Morley sequences and the order property

Advanced Model Theory

March 17, 2022

Reference in the book: Sections 12.3 and 12.8 (VERY loosely).

1 Morley sequences

Fix a monster model $\mathbb{M}$ and a small set A. Let $p \in S_n(\mathbb{M})$ be an A-invariant type for some small A.

Definition 1. A *Morley sequence* of p over A is a sequence $\bar{b}_1, \bar{b}_2, \ldots$ where

$$\bar{b}_i \models p \upharpoonright A\bar{b}_1 \cdots \bar{b}_{i-1}.$$

For example, if p is the transcendental 1-type in a strongly minimal theory, then a Morley sequence of p over A is a sequence $b_1, b_2, \ldots \in \mathbb{M}$ such that $b_1 \notin \operatorname{acl}(A)$, $b_2 \notin \operatorname{acl}(Ab_1)$, $b_3 \notin \operatorname{acl}(Ab_2)$, ...

Definition 2. Let $(I, \leq)$ be an infinite linear order (often $\mathbb{N}$). Let $(\bar{b}_i : i \in I)$ be a sequence in $\mathbb{M}$. Then $(\bar{b}_i : i \in I)$ is *A-indiscernible* if for any n, any $i_1 < \cdots < i_n$ in I, any $j_1 < \cdots < j_n$ in I, we have

$$\bar{b}_{i_1} \cdots \bar{b}_{i_n} \equiv_A \bar{b}_{j_1} \cdots \bar{b}_{j_n}$$

In other words, any two subsequences of the same length have the same type over A.

Example. Taking $n = 1$ in the definition, $\bar{b}_i \equiv_A \bar{b}_j$ for any $i, j \in I$. All elements in the sequence have the same type.

Example. In DLO, if $b_1 < b_2 < \cdots$, then $(b_i : i < \omega)$ is indiscernible (over $\varnothing$). This is true because:

> If $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$, then $\operatorname{tp}(\bar{a}) = \operatorname{tp}(\bar{b})$ by quantifier elimination or back-and-forth methods.

Definition 3 (*Skipped in class*)**.** Let $(I, \leq)$ be an infinite set. Let $(\bar{b}_i : i \in I)$ be a sequence. Then $(\bar{b}_i : i \in I)$ is *totally indiscernible* if for any distinct $i_1, \ldots, i_n \in I$ and any distinct $j_1, \ldots, j_n \in I$,

$$\bar{b}_{i_1} \cdots \bar{b}_{i_n} \equiv_A \bar{b}_{j_1} \cdots \bar{b}_{j_n}$$

Example. If $(b_1, b_2, \ldots)$ is indiscernible, then $\mathrm{tp}(b_1b_2) = \mathrm{tp}(b_1b_3) = \mathrm{tp}(b_2b_3) = \cdots$, but $\mathrm{tp}(b_2b_1)$ could be different from $\mathrm{tp}(b_1b_2)$. But if the sequence is *totally* indiscernible, then $\mathrm{tp}(b_1b_2) = \mathrm{tp}(b_2b_1)$.

Theorem 4. *If $(\bar{b}_i : i < \omega)$ is a Morley sequence of p over A, then $(\bar{b}_i : i < \omega)$ is A-indiscernible.*

Proof. If $i_1 < \cdots < i_n$, then $\bar{b}_{i_j} \models p \restriction A\bar{b}_{i_1} \cdots \bar{b}_{i_{j-1}}$ for each j, and so $\mathrm{tp}(\bar{b}_{i_1}, \ldots, \bar{b}_{i_n}/A) = (\underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}) \restriction A$. This doesn't depend on the choice of $i_1, \ldots, i_n$. $\square$

2 The order property

Fix some complete theory T and monster model $\mathbb{M}$.

Definition 5. Let $\varphi(\bar{x}, \bar{y})$ be a formula. Then $\varphi(\bar{x}, \bar{y})$ has the *order property* if there are $(\bar{a}_i : i \in \mathbb{Z})$ and $(\bar{b}_i : i \in \mathbb{Z})$ such that

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

Example. In DLO, the formula $\varphi(x, y) = (x < y)$ has the order property: take $a_i = b_i = i$. Then $a_i < b_j \iff i < j$.

Remark 6. Let $\varphi^T(\bar{y}; \bar{x}) = \varphi(\bar{x}; \bar{y})$. If $\varphi(\bar{x}; \bar{y})$ has the order property, witnessed by $\bar{a}_i$ and $\bar{b}_j$, then

$$\mathbb{M} \models \neg\varphi(\bar{a}_{-i}, \bar{b}_{1-j}) \iff -i \geq 1 - j \iff i \leq j - 1 \iff i < j$$
$$\mathbb{M} \models \varphi^T(\bar{b}_{-i}, \bar{a}_{-j}) \iff \mathbb{M} \models \varphi(\bar{a}_{-j}, \bar{b}_{-i}) \iff -j < -i \iff i < j.$$

Therefore $\neg\varphi$ and φ^T have the order property.

3 Instability from the order property

Lemma 7. *For any cardinal $\lambda \geq \aleph_0$, there is a linear order $(I, <)$ and a subset $S \subseteq I$ such that $|S| \leq \lambda$, $|I| > \lambda$, and S is dense in I: if $a < b$ in I then there is $x \in S$ with $a \leq x \leq b$.*

Proof. From Lemma 9 in the March 3 notes, there is a cardinal μ such that $|2^\mu| > \lambda$ but $|2^{<\mu}| \leq \lambda$, where 2^μ is the set of binary strings of length μ and $2^{<\mu}$ is the set of binary strings of length strictly less than μ. Let $I = 2^\mu \cup 2^{<\mu}$ and let $S = 2^{<\mu}$. Order I lexicographically, by padding strings in $2^{<\mu}$ on the right with a symbol u such that $0 < u < 1$. For example, $010 \in 2^{<\mu}$ becomes $010uuu\ldots \in \{0, u, 1\}^\mu$, so it is ordered after $0100\ldots$ and before $0101\ldots$. If $a, b \in 2^\mu$ and $a < b$, then a starts with $\tau 0$ and b starts with $\tau 1$ for some $\tau \in 2^\mu$. Then $a < \tau < b$, because $\tau 0 \ldots < \tau u \ldots < \tau 1 \ldots$ $\square$

Theorem 8. *If some formula $\varphi(\bar{x};\bar{y})$ has the order property, then T is unstable: it is not λ-stable for any λ.*

Proof. We show λ-stability fails. Take $I \supseteq S$ as in Lemma 7, with $|I| > \lambda$ and $|S| \leq \lambda$ and S dense in I. By compactness[1], there are $(\bar{a}_i : i \in I)$ and $(\bar{b}_i : i \in I)$ such that $\mathbb{M} \models \varphi(\bar{a}_i;\bar{b}_j) \iff i < j$.

Let $C = \{\bar{b}_j : j \in S\}$. We claim that this map is an injection:

$$\begin{aligned} I \setminus S &\to S_n(C) \\ i &\mapsto \mathrm{tp}(\bar{a}_i/C), \end{aligned}$$

in which case $|C| \leq \lambda$ but $|S_n(C)| \geq |I \setminus S| > \lambda$, and λ-stability fails.

Suppose $i_1, i_2 \in I \setminus S$ and $i_1 \neq i_2$. Without loss of generality, $i_1 < i_2$. Then there is $j \in S$ such that $i_1 < j < i_2$. Then

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \bar{b}_j) \text{ but } \mathbb{M} \models \neg\varphi(\bar{a}_{i_2}, \bar{b}_j)$$

and $\bar{b}_j \in C$, and so $\mathrm{tp}(\bar{a}_{i_1}/C) \neq \mathrm{tp}(\bar{a}_{i_2}/C)$. $\square$

4 The order property from instability

Lemma 9. *If $\varphi(\bar{x};\bar{y})$ does not have the order property, then there is n_φ such that there do not exist $(\bar{a}_i : i < n_\varphi)$ and $(\bar{b}_i : i < n_\varphi)$ such that*

$$\mathbb{M} \models \varphi(\bar{a}_i;\bar{b}_j) \iff i < j.$$

Proof. Compactness. (Add new constant symbols $\bar{a}_i$ and $\bar{b}_i$ for $i \in \mathbb{Z}$. If n_φ didn't exist, then $\{\varphi(\bar{a}_i;\bar{b}_j) : i < j \in \mathbb{Z}\} \cup \{\neg\varphi(\bar{a}_i;\bar{b}_j) : i \geq j \in \mathbb{Z}\}$ is consistent, hence realized in $\mathbb{M}$.) $\square$

Remark: (added after class). If $(I, \leq)$ is a small linear order, say that $\varphi(\bar{x},\bar{y})$ has OP_I if there are $(\bar{a}_i : i \in I)$ and $(\bar{b}_i : i \in I)$ such that $\mathbb{M} \models \varphi(\bar{a}_i, \bar{a}_j) \iff i < j$ for all $i, j \in I$. Then the usual order property (as we defined it) is $\mathrm{OP}_{\mathbb{Z}}$. Lemma 9 says that if φ satisfies $\mathrm{OP}_{(n,\leq)}$ for all finite n, then φ satisfies $\mathrm{OP}_{\mathbb{Z}}$. Here is a more precise explanation of the argument. First, two observations:

1. If I and I' are two isomorphic linear orders, then OP_I is equivalent to $\mathrm{OP}_{I'}$.
2. If φ satisfies OP_{I_0} for every finite $I_0 \subseteq_f I$, then φ satisfies OP_I, by compactness.

To prove Lemma 9, suppose φ has $\mathrm{OP}_{(n,\leq)}$ for every finite n. Every finite linear order is isomorphic to some $(n, \leq)$, and so φ has OP_I for any finite I. Now if I is arbitrary, then φ has OP_I by the second observation above. So this actually proves something stronger: if φ has $\mathrm{OP}_{(n,\leq)}$ for all finite n, then φ has OP_I for any infinite I (including $I = \mathbb{Z}$ among other things). Conversely, if φ has OP_I for some infinite I, then for any finite n we can find $I_0 \subseteq_f I$ with $I_0 \cong (n, \leq)$, and so φ has OP_{I_0} and the equivalent property $\mathrm{OP}_{(n,\leq)}$. In summary, the following are equivalent:

[1] See the newly-added remark following Lemma 9 below. Here we're saying that φ satisfies the OP_I of that remark.

1. φ has the order property.

2. φ has $\mathrm{OP}_{(n,\leq)}$ for all finite n.

3. φ has OP_I for any I.

4. φ has OP_I for some infinite I.

Lemma 10. *Suppose $\varphi(\bar{x};\bar{y})$ doesn't have the order property. Let n_φ be as in Lemma 9. Let $\bar{a}_1, \bar{a}_2, \ldots$ be an indiscernible sequence. Then there is no $\bar{b} \in \mathbb{M}$ such that*

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}) \text{ for } 0 \leq i < n_\varphi$$
$$\mathbb{M} \models \neg\varphi(\bar{a}_i, \bar{b}) \text{ for } n_\varphi \leq i < 2n_\varphi.$$

Proof. Let $n = n_\varphi$. Suppose such a $\bar{b}$ exists. For $0 \leq j < n$, we have

$$(\bar{a}_{n-j}, \bar{a}_{n-j+1}, \ldots, \bar{a}_{n-j+(n-1)}) \equiv (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1})$$

by indiscernibility. Therefore there is $\sigma_j \in \mathrm{Aut}(\mathbb{M})$ such that

$$\sigma_j(\bar{a}_{n-j}, \bar{a}_{(n-j)+1}, \ldots, \bar{a}_{(n-j)+(n-1)}) = (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}).$$

Let $\bar{b}_j = \sigma_j(\bar{b})$. For $i, j < n$,

$$\mathbb{M} \models \varphi(\bar{a}_i; \bar{b}_j) \iff \mathbb{M} \models \varphi(\sigma_j(\bar{a}_{(n-j)+i}), \sigma_j(\bar{b}))$$
$$\iff \mathbb{M} \models \varphi(\bar{a}_{(n-j)+i}, \bar{b}) \iff n - j + i < n \iff i < j.$$

This contradicts the choice of $n = n_\varphi$. $\square$

Lemma 11. *Suppose $\varphi(x_1, \ldots, x_n; \bar{y})$ does not have the order property. Suppose $N \geq \max(n_\varphi, n_{\neg\varphi})$. Let p be an A-invariant global type. Let $(\bar{a}_i : i < \omega)$ be a Morley sequence of p over A. Suppose $\bar{b} \in \mathbb{M}$.*

1. *If $\neg\varphi(\bar{x}; \bar{b}) \in p(\bar{x})$, then $\mathbb{M} \models \neg\varphi(\bar{a}_i, \bar{b})$ for a majority of $i \in \{0, 1, \ldots, 2N-2\}$.*

2. *If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b})$ for a majority of $i \in \{0, 1, \ldots, 2N-2\}$.*

Proof. We prove (1); (2) is similar. Suppose $\neg\varphi(\bar{x}; \bar{b}) \in p(\bar{x})$, but $\mathbb{M} \models \varphi(\bar{a}_i; \bar{b})$ for 50% of $i < 2N - 1$. Then there are $i_0 < \cdots < i_{N-1} < 2N - 1$ such that $\mathbb{M} \models \varphi(\bar{a}_{i_j}; \bar{b})$ for $j < N$. Let $(\bar{c}_i : i < \omega)$ be a Morley sequence of p over $A \cup \{\bar{a}_i : i < \omega\} \cup \{\bar{b}\}$. Then $\bar{c}_i$ realizes the type $p \restriction A\bar{b}$ which contains the formula $\neg\varphi(\bar{x}; \bar{b})$, and so $\mathbb{M} \models \neg\varphi(\bar{c}_i; \bar{b})$ for all i. Finally,

$$\bar{a}_{i_0}, \ldots, \bar{a}_{i_{N-1}}, \bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots$$

is a Morley sequence of p over A, hence indiscernible. But

$$\mathbb{M} \models \varphi(\bar{a}_{i_j}, \bar{b}) \text{ for } 0 \leq j < N$$
$$\mathbb{M} \models \neg\varphi(\bar{c}_i, \bar{b}) \text{ for } 0 \leq i < N,$$

so this contradicts Lemma 10. $\square$

Proposition 12. *Suppose $\varphi(x_1, \ldots, x_n; \bar{y})$ doesn't have the order property. If M is a small model and $p \in S_n(M)$, then the relation $d_p\varphi(\bar{y})$ is definable. That is, there is a formula defining the set*

$$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}.$$

Proof. Take $q \in S_n(M)$ a global coheir of p (March 10, Theorem 5). Then q is M-invariant (March 10, Theorem 17). Let $(\bar{a}_i : i < \omega)$ be a Morley sequence of q over M. By Lemma 11, $d_q\varphi(\bar{y})$ is definable from the Morley sequence by majority voting:

$$\varphi(\bar{x}; \bar{b}) \in q(\bar{x}) \iff \mathbb{M} \models \bigvee_S \bigwedge_{i \in S} \varphi(\bar{a}_i; \bar{b}).$$

where S ranges over $\{S \subseteq 2N - 1 : |S| \geq N\}$. Now $d_q\varphi$ is definable and M-invariant, hence M-definable (March 10, Lemma 10). Then $d_p\varphi$ is (M-)definable: it's the restriction of $d_q\varphi$ to M. $\square$

Theorem 13. *Fix n. Suppose no formula $\varphi(x_1, \ldots, x_n; \bar{y})$ has the order property. Then for any $M \models T$ and $p \in S_n(M)$, p is definable.*

Corollary 14. *The following are equivalent:*

1. *All types over models are definable.*
2. *All 1-types over models are definable.*
3. *No formula $\varphi(\bar{x}; \bar{y})$ has the order property.*
4. *No formula $\varphi(x; \bar{y})$ has the order property.*
5. *T is λ-stable for at least one λ.*

Proof. Similar to Theorem 2 on March 10, but using today's Theorem 8 and Theorem 13. $\square$

Example: (added after class). We can give another proof that strongly minimal theories are stable. Suppose T is strongly minimal but not stable. By condition (4) above, some formula $\varphi(x; \bar{y})$ has the order property. Therefore there are $a_i, \bar{b}_j$ for $i, j \in \mathbb{Z}$ such that

$$\mathbb{M} \models \varphi(a_i; \bar{b}_j) \iff i < j.$$

It's not hard to see that the a_i are pairwise distinct (if $i < i'$, then $\varphi(\bar{x}; \bar{b}_{i'})$ distinguishes between a_i and $a_{i'}$). Note that

$$\begin{aligned} \{a_{-1}, a_{-2}, a_{-3}, \ldots\} &\subseteq \varphi(\mathbb{M}; \bar{b}_0) \\ \{a_0, a_1, a_2, \ldots\} &\subseteq \mathbb{M} \setminus \varphi(\mathbb{M}; \bar{b}_0). \end{aligned}$$

Therefore the set $\varphi(\mathbb{M}; \bar{b}_0)$ is neither finite nor cofinite, contradicting strong minimality.

Fact 15. *$\varphi(\bar{x}; \bar{y})$ has the order property iff it has the dichotomy property.*

5 Commuting types

Theorem 16. *Assume T is stable. Let p, q be global A-invariant types. Then $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$.*

Proof. Suppose an $L(\mathbb{M})$-formula $\varphi(\bar{x}, \bar{y})$ is in $(p \otimes q)(\bar{x}, \bar{y})$ but not $(q \otimes p)(\bar{y}, \bar{x})$. Take a small set $B \supseteq A$ over which φ is defined. Then p, q are B-invariant. Replacing A with B, we may assume $\varphi(\bar{x}; \bar{y})$ is an $L(A)$-formula.

Let $(\bar{b}_1, \bar{a}_1; \bar{b}_2, \bar{a}_2; \bar{b}_3, \bar{a}_3; \ldots)$ be a Morley sequence of $q \otimes p$ over A. In other words

$$\begin{aligned} \bar{b}_1 &\models q \restriction A, & \bar{a}_1 &\models p \restriction A\bar{b}_1 \\ \bar{b}_2 &\models q \restriction A\bar{b}_1\bar{a}_1, & \bar{a}_2 &\models p \restriction A\bar{b}_1\bar{a}_1\bar{b}_2 \\ \bar{b}_3 &\models q \restriction A\bar{b}_1\bar{a}_1\bar{b}_2\bar{a}_2, & \bar{a}_3 &\models p \restriction A\bar{b}_1\bar{a}_1\bar{b}_2\bar{a}_2\bar{b}_3, \\ & & \cdots \end{aligned}$$

If $i < j$, then $(\bar{a}_i, \bar{b}_j) \models (p \otimes q) \restriction A$, and so $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j)$. On the other hand if $j \le i$, then $(\bar{b}_j, \bar{a}_i) \models (q \otimes p) \restriction A$, and so $\mathbb{M} \models \neg\varphi(\bar{a}_i, \bar{b}_j)$. Therefore

$$\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

It follows that φ has the order property, a contradiction.[2] $\square$

Example. Suppose T is strongly minimal, and $p, q \in S_1(\mathbb{M})$ are both the transcendental 1-type. By Theorem 16, $(p \otimes q)(x, y) = (q \otimes p)(y, x)$. Concretely, this means the following are equivalent for $a, b \in \mathbb{M}$ and $C \subseteq \mathbb{M}$:

$$\begin{aligned} & a \notin \operatorname{acl}(C) \text{ and } b \notin \operatorname{acl}(Ca) \\ \iff & b \notin \operatorname{acl}(C) \text{ and } a \notin \operatorname{acl}(Cb). \end{aligned}$$

This implies that $\operatorname{acl}(-)$ satisfies the "Steinitz exchange property":

$$a \in \operatorname{acl}(Cb) \setminus \operatorname{acl}(C) \implies b \in \operatorname{acl}(Ca).$$

This means that $\operatorname{acl}(-)$ defines a "pregeometry" (a.k.a. "matroid").

6 Indiscernible sequences exist

(This section was added.) An indiscernible sequence $(\bar{b}_i : i \in I)$ is *constant* if $\bar{b}_i = \bar{b}_j$ for all i, j. If $(\bar{b}_i : i \in I)$ is non-constant, then $\bar{b}_i \neq \bar{b}_j$ for all $i < j$, by indiscernibility.

Theorem 17. *Suppose $\mathbb{M}$ is infinite. Then there is a non-constant indiscernible sequence $(b_i : i < \omega)$.*

[2] See the newly added Remark after Lemma 9. We have just shown that φ satisfies $\mathrm{OP}_{\mathbb{N}}$, which implies the usual property $\mathrm{OP}_{\mathbb{Z}}$ as discussed there.

Proof. Because $\mathbb{M}$ is infinite, it is not small, by κ-saturation. Take a small model M and take $a \in \mathbb{M} \setminus M$. Let $p = \mathrm{tp}(a/M) \in S_1(M)$. Let $q \in S_1(\mathbb{M})$ be a coheir of p (possible by March 10, Theorem 5). Then q is M-invariant (March 10, Theorem 17). Let $(b_i : i < \omega)$ be a Morley sequence of q over M. Then $(b_i : i < \omega)$ is M-indiscernible, hence indiscernible, by Theorem 4. It remains to show that the sequence is non-constant.

By indiscernibility it suffices to show that $b_2 \neq b_1$. Suppose $b_2 = b_1$. By definition of "Morley sequence," $b_1 \models q \upharpoonright M$ and $b_2 \models q \upharpoonright Mb_1$. Then $(x = b_1) \in \mathrm{tp}(b_2/Mb_1) = (q \upharpoonright Mb_1) \subseteq q$, so the formula $(x = b_1)$ is part of q. As q is finitely satisfiable in M (by virtue of being a coheir), there is some $a_0 \in M$ which satisfies $x = b_1$. Then $a_0 = b_1$. Now $(x = a_0) \in \mathrm{tp}(b_1/M) = (q \upharpoonright M) = p$, and so the formula $(x = a_0)$ is part of $p = \mathrm{tp}(a/M)$, meaning $a = a_0 \in M$, contradicting the choice of $a \notin M$. $\square$

Next week we will use Theorem 17 to prove (finite) Ramsey's theorem, a theorem in combinatorics. Then we will use Ramsey's theorem to prove a stronger form of Theorem 17 which gives more control over the construction of indiscernible sequences.