Seminar on Topological Dynamics of Definable Group Actions

Section3(part)

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1 transitive point in $2^{\mathbb{Q}}$

Definition 1.

1. By $2^{\mathbb{Q}}$, we mean a set of functions from $\mathbb{Q}$ to $\{0,1\}$, or powerset of $\mathbb{Q}$, each element includes some numbers(maps them to 1) and excludes other numbers(maps to 0).

2. $2^{\mathbb{Q}}$ is the product of countably many discrete spaces $2 = \{0,1\}$. Open sets of $2^{\mathbb{Q}}$ have the form $[\sigma] = \{f : \sigma \subseteq f \in 2^{\mathbb{Q}}\}$,where $\sigma \in 2^{<\mathbb{Q}}$,

3. Consider additive group $(\mathbb{Q}, +)$ acting on $2^{\mathbb{Q}}$. Define $q*f \in 2^{\mathbb{Q}}$ by $(q*f)(x) = f(x-q)$. Or, for any $q \in \mathbb{Q}$, $A \in 2^{\mathbb{Q}}$, $q * A := q + A = \{q + a : a \in A \subseteq \mathbb{Q}\}$.

4. Say G-flow X is point transitive, if there is $x_0 \in X$, such that $G * x_0$ is dense in X, or $\mathrm{cl}(G * x_0)$=X.

Theorem 2. *$2^{\mathbb{Q}}$ is a point transitive $\mathbb{Q}$-flow. There is $\eta \in 2^{\mathbb{Q}}$ such that $\mathbb{Q} * \eta$ is dense in $2^{\mathbb{Q}}$.*

Proof. Say $\mathbb{Q} * \eta$ is dense in $2^{\mathbb{Q}}$, we mean for any $f \in 2^{\mathbb{Q}}$, take its any open neighbourhood $[\sigma] \ni f$, it must meet $\mathbb{Q} * \eta$, i.e., there is some $q \in \mathbb{Q}$ such that $q * \eta \in [\sigma]$.

Every open set concerns with only finitely many points on $\mathbb{Q}$. $\mathbb{Q}$ is countable, thus there is a sequence $\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \cdots$, with each $|\sigma_i| = i$, and for any open sets $[\nu]$, there is some σ_i extends ν, or $[\sigma_i] \subseteq [\nu]$.

It suffices to show for every σ_i in the enumeration, there is η such that any $[\nu]$ generated from $\nu \subseteq \sigma_i$ meets $\mathbb{Q} * \eta$. Induction:

σ_1 case: take such $\eta_1 = \sigma_1$. For any η extends η_1, $0 * \eta \in [\sigma_1]$

σ_{n+1} case: Consider $\nu' \in \sigma_{n+1}$. If $\nu' \in \sigma_n$, by induction hypothesis, there is η_n such that $[\nu']$ meets $\mathbb{Q} * [\eta_n]$.

Otherwise, it has the form $\nu \cup \{(r_{n+1}, s_{n+1})\}$, where $\nu \in \sigma_n$. Thus there is η_n and some $q \in \mathbb{Q}$, $q * [\eta_n]$ meets $[\nu]$. Move η_n sufficiently far away, then add the new element. Define η'_n be such that $\eta'_n(x-p) = \eta_n(x)$ for $x \in$ domain of η_n, and $\eta'_n(r_{n+1} - p - q) = s_{n+1}$. Thus $(p+q) * [\eta'_n]$ meets $[\nu']$. For every distinct ν', we can construct distinct η'_n, move it by p to make them disjoint. Take their union with η_n, we got η_{n+1}.

Hence we have $[\eta_1] \supseteq [\eta_2] \supseteq \cdots$, they are all clopen sets, so their intersection is not empty. Denote it by η, $\mathbb{Q} * \eta$ is dense in $2^{\mathbb{Q}}$. $\square$

$(\mathbb{Q}, +)$ has no proper subgroup of finite index, so there are no finite orbits in $2^{\mathbb{Q}}$ except two 1-element orbits of two constant functions(all maps to 0 or all maps to 1).

Remark 3. Let X be a G-flow.

1. U is weak generic iff for some finite $A \subseteq G$, $X - AU$ is not generic. If U' is also weak generic, then $U \cup U'$ is still weak generic.

2. If $\mathrm{cl}(o(x))$ is a minimal flow of X, then $x \in X$ is almost periodic. $p \in X$ is almost periodic iff for every open $U \ni p$, there is finite $A \subseteq G$ such that $\mathrm{cl}(Gp)$ is covered by AU.

For every $\sigma \in 2^{<\mathbb{Q}}$, $[\sigma]$ is not generic. Collection of sets $[\sigma'] : \sigma' \in 2^{Dom(\sigma)}$ is finite, and its union cover whole $2^{\mathbb{Q}}$, so every open set is weak generic, hence every point in $2^{\mathbb{Q}}$ is weak generic.

If $f \in 2^{\mathbb{Q}}$ is a periodic and piece-wise constant function. For the periodic part, it can only be divided into finitely many pieces, of which function value is constant. Otherwise part of the function could be "small size" $\mathbb{Q}$. Then $\mathrm{cl}(\mathbb{Q} * f)$ is a proper subflow of $2^{\mathbb{Q}}$, hence $\mathrm{cl}(\mathbb{Q} * \eta)$ has proper subflow, η is not almost periodic.

2 Model theoretic corresponding of $\mathbb{Q}$-flow $2^{\mathbb{Q}}$

Identify $\mathbb{Q}$ with orbit $\mathbb{Q} * \eta$, expand group $(\mathbb{Q}, +)$ with some predicates. For $\sigma \in 2^{<\mathbb{Q}}$, define $P_\sigma(x)$ on $\mathbb{Q}$ by:

$$P_\sigma(x) \Leftrightarrow x * \eta \in [\sigma]$$

Let M be model of expansion of $(\mathbb{Q}, +, 0)$ by unary predicates $P_\sigma(x)$.Let $T_\eta = Th(M)$.

Remark 4.

1. For all $q \in \mathbb{Q}$, $P_\sigma(x - q)$ is equivalent in M to $P_{q*\sigma}(x)$.

2. $\neg P_\sigma(x)$ is equivalent to disjunction $\bigvee\{P_\nu(x) : \nu \in 2^X - \{\sigma\}\}$, where $X = Dom(\sigma)$.

3. If $\sigma_1, \sigma_2 \in 2^{\mathbb{Q}}$ are compatible, then $P_{\sigma_1}(x) \wedge P_{\sigma_2}(x)$ is equivalent in M to $P_{\sigma_1 \cup \sigma_2}(x)$, otherwise inconsistent.

η is not unique. Additionally, we require η be "sufficiently generic". (It seems to have an accurate description in forcing, but I'm not familiar with that. Several properties used in QE proof below can be written as forcing condition.)

Proposition 5. *If η is sufficiently generic, then T_η admits elimination of quantifiers.*

Proof. Every formula can be written in a prenex normal form:

$$\psi(\bar{x}) \Leftrightarrow Q_1 y_1 Q_2 y_2 \cdots Q_n y_n \bigvee_j \bigwedge_i \theta_{ij}(\bar{x}, \bar{y})$$

Since existential quantifier can be distributed into disjunction, it suffices to consider

$$\exists y \varphi(\bar{x}, y) = \exists y \bigwedge_i \theta_i(\bar{x}, y)$$

is equivalent in M to a quantifier free formula, where each θ is atomic or negation of an atomic formula.

Each formula in the form $\neg P_\sigma(\cdots)$ is equivalent to $\bigvee P_\nu(\cdots)$, formula $y = t(\bar{x}) \wedge \varphi(\bar{x}, y)$ is equivalent to $\varphi(\bar{x}, t(\bar{x}))$. And denote $P_\sigma(q \cdot x)$ by $P_\sigma^q(x)$.
After these simplifications, atomic formulas are in the form $P_\sigma^q(t(\bar{x}) + y)$ or $y \neq t(\bar{x})$. Now $\varphi(\bar{x}, y)$ equals

$$\bigwedge_{i \in I} P_{\sigma_i}^{q_i}(t_i(\bar{x}) + y) \wedge \bigwedge_{j \in J} y \neq t_j(\bar{x})$$

If $P_\sigma^q(t(\bar{x} + y))$ holds in M, then $(q \cdot (t(\bar{x}) + y)) * \eta \in [\sigma]$, $\eta \in (-q \cdot t(\bar{x}) - q \cdot y) * [\sigma]$. Varying $y \in \mathbb{Q}$, by choosing sufficiently generic η, if there is one such y, then there are infinitely many such y. Thus we assume

$$\exists y (\bigwedge_{i \in I} P_{\sigma_i}^{q_i}(t_i(\bar{x}) + y) \wedge \bigwedge_{j \in J} y \neq t_j(\bar{x})) \Leftrightarrow \exists y (\bigwedge_{i \in I} P_{\sigma_i}^{q_i}(t_i(\bar{x}) + y))$$

Consider the formula $\varphi(\bar{x}, y)$:

$$\bigwedge_{i \in I} P_{\sigma_i}^{q_i}(t_i(\bar{x}) + y)$$

some q_i have same value, rewrite the conjunction and group conjunctive branch with same q value together. Suppose $q^0, \cdots, q^n$ are all distinct q values.

$$\varphi_t(\bar{x}, y) = \bigwedge \{P_{\sigma_i}^{q_i}(t_i(\bar{x}) + y) : q_i = q^t\}, \varphi(\bar{x}, y) = \bigwedge_{t < n} \varphi_t(\bar{x}, y)$$

By genericity of η, for every $\bar{a} \subseteq M$:

1. If for some y $M \models \varphi_t(\bar{a}, y)$, then for every $q_i = q^t$, $M \models P_{\sigma_i}^{q_i}(t_i(\bar{a}+y))$, or $q^t \cdot (t_i(\bar{a} + y)) * \eta \in [\sigma_i]$, $y * \eta \in -t_i(\bar{a}) * [\sigma_i]$. Equivalently, $\{(-t_i(\bar{a}) * \sigma_i : q_i = q^t\}$ is compatible. It's quantifier free expressible.

2. If every $\exists y \varphi_t(\bar{a}, y)$ is true in M, then also $M \models \exists y \varphi(x, y)$ witnessed by infinitely many y.

Thus we prove T_η admits quantifier elimination. $\square$

For conjunction $\bigwedge_i P_{\sigma_i}(t_i(\bar{x}))$, if some x realize it, by genericity of η, there are infinitely many x realizing this formula. So it is either inconsistent in M or non-algebraic. T_η is simple theory of SU-rank 1. It's an expansion of the strongly minimal vector space over $\mathbb{Q}$ by some "semi-generic" predicates.

Consider type $tp(a/M)$, where a is from a monster model. Denote $f_q = \bigcup\{\sigma \in 2^{<\mathbb{Q}} : \models P_\sigma(q \cdot a)\}$. $q \neq 1$, then $tp(a/M)$ is determined by sequence $< f_q : q \in \mathbb{Q}^* >$.

It corresponds to the non-algebraic type $p_f \in S_1(M)$ given by

$$p_f(x) := \{x \neq a : a \in M\} \cup \{P_{f_q|X}(q \cdot x) : X \subseteq_{fin} \mathbb{Q} \wedge q \in \mathbb{Q}*\}$$

Type space $S_1(M)$ is a point transitive $\mathbb{Q}$-flow, all non-algebraic types in $S_1(M)$ are weak generic. So $WGen(M)$ is the set of non-algebraic types in $S_1(M)$.

3 Forcing version of η

Open sets on $2^{\mathbb{Q}}$ has the form $[\sigma]$, where each $\sigma \in 2^{<\mathbb{Q}}$. They form a poset, maximal $[\emptyset] = 2^{\mathbb{Q}}$. There are countably many finite functions σ, so the poset is countable. Thus we can find a generic filter $\{[\sigma]\}$ where these σ are compatible. Take their union, we get η and $\mathbb{Q} * \eta$ is dense in $2^{\mathbb{Q}}$. We can further add some forcing conditions.