ω-saturated models and quantifier elimination

Introductory Model Theory

November 11, 2021

Recommended reading: Poizat's *Course in Model Theory*, Chapter 5.

Definition 1. A formula φ is *quantifier-free* if it has no quantifiers.

Definition 2. A theory T has *quantifier elimination* if for every formula $\varphi(\bar{x})$, there is a quantifier-free formula $\psi(\bar{x})$ such that $T \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Definition 3. If $\bar{a} \in M^n$, then $\mathrm{qftp}^M(\bar{a})$ is the set of quantifier-free L-formulas $\varphi(\bar{x})$ such that $M \models \varphi(\bar{b})$.

Theorem 4. *Let T be a theory. Suppose that*

$$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b}) \implies \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b}) \tag{$*$}$$

whenever $M, N \models T$ and $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then T has quantifier elimination.

Proof. Suppose quantifier elimination fails for $\varphi(\bar{x})$. Let Q be the set of quantifier-free formulas $\psi(\bar{x})$ such that $T \vdash \psi \to \varphi$. If $\psi_1, \ldots, \psi_m \in Q$, then $T \vdash \bigvee_{i=1}^m \psi_i \to \varphi$, so $T \not\vdash \varphi \to \bigvee_{i=1}^m \psi_i$. (Otherwise, φ is equivalent to the quantifier-free formula $\bigvee_{i=1}^m \psi_i$.) Therefore there is $M \models T$ and $\bar{a} \in M^n$ with $M \models \varphi(\bar{a}) \wedge \bigwedge_{i=1}^m \neg\psi_i(\bar{a})$.

By compactness, there is $M \models T$ and $\bar{a} \in M^n$ such that $M \models \varphi(\bar{a})$, but $M \models \neg\psi(\bar{a})$ for all $\psi \in Q$.

Now suppose $\theta_1, \ldots, \theta_m \in \mathrm{qftp}(\bar{a})$. Then $\psi := \bigwedge_{i=1}^m \theta_i \in \mathrm{qftp}(\bar{a})$, so $\psi \notin Q$. Therefore $T \not\vdash \psi \to \varphi$. So there is $N \models T$ and $\bar{b} \in N^n$ such that $N \models \psi(\bar{b}) \wedge \neg\varphi(\bar{b})$. Equivalently, $N \models \bigwedge_{i=1}^m \theta_i(\bar{b})$ but $N \models \neg\varphi(\bar{b})$.

By compactness there is $N \models T$ and $\bar{b} \in N^n$ such that $N \models \neg\varphi(\bar{b})$, and $N \models \theta(\bar{b})$ for all $\theta \in \mathrm{qftp}(\bar{a})$. Then $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. On the other hand, $M \models \varphi(\bar{a})$ and $N \models \neg\varphi(\bar{b})$, so $\mathrm{tp}(\bar{a}) \neq \mathrm{tp}(\bar{b})$, a contradiction. $\square$

Conversely, quantifier elimination implies $(*)$. Suppose $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. For any formula $\varphi(\bar{x})$ there is an equivalent quantifier-free formula $\psi(\bar{x})$, and then

$$M \models \varphi(\bar{a}) \iff M \models \psi(\bar{a}) \iff N \models \psi(\bar{b}) \iff N \models \varphi(\bar{b}).$$

Definition 5. Let M be a structure and A be a subset. The substructure *generated by A* is

$$\langle A \rangle = \{t(\bar{a}) : t(x_1, \dots, x_n) \text{ is an } L\text{-term, } \bar{a} \in A^n\}.$$

The structure $\langle A \rangle$ is the smallest substructure of M containing A.

Theorem 6. *Let M, N be L-structures. If $\bar{a} \in M^n$ and $\bar{b} \in N^n$, then the following are equivalent:*

1. *There is an isomorphism $f : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ such that $f(\bar{a}) = \bar{b}$.*
2. $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$.

Here, $f(\bar{a}) = (f(a_1), \dots, f(a_n))$.

Proof sketch. (1) $\implies$ (2): If $t(\bar{x})$ is an L-term, then $f(t^M(\bar{a})) = t^N(\bar{b})$ by induction on $t(\bar{x})$. If $\varphi(\bar{x})$ is a quantifier-free formula, then $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b})$ by induction on $\varphi(\bar{x})$.

(2) $\implies$ (1): Define $f : \langle \bar{a} \rangle \to \langle \bar{b} \rangle$ by sending $t(\bar{a})$ to $t(\bar{b})$.

- This is well-defined: if $M \models t(\bar{a}) = t'(\bar{a})$, then $N \models t(\bar{b}) = t'(\bar{b})$ because $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$.
- This is a bijection: if we define $g : \langle \bar{b} \rangle \to \langle \bar{a} \rangle$ similarly, then $g = f^{-1}$.
- This is an isomorphism: for example, if R is a 2-ary relation symbol, then $M \models R(t(\bar{a}), t'(\bar{a})) \iff N \models R(t(\bar{b}), t'(\bar{b}))$ because $R(t(\bar{x}), t'(\bar{x}))$ is quantifier-free. $\square$

Theorem 7. *The following are equivalent for a theory T:*

1. *Let M, N be ω-saturated models. Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. For any $\alpha \in M$ there is $\beta \in N$ such that $\mathrm{qftp}(\bar{a}, \alpha) = \mathrm{qftp}(\bar{b}, \beta)$.*
2. *T has quantifier elimination.*

Proof. (1) $\implies$ (2): Assume (1).

Claim. If M, N are ω-saturated and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$ and $\varphi(\bar{x})$ is a formula, then $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b})$.

Proof. By induction on $\varphi(\bar{x})$.

- $\varphi(\bar{x})$ is atomic: then it's quantifier-free, and it follows because $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$.
- $\varphi(\bar{x})$ is $\neg\psi(\bar{x})$. Easy.
- $\varphi(\bar{x})$ is $\psi \wedge \theta$. Easy.
- $\varphi(\bar{x})$ is $\exists y\ \psi(\bar{x}, y)$. If $M \models \varphi(\bar{a})$ then there is α such that $M \models \psi(\bar{a}, \alpha)$. Take $\beta \in N$ such that $\mathrm{qftp}(\bar{a}, \alpha) = \mathrm{qftp}(\bar{b}, \beta)$. By induction, $N \models \psi(\bar{b}, \beta)$. Thus $N \models \varphi(\bar{b})$. This shows $M \models \varphi(\bar{a}) \implies N \models \varphi(\bar{b})$, and the converse follows by swapping M and N. $\square_{\text{Claim}}$

Now we prove (2). Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. We must show that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Replacing M and N with elementary extensions (which doesn't change the types), we may assume M and N are ω-saturated. Then the claim shows $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, proving (2).

(2) $\implies$ (1): Assume (2). In the set up of (1), $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ by quantifier elimination. Let $p(\bar{x}, y) = \mathrm{tp}^M(\bar{a}, \alpha)$. For any $\varphi(\bar{x}, y) \in p(\bar{x}, y)$, we have $M \models \exists y\, \varphi(\bar{a}, y)$ and therefore $N \models \exists y\, \varphi(\bar{b}, y)$. The partial type $p(\bar{b}, y)$ is therefore finitely satisfiable in N. By ω-saturation, it is satisfied by some $\beta \in N$. Then $\mathrm{tp}(\bar{b}, \beta) = p = \mathrm{tp}(\bar{a}, \alpha)$, so $\mathrm{qftp}(\bar{b}, \beta) = \mathrm{qftp}(\bar{a}, \alpha)$. $\square$

Example. Let T be the theory of discrete linear orders without endpoints. For each $n < \omega$, add a binary relation $R_n(x, y)$ saying that $d(x, y) = n$. We will prove quantifier elimination. Suppose M, N are ω-saturated. Let $A \subseteq M$ and $B \subseteq N$ be finitely generated (= finite) substructures and $f : A \to B$ be an isomorphism. Given $\alpha \in M$, we must find $\beta \in N$ and an isomorphism $g : A \cup \{\alpha\} \to B \cup \{\beta\}$ extending f. The local isomorphism f looks like

$$f : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$$
$$f(a_i) = b_i$$

where $a_1 < a_2 < \cdots < a_n$ in M and $b_1 < \cdots < b_n$ in N, and $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$. Let $x = d(a_i, \alpha)$ and $y = d(\alpha, a_{i+1})$. We need β with $b_i < \beta < b_{i+1}$ such that $d(b_i, \beta) = x$ and $d(\beta, b_{i+1}) = y$. There are four cases:

- $x, y < \infty$. Take $\beta \in [b_i, b_{i+1}]$ such that $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = d(b_{i+1}, b_i) - x = d(a_{i+1}, a_i) - x = (x + y) - x = y$.
- $x < \infty = y$. Then $d(b_i, b_{i+1}) = d(a_i, a_{i+1}) = x + y = \infty$. Take $\beta \in [b_i, b_{i+1}]$ such that $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = \infty = y$.
- $x = \infty > y$. Similar.
- $x = y = \infty$. Let $\Sigma(x)$ be the partial type over b_i, b_{i+1} saying

  $$\{b_i < x < b_{i+1}\} \cup \{d(b_i, x) \neq n : n < \omega\} \cup \{d(x, b_{i+1}) \neq n : n < \omega\}.$$

  Then $\Sigma(x)$ is finitely satisfiable, so it is satisfied by some β by ω-saturation. Then $d(b_i, \beta) = \infty = x$ and $d(\beta, b_{i+1}) = \infty = y$.

Therefore T has quantifier-elimination, after adding the symbols R_n.

Theorem 8. *Let T be a theory with quantifier elimination. Let M, N be models of T. Then $M \equiv N$ iff $\langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$.*

Proof. $M \equiv N \iff \mathrm{tp}^M() = \mathrm{tp}^N() \iff \mathrm{qftp}^M() = \mathrm{qftp}^N() \iff \langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$. $\square$

For example, any two discrete linear orders without endpoints are elementarily equivalent. The theory of discrete linear orders without endpoints is complete.

1 Addendum

Lemma 9. *Let M and N be L-structures, with N being ω-saturated. Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$. For any $\alpha \in M$ there is $\beta \in N$ such that $\mathrm{tp}^M(\bar{a}, \alpha) = \mathrm{tp}^N(\bar{b}, \beta)$.*

This was implicit in the (2) $\implies$ (1) direction of Theorem 7, but here is the proof again.

Proof. Let $p(\bar{x}, y) = \mathrm{tp}^M(\bar{a}, \alpha)$. For any $\varphi(\bar{x}, y) \in p(\bar{x}, y)$, we have $M \models \exists y\ \varphi(\bar{a}, y)$ and therefore $N \models \exists y\ \varphi(\bar{b}, y)$. Therefore, the set of formulas $p(\bar{b}, y)$ is finitely satisfiable in N.[1] By ω-saturation, $p(\bar{b}, y)$ is satisfied by some $\beta \in N$. Then $(\bar{b}, \beta)$ satisfies $p(\bar{x}, y) = \mathrm{tp}(\bar{a}, \alpha)$, so $\mathrm{tp}(\bar{b}, \beta) = \mathrm{tp}(\bar{a}, \alpha)$. $\square$

Lemma 10. *Let M and N be L-structures, with N being ω-saturated. Suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$. For any $\bar{\alpha} \in M^m$ there is $\bar{\beta} \in N^m$ such that $\mathrm{tp}^M(\bar{a}, \bar{\alpha}) = \mathrm{tp}^N(\bar{b}, \bar{\beta})$.*

Proof. By induction on m. When $m = 0$, we must choose an empty tuple such that $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$, which is true by assumption. Suppose $m > 0$. By induction there are $\beta_1, \ldots, \beta_{m-1} \in N$ such that $\mathrm{tp}^M(\bar{a}, \alpha_1, \ldots, \alpha_{m-1}) = \mathrm{tp}^N(\bar{b}, \beta_1, \ldots, \beta_{m-1})$. By Lemma 9[2], there is $\beta_m \in N$ such that $\mathrm{tp}^M(\bar{a}, \alpha_1, \ldots, \alpha_m) = \mathrm{tp}^N(\bar{b}, \beta_1, \ldots, \beta_m)$. $\square$

Theorem 11. *Let M be an ω-saturated L-structure. Let A be a finite subset of M. If $p \in S_n(A)$, then p is realized in M.*

This is like the definition of "ω-saturated," except now n can be greater than 1.

Proof. Let $A = \{a_1, \ldots, a_m\}$. Take an elementary extension $N \succeq M$ where p is realized by a tuple $\bar{\alpha} \in N^n$. Note that $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{a})$, because $M \preceq N$. By Lemma 10, there is a tuple $\bar{\beta} \in M^n$ such that $\mathrm{tp}^M(\bar{a}, \bar{\beta}) = \mathrm{tp}^N(\bar{a}, \bar{\alpha})$.

If $\varphi(\bar{x})$ is an $L(A)$-formula, then $\varphi(\bar{x})$ is $\psi(\bar{x}, \bar{a})$ for some L-formula $\psi(x_1, \ldots, x_n; y_1, \ldots, y_m)$. Then

$$M \models \varphi(\bar{\beta}) \iff M \models \psi(\bar{\beta}, \bar{a}) \iff N \models \psi(\bar{\alpha}, \bar{a}) \iff N \models \varphi(\bar{\alpha}),$$

where the middle $\iff$ holds because $\mathrm{tp}^M(\bar{a}, \bar{\beta}) = \mathrm{tp}^N(\bar{a}, \bar{\alpha})$. So we see that for any $L(A)$-formula $\varphi(\bar{x})$, we have $M \models \varphi(\bar{\beta}) \iff N \models \varphi(\bar{\alpha})$. Therefore $\mathrm{tp}^M(\bar{\beta}/A) = \mathrm{tp}^N(\bar{\alpha}/A) = p$. So p is realized in M by the tuple $\bar{\beta}$. $\square$

Theorem 12. *Let M be an ω-saturated L-structure. Let A be a finite subset of M. Let $\Sigma(\bar{x})$ be a set of $L(A)$-formulas in the variables $\bar{x} = (x_1, \ldots, x_n)$. If $\Sigma(\bar{x})$ is finitely satisfiable in M, then $\Sigma(\bar{x})$ is satisfied in M.*

[1] In more detail, if $\varphi_i(\bar{x}, y) \in p(\bar{x}, y)$ for $i = 1, \ldots, m$, let $\varphi = \bigwedge_{i=1}^m \varphi_i$. Then $\varphi(\bar{x}, y) \in p(\bar{x}, y)$, and we saw that $N \models \exists y\ \varphi(\bar{b}, y)$. So there is $\beta \in N$ satisfying the formula $\varphi(\bar{b}, y)$, or equivalently, satisfying the formulas $\varphi_1(\bar{b}, y), \ldots, \varphi_m(\bar{b}, y)$.

[2] The $\bar{a}$ in Lemma 9 is $(\bar{a}, \alpha_1, \ldots, \alpha_{m-1})$ here. The α in Lemma 9 is α_m here. The $\bar{b}$ in Lemma 9 is $(\bar{b}, \beta_1, \ldots, \beta_{m-1})$ here. The β in Lemma 9 is the β_m here.

Proof. Because $\Sigma(\bar{x})$ is finitely satisfiable, it can be extended to a complete type $p(\bar{x}) \in S_n(A)$. By Theorem 11, there is a tuple $\bar{b} \in M^n$ satisfying $p(\bar{x})$, and therefore satisfying $\Sigma(\bar{x})$. $\square$

Theorem 13. *Let M, N be countable ω-saturated L-structures. If $M \equiv N$, then $M \cong N$.*

(Compare with the proof that any two countable dense linear orders are elementarily equivalent.)

Proof. Let $\{c_1, c_2, c_3, \ldots\}$ be an enumeration of M and $\{d_1, d_2, d_3, \ldots\}$ be an enumeration of N. Recursively build sequences

$$a_1, a_2, a_3, \ldots \in M$$
$$b_1, b_2, b_3, \ldots \in N$$

such that $\mathrm{tp}^M(a_1, \ldots, a_i) = \mathrm{tp}^N(b_1, \ldots, b_i)$ and $a_{2i-1} = c_i$ and $b_{2i} = d_i$. Why is this possible?

- At step 0, we need $\mathrm{tp}^M() = \mathrm{tp}^N()$. This is true because $M \equiv N$.
- At step $2i-1$, we have $a_1, \ldots, a_{2i-2} \in M$ and $b_1, \ldots, b_{2i-2} \in N$ such that $\mathrm{tp}^M(a_1, \ldots, a_{2i-2}) = \mathrm{tp}^N(b_1, \ldots, b_{2i-2})$. By Lemma 9 there is some $\beta \in N$ such that $\mathrm{tp}^M(a_1, \ldots, a_{2i-2}, c_i) = \mathrm{tp}^N(b_1, \ldots, b_{2i-2}, \beta)$. Take $a_{2i-1} = c_i$ and $b_{2i-1} = \beta$.
- Step $2i$ is similar, switching M and N.

Now that we have the sequences $\{a_i\}$ and $\{b_i\}$, define a function $f : M \to N$ by $f(a_i) = b_i$. First, we show this is well-defined.

- If $x \in M$, then $x = a_i$ for at least one i. Indeed, $x = c_j$ for some j, and $a_{2j-1} = c_j = x$.
- If $x = a_i = a_j$ for two different i, j, then we need $b_i = b_j$ or else f is ill-defined. But this is true, because if $m = \max(i, j)$ then $\mathrm{tp}^M(a_1, \ldots, a_m)$ contains the formula $x_i = x_j$, and so $\mathrm{tp}^N(b_1, \ldots, b_m)$ also contains the formula $x_i = x_j$, implying that $b_i = b_j$.

Similarly, we could define $g : N \to M$ by $g(b_i) = a_i$. Then clearly g and f are inverses, and therefore f is a bijection. It remains to show that f is an isomorphism. Suppose, for example, that $R(x, y)$ is a binary relation symbol. We claim that

$$M \models R(a_i, a_j) \iff N \models R(b_i, b_j)$$

for any i, j. Indeed, take $m = \max(i, j)$. Then $\mathrm{tp}^M(a_1, \ldots, a_m)$ contains the formula $R(x_i, x_j)$, and so the same formula is in $\mathrm{tp}^N(b_1, \ldots, b_m)$, implying $N \models R(b_i, b_j)$. A similar argument handles n-ary relation symbols and n-ary function symbols (using the formula $y = g(z_1, \ldots, z_n)$). Constant symbols are 0-ary function symbols. So the map $f : M \to N$ is an isomorphism. $\square$

Corollary 14. *Let T be a complete theory. Then T has at most one countable ω-saturated model, up to isomorphism.*

Not all theories have countable ω-saturated models. In fact, the necessary and sufficient criterion is as follows:

Theorem 15. *Let T be a complete theory. Then T has a countable ω-saturated model if and only if $S_n(\varnothing)$ is countable for all n.*

Here, $S_n(\varnothing)$ denotes the space of n-types over the emptyset in some model $M \models T$. (Exercise: this doesn't depend on M.)

Proof sketch. First suppose there is a countable ω-saturated model M. The empty set is finite, so every type in $S_n(\varnothing)$ is realized in M, by Theorem 11. Therefore $S_n(\varnothing)$ is countable.

Conversely, suppose $S_n(\varnothing)$ is countable for any n.

Claim. Let M be a model of T. Let $A = \{a_1, \ldots, a_m\}$ be a finite subset. Let N, K be elementary extensions of M. Suppose $\bar{b} \in N^n$ and $\bar{c} \in K^n$. Then

$$\mathrm{tp}^N(\bar{b}/A) = \mathrm{tp}^K(\bar{c}/A) \iff \mathrm{tp}^N(\bar{b}, \bar{a}) = \mathrm{tp}^K(\bar{c}, \bar{a}).$$

Proof. The left-hand side says that for any $\varphi(y_1, \ldots, y_n) \in L(A)$, we have $N \models \varphi(\bar{b}) \iff K \models \varphi(\bar{c})$. The right-hand side says that for any formula $\psi(y_1, \ldots, y_n, x_1, \ldots, x_m) \in L$, we have $N \models \psi(\bar{b}, \bar{a}) \iff K \models \psi(\bar{c}, \bar{a})$. These are equivalent, as

$$\{\varphi(y_1, \ldots, y_n) \in L(A)\} = \{\psi(\bar{y}, \bar{a}) : \psi(y_1, \ldots, y_n, x_1, \ldots, x_m) \in L\}. \qquad \square_{\text{Claim}}$$

Claim. Let M be a model of T. Let $A = \{a_1, \ldots, a_m\}$ be a finite subset. Then $S_n(A)$ is countable.

Proof. Define a map $f : S_n(A) \to S_{n+m}(\varnothing)$ sending $\mathrm{tp}^N(\bar{b}/A)$ to $\mathrm{tp}^N(\bar{b}, \bar{a})$ for any $N \succeq M$ and $\bar{b} \in N^n$. This is well-defined and injective by the previous claim. $\square_{\text{Claim}}$

Now we can build a countable ω-saturated model of T as follows. First, take some ω-saturated model M^+. For each finite set $A \subseteq M^+$ and type $p \in S_1(A)$, take some element $c_{A,p} \in M$ realizing p. (We have just used the axiom of choice.) Define an increasing chain of countable subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots M^+$$

as follows:

- $A_0 = \varnothing$.
- $A_{i+1} = A_i \cup \{c_{A,p} : A \subseteq_f A_i,\ p \in S_1(A)\}$.

This is countable because A_i has countably many finite subsets, and for each finite subset A, the type space $S_1(A)$ is countable by the second claim.

Define $M = \bigcup_{i=0}^{\infty} A_i$. This is a countable union of countable sets, so it is countable.

Claim. $M \preceq M^+$.

Proof. Use the Tarski-Vaught criterion. Suppose $M^+ \models \exists x\ \varphi(\bar{a}, y)$ for some L-formula $\phi(\bar{x}, y)$ and some tuple $\bar{a} \in M^n$. Take $b \in M^+$ such that $M^+ \models \phi(\bar{a}, b)$. Let $A = \{a_1, \ldots, a_n\}$. For big enough i, we have $A \subseteq_f A_i$. Then $\mathrm{tp}(b/A)$ is realized in $A_{i+1} \subseteq M$. So there is $c \in M$ such that $\mathrm{tp}(c/A) = \mathrm{tp}(b/A)$. In particular, $M^+ \models \phi(\bar{a}, c)$. This completes the proof. $\square_{\text{Claim}}$

Therefore M is a countable model of T. It remains to show that M is ω-saturated. Take a finite subset $A \subseteq M$ and a type $p \in S_1(A)$. Then $A \subseteq A_i$ for large enough i. We chose A_{i+1} to contain some c such that $\mathrm{tp}^{M^+}(c/A) = p$. Because $M \preceq M^+$, we also have $\mathrm{tp}^M(c/A) = p$, so p is realized in M. $\square$

A complete theory T is said to be *small* if $S_n(\varnothing)$ is countable for all n, or equivalently, if a countable ω-saturated model exists.

Fact 16. *If T is a complete theory in a countable language and T is κ-categorical for some κ, then T is small.*

(Fact 16 is really too hard to prove here.)

Example. The complete theory of $(\mathbb{R}, +, \cdot, -, 0, 1, \leq)$ is *not* small. In fact, the map from $\mathbb{R}$ to $S_1(\varnothing)$ sending a to $\mathrm{tp}(a)$ is injective. To see this, note that if $a, b \in \mathbb{R}$ and $a < b$, then there is a rational number $q \in \mathbb{Q}$ such that $a < q < b$. Write q as n/m with $m > 0$. Then $am < n < bm$. There is a formula $\phi_{n,m}(x)$ such that $\mathbb{R} \models \phi_{n,m}(a) \iff am < n$. (Essentially, $\phi_{n,m}(x)$ is the formula $m \cdot x < n$, where m and n mean $\underbrace{1 + \cdots + 1}_{m \text{ times}}$ and $\underbrace{1 + \cdots + 1}_{n \text{ times}}$.) Then a satisfies $\phi_{n,m}$ but b does not. So $\mathrm{tp}(a) \neq \mathrm{tp}(b)$. The injection $a \mapsto \mathrm{tp}(a)$ shows that $S_1(\varnothing)$ is uncountable. So there is no countable ω-saturated structure elementarily equivalent to $\mathbb{R}$.