Stable theories

Advanced Model Theory

March 3, 2022

Reference in the book: Theorem 11.4 in Section 11.1, and Section 11.4

1 Strong heirs from ultrapowers

Recall the structures (M, dp) from last week, where

$$(M, dp) \models d\varphi(\bar{b}) \iff \varphi(\bar{x}, \bar{b}) \in p(\bar{x}).$$

We showed that every elementary extension of (M, dp) has the form (N, dq) for some $N \succeq M$ and some heir $q \sqsupseteq p$. We say that q is a *strong heir* of p if $(N, dq) \succeq (M, dp)$.

Definition 1. Suppose $p \in S_n(M)$, I is a set, and $\mathcal{U}$ is an ultrafilter on I. Let $M^{\mathcal{U}}$ be the ultrapower. Then the *ultrapower type* $p^{\mathcal{U}} \in S_n(M^{\mathcal{U}})$ is the strong heir of p such that $(M^{\mathcal{U}}, dp^{\mathcal{U}}) = (M, dp)^{\mathcal{U}}$.

Here is a more explicit description of $p^{\mathcal{U}}$. If $\varphi(\bar{x}, \bar{y})$ is a formula and $\bar{b} \in M^{\mathcal{U}}$ is the class of $(\bar{b}_i : i \in I)$, then

$$\begin{aligned}
\varphi(\bar{x}, \bar{b}) \in p^{\mathcal{U}} &\iff (M, dp)^{\mathcal{U}} \models d\varphi(\bar{b}) \\
&\iff \{i \in I : (M, dp) \models d\varphi(\bar{b}_i)\} \in \mathcal{U} \\
&\iff \{i \in I : \varphi(\bar{x}, \bar{b}_i) \in p(\bar{x})\} \in \mathcal{U}.
\end{aligned}$$

Proposition 2. *Suppose $M \preceq N$, $p \in S_n(M)$, and $q \in S_n(N)$ is an heir of p. Then there is (a copy of) an ultrapower M^U such that $M \preceq N \preceq M^U$ and $p \subseteq q \subseteq p^U$.*

Proof. Let I be the set of functions from N to M extending $\mathrm{id}_M : M \to M$. Note that if $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$ for some $\bar{b} \in N$, then there is $f \in I$ such that $\varphi(\bar{x}, f(\bar{b})) \in p(\bar{x})$, because $q \sqsupseteq p$.[1] For each L-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N$, let

$$S_{\varphi,\bar{b}} = \{f \in I : \varphi(\bar{x}, f(\bar{b})) \in p(\bar{x})\}.$$

Let $\mathcal{F} = \{S_{\varphi,\bar{b}} : \varphi(\bar{x}, \bar{b}) \in q(\bar{x})\}$.

[1]Something subtle is going on here, but this really does work if you think about it. The reason is that any formula $\varphi(\bar{x}, \bar{b})$ with $\bar{b} \in N$ can be written as $\varphi'(\bar{x}, \bar{b}', \bar{c})$ for some formula φ', where $\bar{b}'$ is a tuple of distinct elements in M, and $\bar{c}$ is a tuple of distinct elements in $N \setminus M$. As $q \sqsupseteq p$, there is $\bar{c}' \in M$ such that $\varphi'(\bar{x}, \bar{b}', \bar{c}') \in p(\bar{x})$. Then we can choose $f \in I$ such that $f(\bar{c}) = \bar{c}'$, because $\bar{c}$ is a tuple of *distinct* elements

Claim. $\mathcal{F}$ has FIP.

Proof. Suppose $\varphi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$ for $1 \leq i \leq n$. Then $\bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$. Take $f \in I$ such that $\bigwedge_{i=1}^n \varphi_i(\bar{x}, f(\bar{b}_i)) \in p(\bar{x})$. Then $f \in \bigcap_{i=1}^n S_{\varphi_i, \bar{b}_i}$. $\square_{\text{Claim}}$

Take $\mathcal{U}$ an ultrafilter on I extending $\mathcal{F}$. Form $M^{\mathcal{U}}$ and $p^{\mathcal{U}}$, and take $\bar{a}'$ realizing $M^{\mathcal{U}}$ in some elementary extension.

Let $g : N \to M^{\mathcal{U}}$ be the function which sends $c \in N$ to the class of $(f(c) : f \in I)$. Note if $c \in M$, then $f(c) = c$ for all f, and so $g(c)$ is the class of the constant tuple $(c : f \in I)$, which we identify with c.

For any L-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N$,

$$\begin{aligned} \varphi(\bar{x}, \bar{b}) \in q(\bar{x}) &\implies S_{\varphi, \bar{b}} \in \mathcal{F} \subseteq \mathcal{U} \\ &\implies \{f \in I : \varphi(\bar{x}, f(\bar{b})) \in p(\bar{x})\} \in \mathcal{U} \\ &\iff \varphi(\bar{x}, g(\bar{b})) \in p^{\mathcal{U}}. \end{aligned}$$

Restricting to formulas that don't involve $\bar{x}$, we see

$$N \models \varphi(\bar{b}) \implies M^{\mathcal{U}} \models \varphi(g(\bar{b})),$$

so $g : N \to M^{\mathcal{U}}$ is an elementary embedding. Replacing $M^{\mathcal{U}}$ by an isomorphic copy, we may assume $M \preceq N \preceq M^{\mathcal{U}}$ and g is the inclusion. Then

$$\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \implies \varphi(\bar{x}, \bar{b}) \in p^{\mathcal{U}}$$

for $L(N)$-formulas $\varphi(\bar{x}, \bar{b})$, and so $p^{\mathcal{U}}$ extends q. $\square$

Corollary 3. *Every heir of p extends to a strong heir of p.*

2 Stability

Let T be a complete L-theory and $\mathbb{M}$ be a monster model.

If α is an ordinal, then 2^α denotes the set of strings of length α in the alphabet $\{0, 1\}$, or equivalently, functions from α to 2. Additionally, $2^{<\alpha}$ denotes $\bigcup_{\beta<\alpha} 2^\beta$.

Definition 4. Fix a formula $\varphi(\bar{x}, \bar{y})$ and an ordinal α. Take a collection of variables $\bar{x}_\sigma$ where $\sigma \in 2^\alpha$ and $\bar{y}_\tau$ where $\tau \in 2^{<\alpha}$. Let D_α be the following set of formulas

$$\begin{aligned} D_\alpha = &\{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 0\} \\ &\cup \{\neg\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 1\}. \end{aligned}$$

Thus D_α is a partial type in the variables $(\bar{x}_\sigma : \sigma \in 2^\alpha)$ and $(\bar{y}_\tau : \tau \in 2^{<\alpha})$.

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not in M. Then $\varphi'(\bar{x}, \bar{b}', \bar{c}')$ is $\varphi'(\bar{x}, f(\bar{b}'), f(\bar{c}'))$ which is equivalent to $\varphi(\bar{x}, f(\bar{b}))$.

For example, if $M = \mathbb{Q}^{alg}$ and $N = \mathbb{C}$ and $\varphi(x; y_1, y_2, y_3)$ is the formula $(y_1 x^2 + y_2 x + y_3 \neq 0)$ and $\bar{b} = (\pi, 2, \pi)$, then the formula $\varphi(x; \bar{b})$ is $(\pi x^2 + 2x + \pi \neq 0)$. This formula is equivalent to $\varphi'(x; b'; c)$ where $\varphi'(x; y; z)$ is $(zx^2 + yx + z \neq 0)$, $b' = 2$, and $c = \pi$. Perhaps $c' = 1000$, so that $\varphi'(x; b', c')$ is $(1000x^2 + 2x + 1000 \neq 0)$. The function $f : \mathbb{C} \to \mathbb{Q}^{alg}$ would then be any function on $\mathbb{C}$ such that $f(x) = x$ for $x \in \mathbb{Q}^{alg}$, and $f(\pi) = 1000$. Then $\varphi(x; f(\bar{b}))$ is $\varphi(x, 1000, 2, 1000)$ which is $(1000x^2 + 2x + 1000 \neq 0)$ as claimed.

Example. D_2 consists of the formulas

$$\begin{aligned}
&\varphi(x_{00}, y), \varphi(x_{00}, y_0) \\
&\varphi(x_{01}, y), \neg\varphi(x_{01}, y_0) \\
&\neg\varphi(x_{10}, y), \varphi(x_{10}, y_1) \\
&\neg\varphi(x_{10}, y), \neg\varphi(x_{11}, y_1)
\end{aligned}$$

We say D_α is "consistent" if it is realized in a model of the complete theory T.

Proposition 5. *For a formula $\varphi(\bar{x}, \bar{y})$, the following are equivalent:*

1. *D_α is consistent for any α.*
2. *D_ω is consistent.*
3. *D_n is consistent for all $n < \omega$.*

Proof. (1) $\Longrightarrow$ (2) is trivial, (2) $\Longrightarrow$ (3) is easy, and (3) $\Longrightarrow$ (1) is by compactness, supposedly[2]. $\square$

Definition 6. A formula $\varphi(\bar{x}, \bar{y})$ has the *dichotomy property* if the equivalent conditions of Proposition 5 hold.

Remark 7. In other words, $\varphi(\bar{x}, \bar{y})$ has the dichotomy property if there are $\bar{a}_\sigma$ for $\sigma \in 2^\omega$ and $\bar{b}_\tau$ for $\tau \in 2^{<\omega}$ such that for any $\tau \in 2^{<\omega}$ and any σ extending τ,

$$\begin{aligned}
\sigma \text{ extends } \tau 0 &\implies \mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau) \\
\sigma \text{ extends } \tau 1 &\implies \mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)
\end{aligned}$$

(This is what it means for D_ω to be consistent.)

Proposition 8. *Fix $T, \mathbb{M}$, and an integer $n < \omega$. Suppose there is a small model $M \preceq \mathbb{M}$ and a type $p \in S_n(M)$ that is not definable. Then some formula $\varphi(x_1, \ldots, x_n; \bar{y})$ has the dichotomy property.*

Proof. Because p is non-definable, there is $N \succeq M$ such that p has two distinct heirs $q_1, q_2 \in S_n(N)$. Take a formula $\varphi(x_1, \ldots, x_n; \bar{b})$ in $q_1(\bar{x}) \setminus q_2(\bar{x})$. Then $\neg\varphi(\bar{x}, \bar{b}) \in q_2(\bar{x})$.

Claim. If $M' \succeq M$ and $p' \in S_n(M')$ is an heir of p, then there is $N' \succeq M'$, $q_1', q_2' \in S_n(N')$ extending p', and $\bar{b}' \in N'$ such that $q_1', q_2' \sqsupseteq p$ and $\varphi(\bar{x}, \bar{b}') \in q_1'$ but $\neg\varphi(\bar{x}, \bar{b}') \in q_2'$.

Proof. By Proposition 2, there is an ultrafilter $M \preceq M' \preceq M^{\mathcal{U}}$ with $p \subseteq p' \subseteq p^{\mathcal{U}}$. Then $M' \preceq M^{\mathcal{U}} \preceq N^{\mathcal{U}}$, and $p \sqsubseteq p^{\mathcal{U}} \sqsubseteq q_i^{\mathcal{U}}$ for $i = 1, 2$. Take $N' = N^{\mathcal{U}}$, $q_i' = q_i^{\mathcal{U}}$, and $\bar{b}'$ to be the image of $\bar{b}$ under the elementary embedding $N \to N^{\mathcal{U}}$. $\square_{\text{Claim}}$

[2]This makes intuitive sense, but the details seem very confusing to write out. We'll see a different proof of things in a future class.

Using this we can build a tree of small models M_τ and types $p_\tau \in S_n(M_\tau)$ for $\tau \in 2^{<\omega}$ and parameters $\bar{b}_\tau \in M_\tau$ such that $p_\tau \sqsupseteq p$, $M_{\tau 0} = M_{\tau 1} \succeq M_\tau$ and

$$\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 0}$$
$$\neg\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 1}$$

As each p_τ is consistent, this shows D_n holds for each n, so $\varphi(\bar{x}, \bar{y})$ has the dichotomy property. $\square$

Lemma 9. *For any infinite cardinal λ, there is a cardinal μ such that $|2^{<\mu}| \leq \lambda$ and $2^\mu > \lambda$.*

Proof. Let S be the class of cardinals μ such that $2^\mu > \lambda$. Then $\lambda \in S$, so S is non-empty. Take $\mu = \min(S)$. Then $\mu \leq \lambda$ and $2^\mu > \lambda$. Note for ordinals α,

$$\alpha < \mu \implies |\alpha| < \mu \implies 2^{|\alpha|} \leq \lambda$$

by choice of μ. As $2^{<\mu} = \bigcup_{\alpha<\mu} 2^\alpha$, we have $|2^{<\mu}| \leq \mu \cdot \lambda = \lambda$. $\square$

Proposition 10. *Suppose some formula $\varphi(x_1, \ldots, x_n; \bar{y})$ has the dichotomy property. For any λ, there is $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$ and $|S_n(A)| > \lambda$.*

Proof. By Lemma 9, there is a cardinal μ with $|2^{<\mu}| \leq \lambda$ and $2^\mu > \lambda$.

Because φ has the dichotomy property, D_μ is consistent. So there are a_σ for $\sigma \in 2^\mu$ and b_τ for $\tau \in 2^{<\mu}$ such that

$$\mathbb{M} \models \varphi(a_\sigma, b_\tau) \text{ if } \sigma \text{ extends } \tau 0$$
$$\mathbb{M} \models \neg\varphi(a_\sigma, b_\tau) \text{ if } \sigma \text{ extends } \tau 1$$

Let $A = \{b_\tau : \tau \in 2^{<\mu}\}$. Then $|A| \leq \lambda$ by choice of λ. But the a_σ have pairwise distinct types over A: if $\sigma \neq \sigma'$, then there is some $\tau \in 2^{<\mu}$ such that σ extends $\tau 0$ and σ' extends $\tau 1$, or vice versa. Then $\mathbb{M} \models \varphi(a_\sigma, b_\tau) \ \wedge \ \neg\varphi(a_{\sigma'}, b_\tau)$, so $\mathrm{tp}(a_\sigma/A) \neq \mathrm{tp}(a_{\sigma'}/A)$. Therefore $|S_n(A)| \geq 2^\mu > \lambda$. $\square$

Lemma 11. *Let λ be an infinite small cardinal. The following are equivalent:*

1. *If $A \subseteq \mathbb{M}$ and $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$.*
2. *If $A \subseteq \mathbb{M}$ and $|A| \leq \lambda$, then $|S_n(A)| \leq \lambda$ for all $n < \omega$.*

(We didn't discuss the proof in class, but we'll discuss it in more detail next week.)

Proof. (2) $\implies$ (1) is trivial. Assume (1). By induction on n, $|S_{n-1}(A)| \leq \lambda$. Then we can find $\bar{b}_\alpha \in \mathbb{M}^{n-1}$ for $\alpha < \lambda$ such that

$$S_{n-1}(A) = \{\mathrm{tp}(\bar{b}_\alpha/A) : \alpha < \lambda\}.$$

For each α, $|A\bar{b}_\alpha| \leq \lambda \implies |S_1(A\bar{b}_\alpha)| \leq \lambda$ by (1). So we can find $c_{\alpha,\beta} \in \mathbb{M}$ for $\beta < \lambda$ such that

$$S_1(A\bar{b}_\alpha) = \{\mathrm{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha) : \beta < \lambda\} \text{ (for } \alpha < \lambda).$$

Claim. If $p \in S_n(A)$ then $p = \text{tp}(\bar{b}_\alpha c_{\alpha,\beta}/A)$ for some $\alpha, \beta < \lambda$.

Proof. Take $(\bar{b}', c') \in \mathbb{M}^n$ realizing p. Then $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}_\alpha/A)$ for some $\alpha < \lambda$. Moving $(\bar{b}', c')$ by an automorphism in $\text{Aut}(\mathbb{M}/A)$, we may assume $\bar{b}' = \bar{b}_\alpha$. Then $\text{tp}(c/A\bar{b}_\alpha) = \text{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha)$ for some $\beta < \lambda$. Moving c' by an automorphism in $\text{Aut}(\mathbb{M}/A\bar{b}_\alpha)$, we may assume $c' = c_{\alpha,\beta}$. Then $p = \text{tp}(\bar{b}_\alpha c_{\alpha,\beta}/A)$. $\square_{\text{Claim}}$

By the claim, $|S_n(A)| \leq \lambda^2 = \lambda$. $\square$

Definition 12. The complete theory T is λ-*stable* if the equivalent conditions of Lemma 11 hold, i.e., $|A| \leq \lambda \implies |S_n(A)| \leq \lambda$.

Example. Suppose T is strongly minimal. Then T is λ-stable for any $\lambda \geq |L|$. To see this, take $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$. By Löwenheim-Skolem there is $A \subseteq M \preceq \mathbb{M}$ with $|M| \leq \lambda$. Every type in $S_1(A)$ extends to a type in $S_1(M)$, so $|S_1(M)| \geq |S_1(A)|$. Finally, $S_1(M)$ only contains $\text{tp}(a/M)$ for $a \in M$ plus the transcendental 1-type, so

$$|S_1(A)| \leq |S_1(M)| \leq \lambda + 1 = \lambda.$$

Theorem 13. *Fix a complete theory T with monster model $\mathbb{M}$, and a positive integer n. The following are equivalent:*

1. *T is λ-stable for some λ.*
2. *No formula $\varphi(x_1, \ldots, x_n; \bar{y})$ has the dichotomy property.*
3. *All n-types over models are definable.*

Proof. (1) $\implies$ (2): Proposition 10.

(2) $\implies$ (3): Proposition 8.

It remains to prove (3) $\implies$ (1). (*We didn't get to this part of the proof in class, and we'll discuss it in more details next week.*) Let $\lambda = 2^{|L|}$. Note $\lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|^2} = 2^{|L|} = \lambda$. Take $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$. By Downward Löwenheim-Skolem, there is a small model $M \preceq \mathbb{M}$ with $A \subseteq M$ and $|M| \leq \lambda$. Every n-type over A extends to an n-type over M, so $|S_n(A)| \leq |S_(M)|$. It remains to show $|M| \leq \lambda \implies |S_n(M)| \leq \lambda$. (That is, we may assume A is a small model M.) By (3), every n-type over M is definable. A definable type is determined by the map $\varphi \mapsto d\varphi$, which is a function from L-formulas to $L(M)$-formulas. So the number of (definable) types over M is at most $|L(M)|^{|L|} \leq \lambda^{|L|} = \lambda$. $\square$

Remark 14. The third condition of Theorem 13 doesn't depend on n (essentially by Lemma 11), so the first and second conditions don't either. In particular, we get the following implications:

- If all 1-types over models are definable, then all n-types over models are definable.
- If no formula $\varphi(x; \bar{y})$ has the dichotomy property, then no formula $\varphi(\bar{x}; \bar{y})$ has the dichotomy property.

Example. Suppose T is strongly minimal. Then T is λ-stable for any $\lambda \geq |L|$. To see this, take $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$. By Löwenheim-Skolem there is $A \subseteq M \preceq \mathbb{M}$ with $|M| \leq \lambda$. Every type in $S_1(A)$ extends to a type in $S_1(M)$, so $|S_1(M)| \geq |S_1(A)|$. Finally, $S_1(M)$ only contains $\mathrm{tp}(a/M)$ for $a \in M$ plus the transcendental 1-type, so

$$|S_1(A)| \leq |S_1(M)| \leq \lambda + 1 = \lambda.$$

Using only the (1) $\implies$ (2) $\implies$ (3) parts of Theorem 13, we see that the following hold in a strongly minimal theory:

- No formula $\varphi(\bar{x}, \bar{y})$ has the dichotomy property.
- Every n-type over a model is definable. (Previously we only knew this for $n = 1$.)