Set Theory2

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目录

1 集合的宇宙

1.1 数理逻辑

在 ZFC 下证明 ZFC $\vdash$ CH，希望将 “ZFC $\vdash$ CH” 表述为一阶句子

一般而言，给定一个 $\mathcal{L}$-理论 T 和一个 $\mathcal{L}$-句子 δ，“$T \vdash \sigma$” 不能用一个 $\mathcal{L}$-句子表示，只能用元语言表述

我们需要在 ZFC 中编码“元语言”

在 ZFC 中可以定义 $\mathcal{N} = (\mathbb{N}, +, \times, 0, 1)$

即存在集合论语言 $\mathcal{L} = \{\in\}$ 中的 **公式**，在 ZFC 的任意模型中可以定义 $\mathbb{N}, +, \times, 0, 1$，以上公式与模型无关

用 $\ulcorner 0 \urcorner$，$\ulcorner 1 \urcorner$，$\ulcorner 2 \urcorner$... 表示 ZFC 中的“自然数”，以区别元语言中的自然数

Theorem 1.1. *如果 $R \subseteq \mathbb{N}^n$ 是一个递归关系。$T \subseteq \mathrm{Th}(\mathcal{N})$ 是包含数论的适当丰富的理论，则存在公式 $\varphi(x_1, \dots, x_n)$ 使得对任意自然数 $m_1, \dots, m_n$ 有*

$$\text{如果}(m_1, \dots, m_n) \in R \text{则} T \vdash \varphi(\ulcorner m_1 \urcorner, \dots, \ulcorner m_n \urcorner)$$
$$\text{如果}(m_1, \dots, m_n) \notin R \text{则} T \vdash \neg\varphi(\ulcorner m_1 \urcorner, \dots, \ulcorner m_n \urcorner)$$

Remark.

1. $T \subseteq \mathrm{Th}(\mathcal{N}) \subseteq \mathrm{ZFC}$
2. φ 是语言 $\{+, \times, 0, 1\}$ 上的公式
3. φ 可以还原为一个 $\{\in\}$ 上的公式
4. $\varphi(\ulcorner m_1 \urcorner, \dots, \ulcorner m_n \urcorner)$ 是一个闭语句

编码

编码函数 $f : X \to \mathbb{N}$

存在解码函数 g, h，对 $a = a_0, \dots, a_n \in X$, $h(f(a)) = n + 1$, $g(f(a), k) = a_k$ (分量)

性质：以上三种函数 f, g, h 均是递归函数 $\Rightarrow$ 都是可表示的

性质:"公式集"的编码集是递归的

性质：如果 $T \subseteq \mathrm{ZFC}$ 是可公理化的，则 T 的证明集的编码集是递归的

Corollary 1.2. *存在一个公式 ψ 和 θ 使得*

$$\mathrm{ZFC} \vdash \psi(n) \Leftrightarrow n \textit{ is a formula}$$
$$\mathrm{ZFC} \vdash \neg\psi(n) \Leftrightarrow n \textit{ is not a formula}$$
$$\mathrm{ZFC} \vdash \theta(n) \Leftrightarrow n \textit{ is a proof in } \mathrm{ZFC}$$
$$\mathrm{ZFC} \vdash \neg\theta(n) \Leftrightarrow n \textit{ is not a proof in } \mathrm{ZFC}$$

称 ψ 定义了公式集，θ 定义了证明集

$\mathrm{FORM} = \{\ulcorner \varphi \urcorner \mid \varphi \text{ formula}\} \subseteq \mathbb{N}$

如果 $T \subseteq \mathrm{ZFC}$ 是可公理化的，则"T 是一致的"是一个一阶表述式"不存在一个有穷的证明序列 $D = (\varphi_1, \dots, \varphi_n)$ 使得 φ_n 形如 $\varphi \wedge \neg\varphi$，记作 $\mathrm{Con}(T)$

Theorem 1.3 (第二不完全)**.** 如果 T 是包含 ZFC 的一个递归公理集，且 T 一致，则

$$T \nvdash Con(T)$$

特别地，ZFC $\nvdash Con(\text{ZFC})$

Theorem 1.4. 对任意可公理化的理论 T，ZFC $\vdash Con(T)$ 当且仅当存在 $M \vDash T$

即不能在 ZFC 里证明 ZFC 有一个模型

需要可公理化来写出 Con(T)，因此因为 ZFC $\nvDash$ Con(T)，我们只能假设这么一个模型

集合论的模型跟集合论没什么关系，就是一个集合带一个二元关系，是关于集合论语言的结构

Definition 1.5. 设 (M, E) 是集合论模型

1. 对任意公式 $\varphi(\bar{x}, y)$，定义 M^n 上的函数

   $$h_\varphi : M^n \to M$$

   满足条件

   $$M \vDash \exists y \varphi(\bar{a}, y) \Rightarrow M \vDash \varphi(\bar{a}, h_\varphi(\bar{a}))$$

   称 h_φ 为 φ 的 Skolem 函数（依赖于选择公理，不同的变量选择有不同的函数）

2. 令 $\mathcal{H} = \{h_\varphi \mid \varphi \text{ formula}\}$ 为 Skolem 函数集合，设 S 是 M 的任意子集，则 $\mathcal{H}(S)$ 表示包含 S 且对 $\mathcal{H}$ 封闭的最小集合，称之为 S 的 Skolem 壳

Lemma 1.6. 令 N 是集合论模型，$S \subseteq N$，如果 $M = \mathcal{H}(S)$，则 $M \prec N$

证明. Induction

对任意 $\bar{a} \in M^n$，有 $M \vDash \varphi(\bar{a}) \Leftrightarrow N \vDash \varphi(\bar{a})$

1. 不含量词，显然成立

2. φ 形如 $\exists y\psi(\bar{x},y)$, $N\vDash\exists y\psi(\bar{a},y)\ \Rightarrow\ N\vDash\psi(\bar{a},h_\psi(\bar{a}))$，by IH, $M\vDash\psi(\bar{a},h_\psi(\bar{a}))\Rightarrow M\vDash\exists y\psi(\bar{a},y)$

$\square$

Theorem 1.7 (Löwenheim–Skolem Theorem)**.**

1.2 层垒的谱系

工作于 ZF^-：ZF − 基础公理

$\alpha\mapsto V_\alpha$ 是 On 到 WF 的 1-1 映射，而 On 是真类

Lemma 1.8. *For any ordinal* α

1. V_α *is transitive*
2. $\xi\leq\alpha\Rightarrow V_\xi\subseteq V_\alpha$
3. *if* κ *is inaccessible, then* $|V_\kappa|=\kappa$

Definition 1.9. For any $x\in\mathrm{WF}$, **rank** of x is

$$\mathrm{rank}(x)=\min\{\beta\mid x\in V_{\beta+1}\}$$

$\mathrm{rank}(x)=\alpha\Rightarrow x\in V_{\alpha+1}\wedge x\notin V_\alpha$

- $x\in V_{\alpha+1}\Leftrightarrow\mathrm{rank}(x)\leq\alpha$
- $x\subseteq V_\alpha\Leftrightarrow\mathrm{rank}(x)\leq\alpha$

Lemma 1.10. 1. $V_\alpha=\{x\in\mathrm{WF}\mid \mathit{rank}(x)<\alpha\}$

2. WF *is transitive*
3. $\forall x,y\in\mathrm{WF}$, $x\in y\Rightarrow \mathit{rank}(x)<\mathit{rank}(y)$
4. $\forall y\in\mathrm{WF}$, $\mathit{rank}(y)=\sup\{\mathit{rank}(x)+1\mid x\in y\}$

证明. 1. by definition, $x \in V_{\text{rank}(x)+1} \setminus V_{\text{rank}(x)}$, $\text{rank}(x) < \alpha \Rightarrow x \in V_{\text{rank}(x)+1} \subseteq V_\alpha$

$\text{rank}(x) \geq \alpha \Rightarrow x \notin V_\alpha$

2. WF is the “union” of transitive sets

3. $y \in V_{\text{rank}(y)+1} \setminus V_{\text{rank}(y)}$, $y \subseteq V_{\text{rank}(y)}$, $x \in y \Rightarrow x \in V_{\text{rank}(y)} \Rightarrow \text{rank}(x) < \text{rank}(y)$

4. by 3, $\sup\{\text{rank}(x) + 1 \mid x \in y\} \leq \text{rank}(y)$.

   induction on $\text{rank}(y) \leq \sup\{\text{rank}(x) + 1 \mid x \in y\}$

   - $\text{rank}(y) = 0$
   - $\text{rank}(y) = \beta + 1$, $y \in V_{\beta+2} \setminus V_{\beta+1}$

     $y \in V_{\beta+2} \Rightarrow y \subseteq V_{\beta+1}$. $y \notin V_{\beta+1} \Rightarrow y \not\subseteq V_\beta \Rightarrow y \setminus V_\beta$ nonempty. Let $x \in y \setminus V_\beta$, $\text{rank}(x) \geq \beta$, $\sup\{\text{rank}(x) + 1 \mid x \in y\} \geq \beta + 1 = \text{rank}(y)$
   - $\text{rank}(y) = \gamma$ for some limit, then $y \subseteq V_\gamma$ and for any $\xi < \gamma$, $y \not\subseteq V_\xi$, let $X_\xi \in y \setminus V_\xi$, then $\text{rank}(X_\xi) \geq \xi$, $\sup\{\text{rank}(x) + 1 \mid x \in y\} \geq \sup\{\xi + 1 \mid \xi < \text{rank}(y)\} \geq \text{rank}(y)$

□

- WF 中的集合按照秩分层
- 在 WF 中基础公理是成立的：$\forall y(y \neq \emptyset \to \exists x \in y(x \cap y = \emptyset))$，因为任何序数集都有最小元，挑一个有最小 rank 的就好了
- WF 类的构造没有用到选择公理
- $\text{On} \subseteq \text{WF}$

Lemma 1.11. *for any ordinal* α

1. $\alpha \in \text{WF}$ *and* $\text{rank}(\alpha) = \alpha$

2. $V_\alpha \cap \mathrm{On} = \alpha$

证明. 1. $0 \in V_1 \setminus V_0 \subset \mathrm{WF}$, $\mathrm{rank}(0) = 0$

If $\alpha \in \mathrm{WF}$ and $\mathrm{rank}(\alpha) = \alpha$. $\alpha \in V_{\alpha+1} \setminus V_\alpha$, $\alpha \subseteq V_{\alpha+1}$. $\alpha+1 = \alpha \cup \{\alpha\} \subseteq V_{\alpha+1}$, $\alpha + 1 \in V_{\alpha+2} \subset \mathrm{WF}$. If $\alpha + 1 \in V_{\alpha+1}$, then $\mathrm{rank}(\alpha + 1) \leq \alpha$, but $\alpha \in \alpha + 1 \Rightarrow \mathrm{rank}(\alpha) = \alpha < \mathrm{rank}(\alpha + 1)$. A contradiction

suppose γ is a limit ordinal and for any $\alpha < \gamma$, $\alpha \in V_{\alpha+1} \setminus V_\alpha$. $\gamma = \bigcup_{\alpha<\gamma} \alpha \subseteq \bigcup_{\alpha<\gamma} V_\alpha = V_\gamma$. Thus $\gamma \in V_{\gamma+1}$, $\mathrm{rank}(\gamma) \leq \gamma$ and $\mathrm{rank}(\gamma) \nless \gamma$.

2. suppose $\beta \in V_\alpha \cap \mathrm{On}$, then $\beta = \mathrm{rank}(\beta) < \alpha$. If $\beta \in \alpha$ and $\mathrm{rank}(\beta) < \alpha$, $\beta \in V_\alpha \cap \mathrm{On}$

□

Lemma 1.12. *1. If $x \in \mathrm{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathrm{WF}$, and their rank $<$ rank$(x) + \omega$*

2. If $x, y \in \mathrm{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathrm{WF}$, and their rank $<$ rank(x) $+$ rank$(y) + \omega$

3. $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$

4. for any set x, $x \in \mathrm{WF} \Leftrightarrow x \subset \mathrm{WF}$

证明. 1. suppose $\mathrm{rank}(x) = \alpha$. $x \in V_{\alpha+1} \setminus V_\alpha$ and $x \subseteq V_\alpha$.

by transitivity, $\bigcup x \subseteq V_\alpha \Rightarrow \bigcup x \in V_{\alpha+1} \subset \mathrm{WF}$. $\mathrm{rank}(\bigcup x) \leq \alpha$

suppose $y \in \mathcal{P}(x)$, $y \subseteq x \Rightarrow y \subseteq V_\alpha \Rightarrow y \in V_{\alpha+1}$. $\mathcal{P}(x) \subseteq V_{\alpha+1}$, $\mathcal{P}(x) \in V_{\alpha+2}$, $\mathrm{rank}(\mathcal{P}(x)) \leq \alpha + 1$.

$\{x\} \in \mathcal{P}(x) \in V_{\alpha+2}$.

2. Suppose $\mathrm{rank}(x) = \alpha, \mathrm{rank}(y) = \beta$, $x \subset V_\alpha$, $y \subset V_\beta$

$x \cup y \subset V_\alpha \cup V_\beta = V_{\max(\alpha,\beta)}$, $\mathrm{rank}(x \cup y) \leq \max(\alpha, \beta)$

$x \cap y \subset V_{\min(\alpha,\beta)}$

$\{x,y\} \subseteq V_{\alpha+1} \cup V_{\beta+1} = V_{\max(\alpha,\beta)+1}$, $\text{rank}(\{x,y\}) = \max(\alpha,\beta)+1$

$(x,y) = \{\{x\},\{x,y\}\} \subset V_{\max(\alpha,\beta)+2}$. $\text{rank}((x,y)) = \max(\alpha,\beta)+2$

$x \times y = \{(a,b) \mid a \in x, b \in y\}$. $a \in x \Rightarrow \text{rank}(a) < \alpha$, $b \in y \Rightarrow \text{rank}(b) < \beta$, $\text{rank}(a,b) < \max(\alpha,\beta)+2$, $(a,b) \in V_{\max(\alpha,\beta)+2}$. $x \times y \subseteq V_{\max(\alpha,\beta)+2}$, $\text{rank}(x \times y) \leq \max(\alpha,\beta)+2$.

$x^y \subseteq \mathcal{P}(x \times y) \subseteq V_{\max(\alpha,\beta)+3}$.

3. $\mathbb{N} = \omega \in V_{\omega+1}$

   $\mathbb{Z}$: let $\sim$ be an equivalence relation on $\omega \times \omega$, $(a,b) \sim (c,d) \Leftrightarrow a+d = b+c$, then $\mathbb{Z} = (\omega \times \omega)/\sim$. Hence $\mathbb{Z}$ is a partition of $\omega \times \omega$ and hence $\mathbb{Z} \subseteq \mathcal{P}(\omega \times \omega)$. $\mathbb{Z} \in V_{\omega+3}$

   $\mathbb{Q}$: let $\sim$ be an equivalence on $\mathbb{Z} \times \mathbb{Z}^+$, $(a,b) \sim (c,d) \Leftrightarrow ad = bc$. $\mathbb{Q} \subseteq \mathcal{P}(\mathbb{Z} \times \mathbb{Z}^+)$, $\mathbb{Q} \in V_{\omega+6}$

   $\mathbb{R}$: set of dedekind cut on $\mathbb{Q}$, $\mathbb{R} \subset \mathcal{P}(\mathbb{Q})$, $\mathbb{R} \in V_{\omega+8}$

4. $\Rightarrow$: WF is transitive

   $\Leftarrow$: x is a set and $x \subset \bigcup_{\alpha \in \text{On}} V_\alpha$.

   Claim: there is an ordinal α s.t. $x \subset V_\alpha$

   Otherwise, let $f : \text{On} \to \mathcal{P}(x)$ s.t. $f(\alpha) = x \setminus V_\alpha$. Then for any $y \in \mathcal{P}(x)$, $f^{-1}(y)$ is a set. $\text{On} = \bigcup_{y \in x} f^{-1}(y)$ and is thus a set, a contradiction

$\square$

AC => Any set has cardinality

Lemma 1.13. *Assume AC* ($V \models \text{ZFC}$)

1. *for any group G, there is a group G' in* WF *s.t. $G \cong G'$*

2. *for any topological space T, there is a topological space T' in* WF *s.t. $T \cong T'$ (homeomorphic)*

证明. 1. suppose $(G, *_G)$ is a group, $G, *_G \in V$. By AC, there is a cardinal α s.t. $|G| = \alpha$, that is, there is a bijection $f : \alpha \to G$. Define $*$: for any $x, y, z \in \alpha$, $x * y = z \Leftrightarrow f(x) *_G f(y) = f(z)$. Then $(\alpha,\ *) \cong (G,\ *_G)$, $* \subseteq \alpha \times \alpha$

□

V 中的任何结构都可以在 WF 中找到同构象（同构是在 V 里看到的）

Definition 1.14. 任意集合 A 上的二元关系 $<$ 是 **良基**的，当且仅当对 A 的任意非空子集 X，X 有 $<$ 下的极小元

Theorem 1.15. *If $A \in$ WF, then $\in$ is a well-founded relation on A*

证明. suppose $X \subseteq A$, $X \neq \emptyset$, $X \subseteq$ WF, then elements of X has ranks and $x \in y \Rightarrow \text{rank}(x) < \text{rank}(y)$. Let x having least rank in X, then x is the $\in$-minimal element in X □

Lemma 1.16. *If A is a transitive set and $\in$ is a well-founded relation on A, then $A \in$ WF*

证明. Just need to prove $A \subset$ WF. If $A \not\subset$ WF, $X = A \setminus \text{WF} \neq \emptyset$. Then X has a $\in$-minimal element x. Then $x \neq \emptyset \in$ WF. For any $y \in x$, $y \in A$. By the minimality of x, $y \in$ WF. Then $x \subset$ WF, $x \in$ WF, a contradiction □

Lemma 1.17. *For any set x, there is a minimal transitive set trcl(x) s.t. $x \subseteq$ trcl(x)*

证明. For any $n \in \omega$ define x_n

$$x_0 = x$$
$$x_{n+1} = \bigcup x_n$$

let trcl$(x) = \bigcup_{n \in \omega} x_n$.

1. trcl$(x)$is transitive

   $a \in \text{trcl}(x) \Rightarrow a \in x_n \Rightarrow a \subseteq x_{n+1} \subseteq \text{trcl}(x)$

2. $\mathrm{trcl}(x)$ is minimal

   If $y \supseteq x$ is transitive, recursively prove for any $n < \omega$, $x_n \subseteq y$.

$\square$

$\mathrm{trcl}(x)$ is the **transitive closure** of x.

Lemma 1.18. *We can prove the following without axiom of power set*

1. *if x is transitive, trcl$(x) = x$*
2. *$y \in x \Rightarrow$ trcl$(y) \subseteq$ trcl(x)*
3. *trcl$(x) = x \cup \bigcup\{$trcl$(y) \mid y \in x\}$*

证明. 2. $y \in x \subset \mathrm{trcl}(x)$. $y \in \mathrm{trcl}(x)$. $\mathrm{trcl}(y) \subseteq \mathrm{trcl}(x)$.

3. $x \cup \bigcup\{\mathrm{trcl}(y) \mid y \in x\} \subseteq \mathrm{trcl}(x)$ by (2)

   $\bigcup\{\mathrm{trcl}(y) \mid y \in x\}$ is transitive. For $y \in x$, $y \subseteq \mathrm{trcl}(y)$. Thus rhs is transitive

$\square$

Theorem 1.19 (In ZF$^-$). *For any set X, TFAE*

1. *$X \in \mathrm{WF}$*
2. *trcl$(X) \in \mathrm{WF}$*
3. *$\in$ is a well-founded relation on trcl(X)*

证明. $1 \to 2$: WF is closed under union $\square$

Theorem 1.20. *If $V \vDash ZF^-$, TFAE*

1. *axiom of foundation $(V \vDash)$ axiom of foundation*
2. *for any set X, $\in$ is a well-founded relation on X*
3. *$V = \mathrm{WF}$*

$V \vDash \text{ZF} \Rightarrow V = \text{WF}(\text{WF} \vDash \text{ZF})$

Goal: $V \vDash \text{ZF}^- \Rightarrow \text{WF} \vDash \text{ZF}^-$ 但是 WF 是一个类，我们并没有定义

我们可以用相对化编码 $\text{WF} \vDash \text{ZF}^-$

1.3 相对化 relativization

工作在 ZF^-

Definition 1.21. **M** class, φ formula, φ 对 **M** 的 **相对化** $\varphi^{\mathbf{M}}$

1. $(x = y)^{\mathbf{M}} := x = y$
2. $(x \in y)^{\mathbf{M}} := x \in y$
3. $(\varphi \to \psi)^{\mathbf{M}} := \varphi^{\mathbf{M}} \to \psi^{\mathbf{M}}$
4. $(\neg\varphi)^{\mathbf{M}} := \neg\varphi^{\mathbf{M}}$
5. $(\forall x\varphi)^{\mathbf{M}} := (\forall x \in \mathbf{M})\varphi^{\mathbf{M}}$

1.4 Exercise

Exercise 1.4.1. 1. $V_\alpha = \{x \in \text{WF} \mid \text{rank}(x) < \alpha\}$

2. WF is transitive
3. $\forall x, y \in \text{WF}, x \in y \Rightarrow \text{rank}(x) < \text{rank}(y)$
4. $\forall y \in \text{WF}, \text{rank}(y) = \sup\{\text{rank}(x) + 1 \mid x \in y\}$

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2. WF is the "union" of transitive sets
3. $y \in V_{\text{rank}(y)+1} \setminus V_{\text{rank}(y)}$, $y \subseteq V_{\text{rank}(y)}$, $x \in y \Rightarrow x \in V_{\text{rank}(y)} \Rightarrow \text{rank}(x) < \text{rank}(y)$

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   induction on $\text{rank}(y) \leq \sup\{\text{rank}(x)+1 \mid x \in y\}$

   - $\text{rank}(y) = 0$
   - $\text{rank}(y) = \beta + 1$, $y \in V_{\beta+2} \setminus V_{\beta+1}$

     $y \in V_{\beta+2} \Rightarrow y \subseteq V_{\beta+1}$. $y \notin V_{\beta+1} \Rightarrow y \nsubseteq V_\beta \Rightarrow y \setminus V_\beta$ nonempty. Let $x \in y \setminus V_\beta$, $\text{rank}(x) \geq \beta$, $\sup\{\text{rank}(x)+1 \mid x \in y\} \geq \beta + 1 = \text{rank}(y)$
   - $\text{rank}(y) = \gamma$ for some limit, then $y \subseteq V_\gamma$ and for any $\xi < \gamma$, $y \nsubseteq V_\xi$, let $X_\xi \in y \setminus V_\xi$, then $\text{rank}(X_\xi) \geq \xi$, $\sup\{\text{rank}(x)+1 \mid x \in y\} \geq \sup\{\xi + 1 \mid \xi < \text{rank}(y)\} \geq \text{rank}(y)$

$\square$