughly speaking, a Morley sequence is a realization of $(p \otimes p \otimes p \otimes \cdots) \upharpoonright A$. A Morley sequence of p over A is always A-indiscernible.

20. Any subsequence of an A-indisceribile sequence is A-indiscernible.

21. Given an A-indiscernible sequence $(\bar{a}_i : i \in I)$ and an extension $(J, \leq)$ of the linear order $(I, \leq)$, we can extend the sequence to an A-indiscernible sequence $(\bar{a}_i : i \in J)$ by choosing $\bar{a}_i$ appropriately for $i \in J \setminus I$.

22. If $(\bar{a}_i : i \in I)$ and $(\bar{b}_i : i \in J)$ are two A-indiscernible sequences with the same EM type over A, and $f : I \to J$ is an isomorphism of linear orders, then there is a partial elementary map sending $\bar{a}_i$ to $\bar{b}_{f(i)}$. For example, if f is an automorphism of $(I, \leq)$, then there is a partial elementary map sending $\bar{a}_i$ to $\bar{a}_{f(i)}$.

23. If $(\bar{a}_i : i \in I_0)$ is an infinite sequence, and A is a small set of parameters, and I is a small linear order, then there is an A-indiscernible sequence $(\bar{b}_i : i \in I)$ extracted from the original sequence, in the sense that $\mathrm{tp}^{EM}(\bar{a}/A) \subseteq \mathrm{tp}^{EM}(\bar{b}/A)$. So, if $\varphi(\bar{x}_1, \ldots, \bar{x}_n)$ was some $L(A)$-formula such that

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n})$$

for every increasing sequence $i_1 < \cdots < i_n$ in I_0, then the analogous thing will hold for the new sequence.

24. In a stable theory, any A-indisceribile sequence is totally A-indiscernible. Therefore, any permutation of an A-indiscernible sequence is A-indiscernible.

25. Assuming A is small, $b \in \mathrm{dcl}(A)$ iff b is fixed by every automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.

26. $\mathrm{dcl}(-)$ is a closure operation, meaning that

$$A \subseteq \mathrm{dcl}(A)$$
$$A \subseteq B \implies \mathrm{dcl}(A) \subseteq \mathrm{dcl}(B)$$
$$\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A).$$

A set A is definably closed if $A = \mathrm{dcl}(A)$. The definable closure of A is the smallest definably closed set containing A.

27. Assuming A is small, $b \in \mathrm{acl}(A)$ iff $\{\sigma(b) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite. That set is also the set of realizations of $\mathrm{tp}(b/A)$.

28. $\mathrm{acl}(-)$ is a closure operation, meaning that

$$A \subseteq \mathrm{acl}(A)$$
$$A \subseteq B \implies \mathrm{acl}(A) \subseteq \mathrm{acl}(B)$$
$$\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A).$$

A set A is algebraically closed if $A = \mathrm{acl}(A)$. The algebraic closure of A is the smallest algebraically closed set containing A.

29. Models are algebraically closed. Moreover, $\mathrm{acl}(A)$ is the intersection of the models containing A. (A is supposed to be small, and a "model" is an elementary substructure of the monster.)

30. A definable set D is "almost A-definable" if the following equivalent conditions hold:

    - $\{\sigma(D) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite. (A is supposed to be small, by the way.)
    - $\{\sigma(D) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.
    - D is M-definable for every model M containing A. (Here, "model" means "elementary substructure of the monster" as always.)
    - D is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable, where $\mathrm{acl}^{\mathrm{eq}}(-)$ is algebraic closure in $\mathbb{M}^{\mathrm{eq}}$.

31. A definable type is "almost A-definable" if it's $\mathrm{acl}^{\mathrm{eq}}(A)$-definable, or equivalently, all the sets
$$\{\bar{b} \in \mathbb{M} : \varphi(\bar{x};\bar{b}) \in p(\bar{x})\}$$
are almost A-definable. Another equivalent condition here is that $\{\sigma(p) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.

For the rest of this section, assume the theory is stable.

32. There's a partial order called the "fundamental order for n-types". To any n-type over a model $p \in S_n(M)$ we can associate a class $[p]$ in the fundamental order for n-types. If q is an extension of p (and both types are over models), then $[q] \leq [p]$, and equality holds iff q is a (co)heir of p.

33. We can also associate to any type over any set $p \in S_n(A)$ an element of the fundamental order $\mathrm{bd}(p)$ called the "bound" of p. When A is a model, $\mathrm{bd}(p)$ is just the class $[p]$ in the fundamental order. When q is an extension of p, then $\mathrm{bd}(q) \leq \mathrm{bd}(p)$, with equivalence iff q is a non-forking extension of p.

34. When q, p are types over models, q is a non-forking extension of p iff q is a (co)heir of p.

35. If $p \in S_n(A)$ and $B \supseteq A$, then there is at least one non-forking extension of p over B.

36. If p_1, p_2, p_3 are types over sets and $p_1 \subseteq p_2 \subseteq p_3$, then p_3 is a non-forking extension of p_1 iff p_3 is a non-forking extension of p_2 and p_2 is a non-forking extension of p_1.

37. If $p \in S_n(A)$ and q is a global type extending p, then q is a non-forking extension of p iff q is almost A-definable.

38. If $p \in S_n(A)$, then the global non-forking extensions of A correspond bijectively to the extensions of p to a type over $\mathrm{acl}^{\mathrm{eq}}(A)$. Any two global non-forking extensions of A are connected by an automorphism over A.

39. An $L(\mathbb{M})$-formula $\varphi(\bar{x})$ forks over A if $\varphi(\bar{x})$ is not contained in any almost A-definable type.

40. If $q \in S_n(B)$ extends $p \in S_n(A)$, then q is a forking extension of p if and only if some formula $\varphi(\bar{x})$ in $q(\bar{x})$ forks over A.

41. If $q \in S_n(B)$ extends $p \in S_n(A)$, then q is a non-forking extension of p if and only if some global type extending q is almost A-definable.

42. $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \bar{b}$ means that $\mathrm{tp}(\bar{a}/C\bar{b})$ is a non-forking extension of $\mathrm{tp}(\bar{a}/C)$.

43. The relation $\mathop{\smile\!\!\!\!\!\mid}$ is symmetric: $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}$.

44. If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \bar{b} \iff \sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\mid}_{\sigma(C)} \sigma(\bar{b})$, since the definition of $\mathop{\smile\!\!\!\!\!\mid}$ respects automorphisms.

45. $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \bar{b}$ is equivalent to $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(C)} \bar{b}$ is equivalent to $\mathrm{acl}^{\mathrm{eq}}(\bar{a}C) \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(C)} \mathrm{acl}^{\mathrm{eq}}(\bar{b}C)$.

46. A sequence $(\bar{a}_i : i \in I)$ is independent over C if

$$\bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_C \{\bar{a}_j : j < i\}$$

for each $i \in I$.

47. A sequence $(\bar{a}_i : i \in I)$ is independent over C if

$$\bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_C \{\bar{a}_j : j \neq i\}$$

for each $i \in I$.

48. A type $p \in S_n(A)$ is *stationary* if p has a unique non-forking extension over any $B \supseteq A$.

49. $p \in S_n(A)$ is stationary iff p has a unique non-forking extension over the monster model.

50. $p \in S_n(A)$ is stationary iff p has a unique extension to $\mathrm{acl}^{\mathrm{eq}}(A)$.

51. If $p \in S_n(A)$ and $q \in S_n(\mathrm{acl}^{\mathrm{eq}}(A))$ is some extension, then q is always a non-forking extension of p.

52. $p \in S_n(A)$ is stationary iff p has an A-definable global extension. In that case, that A-definable global extension is the unique non-forking global extension.

53. Any type over a model is stationary. Any type over $\mathrm{acl}^{\mathrm{eq}}(A)$ is stationary.

54. Suppose $\mathrm{tp}(\bar{a}/C)$, $\mathrm{tp}(\bar{b}/C)$ are stationary. Let p, q be the global C-invariant extensions of these two types. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \bar{b}$ if and only if $(\bar{a}, \bar{b})$ realize $p \otimes q \upharpoonright C$.