

Homework1

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Exercise 1 (0). Show that $[a]_{\sim} = [b]_{\sim}$ if and only if $a \sim b$, and that $[a]_{\sim} \cap [b]_{\sim} = \emptyset$ if $a \not\sim b$.

Proof. If $[a]_{\sim} = [b]_{\sim}$, then for all $x \in [a]_{\sim}$, $x \in [b]_{\sim}$. As $a \in [a]_{\sim}$, $a \in [b]_{\sim}$, that is, $a \sim b$.

If $a \sim b$, then for all $x \in [a]_{\sim}$, as $x \sim a$ and $a \sim b$, then $x \sim b$ and hence $x \in [b]_{\sim}$. Thus $[a]_{\sim} \subset [b]_{\sim}$. Similarly, $[b]_{\sim} \subset [a]_{\sim}$. Hence $[a]_{\sim} = [b]_{\sim}$.

If $[a]_{\sim} \cap [b]_{\sim} = \emptyset$ and suppose $a \sim b$. Then $[a]_{\sim} = [b]_{\sim}$. As $a \in [a]_{\sim}$, $[a]_{\sim}$ is not empty and hence $[a]_{\sim} \cap [b]_{\sim} \neq \emptyset$, a contradiction.

If $a \not\sim b$ and suppose $[a]_{\sim} \cap [b]_{\sim} \neq \emptyset$, let $x \in [a]_{\sim} \cap [b]_{\sim}$. Then $x \sim a$ and $x \sim b$, and so $a \sim b$, a contradiction \square

Exercise 2. Suppose a binary relation (E', \approx) is elementary equivalent to an equivalence relation (E, \sim) . Show that \approx is an equivalence relation

Proof. $S_{\omega}(E', E) \neq \emptyset$.

1. \approx is reflexive. For any $x' \in E'$, as $\emptyset \in S_{\omega}(E', E)$, there is a $x \in E$ such that $\{(x', x)\}$ is a local isomorphism from E' to E , which means that $x' \approx x'$ if and only if $x \sim x$. Since \sim is an equivalence relation, $x \sim x$, hence $x' \approx x'$.
2. \approx is symmetric. For any $x', y' \in E'$ and $x' \approx y'$, as $\emptyset \in S_{\omega}(E', E) \subset S_2(E', E)$, we have a local isomorphism $\{(x', x), (y', y)\}$ from E' onto E . Then $x \sim y$ and hence $y \sim x$ as \sim is symmetric. Consequently $y' \approx x'$.
3. \approx is transitive. For any $x', y', z' \in E'$, $x' \approx y'$ and $y' \approx z'$. As $\emptyset \in S_{\omega}(E', E) \subset S_3(E', E)$, we have a local isomorphism $\{(x', x), (y', y), (z', z)\}$ from E' onto E . Then $x \sim y$ and $y \sim z$ and hence $x \sim z$. Thus we have $x' \approx z'$.

□

Exercise 3. Show that \mathcal{K} is non-empty

Proof. Take an injective function $f : \omega \rightarrow E$, and we define S_n for all nonzero $n \in \omega$ as

$$S_n = \left\{ f(i) \mid i \in \omega \wedge \frac{n(n-1)}{2} \leq i < \frac{n(n+1)}{2} \right\}$$

and $S_\omega = E \setminus f(\omega)$. Then

$$S = \{S_\omega\} \cup \bigcup_{n \in \omega} \{S_n\}$$

forms a partition of E . And for each nonzero $n \in \omega$, $|S_n| = n$.

Let $\sim_S = \{(x, y) \in E^2 \mid \exists \alpha \in \omega + 1 (x \in S_\alpha \wedge y \in S_\alpha)\}$ defined by S . Then $\sim_S \in \mathcal{K}$ and hence \mathcal{K} is non-empty. □

Exercise 4. Suppose that $(E, \sim) \in \mathcal{K}$ and (E', \approx) is elementarily equivalent to (E, \sim) . Show that $(E', \approx) \in \mathcal{K}$.

Proof. From Exercise 2 we know that \approx is an equivalence relation on E' . For each nonzero $n \in \omega$, (E, \sim) has exactly one equivalence class of size n , denoted by $S_n = \{e_1, \dots, e_n\}$. As $\emptyset \in S_\omega(E, E') \subset S_{n+1}(E, E')$, we have a local isomorphism $s \in S_1(E, E')$ such that $s(e_i) = e'_i$ for all $i = 1, 2, \dots, n$ and $e'_i \in E'$. Let $S'_n = \{e'_1, \dots, e'_n\}$, then each pair of elements in S'_n is equivalent in the sense of \approx . If there is $e' \in E' \setminus S'_n$ such that $e' \approx e'_1$, then as s is a 1-isomorphism, there should be some other element e in E such that $e \sim e_1$, which is impossible. Hence S'_n is an equivalence class in E' of size n .

If there is another equivalence class S''_n of size n in E' . As $\emptyset \in S_\omega(E', E) \subset S_{2n+1}(E', E)$, we can construct a 1-isomorphism r with $\text{dom}(r) = S'_n \cup S''_n$. As S'_n and S''_n are two distinct equivalence class, $\text{im}(r)$ is also a union of two distinct equivalence class of size n . But (E, \sim) only has exactly one equivalence class of size n , we get a contradiction. Thus for $n = 1, 2, 3, \dots$, (E', \approx) has exactly one equivalence class of size n . Thus $(E', \approx) \in \mathcal{K}$ □

Exercise 5. Suppose that (E, \sim) and (E', \approx) are both in \mathcal{K} . Let s be a local isomorphism from (E, \sim) to (E', \approx) , and let $p \geq 0$. Suppose that for every $a \in \text{dom}(s)$, the \sim -equivalence class of a has the same size as the \approx -equivalence class of $s(a)$, or both equivalence classes have size greater than p . Then s is a p -isomorphism from (E, \sim) to (E', \cong)

Proof. We prove this by induction on p .

If $p = 0$, then s is a 0-isomorphism by definition

If $p = n + 1$. We now prove that s is a $n + 1$ -isomorphism.

For every $e \in E \setminus \text{dom}(s)$, we choose $e' \in E'$ for different cases:

1. if there is a $x \in E$ such that $e \sim x$, then as $|[x]_{\sim}| = |[s(x)]_{\approx}|$ or both $|[x]_{\sim}|$ and $|[s(x)]_{\approx}|$ are greater than $n + 1$, we are able to choose a $e' \in E' \setminus \text{im}(s)$ with $e' \sim s(x)$.
2. If there is no $x \in E$ with $e \sim x$ and $|[e]_{\sim}| = m \leq n$. If there is $y \in \text{im}(s)$ with $|[y]_{\approx}| = |[e]_{\sim}|$, then there will be a $n - m + 1$ -isomorphism $u \supset s$ maps a subset of $[s^{-1}(y)]_{\sim}$ onto $[y]_{\approx}$, otherwise there will be a contradiction if u is not onto. Also as $n - m + 1 \geq 1$, $[s^{-1}(y)]_{\sim} = [y]_{\approx}$, which leads to another contradiction. Hence there is no $y \in \text{im}(s)$ with $|[y]_{\approx}|$. Then we pick an e' from the equivalence class of size m from E' .
3. If there is no $x \in E$ with $e \sim x$ and $|[e]_{\sim}| > n$, then choose a element e' from a equivalence class $S'_{>n}$ of size greater than n from E' with $S'_{>n} \cap \text{im}(s) = \emptyset$.

Let $t = s \cup \{(e, e')\}$. Then t is a local isomorphism by our construction and the conditions in exercise are satisfied. Hence t is an n -isomorphism by induction. Thus forth condition is satisfied.

The back condition is similar.

Hence s is an $n + 1$ -isomorphism □

Exercise 6. Suppose that (E, \sim) and (E', \approx) are both in \mathcal{K} . Show that (E, \sim) is elementarily equivalent to (E', \approx)

Proof. We need to prove that for every $p \in \omega$, $S_p(E, E')$ is not empty. But for local isomorphism \emptyset , the condition in Exercise 5 are always satisfied and $\emptyset \in S_p(E, E')$. Hence $S_p(E, E')$ are not empty for all $p \in \omega$.

Thus (E, \sim) is elementarily equivalent to (E', \approx) . □

Exercise 7. Construct two equivalence relations (E, \sim) and (E', \approx) s.t. $(E, \sim) \sim_{\omega} (E', \approx)$, but $(E, \sim) \not\sim_{\infty} (E', \approx)$.

Proof. Let $E = \mathbb{Q}$, $E' = \mathbb{R}$, and $\sim = \{(x, y) \mid x - y \in \mathbb{Z}\}$. □