Universal algebra

Introduction to Model Theory
(Third hour)

December 23, 2021

Warning

I'll mostly use terminology and conventions from model theory. Universal algebraists probably use slightly different terminology and conventions.

Section 1

Varieties

Identities

Let L be a language with only function symbols, no relation symbols.

- Constant symbols are 0-ary function symbols.

Definition

An *identity* is a sentence of the form

$$\forall \bar{x}\ (t(\bar{x}) = s(\bar{x})).$$

For example,

$$\forall x, y, z : x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

is an identity.

Varieties

Definition

A *variety* is a class of L-structures defined by a set T of identities.

Warnings:

- T can be infinite.
- Outside of universal algebra, "variety" means something completely different.

Groups

The class of groups is a variety.

- The language consists of a binary function symbol $\cdot$, a unary function symbol $(-)^{-1}$, and a nullary (0-ary) function symbol 1.
- The defining identities are

$$\forall x, y, z : x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x : x \cdot 1 = x$$
$$\forall x : 1 \cdot x = x$$
$$\forall x : x \cdot (x^{-1}) = 1$$
$$\forall x : (x^{-1}) \cdot x = 1.$$

Rings

The class of (commutative unital) rings is a variety.

- The language consists of two binary operations $+, \cdot$, a unary operation $-$, and two nullary operations $0, 1$.
- The defining identities are as follows, omitting the universal quantifiers:

$$
\begin{gathered}
x + (y + z) = (x + y) + z \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
x + y = y + x \qquad x \cdot y = y \cdot x \\
x \cdot 1 = x \qquad x + 0 = x \qquad x + (-x) = 0 \\
x \cdot (y + z) = (x \cdot y) + (x \cdot z).
\end{gathered}
$$

Lattices

The class of bounded lattices is a variety.

- The language consists of two binary operations $\wedge, \vee$, and two nullary operations $\bot, \top$
- The defining identities are

$$\begin{aligned} x \wedge y &= y \wedge x & x \vee y &= y \vee x \\ x \wedge (y \wedge z) &= (x \wedge y) \wedge z & x \vee (y \vee z) &= (x \vee y) \vee z \\ x \wedge x &= x & x \vee x &= x \\ x \wedge (x \vee y) &= x & x \vee (x \wedge y) &= x. \end{aligned}$$

More varieties...

- Abelian groups
- Distributive lattices
- Modular lattices
- Bounded lattices (in the language $\wedge, \vee, \bot, \top$)
- Boolean algebras (in the language $\wedge, \vee, \bot, \top, \neg$)
- Heyting algebras (in the language $\wedge, \vee, \bot, \top, \rightarrow$).

More varieties. . .

- Sets (take the empty language and no identities).
- Magmas: a set with a binary operation.
- Semigroups: a set with an associative operation.
- Monoids: $(M, \cdot, 1)$ where $(M, \cdot)$ is a semigroup and $x \cdot 1 = 1 \cdot x = x$.
- Bands: idempotent semigroups ($x \cdot x = x$).

Section 2

Homomorphisms

Homomorphisms

Let M, N be L-structures.

Definition

A *homomorphism* from M to N is a function $h : M \to N$ such that for any k-ary function symbol f, we have

$$h(f(x_1, \ldots, x_n)) = f(h(x_1), h(x_2), \ldots, h(x_n)).$$

This recovers our earlier definitions of "homomorphism" in ring theory, group theory, and lattice theory.

Fact

If $h : M \to N$ is a homomorphism and $t(x_1, \ldots, x_n)$ is an L-term, then

$$h(t(x_1, \ldots, x_n)) = t(h(x_1), \ldots, h(x_n)).$$

The category of models of T

Let $\mathcal{V}$ be a variety. Then there is a category where

- Objects are structures in $\mathcal{V}$.
- Morphisms are homomorphisms.

Fact

This category has all small limits and colimits.

Isomorphisms, automorphism, endomorphisms

Definition

An *isomorphism* from M to N is a bijective homomorphism.

Definition

An *automorphism* of M is an isomorphism from M to M.

The set of automorphisms is denoted $\mathrm{Aut}(M)$, and is a group.

Definition

An *endomorphism* of M is a homomorphism from M to M.

The set of endomorphisms of M is denoted $\mathrm{End}(M)$, and is a monoid.

Isomorphism

Definition

$M \cong N$ if there is an isomorphism from M to N.

Fact

$\cong$ *is an equivalence relation:*

- $M \cong M$
- $M \cong N \implies N \cong M.$
- $M \cong N \cong N' \implies M \cong N'.$

Homomorphisms and identities

Let φ be an identity $t(\bar{x}) = s(\bar{x})$.

Fact

If $f : M \to N$ is a surjective homomorphism and $M \models \varphi$, then $N \models \varphi$.

Fact

If $f : M \to N$ is an injective homomorphism and $N \models \varphi$, then $M \models \varphi$.

Corollary

If $M \cong N$, then M and N satisfy the same identities.

Homomorphisms and identities

Let φ be an identity $t(\bar{x}) = s(\bar{x})$.

Fact

If $f : M \to N$ is a surjective homomorphism and $M \models \varphi$, then $N \models \varphi$.

Proof.

Given $\bar{a} \in N^n$, write $\bar{a}$ as $f(\bar{b})$ for some $\bar{b} \in M$, and then

$$t(\bar{a}) = t(f(\bar{b})) = \underbrace{f(t(\bar{b})) = f(s(\bar{b}))}_{\text{since } M \models \varphi} = s(f(\bar{b})) = s(\bar{a}).$$

□

Homomorphisms and identities

Let φ be an identity $t(\bar{x}) = s(\bar{x})$.

Fact

If $f : M \to N$ is an injective homomorphism and $N \models \varphi$, then $M \models \varphi$.

Proof.

If $\bar{a} \in M^n$, then

$$f(t(\bar{a})) = \underbrace{t(f(\bar{a})) = s(f(\bar{a}))}_{\text{since } N \models \varphi} = f(s(\bar{a})).$$

Since f is injective, $f(t(\bar{a})) = f(s(\bar{a})) \implies t(\bar{a}) = s(\bar{a})$. □

Section 3

Closure properties of varieties

Substructures

Let M be an L-structure.

Definition

A *substructure* (or *subalgebra*) is a subset $A \subseteq M$ closed under every operation f in L:

$$a_1, \ldots, a_n \in A \implies f(a_1, \ldots, a_n) \in A.$$

Then we can regard A as an L-structure by restricting each operation to A.

The lattice of substructures

Fix an L-structure M.

Fact

- *Any intersection of substructures of M is a substructure of M.*
- *The set of substructures is a complete lattice.*
- *There is a finitary closure operation*

$$X \to \langle X \rangle_M$$

  sending $X \subseteq M$ to the smallest substructure containing X
- *The corresponding closed sets are the substructures of M.*
- *$\langle X \rangle_M$ is exactly*

$$\{t(a_1, \ldots, a_n) : t(x_1, \ldots, x_n) \text{ is an } L\text{-term and } a_1, \ldots, a_n \in X\}.$$

Varieties and substructures

Theorem

If A is a substructure of M, and M satisfies an identity φ, then $A \models \varphi$.

Proof.

The inclusion $A \to M$ is an injective homomorphism. $\square$

Theorem

Let $\mathcal{V}$ be a variety. If $M \in \mathcal{V}$ and A is a substructure of M, then $A \in \mathcal{V}$.

Binary products

Let M, N be L-structures.

Definition

The *product* $M \times N$ is the L-structure on the set $M \times N$ with an n-ary function symbol $f \in L$ interpreted as follows:

$$f((x_1, y_1), \ldots, (x_n, y_n)) = (f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)).$$

Binary products

Example

The product of two rings R, S is the ring $R \times S$ where

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 \cdot y_2)$$
$$-(x, y) = (-x, -y)$$
$$1 = (1, 1)$$
$$0 = (0, 0).$$

Idea

All operations are done coordinate-by-coordinate.

Binary products

Fact

If $\mathcal{V}$ is a variety and $M, N \in \mathcal{V}$, then $M \times N \in \mathcal{V}$.

Corollary

Fields are not a variety in any sensible way, since there are fields of size 2 and 3, but no field of size 6.

Infinite products

Let I be a set.

Definition

An I-tuple is a function with domain I.
$(x_i : i \in I)$ denotes the function $i \mapsto x_i$.

If I is $\{1, 2, \ldots, n\}$, then we can identify n-tuples and I-tuples:

$$(x_1, \ldots, x_n) \text{ corresponds to } (x_i : i \in I)$$

Infinite products

Let $\{S_i\}_{i \in I}$ be a family of *sets*.

Definition

The *product* $\prod_{i \in I} S_i$ is the set of I-tuples $(x_i : i \in I)$ where $x_i \in S_i$.

- Thinking of tuples as functions, $\prod_{i \in I} S_i$ is the set of functions $f : I \to \bigcup_{i \in I} S_i$ such that $\forall i \in I : f(i) \in S_i$.
- When $I = \{1, 2, \ldots, n\}$, we can identify $\prod_{i \in I} S_i$ with $S_1 \times \cdots \times S_n$.

Infinite products

Let $\{M_i\}_{i\in I}$ be a family of L-structures.

Definition

The *product structure* $\prod_{i\in I} M_i$ is the L-structure on $\prod_{i\in I} M_i$ in which each k-ary function symbol operates coordinate-by-coordinate:

$$f(\bar{x}^1, \ldots, \bar{x}^k) = (f(x_i^1, x_i^2, \ldots, x_i^k) : i \in I).$$

When $I = \{1, 2, \ldots, n\}$, we have

$$\prod_{i\in I} M_i \cong M_1 \times \cdots \times M_n.$$

Infinite products

Fact

Let $\mathcal{V}$ be a variety. If $M_i \in \mathcal{V}$ for all $i \in I$, then the product $\prod_{i \in I} M_i$ is in $\mathcal{V}$.

This is analogous to the theorem in model theory:

Fact

Let $\mathcal{K}$ be an elementary class (the set of models of some theory). If $M_i \in \mathcal{K}$ for all $i \in I$, and $\mathcal{U}$ is an ultrafilter on I, then the ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$ is in $\mathcal{K}$.

Jointly injective families

Definition

Let M be an L-structure. Let $\{f_i : M \to N_i\}_{i \in I}$ be a family of homomorphisms. Say that the family is "jointly injective" if for any $x \neq y$ in M, there is $i \in I$ such that $f_i(x) \neq f_i(y)$.

Fact

Let $\{f_i : M \to N_i\}_{i \in I}$ be a jointly injective family of homomorphisms.

- *If every N_i satisfies some identity φ, then $M \models \varphi$.*
- *If every N_i is in some variety $\mathcal{V}$, then M is in $\mathcal{V}$.*

The proof is similar to the case of a single injection.

Jointly injective families

Fact

Let M_i be an L-structure for each $i \in I$.

- *For each $j \in I$, there is a projection homomorphism*

$$\pi_j : \prod_{i \in I} M_i \to M_j$$
$$x \mapsto x_j.$$

- *The family of projections π_j is jointly injective.*
- *Therefore, if each $M_j \in \mathcal{V}$, then the product is in $\mathcal{V}$.*

Nullary (0-ary) products

- If $I = \varnothing$, then $\prod_{i \in I} M_i$ makes sense, and is the L-structure with one element.
- Because varieties are closed under products, this L-structure is in any variety.
  - It's the terminal object in the category.

Congruences

Let M be an L-structure.

Definition

A *congruence* on M is a substructure $E \subseteq M \times M$ such that E is an equivalence relation on M.

Equivalently, a congruence on M is an equivalence relation $\sim$ on M such that for any k-ary function symbol f

$$x_1 \sim y_1 \text{ and } x_2 \sim y_2 \text{ and } \cdots \text{ and } x_k \sim y_k \implies f(x_1, \ldots, x_k) \sim f(y_1, \ldots, y_k)$$

Example

A congruence on a group is an equivalence relation $\sim$ such that

$$x \sim x', y \sim y' \implies xy \sim x'y'$$
$$x \sim y \implies x^{-1} \sim y^{-1}$$
$$1 \sim 1$$

Something silly

What if we replace "equivalence relation" with "function" in the definition of congruence. . . ?

Fact

Let M, N be L-structures. A homomorphism from M to N is the same thing as a substructure $f \subseteq M \times N$ that is a function from M to N.

The lattice of congruences

Fix an L-structure M.

Fact

- *Any intersection of congruences on M is a congruence on M.*
- *The set of congruences is a complete lattice.*
- *There is a finitary closure operation on M^2 sending a relation $R \subseteq M^2$ to the smallest congruence containing R.*
- *The closed sets are exactly the congruences on M.*

The closure operation is not as nice as the one for substructures.

Congruences in rings

Fact

Let I be an ideal in a ring R. There is a congruence $\equiv_I$ on R defined by

$$x \equiv_I y \iff x - y \in I.$$

$x \equiv_I y$ is usually written "$x \equiv y \pmod{I}$".

Fact

All congruences on R arise this way. The lattice of congruences is isomorphic to the lattice of ideals.

Moral of the story: ideals are congruences in ring theory.

Congruences in groups

Fact

Let N be a normal subgroup in a group G. There is a congruence $\equiv_N$ on G defined by

$$x \equiv_N y \iff xy^{-1} \in N.$$

Fact

All congruences on G arise this way. The lattice of congruences is isomorphic to the lattice of normal subgroups.

Moral of the story: normal subgroups are congruences in group theory.

Quotients

Definition

Let $\sim$ be a congruence on an L-structure M. Let $[a]$ denote the $\sim$-class of $a \in M$. The *quotient structure* $M/(\sim)$ is the L-structure on $\{[a] : a \in M\}$, where a k-ary function symbol is defined by

$$f([x_1], \ldots, [x_k]) = [f(x_1, \ldots, x_k)].$$

Fact

This is well-defined: if $[x_i] = [y_i]$ *for* $1 \leq i \leq k$, *then* $[f(x_1, \ldots, x_k)] = [f(y_1, \ldots, y_k)]$.

Remark

This generalizes R/I in ring theory and G/N in group theory.

Quotients and varieties

Fact

Let $\mathcal{V}$ be a variety. If $M \in \mathcal{V}$ and $\sim$ is a congruence on M, then $M/(\sim) \in \mathcal{V}$.

Proof.

The map $x \mapsto [x]$ is a surjective homomorphism $M \to M/(\sim)$. $\square$

Kernels and images

Let $f : M \to N$ be a homomorphism.

- The *image* is $f(M) = \{f(x) : x \in M\}$.
- The *kernel* is the relation $x \sim y \iff f(x) = f(y)$.

Fact (Fundamental theorem of homomorphisms)

The image of f is a substructure of N. The kernel of f is a congruence on M. There is an isomorphism

$$M/\ker(f) \cong \operatorname{im}(f).$$

Closure properties of varieties

Fact

Let $\mathcal{V}$ be a variety. Then $\mathcal{V}$ is closed under isomorphisms, substructures, quotients, and products.

- *If $M \in \mathcal{V}$ and $M \cong N$, then $N \in \mathcal{V}$.*
- *If $M \in \mathcal{V}$ and N is a substructure of M, then $N \in \mathcal{V}$.*
- *If $M \in \mathcal{V}$ and $\sim$ is a congruence on M, then $M/(\sim) \in \mathcal{V}$.*
- *If $M_i \in \mathcal{V}$ for all $i \in I$, then $\prod_{i \in I} M_i \in \mathcal{V}$.*

Birkhoff's theorem

Theorem (Birkhoff)

Let $\mathcal{V}$ be a class of structures closed under isomorphisms, substructures, quotients, and products. Then $\mathcal{V}$ is a variety.

Analogues in model theory:

Theorem (Keisler-Shelah, hard)

$\mathcal{K}$ is an elementary class iff the following hold:

- *If $M \in \mathcal{K}$ and $M \cong N$, then $N \in \mathcal{K}$.*
- *$\mathcal{K}$ is closed under ultraproducts.*
- *If M is a structure and some ultrapower $M^I/\mathcal{U} \in \mathcal{K}$, then $M \in \mathcal{K}$.*

Theorem (easier)

$\mathcal{K}$ is an elementary class defined by universally quantified formulas iff $\mathcal{K}$ is closed under isomorphism, substructure, and ultraproducts.

The variety generated by...

Fact

Let $\mathcal{K}$ be a class of structures. There is a unique smallest variety $\mathcal{V}$ containing $\mathcal{K}$.

- Construction 1: $\mathcal{V}$ is the class of structures generated from $\mathcal{K}$ by isomorphisms, products, quotients, and substructures.
- Construction 2: let T be the set of identities which are true in $\mathcal{K}$. Then $\mathcal{V}$ is the variety defined by T.

The variety generated by...

1. The variety of commutative rings is generated by X, where X is any of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
2. The variety of boolean algebras is generated by powersets.
3. The variety of boolean algebras is generated by the two-element boolean algebra.
    - *This is why the axioms of boolean algebras generate the set of tautologies in propositional logic.*

Section 4

Equational logic

Example

Theorem

If R is a ring, then R satisfies the identity $x \cdot 0 = 0$.

Proof.

$x \cdot 0 = (x \cdot 0) + 0 = (x \cdot 0) + (x + (-x)) = ((x \cdot 0) + x) + (-x) = ((x \cdot 0) + (x \cdot 1)) + (-x) = x \cdot (0 + 1) + (-x) = x \cdot (1 + 0) + (-x) = x \cdot 1 + (-x) = x + (-x) = 0.$ □

Idea

This sort of "one-line proof" is always possible.

"One-step equivalence"

Fix a set of identities T in a functional language L.

Definition

Two L-terms t, s are "one-step equivalent", written $t \leftrightarrow s$, if t and s are directly related by an identity in T.

More precisely, $\leftrightarrow$ is generated by the following:

- If $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ is one of the identities in T, then for any terms $a_1, \ldots, a_n$,

$$t(a_1, \ldots, a_n) \leftrightarrow s(a_1, \ldots, a_n) \leftrightarrow t(a_1, \ldots, a_n).$$

- If $t \leftrightarrow s$ and f is a k-ary function symbol, then

$$f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n) \leftrightarrow f(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n)$$

for any $1 \leq i \leq k$ and any terms $a_1, \ldots, a_n$.

"One-step equivalence"

Example

If T is the set of ring axioms, then

$$(x \cdot 0) + (x \cdot 1) \leftrightarrow x \cdot (0 + 1)$$

by substituting the terms $x, 0, 1$ into the identity $x \cdot (y + z) = x \cdot y + x \cdot z$. Then

$$((x \cdot 0) + (x \cdot 1)) + (-x) \leftrightarrow x \cdot (0 + 1) + (-x)$$

by applying $(\ldots) + (-x)$ to both sides.

Equivalence

Continue to fix a set of identities T.

- Let $\equiv$ be the equivalence relation generated by $\leftrightarrow$.
- In other words, $t \equiv s$ if there exist $n \geq 0$ and terms $a_0, \ldots, a_n$ such that $t = a_0 \leftrightarrow a_1 \leftrightarrow \cdots \leftrightarrow a_n = s$.
- (n can be 0, so $\equiv$ is reflexive)
- ($\leftrightarrow$ is already symmetric.)

Example

When T is the theory of rings, we saw above that $x \cdot 0 \equiv 0$.

Equivalence

Fact (Soundness)

If $t(\bar{x}) \equiv s(\bar{x})$, then the identity $t(\bar{x}) = s(\bar{x})$ holds in any model of T.

Fact (Completeness)

If an identity $t(\bar{x}) = s(\bar{x})$ holds in all models of T, then $t(\bar{x}) \equiv s(\bar{x})$.

- Soundness is "easy"—you check that the rules defining $\leftrightarrow$ and $\equiv$ are sound.
- Completeness is less obvious.

The free L-structure

Let $\mathcal{X} = \{x_1, x_2, \dots\}$ be a set of variables.

Definition

The *free L-structure* on $\mathcal{X}$ is the set of L-terms in the variables $\mathcal{X}$.

Example

When L is the language of rings, the free L-structure on x, y, z contains things like

$$0,\ x + (-x),\ (y + z) \cdot (1 + (-1))$$

none of which are equal.

The free model of T

Let $\mathcal{X} = \{x_1, x_2, \ldots\}$ be a set of variables.

Fact

- *$\equiv$ is a congruence on the free L-structure.*
- *The quotient is a model of T.*
- *$\equiv$ is the smallest congruence with this property.*

This isn't hard; it's practically by definition of $\leftrightarrow$.

Definition

The *free model of T* is the quotient of the free L-structure by $\equiv$.

The free model of T

Definition

The free model of T is the quotient of the free L-structure by $\equiv$.

Suppose T is the theory of rings. The free model of T (i.e., the free ring) on x, y, z contains things like

$$x \cdot x,\ (x \cdot x + y) + (-y),\ 1 \cdot 0 + z$$

The first two are equal, but the third is distinct.

Fact

The free ring on $x_1, \ldots, x_n$ is isomorphic to the ring of polynomials $\mathbb{Z}[x_1, \ldots, x_n]$.

The free model of T

Fact

If $t(x_1, \ldots, x_n) \not\approx s(x_1, \ldots, x_n)$, then $t(X_1, \ldots, X_n) \neq s(X_1, \ldots, X_n)$ in the free model of T on $\{X_1, \ldots, X_n\}$, so the identity $t = s$ does not hold in the free model.

The free group

Fact

- *The free group on three variables* $\{x, y, z\}$ *is the set of strings in the alphabet* $\{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$ *containing no substrings of the form* xx^{-1}, $x^{-1}x$, yy^{-1}, $y^{-1}y$, zz^{-1}, $z^{-1}z$.
- *The identity element is the empty string.*
- *The group operation* $\sigma \cdot \tau$ *consists of concatenating* σ *and* τ, *then removing any instances of the six forbidden strings.*

Example

xyz times $z^{-1}y^{-1}z$ is xz, because $xyzz^{-1}y^{-1}z \mapsto xyy^{-1}z \mapsto xz$.

Why Birkhoff's theorem is true

Suppose $\mathcal{K}$ is closed under products, quotients, substructures, and isomorphisms.

Remark

If $f : M \to N$ is an injective homomorphism and $N \in \mathcal{K}$, then $M \in \mathcal{K}$.

Proof.

$M \cong \mathrm{im}(f) \subseteq N$. □

Remark

If $f : M \to N$ is a surjective homomorphism and $M \in \mathcal{K}$, then $N \in \mathcal{K}$.

Proof.

$N \cong M/(\ker(f))$. □

Why Birkhoff's theorem is true

Suppose $\mathcal{K}$ is closed under products, quotients, substructures, and isomorphisms.

Remark

Suppose $\{f_i : M \to N_i\}$ is a jointly injective family and the N_i are in $\mathcal{K}$. Then M is in $\mathcal{K}$.

Proof.

There is an injective homomorphism $M \to \prod_{i \in I} N_i$. □

Why Birkhoff's theorem is true

Suppose $\mathcal{K}$ is closed under products, quotients, substructures, and isomorphisms.

- Let T be the set of identities which hold in $\mathcal{K}$, and $\mathcal{V}$ be the corresponding variety.
- $\mathcal{K} \subseteq \mathcal{V}$; we want the reverse inclusion.

Fact (Not that hard)

If F is a free model of T, then there is a jointly injective family of maps $\{f_i : F \to M_i\}$ with $M_i \in \mathcal{K}$. Consequently, $F \in \mathcal{K}$.

- If $M \in \mathcal{V}$, let $F(M)$ be the free model of T on the set M.
- The identity map $M \to M$ extends to a surjective homomorphism $F(M) \to M$.
- By the Fact, $F(M) \in \mathcal{K}$, so $M \in \mathcal{K}$.
- Thus $\mathcal{V} \subseteq \mathcal{K}$, and $\mathcal{K}$ is defined by T.

Section 5

More category theory

The universal property of free models

Fact

Let R be a commutative ring. For any $a, b, c \in R$, there is a unique ring homomorphism

$$f : \mathbb{Z}[x, y, z] \to R$$

such that $f(x) = a$, $f(y) = b$, and $f(z) = c$. Specifically f is the map sending $P(x, y, z) \in \mathbb{Z}[x, y, z]$ to $P(a, b, c)$.

More generally,

Fact

Let T be a set of identities. Let $\mathcal{X}$ be a set of variables. Let $F(\mathcal{X})$ be the free model of T on the variables $\mathcal{X}$. For any $M \models T$ and $f : \mathcal{X} \to M$, there is a unique homomorphism $g : F(\mathcal{X}) \to M$ extending f.

The universal property of free models

Let $F : \mathbf{Set} \to \mathbf{Mod}_T$ be the functor from sets to models of T sending X to the free model on X. Then we have an adjunction

$$\mathrm{Hom}_{\mathbf{Mod}_T}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Set}}(X, M).$$

So F is left adjoint to the forgetful functor $\mathbf{Mod}_T \to \mathbf{Set}$.

Presentations

Fix a variety $\mathcal{V}$ defined by an equational theory T.

Definition

If $x_1, x_2, \ldots$ are variables and $t_1, s_1, t_2, s_2, \ldots$ are terms in the x_i, then

$$\langle x_1, x_2, \ldots, \mid t_1 = s_1, t_2 = s_2, \ldots \rangle$$

denotes the quotient of the free model of T on the variables $x_1, x_2, \ldots$ by the congruence generated by $\{(t_1, s_1), (t_2, s_2), \ldots\}$.

Example

In the variety of groups, $\langle a, b \mid a^2 = 1,\ b^2 = 1 \rangle$ denotes the free group on the variables a, b, modulo the congruence $\sim$ generated by $a^2 \sim 1$ and $b^2 \sim 1$.

- Equivalently, this is the free group on the variables a, b, modulo the smallest normal subgroup containing a^2 and b^2.

The universal property of presented models

$$\mathrm{Hom}_{\mathbf{Grp}}(\langle a, b \mid a^2 = b^2 = 1\rangle, G) \cong \{(a, b) \in G^2 : a^2 = b^2 = 1\}.$$

More generally,

$$\begin{gathered}\mathrm{Hom}_{\mathbf{Mod}_T}(\langle x_1, x_2, \ldots \mid t_1(\bar{x}) = s_1(\bar{x}), \ldots\rangle, M) \\ \cong \{(a_1, a_2, \ldots) : a_1, a_2, \ldots \in M,\ t_1(\bar{a}) = s_1(\bar{a}),\ t_2(\bar{a}) = s_2(\bar{a}), \ldots\}.\end{gathered}$$

Presented and finitely presented models

Definition

A model $M \models T$ is *presented* if it has a presentation.
M is *finitely presented* if it has a presentation with finitely many generators and relations.

Fact

Every model is presented; list all the elements of M as generators and all the true equations as relations.

Coproducts

If

$$G = \langle x_1, x_2, \ldots \mid t_1(\bar{x}) = s_1(\bar{x}), \ldots \rangle$$
$$H = \langle y_1, y_2, \ldots \mid u_1(\bar{y}) = v_1(\bar{y}), \ldots \rangle$$

then the category-theoretic coproduct $G + H$ exists and has the presentation

$$\langle x_1, x_2, \ldots, y_1, y_2, \ldots \mid t_1(\bar{x}) = s_1(\bar{x}), \ldots, u_1(\bar{y}) = v_1(\bar{y}), \ldots \rangle$$

This is usually called the *free amalgam* of G and H.

Example

The group $\mathbb{Z}/2\mathbb{Z}$ has the presentation $\langle a \mid a^2 = 1\rangle$. The free amalgam with itself is $\langle a, b \mid a^2 = 1,\ b^2 = 1\rangle$.

More generally, all colimits can be described through presentations.