Morley sequences and the order property

Advanced Model Theory

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Reference in the book: Sections 12.3 and 12.8 (VERY loosely).

1 Morley sequences

Fix a monster model $\mathbb{M}$ and a small set A. Let $p \in S_n(\mathbb{M})$ be an A-invariant type for some small A.

Definition 1. A *Morley sequence* of p over A is a sequence $\bar{b}_1, \bar{b}_2, \ldots$ where

$$\bar{b}_i \models p \restriction A\bar{b}_1 \cdots \bar{b}_{i-1}.$$

For example, if p is the transcendental 1-type in a strongly minimal theory, then a Morley sequence of p over A is a sequence $b_1, b_2, \ldots \in \mathbb{M}$ such that $b_1 \notin \operatorname{acl}(A)$, $b_2 \notin \operatorname{acl}(Ab_1)$, $b_3 \notin \operatorname{acl}(Ab_2)$, ...

Definition 2. Let $(I, \leq)$ be an infinite linear order (often $\mathbb{N}$). Let $(\bar{b}_i : i \in I)$ be a sequence in $\mathbb{M}$. Then $(\bar{b}_i : i \in I)$ is *A-indiscernible* if for any n, any $i_1 < \cdots < i_n$ in I, any $j_1 < \cdots < j_n$ in I, we have

$$\bar{b}_{i_1} \cdots \bar{b}_{i_n} \equiv_A \bar{b}_{j_1} \cdots \bar{b}_{j_n}$$

In other words, any two subsequences of the same length have the same type over A.

Example. Taking $n = 1$ in the definition, $\bar{b}_i \equiv_A \bar{b}_j$ for any $i, j \in I$. All elements in the sequence have the same type.

Example. In DLO, if $b_1 < b_2 < \cdots$, then $(b_i : i < \omega)$ is indiscernible (over $\varnothing$). This is true because:

> If $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$, then $\operatorname{tp}(\bar{a}) = \operatorname{tp}(\bar{b})$ by quantifier elimination or back-and-forth methods.

Definition 3. Let $(I, \leq)$ be an infinite set. Let $(\bar{b}_i : i \in I)$ be a sequence. Then $(\bar{b}_i : i \in I)$ is *totally indiscernible* if for any distinct $i_1, \ldots, i_n \in I$ and any distinct $j_1, \ldots, j_n \in I$,

$$\bar{b}_{i_1} \cdots \bar{b}_{i_n} \equiv_A \bar{b}_{j_1} \cdots \bar{b}_{j_n}$$

Example. If $(b_1, b_2, \ldots)$ is indiscernible, then $\mathrm{tp}(b_1 b_2) = \mathrm{tp}(b_1 b_3) = \mathrm{tp}(b_2 b_3) = \cdots$, but $\mathrm{tp}(b_2 b_1)$ could be different from $\mathrm{tp}(b_1 b_2)$. But if the sequence is *totally* indiscernible, then $\mathrm{tp}(b_1 b_2) = \mathrm{tp}(b_2 b_1)$.

Theorem 4. *If $(\bar{b}_i : i < \omega)$ is a Morley sequence of p over A, then $(\bar{b}_i : i < \omega)$ is A-indiscernible.*

Proof. If $i_1 < \cdots < i_n$, then $\bar{b}_{i_j} \models p \restriction A\bar{b}_{i_1} \cdots \bar{b}_{i_{j-1}}$ for each j, and so $\mathrm{tp}(\bar{b}_{i_1}, \ldots, \bar{b}_{i_n}/A) = (\underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}) \restriction A$. This doesn't depend on the choice of $i_1, \ldots, i_n$. $\square$

2 The order property

Fix some complete theory T and monster model $\mathbb{M}$.

Definition 5. Let $\varphi(\bar{x}, \bar{y})$ be a formula. Then $\varphi(\bar{x}, \bar{y})$ has the *order property* if there are $(\bar{a}_i : i \in \mathbb{Z})$ and $(\bar{b}_i : i \in \mathbb{Z})$ such that

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

Example. In DLO, the formula $\varphi(x, y) = (x < y)$ has the order property: take $a_i = b_i = i$. Then $a_i < b_j \iff i < j$.

Remark 6. If $\varphi(\bar{x}; \bar{y})$ has the OP, witnessed by $\bar{a}_i$ and $\bar{b}_j$, and if $\varphi^T(\bar{y}; \bar{x}) = \varphi(\bar{x}; \bar{y})$, then

$$\mathbb{M} \models \neg\varphi(\bar{a}_{-i}, \bar{b}_{1-j}) \iff -i \geq 1 - j \iff i \leq j - 1 \iff i < j$$
$$\mathbb{M} \models \varphi^T(\bar{b}_{-i}, \bar{a}_{-j}) \iff \mathbb{M} \models \varphi(\bar{a}_{-j}, \bar{b}_{-i}) \iff -j < -i \iff i < j.$$

Therefore $\neg\varphi$ and φ^T have the OP.

3 Instability from the order property

Lemma 7. *For any cardinal $\lambda \geq \aleph_0$, there is a linear order $(I, <)$ and a subset $S \subseteq I$ such that $|S| \leq \lambda$, $|I| > \lambda$, and S is dense in I: if $a < b$ in I then there is $x \in S$ with $a \leq x \leq b$.*

Proof. From Lemma 9 in the March 3 notes, there is a cardinal μ such that $|2^\mu| > \lambda$ but $|2^{<\mu}| \leq \lambda$, where 2^μ is the set of binary strings of length μ and $2^{<\mu}$ is the set of binary strings of length strictly less than μ. Let $I = 2^\mu \cup 2^{<\mu}$ and let $S = 2^{<\mu}$. Order I lexicographically, by padding strings in $2^{<\mu}$ on the right with a symbol u such that $0 < u < 1$. For example, $010 \in 2^{<\mu}$ becomes $010uuu\ldots \in \{0, u, 1\}^\mu$, so it is ordered after $0100\ldots$ and before $0101\ldots$. If $a, b \in 2^\mu$ and $a < b$, then a starts with $\tau 0$ and b starts with $\tau 1$ for some $\tau \in 2^\mu$. Then $a < \tau < b$, because $\tau 0 \ldots < \tau u \ldots < \tau 1 \ldots$ $\square$

Theorem 8. *If some formula $\varphi(\bar{x}; \bar{y})$ has the order property, then T is unstable: it is not λ-stable for any λ.*

Proof. We show λ-stability fails. Take $I \supseteq S$ as in Lemma 7, with $|I| > \lambda$ and $|S| \leq \lambda$ and S dense in I. By compactness, there are $(\bar{a}_i : i \in I)$ and $(\bar{b}_i : i \in I)$ such that $\mathbb{M} \models \varphi(\bar{a}_i; \bar{b}_j) \iff i < j$.

Let $C = \{\bar{b}_j : j \in S\}$. We claim that this map is an injection:

$$\begin{aligned} I \setminus S &\to S_n(C) \\ i &\mapsto \operatorname{tp}(\bar{a}_i/C), \end{aligned}$$

in which case $|C| \leq \lambda$ but $|S_n(C)| \geq |I \setminus S| > \lambda$, and λ-stability fails.

Suppose $i_1, i_2 \in I \setminus S$ and $i_1 \neq i_2$. Without loss of generality, $i_1 < i_2$. Then there is $j \in S$ such that $i_1 < j < i_2$. Then

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \bar{b}_j) \text{ but } \mathbb{M} \models \neg\varphi(\bar{a}_{i_2}, \bar{b}_j)$$

and $\bar{b}_j \in C$, and so $\operatorname{tp}(\bar{a}_{i_1}/C) \neq \operatorname{tp}(\bar{a}_{i_2}/C)$. □

4 The order property from instability

Lemma 9. *If $\varphi(\bar{x}; \bar{y})$ does not have the order property, then there is n_φ such that there do not exist $(\bar{a}_i : i < n_\varphi)$ and $(\bar{b}_i : i < n_\varphi)$ such that*

$$\mathbb{M} \models \varphi(\bar{a}_i; \bar{b}_j) \iff i < j.$$

Proof. Compactness. (Add new constant symbols $\bar{a}_i$ and $\bar{b}_i$ for $i \in \mathbb{Z}$. If n_φ didn't exist, then $\{\varphi(\bar{a}_i; \bar{b}_j) : i < j \in \mathbb{Z}\} \cup \{\neg\varphi(\bar{a}_i; \bar{b}_j) : i \geq j \in \mathbb{Z}\}$ is consistent, hence realized in $\mathbb{M}$.) □

Lemma 10. *Suppose $\varphi(\bar{x}; \bar{y})$ doesn't have the order property. Let n_φ be as in Lemma 9. Let $\bar{b}_1, \bar{b}_2, \ldots$ be an indiscernible sequence. Then there is no $\bar{a} \in \mathbb{M}$ such that*

$$\begin{aligned} &\mathbb{M} \models \varphi(\bar{a}, \bar{b}_i) \textit{ for } 0 \leq i < n_\varphi \\ &\mathbb{M} \models \neg\varphi(\bar{a}, \bar{b}_i) \textit{ for } n_\varphi \leq i < 2n_\varphi. \end{aligned}$$

Proof. Let $n = n_\varphi$. Suppose such a $\bar{a}$ exists. For $0 \leq j < n$, there is $\sigma_j \in \operatorname{Aut}(\mathbb{M})$ such that

$$\sigma_j(\bar{b}_{n-j}, \ldots, \bar{b}_{(n-j)+(n-1)}) = (\bar{b}_0, \ldots, \bar{b}_{n-1})$$

by indiscernibility. Let $\bar{a}_j = \sigma_j(\bar{a})$. For $j, i < n$,

$$\begin{aligned} \mathbb{M} \models \varphi(\bar{a}_j; \bar{b}_i) &\iff \mathbb{M} \models \varphi(\sigma_j(\bar{a}), \sigma_j(\bar{b}_{i+n-j})) \\ \iff \mathbb{M} \models \varphi(\bar{a}, \bar{b}_{i+n-j}) &\iff i + n - j < n \iff i < j. \end{aligned}$$

This contradicts the choice of $n = n_\varphi$. □

Lemma 11. *Suppose $\varphi(x_1, \ldots, x_n; \bar{y})$ does not have the order property. Suppose $N > \max(n_\varphi, n_{\neg\varphi})$. Let p be an A-invariant global type. Let $(\bar{a}_i : i < \omega)$ be a Morley sequence of p over A. Suppose $\bar{b} \in \mathbb{M}$.*

1. *If $\varphi(\bar{x};\bar{b}) \in p(\bar{x})$, then $\mathbb{M} \models \varphi(\bar{a}_i;\bar{b})$ for a majority of $i < 2N$.*

2. *If $\neg\varphi(\bar{x};\bar{b}) \in p(\bar{x})$, then $\mathbb{M} \models \neg\varphi(\bar{a}_i;\bar{b})$ for a majority of $i < 2N$.*

Proof. We prove (2); (1) is similar. Suppose $\neg\varphi(\bar{x};\bar{b}) \in p(\bar{x})$, but $\mathbb{M} \models \varphi(\bar{a}_i;\bar{b})$ for 50% of $i < 2N$. Then there are $j_1 < \cdots < j_N < 2N$ such that $\mathbb{M} \models \varphi(\bar{a}_{j_i};\bar{b})$ for $1 \leq i \leq N$. Let $(\bar{a}'_i : i < \omega)$ be a Morley sequence of p over $A \cup \{\bar{a}_i : i < \omega\} \cup \{\bar{b}\}$. Then $\bar{a}'_i$ realizes the type $p \restriction A\bar{b}$ which contains the formula $\neg\varphi(\bar{x};\bar{b})$, and so $\mathbb{M} \models \neg\varphi(\bar{a}'_i;\bar{b})$ for all i. Finally,

$$\bar{a}_{j_1}, \ldots, \bar{a}_{j_N}, \bar{a}'_0, \bar{a}'_1, \bar{a}'_2, \ldots$$

is a Morley sequence of p over A, hence indiscernible. But

$$\mathbb{M} \models \varphi(\bar{a}_{j_i}, \bar{b}) \text{ for } 1 \leq i \leq N$$
$$\mathbb{M} \models \neg\varphi(\bar{a}'_i, \bar{b}) \text{ for } 0 \leq i < N,$$

so this contradicts Lemma 10. □

Proposition 12. *Suppose $\varphi(x_1,\ldots,x_n;\bar{y})$ doesn't have the order property. If M is a small model and $p \in S_n(M)$, then the relation $d_p\varphi(\bar{y})$ is definable.*

Proof. Take $q \in S_n(M)$ a global coheir of p (March 10, Theorem 5). Then q is M-invariant (March 10, Theorem 17). Let $(\bar{a}_i : i < \omega)$ be a Morley sequence of q over M. By Lemma 11, $d_q\varphi(\bar{y})$ is definable from the Morley sequence by majority voting:

$$\varphi(\bar{x};\bar{b}) \in q(\bar{x}) \iff \mathbb{M} \models \bigvee_S \bigwedge_{i \in S} \varphi(\bar{a}_i;\bar{b}).$$

where S ranges over $\{S \subseteq 2N : |S| > N\}$. Now $d_q\varphi$ is definable and M-invariant, hence M-definable (March 10, Lemma 10). Then $d_p\varphi$ is (M-)definable: it's the restriction of $d_q\varphi$ to M. □

Theorem 13. *Fix n. Suppose no formula $\varphi(x_1,\ldots,x_n;\bar{y})$ has the OP. Then for any $M \models T$ and $p \in S_n(M)$, p is definable.*

Corollary 14. *The following are equivalent:*

1. *All types over models are definable.*

2. *All 1-types over models are definable.*

3. *No formula $\varphi(\bar{x};\bar{y})$ has the OP.*

4. *No formula $\varphi(x;\bar{y})$ has the OP.*

5. *T is λ-stable for at least one λ.*

Proof. Similar to Theorem 2 on March 10, using today's Theorem 8 and Proposition 12. □

Fact 15. *$\varphi(\bar{x};\bar{y})$ has the order property iff it has the dichotomy property.*

5 Commuting types

Theorem 16. *Assume T is stable. Let p, q be global A-invariant types. Then $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$.*

Proof. Suppose an $L(\mathbb{M})$-formula $\varphi(\bar{x}, \bar{y})$ is in $(p \otimes q)(\bar{x}, \bar{y})$ but not $(q \otimes p)(\bar{y}, \bar{x})$. Take a small set $B \supseteq A$ over which φ is defined. Then p, q are B-invariant. Replacing A with B, we may assume $\varphi(\bar{x}; \bar{y})$ is an $L(A)$-formula.

Let $(\bar{a}_1, \bar{b}_1; \bar{a}_2, \bar{b}_2; \bar{a}_3, \bar{b}_3; \ldots)$ be a Morley sequence of $p \otimes q$ over A. In other words

$$\begin{aligned} \bar{a}_1 &\models p \restriction A, & \bar{b}_1 &\models q \restriction A\bar{a}_1 \\ \bar{a}_2 &\models p \restriction A\bar{a}_1\bar{b}_1, & \bar{b}_2 &\models q \restriction A\bar{a}_1\bar{b}_1\bar{a}_2 \\ \bar{a}_3 &\models p \restriction A\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2, & \bar{b}_3 &\models q \restriction A\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2\bar{a}_3, \\ & & \cdots \end{aligned}$$

If $i \leq j$, then $(\bar{a}_i, \bar{b}_j) \models (p \otimes q) \restriction A$, and so $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j)$. On the other hand if $j < i$, then $(\bar{b}_j, \bar{a}_i) \models (q \otimes p) \restriction A$, and so $\mathbb{M} \models \neg\varphi(\bar{a}_i, \bar{b}_j)$. Therefore

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j.$$

It follows that φ has the order property, a contradiction. $\square$

Example. Suppose T is strongly minimal, and $p, q \in S_1(\mathbb{M})$ are both the transcendental 1-type. By Theorem 16, $(p \otimes q)(x, y) = (q \otimes p)(y, x)$. Concretely, this means the following are equivalent for $a, b \in \mathbb{M}$ and $C \subseteq \mathbb{M}$:

$$\begin{aligned} & a \notin \operatorname{acl}(C) \text{ and } b \notin \operatorname{acl}(Ca) \\ \iff & b \notin \operatorname{acl}(C) \text{ and } a \notin \operatorname{acl}(Cb). \end{aligned}$$

This implies that $\operatorname{acl}(-)$ satisfies the "Steinitz exchange property":

$$a \in \operatorname{acl}(Cb) \setminus \operatorname{acl}(C) \implies b \in \operatorname{acl}(Ca).$$