Generic properties of groups

Marcin Petrykowski

Contents

1 Introduction **2**

2 Preliminaries **4**

3 Weak generic types **7**
- 3.1 Introduction 7
- 3.2 Basic properties of weak generic sets and types 8
- 3.3 Characterizations of weak genericity 11
- 3.4 Stationarity 17
- 3.5 Expansions of real closed fields 23
  - 3.5.1 Weak generic types in $(\mathbb{R}, +) \times (\mathbb{R}, +)$ 29
  - 3.5.2 Weak generic types in $(\mathbb{R}_+, \cdot) \times (\mathbb{R}_+, \cdot)$ 38

4 Countable coverings of groups **41**
- 4.1 Introduction 41
- 4.2 The general bound 43
- 4.3 Applications of weak generic types 46
- 4.4 Connections with amenability 51
- 4.5 Examples and model-theory-free versions 53

1 Introduction

In this dissertation we consider groups as first order structures. We deal with some of their model-theoretic and combinatorial properties.

The first part of the thesis is devoted to exploring the notion of weak generic type (Definition 3.1.2). It has been introduced by Newelski in [10] and generalizes the notion of generic type, which is fundamental in model theory of groups. We first establish some basic general properties of weak generic sets and types. Then we investigate such sets and types in groups of especially simple form definable in some o-minimal structures.

In the second part of the dissertation we deal with $\aleph_0$-saturated groups and their coverings consisting of countably many 0-type-definable sets. We examine some combinatorial properties of such coverings. We associate with each $\aleph_0$-saturated group G a natural number k_G (Definition 4.1.2) and show that always $k_G \in \{2, 3\}$. Then we try to establish for which groups G the coefficient k_G equals 2. Weak generic types turn out to be a useful tool in determining the coefficient k_G for various classes of groups.

The main results of this thesis are:

(1) Theorem 3.3.4, which gives a full characterization of definable weak generic sets in groups of the form $(G^n, +)$, $0 < n < \omega$, definable in an o-minimal structure $(G, <, +, \dots)$ which is an expansion of an ordered group $(G, <, +)$,

(2) Theorem 3.4.7 stating the equivalence of stationarity of an o-minimal ordered group $(G, <, +, \dots)$ and stationarity of some weak generic types in the group $(G, +) \times (G, +)$,

(3) Theorem 3.5.6, Theorem 3.5.9 and, more generally, a precise description of weak generic types in groups $(R, +) \times (R, +)$ and $(R_+, \cdot) \times (R_+, \cdot)$ (derived in an o-minimal theory of the form $Th(R, <, +, \cdot, \dots)$ where $(R, <, +, \cdot)$ is a real closed field),

(4) Theorem 4.2.2 and Theorem 4.3.1, which imply that $k_G \leq 3$ for every $\aleph_0$-saturated group G (these results were first obtained by Newelski but their new proof, which appears in Section 4.3, is due to the author of this dissertation),

(5) Theorem 4.3.6, which says that for each $\aleph_0$-saturated group G with generic types the coefficient k_G equals 2,

(6) Theorem 4.4.3 stating the same for every definably amenable group G, which allows to conclude that both stable and amenable ($\aleph_0$-saturated) groups G have $k_G = 2$.

The thesis is organized as follows. Chapter 2 provides mainly some basic information on first order structures (the reader is referred to [2] for more details). In Chapter 3 we introduce the notion of weak generic type in a group and examine it in some special classes of groups. In Chapter 4 we investigate some combinatorial properties of countable coverings of $\aleph_0$-saturated groups consisting of 0-type-definable sets. Section 4.3 is a link between the last two chapters. Each of them is preceded by a short introduction containing a more detailed description of its contents.

An essential part of the material contained in this dissertation appears in the papers [9] and [10], which are common work of Ludomir Newelski and the author.

Acknowledgements

I would like to express my deepest gratitude to my advisor Ludomir Newelski for his guidance throughout my undergraduate and graduate study. Our meetings and discussions during my work on this thesis helped me very much. I also thank Lou van den Dries, Chris Miller and Anand Pillay for their helpful suggestions and answering my numerous questions.

2 Preliminaries

A language L is a collection of function, relation and constant symbols. An L-structure is a set with specified interpretations of the symbols of L. For the sake of notational simplicity, given an L-structure

$$M = (|M|, \{f_i : i \in I\}, \{R_j : j \in J\}, \{c_k : k \in K\}),$$

we often denote its universe $|M|$ by M. If $L \subseteq L'$, M' is an L'-structure with $|M'| = |M|$ and each symbol of L is interpreted in M and M' in the same way, then the structure M' is said to be an expansion of the structure M.

If A is a subset of some L-structure, then $L(A)$ denotes the language L expanded by adding a new constant symbol c_a for every $a \in A$. Using logical symbols and symbols from L we create L-formulas. We usually use the same symbol to denote the language itself and the set of all its formulas. A formula φ is a sentence if it does not contain free variables. A first order L-theory is any set of L-sentences.

Let us fix a language L, an L-theory T and L-structures M and N. If Φ and Ψ are sets of formulas, then $\Phi \vdash \Psi$ denotes the fact that each member of Ψ is a logical consequence of Φ. The compactness theorem implies that for every formula φ and every set of formulas Φ such that $\Phi \vdash \varphi$ we can find a finite subset $\Phi_0 \subseteq \Phi$ with $\Phi_0 \vdash \varphi$. T is said to be consistent if there is no sentence φ with $T \vdash \varphi \wedge \neg\varphi$. T is said to be complete if for any sentence φ we have that $T \vdash \varphi$ or $T \vdash \neg\varphi$. For a sentence φ, $M \models \varphi$ means that φ is true in M. All sentences which hold in M form a consistent complete theory, denoted by $Th(M)$. We say that M is a model of T (written $M \models T$) if $M \models \varphi$ for every sentence $\varphi \in T$. We say that M is a substructure of N (written $M \subseteq N$) if $|M| \subseteq |N|$ and for each function symbol f, each relation symbol R and each constant symbol c from L we have that $f^M = f^N \restriction_{|M|}$, $R^M = R^N \restriction_{|M|}$ and $c^M = c^N$, respectively. Finally, for $M \subseteq N$ we say that M is an elementary substructure of N (written $M \prec N$) if for any $L(M)$-sentence we have that $M \models \varphi$ if and only if $N \models \varphi$.

Any $L(M)$-formula $\varphi(\bar{x}, \bar{a})$ with parameters $\bar{a}$ from M defines a set

$$\varphi(M^n, \bar{a}) = \{\bar{b} \in M^n : M \models \varphi(\bar{b}, \bar{a})\}$$

where $n = |\bar{x}|$. If $\bar{a} \subseteq A$, then we say that the set $\varphi(M^n, \bar{a})$ is A-definable. By 0-definable we mean definable over $\emptyset$. A function $f : A \to B$ (where $A \subseteq M^m$, $B \subseteq M^n$ for some $m, n < \omega$) is definable if its graph is definable as a subset of M^{m+n}.

Assume T is a complete first order L-theory, $A \subseteq M \models T$ and $\bar{x}$ is a tuple of variables of length n. A type $p(\bar{x})$ over A of variables $\bar{x}$ is any collection of formulas of the form $\varphi(\bar{x}) \in L(A)$ which is consistent with T. A type $p(\bar{x})$ is complete over A if for any formula $\varphi(\bar{x}) \in L(A)$ we have that $\varphi(\bar{x}) \in p(\bar{x})$ or $\neg\varphi(\bar{x}) \in p(\bar{x})$. Sometimes we call $p(\bar{x})$ complete over A if the following weaker condition is satisfied: for any formula $\varphi(\bar{x}) \in L(A)$ we have that $p(\bar{x}) \vdash \varphi(\bar{x})$ or $p(\bar{x}) \vdash \neg\varphi(\bar{x})$. If $\bar{a}$ is a tuple from M^n, then

$$tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L(A) : M \models \varphi(\bar{a})\}$$

is a complete n-type over A. $S_n(A)$ denotes the set of all complete n-types over A in M. $S_n(A)$ is naturally equipped with a compact Hausdorff topology with the basis consisting of clopen sets of the form $[\varphi(\bar{x})]$, $\varphi(\bar{x}) \in L(A)$, where

$$[\varphi(\bar{x})] = \{p(\bar{x}) \in S_n(A) : \varphi(\bar{x}) \in p(\bar{x})\}.$$

In general, if $p(\bar{x})$ is a type, then $[p(\bar{x})]$ denotes the set of types containing $p(\bar{x})$. If $p(\bar{x})$ is a type over A, then we call the set of realizations of p in M

$$p(M^n) = \{\bar{a} \in M^n : (\forall\varphi(\bar{x}) \in p(\bar{x}))M \models \varphi(\bar{a})\} = \bigcap_{\varphi(\bar{x})\in p(\bar{x})} \varphi(M^n)$$

type-definable over A. If V is a 0-type-definable subset of M^n (i.e. type-definable over $\emptyset$), then we sometimes identify V with the set

$$[V] = \{tp(\bar{a}) : \bar{a} \in V\} \subseteq S_n(\emptyset).$$

A first order structure M is κ-saturated if for any $A \subseteq M$ with $|A| < \kappa$, $n < \omega$ and $p \in S_n(A)$, p has a realization in M. A theory T is stable if there exists a cardinal number κ such that for each model M of T with $|M| \leq \kappa$ we have that $|S_1(M)| \leq \kappa$. We call a structure M stable if its theory $Th(M)$ is stable. Stable structures have many nice properties (for details see [12]).

We call a first order structure $(M, \cdot, \ldots)$ a group if $(M, \cdot)$ satisfies group axioms. We usually denote it by $(G, \cdot, \ldots)$. A structure of the form $(G, \cdot)$ is called a pure group. If $p(x)$ is a type over G and $g \in G$, then

$$g \cdot p(x) = \{g \cdot \varphi(x) : \varphi(x) \in p(x)\} = \{\varphi(g^{-1} \cdot x) : \varphi(x) \in p(x)\}.$$

A group $(G, \cdot)$ is definable in a structure M if G is a definable subset of M^n for some $n < \omega$ and the group action $\cdot : G \times G \to G$ is a definable function in M (i.e. its graph is a definable subset of M^{3n}).

An infinite totally ordered first order structure $(M, <, \ldots)$ is o-minimal if every definable subset of M is a union of finitely many intervals and points. The property of o-minimality is preserved under elementary extensions of structures. Basic theory of o-minimal structures has been developed in a series of papers [13], [5] and [14].

Let $(M, <, \ldots)$ be an o-minimal structure. We usually say "ultimately" instead of "for all sufficiently large $a \in M$". We denote an open interval with endpoints a and b by (a, b) and a closed one by $[a, b]$. In contrast, $\langle a, b\rangle$ denotes the pair of elements a and b.

If $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$, $a < b$ and $f : (a, b) \to M$ is a definable function, then there are $a = a_1 < \ldots < a_n = b$ such that on each interval (a_i, a_{i+1}) f is either constant or strictly monotone and continuous in the order topology. In particular, every definable function $f : M \to M$ is ultimately continuous and monotone.

Each definable subset A of M^n has a finite partition $A = C_1 \cup \ldots \cup C_k$ into pairwise disjoint cells $C_1, \ldots, C_k$. Cells are definable sets of an especially simple nature (see [5], Definition 1.5). We denote the o-minimal dimension of the set A by $\dim(A)$ and recall that $\dim(A) = \max(\dim(C_1), \ldots, \dim(C_k))$.

3 Weak generic types

3.1 Introduction

In this chapter we assume that G is a group, possibly with some additional structure, or more generally a definable group in a model of a complete first order theory T. When G (or T) is stable, the notion of a generic type of elements of G was introduced by Poizat [15]. It became a fundamental notion in geometric model theory. In the stable case we can define generic types as follows [16, 12].

Definition 3.1.1 *We say that a set $X \subseteq G$ is (left) generic if some finitely many left G-translates of X cover G. We say that a formula $\varphi(x)$ is (left) generic if the set $\varphi(G)$ of elements of G realizing φ is (left) generic. Finally, we say that a type $p(x)$ of elements of G is (left) generic if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is (left) generic.*

In the stable case left generic = right generic (defined by means of right translations) and each partial generic type extends to a complete generic type (over any set of parameters), also generic types may be characterized there by means of forking and translation-invariant ranks.

While Definition 3.1.1 is appealing by its simplicity, it does not work well in the unstable context. For instance, complete generic types in the sense of Definition 3.1.1 may not exist in general. So in simple theories generic types are defined extending from the stable case the definition that refers to forking and ranks. However, in this dissertation we do apply Definition 3.1.1 in the general setting, and to overcome its drawbacks we define a weaker notion, still capturing the sense of a "not-so-small" subset of G. Suppose $X = A \cup B$ is a generic subset of G. If B is not generic, then one could argue that A is "not-so-small". This justifies the following definition (due to Newelski).

Definition 3.1.2 *We say that a set $A \subseteq G$ is weak generic, if for some non-generic $B \subseteq G$ we have that $A \cup B$ is generic. We say that a formula $\varphi(x)$ is weak generic if the set $\varphi(G)$ is weak generic. A type $p(x)$ of elements of G is weak generic if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is weak generic.*

The chapter is organized as follows. In Section 3.2 we establish basic facts concerning weak genericity of formulas and types. In three remaining sections of the chapter we confine our attention to o-minimal structures of the form

$(G, <, +, \ldots)$ where $(G, <, +)$ is an ordered group. Section 3.3 provides a characterization of definable weak generic sets in groups $(G^n, +)$ for $n < \omega$. In Section 3.4 we introduce and examine the notion of stationarity of weak generic types. Finally, in Section 3.5 we apply previously obtained results to the case of groups definable in o-minimal expansions of real closed fields.

3.2 Basic properties of weak generic sets and types

We are interested mainly in definable weak generic sets. For these purposes, the next lemma provides an alternative definition.

Lemma 3.2.1 *Assume that G is a group and X is a definable subset of G. The following are equivalent:*
(1) the set X is weak generic,
(2) for some finitely many elements $a_1, \ldots, a_n \in G$ the set $G \setminus \bigcup_{i=1}^n a_i \cdot X$ is not generic,
(3) for some definable non-generic set $Y \subseteq G$ the set $X \cup Y$ is generic.

Proof. (1) $\Rightarrow$ (2) Assume X is weak generic. Hence for some non-generic set $Y \subseteq G$, the set $X \cup Y$ is generic, meaning that for some finitely many elements $a_1, \ldots, a_n \in G$ we have that

$$\bigcup_{i=1}^{n} a_i \cdot (X \cup Y) = \bigcup_{i=1}^{n} a_i \cdot X \cup \bigcup_{i=1}^{n} a_i \cdot Y = G.$$

This means that

$$G \setminus \bigcup_{i=1}^{n} a_i \cdot X \subseteq \bigcup_{i=1}^{n} a_i \cdot Y.$$

Since the set Y is not generic, also the set $\bigcup_{i=1}^n a_i \cdot Y$ is not generic, which proves (2).

(2) $\Rightarrow$ (3) Let

$$Y = G \setminus \bigcup_{i=1}^{n} a_i \cdot X.$$

We see that Y is definable and not generic so putting $a_{n+1} = e$ (the neutral element of G) we obtain $G = \bigcup_{i=1}^{n+1} a_i \cdot (X \cup Y)$. Hence $X \cup Y$ is generic.

(3) $\Rightarrow$ (1) Trivial. $\square$

In the next lemma we give some basic properties of weak generic sets and types.

Lemma 3.2.2 *(1) If $X, Y \subseteq G$ are not weak generic, then $X \cup Y$ is not weak generic.*
(2) If $p(x)$ is a (partial) weak generic type over $A \subseteq G$, then $p(x)$ may be extended to a complete weak generic type over A.

Proof. (1) Let $Z \subseteq G$ be non-generic. Since Y is not weak generic, we see that $Y \cup Z$ is not generic. Since X is not weak generic, we see that $X \cup Y \cup Z$ is not generic. Hence $X \cup Y$ is not weak generic.

(2) Let $q(x) = \{\varphi(x) \in L(A) : p(x) \cup \{\neg\varphi(x)\}$ is not weak generic$\}$. Then $p \subseteq q$. We shall show that q is a consistent partial type over A. If not, then

$$G \models \neg\exists x \bigwedge_{k=1}^{n} \varphi_k(x)$$

for some $n < \omega$ and $\varphi_1, \ldots, \varphi_n \in q$. By compactness, for each $k \in \{1, \ldots, n\}$ we can find a finite set of formulas $p_k \subseteq p$ such that the type $p_k(x) \cup \{\neg\varphi_k(x)\}$ is not weak generic. Let $\psi(x) = \bigwedge\{p_k(x) : 1 \leq k \leq n\}$ and note that for every $k \in \{1, \ldots, n\}$ the set $\psi(G) \cap \neg\varphi_k(G)$ is not weak generic. By (1), neither is the union

$$\bigcup_{k=1}^{n}(\psi(G) \cap \neg\varphi_k(G)) = \psi(G) \cap \bigcup_{k=1}^{n} \neg\varphi_k(G) = \psi(G) \cap G = \psi(G),$$

contradicting the fact that $p(x) \vdash \psi(x)$. Finally, we take any $r(x) \in S(A)$ with $r \supseteq q$ and the proof is complete. $\square$

We see that (complete) weak generic types exist. By Lemma 3.2.2, the set

$$WGEN(A) = \{p \in S(A) : p \text{ is weak generic}\}$$

is closed and non-empty in $S(A)$. The next lemma explains the relationship between weak generic types and generic types, provided the latter exist.

Lemma 3.2.3 *Assume G is a group and $A \subseteq G$.*
(1) If some weak generic type $p(x) \in S(G)$ is generic, then all weak generic types $q(x) \in S(A)$ are generic.
(2) If for every $p, q \in WGEN(G)$ there is $g \in G$ such that $g \cdot p = q$, then all weak generic types $q(x) \in S(A)$ are generic.
(3) If there is just one weak generic type in $S(A)$, then it is generic.

Proof. (1) Suppose that some weak generic type $q(x) \in S(A)$ is not generic. Then some definable generic set $X \subseteq G$ may be divided into two non-generic definable sets X_1, X_2. Since X is generic, some left G-translate X' of X belongs to $p(x)$. Then the corresponding translates X_1', X_2' of X_1, X_2 are also non-generic and one of them belongs to $p(x)$. Hence $p(x)$ is not generic, a contradiction.

(2) If not, then we can find a formula $\varphi(x) \in L(A)$ which is weak generic but not generic. Note that $\{\neg g \cdot \varphi(x) : g \in G\}$ is a partial weak generic type over G (because for each $m < \omega$ and $g_1, \ldots, g_m \in G$ the set $\bigcup_{i=1}^m g_i \cdot \varphi(G)$ is not generic, which implies that the set $\bigcap_{i=1}^m (G \setminus g_i \cdot \varphi(G))$ is weak generic). Extend the type $\{\neg g \cdot \varphi(x) : g \in G\}$ to some $q(x) \in WGEN(G)$. Next extend $\varphi(x)$ to $p(x) \in WGEN(G)$. Then $(\forall g \in G) g \cdot p \neq q$, a contradiction.

(3) Let $p(x)$ denote the only weak generic type in $S(A)$. For the sake of contradiction assume that some $\varphi(x) \in p(x)$ is not generic. Then $\neg\varphi(x)$ is weak generic and may be extended to some weak generic type $q(x) \in S(A)$. Obviously $p \neq q$, a contradiction. $\square$

By Lemma 3.2.3(1), in the stable case weak generic = generic (for types and definable sets). As an example note that if $G = (G, <, +, \ldots)$ is o-minimal, then there are exactly two complete weak generic types, corresponding to $-\infty$ and $+\infty$, and they are not generic (see Example 3.4.2). Hence in (3) the assumption that there is only one weak generic type can not be weakened.

Lemma 3.2.4 *Assume that $G \prec H$ and $\varphi(x) \in L(G)$.*
(1) If $\varphi(G)$ is weak generic in G, then $\varphi(H)$ is weak generic in H.
(2) If G is $\aleph_0$-saturated and $\varphi(H)$ is weak generic in H, then $\varphi(G)$ is weak generic in G.

Proof. (1) There is a non-generic formula $\psi(x) \in L(G)$ such that the set $\varphi(G) \cup \psi(G)$ is generic in G. Since $G \prec H$, the set $\psi(H)$ is not generic in H and the set $\varphi(H) \cup \psi(H)$ is generic in H. Thus $\varphi(H)$ is weak generic in H.

(2) There is a formula $\psi(x) \in L(H)$ such that $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. We have that $\psi(x) = \psi(x, \bar{b})$ where $\bar{b} \subseteq H$ are all parameters of $\psi(x)$. Let $A \subseteq G$ be a finite set containing all parameters of $\varphi(x)$. By $\aleph_0$-saturation of G, we are able to find in G a tuple $\bar{a} \subseteq G$ such that $tp(\bar{a}/A) = tp(\bar{b}/A)$. Then $\psi(x, \bar{a}) \in L(G)$ has properties needed to deduce the weak genericity of the set $\varphi(G)$ in G. Namely, $\psi(G, \bar{a})$ is not generic in G and $\varphi(G) \cup \psi(G, \bar{a})$ is generic in G. We shall show the first assertion only, the second one may be proved in a similar way. For the

sake of contradiction assume that the set $\psi(G, \bar{a})$ is generic in G. Then for some $0 < n < \omega$ we have that

$$G \models \exists x_1, \ldots, x_n \forall y \exists z (\psi(z, \bar{a}) \wedge \bigvee_{k=1}^{n} y = x_k \cdot z)$$

and the same holds in H since $G \prec H$. As $tp(\bar{a}) = tp(\bar{b})$, we have that

$$H \models \exists x_1, \ldots, x_n \forall y \exists z (\psi(z, \bar{b}) \wedge \bigvee_{k=1}^{n} y = x_k \cdot z).$$

But then $\psi(H) = \psi(H, \bar{b})$ is generic in H, a contradiction. $\square$

All lemmas in this section remain true if we consider a group $(G, \cdot)$ definable in a first order structure M. Then G is a definable subset of M^n for some $n < \omega$ and for every $A \subseteq M$ we define the set $WGEN(A)$ of complete weak generic types over A as the set

$$\{p \in S_n(A) : \text{ for each } \varphi(x_1, \ldots, x_n) \in p,\ G \cap \varphi(M^n) \text{ is weak generic in } G\}.$$

Modifying all previous proofs we easily obtain new ones which comply with the definition above.

3.3 Characterizations of weak genericity

We continue to examine properties of definable weak generic sets. We begin with the following proposition on weak generic sets in groups definable in o-minimal structures.

Proposition 3.3.1 *Assume G is a definable group in an o-minimal structure M and X is a definable weak generic subset of G. Then* $\dim(X) = \dim(G)$.

Proof. For the sake of contradiction suppose that $\dim(X) < \dim(G)$. Take a generic set A and a non-generic set B such that $A = B \cup X$ (where A and B are definable subsets of G, apply Lemma 3.2.1). Choose a finite $S \subseteq G$ with $S \cdot A = G$. Then $G \setminus (S \cdot B) \subseteq S \cdot X$ and

$$\dim(G \setminus S \cdot B) \leq \dim(S \cdot X) = \dim(X) < \dim(G).$$

Hence the set $S \cdot B$ is large in the sense of [11] and it must be generic by Lemma 2.4 there. But then also B is generic, a contradiction. $\square$

Assume G is a group and $X, Y \subseteq G$. We say that the set X is translation disjoint from the set Y if for some $a \in G$ the sets $a \cdot X$ and Y are disjoint.

Lemma 3.3.2 *Assume G is a group and X is a weak generic subset of G. Then for some finite $A \subseteq G$ there is no finite covering of X by sets that are translation disjoint from $A \cdot X$.*

Proof. By the weak genericity of X, we can find a generic superset $Y \supseteq X$ such that the set $Y \setminus X$ is not generic. We have that $G = A \cdot Y$ for some finite $A \subseteq G$. We shall prove that the set A meets conditions of the lemma. For the sake of contradiction assume that for some $X_0, \ldots, X_{n-1} \subseteq G$ and $a_0, \ldots, a_{n-1} \in G$ we have that

$$X = \bigcup_{i<n} X_i \text{ and } \bigwedge_{i<n} (a_i \cdot X_i) \cap (A \cdot X) = \emptyset.$$

Then for each $i < n$, $a_i \cdot X_i \subseteq G \setminus A \cdot X \subseteq A \cdot (Y \setminus X)$. So for each $i < n$, $X_i \subseteq a_i^{-1} \cdot A \cdot (Y \setminus X)$, which implies that $X \subseteq \{a_0^{-1}, \ldots, a_{n-1}^{-1}\} \cdot A \cdot (Y \setminus X)$ and finally

$$G = A \cdot Y = A \cdot (Y \setminus X) \cup A \cdot X \subseteq (A \cup (A \cdot \{a_0^{-1}, \ldots, a_{n-1}^{-1}\} \cdot A)) \cdot (Y \setminus X).$$

The group G is covered by finitely many left translates of the set $Y \setminus X$, a contradiction. $\square$

The next corollary shows that weak genericity is related to generating G. More detailed analysis of this connection appears in Section 4.3.

Corollary 3.3.3 *Assume G is a group and X is a weak generic subset of G. Then the set $X \cdot X^{-1}$ is generic in G.*

Proof. Take a finite $A \subseteq G$ such as in Lemma 3.3.2. Then for each $a \in G$ we have that $a \cdot X \cap A \cdot X \neq \emptyset$, which implies that $a \in A \cdot X \cdot X^{-1}$. So $G = A \cdot X \cdot X^{-1}$, which finishes the proof. $\square$

From now on, let $(G, <, +, \ldots)$ be an o-minimal expansion of an ordered group $(G, <, +)$. Then the group G is commutative, divisible and torsion-free (see Theorem 2.1 in [13]). This justifies denoting the group action by $+$. By $(G^n, +)$ we mean the product of groups $(G, +) \times \ldots \times (G, +)$ (n times). The ordering of G is dense since for every $a, b \in G$ with $a < b$ we have that $a < \frac{a+b}{2} < b$.

Theorem 3.3.4 *Assume that* $(G, <, +, \dots)$ *is an o-minimal expansion of an ordered group* $(G, <, +)$, $0 < n < \omega$ *and* $\varphi(x_1, \dots, x_n) \in L(G)$. *The following are equivalent:*
(1) the formula $\varphi(x_1, \dots, x_n)$ *is weak generic in* $(G^n, +)$,
(2) the formula $\neg\varphi(x_1, \dots, x_n)$ *is not generic in* $(G^n, +)$,
(3) the set $\varphi(G^n)$ *contains arbitrarily large n-dimensional boxes:*

$$(\forall R > 0)(\exists a_1, \dots, a_n \in G)[a_1, a_1 + R] \times \dots \times [a_n, a_n + R] \subseteq \varphi(G^n).$$

Proof. (3) $\Rightarrow$ (2) For the sake of contradiction assume that for some $k < \omega$ and $\langle g_1^1, \dots, g_n^1 \rangle, \dots, \langle g_1^k, \dots, g_n^k \rangle \in G^n$ we have that

$$G^n = \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j \rangle + (G^n \setminus \varphi(G^n))).$$

Put $M = \max\{|g_i^j| : 1 \leq i \leq n, 1 \leq j \leq k\}$. Using (3) we are able to find $a_1, \dots, a_n \in G$ such that

$$[a_1 - M, a_1 + M] \times \dots \times [a_n - M, a_n + M] \subseteq \varphi(G^n).$$

Then

$$\langle a_1, \dots, a_n \rangle \notin \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j \rangle + (G^n \setminus \varphi(G^n))),$$

a contradiction.

(2) $\Rightarrow$ (1) Since the set $G^n = \varphi(G^n) \cup (G^n \setminus \varphi(G^n))$ is generic in $(G^n, +)$ and the set $G^n \setminus \varphi(G^n)$ is not generic in $(G^n, +)$, the set $\varphi(G^n)$ is weak generic in $(G^n, +)$.

(1) $\Rightarrow$ (3) Without loss of generality $n \geq 2$ (see Example 3.4.2). Using Lemma 3.2.2(2) find $p(x_1, \dots, x_n) \in S_n(G)$ such that p is a weak generic type in $(G^n, +)$ and $\varphi \in p$. Extend G to a $|G|^+$-saturated group $H \succ G$. Take $\langle a_1, \dots, a_n \rangle \in H^n$ realizing p and fix a positive $R \in G$. We shall show that the following condition holds:

$$(*)\ (\forall a \in H)(a_n \leq a \wedge a \leq a_n + R \Rightarrow tp(a/Ga_{<n}) = tp(a_n/Ga_{<n}))$$

where $a_{<n}$ stands for $\langle a_1, \dots, a_{n-1} \rangle$.

For the sake of contradiction assume that for some $a \in [a_n, a_n + R]_H$ the types $tp(a/Ga_{<n})$ and $tp(a_n/Ga_{<n})$ are distinct. By the o-minimality of H,

we can find $b \in [a_n, a_n + R]_H$ with $b \in dcl(Ga_{<n})$. Let $\psi(x_1, \ldots, x_{n-1}, y) \in L(G)$ be such that $H \models \psi(a_{<n}, b) \wedge \exists! y \psi(a_{<n}, y)$. As $b - R \leq a_n \leq b$, we have that $\chi \in p$ where

$$\chi(x_1, \ldots, x_n) = \exists! y \psi(x_{<n}, y) \wedge \forall y(\psi(x_{<n}, y) \to (y - R \leq x_n \wedge x_n \leq y)).$$

Since $\chi \in p$, the set $\chi(G^n)$ is weak generic in $(G^n, +)$.

We define $f : G^{n-1} \to G$ as follows. Take $\langle c_1, \ldots, c_{n-1} \rangle \in G^{n-1}$. If there is $c_n \in G$ such that $G \models \chi(c_1, \ldots, c_n)$, then there exists just one $d \in G$ with $G \models \psi(c_1, \ldots, c_{n-1}, d)$ and we put $f(c_1, \ldots, c_{n-1}) = d - R$. Otherwise we put $f(c_1, \ldots, c_{n-1}) = 0$ (the neutral element of G). Then the function f is definable over G and we consider the following formula over G:

$$\delta(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}) \leq x_n \wedge x_n \leq f(x_1, \ldots, x_{n-1}) + R.$$

Since $\chi(G^n) \subseteq \delta(G^n) \subseteq G^n$, the set $\delta(G^n)$ is weak generic in $(G^n, +)$. Let $A \subseteq G^n$ be a finite set chosen for $\delta(G^n)$ as in Lemma 3.3.2. Consider an arbitrary $\langle h_1, \ldots, h_{n-1} \rangle \in H^{n-1}$. Choose $M_{h_{<n}} \in G$ such that

$$\{\langle h_1, \ldots, h_n \rangle : f(h_{<n}) + M_{h_{<n}} \leq h_n \leq f(h_{<n}) + M_{h_{<n}} + R\} \cap (A + \delta(H^n)) = \emptyset.$$

If $tp(\langle h_1, \ldots, h_{n-1} \rangle / G) = tp(\langle h'_1, \ldots, h'_{n-1} \rangle / G)$, then $M_{h_{<n}}$ is good also for $\langle h'_1, \ldots, h'_{n-1} \rangle$. By compactness, for each $q(x_1, \ldots, x_{n-1}) \in S_{n-1}(G)$ we can find a formula $\varphi_q(x_1, \ldots, x_{n-1}) \in L(G)$ and $M_q \in G$ such that for every $\langle h_1, \ldots, h_{n-1} \rangle \in H^{n-1}$ with $H \models \varphi_q(h_1, \ldots, h_{n-1})$ we have that

$$\{\langle h_1, \ldots, h_n \rangle : f(h_{<n}) + M_q \leq h_n \leq f(h_{<n}) + M_q + R\} \cap (A + \delta(H^n)) = \emptyset.$$

Again by compactness, $S_{n-1}(G) = [\varphi_{q_1}] \cup \ldots \cup [\varphi_{q_k}]$ for some $k < \omega$ and $q_1, \ldots, q_k \in S_{n-1}(G)$. For $i \in \{1, \ldots, k\}$ put $X_i = (\varphi_{q_i}(G^{n-1}) \times G) \cap \delta(G^n)$ and $e_i = \langle 0, \ldots, 0, M_{q_i} \rangle \in G^n$. Then $\delta(G^n) = X_1 \cup \ldots \cup X_k$ and for every $i \in \{1, \ldots, k\}$ we have that $(e_i + X_i) \cap (A + \delta(G^n)) = \emptyset$. This contradicts the choice of A and finishes the proof of $(*)$.

By $(*)$, we have that

$$H \models \forall y((a_n \leq y \wedge y \leq a_n + R) \to \varphi(a_1, \ldots, a_{n-1}, y)).$$

Therefore the formula $\forall y((x_n \leq y \wedge y \leq x_n + R) \to \varphi(x_1, \ldots, x_{n-1}, y))$ belongs to $p(x_1, \ldots, x_n) = tp(\langle a_1, \ldots, a_n \rangle / G)$. In general, for each formula $\psi(x_1, \ldots, x_n) \in p(x_1, \ldots, x_n)$, $k \in \{1, \ldots, n\}$ and positive $R \in G$ the formula

$$\forall y((x_k \leq y \wedge y \leq x_k + R) \to \psi(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n))$$

belongs to p. We inductively create formulas $\varphi_k(x_1,\ldots,x_n) \in p(x_1,\ldots,x_n)$, $k \in \{1,\ldots,n\}$. Namely, provided that $\varphi_1(x_1,\ldots,x_n)$,…,$\varphi_{k-1}(x_1,\ldots,x_n)$ have already been defined, let $\varphi_k(x_1,\ldots,x_n)$ be the formula

$$\forall y((x_k \leq y \wedge y \leq x_k+R) \rightarrow (\varphi\wedge\varphi_1\wedge\ldots\wedge\varphi_{k-1})(x_1,\ldots,x_{k-1},y,x_{k+1},\ldots,x_n)).$$

Finally, we take any $\langle g_1,\ldots,g_n\rangle \in (\varphi\wedge\varphi_1\wedge\ldots\wedge\varphi_n)(G^n)$ and see that

$$[g_1,g_1+R]\times\ldots\times[g_n,g_n+R] \subseteq \varphi(G^n),$$

which finishes the proof (note that the formula $(\varphi\wedge\varphi_1\wedge\ldots\wedge\varphi_n)(x_1,\ldots,x_n)$ is consistent since it belongs to p). □

The following corollary shows that groups of the form $(G^n,+)$ definable in $(G,<,+,\ldots)$ share some properties of stable groups.

Corollary 3.3.5 *Assume that $(G,<,+,\ldots)$ is an o-minimal expansion of an ordered group $(G,<,+)$, $0<n,k<\omega$ and $\varphi(x_1,\ldots,x_n,y_1,\ldots,y_k)\in L$.*
(1) There is $\psi_1(y_1,\ldots,y_k)$ such that for every $\langle a_1,\ldots,a_k\rangle \in G^k$ we have that $G \models \psi_1(\bar a)$ if and only if the set $\varphi(G^n,\bar a)$ is weak generic in $(G^n,+)$.
(2) There is $\psi_2(y_1,\ldots,y_k)$ such that for every $\langle a_1,\ldots,a_k\rangle \in G^k$ we have that $G \models \psi_2(\bar a)$ if and only if the set $\varphi(G^n,\bar a)$ is generic in $(G^n,+)$.
(3) There is a natural number N such that for every φ-definable $X \subseteq G^n$ the set X is generic in $(G^n,+)$ if and only if G^n may be covered by at most N left translates of X.

Proof. (1) We define $\psi_1(y_1,\ldots,y_k)$ as

$$\forall r\exists z_1,\ldots,z_n\forall x_1,\ldots,x_n((\bigwedge_{i=1}^{n} z_i \leq x_i\wedge x_i \leq z_i+r) \rightarrow \varphi(x_1,\ldots,x_n,y_1,\ldots,y_k))$$

and apply Theorem 3.3.4.

(2) By Theorem 3.3.4, the set $\varphi(G^n,\bar a)$ is generic in $(G^n,+)$ if and only if the set $\neg\varphi(G^n,\bar a)$ is not weak generic in $(G^n,+)$. So it suffices to apply (1).

(3) To simplify the notation assume that $n=1$. Let $\psi_2(y_1,\ldots,y_k)$ be such as in (2). For the sake of contradiction suppose that for every $N<\omega$ we can find $\langle a_1,\ldots,a_k\rangle \in G^k$ such that the set $\varphi(G,a_1,\ldots,a_k)$ is generic in G but not N-generic. Then the set of formulas

$$\bigcup_{N<\omega}\{\psi_2(y_1,\ldots,y_k)\wedge\forall z_1,\ldots,z_N\exists t\forall x(\varphi(x,y_1,\ldots,y_k)\rightarrow\bigwedge_{i=1}^{N} t\neq z_i\cdot x)\}$$

is a type in variables $y_1, \dots, y_k$ and has a realization $\langle b_1, \dots, b_k \rangle \in H^k$ in some $\aleph_0$-saturated elementary extension H of G. We reach a contradiction as the set $\varphi(H, b_1, \dots, b_k)$ is simultaneously generic and not generic in H. $\square$

We conclude this section with one more consequence of Theorem 3.3.4. It enables us to distinguish weak generic types in groups of the form $(G^n, +)$ from non-weak generic ones.

Corollary 3.3.6 *Assume that* $(G, <, +, \dots)$ *is an o-minimal expansion of an ordered group* $(G, <, +)$*,* $0 < n < \omega$ *and* $p(x_1, \dots, x_n) \in S_n(G)$*. The following are equivalent:*
(1) the type $p(x_1, \dots, x_n)$ *is weak generic in* $(G^n, +)$*,*
(2) $\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) = p(x_1, \dots, x_n)$ *for every* $\langle g_1, \dots, g_n \rangle \in G^n$*.*

Proof. (1) $\Rightarrow$ (2) For the sake of contradiction suppose that

$$\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) \neq p(x_1, \dots, x_n)$$

for some tuple $\langle g_1, \dots, g_n \rangle \in G^n$. Then for some $\varphi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$ we have that $(\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$ (it is so since $p(x_1, \dots, x_n)$, as a complete type, is a set of formulas closed under taking conjunctions). The set $\varphi(G^n)$ is weak generic in $(G^n, +)$ and hence contains arbitrarily large boxes (Theorem 3.3.4). Take any $R > \max(|g_1|, \dots, |g_n|)$ and choose $a_1, \dots, a_n \in G$ such that

$$B = [a_1, a_1 + R] \times \dots \times [a_n, a_n + R] \subseteq \varphi(G^n).$$

We obtain

$$\emptyset \neq (\langle g_1, \dots, g_n \rangle + B) \cap B \subseteq (\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset,$$

a contradiction.

(2) $\Rightarrow$ (1) We shall prove a more general fact. Namely, if G is a group and $p(x) \in S(G)$ is such that for every $g \in G$ we have that $g \cdot p = p$, then p is weak generic in G.

Because if not, then we can find a formula $\varphi(x) \in p(x)$ which is not weak generic in G. Then $\neg\varphi(x)$ is generic in G so there are $m < \omega$ and $g_1, \dots, g_m \in G$ such that $G = \bigcup_{i=1}^m g_i \cdot (G \setminus \varphi(G))$. Thus $\bigcap_{i=1}^m g_i \cdot \varphi(G) = \emptyset$, which contradicts the fact that the formulas $g_1 \cdot \varphi(x), \dots, g_m \cdot \varphi(x)$ belong to the consistent type $p(x)$. $\square$

3.4 Stationarity

In this section we assume that $(G, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(G, <, +)$.

Recall that in a stable group all weak generic types are generic. Moreover, all of them are stationary over any model M. This means that every (weak) generic type $p \in S(M)$ has a unique extension to a (weak) generic type $q \in S(A)$ for each $A \supseteq M$. Stationarity of (weak) generic types plays an important role in the theory of stable groups.

Definition 3.4.1 *We call a weak generic type p over a set A stationary if for every $B \supseteq A$ the type p has just one extension to a complete weak generic type over B.*

In general, weak generic types do not need to be stationary (Example 3.5.8). However, in this section we focus on stationary ones.

Example 3.4.2 We shall prove that the types $p_1(x) = \{x < a : a \in G\}$ and $p_2(x) = \{x > a : a \in G\}$ are the only two weak generic types in $(G, +)$ complete over G and that both of them are stationary.

By the o-minimality of $(G, <, +, \ldots)$, every definable subset of G is a union of finitely many points and intervals. For every $a, b \in G$ the interval (a, b) is not weak generic in $(G, +)$ (apply Lemma 3.2.1(2)). Thus no type in $S_1(G)$ but p_1 and p_2 is weak generic in $(G, +)$.

On the other hand, all intervals of the form $(-\infty, a)$ or $(b, +\infty)$ are weak generic in $(G, +)$ since their complements in G are not generic in $(G, +)$. This gives us the weak genericity of the types p_1 and p_2.

If H is any elementary extension of G, then there are also two complete (over H) weak generic types in $(H, +)$. This means that p_1 and p_2 are stationary.

Definition 3.4.3 *We call an o-minimal structure $(M, <, \ldots)$ stationary if for every elementary extension N of M and N-definable function $g : N \to N$ there exists an M-definable function $f : N \to N$ such that $g(x) \leq f(x)$ for all sufficiently large $x \in N$.*

Remark 3.4.4 *Assume $(M, <, \ldots)$ is a stationary o-minimal structure and $N \succ M$. For every N-definable map $g : N \to N$ with $\lim_{x \to +\infty} g(x) = +\infty$ we can find an M-definable map $f : N \to N$ such that $\lim_{x \to +\infty} f(x) = +\infty$ and $f(x) \leq g(x)$ for all sufficiently large $x \in N$.*

Proof. First of all, assume that g is a bijection. Then g^{-1} (the compositional inverse of g) exists and by the stationarity of $(M,<,\ldots)$, we can find an M-definable function $f : N \to N$ such that ultimately $g^{-1} \leq f$. We have that $\lim_{x\to+\infty} g^{-1}(x) = +\infty$, which implies that $\lim_{x\to+\infty} f(x) = +\infty$. Since f is M-definable, we can choose $a \in M$ such that f is strictly increasing on $(a, +\infty)$. We define a function $f_1 : N \to N$ as follows:

$$f_1(x) = \begin{cases} f(x) & \text{, when } x > a \\ f(a) + x - a & \text{, otherwise.} \end{cases}$$

Then f_1 is an M-definable bijection, hence f_1^{-1} exists and also is M-definable. Moreover, $\lim_{x\to+\infty} f_1^{-1}(x) = +\infty$ and ultimately $f_1^{-1} \leq g$ so f_1^{-1} has the desired properties.

If g is not a bijection, then proceeding as above we can find an N-definable bijection $g_1 : N \to N$ such that ultimately $g_1 = g$. The rest of the proof remains the same. □

Now we turn our attention to weak generic sets and types in the group $(G,+)\times(G,+)$. By the o-minimality of $(G,<,+,\ldots)$, every definable subset of the set $G \times G$ is a union of finitely many cells of dimension 0, 1 or 2. By Proposition 3.3.1, we are interested only in cells of dimension 2. They are of the form

$$C^{f,g}_{a,b} = \{\langle x, y\rangle \in G \times G : a < x \wedge x < b \wedge f(x) < y \wedge y < g(x)\}$$

where $\{-\infty\} \cup G \ni a < b \in G \cup \{+\infty\}$ and $f, g : (a, b) \to G \cup \{-\infty, +\infty\}$ are definable maps such that $f(x) < g(x)$ for each $x \in (a, b)$. If $a, b \in G$, then the cell $C^{f,g}_{a,b}$ is not weak generic in $(G,+) \times (G,+)$ by Theorem 3.3.4. Since we shall consider only weak generic types $p(x, y)$ in $(G,+) \times (G,+)$ such that $\{x > a : a \in G\} \subseteq p(x, y)$, we shall be interested only in weak generic cells of the form $C^{f,g}_{a,b}$ where $a \in G$ and $b = +\infty$.

Definition 3.4.5 *Assume that functions $f, g : G \to G$ are definable.*
(1) We say that $f \ll g$ if $f(x) < g(x)$ for all sufficiently large $x \in G$ and the set

$$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

is weak generic in $(G,+) \times (G,+)$.
(2) We say that $f \sim g$ if the set

$$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

is not weak generic in $(G,+) \times (G,+)$.

Replacing 0 by any other element of the group G does not change the meaning of the definition above since for every $a, b \in G$ the cell $C_{a,b}^{f,g}$ is not weak generic in $(G, +) \times (G, +)$.

It is easy to see that $\sim$ is an equivalence relation on the set of all definable functions from G to G and that equivalence classes of $\sim$ are convex (i.e. if $f, g, h : G \to G$ are definable, $f \sim h$ and ultimately $f(x) \leq g(x) \leq h(x)$, then $f \sim g$ and $g \sim h$).

Definition 3.4.6 *Let $f : G \to G$ be a definable function.*
1) Let $p_f^+(x, y)$ denote the only extension of the type

$$\{x > a : a \in G\} \cup \{y > f(x)\} \cup \{y < g(x) : g \gg f\}$$

to a type which is complete over G and weak generic in $(G, +) \times (G, +)$.
2) Let $p_f^-(x, y)$ denote the only extension of the type

$$\{x > a : a \in G\} \cup \{y < f(x)\} \cup \{y > g(x) : g \ll f\}$$

to a type which is complete over G and weak generic in $(G, +) \times (G, +)$.
3) Let $p_{+\infty}(x, y)$ denote the weak generic type

$$\{x > a : a \in G\} \cup \{y > g(x) : \ g : G \to G \ \textit{definable}\}.$$

4) Let $p_{-\infty}(x, y)$ denote the weak generic type

$$\{x > a : a \in G\} \cup \{y < g(x) : \ g : G \to G \ \textit{definable}\}.$$

The next theorem shows that the stationarity of the types from the definition above is equivalent to the stationarity of the o-minimal group G.

Theorem 3.4.7 *Assume that $(G, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(G, <, +)$. The following are equivalent:*
(1) the types $p_f^+(x, y)$ and $p_f^-(x, y)$ are stationary for each definable function $f : G \to G$,
(2) the type $p_{+\infty}(x, y)$ (or $p_{-\infty}(x, y)$) is stationary,
(3) the structure $(G, <, +, \ldots)$ is stationary.

Proof. (1) $\Rightarrow$ (2) Let $f : G \to G$ be a map constantly equal to 0. It follows from (1) that the type p_f^+ is stationary. But $p_{+\infty}(x, y) = p_f^+(y, x)$ and therefore $p_{+\infty}$ is stationary as well.

(2) $\Rightarrow$ (3) For the sake of contradiction suppose the structure $(G, <, +, \ldots)$ is not stationary. Then there exist an $H \succ G$ and a definable function $g : H \to H$ such that no G-definable map $f : H \to H$ dominates g.

Consider the following partial types over H:

$$p_1(x, y) = p_{+\infty}(x, y) \cup \{y < g(x)\}$$

and

$$p_2(x, y) = p_{+\infty}(x, y) \cup \{y > g(x)\}.$$

To reach a contradiction, it is enough to prove that both of them are weak generic in $(H, +) \times (H, +)$. Let us begin with p_1. We have to show that each formula of the form

$$(\bigwedge_{i=1}^{m} x > a_i) \wedge (\bigwedge_{i=1}^{n} y > f_i(x)) \wedge y < g(x)$$

is weak generic in $(H, +) \times (H, +)$ where $a_1, \ldots, a_m \in G$ and $f_1, \ldots, f_n$ are functions from H to H definable over G. Taking $a = \max(a_1, \ldots, a_n)$ and $f = \max(f_1, \ldots, f_n)$ we can confine our attention to the sets X of the form

$$X = \{\langle x, y\rangle \in H \times H : x > a \wedge y > f(x) \wedge y < g(x)\}$$

where $a \in G$ and $f : H \to H$ is definable over G. Without loss of generality we can assume that f is ultimately non-decreasing.

Consider a map $h : H \to H$ defined as follows: $h(a) = f(2a) + a$ for each $a \in H$. Since h is G-definable, g dominates h. Note that for each large enough $M \in H$ the area between the graphs of f and g in $H \times H$ contains the square whose vertices are

$$\langle M, f(2M)\rangle,\ \langle M, f(2M) + M\rangle,\ \langle 2M, f(2M)\rangle \text{ and } \langle 2M, f(2M) + M\rangle.$$

By Theorem 3.3.4, the set X is weak generic in $(H, +) \times (H, +)$. As a result, the type p_1 is weak generic in $(H, +) \times (H, +)$. It is easy to prove that so is p_2, which contradicts the stationarity of $p_{+\infty}$.

(3) $\Rightarrow$ (1) Take any definable $f : G \to G$. We shall show that both p_f^+ and p_f^- are stationary weak generic types.

By the o-minimality of G, f is ultimately non-negative or ultimately non-positive. It is easy to see that p_f^+ is stationary if and only if p_{-f}^- is stationary and p_f^- is stationary if and only if p_{-f}^+ is stationary. Therefore without loss of generality we can assume that f is ultimately non-negative. Moreover, f is ultimately non-increasing or ultimately non-decreasing. If f is ultimately non-increasing, then $p_f^+ = p_z^+$ and $p_f^- = p_z^-$ where $z : G \to G$ is constantly equal to 0. So we can also assume that f is ultimately non-decreasing.

Consider definable sets:

$$A = \{a \in G : (\exists b > a)(\forall c \in (a, b)) f(c) - f(a) \leq c - a\}$$

and

$$B = \{a \in G : (\exists b > a)(\forall c \in (a, b)) f(c) - f(a) > c - a\}.$$

Note that by the o-minimality of G, we have that $G = A \cup B$ and for some $M \in G$ either $(M, +\infty) \subseteq A$ or $(M, +\infty) \subseteq B$. Enlarge M in order to ensure that f is continuous on $(M, +\infty)$.

Case 1. $(M, +\infty) \subseteq A$. Then f grows "slowly" on $(M, +\infty)$:

$$(*)\ (\forall a > M)(\exists b > 0)(\forall c \in (0, b)) f(a + c) \leq f(a) + c.$$

By $(*)$ and the continuity of f,

$$(**)\ (\forall a > M)(\forall c > 0) f(a + c) \leq f(a) + c.$$

Because if not, then the opposite holds: $(\exists a > M)(\exists c > 0) f(a+c) > f(a)+c$. Let $C = \{c > 0 : f(a+c) > f(a)+c\}$ and $c_0 = \inf(C)$. Assertion $(*)$ implies that $c_0 > 0$. Since f is continuous at c_0, $c_0 \notin C$. Choose $d > c_0$ such that $(c_0, d) \subseteq C$. Since $c_0 \notin C$, $f(a + c_0) \leq f(a) + c_0$. On the other hand, by the continuity of f at $a + c_0$, we have that $f(a + c_0) \geq f(a) + c_0$. Thus $f(a + c_0) = f(a) + c_0$ and for every $e \in (0, d - c_0)$ we have that

$$f(a + c_0 + e) > f(a) + c_0 + e = f(a + c_0) + e,$$

which implies that $a + c_0 \notin A$. But $a + c_0 \in (M, +\infty) \subseteq A$, a contradiction. So $(**)$ holds.

For the sake of contradiction assume that p_f^+ is not stationary. Then for some $H \succ G$ and definable $g : H \to H$ we have that $f \ll g$ and $g \ll h$ for each G-definable $h : H \to H$ with $f \ll h$. Since $\lim_{x \to +\infty}(g(x) - f(x)) = +\infty$, there exists an increasing to $+\infty$ G-definable function $h : H \to H$ such that ultimately $h \leq g - f$ (apply Remark 3.4.4). Enlarging M we can assume that h is increasing on $(M, +\infty)$.

Now fix any positive $R \in H$ and find $a > M$ with $h(a) \geq 2R$. By (**), we have that $f(a+R) \leq f(a)+R$. So the area between the graphs of f and $f+h$ contains the square whose vertices are

$$\langle a, f(a)+R\rangle,\ \langle a, f(a)+2R\rangle,\ \langle a+R, f(a)+R\rangle \text{ and } \langle a+R, f(a)+2R\rangle.$$

As R was arbitrary, we can use Theorem 3.3.4 to conclude that the area between the graphs of f and $f+h$ is weak generic in $(H,+) \times (H,+)$. So $f \ll f+h$ and therefore $g \ll f+h$, which contradicts the fact that ultimately $g \geq f+h$. So the type p_f^+ is stationary. The proof that p_f^- is stationary is analogous and we omit it.

Case 2. $(M,+\infty) \subseteq B$. Then f grows "quickly" on $(M,+\infty)$, which implies that $\lim_{x\to+\infty} f(x) = +\infty$. Proceeding as in Remark 3.4.4 find a definable bijection $f_1 : G \to G$ such that $f_1(a) = f(a)$ for each $a \in (M,+\infty)$. If $g = f_1^{-1}$, then g grows "slowly" on $(M,+\infty)$ and from the previous case we know that the types p_g^+ and p_g^- are stationary. The proof is complete since $p_f^+(x,y) = p_{f_1}^+(x,y) = p_g^-(y,x)$ and $p_f^-(x,y) = p_{f_1}^-(x,y) = p_g^+(y,x)$. $\square$

Theorem 3.4.7 provides us with many examples of stationary weak generic types. Actually, we do not know any example of a non-stationary o-minimal structure of the form $(G,<,+,\ldots)$ where $(G,<,+)$ is an ordered group.

Example 3.4.8 If $(G,<,+)$ is an o-minimal ordered group, then every definable function $f : G \to G$ is ultimately equal to $f_q(x)+a$ for some $a \in G$ and $q \in \mathbb{Q}$ where $f_q(x) = q\cdot x$ for each $x \in G$ (see [3], Corollary 1.7.6). Below we list all weak generic types in $(G,+)\times(G,+)$ that are complete over G and contain the formula $(x > 0)$.

(1) $p_{-\infty}(x,y)$ and $p_{+\infty}(x,y)$.

(2) $p_{f_q}^-(x,y)$ and $p_{f_q}^+(x,y)$, $q \in \mathbb{Q}$.

(3) $\{x > a : a \in G\} \cup \{y > q\cdot x : q \in \mathbb{Q} \wedge q < r\} \cup \{y < q\cdot x : q \in \mathbb{Q} \wedge q > r\}$, $r \in \mathbb{R}\setminus\mathbb{Q}$.

The structure $(G,<,+)$ is stationary since its elementary extensions are all linearly bounded. Thus by Theorem 3.4.7, weak generic types of the form (1) and (2) are stationary. It is easy to see that so are those of the form (3).

3.5 Expansions of real closed fields

In this section $(R, <, +, \cdot, 0, 1, \ldots)$ is an o-minimal expansion of an ordered ring $(R, <, +, \cdot, 0, 1)$. As noted in [13] (Theorem 2.3), such a ring must be a real closed field. Since $(R, <, +, \cdot, 0, 1, \ldots)$ is an o-minimal expansion of the ordered group $(R, <, +)$, all results obtained in the previous sections apply. Throughout this section, R_+ denotes the set $\{a \in R : a > 0\}$.

Definition 3.5.1 *We call a structure* $(R, <, +, \cdot, \ldots)$ *polynomially bounded if for every definable function* $f : R \to R$ *there is* $n \in \mathbb{N}_+$ *such that* $|f(x)| \leq x^n$ *for all sufficiently large* $x \in R$.

Remark 3.5.2 *If a real closed field* $(R, <, +, \cdot, \ldots)$ *is polynomially bounded and o-minimal, then for every definable* $f : R \to R$ *with* $\lim_{x \to +\infty} f(x) = +\infty$ *we can find* $n \in \mathbb{N}_+$ *such that* $f(x) \geq \sqrt[n]{x}$ *for all sufficiently large* $x \in R$.

Proof. We proceed as in the proof of Remark 3.4.4. Since f is ultimately increasing, we are able to find a definable bijection $g : R \to R$ such that $f(x) = g(x)$ for all sufficiently large $x \in R$. We know that the inverse map g^{-1} is ultimately dominated by the polynomial function $x \mapsto x^n$ for some $n \in \mathbb{N}_+$. But this implies that ultimately $f(x) = g(x) \geq \sqrt[n]{x}$ and the proof is complete (note that the expression $\sqrt[n]{x}$ is well defined for every $n \in \mathbb{N}_+$ and $x \in R_+$ as the field R is real closed). $\square$

Assume $(R, <, +, \cdot)$ is a pure real closed field. Since every definable map $f : R \to R$ is semi-algebraic, it follows from Proposition 2.6.1 in [1] that the structure $(R, <, +, \cdot)$ is polynomially bounded (see also Lemma 3.5.5).

Corollary 3.5.3 *Every pure real closed field* $(R, <, +, \cdot)$ *is stationary and so are the weak generic types* $p_f^-(x, y)$ *and* $p_f^+(x, y)$ *for each definable* $f : R \to R$.

Proof. Consider an arbitrary elementary extension S of R and any definable map $f : S \to S$. Since the real closed field $(S, <, +, \cdot)$ is polynomially bounded, there exists $n \in \mathbb{N}_+$ such that ultimately $|f(x)| \leq x^n$. This gives us the stationarity of the structure $(R, <, +, \cdot)$ since the map $x \mapsto x^n$ is definable over R. The second assertion immediately follows from the first one by Theorem 3.4.7. $\square$

On the other hand, the structure $(\mathbb{R}, <, +, \cdot, e^x)$ is not polynomially bounded but it is still an o-minimal expansion of the ordered field of real numbers (for more details see [18]).

Definition 3.5.4 *Assume $(R, +, \cdot, 0, 1)$ is a field, $f, g : R \to R$ and $g(x) \neq 0$ for all sufficiently large $x \in R$. We write $f \approx g$ if and only if*

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

The next lemma will be very useful when we examine weak generic types in groups definable in pure real closed fields.

Lemma 3.5.5 *Assume $(R, <, +, \cdot)$ is a pure real closed field. If a function $f : R \to R$ is definable and ultimately non-zero, then for some $q \in \mathbb{Q}$ and $c \in R \setminus \{0\}$ we have that $f(x) \approx c \cdot x^q$.*

Proof. Let S be an arbitrary $|R|^+$-saturated elementary extension of R. We can find $a \in S$ such that $a > r$ for every $r \in R$. Let

$$T = \{s \in S : |s| < r \text{ for some } r \in R\}.$$

Then T is a convex subring of R,

$$T^* = \{s \in S : \frac{1}{r} < |s| < r \text{ for some } r \in R\}$$

and $(T^*, \cdot)$ is a subgroup of the multiplicative group $(S^*, \cdot)$. The quotient group $(S^*/T^*, *, \mathbf{1})$ may be ordered in the following way:

$$s_1/T^* \leq s_2/T^* \Leftrightarrow \frac{s_1}{s_2} \in T.$$

We define a function $\nu : S \to S^*/T^* \cup \{-\infty\}$ (where for every $s \in S^*$ $-\infty < s/T^*$ and $(-\infty) * s/T^* = -\infty$) as follows:

$$\nu(s) = \begin{cases} -\infty & \text{, when } s = 0 \\ s/T^* & \text{, otherwise.} \end{cases}$$

It is easy to prove that ν is a valuation of the field S i.e. for all $x, y \in S$ we have that

(1) $\nu(x \cdot y) = \nu(x) * \nu(y)$,

(2) $\nu(x + y) \leq \max(\nu(x), \nu(y))$ and

(3) $\nu(x) \neq \nu(y) \Rightarrow \nu(x + y) = \max(\nu(x), \nu(y))$.

Since f is semi-algebraic, by Lemma 2.5.2 in [1] there exists a non-zero polynomial $P(X, Y) \in R[X, Y]$ such that $R \models \forall x(P(x, f(x)) = 0)$. So $S \models \forall x(P(x, f(x)) = 0)$ and, in particular, $P(a, f(a)) = 0$. The polynomial $P(X, Y)$ is of the form

$$P(X, Y) = \sum_{i=1}^{n} r_i \cdot X^{k_i} \cdot Y^{l_i}$$

for some $n \in \mathbb{N}_+$, $r_i \in R \setminus \{0\}$ and $k_i, l_i < \omega$ such that $\langle k_i, l_i \rangle \neq \langle k_j, l_j \rangle$ for every $i \neq j \in \{1, \ldots, n\}$. Thus

$$0 = \sum_{i=1}^{n} r_i \cdot a^{k_i} \cdot f(a)^{l_i}$$

and for some $i \neq j \in \{1, \ldots, n\}$ we have that

$$\nu(r_i \cdot a^{k_i} \cdot f(a)^{l_i}) = \nu(r_j \cdot a^{k_j} \cdot f(a)^{l_j}) \neq -\infty$$

since $f(a) \neq 0$ (if $f(a) = 0$, then $f : R \to R$ would be ultimately equal to 0).

This implies that $\nu(\frac{r_i}{r_j} \cdot a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = \mathbf{1}$ and $\nu(a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = \mathbf{1}$. So $a^{k_i - k_j} \cdot f(a)^{l_i - l_j} \in T^*$. If $l_i = l_j$, then $k_i \neq k_j$ and $a^{k_i - k_j} \in T^*$, which implies that $a \in T^*$, a contradiction. So $l_i \neq l_j$. Letting $q = -\frac{k_i - k_j}{l_i - l_j} \in \mathbb{Q}$ we obtain $\frac{f(a)}{a^q} \in T^*$. Therefore $\frac{1}{r} < |\frac{f(a)}{a^q}| < r$ for some $r \in R$. If $b \in S$ and $b > a$, then $tp(a/R) = tp(b/R)$. Hence for every $b > a$ we have that $\frac{1}{r} < |\frac{f(b)}{b^q}| < r$ and consequently

$$S \models \exists y \forall x (x > y \to \frac{1}{r} < |\frac{f(x)}{x^q}| < r).$$

As $R \prec S$, this implies that $\frac{1}{r} < |\frac{f(x)}{x^q}| < r$ for all sufficiently large $x \in R$. By the o-minimality of R, for some $c \in R$ with $\frac{1}{r} \leq |c| \leq r$ we have that $\lim_{x \to +\infty} \frac{f(x)}{x^q} = c$, which finishes the proof. $\square$

Theorem 3.5.6 *Assume $(R, <, +, \cdot)$ is a pure real closed field. Let*

$$f(x) = \sum_{i=1}^{m} a_i \cdot x^{p_i} \text{ and } g(x) = \sum_{j=1}^{n} b_j \cdot x^{q_j}$$

where $m, n \in \mathbb{N}_+$, $a_1, \ldots, a_m, b_1, \ldots, b_n \in R$, $a_1, b_1 > 0$, $p_1 > \ldots > p_m \in \mathbb{Q}$ and $q_1 > \ldots > q_n \in \mathbb{Q}$. The following are equivalent:
(1) $f \ll f + g$,
(2) $q_1 > \max(0, p_1 - 1)$.

Proof. We define a rate of growth $\mathrm{gr}(f)$ of a definable map $f : R \to R$ as follows: if $f(x) \approx c \cdot x^q$ for some $c \in R \setminus \{0\}$ and $q \in \mathbb{Q}$, then $\mathrm{gr}(f) = q$ (Lemma 3.5.5 implies that $\mathrm{gr}(f)$ is well defined for each ultimately non-zero definable function $f : R \to R$). Obviously $\mathrm{gr}(f \cdot g) = \mathrm{gr}(f) + \mathrm{gr}(g)$ and $\mathrm{gr}(f + g) = \max(\mathrm{gr}(f), \mathrm{gr}(g))$ provided that $\mathrm{gr}(f) \neq \mathrm{gr}(g)$. Actually, gr is a valuation in the Hardy field of germs of definable functions from R to R.

First of all, we shall prove that $((x + c)^q - x^q) \approx c \cdot q \cdot x^{q-1}$ for every $c \in R \setminus \{0\}$ and $q \in \mathbb{Q}_+$. Let $q = \frac{p}{p'}$ where $p, p' \in \mathbb{Z}_+$. For each $x \in R_+$ let $\Delta(x) = (x + c)^q - x^q$ and note that $\lim_{x\to+\infty}(\Delta(x) \cdot x^{-q}) = 0$, which implies that $\mathrm{gr}(\Delta(x)) < q$. Since $(x + c)^p = (\Delta(x) + x^q)^{p'}$, we have that

$$\sum_{i=0}^{p} \binom{p}{i} \cdot x^i \cdot c^{p-i} = \sum_{i=0}^{p'} \binom{p'}{i} \cdot \Delta(x)^i \cdot (x^q)^{p'-i}$$

and

$$L(x) = \sum_{i=0}^{p-1} \binom{p}{i} \cdot x^i \cdot c^{p-i} = \sum_{i=1}^{p'} \binom{p'}{i} \cdot \Delta(x)^i \cdot (x^q)^{p'-i} = R(x).$$

Obviously $\mathrm{gr}(L(x)) = \mathrm{gr}(\binom{p}{p-1} \cdot x^{p-1} \cdot c) = p - 1$. On the other hand, since $\mathrm{gr}(\Delta(x)) < q$, we have that

$$\mathrm{gr}(R(x)) = \mathrm{gr}(\binom{p'}{1} \cdot \Delta(x) \cdot (x^q)^{p'-1}) = \mathrm{gr}(\Delta(x)) + q \cdot (p' - 1).$$

Thus $\mathrm{gr}(\Delta(x)) = p - 1 - q \cdot (p' - 1) = q - 1$ and $\Delta(x) \approx c \cdot q \cdot x^{q-1}$.

(1) $\Rightarrow$ (2) We see that $q_1 > 0$ because otherwise for some $c \in R$ we would have that $|g(x)| \leq c$ for every $x \in R$ and consequently $f \sim f + g$. Now if $p_1 - 1 \leq 0$, then $q_1 > p_1 - 1$ as well, which finishes the proof. So without loss of generality we can assume that $p_1 > 1$.

We know that $f(x) < f(x) + g(x)$ for all sufficiently large $x \in R$ and the set

$$A_f^{f+g} = \{\langle x, y\rangle \in R \times R : x > 0 \wedge f(x) < y \wedge y < f(x) + g(x)\}$$

is weak generic in $(R \times R, +)$. By Theorem 3.3.4, for every $M \in R_+$ there exist $x_M, y_M \in R$ such that

$$\{\langle x, y\rangle \in R \times R : x_M < x \wedge x < x_M + M \wedge y_M < y \wedge y < y_M + M\} \subseteq A_f^{f+g}.$$

This implies that $f(x_M)+g(x_M) \geq f(x_M+M)+M$ for all sufficiently large $M \in R$. It is easy to see that $\lim_{M\to+\infty} x_M = +\infty$.

Put $M_0 = \frac{b_1+1}{a_1 \cdot p_1}$. Then still for all sufficiently large $M \in R$ we have that $f(x_M)+g(x_M) \geq f(x_M+M_0)+M_0$ and by the o-minimality of $(R,<,+,\cdot)$, $f(x)+g(x) \geq f(x+M_0)+M_0$ for all sufficiently large $x \in R$. So ultimately

$$\sum_{i=1}^{m} a_i \cdot x^{p_i} + \sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M_0 + \sum_{i=1}^{m} a_i \cdot (x+M_0)^{p_i}$$

and

$$\sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M_0 + \sum_{i=1}^{m} a_i \cdot ((x+M_0)^{p_i} - x^{p_i}).$$

Finally, comparing the ingredients of the sums with the biggest value of gr we see that ultimately

$$b_1 \cdot x^{q_1} \geq a_1 \cdot ((x+M_0)^{p_1} - x^{p_1}) \approx a_1 \cdot M_0 \cdot p_1 \cdot x^{p_1-1} = (b_1+1) \cdot x^{p_1-1}.$$

Hence $q_1 > p_1 - 1$, which finishes the proof.

(2) $\Rightarrow$ (1) Fix $M \in R_+$. Since $q_1 > \max(0, p_1-1)$, proceeding as above we can show that for all sufficiently large $x \in R$

$$\sum_{i=1}^{m} a_i \cdot x^{p_i} + \sum_{j=1}^{n} b_j \cdot x^{q_j} \geq M + \sum_{i=1}^{m} a_i \cdot (x+M)^{p_i}.$$

This means that ultimately $f(x)+g(x) \geq f(x+M)+M$. Choose $x_M \in R_+$ satisfying the latter inequality and such that f and g are increasing on the interval $(x_M,+\infty)$. Then for $y_M = f(x_M+M)$ we have that

$$\{\langle x,y\rangle \in R \times R : x_M < x \wedge x < x_M+M \wedge y_M < y \wedge y < y_M+M\} \subseteq A_f^{f+g}$$

and it suffices to apply Theorem 3.3.4. $\square$

Example 3.5.7 Let $(R,<,+,\cdot)$ be a pure real closed field and for $a \in \mathbb{R}_+ \setminus \mathbb{Q}$ let

$$p(x,y) = \{x > r : r \in R\} \cup \{y > x^q : a > q \in \mathbb{Q}\} \cup \{y < x^q : a < q \in \mathbb{Q}\}.$$

We shall prove that p is a stationary complete weak generic type in the group $(R,+)\times(R,+)$ and p is not of the form p_f^- or p_f^+ for any definable $f : R \to R$.

The weak genericity of p follows from Theorem 3.5.6. Indeed, the set

$$\{\langle x, y\rangle \in R \times R : x > r \wedge y > x^{q_1} \wedge y < x^{q_2}\}$$

(where $r \in R$, $q_1, q_2 \in \mathbb{Q}_+$ and $q_1 < a < q_2$) is weak generic in $(R,+)\times(R,+)$ since $q_2 > \max(0, q_1 - 1)$.

The stationarity (and the completeness) of p follows from Lemma 3.5.5. Namely, if p were non-stationary, then for some $S \succ R$ (where possibly $S = R$) and definable $f : S \to S$ we would have that ultimately $f(x) > x^q$ for each $q \in \mathbb{Q} \cap (-\infty, a)$ and ultimately $f(x) < x^q$ for each $q \in \mathbb{Q} \cap (a, +\infty)$. By Lemma 3.5.5, $f(x) \approx c \cdot x^q$ for some $q \in \mathbb{Q}$ and $c \in S_+$ ($c > 0$ since f is ultimately increasing). Assume that $q > a$ and take any $r \in \mathbb{Q} \cap (a, q)$. Then ultimately $f(x) > x^r$ (since $q > r$). On the other hand, ultimately $f(x) < x^r$ (since $r \in \mathbb{Q} \cap (a, +\infty)$). If $q < a$, then we reach a contradiction in a similar way. As a result, p is stationary (and complete).

Finally, suppose $p(x, y) = p'(x, y)$ for some definable function $f : R \to R$ and a type $p'(x, y) \in \{p_f^-(x, y), p_f^+(x, y)\}$. Then $f(x) \approx c \cdot x^q$ for some $q \in \mathbb{Q}$ and $c \in R_+$. Without loss of generality $q > a$. Take any $r \in \mathbb{Q} \cap (a, q)$. Then $(y < x^r) \in p(x, y)$ and $(y > x^r) \in p'(x, y)$, a contradiction.

Example 3.5.8 Let $(R, <, +, \cdot, \ldots)$ be an o-minimal polynomially bounded expansion of a real closed field $(R, <, +, \cdot)$ and for $q_0 \in \mathbb{Q}_+$ let

$$p(x, y) = \{x > r : r \in R\} \cup \{y > r \cdot x^{q_0} : r \in R\} \cup \{y < x^q : q_0 < q \in \mathbb{Q}\}.$$

We shall prove that p is a non-stationary complete weak generic type in the group $(R, +) \times (R, +)$.

To prove that p is complete we proceed by way of contradiction. If it were not complete, then we could find a definable function $f : R \to R$ such that ultimately $f(x) < x^q$ for each $q \in \mathbb{Q} \cap (q_0, +\infty)$ and ultimately $f(x) > r \cdot x^{q_0}$ for each $r \in R$ (thus $\lim_{x\to+\infty} \frac{f(x)}{x^{q_0}} = +\infty$). By Remark 3.5.2, $\frac{f(x)}{x^{q_0}} \geq \sqrt[n]{x}$ for some $n \in \mathbb{N}_+$ and all sufficiently large $x \in R$. But then ultimately $f(x) \geq x^{q_0 + \frac{1}{n}}$, a contradiction.

To obtain non-stationarity of p, let S be an $|R|^+$-saturated elementary extension of R and choose any $a \in S$ such that $a > r$ for every $r \in R$. Then

$$\{x > r : r \in R\} \cup \{y > r \cdot x^{q_0} : r \in R\} \cup \{y < x^q : q_0 < q \in \mathbb{Q}\} \cup \{y < a \cdot x^{q_0}\}$$

and

$$\{x > r : r \in R\} \cup \{y > r \cdot x^{q_0} : r \in R\} \cup \{y < x^q : q_0 < q \in \mathbb{Q}\} \cup \{y > a \cdot x^{q_0}\}$$

are two distinct extensions of p to weak generic types in $(S, +) \times (S, +)$.

3.5.1 Weak generic types in $(\mathbb{R},+)\times(\mathbb{R},+)$

Now we give a description of complete (over $\mathbb{R}$) weak generic types in the group $(\mathbb{R},+)\times(\mathbb{R},+)$ derived in the theory $Th(\mathbb{R},<,+,\cdot)$. Their structure turns out to be much more complicated than the structure of weak generic types derived in $Th(\mathbb{R},<,+)$ (see Example 3.4.8).

Let S be a $(2^{\aleph_0})^+$-saturated elementary extension of the field of reals. Choose $a \in S$ such that $a > r$ for every $r \in \mathbb{R}$. Let $b_0 \in S$ be such that $b_0 \neq \sum_{i=1}^n r_i \cdot a^{q_i}$ for all $n \in \mathbb{N}_+$, $r_i \in \mathbb{R}$ and $q_i \in \mathbb{Q}$ (in this case we say that b_0 is non-polynomial over a). We describe a recursive procedure of defining $b_1, b_2, \ldots \in S \setminus \{0\}$, $r_1, r_2, \ldots \in \mathbb{R} \setminus \{0\}$ and $q_1, q_2, \ldots \in \mathbb{Q}_+$ so that $q_1 > q_2 > \ldots$ and $b_n = b_0 - r_1 \cdot a^{q_1} - \ldots - r_n \cdot a^{q_n}$ for every $n \in \mathbb{N}_+$.

First we define b_1, r_1 and q_1. We consider two cases, depending on whether b_0 is positive or negative.

Case P. $b_0 > 0$. Consider the following subsets of $\mathbb{Q}_+$:

$$A = \{q \in \mathbb{Q}_+ : b_0 > r \cdot a^q \text{ for every } r \in \mathbb{R}_+\}$$

and

$$B = \{q \in \mathbb{Q}_+ : b_0 < r \cdot a^q \text{ for every } r \in \mathbb{R}_+\}.$$

The sets A and B are disjoint and there is a unique $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that $A \subseteq (0,c]$, $B \subseteq [c,+\infty)$ and $\mathbb{Q}_+ \cup \{c\} = A \cup B \cup \{c\}$. We define b_1, r_1 and q_1 only in the case where the following condition holds:

(†) $c \in \mathbb{Q}_+$, $A = \mathbb{Q}_+ \cap (0,c)$ and $B = \mathbb{Q}_+ \cap (c,+\infty)$.

Otherwise the procedure stops and no b_1, r_1 and q_1 are defined.

If (†) holds, then we put $q_1 = c$. We have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$ for some $r' < r'' \in \mathbb{R}_+$. Since the ordering $(\mathbb{R},<)$ is Dedekind complete, there exists a unique $r \in \mathbb{R}_+$ such that for every $r', r'' \in \mathbb{R}_+$ with $r' < r < r''$ we have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$. We put $r_1 = r$ and $b_1 = b_0 - r_1 \cdot a^{q_1}$.

Case N. $b_0 < 0$. Here we proceed similarly. Consider the following subsets of $\mathbb{Q}_+$:

$$A = \{q \in \mathbb{Q}_+ : b_0 < r \cdot a^q \text{ for every } r \in \mathbb{R}_-\}$$

and

$$B = \{q \in \mathbb{Q}_+ : b_0 > r \cdot a^q \text{ for every } r \in \mathbb{R}_-\}.$$

The sets A and B are disjoint and there is a unique $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that $A \subseteq (0, c]$, $B \subseteq [c, +\infty)$ and $\mathbb{Q}_+ \cup \{c\} = A \cup B \cup \{c\}$. We define b_1, r_1 and q_1 only in the case where (†) holds (for the new sets A and B). Otherwise the procedure stops and no b_1, r_1 and q_1 are defined.

If (†) holds, then we put $q_1 = c$. We have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$ for some $r' < r'' \in \mathbb{R}_-$. Since the ordering $(\mathbb{R}, <)$ is Dedekind complete, there exists a unique $r \in \mathbb{R}_-$ such that for every $r', r'' \in \mathbb{R}_-$ with $r' < r < r''$ we have that $r' \cdot a^{q_1} < b_0 < r'' \cdot a^{q_1}$. We put $r_1 = r$ and $b_1 = b_0 - r_1 \cdot a^{q_1}$.

Suppose b_i, r_i and q_i have been defined so that $b_i \neq 0$. Again we consider two cases, depending on whether b_i is positive or negative.

Case P. $b_i > 0$. We define the sets A, B and $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ as in the case of $b_0 > 0$. Again if (†) fails, then the procedure stops and b_j, r_j, q_j are not defined for any $j > i$. If (†) holds, then we put $q_{i+1} = c$ and define r_{i+1}, b_{i+1} analogously as in the case of b_0 (in particular, $b_{i+1} = b_i - r_{i+1} \cdot a^{q_{i+1}}$).

Case N. $b_i < 0$. We proceed analogously as in the case of $b_0 < 0$. If (†) fails, then the procedure stops. Otherwise we define b_{i+1}, r_{i+1} and q_{i+1}.

If $b_1, \ldots, b_i$, $q_1, \ldots, q_i$ and $r_1, \ldots, r_i$ are defined, then $q_1 > \ldots > q_i$. We shall only prove that $q_1 > q_2$. We have that

$$b_2 = b_1 - r_2 \cdot a^{q_2} = b_0 - r_1 \cdot a^{q_1} - r_2 \cdot a^{q_2}.$$

Without loss of generality we can assume that $r_2 > 0$. Choose any real number $r \in (0, r_2)$. Then $b_1 > r \cdot a^{q_2}$. If $q_1 \leq q_2$, then also $b_1 > r \cdot a^{q_1}$ and consequently $b_0 = b_1 + r_1 \cdot a^{q_1} > (r_1 + r) \cdot a^{q_1}$, which contradicts the definition of r_1. Hence $q_1 > q_2$.

Secondly, note that $b_k \neq 0$ for every $k \in \{1, \ldots, i\}$. Indeed, otherwise we would have that

$$b_k = b_0 - r_1 \cdot a^{q_1} - \ldots - r_n \cdot a^{q_n} = 0$$

and b_0 would be polynomial over a, a contradiction.

Now we are able to give a description of complete weak generic types in the group $(\mathbb{R}, +) \times (\mathbb{R}, +)$. Suppose that the type $tp(\langle a, b_0 \rangle / \mathbb{R})$ is weak generic in $(\mathbb{R}, +) \times (\mathbb{R}, +)$ (which implies that b_0 is non-polynomial over a) and $a > 0$ (hence $a > r$ for every $r \in \mathbb{R}$). Denote the type $tp(\langle a, b_0 \rangle / \mathbb{R})$ by $p(x, y)$ and note that $\{x > r : r \in \mathbb{R}\} \subseteq p(x, y)$.

Case A. Assume that no b_i are defined for $i > 0$. This happens only if (†) fails. We shall consider one by one all possible cases. It turns out that each of these cases determines uniquely the weak generic type $tp(\langle a, b_0\rangle/\mathbb{R})$.

First we consider the situation where $b_0 > 0$. Let A, B be as in Case P.

Case 1. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y > x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \{y < x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is stationary (Example 3.5.7).

Case 2. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y > r \cdot x^q : r \in \mathbb{R}_+\} \cup \{y < x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is non-stationary (Example 3.5.8).

Case 3. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y > x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup \{y < r \cdot x^q : r \in \mathbb{R}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Every weak generic type of this form is non-stationary (the proof is as in the previous case).

Case 4. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Since $p(x, y)$ is weak generic and $b_0 > 0$, we also have that $b_0 > r$ for every $r \in \mathbb{R}$. Therefore $p(x, y) = p_z^+(x, y)$ where $z : \mathbb{R} \to \mathbb{R}$ is constantly equal to 0. By Theorem 3.4.7, $p(x, y)$ is stationary.

Case 5. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y) = p_{+\infty}(x, y)$. By Theorem 3.4.7, $p(x, y)$ is stationary.

If $b_0 < 0$, then we get the following cases. Let A, B be as in Case N.

Case 1'. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y < -x^q : q \in \mathbb{Q} \cap (0, c)\} \cup \{y > -x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$.

Case 2'. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y < r \cdot x^q : r \in \mathbb{R}_-\} \cup \{y > -x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$.

Case 3'. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y < -x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup \{y > r \cdot x^q : r \in \mathbb{R}_-\}$$

to a complete weak generic type over $\mathbb{R}$.

Case 4'. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Since $p(x, y)$ is weak generic and $b_0 < 0$, we also have that $b_0 < r$ for every $r \in \mathbb{R}$. Therefore $p(x, y) = p_z^-(x, y)$ where $z : \mathbb{R} \to \mathbb{R}$ is constantly equal to 0.

Case 5'. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y) = p_{-\infty}(x, y)$.

The stationarity of the weak generic types in Cases 1'-5' is the same as in the respective Cases 1-5.

Case B. Now assume that for a and b_0 with $tp(\langle a, b_0\rangle/\mathbb{R})$ weak generic the procedure breaks down at some finite step so that b_i, r_i and q_i are defined only for $i \in \{1, \ldots, n\}$. This means that the condition (†) (for the appropriate $A, B \subseteq \mathbb{Q}_+$ and $c \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$) in the recursive procedure fails at step n. Let $f(x) = \sum_{i=1}^{n} r_i \cdot x^{q_i}$ and recall that $r_i \in \mathbb{R}\setminus\{0\}$, $q_i \in \mathbb{Q}_+$ and $q_1 > \ldots > q_n$. We consider one by one all possible cases. It turns out again that each of these cases determines the weak generic type $tp(\langle a, b_0\rangle/\mathbb{R})$.

First we consider the situation where $b_n > 0$. Let A, B be as in Case P.

Case 1. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

$$r(x, y) = \{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^q : q \in \mathbb{Q} \cap (0, c)\} \cup$$

$$\cup \{y - f(x) < x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $c > q_1 - 1$ (if $c \leq q_1 - 1$, then the type $r(x, y)$ is not weak generic by Theorem 3.5.6) and $c < q_n$ (by the definition of c and q_n). Every weak generic type of this form is stationary (the proof as in Example 3.5.7).

Case 2. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > r \cdot x^q : r \in \mathbb{R}_+\} \cup$$

$$\cup \{y - f(x) < x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q \geq q_1 - 1$ (by Theorem 3.5.6) and $q < q_n$ (by the definition of q and q_n). If $q = q_1 - 1$, then $p(x, y)$ is stationary (since then $p(x, y) = p_f^+(x, y)$). If $q > q_1 - 1$, then $p(x, y)$ is non-stationary (the proof as in Example 3.5.8).

Case 3. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup$$

$$\cup \{y - f(x) < r \cdot x^q : r \in \mathbb{R}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q > q_1 - 1$ (by Theorem 3.5.6) and $q < q_n$ (by the definition of q and q_n). Every weak generic type of this form is non-stationary (the proof as in Example 3.5.8).

Case 4. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > 0\} \cup \{y - f(x) < x^q : q \in \mathbb{Q}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q_1 \leq 1$ (by Theorem 3.5.6) and $p(x, y) = p_f^+(x, y)$. By Theorem 3.4.7, $p(x, y)$ is stationary.

Case 5. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y)$ contains the following set of formulas:

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) > x^q : q \in \mathbb{Q}_+\}.$$

Take any rational $q > q_1$. Then $(y - f(x) > x^q) \in p(x, y)$, which implies that $b_0 - f(a) > a^q$. Hence

$$b_0 > a^q + \sum_{i=1}^{n} r_i \cdot a^{q_i},$$

which contradicts the choice of q_1. So Case 5 can not hold.

If $b_n < 0$, then we get the following cases. Let A, B be as in Case N.

Case 1'. *$A = \mathbb{Q}_+ \cap (0, c)$ and $B = \mathbb{Q}_+ \cap (c, +\infty)$ for some $c \in \mathbb{R}_+ \setminus \mathbb{Q}$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < -x^q : q \in \mathbb{Q} \cap (0, c)\} \cup$$

$$\cup \{y - f(x) > -x^q : q \in \mathbb{Q} \cap (c, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $c > q_1 - 1$ and $c < q_n$.

Case 2'. *$A = \mathbb{Q}_+ \cap (0, q]$ and $B = \mathbb{Q}_+ \cap (q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < r \cdot x^q : r \in \mathbb{R}_-\} \cup$$

$$\cup \{y - f(x) > -x^{q'} : q' \in \mathbb{Q} \cap (q, +\infty)\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q \geq q_1 - 1$ and $q < q_n$. If $q = q_1 - 1$, then $p(x, y) = p_f^-(x, y)$.

Case 3'. *$A = \mathbb{Q}_+ \cap (0, q)$ and $B = \mathbb{Q}_+ \cap [q, +\infty)$ for some $q \in \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < -x^{q'} : q' \in \mathbb{Q} \cap (0, q)\} \cup$$

$$\cup \{y - f(x) > r \cdot x^q : r \in \mathbb{R}_-\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q > q_1 - 1$ and $q < q_n$.

Case 4'. *$A = \emptyset$ and $B = \mathbb{Q}_+$.* Then $p(x, y)$ is the only extension of the type

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < 0\} \cup \{y - f(x) > -x^q : q \in \mathbb{Q}_+\}$$

to a complete weak generic type over $\mathbb{R}$. Moreover, $q_1 \leq 1$ and $p(x, y) = p_f^-(x, y)$.

Case 5'. *$A = \mathbb{Q}_+$ and $B = \emptyset$.* Then $p(x, y)$ contains the following set of formulas:

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f(x) < -x^q : q \in \mathbb{Q}_+\}.$$

As can be easily seen, this is impossible (the proof as in Case 5).

The stationarity of the weak generic types in Cases 1'-4' is the same as in the respective Cases 1-4.

Case C. Now assume that for a and b_0 with $tp(\langle a, b_0\rangle/\mathbb{R})$ weak generic the procedure never stops and we have defined b_i, r_i and q_i for all $i \in \mathbb{N}_+$. Recall that $b_i \in S \setminus \{0\}$, $r_i \in \mathbb{R} \setminus \{0\}$, $q_i \in \mathbb{Q}_+$, $q_1 > q_2 > \ldots$ and for every $i \in \mathbb{N}_+$ we have that $b_i = b_0 - r_1 \cdot a^{q_1} - \ldots - r_i \cdot a^{q_i}$.

Let f be the formal power series $\sum_{i=1}^{+\infty} r_i \cdot x^{q_i}$ and for each $n \in \mathbb{N}$ let $f_n(x) = \sum_{i=1}^{n} r_i \cdot x^{q_i}$ (in particular, $f_0 = 0$). We shall prove that the type $p(x, y) = tp(\langle a, b_0\rangle/\mathbb{R})$ is the only extension of the type $p_f(x, y)$ to a complete weak generic type over $\mathbb{R}$ where the type $p_f(x, y)$ is defined as follows:

$$\{x > r : r \in \mathbb{R}\} \cup \{y - f_n(x) > r' \cdot x^{q_{n+1}} : n \in \mathbb{N} \wedge r' \in \mathbb{R} \wedge r' < r_{n+1}\} \cup$$

$$\cup \{y - f_n(x) < r'' \cdot x^{q_{n+1}} : n \in \mathbb{N} \wedge r'' \in \mathbb{R} \wedge r'' > r_{n+1}\}.$$

Indeed, the inclusion $p_f(x, y) \subseteq p(x, y)$ immediately follows from the definition of the series f. For the sake of contradiction assume that the weak generic types $p(x, y) = tp(\langle a, b_0\rangle/\mathbb{R})$ and $p'(x, y) = tp(\langle a', b_0'\rangle/\mathbb{R})$ ($a', b_0' \in S$) are two distinct extensions of the type $p_f(x, y)$ (hence $\langle a, b_0\rangle$ and $\langle a', b_0'\rangle$ produce the same formal power series $f = \sum_{i=1}^{+\infty} r_i \cdot x^{q_i}$). Without loss of generality we can assume that $a = a'$.

Since $p \neq p'$, there exists a definable (and thus semi-algebraic) function $g : \mathbb{R} \to \mathbb{R}$ such that $(y < g(x)) \in p$ and $(y > g(x)) \in p'$ (or $(y > g(x)) \in p$ and $(y < g(x)) \in p'$, the proof in this case is analogous). We shall prove that the map g can be replaced with the map of the form $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ where $n \in \mathbb{N}_+$, $s_i \in \mathbb{R}$ and $p_i \in \mathbb{Q}_+$.

We know that g has a Puiseux expansion of the form

$$(*)\ g(x) = \sum_{i=k}^{+\infty} c_i \cdot x^{\frac{-i}{d}}$$

where $k \in \mathbb{Z}$, $d \in \mathbb{Z}_+$ and $c_i \in \mathbb{R}$ for $i \geq k$ (it is a well-known property of semi-algebraic functions). Equality $(*)$ holds for all sufficiently large $x \in \mathbb{R}$. One can prove that for some $M \in \mathbb{R}_+$ we obtain ultimately

$$|g(x) - \sum_{i=k}^{-1} c_i \cdot x^{\frac{-i}{d}}| \leq M.$$

Let $h(x) = \sum_{i=k}^{-1} c_i \cdot x^{\frac{-i}{d}}$. Then still $(y < h(x)) \in p$ and $(y > h(x)) \in p'$ (by the weak genericity of the types p and p'), which means that $b_0 < h(a)$ and $b_0' > h(a)$.

Choose unique $n \in \mathbb{N}_+$, $s_i \in \mathbb{R} \setminus \{0\}$ and $p_i \in \mathbb{Q}_+$ such that (ultimately) $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ and $p_1 > \ldots > p_n$. Then for each $i \in \{1, \ldots, n\}$ we have that $p_i = q_i$ and $s_i = r_i$. We shall only prove that $p_1 = q_1$ and $s_1 = r_1$, leaving to the reader the part of the proof where $i > 1$.

In order to show that $p_1 = q_1$, we shall consider several possible cases.

If $p_1 < q_1$ and $r_1 > 0$, then

$$h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} < \frac{r_1}{2} \cdot a^{q_1} < b_0 < h(a),$$

a contradiction.

If $p_1 < q_1$ and $r_1 < 0$, then

$$h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} > \frac{r_1}{2} \cdot a^{q_1} > b_0' > h(a),$$

a contradiction.

If $p_1 > q_1$ and $s_1 > 0$, then

$$(r_1 + 1) \cdot a^{q_1} < s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} = h(a) < b_0' < (r_1 + 1) \cdot a^{q_1},$$

a contradiction.

If $p_1 > q_1$ and $s_1 < 0$, then

$$(r_1 - 1) \cdot a^{q_1} > s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} = h(a) > b_0 > (r_1 - 1) \cdot a^{q_1},$$

a contradiction.

Since $r_1 \neq 0$ and $s_1 \neq 0$, the contradictions above show that $p_1 \geq q_1$ and $p_1 \leq q_1$, respectively. Hence $p_1 = q_1$.

For the sake of contradiction assume that $s_1 \neq r_1$. Then either $s_1 < r_1$ or $s_1 > r_1$.

If $s_1 < r_1$, then

$$b_0 < h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} < \frac{s_1 + r_1}{2} \cdot a^{p_1} = \frac{s_1 + r_1}{2} \cdot a^{q_1} < b_0,$$

a contradiction.

If $s_1 > r_1$, then

$$b_0' > h(a) = s_1 \cdot a^{p_1} + \sum_{i=2}^{n} s_i \cdot a^{p_i} > \frac{s_1 + r_1}{2} \cdot a^{p_1} = \frac{s_1 + r_1}{2} \cdot a^{q_1} > b_0',$$

a contradiction. Hence $s_1 = r_1$.

The proof that $p_i = q_i$ and $s_i = r_i$ for each $i \in \{2, \ldots, n\}$ is analogous.

As a result, we obtain

$$(y < \sum_{i=1}^{n} r_i \cdot x^{q_i}) \in p(x, y)$$

and

$$(y > \sum_{i=1}^{n} r_i \cdot x^{q_i}) \in p'(x, y).$$

Thus $b_0 < \sum_{i=1}^{n} r_i \cdot a^{q_i}$ and $b_0' > \sum_{i=1}^{n} r_i \cdot a^{q_i}$. We have that

$$b_n = b_0 - \sum_{i=1}^{n} r_i \cdot a^{q_i} < 0,$$

which implies that $r_{n+1} < 0$, and

$$b_n' = b_0' - \sum_{i=1}^{n} r_i \cdot a^{q_i} > 0,$$

which implies that $r_{n+1} > 0$, a contradiction. Hence $p_f(x, y) \vdash p(x, y)$.

By Theorem 3.5.6, the sequence of the rational exponents $(q_n)_{n \in \mathbb{N}_+}$ has the following property: $q_n > q_1 - 1$ for every $n > 1$. Otherwise, the formal power series f could not be obtained as a final result of the recursive procedure described above (since the type $tp(\langle a, b_0 \rangle / \mathbb{R}) = p(x, y) \supseteq p_f(x, y)$ is weak generic and so must be $p_f(x, y)$).

Finally, proceeding as above we can show that the weak generic type $p(x, y)$ is stationary. The crucial fact needed to prove this is as follows: for every definable (thus semi-algebraic) function $g : S \to S$ there exist a map $h(x) = \sum_{i=1}^{n} s_i \cdot x^{p_i}$ (where $n \in \mathbb{N}$, $s_i \in \mathbb{R}$ and $p_i \in \mathbb{Q}_+$) and a constant $M \in S_+$ such that $|g(x) - h(x)| \leq M$ for all sufficiently large x.

If we are given an arbitrary formal power series $f = \sum_{i=1}^{\alpha} r_i \cdot x^{q_i}$ (where $\alpha \in \mathbb{N} \cup \{+\infty\}$, $r_i \in \mathbb{R} \setminus \{0\}$, $q_i \in \mathbb{Q}_+$ and $q_1 > q_2 > \ldots$), then in each of the above cases f determines a unique complete (over $\mathbb{R}$) weak generic type in the group $(\mathbb{R}, +) \times (\mathbb{R}, +)$. In this way we have described complete weak generic types in $(\mathbb{R}, +) \times (\mathbb{R}, +)$ derived in the theory $Th(\mathbb{R}, <, +, \cdot)$.

3.5.2 Weak generic types in $(\mathbb{R}_+, \cdot) \times (\mathbb{R}_+, \cdot)$

In the remainder of this section we make use of the work of Chris Miller on o-minimal expansions of ordered fields. Below we give its brief summary (the reader is referred to [6] for more details).

Assume $(R, <, +, \cdot, \ldots)$ is an o-minimal expansion of an ordered field $(R, <, +, \cdot)$. A **power function** is a definable endomorphism of the group $(R_+, \cdot)$. Every power function is differentiable on R_+. For each $r \in R$ there is at most one power function f with $f'(1) = r$. We denote such a map by x^r and write a^r for $f(a)$. The field

$$K = \{f'(1) : f \text{ is a power function}\} \subseteq R$$

is called **the field of exponents** of R. We say that the structure R is **power bounded** if for every definable $f : R \to R$ there exists an $r \in K$ such that ultimately $|f(x)| \leq x^r$. An **exponential function** is an isomorphism of the structures $(R, <, +, 0)$ and $(R_+, <, \cdot, 1)$.

The main result of [6] says that either R defines (without parameters) an exponential function or R is power bounded and for each ultimately non-zero definable function $f : R \to R$ there exist an $a \in R \setminus \{0\}$ and a 0-definable power function x^r such that $f(x) \approx a \cdot x^r$.

Theorem 3.5.9 *If $R = (R, <, +, \cdot, \ldots)$ is an o-minimal expansion of a real closed field R, then the following are equivalent:*
(1) all complete (over R) weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ are stationary,
(2) the structure R is power bounded.

Proof. (1) $\Rightarrow$ (2) For the sake of contradiction assume that R is not power bounded. As we mentioned above, this implies that the exponential function $\exp : R \to R_+$ is 0-definable in R. Thus the map

$$(\exp, \exp) : (R, +) \times (R, +) \to (R_+, \cdot) \times (R_+, \cdot)$$

is a 0-definable isomorphism of groups. Hence the groups $(S, +) \times (S, +)$ and $(S_+, \cdot) \times (S_+, \cdot)$ are definably isomorphic for every $S \succ R$ and it suffices to show that some weak generic type in $(R, +) \times (R, +)$ is not stationary. To do this, take an arbitrary $S \succ R$, $a \in S \setminus R$ and let $f : S \to S$ be such that $f(x) = a \cdot x$ for every $x \in S$. We shall prove that the weak generic types p_f^- and p_f^+ are extensions of the same complete weak generic type over R.

Since the structure R does not need to be $\aleph_0$-saturated, Lemma 3.2.4 itself is not sufficient to ensure that the restrictions of the types p_f^- and p_f^+ to the complete types over R are weak generic in $(R,+)\times(R,+)$. Nevertheless, this follows from the corollary following Theorem 3.3.4.

It is enough to show that $f \not\sim g$ for each $g : S \to S$ definable over R. Suppose otherwise. Then for some R-definable $g : S \to S$ we have that $S \models g \sim f$ (note that there is a first order formula $\varphi \in L(S)$ expressing the fact that $g \sim f$; namely, φ says that the area defined by the formula $(x > 0 \wedge f(x) < y \wedge y < g(x))$ contains arbitrarily large squares - apply Theorem 3.3.4). So $S \models \exists c(g(x) \sim c \cdot x)$ and $R \models \exists c(g(x) \sim c \cdot x)$. Take $b \in R$ such that $g(x) \sim b \cdot x$ (in R). Then $g(x) \sim b \cdot x$ in S, hence $f(x) \sim b \cdot x$ (since $\sim$ is transitive) and $a \cdot x \sim b \cdot x$, a contradiction (since for $a \neq b$ arbitrarily large squares may be put into the area between the graphs of the linear maps $x \mapsto a \cdot x$ and $x \mapsto b \cdot x$).

(2) $\Rightarrow$ (1) Note that it is enough to examine those weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ which contain the formula $(x \geq 1 \wedge y \geq 1)$. To prove this, consider $F, G : R_+ \times R_+ \to R_+ \times R_+$ defined as follows: $F(x,y) = \langle x, \frac{1}{y} \rangle$ and $G(x,y) = \langle \frac{1}{x}, y \rangle$ for every $x, y \in R_+$. We see that F, G and $F \circ G$ are 0-definable automorphisms of the group $(R_+, \cdot) \times (R_+, \cdot)$ that map the set $\{\langle x,y \rangle : x \geq 1 \wedge y \geq 1\}$ respectively onto the sets

(a) $\{\langle x,y \rangle : x \geq 1 \wedge 0 < y \leq 1\}$,

(b) $\{\langle x,y \rangle : 0 < x \leq 1 \wedge y \geq 1\}$ and

(c) $\{\langle x,y \rangle : 0 < x \leq 1 \wedge 0 < y \leq 1\}$.

The same holds for any elementary extension S of R, which enables us to "translate" an example of a non-stationary weak generic type to the set of types $[x \geq 1 \wedge y \geq 1]$.

In order to prove that each complete weak generic type in $(R_+, \cdot) \times (R_+, \cdot)$ containing the formula $(x \geq 1 \wedge y \geq 1)$ is stationary, we are going to show that for every $S \succ R$ and every definable function $f : S \to S \cap [1, +\infty)$ we are able to find an R-definable map $g : S \to S$ such that the set

$$\{\langle x,y \rangle \in S \times S : x \geq 1 \wedge y \geq 1 \wedge (f(x) \leq y \leq g(x) \vee f(x) \geq y \geq g(x))\}$$

is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$. So take such S and f. Let $a, r \in S$ be such that $f(x) \approx a \cdot x^r$. Then $a > 0$ and $r \geq 0$. The power function $x^r : S \to S$ is R-definable (as it is definable over $\emptyset$) and we put $g = x^r$.

Choose any $c \in S_+$ such that $\frac{1}{c} \cdot x^r \leq f(x) \leq c \cdot x^r$ for all sufficiently large $x \in S$. Without loss of generality we can assume that it is so on the whole interval $[1, +\infty)$ since for every $M \geq 1$ the set $X_M = [1, M] \times [1, +\infty)$ is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$ (otherwise, by Corollary 3.3.3, the set $X_M \cdot X_M^{-1} = [\frac{1}{M}, M] \times S_+$ would be generic in $(S_+, \cdot) \times (S_+, \cdot)$, which is not the case).

Now it suffices to prove that the set

$$X = \{\langle x, y \rangle \in S \times S : x \geq 1 \wedge y \geq 1 \wedge \frac{1}{c} \cdot x^r \leq y \wedge y \leq c \cdot x^r\}$$

is not weak generic in $(S_+, \cdot) \times (S_+, \cdot)$. Suppose otherwise. Then the set $X \cdot X^{-1}$ is generic in $(S_+, \cdot) \times (S_+, \cdot)$ by Corollary 3.3.3. We claim that

$$X \cdot X^{-1} \subseteq Y = \{\langle x, y \rangle \in S \times S : x > 0 \wedge \frac{1}{c^2} \cdot x^r \leq y \wedge y \leq c^2 \cdot x^r\}.$$

To see this, take any $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in X$. We have that $\frac{1}{c} \cdot x_1^r \leq y_1 \leq c \cdot x_1^r$ and $\frac{1}{c} \cdot x_2^r \leq y_2 \leq c \cdot x_2^r$. So

$$\frac{1}{c^2} \cdot (\frac{x_1}{x_2})^r \leq \frac{y_1}{y_2} \leq c^2 \cdot (\frac{x_1}{x_2})^r$$

and $\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle^{-1} = \langle u, v \rangle$ where $u = \frac{x_1}{x_2}$ and $\frac{1}{c^2} \cdot u^r \leq v \leq c^2 \cdot u^r$. Thus $\langle u, v \rangle \in Y$ and $X \cdot X^{-1} \subseteq Y$. In turn, since for every $x \in (0, 1)$ $c^2 \cdot x^r \leq c^2$, we have that

$$Y \subseteq Z = (S_+ \times S_+) \setminus ((0, 1) \times (c^2, +\infty)).$$

This implies that the set Z is generic in $(S_+, \cdot) \times (S_+, \cdot)$, a contradiction. $\square$

Corollary 3.5.10 *If $R = (R, <, +, \cdot, \ldots)$ is an o-minimal expansion of an archimedean real closed field R, then the following are equivalent:*
(1) all complete (over R) weak generic types in $(R_+, \cdot) \times (R_+, \cdot)$ are stationary,
(2) the structure R is polynomially bounded.

Proof. Recall [6] that R is polynomially bounded if and only if R is power bounded and K is archimedean. But the field K is archimedean as a subfield of the archimedean field R. The assertion of the corollary immediately follows from Theorem 3.5.9. $\square$

The pure field of real numbers $(\mathbb{R}, <, +, \cdot)$ is archimedean and polynomially bounded. By the corollary above, all weak generic types in $(\mathbb{R}_+, \cdot) \times (\mathbb{R}_+, \cdot)$ derived in the theory $Th(\mathbb{R}, <, +, \cdot)$ are stationary.

4 Countable coverings of groups

4.1 Introduction

Assume G is an $\aleph_0$-saturated group (or even a group 0-type-definable in an $\aleph_0$-saturated structure), covered by countably many sets X_n, $n < \omega$. If we assume that the sets X_n are 0-definable, then by compactness, finitely many of them cover G.

Now assume that the sets X_n are only type-definable over $\emptyset$. Then we can not expect that finitely many of them cover G, however we have that

(G) finitely many of them generate G (as a group) in some finite number k of steps (in the sense of Definition 4.1.1).

This result follows directly from a theorem in [8].

We could wonder if there is an upper bound on k in (G) for a given group G, or for all groups. Denote this upper bound by k_G. A natural guess here would be that there is no common finite bound on k good for all G. We have found examples where $k_G \geq 3$ (Example 4.2.6). If G is stable, then $k_G \leq 2$. So the idea was that the smaller k_G is, the more "stable-like" G is. However in Section 4.2 we prove that $k_G \leq 3$ for every group G. In fact, we prove that "$k_G \leq 2.5$", that is in (G) really "2.5 steps" are enough to generate G. Moreover, if G is definably amenable, then $k_G = 2$, that is 2 steps are enough. More precisely, for some $n < \omega$ we have that

$$G = X_{\leq n} \cdot (X_{\leq n})^{-1}$$

where $X_{\leq n} = \bigcup_{i \leq n} X_i$.

These results may be restated also in a "model-theory-free" way. We do it in Section 4.5. Assume G is an arbitrary group, X is a countable compactification of ω (hence a metric space) and $f : G \to \omega$. We can think of f as a partition of G into sets $X_n = f^{-1}[\{n\}]$, $n < \omega$, or a countable colouring of G. Then there is a finite set $Y \subseteq X$ such that G is generated in 3 steps by $f^{-1}[B(Y, \epsilon)]$ for every $\epsilon > 0$. Moreover, in the case where G is amenable we have more:

$$G = f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1}.$$

Here $B(Y, \epsilon)$ is the ball in X around Y, of radius ϵ.

If we think of $f(x)$ as the colour of $x \in G$, we can think of this result as saying that for each $\epsilon > 0$ we can exclude all elements of G with colours outside $B(Y, \epsilon)$, and the rest of G still generates G in 3 or 2 steps, respectively. When ϵ converges to 0, the set $B(Y, \epsilon)$ becomes smaller and smaller and converges to Y.

All these results suggest a possibility of extending some "generic types arguments" (which are the core of stable model theory) to a very general unstable context. Also, they show that a type-definable group resembles a compact topological group in some respects.

Indeed, assume G is a compact topological group, covered by countably many closed sets X_n, $n < \omega$. Then by the Baire category theorem and compactness, some finitely many translations of finitely many of the sets X_n cover G (so G is generated by finitely many of the sets X_n in "1.5 steps").

Below we give a precise definition of the coefficient k_G for a given $\aleph_0$-saturated group G. Determining k_G for various groups G is the main goal of this chapter.

Definition 4.1.1 *Assume G is a group, $A \subseteq G$ and $k < \omega$. We say that A generates G in k steps if every $g \in G$ is of the form $g = a_1 \cdot \ldots \cdot a_k$ for some $a_1, \ldots, a_k \in A \cup A^{-1}$.*

Definition 4.1.2 *Assume G is an $\aleph_0$-saturated group. Let k_G be the minimal number k such that whenever G is covered by countably many 0-type-definable sets X_n, $n < \omega$, then finitely many of them generate G in k steps.*

Note that in Definition 4.1.2 we can equivalently replace "k steps" by "$\leq k$ steps". Indeed, we can assume that e belongs to one of the finitely many of the sets X_n in question.

The chapter is organized as follows. In Section 4.2 we prove that $k_G \leq 3$ for each group G and show that this bound is the best possible. Then we apply weak generic types introduced in Chapter 3 to derive k_G for various classes of groups and we take a closer look at stable groups (Section 4.3). In Section 4.4 we point out connections between the amenability of G and the coefficient k_G. Finally, in Section 4.5 we apply previously obtained results outside model theory.

4.2 The general bound

In this section we assume that G is a group, possibly with some additional structure, which is $\aleph_0$-saturated (or even, more generally, a group 0-type-definable in an $\aleph_0$-saturated structure). Moreover, G is covered by countably many 0-type-definable sets X_n, $n < \omega$.

First of all, we prove that the group G is generated in not more than 3 steps by finitely many of the sets X_n (Theorem 4.2.2). Another proof of this result, which appears in the next section (Theorem 4.3.1), uses the notion of weak generic type introduced in Chapter 3. The proof below (due to Newelski) is needed to conclude Corollary 4.5.2.

An essential tool used in the proof of Theorem 4.2.2 is an open analysis of a compact topological space with respect to a family of subsets covering it. It refines the Cantor-Bendixson analysis and the Baire category theorem. Below we give the definition of this notion, which is due to Newelski (see [8] for more details).

Definition 4.2.1 *Assume $\mathcal{K}$ is a compact space and $\mathcal{A}$ a family of subsets of $\mathcal{K}$ covering $\mathcal{K}$. We define an increasing sequence $\langle Z_\alpha \rangle_{\alpha \in Ord \cup \{-1\}}$ of open subsets of $\mathcal{K}$ as follows:*
(1) $Z_{-1} = \emptyset$,
(2) $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$ for limit α,
(3) $Z_\alpha = \bigcup_{A \in \mathcal{A}} int(Z_\beta \cup A)$ for $\alpha = \beta + 1$.
We call $\langle Z_\alpha \rangle_{\alpha \in Ord \cup \{-1\}}$ the open analysis of $\mathcal{K}$ with respect to $\mathcal{A}$.

Theorem 4.2.2 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then $G = \{g_1, \ldots, g_m\} \cdot X_{\leq n} \cdot X_{\leq n}^{-1}$ for some $m, n < \omega$ and $g_1, \ldots, g_m \in G$.*

Proof. Let $\mathcal{K} = S_1(\emptyset) \cap [G(x)]$ and let $\langle Z_\alpha \rangle_{\alpha \in Ord \cup \{-1\}}$ be the open analysis of $\mathcal{K}$ with respect to the family of closed sets $\mathcal{C} = \{X_n, n < \omega\}$. By the Baire category theorem, $\mathcal{K} = Z_\alpha$ for some $\alpha \in Ord$. Let α be the minimal ordinal number such that Z_α is generic in G (i.e. $G = \{g_1, \ldots, g_m\} \cdot Z_\alpha$ for some $g_1, \ldots, g_m \in G$). By compactness, there exists a clopen set $V \subseteq Z_\alpha$ such that $G = \{g_1, \ldots, g_m\} \cdot V$ and therefore $\alpha = \beta + 1$ for some $\beta \in Ord \cup \{-1\}$.

Since V is compact and $V \subseteq Z_\alpha = Z_{\beta+1} = \bigcup_{l<\omega} int(Z_\beta \cup X_l)$, we have that $V \subseteq \bigcup_{l \leq n} int(Z_\beta \cup X_l)$ for some $n < \omega$ and therefore $V \setminus Z_\beta \subseteq X_{\leq n}$. We consider two possible cases.

Case 1. *For every $g \in G$ there is some $h \in V \setminus Z_\beta$ with $g \cdot h \in \{g_1, \ldots, g_m\} \cdot (V \setminus Z_\beta)$.* This implies that $G = \{g_1, \ldots, g_m\} \cdot X_{\leq n} \cdot X_{\leq n}^{-1}$.

Case 2. *There is some $g \in G$ such that for every $h \in V \setminus Z_\beta$ we have that $g \cdot h \notin \{g_1, \ldots, g_m\} \cdot (V \setminus Z_\beta)$.* In this case we have that

$$V \setminus Z_\beta \subseteq g^{-1} \cdot \{g_1, \ldots, g_m\} \cdot (V \cap Z_\beta),$$

which implies that

$$V \subseteq \{e, g^{-1} \cdot g_1, \ldots, g^{-1} \cdot g_m\} \cdot (V \cap Z_\beta)$$

and consequently

$$G = \{g_1, \ldots, g_m\} \cdot \{e, g^{-1} \cdot g_1, \ldots, g^{-1} \cdot g_m\} \cdot (V \cap Z_\beta).$$

The set $V \cap Z_\beta$ is generic in G, thus so is Z_β, contradicting the choice of $\alpha = \beta + 1$. □

Corollary 4.2.3 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then $G = \{g_1, \ldots, g_m\} \cdot X_{\leq n} \cdot X_{\leq n}$ for some $m, n < \omega$ and $g_1, \ldots, g_m \in G$.*

Proof. For $i, j < \omega$ let $X_{i,j} = X_i \cap X_j^{-1}$ and apply Theorem 4.2.2 to the case where G is covered by the 0-type-definable sets $X_{i,j}$, $i, j < \omega$. By this theorem,

$$G = \{g_1, \ldots, g_m\} \cdot \bigcup_{i,j \leq n} X_{i,j} \cdot \bigcup_{i,j \leq n} X_{i,j}^{-1}$$

for some $n < \omega$ and finitely many $g_1, \ldots, g_m \in G$. Hence

$$G = \{g_1, \ldots, g_m\} \cdot X_{\leq n} \cdot X_{\leq n},$$

which finishes the proof. □

Theorem 4.2.2 shows that $k_G \leq 3$ (or even "$k_G \leq 2.5$") for an arbitrary group G. Below we show that there exist groups G with $k_G = 3$ (Example 4.2.6).

Proposition 4.2.4 *Assume G is an $\aleph_0$-saturated group. The following are equivalent:*
(1) $k_G > 2$,
(2) for some 0-type-definable sets X_n, $n < \omega$, we have that $G = \bigcup_{n<\omega} X_n$ and $G \neq X_{\leq n} \cdot X_{\leq n}^{-1}$ for every $n < \omega$,
(3) for some 0-type-definable sets X_n, $n < \omega$, we have that $G = \bigcup_{n<\omega} X_n$ and for every $n < \omega$ we can find $g \in G$ such that $X_{\leq n} \cap g \cdot X_{\leq n} = \emptyset$.

Proof. (2) $\Leftrightarrow$ (3) Straightforward because for $g \in G$ and $A \subseteq G$ we have that $g \in A \cdot A^{-1} \Leftrightarrow A \cap g \cdot A \neq \emptyset$.

(1) $\Rightarrow$ (2) Since $k_G > 2$, for some 0-type-definable sets X_n, $n < \omega$, we have that $G = \bigcup_{n<\omega} X_n$ and $G \neq (X_{\leq n} \cup X_{\leq n}^{-1}) \cdot (X_{\leq n} \cup X_{\leq n}^{-1})$ for every $n < \omega$, which obviously implies (2).

(2) $\Rightarrow$ (1) Take X_n, $n < \omega$, satisfying (2) and for every $i, j < \omega$ put $X_{i,j} = X_i \cap X_j^{-1}$. Then each $X_{i,j}$ is 0-type-definable and $G = \bigcup_{i,j<\omega} X_{i,j}$. Moreover, for each $n < \omega$ we have that

$$\bigcup_{i,j \leq n} (X_{i,j} \cup X_{i,j}^{-1}) \cdot \bigcup_{i,j \leq n} (X_{i,j} \cup X_{i,j}^{-1}) \subseteq X_{\leq n} \cdot X_{\leq n}^{-1} \neq G,$$

which means that $k_G > 2$. $\square$

Corollary 4.2.5 *Assume that G is a group (not necessarily $\aleph_0$-saturated), $\{X_n : n < \omega\}$ is a partition of G consisting of 0-definable sets and $H \succ G$ is an $\aleph_0$-saturated elementary extension of G. If we have that*

$$(\forall n < \omega)(\exists g \in G) g \cdot (G \setminus X_n) \subseteq X_n,$$

then $k_H > 2$.

Proof. We have that $H = X_{-1} \cup \bigcup_{n<\omega} X_n(H)$ where the set

$$X_{-1} = H \setminus \bigcup_{n<\omega} X_n(H) = \bigcap_{n<\omega} (H \setminus X_n(H))$$

is 0-type-definable. By our assumption, for each $n < \omega$ we are able to find $g_n \in G$ such that $g_n \cdot (G \setminus X_n) \subseteq X_n$. Since the set X_n is definable and $G \prec H$, we also have that $g_n \cdot (H \setminus X_n(H)) \subseteq X_n(H)$. So for each $n < \omega$ we obtain

$$(\bigcup_{i=-1}^{n} X_i(H)) \cap (g_{n+1} \cdot \bigcup_{i=-1}^{n} X_i(H)) \subseteq (\bigcup_{i=-1}^{n} X_i(H)) \cap (g_{n+1} \cdot (H \setminus X_{n+1}(H)) \subseteq$$

$$\subseteq (\bigcup_{i=-1}^{n} X_i(H)) \cap X_{n+1}(H) = \emptyset,$$

which means that $X_{\leq n}(H) \cap g_{n+1} \cdot X_{\leq n}(H) = \emptyset$ for every $n < \omega$. The proof is complete by Proposition 4.2.4. $\square$

Example 4.2.6 Let G be the free group with free generators $\{a_n : n < \omega\}$. For every $n < \omega$ let X_n be the set

$$\{g \in G : g \text{ in reduced form begins with } a_n \text{ or } a_n^{-1}\}.$$

Add e to X_0. Then the sets X_n, $n < \omega$, form a countable partition of G. Moreover, $a_n \cdot (G \setminus X_n) \subseteq X_n$ for every $n < \omega$. Let H be an $\aleph_0$-saturated elementary extension of the structure $(G, \cdot, \{X_n : n < \omega\})$. By the previous corollary, we have that $k_H > 2$ (so $k_H = 3$ by Theorem 4.2.2).

4.3 Applications of weak generic types

In this section we assume that G is an $\aleph_0$-saturated group and H is a $|G|^+$-saturated elementary extension of G.

Below we use weak generic types to derive the coefficient k_G in some cases. We begin with an alternative proof of Theorem 4.2.2, which shows that weak genericity is related to generating G.

Theorem 4.3.1 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then $G = \{g_1, \dots, g_m\} \cdot X_n \cdot X_n^{-1}$ for some $m, n < \omega$ and $g_1, \dots, g_m \in G$.*

Proof. Since

$$WGEN(G) = \bigcup_{n<\omega} (WGEN(G) \cap [X_n])$$

and each of the sets $WGEN(G) \cap [X_n]$ is closed in $WGEN(G)$, by the Baire category theorem, we can find $n < \omega$ and $\varphi(x) \in L(G)$ such that

$$(*)\ \emptyset \neq WGEN(G) \cap [\varphi(x)] \subseteq [X_n].$$

Since the formula $\varphi(x)$ is weak generic, there exist non-generic formula $\psi(x) \in L(G)$ and $g_1, \dots, g_m \in G$ such that $G = \{g_1, \dots, g_m\} \cdot (\varphi \vee \psi)(G)$. Formulas $\{\neg g \cdot \psi(x) : g \in G\}$ form a partial weak generic type over G (see the proof of Lemma 3.2.3(2)).

We can extend the type $\{\neg g \cdot \psi(x) : g \in G\}$ to some $p(x) \in WGEN(G)$. Let $j \in \{1, \dots, m\}$ be such that $g_j \cdot (\varphi \vee \psi)(x) \in p$. Then $g_j \cdot \varphi(x) \in p$ since $g_j \cdot \psi(x) \notin p$. As a result, $\varphi(x) \in g_j^{-1} \cdot p \in WGEN(G)$ and by $(*)$, for $q = g_j^{-1} \cdot p$ we have that $q \in [X_n]$. Fix $a \in q(H)$.

Choose any $g \in G$. For some $i \in \{1, \ldots, m\}$ we have that $g_i \cdot (\varphi \vee \psi)(x)$ belongs to $g \cdot q$. If $g_i \cdot \psi(x) \in g \cdot q$, then

$$\psi(x) \in g_i^{-1} \cdot g \cdot q = g_i^{-1} \cdot g \cdot g_j^{-1} \cdot p \text{ and } g_j \cdot g^{-1} \cdot g_i \cdot \psi(x) \in p,$$

which contradicts the choice of p. So we have that $g_i \cdot \varphi(x) \in g \cdot q$. Therefore $\varphi(x) \in g_i^{-1} \cdot g \cdot q \in WGEN(G)$, which implies that $g_i^{-1} \cdot g \cdot q \in [X_n]$ and $g_i^{-1} \cdot g \cdot a \in X_n(H)$. Now we write g as $g_i(g_i^{-1} \cdot g \cdot a)a^{-1}$ and conclude that $g \in \{g_1, \ldots, g_m\} \cdot X_n(H) \cdot X_n(H)^{-1}$.

So far we proved that $G \subseteq \{g_1, \ldots, g_m\} \cdot X_n(H) \cdot X_n(H)^{-1}$ for some $n < \omega$ and $g_1, \ldots, g_m \in G$. But this implies that

$$G \subseteq \{g_1, \ldots, g_m\} \cdot X_n(G) \cdot X_n(G)^{-1} = \{g_1, \ldots, g_m\} \cdot X_n \cdot X_n^{-1}.$$

Namely, consider an arbitrary $g \in G$. Let $i \in \{1, \ldots, m\}$ and $h_1, h_2 \in X_n(H)$ be such that $g = g_i \cdot h_1 \cdot h_2^{-1}$. Then

$$r(x, y) = X_n(x) \cup X_n(y) \cup \{g = g_i \cdot x \cdot y^{-1}\}$$

is a partial type over $\{g, g_i\}$. Taking any $\langle h_1', h_2' \rangle \in r(G \times G)$ we see that $g = g_i \cdot h_1' \cdot h_2'^{-1} \in \{g_1, \ldots, g_m\} \cdot X_n \cdot X_n^{-1}$. $\square$

The corollary below will have important consequences.

Corollary 4.3.2 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then*

$$G = \bigcup_{i=1}^{m} g_i \cdot X_n \cdot p_i(G) \cdot g_i^{-1}$$

for some $m, n < \omega$, $g_1, \ldots, g_m \in G$ and $p_1, \ldots, p_m \in S(\emptyset)$.

Proof. We proceed as in the previous proof. As $g = g_i(g_i^{-1} \cdot g \cdot a)(a^{-1} \cdot g_i)g_i^{-1}$, we obtain the inclusion $G \subseteq \bigcup_{i=1}^{m} g_i \cdot X_n(H) \cdot p_i(H) \cdot g_i^{-1}$ where $p_i = tp(a^{-1} \cdot g_i/\emptyset)$ for each $i \in \{1, \ldots, m\}$. Finally, we obtain

$$G \subseteq \bigcup_{i=1}^{m} g_i \cdot X_n(G) \cdot p_i(G) \cdot g_i^{-1} = \bigcup_{i=1}^{m} g_i \cdot X_n \cdot p_i(G) \cdot g_i^{-1},$$

which completes the proof. $\square$

Proposition 4.3.3 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then*

$$G = \bigcup_{i=1}^{m} g_i \cdot X_{\leq n} \cdot X_{\leq n}^{-1} \cdot g_i^{-1} = \bigcup_{i=1}^{m} (X_{\leq n} \cdot X_{\leq n}^{-1})^{g_i}$$

for some $m, n < \omega$ and $g_1, \dots, g_m \in G$.

Proof. Take $g_1, \dots, g_m \in G$, $p_1, \dots, p_m \in S(\emptyset)$ and $n < \omega$ as in the previous corollary. For each $i \in \{1, \dots, m\}$ find $n_i < \omega$ such that $p_i^{-1} \in [X_{n_i}]$. Finally, replace n with $\max(n, n_1, \dots, n_m)$. $\square$

An immediate consequence of Proposition 4.3.3 is the next corollary, which implies that $k_G = 2$ for every commutative group G. Another proof of this result (which uses the fact that each commutative group is amenable) appears in Section 4.4.

Corollary 4.3.4 *Assume G is an $\aleph_0$-saturated commutative group covered by countably many 0-type-definable sets X_n, $n < \omega$. Then $G = X_{\leq n} \cdot X_{\leq n}^{-1}$ for some $n < \omega$.*

Thus far we have considered arbitrary 0-type-definable sets $X_n \subseteq G$, possibly in a language larger than that of pure groups. For pure groups we can say more.

Corollary 4.3.5 *Assume G is an $\aleph_0$-saturated group covered by countably many sets X_n, $n < \omega$, which are 0-type-definable in the pure group language. Then $G = X_{\leq n} \cdot X_{\leq n}^{-1}$ for some $n < \omega$.*

Proof. By Proposition 4.3.3, for some $n < \omega$ finitely many conjugates of the set $X_{\leq n} \cdot X_{\leq n}^{-1}$ cover G. But since we are in the pure group structure, for any $g \in G$ we have that $(X_{\leq n} \cdot X_{\leq n}^{-1})^g = X_{\leq n} \cdot X_{\leq n}^{-1}$ (as the map $x \mapsto x^g$ is an automorphism of the structure $(G, \cdot)$). $\square$

Now assume that G is an $\aleph_0$-saturated group where each weak generic type is generic. The theorem below says that in this case $k_G = 2$. Note that by Lemma 3.2.3(1), $WGEN(G) = GEN(G) \Leftrightarrow GEN(G) \neq \emptyset$.

Theorem 4.3.6 *Assume G is an $\aleph_0$-saturated group covered by countably many 0-type-definable sets X_n, $n < \omega$. If G has some generic types, then $G = X_{\leq n} \cdot X_{\leq n}^{-1}$ for some $n < \omega$.*

Proof. Since

$$GEN(G) = \bigcup_{n<\omega}(GEN(G) \cap [X_n]),$$

by the Baire category theorem, we can find $n < \omega$ and $\varphi(x) \in L(G)$ such that

$$(*)\ \emptyset \neq GEN(G) \cap [\varphi(x)] \subseteq [X_n].$$

The formula $\varphi(x)$ is generic so for some finitely many $g_1, \ldots, g_m \in G$ we have that $G = \{g_1, \ldots, g_m\} \cdot \varphi(G)$.

Fix $p(x) \in GEN(G)$, $a \in p(H)$ and for each $i \in \{1, \ldots, m\}$ choose $n_i < \omega$ such that $g_i^{-1} \cdot p \in [X_{n_i}]$ (thus $a^{-1} \cdot g_i \in X_{n_i}(H)^{-1}$).

Choose any $g \in G$. For some $i \in \{1, \ldots, m\}$ we have that $g_i \cdot \varphi(x) \in p \cdot g$. Then

$$\varphi(x) \in g_i^{-1} \cdot p \cdot g = tp(g_i^{-1} \cdot a \cdot g/G) \in GEN(G)$$

so by $(*)$, $g_i^{-1} \cdot a \cdot g \in X_n(H)$. We write g as $(a^{-1} \cdot g_i)(g_i^{-1} \cdot a \cdot g)$ and see that $g \in X_{n_i}(H)^{-1} \cdot X_n(H)$. If we replace n with $\max(n, n_1, \ldots, n_m)$, then $G \subseteq X_{\leq n}(H)^{-1} \cdot X_{\leq n}(H)$. Thus proceeding as in the proof of Theorem 4.3.1 we obtain $G \subseteq X_{\leq n}(G)^{-1} \cdot X_{\leq n}(G) = X_{\leq n}^{-1} \cdot X_{\leq n}$. Finally, repeating the proof with X_n^{-1} in place of X_n, we conclude that $G = X_{\leq n} \cdot X_{\leq n}^{-1}$. □

Proposition 1.6.6(1) from [12] implies that $GEN(G) \neq \emptyset$ for every stable group G. Thus by Theorem 4.3.6, if G is an $\aleph_0$-saturated stable group, then $k_G = 2$. Another proof of this result may be found in [9] (Theorem 2.3).

We shall examine the stable case more precisely. Under some additional assumptions we are able to "decrease" k_G a little more. In the next theorem we use the notion of weight (see Section 1.4.4 in [12]).

Proposition 4.3.7 (Newelski) *Assume G is an $\aleph_0$-saturated stable group covered by countably many 0-type-definable sets X_n, $n < \omega$. Suppose that for some $l < \omega$ and each $a \in G$ weight$(a) \leq l$. Then $G = \{g_1, \ldots, g_m\} \cdot X_n$ for some $m, n < \omega$ and $g_1, \ldots, g_m \in G$.*

Proof. Let $\mathcal{A} = S(acl^{eq}(\emptyset)) \cap [G(x)]$ and $\mathcal{G} = \{p \in \mathcal{A} : p \text{ is generic}\}$. Let $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be the free multiplication of types, defined as follows:

if $a \models p$, $b \models q$ and $a \perp b$, then $p * q = tp(a \cdot b/acl^{eq}(\emptyset))$.

One can show that $(\mathcal{A}, *)$ is a semigroup and $(\mathcal{G}, *) \subseteq (\mathcal{A}, *)$ is a compact topological group (see [7] for more details). As $G = \bigcup_{n<\omega} X_n$, we have that

$\mathcal{G} = \bigcup_{n<\omega}(\mathcal{G} \cap [X_n])$. The Baire category theorem implies that for some $n < \omega$ we can find a non-empty clopen subset $U \subseteq \mathcal{G} \cap [X_n]$. We have that $\mathcal{G} = \bigcup_{p\in\mathcal{G}} p * U$ so by compactness, $\mathcal{G} = \bigcup_{i=1}^{k} p_i * U$ for some $p_1, \ldots, p_k \in \mathcal{G}$. For every $i \in \{1, \ldots, k\}$ let I_i be a finite Morley sequence in p_i of length $l+1$.

Choose $a \in G$. Let q be the generic type in the coset $G^0 a$. Then $q = p_i * u$ where $i \in \{1, \ldots, k\}$ and $u \in U$. So $u = p_i^{-1} * q$. Since $weight(a) \leq l$, there is $b \in I_i$ such that $b \perp a$. Then $b^{-1} \models p_i^{-1}$, $b^{-1} \cdot a \perp a$ and $b^{-1} \cdot a \models u$. As $a = b \cdot (b^{-1} \cdot a)$, we have that $a \in I_i \cdot U(G)$. So we obtain

$$G = (\bigcup_{i=1}^{k} I_i) \cdot U(G) = (\bigcup_{i=1}^{k} I_i) \cdot X_n.$$

Finally, put $m = k(l+1)$ and let $g_1, \ldots, g_m$ be all elements of the finite set $\bigcup_{i=1}^{k} I_i$. $\square$

For an arbitrary stable G, in this proof we may take the I_i of length $|T|^+$. Then still $G = (\bigcup_{i=1}^{k} I_i) \cdot X_n$ so $G = X_{\leq m} \cdot X_n$ for some $m < \omega$.

Proposition 4.3.7 says that in the case where G is stable with bounded finite weight, "$k_G = 1.5$". Example 4.3.8 shows that the assumption on weight is essential. In Section 4.5 we show that the cases where "$k_G = 1.5$" are rare (see Proposition 4.5.9).

Example 4.3.8 We give an example of a stable group with "$k_G > 1.5$". Namely, let $G = (\mathbb{Z}^\omega, +, \{P_n : n < \omega\})$ where $P_n(G) = \{f \in \mathbb{Z}^\omega : f(n) = 0\}$. Put $P_\infty = G \setminus \bigcup_{n<\omega} P_n(G)$. Let H be an $\aleph_0$-saturated elementary extension of G. We shall prove that for every $N < \omega$ the set $P_\infty(H) \cup \bigcup_{n<N} P_n(H)$ is not generic in H.

Note that for each natural K we have that

$$(\forall f_0, \ldots, f_K \in G)(\exists h \in G) \bigwedge_{k\leq K} (P_{N+k}(f_k + h) \wedge \bigwedge_{n<N} \neg P_n(f_k + h)).$$

To prove this, take arbitrary $f_0, \ldots, f_K \in G$. Find an element $h \in G$ such that $h(n) > \max(|f_0(n)|, \ldots, |f_K(n)|)$ for $n < N$ and $h(N+k) = -f_k(N+k)$ for $k \leq K$. Then h has the required properties.

Since $G \prec H$, we have that

$$(\forall f_0, \ldots, f_K \in H)(\exists h \in H) \bigwedge_{k\leq K} (P_{N+k}(f_k + h) \wedge \bigwedge_{n<N} \neg P_n(f_k + h)),$$

which implies that the set $P_\infty(H) \cup \bigcup_{n<N} P_n(H)$ is not generic in H.

4.4 Connections with amenability

Assume G is an $\aleph_0$-saturated group, covered by countably many 0-type-definable sets X_n, $n < \omega$. In [9] and in the previous sections we showed that in general G as a group is generated by some finitely many of the sets X_n in $k = 2.5$ steps. Sometimes $k = 2$ steps suffice, that is, for some $n < \omega$ we have that $G = X_{\leq n} \cdot X_{\leq n}^{-1}$ where $X_{\leq n} = \bigcup_{i \leq n} X_i$. In Section 4.3 we proved that it is so when G is commutative or stable. Here we improve this result by replacing "commutative" with "amenable". Also, we give a uniform proof for both the amenable and the stable case.

Recall [17] that a group G is amenable if there exists a finitely additive measure μ on $\mathcal{P}(G)$ such that μ is left-invariant (i.e. $\mu(g \cdot A) = \mu(A)$ for every $g \in G$ and $A \subseteq G$) and $\mu(G) = 1$. We call any such μ a Banach mean on G. Amenable groups form a large class, including solvable groups and, more generally, all groups without a paradoxical decomposition. However, any non-commutative free group is not amenable.

As we are interested mainly in those subsets of G that are definable (and their intersections), we shall make use of the notion of definably amenable group introduced in [4].

Definition 4.4.1 *We call a group G definably amenable if there exists a finitely additive left-invariant measure μ on the algebra of all definable subsets of G such that $\mu(G) = 1$.*

We shall also use the following lemma on a finitely additive measure.

Lemma 4.4.2 *Assume X is a set and μ is a finitely additive finite measure on an algebra $\mathcal{A}$ of subsets of X. Assume that for some $\epsilon > 0$ we have a family $\{A_n : n < \omega\}$ of sets from $\mathcal{A}$ such that $\mu(A_n) > \epsilon$ for every $n < \omega$. Then for some increasing sequence $(n_k)_{k<\omega}$ of natural numbers we have that $\mu(A_{n_0} \cap \ldots \cap A_{n_i}) > 0$ for each $i < \omega$.*

Proof. First we prove that under the assumptions of the lemma

($*$) there exist $n < \omega$ and $\epsilon' > 0$ such that the set $\{m : \mu(A_n \cap A_m) > \epsilon'\}$ is infinite.

If not, then for every $n < \omega$ and $\epsilon' > 0$ the set $\{m : \mu(A_n \cap A_m) > \epsilon'\}$ is finite. Let k be a positive integer such that $\mu(A_n) > 2/k$ for every $n < \omega$.

Put $n_0 = 0$ and for each $i = 1, \ldots, k-1$ choose recursively $n_i < \omega$ so that $\mu(A_{n_i} \cap A_{n_j}) < 1/(k \cdot i)$ for all $j < i$. We have that

$$1 \geq \mu(\bigcup_{i=0}^{k-1} A_{n_i}) = \sum_{i=0}^{k-1} \mu(A_{n_i} \setminus \bigcup_{j<i} A_{n_j}) > \sum_{i=0}^{k-1} (\frac{2}{k} - \frac{1}{k}) = 1,$$

a contradiction.

Having proved $(*)$, we define recursively numbers n_k, $k < \omega$, so that for each $k < \omega$ we have that

$(**)$ for some $\epsilon' > 0$ the set $Z = \{m : \mu(A_{n_0} \cap \ldots \cap A_{n_k} \cap A_m) > \epsilon'\}$ is infinite.

For $k = 0$ we define n_0 by $(*)$. For the recursion step, suppose we have defined $n_0, \ldots, n_k$ so that $(**)$ holds. We shall define n_{k+1}. Let $X' = A_{n_0} \cap \ldots \cap A_{n_k}$ and for $m \in Z$ let $A'_m = X' \cap A_m$. We see that X' and $A'_m, m \in Z$, satisfy the assumptions of the lemma for the restricted μ so that in this situation $(*)$ holds. Let n_{k+1} be the n furnished by $(*)$. This finishes the construction and the proof. $\square$

Theorem 4.4.3 *If G is an $\aleph_0$-saturated definably amenable group covered by countably many 0-type-definable sets X_n, $n < \omega$, then for some $n < \omega$ we have that $G = X_{\leq n} \cdot X_{\leq n}^{-1}$.*

Proof. For the sake of contradiction assume that for any $n < \omega$ we can find $a_n \in G \setminus X_{\leq n} \cdot X_{\leq n}^{-1}$. By compactness, there is a formula without parameters $\varphi_n(x)$ such that $X_{\leq n} \subseteq \varphi_n(G)$ and $a_n \notin \varphi_n(G) \cdot \varphi_n(G)^{-1}$. We have that

$$G = \neg\varphi_n(G) \cup a_n \cdot \neg\varphi_n(G).$$

Since $\mu(\neg\varphi_n(G)) = \mu(a_n \cdot \neg\varphi_n(G))$, we have that $\mu(\neg\varphi_n(G)) \geq 1/2$ for each $n < \omega$. By Lemma 4.4.2, we can choose an increasing sequence $(n_k)_{k<\omega}$ such that

$$\mu(\neg\varphi_{n_0}(G) \cap \ldots \cap \neg\varphi_{n_k}(G)) > 0$$

for every $k < \omega$. On the other hand, $\{\varphi_{n_k}(G), k < \omega\}$ is a family of 0-definable sets covering G so by $\aleph_0$-saturation of G, we are able to choose a finite subcovering $\{\varphi_{n_k}(G), k \leq K\}$ (for some $K < \omega$). Then we have that $\mu(\neg\varphi_{n_0}(G) \cap \ldots \cap \neg\varphi_{n_K}(G)) = \mu(\emptyset) = 0$, a contradiction. $\square$

We can easily conclude from Theorem 4.4.3 that if G is an $\aleph_0$-saturated group, then both the stability and the amenability of G imply that $k_G = 2$. Indeed, the amenable case is trivial. In the stable one, we construct an appropriate measure on $Def(G)$ as in [10].

The following conjecture is a converse to Theorem 4.4.3. Its model-theory-free counterpart is Conjecture 4.5.14.

Conjecture 4.4.4 *Assume that G is an $\aleph_0$-saturated group. If $k_G = 2$, then G is definably amenable.*

If this conjecture were true, then by Theorem 4.3.6 $\aleph_0$-saturated groups with generics would be definably amenable. It would be an improvement of a result from [4] saying that every group G with finitely satisfiable generics such that $Th(G)$ does not have the independence property is definably amenable.

4.5 Examples and model-theory-free versions

In this section we assume that G is a group without any additional structure. In this model-theory-free context we apply the results from the previous sections.

We recall the notion of Cantor-Bendixson rank on a compact space. Assume X is a compact space. For each ordinal α, $X^{(\alpha)}$ denotes the α-th Cantor-Bendixson derivative of X. For a point $a \in X$ we define its Cantor-Bendixson rank in X (denoted by $CB(a)$) as the minimal α such that $a \notin X^{(\alpha+1)}$. If no such α exists, we put $CB(a) = \infty$. We define the Cantor-Bendixson rank of X as the maximum of $CB(a), a \in X$. For a countable compact space X, $CB(X) < \infty$. In this case there are finitely many elements $a \in X$ with $CB(a) = CB(X)$. Their number is called the CB-multiplicity of X, and denoted by $Mlt_{CB}(X)$.

We shall need the following topological lemma.

Lemma 4.5.1 *Assume X is a countable compact space. Then there exist a countable tree $T \subseteq \omega^{<\omega}$ and a family $\{A_\eta : \eta \in T\}$ of non-empty clopen subsets of X such that the following conditions are satisfied.*
(1) $X = A_\emptyset = A_{\langle 0 \rangle} \cup \ldots \cup A_{\langle n-1 \rangle}$ where $n = Mlt_{CB}(X)$.
(2) If $\eta \in T \setminus \{\emptyset\}$, then $Mlt_{CB}(A_\eta) = 1$.
(3) If $\emptyset \neq \eta \subseteq \nu$ and $\eta \neq \nu$, then $A_\nu \subseteq A_\eta$ and $CB(A_\nu) < CB(A_\eta)$.
(4) If $|A_\eta| > 1$, then $A_\eta = \{a_\eta\} \cup \bigcup_{n<\omega} A_{\eta \frown \langle n \rangle}$ where $CB(a_\eta) = CB(A_\eta)$.
(5) $|A_\eta| = 1$ if and only if η is a leaf in T.

Proof. The proof is by induction on $CB(X)$. Assume the conclusion of the lemma holds for every countable compact space X with $CB(X) < \alpha$. Consider an arbitrary X such that $CB(X) = \alpha$ and let

$$\{a_0, \ldots, a_{n-1}\} = \{x \in X : CB(x) = \alpha\}.$$

Put $A_\emptyset = X$ and choose pairwise disjoint clopen subsets $A_{\langle 0\rangle}, \ldots, A_{\langle n-1\rangle}$ such that $a_i \in A_{\langle i\rangle}$ for $i < n$ and $X = A_{\langle 0\rangle} \cup \ldots \cup A_{\langle n-1\rangle}$. If $CB(X) = 0$, then the proof is complete. Otherwise we proceed as follows.

Fix $i < n$. X is a metric space. Since X is countable, we can choose a decreasing sequence of real numbers $(\epsilon_k)_{k<\omega}$ converging to 0 such that for each $k < \omega$ the open ball $B(a_i, \epsilon_k)$ is closed and the set $B(a_i, \epsilon_k) \setminus B(a_i, \epsilon_{k+1})$ is non-empty. For $k < \omega$ we put $A_{\langle i,k\rangle} = B(a_i, \epsilon_{k-1}) \setminus B(a_i, \epsilon_k)$ where we stipulate $\epsilon_{-1} = \infty$. If necessary, we refine the sets $A_{\langle i,k\rangle}$ to ensure that $Mlt_{CB}(A_{\langle i,k\rangle}) = 1$ for each $k < \omega$. We have that $A_{\langle i\rangle} = \{a_i\} \cup \bigcup_{k<\omega} A_{\langle i,k\rangle}$, and each $A_{\langle i,k\rangle}$ is a countable compact space such that $CB(A_{\langle i,k\rangle}) < \alpha$. In order to finish the proof, we apply the induction hypothesis to each of the sets $A_{\langle i,k\rangle}$. $\square$

The following corollary is a consequence of Theorem 4.2.2 (essentially, it is its restatement in a model-theory-free way).

Corollary 4.5.2 *Assume G is a group, X is a countable compactification of ω and $f : G \to \omega$. Then there are a finite subset $Y \subseteq X$ and finitely many $g_1, \ldots, g_m \in G$ such that for every $\epsilon > 0$ we have that*

$$G = \{g_1, \ldots, g_m\} \cdot f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1}.$$

Proof. Let $\{A_\eta : \eta \in T \subseteq \omega^{<\omega}\}$ be a covering of X such as in Lemma 4.5.1 and let $T' = T \setminus \{\emptyset\}$. For $\eta \in T'$ let $X_\eta = f^{-1}[A_\eta]$. We consider G as a first order structure $(G, \cdot, \{X_\eta : \eta \in T'\})$ for the language of groups extended by countably many unary predicates $X_\eta, \eta \in T'$, interpreted as sets X_η in G. Let $H = (H, \cdot, \{X_\eta(H) : \eta \in T'\})$ be an $\aleph_0$-saturated elementary extension of the structure $(G, \cdot, \{X_\eta : \eta \in T'\})$. We are going to apply Theorem 4.2.2 for certain covering of H by some 0-type-definable sets $p_\eta(H), \eta \in T'$. So for $\eta \in T'$ we define

$$p_\eta(x) = \begin{cases} \{X_\eta(x)\} & \text{, when } \eta \text{ is a leaf} \\ \{X_\eta(x)\} \cup \{\neg X_{\eta\frown\langle n\rangle}(x) : n < \omega\} & \text{, otherwise.} \end{cases}$$

We claim that

$$H = \bigcup_{\eta \in T'} p_\eta(H).$$

To see this, choose any $a \in H$. Then for some $\eta \in T'$ we have that $a \in X_\eta(H)$ and $a \notin X_{\eta^\frown\langle n\rangle}(H)$ for every $n < \omega$. But this means that $a \models p_\eta$ and $a \in p_\eta(H)$.

Therefore $\{p_\eta(H) : \eta \in T'\}$ is a countable covering of H consisting of 0-type-definable sets. By Theorem 4.2.2, we can choose $\eta_1, \ldots, \eta_n \in T'$ and $g_1, \ldots, g_m \in H$ such that

$$H = \{g_1, \ldots, g_m\} \cdot (p_{\eta_1}(H) \cup \ldots \cup p_{\eta_n}(H)) \cdot (p_{\eta_1}(H) \cup \ldots \cup p_{\eta_n}(H))^{-1}.$$

Note that without loss of generality $g_1, \ldots, g_m \in G$. To prove this, we must go back to the proof of Theorem 4.2.2. Since $H = \{g_1, \ldots, g_m\} \cdot V(H)$, V is clopen and $G \prec H$, we can find $g_1, \ldots, g_m \in G$ with $G = \{g_1, \ldots, g_m\} \cdot V(G)$.

To finish the proof, we take $Y = \{a_{\eta_1}, \ldots, a_{\eta_n}\}$ where $a_{\eta_i} \in A_{\eta_i}$ is such that $CB(a_{\eta_i}) = CB(A_{\eta_i})$. $\square$

In turn, Corollary 4.5.3 follows from Theorem 4.4.3.

Corollary 4.5.3 *Assume that G is an amenable group, X is a countable compactification of ω and $f : G \to \omega$. Then there is a finite subset $Y \subseteq X$ such that for every $\epsilon > 0$ we have that*

$$G = f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1}.$$

Proof. We proceed as in the proof of Corollary 4.5.2. Since G is amenable, the group $(G, \cdot, \{X_\eta : \eta \in T'\})$ is definably amenable. Since definable amenability is preserved under elementary extensions (see [4]), $(H, \cdot, \{X_\eta(H) : \eta \in T'\})$ is an $\aleph_0$-saturated definably amenable group. Finally, we apply Theorem 4.4.3 to obtain $H = (p_{\eta_1}(H) \cup \ldots \cup p_{\eta_n}(H)) \cdot (p_{\eta_1}(H) \cup \ldots \cup p_{\eta_n}(H))^{-1}$. $\square$

Assume $\{X_n : n < \omega\}$ is a partition of a group G and X is a countably infinite compact space where we identify the set of isolated points with ω. Let $f : G \to \omega$ be defined by $f(x) = n$ if and only if $x \in X_n$. We can think of $f(x)$ as a colour of x. In this situation Corollary 4.5.2 applies and it yields a finite set $Y \subseteq X$ such that for each $\epsilon > 0$, if we throw out from the group all elements with colours outside $B(Y, \epsilon)$, then the remaining part of G still generates G in "2.5" steps. In this set-up we can think of the sets X_n as being arranged as isolated points of some countable compact space X.

Apparently we can arrange them there in an arbitrary way, and also we have some freedom in the choice of X. One would expect that if the space X is simple (that is, of low CB-rank), then the sets $f^{-1}[B(Y, \epsilon)]$ are "large", and so maybe 2 steps (like in Corollary 4.5.3) should suffice to generate G. It is true when $CB(X) = 1$ (see Example 4.5.4 below). On the other hand, if $CB(X) = 2$, then we can not replace the "2.5" steps in Corollary 4.5.2 by 2 steps. We analyze this case in detail in Example 4.5.5. This will be the main point in constructing a group G with a colouring for which 2 steps are not enough. These examples are also intended to clarify the mutual connection between Theorem 4.2.2 and Corollary 4.5.2.

Example 4.5.4 We consider the case where X is $\omega + 1$ with the order topology (that is, the simplest case of an X with $CB(X) = 1$). Assume $f : G \to \omega \subseteq X$ and for each $n < \omega$ put $X_n = f^{-1}[\{n\}]$. Then the family $\{X_n : n < \omega\}$ is a partition of G. Let $(G, \cdot, \{X_n : n < \omega\}) \prec H$ where H is $\aleph_0$-saturated. Then

$$H = \bigcup_{n<\omega} X_n(H) \cup \bigcap_{n<\omega} (H \setminus X_n(H))$$

so H is covered by countably many 0-type-definable sets. Here a typical finite subset of X is of the form $Y = \{0, \dots, M\} \cup \{\omega\}$ (that is, any finite subset of X is contained in a finite subset of this form). So by Theorem 4.2.2, there is an $M < \omega$ such that for

$$A = \bigcup_{n\leq M} X_n(H) \cup \bigcap_{n<\omega} (H \setminus X_n(H))$$

and some $g_1, \dots, g_m \in H$ we have that

$(*)$ $H = \{g_1, \dots, g_m\} \cdot A \cdot A^{-1}$.

Without loss of generality $g_1, \dots, g_m \in G$ (the proof is as in Corollary 4.5.2).

Again for $Y = \{0, \dots, M\} \cup \{\omega\}$ a typical ϵ-ball around Y in X is of the form $B(Y, \epsilon) = \{0, \dots, M\} \cup \{n \in \omega : n \geq N\} \cup \{\omega\}$ where when ϵ tends to 0, N goes to infinity. Then the set $f^{-1}[B(Y, \epsilon)]$ equals $X_{\leq M} \cup X_{\geq N}$.

By $(*)$ and compactness, for each $N < \omega$ we have that

$$G = \{g_1, \dots, g_m\} \cdot (X_{\leq M} \cup X_{\geq N}) \cdot (X_{\leq M} \cup X_{\geq N})^{-1},$$

which shows that the conclusion of Corollary 4.5.2 holds (it suffices to take $Y = \{0, 1, \dots, M, \omega\}$).

In fact, in the case where $X = \omega + 1$ we have that in Corollary 4.5.2 two steps are enough, that is, for some $M < \omega$ we have that

$$G = (X_{\leq M} \cup X_{\geq N}) \cdot (X_{\leq M} \cup X_{\geq N})^{-1}$$

for every $N < \omega$. To prove this, we consider two cases.

Case 1. *For every $m < \omega$ and $a \in G$, there is some $b \in X_{\geq m}$ with $a \cdot b \in X_{\geq m}$.* This implies that $G = X_{\geq m} \cdot X_{\geq m}^{-1}$ for each $m < \omega$. Hence the set $\bigcap_{n<\omega}(H \setminus X_n(H))$ generates H in 2 steps.

Case 2. *For some $m < \omega$ and $a \in G$, for all $b \in X_{\geq m}$ we have that $a \cdot b \notin X_{\geq m}$.* In this case we have that for each $b \in G$, either $b \in X_{\leq m}$ or $a \cdot b \in X_{\leq m}$ so $G = \{e, a^{-1}\} \cdot X_{\leq m}$ and $H = \{e, a^{-1}\} \cdot X_{\leq m}(H)$.

Example 4.5.5 We consider the case where X is $\omega \cdot \omega + 1$ with order topology. This is the simplest case of X with $CB(X) = 2$. Let $f : G \to \omega$. We may identify ω with the set of isolated points of X, consisting of ordinals $\omega \cdot m + n + 1$, $m, n < \omega$ (for the sake of notational simplicity we disregard 0). In this way f becomes a function $f : G \to X$. For every $m, n < \omega$ we define $X_{m,n} = f^{-1}[\{\omega \cdot m + n + 1\}]$. The family $\{X_{m,n} : m, n < \omega\}$ is a partition of G. Let $Y_m = \bigcup_{n<\omega} X_{m,n}$ and $(G, \cdot, \{X_{m,n} : m, n < \omega\}, \{Y_m : m < \omega\}) \prec H$ where H is $\aleph_0$-saturated. Then

$$H = \bigcup_{m,n<\omega} X_{m,n}(H) \cup \bigcup_{m<\omega} \left(Y_m(H) \setminus \bigcup_{n<\omega} X_{m,n}(H) \right) \cup \bigcap_{m<\omega} (H \setminus Y_m(H))$$

so H is covered by countably many 0-type-definable sets. Here a typical finite subset Y of X is of the form

$$Y^M = \{\omega \cdot m + n + 1 : m, n \leq M\} \cup \{\omega \cdot m : m \leq M\} \cup \{\omega \cdot \omega\}.$$

So by Theorem 4.2.2, there is an $M < \omega$ such that for

$$A = \bigcup_{m,n\leq M} X_{m,n}(H) \cup \bigcup_{m\leq M} \left(Y_m(H) \setminus \bigcup_{n<\omega} X_{m,n}(H) \right) \cup \bigcap_{m<\omega} (H \setminus Y_m(H))$$

and some $g_1, \ldots, g_m \in H$ we have that $H = \{g_1, \ldots, g_m\} \cdot A \cdot A^{-1}$. Again without loss of generality $g_1, \ldots, g_m \in G$. By compactness, for each $N < \omega$ and

$$B = \bigcup_{m,n\leq M} X_{m,n} \cup \bigcup_{m\leq M, n\geq N} X_{m,n} \cup \bigcup_{m\geq N, n<\omega} X_{m,n}$$

we have that $G = \{g_1, \ldots, g_m\} \cdot B \cdot B^{-1}$, which shows that Corollary 4.5.2 holds in the case where $X = \omega \cdot \omega + 1$ (the set Y^M works).

Note that while in Example 4.5.4 for a typical finite $Y \subseteq X$ and small $\epsilon > 0$ the set $B(Y, \epsilon)$ is co-finite in X, in Example 4.5.5 this set is usually co-infinite. This means that if, given a partition $\{X_n : n < \omega\}$ of G, we arrange the sets X_n as isolated points of the space $X = \omega + 1$, then Corollary 4.5.2 allows us to throw out elements of finitely many of the sets X_n so that the rest still generate G (in 2.5 steps). However, if we arrange the sets X_n as the isolated points of the space $X = \omega \cdot \omega + 1$, then Corollary 4.5.2 allows us to throw out elements of infinitely many of the sets X_n so that the rest still generate G in 2.5 steps.

The next proposition will be helpful in constructing an example of a group G and its colouring f for which in Corollary 4.5.2 two steps are not enough. We say that a family $\mathcal{A}$ of sets is 3-disjoint if every 3 distinct members of $\mathcal{A}$ have an empty intersection.

Proposition 4.5.6 *Assume that G is a group and $0 < l < \omega$. Let X^0 denote the set of isolated points of $X = \omega \cdot \omega + 1$. The following statements are equivalent:*
(1) for some $f : G \to X^0 \subseteq X$ it is not true that for some finite $Y \subseteq X$ we have that for each $\epsilon > 0$ G is generated by $f^{-1}[B(Y, \epsilon)]$ in l steps,
(2) there exists a partition $\{X_{m,n} : m, n < \omega\}$ of G such that for each $M < \omega$ we can find $N \geq M$ such that the set

$$\bigcup_{m,n \leq M} X_{m,n} \cup \bigcup_{m \leq M, n \geq N} X_{m,n} \cup \bigcup_{m \geq N, n < \omega} X_{m,n}$$

does not generate G in l steps,
(3) there exists a 3-disjoint family $\mathcal{A} = \{A_n : n < \omega\}$ of subsets of G such that for each $n < \omega$ the set $G \setminus A_n$ does not generate G in l steps.

Proof. (1) $\Leftrightarrow$ (2) We proceed as in Example 4.5.5.

(2) $\Rightarrow$ (3) By induction on k we construct $M_k, N_k < \omega$ such that for each $k < \omega$ we have that $M_k \leq N_k$, $M_{k+1} = N_k + 1$ and the set

$$X_k = \bigcup_{m,n \leq M_k} X_{m,n} \cup \bigcup_{m \leq M_k, n \geq N_k} X_{m,n} \cup \bigcup_{m \geq N_k, n < \omega} X_{m,n}$$

does not generate G in l steps.

Let $M_0 = 0$. Having defined M_k we find an appropriate N_k using (2). Then for $n < \omega$ we put $A_n = G \setminus X_n$ and the family $\mathcal{A} = \{A_n : n < \omega\}$ satisfies the required conditions (note that for every $m_0 < n_0 < \omega$ we have that $A_{m_0} \cap A_{n_0} = \bigcup\{X_{m,n} : M_{m_0} \leq m \leq N_{m_0}, M_{n_0} \leq n \leq N_{n_0}\}$).

(3) $\Rightarrow$ (2) We are given a family $\mathcal{A} = \{A_n : n < \omega\}$ satisfying (3). Without loss of generality $\bigcup \mathcal{A} = G$ (we can add elements of $G \setminus \bigcup \mathcal{A}$ to A_0). We define

$$X_{m,n} = \begin{cases} A_m \cap A_n & \text{, when } m < n, \\ A_m \setminus \bigcup_{k \neq m} A_k & \text{, when } m = n, \\ \emptyset & \text{, when } m > n. \end{cases}$$

Since $\mathcal{A}$ is a 3-disjoint family of sets, the sets $X_{m,n}$, $m, n < \omega$, form a partition of G. Consider any $M < \omega$. Note that for each $N \geq M$ the set

$$\bigcup_{m,n \leq M} X_{m,n} \cup \bigcup_{m \leq M, n \geq N} X_{m,n} \cup \bigcup_{m \geq N, n < \omega} X_{m,n}$$

is contained in the set $G \setminus A_M$. By (3), it does not generate G in l steps. $\square$

Corollary 4.5.7 *There exist a group G and its partition $\{X_{m,n} : m, n < \omega\}$ such that for each $M < \omega$ we can find $N > M$ such that the set*

$$\bigcup_{m,n \leq M} X_{m,n} \cup \bigcup_{m \leq M, n \geq N} X_{m,n} \cup \bigcup_{m \geq N, n < \omega} X_{m,n}$$

does not generate G in 2 steps.

Proof. Consider the free group G with two free generators x, y. For every $n < \omega$ let A_n be the set

$\{w \in G : w$ begins with $x^n \cdot y$ or $x^{-n} \cdot y^{-1}$ or ends with $y \cdot x^n$ or $y^{-1} \cdot x^{-n}\}$.

By Proposition 4.5.6, it suffices to show that the family $\mathcal{A} = \{A_n : n < \omega\}$ is 3-disjoint and the set $G \setminus A_n$ does not generate G in 2 steps for any $n < \omega$. The first claim is obvious. To prove the second one, consider the element $x^n \cdot y \cdot x^n$. Assume that $x^n \cdot y \cdot x^n = u \cdot v$ for some $u, v \in G$. Then either u begins with $x^n \cdot y$ or v ends with $y \cdot x^n$. Hence either $u \notin (G \setminus A_n) \cup (G \setminus A_n)^{-1}$ or $v \notin (G \setminus A_n) \cup (G \setminus A_n)^{-1}$, which finishes the proof. $\square$

However, the covering from Corollary 4.5.7 can not consist of finite sets.

Proposition 4.5.8 *Assume G is a group and $\{X_{m,n} : m, n < \omega\}$ is its partition such that each $X_{m,n}$ is finite. Then there exists $M < \omega$ such that for each $N \geq M$ the set*

$$\bigcup_{m,n \leq M} X_{m,n} \cup \bigcup_{m \leq M, n \geq N} X_{m,n} \cup \bigcup_{m \geq N, n < \omega} X_{m,n}$$

generates G in 2 steps.

Proof. Suppose not. We proceed as in the proof of Proposition 4.5.6 and obtain a family $\mathcal{A} = \{A_n : n < \omega\}$ of subsets of G such that:

(a) for all $n_1 < n_2 < \omega$ we have that $|A_{n_1} \cap A_{n_2}| < \omega$,

(b) for all $n_1 < n_2 < n_3 < \omega$ we have that $A_{n_1} \cap A_{n_2} \cap A_{n_3} = \emptyset$ and

(c) for every $n < \omega$ there is some $a_n \in G$ with $a_n \cdot (G \setminus A_n) \subseteq A_n$ and $(G \setminus A_n) \cdot a_n \subseteq A_n$.

Note that (c) holds because $G \setminus A_n$ does not generate G in 2 steps. Consider arbitrary $m < n < \omega$ and $g \in G$. By (c), the following implication holds:

$$g \notin A_m \cup A_n \Rightarrow (a_m \cdot g \in A_m \cap A_n \vee g \cdot a_n \in A_m \cap A_n \vee a_m \cdot g \cdot a_n \in A_m \cap A_n).$$

As this is true for any $g \in G$, we have that

$$G \setminus (A_m \cup A_n) \subseteq a_m^{-1}(A_m \cap A_n) \cup (A_m \cap A_n)a_n^{-1} \cup a_m^{-1}(A_m \cap A_n)a_n^{-1}$$

and (a) implies that $|G \setminus (A_m \cup A_n)| < \omega$. Hence for all $m < n < \omega$ the set $A_m \cup A_n$ is co-finite so, in particular, the set $(A_0 \cup A_1) \cap (A_2 \cup A_3) \cap (A_4 \cup A_5)$ is non-empty, contradicting (b). $\square$

Proposition 4.5.6 and Corollary 4.5.7 show that the free group of rank 2 may be "coloured" in such a way that in Corollary 4.5.2 two steps are not enough.

Proposition 4.5.8 may be strengthened to show that if $CB(X) = 2$, f maps G into the set of isolated points of X and the sets $f^{-1}[\{a\}]$, $a \in X$, are finite, then in Corollary 4.5.2 two steps suffice. However, in the case where $CB(X) = 3$ one can modify the example from Corollary 4.5.7 to give an example of such an f for which in Corollary 4.5.2 two steps are not enough.

The next proposition says that "$k_G > 1.5$" in most cases. This shows that the bound $k_G \leq 2$ for amenable groups is the best possible.

Proposition 4.5.9 *Assume G is an infinite group. There exists a partition $\{X_n : n < \omega\}$ of G such that for each $n < \omega$ the set $G \setminus X_n$ is not generic. In particular, the set $X_{\leq n}$ is not generic for any $n < \omega$.*

Proof. If $A \subseteq G$, then the following are equivalent:

(1) A is not generic,

(2) $(\forall n < \omega)(\forall g_1, \ldots, g_n \in G)(\exists g \in G) g \notin \{g_1, \ldots, g_n\} \cdot A$,

(3) $(\forall n < \omega)(\forall g_1, \ldots, g_n \in G)(\exists g \in G)(\{g_1, \ldots, g_n\}^{-1} \cdot g) \cap A = \emptyset$.

Let $|G| = \kappa$ and enumerate the family of finite subsets of G as $\{G_\alpha : \alpha < \kappa\}$. We shall construct a family $\{Y_{\langle \alpha, n\rangle} : \langle \alpha, n \rangle \in \kappa \times \omega\}$ of finite subsets of G such that

(a) for $\langle \alpha, n\rangle \neq \langle \beta, m \rangle$ we have that $Y_{\langle \alpha, n\rangle} \cap Y_{\langle \beta, m\rangle} = \emptyset$ and

(b) $Y_{\langle \alpha, n\rangle} = G_\alpha^{-1} \cdot g$ for some $g \in G$.

Let $f : \kappa \to \kappa \times \omega$ be a bijection with coordinates f_1 and f_2, that is $f(\alpha) = \langle f_1(\alpha), f_2(\alpha)\rangle$ for every $\alpha < \kappa$. We define recursively pairwise disjoint finite sets $Z_\alpha \subseteq G$, $\alpha < \kappa$.

Let $\beta < \kappa$ and assume we have defined the sets Z_α, $\alpha < \beta$. Consider the set $Z = \bigcup\{Z_\alpha : \alpha < \beta\}$. Since $|Z| < \kappa$, Z is not generic in G. In particular, $G_{f_1(\beta)} \cdot Z \neq G$. Choose any $g \in G \setminus (G_{f_1(\beta)} \cdot Z)$. Then $(G^{-1}_{f_1(\beta)} \cdot g) \cap Z = \emptyset$ and we put $Z_\beta = G^{-1}_{f_1(\beta)} \cdot g$.

The sets $Y_{\langle \alpha, n\rangle} = Z_{f^{-1}(\langle \alpha, n\rangle)}$ ($\alpha < \kappa$, $n < \omega$) form a family satisfying (a) and (b).

Now for $n < \omega$ we put $X_n = \bigcup\{Y_{\langle \alpha, n\rangle} : \alpha < \kappa\}$. By (a) and (b), the sets X_n, $n < \omega$, are pairwise disjoint and for each $n < \omega$ the set $G \setminus X_n$ is not generic. To complete the construction of a partition of G, we add to X_0 all elements of the set $G \setminus \bigcup\{X_n : n < \omega\}$. □

In order to sum up results of this section, we consider the class of groups defined as follows.

Definition 4.5.10 *Let* K2 *be the class of all groups G such that whenever X is a countable compactification of ω and $f : G \to \omega$ then there is a finite subset $Y \subseteq X$ such that for every $\epsilon > 0$ we have that*

$$G = f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1}.$$

Proposition 4.5.11 *If $G \in$ K2 and $H < G$, then $H \in$ K2.*

Proof. Let X be a countable compactification of ω and $f : H \to \omega \subseteq X$ be any colouring of H. Choose a subset $\{g_i : i \in I\} \subseteq G$ such that $\{H \cdot g_i : i \in I\}$ is a partition of G. Define a function $f_1 : G \to \omega$ as follows. For $g \in G$ take unique $h \in H$ and $i \in I$ with $g = h \cdot g_i$ and put $f_1(h \cdot g_i) = f(h)$. Since $G \in$ K2, for some finite $Y \subseteq X$ we have that

$$(\forall \epsilon > 0)\ G = f_1^{-1}[B(Y, \epsilon)] \cdot (f_1^{-1}[B(Y, \epsilon)])^{-1}.$$

We shall prove that the same will hold when we replace f_1 and G with f and H, respectively. To do this, take an arbitrary $h \in H$ and $\epsilon > 0$. For some $h_1, h_2 \in H$ and $i, j \in I$ such that

$$\{h_1 \cdot g_i, h_2 \cdot g_j\} \subseteq f_1^{-1}[B(Y, \epsilon)]$$

we have that $h = h_1 \cdot g_i \cdot (h_2 \cdot g_j)^{-1} = h_1 \cdot g_i \cdot g_j^{-1} \cdot h_2^{-1}$ and $g_i \cdot g_j^{-1} = h_1^{-1} \cdot h \cdot h_2 \in H$, which implies that $g_i = g_j$ and $h = h_1 \cdot h_2^{-1}$. But $f(h_1) = f_1(h_1 \cdot g_i)$ and $f(h_2) = f_1(h_2 \cdot g_j)$ are in $B(Y, \epsilon)$, hence $h \in f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1}$ and finally we obtain

$$(\forall \epsilon > 0)\ H = f^{-1}[B(Y, \epsilon)] \cdot (f^{-1}[B(Y, \epsilon)])^{-1},$$

which finishes the proof. $\square$

Definition 4.5.12 *(1) Let* AG *denote the class of all amenable groups.*
(2) Let NF *denote the class of groups without a free subgroup of rank 2.*

It is easy to show that AG $\subseteq$ NF. For more details on classes AG and NF see [17]. The following corollary establishes a connection between classes AG, K2 and NF.

Corollary 4.5.13 *(1)* AG $\subseteq$ K2.
(2) K2 $\subseteq$ NF.

Proof. (1) This is exactly the assertion of Corollary 4.5.3.

(2) Let F_2 denote the free group of rank 2. By Proposition 4.5.6 and Corollary 4.5.7, we have that $F_2 \notin$ K2. Consider a group $G \notin$ NF. Then $F_2 < G$ and by Proposition 4.5.11, $G \notin$ K2. Thus K2 $\subseteq$ NF. $\square$

We end this section with a model-theory-free counterpart of Conjecture 4.4.4.

Conjecture 4.5.14 K2 = AG.

References

[1] Jacek Bochnak, Michel Coste, Marie-Françoise Roy, **Real algebraic geometry,** Springer-Verlag, Berlin 1998.

[2] Chen Chung Chang, H. Jerome Keisler, **Model theory,** North-Holland, Amsterdam 1973.

[3] Lou van den Dries, **Tame topology and o-minimal structures,** Cambridge University Press, Cambridge 1998.

[4] Ehud Hrushovski, Yaa'cov Peterzil, Anand Pillay, *Groups, measures and the NIP,* preprint 2005.

[5] Julia Knight, Anand Pillay, Charles Steinhorn, *Definable sets in ordered structures. II,* Transactions of the American Mathematical Society 295 (1986), 593-605.

[6] Chris Miller, *A growth dichotomy for o-minimal expansions of ordered fields,* **Logic: from foundations to applications,** 385-399, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York 1996.

[7] Ludomir Newelski, *M-gap conjecture and m-normal theories,* Israel Journal of Mathematics 106 (1998), 285-311.

[8] Ludomir Newelski, *The diameter of a Lascar strong type,* Fundamenta Mathematicae 176 (2003), 157-170.

[9] Ludomir Newelski, Marcin Petrykowski, *Coverings of groups and types,* Journal of the London Mathematical Society 71 (2005), 1-21.

[10] Ludomir Newelski, Marcin Petrykowski, *Weak generic types and coverings of groups,* Fundamenta Mathematicae, accepted.

[11] Anand Pillay, *On groups and fields definable in o-minimal structures,* Journal of Pure and Applied Algebra 53 (1988), 239-255.

[12] Anand Pillay, **Geometric stability theory,** Oxford Science Publications, The Clarendon Press, Oxford University Press, New York 1996.

[13] Anand Pillay, Charles Steinhorn, *Definable sets in ordered structures. I,* Transactions of the American Mathematical Society 295 (1986), 565-592.

[14] Anand Pillay, Charles Steinhorn, *Definable sets in ordered structures. III,* Transactions of the American Mathematical Society 309 (1988), 469-476.

[15] Bruno Poizat, *Groupes stables, avec types génériques réguliers,* Journal of Symbolic Logic 48 (1983), 339-355.

[16] Frank Wagner, **Simple theories,** Kluwer, Dordrecht 2000.

[17] Stan Wagon, **The Banach-Tarski paradox,** Cambridge University Press, Cambridge 1985.

[18] Alex Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,* Journal of the American Mathematical Society 9 (1996), 1051-1094.