Cantor-Bendixson Rank

Thursday, May 5, 2022 9:27 AM

References: §13.2 : Cantor rank
§15.2 : independence
§17.3 : Morley rank

Слава Україні
Героям слава

§1 Independent sequences

__Def 1.1__ $(A_i : i \in I)$ is __independent__ over B
if $A_i \underset{B}{\perp\!\!\!\perp} A_{\neq i}$ where $A_{\neq i} = \{A_j : j \in I, j \neq i\}$.

__Ex 1.2__ (A_1, A_2) is independent$/B$ $\iff$ $A_1 \underset{B}{\perp\!\!\!\perp} A_2$ ~~and~~ $\cancel{A_2 \underset{B}{\perp\!\!\!\perp} A_1}$

__Fact 1.3__ if $T=$ $\mathbb{R}$-vector spaces, if $v_1, \dots v_n \in M \setminus \{0\}$
then $(v_1, \dots, v_n)$ is independent $/\emptyset$ iff

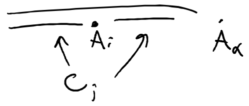
$$\forall \bar{x} \in \mathbb{R}^n \setminus \{\bar{0}\} : \quad x_1 v_1 + \dots + x_n v_n \neq 0$$

__Prop 1.4__ $(A_i : i \in I)$ is independent $/B$ $\iff$ $\left[\forall I_0 \subseteq_{fin} I,\ (A_i : i \in I_0) \text{ is indep.} /B.\right]$.

__Proof__ Finite character & monotonicity of $\perp\!\!\!\perp$

__Lemma 1.5__ Let $(A_i : i \leq \alpha)$ be a seq.
Let $A_{<\alpha} = (A_i : i < \alpha)$
Suppose $A_{<\alpha}$ is indep. $/B$ and $A_\alpha \underset{B}{\perp\!\!\!\perp} A_{<\alpha}$
Then $A_{\leq \alpha} = (A_i : i \leq \alpha)$ is indep $/B$.

__Proof__ Want to show if $i \leq \alpha$, then $A_i \underset{B}{\perp\!\!\!\perp} \{A_j : j \leq \alpha, j \neq i\}$.
For $i = \alpha$, this is assumed.
Suppose $i < \alpha$.
Let $C_i = \{A_j : j < \alpha, j \neq i\}$.



Want $A_i \underset{B}{\perp\!\!\!\perp} A_\alpha C_i$

We know: $A_i \underset{B}{\perp\!\!\!\perp} C_i$ because $A_{<\alpha}$ is independent

We know $A_\alpha \underset{B}{\perp\!\!\!\perp} A_i C_i$

$tp(A_\alpha / A_i C_i B) \sqsupseteq tp(A_\alpha / B)$

Base monotonicity

$\to A_\alpha \underset{BC_i}{\perp\!\!\!\perp} A_i$

$tp(A_\alpha / A_i C_i B) \sqsupseteq tp(A_\alpha / BC_i) \sqsupseteq tp(A_\alpha / B)$

Symmetry

$\to A_i \underset{BC_i}{\perp\!\!\!\perp} A_\alpha$ — Transitivity

$tp(A_i / A_\alpha B C_i) \sqsupseteq tp(A_i / BC_i) \sqsupseteq tp(A_i / B)$

$\to A_i \underset{B}{\perp\!\!\!\perp} C_i A_\alpha$. $\square$

__Prop 1.6__ If $(A_i : i < \alpha)$ is a sequence and $A_i \underset{B}{\perp} \{A_j : j<i\}$ for each i,

then $(A_i : i<\alpha)$ is independent.

Proof Lemma 1.5 and induction. $\square$

Example (A_1, A_2, A_3, A_4) is indep $/B \iff A_2 \underset{B}{\perp} A_1$ and $A_3 \underset{B}{\perp} A_1 A_2$ and $A_4 \underset{B}{\perp} A_1 A_2 A_3$

Example 1.7 If $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots$ is a Morley seq. $/B$

then $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots)$ is indep. $/B$

p is B-definable

$\bar{a}_i \models p \upharpoonright B\bar{a}_1\bar{a}_2 \ldots \bar{a}_{i-1}$

$tp(\bar{a}_i / B\bar{a}_1\bar{a}_2 \ldots \bar{a}_{i-1}) \subseteq p$

$tp(\bar{a}_i / B\bar{a}_1 \ldots \bar{a}_{i-1})$ has a B-definable global extension

$tp(\bar{a}_i / B\bar{a}_1 \cdots \bar{a}_{i-1}) \sqsupseteq tp(\bar{a}_i / B)$

$a_i \underset{B}{\perp} \bar{a}_1 \ldots \bar{a}_{i-1}$.

More generally, if

$p_1, \ldots p_n$ are B-definable

and $(\bar{a}_1, \ldots, \bar{a}_n) \models (p_1 \otimes \cdots \otimes p_n) \upharpoonright B$

then $(\bar{a}_1, \ldots \bar{a}_n)$ is indep. $/B$.

---

§2 $\boxed{\text{Suppose } T \text{ is strongly minimal.}}$

If $A \subseteq \mathbb{M}$, $b, b' \in \mathbb{M} \setminus acl(A)$

then $tp(b/A) = tp(b'/A)$.

Otherwise $\exists$ A-def. $D \subseteq \mathbb{M}$ with $b \in D$, $b' \notin D$

If D finite $\Rightarrow b \in acl(A)$

D cofinite $\Rightarrow b' \in acl(A)$ } no.

Def The transcendental type over A is $tp(b/A)$ for $b \notin acl(A)$

$b \models p \iff b \notin acl(A)$.

Let $p \in S_1(\mathbb{M})$ be transcendental.

$p \upharpoonright A$ is $tp(b/A)$ where $b \in N \succeq \mathbb{M}$, $b \notin acl(\mathbb{M}) = \mathbb{M} \supseteq acl(A)$

so $tp(b/A)$ is transcendental.

So $p \upharpoonright A$ is the transcendental type $/A$.

p is $\emptyset$-invariant, $\emptyset$-definable

Def b is transcendental if $b \notin acl(\emptyset)$.

Lemma 2.1 If b is transcendental $b \underset{\emptyset}{\perp} C$ iff $b \notin acl(C)$.

Proof $tp(b/\emptyset)$ is stationary so

$tp(b/C) \sqsupseteq tp(b/\emptyset) \iff tp(b/C) = p \upharpoonright C \iff b \notin acl(C)$. $\square$

Prop 2.2 A sequence $(b_i : i<\alpha)$ ~~is~~ of transcendentals is indep. $/\emptyset$ if $b_i \notin acl(\{b_j : j<i\})$.

Proof 1.6, 2.1 $\square$.

Remark 2.3 If $p \in S_1(\mathcal{M})$ transcendental, if $b_1, \dots b_n$ ~~are transcen~~ $\in \mathcal{M} \setminus acl(\emptyset)$ then $(b_1, \dots b_n)$ is indep$/\emptyset$ iff $(b_1, \dots b_n) \models p^{\otimes n} \restriction \emptyset$

$$\underbrace{p \otimes p \otimes \cdots \otimes p}_{n \text{ times}}$$

Lemma 2.4 If $I_1, I_2 \subseteq \mathcal{M} \setminus acl(\emptyset)$ are indep. sets. and if $f: I_1 \to I_2$ a bijection, then f is a P.E.M.

Proof If $b_1, \dots b_n \in I_1$ then $b_1, \dots b_n \models p^{\otimes n} \restriction \emptyset$

and $f(b_1), \dots f(b_n) \in I_2$ so $f(\bar{b}) \models p^{\otimes n} \restriction \emptyset$

$\bar{b} \equiv_{\emptyset} f(\bar{b})$. $\square$.

Def 2.5 ~~A basi~~ If $M \preccurlyeq \mathcal{M}$, a basis of M is a maximal indep. set $I \subseteq M \setminus acl(\emptyset)$.

bases exist by Zorn's Lemma.

Prop 2.6 If B is a basis of M, then $acl(B) = M$

Proof Otherwise $acl(B) \subsetneq M$, take $c \in M \setminus acl(B)$

$B \cup \{c\}$ is independent $\Rightarrow\Leftarrow$. $\square$

Theorem 2.7 T (strongly min.) is λ-categorical for $\lambda > |L|$.

Proof Suppose $M_1, M_2 \preccurlyeq \mathcal{M}$ $|M_i| = \lambda$ for $i=$

for $i=1,2$. Take bases $B_i \subseteq M_i$

$|B_i| \le |M_i| = \lambda$

$\lambda = |M_i| = |acl(B_i)| \le |B_i| + |L|$

So $|B_i| = \lambda$ $|B_1| = |B_2|$ Take $f: B_1 \to B_2$ a bijection Lemm 2.4 $\Rightarrow$ f is a P.E.M.

$\exists \sigma \in Aut(\mathcal{M})$, $\sigma \supseteq f$

$\sigma(M_1) = \sigma(acl(B_1)) = acl(\sigma(B_1)) = acl(B_2) = M_2$

$\sigma: M_1 \cong M_2$. $\square$

$\sigma(M_1) = \sigma(\mathrm{acl}(B_1)) = \mathrm{acl}(\sigma(B_1)) = \mathrm{acl}(B_2) = M_2$

$\sigma : M_1 \xrightarrow{\cong} M_2$. $\square$.

<u>Def</u> $\dim(M) = |B|$ for any basis $B \subseteq M$.

<u>Fact</u> This is well-defined

~~AA is~~ $\dim(M_1) = \dim(M_2) \Rightarrow M_1 \cong M_2$.

Resume @ 10:50.

No longer assume T is stable

<u>§3</u> Topology on type spaces.

Fix $A \subseteq \mathbb{M}$, n . $S_n(A)$

If $\varphi \in L(A)$, $[\varphi] = \{p \in S_n(A) : p(x) \vdash \varphi(x)\}$

If $D = \varphi(\mathbb{M}^n)$, $[D] = [\varphi]$.

$[D_1 \cap D_2] = [D_1] \cap [D_2]$

$[D_1 \cup D_2] = [D_1] \cup [D_2]$

$[\mathbb{M}^n \setminus D] = S_n(A) \setminus [D]$.

~~Clopen sets $= [\varphi]$ for some φ.~~

$U \subseteq S_n(A)$ is clopen if $U = [\varphi]$ for some $\varphi \in L(A)$

Iso of bool. algebras:

$$\{\text{A-definable sets } D \subseteq \mathbb{M}^n\} \xrightarrow{\cong} \{\text{clopen sets in } S_n(A)\}.$$

<u>Fact 3.1</u> (Compactness) If $\mathcal{F}$ is a collection of clopen subsets of $S_n(A)$ and $\mathcal{F}$ has FIP. then $\bigcap \mathcal{F} \neq \emptyset$.

<u>Fact 3.2</u> (Compactness) If $\mathcal{F}$ is a collection of clopen sets and $\bigcup \mathcal{F} = S_n(A)$ then $\exists \mathcal{F}_0 \subseteq \mathcal{F}$ $\bigcup \mathcal{F}_0 = S_n(A)$.

<u>Def 3.3</u> $X \subseteq S_n(A)$ is <u>closed</u> if $X = \bigcap \mathcal{F}$, $\mathcal{F}$ a collection of clopen sets

<u>open</u> if $X = \bigcup \mathcal{F}$, ———.

<u>Remark 3.4</u> If $\Sigma(\bar{x})$ is a set of $L(A)$-formulas

$$\begin{aligned}\bigcap_{\varphi \in \Sigma} [\varphi] &= \{p \in S_n(A) : \forall \varphi \in \Sigma : p(\bar{x}) \vdash \varphi(\bar{x})\} \\ &= \{p \in S_n(A) : p \vdash \Sigma\} \\ &= \text{completions of } \Sigma \\ &= \{\mathrm{tp}(\bar{b}/A) : \bar{b} \models \Sigma\}.\end{aligned}$$

Closed sets in $S_n(A)$ $\longleftrightarrow$ partial types $/A$

<u>Fact 3.5</u> $\{p\}$ is closed.

Fact 3.6 X is clopen $\iff$ X is closed and open.

Fact 3.7 If $\mathcal{F}$ is a collection of closed sets & $\mathcal{F}$ has ~~F~~ FIP, then $\bigcap \mathcal{F} \neq \emptyset$.

Cor 3.8 If $\mathcal{F}$ is a nonempty collection of nonempty closed sets in $S_n(A)$, ...

1) If $\mathcal{F}$ is a chain $(\forall X, Y \in \mathcal{F},\ X \subseteq Y \text{ or } Y \subseteq X)$

or

2) If $\mathcal{F}$ is filtered $(\forall X, Y \in \mathcal{F}\ \exists Z \in \mathcal{F}.\ Z \subseteq X \cap Y)$

then $\mathcal{F}$ has FIP $\Rightarrow \bigcap \mathcal{F} \neq \emptyset$.

Prop 3.9 (Total separation) If $p, q \in S_n(A)$ $p \neq q$ then $\exists\, U \subseteq S_n(A)$ clopen, $p \in U$, $q \notin U$.

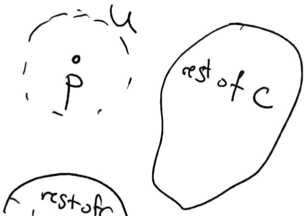
Proof Take φ ~~s.t.~~ $p \vdash \varphi$
$q \vdash \neg\varphi$

then $p \in [\varphi]$ $q \notin [\varphi]$. $\square$

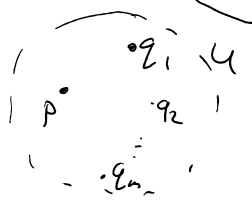
Proposition 3.10 If $C \subseteq S_n(A)$ closed, and $p \in C$, then (1) or (2):

(1) $\exists$ clopen U $p \in U$, $U \cap C = \{p\}$ isolated point

(2) $\forall$ clopen $U \ni p$, $|U \cap C| = \infty$. accumulation point



Proof If (2) false then $\exists$ clopen U
$U \cap C = \{p, q_1, \dots q_m\}$.



Prop 3.9 $\Rightarrow$ V_i clopen
$p \in V_i$ $q_i \notin V_i$

$U \cap \bigcap_{i=1}^{m} V_i$ works

$C \cap U \cap \bigcap_{i=1}^{m} V_i = \{p\}$ (1) holds. $\square$.

Def 3.11 The derived set C' of $C = \{p \in C : p$ is an accumulation point$\}$.

$C =$  $C' =$

$\{1 - 1/n : n < \omega\} \cup \{1\}$ $\{1\}$.

Prop 3.12 If C closed then C' is closed

Proof $\forall p \in C$ isolated, take U_p clopen with $U_p \cap C = \{p\}$.

$$S_n(A) \setminus C' = \underbrace{(S_n(A) \setminus C)}_{\text{open}} \cup \underbrace{\bigcup_{p \in C \setminus C'} U_p}_{\text{open}}.$$

C' is closed.

<u>Prop 3.13</u> If $C \subseteq S_n(A)$ closed & $|C| = \infty$ then $C' \neq \emptyset$.

<u>Proof</u> Otherwise, every $p \in C$ is isolated.

$\forall p \in S_n(A)$ $\exists U_p$ clopen $U_p \cap C = \emptyset$ or $\{p\}$.

If $p \in S_n(A) \setminus C$ $\exists$ clopen U_p with $p \in U_p \subseteq S_n(A) \setminus C$. $U_p \cap C = \emptyset$.

$\bigcup_{p \in S_n(A)} U_p = S_n(A)$ (because $p \in U_p$)

By compactness, $\exists$ ~~po~~ $p_1, \ldots p_m$, $S_n(A) = \bigcup_{i=1}^m U_{p_i}$.

$C = C \cap S_n(A) = \bigcup_{i=1}^m (C \cap U_{p_i})$ ← finite, C is finite $\Rightarrow \Leftarrow$. $\square$.

<u>Def 3.14</u> $K \subseteq S_n(A)$ is <u>perfect</u> if K is closed & $K' = K$

$S_n(A)$ is <u>scattered</u> if $\nexists K \subseteq S_n(A)$ s.t. $K \neq \emptyset$, $K' = K$.

<u>§4</u> <u>Cantor-Bendixson rank</u>

Continue to fix $S_n(A)$.

~~Def~~ For ordinal α, define $E_\alpha \subseteq S_n(A)$ closed by recursion:

$E_0 = S_n(A)$

$E_{\alpha+1} = E_\alpha'$

$E_\beta = \bigcap_{\alpha < \beta} E_\alpha$ for limit ordinals β.

The chain $(E_\alpha : \alpha \in \mathrm{Ord})$ can only decrease ~~$2^{|S_n(A)|}$~~ $|S_n(A)|$ times

So $E_\alpha = E_{\alpha+1}$ for some α.

$E_\alpha' = E_\alpha$. $E_\beta = E_\alpha$ $\forall \beta \geq \alpha$. by induction

Let $E_\infty = E_\alpha$ for all sufficiently large α.

Note $E_\infty = E_\infty'$.

<u>Remark 4.1</u> ∞ is a formal symbol with $\infty > \alpha$ $\forall \alpha \in \mathrm{Ord}$.

Warning: $\alpha < \infty$ doesn't mean $\alpha < \omega$.

→ <u>Def 4.2</u> $R(p) = \max\{\alpha \mid p \in E_\alpha\}$ (possibly ∞) (Cantor-Bendixson rank)

→ If $C \subseteq S_n(A)$ closed, $R(C) = \max\{\alpha \mid E_\alpha \cap C \neq \emptyset\}$.

This is well-defined because if β is a limit ordinal and $E_\alpha \cap C \neq \emptyset$ $\forall \alpha < \beta$ then $\underbrace{\bigcap_{\alpha<\beta} E_\alpha \cap C}_{= E_\beta \cap C} \neq \emptyset$ by Cor 3.8.

So $\{\alpha \mid E_\alpha \cap C \neq \emptyset\}$ is "closed".

Note ~~p~~ $p \in E_\alpha \iff \alpha \le R(p)$

$E_\alpha = \{p \in S_n(A) : R(p) \ge \alpha\}$.

Remark 4.3 $R(p)$ is characterized by ...

$R(p) \ge \alpha+1 \iff p$ is an accumulation point of $\{q \in S_n(A) : R(q) \ge \alpha\}$.

$R(p) = 0 \iff p$ is isolated

Remark 4.4 $R(C) = \max\{p \in C : R(p)\}$.

Why? $\alpha \le R(C) \iff E_\alpha \cap C \neq \emptyset \iff \exists p \in C : p \in E_\alpha$

$\iff \exists p \in C : R(p) \ge \alpha$

$\iff \alpha \le \cancel{\max} \max_{p \in C} R(p)$. $\square$

Def 4.5 If $\varphi(x) \in L(A)$, $R(\varphi)$ is $R([\varphi])$

If $\Sigma(x) \subseteq L(A)$, $R(\Sigma) = R(\bigcap_{\varphi \in \Sigma} [\varphi])$

If $D \subseteq M^n$ A-def $R(D) = R([\varphi])$

$D = \varphi(M^n)$ $\quad = R([D])$

Warning: all these depend on A.

Prop 4.6 Suppose $C_1, C_2 \subseteq S_n(A)$ closed.

1.) If $C_1 \subseteq C_2$ then $R(C_1) \le R(C_2)$

2) $R(C_1 \cup C_2) = \max(R(C_1), R(C_2))$.

If Σ, Ψ are partial types $/A$

3) If $\Sigma \vdash \Psi$ then $R(\Sigma) \le R(\Psi)$

If $\varphi, \psi \in L(A)$

4) $R(\varphi \vee \psi) = \cancel{\le} \max(R(\varphi), R(\psi))$

Proof: Clear.

1, 2 are true by Rem 4.4

~~3, 4~~

$1 \Rightarrow 3$

$2 \Rightarrow 4$

Prop 4.7

1) If $C \subseteq S_n(A)$ closed, then

$R(C) = \min\{R(U) : U \supseteq C, U \text{ clopen}\}$

2) If $\Sigma(x) \subseteq L(A)$

then $R(\Sigma) = \min\{R(\varphi) : \Sigma \vdash \varphi\}$.

Resume 11:41.

1) if $C = \bigcap_{i \in I} U_i$ filtered intersection ($\{U_i\}$ is filtered $\forall i, j \in I$ $\exists k$ $U_k \subseteq U_i \cap U_j$)

then $R(C) = \min_{i \in I} R(U_i)$

2) $R(\Sigma) = \min_{\Sigma_0 \subseteq_f \Sigma} R(\Sigma_0)$.

$[\Sigma] = \bigcap_{\Sigma_0 \subseteq_f \Sigma} [\Sigma_0]$ filtered isection of clopens.

Proof

$(1) \Rightarrow (2)$, Just need (1)

If $C = \bigcap_{i \in I} U_i$ then $C \subseteq U_i \ \forall i \in I$

$R(C) \le R(U_i)$

$R(C) \le \min_{i \in I} R(U_i)$

Just need $R(C) \geq \min_{i \in I} R(U_i)$

Trivial if $R(C) = \infty$

Suppose $R(C) = \alpha < \infty$. Suppose $\alpha < \min_{i \in I} R(U_i)$

$C \cap E_{\alpha+1} = \emptyset$. $\quad \bigcap_{i \in I} U_i \cap E_{\alpha+1} = \emptyset$

By Cor 3.8, $U_i \cap E_{\alpha+1} = \emptyset$ for some $i \in I$

$R(U_i) < \alpha+1 \qquad R(U_i) \leq \alpha$

$\min_{i \in I} R(U_i) \leq \alpha = R(C)$. $\square$

Def: $R(C) = -\infty$
$\iff C = \emptyset$.

Prop 4.8 If $C \subseteq S_n(A)$ closed & $-\infty < R(C) < \infty$
then, $\{p \in C : R(p) = R(C)\}$ is finite. and non-empty

Proof Let $\alpha = R(C)$. If $E_\alpha \cap C$ is infinite, $\exists q \in (E_\alpha \cap C)'$ by Prop 3.13

$q \in (E_\alpha \cap C)' \subseteq E_\alpha' = E_{\alpha+1}$

$q \in (E_\alpha \cap C)' \subseteq C' \subseteq C$. So $q \in C \cap E_{\alpha+1}$

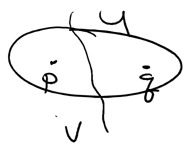
$R(C) \geq \alpha+1 \Rightarrow \Leftarrow$. $\square$

Lemma 4.9 If $U \subseteq S_n(A)$ clopen & $R(U) \geq \alpha+1$, then $\exists U_1, U_2$ clopen

$U = U_1 \sqcup U_2$ (means: $U = U_1 \cup U_2$,
~~the~~ $U_1 \cap U_2 = \emptyset$).

and $R(U_1) \geq \alpha+1$, $R(U_2) \geq \alpha$.

Proof Take $p \in E_{\alpha+1} \cap U$. $\quad p \in E_\alpha'$



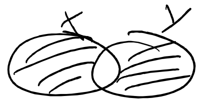
$U \cap E_\alpha$ is infinite, $\supsetneq \{p\}$. $\exists q \in U \cap E_\alpha \quad q \neq p$.

By Prop 3.9 $\exists$ clopen $V \quad p \in V \quad q \notin V$

$U_1 = U \cap V \qquad U_2 = U \setminus V$.

$p \in U_1$, $p \in E_{\alpha+1}$, $R(U_1) \geq \alpha+1$, $q \in U_2$, $q \in E_\alpha$, $R(U_2) \geq \alpha$. $\square$

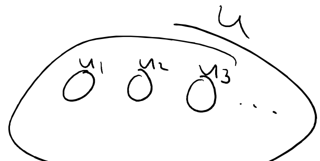
If X, Y sets let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$



Prop 4.10 If $U \subseteq S_n(A)$ clopen, TFAE.

1) $R(U) \geq \alpha+1$

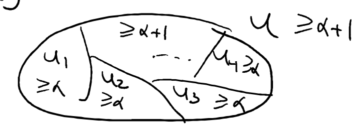
2) $\exists$ clopen $U_1, U_2, U_3, \ldots \subseteq U \quad U_i \cap U_j = \emptyset$ for $i \neq j$
and $R(U_i) \geq \alpha$.



3) $\exists$ clopen $U_1, U_2, U_3, \ldots \subseteq U$ such that $R(U_i \triangle U_j) \geq \alpha$.

(idea: modulo the ideal ~~of~~ $\{U : R(U) < \alpha\}$,
there are inf. many subsets of U)

Proof $1 \Rightarrow 2$ Iterate Prop 4.9



$2 \Rightarrow 3$ If $U_i \cap U_j = \emptyset$

then $U_i \triangle U_j = U_i \cup U_j$

$R(U_i \triangle U_j) = R(U_i \cup U_j) \geq R(U_i) \geq \alpha$

<u>$3 \Rightarrow 1$</u> Assume 3. $\quad U_1 \triangle U_2 \subseteq U$

$R(U) \geq R(U_1 \triangle U_2) \geq \alpha$

Suppose $R(U) < \alpha+1$ then $R(U) = \alpha$

By Prop 4.8, $\quad U \cap E_\alpha = \{q_1, \dots q_m\}$.

$\omega \longrightarrow Pow(U \cap E_\alpha)$

$i \longmapsto U_i \cap E_\alpha$.

cannot be injective

So $\exists\, i<j \quad U_i \cap E_\alpha = U_j \cap E_\alpha$

$(U_i \triangle U_j) \cap E_\alpha = \emptyset$

$R(U_i \triangle U_j) < \alpha \Rightarrow\Leftarrow$. $\square$.

Prop 4.10 (2), 4.7 give alt. def. of $R(-)$ ...

If $D \subseteq M^n$ is A-definable ...

$R(D) \geq 0 \iff D \neq \emptyset$

$R(D) \geq \alpha+1 \iff \exists$ A-definable $D_1, D_2, D_3, \dots \subseteq D$

$D_i \cap D_j = \emptyset \qquad R(D_i) \geq \alpha$.

(Also $R(D) \geq \beta \iff R(D) \geq \alpha \quad \forall \alpha<\beta$, when β is a limit ordinal)

$R(\varphi) = R(\varphi(M^n))$ for $\varphi(x_1, \dots x_n) \in L(A)$

$\Sigma(x_1, \dots x_n) \subseteq L(A)$

$R(\Sigma) = \min\limits_{\Sigma_0 \subseteq_{fin} \Sigma} R(\Sigma_0)$ where $R(\Sigma_0) = R\left(\bigwedge\limits_{\varphi \in \Sigma_0} \varphi\right)$.

(Morley rank is Cantor-Bendixson rank when $A = M$).

---

§5 When is $S_n(A)$ scattered?

<u>Prop 5.1</u> $R(S_n(A)) < \infty \iff S_n(A)$ is scattered (no non-empty perfect sets)

<u>Proof</u> If $R(S_n(A)) = \infty$ then $E_\infty \neq \emptyset$, E_∞ is a perfect non-empty set, $S_n(A)$ is not scattered.

Conversely: suppose $S_n(A)$ not scattered, $\exists\, \emptyset \neq K \subseteq S_n(A)$ K perfect, $K = K'$

Then $K \subseteq E_\alpha$ by induction on α.

$\alpha = 0$ ✓ $\qquad K \subseteq E_\alpha \Rightarrow K = K' \subseteq E_\alpha' = E_{\alpha+1}$

if $K \subseteq E_\alpha \;\forall \alpha<\beta \quad K \subseteq \bigcap\limits_{\alpha<\beta} E_\alpha = E_\beta$.

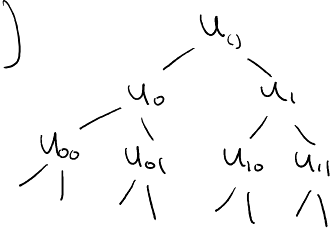
$K \subseteq E_\infty, \qquad E_\infty \neq \emptyset, \qquad R(S_n(A)) = \infty$. $\square$

<u>Lemma 5.2</u> If $U \subseteq S_n(A)$ clopen & $R(U) = \infty$

then $\exists$ clopen U_1, U_2, $\quad U = U_1 \sqcup U_2$, $\quad R(U_1) = \infty = R(U_2)$.

Proof Like Lem 4.9.

Lemma 5.3 1) $R(S_n(A)) = \infty \iff \big(\exists\ (U_\sigma : \sigma \in 2^{<\omega})$ such that $U_\sigma = U_{\sigma 0} \sqcup U_{\sigma 1}\ \forall \sigma$, $U_\sigma \neq \emptyset$, U_σ clopen $\big)$



2) If $R(S_n(A)) = \infty$, then $|S_n(A)| \geq 2^{\aleph_0}$

3) If $R(S_n(A)) < \infty$, then $|S_n(A)| \leq |L(A)| = |L| + |A|$

Proof 1) If $R(S_n(A)) = \infty$, get tree by iterating Lem 5.2 start with $U = S_n(A)$

Conversely, suppose the tree exists.

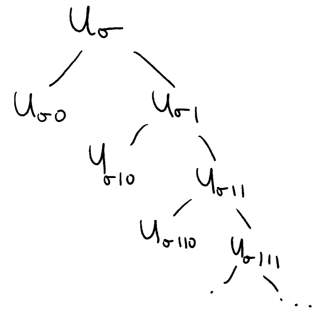
Claim If $\sigma \in 2^{<\omega}$, $\alpha \in \mathrm{Ord}$, then $R(U) \geq \alpha$

Proof By induction on α.

$\alpha = 0$: $U_\sigma \neq \emptyset$.

α limit ordinal : easy.

$\alpha + 1$: by induction, $R(U_\tau) \geq \alpha\ \ \forall \tau \in 2^{<\omega}$.



$U_{\sigma 0}, U_{\sigma 10}, U_{\sigma 110}, \dots \subseteq U_\sigma$
clopen, disjoint

By Prop 4.10, $R(U_\sigma) \geq \alpha + 1$. $\square_{\text{claim}}$.

By Claim, $R(U) = \infty$ $R(S_n(A)) \geq R(U) = \infty$. ✓

2) If $R(S_n(A)) = \infty$ take a tree $(U_\sigma : \sigma \in 2^{<\omega})$ as in (1)

$\forall \tau \in 2^\omega$, $\bigcap_{i=0}^{\infty} U_{\tau \restriction i} \neq \emptyset$ by Cor 3.8

Take $p_\tau \in \bigcap_{i=0} U_{\tau\restriction i}$. $\boxed{\tau \mapsto p_\tau \text{ injective}}$

If $\tau \neq \tau'$ $\exists n$ $\tau\restriction n \neq \tau' \restriction n$

$|S_n(A)| \geq |2^\omega| = 2^{\aleph_0}$ $p_\tau \in U_{\tau\restriction n}$ $p_{\tau'} \in U_{\tau'\restriction n}$ $U_{\tau\restriction n} \cap U_{\tau'\restriction n} = \emptyset$ so $p_\tau \neq p_{\tau'}$.

3) If $R(S_n(A)) < \infty$.

If $p \in S_n(A)$, $R(p) = \alpha < \infty$,

$p \in E_\alpha$ $p \notin E_\alpha' = E_{\alpha+1}$

p is isolated in E_α

$\exists$ clopen U_p $U_p \cap E_\alpha = \{p\}$.

Note p is the point of max rank in U_p.

$p \mapsto U_p$ is injective.

$p \mapsto U_p$ is injective.

$|S_n(A)| \leq |\{\text{clopens}\}| \leq |L(A)|$. $\square$

<u>Theorem 5.5</u> If L is countable & T is stable,
$\lambda_0(T) = \aleph_0$ or $2^{\aleph_0}$

<u>Proof</u>: T is $2^{\aleph_0}$-stable by Lemma 1 March 10.
$\lambda_0 \leq 2^{\aleph_0}$

If $\aleph_0 < \lambda_0 < 2^{\aleph_0}$ ...

T is not $\aleph_0$-stable, so $\exists A \subseteq M$ $|A| \leq \aleph_0 \leq \lambda_0$
$|S_n(A)| > \aleph_0$.

$|S_n(A)| \not\leq |L(A)|$. So $R(S_n(A)) = \infty$

So $|S_n(A)| \geq 2^{\aleph_0} > \lambda_0$

So T is not λ_0-stable $\Rightarrow\Leftarrow$. $\square$.