Homework 1 Solutions

Introductory to Model Theory

Autumn 2021

0. *If* $[a]_\sim = [b]_\sim$*, then* $a \in [b]_\sim$ *and* $a \sim b$*. If* $a \sim b$*, then for every* $c \in [a]_\sim$*,* $c \sim a \sim b$*, so* $c \in [b]_\sim$ *and* $[a]_\sim \subseteq [b]_\sim$*. The proof of* $[b]_\sim \subseteq [a]_\sim$ *is similar and then* $[a]_\sim = [b]$*.*

If $a \not\sim b$*, then for any* $c \in [a]_\sim$ *and* $d \in [b]_\sim$*, we have* $c \not\sim b$ *and* $d \not\sim a$*, so* $[a] \sim \cap [b]_\sim = \emptyset$*.* $\square$

1. *By hypothesis,* $\emptyset \subseteq S_3(E, E')$*. We check every property of equivalence relation.*

Reflexivity: For any $a' \in E'$*, we have* $a \in E$ *s.t.* $\{(a', a)\}$ *is a* 2*-isomorphism. Since* $a' \approx a' \iff a \sim a$*, we have* $a' \sim a'$*.*

Symmetry: For any $a', b' \in E'$*, we have* $a, b \in E$ *s.t.* $\{(a', a), (b', b)\}$ *is a* 1*-isomorphism. So* $a' \approx b' \iff a \sim b \iff b \sim a \iff b' \approx a'$*.*

Transitivity: For any $a', b', c' \in E'$*, we have* $a, b, c \in E$ *s.t.* $\{(a', a), (b', b)\}$ *is a* 0*-isomorphism. If* $a' \approx b'$ *and* $b' \approx c'$*, we have* $a \sim b$ *and* $b \sim c$*, thus* $a \sim c$*, and* $a' \approx c'$*.* $\square$

3. *Let* $E = \{(a, b) \in \mathbb{N}^2 | \, k(k+1)/2 \leq a, b < (k+1)(k+2)/2$ *for some* $k \in \mathbb{N}\}$*, then* $E \in \mathcal{K}$*.* $\square$

4. *By problem 1, we have* $(E', \approx)$ *is an equivalence relation. Now we are going to check there is exactly one equivalence class of size of every* $n \in \mathbb{N}$*.*

For any $n \in \mathbb{N}$*, let* $m = 2n + 1$ *and choose* C *as the equivalence class of size* n *in* $(E, \sim)$*, we have* $C' \in E'$ *and* $C \to C'$ *be a* $(m - n)$*-isomorphism. For any*

$x \in E\backslash C, y \in C$, $x \not\sim y$, so for any $x' \in E'\backslash C', y' \in C'$, $x' \not\sim y'$, which means C' is an equivalence class of size n.

Assume there are two equivalence class of size n, namely C_1' and C_2'. We have C_1 and C_2 and $C_1 \cup C_2 \to C_1' \cup C_2'$ be a $(m-2n)$-isomorphism. Since there is only one equivalence class of size n in $(E, \sim)$, we may assume C_1 is a proper subset of a bigger equivalence class. Let $x \in E$ s.t. $x \sim y$ and $x \neq y$ for any $y \in C_1$, then we have $y' \in E'$ s.t. $x' \sim y'$ and $x' \neq y'$ for any $y' \in C_1'$, which contradicts that C_1' is a equivalence class of size n. □

5. *We prove it by induction on p. The case of $p = 0$ is trivial as s is a local isomorphism. Suppose the case of p is correct, and for every $a \in dom(s)$, $i \leq p+q+1$, $P_i(a) \iff P_i(s(a))$, we need to check forth and back condition.*

For any $x \in E\backslash dom(s)$, there are several cases.

- *$x \sim y$ for some $y \in dom(s)$.*

  Let $y' = s(y)$, then for every $i \leq p+q+1$, $P_i(y) \iff P_i(y')$, so $|[y]_\sim| \leq p+q+1 \iff |[y']_\approx| \leq p+q+1$.

  Let $k = |[y]_\sim \cap dom(s)|$, because $x \sim y$ and $k \leq |dom(s)| = q < p+q+1$, $P_i(y) \iff P_i(y')$ for any $i \leq k$. Since $|[y]_\sim| \geq k+1$, $P_i(y)$ is false for any $i \leq k$, so $|[y']_\approx| \geq k+1 > k = |[y']_\approx \cap dom(s)|$ and we have $x' \notin im(s)$ s.t. $x' \approx y'$ and for every $i \leq p+q+1$, $P_i(x) \iff P_i(y) \iff P_i(y') \iff P_i(x')$.

- *$x \not\sim y$ for any $y \in dom(s)$.*

  There are two cases here.

  - $|[x]_\sim| = n \leq p+q+1$.

    There is only one equivalence class of size n in E and E'. No $y \in dom(s)$ is in $[x]_\sim$, so $P_n(y)$ is false for every $y \in dom(s)$, $P_n(s(y))$ is false for every $y \in dom(s)$, and No $y' \in im(s)$ is in a equivalence class of size n. We can choose $x' \in E'$ in a equivalence class of size n, then for every $i \leq p+q+1$, $P_i(x) \iff P_i(x')$.

- $|[x]_\sim| > p+q+1$.

  Now $P_i(x)$ *is false for every* $i \leq p+q+1$. *As* $dom(s) < \infty$, *we can choose some* $x' \in E'$ *in a equivalence class of size bigger than* $p+q+1$ *and* $x' \not\sim y'$ *for any* $y' \in im(s)$. *Then for every* $i \leq p+q+1$, $P_i(x) \iff P_i(x')$ *because* $P_i(x)$ *and* $P_i(x')$ *both are false for* $i \leq p+q+1$.

Let $s' = s \cup \{(x, x')\}$, *then for every* $a \in dom(s)$, $i \leq p+q+1$, $P_i(a) \iff P_i(s'(a))$. *Note* $|dom(s')| = q+1$, *and by induction hypothesis, we have* s' *is a* p*-isomorphism, which shows the forth condition works and the back condition is similar. So* s *is a* $(p+1)$*-isomorphism.* □

6. *Let* $s = \emptyset$ *and apply the previous problem, we have* s *is a* p*-isomorphism for every* $p \in \mathbb{N}$, *so* $s \in S_\omega(E, E')$. □

7. *Let* $E = \{(a,b) \in \mathbb{N}^2 |\ k(k+1)/2 \leq a, b < (k+1)(k+2)/2$ *for some* $k \in \mathbb{N}\}$ *and* $E' = E \cup \{(a,b) \in \mathbb{Z}^2 | a, b < 0\} \subseteq \mathbb{Z}^2$. *Then* E *and* E' *are both countable and in* $\mathcal{K}$. *Then* $E \sim_\omega E'$ *by the previous problem but they are not isomorphic as there is a infinite class in* E' *but not in* E. *So* $E \not\sim_\infty E'$.

□