Forking and stability spectra

Advanced model theory

April 21–28, 2022

Reference in the book: Sections 13.2, 15.1, 15.2, 16.1, 16.2, 16.5, but we're using very different approaches from the book.

1 Elimination of imaginaries in Peano Arithmetic and ACF

Theorem 1.1. *If T is a completion of Peano Arithmetic, then T has uniform elimination of imaginaries.*

Proof. Fix 0-interpretable D/E where $D \subseteq \mathbb{M}^n$. Take lexicographic order on $\mathbb{M}^n$. The induction axiom implies $\min(X)$ exists for any non-empty definable $X \subseteq \mathbb{M}^n$. Let $f : D/E \to \mathbb{M}^n$ be $f(X) = \min(X)$. Then f is a 0-interpretable injection. $\square$

Next consider ACF_0. Fix a monster model $\mathbb{M}$.

Fact 1.2. *If $S \subseteq \mathbb{M}^n$ is finite, then S has a code.*

The $n = 1$ case was (April 7, Example 31). Here is a proof of $n = 2$. (The general case is similar.)

Proof. For $q \in \mathbb{Q}$ let $\pi_q : \mathbb{M}^2 \to \mathbb{M}$ be $\pi_q(x, y) = y - qx$. Each $\pi_q(S)$ is a finite subset of $\mathbb{M}$, so has a code in $\mathbb{M}$ by (April 7, Example 31). Let $A = \{\ulcorner \pi_q(S) \urcorner : q \in \mathbb{Q}\} \subseteq \mathbb{M}$.

Claim. If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $\sigma(S) = S \iff \sigma \in \mathrm{Aut}(\mathbb{M}/A)$.

Proof. $\Rightarrow$: Easy: if $\sigma(S) = S$ then $\sigma(\pi_q(S)) = \pi_q(\sigma(S)) = \pi_q(S)$, and so $\sigma(\ulcorner \pi_q(S) \urcorner) = \ulcorner \pi_q(S) \urcorner$. Therefore σ fixes A pointwise.

$\Leftarrow$: Suppose $S' = \sigma(S) \neq S$. Then $S' \not\subseteq S$ and $S \not\subseteq S'$ (since $|S'| = |S|$). Therefore $S' \cup S \supsetneq S$. The map $\pi_q : S' \cup S \to \mathbb{M}$ is injective for all but finitely many $q \in \mathbb{Q}$. (Consider the finite set of lines through two points in $S \cup S'$, and take q not equal to the slope of any of these lines.) Fix a $q \in \mathbb{Q}$ such that π_q is injective on $S' \cup S$. Then

$$|\pi_q(S) \cup \pi_q(S')| = |\pi_q(S \cup S')| = |S \cup S'| > |S| \geq |\pi_q(S)|.$$

Therefore $\pi_q(S) \neq \pi_q(S') = \sigma(\pi_q(S))$, and so σ doesn't fix $\ulcorner \pi_q(S) \urcorner \in A$. $\square_{\text{Claim}}$

Then S is A-invariant hence A-definable. Take $\bar{b} \in A$ defining S. For any σ,

$$\sigma(\bar{b}) = \bar{b} \implies \sigma(S) = S \implies \sigma \in \mathrm{Aut}(\mathbb{M}/A) \implies \sigma(\bar{b}) = \bar{b}.$$

Therefore $\bar{b}$ codes S. $\square$

Lemma 1.3. *If $A \subseteq \mathbb{M}^{\mathrm{eq}}$ and $D \subseteq \mathbb{M}^n$ is non-empty and A-definable then there is $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$ with $\bar{b} \in D$.*

Proof. By induction on n.

- $n = 1$
  - D is finite. Then $D \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. Take any $b \in D$.
  - D is cofinite. Then $D \cap \mathbb{Q} \neq \varnothing$. Take any $b \in D \cap \mathbb{Q}$.
- $n > 1$: Let $D' = \{\bar{b} \in \mathbb{M}^{n-1} : \exists c \in \mathbb{M}\ (\bar{b}, c) \in D\}$. Then D' is non-empty and A-definable. By induction there is $\bar{b} \in D'$, $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$. Let $D'' = \{c \in \mathbb{M} : (\bar{b}, c) \in D\}$. Then D'' is non-empty and $A\bar{b}$-definable. By induction there is $c \in D''$ with

$$c \in \mathrm{acl}^{\mathrm{eq}}(A\bar{b}) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{acl}^{\mathrm{eq}}(A).$$

  Then $(\bar{b}, c) \in \mathrm{acl}^{\mathrm{eq}}(A)$ and $(\bar{b}, c) \in D$. $\square$

Theorem 1.4. ACF_0 *has uniform elimination of imaginaries.*

Proof. $\mathrm{ACF}_0 \vdash 0 \neq 1$, so it suffices to show elimination of imaginaries by (April 7, Theorem 26). Take e in a 0-interpretable set D/E. By (April 7, Example 32), e is the code of an E-equivalence class X (namely $X = e$). By Lemma 1.3 there is $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(e)$ with $\bar{a} \in X$. Let $S = \{\sigma(\bar{a}) : \sigma \in \mathrm{Aut}(\mathbb{M}/e)\}$. Note $S \subseteq X$ because $\bar{a} \in X$ and X is e-invariant. By (April 7, Proposition 14), S is finite and e-definable. Take $\ulcorner S \urcorner \in \mathbb{M}^m$ by Fact 1.2. Then $\ulcorner S \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(e)$. On the other hand, X is the unique E-equivalence class containing S, so X and e are $\ulcorner S \urcorner$-definable, and $e \in \mathrm{dcl}^{\mathrm{eq}}(\ulcorner S \urcorner)$.[1] Thus e is interdefinable with $\ulcorner S \urcorner \in \mathbb{M}^m$. $\square$

With some modifications to the proof, one can also handle algebraically closed fields of positive characteristic:

Fact 1.5. ACF *has uniform elimination of imaginaries.*

[1]More precisely, if $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner$, then $\sigma(S) = S$. As D, E are 0-definable, $\sigma(X)$ is some E-equivalence class. But $S \subseteq X \implies \sigma(S) \subseteq \sigma(X)$. So $S \subseteq \sigma(X)$. Therefore $\sigma(X)$ must be the same E-equivalence class as X. Then $\sigma(X) = X$. This argument shows $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner \implies \sigma(X) = X$, which means X is $\ulcorner S \urcorner$-definable.

2 Stability and M^{eq}

Theorem 2.1. *If $\mathbb{M}$ is λ-stable, then $\mathbb{M}^{\text{eq}}$ is λ-stable.*

Proof. Suppose $A \subseteq \mathbb{M}^{\text{eq}}$ has $|A| \leq \lambda$. Suppose for a contradiction that $|S_{\bar{x}}(A)| \geq \lambda^+$. Each imaginary is definable over finitely many reals, so there is $B \subseteq \mathbb{M}$ with $|B| \leq \lambda$ and $A \subseteq \text{dcl}^{\text{eq}}(B)$. Then $\text{tp}(e/B)$ determines $\text{tp}(e/A)$, so $|S_{\bar{x}}(A)| \leq |S_{\bar{x}}(B)|$. Replacing A with B, we may assume $A \subseteq \mathbb{M}$. Let X be the product of sorts where $\bar{x}$ lives. By definition of $\mathbb{M}^{\text{eq}}$ there is a product of sorts Y in $\mathbb{M}$ and a 0-definable surjection $\pi : Y \to X$. Then $\text{tp}(\bar{c}/A)$ determines $\text{tp}(\pi(\bar{c})/A)$, so $|S_{\bar{x}}(A)| \leq |S_Y(A)|$. By λ-stability, $|S_Y(A)| \leq \lambda$. $\square$

3 Almost A-definability

Proposition 3.1. *If $A \subseteq \mathbb{M}$, then $A = \bigcap\{M : M \preceq \mathbb{M},\ M \supseteq A\}$.*

Proof. By (April 7, Prop. 18), $\text{acl}(M) = M$ if $M \preceq \mathbb{M}$. If $b \in \text{acl}(A)$, then $b \in \text{acl}(A) \subseteq \text{acl}(M) = M$, so $\text{acl}(A) \subseteq \bigcap_{M \supseteq A} M$.

Conversely, suppose $b \notin \text{acl}(A)$. Take some small $M_0 \preceq \mathbb{M}$ with $M_0 \supseteq A$. By (April 7, Prop. 14), $\{\sigma(b) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is large, hence not contained in M_0. Take σ such that $\sigma(b) \notin M_0$, so $b \notin \sigma^{-1}(M_0)$. If $M = \sigma^{-1}(M_0)$, then $A \subseteq M \preceq \mathbb{M}$ and $b \notin M$. $\square$

Proposition 3.2. *If $D \subseteq \mathbb{M}^n$ is definable and $A \subseteq \mathbb{M}$ is small, the following are equivalent:*

1. *D is $\text{acl}^{\text{eq}}(A)$-definable.*
2. *$\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is finite.*
3. *$\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is small.*
4. *D is M-definable for every small $M \preceq \mathbb{M}$ with $M \supseteq A$.*

Proof. Let $\ulcorner D \urcorner \in \mathbb{M}^{\text{eq}}$ be a code for D. The four conditions say

1. $\ulcorner D \urcorner \in \text{acl}^{\text{eq}}(A)$.
2. $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is finite.
3. $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is small.
4. $\ulcorner D \urcorner \in M^{\text{eq}}$ for every $M \preceq \mathbb{M}$ and $M \supseteq A$.

These are equivalent by (April 7, Prop. 14) and Proposition 3.1. $\square$

We say D is "almost A-definable" if those equivalent conditions hold.

Proposition 3.3. *If $p \in S_n(\mathbb{M})$ is a definable type and $A \subseteq \mathbb{M}$ is small, the following are equivalent:*

1. *p is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.*

2. *$\{\sigma(p) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.*

3. *p is M-definable for every small $M \preceq \mathbb{M}$ with $M \supseteq A$.*

Proof. Apply Proposition 3.2 to the definable sets $D_\varphi := \{\bar{b} \in \mathbb{M} : \varphi(\bar{x}; \bar{b}) \in p(\bar{x})\}$. □

Note condition (3) is equivalent to p being "Lascar A-invariant" as in the March 31 notes.

4 Theorem of the bound revisited

Assume for the rest of these notes that the theory is stable.

Recall the fundamental order, $[p]$, $\mathrm{Ex}_M(p)$, and $\mathrm{Bd}(p)$ from the March 31 notes. (We only need the first three sections of those notes, and the important things from those notes are Proposition 9, Corollary 13, and Lemma 15.)

Lemma 4.1. *Suppose $p \in S_n(A)$ and $q \in S_n(\mathbb{M})$ is an extension with $[q] \in \mathrm{Bd}(p)$.*

1. *If $A \subseteq M \preceq \mathbb{M}$, then q is M-definable.*

2. *q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.*

Proof. 1. Let $r = q \restriction M$. So $p \in S_n(A), r \in S_n(M), q \in S_n(\mathbb{M}), p \subseteq r \subseteq q$. If $r \sqsubseteq q$, then q is M-definable. If $r \not\sqsubseteq q$, then $[r] > [q]$, contradicting the fact that $[q]$ is a bound of p.

2. Let M vary, use Proposition 3.3(3). □

Corollary 4.2. *If $p \in S_n(A)$, there is an $\mathrm{acl}^{\mathrm{eq}}(A)$-definable extension $q \in S_n(\mathbb{M})$.*

Proof. Take $\beta \in \mathrm{Bd}(p) \subseteq \mathrm{Ex}_{\mathbb{M}}(p)$. Take an extension $q \in S_n(\mathbb{M})$ with $[q] = \beta$. □

Proposition 4.3. *Suppose $A = \mathrm{acl}^{\mathrm{eq}}(A)$. Suppose $p \in S_n(A)$, and $q_1, q_2 \in S_n(\mathbb{M})$ are two A-definable extensions. Then $q_1 = q_2$.*

Proof. If not, take $\varphi(\bar{x}; \bar{b}) \in q_1(\bar{x})$, $\neg\varphi(\bar{x}; \bar{b}) \in q_2(\bar{x})$. By Corollary 4.2, $\mathrm{tp}(\bar{b}/A)$ has an A-definable global extension $r(\bar{x})$. Take $\bar{c} \models q_1 \restriction A\bar{b}$. Then $\varphi(\bar{x}; \bar{b}) \in (q_1 \restriction A\bar{b})$, so $\mathbb{M} \models \varphi(\bar{c}; \bar{b})$. Also,

$$\bar{b} \models r \restriction A \text{ and } \bar{c} \models q_1 \restriction A\bar{b}$$
$$(\bar{b}, \bar{c}) \models (r \otimes q_1) \restriction A$$
$$(\bar{c}, \bar{b}) \models (q_1 \otimes r) \restriction A$$

(because any two types commute in a stable theory, March 17, Thm. 16)

$$\bar{c} \models q_1 \restriction A \text{ and } \bar{b} \models r \restriction A\bar{c}$$
$$\bar{c} \models q_2 \restriction A \text{ and } \bar{b} \models r \restriction A\bar{c}$$

(because $q_1 \restriction A = p = q_2 \restriction A$)

$$(\bar{c}, \bar{b}) \models (q_2 \otimes r) \restriction A$$
$$(\bar{b}, \bar{c}) \models (r \otimes q_2) \restriction A$$
$$\bar{b} \models r \restriction A \text{ and } \bar{c} \models q_2 \restriction A\bar{b}.$$

But $q_2 \restriction A\bar{b}$ contains the formula $\neg\varphi(\bar{x}; \bar{b})$, so $\mathbb{M} \models \neg\varphi(\bar{c}; \bar{b})$, a contradiction. □

Proposition 4.4. *Suppose $A = \operatorname{acl}^{\mathrm{eq}}(A)$ and $p \in S_n(A)$.*

1. *$\operatorname{Bd}(p)$ contains a single point β.*
2. *If $q \in S_n(\mathbb{M})$ extends p, then q is A-definable iff $[q] = \beta$.*
3. *There is a unique extension $q \in S_n(\mathbb{M})$ with those two properties.*

Proof.

1. Take $\beta_1, \beta_2 \in \operatorname{Bd}(p)$. Take global extensions q_1, q_2 with $[q_i] = \beta_i$. By Lemma 4.1(2), q_1, q_2 are A-definable. By Proposition 4.3, $q_1 = q_2$, so $\beta_1 = \beta_2$.
2. If $[q] = \beta$, then q is A-definable by Lemma 4.1(2). Conversely, suppose q is A-definable. Take an extension $q' \in S_n(\mathbb{M})$ with $[q'] = \beta$. By Lemma 4.1(2), q' is A-definable. By Proposition 4.3, $q = q'$, and so $[q] = \beta$.
3. Existence is Corollary 4.2, uniqueness is Proposition 4.3. □

Proposition 4.5 ("Theorem of the bound" plus other stuff)**.** *Suppose $p \in S_n(A)$.*

1. *$\operatorname{Bd}(p)$ contains a single point β.*
2. *If $q \in S_n(\mathbb{M})$ extends p, then q is $\operatorname{acl}^{\mathrm{eq}}(A)$-definable iff $[q] = \beta$.*
3. *If $X = \{q \in S_n(\mathbb{M}) : q \supseteq p,\ q \text{ is } \operatorname{acl}^{\mathrm{eq}}(A)\text{-definable}\}$, and $Y = \{q \in S_n(\operatorname{acl}^{\mathrm{eq}}(A)) : q \supseteq p\}$, then there is a bijection*

$$X \to Y$$
$$q \mapsto q \restriction \operatorname{acl}^{\mathrm{eq}}(A).$$

4. *If $q_1, q_2 \in S_n(\mathbb{M})$ are $\operatorname{acl}^{\mathrm{eq}}(A)$-definable extensions of p, then there is $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$ such that $\sigma(q_1) = q_2$.*

Proof.

3. Proposition 4.4(3) gives a bijection between types over $\operatorname{acl}^{\mathrm{eq}}(A)$ and $\operatorname{acl}^{\mathrm{eq}}(A)$-definable global types.
4. Take $\bar{c}_i \models q_i \restriction \operatorname{acl}^{\mathrm{eq}}(A)$ for $i = 1, 2$. Then $\operatorname{tp}(\bar{c}_i/A) = p$ for $i = 1, 2$, so there is $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$ with $\sigma(\bar{c}_1) = \bar{c}_2$. Then

$$\sigma(\operatorname{tp}(c_1/\operatorname{acl}^{\mathrm{eq}}(A))) = \operatorname{tp}(c_2/\operatorname{acl}^{\mathrm{eq}}(A))$$
$$\sigma(q_1 \restriction \operatorname{acl}^{\mathrm{eq}}(A)) = q_2 \restriction \operatorname{acl}^{\mathrm{eq}}(A)$$
$$\sigma(q_1) \restriction \operatorname{acl}^{\mathrm{eq}}(A) = q_2 \restriction \operatorname{acl}^{\mathrm{eq}}(A).$$

   By the bijection in part (3), $\sigma(q_1) = q_2$.

1. Take $\beta_1, \beta_2 \in \mathrm{Bd}(p)$. Take $q_1, q_2 \in S_n(\mathbb{M})$ extending p with $[q_i] = \beta_i$. By Lemma 4.1(2), q_1, q_2 are $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. By part (4), there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(q_1) = q_2$. Then $\beta_1 = [q_1] = [q_2] = \beta_2$.

2. If $[q] = \beta \in \mathrm{Bd}(p)$, then q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable by Lemma 4.1(2). Conversely, suppose q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. Take $q' \in S_n(\mathbb{M})$ extending p with $[q'] = \beta$. By Lemma 4.1(2), q' is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. By part (4), there is $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(q') = q$, so $[q'] = [q] = \beta$. ☐

Remark 4.6. The *strong type* of $\bar{b}$ over A is $\mathrm{tp}(\bar{b}/\mathrm{acl}^{\mathrm{eq}}(A))$. The strong type is sometimes denoted $\mathrm{stp}(\bar{b}/A)$. A set X is $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant iff

$$\mathrm{stp}(\bar{b}/A) = \mathrm{stp}(\bar{c}/A) \implies (\bar{b} \in X \iff \bar{c} \in X).$$

Proposition 4.5(3) says that there is a bijection between strong types over A and $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant global types.

One can define strong types without using $\mathbb{M}^{\mathrm{eq}}$. It turns out that $\mathrm{stp}(\bar{b}/A) = \mathrm{stp}(\bar{c}/A)$ if and only if $\bar{b}E\bar{c}$ for every A-definable "finite" equivalence relation E on $\mathbb{M}^n$. Here, we say that E is "finite" if $\mathbb{M}^n/E$ is finite. This is the approach used in the textbook. But it is more enlightening to use $\mathbb{M}^{\mathrm{eq}}$.

5 Forking revisited

Let $\mathrm{bd}(p)$ denote the unique bound of $p \in S_n(A)$.

Remark 5.1. If $p \in S_n(A)$ and $q \in S_n(B)$ is an extension, then $\mathrm{bd}(q) \le \mathrm{bd}(p)$. Indeed, $\mathrm{bd}(q) = \max \mathrm{Ex}_{\mathbb{M}}(q)$ and $\mathrm{bd}(p) = \max \mathrm{Ex}_{\mathbb{M}}(p)$ and $\mathrm{Ex}_{\mathbb{M}}(q) \subseteq \mathrm{Ex}_{\mathbb{M}}(p)$.

Remark 5.2. If $p \in S_n(M)$ and $M \preceq \mathbb{M}$, then $\mathrm{bd}(p) = \max \mathrm{Ex}_M(p) = \max\{[p]\} = [p]$.

Definition 5.3. Let $q \in S_n(B)$ extend $p \in S_n(A)$. Then q is a *non-forking extension* of p, written $q \sqsupseteq p$, if $\mathrm{bd}(q) = \mathrm{bd}(p)$, and q is a *forking extension* of p, written $q \not\sqsupseteq p$, if $\mathrm{bd}(q) < \mathrm{bd}(p)$.

When A, B are models, this agrees with the previous notation $q \sqsupseteq p$ for q being an heir/coheir of p.

Proposition 5.4 (Full Transitivity)**.** *Suppose $A_1 \subseteq A_2 \subseteq A_3$ and $p_i \in S_n(A_i)$ for $i = 1, 2, 3$ with $p_1 \subseteq p_2 \subseteq p_3$. Then $p_1 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_2$ and $p_2 \sqsubseteq p_3$.*

Proof. Obvious. ☐

Proposition 5.5 (Extension)**.** *If $p \in S_n(A)$ and $B \supseteq A$, then there is at least one $q \in S_n(B)$ with $q \sqsupseteq p$.*

Proof. Take a small model $M \supseteq B$. Then $\mathrm{bd}(p) = \max \mathrm{Ex}_M(p)$ so there is $r \in S_n(M)$ extending p with $[r] = \mathrm{bd}(p)$. Let $q = r \upharpoonright B$. Then $\mathrm{bd}(r) = \mathrm{bd}(p)$, so $r \sqsupseteq p$. By full transitivity, $q \sqsupseteq p$. □

Proposition 5.6. *If $p \in S_n(A)$ and $q \in S_n(\mathbb{M})$ is a global extension, then $q \sqsupseteq p$ iff $[q] = \mathrm{bd}(p)$ iff q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.*

Proof. The first iff is by Remark 5.2, the second is by Proposition 4.5(2). □

In light of Proposition 5.6, Proposition 4.5 is really a statement about global non-forking extensions. In particular, global non-forking extensions of p correspond to extensions of p to $\mathrm{acl}^{\mathrm{eq}}(A)$, and any two extensions are conjugate over A.

Proposition 5.7. *If $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \sqsupseteq p$ iff some global extension of q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.*[2]

Proof. If $q \sqsupseteq p$, by extension there is $r \in S_n(\mathbb{M})$ with $r \sqsupseteq q$, and then $r \sqsupseteq p$ by transitivity and r is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable by Proposition 5.6.

Conversely, if $r \supseteq q$ and r is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable, then $r \sqsupseteq p$ by Proposition 5.6, so $q \sqsupseteq p$ by full transitivity. □

Proposition 5.8. *If $p \in S_n(A)$ and $q \in S_n(\mathrm{acl}^{\mathrm{eq}}(A))$ is an extension, then $q \sqsupseteq p$.*

Proof. By Corollary 4.2, q has a global $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant extension. □

Definition 5.9. An $L(\mathbb{M})$-formula $\varphi(\bar{x})$ *forks over A* if no global type containing $\varphi(\bar{x})$ is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.

Proposition 5.10 (Finite character)**.** *If $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \not\sqsupseteq p$ iff some $\varphi(\bar{x}) \in q(\bar{x})$ forks over A.*

Proof. Let $\Sigma_A(\bar{x})$ be the global partial type

$$\{\varphi(\bar{x};\bar{b}) \leftrightarrow \varphi(\bar{x};\bar{c}) : \varphi \in L,\ \bar{b} \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} \bar{c}\}.$$

A global type $r \in S_n(\mathbb{M})$ extends $\Sigma_A(\bar{x})$ iff r is $\mathrm{acl}^{\mathrm{eq}}(A)$ -invariant iff r is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. Therefore...

1. An $L(\mathbb{M})$-formula $\varphi(\bar{x})$ forks over A iff $\Sigma_A(\bar{x}) \cup \{\varphi(\bar{x})\}$ is inconsistent (Definition 5.9)

2. $q \sqsupseteq p$ iff $q(\bar{x}) \cup \Sigma_A(\bar{x})$ is consistent (Proposition 5.7).

Now use compactness: $q(\bar{x}) \cup \Sigma_A(\bar{x})$ is inconsistent iff some finite subset is inconsistent. □

We say that $q \in S_n(B)$ *forks over A* if q contains a formula which forks over A.

[2] This criterion is an alternative way to define $\sqsupseteq$.

6 Stationary types

Lemma 6.1. *The following are equivalent for $p \in S_n(A)$:*

1. *p has a unique non-forking extension over $\mathbb{M}$.*
2. *p has a unique non-forking extension over any $B \supseteq A$.*
3. *p has a unique extension to $\mathrm{acl}^{\mathrm{eq}}(A)$.*
4. *p has an A-definable extension over $\mathbb{M}$.*

Proof. (1) $\implies$ (2): if q_1, q_2 are two non-forking extensions over B, then they have global non-forking extensions r_1, r_2. By (1), $r_1 = r_2$, so $q_1 = r_1 \restriction B = r_2 \restriction B = q_2$.

(2) $\implies$ (3): take $B = \mathrm{acl}^{\mathrm{eq}}(A)$, use Proposition 5.8.

(3) $\iff$ (1): Proposition 4.5(3).

(1) $\implies$ (4): if q is the unique non-forking extension, then $\mathrm{Aut}(\mathbb{M}/A)$ fixes q by symmetry, so q is A-invariant and A-definable.

(4) $\implies$ (1): Let q be the A-definable extension. Then q is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable (= non-forking). If q' is any other non-forking extension, then $q' = \sigma(q) = q$ for some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, by Proposition 4.5(4). $\square$

We say p is *stationary* if it satisfies these equivalent properties.

Example 6.2.

1. If $p \in S_n(M)$ and $M \preceq \mathbb{M}$, then p is stationary.
2. More generally, if $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $p \in S_n(A)$, then p is stationary. (It has an A-definable extension by Corollary 4.2.) In the language of Remark 4.6, strong types are stationary.
3. If $p \in S_n(A)$ is stationary and $q \in S_n(B)$ is a non-forking extension, then q is stationary. (By transitivity, every global non-forking extension of q is a global non-forking extension of p. So q has no more global non-forking extensions than p.)
4. In a strongly minimal theory, the transcendental type over A is stationary. (The global transcendental type is $\varnothing$-definable and extends it.)
5. In $\mathbb{C} \models \mathrm{ACF}$, $\mathrm{tp}(i/\mathbb{Q}) = \mathrm{tp}(-i/\mathbb{Q})$ is *not* stationary, because it has two extensions to $\mathrm{acl}(\mathbb{Q}) = \mathbb{Q}^{alg}$, namely $\mathrm{tp}(i/\mathbb{Q}^{alg})$ and $\mathrm{tp}(-i/\mathbb{Q}^{alg})$.

Lemma 6.3. *Let $p \in S_n(A)$ be stationary, let $q \in S_n(B)$ be an extension of p, and let $p^{\mathbb{M}} \in S_n(\mathbb{M})$ be the unique A-definable global extension of p. Then $q \sqsupseteq p$ iff $q = p^{\mathbb{M}} \restriction B$.*

Proof. Note $p^{\mathbb{M}} \sqsupseteq p$. If $q = p^{\mathbb{M}} \restriction B$, then $q \sqsupseteq p$ by full transitivity. Conversely, if $q \sqsupseteq p$, take some $r' \in S_n(\mathbb{M})$ with $r' \sqsupseteq q \sqsupseteq p$. Then $r' = p^{\mathbb{M}}$ by stationarity, so $q \subseteq r' = p^{\mathbb{M}}$, and $q = p^{\mathbb{M}} \restriction B$. $\square$

7 Local character

Definition 7.1. $\kappa_n(T)$ is the smallest infinite cardinal such that there is no descending chain $(\beta_\alpha : \alpha < \kappa)$ of length κ in the fundamental order for n-types. $\kappa(T)$ denotes $\kappa_1(T)$.

Later we will see that $\kappa_n(T) = \kappa(T)$.

Remark 7.2. $\kappa_n(T) \leq |L|^+$. Otherwise, there is a descending chain $(\beta_\alpha : \alpha < |L|^+)$. If $\beta_\alpha = [p_\alpha]$, then $[p_\alpha] \subsetneq [p_{\alpha+1}]$, and we can take $\varphi_\alpha \in [p_{\alpha+1}] \setminus [p_\alpha]$. The φ_α are pairwise distinct, and there are $|L|^+$ of them, which is absurd.

For example, in a countable language, $\kappa_n(T)$ is either $\aleph_0$ or $\aleph_1$.

Proposition 7.3 (Local character)**.** *If $p \in S_n(A)$, then there is $B \subseteq A$ with $|B| < \kappa_n(T)$ such that $p \sqsupseteq (p \upharpoonright B)$.*

Proof. Suppose not. Let $\kappa = \kappa_n(T)$. Recursively build a sequence $(\bar{b}_\alpha : \alpha < \kappa)$ as follows. At step $\alpha < \kappa$, let $B_\alpha = (\bar{b}_\beta : \beta < \alpha)$. Then $B_\alpha \subseteq A$ and $|B_\alpha| < \kappa$, so $p \not\sqsupseteq (p \upharpoonright B_\alpha)$. Take some formula $\varphi_\alpha(\bar{x}; \bar{b}_\alpha)$ forking over B_α.

Then $(B_\alpha : \alpha < \kappa)$ is an increasing chain of subsets of A, and $p \upharpoonright B_{\alpha+1}$ contains $\varphi_\alpha(\bar{x}; \bar{b}_\alpha)$ which forks over B_α, and so

$$p \upharpoonright B_{\alpha+1} \not\sqsupseteq p \upharpoonright B_\alpha$$
$$\mathrm{bd}(p \upharpoonright B_{\alpha+1}) < \mathrm{bd}(p \upharpoonright B_\alpha).$$

So we have a descending chain of length κ in the fundamental order, a contradiction. □

8 Stability spectra

Lemma 8.1. *Let M be a model. Let λ be a cardinal. Let β, β' be elements in the fundamental order with $\beta > \beta'$. Suppose $N \succeq M$ is sufficiently saturated and strongly homogeneous. If $p \in S_n(M)$ with $[p] = \beta$, then there are at least λ extensions $q \in S_n(N)$ with $[q] = \beta'$.*

Proof. Because $\beta' \leq [p]$, there is an extension $N_0 \succeq M$ and a $q_0 \in S_1(N_0)$ with $[q_0] = \beta'$, by (March 31, Prop. 7). Embed N_0 into N and take $q \in S_1(N)$ the heir of q_0. Then $[q] = \beta' < \beta = [p]$, so $q \not\sqsupseteq p$, so q forks over M, and q is not M-definable. By (April 7, Prop. 14), $\{\sigma(q) : \sigma \in \mathrm{Aut}(N/M)\}$ is large, hence bigger than λ. Each conjugate $\sigma(q)$ also has class β' and extends p. □

Proposition 8.2. *Suppose $\lambda^\mu > \lambda$ for some $\mu < \kappa(T)$. Then T is not λ-stable.*

Proof. Take μ minimal with $\lambda^\mu > \lambda$. Note $\lambda^\lambda \geq 2^\lambda > \lambda$, so $\mu \leq \lambda$.

Take a descending sequence $(\beta_\alpha : \alpha < \mu)$ in the fundamental order. Take a type $p \in S_1(M_0)$ with $[p] = \beta_0$. Build an increasing chain of models $(M_\alpha : \alpha < \mu)$ with $M_{\alpha+1}$ being highly saturated and strongly homogeneous over M_α and $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$ for limit ordinals

γ. By Lemma 8.1, we can find distinct $p_i \in S_1(M_1)$ for $i < \lambda$ with $p_i \supseteq p$ and $[p_i] = \beta_1$. Then for each p_i we can find distinct $p_{ij} \in S_1(M_2)$ for $j < \lambda$ with $p_{ij} \supseteq p_i$ and $[p_{ij}] = \beta_2$. Continuing on in this fashion[3], we get a tree of types p_σ for $\sigma \in \lambda^{\le \mu}$ where p_σ is a type over M_α if $\sigma \in \lambda^\alpha$, where $p_\tau \supseteq p_\sigma$ if τ extends σ, and where $p_{\sigma i} \neq p_{\sigma j}$ for $i < j < \lambda$.

Fix $\alpha < \mu$. Note that $|\lambda^\alpha| = \lambda^{|\alpha|} \le \lambda$ by choice of μ. For each $\sigma \neq \tau \in \lambda^\alpha$, take some $L(M_\alpha)$-formula $\varphi(x; b)$ distinguishing p_σ from p_τ, and collect all the b in a set $B_\alpha \subseteq M_\alpha$. Then $|B_\alpha| \le |\lambda^\alpha|^2 \le \lambda^2 = \lambda$, and the $p_\sigma \restriction B_\alpha$ are pairwise distinct as σ ranges over λ^α.

Let $B = \bigcup_{\alpha<\mu} B_\alpha$. Then $|B| \le \mu \cdot \lambda = \lambda$. We claim there is an injection

$$\begin{aligned} \lambda^\mu &\to S_1(B) \\ \sigma &\mapsto p_\sigma \restriction B, \end{aligned}$$

which would imply $|S_1(B)| \ge \lambda^\mu > \lambda$, contradicting λ-stability. Suppose $\sigma, \tau \in \lambda^\mu$ with $\sigma \neq \tau$. Then there is $\rho \in \lambda^\alpha$ and $i \neq j \in \lambda$ such that σ extends ρi and τ extends ρj. Then the restrictions of $p_\sigma \restriction B$ and $p_\tau \restriction B$ to $B_{\alpha+1}$ are $p_{\rho i} \restriction B_{\alpha+1}$ and $p_{\rho j} \restriction B_{\alpha+1}$, which differ because of how we chose $B_{\alpha+1}$. Therefore $p_\sigma \restriction B \neq p_\tau \restriction B$. $\square$

Corollary 8.3. *If T is λ-stable, then $\lambda \ge \kappa(T)$.*

Proof. Otherwise, take $\mu = \lambda < \kappa(T)$. Then $\lambda < 2^\lambda \le \lambda^\lambda = \lambda^\mu$, and $\mu < \kappa(T)$, so T is not λ-stable. $\square$

Lemma 8.4. *Suppose T is λ-stable and $A \subseteq \mathbb{M}$ with $|A| \le \lambda$. Then $|S_1(\mathrm{acl}^{\mathrm{eq}}(A))| \le \lambda$.*

Proof. Build an increasing sequence $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of length ω, where A_{n+1} is obtained by adding to A_n one realization of each 1-type over A_n. If $|A_n| \le \lambda$, then $|S_1(A_n)| \le \lambda$ by λ-stability, so $|A_{n+1}| \le \lambda$. By induction, $|A_n| \le \lambda$ for all n. Let $M = \bigcup_n A_n$. Then $|M| \le \aleph_0 \cdot \lambda = \lambda$.

We claim $M \preceq \mathbb{M}$. Use the Tarski-Vaught test. Let $D \subseteq \mathbb{M}$ be M-definable and non-empty. Then D is A_n-definable for some n. Take $b \in D$. Then $\mathrm{tp}(b/A_n)$ is realized by some $b' \in A_{n+1} \subseteq M$. Then $b \equiv_{A_n} b' \implies b' \in D$, so $D \cap M \neq \varnothing$.

Thus $M \preceq \mathbb{M}$ and $|M| \le \lambda$. By λ-stability, $|S_1(M)| \le \lambda$. As $\mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$, we have a surjective restriction map $S_1(M) \to S_1(\mathrm{acl}^{\mathrm{eq}}(A))$, so $|S_1(\mathrm{acl}^{\mathrm{eq}}(A))| \le |S_1(M)| \le \lambda$. $\square$

Definition 8.5. $\lambda_0(T)$ is the smallest infinite cardinal λ such that T is λ-stable.

Theorem 8.6. *T is λ-stable if and only if $\lambda \ge \lambda_0(T)$ and $\lambda^\mu \le \lambda$ for all $\mu < \kappa(T)$.*

Proof. If $\lambda < \lambda_0(T)$, then T isn't λ-stable by definition of $\lambda_0(T)$. If $\lambda^\mu > \lambda$ for some $\mu < \kappa(T)$, then T isn't λ-stable by Proposition 8.2.

So assume $\lambda \ge \lambda_0(T)$ and $\lambda^\mu \le \lambda$ for every $\mu < \kappa(T)$. We claim T is λ-stable.

By Corollary 8.3, $\lambda \ge \lambda_0 \ge \kappa$.

[3] Sweeping some details under the rug at limit ordinals… (We should have assumed that $\beta_\gamma = \bigcup_{\alpha<\gamma} \beta_\alpha$ when γ is a limit ordinal.)

Fix a set A with $|A| \leq \lambda$. We claim $|S_1(A)| \leq \lambda$. If $p \in S_1(A)$, then there is a subset $B \subseteq A$ with $\mu := |B| < \kappa$ and $p \sqsupseteq p \restriction B$ by Proposition 7.3. The number of possibilities for μ is $\leq \kappa$. Given μ, the number of possibilities for B is $\leq |A|^\mu \leq \lambda^\mu \leq \lambda$. So the number of possibilities for B is $\leq \kappa \cdot \lambda = \lambda$.

If c realizes p, then $p = \mathrm{tp}(c/A) \sqsupseteq \mathrm{tp}(c/B)$. Note that $\mathrm{tp}(c/A)$ is determined by $\mathrm{tp}(c/\operatorname{acl}^{\mathrm{eq}}(A))$, which is determined by $\mathrm{tp}(c/\operatorname{acl}^{\mathrm{eq}}(B))$ because types over algebraically closed sets are stationary. As $|B| < \kappa \leq \lambda_0$, Lemma 8.4 gives $|S_1(\operatorname{acl}^{\mathrm{eq}}(B))| \leq \lambda_0$. Thus, given B, there are $\leq \lambda_0$ possibilities for $p \restriction B$ and p. All together, there are $\leq \kappa \cdot \lambda \cdot \lambda_0 = \lambda$ possibilities for p. $\square$

Remark 8.7. Theorem 8.6 shows that $\kappa(T)$ and $\lambda_0(T)$ determine the *stability spectrum* of T, the set $S = \{\lambda : T \text{ is } \lambda\text{-stable}\}$. Conversely, Poizat claims that one can conversely recover $\kappa(T)$ and $\lambda_0(T)$ from the stability spectrum. I believe this is actually false.

9 Superstability

Definition 9.1. T is *superstable* if $\kappa(T) = \aleph_0$, i.e., there are no infinite descending chains in the fundamental order.

If $p \in S_n(A)$ and T is superstable, there is a *finite* subset $A_0 \subseteq A$ such that $p \sqsupseteq p \restriction A_0$, by Proposition 7.3.

Proposition 9.2. *T is superstable if and only if T is λ-stable for all sufficiently large λ.*

Proof. Suppose $\kappa(T) = \aleph_0$. By Theorem 8.6, T is λ-stable iff $\lambda \geq \lambda_0(T)$ and $\lambda^\mu \leq \lambda$ for $\mu < \aleph_0$. But $\lambda^\mu \leq \lambda$ for finite μ, so the second condition always holds. Then T is λ-stable for all $\lambda \geq \lambda_0(T)$.

Suppose $\kappa(T) > \aleph_0$. If $\lambda = \aleph_{\alpha+\omega}$ for some ordinal α, then $\mathrm{cof}(\lambda) = \omega$, so $\lambda^{\aleph_0} > \lambda$ by König's lemma (see Corollary 8.16 in the textbook). By Theorem 8.6 (or Proposition 8.2), T is not λ-stable. There are arbitrarily high cardinals of the form $\aleph_{\alpha+\omega}$. $\square$

Now suppose T is a stable theory in a countable language. By Remark 7.2, we know $\kappa(T)$ is $\aleph_0$ or $\aleph_1$. By Corollary 8.3, $\lambda_0(T) \geq \kappa(T)$. By the proof of (March 10, Lemma 1), we know T is $2^{\aleph_0}$-stable, so $\lambda_0(T) \leq 2^{\aleph_0}$. By definition, $\kappa(T)$ and $\lambda_0(T)$ are infinite. So we know:

$$\aleph_0 \leq \kappa(T) \leq \aleph_1$$
$$\aleph_0 \leq \lambda_0(T) \leq 2^{\aleph_0}$$
$$\kappa(T) \leq \lambda_0(T).$$

Later using Morley rank we will prove the following:

Fact 9.3. *$\lambda_0(T)$ is $\aleph_0$ or $2^{\aleph_0}$.*

So there are three possibilities:

Name	$\kappa(T)$	$\lambda_0(T)$	stability spectrum
ω-stable	$\aleph_0$	$\aleph_0$	$\{\lambda : \lambda \geq \aleph_0\}$
Superstable but not ω-stable	$\aleph_0$	$2^{\aleph_0}$	$\{\lambda : \lambda \geq 2^{\aleph_0}\}$
Stable but not superstable	$\aleph_1$	$2^{\aleph_0}$	$\{\lambda : \lambda = \lambda^{\aleph_0}\}$

Fact 9.4.

1. *Strongly minimal theories like ACF are ω-stable (March 3, example on p. 5).*
2. *$(\mathbb{Z}, +)$ is superstable but not ω-stable.*
3. *Separably closed fields (other than ACF) are stable but not superstable.*

10 The forking calculus

Everything we have done so far with the fundamental order, theorem of the bound, and non-forking (except §7–9) works with not just n-types, but α-types where α is possibly infinite.[4]

Definition 10.1. Let $\bar{a}, \bar{b}$ be tuples, possibly infinite, and let $C \subseteq \mathbb{M}$ be a set. Then $\bar{a}$ and $\bar{b}$ are *independent* over C, written $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$, if $\operatorname{tp}(\bar{a}/C\bar{b}) \sqsupseteq \operatorname{tp}(\bar{a}/C)$.

Lemma 10.2. *Suppose $C = \operatorname{acl}^{\mathrm{eq}}(C)$, and $\bar{a}, \bar{b}$ are tuples. Let p, q be the global C-definable extensions of $\operatorname{tp}(\bar{a}/C)$ and $\operatorname{tp}(\bar{b}/C)$.*

1. *$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$ iff $(\bar{b}, \bar{a}) \models (q \otimes p) \restriction C$.*
2. *$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}$.*

Proof.

$$\begin{aligned}
&(\bar{b}, \bar{a}) \models (q \otimes p) \restriction C \\
&\iff \bar{b} \models q \restriction C \text{ and } \bar{a} \models p \restriction C\bar{b} && \text{(by definition)} \\
&\iff \bar{a} \models p \restriction C\bar{b} && \text{(because } \bar{b} \models q \restriction C) \\
&\iff \operatorname{tp}(\bar{a}/C\bar{b}) \subseteq p \\
&\iff \operatorname{tp}(\bar{a}/C\bar{b}) \sqsupseteq \operatorname{tp}(\bar{a}/C) && \text{(by Lemma 6.3)} \\
&\iff \bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}.
\end{aligned}$$

This proves (1), and then (2) follows because

$$(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C \iff (\bar{b}, \bar{a}) \models (q \otimes p) \restriction C$$

as all types commute in stable theories (March 17, Thm. 16). $\square$

[4] There is an alternative approach to this section only using n-types with finite n. It's a little clumsier, but if anyone is interested, I can write up notes on it.

Lemma 10.3. *Suppose C is arbitrary and $\bar{a}, \bar{b}$ are tuples.*

1. $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b} \iff \bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{\operatorname{acl}^{\mathrm{eq}}(C)} \bar{b}$.
2. $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a}$.

Proof. By Lemma 10.2, it suffices to prove (1). Note

$$\operatorname{tp}(\bar{a}/C) \subseteq \operatorname{tp}(\bar{a}/C\bar{b}) \sqsubseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C\bar{b}))$$
$$\operatorname{tp}(\bar{a}/C) \sqsubseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C)) \subseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C)\bar{b}) \sqsubseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C\bar{b}))$$

by Proposition 5.8. By full transitivity,

$$\operatorname{tp}(\bar{a}/C) \sqsubseteq \operatorname{tp}(\bar{a}/C\bar{b}) \iff \operatorname{tp}(\bar{a}/C) \sqsubseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C\bar{b})) \iff \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C)) \sqsubseteq \operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(C)\bar{b}).$$

□

Lemma 10.4. *Suppose $A, B, C \subseteq \mathbb{M}$. Let $\bar{a}, \bar{a}'$ be two tuples enumerating A and let $\bar{b}, \bar{b}'$ be two tuples enumerating B. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b} \iff \bar{a}' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b}'$.*

Proof. First, of all,

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b} \iff \bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b}',$$

because both sides say $\operatorname{tp}(\bar{a}/CB) \sqsupseteq \operatorname{tp}(\bar{a}/C)$. A similar argument shows $\bar{b}' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a} \iff \bar{b}' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a}'$. By symmetry, $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b}' \iff \bar{a}' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b}'$. □

Definition 10.5. $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$ means $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{b}$, where $\bar{a}, \bar{b}$ are tuples enumerating A, B, respectively.

Proposition 10.6. $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ *satisfies these properties:*

(Symmetry) $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B \iff B \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C A$

(Monotonicity) *If $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$ and $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B'$.*

(Transitivity) *If $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{CB} B'$, then $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C BB'$.*

(Base monotonicity) *If $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C BB'$, then $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{CB} B'$.*

(Normality) *$A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C CB$.*

(Invariance) *If $\sigma \in \operatorname{Aut}(\mathbb{M})$, then $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B \implies \sigma(A) \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{\sigma(C)} \sigma(B)$.*

(Extension) *Given A, B, C, there is $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$.*

(Finite character) *If $A_0 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$.*

Proof. Symmetry was Lemma 10.3(2). As $C \subseteq CB \subseteq CBB'$, full transitivity (Proposition 5.4) shows that

$$\operatorname{tp}(A/CBB') \sqsupseteq \operatorname{tp}(A/C) \iff (\operatorname{tp}(A/CBB') \sqsupseteq \operatorname{tp}(A/CB) \text{ and } \operatorname{tp}(A/CB) \sqsupseteq \operatorname{tp}(A/C))$$

or equivalently

$$A \mathop{\smile\!\!\!\!\!\!\,|}_{C} BB' \iff (A \mathop{\smile\!\!\!\!\!\!\,|}_{CB} B' \text{ and } A \mathop{\smile\!\!\!\!\!\!\,|}_{C} B).$$

This gives Transitivity, Base monotonicity, and Monotonicity on the right. Monotonicity on the left follows by symmetry.

Normality and invariance are obvious from the definition.

Extension comes from Proposition 5.5.

Proposition 5.10 shows that $\operatorname{tp}(A/BC)$ forks over C iff some formula in it forks over C iff $\operatorname{tp}(A/B_0C) \not\sqsupseteq \operatorname{tp}(A/C)$ for some finite $B_0 \subseteq B$. This gives Finite Character on the right, and Finite Character on the left follows by symmetry. □

11 Examples related to independence

Proposition 11.1. *Let p, q be global C-invariant types. Let $\bar{a}$ and $\bar{b}$ be realizations of $p \restriction C$ and $q \restriction C$, respectively. Then $(\bar{a}, \bar{b}) \models (p \otimes q) \restriction C$ iff $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_C \bar{b}$.*

Proof. By Lemma 6.1(4), $\operatorname{tp}(\bar{a}/C)$ and $\operatorname{tp}(\bar{b}/C)$ are stationary. The rest of the proof is like Lemma 10.2(1). □

Proposition 11.2. *The following are equivalent:*

1. $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$
2. $A \mathop{\smile\!\!\!\!\!\!\,|}_\varnothing B$ *after naming C as constants.*

Proof. Use the characterization of nonforking from Proposition 5.7. □

Proposition 11.3. $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_B \bar{a} \iff \bar{a} \in \operatorname{acl}^{\mathrm{eq}}(B)$.

Proof. $\Rightarrow$: if $\operatorname{tp}(\bar{a}/B\bar{a})$ doesn't fork over B, it has a global $\operatorname{acl}^{\mathrm{eq}}(B)$-invariant extension $p(\bar{x})$. Then $(\bar{x} = \bar{a}) \in \operatorname{tp}(\bar{a}/B\bar{a}) \subseteq p(\bar{x})$, so p is the constant type $\operatorname{tp}(\bar{a}/\mathbb{M})$. In order to be $\operatorname{acl}^{\mathrm{eq}}(B)$-invariant, $\{\sigma(\bar{a}) : \sigma \in \operatorname{Aut}(\mathbb{M}/B)\}$ must be small, so $\bar{a} \in \operatorname{acl}^{\mathrm{eq}}(B)$.

$\Leftarrow$: by Extension (in Proposition 10.6), there is some $\sigma \in \operatorname{Aut}(\mathbb{M})$ such that $\sigma(\operatorname{acl}^{\mathrm{eq}}(B)) \mathop{\smile\!\!\!\!\!\!\,|}_B \operatorname{acl}^{\mathrm{eq}}(B)$. But $\sigma(\operatorname{acl}^{\mathrm{eq}}(B)) = \operatorname{acl}^{\mathrm{eq}}(B)$, and so $\operatorname{acl}^{\mathrm{eq}}(B) \mathop{\smile\!\!\!\!\!\!\,|}_B \operatorname{acl}^{\mathrm{eq}}(B)$. If $\bar{a} \in \operatorname{acl}^{\mathrm{eq}}(B)$, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_B \bar{a}$ by Monotonicity. □

Fact 11.4. *Let T be the theory of non-zero vector spaces over $\mathbb{R}$. If $A \subseteq \mathbb{M}$, let $\operatorname{span}(A)$ denote the $\mathbb{R}$-linear span of A, the set of things of the form $\sum_{i=1}^n x_i a_i$ with $n \geq 0$, $x_i \in \mathbb{R}$, and $a_i \in A$. Then*

1. *T has quantifier elimination.*

2. T is strongly minimal, and therefore stable.

3. $A \mathop{\smile\!\!\!\!\!\!|}_{\varnothing} B \iff \operatorname{span}(A) \cap \operatorname{span}(B) = \{0\}$.

4. $A \mathop{\smile\!\!\!\!\!\!|}_{C} B \iff \operatorname{span}(AC) \cap \operatorname{span}(BC) = \operatorname{span}(C)$.

12 A loose end

Proposition 12.1. *$\kappa_n(T)$ (Definition 7.1) doesn't depend on n.*

Proof. First we reformulate $\kappa_n(T)$.

Claim. $\kappa_n(T)$ is the smallest cardinal κ such that there is no tuple $\bar{a} \in \mathbb{M}^n$ and increasing sequence $(C_\alpha : \alpha < \kappa)$ with $\operatorname{tp}(\bar{a}/C_{\alpha+1}) \not\supseteq \operatorname{tp}(\bar{a}/C_\alpha)$ for each $\alpha < \kappa$.

Proof. Such a chain gives a descending sequence $(\operatorname{bd}(\operatorname{tp}(\bar{a}/C_\alpha)) : \alpha)$ in the fundamental order on n-types.

Conversely, given a descending sequence $(\beta_\alpha : \alpha < \kappa)$ in the fundamental order, we get a forking chain as in the proof of Proposition 8.2. $\square_{\text{Claim}}$

We show $\kappa_1(T) = \kappa_2(T)$; the proof that $\kappa_n(T) = \kappa_1(T)$ is similar using induction.

First we claim $\kappa_1(T) \le \kappa_2(T)$. Given a forking chain $(\operatorname{tp}(a/C_\alpha) : \alpha < \kappa)$, we have $a \not\mathop{\smile\!\!\!\!\!\!|}_{C_\alpha} C_{\alpha+1}$, so $(a,a) \not\mathop{\smile\!\!\!\!\!\!|}_{C_\alpha} C_{\alpha+1}$, and so $(\operatorname{tp}(a,a/C_\alpha) : \alpha < \kappa)$ is a forking chain of 2-types.

Conversely, let $(\operatorname{tp}(a,b/C_\alpha) : \alpha < \kappa)$ be a forking chain of 2-types. Then $ab \not\mathop{\smile\!\!\!\!\!\!|}_{C_\alpha} C_{\alpha+1}$ for each α. By left transitivity, for each α either

$$a \not\mathop{\smile\!\!\!\!\!\!|}_{C_\alpha} C_{\alpha+1} \qquad \text{or} \qquad b \not\mathop{\smile\!\!\!\!\!\!|}_{C_\alpha a} C_{\alpha+1} a.$$

That is, at least one of the following holds:

$$\operatorname{bd}(\operatorname{tp}(a/C_{\alpha+1})) < \operatorname{bd}(\operatorname{tp}(a/C_\alpha))$$
$$\operatorname{bd}(\operatorname{tp}(b/C_{\alpha+1}a)) < \operatorname{bd}(\operatorname{tp}(b/C_\alpha a)).$$

So one of the sequences $(\operatorname{bd}(\operatorname{tp}(a/C_\alpha)) : \alpha < \kappa)$ or $(\operatorname{bd}(\operatorname{tp}(b/C_\alpha a)) : \alpha < \kappa)$ decreases at least κ times, giving a descending chain of length κ in the fundamental order on 1-types. $\square$

13 Independent sequences

Definition 13.1. A family $(A_i : i \in I)$ is *independent* over B if $A_i \mathop{\smile\!\!\!\!\!\!|}_B A_{\neq i}$ for each $i \in I$, where $A_{\neq i} = \{A_j : j \neq i\}$.

Example 13.2. A_1, A_2 are independendt over B if $A_1 \mathop{\smile\!\!\!\!\!\!|}_B A_2$ and $A_2 \mathop{\smile\!\!\!\!\!\!|}_B A_1$. By symmetry, this just means $A_1 \mathop{\smile\!\!\!\!\!\!|}_B A_2$.

Fact 13.3. *In the theory of* $\mathbb{R}$*-vector spaces, if* $v_1, \dots, v_n$ *are non-zero vectors, then* $v_1, \dots, v_n$ *is independent over* $\varnothing$ *iff* $v_1, \dots, v_n$ *are linearly independent (in the sense of linear algebra), meaning that*

$$x_1 v_1 + \cdots + x_n v_n \neq 0.$$

for non-zero $\bar{x} \in \mathbb{R}^n$*.*

Proposition 13.4. $(A_i : i \in I)$ *is independent over* B *iff* $(A_i : i \in I_0)$ *is independent over* B *for every finite* $I_0 \subseteq I$*.*

Proof. Monotonicity and finite character. □

Lemma 13.5. *Suppose* $(A_i : i < \alpha)$ *is independent over* B *and* $A_\alpha \mathop{\smile\!\!\!\!\!\!\mid}_B A_{<\alpha}$ *where* $A_{<\alpha} = \{A_i : i < \alpha\}$*. Then* $(A_i : i \leq \alpha)$ *is independent over* B*.*

Proof. We must show $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B \{A_j : j \leq \alpha,\ j \neq i\}$ for each $i \leq \alpha$. When $i = \alpha$, this is assumed. Suppose $i < \alpha$. Let $C_i = \{A_j : j < \alpha,\ j \neq i\}$. We want to show $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B C_i A_\alpha$. We know $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B C_i$ because $(A_i : i < \alpha)$ is independent. We know $A_\alpha \mathop{\smile\!\!\!\!\!\!\mid}_B A_i C_i$ by assumption. By base monotonicity, $A_\alpha \mathop{\smile\!\!\!\!\!\!\mid}_{BC_i} A_i$. By symmetry, $A_i \mathop{\smile\!\!\!\!\!\!\mid}_{BC_i} A_\alpha$. By transitivity, the known facts $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B C_i$ and $A_i \mathop{\smile\!\!\!\!\!\!\mid}_{BC_i} A_\alpha$ imply $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B C_i A_\alpha$, as desired. □

Proposition 13.6. *If* $(A_i : i < \alpha)$ *is a sequence and* $A_i \mathop{\smile\!\!\!\!\!\!\mid}_B A_{<i}$ *for each* $i < \alpha$*, where* $A_{<i} = \{A_j : j < i\}$*, then* $(A_i : i < \alpha)$ *is independent over* B*.*

Proof. By induction on α using Lemma 13.5. □

Example 13.7. If $\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots$ is a Morley sequence over B, then $\bar{a}_1, \bar{a}_2, \dots$ is independent over B, because $\mathrm{tp}(\bar{a}_i / B\bar{a}_{<i})$ has a B-definable global extension, implying it doesn't fork over B, implying $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\mid}_B \bar{a}_{<i}$.

More generally, if $p_1, \dots, p_n$ are B-invariant types and $(\bar{a}_1, \dots, \bar{a}_n)$ realizes $(p_1 \otimes \cdots \otimes p_n) \restriction B$, then $\bar{a}_1, \dots, \bar{a}_n$ is independent over B.

14 Bases in strongly minimal theories

Suppose T is strongly minimal. If $A \subseteq \mathbb{M}$, then there is a unique type $p \in S_1(A)$ such that $b \models p \iff b \notin \mathrm{acl}(A)$. If such a type did not exist, there would be two $b, b' \notin \mathrm{acl}(A)$ with $\mathrm{tp}(b/A) \neq \mathrm{tp}(b'/A)$. Then there is an A-definable set $D \subseteq \mathbb{M}$ with $b \in D$, $b' \in \mathbb{M} \setminus D$. By strong minimality, D or $\mathbb{M} \setminus D$ is finite. Then b or b' is algebraic.

This type is called the *transcendental type* over A. If $p \in S_1(\mathbb{M})$ is the global transcendental type, then $p \restriction A$ is the transcendental type over A, because if $N \succeq \mathbb{M}$ and $b \in N \setminus \mathrm{acl}(\mathbb{M})$, then $b \notin \mathrm{acl}(A)$ and so $\mathrm{tp}(b/\mathbb{M}) \restriction A = \mathrm{tp}(b/A)$ is the transcendental type over A.

The transcendental type over A has a global $\varnothing$-definable extension, so it is stationary (Lemma 6.1). Say that $b \in \mathbb{M}$ is *transcendental* if $b \notin \mathrm{acl}(\varnothing)$.

Lemma 14.1. *If* b *is transcendental, then* $b \mathop{\smile\!\!\!\!\!\!\mid}_\varnothing C$ *iff* $b \notin \mathrm{acl}(C)$*.*

Proof. Let p be the global transcendental type. By Lemma 6.3, $b \mathop{\smile\!\!\!\!\!\!|}_{\varnothing} C$ holds iff $\operatorname{tp}(b/C) = p \upharpoonright C$, which just means that $b \notin \operatorname{acl}(C)$. $\square$

Proposition 14.2. *A sequence of transcendentals $(b_i : i < \alpha)$ is independent over $\varnothing$ iff $b_i \notin \operatorname{acl}(\{b_j : j < i\})$ for each i.*

Proof. Proposition 13.6. $\square$

Remark 14.3. Let p be the global transcendental type. A finite sequence of transcendentals $(b_1, \ldots, b_n)$ is independent iff $b_i \notin \operatorname{acl}(b_1, \ldots, b_{i-1})$ for each i, iff $b_i \models p \upharpoonright b_1 b_2 \cdots b_{i-1}$ iff $(b_1, \ldots, b_n) \models p^{\otimes n} \upharpoonright \varnothing$. Independent sequences of transcendentals are just Morley sequences of p.

Lemma 14.4. *Let I_1, I_2 be two independent sets. Let $f : I_1 \to I_2$ be a bijection. Then f is a partial elementary map.*

Proof. Suppose $b_1, \ldots, b_n \in I_1$ map to $c_1, \ldots, c_n \in I_2$. Then $\operatorname{tp}(b_1, \ldots, b_n) = p^{\otimes n} \upharpoonright \varnothing = \operatorname{tp}(c_1, \ldots, c_n)$. $\square$

Definition 14.5. Suppose $M \preceq \mathbb{M}$. A *basis* of M is a maximal independent set of transcendentals in M.

Every $M \preceq \mathbb{M}$ has a basis by Zorn's lemma and Proposition 13.4.

Proposition 14.6. *Let B be a basis of $M \preceq \mathbb{M}$. Then $M = \operatorname{acl}(B)$.*

Proof. Otherwise, take $c \in M \setminus \operatorname{acl}(B)$. Then $c \mathop{\smile\!\!\!\!\!\!|}_{\varnothing} B$ by Lemma 14.1, so Lemma 13.5 shows $B \cup \{c\}$ is independent, contradicting maximality. $\square$

Theorem 14.7. *The strongly minimal theory T is κ-categorical for all $\kappa > |L|$.*

Proof. Suppose $M_1, M_2 \preceq \mathbb{M}$ have $|M_1| = |M_2| = \kappa > |L|$. Take a basis $B_i \subseteq M_i$ for $i = 1, 2$. Then $|B_i| \leq |M_i| = \kappa$. If $|B_i| < \kappa$, then $|M_i| = |\operatorname{acl}(B_i)| \leq |B_i| + |L| < \kappa$, a contradiction. Therefore $|B_1| = |B_2| = \kappa$. Take a bijection $f : B_1 \to B_2$. By Lemma 14.4, f is a partial elementary map, and so it extends to an automorphism $\sigma \in \operatorname{Aut}(\mathbb{M})$. Then $\sigma(M_1) = \sigma(\operatorname{acl}(B_1)) = \operatorname{acl}(\sigma(B_1)) = \operatorname{acl}(B_2) = M_2$. Therefore $M_1 \cong M_2$. $\square$

Later, we will use *ranks* to see the following:

- If B, B' are bases of a model M, then $|B| = |B'|$. The *dimension* of M is defined to be $|B|$ for any basis M.
- M is determined up to isomorphism by its dimension.

(The argument above handles the case where the dimension is $> |L|$.)