Cook-Levin Theorem

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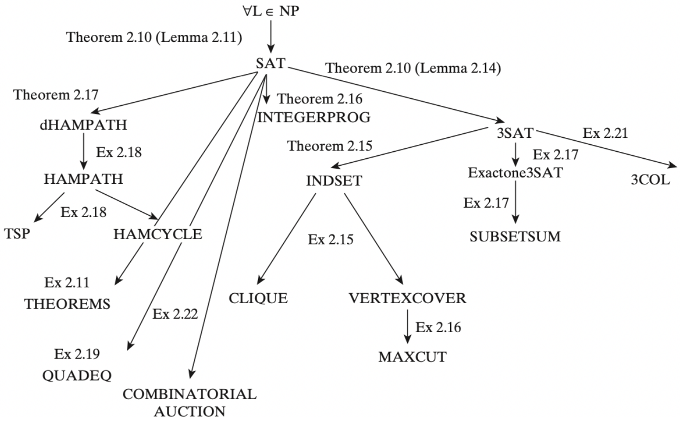
Outline

Theorem (Cook-Levin Theorem)

1. *SAT is **NP**-complete*
2. *3SAT is **NP**-complete*

The web of reductions



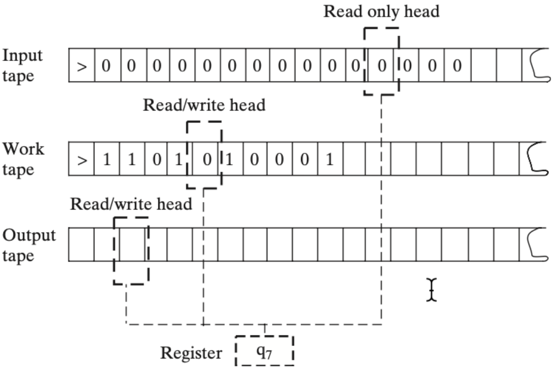
Turing machine

Definition

A TM M is described by a tuple (Γ, Q, δ) containing

- A finite set Γ of the symbols that M's tapes can contain. We assume that Γ contains a designated "blank" symbol, denoted $\square$; a designated "start" symbol, denoted $\triangleright$; and the numbers 0 and 1. We call Γ the alphabet of M
- A finite set Q of possible states M' register can be in. We assume that Q contains a designated start state, denoted q_{start}, and a designated halting state, denoted q_{halt}
- A function $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{\text{L,S,R}\}^k$, where $k \geq 2$, describing the rules M use in performing each step. This function is called the transition function of M

Turing machine



Efficiency and running time

Definition (Computing a function and running time)

Let $f : \{0,1\}^* \to \{0,1\}^*$ and let $T : \mathbb{N} \to \mathbb{N}$ be some functions, and let M be a Turing machine. We say that M computes f if for every $x \in \{0,1\}^*$ whenever M is initialized to the start configuration on input x, then it halts with $f(x)$ written on its output tape. We say M computes f in $T(|x|)$-time if its computation on every input x requires at most $T(|x|)$ steps

The class P

A complexity class is a set of functions that can be computed within given resource bounds. We say that a machine decides a language $L \subseteq \{0,1\}^*$ if it computes the function $f_L : \{0,1\}^* \to \{0,1\}$ where $f_L(x) = 1 \Leftrightarrow x \in L$

Definition

Let $T : \mathbb{N} \to \mathbb{N}$ be some function. A language L is in **DTIME**$(T(n))$ iff there is a deterministic Turing machine that runs in time $c \cdot T(n)$ for some constant $c > 0$ and decides L

Definition

P $= \bigcup_{c \geq 1}$ **DTIME**(n^c)

The class **NP**

Definition

A language $L \subseteq \{0,1\}^*$ is in **NP** if there exists a polynomial function $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M (called the verifier for L) such that for every $x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x,u) = 1$, then we call u a certificate for x w.r.t. L and M

Non-deterministic Turing machine

Definition

Non-deterministic Turing machine has two transition function δ_0 and δ_1, and a special state denoted by q_{accept}. When an NDTM M computes a function, we envision that at each computational step M makes an arbitrary choice at to which of its two transition functions to apply. For every input x, we say that $M(x) = 1$ if there exists some sequence of this choices that would make M reach q_{accept} on input x. We say that M runs in $T(n)$ time if for every input $x \in \{0,1\}^*$ and every sequence of nondeterministic choices, M reaches the halting state or q_{accept} within $T(|x|)$ steps

The class **NP**

Definition

For every function $T : \mathbb{N} \to \mathbb{N}$ and $L \subseteq \{0,1\}^*$, we say that $L \in \mathbf{NTIME}(T(n))$ if there is a constant $c > 0$ and a $c \cdot T(n)$-time NDTM M s.t. for every $x \in \{0,1\}^*$, $x \in L \Leftrightarrow M(x) = 1$

Theorem

$\boldsymbol{NP} = \bigcup_{c \in \mathbb{N}} \boldsymbol{NTIME}(n^c)$

Proof.

The main idea is that the sequence of nondeterministic choices made by an accepting computation of an NDTM can be viewed as a certificate that the input is in the language, and vice versa □

Reducibility

Definition

A language $L \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to a language $L' \subseteq \{0,1\}^*$ (sometimes shortened to just "polynomial-time reducible"), denoted by $L \leq_p L'$ if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ s.t. for every $x \in \{0,1\}^*$, $x \in L$ iff $f(x) \in L'$
We say that L' is **NP**-hard if $L \leq_p L'$ for every $L \in$ **NP**. We say that L' is **NP**-complete if L' is **NP**-hard and $L' \in$ **NP**

Goal

We denote by SAT the language of all satisfiable CNF (conjunction normal form) formulae and by 3SAT the language of all satisfiable 3CNF formulae

Theorem (Cook-Levin Theorem)

1. *SAT is **NP**-complete*
2. *3SAT is **NP**-complete*

Oblivious Turing machine

Definition

Define a TM M to be oblivious if its head movements do not depend on the input but only on the input length. That is, M is oblivious if for every input $x \in \{0,1\}^*$ and $i \in \mathbb{N}$, the location of each of M's heads at the ith step of execution on input x is only a function of $|x|$ and i.

Theorem

For any Turing machine M that decides a language in time $T(n)$, there exists an oblivious Turing machine that decides the same language in $T(n)^2$

A lemma

Lemma

For every Boolean function $f : \{0,1\}^l \to \{0,1\}$, there is an l-variable CNF formula φ of size $l2^l$ s.t. $\varphi(u) = f(u)$ for every $u \in \{0,1\}^l$, where the size of a CNF formula is defined to be the number of $\wedge/\vee$ symbols it contains

Proof.

For every $v \in \{0,1\}^l$, there exists a clause $C_v(z_1, \ldots, z_l)$ s.t. $C_v(v) = 0$ and $C_v(u) = 1$ for every $u \neq v$.
We let φ be the AND of all the clauses C_v for v s.t. $f(v) = 0$

$$\varphi = \bigwedge_{v: f(v)=0} C_v(z_1, \ldots, z_l)$$

Note that φ has size at most $l2^l$. □

Main lemma

Lemma

*SAT is **NP**-hard*

Proof.

Let L be an **NP** language. By definition, there is a polynomial time TM M s.t. for every $x \in \{0,1\}^*$, $x \in L \Leftrightarrow M(x,u) = 1$ for some $u \in \{0,1\}^{p(|x|)}$, where $p : \mathbb{N} \to \mathbb{N}$ is some polynomial. We show L is polynomial-time Karp reducible to $\mathtt{SAT}$ by describing a polynomial-time transformation $x \to \varphi_x$ from strings to CNF formulae s.t. $x \in L$ iff φ_x is satisfiable. Equivalently

$$\varphi_x \in \mathtt{SAT} \quad \text{iff} \quad \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x \circ u) = 1$$

where $\circ$ denotes concatenation $\square$

Assumption

Assume

1. M only has two tapes - an input tape and a work/output tape
2. M is an oblivious TM in the sense that its head movement does not depend on the contents of its tapes. That is, M's computation takes the same time for all inputs of size n, and for every i the location of M's head at the ith step depends only on i and the length of the input

Proof

Denote by Q the set of M's possible states and by Γ its alphabet. The snapshot of M's execution on some input y at a particular step i is the triple $\langle a, b, q\rangle \in \Gamma \times \Gamma \times Q$ s.t. a, b are the symbols read by M's heads from the two tapes and q is the state M is in at the ith step. Clearly the snapshot can be encoded as a binary string. Let c denote the length of this string, which is some constant depending upon $|Q|$ and $|\Gamma|$

Proof

For every $y \in \{0,1\}^*$, the snapshot of M's execution on input y at the ith step depends on its state in the $(i-1)$st step and the contents of the current cells of its input and work tapes.
And it suffices to check that for each $i \leq T(n)$, the snapshot z_i is correct given the snapshot for the previous $i-1$ steps.

Proof

However, since the TM can only read/modify one bit at a time, to check the correctness of z_i it suffices to look at only *two* of the previous snapshots. Specifically, to check z_i we need to only look at the following: z_{i-1}, $y_{\text{inputpos}(i)}$, $z_{\text{prev}(i)}$.
Here y is a shorthand for $x \circ u$. inputpos(i) denotes the location of M's input tape head at the ith step. prev(i) is the last step before i when M's head was in the same cell on its work tape that it is during step i.

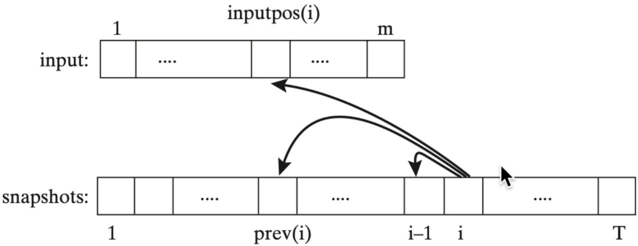
Proof

Since M is a deterministic TM, for every triple of values to $z_{i-1}, y_{\mathsf{inputpos}(i)}$, $z_{\mathsf{prev}(i)}$, there is at most one value of z_i that is correct. Thus there is some function F that maps $\{0,1\}^{2c+1}$ to $\{0,1\}^c$ s.t. a correct z_i satisfies

$$z_i = F(z_{i-1}, z_{\mathsf{prev}(i)}, y_{\mathsf{inputpos}(i)})$$

Because M is oblivious, the values inputpos(i) and prev(i) do not depend on the particular input i. These indices can be computed in polynomial-time by simulating M on a trivial input.

Proof



Proof

Now $M(x \circ u) = 1$ for some $u \in \{0,1\}^{p(n)}$ iff there exists a string $y \in \{0,1\}^{n+p(n)}$ and a sequence of strings $z_1, \ldots, z_{T(n)} \in \{0,1\}^c$ (where $T(n)$ is the number of steps M takes on inputs of length $n + p(n)$) satisfying the following conditions

1. The first n bits of y are equal to x
2. The string z_1 encodes the initial snapshot of M. That is, z_1 encodes the triple $\langle \triangleright, \square, q_{\mathsf{start}} \rangle$.
3. For every $i \in \{2, \ldots, T(n)\}$, $z_i = F(z_{i-1}, z_{\mathsf{prev}(i)}, y_{\mathsf{inputpos}(i)})$.
4. The last string $z_{T(n)}$ encodes a snapshot where the machine halts and outputs 1

Analysis

- The formula φ_x will take variables $y \in \{0,1\}^{n+p(n)}$ and $z \in \{0,1\}^{cT(n)}$.
- Condition 1 can be expressed as a CNF formula of size $4n$
- Conditions 2 and 4 each depend on c variables and hence can be expressed by CNF formulae of size $c2^c$
- Condition 3, which is an AND of $T(n)$ conditions each depending on at most $3c+1$ variables, can be expressed as a CNF formula of size at most $T(n)(3c+1)2^{3c+1}$.
- ALL these conditions can be expressed as a CNF formula of size $d(n+T(n))$ where d is some constant
- this CNF formula can be computed in time polynomial in the running time of M.

3SAT

Lemma

SAT $\leq_p$ *3SAT*

Proof.

Suppose φ is a 4CNF. Let C be a clause of φ, say $C = u_1 \vee \bar{u}_2 \vee \bar{u}_3 \vee u_4$. We add a new variable z to the φ and replace C with the pair $C_1 = u_1 \vee \bar{u}_2 \vee z$ and $C_2 = \bar{u}_3 \vee u_4 \vee \bar{z}$. If C is true, then there is an assignment to z that satisfies both C_1 and C_2. If C is false, then no matter what value we assign to z either C_1 or C_2 will be false.
For every clause C of size $k > 3$, we change it into an equivalent pair of clauses C_1 of size $k - 1$ and C_2 of size 3. $\square$

The web of reductions