Introduction To Model Theory

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1 Back-and-forth Equivalence I

Convention: Relations and functions are sets of pairs (x, y)

Definition 1.1. A **binary relation** is a pair (E, R) where E is a set and $R \subseteq E^2$. We call E the **universe** of the relation. For $a, b \in E$, write aEb if $(a, b) \in R$

We abbreviate (E, R) as R or E, if E or R is clear

Example 1.1. $(\mathbb{R}, <)$, $(\mathbb{R}, =)$, $(\mathbb{R}, \geq)$,$(\mathbb{Z}, <)$

Definition 1.2. A binary relation R is said to be

- **reflexive** if aRa $(\forall a \in E)$
- **symmetric** if $aRb \Rightarrow bRa$ $(\forall a, b \in E)$
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$ $(\forall a, b, c \in E)$
- **antisymmetric** if $aRb \wedge bRa \Rightarrow a = b$ $(\forall a, b \in E)$
- **total** if $aRb \vee bRa$ $(\forall a, b \in E)$
- an **equivalence relation** if it's reflexive, symmetric and transitive
- a **partial order** if it's reflexive, antisymmetric and transitive
- a **linear order** if it's a total partial order

Example 1.2. $=$ is an equivalence relation
$\subseteq$ is a partial order
$\leq$ is a linear order

Definition 1.3. An **isomorphism** from (E, R) to (E', R') is a bijection $f : E \to E'$ s.t. for any $a, b \in E$, $aRb \Leftrightarrow f(a)R'f(b)$. Two binary relations (E, R) and (E', R') are **isomorphic** ($\cong$) if there is an isomorphism between them

Example 1.3. $f : (\mathbb{Z}, <) \to (2\mathbb{Z}, >)$ and $f(x) = -2x$ is an isomorphism. $x < y \Leftrightarrow -2x > -2y$

$\cong$ is an equivalence relation

Definition 1.4. A **local isomorphism** from R to R' is an isomorphism from a finite restriction of R to a finite restriction of R'. The set of local isomorphisms from R to R' is denoted $S_0(R, R')$. For $f \in S_0(R, R')$, $\text{dom}(f)$ and $\text{im}(f)$ denote the domain and range of f

Example 1.4. $(\mathbb{Z}, <)$ is a restriction of $(\mathbb{R}, <)$

Example 1.5. Suppose $R = R' = (\mathbb{Z}, <)$, there is $f \in S_0(R, R')$ given by $\text{dom}(f) = \{1, 2, 3\}$ and $\text{im}(f) = \{10, 20, 30\}$ and $f(1) = 10$,$f(2) = 20$, $f(3) = 30$

Definition 1.5. Let f, g be local isomorphisms from R to R'. Then f is a **restriction** of g if $f \subseteq g$ and f is an **extension** of g if $f \supseteq g$.

Example 1.6. $g : \{0, 1, 2, 3\} \to \{5, 10, 20, 30\}$, g extends f in the previous example

Definition 1.6. Let R, R' be binary relations with universe E, E'. A **Karpian family** for (R, R') is a set $K \subseteq S_0(R, R')$ satisfying the following two conditions for any $f \in K$

1. (**forth**) if $a \in E$ then there is $g \in K$ with $g \supseteq f$ and $a \in \text{dom}(g)$
2. (**back**) if $b \in E'$ then there is $g \in K$ with $g \supseteq f$ and $b \in \text{im}(g)$

R and R' are **∞-equivalent**, write $R \sim_\infty R'$, if there is a non-empty Karpian family

Proposition 1.7. *If $f : (E, R) \to (E', R')$ an isomorphism and $K = \{g \subseteq f : g \text{ is finite}\}$, then K is Karpian and $R \sim_\infty R'$*

Proof. Suppose $g \in K$

- (forth) Suppose $a \in E$, take $b = f(a)$ and let $h = g \cup \{(a, b)\}$. Then $h \subseteq f$, so $h \in K$, $h \supseteq g$, $a \in \text{dom}(h)$
- (back) similarly

$\square$

Proposition 1.8. *If (E, R) and (E', R') are countable and $R \sim_\infty R'$, then $R \cong R'$*

Proof. Let $K \subseteq S_0(R, R')$ be Karpian, $K \neq \emptyset$, $E = \{e_1, e_2, e_3, \dots\}$, $E' = \{e'_1, e'_2, e'_3, \dots\}$

Recursively build $f_1 \subseteq f_2 \subseteq \cdots$, $f_i \in K$

Let f_1 be anything in K as K is non-empty.

f_{2i} some extension of f_{2i-1} with $e_i \in \mathrm{dom}(f_{2i})$

f_{2i+1} some extension of f_{2i} with $e'_i \in \mathrm{im}(f_{2i+1})$

Now let $g = \bigcup_{i=1}^{\infty} f_i$, then g is an isomorphism □

Definition 1.9. A **dense linear order without endpoints** (DLO) is a linear order $(C, \leq)$ satisfying

1. $C \neq \emptyset$
2. $\forall x, y \in C$, $x < y \Rightarrow \exists z \in C\ x < z < y$
3. $\forall x \in C$, $\exists y, z \in C\ y < x < z$

Example 1.7. $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$

non-example: $(\mathbb{Z}, \leq)$, $([0,1], \leq)$

Proposition 1.10. *Let $(C, \leq)$ and $(C', \leq)$ be DLO's. Then $S_0(C, C')$ is Karpian. So $C \sim_\infty C'$*

Proof. Let $f \in S_0(C, C')$, $\mathrm{dom}(f) = \{a_1, \dots, a_n\}$, $a_1 < \cdots < a_n$ and $\mathrm{im}(f) = b_1, \dots, b_n$, $b_1 < \cdots < b_n$. Since f is a local isomorphism, $f(a_i) = b_i$

- (forth) Suppose $a \in C$. We want $b \in C'$ s.t. $f \cup \{(a, b)\} \in S_0(C, C')$.
  - if $a_i < a < a_{i+1}$. We take $b \in C'$ s.t. $b_i < b < b_{i+1}$ since dense
  - if $a < a_1$. We take $b \in C'$ s.t. $b < b_1$ since no endpoints
  - if $a > a_n$, take $b \in C'$ s.t. $b > b_n$
  - if $a = a_i$, take $b = b_i$
- (back) similar

□

Proposition 1.11. *If $(C, \leq)$ and $(C', \leq)$ are countable DLOs, then $C \sim_\infty C'$, so $C \cong C'$*

Hence

$$\begin{aligned}
(\mathbb{Q}, \leq) &\cong (\mathbb{Q} \setminus \{0\}, \leq) \\
&\cong (\mathbb{Q} \cup \{\sqrt{2}\}, \leq) \\
&\cong (\mathbb{Q} \cap (0,1), \leq)
\end{aligned}$$

Definition 1.12. Let R, R' be binary relations with universe E, E'

- A **0-isomorphism** from R to R' is a local isomorphism from R to R'
- For $p > 0$, a p**-isomorphism** from R to R' is a local isomorphism f from R to R' satisfying the following two conditions
  1. (**forth**) For any $a \in E$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $a \in \mathrm{dom}(g)$
  2. (**back**) For any $b \in E'$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $b \in \mathrm{im}(g)$
- An ω**-isomorphism** from R to R' is a local isomorphism f from R to R' s.t. f is a p-isomorphism for all $p < \omega$

The set of p-isomorphisms from R to R' is denoted $S_p(R, R')$

Example 1.8. Suppose $R = R' = (\mathbb{Z}, <)$, $f : \{2, 4\} \to \{1, 2\}$ is a local isomorphism with $f(2) = 1$ and $f(4) = 2$. Then $f \notin S_1(\mathbb{Z}, \mathbb{Z})$ (forth) fails. For $a = 3$, there is no b s.t. $1 < b < 2$

$g : \{2, 4\} \to \{1, 5\}$ is a 1-isomorphism but not a 2-isomorphism

Proposition 1.13. *If $f \in S_p(R, R')$ and $g \subseteq f$, then $g \in S_p(R, R')$*

Proof. if $p = 0$ easy

if $p > 0$ (forward), $\forall a \in E$, $\exists h \in S_{p-1}(R, R')$ has $a \in \mathrm{dom}(h)$ and $h \supseteq f \supseteq g$ □

Proposition 1.14. $S_p(R, R') \neq \emptyset$ *iff* $\emptyset \in S_p(R, R')$

Proof. $\Leftarrow$ immediate

$\Rightarrow$. Suppose $f \in S_p(R, R')$. Then $\emptyset \subseteq f$. Hence $\emptyset \in S_p(R, R')$. □

Definition 1.15. R and R' are p**-equivalent**, written $R \sim_p R'$, if there is a p-isomorphism from $R \to R'$

R and R' are ω**-equivalent** or **elementarily equivalent**, written $R \sim_\omega R'$ or $R \equiv R'$, if there is an ω-isomorphism from R to R'

Note: $R \sim_\omega R'$ iff $S_\omega(R, R') \neq \emptyset$ iff $\emptyset \in S_\omega(R, R')$ iff $\forall p\ \emptyset \in S_p(R, R')$ iff $\forall p\ R \sim_p R'$

Definition 1.16. Let R, R' be binary relations with universe E, E'. The Ehrfenfeucht-Fraïssé game of length n, denoted $\mathrm{EF}_n(R, R')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are n rounds
- In the ith round, the Spoiler chooses either an $a_i \in E$ or a $b_i \in E'$
- The Duplicator responds with a $b_i \in E'$ or an $a_i \in E$ respectively
- At the ends of the game, the Duplicator wins

  $$\{(a_i, b_i), \ldots, (a_n, b_n)\}$$

  is a local isomorphism from R to R'
- Otherwise, the Spoiler wins

Example 1.9. For $\mathrm{EF}_3(\mathbb{Q}, \mathbb{R})$

$\mathbb{Q}$	$\mathbb{R}$
S:$a_1 = 7$	D:$b_1 = 7$
D:$a_2 = 1.4$	S:$b_2 = \sqrt{2}$
D:$a_3 = -10$	S:$b_3 = 1.41$

So D wins

Example 1.10. $\mathrm{EF}_3(\mathbb{R}, \mathbb{Z})$

$\mathbb{R}$	$\mathbb{Z}$
D:$a_1 = 1$	S:$b_1 = 1$
D:$a_2 = 1.1$	S:$b_2 = 2$
S:$a_3 = 1.01$	

D fails

Proposition 1.17. *$EF_n(R, R')$ is a win for Duplicator iff $R \sim_n R'$*

Proposition 1.18. *In $EF_n(R, R')$ if moves so far are $a_1, b_1, \ldots, a_i, b_i$, $p = n - 1$, $f = \{(a_1, b_1), \ldots, (a_i, b_i)\}$. Then Duplicator wins iff $f \in S_p(R, R')$*

2 Back-and-forth Equivalence II

Definition 2.1. Let (M, R), (M', R') be binary relations.. The Ehrenfeucht-Fraïssé game of length n, denoted $\mathrm{EF}_n(M, M')$ is played as follows

- There are two players, the Duplicator and Spoiler

- There are n rounds
- In the ith round, the Spoiler chooses either an $a_i \in M$ or a $b_i \in M'$
- The Duplicator responds with a $b_i \in M'$ or an $a_i \in M$ respectively
- At the ends of the game, the Duplicator wins

$$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

  is a local isomorphism from R to R'
- Otherwise, the Spoiler wins

Lemma 2.2. *Suppose we are playing EF*$_n(M, M')$ *and there have been q rounds so far, with $p = n - q$ rounds remaining. Suppose the moves so far are $(a_1, b_1), \dots, (a_n, b_n)$. Let $f = \{(a_1, b_1), \dots, (a_q, b_q)\}$. Then the following are equivalent*

- *Duplicator has a winning strategy*
- *f is a p-isomorphism*

Proof. By induction on p.

if $p = 0$, then the game is over, so Duplicator wins iff $f \in S_0(M, M')$

$p > 0$. If f isn't a local isomorphism, then Duplicator will definitely lose, and f isn't a p-isomorphism. So we may assume $f \in S_0(M, M')$. Then the following are equivalent

- Duplicator wins
- For any $a_{q+1} \in M$, there is a $b_{q+1} \in M'$ s.t. Duplicator wins in the position $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$, AND for any $b_{q+1} \in M'$, there is a $a_{q+1} \in M$ s.t. Duplicator wins in the position $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$,
- For any $a_{q+1} \in M$ there is a $b_{q+1} \in M'$ s.t. $f \cup \{(a_{q+1}, b_{q+1})\} \in S_{p-1}(M, M')$ (by induction) , AND ...
- For any $a_{q+1} \in M$, there is $g \in S_{p-1}(M, M')$ s.t. $g \supseteq f$ and $a_{q+1} \in \mathrm{dom}(g)$, AND ....
- $f \in S_p(M, M')$

□

Theorem 2.3. *If M is p-equivalent to M', then EF*$_p(M, M')$ *is a win for the Duplicator. Otherwise it is a win for the Spoiler*

Proof. We need to prove $\emptyset \in \mathrm{EF}_p(M, M')$ $\square$

Theorem 2.4. *Every $(p+1)$-isomorphism is a p-isomorphism*

Proof. By induction on p.
$p = 0$: every 1-isomorphism is a 0-isomorphism. $\square$

So $S_0(M, M') \supseteq S_1(M, M') \supseteq S_2(M, M') \supseteq \cdots$ In terms of the Ehfrenfeucht-Fraïssé game

Theorem 2.5. *Suppose $s \in S_p(M, M')$ and $t \in S_p(M', M'')$ and* $\mathrm{dom}(t) = \mathrm{im}(s)$. *Then $u := t \circ s \in S_p(M, M'')$*

Corollary 2.6. *If $M \sim_p M'$ and $M' \sim_p M''$, then $M \sim_p M''$*

Proof. $\emptyset \in S_p(M, M')$ and $\emptyset \in S_p(M', M'')$, hence $\emptyset \in S_p(M, M'')$ $\square$

Theorem 2.7. *Suppose $s \in S_p(M, M')$. Then $s^{-1} \in S_p(M, M')$*

Proof. Since $s \in S_p(M, M')$, s is a local isomorphism from M onto M'. As s is an bijection, s^{-1} is also a bijection. $\square$

Corollary 2.8. *If $M \sim_p M'$, then $M' \sim_p M$*

$\sim_p$ is an equivalence relation

Theorem 2.9. *Let K be a Karpian family for (M, R) and (M', R'). Then $K \subseteq S_p(M, M')$ for all p. (also for all α)*

Corollary 2.10. *If M, M' are DLOs, then $S_0(M, M') = S_p(M, M')$ for all p. $M \sim_\omega M'$*

Corollary 2.11. $A \cong B \Longrightarrow A \sim_\infty B \Longrightarrow A \sim_\omega B \Rightarrow A \sim_p B$

Corollary 2.12. *$\sim_p$ and $\sim_\omega$ are equivalence relations*

Theorem 2.13. *Suppose $(\mathbb{Q}, \leq) \sim_\omega (C, R)$. Then (C, R) is a DLO*

Proof. Suppose (C, R) is not a DLO and break into cases

- R is not reflexive. As $\emptyset \in S_1(\mathbb{Q}, C)$. Spoiler chooses $b_1 \in C$ s.t. $(b_1, b_1) \notin R$. Then duplicator must choose $a_1 \in \mathbb{Q}$ s.t. $a_1 \not\leq a_1$, impossible

- R is antisymmetric. $\emptyset \in S_2(\mathbb{Q}, C)$. Let $b_1, b_2 \in C$ s.t. $b_1 R b_2$ and $b_2 R b_1$. We want to show that $b_1 = b_2$. Since $\emptyset \in S_2(\mathbb{Q}, C)$, we have a local isomorphism $\{(a_1, b_1), (a_2, b_2)\} \in S_0(\mathbb{Q}, C)$. Hence $a_1 \leq a_2$ and $a_2 \leq a_1$. As so $a_1 = a_2$. As this is a bijection, $b_1 = b_2$.

- R is transitive. $\emptyset \in S_3(\mathbb{Q}, C)$. Let $b_1, b_2, b_3 \in C$ s.t. $b_1 R b_2$ and $b_2 R b_3$. □□□ □$a_1, a_2, a_3 \in \mathbb{Q}$ s.t. $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \in S_0(\mathbb{Q}, C)$.
- R is total. □□□$S_2(\mathbb{Q}, C)$.
- (C, R) has no maximum. $\forall b_1 \in C$
- (C, R) has no minimum
- (C, R) is dense. For any $b_1 \neq b_2 \in C$ s.t. $b_1 R b_2$. $S_3(\mathbb{Q}, C)$

$\square$

Corollary 2.14. *The class of DLOs is the $\sim_\omega$-equivalence class of $(\mathbb{Q}, \leq)$*

Definition 2.15. A linear order $(C, \leq)$ is **discrete** without endpoints if $C \neq \emptyset$ and

$$\forall a \exists b : a \lhd b$$
$$\forall b \exists a : a \lhd b$$

where $a \lhd b$ means $a < b$ and not $\exists c : a < c < b$

Example 2.1. $(\mathbb{Z}, \leq)$. So is $(C, \leq)$, where

$$\begin{aligned} C = &\{\dots, -3, -2, -1\} \cup \\ &\{-1/2, -1/3, -1/4, -1/5, \dots\} \cup \\ &\{\dots, 1/5, 1/4, 1/3, 1/2\} \cup \\ &\{1, 2, 3, \dots\} \end{aligned}$$

Definition 2.16. Let $(C, <)$ be discrete. If $a \leq b \in C$, then $d(a, b)$ is the size of $[a, b) = \{x \in C : a \leq x < b\}$ or ∞ if infinite. If $a > b$, then $d(a, b) = d(b, a)$ (definition)

$d(a, b) = 0 \Leftrightarrow a = b$

Lemma 2.17. *Let $(C, <)$ and $(C', <)$ be discrete linear orders without endpoints. Suppose $a_1 < \dots < a_n$ in C and $b_1 < \dots < b_n$ in C'. Let f be the local isomorphism $f(a_i) = b_i$. Suppose that for every $1 \leq i < n$, we have*

$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or } d(a_i, a_{i+1}) \geq 2^p \leq d(b_i, b_{i+1})$$

Then f is a p-isomorphism

IDEA: a 0-isomorphism needs to respect the order. A 1-isomorphism needs to respect the order plus the relation $d(x, y) = 1$ (to make sure we can find the point). A 2-isomorphism needs to respect the order plus the relation $d(x, y) = i$ for $i = 1, 2, 3$. A 3-isomorphism needs to respect the order plus the relations $d(x, y) = i$ for $i = 1, 2, 3, \dots, 7$

this is like binary search algorithm:D

Proof.

- $a_i < a < a_{i+1}$
  - if $d(a_i, a_{i+1} = d(b_i, b_{i+1}))$
    which means they are finite

□

Theorem 2.18. *Let $(C, \leq)$ and $(C', \leq')$ be discrete linear orders without points. Then $\emptyset$is a p-equivalence from $(C, \leq)$ to $(C', \leq)$ for all p. Therefore $(C, \leq) \sim \omega(C', \leq)$.*

Remark. If $(\mathbb{Z}, \leq) \sim_\omega (C, R)$, then (C, R) is a dense linear order

Definition 2.19. Let (M, R), (M', R') be binary relations.. The **infinite Ehfrenfeucht-Fraïssé game**, denoted $\text{EF}_\infty(M, M')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are infinitely many rounds (indexed by ω)
- In the ith round, the Spoiler chooses either an $a_i \in M$ or a $b_i \in M'$
- The Duplicator responds with a $b_i \in M'$ or an $a_i \in M$ respectively
- if $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is not a local isomorphism, then the Spoiler immediately wins
- The Duplicator wins if the Spoiler has not won by the end of the game

Theorem 2.20. *TFAE*

1. *$R \sim_\infty R'$, i.e., there is a non-empty Karpian family K*
2. *Duplicator has a winning strategy for EF$_\infty(M, M')$*
3. *Spoiler does not have a winning strategy for EF$_\infty(M, M')$*

Proof. $1 \to 2$. Karpian family is the winning strategy □

3 Connections to Back-and-Forth Technique

Theorem 3.1 (Fraïssé's Theorem)**.** *Let (M, R) and (N, S) be m-ary relations, let $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then $\bar{a}$ and $\bar{b}$ are p-equivalent iff*

$$(M, R) \vDash f(\bar{a}) \Longleftrightarrow (N, S) \vDash f(\bar{b})$$

for any formula $f(\bar{x})$ with quantifier rank at most p

Proof. $\Rightarrow$. Induction on p. If $\bar{a} \sim_0 \bar{b}$, then by definition, they satisfy the same atomic formulas. Therefore they satisfy the same quantifier-free formulas.

Suppose that $\bar{a} \sim_{p+1} \bar{b}$. The formula $f := (\exists y)g(\bar{x}, y)$ has quantifier rank at most $p + 1$. So $g(\bar{x}, y)$ is a formula of quantifier rank at most p. $(M, R) \vDash f(\bar{a})$ iff there is a $c \in M$ s.t. $(M, R) \vDash g(\bar{a}, c)$. Then there is a $d \in N$ s.t. $\bar{a}c \sim_p \bar{b}d$. By IH, $(N, S) \vDash g(\bar{b}, d)$ and thus $(N, S) \vDash (\exists y)g(\bar{b}, y)$. Another direction is similar $\square$

To prove the converse we need the following lemma

Lemma 3.2. *If the arity m of a relation, and the integers n and p are fixed, there is only finite number $C(n, p)$ of p-equivalence classes of n-tuples*

$(M, R_1, \bar{a}_1), \dots, (M, R_n, \bar{a}_n)$. For any (M, R) and $\bar{a} \in M$, $\exists 1 \leq i \leq n$ s.t. $\bar{a} \sim_p \bar{a}_i$

Proof. Induction on p. If $p = 0$, then consider a set of symbols $X = \{x_1, \dots, x_n\}$. There are at most finitely many m-ary relations defined on X. Also there are at most finitely many ways to interpret the relation "=" on X. Let (M, R) and (N, S) be m-ary relations, $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. Let $R_A = R \cap A^m$ and $S_B = S \cap B^m$. If $p = 0$, $\bar{a} \sim_0 \bar{b}$ iff R_A is isomorphic to R_B via $a_i \mapsto b_i$, $i = 1, \dots, n$. So there are at most finitely many 0-equivalence classes of n-tuples

By IH, there exists relations $\{(M_k, R_k) \mid k \leq C(n + 1, p)\}$ and $\{\bar{d}_k \in M_k^{n+1} \mid k \leq C(n + 1, p)\}$ s.t. each $n + 1$-tuple is p-equivalent to some $\bar{d}_k$. Now consider an arbitrary relation (M, R) and an n-tuple $\bar{a}$, we define $[\bar{a}] = \{k \mid \exists c \in M(\bar{a}c \sim_p \bar{d}_k)\}$. For any relation (N, S) and $\bar{b} \in N^n$, $\bar{a} \sim_{p+1} \bar{b} \Leftrightarrow [\bar{a}] = [\bar{b}]$ $\square$

Proof (*continued*). We now show that if $\bar{a}$ and $\bar{b}$ satisfy the same formulas of QR at most p, then $\bar{a} \sim_p \bar{b}$.

Claim: For each p-equivalence class C, there is a formula f_C of QR p s.t. the tuples in C are exactly those satisfy f_C. $(M, R, \bar{a}) \in C \Leftrightarrow R \vDash f_C(\bar{a})$.

Induction on p. If $p = 0$, given an n-tuple $\bar{a}$, there are finitely many atomic formulas with variables $x_1, \ldots, x_n$. $n^2 + n^m$. $\{x_i = x_j \mid i, j \leq n\}$ and $\{r(x_{i_1}, \ldots, x_{i_m}) \mid i_j \leq n\}$.

Let f_C be the conjunction of those satisfied by $\bar{a}$ and negation of the others. Then f_C characterizes the 0-equivalence class of $\bar{a}$. (characterizes $R|_{\{a_1,\ldots,a_n\}}$)

Now prove $p+1$. Let $\bar{a}$ be an n-tuple of (M, R). Let $f_1(\bar{x}, y), \ldots, f_k(\bar{x}, y)$ characterize all the p-equivalence classes $C_1, \ldots, C_k$ on $n+1$-tuples. Let $\langle \bar{a} \rangle = \{i \leq k \mid (M, R) \vDash (\exists y) f_i(\bar{a}, y)\}$. $\langle \bar{a} \rangle = [\bar{a}]$

Let $f_C(\bar{x}) = \bigwedge_{i \in \langle \bar{a} \rangle} (\exists y) f_i(\bar{x}, y) \wedge \bigwedge_{i \notin \langle \bar{a} \rangle} \neg (\exists y) f_i(\bar{x}, y)$. $\bar{b} \sim_{p+1} \bar{a}$ iff $[\bar{a}] = [\bar{b}]$ iff $\langle \bar{a} \rangle = \langle \bar{b} \rangle$ iff $f_C(\bar{b})$ holds $\square$

bracket system

4 Compactness

4.1 Ultraproducts

If I is a nonempty set, a **filter** is a set F of subsets of I s.t.

- $I \in F, \emptyset \in F$
- if $X, Y \in F$, then $X \cap Y \in F$
- if $X \in F$ and $X \subset Y$, then $Y \in F$

A **filter prebase** B is a set of subsets of I contained in a filter; this means that the intersection of a finite number of elements of B is never empty. The filter F_B consisting of subsets of I containing a finite intersection of elements of B is the smallest filter containing B; we call it the filter **generated** by B. If, in addition, the intersection of two elements of B is always in B, we call B a **filter base**

Example 4.1. Let J be a set and I the set of finite subsets of J; for every $i \in I$, let $I_i = \{j : j \in I, j \supset i\}$, and let B be the set of all the I_i. Then $I_i \cap I_j = I_{i \cup J}$; B is closed under finite intersections and does contain $\emptyset$; It is therefore a filter base.

Theorem 4.1. *A filter F of subsets of I is an ultrafilter iff for every subset A of I, either A or its complement $I - A$ is in F*

Theorem 4.2. *Let U be an ultrafilter of subsets of I. If I is covered by finitely many subsets $A_1, \ldots, A_n$, then one of the A_i is in U; moreover, if the A_i are pairwise disjoint, exactly one of the A_i is in U*

Ultrafilter and Compactness

A topological space X is compact if and only if every ultrafilter in X is convergent

4.2 Applications of Compactness

Lemma 4.3. *If M and N are elementarily equivalent structures, then M can be embedded into an ultraproduct of N*

Proof. Let I be the set of injections from finite subset of M to N. If $f(\bar{a})$ is a formula with parameters $\bar{a}$ in M, $M \vDash f(\bar{a})$, let $I_{f(\bar{a})}$ denote the set of such injections s whose universe contains $\bar{a}$ and s.t. $N \vDash f(s(\bar{a}))$. The set $I_{f(\bar{a})}$ is never empty, as $M \vDash f(\bar{a})$, so $M \vDash \exists \bar{x}(f(\bar{x}) \wedge D(\bar{x}))$, where D is the conjunction of the formulas $x_i = x_j$ if $a_i = a_j$, and $x_i \neq x_j$ otherwise, and N also satisfies this formula. On the other hand, $I_{f(\bar{a})} \cap I_{g(\bar{b})} = I_{f(\bar{a}) \wedge g(\bar{b})}$, so the $I_{f(\bar{a})}$ form a filter base, which can be extended to an ultrafilter

Define a function S from M to N^U as follows: If $a \in M$, the ith coordinate of Sa is ia if i is defined at a, and any element of N otherwise (We are excluding the case of empty universes, which is trivial.) Note that $\{i : i \text{ is defined at } a\} = I_{a=a}$, and that changing the coordinates outside of $I_{a=a}$ will not change Sa modulo U, so S is well-defined. If $a = b$, then $S(a) = S(b)$ iff $\{i : N \vDash i(a) = i(b)\} = I_{a=b} \in U$. If $a \neq b$, then $I_{a \neq b} \in U$, hence S is an injection.

$N^U \vDash \phi(S(\bar{a}))$ iff $\{i : N \vDash \phi(i(\bar{a}))\} \in U$. If $M \vDash \phi(\bar{a})$, then $\{i : N \vDash \phi(i(\bar{a}))\} = I_{\phi(\bar{a})}$. □

5 Quantifier elimination

Theorem 5.1. *If two structures M and N are elementarily equivalent and ω-saturated, they are ∞-equivalent: More precisely, two tuples of the same type (over $\emptyset$), one in M and the other in N, can be matched up by an infinite back-and-forth construction*

If M is ω-saturated, then for every $\bar{a}$ of M and every p of $S_n(\bar{a})$, p is realised in M

An ω-saturated model therefore realises all absolute n-types for all n. This condition, however, is not sufficient for a model to be ω-saturated. Example: let T be the theory of discrete order without endpoints; M is ω-saturated iff it has the form $\mathbb{Z} \times \mathbb{C}$ where $\mathbb{C}$ is a dense chain without endpoints, while it realizes all pure n-types iff it has the form $\mathbb{Z} \times \mathbb{C}$ where $\mathbb{C}$ is an infinite chain

If T is a complete theory and M is an ω-saturated model of T, then every denumerable model N of T can be elementarily embedded in M. In fact, if $N = \{a_0, a_1, \dots, a_n, \dots\}$, we can successively realize, in M, the type of a_0, then the type of a_1 over a_0, ..., the type of a_{n+1} over $(a_0, \dots, a_n)$, ...

As two denumerable, elementarily equivalent, ω-saturated structures are isomorphic. Under what conditions does a complete theory T have a (unique) ω-saturated denumerable model? That happens iff for every n, $S_n(T)$ is (finite or) denumerable. (Here, we do not assume that T is denumerable)

In fact, this condition further implies that for every $\bar{a} \in M$, $S_1(\bar{a})$ is denumerable (because to say that b and c have the same type over $\bar{a}$ is to say that $\bar{a}b$ and $\bar{a}c$ have the same type over $\emptyset$) . It is clearly necessary, because a denumerable model can realize only denumerable many n-types. To see that it is sufficient: Let A_1 be a denumerable subset of M that realizes all 1-types over $\emptyset$; then let A_2 be a denumerable subset of M that realises all 1-types over finite subsets of A_1; etc. Let $A = \bigcup A_n$. A satisfies Tarski's test so it is an elementary submodel of M

Theorem 5.2. *Let T be a theory, not necessarily complete, and let F be a nonempty set of formulas $f(\bar{x})$ in the language L of T, having for free variables only $\bar{x} = (x_1, \dots, x_n)$, s.t. two n-tuples from models of T have the same type whenever they satisfy the same formulas of F. Then for every formula $g(\bar{x})$ of L in these variables, there is some $f(\bar{x})$ that is a Boolean combination of elements of F s.t. $T \vDash \forall \bar{x}(f(\bar{x}) \leftrightarrow g(\bar{x}))$*

Proof. Consider the clopen set $[g(\bar{x})]$ in $S_n(T)$. If $[g] = \emptyset$, then $[g] = [f \wedge \neg f]$, and if $[g] = S_n(T)$, then $[g] = [f \vee \neg f]$, where f is an arbitrary element of F, which is nonempty. Consider $p \in [g]$ and $q \notin [g]$. There is $f_{p,q} \in F$ s.t. $p \vDash f_{p,q}(\bar{x})$ and $q \vDash \neg f_{p,q}(\bar{x})$ If p and q are different, then they are realised by two tuples satisfying different formulas of F. Here we consider the model amalgamated by the model realising p and the model realising q. Thus such $f_{p,q}$ exists

Keeping p fixed and varying q, all the $[f_{p,q}]$ and $\neg[g]$ form a family of closed sets whose intersection is empty; $\bigcup[\neg f_{p,q}] \supset [\neg g]$. by compactness, one of its finite subfamilies must have empty intersection, meaning that for some $h_p = f_{p,q} \wedge \cdots \wedge f_{p,q_n} \in [h_p] \subset [g]$

Now when we vary p, $[g]$ is a compact set that is covered by the open sets $[h_p]$, so a finite number of them are enough to cover it; the disjunction of these h_p, module T, is equivalent to g $\square$

Note that if we want that every sentence be equivalent module T to a quantifier-free sentence; that requires, naturally, that the set of sentences without quantifiers be nonempty, meaning that the language **involves** constant symbols, or else nullary relation symbols.

A theory T is **model complete** if it has the following property: If $M, N \vDash T$ and if $N \subseteq M$, then $N \preceq M$

Two theories T_1 and T_2 in the same language L, are **companions** if every model of one can be embedded into a model of the other

Theorem 5.3. *Two theories are companions of each other iff they have the same universal consequences (a sentence being called **universal** if it is of the form $\forall x_1, \ldots, x_n \ f(x_1, \ldots, x_n)$ with f quantifier-free)*

Proof. A universal sentence f that is true in a structure is always true in its substructure; if $T_1 \vDash f$ and if there is a model of T_2 that doesn't satisfy f, it cannot be extended to a model of T_1

Conversely, suppose that T_1 and T_2 have the same universal consequences, and let $M_1 \vDash T_1$. We name each element of M_1 by a new constant, and let $D(M_1)$ be the set of all *quantifier-free* sentences in the new language that are true in M_1. If $D(M_1) \vDash f(a_1, \ldots, a_n)$, then $M \vDash \exists \bar{x} \ f(\bar{x})$, so $\forall \bar{x} \neg f(\bar{x})$ is not a consequence of T_1, and therefore not of T_2. There is therefore some model $M_2 \vDash T_2$ with $\bar{b} \in M_2$ s.t. $M_2 \vDash f(\bar{b})$. By compactness, this means that $D(M_1) \cup T_2$ is consistent, in other words, that M_1 embeds into a model of T_2 $\square$

A theory T therefore has a minimal companion, which we shall denote by $T_\forall$, which is axiomatized by the universal consequences of T.

A theory T' is a **model companion** of T if it is a companion of T that is model complete

Theorem 5.4. *A theory has at most one model companion*

Proof. Let T_1 and T_2 be model companions of T. Therefore T_1 and T_2 are companions. Let $M_1 \vDash T_1$; it embeds into a $N_1 \vDash T_2$, which embeds into a $M_2 \vDash T_1$. We get a chain $M_1 \subset N_1 \subset M_2 \subset N_2 \subset \cdots \subset M_n \subset N_n \subset \cdots$, whose limit we call P. As T_1 is model complete, the chain of M_n is elementary, and P is an elementary extension of M_1; similarly $N_1 \preceq P$. Therefore $M_1$is also a model of T_2; by symmetry T_1 and T_2 have the same models, meaning $T_1 = T_2$ $\square$

We say that T' is a **model completion** of T if it is a model companion of T and also the following condition is satisfied: if $M \models T$, embeds into a model $M_1 \models T'$ and into a model $M_2 \models T'$, then a tuple $\bar{a}$ of M satisfies the same formulas in M_1 and in M_2

Naturally a model complete theory is its own model completion, and it is clear that a theory that admits quantifier elimination is the model completion of every one of its companions. A theory is the model completion of every one of its companions iff it is the model completion of the weakest of them all, $T_\forall$

In the particular case where for every $n > 0$ we can take for F the quantifier-free formulas, we say that the theory T **eliminates quantifiers** or **admits quantifier elimination**.

Theorem 5.5. *The model completion of a universal theory* (*i.e., one that is axiomatized by universal sentences*) *admits quantifier elimination*

Proof. Let $\bar{a}$ and $\bar{b}$ satisfying the same quantifier-free formulas, be in two models M_1 and M_2 of this theory T', and let $N_1 \subseteq M_1$, $N_2 \subseteq M_2$ generated by $\bar{a}$ and $\bar{b}$ respectively. $\square$

DLO has quantifier elimination

Facts. In DLO, any 0-isomorphism is an ω-isomorphism.

Suppose qftp$(\bar{a})$ = qftp$(\bar{b})$, want tp$(\bar{a})$ = tp$(\bar{b})$

$\exists f : \langle\bar{a}\rangle_{\mathfrak{M}} \to \langle\bar{b}\rangle_{\mathfrak{N}}$ an isomorphism by Theorem 6, $f \in S_0(\mathfrak{M}, \mathfrak{N}) = S_\omega(\mathfrak{M}, \mathfrak{N})$. Then by Fraïssé's theorem, tp$(\bar{a})$ = tp$(\bar{b})$

$M \equiv N \Leftrightarrow \langle\emptyset\rangle_M \cong \langle\emptyset\rangle_N \Leftrightarrow char(M) = char(N)$

same characteristic determine same minimal subring

$M^n / \operatorname{Aut}(M/A) \cong S_n(A)$

Algebraically closed fields are axiomatized by the field axioms plus the axiom schema

$$\forall y_0, \dots, y_n \left(y_n \neq 0 \to \exists x \sum_{i=0}^{n} y_i x^i = 0 \right)$$

Lemma 5.6. *If $K \models$ ACF, then K is infinite*

Proof. If $K = \{a_1, \dots, a_n\}$, then $P(x) = 1 + \prod_{i=1}^{n}(x - a_i)$ has no root in K $\square$

If $M \vDash \mathsf{ACF}$ and K is a subfield, then K^{alg} denotes the set of $a \in M$ algebraic over K

Lemma 5.7. *Given uncountable $M, N \vDash$ ACF, suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. Suppose $\alpha \in M$. Then there is $\beta \in N$ s.t. $\mathrm{qftp}^M(\bar{a}, \alpha) = \mathrm{qftp}^N(\bar{b}, \beta)$*

Proof. Let $A = \langle \bar{a} \rangle_M$ and $B = \langle \bar{b} \rangle_N$. There is an isomorphism $f : A \to B$ and we can extend f to an isomorphism $f : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$ (Note that A and B are subrings since they are only closed under multiplication and addition). Moving N by an isomorphism we may assume $\mathrm{Frac}(A) = \mathrm{Frac}(B)$ and $f = id_{\mathrm{Frac}(A)}$. (In particular, $\bar{a} = \bar{b}$). let $K = \mathrm{Frac}(A)$. Let $K = \mathrm{Frac}(A)$

Claim. There is $\beta \in N$ with $I(\alpha) = I(\beta)$ in K

Suppose α is algebraic over K with minimal polynomial $P(x)$. Take $\beta \in N$ with $P(\beta) = 0$. Let $Q(x)$ be the minimal polynomial over β over K. Then $P(x) \in Q(x) \cdot K[x]$. But $P(x)$ is irreducible, so $P(x) = Q(x)$. Then $I(\alpha) = I(\beta)$

suppose α is transcendental, since there are only countable many solutions, there is transcendental $\beta \in N$. Then $I(\alpha) = I(\beta) = 0$

Take such β, let $I = I(\alpha) = I(\beta)$

- If $P(x) \in K[x]$, $P(\alpha) = 0 \Leftrightarrow P(x) \in I \Leftrightarrow P(\beta) = 0$
- If $P(x), Q(x) \in K[x]$, then $P(\alpha) = Q(\alpha) \Leftrightarrow (P - Q)(\alpha) = 0 \Leftrightarrow (P - Q)(\beta) = 0 \Leftrightarrow P(\beta) = Q(\beta)$
- Hence if $\varphi(x)$ is an atomic $\mathcal{L}(K)$-formula, then $M \vDash \varphi(\alpha) \Leftrightarrow N \vDash \varphi(\beta)$
- so is quantifier-free $\varphi(x) \in \mathcal{L}(K)$

$\square$

Lemma 5.8. *Lemma 5.7 holds if we replace "uncountable" with "ω-saturated"*

Proof. Take uncountable $M' \succeq M$ and $N' \succeq N$, this is possible since models of ACF are infinite. By Lemma 5.7, there is $\beta_0 \in N'$ s.t. $\mathrm{qftp}(\bar{a}, \alpha) = \mathrm{qftp}(\bar{b}, \beta_0)$. By ω-saturation, we can find $\beta \in N$ s.t. $\mathrm{tp}(\beta/\bar{b}) = \mathrm{tp}(\beta_0/\bar{b})$. Then $\mathrm{tp}(\bar{b}, \beta) = \mathrm{tp}(\bar{b}, \beta_0)$ $\square$

Theorem 5.9. *ACF has quantifier elimination*

Theorem 5.10. *Suppose $M, N \vDash$ ACF, then $M \equiv N \Leftrightarrow \mathrm{char}(M) = \mathrm{char}(N)$*

Proof. TFAE

- $M \equiv N$
- for every sentence φ, $M \vDash \varphi \Leftrightarrow N \vDash \varphi$
- for every quantifier-free sentence φ, $M \vDash \varphi \Leftrightarrow N \vDash \varphi$
- for every atomic sentence φ, $M \vDash \varphi \Leftrightarrow N \vDash \varphi$
- for any terms t_1, t_2, $M \vDash t_1 = t_2 \Leftrightarrow N \vDash t_1 = t_2$
- for any term t, $M \vDash t = 0 \Leftrightarrow N \vDash t = 0$
- for any $n \in \mathbb{Z}$, $M \vDash n = 0 \Leftrightarrow N \vDash n = 0$
- $\{n \in \mathbb{Z} : n^M = 0\} = \{n \in \mathbb{Z} : n^N = 0\}$
- $\mathrm{char}(M) = \mathrm{char}(N)$

$\square$

Corollary 5.11. *ACF_p is complete for each p*

Corollary 5.12. *$\mathbb{C}$ is completely axiomatized by ACF_0*

Lemma 5.13. *Let M be algebraically closed. Let K be a field. Let $\varphi(x)$ be an $\mathcal{L}(K)$-formula in one variable. Let $D = \varphi(M)$. Then there is a finite subset $S \subseteq K^{alg}$ s.t. $D = S$ or $D = M \setminus S$, that is, either $D \subseteq K^{alg}$ or $M \setminus K \subseteq K^{alg}$*

Proof. By Q.E., we may assume φ is quantifier-free. Then φ is a boolean combination of atomic formulas

Let $\mathcal{F} = \{S : S \subseteq_f K^{\mathrm{alg}}\} \cup \{M \setminus S : S \subseteq_f K^{\mathrm{alg}}\}$. Note that $\mathcal{F}$ is closed under boolean combinations. So we may assume φ is an atomic formula

Then $\varphi(x)$ is $(P(x) = 0)$ for some $P(x) \in K[x]$. If $P(x) \equiv 0$, then $\varphi(M) = M \in \mathcal{F}$. Otherwise $\varphi(M) \subseteq_f K^{\mathrm{alg}}$, so $\varphi(M) \in \mathcal{F}$ $\square$

Lemma 5.14. *Suppose $M \preceq N \vDash \mathsf{ACF}$ and K is a subfield of M. Suppose $c \in N$ is algebraic over K. Then $c \in M$*

Proof. Let $P(x)$ be the minimal polynomial of c over K. Let $b_1, \dots, b_n$ be the roots of $P(x)$ in M. Then

$$M \vDash \forall x \left(P(x) = 0 \to \bigvee_{i=1}^{n} x = b_i \right)$$

so the same holds in N. Then $P(c) = 0 \Rightarrow c \in \{b_1, \dots, b_n\} \subseteq M$ $\square$

Theorem 5.15. *If $M \vDash$ ACF and K is a subfield, then K^{alg} is a subfield of M and $(K^{alg})^{alg} = K^{alg}$*

Proof. Suppose $a, b \in K^{\text{alg}}$. We claim $a + b \in K^{\text{alg}}$. Let $P(x)$ and $Q(y)$ be the minimal polynomials of a, b over K. Let $\varphi(z)$ be the $\mathcal{L}(K)$-formula

$$\exists x, y(P(x) = 0 \wedge Q(y) = 0 \wedge x + y = z)$$

Then $M \vDash \varphi(a + b)$ and $\varphi(M) = \{x + y : P(x) = 0 = Q(y)\}$ is finite. Thus $a + b \in \varphi(M) \subseteq K^{\text{alg}}$

A similar argument shows K^{alg} is closed under the field operations, so K^{alg} is a subfield of M □

Theorem 5.16. *Suppose $M \vDash$ ACF and K is a subfield. TFAE*

1. $K = K^{alg}$
2. $K \vDash$ ACF
3. $K \preceq M$

Proof. $1 \to 2$: suppose $P(x) \in K[x]$ has degree > 0. Then there is $c \in M$ s.t. $P(c) = 0$. By definition, $c \in K^{\text{alg}} = K$

$2 \to 3$: quantifier elimination

$3 \to 1$. 5.14 □

Corollary 5.17. *If $M \vDash$ ACF and K is a subfield, then $K^{alg} \vDash$ ACF*

K^{alg} is called the **algebraic closure** of K. It is independent of M:

Theorem 5.18. *Let M, N be two algebraically closed fields extending K. Let $(K^{alg})_M$ and $(K^{alg})_N$ be K^{alg} in M and N, respectively. Then $(K^{alg})_M \cong (K^{alg})_N$*

6 Saturated Models

Lemma 6.1. *Let $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq \cdots$ be an increasing chain of sets indexed by $\alpha < \kappa$ for some regular cardinal κ. If $A \subseteq \bigcup_{\alpha<\kappa} S_\alpha$ and $|A| < \kappa$, then $A \subseteq S_\alpha$ for some $\alpha < \kappa$*

Proof. define $f : A \to \kappa$ by $f(x) = \min\{\alpha : x \in S_\alpha\}$. Then $|f(A)| \leq |A| < \kappa$, so $\alpha := \sup f(A) < \kappa$. For any $x \in A$, we have $f(x) \leq \alpha$ and so $x \in S_{f(x)} \subseteq S_\alpha$ □

Theorem 6.2. *If M is a structure and κ is a cardinal, there is a κ-saturated $N \succeq M$*

Proof. Build an elementary chain

$$M_0 \preceq M_1 \preceq \cdots \preceq M_\alpha \preceq \cdots$$

of length κ^+, where

1. $M_0 = M$
2. $M_{\alpha+1}$ is an elementary extension of M_α realizing every type in $S_1(M_\alpha)$
3. If α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$

Let $N = \bigcup_{\alpha<\kappa^+} M_\alpha$. If $A \subseteq N$ and $|A| < \kappa$, then $A \subseteq M_\alpha$ for some $\alpha < \kappa^+$ □

Theorem 6.3. *Suppose M is κ-saturated. If $A \subseteq M$ and $|A| < \kappa$, then every $p \in S_n(A)$ is realized in M*

Proof. Take $N \succeq M$ containing a realization $\bar{a}$ of p. We can extend the partial elementary map to$\mathrm{id}_A : A \to A$ to $f : A \cup \{a_1, \dots, a_n\} \to B$ where $B \subseteq M$. Then $\mathrm{tp}^M(f(\bar{a})/A) = \mathrm{tp}^N(\bar{a}/A) = p$, so $f(\bar{a})$ realizes p in M □

Lemma 6.4. *For any M there is an elementary extension $N \succeq M$ with the following properties:*

- *Every type over M is realized in N*
- *If $A, B \subseteq M$ and $f : A \to B$ is a partial elementary map, then there is $\sigma \in Aut(N)$ with $\sigma \supseteq f$*

Proof. Build an elementary chain

$$M = M_0 \preceq M_1 \preceq \cdots$$

of length ω, where M_{i+1} is $|M_i|^+$-saturated. Every $p \in S_n(M)$ is realized in M_1

For the second point, let $f : A \to B$ be given. Recursively build an increasing chain of partial elementary maps f_n with $\mathrm{dom}(f_n), \mathrm{im}(f_n) \subseteq M_n$ as follows:

- $f_0 = f$
- If $n > 0$ is odd, then f_n is a partial elementary map extending f_{n-1} with $\mathrm{dom}(f_n) = M_{n-1}$ and $\mathrm{im}(f_n) \subseteq M_n$

- If $n > 0$ is even, then f_n is a partial elementary map extending f_{n-1} with $\mathrm{dom}(f_n) \subseteq M_n$ and $\mathrm{im}(f_n) = M_{n-1}$

$\square$

Theorem 6.5. *If M is a structure and κ is a cardinal, there is a strongly κ-homogeneous κ-saturated $N \succeq M$*

Proof. Build an elementary chain

$$M_0 \preceq M_1 \preceq \cdots \preceq M_\alpha \preceq \cdots$$

of length κ^+. $\square$

Lemma 6.6. *Let M be a κ-saturated L-structure. For $L_0 \subseteq L$, the reduct $M \restriction L_0$ is κ-saturated*

Lemma 6.7. *Let M be an L-structure and κ be a cardinal. There is an L-structure $N \succeq M$ s.t. for every $L_0 \subseteq L$, the reduct $N \restriction L_0$ is κ-saturated and κ-strongly homogeneous*

Definition 6.8. Let T be an $L(R)$-theory

1. R is **implicitly defined** in T if for every L-structure M, there is at most one $R \subseteq M^n$ s.t. $(M, R) \vDash T$
2. R is **explicitly defined** in T if there is an L-formula $\phi(x_1, \dots, x_n)$ s.t. $T \vdash \forall \bar{x}(R(\bar{x}) \leftrightarrow \phi(\bar{x}))$

Lemma 6.9. *Suppose R is not explicitly defined in T. Then there are $M, N \vDash T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$ s.t.*

- $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$
- *$M \vDash R(\bar{a})$ and $N \vDash \neg R(\bar{b})$*

Proof. Suppose not. Let $S = \{\mathrm{tp}^L(\bar{a}) : M \vDash T, \bar{a} \in M^n\}$. For $p \in S$, one of two things happends

1. Every realization of p satisfies R
2. Every realization of p satisfies $\neg R$

Otherwise we can find a realization $\bar{a}$ satisfying R and a realization $\bar{b}$ satisfying $\neg R$, as desired.

By compactness, for each $p \in S$ there is an L-formula $\phi_p(\bar{x}) \in p(\bar{x})$ s.t. one of two things happens

1. $T \cup \{\phi_p(\bar{x})\} \vdash R(\bar{x})$
2. $T \cup \{\phi_p(\bar{x})\} \vdash \neg R(\bar{x})$

Let $\Sigma(\bar{x}) = T \cup \{\neg\phi_p(\bar{x}) : p \in S\}$. If $\Sigma(\bar{x})$ is consistent, there is $M \vDash T$ and $\bar{a} \in M^n$ satisfying $\Sigma(\bar{x})$. Let $p = \text{tp}^L(\bar{a})$, so it satisfies ϕ_p but it also satisfies $\neg\phi_p$, a contradiction

Therefore $\Sigma(\bar{x})$ is inconsistent. By compactness there are $p_1, \dots, p_n, q_1, \dots, q_m \in S$ s.t.

$$T \vdash \bigvee_{i=1}^{n} \phi_{p_i}(\bar{x}) \vee \bigvee_{i=1}^{m} \phi_{q_i}(\bar{x})$$
$$T \cup \{\phi_{p_i}(\bar{x})\} \vdash R(\bar{x}) \quad \text{for } i = 1, \dots, n$$
$$T \cup \{\phi_{q_i}(\bar{x})\} \vdash \neg R(\bar{x}) \quad \text{for } i = 1, \dots, n$$

Then $T \vdash \forall \bar{x}(R(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \phi_{p_i}(\bar{x}))$. The $\leftarrow$ is by the choice of the ϕ_{p_i}. The $\rightarrow$ is because if none of the ϕ_{p_i} hold, then one of the ϕ_{q_i} holds, and then $\neg R$ must hold.

Finally $\vee_{i=1}^{n} \phi_{p_i}(\bar{x})$ is an explicit definition of R

If $m = 0$, then $T \vdash R(\bar{x})$, if $n = 0$, then $T \vdash \neg R(\bar{x})$ □

Theorem 6.10 (beth)**.** *If R is implicitly defined in T, then R is explicitly defined in T*

Proof. **Case 1**: T is complete.

If R is not explicitly defined, we obtain $M, N \vDash T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$ with $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$ but $M \vDash R(\bar{a})$ and $N \vDash \neg R(\bar{a})$. Since T is complete, we have $M \equiv N$. By elementary amalgamation, we may find elementary embeddings $M \to N'$, $N \to N'$. Replacing M and N by N' and N', we may choose $M = N$. By Lemma 6.7, we may replace M with an elementary extension and assume M and $M \upharpoonright L$ are $\aleph_0$-saturated and $\aleph_0$-strongly homogeneous. The fact that $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$ implies that there is an automorphism $\sigma \in \text{Aut}(M \upharpoonright L)$ with $\sigma(\bar{a}) = \bar{b}$. Let $R' = \sigma(R)$. Let $M' = (M \upharpoonright L, R')$. Then σ is an isomorphism from M to M', so $M' \vDash T$. But $M' \upharpoonright L = M \upharpoonright L$. Because R is implicitly defined, $R = R'$. But then

$$\bar{a} \in R \Leftrightarrow \sigma(\bar{a}) \in \sigma(R) \Leftrightarrow \bar{b} \in R' \Leftrightarrow \bar{b} \in R$$

contradicting the fact that $M \vDash R(\bar{a})$ and $M \vDash \neg R(\bar{b})$

Case 2: T is not complete. Any completion of T implicitly defines R. By Case 1, any completion of T explicitly defines R. So in any model $M \vDash T$, there is an L-formula ϕ_M s.t. $M \vDash \forall \bar{x}(R(\bar{x}) \leftrightarrow \phi_M(\bar{x}))$

Assume R is not explicitly defined, there are $M, N \vDash T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$, with $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$ and $M \vDash R(\bar{a})$ and $N \vDash \neg R(\bar{a})$. Let T' be the L-theory obtained from T by replacing every R with ϕ_M. Then $M \vDash T'$. The type $\mathrm{tp}^L(\bar{a})$ contains the following

- $\phi_M(\bar{x})$
- sentences in T'

So $N \vDash \phi_M(\bar{b})$ and $N \vDash T'$.

Let $R' = \{\bar{c} \in N^n : N \vDash \phi_M(\bar{c})\}$. Then $(N \upharpoonright L, R') \vDash T$ because $N \vDash T'$. Therefore $R' = R$ because R is implicitly defined. But $N \vDash \phi_M(\bar{b})$ and $N \vDash \neg R(\bar{b})$, a contradiction $\square$

Theorem 6.11. *Let T be a complete theory. Then T has a countable ω-saturated model iff T is small*

Proof. $\Rightarrow$: trivial

$\Leftarrow$: Suppose $S_n(T)$ is countable for any n. Take some ω-saturated model M^+. For each finite set $A \subseteq M^+$ and type $p \in S_1(A)$, take some element $c_{A,p} \in M$ realizing p. Define an increasing chain of countable subsets $A_0 \subseteq A_1 \subseteq \cdots M^+$ as follows

- $A_0 = \emptyset$
- $A_{i+1} = A_i \cup \{c_{A,p} : A \subseteq_f A_i, p \in S_1(A)\}$

each A_i is countable, and define $M = \bigcup_{i=0}^{\infty} A_i$, which is countable

Now we only need to prove that M is ω-saturated and $M \preceq M^+$ $\square$

7 Prime models

7.1 Omitting types theorem

Theorem 7.1 (Baire Category Theorem for $S_n(A)$)**.** *Let $U_1, U_2, \dots$ be dense open sets. Then $\bigcap_{i=1}^{\infty} U_i$ is dense*

Lemma 7.2. *$S_n(A)$ is finite iff all types in $S_n(A)$ are isolated*

Proof. If each $p \in S_n(A)$ is isolated. The family $\{\{p\} : p \in S_n(A)\}$ covers $S_n(A)$, so there is a finite cover. This is impossible unless $S_n(A)$ is finite $\square$

Definition 7.3. A set $X \subseteq S_n(A)$ is **comeager** if $X \supseteq \bigcap_{i=1}^{\infty} U_i$ for some dense open sets U_i

Work in $S_\omega(T)$.

Lemma 7.4. *If $X_1, X_2, \dots$ are comeager, then $\bigcap_{i=1}^{\infty} X_i$ is comeager*

Lemma 7.5. *For any formula $\phi(x_0, \dots, x_n, y)$, there is a dense open set Z_ϕ s.t. if $M \vDash T$, $\bar{c} \in M^\omega$, $\text{tp}^M(\bar{c}) \in Z_\phi$ and $M \vDash \exists y \phi(c_0, \dots, c_n, y)$, then there is $i < \omega$ s.t. $M \vDash \phi(c_0, \dots, c_n, c_i)$*

Proof. Take $A = [\neg \exists y \phi(x_0, \dots, x_n, y)]$ and $B_i = [\phi(x_0, \dots, x_n, x_i)]$ for $i < \omega$. Let $Z_\phi = A \cup \bigcup_{i=0}^{\infty} B_i$, which is open. If $p = \text{tp}^M(\bar{c}) \in Z_\phi$ and $M \vDash \exists y \phi(c_0, \dots, c_n, y)$ then $p \notin A$, so there is $i < \omega$ s.t. $p \in B_i$ meaning $M \vDash \phi(c_0, \dots, c_n, c_i)$

It remains to show that Z_ϕ is dense. Take non-empty $[\psi] \subseteq S_\omega(T)$; we claim $Z_\phi \cap [\psi] \neq \emptyset$. Take $p = \text{tp}^M(\bar{e}) \in [\psi]$. We may assume $p \notin Z_\phi$, or we are done. Then $p \notin A$, so $M \vDash \exists y \phi(e_0, \dots, e_n, y)$. Take $b \in M$ s.t. $M \vDash \phi(e_0, \dots, e_n, b)$. Take $i > n$ large enough that x_i doesn't appear in ϕ. Let $\bar{c} = (e_0, \dots, e_{i-1}, b, e_{i+1}, e_{i+2}, \dots)$. We have $M \vDash \psi(\bar{e})$ because $\text{tp}(\bar{e}) \in [\psi]$ and therefore $M \vDash \psi(\bar{c})$, so $\text{tp}(\bar{c}) \in [\psi]$. Also $M \vDash \phi(c_0, \dots, c_n, c_i)$ ☐

Proposition 7.6. *There is a comeager set $W \subseteq S_\omega(T)$ s.t. if $\text{tp}^M(\bar{c}) \in W$, then $\{c_i : i < \omega\} \preceq M$*

Proof. Let $W = \bigcap_\phi Z_\phi$. Suppose $\text{tp}^M(\bar{c}) \in M$. Then for any $\phi(x_0, \dots, x_n, y)$, if $M \vDash \exists y \phi(c_0, \dots, c_n, y)$, then there is $i < \omega$ s.t. $M \vDash \phi(c_0, \dots, c_n, c_i)$. By Tarski-Vaught, $\{c_i : i < \omega\} \preceq M$. ☐

Lemma 7.7. *Let $p \in S_n(T)$ be non-isolated. For any $(j_1, \dots, j_n) \in \mathbb{N}^n$, there is a dense open set $V_{p,\bar{j}} \subseteq S_\omega(T)$ s.t. $\text{tp}^M(\bar{c}) \in V_{p,\bar{j}} \Leftrightarrow \text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$*

Proof. Let $V_{p,\bar{j}} = V = \bigcup_{\phi \in p} [\neg \phi(x_{j_1}, \dots, x_{j_n})]$. If $\text{tp}^M(\bar{c}) \in V$, then there is some $\phi \in p$ s.t. $M \vDash \neg \phi(c_{j_1}, \dots, c_{j_n})$, and so $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$. Conversely, if $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$, there is $\phi \in p$ s.t. $M \vDash \neg \phi(c_{j_1}, \dots, c_{j_n})$, and then $\text{tp}^M(\bar{c}) \in V$

It remains to show that V is dense. Suppose $[\psi] \subseteq S_\omega(T)$ is non-empty. Take $q = \text{tp}^M(\bar{e}) \in [\psi]$. We may assume $q \notin V$. By choice of V, $\text{tp}^M(e_{j_1}, \dots, e_{j_n}) = p$. Take m large enough so that $m \geq \max(j_1, \dots, j_n)$ and ψ is a formula in $x_0, \dots, x_m$. Let $\phi(y_1, \dots, y_n)$ be

$$\exists x_0, \dots, x_m \ \psi(x_0, \dots, x_m) \wedge \bigwedge_{i=1}^{n} (y_i = x_{j_i})$$

Then $(e_{j_1}, \dots, e_{j_n})$ satisfies ϕ, and so $\phi \in p$. As p is non isolated, there is $N \models \phi(d_1, \dots, d_n)$ with $\mathrm{tp}^N(d_1, \dots, d_n) \neq p$. By definition of ϕ there are $c_0, \dots, c_m \in N$ with $N \models \psi(c_0, \dots, c_m)$ and $(d_1, \dots, d_n) = (c_{j_1}, \dots, c_{j_n})$. Choose $c_{m+1}, c_{m+2}, \dots \in N$ arbitrarily. Then $\bar{c} = (c_i : i < \omega) \in N^\omega$ and $\mathrm{tp}(\bar{c}) \in [\psi]$, and $\mathrm{tp}(c_{j_1}, \dots, c_{j_n}) = \mathrm{tp}(d_1, \dots, d_n) \neq p$, so $\mathrm{tp}(\bar{c}) \in V$, showing $V \cap [\psi] \neq \emptyset$ ☐

Proposition 7.8. *Let $p \in S_n(T)$ be non-isolated. There is a comeager set $V_p \subseteq S_\omega(T)$ s.t. if $\mathrm{tp}^M(\bar{c}) \in V_p$, then p is not realized by a tuple in $\{c_i : i < \omega\}$*

Proof. Let $V_p = \bigcap_{\bar{j} \in \mathbb{N}^n} V_{p,\bar{j}}$. If $\mathrm{tp}^M(\bar{c}) \in V_p$, then for any $j_1, \dots, j_n \in \mathbb{N}$

$$\mathrm{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$$

☐

Theorem 7.9 (Omitting types theorem)**.** *Let Π be a countable set of pairs (p, n), where $n < \omega$ and p is a non-isolated type in $S_n(T)$. There is a countable model $M \models T$ omitting p for every $(p, n) \in \Pi$*

Proof. The set $Q = W \cap \bigcap_{(p,n)\in\Pi} V_p$ is comeager, hence non-empty. Take $\mathrm{tp}^N(\bar{c}) \in Q$. Then $M := \{c_i : i < \omega\} \preceq N$ because $\mathrm{tp}^N(\bar{c}) \in W$. For $(p, n) \in \Pi$, M omits p because $\mathrm{tp}(\bar{c}) \in V_p$ ☐

Theorem 7.10 (Ryll-Nardzewski)**.** *Let T be a complete theory in a countable language. Then T is ω-categorical iff $S_n(T)$ is finite for every $n < \omega$*

Proof. Suppose $S_n(T)$ is infinite for some n. By 7.2 there is a non-isolated $p \in S_n(T)$. By 7.9 there is a countable model $M_0 \models T$ omitting p. Take an elementary extension $M_1 \succeq M_0$ where p is realized by $\bar{a} \in M_1^n$. By Löwenheim–Skolem Theorem we may assume M_1 is countable. Then $M_1 \ncong M_0$ ☐

8 Heirs and definable types

8.1 Definable types

Definition 8.1. $p(\bar{x})$ is a **definable type** if for every formula $\varphi(\bar{x}; \bar{y})$ the set

$$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is definable, defined by some $L(M)$-formula $d\varphi(\bar{y})$

Proposition 8.2. *If T is strongly minimal and $M \vDash T$, there is a 1-type $p(x) \in S_1(M)$ s.t.*

$$\varphi(x, \bar{b}) \in p(x) \Leftrightarrow \exists^\infty a \in M : M \vDash \varphi(a, \bar{b})$$

Moreover, $p = \mathrm{tp}(c/M)$ for any $N \succeq M$ and $c \in N \setminus M$

Proof. Take $N \succ M$ and $c \in N \setminus M$; let $p(x) = \mathrm{tp}(c/M)$. We must show that

$$N \vDash \varphi(c, \bar{b}) \Leftrightarrow \exists^\infty a \in M : M \vDash \varphi(a, \bar{b})$$

$\Rightarrow$: if not, $N \vDash \varphi(c, \bar{b})$ but $\varphi(M, \bar{b})$ is finite, then $c \in M$

$\Leftarrow$: if $N \vDash \neg\varphi(c, \bar{b})$, then $\neg\varphi(M, \bar{b})$ is infinite and so $\varphi(M, \bar{b})$ is finite □

$p(x)$ is called the **transcendental 1-type**

Proposition 8.3. *If T is strongly minimal*

1. *T eliminates the $\exists^\infty$ quantifier*

2. *If $M \vDash T$, the transcendental 1-type $p \in S_1(M)$ is definable*

Proof. 1. For any $\varphi(x, y)$, there is $n_\varphi < \omega$ s.t. for every $M \vDash T$ and $\bar{b} \in M$

$$\left|\varphi(M, \bar{b})\right| < n_\varphi \text{ or } \left|\neg\varphi(M, \bar{b})\right| < n_\varphi$$

2. For each $\varphi(x, \bar{y})$, $d\varphi(\bar{y})$ is the formula $\exists^\infty x\ \varphi(x, \bar{y})$

□

Corollary 8.4. *If $p \in S_1(M)$ and M is strongly minimal, then p is definable*

Definition 8.5. A theory T is **stable** if all n-types over models are definable

8.2 Heirs and strong heirs

Suppose $M \preceq N$ and $p \in S_n(M)$. An **extension** or **son** of p is $q \in S_n(N)$ with $q \supseteq p$, i.e., $p = q \restriction M$

Definition 8.6 (Heirs). $q \in S_n(N)$ is an **heir** of p, written $p \sqsubseteq q$, if for any $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$ with $\bar{b} \in M$ and $\bar{c} \in N$, there is $\bar{c}' \in M$ with $\varphi(\bar{x}, \bar{b}, \bar{c}) \in p(\bar{x})$

Lemma 8.7. *Suppose $M_1 \preceq M_2 \preceq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$, with $p_1 \subseteq p_2 \subseteq p_3$*

1. *If $p_1 \sqsubseteq p_2 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_3$*

2. *If $p_1 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_2$*

Definition 8.8. If $p \in S_n(M)$, then (M, dp) is the expansion of M be relation symbols $d\varphi(\bar{y})$ for each $\varphi(\bar{x}, \bar{y})$, interpreted as follows:

$$(M, dp) \vDash d\varphi(\bar{b}) \Leftrightarrow \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$$

Remark. p is definable iff the new relations in (M, dp) are definable in the old structure M

Remark. The class of structures of the form (M, dp) with $M \vDash T$ and $p \in S_n(M)$ is an elementary class, axiomatized by T plus the following:

$$\forall \bar{y}_1 \dots \bar{y}_m \left(\bigwedge_{i=1}^{m} d\varphi_i(\bar{y}) \to \exists \bar{x} \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, \bar{y}_i) \right) \text{ for formulas } \varphi_1(\bar{x}, \bar{y}_1), \dots, \varphi_n(\bar{x}, \bar{y}_n)$$

$$\forall \bar{y}(d\varphi(\bar{y}) \vee d\neg\varphi(\bar{y})) \text{ for each formula } \varphi(\bar{x}, \bar{y})$$

Any model of such theory has an underlying p

Lemma 8.9. *If $(M, dp) \preceq (N, dq)$, then $M \preceq N$ and $p \sqsubseteq q$*

Proof. $(N, dq) \succeq (M, dp)$ implies $N \succeq M$. Then:

- $q \supseteq p$: if $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ (with $\bar{b} \in M$), then $(M, dp) \vDash d\varphi(\bar{b})$, so $(N, dq) \vDash d\varphi(\bar{b})$, and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$
- $q \sqsupseteq p$: suppose $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$, with $\bar{b} \in M$ and $\bar{c} \in N$. Then $(N, dq) \vDash d\varphi(\bar{b}, \bar{c})$, and $(N, dq) \vDash \exists \bar{z}\ d\varphi(\bar{b}, \bar{z})$. Then $(M, dp) \vDash \exists \bar{z}\ d\varphi(\bar{b}, \bar{z})$

□

Corollary 8.10. *If $p \in S_n(M)$, then there is $M_0 \preceq M$ with $|M_0| \leq |T|$, s.t. $p \sqsupseteq (p \restriction M_0)$*

Proof. Apply downward Löwenheim–Skolem theorem to (M, dp) to find $(M_0, dq) \preceq (M, dp)$ with $|M_0| \leq |T|$. Then $q = p \restriction M_0$ and $p \sqsupseteq q$ □

Definition 8.11. If $M \preceq N$ and $p \in S_n(M)$ and $q \in S_n(N)$, then q is a **strong heir** of p if $(N, dq) \succeq (M, dp)$

Proposition 8.12 (Types have heirs)**.** *Suppose $M \preceq N$ and $p \in S_n(M)$*

1. *There is $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p*
2. *There is $q \in S_n(N)$ an heir of p*

Proof. 1. Let $\bar{c}$ be an infinite tuple enumerating N. Then $\text{tp}^L(\bar{c}/M)$ is finitely satisfiable in M, hence finitely satisfiable in the expansion (M, dp). Therefore it is satisfied in some $(N', dq) \succeq (M, dp)$. So there is $\bar{e}$ in N' with $\text{tp}^L(\bar{e}/M) = \text{tp}^L(\bar{c}/M)$. Then the map $f(c_i) = e_i$ is an L-elementary embeddings of N into N extending $\text{id}_M : M \to M$. Moving N' by an isomorphism, we may assume $N' \succeq N$

2. Take $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p. Let $q = q' \restriction N$. Then $q' \supseteq q \supseteq p$ and $q' \sqsupseteq p$, so $q \sqsupseteq p$.

$\square$

8.3 Heirs and definable types

Proposition 8.13. *Let $p \in S_n(M)$ be definable and $N \succeq M$*

1. *p has a unique heir $q \in S_n(N)$*
2. *For $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N$*

$$\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Leftrightarrow N \vDash d_p\varphi(\bar{b}) \qquad (*)$$

3. *In particular, q is definable with $d_q\varphi = d_p\varphi$ for all φ*

Proof. **Claim**. If $q \in S_n(N)$ and $q \sqsupseteq p$, then q satisfies (*)

Take $\bar{a} \in N' \succeq N$ realizing q. If (*) fails then

$$(\varphi(\bar{x}, \bar{b})) \in q(\bar{x}) \not\Leftrightarrow N \vDash d_p\varphi(\bar{b})$$
$$N' \vDash \neg(\varphi(\bar{a}, \bar{b}) \leftrightarrow d_p\varphi(\bar{b}))$$
$$\neg(\varphi(\bar{x}, \bar{b}) \leftrightarrow d_p\varphi(\bar{b})) \in q(\bar{x})$$

As $q \sqsupseteq p$, there is $b' \in M$ s.t.

$$\neg(\varphi(\bar{x}, \bar{b}') \leftrightarrow d_p\varphi(\bar{b}')) \in p(\bar{x})$$
$$N' \vDash \neg(\varphi(\bar{a}, \bar{b}') \leftrightarrow d_p\varphi(\bar{b}'))$$
$$\varphi(\bar{x}, \bar{b}') \in p(\bar{x}) \not\Leftrightarrow M \vDash d_p\varphi(\bar{b}')$$

a contradiction

There is at least one heir, and at most one heir satisfying (*) $\square$

Example 8.1. Suppose T is strongly minimal and $M \preceq N$ are models of T. Let p and q be the transcendental 1-types over M and N. For any $\varphi(x, \bar{y})$

$$d_p\varphi(\bar{y}) \equiv (\exists^\infty x\ \varphi(x, \bar{y})) \equiv d_q\varphi(\bar{y})$$

so q is the unique heir of p

Proposition 8.14. *TFAE for $p \in S_n(M)$*

1. *p is definable*

2. *For every $N \succeq M$, p has a unique heir over N*

Proof. Suppose p has unique heirs. Then for any $N \succeq M$, p has at most one strong heir over N. Therefore there is at most one way to expand N to an elementary extension of (M, dp). Then the elementary diagram (M, dp) implicityly defines the relations $d\varphi$. By Beth's implicit definability theorem, (M, dp) is a expansion of M by definable relations, meaning p is definable $\square$

Proposition 8.15. *Suppose $M_1 \preceq M_2 \preceq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$ with $p_1 \subseteq p_2 \subseteq p_3$. Suppose p_1 is definable. Then $p_1 \sqsubseteq p_2 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_3$*

Proof. We only need to show the implication $p_1 \sqsubseteq p_3 \Rightarrow p_2 \sqsubseteq p_3$. Suppose $p_1 \sqsubseteq p_3$. Take $p_2' \sqsupseteq p_1$ and $p_3' \sqsupseteq p_2'$. By the uniqueness of heirs of definable types, $p_2' = p_2$ and p_2 is definable. Then $p_3' = p_3$ $\square$

8.4 Types in ACF

A **positive quantifier free formula** is a quantifier-free formula that doesn't use $\neg$

Fix a model $M \vDash \mathsf{ACF}$

Definition 8.16. A set $V \subseteq M^n$ is an **algebraic set** if

$$V = \varphi(M^n; \bar{b}) = \{\bar{a} \in M^n : M \vDash \varphi(\bar{a}, \bar{b})\}$$

where φ is positive quantifier free.

Remark. V is an algebraic set iff V is defined by finitely many polynomial equations

$$V = \{\bar{a} \in M^n : P_1(\bar{a}) = \cdots = P_m(\bar{a}) = 0\}$$

Lemma 8.17.

1. *M^n and $\emptyset$ are algebraic sets*

2. *If $V, W \subseteq M^n$ are algebraic sets, then $V \cap W$ and $V \cup W$ are algebraic sets*

3. *Any finite subset of M^n is an algebraic set*

Fact 8.18 (Quantifier elimination)**.** *Every definable set $D \subseteq M^n$ is a finite boolean combination of algebraic sets*

Fact 8.19 (Consequence of Hilbert's basis theorem)**.** *The class of algebraic sets has the descending chain condition (DCC): there is no infinite chain of algebraic sets* $V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$

Corollary 8.20. *If $\mathcal{S}$ is a non-empty collection of algebraic sets, then $\mathcal{S}$ contains at least one minimal element*

Corollary 8.21. *An infinite intersection $\bigcap_{i \in I} V_i$ of algebraic sets is an algebraic set*

Corollary 8.22. *If $S \subseteq K[\bar{x}]$ is any set of polynomials, possibly infinite, then the subset of M^n defined by S is an algebraic set. All algebraic sets arise this way*

Corollary 8.23 (Noetherian induction)**.** *Let $\mathcal{S}$ be a class of algebraic sets. Suppose the following holds*

If X is an algebraic set, and every algebraic set $Y \subsetneq X$ is in $\mathcal{S}$, then $X \in \mathcal{S}$

Then every algebraic set is in $\mathcal{S}$

Definition 8.24. An algebraic set V is **reducible** if $V = W_1 \cup W_2$ for algebraic sets $W_1, W_2 \subsetneq V$. A **variety** is a non-empty irreducible algebraic set

Remark. If V is an algebraic variety, then the set of algebraic proper subsets of V is closed under finite unions

Proposition 8.25. *If V is an algebraic set, then V is a finite union of varieties*

Proof.

- $V = \emptyset$: V is a union of zero varieties
- V is irreducible: V is a union of one variety
- V is reducible: $V = X \cup Y$ where $X, Y \subsetneq V$. By Noetherian induction!

$\square$

Definition 8.26. The **generic type** of V is the type generated by the following formulas

1. $x \in V$
2. $x \notin W$ for each algebraic proper subset $W \subsetneq V$

We will write this type as $p_V(\bar{x})$

Note that $x \in V$ and $x \notin W$ is all definable

Proposition 8.27. *Let V be a variety*

1. $p_V(\bar{x})$ *is a consistent complete type*

2. *If W is an algebraic set, then $p_V(\bar{x}) \vdash \bar{x} \in W \Leftrightarrow W \supseteq V$*

Proof. Finite satisfiability: given finitely many proper algebraic subsets $W_1, \dots, W_m \subsetneq V$, we have $V \supsetneq \bigcup_{i=1}^{m} W_i$, so there is $\bar{a} \in V$ and $\bar{a} \notin W_i$ for $1 \leq i \leq m$

1. If $W \supseteq V$, then $p_V(\bar{x}) \vdash \bar{x} \in V \vdash \bar{x} \in W$. If $W \not\supseteq V$, then $(W \cap V) \subsetneq V$, so $p_V(\bar{x}) \vdash \bar{x} \notin W \cap V$. But $p_V(\bar{x}) \vdash \bar{x} \in V$ so $p_V(\bar{x}) \vdash \bar{x} \notin W$

Completeness: by 2, for any positive quantifier-free formula $\varphi(\bar{x})$

$$p_V(\bar{x}) \vdash \varphi(\bar{x}) \text{ or } p_V(\bar{x}) \vdash \neg\varphi(\bar{x})$$

□

Theorem 8.28. *The map $V \mapsto p_V$ is a bijection from the set of varieties $V \subseteq M^n$ to $S_n(M)$*

Proof. Injectivity: suppose V, W are varieties and $V \neq W$. WLOG, $V \not\subseteq W$. Then $p_W(\bar{x}) \vdash \bar{x} \in W$ but $p_V(\bar{x}) \nvdash \bar{x} \in W$, so $p_V \neq p_W$

Surjectivity: fix $p \in S_n(M)$. Take V a minimal algebraic set s.t. $p(\bar{x}) \vdash \bar{x} \in V$. (There is at least one such V, namely M^n). V is non-empty because p is consistent. If V is reducible as $V = X \cup Y$ for smaller algebraic sets X, Y, then $p(\bar{x}) \vdash \bar{x} \in X$ or $p(\bar{x}) \vdash \bar{x} \in Y$ by completeness, contradicting the choice of V. Thus V is a variety. By choice of V, $p(\bar{x}) \vdash \bar{x} \in V$. □

Proposition 8.29. *$N \succeq M$, let $V \subseteq M^n$ be a variety, defined by a formula φ*

1. *φ defines a variety $V_N \subseteq N^n$*

2. *V_N depends only on V, not on the choice of φ*

Proof. Take ψ a positive quantifier-free formula defining V. Then $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is satisfied by M, and therefore by N. Let $V_N = \psi(N)$. As ψ is positive quantifier free, V_N is an algebraic set. As $M \models \exists \bar{x}\psi(\bar{x})$, V_N is non-empty. If $V_N = W_1 \cup W_2$ where W_1, W_2 are algebraic proper subsets of V_N defined by $\theta_i(\bar{x}, \bar{b}_i)$ for some positive quantifier-free L-formula θ_i and tuple of parameters $\bar{b}_i \in N$. Then

$$N \models \exists \bar{y}_1 \bar{y}_2 \left(\forall \bar{x} \left(\psi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{2} \theta_i(\bar{x}, \bar{y}_i) \right) \wedge \bigwedge_{i=1}^{2} \exists \bar{x}(\psi(\bar{x}) \wedge \neg\theta_i(\bar{x}, \bar{y}_i)) \right)$$

which implies V is reducible □

Theorem 8.30. *Let $M \preceq N$ be models of ACF. Let $V \subseteq M^n$ be a variety, and let $V_N \subseteq N^n$ be its extension. Then $p_{V_N} \in S_n(N)$ is the unique heir of $p_V \in S_n(M)$*

Proof. Let $q \in S_n(N)$ be an heir of p_V. Let φ be an $L(M)$-formula defining V and V_N. Then $\varphi(\bar{x}) \in p_V(\bar{x}) \subseteq q(\bar{x})$, so $q(\bar{x}) \vdash \bar{x} \in V_N$. Suppose $q(\bar{x}) \nvdash \bar{x} \notin W$ for some algebraic $W \subsetneq V_N$, $q(\bar{x}) \vdash \bar{x} \in W$. Let $\psi(\bar{x}, \bar{b})$ be a positive quantifier-free formula defining W. Let $\theta(\bar{b})$ be the $L(M)$-formula

$$\forall \bar{x}(\psi(\bar{x}, \bar{b}) \to \varphi(\bar{x})) \wedge \exists \bar{x}(\varphi(\bar{x}) \wedge \neg\psi(\bar{x}, \bar{b}))$$

which says $\psi(M^n, \bar{b}) \subsetneq \varphi(M^n)$. $N \vDash \theta(\bar{b})$ since $W \subsetneq V$. Then $q(\bar{x}) \vdash \psi(\bar{x}, \bar{b}) \wedge \theta(\bar{b})$. Because $q \sqsupseteq p_V$, there is $\bar{b}' \in M$ s.t.

$$p_V(\bar{x}) \vdash \psi(\bar{x}, \bar{b}') \wedge \theta(\bar{b}')$$

Thus we find an algebraic proper subset of V □

General fact: If $q \sqsupseteq p$, suppose $\forall \bar{b}(\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in p(\bar{x}))$, then $\forall \bar{b} \in N$, $\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in q(\bar{x})$

8.5 1-types in DLO

9 Stable Theories

9.1 Strong heirs from ultrapowers

Definition 9.1. If $p \in S_n(M)$, I set, $\mathcal{U}$ ultrafilter on I, $M^{\mathcal{U}} = M^I/\mathcal{U}$. The **ultrapower type** $p^{\mathcal{U}} \in S_n(M^{\mathcal{U}})$ is the strong heir of p s.t. $(M^{\mathcal{U}}, dp^{\mathcal{U}}) = (M, dp)^{\mathcal{U}}$

$p^{\mathcal{U}}$ is a strong heir of p
If $\varphi(\bar{x}, \bar{y}) \in L$, $\bar{b} \in M^{\mathcal{U}}$ represented by $(\bar{b} : i \in I) \in M^I$,
$\varphi(\bar{x}, \bar{b}) \in p^{\mathcal{U}} \Leftrightarrow (M, dp)^{\mathcal{U}} \vDash d\varphi(\bar{b}) \Leftrightarrow \{i \in I \mid (M, dp) \vDash d\varphi(\bar{b}_i)\} \in \mathcal{U} \Leftrightarrow \{i \in I \mid \varphi(x, \bar{b}_i) \in p(x)\} \in \mathcal{U}$

Proposition 9.2. *Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $q \sqsupseteq p$. Then there is I, ultrafilter $\mathcal{U}$ on I s.t. (for some copy of $M^{\mathcal{U}}$, moved by isomorphism), $M \preceq N \preceq M^{\mathcal{U}}$, $p \subseteq q \subseteq p^{\mathcal{U}}$*

Proof. Let $I = \{f : N \to M \mid f \supseteq \mathrm{id}_M\}$.
Note that if $\phi(\bar{x}, \bar{b}) \in q(\bar{x})$, $\bar{b} \in N$, there is $f \in I$, $\phi(\bar{x}, f(\bar{b})) \in p(\bar{x})$. (has some duplicate variable problem, if $b_1 = b_2$, but $c_1 \neq c_2$, but maybe we could take some equivalent formulas)

For each $\phi(\bar{x},\bar{b})$, $\bar{b}\in N$, let $S_{\varphi,\bar{b}}=\{f\in I\mid\phi(\bar{x},f(\bar{b}))\in p(\bar{x})\}$. Let $\mathcal{F}=\{S_{\phi,\bar{b}}\mid\phi(\bar{x},\bar{b})\in q(\bar{x})\}$

Claim $\mathcal{F}$ has F.I.P

Suppose $\phi_i(\bar{x},\bar{b}_i)\in q(\bar{x})$, $1\le i\le m$. So $\bigwedge_{i=1}^m\phi_i(\bar{x},\bar{b}_i)\in q(\bar{x})$, then there is $f\in I$ s.t. $\bigwedge_{i=1}^m\phi_i(\bar{x},f(\bar{b}_i)\in p(\bar{x}))$. Then $f\in S_{\varphi_i,\bar{b}_i}$, so $\bigcap_{i=1}^n S_{\phi_i,b_i}\ne\emptyset$

Thus there is $\mathcal{U}\supseteq\mathcal{F}$. Form $M^{\mathcal{U}}$, $p^{\mathcal{U}}$. Let $g:N\to M^{\mathcal{U}}$ as follows. If $c\in N$, $g(c)=[(f(c):f\in I)]$. Note if $c\in M$, then $f(c)=c$ for all f, and so $g\mid M=\mathrm{id}_M$

For any $\phi(\bar{x},\bar{y})$, $\bar{b}\in N$, $\phi(\bar{x},\bar{b})\in q(\bar{x})\Rightarrow S_{\phi,\bar{b}}\in\mathcal{F}\Rightarrow S_{\phi,\bar{b}}\in\mathcal{U}\Rightarrow\{f\in I\mid\phi(\bar{x},f(\bar{b}))\in p(\bar{x})\}\in\mathcal{U}\Leftrightarrow\phi(\bar{x},g(\bar{b}))\in p^{\mathcal{U}}$

So $g:N\to M^{\mathcal{U}}$, $\phi(\bar{x},\bar{b})\in q(\bar{x})\Rightarrow\phi(\bar{x},g(\bar{b}))\in p^{\mathcal{U}}$. $N\vDash\phi(\bar{b})\Rightarrow M^{\mathcal{U}}\vDash\phi(g(\bar{b}))$. WLOG, $N\preceq M^{\mathcal{U}}$ and $g\supseteq\mathrm{id}_N$. $\phi(\bar{x},\bar{b})\in q(\bar{x})\Rightarrow\phi(\bar{x},\bar{b})\in p^{\mathcal{U}}$. $q\subseteq p^{\mathcal{U}}$. □

Since we can prove compactness by ultrapower. Everything we get from compactness can be got by some ultrapower

Corollary 9.3. *Every heir of p extends to a strong heir of p*

9.2 Stability

Definition 9.4. If α is an ordinal, then $2^\alpha=$ strings of length α in alphabet $\{0,1\}$

Definition 9.5. $\varphi(\bar{x},\bar{y})$ be a formula. For α an ordinal, take variables $\bar{x}_\sigma$ for $\sigma\in 2^\alpha$, $\bar{y}_\tau$ for $\tau\in 2^{<\alpha}$.

$D_\alpha=\{\varphi(\bar{x}_\sigma,\bar{y}_\tau):\sigma\text{ extends }\tau 0\}\cup\{\neg\varphi(\bar{x}_\sigma,\bar{y}_\tau):\sigma\text{ extends }\tau 1\}$

$\varphi(\bar{x},\bar{y})$ has the **dichotomy property** if

1. D_ω is consistent
2. D_n is consistent for all $n\in\omega$
3. D_α is consistent for all α

1-3 are equivalent

Example 9.1. D_2 is $\varphi(x_{00},y)$, $\varphi(x_{00},y_0)$, $\varphi(x_{01},y)$, $\neg\varphi(x_{01},y_0)$ and so on

y / \ y0 y1 / \ / \ x00 x01 x10 x11

Proposition 9.6. *Fix $T,\mathbb{M}$, and an integer $n<\omega$. Suppose there is a small model $M\preceq\mathbb{M}$ and a type $p\in S_n(M)$ that is not definable, then some formula $\varphi(x_1,\dots,x_n,\bar{y})$ has the dichotomy property*

Proof. Because p is not definable, there is an $N \succeq M$, $q_1, q_2 \in S_n(N)$, $q_1, q_2 \sqsupseteq p$ and $q_1 \neq q_2$. There is $\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x}) \setminus q_2(\bar{x})$, $\bar{b} \in N$.

Claim If $M' \succeq N$, $p' \in S_n(M')$, $p' \sqsupseteq p$, then there is some $N' \succeq M'$, $q_1', q_2' \in S_n(N')$, $q_1', q_2' \supseteq p'$, $q_1', q_2' \sqsupseteq p$. and there is $\bar{b}' \in N'$, $\varphi(\bar{x}, \bar{b}') \in q_1'$, $\neg\varphi(\bar{x}, \bar{b}') \in q_2$

There is $M^{\mathcal{U}}$ s.t. $M \preceq M' \preceq M^{\mathcal{U}}$, $p \subseteq p' \subseteq p^{\mathcal{U}}$. Then $M' \preceq M^{\mathcal{U}} \preceq N^{\mathcal{U}}$ and $p \sqsubseteq p^{\mathcal{U}} \sqsubseteq q_i^{\mathcal{U}}$ for $i = 1, 2$. Take $N' = N^{\mathcal{U}}$, $q_i' = q_i^{\mathcal{U}}$, and $\bar{b}'$ to be the image of $\bar{b}$ under the elementary embedding $N \to N^{\mathcal{U}}$

Recursively build a tree of (M,p) / \ (M0,p0) (M1,p1)

build $(M_\tau, p_\tau, \varphi(x, b_\tau))$ for $\tau \in 2^{<\omega}$

$M_{()} = M$, $p_\tau \sqsupseteq p$. $M_{\tau 0} = M_{\tau 1} \succeq M_\tau$, $\bar{b}_\tau \in M_{\tau 0}$, $\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 0}(\bar{x})$, $\neg\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau 1}(\bar{x})$.

Then φ has dichotomy □

working in $\mathbb{M}$

Proposition 9.7. *If some $\varphi(x_1, \ldots, x_n, \bar{y})$ has dichotomy property, then for every cardinal $\lambda \geq \aleph_0$, there is $A \subseteq \mathbb{M}$, $|A| \leq \lambda$, $|S_n(A)| > \lambda$*

Proof. take smallest cardinal μ s.t. $2^\mu > \lambda$, $\mu \leq \lambda$. note that $|2^{<\mu}| = \left|\bigcup_{\alpha<\mu} 2^\alpha\right| \leq \lambda$.

φ has dichotomy proposition, so D_μ is consistent. In the monster, there are $\bar{a}_\sigma$ for $\sigma \in 2^\mu$, $\bar{b}_\tau$ for $\tau \in 2^{<\mu}$ s.t. if σ extends $\tau 0$ then $\mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau)$ and if σ extends $\tau 1$ then $\mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$. Let $A = \{\bar{b}_\tau : \tau \in 2^{<\mu}\}$. Then $|A| \leq \lambda$ but $\mathrm{tp}(a_\sigma/A) \neq \mathrm{tp}(a_{\sigma'}/A)$ for $\sigma \neq \sigma'$. Thus $|S_n(A)| \geq 2^\mu > \lambda$. □

Lemma 9.8. *for λ infinite, TFAE*

1. *$\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $\forall n$, $|S_n(A)| \leq \lambda$*
2. *$\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$*

Proof. $2 \to 1$: By induction on n, $|S_{n-1}(A)| \leq \lambda$. Then we can find $\bar{b}_\alpha \in \mathbb{M}^{n-1}$ for $\alpha < \lambda$ s.t.

$$S_{n-1}(A) = \{\mathrm{tp}(\bar{b}_\alpha/A) : \alpha < \lambda\}$$

For each α, $\left|A\bar{b}_\alpha\right| \leq \lambda \Rightarrow \left|S_1(A\bar{b}_\alpha)\right| \leq \lambda$. So we can find $c_{\alpha,\beta} \in \mathbb{M}$ for $\beta < \lambda$ s.t.

$$S_1(A\bar{b}_\alpha) = \{\mathrm{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha) : \beta < \lambda\} (\text{for } \alpha < \lambda)$$

Claim: if $p \in S_n(A)$ then $p = \mathrm{tp}(\bar{b}_\alpha c_{\alpha,\beta}/A)$ for some $\alpha, \beta < \lambda$

Take $(\bar{b}', c') \in \mathbb{M}^n$ realizing p. Then $\mathrm{tp}(\bar{b}'/A) = \mathrm{tp}(\bar{b}_\alpha/A)$ for some $\alpha < \lambda$. Moving $(\bar{b}', c')$ by an automorphism in $\mathrm{Aut}(\mathbb{M}/A)$, we may assume $\bar{b}' =$

$\bar{b}_\alpha$. Then $\text{tp}(c/A\bar{b}_\alpha) = \text{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha)$ for some $\beta < \lambda$. Moving c' by an automorphism in $\text{Aut}(\mathbb{M}/A\bar{b}_\alpha)$, we may assume $c' = c_{\alpha,\beta}$

By the claim, $|S_n(A)| \leq \lambda^2 = \lambda$ $\square$

Definition 9.9. T is λ-stable if $|A| \leq \lambda \Rightarrow |S_1(A) \leq \lambda|$

Proposition 9.10. *If $\lambda \geq |L|$, TFAE*

1. *$\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $\forall n$, $|S_n(A)| \leq \lambda$*

2. *$\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$*

3. *If $M \preceq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_1(M)| \leq \lambda$*

4. *If $M \preceq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$*

Proof. $3 \to 1$: Let $A \subseteq \mathbb{M}$, $|A| \leq \lambda$, using downward Löwenheim–Skolem Theorem to get a model $A \subseteq M \preceq \mathbb{M}$ and $|A| + |L| = |M|$

$4 \to 2$: similar $\square$

Example 9.2. strongly minimal theory is λ-stable for $\lambda \geq |L|$

Given $A \subseteq \mathbb{M}$, $\exists M \preceq \mathbb{M}$, $|M| \leq \lambda$. $S_1(M)$=const types + transcendental types, so $|S_1(M)| = |M| + 1$

λ-stable $\Rightarrow$ no φ has D.P $\Rightarrow$ all types are definable

Lemma 9.11. *Suppose $\forall M \preceq \mathbb{M}$, $\forall p \in S_1(M)$ is definable. Then T is λ-stable for some λ*

Proof. Take $\lambda = 2^{|L|} > |L|$. Suppose $M \preceq \mathbb{M}$ and $|M| \leq \lambda$. $p \in S_1(M)$ is determined by $\varphi \in L \mapsto d_p\varphi \in L(M)$, $|S_1(M)| \leq |L(M)|^{|L|} \leq \lambda^{|L|} = 2^{|L|}$ $\square$

Theorem 9.12. *TFAE*

1. *T is λ-stable for some λ*

2. *no formula $\varphi(\bar{x}, \bar{y})$ has D.P.*

3. *no $\varphi(x, \bar{y})$ has D.P.*

4. *$M \models T$, $p \in S_1(M) \Rightarrow p$ is definable*

5. *$M \models T$, $p \in S_n(M) \Rightarrow p$ is definable*

Proof. $5 \to 1$: Let $\lambda = 2^{|L|}$. Note that $\lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|} = \lambda$. Take $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$. By downward Löwenheim–Skolem Theorem, there is a small model $M \preceq \mathbb{M}$ with $A \subseteq M$ and $|M| \leq \lambda$. Every n-type over A extends to an n-type over M, so $|S_n(A)| \leq |S_n(M)|$. It remains to show $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$. (That is, we may assume A is a small model M). By (5), every n-type over M is definable. A definable type is determined by the map $\varphi \mapsto d\varphi$, which is a function from L-formulas to $L(M)$-formulas. So the number of (definable) types over M is at most $|L(M)|^{|L|} \leq \lambda^{|L|} = \lambda$ $\square$

9.3 The dichotomy property and definability of types

We will prove

If no formula has the dichotomy property, then all types over *arbitrary sets* are definable

9.3.1 The dichotomy property

Fix a complete theory T and monster model $\mathbb{M}$. Fix a formula $\varphi(\bar{x}; \bar{y})$

Definition 9.13. "D_α is consistent" if there are $(\bar{a}_\sigma : \sigma \in 2^\alpha)$ and $(\bar{b}_\tau : \tau \in 2^{<\alpha})$ s.t.

$$\sigma \sqsupseteq \tau 0 \Rightarrow \mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau)$$
$$\sigma \sqsupseteq \tau 1 \Rightarrow \mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$$

Remark. "D_α" is the name for the set of formulas $\{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 0\} \cup \{\neg\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 1\}$

Lemma 9.14. *If D_n is consistent for all $n < \omega$, then D_ω is consistent*

Proof. Let F be a finite fragment of D_ω. By compactness, it suffices to show that F is consistent. Take n bigger than the length of τ for any $\bar{y}_\tau$ appearing in F. Because D_n is consistent, there are $(\bar{a}^0\sigma : \sigma \in 2^n)$ and $(\bar{b}^0_\tau : \tau \in 2^{<n})$. Define $\square$

9.3.2 φ-types

Continue to fix $T, \mathbb{M}, \varphi(\bar{x}; \bar{y})$. Let n be the length of the variable tuple $\bar{x}$

Definition 9.15. If $B \subseteq \mathbb{M}$ is a set and $\bar{a} \in \mathbb{M}^n$, then $\mathrm{tp}^\varphi(\bar{a}/B)$ is the partial type

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \varphi(\bar{a}, \bar{b})\} \cup \{\neg\varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \neg\varphi(\bar{a}, \bar{b})\}$$

Definition 9.16. $S_\varphi(B) = \{\text{tp}^\varphi(\bar{a}/B) : \bar{a} \in \mathbb{M}^n\}$

Definition 9.17. A φ-type $p \in S_\varphi(B)$ is **definable** if there is an $L(B)$-formula $\psi(\bar{y})$ s.t.

$$\forall \bar{b} \in B, \quad \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \vDash \psi(\bar{b})$$

Theorem 9.18. *Suppose φ does not have the dichotomy property. Then every φ-type over any set is definable*

9.3.3 Proof of Theorem 9.18

Definition 9.19. Let $\Sigma(\bar{x})$ be a small set of $L(\mathbb{M})$-formulas. Define "$R_{\varphi,2}(\Sigma(\bar{x})) \geq n$" by recursion on n:

- $R_{\varphi,2}(\Sigma(\bar{x})) \geq 0$ iff $\Sigma(\bar{x})$ is consistent
- $R_{\varphi,2}(\Sigma(\bar{x})) \geq n+1$ iff there is $\bar{b} \in \mathbb{M}$ s.t.

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) \geq n$$

Lemma 9.20. *$R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ iff there are $(\bar{a}_\sigma : \sigma \in 2^n)$ and $(\bar{b}_\tau : \tau \in 2^{<n})$ s.t.*

- *If σ extends $\tau 0$, then $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds*
- *If σ extends $\tau 1$, then $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds*
- *Each $\bar{a}_\sigma$ satisfies $\Sigma(\bar{x})$*

Definition 9.21. $R_{\varphi,2}(\Sigma(\bar{x}))$ is the largest n s.t. $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$, or $-\infty$ if there is no such n, or $+\infty$ if $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ for all n

$R_{\varphi,2}(\Sigma(\bar{x}))$ is called the "φ-2-rank" or "$R_{\varphi,2}$-rank" of Σ

Remark (Monotonicity). If $\Sigma(x) \vdash \Sigma'(x)$, then $R_{\varphi,2}(\Sigma(\bar{x})) \leq R_{\varphi,2}(\Sigma'(\bar{x}))$

Remark. From Lemma 9.20 we see that $R_{\varphi,2}(\{\bar{x} = \bar{x}\}) \geq n$ iff "D_n" is consistent. By Lemma 9.14,

$$R_{\varphi,2}(\{\bar{x} = \bar{x}\}) = +\infty \text{ iff } \varphi \text{ has the dichotomy property}$$

In particular, if φ does not have the dichotomy property, then $R_{\varphi,2}(\{\bar{x} = \bar{x}\})$ is finite. By Monotonicity, $R_{\varphi,2}(\Sigma(\bar{x}))$ is finite for any $\Sigma(\bar{x})$

Remark (Definability). Suppose $\Sigma(\bar{x})$ is a finite partial type over $A \subseteq \mathbb{M}$ and suppose $n < \omega$. Then the set

$$\{\bar{b} \in \mathbb{M} : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n\}$$

is A-definable

Now we can prove Theorem 9.18. Suppose $p \in S_\varphi(B)$. Take a finite subtype $\Sigma(\bar{x}) \subseteq_f p(\bar{x})$ minimizing $R_{\varphi,2}(\Sigma(\bar{x}))$. Then $\Sigma(\bar{x})$ is a partial type over B. Let $k = R_{\varphi,2}(\Sigma(\bar{x}))$

Claim. If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) = k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) < k$$

Proof. Monotonicity gives

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \leq k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) \leq k$$

If the first inequality is sharp, then it contradicts the minimality of $\Sigma(\bar{x})$. If the second inequality is not sharp ,then

$$R_{\varphi,2}(\Sigma(\bar{x})) \geq k + 1$$

□

Claim. If $\neg\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) < k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) = k$$

Combining the two claims, we see that the set

$$\{\bar{b} \in B : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is exactly

$$\{\bar{b} \in B : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k\}$$

By Definability, there is an $L(B)$-formula $\psi(\bar{x})$ s.t.

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

Therefore p is a definable φ-type

9.3.4 Remarks on the proof

1. In the proof of theorem 9.18, the finite subtype $\Sigma(\bar{x}) \subseteq p(\bar{x})$ chosen to minimize $R_{\varphi,2}(\Sigma(\bar{x}))$ actually has

$$R_{\varphi,2}(\Sigma(\bar{x})) = R_{\varphi,2}(p(\bar{x}))$$

This comes from the following

Fact 9.22. (a) *If $n < \omega$ and if $R_{\varphi,2}(\Sigma_0(\bar{x})) \geq n$ for every finite subtype $\Sigma_0(\bar{x})$, then $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$*

(b) *$R_{\varphi,2}(\Sigma(\bar{x}))$ is the minimum of $R_{\varphi,2}(\Sigma_0(\bar{x}))$ as Σ_0 ranges over finite subtypes of Σ.*

2. We have discussed $R_{\varphi,2}$ for partial types, but we can also define it for definable sets. If D is a definable set, defined by a finite type $\Sigma(\bar{x})$, then $R_{\varphi,2}(D) := R_{\varphi,2}(\Sigma(\bar{x}))$. The

9.3.5 Consequences of Theorem 9.18

Theorem 9.23. *Suppose that no formula $\varphi(x_1, \dots, x_n; \bar{y})$ has the dichotomy property. For any model $M \vDash T$ and any $p \in S_n(M)$, p is definable*

Proof. Take $\bar{a} \in \mathbb{M}$ realizing p. Let $\varphi(\bar{x}; \bar{y})$ be a formula. By Theorem 9.18, $\mathrm{tp}^{\varphi}(\bar{a}/M)$ is definable, therefore p is definable □

Remark. When $A \subseteq \mathbb{M}$ is arbitrary, one says that $p \in S_n(A)$ is **definable** if for any $\varphi(\bar{x}; \bar{y})$ there is an $L(A)$-formula $\psi(\bar{y})$ s.t. for $\bar{b} \in A$

$$\varphi(\bar{x}; \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \vDash \psi(\bar{b})$$

The proof of Theorem 9.23 shows more generally that

If no formula has the dichotomy property, then any $p \in S_n(A)$ is definable for any A

Therefore in a stable theory, any type over *any* set is definable

Warning 9.24. *Definable types over arbitrary sets are not as well-behaved as definable types over models. For example*

1. *A definable type over $A \subseteq \mathbb{M}$ can have more than one A-definable extension to $\mathbb{M}$: In DLO, there is only one 1-type over $\emptyset$. It has two different $\emptyset$-definable extensions to $\mathbb{M}$: the types at $+\infty$ and $-\infty$*

2. *A definable type over $A \subseteq \mathbb{M}$ can have no A-definable extensions to $\mathbb{M}$: In ACF,* $\text{tp}(\sqrt{2}/\mathbb{Q})$ *is definable (because ACF is stable), but there is no $\mathbb{Q}$-definable extension to the monster model $\mathbb{M}$. Indeed, there are exactly two extensions to $\mathbb{M}$, namely* $\text{tp}(\sqrt{2}/\mathbb{M})$ *and* $\text{tp}(-\sqrt{2}/\mathbb{M})$. *These are exchanged by some automorphisms in $Aut(\mathbb{M}/\mathbb{Q})$ so neither one can be $\mathbb{Q}$-definable*

Theorem 9.25. *Assume T is stable and $M \vDash T$. Let $D \subseteq M^n$ be $\emptyset$-definable. If $X \subseteq D$ is definable (with parameters from M), then X is definable over parameters from D*

Proof. Let $\psi(\bar{y})$ be a formula defining D and let $\varphi(\bar{a};\bar{y})$ be a formula defining X. Then $\text{tp}^{\varphi}(\bar{a}/D)$ is definable by Theorem 9.18. Therefore there is a formula $\theta(\bar{y})$ with parameters from D s.t. if $\bar{b} \in D$, then

$$(M \vDash \varphi(\bar{a};\bar{b})) \Leftrightarrow \varphi(\bar{x};\bar{b}) \in \text{tp}^{\varphi}(\bar{a}/D) \Leftrightarrow (M \vDash \theta(\bar{b}))$$

In other words, if $\psi(\bar{y})$ holds, then $\varphi(\bar{a};\bar{y})$ is equivalent to $\theta(\bar{y})$. Therefore X is defined by $\psi(\bar{y}) \wedge \theta(\bar{y})$, a formula with parameters from D □

Fact 9.26. *Let M be a stable structure and $D \subseteq M^n$ be A-definable. Let N be the structure whose domain is D, with an m-ary relation symbol for each A-definable $X \subseteq D^m$. Then the definable subsets of D^m in M agree with the definable subsets of D^m in N*

9.4 Coheirs

Definition 9.27. If $M \preceq N$, if $p \in S_n(M)$, if $q \in S_n(N)$, then q is a **coheir** of p if $q \supseteq p$ and q is finitely satisfiable in M (for any $\phi(x) \in q(x)$, there is $a \in M$ s..t $N \vDash \phi(a)$)

Example 9.3. $\mathbb{Q}^{\text{alg}} \preceq \mathbb{C}$, $q = \text{tp}(\pi/\mathbb{C})$, $p = \text{tp}(\pi/\mathbb{Q}^{\text{alg}})$. $q \supseteq p$, but q isn't a coheir since $x = \pi \in q(x)$

Example 9.4. If $M \preceq N$ strongly minimal, $q(x) \in S_1(N)$, $p(x) \in S_1(M)$ is the transcendental 1-type, $p \subseteq q$, then q is a coheir of p,

If $\varphi(x) \in q(x)$, then $\varphi(N)$ is cofinite and M is infinite, so $\varphi(N) \cap M \neq \emptyset$

Lemma 9.28. *If $M \preceq N$, $\Sigma(\bar{x})$ partial type over N, $\Sigma(\bar{x})$ is f.sat. in M, then $\exists q(\bar{x}) \in S_n(N)$, $q(\bar{x})$ is fsat. in M*

Proof. Let $\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(N) : \forall \bar{a} \in M, N \vDash \psi(\bar{a})\}$

If $\bar{a} \in M$, then $\bar{a}$ satisfies Ψ

Claim $\Sigma(\bar{x})$ fsat in $M \Rightarrow \Sigma \cup \Psi$ is fsat $\Rightarrow q \in S_n(N)$, $q \supseteq \Sigma \cup \Psi$

If q isn't fast. in M then $\varphi(\bar{x}) \in q(\bar{x})$, $\varphi(\bar{x})$ not sat. in M □

Theorem 9.29. *If $p \in S_n(M)$, $N \succeq M$, then $\exists q \in S_n(N)$, q is a coheir of p*

Theorem 9.30. *Suppose $M_1 \preceq M_2 \preceq M_3$, $p_1 \in S_n(M_1)$, $p_2 \in S_n(M_2)$, p_2 is a coheir of p_1. Then $\exists p_3 \in S_n(M_3)$, p_3 is a coheir of $p_1$and p_2*

9.5 Coheir Independence

9.5.1 Coheir independence

Definition 9.31. Let M be a small model, $\bar{a}, \bar{b}$ small tuples (possibly infinite). Then $\bar{a}$is **coheir independent** from $\bar{b}$ over M, written

$$\bar{a} \mathop{\downarrow}^{u}_{M} \bar{b}$$

if $\text{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M

Remark. The relation $A \downarrow^u_M B$ is finitary w.r.t. the arguments A and B, in the following sense. $A \downarrow^u_M B$ holds iff the following does:

For any finite tuple $\bar{a} \in A$ and any finite tuple $\bar{b} \in B$, we have $\bar{a} \downarrow^u_M \bar{b}$
Since a formula $\varphi(\bar{x}, \bar{y})$ can only refer to finitely many variables

Remark. The relation $\downarrow^u$ can be used to define heirs and coheirs, as follows. Suppose M, N are small models with $M \preceq N$. Suppose $p \in S_n(M)$ and $q \in S_n(N)$ with $q \supseteq p$. Take $\bar{a} \in \mathbb{M}^n$ realizing q

1. $q = \text{tp}(\bar{a}/N)$ is a coheir of $p = \text{tp}(\bar{a}/M)$ iff $\bar{a} \downarrow^u_M N$
2. $q = \text{tp}(\bar{a}/N)$ is an heir of $p = \text{tp}(\bar{a}/M)$ iff $N \downarrow^u_M \bar{a}$

9.5.2 Existence

Lemma 9.32. *Let M be a small model and $\bar{a}, \bar{b}$ be tuples, possibly infinite*

1. *There is $\sigma \in Aut(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) \downarrow^u_M \bar{b}$*
2. *There is $\sigma \in Aut(\mathbb{M}/M)$ s.t. $\bar{a} \downarrow^u_M \sigma(\bar{b})$*

Proof. 1. Let α be the length of $\bar{a}$ and $\bar{x}$ be an α-tuple of variables. Let

$$\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(M\bar{b}) : \psi(\bar{x}) \text{ is satisfied by every } \bar{a}' \in M^\alpha\}$$

If $\varphi(\bar{x}) \in \text{tp}(\bar{a}/M)$, then there is $\bar{a}' \in M^\alpha$ satisfying $\varphi(\bar{x})$ because $\text{tp}(\bar{a}/M)$ is finitely satisfiable in M. Then $\bar{a}'$ satisfies $\{\varphi(\bar{x})\} \cup \Psi(\bar{x})$.

This shows $\text{tp}(\bar{a}/M) \cup \Psi(\bar{x})$ is finitely satisfiable, hence realized by some $\bar{a}' \in \mathbb{M}^\alpha$

Then $\bar{a}'$ realizes $\text{tp}(\bar{a}/M)$, so $\text{tp}(\bar{a}'/M) = \text{tp}(\bar{a}/M)$, and there is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) = \bar{a}'$. Finally $\bar{a}' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \bar{b}$ by choice of $\Psi(\bar{x})$: if $\varphi(\bar{x}) \in \text{tp}(\bar{a}'/M\bar{b})$ and $\varphi(\bar{x})$ isn't satisfiable in M, then $M \vDash \neg\exists\bar{x}\varphi(\bar{x})$ and $M \vDash \forall\bar{x}\neg\varphi(\bar{x})$, hence $\neg\varphi(\bar{x}) \in \Psi(\bar{x})$ and $\bar{a}$ doesn't satisfy $\varphi(\bar{x})$, a contradiction

2. By 1, there is $\tau \in \text{Aut}(\mathbb{M}/M)$ s.t. $\tau(\bar{a}) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \bar{b}$. Let $\sigma = \tau^{-1}$. Then $\sigma(\tau(\bar{a})) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_{\sigma(M)} \sigma(\bar{b})$, or equivalently, $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \sigma(\bar{b})$

□

Corollary 9.33. *Suppose $p \in S_n(M)$ and $N \succeq M$*

1. *There is $q \in S_n(M)$ s.t. q is a coheir of p*
2. *There is $q \in S_n(M)$ s.t. q is an heir of p*

Proof. 1. Take $\bar{a} \in \mathbb{M}^n$ realizing p. Let $\bar{b}$ enumerate N. By Lemma, there is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \bar{b}$, i.e., $\sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M N$. Thus $\text{tp}(\sigma(\bar{a})/N)$ is a coheir of $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M) = p$

2. Similarly we have $N \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \sigma(\bar{a})$, and thus $\text{tp}(\sigma(\bar{a})/N)$ is an heir of $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M)$

□

9.5.3 "u" for "ultrafilter"

Proposition 9.34. *Let $\bar{a}$ be an α-tuple in $\mathbb{M}$. Let M be a small model and B a small set. TFAE*

1. $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M B$
2. *There is an ultrafilter $\mathcal{U}$ on the set M^α s.t. for any $L(MB)$-formula $\varphi(\bar{x})$*

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Leftrightarrow \{\bar{a}' \in M^\alpha : \mathbb{M} \vDash \varphi(\bar{a}')\} \in \mathcal{U}$$

Proof. $\Rightarrow$: For $\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)$, let $I = M^\alpha$ and $\mathcal{F} = \{\varphi(M^\alpha) : \varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)\}$. We claim that $\mathcal{F}$ has FIP. Let $\mathcal{U}$ be an ultrafilter on M^α extending $\mathcal{F}$. Then for any $L(MB)$-formula

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \in \mathcal{F} \Rightarrow \varphi(M^\alpha) \in \mathcal{U} \Leftrightarrow \{\bar{a}' \in M : \mathbb{M} \vDash \varphi(\bar{a}')\} \in \mathcal{U}$$

Then

$$\varphi(\bar{x}) \notin \text{tp}(\bar{a}/MB) \Rightarrow \neg\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \notin \mathcal{U}$$

$\Leftarrow$: □

Proposition 9.35. *Suppose $p \in S_n(M)$ and $N \succeq M$*

1. *If $q \in S_n(N)$ is a coheir of p, then there is an ultrafilter $\mathcal{U}$ on M^n s.t.*
$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\} \tag{$\star$}$$
2. *Conversely, if $\mathcal{U}$ is an ultrafilter on M^n and we define $q(\bar{x})$ according to $(\star)$, then $q(\bar{x}) \in S_n(N)$ and q is a coheir of p*

Proof. 1. Take $\bar{a}$ realizing q and p, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^u_M N$. Apply proposition 9.34

2. It suffices to show that q is finitely satisfiable in M and complete

□

Corollary 9.36 (Coheirs extend)**.** *Suppose $M \preceq N \preceq N'$ and $p \in S_n(M)$ and $q \in S_n(N)$ is a coheir of p, then is $q' \in S_n(N')$ with $q' \supseteq q$ and q' is a coheir of p*

Proof. By proposition 9.35 there is an ultrafilter $\mathcal{U}$ on M^n s.t.

$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\}$$

Take $q'(\bar{x}) = \{\varphi(\bar{x}) \in L(N') : \varphi(M^n) \in \mathcal{U}\}$ □

Remark. Suppose $q \in S_n(N)$ is an heir of $p \in S_n(M)$. Then $N \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{a}$ for a realization $\bar{a}$. Proposition 9.34 gives an ultrafilter $\mathcal{U}$ and tells us something., ultimate conclusion is

There is an ultrapower $M^{\mathcal{U}} \succeq N$ s.t. $p^{\mathcal{U}} \supseteq q$

9.5.4 Symmetry

Suppose $q \in S_n(N)$ is an extension of $p \in S_n(M)$.

In stable theory, coheir and heir are the same thing, so for any $q \in S_n(N)$ and $p \in S_n(M)$, $M \preceq N$

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^u_M N \Leftrightarrow N \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{a}$$

Theorem 9.37. *If T is stable, then*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{b} \Leftrightarrow \bar{b} \mathop{\smile\!\!\!\!\!\!\mid}^u_M \bar{a}$$

Proof. It suffices to prove $\Rightarrow$. Let α be the length of $\bar{a}$. Take a small model N containing M and $\bar{b}$. By the method of 9.36, one can find a type $q \in S_\alpha(N)$ extending tp$(\bar{a}/M\bar{b})$ finitely satisfiable in M. Take $\bar{a}'$ realizing q. Then $\bar{a}' \mathop{\downfree}^u_M N$. Also tp$(\bar{a}'/M\bar{b}) = q \upharpoonright (M\bar{b}) =$ tp$(\bar{a}/M\bar{b})$, so there is $\sigma \in \mathrm{Aut}(\mathbb{M}/M\bar{b})$ s.t. $\sigma(\bar{a}') = \bar{a}$. Then

$$\bar{a}' \mathop{\downfree}^u_M N \Rightarrow \sigma(\bar{a}') \mathop{\downfree}^u_{\sigma(M)} \sigma(N) \Leftrightarrow \bar{a} \mathop{\downfree}^u_M \sigma(N)$$

Replacing N with $\sigma(N)$, we may assume $\bar{a} \mathop{\downfree}^u_M N$. Therefore we have $N \mathop{\downfree}^u_M \bar{a}$. As $\bar{b} \in N$, this implies $\bar{b} \mathop{\downfree}^u_M \bar{a}$ ☐

9.5.5 Finitely satisfiable types commute with definable types

Recall that if $M \preceq N \preceq \mathbb{M}$, then

$$N \mathop{\downfree}^u_M \bar{a} \Leftrightarrow \mathrm{tp}(\bar{a}/N) \sqsupseteq \mathrm{tp}(\bar{a}/M)$$

Therefore the following lemma generalizes the fact that definable types have unique types

Lemma 9.38. *Let M be a small model. Suppose* tp$(\bar{a}/M)$ *is definable and* $\bar{b} \mathop{\downfree}^u_M \bar{a}$. *Then* tp$(\bar{a}/M\bar{b})$ *is* $p \upharpoonright M\bar{b}$, *where p is the M-definable global type extending* tp$(\bar{a}/M)$

Proof. We must show that for any L-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and any $\bar{c} \in M$,

$$\varphi(\bar{x}, \bar{b}, \bar{c}) \in \mathrm{tp}(\bar{a}/M\bar{b}) \Leftrightarrow \mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})$$

Otherwise, these things are true

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \not\Leftrightarrow \mathbb{M} \models (d_p(\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})$$
$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})$$
$$(\varphi(\bar{a}, \bar{y}, \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{y}, \bar{c})) \in \mathrm{tp}(\bar{b}/M\bar{a})$$

As $\bar{b} \mathop{\downfree}^u_{M'}$, there is $\bar{b}' \in M$ s.t.

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) \not\leftrightarrow (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})$$
$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) \not\Leftrightarrow \mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})$$
$$\varphi(\bar{x}, \bar{b}', \bar{c}) \in \mathrm{tp}(\bar{a}/M) \not\Leftrightarrow \mathbb{M} \models (d_p\bar{x})\varphi(\bar{x}, \bar{b}', \bar{c})$$

A contradiction ☐

Lemma 9.39. *Let $p \in S_n(\mathbb{M})$ be finitely satisfiable in a small model M. If $\bar{a} \models p \upharpoonright M\bar{b}$, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \bar{b}$*

Theorem 9.40. *Let p, q be global types. Suppose p is definable over some small set A. (p is A-invariant) Suppose q is finitely satisfiable in some small set B (q is B-invariant by 9.49). Then p and q commute*

Proof. Otherwise, there is an $L(\mathbb{M})$-formula $\varphi(\bar{x}, \bar{y})$ s.t.

$$(p \otimes q)(\bar{x}, \bar{y}) \vdash \varphi(\bar{x}, \bar{y})$$
$$(q \otimes p)(\bar{y}, \bar{x}) \vdash \neg\varphi(\bar{x}, \bar{y})$$

The formula φ uses only finitely many parameters $\bar{c}$ from $\mathbb{M}$. By Löwenheim–Skolem Theorem there is a small model M containing $AB\bar{c}$. Then $\varphi(\bar{x}, \bar{y})$ is an $L(M)$-formula. Also, p is M-definable and q is finitely satisfiable in M. Note that p, q and $p \otimes q$, $q \otimes p$ are M-invariant types. Take $(\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright M$ and $\bar{a} \models p \upharpoonright M$, $\bar{b} \models q \upharpoonright M\bar{a}$. By Lemma 9.39, $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M \bar{a}$

Now $\mathrm{tp}(\bar{a}/M)$ is the definable type $p \upharpoonright M$, so by Lemma 9.39

$$\bar{a} \models p \upharpoonright M\bar{b}$$

Thus $(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$

It follows that $(q \otimes p)(\bar{y}, \bar{x})$ and $(p \otimes q)(\bar{x}, \bar{y})$ have the same restriction to M. Then φ leads to a contradiction □

9.5.6 Types commute in stable theories

Assume the theory T is stable

Proposition 9.41 (Assuming stability)**.** *Let $p \in S_n(\mathbb{M})$ be a global type and M be a small model. TFAE*

1. *p is finitely satisfiable in M*
2. *p is M-invariant*
3. *p is M-definable*

Proof. $1 \to 2$: 9.49
$2 \to 3$: 9.51 □

Theorem 9.42 (Assuming stability)**.** *Let $p(\bar{x})$, $q(\bar{y})$ be two invariant global types. Then p and q commute*

Proof. The types p and q are invariant over small sets A and B respectively. Take a small model M containing $A \cup B$. Then p and q are M-invariant. By Proposition 9.41, p is M-definable and p is finitely satisfiable in M. Therefore p and q commute by Theorem 9.40 $\square$

9.5.7 Morley products and $\mathop{\smile\!\!\!\!\!\!\mid}^{u}$

Let M be a small model. If p and q are M-definable types, then the Morley product $p \otimes q$ is also M-definable by 9.63. Since M-definable global types corresponds to (M-)definable types over M (Proposition 9.48), we can regard $\otimes$ as an operation on definable types over M

If T is stable, then *all* types over M are definable, and we get an operation

$$S_n(M) \times S_n(M) \to S_{m+n}(M)$$
$$(p, q) \mapsto p \otimes q$$

The following theorem shows that, at least in stable theories, there is a very close connection between the Morley product $p \otimes q$ and the coheir independence relation $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} \bar{b}$

Theorem 9.43. *Assume T is stable. Let $M \preceq \mathbb{M}$ be a small model and $\bar{a}, \bar{b}$ be tuples in $\mathbb{M}$. Then*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} \bar{b} \Leftrightarrow \mathrm{tp}(\bar{b}, \bar{a}/M) = \mathrm{tp}(\bar{b}/M) \otimes \mathrm{tp}(\bar{a}/M)$$

Proof. First suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} \bar{b}$. Then $\mathrm{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M. By Lemma 9.28, there is a global type p which is finitely satisfiable in M and extends $\mathrm{tp}(\bar{a}/M\bar{b})$. By Proposition 9.41, p is M-definable. Then p is the unique M-definable global extension of the definable type $\mathrm{tp}(\bar{a}/M)$. Let q be the unique M-definable global extension of the definable type $\mathrm{tp}(\bar{b}/M)$. Then

$$\bar{b} \models q \restriction M \quad \text{and} \quad \bar{a} \models p \restriction M\bar{b}$$

because p extends $\mathrm{tp}(\bar{a}/M\bar{b})$. Therefore

$$(\bar{b}, \bar{a}) \models (q \otimes p) \restriction M$$

or equivalently, $\mathrm{tp}(\bar{b}, \bar{a}/M) = (q \otimes p) \restriction M$.

Conversely, suppose $\mathrm{tp}(\bar{b}, \bar{a}/M) = \mathrm{tp}(\bar{b}/M) \otimes \mathrm{tp}(\bar{a}/M)$. Let q be the unique M-definable global extension of the definable type $\mathrm{tp}(\bar{b}/M)$ and let p be the unique M-definable global extension of the definable type $\mathrm{tp}(\bar{a}/M)$ by 9.48. Then

$$(\bar{b}, \bar{a}) \models (q \otimes p) \restriction M$$

or equivalently

$$\bar{b} \models q \restriction M \quad \text{and} \quad \bar{a} \models p \restriction M\bar{b}$$

By Proposition 9.41 p is finitely satisfiable in M, and so

$$\bar{a} \models p \restriction M\bar{b} \Rightarrow \bar{a} \mathop{\downarrow}\limits_{M}^{u} \bar{b}$$

by Lemma 9.39 $\square$

9.6 Invariant types

Lemma 9.44. *If $X \subseteq \mathbb{M}^n$, TFAE*

1. *$\sigma(X) = X$ if $\sigma \in Aut(\mathbb{M}/A)$*
2. *If $\bar{a}, \bar{b} \in \mathbb{M}^n$, $\bar{a} \equiv_A \bar{b} \Rightarrow (\bar{a} \in X \Leftrightarrow \bar{b} \in X)$*
3. *There is $f : S_n(A) \to \{0, 1\}$ s.t. $\bar{a} \in X \Leftrightarrow f(\mathrm{tp}(\bar{a}/A)) = 1$*

Proof. rewrite (2) as

- If $\bar{a}, \bar{b} \in \mathbb{M}^n$, $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\sigma(\bar{a}) = \sigma(\bar{b})$, then $\bar{a} \in X \Leftrightarrow \bar{b} \in X$
- If $\bar{a} \in M$, $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\bar{a} \in X \Leftrightarrow \sigma(\bar{a}) \in X$

$\square$

Definition 9.45. $X \subseteq \mathbb{M}^n$ is **A-invariant** if $\forall \sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\sigma(X) = X$

Example 9.5. If X is A-definable, then X is A-invariant

Lemma 9.46. *If $D \subseteq \mathbb{M}^n$ is definable and A-invariant, then D is A-definable*

Proof. Step 1: If $\bar{b} \in D$ then $\mathrm{tp}(\bar{b}/A) \vdash \bar{x} \in D$, by compactness, there is $\varphi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$ s.t. $\varphi(\bar{x}) \vdash \bar{x} \in D$, $\varphi(\mathbb{M}^n) \subseteq D$

Step 2: So then D is covered by A-definable subsets of D. By compactness, D is covered by finitely many of them, which implies D is A-definable

Suppose $D = \psi$, then $[\psi] = \bigcup [\varphi_i]$ $\square$

Definition 9.47. p is **A-definable** if $\forall \varphi$, $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is A-definable

Remark.
1. p is A-definable $\Rightarrow$ p is A-invariant
2. If p is definable, then p is A-invariant $\Leftrightarrow$ p is A-definable

3. If p is definable thne p is A-definable for some small A

   Each $d_p\varphi$ uses only finitely many parameters

Proposition 9.48. *Suppose $M \preceq \mathbb{M}$, small*

1. *If $p \in S_n(M)$ definable and $p^{\mathbb{M}}$ is its heir over $\mathbb{M}$, then $p^{\mathbb{M}} \in S_n(\mathbb{M})$ is M-definable*

2. *$p \mapsto p^{\mathbb{M}}$ is a bijection from definable types over M to M-definable types over $\mathbb{M}$*

Proof.
1. $p^{\mathbb{M}}$ has the same definition as p, so it's M-definable

2. $q \mapsto q \upharpoonright M$ is an inverse to $p \mapsto p^{\mathbb{M}}$

$\square$

Warning: an M-invariant type p is not determined by $p \upharpoonright M$. If $A \subseteq \mathbb{M}$, A-definable type p is not determined by $p \upharpoonright A$. Only works for models CHECK

Theorem 9.49. *Suppose $M \preceq \mathbb{M}$ and $p \in S_n(M)$*

1. *If $q \in S_n(\mathbb{M})$ and q is a coheir of p, then q is M-invariant*

2. *$\exists q \in S_n(\mathbb{M})$, $p \subseteq q$ is M-invariant*

Proof. If q is a coheir of p, but q is not M-invariant, then $\exists \bar{b}, \bar{c}$, $\bar{b} \equiv_M \bar{c}$, $\varphi(\bar{x}, \bar{b}) \in q$, $\varphi(\bar{x}, \bar{c}) \notin q$. Then $\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$. Because q is fsat. in M, $\exists \bar{a} \in M$, $M \models \varphi(\bar{a}, \bar{b}) \wedge \neg\varphi(\bar{a}, \bar{c})$, so $\bar{b} \not\equiv_M \bar{c}$ $\square$

In stable theories:

Lemma 9.50. *If T is stable and p is A-invariant, then p is A-definable*

Theorem 9.51. *Suppose T stable, $M \preceq \mathbb{M}$ small, $p \in S_n(M)$. Let $p^{\mathbb{M}}$ the global heir.*

1. *$p^{\mathbb{M}}$ is the only M-invariant global type extending p*

2. *$p^{\mathbb{M}}$ is the only global coheir of p*

3. *If $M \preceq N \preceq \mathbb{M}$ and q is the heir of p over N, then q is the unique coheir of p over N*

Proof.
1. M-invariant $\Leftrightarrow$ M-definable

2. there is some coheir of p. Any coheir is M-invariant, so $p^{\mathbb{M}}$ is the only coheir

$\square$

Corollary 9.52. *In a stable theory, coheirs are unique and coheir=heir*

Corollary 9.53. *In a stable theory, "coheir" is transitive*

9.7 Morley sequence

Lemma 9.54. *If p, q are A-invariant global types, $p \in S_n(\mathbb{M})$, $q \in S_m(\mathbb{M})$, then there is $r \in S_{n+m}(A)$ s.t. $(\bar{b}, \bar{c}) \models r$ iff*

$$\bar{b} \models p \upharpoonright A \quad \textit{and} \quad c \models q \upharpoonright (A\bar{b}) \tag{$\star$}$$

Proof. Let $X = \{(\bar{b}, \bar{c}) : \bar{b} \models p \upharpoonright A \text{ and } \bar{c} \models q \upharpoonright A\bar{b}\}$. If $(\bar{b}, \bar{c}) \in X$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, then $\sigma(\bar{b}) \models \sigma(p \upharpoonright A) = p \upharpoonright A$ and $\sigma(\bar{c}) \models q \upharpoonright A\sigma(\bar{b})$. So $\sigma(\bar{b}, \bar{c}) \in X$, X is A-invariant

Fix $\bar{b}_0 \models p \upharpoonright A$, $\bar{c}_0 \models q \upharpoonright A\bar{b}_0$, so $(\bar{b}_0, \bar{c}_0) \in X$. Let $r = \mathrm{tp}(\bar{b}_0, \bar{c}_0/A)$. If $(\bar{b}, \bar{c}) \models r$, then $(\bar{b}, \bar{c}) \in X$

Conversely, if $(\bar{b}, \bar{c}) \in X$, want $(\bar{b}, \bar{c}) \models r$, i.e., $(\bar{b}, \bar{c}) \equiv_A (\bar{b}_0, \bar{c}_0)$

$\bar{b} \models p \upharpoonright A = \mathrm{tp}(\bar{b}_0/A)$ so $\bar{b} \equiv_A \bar{b}_0$, $\exists \sigma \in \mathrm{Aut}(A)$, $\sigma(\bar{b}) = \bar{b}_0$. Replace $(\bar{b}, \bar{c})$ with $(\sigma(\bar{b}), \sigma(\bar{c})) = (\bar{b}_0, \sigma(\bar{c}))$.

WMA $\bar{b} = \bar{b}_0$. Then $\bar{c}$ and $\bar{c}_0$ both satisfy $q \upharpoonright A\bar{b}_0$. Move $\bar{c}$ by $\tau \in \mathrm{Aut}(\mathbb{M}/A\bar{b}_0)$, we may assume $\bar{c} = \bar{c}_0$. Then $\bar{c} \equiv_{A\bar{b}_0} \bar{c}_0 \Rightarrow \bar{b}\bar{c} \equiv_A \bar{b}_0\bar{c}_0$ $\square$

Proposition 9.55. *If $p \in S_n(\mathbb{M})$, $q \in S_m(\mathbb{M})$ and both are A-invariant, then there is A-invariant $p \otimes q \in S_{n+m}(\mathbb{M})$ s.t. for any small $A' \supseteq A$,*

$$(\bar{b}, \bar{c}) \models (p \otimes q) \upharpoonright A' \Leftrightarrow b \models p \upharpoonright A' \textit{ and } \bar{c} \models q \upharpoonright A'\bar{b}$$

Proof. Note p, q are A'-invariant for any A'-invariant, so lemma gives $r_{A'} \in S_{n+m}(A')$ for each $A' \supseteq A$ s.t. $(\bar{b}, \bar{c}) \models r_{A'} \Leftrightarrow$ the condition

If $A'' \supseteq A' \supseteq A$, if $(\bar{b}, \bar{c}) \models r_{A''}$ then $(\bar{b}, \bar{c}) \models r_{A'}$ so $r_{A'} \models r_{A'} \upharpoonright A'$.

Let $p \otimes q = \bigcup_{A'} r_{A'}$, then $p \otimes q \in S_{n+m}(\mathbb{M})$ and $r_{A'} = p \otimes q \upharpoonright A'$ $\square$

If $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, then $\sigma(p \otimes q) = \sigma(p) \otimes \sigma(q) = p \otimes q$, so $p \otimes q$ is A-invariant

Fact 9.56. *If $p \in S_n(M)$ A-invariant where M is $|A|^+$-saturated and $N \succeq M$, then p has a unique A-invariant extension over N*

Fact 9.57. *If $p, q \in S_{n+m}(\mathbb{M})$ A-invariant, take $\bar{b} \models p$, $\bar{b} \in \mathbb{M}_1 \succeq \mathbb{M}$, take $\bar{c} \models q \upharpoonright \mathbb{M}_1$ then* $\mathrm{tp}(\bar{b}, \bar{c}/\mathbb{M}) = p \otimes q$

Definition 9.58. The **(Morley) product** of invariant types p, q is $p \otimes q$

If p, q are A-invariant, then $(\bar{b}, \bar{c}) \vDash (p \otimes q) \upharpoonright A \Leftrightarrow \bar{b} \vDash p \upharpoonright A$ and $\bar{c} \vDash q \upharpoonright A\bar{b}$

Definition 9.59. $\mathrm{acl}(A) = \bigcup\{\varphi(\mathbb{M}) : \varphi(x) \in L(A), |\varphi(\mathbb{M})| < \infty\}$

Fact 9.60. *In ACF, if K a subfield of $\mathbb{M}$, then $\mathrm{acl}(K)$ is K^{alg}*

Fact 9.61. *In any theory T, $\mathrm{acl}(-)$ is a finitary closure operation*

Example 9.6. If T is strongly minimal and $p \in S_1(\mathbb{M})$ transcendental 1-type, what is $p \otimes p$

$b \vDash p \upharpoonright A \Leftrightarrow b \notin \mathrm{acl}(A)$

Therefore $(b, c) \vDash (p \otimes p) \upharpoonright A$ iff $b \vDash p \upharpoonright A$ and $c \vDash p \upharpoonright Ab$ iff $b \notin \mathrm{acl}(A)$ and $c \notin \mathrm{acl}(Ab)$

idea: b, c are algebraically independent over A

In stable theories, $(p \otimes q)(x, y)$ is the "most free" completion of $p(\bar{x}) \cup q(\bar{y})$

Example 9.7. Suppose $\mathbb{M} \vDash$ ACF. let p_V denote generic type of a variety $V \subseteq \mathbb{M}$ $\{x \in V\} \cup \{x \notin W : W \subsetneq V, W \text{ algebraic}\}$

If $V \subseteq \mathbb{M}^n$, $W \subseteq \mathbb{M}^m$ varieties, then $V \times W$ is a variety, and $p_V \otimes p_W = p_{V \times W}$

Proof. $p_V \otimes p_W = p_Z$ for some variety $Z \subseteq \mathbb{M}^{n+m}$. Take small $M \preceq \mathbb{M}$ s.t. V, W, Z are M-definable. Take $\bar{a} \vDash p_V \upharpoonright M$, take small $N \preceq \mathbb{M}$, $N \supseteq M\bar{a}$. Take $\bar{b} \vDash p_W \upharpoonright N$, so $(\bar{a}, \bar{b}) \vDash p_V \otimes p_W \upharpoonright M = p_Z \upharpoonright M$.

"$x \in V \in p_V \upharpoonright M$", $\bar{a} \in V$, $\bar{b} \in W$, so $(\bar{a}, \bar{b}) \in V \times W$.

Fact: $p_Z(\bar{x}) \vdash \bar{x} \in U \Leftrightarrow Z \subseteq U$ for U algebraic

So $(\bar{a}, \bar{b}) \in V \otimes W \Leftrightarrow Z \subseteq V \times W$

Suppose $Z \subsetneq V \times W$. Take $(\bar{a}_0, \bar{b}_0) \in V \times W \setminus Z$. Let $Z_{\bar{a}} = \{\bar{y} \in M : (\bar{a}, \bar{y}) \in Z\}$, then $Z_{\bar{a}}$ is an algebraic set over $N \supseteq M_{\bar{a}}$

L ☐

Definition 9.62. invariant types p, q "commute" if $p \otimes q(\bar{x}, \bar{y}) = q \otimes p(\bar{y}, \bar{x})$

Example 9.8. In ACF, any two types commutes

$p_V \otimes p_W = p_{V \times W} = p_W \otimes p_V$

If p is a definable type and $\varphi(\bar{x}, \bar{y})$ is a formula, then $(d_p\bar{x})\varphi(\bar{x}, \bar{y})$ means $d\varphi(\bar{y})$, the formula defining $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$

$d_p\bar{x}$ works like quantifier, free variables in $(d_p\bar{x})\varphi(\bar{x}, \bar{y})$ are $\bar{y}$

Example 9.9. Suppose $\mathbb{M} \vDash T$ strongly minimal, let $p =$ transcendental 1-type, $\varphi()$

Proposition 9.63. *If p, q are A-definable global types, then $p \otimes q$ is A-definable and $(d_{p\otimes q}(\bar{x}, \bar{y}))\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{z})$*

Proof. Fix $\bar{c} \in \mathbb{M}$, take $M \preceq \mathbb{M}$ s.t. $\bar{c} \in M$ and $M \supseteq A$, so p, q are M-definable. Take $\bar{a} \vDash p \restriction M$ and $\bar{b} \vDash q \restriction M\bar{a}$, so $(\bar{a}, \bar{b}) \vDash (p \otimes q) \restriction M$. So

$$\begin{aligned}
\varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q &\Leftrightarrow \varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q \restriction M \\
&\Leftrightarrow \mathbb{M} \vDash \varphi(\bar{a}, \bar{b}, \bar{c}) \\
&\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \restriction M\bar{a} \\
&\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \\
&\Leftrightarrow \mathbb{M} \vDash (d_q\bar{y})\varphi(\bar{a}, \bar{y}, \bar{c}) \\
&\Leftrightarrow (d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \in p(\bar{x}) \\
&\Leftrightarrow (d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{c})
\end{aligned}$$

□

Example 9.10. in a strongly minimal theory, if $p \in S_1(\mathbb{M})$ is transcendental and $q = p \otimes p$ then $(d_q(x, y))\varphi(x, y, \bar{z})$ is $\exists^\infty x \exists^\infty y \varphi(x, y, \bar{z})$

Two definable types p, q commute iff $(d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_q\bar{y})(d_p\bar{x})\varphi(\bar{x}, \bar{y}, \bar{z})$
Let A-invariant $p \in S_n(\mathbb{M})$

Definition 9.64. A **Morley sequence** of p over A is a sequence $\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots \in \mathbb{M}^n$ s.t.

$$\bar{b}_1 \vDash p \restriction A, \bar{b}_2 \vDash p \restriction A\bar{b}_1, \dots, \bar{b}_i \vDash p \restriction A\bar{b}_1 \dots \bar{b}_{i-1} \dots$$

So $(\bar{b}_1, \dots, \bar{b}_n) \vDash \underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}$

Example 9.11. If T is strongly minimal, p is transcendental 1-type, a Morley sequence over A is $b_1, b_2, \dots$ s.t. $b_1 \notin \operatorname{acl}(A)$, $b_2 \notin \operatorname{acl}(Ab_1)$,...

Example 9.12. In DLO, in $(\mathbb{R}, \leq)$, $1, 2, 3, 4, \dots$ is indiscernible
An increasing sequence is indiscernible in DLO

Theorem 9.65. *If $p \in S_n(\mathbb{M})$ A-invariant and $(\bar{b}_i : i < \omega)$ is a Morley sequence of p over A, then it is A-indiscernible*

9.8 Order Property

Remark. If φ has O.P., then $\neg\varphi$

Lemma 9.66. *For any infinite $\lambda \geq \aleph_0$ there is a linear order $(I, \leq)$ and $S \subseteq I$ s.t. $|I| > \lambda$, $|S| \leq \lambda$, S is dense in I*

Proof. there is μ s.t. $|2^\mu| > \lambda$ and $|2^{<\mu}| \leq \lambda$.
Let $I = 2^\mu \cup 2^{<\mu}$ and $S = 2^{<\mu}$ □

Theorem 9.67. *If $\varphi(\bar{x}, \bar{y})$ has O.P., then T is not λ-stable for any λ*

Proof. Take $I \supseteq S$ s.t. S dense in I, $|S| \leq \lambda$, $|I| > \lambda$
$\bar{a}_i, \bar{b}_j$, $i, j \in \mathbb{Z}$, $\varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$. By compactness, we can take any linear order. There is $\bar{a}_i, \bar{b}_j$ for $i, j \in I$ s.t. $\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$
Let $C = \{\bar{b}_j : j \in S\}$, $|C| \leq \lambda$.
Claim $I \setminus S \to S_n(C)$, $i \mapsto \mathrm{tp}(\bar{a}_i/C)$ is an injection
If $i_1 < i_2$, then there is $j \in S$, $i_1 < j < i_2$ then $\varphi(\bar{a}_i, \bar{b}_j) \wedge \neg\varphi(\bar{a}_{i_2}, \bar{b}_j)$, $\bar{b}_j \in C$, so $\bar{a}_{i_1} \not\equiv_C \bar{a}_{i_2}$
$|S_n(C)| \geq |I \setminus S| > \lambda$ □

Lemma 9.68. *Suppose $\varphi(\bar{x}, \bar{y})$ doesn't have O.P. Let n_φ be from Lemma 9. Let $\bar{b}_1, \bar{b}_2, \ldots$ be indiscernible (over $\emptyset$). Then there is no $\bar{a}$ s.t. $\mathbb{M} \vDash \varphi(\bar{a}, \bar{b}_i)$ for $0 \leq i < n_\varphi$ s.t.*

Proof. $n = n_\varphi$. Suppose $\bar{a}$ exists, for $0 \leq$ □

Lemma 9.69. *Suppose $\varphi(x_1, \ldots, x_n; \bar{y})$ doesn't have O.P.. Take $N > \max(n_\varphi, n_{\neg\varphi})$. let p be an A-invariant type over $\mathbb{M}$. Let $a_1, a_2, \ldots$ be a Morley sequence of p over A*

1. *If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then $\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$*
2. *If $\varphi(\bar{x}, \bar{b}) \notin p(\bar{x})$, then $\mathbb{M} \vDash \neg\varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$*

Example 9.13. If T is strongly minimal then T is stable if $\varphi(x, \bar{y})$ has the O.P., then there is $a_i, \bar{b}_i \in \mathbb{M}$ $\mathbb{M} \vDash \varphi(a_i, \bar{b}_j) \Leftrightarrow i < j$ for $i, j \in \mathbb{Z}$
So $\varphi(\mathbb{M}, \bar{b}_0)$ is neither finite or cofinite

Theorem 9.70. *If T is stable and p and q are global types (all types are definable and hence invariant for some A), then $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$*

Proof. Suppose not. Take $\varphi(\bar{x}, \bar{y}) \in L(\mathbb{M})$. $\varphi(\bar{x}, \bar{y}) \in (p \otimes q)(\bar{x}, \bar{y})$, $\varphi(\bar{x}, \bar{y}) \notin (q \otimes p)(\bar{y}, \bar{x})$.
Take A s.t. p, q are A-definable and $\varphi(\bar{x}, \bar{y}) \in L(A)$
Take $p \otimes q \otimes p \otimes q \otimes \cdots$
$((b_i, c_i) : i \in \omega)$ a Morley sequence of $p \otimes q$ over A
If $i \leq j$, $(b_i, c_j) \vDash p \otimes q \restriction A$, $\mathbb{M} \vDash \varphi(b_i, c_j)$
If $i > j$, $(c_j, b_i) \vDash q \otimes p \restriction A$ $\mathbb{M} \vDash \neg\varphi(b_i, c_j)$ □

9.9 Ramsey's theorem and indiscernible sequences

Definition 9.71. X set, C a set of "colors", then $f : [X]^\kappa \to C$ is a coloring of κ-elements subsets of X

Definition 9.72. $Y \subseteq X$ is **homogeneous** if $f \upharpoonright [Y]^\kappa$ is constant

Definition 9.73. If N, m, n, k are cardinals, $N \to (m)^n_k$ means that if $|X| = N$, $|C| = k$, $f : [X]^n \to C$, then there is $Y \subseteq X$, Y is homogeneous and has size m

Fact 9.74 (Friends and strangers theorem)**.** *$|X| = 6$, $|C| = 2$ and $f : [X]^2 \to C$, then there is $Y \subseteq X$ homogeneous and size 3*

Theorem 9.75 (Finite Ramsey's theorem)**.** *If $n, m, k \in \omega$ then there is $N < \omega$ s.t. $N \to (m)^n_k$*

Proof. Let $L = \{R_1, \dots, R_k\}$, R_i is an n-ary predicate (relation) symbol. T is the L-theory that says:

- If $R_i(\bar{x})$ then $\bar{x}$ is distinct
- If $\bar{x}$ is distinct then $R_i(\bar{x})$ holds for exactly one i
- If $\bar{y}$ is a permutation of $\bar{x}$, $R_i(\bar{x}) \leftrightarrow R_i(\bar{y})$

A model of T is a set M and a coloring of $[M]^n$

Let φ be the formula s.t. $M \vDash \varphi \Leftrightarrow$ there is a homogeneous $Y \subseteq M$, $|Y| = m$

$$\exists y_1, \dots, y_m \bigwedge_{1 \le i_1 < \dots < i_n \le m} \bigwedge_{1 \le j_1 < \dots < j_n \le m} \text{same color}$$

Suppose $N \not\to (m)^n_k$, then $\exists M \vDash T$ $|M| = N$ and $M \nvDash \varphi$. Suppose $N \not\to (m)^n_k$ for any $N < \omega$, then by compactness, $T \cup \{\neg\varphi\}$ has infinite models. By theorem 17 last week, there is $M \vDash T \cup \{\neg\varphi\}$, indiscernible sequence $a_1, a_2, \dots \in M$ not constant, but indiscernibility $\Rightarrow \{a_1, a_2, \dots\}$ is homogeneous, $\{a_1, \dots, a_m\}$ is homogeneous $\square$

Fact 9.76 (Infinite Ramsey's theorem)**.** *$\aleph_0 \to (\aleph_0)^n_k$ for $n, k \in \omega$*

extracting indiscernibles

Working $\mathbb{M} \vDash T$. If $(I, \le)$ is a linear order and $(\bar{a}_i : i \in I)$ is a sequence in $\mathbb{M}$ and if $B \subseteq \mathbb{M}$

Definition 9.77. $\text{tp}^{\text{EM}}(\bar{a}/B) = \{\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B) : \forall i_1 < \dots < i_n \in I, \mathbb{M} \vDash \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n})\}$, the **Ehrenfeucht-Mostowski type** over B

Remark. tp^{EM} is really a sequence of partial types over B, $\Sigma_1, \Sigma_2, \dots$

Example 9.14. In $(\mathbb{R}, \leq)$, 1,1,2,2,3,3,4,4,...
$(x_1 \leq x_2) \in \text{tp}^{\text{EM}}(\dots)$
$x_1 < x_2 \notin \text{tp}^{\text{EM}}$

Remark. If $(\bar{a}_i : i \in I)$ is a sequence, $I_0 \subseteq I$, then $\text{tp}^{\text{EM}}((\bar{a}_i : i \in I)/B) \subseteq \text{tp}^{\text{EM}}((\bar{a}_i : i \in I_0)/B)$

Definition 9.78. If $\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B)$, $(\bar{a}_i : i \in I)$ is "φ-indiscernible" if $\forall i_1 < \dots < i_n$, $\forall j_1 < \dots < j_n$,

$$\mathbb{M} \vDash \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) \leftrightarrow \varphi(\bar{a}_{j_1}, \dots, \bar{a}_{j_n})$$

Remark. $(\bar{a}_i : i \in I)$ is B-indiscernible iff it is φ-indiscernible for all $\varphi \in L(B)$

Definition 9.79. If Δ is a set of formulas, $\bar{a}$ is **Δ-indiscernible** if it is φ-indiscernible for all $\varphi \in \Delta$

Lemma 9.80. *Let $(\bar{a}_i : i \in I)$ be infinite*

1. *If $m < \omega$, Δ is a finite set of L-formulas, then there is Δ-indiscernible subsequence of length m*
2. *If $(J, \leq)$ is a linear order, Δ a set of formulas, then there is $(\bar{b}_j : j \in J) \in \mathbb{M}$ s.t. $\bar{b}$ is Δ-indiscernible and* $\text{tp}^{\text{EM}}(\bar{b}) \supseteq \text{tp}^{\text{EM}}(\bar{a})$

Proof.
1. By induction on $|\Delta|$.

   $|\Delta| = 0$, take any subsequence of length m

   $|\Delta| > 0$, $\Delta = \Delta_0 \cup \{\varphi\}$, $\varphi(x_1, \dots, x_n)$. Ramsey: there is $N \to (m)^n_2$, by induction there is subsequence $(\bar{b}_i : i < N)$ Δ_0-indiscernible. Define $f : [N]^n \to \{0, 1\}$ by

   $$f(\{i_1, \dots, i_n\}) = \begin{cases} 1 & \mathbb{M} \vDash \varphi(b_{i_1}, \dots, b_{i_n}) \\ 0 & \text{otherwise} \end{cases}$$

   there is subsequence $(\bar{c}_i : i < m)$ that is homogeneous, φ-indiscernible
2. By compactness, we may assume J is finite, Δ is finite. By part 1

$\square$

Theorem 9.81. *If $(\bar{a}_i : i \in I)$ an infinite sequence, B is a set of parameters, $(J, \leq)$ infinite linear order, then there is B-indiscernible sequence $(\bar{b}_j : j \in J)$ with* $\text{tp}^{\text{EM}}(\bar{b}/B) \supseteq \text{tp}^{\text{EM}}(\bar{a}/B)$

Proof. Apply Lemma 9.80 with $\Delta = \{\text{all the } L(B)\text{-formulas}\}$ $\square$

"Extracting indiscernible sequences"

Example 9.15 (=Theorem 17 last week)**.** If $|\mathbb{M}| = \infty$, take distinct $a_0, a_1, a_2, \dots \in \mathbb{M}$, $x_1 \neq x_2 \in \mathrm{tp}^{\mathrm{EM}}(\bar{a})$. Take $b_0, b_1, \dots$ indiscernible, extracted from $\bar{a}$, then $(x_1 \neq x_2) \in \mathrm{tp}^{\mathrm{EM}}(\bar{a}) \subseteq \mathrm{tp}^{\mathrm{EM}}(\bar{b})$, so $b_i \neq b_j$ for $i < j$. So $\bar{b}$ is a non-constant indiscernible sequence

Example 9.16. Suppose $\mathbb{M} \succeq (\mathbb{R}, +, \cdot, \leq, 0, 1, -)$. Suppose $b_1, b_2, b_3, \dots$ is indiscernible, extracted from $1, 2, 3, \dots$

$x_1 > 0 \in \mathrm{tp}^{\mathrm{EM}}(\bar{a}) \subseteq \mathrm{tp}^{\mathrm{EM}}(\bar{b})$

$x_2 - x_1 \geq 1 \in \mathrm{tp}^{\mathrm{EM}}(\bar{b})$

Remark. $(\bar{a}_i : i \in I)$ is B-indiscernible iff $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$ is "complete", i.e., $\forall \varphi(x_1, \dots, x_n) \in L(B)$, $\varphi \in \mathrm{tp}^{\mathrm{EM}}$ or $\neg\varphi \in \mathrm{tp}^{\mathrm{EM}}$

Theorem 9.82. *If $(\bar{a}_i : i \in I)$ is B-indiscernible, if $(J, \leq)$ is a linear order, then there is B-indiscernible $(\bar{b}_j : j \in J)$ with $\mathrm{tp}^{\mathrm{EM}}(\bar{b}/B) = \mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$*

Remark. If $(\bar{a}_i : i \in I)$ is B-indiscernible, then $\mathrm{tp}(\bar{a}/B)$ is determined by $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$ and $(I, \leq)$

$\mathbb{M} \vDash \varphi(a_{i_1}, \dots, a_{i_n}) \Leftrightarrow \varphi \in \mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$

So if $(\bar{a}_i : i \in I)$, $\bar{b}_i : i \in I$ both B-indiscernible and $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/B) = \mathrm{tp}^{\mathrm{EM}}(\bar{b}/B)$, then $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{b}/B)$

Theorem 9.83 (extending indiscernibles)**.** *If $(\bar{a}_i : i \in I)$ is B-indiscernible, if $(J, \leq)$ extends $(I, \leq)$, then $\exists \bar{a}_j$ for $j \in J \setminus I$ s.t. $(\bar{a}_j : j \in J)$ is B-indiscernible*

Proof. extract B-indiscernible $(\bar{c}_j : j \in J)$ from $(\bar{a}_i : i \in I)$, $\mathrm{tp}^{\mathrm{EM}}(\bar{c}/B) = \mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$

the subsequence $(\bar{c}_i : i \in I)$ has same EM-type as

there is $\sigma \in \mathrm{Aut}(\mathbb{M}/B)$ s.t. $\sigma(\bar{c}_i) = \bar{a}_i$ for $i \in I$. Define $\bar{a}_j := \sigma(\bar{c}_j)$ for $j \in J \setminus I$ $\square$

Theorem 9.84. *If $\varphi(\bar{x}, \bar{y}) \in L$, TFAE*

1. *φ has O.P., $\bar{a}_i, \bar{b}_i$, $i \in \mathbb{Z}$, $\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$*
2. *same as (1) but $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$ is indiscernible*
3. *There is an indiscernible $(\bar{a}_i : i \in \mathbb{Z})$ some $\bar{b}$ s.t. $\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$*

Proof. $1 \to 2$: extract an indiscernible sequence from

$2 \to 3$: take $\bar{b} = \bar{b}_0$

$3 \to 1$: For any $j \in \mathbb{Z}$, $(\bar{a}_i : i \in \mathbb{Z}) \equiv_B (\bar{a}_{i+j} : i \in \mathbb{Z})$, there is $\sigma_j \in \mathrm{Aut}(\mathbb{M})$, $\sigma_j(\bar{a}_i) = \bar{a}_{i+j}$. Let $\bar{b}_j = \sigma_j(\bar{b})$. Then $\bar{a}_i\bar{b}_j = \sigma(\bar{a}_{i-j}\bar{b})$

$\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow \mathbb{M} \vDash \varphi(\bar{a}_{i-j}, \bar{b}) \Leftrightarrow i - j < 0 \Leftrightarrow i < j$ ☐

Corollary 9.85. *T is unstable $\Leftrightarrow$ there is $\varphi(\bar{x}, \bar{y})$ with O.P. $\Leftrightarrow$ $(\bar{a}_i : i \in \mathbb{Z})$, $\varphi(\bar{x}, \bar{y})$, $\bar{b}$ s.t. $\varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$*

Total indiscernibility

Example 9.17. In DLO, 1,2,3,4,... is indiscernible but not totally indiscernible

In a totally

Proposition 9.86. *If T is unstable, then $\exists$ indiscernible sequence that isn't totally indiscernible*

Proof. Take φ with O.P., take $(\bar{a}_i\bar{b}_i : i \in \mathbb{Z})$ witnessing O.P., then $\varphi(a_1, b_2) \wedge \neg\varphi(a_2, b_1)$, so $(\bar{a}_i\bar{b}_i : i \in \mathbb{Z})$ isn't totally indiscernible ☐

Definition 9.87. $\mathrm{tp}(a_1, \dots, a_n/B)$ is **symmetric** if $\forall$ permutation $\sigma \in S(n)$ $\bar{a}_1, \dots, \bar{a}_n \equiv_B \bar{a}_{\sigma(1)}, \dots, \bar{a}_{\sigma(n)}$

Remark. Let σ_i be the permutation swapping i and $i+1$ and fixing everything else.

$\mathrm{tp}(\bar{a}_1, \dots, \bar{b}_n/B)$ is symmetric iff it holds for each σ_i

Remark. Let $(\bar{a}_i : i \in I)$ be B-indiscernible. Let $p_n = \mathrm{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/B)$ for any $i_1 < \dots < i_n$. Then $(\bar{a}_i : i \in I)$ is totally B-indiscernible iff each p_n is symmetric

Remark. If $(\bar{a}_i : i \in I)$ is B-indiscernible, then $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/B)$ determines whether $\bar{a}$ is totally B-indiscernible

$\mathrm{tp}^{\mathrm{EM}}$ is $p_1, p_2, \dots$

Lemma 9.88. *Let $(\bar{a}_i : i \in \mathbb{Z})$ be B-indiscernible. Let $C = \{\bar{a}_i : i \notin \{0, 1\}\}$. If $\bar{a}_0\bar{a}_1 \equiv_{BC} \bar{a}_1\bar{a}_0$. Then $(\bar{a}_i : i \in \mathbb{Z})$ is totally B-indiscernible*

Proof. there is $\sigma_0 \in \mathrm{Aut}(\mathbb{M}/BC)$, $\sigma_0(\bar{a}_0) = \bar{a}_1$, $\sigma(\bar{a}_1) = \bar{b}_0$

By indiscernibility, there is $\alpha_i \in \mathrm{Aut}(\mathbb{M}/B)$ s.t. α_i swaps $\bar{a}_i$, $\bar{a}_{i+1}$ fixes $\bar{a}_j$ for $j \notin \{i, i+1\}$. This means $\bar{a}_1 \dots \bar{a}_n \equiv_B \bar{a}_{\sigma_i(1)} \dots \bar{a}_{\sigma_i(n)}$ so $\mathrm{tp}(\bar{a}_1, \dots, \bar{a}_n/B)$ is symmetric ☐

Proposition 9.89. *If $\mathbb{M}$ is stable and $A \subseteq \mathbb{M}$ small, then $\mathbb{M}$ is stable as an $L(A)$-structure*

Proof. Otherwise, there is $L(A)$-formula $\varphi(\bar{x}, \bar{y})$ with the O.P. $\varphi(\bar{x}, \bar{y}, \bar{c})$ for some $\bar{c} \in A$, $\bar{b}_i\bar{c}$ is the new $\bar{b}$ □

Theorem 9.90. *TFAE*

1. *T is stable*
2. *every indiscernible sequence is totally indiscernible*
3. *B-indiscernible $\Rightarrow$ totally B-indiscernible*

Proof. $3 \to 2$: trivial

$1 \to 3$: Suppose T stable but $(\bar{a}_i : i \in I)$ B-indiscernible not totally B-indiscernible

Extract $(\bar{a}'_i : i \in I)$ from $(\bar{a}_i : i \in I)$ some □

Corollary 9.91. *If T is stable, if $(\bar{a}_i : i \in I)$ is indiscernible, if D is definable, $\{i \in I : \bar{a}_i \in D\}$ is finite or cofinite in I*

Proof. Suppose not. Take $i_1, i_2, \dots \in I$ s.t. $a_{i_1}, a_{i_2}, \dots \notin D$, □

10 Fundamental Order and Forking

10.1 The fundamental order

Fix $n < \omega$

Definition 10.1. If $M \preceq \mathbb{M}$, $p \in S_n(M)$, $\varphi(x_1, \dots, x_n; \bar{y})$. p **represents** φ if $\exists \bar{b} \in M\ \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$. p **omits** φ otherwise

The **class** of p is $[p] = \{\varphi : p \text{ represents } \varphi\}$

$[p] \leq [q]$ if $[p] \supseteq [q]$

The **fundamental order** is $\{[p] : M \preceq \mathbb{M}, p \in S_n(M)\}$, with $\leq$ (depends on n). $p \leq q$ means $[p] \leq [q]$

Remark. $\leq$ is a partial order on the fundamental order but a preorder on the class $\{p : M \vDash T, p \in S_n(M)\}$

$[p]$ is not a standard notation

Example 10.1. $n = 1$, $\varphi(x, y) := x = y$. $p \in S_1(M)$ represents p iff $\exists b \in M$, $x = b \in p(x)$ iff p is a constant type

Example 10.2. $n = 1$, $T = \mathsf{DLO}$, there are four classes:

1. constant types

2. types at $+\infty$

3. types at $-\infty$

4. others

$$\begin{tikzcd}
2 \arrow[dr] & & 3 \arrow[dl] \\
& 4 \arrow[d] & \\
& 1 &
\end{tikzcd}$$

$x = y$ is represented in 1

$x < y$ is represented in 1,3,4 $\mathrm{tp}(2/\mathbb{R})$ has $x < 3$, $\mathrm{tp}(-\infty/\mathbb{R})$ has $x < 0$, $\mathrm{tp}(\sqrt{2}/\mathbb{Q})$ has $x < 2$, $\mathrm{tp}(+\infty/R)$ doesn't have $x < b$

$x > y$ is represented in 1,2,4

$\mathrm{tp}(\sqrt{2}/\mathbb{Q})$ and $\mathrm{tp}(0^+/\mathbb{R})$ have the same class

Goal: in a stable theory: if q is an extension of p, then if $q \sqsupseteq p$, then $[q] = [p]$, if $q \sqsupseteq p$, then $[q] < [p]$

Proposition 10.2. *Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $p \subseteq q$*

1. *$[q] \leq [p]$*

2. *$[q] = [p]$ iff for any L-formula $\varphi(\bar{x}, \bar{y})$, if $\bar{b} \in N$ and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$, then $\exists \bar{b}' \in M$ $\varphi(\bar{x}, \bar{b}') \in p$*

3. *if $q \sqsupseteq p$, then $[q] = [p]$*

Proof. 1. every formula φ represented by p is represented by q

2. $[q] = [p] \Leftrightarrow [q] \geq [p] \Leftrightarrow [q] \subseteq [p] \Leftrightarrow$ this condition

3.

$\square$

Remark. Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $p \subseteq q$

1. $[q] = [p]$ means that $\forall \varphi(\bar{x}, \bar{y}) \in L$, $\exists \bar{b} \in N$, $\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \exists \bar{b} \in M \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$

2. but $q \sqsupseteq p$ considers $L(M)$-formulas

$q \sqsupseteq p$ iff $[q] = [p]$ in $L(M)$

Proposition 10.3. *$M, N \preceq \mathbb{M}$, $p \in S_n(M)$, $q \in S_n(N)$, then $[p] \geq [q]$ iff $\exists$ ultrafilter $\mathcal{U}$ and elementary embedding $M \to N^{\mathcal{U}}$ making $q^{\mathcal{U}} \supseteq p$*

Proof. $\Rightarrow$ similar to 9.2

$\Leftarrow$: $[q^{\mathcal{U}}] = [q]$ because $q^{\mathcal{U}} \sqsupseteq q$, $[q^{\mathcal{U}}] \leq [p]$ because $q^{\mathcal{U}} \supseteq p$ $\square$

10.2 The fundamental order in stable theory

Assume T is stable

Lemma 10.4. *Suppose $M \preceq N \preceq \mathbb{M}$, $p \in S_n(M)$, $q_1, q_2 \in S_n(N)$, $q_1, q_2 \supseteq p$ and $[q_1] = [p] = [q_2]$. Then $q_1 = q_2$.*

In other words, there is at most one extension of p to N with the same class as p

Proof. similar to 9.6

Suppose $q_1 \neq q_2$, $\exists \varphi(\bar{x}, \bar{b})$ s.t. $\varphi \in q_1, \neg\varphi \in q_2$

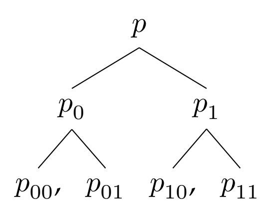
Let $\beta = [p]$

Claim: If $M' \preceq \mathbb{M}$, $p' \in S_n(M')$, $[p'] = \beta$, then $\exists N' \succeq M'$, $\exists q_1', q_2' \in S_n(N')$, $q_1', q_2' \supseteq p'$, $[q_1'] = [q_2'] = \beta$ and $\exists \bar{b}' \in N'$, $\varphi(\bar{x}, \bar{b}') \in q_1'$ and $\neg\varphi \in q_2'$

$[p'] \geq [p]$, so there $\mathcal{U}$, elementary embedding $M' \to M^{\mathcal{U}}$ s.t. $p^{\mathcal{U}} \supseteq p'$. Then we have $M' \to M^{\mathcal{U}} \to N^{\mathcal{U}}$

$[q_1^{\mathcal{U}}] = [q_1] = \beta = [q_2] = [q_2^{\mathcal{U}}]$. Let $q_i' = q_i^{\mathcal{U}}$, $N' = N^{\mathcal{U}}$

Using the claim, we can build a tree of types



where $p_{\sigma 0}$ and $p_{\sigma 1}$ are extensions of p_σ differing by a formula $\varphi(\bar{x}, \bar{b}_\sigma)$. Then φ has the dichotomy property $\square$

Proposition 10.5. *If $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $q \supseteq p$*

1. $q \sqsupseteq p \Leftrightarrow [q] = [p]$
2. $q \not\sqsupseteq p \Leftrightarrow [q] < [p]$

Proof. Let q' be the heir of p, $q' \in S_n(N)$

If $q \sqsupseteq p$, then $q = q'$

If $[q] = [p]$, then $[q] = [q'] = [p]$ so Lemma 10.4 shows $q = q'$ $\square$

10.3 bounds

T is stable

Fix $A \subseteq \mathbb{M}$, $p \in S_n(A)$

Definition 10.6. If $M \preceq \mathbb{M}$, $M \supseteq A$, then $\mathrm{Ex}_M(p) = \{[q] : q \in S_n(M), q \supseteq p\}$

Lemma 10.7. *Every chain in* $\mathrm{Ex}_M(p)$ *has an upper bound*

Proof. Let $F = \{q \in S_n(M) : q \supseteq p\}$. Suppose $\{[q_i] : i \in I\}$ is a chain, $q_i \in F$, $(I, \leq)$ a linear order, $[q_i] \leq [q_j]$ for $i \leq j$

If $i \leq j$, q_i omits φ, then q_j omits φ

Let $\Sigma(\bar{x}) = \{\neg\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y})$ omitted by some $q, \bar{b} \in M\}$

Claim: $p(\bar{x}) \cup \Sigma(\bar{x})$ is consistent

suppose $\varphi_1, \ldots, \varphi_m$, φ_j is omitted by q_{i_j}, $i_j \in I$, $\bar{b}_1, \ldots, \bar{b}_m \in M$. Want $p \cup \{\neg\varphi_j(\bar{x}, \bar{b}_j) : 1 \leq j \leq m\}$ consistent

Take $q(\bar{x}) \in S_n(M)$ a completion of $p(\bar{x}) \cup \Sigma(\bar{x})$. Then $q \in F$, so $[q] \in \mathrm{Ex}_M(p)$. ☐

Definition 10.8. $\mathrm{Bd}_M(p) = \{\text{maximal } \beta \in \mathrm{Ex}_M(p)\}$

Elements of $\mathrm{Bd}_M(p)$ are called **bounds** of p

Corollary 10.9. $\forall \beta \in \mathrm{Ex}_M(p)$, $\exists \beta' \in \mathrm{Bd}_M(p)$, $\beta' \geq \beta$, *and* $\mathrm{Bd}_M(p)$ *is not empty*

Example 10.3. Suppose $A \preceq \mathbb{M}$, $p \in S_n(A)$, A is a model

Claim: $[p] = \max \mathrm{Ex}_M(p)$, so $\mathrm{Bd}_M(p) = \{[p]\}$

Take $q \in S_n(M)$, $q \sqsupseteq p$, then $[q] = [p]$, $[q] \in \mathrm{Ex}_M(p)$. If $r \in S_n(M)$, $r \supseteq p$, then $[r] \leq [p]$, so if $p \in \mathrm{Ex}_M(p)$ then $\beta \leq [p]$

Lemma 10.10. *Suppose* $M, N \preceq \mathbb{M}$, $M, N \supseteq A$, $p \in S_n(A)$

1. $\forall \beta \in Ex_M(p)$, $\exists \beta' \in \mathrm{Ex}_N(p)$, $\beta' \geq \beta$
2. $\mathrm{Bd}_M(p) = \mathrm{Bd}_N(p)$

Proof.
1. Take $M' \preceq \mathbb{M}$, $M' \supseteq M \cup N$, $\beta \in \mathrm{Ex}_M(p)$ means $\exists q \in S_n(M)$, $q \supseteq p$, $[q] = \beta$

   Let $q' \in S_n(M')$ be $q' \sqsupseteq q$

   Let $r = q' \restriction N$. Then $r \supseteq p$, so $[r] \in \mathrm{Ex}_N(p)$. $[r] \geq [q'] = [q] = \beta$

2. suppose $\beta \in \mathrm{Bd}_M(p)$

   - by 1, there is $\beta' \in \mathrm{Ex}_N(p)$ with $\beta \leq \beta'$

- by Corollary 10.9, there is $\beta'' \in \text{Bd}_N(p)$ with $\beta' \leq \beta''$
- By 1, there is $\beta''' \in \text{Ex}_M(p)$ with $\beta'' \leq \beta'''$

Then $\beta \leq \beta' \leq \beta'' \leq \beta''' \in \text{Ex}_M(p)$. Therefore

$$\beta = \beta' = \beta'' = \beta'''$$

This shows $\text{Bd}_M(p) \subseteq \text{Bd}_N(p)$

□

Since $\text{Bd}_M(p)$ doesn't depend on M, we write it as $\text{Bd}(p)$

10.4 Theorem of the bound

T is stable

Definition 10.11. $p \in S_n(\mathbb{M})$ is **Lascar A-invariant** if p is M-invariant for every $A \subseteq M \preceq \mathbb{M}$

weaker than being A-invariant in stable theory

Lemma 10.12. *If $A \subseteq M \preceq \mathbb{M}$, $p \in S_n(A)$, $q \in S_n(M)$, $q \supseteq p$, $[q] \in \text{Bd}(p)$. Let $q^{\mathbb{M}}$ be the global heir of q. Then $q^{\mathbb{M}}$ is Lascar A-invariant*

Proof. By 10.2, $[q^{\mathbb{M}}] = [q] \in \text{Bd}(p)$. If $q^{\mathbb{M}}$ isn't Lascar A-invariant, there is small $N \supseteq A$ $q^{\mathbb{M}}$ isn't N-invariant, not N-definable. Then $q^{\mathbb{M}} \not\supseteq q^{\mathbb{M}} \restriction N$ (or else $q^{\mathbb{M}}$ would be N-definable 9.48). By Proposition 10.5, $[q^{\mathbb{M}} \restriction N] > [q^{\mathbb{M}}] = [q]$

Let $r = q^{\mathbb{M}} \restriction N$, $r \supseteq p$, so $[r] \in \text{Ex}_N(p)$, $[q] \in \text{Bd}(p) = \text{Bd}_N(p)$ is maximal in $\text{Ex}_N(p)$, but $[r] > [q]$, $[r] \in \text{Ex}_N(p)$ □

Lemma 10.13. *Fix $\bar{b}$ and A, then $\exists M \supseteq A$, $M \preceq \mathbb{M}$, the global heir of* $\text{tp}(\bar{b}/M)$ *is Lascar A-invariant. Also given $\beta \in$* $\text{Bd}(\text{tp}(\bar{b}/A))$*, can make* $\text{tp}(\bar{b}/M)$ *and it's heir have class β*

Proof. Take $\beta \in \text{Bd}(p)$, $p = \text{tp}(\bar{b}/A)$. Take $M \supseteq A$ $M \preceq \mathbb{M}$. Take $q \in S_n(M)$, $[q] = \beta$. Take $\bar{b}_0 \vDash q$, $\text{tp}(\bar{b}_0/A) = \text{tp}(\bar{b}/A)$. There is $\sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(\bar{b}_0) = \bar{b}$.

Move M, q, b_0 by σ, We may assume $\bar{b}_0 = \bar{b}$, so $\text{tp}(\bar{b}/M) = q$, $[q] = \beta$.

By 10.12, $q^{\mathbb{M}}$ is Lascar A-invariant □

Lemma 10.14. *Fix $\bar{b}$, A. Suppose $M_1, M_2 \preceq \mathbb{M}$, $M_1, M_2 \supseteq A$. Let $p_i \in S_n(\mathbb{M})$ be the heir of* $\text{tp}(\bar{b}/M_i)$*. Suppose p_1, p_2 are Lascar A-invariant, then $p_1 = p_2$*

Proof. Suppose $p_1 \neq p_2$. Take $\varphi(\bar{x}, \bar{c}) \in p_1(\bar{x})$, $\neg\varphi(\bar{x}, \bar{c}) \in p_2$.

Lemma 10.13 shows there is $M_3 \preceq \mathbb{M}$, $M_3 \supseteq A$ s.t. $\text{tp}(\bar{c}/M_3) \sqsubseteq r \in S_n(\mathbb{M})$ and r is Lascar A-invariant.

Take $\bar{e} \vDash r \upharpoonright M_1 M_2 M_3 \bar{b}$. Note $\bar{b} \vDash p_1 \upharpoonright M_1$ and $\bar{e} \vDash r \upharpoonright M_1 \bar{b}$. Then $(\bar{b}, \bar{e}) \vDash (p_1 \otimes r) \upharpoonright M_1$ since p_1, r are M_1-invariant. In stable theory, product commutes. Therefore $(\bar{e}, \bar{b}) \vDash (r \otimes p_1) \upharpoonright M_1$. Then $\bar{b} \vDash p_1 \upharpoonright M_1 e$.

$\bar{e} \vDash r \upharpoonright M_3 = \text{tp}(\bar{c}/M_3)$, $\bar{e} \equiv_{M_3} \bar{c}$, p_1 is M_3-invariant. Hence $\varphi(\bar{x}, \bar{e}) \in p_1$. So $\mathbb{M} \vDash \varphi(\bar{c}, \bar{e})$

Same argument with p_2, get $\mathbb{M} \vDash \neg\varphi(\bar{c}, \bar{e})$, a contradiction ☐

Theorem 10.15. *If $p \in S_n(A)$, $|\text{Bd}(p)| = 1$*

Proof. Take $\bar{b} \vDash p$, $\beta_1, \beta_2 \in \text{Bd}(p)$. Lemma 10.13, there is $A \subseteq M_1, M_2 \preceq \mathbb{M}$ s.t. $[\text{tp}(\bar{b}/M_i)] = \beta$ if $p_i = \text{tp}(\bar{b}/M_i)$, $p_i^{\mathbb{M}}$ is Lascar A-invariant.

Lemma 10.14 $p_1^{\mathbb{M}} = p_2^{\mathbb{M}}$ ☐

Definition 10.16. $\text{bd}(p)$ =the bound of p

example

10.5 Non-forking extensions

Assume stability

Proposition 10.17. *If $A \subseteq B$, $p \in S_n(A)$, $q \in S_n(B)$, $p \subseteq q$, then* $\text{bd}(q) \leq \text{bd}(p)$

Proof. Take $M \supseteq B$, $M \preceq \mathbb{M}$, $r \in S_n(M)$ extending q with $[r] = \text{bd}(q)$. Then r extends p, so $[r] \in \text{Ex}_M(p)$. As $\text{bd}(p)$ is the maximum of $\text{Ex}_M(p)$ we must have $[r] \leq \text{bd}(p)$ ☐

Definition 10.18. If $A \subseteq B$, $p \in S_n(A)$, $q \in S_n(B)$, $q \supseteq p$, q is a **nonforking extension** of p iff $\text{bd}(q) = \text{bd}(p)$

Proposition 10.19. *If $M \preceq N$ and $q \in S_n(N)$ extends $p \in S_n(M)$, then q is a non-forking extension of p iff q is an heir of p*

Proposition 10.19 ensures the notation $q \sqsupseteq p$ is unambiguous

Proof. $\text{bd}(p) = [p]$ and $\text{bd}(q) = [q]$ ☐

Proposition 10.20 (Full transitivity)**.** *Suppose $A_1 \subseteq A_2 \subseteq A_3$ and $p_i \in S_n(A_i)$ for $i = 1, 2, 3$ with $p_1 \subseteq p_2 \subseteq p_3$. Then $p_1 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_2$ and $p_2 \sqsubseteq p_3$*

Proposition 10.21 (Extension)**.** *If $p \in S_n(A)$ and $B \supseteq A$, then there is at least one $q \in S_n(B)$ with $q \sqsupseteq p$*

Proof. Take a small model $M \supseteq B$. Then $\mathrm{bd}(p) \in \mathrm{Bd}(p) \subseteq \mathrm{Ex}_M(p)$, so there is $r \in S_n(M)$ extending p with $[r] = \mathrm{bd}(p)$. Let $q = r \restriction B$. Then $\mathrm{bd}(r) = \mathrm{bd}(p)$, so $r \sqsupseteq p$. By full transitivity, $q \sqsupseteq p$ $\square$

10.6 Forking formulas and Lascar invariance

Lemma 10.22. *If $A \subseteq M \preceq \mathbb{M}$ and if the global heir of* $\mathrm{tp}(\bar{b}/M)$ *is Lascar A-invariant, then* $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A)$

Proof. Let β be the bound of $\mathrm{tp}(\bar{b}/A)$. By Lemma 10.13 there is a small model $M' \supseteq A$ s.t. the global heir of $\mathrm{tp}(\bar{b}/M')$ is Lascar A-invariant and has class β. By Lemma 10.14 $\mathrm{tp}(\bar{b}/M')$ and $\mathrm{tp}(\bar{b}/M)$ have the same global heir. By Proposition 10.2 they have the same class. Then the class of $\mathrm{tp}(\bar{b}/M)$ is $\beta = \mathrm{bd}(\mathrm{tp}(\bar{b}/A))$, implying $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A)$ $\square$

Proposition 10.23 (Forking and Lascar A-invariance)**.** *If p is a global type and $A \subseteq \mathbb{M}$, then $p \sqsupseteq (p \restriction A)$ iff p is Lascar A-invariant*

Proof. First suppose $p \sqsupseteq (p \restriction A)$. For any small model $M \supseteq A$, we have $p \sqsupseteq (p \restriction M)$ by Full transitivity, which then means p is the heir of $p \restriction M$ by Proposition 10.19. Then p is M-definable, so p is Lascar A-invariant

Conversely, suppose p is Lascar A-invariant. Take a small model $M \supseteq A$ and take $\bar{b} \models p \restriction M$. Then p is M-definable, so p is the global heir of $p \restriction M = \mathrm{tp}(\bar{b}/M)$. By Lemma 10.22, $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A) = p \restriction A$. But p is the heir of $\mathrm{tp}(\bar{b}/M)$, so $p \sqsupseteq \mathrm{tp}(\bar{b}/M) \sqsupseteq p \restriction A$. By transitivity we have $p \sqsupseteq (p \restriction A)$ $\square$

Corollary 10.24. *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \sqsupseteq p$ iff some global extension of q is Lascar A-invariant*

Definition 10.25. An $L(\mathbb{M})$-formula $\varphi(\bar{x})$ **forks over** A if every global type containing it fails to be Lascar A-invariant

Proposition 10.26 (Finite Character)**.** *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \not\sqsupseteq p$ (q is a forking extension of p) iff some formula in q forks over A*

Proof. For any model M, let $\Sigma_M(\bar{x})$ be the global partial type

$$\{\varphi(\bar{x};\bar{b}) \leftrightarrow \varphi(\bar{x};\bar{c}) : \varphi \in L, \bar{b} \equiv_M \bar{c}\}$$

A global type $p \in S_n(\mathbb{M})$ extends Σ_M iff it is M-invariant, iff it is M-definable. Define $\Sigma_A(\bar{x})$ to be the union of $\Sigma_M(\bar{x})$ for M ranging over small models containing A. Then $p \in S_n(\mathbb{M})$ extends $\Sigma_A(\bar{x})$ iff it is Lascar A-invariant. Therefore an $L(\mathbb{M})$-formula $\psi(\bar{x})$ forks over A iff $\Sigma_A(\bar{x}) \cup \{\psi(\bar{x})\}$ is inconsistent. By Corollary 10.24, $q \not\supseteq p$ iff $\Sigma_A(\bar{x}) \cup q(\bar{x})$ is inconsistent. Then the result follows by compactness $\square$

Intuition if φ forks over A, then $\varphi(\mathbb{M})$ is "small", and $\{\varphi(\mathbb{M}) : \varphi \text{ forks over } A\}$ is an ideal

10.7 The dichotomy property and the fundamental order

Lemma 10.27. *Assume stability. Suppose $M \preceq N \preceq \mathbb{M}$, $p \in S_n(M)$, and $q_1, q_2 \in S_n(N)$ are exteqqeQ*

11 Algebraic closure and imaginaries

11.1 Many-sorted logic

11.1.1 First approximation: many-sorted structures

Definition 11.1. A (**single sorted**) **structure** consists of a set M and a collection of **functions**, **relations** and **constants**. Each function is a function $f : M^{n_f} \to M$ for some number n_f called the **arity** of f. Each relation is a relation $R \subseteq M^{n_R}$ for some n_R called the **arity** of R. Each constant is an element of M

Definition 11.2. A **many-sorted structure** consists of a collection of **sorts**, **functions**, **relations** and **constants**. Each sort is a set. Each function is a function $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ where $X_1, X_2, \dots, X_n, Y$ are sorts. Each relation is a relation $R \subseteq X_1 \times \cdots \times X_n$ where $X_1, \dots, X_n$ are sorts. Each constant is an element of a sort

This approach works if we only need to consider definable sets and formulas within a fixed structure. If we want to talk about theories or elementary equivalence, we need to define many-sorted languages before we can properly define many-sorted structures.

11.1.2 Many-sorted languages

Definition 11.3. A **many-sorted language** consists of the following data:

1. A set $\mathcal{S}$ of **sorts**

2. A set $\mathcal{F}$ of **function symbols**: for each $f \in \mathcal{F}$, a finite non-empty list of sorts $(X_1, \dots, X_n, Y)$, called the **signature** of f

3. A set $\mathcal{R}$ of **relation symbols**: for each $R \in \mathcal{R}$, a finite list of sorts $(X_1, \dots, X_n)$, called the **signature** of R

4. A set $\mathcal{C}$ of **constant symbols**: for each $c \in \mathcal{C}$, a sort X, called the **signature** of c

11.2 Definable closure

Work in a monster model $\mathbb{M}$

Fact 11.4. *If $\mathcal{F}$ is a small family of definable $D \subseteq \mathbb{M}^n$, suppose $X \subseteq \mathbb{M}^n$ definable. Suppose X is "infinite boolean combination" of $\mathcal{F}$, i.e., if $\bar{a}, \bar{a} \in \mathbb{M}^n$ and $\forall D \in \mathcal{F}$, $\bar{a} \in D \Leftrightarrow \bar{b} \in D$, then $\bar{a} \in X \Leftrightarrow \bar{b} \in X$. Then X is a (finite) boolean combination of sets in $\mathcal{F}$*

Proof. WLOG, $\mathcal{F}$ is closed under finite boolean combination

- If $\bar{a} \in X$, $\exists D \in \mathcal{F}$, $\bar{a} \in D$
- X is a finite union of things in $\mathcal{F}$

$\square$

Fact 11.5 (9.46)**.** *If $D \subseteq \mathbb{M}^n$ definable, $A \subseteq \mathbb{M}$ small, then D is A-definable iff D is A-invariant*

Definition 11.6. If $D_1, \dots, D_{n+1}$ are A-definable, and $f : D_1 \times \cdots \times D_n \to D_{n+1}$, then f is A**-definable** if $\Gamma(f) = \{(\bar{a}, b) : b = f(\bar{a})\}$ is definable

Definition 11.7. If $A \subseteq \mathbb{M}$, $\mathrm{dcl}(A) = \{b \in \mathbb{M} : \{b\}$ is A-definable$\}$

Example 11.1. In a field, $a \div b \in \mathrm{dcl}(\{a, b\})$ because $\{a \div b\}$ is $\varphi(\mathbb{M})$, $\varphi(x) := bx = a$

Note: if $\bar{b} \in \mathbb{M}^n$, $\bar{b} \in \mathrm{dcl}(A)^n \Leftrightarrow \{\bar{b}\} \subseteq \mathbb{M}^n$ is A-definable

Proposition 11.8. *If $\bar{b} \in \mathbb{M}^n$, $A \subseteq \mathbb{M}$, TFAE*

1. *$\bar{b} \in \mathrm{dcl}(A)$, i.e., $\{\bar{b}\}$ is A-definable*
2. *$\forall \sigma \in Aut(\mathbb{M}/A)$, $\sigma(\bar{b}) = \bar{b}$, i.e. $\{\bar{b}\}$ is A-invariant*

3. $\bar{b}$ *is the only realization of* $\operatorname{tp}(\bar{b}/A)$

Proof. $1 \Leftrightarrow 2$ by fact since $\{\bar{b}\}$ is definable
$2 \Leftrightarrow 3$: let $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \operatorname{Aut}(\mathbb{M}/A)\}$. □

Proposition 11.9. *1.* $A \subseteq \operatorname{dcl}(A)$

2. $A \subseteq B \Rightarrow \operatorname{dcl}(A) \subseteq \operatorname{dcl}(B)$

3. $\operatorname{dcl}(\operatorname{dcl}(A)) = \operatorname{dcl}(A)$

4. D *is* A*-definable* $\Leftrightarrow$ D *is* $\operatorname{dcl}(A)$*-definable*
Conditions 1-3 say that $\operatorname{dcl}(-)$ *is an abstract "closure operator"*

Proof. 4. If D is $\operatorname{dcl}(A)$-definable, $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$, $b \in \operatorname{dcl}(A)$, $\sigma(b) = b$ by Proposition 11.8, $\sigma \in \operatorname{Aut}(\mathbb{M}/\operatorname{dcl}(A))$, $\sigma(D) = D$, so D is A-invariant

3. take $b \in \operatorname{dcl}(\operatorname{dcl}(A))$, $\{b\}$ is $\operatorname{dcl}(A)$-definable, $\{b\}$ is A-definable, $b \in \operatorname{dcl}(A)$

□

Definition 11.10. A is **definably closed** if $\operatorname{dcl}(A) = A$

Proposition 11.11. $\operatorname{dcl}(A)$ *is the smallest definably closed set containing* A

Proof. $\operatorname{dcl}(A)$ is definably closed
$\operatorname{dcl}(A) \supseteq A$
Now suppose $B = \operatorname{dcl}(B)$ and $B \supseteq A$, by "monotonicity", $\operatorname{dcl}(B) \supseteq \operatorname{dcl}(A)$, so $B \supseteq \operatorname{dcl}(A)$ □

Fact 11.12. *if* $M \models ACF_0$*, if* $A \subseteq M$*, then* $A = \operatorname{dcl}(A) \Leftrightarrow A$ *is a subfield of* M
$\Rightarrow$*: easy* $\Leftarrow$*: harder*
It fails in ACF_p*, in* ACF_p*,* $p > 0$*,* $K \subseteq M$ *is definably closed* $\Leftrightarrow \forall x \in K$*,* $\sqrt[p]{x} \in K$

Definition 11.13. $\bar{a}, \bar{b}$ are **interdefinable** iff $\operatorname{dcl}(\bar{a}) = \operatorname{dcl}(\bar{b})$ iff $\bar{a} \in \operatorname{dcl}(\bar{b})$ and $\bar{b} \in \operatorname{dcl}(\bar{a})$

Lemma 11.14. $\operatorname{dcl}(\bar{a}) = \operatorname{dcl}(\bar{b}) \Leftrightarrow Aut(\mathbb{M}/\bar{a}) = Aut(\mathbb{M}/\bar{b})$

Proof. By Proposition 11.8, $\operatorname{dcl}(\bar{a}) \subseteq \operatorname{dcl}(\bar{b}) \Leftrightarrow \bar{a} \in \operatorname{dcl}(\bar{b}) \Leftrightarrow \operatorname{Aut}(\mathbb{M}/\bar{b}) \subseteq \operatorname{Aut}(\mathbb{M}/\bar{a})$ □

Lemma 11.15. *If* $\operatorname{dcl}(\bar{a}) = \operatorname{dcl}(\bar{b})$*, then* $\exists$ $\emptyset$*-definable bijection* $f : X \to Y$ *s.t.* $f(\bar{a}) = \bar{b}$

Proof. $\bar{a} \in \mathrm{dcl}(\bar{b})$, so there is L-formula $\varphi_1(\bar{x}, \bar{y})$ s.t. $\{\bar{a}\} = \varphi_1(\mathbb{M}, \bar{b})$, also there is $\{\bar{b}\} = \varphi_2(\bar{a}, \mathbb{M})$

Let $\varphi = \varphi_1 \wedge \varphi_2$, can replace φ_1, φ_2 with φ.

Let $\psi(\bar{x}, \bar{y})$ be $\varphi(\bar{x}, \bar{y}) \wedge \exists! \bar{w}\ \varphi(\bar{w}, \bar{y}) \wedge \exists! \bar{z}\ \varphi(\bar{x}, \bar{z})$. Then $\psi(\bar{x}, \bar{y})$ defines a bijection and $f(\bar{a}) = \bar{b}$

We can extract the definition of $f(\bar{a}) = \bar{b}$ and fill f with garbage $\square$

11.3 Algebraic closure

Definition 11.16. $\mathrm{acl}(A) = \bigcup\{D \subseteq \mathbb{M}^1 : D \text{ is } A\text{-definable}, |D| < \infty\}$

Example 11.2. In fields, $\sqrt{a} \in \mathrm{acl}(a)$ because $\{\sqrt{a}, -\sqrt{a}\}$ is $\{a\}$-definable

Note: $\bar{b} \in \mathrm{acl}(A) \Leftrightarrow$ there is A-definable $D \subseteq \mathbb{M}^n$, $\bar{b} \in D$, $|D| < \infty$

If $\bar{b} \in D \subseteq \mathbb{M}^n$, then let $D_i = \pi_i(D)$, $\pi_i(\bar{x}) = x$,

Proposition 11.17. *If $\bar{b} \in \mathbb{M}^n$, $A \subseteq \mathbb{M}$ small, let $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in Aut(\mathbb{M}/A)\}$*

1. *If $\bar{b} \in \mathrm{acl}(A)$, then S is finite and A-definable*
2. *If $\bar{b} \notin \mathrm{acl}(A)$, then S is large (S is not small)*

Proof.

1. $\bar{b} \in \mathrm{acl}(A)$, there is D finite, A-definable, $\bar{b} \in D$. $\sigma(\bar{b}) \in \sigma(D) = D$ for $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, so $S \subseteq D$, S is finite, then S is definable. Also S is A-invariant

   for each $a \in D \setminus S$, we have a $L(A)$-formula φ_a s.t. $\mathbb{M} \models \neg\varphi_a(a) \wedge \varphi_a(b)$. Then $\bigwedge_{a \in D \setminus S} \varphi_a$ extract S from D

2. If $\bar{b} \notin \mathrm{acl}(A)$ but S is small. Let $\Sigma(\bar{x}) = \mathrm{tp}(\bar{b}/A) \cup \{\bar{x} \neq \bar{c} : \bar{c} \in S\}$. $\Sigma(\bar{x})$ is inconsistent. By Compactness, there is $\psi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$, $\bar{c}_1, \dots, \bar{c}_m \in S$, $\{\psi(\bar{x}), x \neq \bar{c}_1, \dots, x \neq \bar{c}_n\}$ is inconsistent. Thus $D = \psi(\mathbb{M}) \subseteq \{\bar{c}_1, \dots, \bar{c}_m\}$. D is A-definable, $\bar{b} \in D$, D is finite, so $\bar{b} \in \mathrm{acl}(A)$

$\square$

Proposition 11.18.

1. *$A \subseteq \mathrm{acl}(A)$*
2. *$A \subseteq B \Rightarrow \mathrm{acl}(A) \subseteq \mathrm{acl}(B)$*
3. *$\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$*

Proof. 3. Take $b \in \mathrm{acl}(\mathrm{acl}(A))$, then $b \in \varphi(\mathbb{M}, \bar{c}) = D$, D is finite, D is $\mathrm{acl}(A)$-definable, $\bar{c} \in \mathrm{acl}(A)$

$\mathcal{F} = \{\sigma(D) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\} = \{\varphi(\mathbb{M}, \sigma(\bar{c})) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite by Proposition 11.17, $\sigma(D)$ is finite as D is finite. Therefore $D' = \bigcup \mathcal{F}$ is finite, A-invariant, $b \in D'$ as $b \in D \in \mathcal{F}$. Therefore $b \in \mathrm{acl}(A)$

□

Definition 11.19. A is **algebraically closed** if $A = \mathrm{acl}(A)$

Proposition 11.20. $\mathrm{acl}(A)$ *is the smallest algebraically closed set containing* A

Proposition 11.21. *If* $M \preceq \mathbb{M}$, *then* $\mathrm{acl}(M) = M$

Proof. Otherwise, $\mathrm{acl}(M) \supsetneq M$, take $b \in \mathrm{acl}(M) \setminus M$. $S = \{\sigma(b) : \sigma \in \mathrm{Aut}(\mathbb{M}/M)\}$. By Proposition 11.17, S is finite, M-definable. M is M-invariant so $S \cap M = \emptyset$, contradicting to Tarski-Vaught criterion □

Proposition 11.22. *If* $M \models$ *ACF and* K *is a subfield. TFAE*

1. $K = \mathrm{acl}(K)$
2. $K \models$ *ACF*
3. $K \preceq M$

Idea: in ACF, field theoretic algebraic closure = model theoretic algebraic closure

Proof. 1 → 2: Take $P(x)$, $P \not\equiv 0$, $M \models$ ACF, $P(x) = c \cdot (x - r_1) \dots (x - r_n)$, $c \in K$, $r_1, \dots, r_n \in M$. $D = \{r_1, \dots, r_n\}$ is K-definable. $K = \mathrm{acl}(K)$ implies $D \subseteq \mathrm{acl}(K) = K$

2 → 3: q.e. □

Fact 11.23. *If* T *is strongly minimal,* $A \subseteq M \models T$

$$A \preceq M \Leftrightarrow |A| = \infty \text{ and } A = \mathrm{acl}(A)$$

11.4 Imaginaries

Definition 11.24. An **A-interpretable set** is X/E where X is A-definable and $E \subseteq X^2$ is A-definable equivalence relation on X

Interpretable = $\mathbb{M}$-interpretable. 0-interpretable = $\emptyset$-interpretable

Definition 11.25. If M is a structure, M^{eq} is the expansion of M by

- A new sort for every 0-interpretable D/E
- Relation symbols for $D \to D/E$ (i.e., if $D \subseteq M^n$, add $R \subseteq M^n \times (D/E)$ where $R(a_1, \dots, a_n, b) \Leftrightarrow \bar{a} \in D, [a]_E = b$) If $D \subsetneq M^n$ can't add a function symbol

M is called the **home sort** (when M is one-sorted)

M^{eq} is the expansion of M obtained by adding each 0-interpretable set as a new sort, with enough data to connect the new sorts to the old sorts

If $\mathbb{M}$ is a monster model, then $\mathbb{M}^{\text{eq}}$ is a monster model(7) with the "same" automorphism group(6) and the "same" small models(5). If we restrict our attention to the original sorts from M, then $\mathbb{M}^{\text{eq}}$ and $\mathbb{M}$ have the same definable sets(1) (4) and the same partial elementary maps(2). However, $\mathbb{M}^{\text{eq}}$ have some new elements, and the definable sets in $\mathbb{M}^{\text{eq}}$ correspond exactly to the interpretable sets in the original structure $\mathbb{M}$ (8)(9). On the other hand, the new elements of $\mathbb{M}^{\text{eq}}$ are definable from the old elements(3). So $\mathbb{M}^{\text{eq}}$ **is a way of coverting interdefinable sets into definable sets while preserving most other things**

Fact 11.26. *1. If $X \subseteq M^n$, X is 0-definable in $M \Leftrightarrow X$ is 0-definable in M^{eq}. In other words, M^{eq} doesn't define any new sets on the original sorts of M*

2. *If $A, B \subset M$, $f : A \to B$ bijection, then f is a partial elementary map in M $\Leftrightarrow f$ is a partial elementary map in M^{eq}*

3. *In M^{eq}*, $\text{dcl}(M) = M^{\text{eq}}$

4. *Consequently, any M^{eq}-definable set $X \subseteq M^n$ is M-definable in M^{eq}, and therefore M-definable in M*

5. *If $N \preceq M$ then $N^{\text{eq}} \preceq M^{\text{eq}}$ (more precisely, $N^{\text{eq}} \cong \text{dcl}_{M^{\text{eq}}}(N) \preceq M^{\text{eq}}$) This gives an $\preceq$-preserving bijection.*

   Moreover, all elementary substructures of M arise this way. This yields an order-preserving bijection between the elementary substructures of M and the elementary substructures of M^{eq}. In particular, all elementary substructures of M^{eq} arise this way

6. *If $\sigma \in Aut(M)$, σ acts on M^{eq} in a natural way. σ induces $\hat{\sigma} \in Aut(M^{\text{eq}})$. This gives an isomorphism $Aut(M) \cong Aut(M^{\text{eq}})$*

   5 and 6 come from equivalence of categories $\text{Mod}\, T \to \text{Mod}\, T^{\text{eq}}$

7. *M is κ-saturated and strongly κ-homogeneous $\Leftrightarrow M^{\text{eq}}$ is κ-saturated and strongly κ-homogeneous*

8. *If D/E is 0-interpretable in M, then D/E is 0-definable in M^{eq}*

9. *If X is 0-definable in M^{eq}, then there is 0-interpretable D/E in M and a 0-definable bijection $f : X \to D/E$ in M^{eq}*

Ideas (8-9): interpretable in $M \Leftrightarrow$ definable in M^{eq}

Proof. Only some remarks

1. More generally, one can show that $X \subseteq M^n \times \prod_{j=1}^m (D_j/E_j)$ is 0-definable in M^{eq} iff $\tilde{X} \subseteq M^n \times \prod_{j=1}^m D_j$ is 0-definable in M where $\tilde{X} = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n, [b_1]_E, \dots, [b_M]_{E_M}) \in X\}$.

3. using the definable functions $D \to D/E$

4. Note (1) means that if $\bar{x}$ is a tuple of variables in the old sorts of M, then any L^{eq}-formula $\phi(x)$ is equivalent to an L-formula

5. Behind the scenes, there is a theory T^{eq} and $M \vDash T \Rightarrow M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$. Moreover, all models of T^{eq} have the form M^{eq} up to isomorphism. Finally, elementary embeddings $M \to N$ correspond bijectively to elementary embeddings $M^{\mathrm{eq}} \to N^{\mathrm{eq}}$

7. $\Leftarrow$ is easier by (2). If you just want a monster model $\mathbb{M}$ s.t. $\mathbb{M}^{\mathrm{eq}}$ is a monster model, you can do the following: take $M \vDash T$, construct M^{eq}, take some monster elementary extension $U \succeq M^{\mathrm{eq}}$, then check that $U$is $\mathbb{M}^{\mathrm{eq}}$ for some $\mathbb{M} \succeq M$

   If M is κ-saturated and strongly κ-homogeneous. κ-saturation is not hard in terms of a compactness-like property: if $|A| < \kappa$ and a collection of A-definable sets has FIP, then it has non-empty intersection. To transfer this from M to M^{eq}, one takes the A-definable sets in M^{eq} and lifts them to A-definable sets in M using the maps $D \to D/E$

9. This comes down to the following things. First, if D/E and D'/E' are two interpretable sets, then $(D/E) \times (D'/E')$ "is" an interpretable set, namely $(D \times D')/E''$ where $(a,b)E''(c,d) \Leftrightarrow aEc \wedge bE'd$. Secondly if X is a definable subset of D/E, then X "is" an interpretable set D'/E', where $D' = \{a \in D : [a]_E \in X\}$ and E' is the restriction of E to D'

□

From now on, we use the word "interpretable" to mean "definable in $\mathbb{M}^{\mathrm{eq}}$" and "definable" to mean "definable" in $\mathbb{M}$

An **imaginary** is an element of $\mathbb{M}^{\mathrm{eq}}$

11.5 Elimination of imaginaries

Definition 11.27. T has **elimination of imaginaries** (EI) if $\forall a \in \mathbb{M}^{\mathrm{eq}}$, $\exists \bar{b} \in \mathbb{M}$, $\mathrm{dcl}(a) = \mathrm{dcl}(\bar{b})$

Definition 11.28. T has **uniform EI** if

1. If D/E is 0-interpretable, then there is 0-definable Y, there is bijection $f : D/E \to Y$, 0-interpretable (= 0-definable in M^{eq})

2. If D/E is 0-interpretable, then there is Y 0-definable, 0-definable surjection $g : D \to Y$ s.t. $g(x) = g(y)$ iff $E(x, y)$

$1 \Leftrightarrow 2$

Note: uniform EI implies EI

If $e \in D/E$, there is Y 0-definable, bijection $f : D/E \to Y$, e interpretable with $f(e) \in \mathbb{M}^n$

Lemma 11.29. *If T has EI, if D/E is 0-interpretable, then there is 0-interpretable $X_i \subseteq D/E$, $X_i \cap X_j = \emptyset$, $D/E = \bigcup_{i=1}^n X_i$, each X_i has 0-interpretable bijection to a 0-definable set $f_i : X_i \to Y_i$*

Proof. Say 0-interpretable $X \subseteq D/E$ is "good" if there is 0-definable Y, 0-interpretable bijection $f : X \to Y$.

If $X' \subseteq X$, X is "good", X' is 0-interpretable, then X' is good

Claim D/E is covered by good sets

If $e \in D/E$, E.I. implies there is $\bar{b} \in \mathbb{M}^m$, $\mathrm{dcl}^{\mathrm{eq}}(e) = \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$. Lemma **??** implies there is 0-interpretable bijection $f : X \to Y$, $f(e) = \bar{b}$, X is good

There are at most $|L|$-many good sets. By saturation, $D/E = \bigcup_{i=1}^n X_i$ (class of good sets is small)

Replace X_i with $X_i \setminus (X_1 \cup \cdots \cup X_{i-1})$, we may assume the X_I are pairwise disjoint □

Theorem 11.30. *Suppose T has one-sort and $|\mathrm{dcl}(\emptyset)| \geq 2$, then T has E.I. $\Leftrightarrow$ T has uniform E.I.*

Proof. $\Rightarrow$: Take D/E 0-interpretable. Lemma 11.29 gives $D/E = \coprod_{i=1}^n X_i$, $f_i : X_i \to Y_i$, Y_i 0-definable

Fix $a, b \in \mathrm{dcl}(\emptyset)$

By replacing y_i with $y_i \times \{(a, a, \dots, a)\}$. WMA there is m s.t. $Y_i \subseteq \mathbb{M}^m$ $\forall i$. Take $N \gg 0$, $2^N > n$, take distinct $\bar{c}_1, \dots, \bar{c}_n \in \{a, b\}^N$

Replacing Y_i with $Y_i \times \{\bar{c}_i\}$, now Y_is are disjoint □

Example 11.3. DLO has E.I., doesnt have uniform E.I.

$D = M^2$, E two class $\{(x, y) : x = y\}$ and $\{(x, y) : x \neq y\}$

uniform E.I. would imply $D/E \leftrightarrow Y$, Y is 0-definable. But there is no 0-definable Y with two elements

Remark. $\mathbb{M}^{\text{eq}}$ has uniform E.I.

If D/E is 0-interpretable and $E' \subseteq (D/E) \times (D/E)$ is a 0-interpretable equivalence relation on D/E, then $(D/E)/E'$ is also 0-interpretable. In fact, it's D/E'' where

$$E''(\bar{a}, \bar{b}) \Leftrightarrow E'([\bar{a}]_E, [\bar{b}]_E)$$

Therefore $\mathbb{M}^{\text{eq}} \approx (\mathbb{M}^{\text{eq}})^{\text{eq}}$

Example 11.4. DLO is an example of a theory with elimination of imaginaries but not uniform elimination of imaginaries

Let E be the equivalence relation on $\mathbb{M}^2$ with two classes, one of which is the line $y = x$ and the other is its complement. If there was a 0-interpretable bijection from $\mathbb{M}^2/E$ to $Y \subseteq \mathbb{M}^n$, then Y would contain two elements, both of which are in dcl($\emptyset$). But dcl($\emptyset$) $= \emptyset$, so Y cannot have any elements unless $n = 0$ and when $n = 0$ the set Y can only have one element.

11.6 Codes

Definition 11.31. A real tuple or imaginary $e$is a **code** for D (e **codes** D) if $\{\sigma \in \text{Aut}(\mathbb{M}) : \sigma(D) = D\} = \text{Aut}(\mathbb{M}^{\text{eq}}/e) = \text{Aut}(\mathbb{M}/e)$

Remark. If e, e' code D, then $\text{Aut}(\mathbb{M}/e) = \text{Aut}(\mathbb{M}/e')$, so $\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(e')$ by Lemma 11.14

Remark. If e codes D, then D is A-definable $\Leftrightarrow e \in \text{dcl}^{\text{eq}}(A)$

Proof. TFAE

- D is A-definable
- D is A-invariant
- $\forall \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(D) = D$
- $\forall \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(e) = e$
- $e \in \text{dcl}^{\text{eq}}(A)$

∎

Example 11.5. Suppose $T = \mathsf{ACF}$ and $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$. Let $P(x) = \prod_{i=1}^{n}(x - r_i)$. Write $P(x)$ as $x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$. Then $(c_0, \dots, c_{n-1})$ is a code for S. Indeed

$$\begin{aligned}\sigma(\bar{c}) = \bar{c} &\Leftrightarrow \sigma(P(x)) \equiv P(x) \\ &\Leftrightarrow \prod_{i=1}^{n}(x - \sigma(r_i)) \equiv \prod_{i=1}^{n}(x - r_i) \\ &\Leftrightarrow \{\sigma(r_1), \dots, \sigma(r_n)\} = \{r_1, \dots, r_n\} \\ &\Leftrightarrow \sigma(S) = S\end{aligned}$$

Example 11.6. If D/E is 0-interpretable and $e \in D/E$, then e is an E-equivalence class $X = E(\mathbb{M}, \bar{a})$, and $\sigma(e) = \sigma(X)$ for all σ. Therefore σ codes X

Lemma 11.32. *Let $\varphi(\bar{x}, \bar{y})$ be a formula. Let $f(\bar{y})$ be a 0-definable function s.t.*

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f(\bar{b}) = f(\bar{c})$$

Then $f(\bar{b})$ is a code for $\varphi(\mathbb{M}, \bar{b})$, for each $\bar{b}$

Proposition 11.33. *TFAE*

1. *T has uniform elimination of imaginaries*
2. *For any formula $\varphi(\bar{x}; \bar{y})$, there is a 0-definable function $f_\varphi(\bar{y})$ s.t.*

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f_\varphi(\bar{b}) = f_\varphi(\bar{c})$$

Proof. $1 \to 2$: apply uniform E.I. to $\mathbb{M}^n/E$, where $E(\bar{b}, \bar{c}) \Leftrightarrow (\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}))$

$2 \to 1$: given a 0-interpretable set D/E,

$$E(\bar{b}, \bar{c}) \Leftrightarrow E(\mathbb{M}, \bar{b}) = E(\mathbb{M}, \bar{c}) \Leftrightarrow f_E(\bar{b}) = f_E(\bar{c})$$

for $\bar{b}, \bar{c} \in D$. So we have a 0-definable function on D satisfying condition 2 of Definition ☐

Corollary 11.34. *If T has uniform elimination of imaginaries, then every definable set has a code in $\mathbb{M}$*

Corollary 11.35. *Every definable set has a code in $\mathbb{M}^{\mathrm{eq}}$*

Proposition 11.36. *TFAE*

1. *T has elimination of imaginaries*

2. *Every definable $D \subseteq \mathbb{M}^n$ has a code in $\mathbb{M}$*

Proof. $1 \to 2$: given D take a code $e \in \mathbb{M}^{\mathrm{eq}}$, then take $\bar{b} \in \mathbb{M}^m$ interdefinable with e. Then

$$\mathrm{Aut}(\mathbb{M}/\bar{b}) = \mathrm{Aut}(\mathbb{M}/e) = \{\sigma \in \mathrm{Aut}(\mathbb{M}) : \sigma(D) = D\}$$

so $\bar{b}$ is a code for D

$2 \to 1$: if $e \in D/E \subseteq \mathbb{M}^{\mathrm{eq}}$, then e codes a definable set X □

Corollary 11.37. $\mathrm{dcl}^{\mathrm{eq}}(e)$ *is the smallest definably closed set defining D*

11.7 Elimination of imaginaries and naming parameters

Proposition 11.38. *Uniform elimination of imaginaries is preserved by naming parameters*

Proof. Fix D/E, $D \subseteq \mathbb{M}^n$ □

12 Forking and stability spectra

12.1 EI in PA and ACF

$\mathbb{Q}$ is 0-definable

Theorem 12.1. *If complete $T \supseteq \mathsf{PA}$ (e.g., $T = \mathrm{Th}(\mathbb{N})$), then T has uniform E.I.*

Proof. Fix interpretable D/E. Want $D/E \to Y$ definable bijection, or $D/E \to \mathbb{M}^n$ definable injection

Take $f : D/E \to \mathbb{M}^n$, $f(X) = \min(X)$, min is w.r.t. lexicographic order on $\mathbb{M}^n$. $\mathsf{PA} \Rightarrow \mathbb{M}, \mathbb{M}^n$ are definably well-ordered □

Consider $T = \mathsf{ACF}_0$

Fact 12.2. *If $S \subseteq_f \mathbb{M}^n$, then $\exists \ulcorner S \urcorner \in \mathbb{M}$*

Proof. $n = 1$, if $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$ form $P(x) = \prod_{i=1}^m (x - r_i)$. Then $P(x) = x^m + c_{m-1}x^{m-1} + \dots + c_1 x + c_0$, $\bar{c}$ is a code for S

$n = 2$, for $q \in \mathbb{Q}$, let $\pi_q : \mathbb{M}^2 \to \mathbb{M}$, $(x, y) \mapsto y - qx$. Let $A = \{\ulcorner \pi_q(S) \urcorner : q \in \mathbb{Q}\} \subseteq \mathbb{M}$

Claim: If $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(S) = S \Leftrightarrow \sigma \in \mathrm{Aut}(\mathbb{M}/A)$