Coheirs and invariant types

Thursday, March 10, 2022 9:55 AM

T complete, $\mathbb{M}$ monster

Lemma 11 Let $\lambda \geq \aleph_0$

$(1) \Leftrightarrow (2)$

1.) If $A \subseteq \mathbb{M}$, $|A| \leq \lambda \Rightarrow |S_1(A)| \leq \lambda$

2.) If $A \subseteq \mathbb{M}$, $n < \omega$, $|A| \leq \lambda \Rightarrow |S_n(A)| \leq \lambda$

Proof $(2) \Rightarrow (1)$ trivial

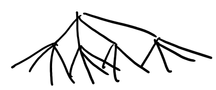
Assume (1). Prove (2). Assume $|A| \leq \lambda$

By induction, $|S_{n-1}(A)| \leq \lambda$.

$S_{n-1}(A) = \{ tp(\bar{b}_\alpha / A) : \alpha < \lambda \}$ (for some $\bar{b}_\alpha$)

For each α, $|S_1(A\bar{b}_\alpha)| \leq \lambda$ by (1). $A\bar{b}_\alpha = A \cup \{ b_{\alpha,1}, \dots b_{\alpha,n-1} \}$

$S_1(A\bar{b}_\alpha) = \{ tp(c_{\alpha,\beta} / A\bar{b}_\alpha) : \beta < \lambda \}$.



Claim If $(\bar{d}, e) \in \mathbb{M}^{n-1}$ then $\exists \alpha, \beta$ such that

$$tp(\bar{d}e/A) = tp(\bar{b}_\alpha, c_{\alpha,\beta}/A)$$

Proof $tp(\bar{d}/A) \in S_{n-1}(A)$
$\quad = tp(\bar{b}_\alpha/A)$ for some α

$\exists \sigma \in Aut(\mathbb{M}/A)$ $\sigma(\bar{d}) = \bar{b}_\alpha$

$tp(\sigma(e)/A\bar{b}_\alpha) \in S_1(A\bar{b}_\alpha)$
$\quad = tp(c_{\alpha,\beta}/A\bar{b}_\alpha)$ for some β.

$\exists \tau \in Aut(\mathbb{M}/A\bar{b}_\alpha)$ $\tau(\sigma(e)) = c_{\alpha,\beta}$

$$(\bar{d}, e) \xrightarrow{\sigma} (\bar{b}_\alpha, \sigma(e)) \xrightarrow{\tau} (\bar{b}_\alpha, c_{\alpha,\beta})$$

Then $\tau \circ \sigma \in Aut(\mathbb{M}/A)$ and $(\tau \circ \sigma)(\bar{d}e) = (\bar{b}_\alpha, c_{\alpha,\beta})$

So $tp(\bar{d}, e/A) = tp(\bar{b}_\alpha, c_{\alpha,\beta}/A)$. $\square$ claim

the Claim $\Rightarrow$ $S_n(A) = \{ tp(b_\alpha, c_{\alpha,\beta}/A) : \alpha, \beta < \lambda \}$

$|S_n(A)| \leq \lambda^2 = \lambda$. $\square$

Definition T is <u>λ-stable</u> if $\forall$ $A \subseteq \mathbb{M}$, $|A| \leq \lambda$ then $|S_n(A)| \leq \lambda$.

If $\lambda \geq |L|$, then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$

1.) If $A \subseteq \mathbb{M}$, $|A| \leq \lambda \Rightarrow |S_1(A)| \leq \lambda$

2.) If $A \subseteq \mathbb{M}$, $n < \omega$, $|A| \leq \lambda \Rightarrow |S_n(A)| \leq \lambda$

(*) 3.) If $M \preceq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_1(M)| \leq \lambda$

4.) If $M \preceq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$

$$\begin{tikzcd} (1) \arrow[r, leftrightarrow] \arrow[d, shift left] & (2) \arrow[d, shift left] \\ (3) \arrow[u, shift left] & (4) \arrow[u, shift left] \end{tikzcd}$$

Proof $(1) \Leftrightarrow (2)$ above.

(3) $\Rightarrow$ (1). Assume (3). Let $A \subseteq \mathcal{M}$, $|A| \leq \lambda$.

Lövenheim-Skolem $\Rightarrow$ $\exists\, M \preceq \mathcal{M}$, $M \supseteq A$, $|M| \leq |A| + |L| \leq \lambda$

$S_1(M) \xrightarrow[\text{restriction}]{} S_1(A)$ is surjective. $|S_1(A)| \leq |S_1(M)| \leq \lambda$ by (3).

(4) $\Rightarrow$ (2) similar.

<u>Lemma 1</u> Suppose: $\forall M \preceq \mathcal{M}$, $\forall p \in S_1(M)$ is definable.

Then T is λ-stable for some λ.

<u>Proof</u> Take $\lambda = 2^{|L|} > |L|$. ~~Sup~~ Suppose $M \preceq \mathcal{M}$, $|M| \leq \lambda$.

Every type in $S_1(M)$ is definable.

$p \in S_1(M)$ is determined by $\varphi \mapsto d_p\varphi$. $|L(M)| \leq \lambda$

L-fmla $\to$ $L(M)$-fmla.

$$|S_1(M)| \leq |L(M)|^{|L|} \leq \lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|^2} = 2^{|L|} = \lambda. \qquad \square$$

<u>Theorem 2</u> TFAE

1) $M \models T$, $p \in S_n(M) \Rightarrow p$ is definable.
2) $M \models T$, $p \in S_1(M) \Rightarrow p$ is definable.
3) No $\varphi(\bar{x}, \bar{y})$ has the dichotomy property
4) No $\varphi(x, \bar{y})$ has the dichotomy property
5) T is λ-stable for some λ.

<u>Proof</u>

(3) $\Rightarrow$ (4) trivial
(1) $\Rightarrow$ (2) trivial

(3) $\Rightarrow$ (1) <u>Prop 8 last week</u>
(4) $\Rightarrow$ (2)
(2) $\Rightarrow$ (5) Lemma 1.
(5) $\Rightarrow$ (3) Prop 10.

$$\begin{tikzcd} & & (5) \arrow[dll, bend right, "\checkmark"'] \\ (3) \arrow[r, Rightarrow, "\checkmark"] \arrow[d, Rightarrow, "\checkmark"] & (4) \arrow[d, Rightarrow, "\checkmark"] & \\ (1) \arrow[r, Rightarrow, "\checkmark"] & (2) \arrow[uur, Rightarrow, "\checkmark"'] & \end{tikzcd}$$

$\square$

T is STABLE if those conditions hold.

<u>Fact</u> If T is not stable, then for $\lambda > \aleph_0$, T has 2^λ-many models of size λ.
$|L| = \aleph_0$

<u>Coheirs</u>

<u>Def 3</u> If $M \preceq N$, if $p \in S_n(M)$, if $q \in S_n(N)$, then q is a <u>coheir</u> of p if $q \supseteq p$ and ~~q is finitely s~~ for any $\varphi(\bar{x}) \in q(\bar{x})$, $\exists\, \bar{a} \in M$, $N \models \varphi(\bar{a})$.

(L(N) - formula)

If $\varphi_1, \dots \varphi_m \in q$, then $\bigwedge_{i=1}^{m} \varphi_i(\bar{x}) \in q(\bar{x})$, so $\exists \bar{a} \in M : N \models \bigwedge_{i=1}^{m} \varphi_i(\bar{a})$, so

$\{\varphi_1(\bar{x}), \dots \varphi_m(\bar{x})\}$ is satisfiable in M.

q is a coheir (of $q \upharpoonright M$) iff q is f.sat. in M.

Example $\mathbb{Q}^{alg} \preccurlyeq \mathbb{C}$ $\quad q = tp(\pi / \mathbb{C})$ $\quad p = tp(\pi / \mathbb{Q}^{alg})$.

$q \supseteq p$, but q isn't a coheir

$(x = \pi) \in q(x)$, not satisfiable in $\mathbb{Q}^{alg}$.

Example If $M \preccurlyeq N$ strongly minimal, ~~q(x) is~~ $q(x) \in S_1(N)$, $p(x) \in S_1(M)$ transcendental

then q is a coheir of p. $q \supseteq p$ ✓

q is fsat in M: if $\varphi(x) \in q(x)$, then ~~q~~ $\varphi(N)$ is cofinite

M is infinite, so $M \cap \varphi(N) \neq \emptyset$. $\exists a \in M . N \models \varphi(a)$.

Lemma 4 If $M \preccurlyeq N$, $\Sigma(\bar{x})$ partial type over N, $\Sigma(\bar{x})$ is f.sat. in M

THEN $\exists\ q(\bar{x}) \in S_n(N)$, $q(\bar{x})$ is fsat. in M.

Proof Let $\Psi(\bar{x}) = \{ \psi(\bar{x}) \in L(N) : \forall \bar{a} \in M \quad N \models \psi(\bar{a}) \}$

If $\bar{a} \in M$, then $\bar{a}$ satisfies Ψ

$\Sigma(\bar{x})$ fsat in M $\Rightarrow$ $\Sigma(\bar{x}) \cup \Psi(\bar{x})$ is fsat. $\Rightarrow$ $q \in S_n(N)$ $\quad q \supseteq \Sigma \cup \Psi$.

If q isn't fsat. in M then $\varphi(\bar{x}) \in q(\bar{x})$, $\varphi(\bar{x})$ not sat. in M.

$\neg\varphi(x) \in \Psi(\bar{x}) \subseteq q(\bar{x})$, ~~so~~ NO! $\square$

Theorem 5 If $p \in S_n(M)$, $N \succcurlyeq M$, then $\exists q \in S_n(N)$,

q is a coheir of p.

Proof Lemma 4 with $\Sigma = p$.

Theorem 6 Suppose $M_1 \preccurlyeq M_2 \preccurlyeq M_3$, $p_1 \in S_n(M_1)$, $p_2 \in S_n(M_2)$

p_2 is a coheir of p_1. THEN $\exists p_3 \in S_n(M_3)$, $\underbrace{p_3 \text{ is a coheir of } p_1}_{p_3 \text{ is f.sat. in } M_1,\ p_3 \supseteq p_2}$ and p_2

Proof Apply Lemma 4 to $\Sigma = p_2$

$M = M_1 \quad N = M_3$. $\square$

Warning "coheir" isn't transitive.

Invariant types

$\mathbb{M}$ monster model If $A \subseteq \mathbb{M}$ small

$\underbrace{tp(\bar{a}/A) = tp(\bar{b}/A)}_{\bar{a} \equiv_A \bar{b} \text{ means this}} \iff \exists \sigma \in \underbrace{Aut(\mathbb{M}/A)}_{} \ [\sigma(\bar{a}) = \bar{b}]$

$\{ \sigma \in Aut(\mathbb{M}) : \sigma$ fixes A pointwise $\forall x \in A \quad \sigma(x) = x \}$.

Lemma 8 If $X \subseteq \mathbb{M}^n$, TFAE

1) $\sigma(X) = X$ if $\sigma \in Aut(\mathbb{M}/A)$
2) If $\bar{a}, \bar{b} \in \mathbb{M}^n$, $\bar{a} \equiv_A \bar{b} \Rightarrow (\bar{a} \in X \iff \bar{b} \in X)$
3) There is $f : S_n(A) \to \{0,1\}$ such that $\bar{a} \in X \iff f(tp(\bar{a}/A)) = 1$.

Proof $(2) \iff (3)$ easy.

$(1) \iff (2)$: Rewrite (2) as

- If $\bar{a}, \bar{b} \in \mathbb{M}$, $\sigma \in Aut(\mathbb{M}/A)$, and $\sigma(\bar{a}) = \bar{b}$, then $\bar{a} \in X \iff \bar{b} \in X$.

i.e.,

- If $\bar{a} \in \mathbb{M}$, $\sigma \in Aut(\mathbb{M}/A)$ then $\bar{a} \in X \iff \sigma(\bar{a}) \in X$

i.e.

- $X = \sigma^{-1}(X)$
- $X = \sigma(X)$. $\square$

Def 9 $X \subseteq \mathbb{M}^n$ is Aut($\mathbb{M}$/A)-invariant or A-invariant if Lemma 8 (1)–(3) hold.

Example If X is A-definable ($X = \varphi(\mathbb{M}^n)$, $\varphi \in L(A)$) then X is A-invariant ($tp(\bar{b}/A)$ determines whether $\mathbb{M} \models \varphi(\bar{b})$.)

Lemma 10 If $D \subseteq \mathbb{M}^n$ is definable and A-invariant, then D is A-definable.

Proof: Compactness ...

Step 1 If $\bar{b} \in D$ then $tp(\bar{b}/A) \vdash \bar{x} \in D$

by "compactness", $\exists\, \varphi(\bar{x}) \in tp(\bar{b}/A)$ $\varphi(\bar{x}) \vdash \bar{x} \in D$.

$\varphi(\mathbb{M}^n) \subseteq D$

Step 2 So then D is covered by A-definable ~~sets~~ subsets of D.

by "compactness", D is covered by finitely many of them,

$\Rightarrow D$ is A-definable. $\square$

Def 11 A global type is $p \in S_n(\mathbb{M})$.

Def 12 $p \in S_n(\mathbb{M})$ is A-invariant (for small $A \subseteq \mathbb{M}$) if $\sigma(p) = p$ $\forall \sigma \in Aut(\mathbb{M}/A)$.

Equivalently, $\forall\, \varphi(\bar{x}, \bar{y})$, the set

$\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is A-invariant.

Equivalently, if $\bar{b} \equiv_A \bar{c}$ then $\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \varphi(\bar{x}, \bar{c}) \in p(\bar{x})$.

Def 13 p is A-definable if $\forall \varphi$, $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is A-definable.

Remark 14 1) p is A-definable $\Rightarrow$ p is A-invariant

2) If p is definable, then p is A-invar. $\Leftrightarrow$ p is A-definable.

3) If p is definable then p is A-definable for some small A $|A| \leq |L|$.

Each $d_p\varphi$ uses only fin. many parameters

Prop 15 Suppose $M \preceq \mathbb{M}$ small.

1) If $p \in S_n(M)$ definable and $p^{\mathbb{M}}$ is its heir over $\mathbb{M}$, then $p^{\mathbb{M}} \in S_n(\mathbb{M})$ is M-definable.

2) $p \mapsto p^{\mathbb{M}}$ is a bijection from definable types over M ~~to~~ to M-definable types over $\mathbb{M}$.

Proof 1) $p^{\mathbb{M}}$ ~~is~~ has same def. as p, so it's M-definable

2) $q \mapsto q \restriction M$ is an inverse to $p \mapsto p^{\mathbb{M}}$

If q is M-definable, then $q \restriction M$ is definable and

$$q = (q \restriction M)^{\mathbb{M}}.$$

Warning 16 an M-invariant type p is not determined by $p \restriction M$

If $A \subseteq \mathbb{M}$, A-definable type p is not determined by $p \restriction A$.

Theorem 17 Suppose $M \preceq \mathbb{M}$, $p \in S_n(M)$

1) If $q \in S_n(\mathbb{M})$ and q is a coheir of p, then q is M-invariant.

2) $\exists\, q \in S_n(\mathbb{M})$ q is M-invariant, $q \supseteq p$.

Proof (1) $\Rightarrow$ (2) by Theorem 5.

(1): If q a coheir of p, but q is n't M-invariant, then $\exists\, \bar{b}, \bar{c}, \varphi$

$\bar{b} \equiv_M \bar{c}$ but $\varphi(x, \bar{b}) \in q(\bar{x})$ $\varphi(\bar{x}, \bar{c}) \notin q(\bar{x})$.

$\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$.

q is fsat. in M, $\exists\, \bar{a} \in M$ $\mathbb{M} \models \varphi(\bar{a}, \bar{b}) \wedge \neg\varphi(\bar{a}, \bar{c})$.

$\Rightarrow$ $\bar{b} \not\equiv_M \bar{c}$ since $\varphi(\bar{a}, \bar{y})$ is true for $\bar{b}$, false for $\bar{c}$. $\square$

In stable theories:

Lemma 19: If T is stable and p is A-invariant, then p is A-definable.

$\square$.

Theorem 20 Suppose T stable, $M \preceq \mathbb{M}$ small, $p \in S_n(M)$.

Let $p^{\mathbb{M}}$ be the global heir.

1) $p^{\mathbb{M}}$ is the only M-invariant global type extending p
2) $p^{\mathbb{M}}$ is the only global coheir of p
3) If $M \preceq N \preceq \mathbb{M}$ and q is the heir of p over N, then q is the unique coheir of p over N.

<u>Proof</u> 1) M-invar $\Leftrightarrow$ M-definable, use Prop 15
$p^{\mathbb{M}}$ is the only M-definable type $\supseteq p$.

2) $\exists$ <u>some</u> coheir of p. Any coheir is M-invariant (Theorem 17)
so $p^{\mathbb{M}}$ is the only coheir.

3) If q' is some coheir over N, Theorem 6 gives $r \in S_n(\mathbb{M})$
$r \supseteq q'$, r is a coheir of p and q'.
By (2), $r = p^{\mathbb{M}}$, then $q' = (p^{\mathbb{M}} \upharpoonright N) = q$. $\leftarrow$ the heir over N.

<u>Cor 21</u> In a stable theory, coheirs are unique and coheirs = heirs.
<u>Cor 22</u> In a stable theory, "coheir" is transitive. (because "heir" is.)

<u>Notation</u> If $A \subseteq \mathbb{M}$, $p \in S_n(A)$ then $\bar{b} \models p$ means
$\bar{b}$ satisfies p, ie $tp(\bar{b}/A) = p$.

<u>Lemma 23</u> If p, q are A-invariant global types $p \in S_n(\mathbb{M})$, $q \in S_m(\mathbb{M})$
then there is $r \in S_{n+m}(A)$ such that $(\bar{b}, \bar{c}) \models r$ iff

$$\bar{b} \models p \upharpoonright A \quad \text{and} \quad \bar{c} \models q \upharpoonright (A\bar{b}) \qquad (*).$$

<u>Proof</u> Let $X = \{(\bar{b}, \bar{c}) : \bar{b} \models p \upharpoonright A \text{ and } \bar{c} \models q \upharpoonright A\bar{b}\}$
if $(\bar{b}, \bar{c}) \in X$ and $\sigma \in Aut(\mathbb{M}/A)$, then
$\sigma(\bar{b}) \models p \upharpoonright A$ and $\sigma(\bar{c}) \models q \upharpoonright A\sigma(\bar{b})$,
so $\sigma(\bar{b}, \bar{c}) \in X$.
X is A-invariant.
Fix $\bar{b}_0 \models p \upharpoonright A$, $\bar{c}_0 \models q \upharpoonright A\bar{b}_0$ so $(\bar{b}_0, \bar{c}_0) \in X$
Let $r = tp(\bar{b}_0 \bar{c}_0 / A)$. If $(\bar{b}, \bar{c}) \models r$ then $(\bar{b}, \bar{c}) \in X$.
Conversely if $(\bar{b}, \bar{c}) \in X$, want $(\bar{b}, \bar{c}) \models r$ i.e. $(\bar{b}, \bar{c}) \equiv_A (\bar{b}_0, \bar{c}_0)$.
$\bar{b} \models p \upharpoonright A = tp(\bar{b}_0 / A)$ so $\bar{b} \equiv_A \bar{b}_0$, $\exists\, \sigma \in Aut(A)$
$\sigma(\bar{b}) = \bar{b}_0$. Replace $(\bar{b}, \bar{c})$ with $(\sigma(\bar{b}), \sigma(\bar{c})) = (\bar{b}_0, \sigma(\bar{c}))$.
WMA $\bar{b} = \bar{b}_0$. Then $\bar{c}$ and $\bar{c}_0$ both satisfy $q \upharpoonright A\bar{b}_0$
so $\bar{c} \equiv_{A\bar{b}_0} \bar{c}_0$ Move $\bar{c}$ by $\tau \in Aut(\mathbb{M} / A\bar{b}_0)$,

we may assume $\bar{c} = \bar{c}_0$. Then $(b, \bar{c}) = (b_0, \bar{c}_0)$

$b\bar{c} = \bar{b}_0 \bar{c} \equiv_A \bar{b}_0 \bar{c}_0$. $\square$

Prop 24 If $p \in S_n(\mathcal{M})$, $q \in S_m(\mathcal{M})$ and both are A-invariant, then $\exists$ A-invariant $p \otimes q \in S_{n+m}(\mathcal{M})$ such that for any small A' with $A \subseteq A' \subseteq \mathcal{M}$,

$$(**) \quad (\bar{b}, \bar{c}) \models (p \otimes q) \restriction A' \iff [\, \bar{b} \models p \restriction A' \text{ and } \bar{c} \models q \restriction A'\bar{b} \,].$$

Proof ~~Lemma 23~~ Note p, q are A'-invariant for $A' \supseteq A$

so Lemma 23 gives $r_{A'} \in S_{n+m}(A')$ for each $A' \supseteq A$

such that $\quad (\bar{b}, \bar{c}) \models r_{A'} \iff [\, \bar{b} \models p \restriction A' \text{ and } \bar{c} \models q \restriction A'\bar{b} \,]$.

Note if ~~the~~ $A'' \supseteq A' \supseteq A$, if $(\bar{b}, \bar{c}) \models r_{A''} \Rightarrow (\bar{b}, \bar{c}) \models r_{A'}$

So $r_{A'} = r_{A''} \restriction A'$.

Let $p \otimes q = \bigcup_{A'} r_{A'}$. Then $p \otimes q \in S_{n+m}(\mathcal{M})$ and $r_{A'} = (p \otimes q) \restriction_{A'}$ for $A' \supseteq A$.

By def. of $r_{A'}$, $(**)$ holds.

If $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$, $\sigma(p \otimes q) = \sigma(p) \otimes \sigma(q) = p \otimes q$.

So $p \otimes q$ is A-invariant.

~~Ex~~ **Fact** If $p \in S_n(M)$ p is "A-invariant" where M is $|A|^+$-saturated (if $\bar{b}, \bar{c} \in M$, $\bar{b} \equiv_A \bar{c} \Rightarrow (\varphi(x, \bar{b}) \in p \iff \varphi(x, \bar{c}) \in p)$)

and $N \succeq M$, then p has a unique "A-invariant" extension over N.

$p|N$ denotes this extension.

Fact If $p, q \in S_{n \text{ or } m}(\mathcal{M})$ A-invariant, Take $\bar{b} \models p$ $\quad \bar{b} \in \mathcal{M}_1^* \succeq \mathcal{M}$.

Take ~~$c \models p|M$~~ $\bar{c} \models q|\mathcal{M}_1$ then $tp(\bar{b}, \bar{c}/\mathcal{M}) = p \otimes q$

Def ~~$p \otimes q$ is~~ The (Morley) product of invariant types p, q is $p \otimes q$ from Prop. 24.

If p, q are A-invar. then $\quad (\bar{b}, \bar{c}) \models (p \otimes q) \restriction A \iff (\bar{b} \models p \restriction A \text{ and } \bar{c} \models q \restriction A\bar{b})$

Example If T is strongly minimal and $p = q =$ transcendental 1-type, what ~~$p \otimes$~~ is $(p \otimes p)(x, y)$?

Def (In any theory) $\mathrm{acl}(A) = \bigcup \{ \varphi(\mathcal{M}) : \varphi(x) \in L(A), |\varphi(\mathcal{M})| < \infty \}$.

$b \in \mathrm{acl}(A)$ if $\exists$ $L(A)$-fmla $\varphi(x)$, $b \in \varphi(\mathcal{M}) \subseteq_f \mathcal{M}$.

example: in ACF, $\sqrt{2} \in acl(\mathbb{Q})$ because $\varphi(x) = (x^2 - 2 = 0)$.

Fact in ACF, if K subfield of $\mathbb{M}$ then $acl(K)$ is K^{alg} field theoretic alg closure

Fact In any theory T, $acl(-)$ is a finitary closure operation.

$$X \subseteq Y \Rightarrow acl(X) \subseteq acl(Y)$$
$$X \subseteq acl(X)$$
$$acl(acl(X)) = acl(X)$$
$$acl(X) = \bigcup_{X_0 \subseteq_f X} acl(X_0).$$

"transcendental" means "not algebraic".

If T is strongly minimal & $p \in S_1(\mathbb{M})$ transcendental

$b \models p \restriction A \iff b \notin acl(A)$.

$\varphi(x) \in p \iff \varphi(\mathbb{M})$ is cofinite $\iff \varphi(\mathbb{M})$ not finite

$p \restriction A = \{\varphi(x) \in L(A) : \varphi(\mathbb{M})$ is cofinite$\}$

Therefore $(b, c) \models (p \otimes p) \restriction A$ iff $b \models p \restriction A$ and $c \models p \restriction Ab$

iff $b \notin acl(A)$ and $c \notin acl(Ab)$.

idea: b, c are algebraically independent over A.

Idea (in stable theories), $(p \otimes q)(\bar{x}, \bar{y})$ is the "most free" completion of $p(\bar{x}) \cup q(\bar{y})$.

Example Suppose $\mathbb{M} \models ACF$. Let p_V denote generic type of a variety $V \subseteq \mathbb{M}^n$. $\{x \in V\} \cup \{x \notin W : W \subsetneq V,\ W$ algebraic$\}$

If $V \subseteq \mathbb{M}^n$, $W \subseteq \mathbb{M}^m$ varieties, then $V \times W$ is a variety and $p_V \otimes p_W$ is $p_{V \times W}$.

Proof $p_V \otimes p_W = p_Z$ for some variety $Z \subseteq \mathbb{M}^{n+m}$. Goal: $Z = V \times W$.

Take small $M \preceq \mathbb{M}$ s.t. V, W, Z are M-definable.

Take $\bar{a} \models p_V \restriction M$, take small $N \preceq \mathbb{M}$ $N \supseteq M\bar{a}$,

Take $\boxed{\bar{b} \models p_W \restriction N}$ ($\Rightarrow \bar{b} \models p_W \restriction M\bar{a}$, so $(\bar{a}, \bar{b}) \models p_V \otimes p_W \restriction M = p_Z \restriction M$).

$\bar{a} \in V$ ("$\bar{x} \in V$" $\in p_V \restriction M$). $\bar{b} \in W$, so $(\bar{a}, \bar{b}) \in V \times W$.

(Fact $p_Z(\bar{x}) \vdash \bar{x} \in U \iff Z \subseteq U$, for U algebraic)

So $(\bar{a}, \bar{b}) \in V \times W \Rightarrow Z \subseteq V \times W$.

Suppose $Z \subsetneq V \times W$. Take $(\bar{a}_0, \bar{b}_0) \in V \times W \setminus Z$, $\bar{a}_0, \bar{b}_0 \in M$.

Let $Z_{\bar{a}} = \{\bar{y} \in M : (\bar{a}, \bar{y}) \in Z\}$.

Then $Z_{\bar{a}}$ is an algebraic set over $N \supseteq M\bar{a}$, $(\bar{a}, \bar{b}) \in Z \Rightarrow \bar{b} \in Z_{\bar{a}}$.

$\bar{b} \models p_W \restriction N \Rightarrow W \subseteq Z_{\bar{a}}$.

Then $\bar{b}_0 \in W \subseteq Z_{\bar{a}}$ $\quad (\bar{a}, \bar{b}_0) \in Z$.

Let $Z^{\bar{b}_0} = \{ \bar{x} \in M ; \ (\bar{x}, \bar{b}_0) \in Z \}$. $\quad Z^{\bar{b}_0}$ is M-definable algebraic.

$\bar{a} \in Z^{\bar{b}_0}$, $\quad \bar{a} \models p_V \restriction M$ $\quad$ So $V \subseteq Z^{\bar{b}_0}$.

$\bar{a}_0 \in V \subseteq Z^{\bar{b}_0}$ so $(\bar{a}_0, \bar{b}_0) \in Z \Rightarrow\Leftarrow$. $\quad Z = V \times W$. $\quad \square$.

<u>Def</u> ~~p~~ invariant types p, q "<u>commute</u>" if $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$.

That is, for $C \subseteq M$ with p, q C-invariant,

$$\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a} \Rightarrow (\bar{b} \models q \restriction C \text{ and}) \ \bar{a} \models p \restriction C\bar{b}$$

<u>Example</u> In ACF, any two types commute.

$$(p_V \otimes p_W)(\bar{x}, \bar{y}) = p_{V \times W}(\bar{x}, \bar{y}) = p_{W \times V}(\bar{y}, \bar{x}) = (p_W \otimes p_V)(\bar{y}, \bar{x}).$$

$(d_p x)\, \varphi(\bar{x}, \bar{y}) = \psi(\bar{y})$ $\qquad d\varphi(\bar{y})$ is $\exists^\infty x \ \varphi(x, \bar{y})$

$\exists^\infty x$

$(d_{p \otimes q}\, x, y)\, \varphi(x, y, z)$ is $(d_p x)(d_p y)\, \varphi(x, y, z)$.

$\exists^\infty x \exists^\infty y$

$\exists^\infty y \exists^\infty x$.