Homework11

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Exercise 1. Show that the field $\mathbb{C}$ is strongly $|\mathbb{C}|$-homogeneous

Proof. For any partial elementary map $f : A \to B$ where $A, B \subseteq \mathbb{C}$ and $|A| < \mathfrak{c}$, we can enumerate $\mathbb{C}$ as $(c_\alpha : \alpha < \mathfrak{c})$. We build a sequence of $(f_\alpha : \alpha < \mathfrak{c})$

$$f = f_0 \subset f_1 \subset f_2 \subset \cdots$$

s.t. $|\mathrm{dom}(f_\alpha)| < \mathfrak{c}$ and each f_α is partial elementary for each $\alpha < \mathfrak{c}$. If $\alpha = \beta \cdot \omega + 2n + 1$ where $n \in \omega$, since $\mathrm{dom}(f_{\beta+2n}) < \mathfrak{c}$, there is $b \in \mathbb{C}$ s.t. $f_{\beta+2n} \cup \{(c_{\beta+n}, b)\}$ is partial elementary. Let $f_\alpha = f_{\beta+2n} \cup \{(c_{\beta+1}, b)\}$. If $\alpha = \beta \cdot \omega + 2n + 2$, there is $b' \in \mathbb{C}$ s.t. $f^{-1}_{\beta+2n+1} \cup \{(b', c_{\beta+1})\}$ is partial elementary. Let $f_\alpha = f_{\beta+2n+1} \cup \{(c_{\beta+1}, b')\}$. If α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then $|\mathrm{dom}(f_\alpha)| \le |A| + |\alpha| < \mathfrak{c}$.

Let $g = \bigcup_{\alpha<\mathfrak{c}} f_\alpha$. Then g is partial elementary with $\mathrm{dom}(g) = \mathrm{im}(g) = \mathfrak{c}$. Thus $g \in \mathrm{Aut}(\mathbb{C})$ □

Exercise 2. Show that the field $\mathbb{R}$ is strongly κ-homogeneous for any cardinal κ

Proof. Suppose a partial elementary map $f : A \to B$ where $A, B \subseteq \mathbb{R}$. Since we are working in a field, $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1)$.

Given $a \in A$, first we show that $f(a) = a$.

First note that for any integers $n \in \mathbb{N}$,

$$n := \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

Then integers are definable via 1 since $\mathbb{R}$ is an abelian group and has a unique additive inverse. Then since $\mathbb{R}$ has unique multiplicative inverse, each $q \in \mathbb{Q}$ is expressble via 1.

Also, we can define $a \le b$ by $\exists z(a + z \cdot z = b)$. Thus for all $q \in \mathbb{Q}$, if $q < a$, then $q < x \in \text{tp}(a)$ and if $q > a$, then $q > x \in \text{tp}(a)$. Thus $f(a) = a$ as $\text{tp}(a) = \text{tp}(f(a))$.

Thus we can extend f to the identity function of $\mathbb{R}$. □

Exercise 3. Let $S = \{0, 1\} \times \mathbb{Z}$ and let $\le$ be the lexicographic order on S:

$$(0, x) < (1, y)$$
$$(0, x) \le (0, y) \Leftrightarrow x \le y$$
$$(1, x) \le (1, y) \Leftrightarrow x \le y$$

Show that $(S, \le)$ is not strongly ω-homogeneous, but some expansion of $(S, \le)$ is strongly ω-homogeneous

Proof. Consider the map $f = \{((0, 0), (1, 0))\}$. f is a partial elementary if and only if $(S, (0, 0)) \equiv (S, (1, 0))$, which is true by Theorem 1.8 on book. If there is $f \subset \sigma \in \text{Aut}(S)$, then $d((0, 0), (1, 0)) = \infty$ but $d(\sigma(0, 0), \sigma(1, 0))$ is finite, a contradiction. Thus $(S, \le)$ is not strongly ω-homogeneous.

We claim that for any $a, b \in \mathbb{Z}$, $(S, \le, (0, a), (1, b))$ is strongly ω-homogeneous. Note that identity function is the only automorphism since automorphism needs to respect the distance function and $\le$ relation, which will uniquely determine an element given $(0, a)$ and $(1, b)$. Thus any finite elementary map is a subset of the identity function and we can extend it. □

$T = \text{Th}(\mathbb{R}, +, \cdot, 0)$

Exercise 4. Suppose $M \models T$. Show that there is at most one linear order $\le$ on M s.t. the following hold

- If $x \le y$, then $x + z \le y + z$
- If $x \le y$ and $0 \le z$, then $xz \le yz$

Proof. Suppose there is a linear order R on M s.t.

- $\forall z(xRy \to (x + z)R(y + z))$
- $xRy \wedge 0Rz \to xzRyz$

Fist note that $0Ra \leftrightarrow (-a)R0$. If $0Ra$, then $(-a)R0$ and so $(-a^2)R0$. If $aR0$, then $0R(-a)$ and so $(-a^2)R0$. Therefore for all a, $(-a^2)R0$ and so $0Ra^2$. Hence for all a, $0 \le a \Leftrightarrow a = b^2 \Leftrightarrow 0Ra$. Then $a \le b \Leftrightarrow (a - b) \le 0 \Leftrightarrow (a - b)R0 \Leftrightarrow aRb$. □

Proof. Write down an explicit definition of $\leq$ in $(\mathbb{R}, +, \cdot)$ $\square$

Proof. Let $\phi(x, y) := \exists z(x + z \cdot z = y)$.

- If $\phi(x, y)$, then let $z \in \mathbb{R}$ s.t. $x+z\cdot z = y$, for any $c \in \mathbb{R}$, $x+c+z\cdot z = y+c$, hence $\phi(x + c, y + c)$
- If $\phi(x, y)$ and $\phi(0, z)$, then there is $a, b \in \mathbb{R}$ s.t. $x + a \cdot a = y$ and $0+b\cdot b = z$, thus $yz = (x+a\cdot a)z = xz+a\cdot a\cdot b\cdot b = xz+(a\cdot b)\cdot(a\cdot b)$, hence $\phi(xz, yz)$
- If $x \leq y$, $\phi(x, y)$ is clear

$\square$