The fundamental order and forking

Advanced model theory

March 31, 2022

Reference in the book: Sections 13.1, 15.1, and 15.2. (But the treatment of the Theorem of the Bound is completely different.)

1 The fundamental order

Fix a complete L-theory T and monster model $\mathbb{M}$. Fix some $n < \omega$.

Definition 1. If $M \preceq \mathbb{M}$ and $p \in S_n(M)$ and $\varphi(x_1, \ldots, x_n; \bar{y})$ is an L-formula, then p *represents* φ if there is $\bar{b} \in M$ such that $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$. Otherwise, p *omits* φ. The *class* $[p]$ is the set $\{\varphi : p \text{ represents } \varphi\}$. If $p \in S_n(M)$ and $q \in S_n(N)$, then $p \leq q$ or $[p] \leq [q]$ means $[p] \supseteq [q]$: every formula represented by q is represented by p. The *fundamental order* is $\{[p] : M \models T,\ p \in S_n(M)\}$ with the order $\leq$.

Remark 2. The notation "$[p]$" isn't standard. The fundamental order depends on n. The relation $\leq$ defines a preorder on the class of n-types over models, and a partial order on the fundamental order.

Example 3. Suppose $n = 1$. The formula $\varphi(x, y) \equiv (x = y)$ is represented in $p \in S_1(M)$ iff there is $b \in M$ such that $(x = b) \in p(x)$, i.e., p is a constant type.

Example 4. Suppose $T = \mathrm{DLO}$ and $n = 1$. One can prove that there are four classes in the fundamental order:

1. The class of realized types.
2. The class of the type τ^+ at $+\infty$.
3. The class of the type τ^- at $-\infty$.
4. The class of all other types:

They are arranged like so:

$$\begin{tikzcd} (2) \arrow[dr, dash] & & (3) \arrow[dl, dash] \\ & (4) \arrow[d, dash] & \\ & (1) & \end{tikzcd}$$

For example, the formula $x = y$ is only represented in the bottom class (1), the formula $x < y$ is only represented in the classes (1), (3), and (4), and the formula $x > y$ is only represented in the classes (1), (2), and (4).

Proposition 5. *Suppose $M \preceq N$ and $q \in S_n(N)$ is an extension of $p \in S_n(M)$.*

1. $[q] \leq [p]$.
2. $[q] = [p]$ *iff for any L-formula $\varphi(\bar{x}; \bar{y})$ and $\bar{b} \in N$ such that $\varphi(\bar{x}; \bar{b}) \in q(\bar{x})$, there is $\bar{b}' \in M$ such that $\varphi(\bar{x}; \bar{b}') \in p(\bar{x})$.*
3. *If $q \sqsupseteq p$ then $[q] = [p]$.*

Proof. 1. Every formula represented in p is represented in q, so $[q] \supseteq [p]$.

2. $[q] = [p] \iff [q] \geq [p] \iff [q] \subseteq [p]$, and $[q] \subseteq [p]$ is the listed condition.

3. The condition in (2) is weaker than the definition of "heir." □

Remark 6. Let $q \in S_n(N)$ be an extension of $p \in S_n(M)$.

1. $[q] = [p]$ means that for any L-formula $\varphi(\bar{x}; \bar{y})$,

$$\exists \bar{b} \in N \ (\varphi(\bar{x}, \bar{b}) \in q(\bar{x})) \implies \exists \bar{b}' \in M \ (\varphi(\bar{x}, \bar{b}) \in p(\bar{x})).$$

2. $q \sqsupseteq p$ means that for any $L(M)$-formula $\varphi(\bar{x}; \bar{y})$,

$$\exists \bar{b} \in N \ (\varphi(\bar{x}, \bar{b}) \in q(\bar{x})) \implies \exists \bar{b}' \in M \ (\varphi(\bar{x}, \bar{b}) \in p(\bar{x})).$$

In particular, $q \sqsupseteq p$ means that "$[q] = [p]$ if we expand to the language $L(M)$."

Proposition 7. *Suppose $M, N \preceq \mathbb{M}$ and $p \in S_n(M)$ and $q \in S_n(N)$. Then $[p] \geq [q]$ iff there is an ultrafilter $\mathcal{U}$ and an elementary embedding $M \to N^{\mathcal{U}}$ making $q^{\mathcal{U}}$ an extension of p.*

Proof. ($\Rightarrow$): similar to Proposition 2 in the March 3 notes. (Remove the words "extending $\mathrm{id}_M : M \to M$" in the first line, and then the same proof works.)

($\Leftarrow$): $[q^{\mathcal{U}}] = [q]$ because $q^{\mathcal{U}} \sqsupseteq q$, and $[q^{\mathcal{U}}] \leq [p]$ because $q^{\mathcal{U}} \supseteq p$. □

2 The fundamental order in a stable theory

For the rest of the lecture, we assume T is stable.

Lemma 8. *Suppose $M \preceq N \preceq \mathbb{M}$, $p \in S_n(M)$, and $q_1, q_2 \in S_n(N)$ are extensions of p. If $[q_1] = [q_2] = [p]$ then $q_1 = q_2$. In other words, there is at most one extension of p to N with the same class as p.*

Proof. (Compare with the proof of Proposition 8 in the March 3 notes.)

Suppose $q_1 \neq q_2$. Take $\varphi(\bar{x}, \bar{b})$ such that

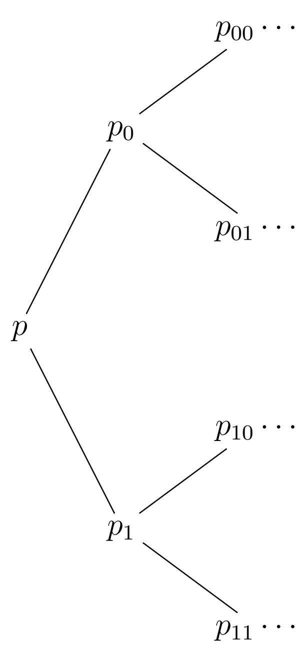
$$\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x})$$
$$\neg\varphi(\bar{x}, \bar{b}) \in q_2(\bar{x}).$$

Let $\beta = [q_1] = [q_2] = [p]$.

Claim. If $p' \in S_n(M')$ and $[p'] = \beta$, then there is $N' \succeq M'$, $q_1', q_2' \in S_n(N')$ extending p', and $b' \in N'$ such that $[q_1'] = [q_2'] = \beta$ and $\varphi(\bar{x}, \bar{b}') \in q_1'(\bar{x})$ and $\neg\varphi(\bar{x}, \bar{b}') \in q_2'(\bar{x})$.

Proof. As $[p'] = \beta \leq [p] = \beta$, there is an ultrafilter $\mathcal{U}$ and an elementary embedding $M' \to M^{\mathcal{U}}$ making $p^{\mathcal{U}}$ extend p'. Then $[q_i^{\mathcal{U}}] = [q_i] = \beta$ for $i = 1, 2$. Take $N' = N^{\mathcal{U}}$, $q_i' = q_i^{\mathcal{U}}$, and take $\bar{b}'$ the image of $\bar{b}$ under $N \to N^{\mathcal{U}}$. $\square_{\text{Claim}}$

Using the claim, we can build a tree of extensions $(p_\sigma : \sigma \in 2^{<\omega})$



where $p_{\sigma 0}$ and $p_{\sigma 1}$ are two extensions of p_σ differing by a formula $\varphi(\bar{x}, \bar{b}_\sigma)$. Then φ has the dichotomy property, contradicting stability. $\square$

In a stable theory, all types are definable, so types over models have unique heirs.

Proposition 9. *Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, and $q \supseteq p$.*

1. $q \sqsupseteq p \iff [q] = [p]$.
2. $q \not\sqsupseteq p \iff [q] < [p]$.

Proof. Let q' be the heir of p over M. Then $[q'] = [p]$ by Proposition 5. If $q \sqsupseteq p$, then $q = q'$ so $[q] = [q'] = [p]$. Conversely if $[q] = [p] = [q']$, then $q = q' \sqsupseteq p$ by Lemma 8. This proves (1). Then (2) holds because $[q] \leq [p]$ by Proposition 5. $\square$

For the rest of the lecture, the goal is to find an analogue of this picture for types over arbitrary sets (not models).

3 Bounds

Continue to assume T is stable. Fix $A \subseteq \mathbb{M}$ and an n-type $p \in S_n(A)$.

Definition 10. If M is a small model containing A, then $\mathrm{Ex}_M(p) = \{[q] : q \in S_n(M),\ q \supseteq p\}$.

Lemma 11. *Every chain in $\mathrm{Ex}_M(p)$ has an upper bound in $\mathrm{Ex}_M(p)$.*

Proof. Let $F = \{q \in S_n(M) : q \supseteq p\}$. Let I be a linear order and $(q_i : i \in I)$ be a sequence in F such that $[q_i] \leq [q_j]$ for $i \leq j$. If $i \leq j$, then every formula φ omitted (not represented) by q_i is omitted by q_j. Let $\Sigma(\bar{x})$ be

$$\{\neg\varphi(\bar{x}; \bar{b}) : \varphi(\bar{x}, \bar{y}) \text{ is omitted by some } q_i,\ \bar{b} \in M\}.$$

Claim. $p(\bar{x}) \cup \Sigma(\bar{x})$ is consistent.

Proof. Suppose $\varphi_j(\bar{x}; \bar{y})$ is omitted by q_{i_j} and suppose $\bar{b}_j \in M$ for $1 \leq j \leq m$. Without loss of generality, $i_1 \leq \cdots \leq i_m$. Then $\varphi_j(\bar{x}; \bar{y})$ is omitted by q_{i_m} for all j, because $i_j \leq i_m$. Then $q_{i_m}(\bar{x})$ extends $p(\bar{x}) \cup \{\neg\varphi_j(\bar{x}; \bar{b}_j) : 1 \leq j \leq m\}$, which must be consistent. $\square_{\text{Claim}}$

Take $q(\bar{x}) \in S_n(M)$ a completion of $p(\bar{x}) \cup \Sigma(\bar{x})$. Then $q \in F$, so $[q] \in \mathrm{Ex}_M(p)$. By choice of Σ, any formula omitted by q_i is omitted by q, so $[q_i] \leq [q]$. $\square$

Definition 12. A *bound* of p is a maximal element of $\mathrm{Ex}_M(p)$, i.e., a $\beta \in \mathrm{Ex}_M(p)$ such that there is no $\beta' \in \mathrm{Ex}_M(p)$ with $\beta' > \beta$. The set of bounds is denoted $\mathrm{Bd}_M(p)$.

(The notation "$\mathrm{Ex}_M(p)$" and "$\mathrm{Bd}_M(p)$" isn't standard.)

Corollary 13. *Every element of $\mathrm{Ex}_M(p)$ is bounded above by a maximal element: if $\beta \in \mathrm{Ex}_M(p)$, then there is $\beta' \in \mathrm{Bd}_M(p)$ with $\beta' \geq \beta$.*

Proof. Zorn's lemma plus Lemma 11. $\square$

Example 14. Consider the case where A is a model. Then $[p]$ makes sense (as $p \in S_n(A)$). We claim $[p]$ is the maximum element of $\mathrm{Ex}_M(p)$.

- $[p] \in \mathrm{Ex}_M(p)$ because if $q \in S_n(M)$ is the heir of p, then $[p] = [q] \in \mathrm{Ex}_M(p)$.
- If $q \in S_n(M)$ is an extension of p, then $[q] \le [p]$ by Proposition 5. Therefore every element of $\mathrm{Ex}_M(p)$ is $\le [p]$.

Because $[p]$ is the maximum, $[p]$ is the unique bound of p, i.e., $\mathrm{Bd}_M(p) = \{[p]\}$.

Lemma 15. *Suppose $M, N \preceq \mathbb{M}$ both contain A, and $p \in S_n(A)$.*

1. *If $\beta \in \mathrm{Ex}_M(p)$, then there is $\beta' \in \mathrm{Ex}_N(p)$ with $\beta' \ge \beta$.*
2. *$\mathrm{Bd}_M(p) = \mathrm{Bd}_N(p)$.*

Proof. 1. Take a small model $M' \supseteq M \cup N$. Take $q \in S_n(M)$ extending p with $[q] = \beta$. Let q' be the heir of q over M', and let $r = q' \restriction N$. Then

$$\beta = [q] = [q'] \le [r] =: \beta' \in \mathrm{Ex}_N(p),$$

because $q' \sqsupseteq q$, $q' \supseteq r$, and r extends p.

2. Suppose $\beta \in \mathrm{Bd}_M(p)$.
   - By part (1), there is $\beta' \in \mathrm{Ex}_M(p)$ with $\beta \le \beta'$.
   - By Corollary 13, there is $\beta'' \in \mathrm{Bd}_M(p)$ with $\beta' \le \beta''$.
   - By part (1) (with M and N exchanged), there is $\beta_3 \in \mathrm{Ex}_M(p)$ with $\beta'' \le \beta_3$.

   Then $\beta \le \beta' \le \beta'' \le \beta_3 \in \mathrm{Ex}_M(p)$. As β was maximal in $\mathrm{Ex}_M(p)$, we must have

   $$\beta = \beta' = \beta'' = \beta_3.$$

   Then $\beta = \beta'' \in \mathrm{Bd}_N(p)$. This shows $\mathrm{Bd}_M(p) \subseteq \mathrm{Bd}_N(p)$. The reverse inclusion follows by symmetry. $\square$

Since $\mathrm{Bd}_M(p)$ doesn't depend on M, we write it as $\mathrm{Bd}(p)$, and we can talk about "the bounds" of $p \in S_n(A)$ without specifying M.

4 Theorem of the bound

Continue to assume stability.

Definition 16. A global type $p \in S_n(\mathbb{M})$ is *Lascar A-invariant* if it is M-invariant for all small models $M \supseteq A$.

This is a weaker condition than being A-invariant. Since we're assuming stability, "M-invariant" means the same thing as "M-definable".

Lemma 17. *Suppose $p \in S_n(A)$ and M is a small model containing A and $q \in S_n(M)$ is an extension of p such that $[q] \in \mathrm{Bd}(p)$. Then the global heir of q is Lascar A-invariant.*

Proof. Let $q^{\mathbb{M}}$ denote the global heir. Note $[q^{\mathbb{M}}] = [q] \in \mathrm{Bd}(p)$ by Proposition 5. Suppose $q^{\mathbb{M}}$ is not Lascar A-invariant. Then there is a small model $N \supseteq A$ such that $q^{\mathbb{M}}$ is not N-invariant, i.e., not N-definable. Let $r = q^{\mathbb{M}} \restriction N$. Then $q^{\mathbb{M}}$ is not the heir of r (or else $q^{\mathbb{M}}$ would be N-definable), so $[r] > [q^{\mathbb{M}}] = [q]$ by Proposition 9. But $r \supseteq p$, so $[r] \in \mathrm{Ex}_N(p)$. This contradicts the fact that $[q] \in \mathrm{Bd}(p) = \mathrm{Bd}_N(p)$. □

Lemma 18. *Fix $\bar{b}$ and A. There is a small model $M \supseteq A$ such that the global heir of $\mathrm{tp}(\bar{b}/M)$ is Lascar A-invariant. Moreover, if $p = \mathrm{tp}(\bar{b}/A)$ and $\beta \in \mathrm{Bd}(p)$, we can choose $\mathrm{tp}(\bar{b}/M)$ and its global heir to have class β.*

Proof. Let $p = \mathrm{tp}(\bar{b}/A)$. Take a small model $M \supseteq A$, a bound $\beta \in \mathrm{Bd}(p) = \mathrm{Bd}_M(p)$, an extension $q \in S_n(M)$ with $[q] = \beta$, and a realization $\bar{b}_0 \models q$. Then $\mathrm{tp}(\bar{b}_0/A) = q \restriction A = p = \mathrm{tp}(\bar{b}/A)$. Take $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(\bar{b}_0) = \bar{b}$. Moving $M, q, \bar{b}_0$ by σ (which doesn't change β), we may assume $\bar{b} = \bar{b}_0$. Then q is an extension of p with $[q] = \beta \in \mathrm{Bd}(p)$, so by Lemma 17, the global heir of $q = \mathrm{tp}(\bar{b}/M)$ is Lascar A-invariant. The global heir has class β by Proposition 5. □

The global heir in Lemma 18 doesn't depend on the choice of M:

Lemma 19. *Let $\bar{b}, A$ be given. Suppose M_1, M_2 are two small models containing A such that the global heir of $\mathrm{tp}(\bar{b}/M_i)$ is Lascar A-invariant for $i = 1, 2$. Then the two global heirs are equal.*

Proof. Let p_i be the global heir of $\mathrm{tp}(\bar{b}/M_i)$, for $i = 1, 2$. Suppose $\varphi(\bar{x}, \bar{c}) \in p_1$ but $\neg\varphi(\bar{x}, \bar{c}) \in p_2$. By Lemma 18 there is a third model $M_3 \supseteq A$ such that the global heir r of $\mathrm{tp}(\bar{c}/M_3)$ is Lascar A-invariant. Note p_1, p_2, r are M_i-invariant for any i. Take $\bar{e}$ realizing $r \restriction M_1 M_2 M_3 \bar{b}$. Then

$$\bar{b} \models p_1 \restriction M_1 \text{ and } \bar{e} \models r \restriction M_1 \bar{b},$$

so $(\bar{b}, \bar{e}) \models (p_1 \otimes r) \restriction M_1$, as p_1 and r are M_1-invariant. By stability, all types commute (March 17, Theorem 16), so $(\bar{e}, \bar{b}) \models (r \otimes p_1) \restriction M_1$. In particular,

$$\bar{b} \models p_1 \restriction M_1 \bar{e}. \tag{$*$}$$

Now $\bar{e} \models r \restriction M_3 = \mathrm{tp}(\bar{c}/M_3)$, so $\bar{e} \equiv_{M_3} \bar{c}$. As p_1 is M_3-invariant, $\varphi(\bar{x}, \bar{c}) \in p_1(\bar{x})$ implies $\varphi(\bar{x}, \bar{e}) \in p_1(\bar{x})$. By $(*)$,

$$\mathbb{M} \models \varphi(\bar{b}, \bar{e}).$$

A similar argument using p_2 and M_2 instead of p_1 and M_1 shows $\mathbb{M} \models \neg\varphi(\bar{b}, \bar{e})$, a contradiction. □

Theorem 20 (Theorem of the bound)**.** *If $p \in S_n(A)$, then p has a unique bound.*

Proof. Take a realization $\bar{b}$ of p. Take $\beta_1, \beta_2 \in \mathrm{Bd}(p)$. For $i = 1, 2$, we can find a small model $M_i \supseteq A$ such that the global coheir $p_i \sqsupseteq \mathrm{tp}(\bar{b}/M_i)$ is lascar A-invariant and has $[p_i] = \beta_i$. By Lemma 19, $p_1 = p_2$, and so $\beta_1 = \beta_2$. $\square$

Therefore we can talk about "the bound" of a type. Let $\mathrm{bd}(p)$ denote the bound of p. By Example 14, $\mathrm{bd}(p) = [p]$ when A is a model.

5 Nonforking extensions

Continue to assume stability.

Proposition 21. *Suppose $A \subseteq B$, and $q \in S_n(B)$ is an extension of $p \in S_n(A)$. Then* $\mathrm{bd}(q) \leq \mathrm{bd}(p)$.

Proof. Take a small model $M \supseteq B$ and take $r \in S_n(B)$ extending q with $[r] = \mathrm{bd}(q)$. (This is possible as $\mathrm{bd}(q) \in \mathrm{Bd}(q) \subseteq \mathrm{Ex}_M(q)$.) Then r extends p, so $[r] \in \mathrm{Ex}_M(p)$. As $\mathrm{bd}(p)$ is the maximum of $\mathrm{Ex}_M(p)$, we must have $[r] \leq \mathrm{bd}(p)$. $\square$

Definition 22. An extension $q \supseteq p$ is *nonforking* if $\mathrm{bd}(q) = \mathrm{bd}(p)$, and *forking* if $\mathrm{bd}(q) < \mathrm{bd}(p)$. We write $q \sqsupseteq p$ to indicate that q is a nonforking extension of p.

Proposition 23. *If $M \preceq N$ and $q \in S_n(N)$ extends $p \in S_n(M)$, then q is a non-forking extension of p iff q is an heir of p.*

Proof. By Example 14, $\mathrm{bd}(p) = [p]$ and $\mathrm{bd}(q) = [q]$. Then this is just Proposition 9. $\square$

Proposition 23 ensures the notation $q \sqsupseteq p$ is unambiguous.

Proposition 24 (Full transitivity)**.** *Suppose $A_1 \subseteq A_2 \subseteq A_3$ and $p_i \in S_n(A_i)$ for $i = 1, 2, 3$ with $p_1 \subseteq p_2 \subseteq p_3$. Then $p_1 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_2$ and $p_2 \sqsubseteq p_3$.*

Proof. Obvious. $\square$

Proposition 25 (Extension)**.** *If $p \in S_n(A)$ and $B \supseteq A$, then there is at least one $q \in S_n(B)$ with $q \sqsupseteq p$.*

Proof. Take a small model $M \supseteq B$. Then $\mathrm{bd}(p) \in \mathrm{Bd}(p) \subseteq \mathrm{Ex}_M(p)$, so there is $r \in S_n(M)$ extending p with $[r] = \mathrm{bd}(p)$. Let $q = r \restriction B$. Then $\mathrm{bd}(r) = \mathrm{bd}(p)$, so $r \sqsupseteq p$. By full transitivity, $q \sqsupseteq p$. $\square$

6 Forking formulas and Lascar invariance

Continue to assume T is stable.

Lemma 26. *If $A \subseteq M \preceq \mathbb{M}$ and if the global heir of* $\mathrm{tp}(\bar{b}/M)$ *is Lascar A-invariant, then* $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A)$.

Proof. Let β be the bound of $\mathrm{tp}(\bar{b}/A)$. By Lemma 18 there is a small model $M' \supseteq A$ such that the global heir of $\mathrm{tp}(\bar{b}/M')$ is Lascar A-invariant and has class β. By Lemma 19, $\mathrm{tp}(\bar{b}/M')$ and $\mathrm{tp}(\bar{b}/M)$ have the same global heir. By Proposition 5, they have the same class. Then the class of $\mathrm{tp}(\bar{b}/M)$ is $\beta = \mathrm{bd}(\mathrm{tp}(\bar{b}/A))$, implying $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A)$. $\square$

Proposition 27 (Forking and Lascar A-invariance)**.** *If p is a global type and $A \subseteq \mathbb{M}$, then $p \sqsupseteq (p \restriction A)$ iff p is Lascar A-invariant.*

Proof. First suppose $p \sqsupseteq (p \restriction A)$. For any small model $M \supseteq A$, we have $p \sqsupseteq (p \restriction M)$ by Full Transitivity, which then means p is the heir of $p \restriction M$ by Proposition 23. Then p is M-definable, so p is Lascar A-invariant.

Conversely, suppose p is Lascar A-invariant. Take a small model $M \supseteq A$ and take $\bar{b} \models p \restriction M$. Then p is M-definable, so p is the global heir of $p \restriction M = \mathrm{tp}(\bar{b}/M)$. By Lemma 26, $\mathrm{tp}(\bar{b}/M) \sqsupseteq \mathrm{tp}(\bar{b}/A) = (p \restriction A)$. But p is the heir of $\mathrm{tp}(\bar{b}/M)$, so $p \sqsupseteq \mathrm{tp}(\bar{b}/M) \sqsupseteq (p \restriction A)$. Apply transitivity to see $p \sqsupseteq (p \restriction A)$. $\square$

Corollary 28. *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \sqsupseteq p$ iff some global extension of q is Lascar A-invariant.*

Proof. By full transitivity and extension, $q \sqsupseteq p$ iff there is a global extension $r \supseteq q$ such that $r \sqsupseteq p$, i.e., $r \sqsupseteq (r \restriction A)$. $\square$

Definition 29. An $L(\mathbb{M})$-formula $\varphi(\bar{x})$ *forks over A* if every global type containing it fails to be Lascar A-invariant.

Proposition 30 (Finite character)**.** *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \not\sqsupseteq p$ (q is a forking extension of p) iff some formula in q forks over A.*

Proof. For any model M, let $\Sigma_M(\bar{x})$ be the global partial type

$$\{\varphi(\bar{x};\bar{b}) \leftrightarrow \varphi(\bar{x};\bar{c}) : \varphi \in L,\ \bar{b} \equiv_M \bar{c}\}.$$

A global type $p \in S_n(\mathbb{M})$ extends Σ_M iff it is M-invariant, iff it is M-definable. Define $\Sigma_A(\bar{x})$ to be the union of $\Sigma_M(\bar{x})$ for M ranging over small models containing A. Then $p \in S_n(\mathbb{M})$ extends $\Sigma_A(\bar{x})$ iff it is Lascar A-invariant. Therefore, an $L(\mathbb{M})$-formula $\psi(\bar{x})$ forks over A iff $\Sigma_A(\bar{x}) \cup \{\psi(\bar{x})\}$ is inconsistent. By Corollary 28, $q \not\sqsupseteq p$ iff $\Sigma_A(\bar{x}) \cup q(\bar{x})$ is inconsistent. Then the result follows by compactness. $\square$