A Course in Model Theory

Katin Tent & Martin Ziegler

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1 The Basics

1.1 Structures

Definition 1.1. A **language** L is a set of constants, function symbols and relation symbols

Definition 1.2. Let L be a language. An L-**structure** is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$ where

A	if a non-empty set, the **domain** or **universe** of $\mathfrak{A}$
$z^{\mathfrak{A}} \in A$	if Z is a constant
$Z^{\mathfrak{A}} : A^n \to A$	if Z is an n-ary function symbol
$Z^{\mathfrak{A}} \subseteq A^n$	if Z is an n-ary relation symbol

Definition 1.3. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. A map $h : A \to B$ is called a **homomorphism** if for all $a_1, \dots, a_n \in A$

$$\begin{array}{rcl} h(c^{\mathfrak{A}}) & = & c^{\mathfrak{B}} \\ h(f^{\mathfrak{A}}(a_1, \dots, a_n)) & = & f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \\ R^{\mathfrak{A}}(a_1, \dots, a_n) & \Rightarrow & R^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \end{array}$$

We denote this by

$$h : \mathfrak{A} \to \mathfrak{B}$$

If in addition h is injective and

$$R^{\mathfrak{A}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n))$$

for all $a_1, \dots, a_n \in A$, then h is called an (isomorphic) **embedding**. An **isomorphism** is a surjective embedding

Definition 1.4. We call $\mathfrak{A}$ a **substructure** of $\mathfrak{B}$ if $A \subseteq B$ and if the inclusion map is an embedding from $\mathfrak{A}$ to $\mathfrak{B}$. We denote this by

$$\mathfrak{A} \subseteq \mathfrak{B}$$

We say $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$

Lemma 1.5. *Let $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ be an isomorphism and $\mathfrak{B}$ an extension of $\mathfrak{A}$. Then there exists an extension $\mathfrak{B}'$ of $\mathfrak{A}'$ and an isomorphism $g : \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}'$ extending h*

For any family $\mathfrak{A}_i$ of substructures of $\mathfrak{B}$, the intersection of the A_i is either empty or a substructure of $\mathfrak{B}$. Therefore if S is any non-empty subset of $\mathfrak{B}$, then there exists a smallest substructure $\mathfrak{A} = \langle S \rangle^{\mathfrak{B}}$ which contains S. We call the $\mathfrak{A}$ the substructure **generated** by S

Lemma 1.6. *If $\mathfrak{a} = \langle S \rangle$, then every homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is determined by its values on S*

Definition 1.7. Let $(I, \leq)$ be a **directed partial order**. This means that for all $i, j \in I$ there exists a $k \in I$ s.t. $i \leq k$ and $j \leq k$. A family $(\mathfrak{A}_i)_{i\in I}$ of L-structures is called **directed** if

$$i \leq j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j$$

If I is linearly ordered, we call $(\mathfrak{A}_i)_{i\in I}$ a **chain**

If a structure $\mathfrak{A}_1$ is isomorphic to a substructure $\mathfrak{A}_0$ of itself,

$$h_0 : \mathfrak{A}_0 \xrightarrow{\sim} \mathfrak{A}_1$$

then Lemma 1.5 gives an extension

$$h_1 : \mathfrak{A}_1 \xrightarrow{\sim} \mathfrak{A}_2$$

Continuing in this way we obtain a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq ...$ and an increasing sequence $h_i : \mathfrak{A}_i \xrightarrow{\sim} \mathfrak{A}_{i+1}$ of isomorphism

Lemma 1.8. *Let $(\mathfrak{A}_i)_{i\in I}$ be a directed family of L-structures. Then $A = \bigcup_{i\in I} A_i$ is the universe of a (uniquely determined) L-structure*

$$\mathfrak{A} = \bigcup_{i\in I} \mathfrak{A}_i$$

which is an extension of all $\mathfrak{A}_i$

A subset K of L is called a **sublanguage**. An L-structure becomes a K-structure, the **reduct**.

$$\mathfrak{A}\restriction K = (A, (Z^{\mathfrak{A}})_{Z\in K})$$

Conversely we call $\mathfrak{A}$ an **expansion** of $\mathfrak{A}\restriction K$.

1. Let $B \subseteq A$, we obtain a new language

   $$L(B) = L \cup B$$

   and the $L(B)$-structure

   $$\mathfrak{A}_B = (\mathfrak{A}, b)_{b\in B}$$

   Note that $\mathbf{Aut}(\mathfrak{A}_B)$ is the group of automorphisms of $\mathfrak{A}$ fixing B elementwise. We denote this group by $\mathbf{Aut}(\mathfrak{A}/B)$

Let S be a set, which we call the set of sorts. An S-sorted language L is given by a set of constants for each sort in S, and typed function and relations. For any tuple $(s_1, \dots, s_n)$ and $(s_1, \dots, s_n, t)$ there is a set of relation symbols and function symbols respectively. An S-sorted structure is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$, where

A	if a family $(A_s)_{s \in S}$ of non-empty sets
$Z^{\mathfrak{A}} \in A_s$	if Z is a constant of sort $s \in S$
$Z^{\mathfrak{A}} : A_{s_1} \times \cdots \times A_{s_n} \to A_t$	if Z is a function symbol of type $(s_1, \ldots, s_n, t)$
$Z^{\mathfrak{A}} \subseteq A_{s_1} \times \cdots \times A_{s_n}$	if Z is a relation symbol of type $(s_1, \ldots, s_n)$

Example 1.1. Consider the two-sorted language L_{Perm} for permutation groups with a sort x for the set and a sort g for the group. The constants and function symbols for L_{Perm} are those of L_{Group} restricted to the sort g and an additional function symbol φ of type (x, g, x). Thus an L_{Perm}-structure (X, G) is given by a set X and an L_{Group}-structure G together with a function $X \times G \to X$

1.2 Language

Lemma 1.9. *Suppose $\overrightarrow{b}$ and $\overrightarrow{c}$ agree on all variables which are free in φ. Then*

$$\mathfrak{A} \vDash \varphi[\overrightarrow{b}] \Leftrightarrow \mathfrak{A} \vDash \varphi[\overrightarrow{c}]$$

We define

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]$$

by $\mathfrak{A} \vDash \varphi[\overrightarrow{b}]$, where $\overrightarrow{b}$ is an assignment satisfying $\overrightarrow{b}(x_i) = a_i$. Because of Lemma 1.9 this is well defined.

Thus $\varphi(x_1, \ldots, x_n)$ defines an n-ary relation

$$\varphi(\mathfrak{A}) = \{\bar{a} \mid \mathfrak{A} \vDash \varphi[\bar{a}]\}$$

on A, the **realisation set** of φ. Such realisation sets are called **0-definable subsets** of A^n, or 0-definable relations

Let B be a subset of A. A B**-definable** subset of $\mathfrak{A}$ is a set of the form $\varphi(\mathfrak{A})$ for an $L(B)$-formula $\varphi(x)$. We also say that φ are defined **over** B and that the set $\varphi(\mathfrak{A})$ is defined by φ. We call two formulas **equivalent** if in every structure they define the same set.

Atomic formulas and their negations are called **basic**. Formulas without quantifiers are Boolean combinations of basic formulas. It is convenient to allow the empty conjunction and the empty disjunction. For that we introduce two new formulas: the formula $\top$, which is always true, and the formula $\perp$, which is always false. We define

$$\bigwedge_{i<0} \pi_i = \top$$
$$\bigvee_{i<0} \pi_i = \perp$$

A formula is in **negation normal form** if it is built from basic formulas using $\wedge, \vee, \exists, \forall$

Lemma 1.10. *Every formula can be transformed into an equivalent formula which is in negation normal form*

Proof. Let $\sim$ denote equivalence of formulas. We consider formulas which are built using $\wedge, \vee, \exists, \forall, \neg$ and move the negation symbols in front of atomic formulas using

$$\begin{aligned}
\neg(\varphi \wedge \psi) &\sim (\neg\varphi \vee \neg\psi) \\
\neg(\varphi \vee \psi) &\sim (\neg\varphi \wedge \neg\psi) \\
\neg\exists x\varphi &\sim \forall x\neg\varphi \\
\neg\forall x\varphi &\sim \exists x\neg\varphi \\
\neg\neg\varphi &\sim \varphi
\end{aligned}$$

□

Definition 1.11. A formula in negation normal form which does not contain any existential quantifier is called **universal**. Formulas in negation normal form without universal quantifiers are called **existential**

Lemma 1.12. *Let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then for all existential formulas $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in A$ we have*

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Rightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

For universal φ, the dual holds

$$\mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)] \Rightarrow \mathfrak{A} \models \varphi[a_1, \dots, a_n]$$

Let $\mathfrak{A}$ be an L-structure. The **atomic diagram** of $\mathfrak{A}$ is

$$\mathrm{Diag}(\mathfrak{A}) = \{\varphi \text{ basic } L(A)\text{-sentence} \mid \mathfrak{A}_A \models \varphi\}$$

Lemma 1.13. *The models of* $\mathrm{Diag}(\mathfrak{A})$ *are precisely those structures* $(\mathfrak{B}, h(a))_{a\in A}$ *for embeddings $h : \mathfrak{A} \to \mathfrak{B}$*

Proof. The structures $(\mathfrak{B}, h(a))_{a\in A}$ are models of the atomic diagram by Lemma **??**. For the converse, note that a map h is an embedding iff it preserves the validity of all formulas of the form

$$\begin{aligned}&(\neg)x_1 \doteq x_2\\&c \doteq x_1\\&f(x_1, \dots, x_n) \doteq x_0\\&(\neg)R(x_1, \dots, x_n)\end{aligned}$$

□

Exercise 1.2.1. Every formula is equivalent to a formula in prenex normal form:

$$Q_1 x_1 \dots Q_n x_n \varphi$$

The Q_i are quantifiers and φ is quantifier-free

Proof.

$$\begin{aligned}&(\forall x)\phi \wedge \psi \vDash\Dashv \forall x(\phi \wedge \psi) \text{ if } \exists x\top (\text{at least one individual exists})\\&(\forall x\phi) \vee \psi \vDash\Dashv \forall x(\phi \vee \psi)\\&(\exists x\phi) \wedge \psi \vDash\Dashv \exists x(\phi \wedge \psi)\\&(\exists x\phi) \vee \psi \vDash\Dashv \exists x(\phi \vee \psi) \text{ if } \exists x\top\\&\neg\exists x\phi \vDash\Dashv \forall x\neg\phi\\&\neg\forall x\phi \vDash\Dashv \exists x\neg\phi\\&(\forall x\phi) \to \psi \vDash\Dashv \exists x(\phi \to \psi) \text{ if } \exists x\top\\&(\exists x\phi) \to \psi \vDash\Dashv \forall x(\phi \to \psi)\\&\phi \to (\exists x\psi) \vDash\Dashv \exists x(\phi \to \psi) \text{ if } \exists x\top\\&\phi \to (\forall x\psi) \vDash\Dashv \forall x(\phi \to \psi)\end{aligned}$$

□

1.3 Theories

Definition 1.14. An L**-theory** T is a set of L-sentences

A theory which has a model is a **consistent** theory. We call a set Σ of L-formulas **consistent** if there is an L-structure and **an assignment** $\vec{b}$ **s.t.** $\mathfrak{A} \vDash [\vec{b}]$ for all $\varphi \in \Sigma$

Lemma 1.15. *Let T be an L-theory and L' be an extension of L. Then T is consistent as an L-theory iff T is consistent as a L'-theory*

Lemma 1.16. 1. *If $T \models \varphi$ and $T \models (\varphi \to \psi)$, then $T \models \psi$*

2. *If $T \models \varphi(c_1, \dots, c_n)$ and the constants $c_1, \dots, c_n$ occur neither in T nor in $\varphi(x_1, \dots, x_n)$, then $T \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$*

Proof. 2. Let $L' = L \setminus \{c_1, \dots, c_n\}$. If the L'-structure is a model of T and $a_1, \dots, a_n$ are arbitrary elements, then $(\mathfrak{A}, a_1, \dots, a_n) \models \varphi(c_1, \dots, c_n)$. This means $\mathfrak{A} \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$.

$\square$

S and T are called **equivalent**, $S \equiv T$, if S and T have the same models

Definition 1.17. A consistent L-theory T is called **complete** if for all L-sentences φ

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi$$

Definition 1.18. For a complete theory T we define

$$|T| = \max(|L|, \aleph_0)$$

The typical example of a complete theory is the theory of a structure $\mathfrak{A}$

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$$

Lemma 1.19. *A consistent theory is complete iff it is maximal consistent, i.e., if it is equivalent to every consistent extension*

Definition 1.20. Two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are called **elementary equivalent**

$$\mathfrak{A} \equiv \mathfrak{B}$$

if they have the same theory

Lemma 1.21. *Let T be a consistent theory. Then the following are equivalent*

1. *T is complete*
2. *All models of T are elemantarily equivalent*
3. *There exists a structure $\mathfrak{A}$ with $T \equiv \mathrm{Th}(\mathfrak{A})$*

Proof. $1 \to 3 \to 2 \to 1$ $\square$

2 Elementary Extensions and Compactness

2.1 Elementary substructures

Let $\mathfrak{A}, \mathfrak{B}$ be two L-structures. A map $h : A \to B$ is called **elementary** if for all $a_1, \dots, a_n \in A$ we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

which is actually saying $(\mathfrak{A}, a)_{a \in A} \equiv (\mathfrak{B}, a)_{a \in A}$. We write

$$h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$$

Lemma 2.1. *The models of* $\mathrm{Th}(\mathfrak{A}_A)$ *are exactly the structures of the form* $(\mathfrak{B}, h(a))_{a \in A}$ *for elementary embeddings* $h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$

We call $\mathrm{Th}(\mathfrak{A}_A)$ the **elemantary diagram** of $\mathfrak{A}$

A substructure $\mathfrak{A}$ of $\mathfrak{B}$ is called **elementary** if the inclusion map is elementary. In this case we write

$$\mathfrak{A} \prec \mathfrak{B}$$

Theorem 2.2 (Tarski's Test)**.** *Let $\mathfrak{B}$ be an L-structure and A a subset of B. Then A is the universe of an elementary substructure iff every $L(A)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ can be satisfied by an element of A*

Proof. If $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \models \exists \varphi(x)$, we also have $\mathfrak{A} \models \exists x \varphi(x)$ and there eexists $a \in A$ s.t. $\mathfrak{A} \models \varphi(a)$. Thus $\mathfrak{B} \models \varphi(a)$

Conversely, suppose that the condition of Tarski'test is satisfied. First we show that A is the universe of a substructure $\mathfrak{A}$. The $L(A)$-formula $x \doteq x$ is satisfiable in $\mathfrak{A}$, so A is not empty. If $f \in L$ is an n-ary function symbol $(n \geq 0)$ and $a_1, \dots, a_n$ is from A, we consider the formula

$$\varphi(x) = f(a_1, \dots, a_n) \doteq x$$

Since $\varphi(x)$ is always satisfied by an element of A, it follows that A is closed under $f^{\mathcal{B}}$

Now we show, by induction on ψ, that

$$\mathfrak{A} \models \psi \Leftrightarrow \mathfrak{B} \models \psi$$

for all $L(A)$-sentences ψ.

For $\psi = \exists x \varphi(x)$. If ψ holds in $\mathfrak{A}$. If ψ holds in $\mathfrak{A}$, there exists $a \in A$ s.t. $\mathfrak{A} \models \varphi(a)$. The induction hypothesis yields $\mathfrak{B} \models \varphi(x)$, thus $\mathfrak{B} \models \psi$. For the converse suppose ψ holds in $\mathfrak{B}$. Then $\varphi(x)$ is satisfied in $\mathcal{B}$ and by Tarski's test we find $a \in A$ s.t. $\mathfrak{B} \models \varphi(a)$. By induction $\mathfrak{A} \models \varphi(a)$ and $\mathfrak{A} \models \psi$ $\square$

We use Tarski's Test to construct small elementary substructures

Corollary 2.3. *Suppose S is a subset of the L-structure $\mathfrak{B}$. Then $\mathfrak{B}$ has a elementary substructure $\mathfrak{A}$ containing S and of cardinality at most*

$$\max(|S|, |L|, \aleph_0)$$

Proof. We construct A as the union of an ascending sequence $S_0 \subseteq S_1 \subseteq \dots$ of subsets of B. We start with $S_0 = S$. If S_i is already defined, we choose an element $a_\varphi \in B$ for every $L(S_i)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ and define S_{i+1} to be S_i together with these a_φ.

An L-formula is a finite sequence of symbols from L, auxiliary symbols and logical symbols. These are $|L| + \aleph_0 = \max(|L|, \aleph_0)$ many symbols and there are exactlymax$(|L|, \aleph_0)$ many L-formulas

Let $\kappa = \max(|S|, |L|, \aleph_0)$. There are κ many $L(S)$-formulas: therefore $|S_1| \leq \kappa$. Inductively it follows for every i that $|S_i| \leq \kappa$. Finally we have $|A| \leq \kappa \cdot \aleph_0 = \kappa$ □

A directed family $(\mathfrak{A}_i)_{i \in I}$ of structures is **elementary** if $\mathfrak{A}_i \prec \mathfrak{A}_j$ for all $i \leq j$

Theorem 2.4 (Tarski's Chain Lemma)**.** *The union of an elementary directed family is an elementary extension of all its members*

Proof. Let $\mathfrak{A} = \bigcup_{i \in I} (\mathfrak{A}_i)_{i \in I}$. We prove by induction on $\varphi(\bar{x})$ that for all i and $\bar{a} \in \mathfrak{A}_i$

$$\mathfrak{A}_i \vDash \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \vDash \varphi(\bar{a})$$

□

Exercise 2.1.1. Let $\mathfrak{A}$ be an L-structure and $(\mathfrak{A}_i)_{i \in I}$ a chain of elementary substructures of $\mathfrak{A}$. Show that $\bigcup_{i \in I} A_i$ is an elementary substructure of $\mathfrak{A}$.

Exercise 2.1.2. Consider a class $\mathcal{C}$ of L-structures. Prove

1. Let $\mathrm{Th}(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \vDash \varphi \text{ for all } \mathfrak{A} \in \mathcal{C}\}$ be the **theory of** $\mathcal{C}$. Then $\mathfrak{M}$ is a model of Th(C) iff $\mathfrak{M}$ is elementary equivalent to an ultraproduct of elements of $\mathcal{C}$

2. Show that $\mathcal{C}$ is an elementary class iff $\mathcal{C}$ is closed under ultraproduct and elementary equivalence

3. Assume that $\mathcal{C}$ is a class of finite structures containing only finitely many structures of size n for each $n \in \omega$. Then the infinite models of $\mathrm{Th}(\mathcal{C})$ are exactly the models of

$$\mathrm{Th}_a(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \vDash \varphi \text{ for all but finitely many } \mathfrak{A} \in \mathcal{C}\}$$

Proof. Chang&Keisler p220 □

2.2 The Compactness Theorem

We call a theory T **finitely satisfiable** if every finite subset of T is consistent

Theorem 2.5 (Compactness Theorem)**.** *Finitely satisfiable theories are consistent*

Let L be a language and C a set of new constants. An $L(C)$-theory T' is called a **Henkin theory** if for every $L(C)$-formula $\varphi(x)$ there is a constant $c \in C$ s.t.

$$\exists x \varphi(x) \to \varphi(c) \in T'$$

The elements of C are called **Henkin constants** of T'

An L-theory T is **finitely complete** if it is finitely satisfiable and if every L-sentence φ satisfies $\varphi \in T$ or $\neg\varphi \in T$

Lemma 2.6. *Every finitely satisfiable L-theory T can be extended to a finitely complete Henkin Theory T^**

Note that conversely the lemma follows directly from the Compactness Theorem. Choose a model $\mathfrak{A}$ of T. Then $\mathrm{Th}(\mathfrak{A}_A)$ is a finitely complete Henkin theory with A as a set of Henkin constants

Proof. We define an increasing sequence $\emptyset = C_0 \subseteq C_1 \subseteq \cdots$ of new constants by assigning to every $L(C_i)$-formula $\varphi(x)$ a constant $c_{\varphi(x)}$ and

$$C_{i+1} = \{c_{\varphi(x)} \mid \varphi(x) \text{ a } L(C_i)\text{-formula}\}$$

Let C be the union of the C_i and T^H the set of all Henkin axioms

$$\exists x \varphi(x) \to \varphi(c_{\varphi(x)})$$

for $L(C)$-formulas $\varphi(x)$. It is easy to see that one can expand every L-structure to a model of T^H. Hence $T \cup T^H$ is a finitely satisfiable Henkin theory. Using the fact that the union of a chain of finitely satisfiable theories is also finite satisfiable, we can apply Zorn's Lemma and get a maximal finitely satisfiable $L(C)$-theory T^* which contains $T \cup T^H$. As in Lemma 1.19 we show that T^* is finitely complete: if neither φ nor $\neg\varphi$ belongs to T^*, neither $T^* \cup \{\varphi\}$ nor $T^* \cup \{\neg\varphi\}$ would be finitely satisfiable. Hence there would be a finite subset Δ of T^* which would be consistent neither with φ nor with $\neg\varphi$. Then Δ itself would be inconsistent and T^* would not be finite satisfiable. This proves the lemma. □

Lemma 2.7. *Every finitely satisfiable L-theory T can be extended to a finitely complete Henkin theory T^**

Lemma 2.8. *Every finitely complete Henkin theory T^* has a model $\mathfrak{A}$ (unique up to isomorphism) consisting of constants; i.e.,*

$$(\mathfrak{A}, a_c)_{c \in C} \models T^*$$

with $A = \{a_c \mid c \in C\}$

Proof. Since T^* is finite complete, every sentence which follows from a finite subset of T^* belongs to T^*

Define for $c, d \in C$

$$c \simeq d \Leftrightarrow c \dot{=} d \in T^*$$

$\simeq$ is an equivalence relation. We denote the equivalence class of c by a_c, and set

$$A = \{a_c \mid c \in C\}$$

We expand A to an L-structure $\mathfrak{A}$ by defining

$$R^{\mathfrak{A}}(a_{c_1}, \dots, a_{c_n}) \Leftrightarrow R(c_1, \dots, c_n) \in T^* \qquad (\star)$$

$$f^{\mathfrak{A}}(a_{c_1}, \dots, a_{c_n}) \Leftrightarrow f(c_1, \dots, c_n) \dot{=} c_0 \in T^* \qquad (\star\star)$$

We have to show that this is well-defined. For $(\star)$ we have to show that

$$a_{c_1} = a_{d_1}, \dots, a_{c_n} = a_{d_n}, R(c_1, \dots, c_n) \in T^*$$

implies $R(d_1, \dots, d_n) \in T^*$, which is obvious.

For $(\star\star)$, we have to show that for all $c_1, \dots, c_n$ there exists c_0 with $f(c_1, \dots, c_n) \dot{=} c_0 \in T^*$.

Let $\mathfrak{A}^*$ be the $L(C)$-structure $(\mathfrak{A}, a_c)_{c \in C}$. We show by induction on the complexity of φ that for every $L(C)$-sentence φ

$$\mathfrak{A}^* \models \varphi \Leftrightarrow \varphi \in T^*$$

$\square$

Corollary 2.9. *We have $T \models \varphi$ iff $\Delta \models \varphi$ for a finite subset Δ of T*

Corollary 2.10. *A set of formulas $\Sigma(x_1, \dots, x_n)$ is consistent with T if and only if every finite subset of Σ is consistent with T*

Proof. Introduce new constants $c_1, \dots, c_n$. Then Σ is consistent with T is and only if $T \cup \Sigma(c_1, \dots, c_n)$ is consistent. Now apply the Compactness Theorem $\square$

Definition 2.11. Let $\mathfrak{A}$ be an L-structure and $B \subseteq A$. Then $a \in A$ **realises** a set of $L(B)$-formulas $\Sigma(x)$ if a satisfied all formulas from Σ. We write

$$\mathfrak{A} \models \Sigma(a)$$

We call $\Sigma(x)$ **finitely satisfiable** in $\mathfrak{A}$ if every finite subset of Σ is realised in $\mathfrak{A}$

Lemma 2.12. *The set $\Sigma(x)$ is finitely satisfiable in $\mathfrak{A}$ iff there is an elementary extension of $\mathfrak{A}$ in which $\Sigma(x)$ is realised*

Proof. By Lemma 2.1 Σ is realised in an elementary extension of $\mathfrak{A}$ iff Σ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. So the lemma follows from the observation that a finite set of $L(A)$-formulas is consistent with $\mathrm{Th}(\mathfrak{A}_A)$ iff it is realised in $\mathfrak{A}$ $\square$

Definition 2.13. Let $\mathfrak{A}$ be an L-structure and B a subset of A. A set $p(x)$ of $L(B)$-formulas is a **type** over B if $p(x)$ is maximal finitely satisfiable in $\mathfrak{A}$. We call B the **domain** of p. Let

$$S(B) = S^{\mathfrak{A}}(B)$$

denote the set of types over B.

Every element a of $\mathfrak{A}$ determines a type

$$\mathrm{tp}(a/B) = tp^{\mathfrak{A}}(a/B) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a), \varphi \text{ an } L(B)\text{-formula}\}$$

So an element a realises the type $p \in S(B)$ exactly if $p = \mathrm{tp}(a/B)$. If $\mathfrak{A}'$ is an elementary extension of $\mathfrak{A}$, then

$$S^{\mathfrak{A}}(B) = S^{\mathfrak{A}'}(B) \quad \text{and} \quad \mathrm{tp}^{\mathfrak{A}'}(a/B) = \mathrm{tp}^{\mathfrak{A}}(a/B)$$

If $\mathfrak{A}' \models p(x)$ then $\mathfrak{A}' \models \exists x p(x)$, so $\mathfrak{A} \models \exists x p(x)$.

We use the notation $\mathrm{tp}(a)$ for $\mathrm{tp}(a/\emptyset)$

Maximal finitely satisfiable sets of formulas in $x_1, \dots, x_n$ are called ***n*-types** and

$$S_n(B) = S_N^{\mathfrak{A}}(B)$$

denotes the set of n-types over B.

$$\mathrm{tp}(C/B) = \{\varphi(x_{c_1}, \dots, x_{c_n}) \mid \mathfrak{A} \models \varphi(c_1, \dots, c_n), \varphi \text{ an } L(B)\text{-formula}\}$$

Corollary 2.14. *Every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ in which all types over A are realised*

Proof. We choose for every $p \in S(A)$ a new constant c_p. We have to find a model of

$$\mathrm{Th}(\mathfrak{A}_A) \cup \bigcup_{p \in S(A)} p(c_p)$$

This theory is finitely satisfiable since every p is finitely satisfiable in $\mathfrak{A}$.

Or use Lemma 2.12. Let $(p_\alpha)_{\alpha<\lambda}$ be an enumeration of $S(A)$. Construct an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\beta \prec \ldots (\beta \leq \lambda)$$

s.t. each p_α is realised in $\mathfrak{A}_{\alpha+1}$ (by recursion theorem on ordinal numbers)

Suppose that the elementary chain $(\mathfrak{A}_{\alpha'})_{\alpha'<\beta}$ is already constructed. If β is a limit ordinal, we let $\mathfrak{A}_\beta = \bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$, which is elementary by Lemma 2.4. If $\beta = \alpha + 1$ we first note that p_α is also finitely satisfiable in $\mathfrak{A}_\alpha$, therefore we can realise p_α in a suitable elementary extension $\mathfrak{A}_\beta \succ \mathfrak{A}_\alpha$ by Lemma 2.12. Then $\mathfrak{B} = \mathfrak{A}_\lambda$ is the model we were looking for □

2.3 The Löwenheim-Skolem Theorem

Theorem 2.15 (Löwenheim-Skolem)**.** *Let $\mathfrak{B}$ be an L-structure, S a subset of B and κ an infinite cardinal*

1. *If*

$$\max(|S|, |L|) \leq \kappa \leq |B|$$

then $\mathfrak{B}$ has an elementary substructure of cardinality κ containing S

2. *If $\mathfrak{B}$ is infinite and*

$$\max(|\mathfrak{B}|, |L|) \leq \kappa$$

then $\mathfrak{B}$ has an elementary extension of cardinality κ

Proof. 1. Choose a set $S \subseteq S' \subseteq B$ of cardinality κ and apply Corollary 2.3

2. We first construct an elementary extension $\mathfrak{B}'$ of cardinality at least κ. Choose a set C of new constants of cardinality κ. As $\mathfrak{B}$ is infinite, the theory

$$\mathrm{Th}(\mathfrak{B}_B) \cup \{\neg c \doteq d \mid c, d \in C, c \neq d\}$$

is finitely satisfiable. By Lemma 2.1 any model $(\mathfrak{B}'_B, b_c)_{c\in C}$ is an elementary extension of $\mathcal{B}$ with κ many different elements (b_c)

Finally we apply the first part of the theorem to $\mathcal{B}'$ and $S = B$

□

Corollary 2.16. *A theory which has an infinite model has a model in every cardinality $\kappa \geq \max(|L|, \aleph_0)$*

Definition 2.17. Let κ be an infinite cardinal. A theory T is called **κ-categorical** if for all models of T of cardinality κ are isomorphic

Theorem 2.18 (Vaught's Test)**.** *A κ-categorical theory T is complete if the following conditions are satisfied*

1. *T is consistent*
2. *T has no finite model*
3. *$|L| \leq \kappa$*

Proof. We have to show that all models $\mathfrak{A}$ and $\mathfrak{B}$ of T are elemantarily equivalent. As $\mathfrak{A}$ and $\mathfrak{B}$ are infinite, $\mathrm{Th}(\mathfrak{A})$ and $\mathrm{Th}(\mathfrak{B})$ have models $\mathfrak{A}'$ and $\mathfrak{B}'$ of cardinality κ. By assumption $\mathfrak{A}'$ and $\mathfrak{B}'$ are isomorphic, and it follows that

$$\mathfrak{A} \equiv \mathfrak{A}' \equiv \mathfrak{B}' \equiv \mathfrak{B}$$

□

Example 2.1.
1. The theory DLO of dense linear orders without endpoints is $\aleph_0$-categorical and by Vaught's test complete. Let $A = \{a_i \mid i \in \omega\}$, $B = \{b_i \mid i \in \omega\}$. We inductively define sequences $(c_i)_{i<\omega}$, $(d_i)_{i<\omega}$ exhausting A and B. Assume that $(c_i)_{i<m}, (d_i)_{i<m}$ have defined so that $c_i \mapsto d_i, i < m$ is an order isomorphism. If $m = 2k$ let $c_m = a_j$ where a_j is the element with minimal index in $\{a_i \mid i \in \omega\}$ not occurring in $(c_i)_{i<m}$. Since $\mathfrak{B}$ is a dense linear order without endpoints there is some element $d_m \in \{b_i \mid i \in \omega\}$ s.t. $(c_i)_{i\leq m}$ and $(d_i)_{i\leq m}$ are order isomorphic. If $m = 2k+1$ we interchange the roles of $\mathfrak{A}$ and $\mathfrak{B}$
2. For any prime p or $p = 0$, the theory ACF_p of algebraically closed fields of characteristic p is κ-categorical for any $\kappa > \aleph_0$

Consider the Theorem 2.18 we strengthen our definition

Definition 2.19. Let κ be an infinite cardinal. A theory T is called **κ-categorical** if it is complete, $|T| \leq \kappa$ and, up to isomorphism, has exactly one model of cardinality κ

3 Quantifier Elimination

3.1 Preservation theorems

Lemma 3.1 (Separation Lemma). *Let T_1, T_2 be two theories. Assume $\mathcal{H}$ is a set of sentences which is closed under $\wedge, \vee$ and contains $\perp$ and $\top$. Then the following are equivalent*

1. *There is a sentence $\varphi \in \mathcal{H}$ which separates T_1 from T_2. This means*
$$T_1 \models \varphi \quad \textit{and} \quad T_2 \models \neg\varphi$$
2. *All models $\mathfrak{A}_1$ of T_1 can be separated from all models $\mathfrak{A}_2$ of T_2 by a sentence $\varphi \in \mathcal{H}$. This means*
$$\mathfrak{A}_1 \models \varphi \quad \textit{and} \quad \mathfrak{A}_2 \models \neg\varphi$$

For 1, suppose $T_1 = T \cup \{\psi\}$ and $T_2 = T \cup \{\neg\psi\}$. If $T_1 \models \varphi$ and $T_2 \models \neg\varphi$, then $T \models \psi \to \varphi$ and $T \models \neg\psi \to \neg\varphi$ which is equivalent to $T \models \varphi \to \psi$. Thus we have $T \models \varphi \leftrightarrow \psi$.

Proof. $2 \to 1$. For any model $\mathfrak{A}_1$ of T_1 let $\mathcal{H}_{\mathfrak{A}_1}$ be the set of all sentences from $\mathcal{H}$ which are true in $\mathfrak{A}_1$. (2) implies that $\mathcal{H}_{\mathfrak{A}_1}$ and T_2 cannot have a common model. By the Compactness Theorem there is a finite conjunction $\varphi_{\mathfrak{A}_1}$ of sentences from $\mathcal{H}_{\mathfrak{A}_1}$ inconsistent with T_2. Clearly
$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$
is inconsistent. Again by compactness T_1 implies a disjunction φ of finitely many of the $\varphi_{\mathfrak{A}_1}$ (Corollary 2.10) and
$$T_1 \models \varphi \quad \text{and} \quad T_2 \models \neg\varphi$$
□

For structures $\mathfrak{A}, \mathfrak{B}$ and a map $f : A \to B$ preserving all formulas from a set of formulas Δ, we use the notation
$$f : \mathfrak{A} \to_\Delta \mathfrak{B}$$
We also write
$$\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$$
to express that all sentences from Δ true in $\mathfrak{A}$ are also true in $\mathfrak{B}$

Lemma 3.2. *Let T be a theory, $\mathfrak{A}$ a structure and Δ a set of formulas, closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent*

1. *All sentences $\varphi \in \Delta$ which are true in $\mathfrak{A}$ are consistent with T*

2. *There is a model $\mathfrak{B} \models T$ and a map $f : \mathfrak{A} \to_\Delta \mathfrak{B}$*

Proof. $2 \to 1$. Assume $f : \mathfrak{A} \to_\Delta \mathfrak{B} \models T$. If $\varphi \in \Delta$ is true in $\mathfrak{A}$, it is also true in $\mathfrak{B}$ and therefore consistent with T.

$1 \to 2$. Consider $\mathrm{Th}_\Delta(\mathfrak{A}_A)$, the set of all sentences $\delta(\bar{a})$ ($\delta(\bar{x}) \in \Delta$), which are true in $\mathfrak{A}_A$. The models $(\mathfrak{B}, f(a)_{a \in A})$ of this theory correspond to maps $f : \mathfrak{A} \to_\Delta \mathfrak{B}$. **This means that we have to find a model of** $T \cup \mathrm{Th}_\Delta(\mathfrak{A}_A)$. To show finite satisfiability it is enough to show that $T \cup D$ is consistent for every finite subset D of $\mathrm{Th}_\Delta(\mathfrak{A}_A)$. Let $\delta(\bar{a})$ be the conjunction of the elements of D. Then $\mathfrak{A}$ is a model of $\varphi = \exists \bar{x} \delta(\bar{x})$, so by assumption T has a model $\mathfrak{B}$ which is also a model of φ. This means that there is a tuple $\bar{b}$ s.t. $(\mathfrak{B}, \bar{b}) \models \delta(\bar{a})$ $\square$

Lemma 3.2 applied to $T = \mathrm{Th}(\mathfrak{B})$ shows that $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$ iff there exists a map f and a structure $\mathfrak{B}' \equiv \mathfrak{B}$ s.t. $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$

Theorem 3.3. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *There is a universal sentence which separates T_1 from T_2*

2. *No model of T_2 is a substructure of a model of T_1*

Proof. $1 \to 2$. Let φ be a universal sentence which separates T_1 and T_2. Let $\mathfrak{A}_1$ be a model of T_1 and $\mathfrak{A}_2$ a substructure of $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is a model of φ, $\mathfrak{A}_2$ is also a model of φ. Therefore $\mathfrak{A}_2$ cannot be a model of T_2

$2 \to 1$. Here we add some details for the proof $2 \to 1$. If T_1 and T_2 cannot be separated by a universal sentence, then they have models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which cannot be separated by a universal sentence. That is, for all universal sentence φ, if $\mathfrak{A}_1 \models \varphi$ then $\mathfrak{A}_2 \models \varphi$. Thus $\mathfrak{A}_1 \Rightarrow_\forall \mathfrak{A}_2$, here $\Rightarrow_\forall$ means for all universal sentence.

Now note that

$$\mathfrak{A}_1 \models \varphi \to \mathfrak{A}_2 \models \varphi \quad \Leftrightarrow \quad \mathfrak{A}_2 \models \neg\varphi \to \mathfrak{A}_2 \models \neg\varphi$$

and $\neg\varphi$ is an existential sentence. Hence we have

$$\mathfrak{A}_2 \Rightarrow_\exists \mathfrak{A}_1$$

The reason that we want to use $\exists$ is that it holds in the substructure case and we could imagine that $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$ (I guess this is our intuition). Now by Lemma 3.2 we have $\mathfrak{A}_1' \equiv \mathfrak{A}_1$ and a map $f : \mathfrak{A}_2 \to_\exists \mathfrak{A}_1'$. Apparently $\mathfrak{A}_1' \vDash \mathrm{Diag}(\mathfrak{A}_2)$. Hence $\mathfrak{A}_1'$ is a model of T_1 and T_2 □

Definition 3.4. For any L-theory T, the formulas $\varphi(\bar{x}), \psi(\bar{x})$ are said to be **equivalent** modulo T (or relative to T) if $T \vDash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

Corollary 3.5. *Let T be a theory*

1. *Consider a formula $\varphi(x_1, \dots, x_n)$. The following are equivalent*
   - (a) *$\varphi(x_1, \dots, x_n)$ is, modulo T, equivalent to a universal formula*
   - (b) *If $\mathfrak{A} \subseteq \mathfrak{B}$ are models of T and $a_1, \dots, a_n \in A$, then $\mathfrak{B} \vDash \varphi(a_1, \dots, a_n)$ implies $\mathfrak{A} \vDash \varphi(a_1, \dots, a_n)$*
2. *We say that a theory which consists of universal sentences is universal. Then T is equivalent to a universal theory iff all substructures of models of T are again models of T*

Proof. 1. Assume (2). We extend L by an n-tuple $\bar{c}$ of new constants $c_1, \dots, c_n$ and consider theory

$$T_1 = T \cup \{\varphi(\bar{c})\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi(\bar{c})\}$$

Then (2) says the substructures of models of T_1 cannot be models of T_2. By Theorem 3.3 T_1 and T_2 can be separated by a universal $L(\bar{c})$-sentence $\psi(\bar{c})$. By Lemma 1.16, $T_1 \vDash \psi(\bar{c})$ implies

$$T \vDash \forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$$

and from $T_2 \vDash \neg\psi(\bar{c})$ we see

$$T \vDash \forall \bar{x}(\neg\varphi(\bar{x}) \to \neg\psi(\bar{x}))$$

2. Suppose a theory T has this property. Let φ be an axiom of T. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, it is not possible for $\mathfrak{B}$ to be a model of T and for $\mathfrak{A}$ to be a model of $\neg\varphi$ at the same time. By Theorem 3.3 there is a universal sentence ψ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of T follow from

$$T_\forall = \{\psi \mid T \vDash \psi, \psi \text{ universal}\}$$

□

An $\forall\exists$-formula is of the form

$$\forall x_1 \ldots x_n \psi$$

where ψ is existential

Lemma 3.6. *Suppose φ is an $\forall\exists$-sentence, $(\mathfrak{A}_i)_{i\in I}$ is a directed family of models of φ and $\mathfrak{B}$ the union of the $\mathfrak{A}_i$. Then $\mathfrak{B}$ is also a model of φ.*

Proof. Write

$$\varphi = \forall \bar{x} \psi(\bar{x})$$

where ψ is existential. For any $\bar{a} \in B$ there is an A_i containing $\bar{a}$, clearly $\psi(\bar{a})$ holds in $\mathfrak{A}_i$. As $\psi(\bar{a})$ is existential it must also hold in $\mathfrak{B}$ $\square$

Definition 3.7. We call a theory T **inductive** if the union of any directed family of models of T is again a model

Theorem 3.8. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *there is an $\forall\exists$-sentence which separates T_1 and T_2*
2. *No model of T_2 is the union of a chain (or of a directed family) of models of T_1*

Proof. $1 \to 2$. Assume φ is a $\forall\exists$-sentence which separates T_1 from T_2, $(\mathfrak{A}_i)_{i\in I}$ is a directed family of models of φ, by Lemma 3.6 $\mathfrak{B}$ is also a model of φ. Since $\mathfrak{B} \vDash \varphi$, $\mathfrak{B}$ cannot be a model of T_2

$2 \to 1$. If (1) is not true, T_1, T_2 have models which cannot be separated by an $\forall\exists$-sentence. Since $\exists\forall$-formulas are equivalent to negated $\forall\exists$-formulas (since $\forall$ is too strong), we have

$$\mathfrak{B}^0 \Rightarrow_{\exists\forall} \mathfrak{A}$$

By Lemma 3.2 there is a map

$$f : \mathfrak{B}^0 \to_{\forall} \mathfrak{A}^0$$

with $\mathfrak{A}^0 \equiv \mathfrak{A}$ (since $\mathfrak{B}^0 \to_{\exists\forall} \mathfrak{A}^0$). We can assume that$\mathfrak{B}^0 \subseteq \mathfrak{A}^0$ and f is the inclusion map. Then

$$\mathfrak{A}^0_B \Rightarrow_{\exists} \mathfrak{B}^0_B$$

Applying Lemma 3.2 again, we obtain an extension $\mathfrak{B}_B^1$ of $\mathfrak{A}_B^0$ with $\mathfrak{B}_B^1 \equiv \mathfrak{B}_B^0$, i.e. $\mathfrak{B}^0 \prec \mathfrak{B}^1$. Hence we have an infinite chain

$$\mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq^1 \mathfrak{B}^1 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{B}^2 \subseteq \cdots$$
$$\mathfrak{B}^0 \prec \mathfrak{B}^1 \prec \mathfrak{B}^2 \prec \cdots$$
$$\mathfrak{A}^i \equiv \mathfrak{A}$$

Let $\mathfrak{B}$ be the union of the $\mathfrak{A}^i$. Since $\mathfrak{B}$ is also the union of the elementary chain of the $\mathfrak{B}^i$, it is an elementary extension of $\mathfrak{B}^0$ and hence a model of T_2. But the $\mathfrak{A}^i$ are models of T_1, so (2) does not hold □

Corollary 3.9. *Let T be a theory*

1. *For each sentence φ the following are equivalent*

   (*a*) *φ is, modulo T, equivalent to an $\forall\exists$-sentence*

   (*b*) *If*

   $$\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \cdots$$

   and their union $\mathfrak{B}$ are models of T, then φ holds in $\mathfrak{B}$ if it is true in all the $\mathfrak{A}^i$

2. *T is inductive iff it can be axiomatised by $\forall\exists$-sentences*

Proof. 1. Theorem 3.8 shows that $\forall\exists$-formulas are preserved by unions of chains. Hence (a)⇒(b). For the converse consider the theories

$$T_1 = T \cup \{\varphi\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi\}$$

Part (b) says that the union of a chain of models of T_1 cannot be a model of T_2. By Theorem 3.8 we can separate T_1 and T_2 by an $\forall\exists$-sentence ψ. Hence $T \cup \{\varphi\} \vDash \psi$ and $T \cup \{\neg\varphi\} \vDash \neg\psi$

2. Clearly $\forall\exists$-axiomatised theories are inductive. For the converse assume that T is inductive and φ is an axiom of T. Ifpp $\mathfrak{B}$ is a union of models of T, it cannot be a model of $\neg\varphi$. By Theorem 3.8 there is an $\forall\exists$-sentence ψ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of T follows from

$$T_{\forall\exists} = \{\psi \mid T \vDash \psi, \psi\ \forall\exists\text{-formula}\}$$

□

Exercise 3.1.1. Let X be a topological space, Y_1 and Y_2 quasi-compact subsets, and $\mathcal{H}$ a set of clopen subsets. Then the following are equivalent

1. There is a positive Boolean combination B of elements from $\mathcal{H}$ s.t. $Y_1 \subseteq B$ and $Y_2 \cap B = \emptyset$
2. For all $y_1 \in Y_1$ and $y_2 \in Y_2$ there is an $H \in \mathcal{H}$ s.t. $y_1 \in H$ and $y_2 \notin H$

Proof. $2 \to 1$. Consider an element $y_1 \in Y_1$ and $\mathcal{H}_{y_1}$, the set of all elements of $\mathcal{H}$ containing y_1. 2 implies that the intersection of the sets in $\mathcal{H}_{y_1}$ is disjoint from Y_2. So a finite intersection h_{y_1} of elements of $\mathcal{H}_{y_1}$ is disjoint from Y_2. The $h_{y_i}, y_1 \in Y_1$, cover Y_1. So Y_1 is contained in the union H of finitely many of the h_{y_i}. Hence H separates Y_1 from Y_2 □

3.2 Quantifier elimination

Definition 3.10. A theory T has **quantifier elimination** if every L-formula $\varphi(x_1, \dots, x_n)$ in the theory is equivalent modulo T to some quantifier-free formula $\rho(x_1, \dots, x_n)$

It's easy to transform any theory T into a theory with quantifier elimination if one is willing to expand the language: just enlarge L by adding an n-place relation symbol R_φ for every L-formula $\varphi(x_1, \dots, x_n)$ and T by adding all axioms

$$\forall x_1, \dots, x_n (R_\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n))$$

The resulting theory, the **Morleyisation** T^m of T, has quantifier elimination

A **prime structure** of T is a structure which embeds into all models of T

Lemma 3.11. *A consistent theory T with quantifier elimination which possess a prime structure is complete*

Proof. If $\mathfrak{M}, \mathfrak{N} \vDash T$ and $\mathfrak{M} \vDash \varphi$ and $\mathfrak{N} \vDash \neg\varphi$. Suppose prime structure is $\mathfrak{H}$, then $\mathfrak{H} \vDash \varphi$ and $\mathfrak{H} \vDash \neg\varphi$ since we have quantifier elimination □

Definition 3.12. A **simple existential formula** has the form

$$\varphi = \exists y \rho$$

for a quantifier-free formula ρ. If ρ is a conjunction of basic formulas, φ is called **primitive existential**

Lemma 3.13. *The theory T has quantifier elimination iff every primitive existential formula is, modulo T, equivalent to a quantifier-free formula*

Proof. We can write every simple existential formula in the form $\exists y \bigvee_{i<n} \rho_i$ for ρ_i which are conjunctions of basic formulas. This shows that every simple existential formula is equivalent to a disjunction of primitive existential formulas, namely to $\bigvee_{i<n}(\exists y \rho_i)$. We can therefore assume that every simple existential formula is, modulo T, equivalent to a quantifier-free formula

We are now able to eliminate the quantifiers in arbitrary formulas in prenex normal form (Exercise 1.2.1)

$$Q_1 x_1 \dots Q_n x_n \rho$$

if $Q_n = \exists$, we choose a quantifier-free formula ρ_0 which, modulo T, is equivalent to $\exists x_n \rho$ and proceed with the formula $Q_1 x_1 \dots Q_{n-1} x_{n-1} \rho_0$. If $Q_n = \forall$, we find a quantifier-free ρ_1 which is, modulo T, equivalent to $\exists x_n \neg \rho$ and proceed with $Q_1 x_1 \dots Q_{n-1} x_{n-1} \neg \rho_1$ □

Theorem 3.14. *For a theory T the following are equivalent*

1. *T has quantifier elimination*
2. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T with a common substructure $\mathfrak{A}$ we have*

$$\mathfrak{M}^1_A \equiv \mathfrak{M}^2_A$$

3. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T with a common substructure $\mathfrak{A}$ and for all primitive existential formulas $\varphi(x_1, \dots, x_n)$ and parameter $a_1, \dots, a_n$ from A we have*

$$\mathfrak{M}^1 \models \varphi(a_1, \dots, a_n) \Rightarrow \mathfrak{M}^2 \models \varphi(a_1, \dots, a_n)$$

   (*this is exactly the equivalence relation*)

If L has no constants, $\mathfrak{A}$ is allowed to be the empty "structure"

Proof. $1 \to 2$. Let $\varphi(\bar{a})$ be an $L(A)$-sentence which holds in $\mathfrak{M}^1$. Choose a quantifier-free $\rho(\bar{x})$ which is, modulo T, equivalent to $\varphi(\bar{x})$. Then

$$\begin{array}{llllllll}
\mathfrak{M}^1 & \models & \varphi(\bar{a}) & \Rightarrow & \mathfrak{M}^1 & \models & \rho(\bar{a}) & \\
 & & & \Rightarrow & \mathfrak{A} & \models & \rho(\bar{a})) & \Rightarrow \\
\mathfrak{M}^2 & \models & \rho(\bar{a})) & \Rightarrow & \mathfrak{M}^2 & \models & \varphi(\bar{a}) &
\end{array}$$

$3 \to 1$. Let $\varphi(\bar{x})$ be a primitive existential formula. In order to show that $\varphi(\bar{x})$ is equivalent, modulo T, to a quantifier-free formula $\rho(\bar{x})$ we extend L by an n-tuple $\bar{c}$ of new constants $c_1, \dots, c_n$. **We have to show that we can separate** $T \cup \{\varphi(\bar{c})\}$ **and** $T \cup \{\neg\varphi(\bar{c})\}$ **by a quantifier free sentence** $\rho(\bar{c})$. Then $T \models \varphi(\bar{c}) \to \rho(\bar{c})$ and $T \models \neg\varphi(\bar{c}) \to \neg\rho(\bar{c})$. Hence $T \models \varphi(\bar{c}) \leftrightarrow \rho(\bar{c})$.

We apply the Separation Lemma ($\mathcal{H}$ hear is the set of quantifier-free sentence). Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be two models of T with two distinguished n-tuples $\bar{a}^1$ and $\bar{a}^2$. Suppose that $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ satisfy the same quantifier-free $L(\bar{c})$-sentences. We have to show that

$$\mathfrak{M}^1 \vDash \varphi(\bar{a}^1) \Rightarrow \mathfrak{M}^2 \vDash \varphi(\bar{a}^2)$$

Now suppose $\mathfrak{A}_1 \vDash T \cup \{\varphi(\bar{c})\}$ and $\mathfrak{A}_2 \vDash T \cup \{\neg\varphi(\bar{c})\}$, they can't satisfy the same quantifier-free $L(\bar{c})$-sentences. Thus we finish the proof

Consider the substructure $\mathfrak{A}^i = \langle \bar{a}^i \rangle^{\mathfrak{M}^i}$, generated by $\bar{a}^i$. If we can show that there is an isomorphism

$$f : \mathfrak{A}^1 \to \mathfrak{A}^2$$

taking $\bar{a}$ to $\bar{a}$, we may assume that $\mathfrak{A}^1 = \mathfrak{A}^2 = \mathfrak{A}$ and $\bar{a}^1 = \bar{a}^2 = \bar{a}$.

Every element of $\mathfrak{A}^1$ has the form $t^{\mathfrak{M}^1}[\bar{a}^1]$ for an L-term $t(\bar{x})$. The isomorphism f to be constructed must satisfy

$$f(t^{\mathfrak{M}^1}[\bar{a}^1]) = t^{\mathfrak{M}^2}[\bar{a}^2]$$

We define f by this equation and have to check that f is well defined and injective. Assume

$$s^{\mathfrak{M}^1}[\bar{a}^1] = t^{\mathfrak{M}^1}[\overline{af1}]$$

Then $\mathfrak{M}^1, \bar{a}^1 \vDash s(\bar{c}) \doteq t(\bar{c})$, and by out assumption it also holds in $(\mathfrak{M}^2, \bar{a}^2)$, which means

$$s^{\mathfrak{M}^2}[\bar{a}^2] = t^{\mathfrak{M}^2}[\bar{a}^2]$$

Swapping the two sides yields injectivity.

Surjectivity is clear. It remains to show that f commutes with the interpretation of the relation symbols. Now

$$\mathfrak{M}^1 \vDash R\left[t_1^{\mathfrak{M}^1}[\bar{a}^1], \dots, t_m^{\mathfrak{M}^1}[\bar{a}^1]\right]$$

is equivalent to $(\mathfrak{M}^1, \bar{a}^1) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which is equivalent to $(\mathfrak{M}^2, \bar{a}^2) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which in turn is equivalent to

$$\mathfrak{M}^2 \vDash R\left[t_1^{\mathfrak{M}^2}[\bar{a}^2], \dots, t_m^{\mathfrak{M}^2}[\bar{a}^2]\right]$$

$\square$

Note that (2) of Theorem 3.14 is saying that T is **substructure complete**; i.e., for any model $\mathfrak{M} \vDash T$ and substructure $\mathfrak{A} \subseteq \mathfrak{M}$ the theory $T \cup \mathrm{Diag}(\mathfrak{A})$ is complete

Definition 3.15. We call T **model complete** if for all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T

$$\mathfrak{M}^1 \subseteq \mathfrak{M}^2 \Rightarrow \mathfrak{M}^1 \prec \mathfrak{M}^2$$

T is model complete iff for any $\mathfrak{M} \models T$ the theory $T \cup \mathrm{Diag}(\mathfrak{M})$ is complete

Note that if $\mathfrak{M}_1 \models \mathrm{Diag}(\mathfrak{M})$, then there is an embedding $h : \mathfrak{M} \to \mathfrak{M}_1$ and $\mathfrak{M}_1$ is isomorphic to an extension $\mathfrak{M}_1'$ of $\mathfrak{M}$. Then we have $\mathfrak{M} \subseteq \mathfrak{M}_1'$.

So here we are actually saying that all embeddings are elementary

Lemma 3.16 (Robinson's Test)**.** *Let T be a theory. Then the following are equivalent*

1. *T is model complete*
2. *For all models $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ of T and all existential sentences φ from $L(M^1)$*

$$\mathfrak{M}^2 \models \varphi \Rightarrow \mathfrak{M}^1 \models \varphi$$

3. *Each formula is, modulo T, equivalent to a universal formula*

Proof. $1 \leftrightarrow 3$. Corollary 3.5

(2) and Corollary 3.5 shows that all existential sentences are, modulo T, equivalent to a universal sentence. Then by induction we can show 3. (Details) $\square$

If $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ satisfies (2), we call $\mathfrak{M}^1$ **existentially closed** in $\mathfrak{M}^2$. We denote this by

$$\mathfrak{M}^1 \prec_1 \mathfrak{M}^2$$

Definition 3.17. Let T be a theory. A theory T^* is a **model companion** of T if the following three conditions are satisfied

1. Each model of T can be extended to a model of T^*
2. Each model of T^* can be extended to a model of T
3. T^* is model complete

Theorem 3.18. *A theory T has, up to equivalence, at most one model companion T^**

Proof. If T^+ is another model companion of T, every model of T^+ is contained in a model of T^* and conversely. Let $\mathfrak{A}_0 \models T^+$. Then $\mathfrak{A}_0$ can be embedded in a model $\mathfrak{B}_0$ of T^*. In turn $\mathfrak{B}_0$ is contained in a model $\mathfrak{A}_1$ of T^+. In this way we find two elementary chains $(\mathfrak{A}_i)$ and $(\mathfrak{B}_i)$, which have a common union $\mathfrak{C}$. Then $\mathfrak{A}_0 \prec \mathfrak{C}$ and $\mathfrak{B}_0 \prec \mathfrak{C}$ implies $\mathfrak{A}_0 \equiv \mathfrak{B}_0$ since T are all sentences. Thus $\mathfrak{A}_0$ is a model of T^* $\square$

Existentially closed structures and the Kaiser hull

Let T be an L-theory. It follows from 3.3 that the models of $T_\forall = \{\varphi \mid T \models \varphi$ where φ is universal$\}$ are the substructures of models of T. The conditions (1) and (2) in the definition of "model companion" can therefore be expressed as

$$T_\forall = T^*_\forall$$

(1 and 2 says $\mathrm{Mod}(T_\forall) = \mathrm{Mod}(T^*_\forall)$) Hence the model companion of a theory T depends only on $T_\forall$. (Note that $T_\forall$ is model complete)

Definition 3.19. An L-structure $\mathfrak{A}$ is called T**-existentiallay closed** (or T**-ec**) if

1. $\mathfrak{A}$ can be embedded in a model of T
2. $\mathfrak{A}$ is existentially closed in every extension which is a model of T

A structure $\mathfrak{A}$ is T-ec exactly if it is $T_\forall$-ec. Since every model of $\mathfrak{B}$ of $T_\forall$ can be embedded in a model $\mathfrak{M}$ of T and $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$ and $\mathfrak{A} \prec_1 \mathfrak{M}$ implies $\mathfrak{A} \prec_1 \mathfrak{B}$

Lemma 3.20. *Every model of a theory T can be embedded in a T-ec structure*

Proof. Let $\mathfrak{A}$ be a model of $T_\forall$. We choose an enumeration $(\varphi_\alpha)_{\alpha<\kappa}$ of all existential $L(A)$-sentences and construct an ascending chain $(\mathfrak{A}_\alpha)_{\alpha\leq\kappa}$ of models of $T_\forall$. We begin with $\mathfrak{A}_0 = \mathfrak{A}$. Let $\mathfrak{A}_\alpha$ be constructed. If φ_α holds in an extension of $\mathfrak{A}_\alpha$ which is a model of T we let $\mathfrak{A}_{\alpha+1}$ be such a model. Otherwise we set $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$. For limit ordinals λ we define $\mathfrak{A}_\lambda$ to be the union of all $\mathfrak{A}_\alpha$. $\mathfrak{A}_\lambda$ is again a model of $T_\forall$

The structure $\mathfrak{A}^1 = \mathfrak{A}_\kappa$ has the following property: every existential $L(A)$-sentence which holds in an extension of $\mathfrak{A}^1$ that is a model of T holds in $\mathfrak{A}^1$. Now in the same manner, we construct $\mathfrak{A}^2$ from $\mathfrak{A}^1$, etc. The union $\mathfrak{M}$ of the chain $\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \ldots$ is the desired T-ec structure □

Every elementary substructure $\mathfrak{N}$ of a T-ec structure $\mathfrak{M}$ is again T-ec: Let $\mathfrak{N} \subseteq \mathfrak{A}$ be a model of T. Since $\mathfrak{M}_N \Rightarrow_\exists \mathfrak{A}_N$, there is an embedding of $\mathfrak{M}$ in an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ which is the identity on N. Since $\mathfrak{M}$ is existentially closed in $\mathfrak{B}$, it follows that $\mathfrak{N}$ is existentially closed in $\mathfrak{B}$ and therefore also in $\mathfrak{A}$

$$\begin{tikzcd}
& \mathfrak{B} & \\
\mathfrak{A} \arrow[ur, "\prec"] & & \mathfrak{M} \arrow[ul, "\prec_1"'] \\
& \mathfrak{N} \arrow[ul, "\prec_1"] \arrow[ur, "\prec"'] &
\end{tikzcd}$$

Lemma 3.21. *Let T be a theory. Then there is a biggest inductive theory T^{KH} with $T_\forall = T^{\mathrm{KH}}_\forall$. We call T^{KH} the* ***Kaiser hull*** *of T*

Proof. Let T^1 and T^2 be two inductive theories with $T^1_\forall = T^2_\forall = T_\forall$. We have to show that $(T^1 \cup T^2)_\forall = T_\forall$. Note that for every model $\mathfrak{A} \models T^1$ and $\mathfrak{B} \models T^2$ we have $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$ and vice versa. Then we have the embeddings just like model companions. Let $\mathfrak{M}$ be a model of T, as in the proof of 3.18 we extend $\mathfrak{M}$ by a chain $\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \cdots$ of models of T^1 and T^2. The union of this chain is a model of $T^1 \cup T^2$ $\square$

Lemma 3.22. *The Kaiser hull T^{KH} is the $\forall\exists$-part of the theory of all T-ec structures*

Proof. Let T^* be the $\forall\exists$-part of the theory of all T-ec structures. Since T-ec structures are models of $T_\forall$, we have $T_\forall \subseteq T^*_\forall$. It follows from 3.20 that $T^*_\forall \subseteq T_\forall$. Hence T^* is contained in the Kaiser Hull.

It remains to show that every T-ec structure $\mathfrak{M}$ is a model of the Kaiser hull. Choose a model $\mathfrak{N}$ of T^{KH} which contains $\mathfrak{M}$. Then $\mathfrak{M} \prec_1 \mathfrak{N}$. This implies $\mathfrak{N} \Rightarrow_{\forall\exists} \mathfrak{M}$ and therefore $\mathfrak{M} \models T^{KH}$ $\square$

This implies that T-ec strctures are models of $T_{\forall\exists}$

Theorem 3.23. *For any theory T the following are equivalent*

1. *T has a model companion T^**
2. *All models of K^{KH} are T-ec*
3. *The T-ec structures form an elementary class.*

If T^* exists, we have

$$T^* = T^{\mathrm{KH}} = \text{ theory of all } T\text{-ec structures}$$

Exercise 3.2.1. Let L be the language containing a unary function f and a binary relation symbol R and consider the L-theory $T = \{\forall x \forall y (R(x,y) \to (R(x, f(y))))\}$. Showing the follow

1. For any T-structure $\mathfrak{M}$ and $a, b \in M$ with $b \notin \{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ we have $\mathfrak{M} \models \exists z(R(z,a) \land \neg R(z,b))$

2. Let $\mathfrak{M}$ be a model of T and a an element of M s.t. $\{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ is infinite. Then in an elementary extension $\mathfrak{M}'$ there is an element b with $\mathfrak{M}' \models \forall z(R(z,a) \to R(z,b))$

3. The class of T-ec structures is not elementary, so T does not have a model companion

Exercise 3.2.2. A theory T with quantifier elimination is axiomatisable by sentences of the form

$$\forall x_1 \dots x_n \psi$$

where ψ is primitive existential formula

3.3 Examples

Infinite sets. The models of the theory Infset of **infinite sets** are all infinite sets without additional structure. The language $L_\emptyset$ is empty, the axioms are (for $n = 1, 2, \dots$)

- $\exists x_0 \dots x_{n-1} \bigwedge_{i<j<n} \neg x_i \doteq x_j$

Theorem 3.24. *The theory Infset of infinite sets has quantifier elimination and is complete*

Proof. Since the language is empty, the only basic formula is $x_i = x_j$ and $\neg(x_i = x_j)$. By Lemma 3.13 we only need to consider primitive existential formulas. □

Dense linear orderings.

$$\begin{aligned}
&\forall a, b(a \leq b \land b \leq a \to a \doteq b) \\
&\forall a, b, c(a \leq b \land b \leq c \to a \leq c) \\
&\forall a, b(a \leq b \lor b \leq a) \\
&\forall a, b \exists c(a < b \to a < c < b)
\end{aligned}$$

Theorem 3.25. *DLO has quantifier elimination*

Proof. Let A be a finite common substructure of the two models O_1 and O_2. We choose an ascending enumeration $A = \{a_1, \dots, a_n\}$. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence, which is true in O_1 and assume $O_1 \models \rho(b_1)$. We want to extend the order preserving map $a_i \mapsto a_i$ to an order preserving map $A \cup \{b_1\} \to O_2$. For this we have an image b_2 of b_1. There are four cases

1. $b_1 \in A$, we set $b_2 = b_1$
2. $b_1 \in (a_i, a_{i+1})$. We choose b_2 in O_2 with the same property
3. b_1 is smaller than all elements of A. We choose a $b_2 \in O_2$ of the same kind
4. b_1 is bigger than all a_i. Choose b_2 in the same manner

This defines an isomorphism $A \cup \{b_1\} \to A \cup \{b_2\}$, which show that $O_2 \vDash \rho(b_2)$ □

Modules. Let R be a (possibly non-commutative) ring with 1. An R-module

$$\mathfrak{M} = (,0,+,-,r)_{r \in R}$$

is an abelian group $(M, 0, +, -)$ together with operations $r : M \to M$ for every ring element $r \in R$. We formulate the axioms in the language $L_{Mod}(R) = L_{AbG} \cup \{r \mid r \in R\}$. The theory $\mathsf{Mod}(R)$ of R-modules consists of

$$\begin{aligned}
&\mathsf{AbG}\\
&\forall x, y\ r(x+y) \doteq rx + ry\\
&\forall x\ (r+s)x \doteq rx + sx\\
&\forall x\ (rs)x \doteq r(sx)\\
&\forall x\ 1x \doteq x
\end{aligned}$$

for all $r, s \in R$. Then $\mathsf{Infset} \cup \mathsf{Mod}(R)$ is the theory of all infinite R-modules

A module over fields is a vector space

Theorem 3.26. *Let K be a field. Then the theory of all infinite K-vector spaces has quantifier elimination and is complete*

Proof. Let A be a common finitely generated substructure (i.e., a subspace) of the two infinite K-vector spaces V_1 and V_2. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence which holds in V_1. Choose a b_1 from V_1 which satisfies $\rho(y)$. If b_1 belongs to A, we finished. If not, we choose a $b_2 \in V_2 \setminus A$. Possibly we have to replace V_2 by an elementary extension. The vector spaces $A + Kb_1$ and $A + Kb_2$ are isomorphic by an isomophism which maps b_1 to b_2 and fixes A elementwise. Hence $V_2 \vDash \rho(b_2)$

The theory is complete since a quantifier-free sentence is true in a vector space iff it is true in the zero-vector space. □

Definition 3.27. An **equation** is an $L_{Mod}(R)$-formula $\gamma(\bar{x})$ of the form

$$r_1 x_1 + \cdots + r_m x_m = 0$$

A **positive primitive** formula (**pp**-formula) is of the form

$$\exists \bar{y}(\gamma_1 \wedge \cdots \wedge \gamma_n)$$

where the $\gamma_i(\overline{xy})$ are equations

Theorem 3.28. *For every ring R and any R-module M, every $L_{Mod}(R)$-formula is equivalent (modulo the theory of M) to a Boolean combination of positive primitive formulas*

Algebraically closed fields.

Theorem 3.29 (Tarski)**.** *The theory ACF of algebraically closed fields has quantifier elimination*

Proof. Let K_1 and K_2 be two algebraically closed fields and R a common subring. Let $\exists y \rho(y)$ be a simple existential sentence with parameters in R which hold in K_1. We have to show that $\exists y \rho(y)$ is also true in K_2.

Let F_1 and F_2 be the quotient fields of R in K_1 and K_2, and let $f : F_1 \to F_2$ be an isomorphism which is the identity on R. Then f extends to an isomorphism $g : G_1 \to G_2$ between the relative algebraic closures G_i of F_i in K_i. $\square$

4 Countable Models

4.1 The omitting types theorem

Definition 4.1. Let T be an L-theory and $\Sigma(x)$ a set of L-formulas. A model $\mathfrak{A}$ of T not realizing $\Sigma(x)$ is said to **omit** $\Sigma(x)$. A formula $\varphi(x)$ **isolates** $\Sigma(x)$ if

1. $\varphi(x)$ is consistent with T
2. $T \models \forall x(\varphi(x) \to \sigma(x))$ for all $\sigma(x) \in \Sigma(x)$

A set of formulas is often called a **partial type**.

Theorem 4.2 (Omitting Types)**.** *If T is countable and consistent and if $\Sigma(x)$ is not isolated in T, then T has a model which omits $\Sigma(x)$*

If $\Sigma(x)$ is isolated by $\varphi(x)$ and $\mathfrak{A}$ is a model of T, then $\Sigma(x)$ is realised in $\mathfrak{A}$ by all realisations $\varphi(x)$. Therefore the converse of the theorem is true for **complete** theories T: if $\Sigma(x)$ is isolated in T, then it is realised in every model of T

Proof. We choose a countable set C of new constants and extend T to a theory T^* with the following properties

1. T^* is a Henkin theory: for all $L(C)$-formulas $\psi(x)$ there exists a constant $c \in C$ with $\exists x \psi(x) \to \psi(c) \in T^*$

2. for all $c \in C$ there is a $\sigma(x) \in \Sigma(x)$ with $\neg\sigma(c) \in T^*$

We construct T^* inductively as the union of an ascending chain

$$T = T_0 \subseteq T_1 \subseteq T_1 \subseteq \ldots$$

of consistent extensions of T by finitely many axioms from $L(C)$, in each step making an instance of (1) or (2) true.

Enumerate $C = \{c_i \mid i < \omega\}$ and let $\{\psi_i(x) \mid i < \omega\}$ be an enumeration of the $L(C)$-formulas

Assume that T_{2i} is the already constructed. Choose some $c \in C$ which doesn't occur in $T_{2i} \cup \{\psi_i(x)\}$ and set $T_{2i+1} = T_{2i} \cup \{\exists x \psi_i(x) \to \psi_i(c)\}$.

Up to equivalence T_{2i+1} has the form $T \cup \{\delta(c_i, \bar{c})\}$ for an L-formula $\delta(x, \bar{y})$ and a tuple $\bar{c} \in C$ which doesn't contain c_i. Since $\exists \bar{y} \delta(x, \bar{y})$ doesn't isolate $\Sigma(x)$, for some $\sigma \in \Sigma$ the formula $\exists \bar{y} \delta(x, \bar{y}) \wedge \neg\sigma(x)$ is consistent with T. Thus $T_{2i+2} = T_{2i+1} \cup \{\neg\sigma(c_i)\}$ is consistent

Take a model $(\mathfrak{A}', a_c)_{c \in C}$ of T^*. Since T^* is a Henkin theory, Tarski's Test 2.2 shows that $A = \{a_c \mid c \in C\}$ is the universe of an elementary substructure $\mathfrak{A}$ (Lemma 2.7). By property (2), $\Sigma(x)$ is omitted in $\mathfrak{A}$ □

Corollary 4.3. *Let T be countable and consistent and let*

$$\Sigma_0(x_0, \ldots, x_{n_0}), \Sigma_1(x_1, \ldots, x_{n_1}), \ldots$$

be a sequence of partial types. If all Σ_i are not isolated, then T has a model which omits all Σ_i

Proof. If $\Sigma_0(x), \Sigma_1(x), \ldots$. Then $T_{2i+2} = T_{2i+1} \cup \{\neg\sigma_m(c_{mn})\}$

If $\Sigma(x_1, \ldots, x_n)$, then $T_{2i+1} = T_{2i} \cup \{\exists \bar{x} \psi_i(\bar{x}) \to \psi_i(\bar{c})\}$.

Combine the two case □

4.2 The space of types

Fix a theory T. An n**-type** is a maximal set of formulas $p(x_1, \dots, x_n)$ consistent with T. We denote by $S_n(T)$ the set of all n-types of T. We also write $S(T)$ for $S_1(T)$. $S_0(T)$ is all complete extensions of T

If B is a subset of an L-structure $\mathfrak{A}$, we recover $S_n^{\mathfrak{A}}(B)$ as $S_n(\mathrm{Th}(\mathfrak{A}_B))$. In particular, if T is complete and $\mathfrak{A}$ is any model of T, we have $S^{\mathfrak{A}}(\emptyset) = S(T)$

For any L-formula $\varphi(x_1, \dots, x_n)$, let $[\varphi]$ denote the set of all types containing φ.

Lemma 4.4. *1. $[\varphi] = [\psi]$ iff φ and ψ are equivalent modulo T*

2. The sets $[\varphi]$ are closed under Boolean operations. In fact $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, $[\varphi] \cup [\psi] = [\varphi \vee \psi]$, $S_n(T) \setminus [\varphi] = [\neg\varphi]$, $S_n(T) = [\top]$ and $\emptyset = [\bot]$

It follows that the collection of sets of the form $[\varphi]$ is closed under finite intersection and includes $S_n(T)$. So these sets form a basis of a topology on $S_n(T)$

Lemma 4.5. *The space $S_n(T)$ is 0-dimensional and compact*

Proof. Being 0-dimensional means having a basis of clopen sets. Our basic open sets are clopen since their complements are also basic open

If p and q are two different types, there is a formula φ contained in p but not in q. It follows that $[\varphi]$ and $[\neg\varphi]$ are open sets which separate p and q. This shows that $S_n(T)$ is Hausdorff

To prove compactness, we need to show that any collection of closed subsets of X with the finite intersection property has nonempty intersection. Could check this

Consider a family $[\varphi_i]$ $(i \in I)$, with the finite intersection property.This means that $\varphi_{i_i} \wedge \dots \wedge \varphi_{i_k}$ are consistent with T. So Corollary 2.10 $\{\varphi_i \mid i \in I\}$ is consistent with T and can be extended to a type p, which then belongs to all $[\varphi_i]$. □

Lemma 4.6. *All clopen subsets of $S_n(T)$ has the form $[\varphi]$*

Proof. It follows from Exercise 3.1.1 that we can separate any two disjoint closed subsets of $S_n(T)$ by a basic open set. □

The Stone duality theorem asserts that the map

$$X \mapsto \{C \mid C \text{ clopen subset of } X\}$$

yields an equivalence between the category of 0-dimensional compact spaces and the category of Boolean algebras. The inverse map assigns to every Boolean algebra to its **Stone space**

Definition 4.7. A map f from a subset of a structure $\mathfrak{A}$ to a structure $\mathfrak{B}$ is **elementary** if it preserves the truth of formulas; i.e., $f : A_0 \to B$ is elementary if for every formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} \in A_0$ we have

$$\mathfrak{A} \vDash \varphi(\bar{a}) \Rightarrow \mathfrak{B} \vDash \varphi(f(\bar{a}))$$

Lemma 4.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures, A_0 and B_0 subsets of A and B, respectively. Any elementary map $A_0 \to B_0$ induces a continuous surjective map $S_n(B_0) \to S_n(A_0)$*

Proof. If $q(\bar{x}) \in S_n(B_0)$, we define

$$S(f)(q) = \{\varphi(x_1, \dots, x_n, \bar{a}) \mid \bar{a} \in A_0, \varphi(x_1, \dots, x_n, f(\bar{a})) \in q(\bar{x})\}$$

If $\varphi(\bar{x}, f(\bar{a})) \notin q(\bar{x})$, then $\mathfrak{B} \nvDash \varphi(\bar{x}, \bar{a})$. Therefore $\mathfrak{A} \nvDash \varphi(\bar{x}, \bar{a})$. $S(f)$ defines a map from $S_n(B_0)$ to $S_n(A_0)$. Moreover, it is surjective since $\{\varphi(x_1, \dots, x_n, f(\bar{a})) \mid \varphi(x_1, \dots, x_n, a) \in p\}$ is finitely satisfiable for all $p \in S_n(A_0)$. And $S(f)$ is continuous since $[\varphi[x_1, \dots, x_n, f(\bar{a})]]$ is the preimage of $[\varphi(x_1, \dots, x_n, \bar{a})]$ under $S(f)$ □

There are two main cases

1. An elementary bijection $f : A_0 \to B_0$ defines a homeomorphism $S_n(A_0) \to S_n(B_0)$. We write $f(p)$ for the image of p
2. If $\mathfrak{A} = \mathfrak{B}$ and $A_0 \subseteq B_0$, the inclusion map induces the **restriction** $S_n(B_0) \to S_n(A_0)$. We write $q \restriction A_0$ for the restriction of q to A_0. We call q an extension of $q \restriction A_0$)

   Lemma 4.9. *A type p is isolated in T iff p is an isolated point in $S_n(T)$. In fact, φ isolates p iff $[\varphi] = \{p\}$. That is, $[\varphi]$ is an* ***atom*** *in the Boolean algebra of clopen subsets of $S_n(T)$*

   Proof. If φ isolates p. Then $\varphi \in p$ and hence $[\varphi] = \{\varphi\}$.

   If $[\varphi] = \{p\}$, then $\varphi \in p$. What's more, $\mathfrak{M} \vDash \varphi \Leftrightarrow \mathfrak{M} \vDash p$ in T

   The set $[\varphi]$ is a singleton iff $[\varphi]$ is non-empty and cannot be divided into two non-empty clopen subsets $[\varphi \wedge \psi]$ and $\varphi \wedge \neg\psi$. This means that for all ψ either ψ or $\neg\psi$ follows from φ modulo T. So $[\varphi]$ is a singleton iff φ generates the type

   $$\langle\varphi\rangle = \{\psi(\bar{x}) \mid T \vDash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$$

   □

We call a formula $\varphi(x)$ **complete** if

$$\{\psi(\bar{x}) \mid T \vDash \forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))\}$$

is a type.

Corollary 4.10. *A formula isolates a type iff it is complete*

Exercise 4.2.1.

1. Closed subsets of $S_n(T)$ have the form $\{p \in S_n(T) \mid \Sigma \subseteq p\}$, where Σ is any set of formulas
2. Let T be countable and consistent. Then any meagre[1] subset X of $S_n(T)$ can be omitted, i.e., there is a model which omits all $p \in X$

Proof.

1. The sets $[\varphi]$ are a basis for the closed subsets of $S_n(T)$. So the closed sets of $S_n(T)$ are exactly the intersections $\bigcap_{\varphi \in \Sigma}[\varphi] = \{p \in S_n(T) \mid \Sigma \subseteq p\}$
2. The set X is the union of a sequence of countable nowhere dense sets X_i. We may assume that X_i are closed, i.e., of the form $\{p \in S_n(T) \mid \Sigma_i \subseteq p\}$. That X_i has no interior means that Σ_i is not isolated. The claim follows now from Corollary 4.3

□

Exercise 4.2.2. Consider the space $S_\omega(T)$ of all complete types in variables $v_0, v_1, \ldots$. Note that $S_\omega(T)$ is again a compact space and therefore not meagre by Baire's theorem

1. Show that $\{\mathrm{tp}(a_0, a_1, \dots) \mid$ the a_i enumerate a model of $T\}$ is comeagre in $S_\omega(T)$

4.3 $\aleph_0$-categorical theories

Theorem 4.11. *Let T be a countable complete theory. Then T is $\aleph_0$-categorical iff for every n there are only finitely many formulas $\varphi(x_1, \dots, x_n)$ up to equivalence relative to T*

Definition 4.12. An L-structure $\mathfrak{A}$ is **ω-saturated** if all types over finite subsets of A are realised in $\mathfrak{A}$

[1] A subset of a topological space is **nowhere dense** if its closure has no interior. A countable union of nowhere dense sets is meagre

The types in the definition are meant to be 1-types. On the other hand, it is not hard to see that an ω-saturated structure realises all n-types over finite sets (Exercise **??**) for all $n \geq 1$. The following lemma is a generalisation of the $\aleph_0$-categoricity of DLO.

Lemma 4.13. *Two elementarily equivalent, countable and ω-saturated structures are isomorphic*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are as in the lemma. We choose enumerations $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$. Then we construct an ascending sequence $f_0 \subseteq f_1 \subseteq \cdots$ of finite elementary maps

$$f_i : A_i \to B_i$$

between finite subsets of $\mathfrak{A}$ and $\mathfrak{B}$. We will choose the f_i in such a way that A is the union of the A_i and B the union of the B_i. The union of the f_i is then the desired isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$

The empty map $f_0 = \emptyset$ is elementary since $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. Assume that f_i is already constructed. There are two cases:

$i = 2n$; We will extend f_i to $A_{i+1} = A_i \cup \{a_n\}$. Consider the type

$$p(x) = \mathrm{tp}(a_n/A_i) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a_n), \varphi(x) \text{ a } L(A_i)\text{-formula}\}$$

Since f_i is elemantarily, $f_i(p)(x)$ is in $\mathfrak{B}$ a type over B_i. Since $\mathfrak{B}$ is ω-saturated, there is a realisation b' of this type. So for $\bar{a} \in A_i$

$$\mathfrak{A} \models \varphi(a_n, \bar{a}) \Rightarrow \mathfrak{B} \models \varphi(b', f_i(\bar{a}))$$

This shows that $f_{i+1}(a_n) = b'$ defines an elementary extension of f_i

$i = 2n + 1$; we exchange $\mathfrak{A}$ and $\mathfrak{B}$ □

Proof of Theorem 4.11. Assume that there are only finitely many $\varphi(x_1, \dots, x_n)$ relative to T for every n. By Lemma 4.13 it suffices to show that all models of T are ω-saturated. Let $\mathfrak{M}$ be a model of T and A an n-element subset. If there are only N many formulas, up to equivalence, in the variable $x_1, \dots, x_{n+1}$, there are, up to equivalence in $\mathfrak{M}$, at most N many $L(A)$-formulas $\varphi(x)$. Thus, each type $\varphi(x) \in S(A)$ is isolated (w.r.t. $\mathrm{Th}(\mathfrak{M}_A)$) by a smallest formula $\varphi_p(x)$ (obviously conjunction). Each element of M which realises $\varphi_p(x)$ also realises $p(x)$, so $\mathfrak{M}$ is ω-saturated.

Conversely, if there are infinitely many $\varphi(x_1, \dots, x_n)$ modulo T for some n, then - as the type space $S_n(T)$ is compact - there must be some non-isolated type p. By the Omitting Types Theorem there is a countable model of T in which this type is not realised. On the other hand, there also exists a countable model of T realizing this type. So T is not $\aleph_0$-categorical □

The proof shows that a countable complete theory with infinite models is $\aleph_0$-categorical iff all countable models are ω-saturated

Definition 4.14. An L-structure $\mathfrak{M}$ is ω**-homogeneous** if for every elementary map f_0 defined on a finite subset A of M and for any $a \in M$ there is some element $b \in M$ s.t.

$$f = f_0 \cup \{\langle a, b\rangle\}$$

is elementary

$f = f_0 \cup \{\langle a, b\rangle\}$ is elementary iff b realises $f_0(\mathrm{tp}(a/A))$

Corollary 4.15. *Let $\mathfrak{A}$ be a structure and $a_1, \dots, a_n$ elements of $\mathfrak{A}$. Then* $\mathrm{Th}(\mathfrak{A})$ *is $\aleph_0$-categorical iff* $\mathrm{Th}(\mathfrak{A}, a_1, \dots, a_n)$ *is $\aleph_0$-categorical*

Example 4.1. The following theories and $\aleph_0$-categorical

1. Infset (saturated)
2. For every finite field $\mathbb{F}_q$, the theory of infinite $\mathbb{F}_q$-vector spaces. (Vector spaces over the same field and of the same dimension are isomorphic)
3. The theory DLO of dense linear orders without endpoints. This follows from Theorem 4.11 since DLO has quantifier elimination: for every n there are only finitely many (say N_n) ways to order n elements. Each of these possibility corresponds to a complete formula $\psi(x_1, \dots, x_n)$. Hence there are up to equivalence, exactly 2^{N_n} many formulas $\varphi(x_1, \dots, x_n)$

Definition 4.16. A theory T is **small** if $S_n(T)$ are at most countable for all $n < \omega$

Lemma 4.17. *A countable complete theory is small iff it has a countable ω-saturated model*

Proof. If T has a finite model $\mathfrak{A}$, T is small and $\mathfrak{A}$ is ω-saturated (countable assignment). So we may assume that T has infinite models $\square$

5 TODO Don't understand

Lemma 3.22

Exercise 3.2.2

theorem 4.11 need to enhance my TOPOLOGY and ALGEBRA!!!