Category Theory In Context

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November 23, 2021

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1 Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Definition 1.1. A **category** consists of

- a collection of **objects** $X, Y, Z, \ldots$
- a collection of **morphisms** $f, g, h, \ldots$

so that

- Each morphism has specified **domain** and **codomain** objects; the notation $f : X \to Y$ signifies that f is a morphism with domain X and codomain Y
- Each object has a designated **identity morphism** $1_X : X \to X$

- For any pair of morphisms f, g with the codomain of f equal to the domain of g, there exists a specified **composite morphism** gf whose domain is equal to the domain of f and whose codomain is equal to the codomain of g, i.e., :

$$f : X \to Y, \quad g : Y \to Z \qquad \rightsquigarrow \qquad gf : X \to Z$$

This data is subject to the following two axioms

- For any $f : X \to Y$, the composites $1_Y f$ and $f 1_X$ are both equal to f
- For any composable triple of morphisms f, g, h, the composites $h(gf)$ and $(hg)f$ are equal and hence denoted by hgf.

$$f : X \to Y, \quad g : Y \to Z, \quad h : Z \to W \qquad \rightsquigarrow \qquad hgf : X \to W$$

Example 1.1. 1. For any language $\mathcal{L}$ and any theory T of $\mathcal{L}$, there is a category $\mathbf{MODEL}_T$ whose objects are models of T. Morphisms is just homomorphisms

2. For a fixed unital but not necessarily commutative ring R, Mod_R is the category of left R-modules and R-modules homomorphisms. This category is denoted by $\mathbf{Vect}_{\Bbbk}$ when the ring happens to be a field $\Bbbk$ and abbreviated as **Ab** in the case of $\mathrm{Mod}_{\mathbb{Z}}$, as a $\mathbb{Z}$-module is precisely ab abelian group

Concrete categories are those whose objects have underlying sets and whose morphisms are functions between underlying sets

Abstract categories

Example 1.2. 1. A group defines a category $\mathrm{B}G$ with a single object

2. A category is **discrete** if every morphism is an identity

Definition 1.2. A category is **small** if it has only a set's worth of arrows

Both $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ are sets

Thus it has only a set's worth of objects

Definition 1.3. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms

The set of arrows between a pair of fixed objects in a locally small category is typically called a **hom-set**

Definition 1.4. An **isomorphism** in a category is a morphism $f : X \to Y$ for which there exists a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_X$, denoted by $X \cong Y$

An **endomorphism** is a morphism whose domain equals its codomain

Definition 1.5. A **groupoid** is a category in which every morphism is an isomorphism

Lemma 1.6. *Any category $\mathcal{C}$ contains a **maximal groupoid**, the subcategory containing all of the objects and only those morphisms that are isomorphisms*

Exercise 1.1.1.
1. Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows: x$ s.t. $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism
2. Show that a morphism can have at most one inverse isomorphism

Proof.
1. $g = 1_x g = (hf)g = h(fg) = h1_y = h$
2. From 1

□

Exercise 1.1.2. For any category $\mathcal{C}$ and any object $c \in \mathcal{C}$, show that

1. There is a category $c/\mathcal{C}$ whose objects are morphisms $f : c \to x$ with domain c in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes.

2. There is a category $\mathcal{C}/c$ whose objects are morphisms $f : x \to c$ with codomain c in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes

The category $c/\mathcal{C}$ and $\mathcal{C}/c$ are called **slice categories** of $\mathcal{C}$ **under** and **over** c, respectively

1.2 Duality

Definition 1.7. Let $\mathcal{C}$ be any category. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ has

- the same objects as in $\mathcal{C}$
- a morphism f^{op} in $\mathcal{C}^{\mathrm{op}}$ for each a morphism f in $\mathcal{C}$ so that the domain of f^{op} is defined to be the codomain of f and the codomain of f^{op} is defined to be the domain of f
- For each object X, the arrow 1_X^{op} serves as its identity in $\mathcal{C}^{\mathrm{op}}$
- A pair of morphisms $f^{\mathrm{op}}, g^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ is composable precisely when the pair g, f is composable in $\mathcal{C}$. We then define $g^{\mathrm{op}} \circ f^{\mathrm{op}}$ to be $(f \circ g)^{\mathrm{op}}$: i.e.
$$\mathrm{dom}(f^{\mathrm{op}}) = \mathrm{cod}(f) = \mathrm{dom}(g) = \mathrm{cod}(g^{\mathrm{op}})$$

Lemma 1.8. *T.F.A.E.*

1. *$f : x \to y$ is an isomorphism*
2. *For all objects $c \in \mathcal{C}$, post-composition with f defines a bijection*
$$f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$$
3. *For all objects $c \in \mathcal{C}$, pre-composition with f defines a bijection*
$$f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$$

Lemma 1.8 asserts that isomorphisms in a locally small category are defined *representably* in terms of isomorphisms in the category of sets.

Proof. $2 \to 1$. Let $c = y$, since f_* in an bijection, there must be an element $g \in \mathrm{Hom}(y, x)$ s.t. $f_*(g) = 1_y$. Hence $fg = 1_y$. Thus $gf, 1_x$ have common image under f_*, thus $gf = 1_x$. Whence f and g are inverse isomorphisms $\square$

Definition 1.9. A morphism $f : x \to y$ in a category is

1. a **monomorphism** if for any parallel morphisms $h, k : w \rightrightarrows x$, $fg = fk$ implies that $h = k$

2. an **epimorphism** if for any parallel morphisms $h, k : w \rightrightarrows x$, $hf = kf$ implies that $h = k$

Also, we can re-express it

1. $f : x \to y$ is a monomorphism in $\mathcal{C}$ iff for all objects $c \in \mathcal{C}$, $f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$ is injective

2. $f : x \to y$ is an epimorphism in $\mathcal{C}$ iff for all $c \in \mathcal{C}$, $f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$ is injective

Example 1.3. Suppose that $x \xrightarrow{s} y \xrightarrow{r} x$ are morphisms s.t. $rs = 1_x$. The map s is a **section** or **right inverse** to r, while the map r defines a **retraction** or **left inverse** to s. The maps s and r express the object x as a **retract** of the object y

In this case, s is always a monomorphism and, dually, r is always an epimorphism. To ackowledge the presence of these one-sided inverses, s is said to be a **split monomorphism** and r is said to be a **split epimorphism**

Example 1.4. By the previous example, an isomorphism is necessarily both monic and epic, but the converse need not hold in general. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic in the category **Rng**, but this map is not an isomorphism: there are no ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$

Lemma 1.10. *1. If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monomorphisms, then so is $gf : x \rightarrowtail z$*

2. If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is monic, then f is monic

Dually

1. If $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epimorphisms, then so is $gf : x \twoheadrightarrow z$

2. If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is epic, then g is epic

Exercise 1.2.1. 1. Show that a morphism $f : x \to y$ is a split epimorphism in a category $\mathcal{C}$ iff for all $c \in \mathcal{C}$, the post-composition function $f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$ is surjective

2. Show that a morphism $f : x \to y$ is a split monomorphism in a category $\mathcal{C}$ iff for all $c \in \mathcal{C}$, the post-composition function $f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$ is surjective

Exercise 1.2.2. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism

Proof. Suppose $y \xrightarrow{g} x \xrightarrow{f} y$ and $fg = 1_y$, then $fgf = f = f \circ 1_x$. Since f is mono, $gf = 1_x$ □

1.3 Functoriality

Definition 1.11. A **functor** $F : \mathcal{C} \to \mathcal{D}$, between categories $\mathcal{C}$ and $\mathcal{D}$, consists of the following data:

- An object $Fc \in \mathcal{D}$, for each objects $c \in \mathcal{C}$
- A morphism $Ff : Fc \to Fc' \in \mathcal{D}$, for each morphism $f : c \to c' \in \mathcal{C}$

Functoriality axioms

- For any composable pair $f, g \in \mathcal{C}$, $Fg \circ Ff = F(g \circ f)$
- For each object $c \in \mathcal{C}$, $F(1_c) = 1_{Fc}$

Definition 1.12. A **contravariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$

- A morphism $Ff : Fc' \to Fc \in \mathcal{D}$ for each morphism $f : c \to c' \in \mathcal{C}$
- For any composable pair $f, g \in \mathcal{C}$, $Ff \circ Fg = F(g \circ f)$

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$$

$$\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto"] & Fc \\
c' \arrow[r, mapsto] & Fc' \arrow[u, "Ff"']
\end{tikzcd}$$

Lemma 1.13. *Functors preserve isomorphisms*

Proof. Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and an isomorphism $f : x \to y$ in $\mathcal{C}$ with inverse $g : y \to x$. Then

$$F(g)F(f) = F(gf) = F(1_x) = 1_{Fx}$$

Thus $Fg : Fy \to Fx$ is a left inverse to $Ff : Fx \to Fy$ □

Definition 1.14. If $\mathcal{C}$ is locally small, then for any object $c \in \mathcal{C}$ we may define a pair of covariant and contravariant **functors represented by** c:

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{Hom}(c,-)"] & & \textbf{Sets} \\
x \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}(c,x) \arrow[d, "f_*"] \\
y & \longmapsto & \mathrm{Hom}(c,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, "\mathrm{Hom}(-,c)"] & & \textbf{Sets} \\
x \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}(x,c) \\
y & \longmapsto & \mathrm{Hom}(y,c) \arrow[u, "f^*"]
\end{tikzcd}
$$

Post-composition defines a **covariant** action on hom-sets

Definition 1.15. For any categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathcal{C} \times \mathcal{D}$, their **product**, whose

- objects are ordered pairs (c, d), where c is an object of $\mathcal{C}$ and d is an object of $\mathcal{D}$
- morphisms are ordered pairs $(f, g) : (c, d) \to (c', d')$, where $f : c \to c' \in \mathcal{C}$ and $g : d \to d' \in \mathcal{D}$ and
- in which composition and identities are defined componentwise

Definition 1.16. If $\mathcal{C}$ is locally small, then there is a **two-sided represented functor**

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \textbf{Sets}$$

A pair of objects (x, y) is mapped to the hom-set $\mathrm{Hom}(x, y)$. A pair of morphisms $f : w \to x$ and $h : y \to z$ is sent to the function

$$\mathrm{Hom}(x, y) \xrightarrow{(f^*, h_*)} \mathrm{Hom}(w, z)$$

$$g \longmapsto hgf$$

An **isomorphism of categories** is given by a pair of inverse functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ s.t. the composites Gf and FG, respectively, equal the identity functors on $\mathcal{C}$ and $\mathcal{D}$

1.4 Naturality

Definition 1.17. Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of

- an arrow $\alpha_c : Fc \to Gc$ in $\mathcal{D}$ for each object $c \in \mathcal{C}$, the collection of which define the **components** of the natural transformation s.t. for any morphism $f : c \to c'$ in $\mathcal{C}$, the following square of morphisms in $\mathcal{D}$

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

  commutes

A **natural isomorphism** is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_c is an isomorphism. In this case, the natural isomorphism may be depicted as $\alpha : F \cong G$

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

Example 1.5. Consider morphism $f : w \to x$ and $h : y \to z$ in a locally small category C. Post-composition by h and pre-composition by f define functions between hom-sets

$$\begin{tikzcd} C(x,y) \arrow[r, "h\circ-"] \arrow[d, "-\circ f"] & C(x,z) \arrow[d, "-\circ f"] \\ C(w,y) \arrow[r, "h\circ-"'] & C(w,z) \end{tikzcd}$$

$h \circ -$ is denoted by h_* and $- \circ f$ is denoted by f^*. By interpreting the horizontal arrows as the image of h under the actions of the functors $C(x, -)$ and $C(w, -)$, the square demonstrates that there is a natural transformation

$$f^* : C(x, -) \Rightarrow C(w, -)$$

Exercise 1.4.1. Given a pair of functors $F : \mathbf{A} \times \mathbf{B} \times \mathbf{B}^{\mathrm{op}} \to \mathbf{D}$ and $G : \mathbf{A} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, a family of morphisms

$$\alpha_{a,b,c} : F(a, b, b) \to G(a, c, c)$$

in **D** defines the components of an **extranatural transformation** $\alpha : F \Rightarrow G$ if for any $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$ the following diagram commutes

$$\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "F(f,1_b,1_b)"] & G(a,c,c) \arrow[d, "G(f,1_c,1_c)"] & F(a,b,b') \arrow[r, "F(1_a,1_b,g)"] \arrow[d, "F(1_a,g,1_{b'})"] & F(a,b,b) \arrow[d, "\alpha_{a,b,c}"] & F(a,b,b) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "F(f,1_b,1_b)"] & G(a,c,c) \arrow[d, "G(f,1_c,1_c)"] \\
F(a',b,b) \arrow[r, "\alpha_{a',b,c}"'] & G(a',c,c) & F(a,b',b') \arrow[r, "\alpha_{a,b',c}"'] & G(a,c,c) & F(a',b,b) \arrow[r, "\alpha_{a',b,c}"'] & G(a',c,c)
\end{tikzcd}$$

1.5 Equivalence of categories

Let $\mathbb{1}$ denote the discrete category with a single object and let $\mathbb{2}$ denote the category with two objects $0, 1 \in \mathbb{2}$ and a single non-identity arrow $0 \to 1$. There are two evident functors $i_0, i_1 : \mathbb{1} \rightrightarrows \mathbb{2}$ whose subscripts designate the objects in their image

Lemma 1.18. *Fixing a parallel pair of functors $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$, natural transformations $\alpha : F \Rightarrow G$ correspond bijectively to functors $H : \mathbf{C} \times \mathbb{2} \to \mathbf{D}$ s.t. H restricts along i_0 and i_1 to the functors F, G, i.e., so that*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathbf{C} \times \mathbb{2} \arrow[d, "H"] & \mathbf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\
& \mathbf{D} &
\end{tikzcd}$$

commutes

Hence i_0 denotes the functor defined on objects by $c \mapsto (c, 0)$

Definition 1.19. An **equivalence of categories** consists of functors $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ together with natural isomorphisms $\eta : 1_{\mathbf{C}} \cong GF$, $\epsilon : FG \cong 1_{\mathbf{D}}$. Categories **C** and **D** are **equivalent**, written $\mathbf{C} \simeq \mathbf{D}$, if there exists an equivalence between them

Lemma 1.20. *If $\mathbf{C} \simeq \mathbf{D}$ and $\mathbf{D} \simeq \mathbf{E}$, then $\mathbf{C} \simeq \mathbf{E}$*

Definition 1.21. A functor $F : \mathbf{C} \to \mathbf{D}$ is

- **full** if for each $x, y \in \mathbf{C}$, the map $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is surjective
- **faithful** if for each $x, y \in \mathbf{C}$, the map $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is injective

- **essentially surjective on objects** if for every object $d \in \mathbf{D}$ there is $c \in \mathbb{C}$ s.t. d is isomorphic to Fc

Lemma 1.22. *Any morphism $f : a \to b$ and fixed isomorphism $a \cong a'$ and $b \cong b'$ determine a unique morphism $f' : a' \to b'$ so that any of - or, equivalently, all of - the following four diagrams commutes*

$$\begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd} \quad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd}$$

Theorem 1.23 (characterizing equivalences of categories)**.** *A functor defining an equivalence of categories is full, faithful and essentially surjective on objects. Assuming the axiom of choice, any functor with these properties defines an equivalence of categories*

Proof. First suppose that $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$, $\eta : 1_{\mathbf{C}} \cong GF$ and $\epsilon : FG \cong 1_{\mathbf{D}}$ define an equivalence of categories. For any $d \in \mathbf{D}$, the component of the natural isomorphism $\epsilon_d : FGd \cong d$ demonstrates that F is essentially surjective. Consider a parallel pair $f, g : c \rightrightarrows c'$ in $\mathbf{C}$. If $Ff = Fg$, then both f and g define an arrow $c \to c'$ making the diagram

$$\begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, dashed, "f \text{ or } g"'] & GFc \arrow[d, dashed, "GFf=GFg"] \\ c' \arrow[r, "cong", "\eta_{c'}"'] & GFc' \end{tikzcd}$$

that expresses the naturality of η commute. Lemma implies that there is a unique arrow $c \to c'$ with this property, whence $f = g$. Thus F is faithful and by symmetry, so is G. Given $k : Fc \to Fc'$, by Lemma 1.22, Gk and the isomorphism η_c and $\eta_{c'}$ define a unique $h : c \to c'$ for which both Gk and GFh make the diagram

$$\begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, dashed, "h"'] & GFc \arrow[d, dashed, "Gk \text{ or } GFh"] \\ c' \arrow[r, "\cong", "\eta_{c'}"'] & GFc' \end{tikzcd}$$

commute. By Lemma 1.22, $GFh = Gk$

For the converse, suppose now that $F : \mathbf{C} \to \mathbf{D}$ is full, faithful and essentially surjective on objects. Using essential surjectivity and the axiom of choice, choose, for each $d \in \mathbf{D}$, an object $Gd \in \mathbf{C}$ and an isomorphism

$\epsilon_d : FGd \cong d$. For each $l : d \to d'$, Lemma 1.22 defines a unique morphism making the square

$$\begin{tikzcd} FGd \arrow[r, "\epsilon_d", "\cong"'] \arrow[d, dashed] & d \arrow[d, "l"] \\ FGd' \arrow[r, "\cong", "\epsilon_{d'}"'] & d' \end{tikzcd}$$

commute. Since F is fully faithful, there is a unqiue morphism $Gd \to G'$ with this image under F, which we define to be Gl. □

A category is **connected** if any pair of objects can be connected by a finite zig-zag of morphisms

Proposition 1.24. *Any connected groupoid is equivalent, as a category, to the automorphism group of any of its objects.*

Proof. Choose any object g of a connected groupoid $\mathbf{G}$ and let $G = \mathbf{G}(g, g)$ denote its automorphism group. The inclusion $\mathbf{B}G \hookrightarrow \mathbf{G}$ mapping the unique object of $\mathbf{B}G$ to $g \in \mathbf{G}$ is full and faithful, by definition, and essentially surjective, since $\mathbf{G}$ was assumed to be connected. Apply Theorem 1.23 □

Definition 1.25. A category **C** is **skeletal** if it contains just one object in each isomorphism class. The **skeleton** sk **C** of a category **C** is the unique skeletal category that is equivalent to **C**

1.6 The art of the diagram chase

Definition 1.26. A **monoid** is an object $M \in$ **Sets** together with a pair of morphisms $\mu : M \times M \to M$ and $\eta : 1 \to M$ so that the following diagrams commute:

$$\begin{tikzcd} M \times M \times M \arrow[r, "1_M \times \mu"] \arrow[d, "\mu \times 1_M"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta \times 1_M"] \arrow[dr, "1_M"'] & M \times M \arrow[d, "\mu"] & M \arrow[l, "1_M \times \eta"'] \arrow[dl, "1_M"] \\ & M & \end{tikzcd}$$

μ defines a binary "multiplication" operation on M. η identifies an element $\eta \in M$

Definition 1.27. A **diagram** in a category **C** is a functor $F : \mathbf{J} \to \mathbf{C}$ whose domain, the **indexing category**, is a small category

Lemma 1.28. *Functors preserve commutative diagrams*

Lemma 1.29. *Suppose $f_1, \dots, f_n$ is a composable sequence - a "path" - of morphisms in a category. If the composite $f_k f_{k-1} \dots f_{i+1} f_i$ equals $g_m \cdots g_1$, for another composable sequence of morphisms $g_1, \dots, g_m$, then $f_n \cdots f_1 = f_n \cdots f_{k+1} g_m \cdots g_1 f_{i-1} \cdots f_1$*

Lemma 1.30. *For any commutative square $\beta\alpha = \delta\gamma$ in which each of the morphisms is an isomorphism, then the inverses define a commutative square $\alpha^{-1}\beta^{-1} = \gamma^{-1}\delta^{-1}$*

Definition 1.31. An object $i \in \mathbf{C}$ is **initial** if for every $c \in \mathbf{C}$ there is a unique morphism $i \to c$. Dually, an object $t \in \mathbf{C}$ is **terminal** if for every $c \in \mathbf{C}$ there is a unique morphism $c \to t$

Lemma 1.32. *Let $f_1, \dots, f_n$ and $g_1, \dots, g_m$ be composable sequences of morphisms so that the domain of f_1 equals the domain of g_1 and the codomain of f_n equals the codomain of g_m. If this common codomain is a terminal object, or if this common domain is an initial object, then $f_n \cdots f_1 = g_m \cdots g_1$*

Definition 1.33. A **concrete category** is a category $\mathbf{C}$ equipped with a faithful functor $U : \mathbf{C} \to \mathbf{Sets}$

Lemma 1.34. *If $U : \mathbf{C} \to \mathbf{D}$ is faithful, then any diagram in $\mathbf{C}$ whose image commutes in $\mathbf{D}$ also commutes in $\mathbf{C}$*

Lemma 1.35. *Consider morphisms with the induced sources and targets*

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\
a' \arrow[r, "k"'] & b' \arrow[r, "m"'] & c'
\end{tikzcd}$$

and suppose that the outer rectangle commutes. This data defines a commutative rectangle if either

1. *the right-hand square commutes and m is a monomorphism*

2. *the left-hand square commutes and f is an epimorphism*

1.7 The 2-category of categories

For any fixed pair of categories $\mathbf{C}$ and $\mathbf{D}$, there is a **functor category** D^{C} whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations.

Lemma 1.36 (vertical composition)**.** *Suppose $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations between parallel functors $F, G, H : \mathbf{C} \to \mathbf{D}$. Then there is a natural transformation $\beta \cdot \alpha : F \Rightarrow H$ whose composites*

$$(\beta \cdot \alpha)_c := \beta_c \cdot \alpha_c$$

are defined to be the composites of the components of α and β

Proof.

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"] & Gc \arrow[r, "\beta_c"] \arrow[d, "Gf"] & Hc \arrow[d, "Hf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"] & Hc'
\end{tikzcd}$$

□

Corollary 1.37. *For any pair of categories **C** and **D**, the functors from **C** to **D** and natural transformations between them define a category $\mathbf{D}^{\mathbf{C}}$*

The composition operation defined in Lemma 1.36 is called **vertical composition**. Drawing the parallel functors horizontally, a composable pair of natural transformations in the category $\mathbf{D}^{\mathbf{C}}$ fits into a **pasting diagram**

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & \mathsf{D}
\end{tikzcd}$$

There is also a **horizontal composition** operation defined by the follow-

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "HF"] \arrow[r, bend right=60, "KG"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & \mathsf{E}
\end{tikzcd}$$

ing lemma

Lemma 1.38 (horizontal composition)**.** *Given a pair of natural transformations there is a natural transformation $\beta * \alpha : HF \Rightarrow KG$ whose component at $c \in C$ is defined as the composite of the following commutative square*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} \end{tikzcd}$$

$$\begin{tikzcd} HFc \arrow[r, "\beta_{Fc}"] \arrow[d, "H\alpha_c"'] \arrow[dr, dashed, "(\beta*\alpha)_c" description] & KFc \arrow[d, "K\alpha_c"] \\ HGc \arrow[r, "\beta_{Gc}"'] & KGc \end{tikzcd}$$

$$\begin{tikzcd} HFc \arrow[r, "H\alpha_c"] \arrow[d, "HFf"'] & HGc \arrow[r, "\beta_{Gc}"] \arrow[d, "HGf"'] & KGc \arrow[d, "KGf"] \\ HFc' \arrow[r, "H\alpha_{c'}"'] & HGc' \arrow[r, "\beta_{Gc'}"'] & KGc' \end{tikzcd}$$

Proof. □

Lemma 1.39 (middle four interchange)**.** *Given functors and natural transformations*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & \mathsf{D} \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2ex, "\Downarrow\delta"] & \mathsf{E} \end{tikzcd}$$

the natural transformation $JF \Rightarrow LH$ defined by first composing vertically and then composing horizontally equals the natural transformation defined by first composing horizontally and then composing vertically

Definition 1.40. A **2-category** is comprised of

- objects, e.g., the categories **C**
- 1-morphisms between pairs of objects, e.g., the functors $\mathbf{C} \xrightarrow{F} \mathbf{D}$

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow \beta \cdot \alpha"] & \mathsf{D} \arrow[r, bend left, "J"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow \delta \cdot \gamma"] & \mathsf{E} & = & \mathsf{C} \arrow[r, bend left=50, "JF"] \arrow[r, bend left=25, phantom, "\Downarrow \gamma * \alpha"] \arrow[r, "KG" description] \arrow[r, bend right=25, phantom, "\Downarrow \delta * \beta"] \arrow[r, bend right=50, "LH"'] & \mathsf{E}
\end{tikzcd}
$$

- 2-morphisms between parallel pairs of 1-morphisms, e.g., the natural transformations

so that

- the objects and 1-morphisms form a category, with identities $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$
- For each fixed pair of objects $\mathbf{C}$ and $\mathbf{D}$, the 1-morphisms $F : \mathbf{C} \to \mathbf{D}$ and 2-morphisms between such form a category under an operation called vertical composition
- There is also a category whose objects are the objects in which a morphism from $\mathbf{C}$ to $\mathbf{D}$ is a 2-cell

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "\alpha"'] \arrow[r, phantom, "\Downarrow G"] & \mathbf{D}
\end{tikzcd}
$$

2 TODO CHECK

1.38