Week5

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Exercise 1. Show that $a_1, a_2, \ldots$ is not totally indiscenible

Proof. If $a_i = a_j$ and $i < j$, then since $a_i a_j \equiv a_m a_n$ for any $m < n$, $a_1, a_2, \ldots$ is a constant sequence. Because $a_1, a_2, \ldots$ is a non-constant indiscenible sequence, either $a_1 < a_2$ or $a_1 > a_2$. We may assume $a_1 < a_2$. Then $a_1 a_2 \not\equiv a_2 a_1$ since $x < y \in \mathrm{tp}(a_1 a_2)$ but $x > y \in \mathrm{tp}(a_2 a_1)$ □

Exercise 2. Show that $a_1 a_2 > 0$

Proof. If $a_i = 0$, then $x = 0 \in \mathrm{tp}(a_i)$. But since $a_i \equiv a_j$ for any j, $a_1, a_2, \ldots$ is a constant sequence, a contradiction.

If $a_1 a_2 < 0$, then $a_2 a_3 < 0$ and so $a_1 a_2^2 a_3 > 0$ which implies $a_1 a_3 > 0$. But $a_1 a_2 \equiv a_1 a_3$, we get a contradiction. Hence $a_1 a_2 > 0$ □

Exercise 3. Suppose $a_2 - a_1 \geq 1$. Show that $a_2 - a_1 \geq 7$

Proof. we have $a_8 - a_7 \geq 1, a_7 - a_6 \geq 1, \ldots, a_2 - a_1 \geq 1$, and so $a_8 - a_1 \geq 7$. Hence $a_2 - a_1 \geq 7$ □

Exercise 4. Show that at least one of the following is true: $a_2 < (1.01) \cdot a_1$ or $a_2 > 200 \cdot a_1$

Proof. Assume $a_2 \geq (1.01) \cdot a_1$ and$a_2 \leq 200 \cdot a_1$.

Claim: $a_{2n} \geq (1.01)^{2n-1} a_1$

If $a_{2n} \geq (1.01)^{2n-1} a_1$, then $a_{2n+2} \geq (1.01) \cdot a_{2n+1}$, $a_{2n+1} \geq (1.01) \cdot a_{2n}$, and so $a_{2n+2} a_{2n+1} a_{2n} \geq (1.01)^{2n+1} a_1 a_{2n} a_{2n+1}$. Since $a_{2n+1} a_{2n} > 0$, $a_{2n+2} \geq (1.01)^{2n+1} a_1$.

Hence if we take N large enough, then $a_{2N} \geq (1.01)^{2N-1} a_1 > 200 \cdot a_1$. Then by indiscernibility, $a_2 > 200 \cdot a_1$, a contradiction □

Exercise 5. Show that $a_i + a_j \neq a_k$ for any i, j, k

Proof. Without loss of generality, we may assume that $\{a_i, a_j, a_k\} = \{a_1, a_2, a_3\}$ and $a_1 > 0$

1. If $a_1 + a_2 = a_3$. Then $a_3 - a_2 = a_1$ and there is $q \in \mathbb{Q}$ s.t. $q \leq a_1 < 1+q$ where $q > 0$. Then $a_3 - a_2 \geq q$. Take $N = \ulcorner \frac{1+q}{q} \urcorner$, since $a_{N+2} - a_{N+1} \geq q, a_{N+1} - a_N \geq q, a \ldots, a_3 - a_2 \geq q$, we have $a_{N+2} - a_2 \geq Nq \geq 1 + q > a_1$. Hence $a_3 - a_2 > a_1$, a contradiction

2. If $a_1 + a_3 = a_2$, take $a_2 - a_1 = a_3$ and we can prove similarly

3. If $a_2 + a_3 = a_1$. Then $a_2 - a_1 = -a_3$ and there $q \in \mathbb{Q}$ s.t. $-1 - q < -a_3 \leq -q$ where $q > 0$. Similarly we can prove that $a_2 - a_1 < -a_3$.

Therefore $a_i + a_j \neq a_k$ $\square$

Exercise 6. Show that there is an indiscenible sequence $b_1, b_2, b_3, \ldots$ s.t. $b_2 > 200 \cdot b_1$

Proof. Let $(a_i : i \in \mathbb{N} \setminus \{0\})$ be an infinite sequence s.t. $a_i = 201^i$, $i \in \mathbb{N}$. Then by Theorem 10 in the notes, there is an indiscenible sequence $(b_j : j \in \mathbb{N} \setminus \{0\})$ with $\mathrm{tp}^{EM}(\bar{b}) = \mathrm{tp}^{EM}(\bar{a})$. Since in $\bar{a}$, for any $j > i$, $a_j > 200 \cdot a_i$, therefore $b_m > 200 \cdot b_n$ for any $m > n$. Particularly, $b_2 > 200 \cdot b_1$. $\square$