Set Theory

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1 Ordinal Numbers

1.1 Linear and Partial Ordering

Definition 1.1. A binary relation $<$ on a set P is a **partial ordering** of P if

1. $p \not< p$ for any $p \in P$
2. if $p < q$ and $q < r$ then $p < r$

$(P, <)$ is called a **partially ordered set**. A partial ordering $<$ of P is a **linear ordering** if moreover

3. $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$

   If $<$ is a partial ordering, then $\leq$ is also a partial ordering

if $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is **order-preserving** if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called **increasing**

1.2 Well-Ordering

Definition 1.2. A linear ordering $<$ of a set P is a **well-ordering** if every nonempty subset of P has a least element

Lemma 1.3. *If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction $\square$

Corollary 1.4. *The only automorphism of a well-ordered set is the identity*

Proof. By Lemma 1.3, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x $\square$

Corollary 1.5. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique*

if W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an **initial segment** of W

Lemma 1.6. *No well-ordered set is isomorphic to an initial segment of itself*

Proof. If $\text{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 1.3 $\square$

Theorem 1.7. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds*

1. *W_1 is isomorphic to W_2*
2. *W_1 is isomorphic to an initial segment of W_2*
3. *W_2 is isomorphic to an initial segment of W_1*

Proof. For $u \in W_i$, $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}$$

Using Lemma 1.6, f is a injective: if $f(x_1) = f(x_2) = y$, then $W_1(x_1) \cong W_2(y) \cong W_1(x_2)$, and $x_1 < x_2$ or $x_2 < x_1$ fail. If h is an isomorphism between $W_1(x)$ and $W_2(y)$, and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving

If $\text{dom}(f) = W_1$ and $\text{ran}(f) = W_2$, then case 1 holds

if $y_1 < y_2$ and $y_2 \in \text{ran}(f)$, then $y_1 \in \text{ran}(f)$. Thus if $\text{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \text{ran}(f)$, we have $\text{ran}(f) = W_2(y_0)$. Necessarily, $\text{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where x_0=the least element of $W_1 - \text{dom}(f)$ $\square$

if W_1 and W_2 are isomorphic, we say that they have the same **order-type**.

1.3 Ordinal Numbers

Definition 1.8. A set T is **transitive** if every element of T is a subset of T

Definition 1.9. A set is an **ordinal number** (an **ordinal**) if it is transitive and well-ordered by $\in$

Define

$$\alpha < \beta \quad \text{iff} \quad \alpha \in \beta$$

Lemma 1.10. *1. $0 = \emptyset$ is an ordinal*

2. *if α is an ordinal and $\beta \in \alpha$, then β is an ordinal*
3. *if $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$*
4. *if α, β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$*

Proof. 1,2 by definition

3. if $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ: for $\eta \in \alpha$, $\eta \neq \gamma$ and $\gamma \notin \eta$, hence $\eta \in \gamma$ since ordinals are well-ordered by $\in$. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.
4. $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$ and $\gamma \in \beta$, by 3. Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a **strict** ordering of α)

□

Using Lemma 1.10 one gets the followings

1. $<$ is a linear ordering of the class Ord
2. for each α, $\alpha = \{\beta : \beta < \alpha\}$
3. if C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$
4. if X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$

5. for every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$. In view of 4, the class Ord is a proper class; otherwise consider $\sup \mathrm{Ord} + 1$

Theorem 1.11. *Every well-ordered set is isomorphic to a unique ordinal number*

Proof. The uniqueness follows from Lemma 1.6. Given a well-ordered set W, define $F(x) = \alpha$ is α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exists). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ □

0 is a limit ordinal and define $\sup \emptyset = 0$

Definition 1.12 (Natural Numbers)**.** We denote the least nonzero limit ordinal ω (or $\mathbb{N}$). The ordinals less than ω are call **finite ordinals**, or **natural numbers**

1.4 Induction and Recursion

Theorem 1.13 (Transfinite Induction)**.** *Let C be a class of ordinals and assume that*

1. $0 \in C$
2. *if $\alpha \in C$, then $\alpha + 1 \in C$*
3. *if α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$*

Then C is the class of all ordinals

Proof. Otherwise, let α be the least $\alpha \notin C$ and apply 1,2 and 3. □

A function whose domain is the set $\mathbb{N}$ is called an (**infinite**) **sequence** (A **sequence in** X is a function $f : \mathbb{N} \to X$). The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

A **finite sequence** is a function s s.t. $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \mathbb{N}$; then s is a **sequence of length** n

A **transfinite sequence** is a function whose domain is an ordinal

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α**-sequence** or a **sequence of length** α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an **enumeration** of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}$$

Sometimes we call a "sequence"

$$\langle a_\alpha : \alpha \in \text{Ord} \rangle$$

a function (a proper class) on Ord

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequence), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

s.t.

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$

Theorem 1.14 (Transfinite Recursion)**.** *Let G be a function (on V), then* (1) *below defines a unique function F on Ord s.t.*

$$F(\alpha) = G(F \restriction \alpha)$$

for each α

In other words, if we let $a_\alpha = F(\alpha)$, then for each α

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

(Note that we tacitly use Replacement: $F \restriction \alpha$ is a set for each α)

Corollary 1.15. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ s.t. $\operatorname{ran}(G) \subset X$ there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta \rangle$ in X s.t. $a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$ for every $\alpha < \theta$*

Proof. Let

$$F(\alpha) = x \leftrightarrow \text{there is a sequence } \langle a_\xi : \xi < \alpha \rangle \text{ s.t.:} \tag{1}$$

$$1.\ (\forall \xi < \alpha) a_\xi = G(\langle a_\eta : \eta < \xi \rangle)$$

$$2.\ x = G(\langle a_\xi : \xi < \alpha \rangle)$$

□

2 Question

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