Clarification of invariant types and Morley products

Advanced Model Theory

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This document aims to clarify some issues around invariant types and Morley products which came up in Homework #4.

1. The first section clarifies what a "constant type" is.
2. The second section reviews how we have defined invariant types and Morley products in class, using an approach where we stay inside a fixed monster model $\mathbb{M}$.
3. The third section describes an alternate approach where we step outside the monster model. This is closer to the approach in Poizat's textbook. Some of you seem to have been implicitly using this approach on Homework #4.

Throughout all the sections, there are some **Warnings** to guard against common mistakes.

The first two sections review what we discussed in class, and I would encourage everyone to read them. The third section contains new material. I would encourage you to read it if you wrote things like $p|\mathbb{M}a$ on your solutions to Homework #4.

If you don't have time to read things, here are the main comments I have on the homework:

- Global types aren't usually realized in the monster model. If you realize a global type, remember that the realization is probably not in $\mathbb{M}$, and be careful.
- If p is a global A-invariant type, then p is a type over $\mathbb{M}$. If $\bar{a}$ is a tuple outside the monster model, then the restriction $p \restriction \mathbb{M}\bar{a}$ doesn't make sense, because $\mathbb{M}\bar{a}$ isn't a subset of $\mathbb{M}$. (But see §3.)
- We have only discussed the equivalence

  $$(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C \iff \big(\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a}\big)$$

  in the case where $\bar{a}, \bar{b} \in \mathbb{M}$ and C is small. When $\bar{a} \notin \mathbb{M}$, the right hand side doesn't even make sense.
- If p is a global type (a type over the monster), then we shouldn't talk about p being A-invariant unless A is small. In particular, we shouldn't talk about p being $\mathbb{M}$-invariant.

1 Constant types

Following Poizat's convention, a "type" means a complete type, and a "partial type" means a partial type. The space of n-types over a set A in a structure M is denoted $S_n(A)$. (Often $A = M$. Note that $S_n(A)$ doesn't change if we replace M with an elementary extension.)

If M is a model (possibly small, possibly the monster, possibly unrelated to $\mathbb{M}$), we say that $p(\bar{x}) \in S_n(M)$ is a *constant type* if it contains the formula $\bar{x} = \bar{a}$ for some tuple $\bar{a} \in M$.

Warning. *The notation $\bar{a} \in M$ really means $\bar{a} \in M^n$.*

Note that the following are equivalent for a model M and a type $p \in S_n(M)$:

1. p is a constant type
2. p is realized by a tuple in M
3. p is $\mathrm{tp}(\bar{a}/M)$ for some $\bar{a} \in M$.

Therefore, constant types are also called "realized types" by some authors.[1] This can be confusing, because if $p \in S_n(M)$ is a non-constant type, then p is realized by some $\bar{a}$ in an elementary extension $N \succeq M$. For this reason, I prefer the term "constant type."

Warning. *Suppose $p \in S_n(M)$, and N is an elementary extension containing a realization $\bar{a}$ of p. Then $\mathrm{tp}(\bar{a}/N)$ is an extension of $p = \mathrm{tp}(\bar{a}/M)$. However, we should not really identify p with $\mathrm{tp}(\bar{a}/N)$. For example, $\mathrm{tp}(\bar{a}/N)$ is not usually an heir or coheir of p. Likewise, $\mathrm{tp}(\bar{a}/N)$ is a constant type, but p is usually not.*

Warning. *If $p \in S_n(A)$ and A is not a model, then the notion of "constant type" is a little ambiguous. For example, in* ACF*, suppose $A = \{1, 2, 3\}$. Consider $p = \mathrm{tp}(\frac{1}{2}/A)$. Then p does not contain any formula of the form $x = a$ (because $\frac{1}{2} \notin A$). On the other hand, p contains the formula $x + x = 1$, which is logically equivalent to $x = \frac{1}{2}$. So p is a bit like a constant type.*

2 Approach 1: fixed monster model

Fix a cardinal number κ that is pretty big. Work in a model $\mathbb{M}$ that is κ-saturated and κ-strongly homogeneous.[2] We call $\mathbb{M}$ the *monster model.* A set $A \subseteq \mathbb{M}$ is *small* if $|A| < \kappa$. The monster model $\mathbb{M}$ is usually not small. A *small model* means a small elementary substructure $M \preceq \mathbb{M}$.

We are interested in two kinds of types:

- Types over small sets, especially small models.

[1] For example, on page 57 of the textbook, Poizat introduces this terminology and says it is "regrettable".

[2] Some authors additionally assume that $\kappa = |\mathbb{M}|$. This can be arranged <u>if</u> you make special assumptions about the set-theoretic universe, which we are *not* doing.

- Types over the monster model $\mathbb{M}$, also called *global types*.

Warning. *It's important to distinguish global types from types over small sets. They behave very differently. For example:*

- *Types over small sets are always realized in* $\mathbb{M}$.
- *Global types are usually not realized in* $\mathbb{M}$. *On the other hand, a global type is realized in some further elementary extension* $N \succeq \mathbb{M}$.

If $A \subseteq B \subseteq \mathbb{M}$, and $p \in S_n(B)$, then $p \restriction A$ denotes the restriction of p to A, a type in $S_n(A)$. Note that $\operatorname{tp}(\bar{c}/A)$ is the restriction of $\operatorname{tp}(\bar{c}/B)$ for any $\bar{c} \in \mathbb{M}$.

Remark 1. One can think of a global type as a compatible family of types over small sets. More precisely, one can think of a global type $p \in S_n(\mathbb{M})$ as a map that assigns to each small set $A \subseteq \mathbb{M}$ a complete n-type $p_A \in S_n(A)$, namely $p_A = p \restriction A$. Such a map $A \mapsto p_A$ is compatible iff $A \subseteq B \implies p_A \subseteq p_B$ for small subsets $A \subseteq B \subseteq \mathbb{M}$.

Warning. *Let* $\Gamma(\bar{x})$ *be a partial type over* $\mathbb{M}$. *Suppose* $\Gamma(\bar{x})$ *generates a complete global type* $p(\bar{x}) \in S_n(\mathbb{M})$. *If* A *is a small set, let* $\Gamma \restriction A$ *be the set of* $L(A)$*-formulas in* Γ. *Then* $p \restriction A$ *contains all the formulas in* $\Gamma \restriction A$, *but* $p \restriction A$ *is not necessarily generated by* $\Gamma \restriction A$. *For example, suppose we are in a monster model* $\mathbb{M} \models \mathrm{ACF}_0$, *and* $\Gamma(x)$ *is the set of formulas* $\{x \neq a : a \in \mathbb{M}\}$. *Then* $\Gamma(x)$ *generates a complete type* $p(x)$. *(The type* p *is the transcendental 1-type over* $\mathbb{M}$.*) Take* $A = \mathbb{Q}$. *Then* $\Gamma \restriction \mathbb{Q}$ *is the set of formulas* $\{x \neq a : a \in \mathbb{Q}\}$. *I claim that* $\Gamma \restriction \mathbb{Q}$ *does not generate* $p \restriction \mathbb{Q}$. *Note* p *contains the two* $L(\mathbb{M})$*-formulas* $x \neq \sqrt{2}$ *and* $x \neq -\sqrt{2}$. *Therefore it contains their conjunction* $(x \neq \sqrt{2}) \wedge (x \neq -\sqrt{2})$ *and the logically equivalent statement* $x^2 \neq 2$. *The formula* $x^2 \neq 2$ *is an* $L(\mathbb{Q})$ *formula so it is part of* $p \restriction \mathbb{Q}$. *Therefore* $\sqrt{2}$ *does not satisfy* $p \restriction \mathbb{Q}$. *On the other hand,* $\sqrt{2}$ *satisfies* $\Gamma \restriction \mathbb{Q}$, *because* $\sqrt{2} \neq a$ *for any* $a \in \mathbb{Q}$. *Therefore* $\Gamma \restriction \mathbb{Q}$ *does not logically imply* $p \restriction \mathbb{Q}$.

If A is a small set and $p \in S_n(\mathbb{M})$ is a global type, then p is *A-invariant* if the following equivalent conditions hold:

- For any $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$, $\sigma(p) = p$.
- If $\bar{b} \equiv_A \bar{c}$ and $\varphi(\bar{x}, \bar{y})$ is an L-formula, then

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \varphi(\bar{x}, \bar{c}) \in p(\bar{x}).$$

Warning. *We only consider* A*-invariant types* p *when* A *is small and* p *is a global type (a type over the monster model). In particular, we shouldn't talk about* A*-invariant types over small sets. We shouldn't talk about* $\mathbb{M}$*-invariant types (unless we've gone to a bigger,* $|\mathbb{M}|^+$*-saturated monster model).*

Remark 2. If we adopt the view of Remark 1, then a global type p is A-invariant if the restrictions $p \restriction B$ depend only on $\operatorname{tp}(B/A)$. More precisely, the condition that p is A-invariat can be rephrased as follows:

If B, B' are two small sets and $f : B \to B'$ is a bijection that is a partial $L(A)$-elementary map, then f maps $p \restriction B$ to $p \restriction B'$.

Fact 3. *if p is A-invariant and $A \subseteq B \subseteq \mathbb{M}$ and B is small, then p is B-invariant.*

We say that $p \in S_n(\mathbb{M})$ is an *invariant type* if it is A-invariant for some small $A \subseteq \mathbb{M}$.

Fact 4. *If A is a small set and $p \in S_n(\mathbb{M})$ is finitely satisfiable in A, then p is A-invariant.*

If M is a model (possibly small, possibly the monster) and $A \subseteq M$ and $p \in S_n(M)$, we say that p is A-*definable* if for any L-formula $\varphi(\bar{x}, \bar{y})$, there is an $L(A)$-formula $\psi(\bar{y})$ such that for $\bar{b} \in M$,

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff M \models \psi(\bar{b}).$$

We say that $p \in S_n(M)$ is *definable* if it is M-definable.

Remark 5. There are different notational conventions for the map $\varphi \mapsto \psi$. Poizat writes $\psi(\bar{y})$ as $d\varphi(\bar{y})$. Sometimes people write $d_p\varphi(\bar{y})$ to make the dependence on $\bar{y}$ explicit. I like to write $(d_p\bar{x})\varphi(\bar{x}, \bar{y})$ to make it clear which variables we are removing. (Then $(d_p\bar{x})$ works like a quantifier, binding the $\bar{x}$ appearing in $\varphi(\bar{x}, \bar{y})$.) On homework and exams, you may use any of these notational conventions.

Warning. *The definition of a definable type $p \in S_n(M)$ is only allowed to use parameters from M. In particular, you cannot take a realization $\bar{a}$ in a further elementary extension and then "define" p via the fact that*

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \mathbb{M} \models \varphi(\bar{a}, \bar{b})$$

for L-formulas $\varphi(\bar{x}, \bar{y})$ and parameters $\bar{b} \in M$. In other words, don't try and take $d\varphi(\bar{y})$ to be $\varphi(\bar{a}, \bar{y})$ where $\bar{a}$ is a realization of p. (Usually, this isn't allowed because $\bar{a}$ is not in M.)

Warning. *We only consider definable types over models ($p \in S_n(M)$). We don't consider definability of types over random sets ($p \in S_n(A)$), though this comes up in some of the optional extra notes. (Definable types over sets don't behave as well as definable types over models.)*

Remark 6. Definable types are connected to invariant types:

1. If a global type $p \in S_n(\mathbb{M})$ is A-definable for a small set A, then it is A-invariant.

2. If a global type $p \in S_n(\mathbb{M})$ is definable (i.e., $\mathbb{M}$-definable), then it is A-definable for some small A. (The definitions only use $|L|$-many parameters, which can be collected into some small set A.)

3. By the preceding two points, any global definable type is an invariant type.

4. A global type is A-definable iff it is definable and A-invariant.

5. If $A \subseteq M \subseteq \mathbb{M}$ and M is small, then there is a bijective correspondence between A-definable types over M and A-definable global types. In one direction $p \in S_n(\mathbb{M})$ maps to $p \restriction A \in S_n(M)$. In the other direction, $p \in S_n(M)$ maps to its global heir.

Warning. *Unlike invariant types, we consider definable types over both small sets and the monster.*

Let $p \in S_n(\mathbb{M})$ and $q \in S_m(\mathbb{M})$ be invariant global types. Then p is A-invariant and q is B-invariant for some small sets A, B. There is a unique invariant global type called $p \otimes q \in S_{n+m}(\mathbb{M})$ characterized by the following property:

If $C \supseteq A \cup B$ and C is small, then for any $\bar{a}, \bar{b} \in \mathbb{M}$,

$$(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C \iff \big(\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a}\big).$$

The following weaker property also characterizes $p \otimes q$:

If C is a sufficiently large small set, then for any $\bar{a}, \bar{b} \in \mathbb{M}$,

$$(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C \iff \big(\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a}\big).$$

In particular, $p \otimes q$ doesn't depend on the choice of A, B. (If p is A-invariant, then it is A'-invariant for some other $A' \neq A$, such as any superset $A' \supset A$.)

Remark 7. The property

$$(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C \iff \big(\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a}\big) \tag{$*$}$$

characterizes $(p \otimes q) \restriction C$. This is a natural thing to do if we think of global types as families of types over small sets as in Remark 1.

Warning. *We are only asserting ($*$) for small $C \subseteq \mathbb{M}$. I suppose it also holds for non-small $C \subseteq \mathbb{M}$, but then there's no guarantee you could find $\bar{a}, \bar{b}$ satisfying ($*$).*

Also, since q is a type over $\mathbb{M}$, line ($*$) doesn't make sense unless $\bar{a} \in \mathbb{M}$. (Otherwise, what is $q \restriction C\bar{a}$?) In particular, $\bar{a}$ shouldn't usually be a realization of p, since a realization of p usually won't live inside $\mathbb{M}$.

Fact 8. *If p, q are A-invariant global types, then $p \otimes q$ is A-invariant. Therefore, if p is A-invariant and q is B-invariant, then $p \otimes q$ is $(A \cup B)$-invariant.*

Fact 9. *If p, q are global definable types, then $p \otimes q$ is definable (not just invariant).*

3 Approach 2: varying the monster

Definition 10. Suppose M is a model and $A \subseteq M$. Suppose every n-type over A is realized in M for every $n < \omega$. (For example, this holds if M is κ-saturated and $|A| < \kappa$.) We say that $p \in S_n(M)$ is *A-invariant* if the following property holds:

- If $\bar{b}, \bar{c} \in M$ and $\bar{b} \equiv_A \bar{c}$ and $\varphi(\bar{x}, \bar{y})$ is a formula, then

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \varphi(\bar{x}, \bar{c}) \in p(\bar{x}).$$

For example, when M is the monster model and A is a small set, this generalizes the notion of "A-invariance" in Section 2.

Warning. *We only want to consider A-invariant types over models M such that M realizes every type over A. For example, we don't want to consider M-invariant types over M.*

Remark 11. Suppose M is κ-saturated and strongly κ-homogeneous, $A \subseteq M$, $|A| < \kappa$, and $p \in S_n(M)$. Then p is A-invariant iff

$$\forall \sigma \in \mathrm{Aut}(M/A)\ (\sigma(p) = p).$$

This explains the terminology "invariant." But this equivalence needn't hold in general without strong homogeneity. So perhaps the name "invariance" is a misnomer in the general case. For this reason, Poizat says "special" instead of "invariant."

Lemma 12. *Suppose $p \in S_n(M)$ is A-invariant*[3] *Let $N \succeq M$ be an elementary extension.*[4] *Then there is a unique $q \in S_n(M)$ extending p such that q is A-invariant.*

Proof. Given a formula $\varphi(\bar{x}, \bar{y})$ and a parameter $\bar{c} \in N$, we want to determine whether $\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$. Let $\bar{b}$ be a realization of $\mathrm{tp}(\bar{c}/A)$ inside M. Define $\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$ to hold iff $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$. This doesn't depend on the choice of $\bar{b}$ by A-invariance of p.

Now we have a well-defined set of $L(N)$-formulas $q(\bar{x})$ such that if $\bar{c} \in N$ and $\bar{b} \in M$ and $\bar{c} \equiv_A \bar{b}$, then

$$\varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \iff \varphi(\bar{x}, \bar{b}) \in p(\bar{x}). \tag{†}$$

First, we check $q(\bar{x})$ is finitely satisfiable. Otherwise, take a contradictory set of formulas $\{\varphi_i(\bar{x}, \bar{c}_i) : 1 \leq i \leq m\}$ in $q(\bar{x})$. Take $(\bar{b}_1, \ldots, \bar{b}_m) \in M$ realizing $\mathrm{tp}(\bar{c}_1, \ldots, \bar{c}_m/A)$. Then $\{\varphi_i(\bar{x}, \bar{b}_i) : 1 \leq i \leq m\} \subseteq p(\bar{x})$. As $p(\bar{x})$ is consistent,

$$M \models \exists \bar{x} \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, \bar{b}_i),$$

which then implies

$$N \models \exists \bar{x} \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, \bar{c}_i)$$

[3] ...and so M realizes every n-type over A for finite n.

[4] Note N realizes every n-type over A for finite n, because $N \supseteq M$.

because $(\bar{b}_1, \ldots, \bar{b}_m) \equiv (\bar{c}_1, \ldots, \bar{c}_m)$. This contradicts the fact that $\{\varphi_i(\bar{x}, \bar{c}_i) : 1 \leq i \leq m\}$ was contradictory.

Then $q(\bar{x})$ is finitely satisfiable, i.e., a partial type over N. It is also complete: if $\varphi(\bar{x}, \bar{c})$ is an $L(N)$-formula, take $\bar{b} \in M$ realizing $\mathrm{tp}(\bar{c}/A)$. Then

$$\begin{gathered}\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \text{ or } \neg\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \\ \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \implies \varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \\ \neg\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \implies \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x}).\end{gathered}$$

So one of $\varphi(\bar{x}, \bar{c})$ or $\neg\varphi(\bar{x}, \bar{c})$ is in $q(\bar{x})$. Thus $q(\bar{x})$ is a complete type.

Next, $q(\bar{x})$ extends $p(\bar{x})$ because if $\bar{c} \in M$, then $\bar{c} \equiv_A \bar{c}$ and so (†) gives

$$\varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \iff \varphi(\bar{x}, \bar{c}) \in p(\bar{x}).$$

Finally, $q(\bar{x})$ is A-invariant by construction—if $\bar{c} \equiv_A \bar{c}'$, then take $\bar{b} \in M$ realizing $\mathrm{tp}(\bar{c}/A) = \mathrm{tp}(\bar{c}'/A)$. Then (†) gives

$$\begin{gathered}\varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \iff \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \\ \varphi(\bar{x}, \bar{c}') \in q(\bar{x}) \iff \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\end{gathered}$$

and so

$$\varphi(\bar{x}, \bar{c}) \in q(\bar{x}) \iff \varphi(\bar{x}, \bar{c}') \in q(\bar{x}).$$

This proves existence. For uniqueness, note that any A-invariant $q \in S_n(N)$ extending p must satisfy (†), which uniquely determines q. $\square$

Example. If $p \in S_n(M)$ is A-definable, then the unique A-invariant extension to N is the heir of p, the A-definable type over N with the same definition schema as p.

Warning. *In Definition 10 and Lemma 12, it's really necessary to assume that M realizes every n-type over A. Here's what goes wrong if we omit this condition. Embed $M = (\mathbb{R}, \leq)$ into a monster model $N \models \mathrm{DLO}$. Let $A = \mathbb{Q} \subseteq M$. No two elements of $\mathbb{R}$ have the same type over $\mathbb{Q}$. Therefore, any type $p \in S_n(\mathbb{R})$ satisfies the definition of "$\mathbb{Q}$-invariance," if we ignore the fact that $\mathbb{R}$ doesn't realize all types over $\mathbb{Q}$. Then, one can prove the following:*

1. *If $\alpha \in \mathbb{R}$ is irrational, then the constant type $\mathrm{tp}(\alpha/\mathbb{R})$ has no $\mathbb{Q}$-invariant extension to N. (The only extension to N is the constant type $\mathrm{tp}(\alpha/N)$, which turns out to not be $\mathbb{Q}$-invariant.)*

2. *Suppose $\beta \in N$ is greater than every element of $\mathbb{R}$. Then $\mathrm{tp}(\beta/\mathbb{R})$ is the type at $+\infty$ over $\mathbb{R}$. It turns out that $\mathrm{tp}(\beta/\mathbb{R})$ has two distinct $\mathbb{Q}$-invariant extensions to N. (One is the coheir of $\mathrm{tp}(\beta/\mathbb{R})$ and the other is the heir.)*

Fix a model M, a subset $A \subseteq M$ such that every n-type over A is realized in M for all finite n, and fix an A-invariant type p. For any elementary extension $N \succeq M$, let $p|N$ denote the A-invariant extension of N as in Lemma 12. More generally, if $B \subseteq N$, let $p|B$ denote the restriction of $p|N$ to B. For example, if $B \subseteq M$ then $p|B = p \restriction B$.

Remark 13. We can think of $p|B$ as the type over B characterized by the fact that

$$\varphi(\bar{x}, \bar{c}) \in p|B \iff \varphi(\bar{x}, \bar{c}') \in p$$
$$\text{for } \bar{c}' \in M \text{ realizing } \mathrm{tp}(\bar{c}/A).$$

In particular, $p|B$ doesn't depend on the choice of N, in some sense. More precisely, $p|B$ doesn't change if we replace N with an elementary extension.

Remark 14. The point is that an A-invariant type p determines a family of types $p|B$ for each B, just like in Remark 1.

Here is how one defines Morley products from this point of view:

Definition 15. Let p, q be A-invariant types in M.[5] Take $\bar{a}$ realizing $p|M$ ($= p$) and $\bar{b}$ realizing $q|M\bar{a}$. (The realizations $\bar{a}, \bar{b}$ live in an elementary extension of M.) Define $p \otimes q$ to be $\mathrm{tp}(\bar{a}, \bar{b}/M)$.

Lemma 16. *This is well-defined, independent of the choice of $\bar{a}, \bar{b}$. Moreover, $p \otimes q$ is A-invariant.*

Proof. Take $\bar{a}$ realizing $p|M$ and $\bar{b}$ realizing $q|M\bar{a}$. Take $\bar{a}'$ realizing $p|M$ and $\bar{b}'$ realizing $q|M\bar{a}'$. Without loss of generality these realizations live in a monster model $\mathbb{M} \succeq M$.

Claim. $\mathrm{tp}(\bar{a}, \bar{b}/M) = \mathrm{tp}(\bar{a}', \bar{b}'/M)$.

Proof. Otherwise, there is $\bar{c} \in M$ and a formula $\varphi(\bar{x}, \bar{y}, \bar{c})$ such that

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \ \wedge \ \neg\varphi(\bar{a}', \bar{b}', \bar{c}).$$

This implies

$$\varphi(\bar{a}, \bar{y}, \bar{c}) \in \mathrm{tp}(\bar{b}/M\bar{a}) = q|M\bar{a} \subseteq q|\mathbb{M}$$
$$\neg\varphi(\bar{a}', \bar{y}, \bar{c}) \in \mathrm{tp}(\bar{b}'/M\bar{a}) = q|M\bar{a}' \subseteq q|\mathbb{M}.$$

Note $\bar{a}, \bar{a}'$ both realize p, so they have the same type over M: $\bar{a} \equiv_M \bar{a}'$ This implies $\bar{a} \equiv_{A\bar{c}} \bar{a}'$, because $A\bar{c}$ is a subset of M. This is equivalent to $\bar{a}\bar{c} \equiv_A \bar{a}'\bar{c}$. As $q|\mathbb{M}$ is A-invariant, we see

$$\varphi(\bar{a}, \bar{y}, \bar{c}) \in q|\mathbb{M} \iff \varphi(\bar{a}', \bar{y}, \bar{c}) \in q|\mathbb{M},$$

contradicting what is written above. $\square_{\text{Claim}}$

This shows that $p \otimes q$ is well-defined. Next, we claim that $p \otimes q$ is A-invariant. Otherwise,

$$p \otimes q \vdash \varphi(\bar{x}, \bar{y}, \bar{c}) \ \wedge \ \neg\varphi(\bar{x}, \bar{y}, \bar{c}')$$

for two $\bar{c}, \bar{c}' \in M$ with $\bar{c} \equiv_A \bar{c}'$. By definition of $p \otimes q$, this means

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \ \wedge \ \neg\varphi(\bar{a}, \bar{b}, \bar{c}').$$

[5]...where $M \supseteq A$ and every finite type over A is realized in M, as usual.

This implies

$$\varphi(\bar{a}, \bar{y}, \bar{c}) \in \operatorname{tp}(\bar{b}/M\bar{a}) = q|M\bar{a} \subseteq q|\mathbb{M}$$
$$\neg\varphi(\bar{a}, \bar{y}, \bar{c}') \in \operatorname{tp}(\bar{b}/M\bar{a}) = q|M\bar{a}' \subseteq q|\mathbb{M}.$$

If $\bar{a}\bar{c} \equiv_A \bar{a}\bar{c}'$, this contradicts A-invariance of $q|\mathbb{M}$. Otherwise, there is a formula $\psi(\bar{x}, \bar{z}, \bar{e})$ with $\bar{e} \in A$ such that

$$\mathbb{M} \models \psi(\bar{a}, \bar{c}, \bar{e}) \ \wedge \ \neg\psi(\bar{a}, \bar{c}', \bar{e}).$$

Because $\bar{a}$ realizes p, this implies

$$\psi(\bar{x}, \bar{c}, \bar{e}) \in p$$
$$\neg\psi(\bar{x}, \bar{c}', \bar{e}) \in p.$$

But $\bar{c} \equiv_A \bar{c}'$, so this contradicts A-invariance of p. □

Finally, we show that this definition of Morley product agrees with the one we have used before:

Lemma 17. *Suppose $\mathbb{M}$ is a monster model and A is a small set. Let p, q be global A-invariant types. Then the $p \otimes q$ of Definition 15 agrees with $p \otimes q$ of Section 2.*

Proof. Take an $|\mathbb{M}|^+$-saturated elementary extension $N \succeq \mathbb{M}$. Take $\bar{a}_0 \in N$ realizing $p = p|\mathbb{M}$, and take $\bar{b}_0 \in N$ realizing $q|\mathbb{M}\bar{a}_0$. Let $r = \operatorname{tp}(\bar{a}_0, \bar{b}_0/\mathbb{M})$. Then r is the Morley product as defined in Definition 15. Let s be the Morley product as defined in Section 2, i.e., as defined in class. Suppose $s \neq r$. Take a formula $\varphi(\bar{x}, \bar{y}, \bar{c})$ with $\bar{c} \in \mathbb{M}$ such that

$$\varphi(\bar{x}, \bar{y}, \bar{c}) \in s(\bar{x}, \bar{y}) \iff\!\!\!\!\!\!\!\!/\;\;\;\; \varphi(\bar{x}, \bar{y}, \bar{c}) \in r(\bar{x}, \bar{y}).$$

Let B be the small set $A\bar{c} \subseteq \mathbb{M}$. Take $\bar{a} \models p \restriction B$ and $\bar{b} \models q \restriction B\bar{a}$. Then $(\bar{a}, \bar{b}) \models s \restriction B$.

Note $\bar{a}, \bar{a}_0$ both realize $p \restriction B$, as $B \subseteq \mathbb{M}$. Therefore $\bar{a} \equiv_B \bar{a}_0$, which means $\bar{a}\bar{c} \equiv_A \bar{a}_0\bar{c}$. We have the following equivalences:

$$\varphi(\bar{x}, \bar{y}, \bar{c}) \in s(\bar{x}, \bar{y}) \iff \varphi(\bar{x}, \bar{y}, \bar{c}) \in s \restriction B$$
$$\iff \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \iff \varphi(\bar{a}, \bar{y}, \bar{c}) \in q \restriction B\bar{a}$$
$$\iff \varphi(\bar{a}, \bar{y}, \bar{c}) \in q|N \iff \varphi(\bar{a}_0, \bar{y}, \bar{c}) \in q|N$$
$$\iff \varphi(\bar{a}_0, \bar{y}, \bar{c}) \in q|\mathbb{M}\bar{a}_0 \iff N \models \varphi(\bar{a}_0, \bar{b}_0, \bar{c})$$
$$\iff \varphi(\bar{x}, \bar{y}, \bar{c}) \in r(\bar{x}, \bar{y})$$

(using the facts that $s \restriction B = \operatorname{tp}(\bar{a}, \bar{b}/B)$, $q \restriction B\bar{a} = \operatorname{tp}(\bar{b}/B\bar{a})$, $q|N$ is A-invariant, $q|\mathbb{M}\bar{a}_0 = \operatorname{tp}(\bar{b}_0/\mathbb{M}\bar{a}_0)$, and $r = \operatorname{tp}(\bar{a}_0, \bar{b}_0/\mathbb{M})$). This contradicts the choice of φ. □