# Seminar on Topological Dynamics of Definable Group Actions

Section 3 - Example 2

April 7, 2022

## 1 Review

- There is a $\mathbb{Q}$-flow $2^{\mathbb{Q}}$ acted by right shift $k * f(n) \equiv f(n-k)$.
- Functions which are periodic and piecewise constant are dense in $2^{\mathbb{Q}}$. Each of them is almost periodic. Hence every point in $2^{\mathbb{Q}}$ is weakly generic.
- There is a sufficiently generic function $\eta \in 2^{\mathbb{Q}}$, which has a dense orbit in $2^{\mathbb{Q}}$ and some other nice propertices.
- Fix such $\eta$, let $P_\sigma(x) \iff x * \eta \in [\sigma]$ be a unary predicate for any $\sigma \subseteq_{fin} \mathbb{Q}$, then $M = (\mathbb{Q}, +, P_\sigma)_\sigma$ has quantifier elimination. Let $T_\eta$ be the theory of $M$.

## 2 Weakly generic types

Every element $m \in M$ is definable as $y + P_\sigma(M) = P_{m*\sigma}(M)$. Hence $M$ is a prime model of $T_\eta$. Any model is a vector space over $\mathbb{Q}$.

We regard a point in $\Pi_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ as $\bar{f} = \{f_q : f_q \in 2^{\mathbb{Q}}, q \in \mathbb{Q}^*\}$, a sequence(family) of functions in $2^{\mathbb{Q}}$, the topology is still the product topology. The action $n * \bar{f}$ is given by $(qa * f_q)(k) = f_q(k - qa)$. There is a isomorphism between non-algebraic types in $S_1(M)$ and $\Pi_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$, which is similar to the case in Example 1 that $S_\Delta(M) \cong 2^{\mathbb{Z}}$.

**Lemma 1.** *For $\bar{f} = \{f_q : f_q \in 2^{\mathbb{Q}}, q \in \mathbb{Q}^*\}$, let $p_{\bar{f}} \equiv \{x \neq a\ a \in M\} \cup \{P_{f_q|X}(q \cdot x) : X \subseteq_{fin} \mathbb{Q}, q \in \mathbb{Q}^*\}$. Then $\bar{f} \mapsto p_{\bar{f}}$ is an isomorphism from between non-algebraic types in $S_1(M)$ and $\Pi_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$.*

Note every constant type is not weakly generic, and every point in $\Pi_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ is weakly generic for the same reason as $2^{\mathbb{Q}}$. We see that non-algebraic types in $S_1(M)$ are exactly weakly generic types in $S_1(M)$. So we can use the notation $WGen(M)$ for it.

Let $N$ be any model of $T_\eta$, we can talk about $\Pi_{q\in\mathbb{Q}^*} 2^N$ similarly. There is also $\bar{\eta}_N \in \Pi_{q\in\mathbb{Q}^*} 2^N$ such that the $N$-orbit is dense in $\Pi_{q\in\mathbb{Q}^*} 2^N$, and $\bar{\eta}_N$ is sufficiently generic.

**Remark 2.** Note $N$ is not constructed via $\bar{\eta}_N$, but $N$ inherit elementary property of genericity from $M$.

We are going to show there is also a isomorphism between $WGen(N)$ and $\Pi_{q\in\mathbb{Q}^*}2^N$. Assume $N = M \oplus V$ for some subspace $V$ of $N$ for the rest of this section. We may regard $f_q \in 2^N$ as the union of $v * f_{q,v}$ where $f_{q,v} \in 2^{\mathbb{M}}$.

**Lemma 3.** *For every $f \in 2^{\subset N}$ there are unique $f_v \in 2^{\subset M}$, $v \in V$, such that $f$ is the disjoint union of the functions $v * f_v$, $v \in V$.*

*Proof.* Let $f_v = (-v) * f|(v + M)$. □

**Lemma 4.** *(1) Every non-algebraic atomic formula over $N$ in variable $x$ is equivalent to $P_\sigma(q(x - v))$ for some $\sigma \in 2^{<\mathbb{Q}}$, $q \in \mathbb{Q}^*$ and $v \in V$.*
*(2) The negation of each formula as in (1) is equivalent in $N$ to a disjunction of formulas of this kind.*
*(3) If $(v_i, q_i)$, $i < n$, are some finitely many pairwise distinct elements of $V \times \mathbb{Q}^*$, and for each $i < n$ we have some finitely many ompatible functions $\sigma_{i,j} \in 2^{<\mathbb{Q}}$, $j < m_i$, then the formula*

$$\bigwedge_{i<n} \bigwedge_{j<m_i} P_{\sigma_{i,j}}(q_i(x - v_i)).$$

**Lemma 5.** *For $\bar{f} = \{f_q : f_q \in 2^N, q \in \mathbb{Q}^*\}$, let $p_{\bar{f}} \equiv \{x \neq n\ a \in N\} \cup \{P_\sigma(q(x - v)) : \sigma = f_{q,v}|X, X \subseteq_{fin} N, q \in \mathbb{Q}^*\}$. Then $\bar{f} \mapsto p_{\bar{f}}$ is an isomorphism from between non-algebraic types in $S_1(N)$ and $\Pi_{q\in\mathbb{Q}^*}2^N$.*

Also, non-algebraic types in $S_1(N)$ are exactly weakly generic types in $S_1(N)$, i.e. $WGen(N)$. By the isomorphism, $p_{\bar{\eta}_N}$ is not almost periodic. So also in $WGen(N)$, not all types are almost periodic. We will consider some specific examples of types in $WGen(N)$ that are not almost periodic later.

# 3 Self-replicating function

**Definition 6.** (1) Assume $N' = N \oplus W$ for some $\mathbb{Q}$-vector space $W$ and $\bar{f} = \langle f_q \rangle \in \Pi_{q\in\mathbb{Q}^*}2^N$. We define the $W$-prolongation $\bar{f}'$ of $\bar{f}$ as the sequence $\langle f'_q \rangle \in \Pi_{q\in\mathbb{Q}^*}2^{N'}$ given by $f'_q(n + w) = f_q(n)$ where $n \in N$ and $w \in W$.
(2) We say that $\bar{f} \in \Pi_{q\in\mathbb{Q}^*}2^{N'}$ has countable support if for some countable subspace $N'$ of $N$, $N = N' \oplus W$, $\bar{f}$ is a $W$-prolongation of some $\bar{f}' \in \Pi_{q\in\mathbb{Q}^*}2^{N'}$.
(3) We say that $p \in WGen(N)$ has countable support $p = p_{\bar{f}}$ for some $\bar{f}$ with countable support.

**Proposition 7.** *There are types $p \in WGen(N)$ that have countable support and are not almost periodic.*

*Proof.* Choose $\eta' = \bigcup_{v\in V} v * \eta$ where $\eta$ is the sufficiently generic function in $2^{\mathbb{Q}}$. Let $\bar{\eta'} = \langle \eta'_q \rangle$ where $\eta'_q = \eta'$, then choose $p = p_{\bar{\eta'}}$. □

It's more convinient to work in $\Pi_{q\in\mathbb{Q}^*}2^N$ than in $WGen(N)$. We introduce a technical property of $\bar{f}$ implying $\bar{f}$ is almost periodic. It's like a generalization of periodic and piecewise-constant function from $M$ to $N$.

**Definition 8.** (1) We say that $f \in 2^N$ is self-replicating if the following holds:
For every finite $X \subseteq N$ there is a finite $Y \subseteq N$ such that for any $n \in N$, there is some $m \in N$ with $m * (f|X) =\subseteq f|(n+Y)$.
(2) We say that $\bar{f} \in \Pi_{q\in\mathbb{Q}^*}2^N$ is self-replicating if the following holds:
For every finite sequence $q_i \in \mathbb{Q}^*$, $i \in I$, of distinct numbers, for every finite set $X \subseteq N$, there is a finite $Y \subseteq N$ such that for evrey $n \in N$ there is $m \in N$ such that for every $i \in I$, $q_i m * (f_{q_i}|X) \subseteq f_{q_i}|(q_i n + Y)$.

**Lemma 9.** *(1) Assume $f \in 2^N$ is self-replicating, then $f$ is alomost periodic.*
*(2) Assume $\bar{f} \in \Pi_{q\in\mathbb{Q}^*}2^N$ is self-replicating, then $\bar{f}$ is alomost periodic.*

*Proof.* (1) Suppose $g \in cl(N * f)$, we need to show $f \in cl(N * g)$. Assume $X$ is finite, choose a finite $Y$ provided by the self-replication of $f$. Since $g \in cl(N * f)$, there is some $n \in N$ such that $g|Y \subseteq n * f$. Choose an $m \in N$ with $m * (f|X) \subseteq f|(-n+Y)$, then $((-n) * g)|(-n+Y) = f|(-n+Y)$ contains $m * (f|X)$, so $f|X \subseteq (-(n+m)) * g$.
(2) Similar. □

**Lemma 10.** *Assume $N' = N \oplus W$ and $\bar{f} = \langle f_q \rangle \in \Pi_{q\in\mathbb{Q}^*}2^N$ is self-replicating. Then the $W$-prolongation of $\bar{f}$ is also self-replicating.*

*Proof.* Easy. □

**Lemma 11.** *(1) The set of self-replicating functions $f \in 2^N$ is dense in $2^N$.*
*(2) The set of self-replicating sequences $\bar{f} \in \Pi_{q\in\mathbb{Q}^*}2^N$ is dense in $\Pi_{q\in\mathbb{Q}^*}2^N$.*

*Proof.* The proof of (2) is similar to that of (1), so we only prove (1). When $dim_{\mathbb{Q}}(N)$ is finite, assume $Z \subseteq_{fin} N$ and $\sigma \in 2^Z$. We need to extend $\sigma$ into a self-replicating $f \in 2^N$. We may assume

$$Z = \left\{ \sum l_i b_i : l_i \in \mathbb{N}, l_i < k \right\}$$

where $\mathcal{B} = \{b_i : i \in I\}$ is a set of bases, $k \in \mathbb{N}$. Then we define $f$ as

$$f\left(\sum (l_i + t_i k + r_i) b_i\right) = \sigma\left(\sum l_i b_i\right)$$

where $t_i, l_i \in \mathbb{Z}$, $0 \le l_i < k$ and $r_i \in \mathbb{Q} \cap [0, 1)$.

To check self-replication, for an enlarged

$$X = \left\{ \sum \frac{l_i}{l} b_i : l_i \in \mathbb{N}, 0 \le \frac{l_i}{l} < tk \right\}$$

where $l, t \in \mathbb{N}$, we find

$$Y = \left\{ \sum \frac{l_i}{l} b_i : l_i \in \mathbb{N}, 0 \le \frac{l_i}{l} < (t+1)k \right\}$$

When $dim_{\mathbb{Q}}(N)$ is infinite, suppose we are going to deal with $Z \subseteq_{fin} N$ and $\sigma \in 2^Z$. Let $N = W_0 \oplus W_1$ where $W_0$ is finite and $Z \subseteq W_0$. There is self-replicating $f_0 \in 2^{W_0}$ extending $\sigma$. Let $f \in 2^N$ be the $W_1$-prolongation of $f_0$, then $f$ is self-replicating by lemma 10. □

**Proposition 12.** *Assume $N$ is countable and $N'$ is a $\mathbb{Q}$-vector space properly extending $N$.*
*(1) Let $f \in 2^N$, there is a self-replicating $f' \in 2^{N'}$ extending $f$.*
*(2) Let $\bar{f} \in \Pi_{q\in\mathbb{Q}^*} 2^N$, there is a self-replicating $\bar{f}' \in \Pi_{q\in\mathbb{Q}^*} 2^{N'}$ extending $\bar{f}$.*

*Proof.* By lemma 10, we may assume $N' = N \oplus W$ where $dim_{\mathbb{Q}}(W) = 1$, or just assume $W = \mathbb{Q}$. We present $N$ as a incresing chain of finite sets $(X_j)_{j<\omega}$. By the proof of lemma 11, for each $j$ we have a self-replicating $f_j \in 2^N$. We may assume the self-replication is uniform, meaning that for every $j' < \omega$ and finite $X \subseteq N$, there is a finite $Y \subseteq N$ such that for every $n \in N$, there is an $m \in N$ such that for every $j < j'$ we have that $m * (f_j|X) \subseteq f_j(n + Y)$.

We define $f' \in 2^N$ as (a) For $n' = n+r$, where $n \in N$ and $r \in \mathbb{Q}\cap[0,1)$, let $f'(n') = f(n)$. So $f'$ extends $f$. (b) For $n' = n + k + r$, with $n \in N$, $k \in \mathbb{Z}\backslash\{0\}$, $r \in \mathbb{Q} \cap [0,1)$, let $f'(n') = f_t(k)$, where $2^t$ is the highest power of 2 dividing $k$.

To check self-replication, for an enlarged $X \subseteq_{fin} N'$, let $X = X_t \times X^*$ where $X_t \subseteq N$ is the enlarged finite set in the proof of lemma 11 (we omit parameter $l$), and

$$X^* = \left\{ \frac{i}{s} : i \in \mathbb{N}, -2^t \leq \frac{i}{s} \leq 2^t \right\} \tag{1}$$

for some $s, t \in \mathbb{N}$.

Let $j' = t + 1$, $Y_0$ be a uniform choice on $(f_j)_{j<j'}$,

$$Y^* = \left\{ \frac{i}{s} : i \in \mathbb{N}, -2^{t+3} \leq \frac{i}{s} \leq 2^{t+3} \right\}. \tag{2}$$

Then we choose $Y = Y_0 \times Y$. □

**Proposition 13.** *Assume $p \in WGen(N)$ has countable support. Assume $N'$ is a properly elementary extension of $N$. Then there is an almost periodic type $p' \in WGen(N')$ extending $p$.*

*Proof.* We may assume $N = N_0 \oplus W$ such that $N_0$ is countable, $p|N_0 = p_{\bar{f}_0}$ for some $f_0 \in \Pi_{q\in\mathbb{Q}^*} 2^{N_0}$ and $p = p_{\bar{f}}$ for the $W$-prolongation $\bar{f} \in \Pi_{q\in\mathbb{Q}^*} 2^N$ of $\bar{f}_0$.

Suppose $N' = N \oplus W'$, let $N_1 = N_0 \oplus W$, by proposition 12, there is self-replicating $f_1 \in \Pi_{q\in\mathbb{Q}^*} 2^{N_1}$ extending $\bar{f}_0$. Let $f' \in \Pi_{q\in\mathbb{Q}^*} 2^{N'}$ be the $W$-prolongation of $f_1$, then $p' = p_{\bar{f}'}$ is what we want. □

**Remark 14.** We didn't require any satruation for models to get these results. Even when $N$ is $\omega$-saturated, many types in $WGen(N)$ are not almost periodic and have almost periodic extension in any properly elementary extension $N' \succ N$. This shows that the notion of almost periodic is not down-absolute even between $\omega$-saturated models. Maybe this suggests weakly generic types are the correct generalization of generic types.

In this example, it's still not clear whether we can drop the condition that $p$ is countable support in proposition 13. This example is a simple unstable structure, we don't know any o-minimal example.