Morley Rank

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Def 6.1 Morley rank $RM(-)$ is Cantor-Bendixson rank
rang Morley

$R(-)$ for $A = \mathbb{M}$

Cantor-Bendixson rank in $S_n(\mathbb{M})$

If $p \in S_n(\mathbb{M})$

$RM(p) \geq 0$ always

$RM(p) \geq \alpha+1 \iff p \in (\{q \in S_n(\mathbb{M}) : RM(q) \geq \alpha\})'$.

If $\Sigma(\bar{x})$ is a partial type over $\mathbb{M}$

$[\Sigma] = \{p \in S_n(\mathbb{M}) : p \supseteq \Sigma\} \subseteq S_n(\mathbb{M})$

$RM(\Sigma) = \max\limits_{\substack{p \in S_n(\mathbb{M}) \\ p \supseteq \Sigma}} RM(p)$. $\leftarrow$ If $\Sigma \in S_n(\mathbb{M})$ these agree

If $\varphi(x_1, \ldots x_n) \in L(\mathbb{M})$. $RM(\varphi) = RM(\{\varphi\}) = RM([\varphi])$

If D is definable, $D = \varphi(\mathbb{M}^n)$, then $RM(D) = RM(\varphi)$

$$\{D \subseteq \mathbb{M}^n : D \text{ is definable}\} \xrightarrow{\cong} \{X \subseteq S_n(\mathbb{M}) : X \text{ is clopen}\}$$
$$D \mapsto [D]$$
$$\varphi(\mathbb{M}^n) \mapsto [\varphi].$$

$$\left[\, RM(\Sigma) = \min_{\Sigma_0 \subseteq_{fin} \Sigma} RM(\Sigma_0) = \min_{\Sigma \vdash \varphi} RM(\varphi). \,\right]$$

$$\begin{cases} RM(D) \geq 0 \iff D \neq \emptyset \\ RM(D) \geq \alpha+1 \iff \exists \text{ definable } D_1, D_2, D_3, \ldots \subseteq D, \quad D_i \cap D_j = \emptyset \text{ for } i \neq j \\ \qquad\qquad\qquad\qquad\qquad\qquad RM(D_i) \geq \alpha. \\ RM(D) \geq \beta \iff [\forall \alpha < \beta \;\; RM(D) \geq \alpha.] \text{ if } \beta \text{ is a limit} \end{cases}$$

Lemma 6.2 If $M \preceq \mathbb{M}$ is $\aleph_0$-saturated, if $\Sigma(\bar{x})$ is a partial type over M, then $RM(\Sigma) = (R(\Sigma)$ over $M)$
$[\Sigma] \subseteq S_n(M)$

Let $R(-) =$ CB-rank$/M$.

Proof By Prop 4.7(2), we may assume Σ is ~~finite~~ $\{\varphi\}$.

$D \subseteq \mathbb{M}^n$ D is M-definable, want $RM(D) = R(D)$

Claim If $\alpha \in \mathrm{Ord}$, D is M-definable, then
$RM(D) \geq \alpha \iff R(D) \geq \alpha$.

Proof By induction on α.

$\alpha = 0$, $RM(D) \geq 0 \iff D \neq \emptyset \iff R(D) \geq 0$

limit ordinal α case: easy.

$\alpha = 0$, $\quad RM(D) \geq 0 \iff D \neq \emptyset \iff R(D) = 0$

limit ordinal ~~is~~ case: easy.

$\alpha+1$: $\quad RM(D) \geq \alpha+1 \iff \exists$ definable $D_1, D_2, \ldots \subseteq D$, disjoint, $RM(D_i) \geq \alpha$

$R(D) \geq \alpha+1 \iff \exists$ $\mathbb{M}$-definable $D_1, D_2, \cdots \subseteq D$, disjoint, $R(D_i) \geq \alpha$ (i.e., $RM(D_i) \geq \alpha$, [by induction]).

$R(D) \geq \alpha+1 \Rightarrow RM(D) \geq \alpha+1$.

D

D_1 D_2 D_3 . . .

$RM(D) \geq \alpha+1 \Rightarrow \ldots$

Let $D = \psi(\mathbb{M}^n, \bar{b})$ $\quad \bar{b} \in \mathbb{M}$

$D_i = \varphi_i(\mathbb{M}^n, \bar{c}_i)$ $\quad \bar{c}_i \in \mathbb{M}$.

By $\aleph_0$-saturation, find $\bar{c}_i' \in M$ such that

$$\bar{c}_1 \bar{c}_2 \bar{c}_3 \ldots \equiv_{\bar{b}} \bar{c}_1' \bar{c}_2' \bar{c}_3' \ldots$$

Let $D_i' = \varphi_i(\mathbb{M}^n, \bar{c}_i')$, $\quad D_i'$ are M-definable

$\exists \sigma \in Aut(\mathbb{M})$ $\quad \sigma(\bar{b})$, $\quad \sigma(\bar{c}_i) = \bar{c}_i'$

$\sigma(D_i) = D_i'$, $\quad \sigma(D) = D$.

$RM(D_i') = RM(D_i) \geq \alpha$ $\quad D_i' \cap D_j' = \emptyset$ for $i \neq j$. $\quad D_i' \subseteq D$. $\quad \square$

So Cantor-Bendixson rank in $S_n(M)$ is Morley Rank when M is $\aleph_0$-sat.

There can be $A \subseteq \mathbb{M}$ such that CB-rank in $S_n(A) \neq$ Morley rank

Ex it can happen that $|\mathbb{M}| = \infty$, but $S_1(A) = \{p\}$. ~~(maybe $A = \emptyset$)~~

$R(S_1(A)) = R(x=x) = R(\mathbb{M}') = 0$

but $RM(\mathbb{M}') > 0$ because $\{a_1\}, \{a_2\}, \{a_3\}, \ldots$ disjoint subsets of $\mathbb{M}$.

Example 6.3 $\quad$ If D is definable…

1) $RM(D) \geq 0 \iff D \neq \emptyset$

2) $RM(D) \geq 1 \iff \exists$ disjoint definable subset $D_1, D_2, D_3, \ldots \subseteq D$ such that $D_i \neq \emptyset$.

$\iff |D| = \infty$.

3) $RM(D) \geq 2 \iff \exists$ disjoint definable subsets $D_1, D_2, D_3, \ldots \subseteq D$ such that $|D_i| = \infty$.

If T is strongly minimal, then $RM(\mathbb{M}) = 1$

Later: $RM(\mathbb{M}^n) = n$.

Prop 6.4 $\quad$ If $f: X \to Y$ is definable $\quad$ ($X, Y, \Gamma(f)$ are def. sets)

1) If f is surjective, then $RM(X) \geq RM(Y)$

2) If f is bijective, then $RM(X) = RM(Y)$

3) If f is injective, then $RM(X) \le RM(Y)$

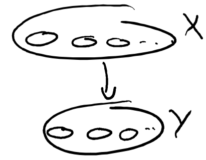
Proof 1) Claim If $\alpha \in \mathrm{Ord}$, if $f: X \to Y$ def. surjection, if $RM(Y) \geq \alpha$ then $RM(X) \geq \alpha$.

Proof: By induction on α.

$\alpha = 0$: $Y \neq \emptyset \Rightarrow X \neq \emptyset$.

limit ordinals: easy

$\alpha+1$: If $RM(Y) \geq \alpha+1$. $\exists$ def. $D_1, D_2, \ldots \subseteq Y$ disjoint $RM(D_i) \geq \alpha$



$f^{-1}(D_i) \subseteq_{\mathrm{def}} X$, disjoint

$f^{-1}(D_i) \cap f^{-1}(D_j) = \emptyset$

$f^{-1}(D_i) \to D_i$ surjective, by induction

$RM(D_i) \geq \alpha \Rightarrow RM(f^{-1}(D_i)) \geq \alpha$.

$\{f^{-1}(D_i)\}_i$ shows that $RM(X) \geq \alpha+1$. $\square_{\text{claim}}$

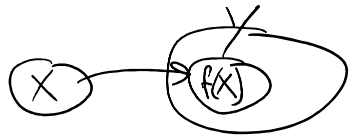
Take $\alpha = RM(Y)$, Claim $\Rightarrow$ $RM(X) \geq \alpha = RM(Y)$. ✓.

2) If $f: X \to Y$ bijection, then ~~$f^{-1}: Y \to X$~~

$RM(X) \geq RM(Y)$ by (1),

$f^{-1}: Y \to X$ a bijection, so $RM(Y) \geq RM(X)$. ✓.

3) If $f: X \to Y$ injective

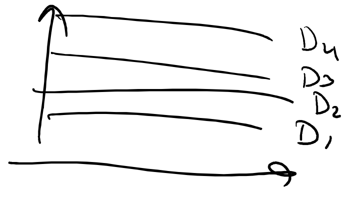


$f: X \to f(X)$ is a bijection

$RM(X) = RM(f(X)) \leq RM(Y)$. ✓ $\square$.

Example 6.5 If $|\mathcal{M}| = \infty$, then $RM(\mathcal{M}^n) \geq n$.

Proof $n = 0$: $\mathcal{M}^n \neq \emptyset$.



$n+1$: take $a_1, a_2, \ldots \in \mathcal{M}$ distinct

then $D_i = \mathcal{M}^n \times \{a_i\} \subseteq \mathcal{M}^{n+1}$

$\pi: D_i \to \mathcal{M}^n$ definable bijection

$RM(D_i) = RM(\mathcal{M}^n) \geq n$ by induction

$D_i \cap D_j = \emptyset$ for $i \neq j$

So $RM(\mathcal{M}^{n+1}) \geq n+1$. $\square$

Lemma 6.6 Let $f: X \to Y$ be definable, finite fibers

$[\forall \bar{b} \in Y,\ f^{-1}(\bar{b}) = \{\bar{a} \in X : f(\bar{a}) = \bar{b}\}$ is finite $]$

Then $RM(X) \leq RM(Y)$.

<u>Proof</u> If $\alpha \in \mathrm{Ord}$, $\alpha \le RM(X) \Rightarrow \alpha \le RM(Y)$.

by induction on α.

$\alpha = 0$: $X \ne \emptyset \Rightarrow Y \ne \emptyset$. True, or else $f : X \to Y$ can't exist.

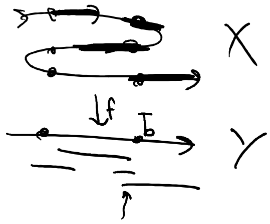
take $\bar{a} \in X$, $f(\bar{a}) \in Y$. ~~no function from~~

limit ordinal: easy

$\alpha+1$: If $RM(X) \ge \alpha+1$, take $D_1, D_2, \ldots \subseteq X$ disjoint, $RM(D_i) \ge \alpha$.

By induction $RM(f(D_i)) \ge \alpha$

$f(D_i)$ no longer disjoint



By compactness $\exists k$ s.t. $|f^{-1}(\bar{b})| \le k$ $\forall \bar{b} \in Y$

$\{f(D_i)\}_{i \in \omega}$ is $(k+1)$-inconsistent.

if $\bar{b} \in \bigcap_{i=1}^{k+1} f(D_i)$, say, then $\exists \bar{a}_i \in D_i$ $f(\bar{a}_i) = b$

$\bar{a}_1, \ldots \bar{a}_{k+1}$ distinct, $|f^{-1}(\bar{b})| \ge k+1$ $\Rightarrow\!\!\Leftarrow$.

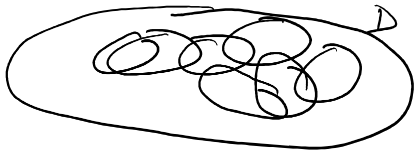
So by Lemma 4.12, $RM(Y) \ge \alpha+1$. $\blacksquare$

(for $k < \omega$)

<u>Def 4.11</u> A family $\mathcal{F}$ is <u>k-inconsistent</u> if $\forall$ distinct $D_1, D_2, \ldots D_k \in \mathcal{F}$, $\bigcap_{i=1}^{k} D_i = \emptyset$

2-inconsistent = disjoint.

Lemma 4.12 If D is A-definable, if $D_1, D_2, D_3 \ldots \subseteq D$ are A-def., $R(D_i) \ge \alpha$, and $\{D_1, D_2, \ldots\}$ is k-inconsistent then $R(D) \ge \alpha+1$.



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<u>Proof</u>: $D \supseteq D_1$ so $R(D) \ge R(D_1) \ge \alpha$.

If $R(D) \not\ge \alpha+1$, then $R(D) = \alpha$.

Let $E_\alpha = \{p \in S_n(A) : R(p) = \alpha\}$, $[D] \cap E_\alpha \ne \emptyset$

$|D \cap E_\alpha| < \infty$ (else $(D \cap E_\alpha)' \ne \emptyset$, if $q \in (D \cap E_\alpha)'$ then $q \in D \cap E_{\alpha+1}$, no).

Consider $\omega \longrightarrow \mathrm{Pow}([D] \cap E_\alpha)$

$i \longmapsto [D_i] \cap E_\alpha$.

~~Not injective, so $\exists\, i \ne j$ $[D_i] \cap E_\alpha = [D_j] \cap E_\alpha$~~

Some fiber is infinite

Pass to a subsequence, WMA $[D_i] \cap E_\alpha = [D_1] \cap E_\alpha$

$\bigcap_{i=1}^{k} D_i = \emptyset$

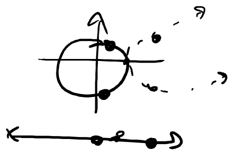
$\bigcap_{i=1}^{k} [D_i] = \left[\bigcap_{i=1}^{k} D_i\right] = [\emptyset] = \emptyset$

$\bigcap_{i=1}^{k}[D_i] = \left[\bigcap_{i=1}^{k} D_i\right] = [\emptyset] = \emptyset$

$\bigcap_{i=1}^{k}([D_i]\cap E_\alpha) = \emptyset \quad = \quad [D_1]\cap E_\alpha$

$R(D_1)<\alpha$. $\Rightarrow\Leftarrow$. $\square$.

<u>Example 6.7</u> In ACF_0, the set $D=\{(x,y) : x^2+y^2=1\}$ has $RM(D)=1$ because: D is infinite, $RM(D)\geq 1$.



$D \to M^1$ has finite fibes

$(x,y)\mapsto x$

so $RM(D)\leq RM(M^1)=1$.

<u>Def 6.8</u> Fix $\bar{a}\in M^n$, $B\subseteq M$, $RM(\bar{a}/B) = RM(\underbrace{tp(\bar{a}/B)}_{\text{incomplete type over } M})$.

<u>Prop 6.9</u> Fix $B\subseteq M$

1.) If $\bar{a}\in M^n$, then $RM(\bar{a}/B) = \min\{RM(X) : X\subseteq M^n, X \text{ is } B\text{-def.}, \bar{a}\in X\}$.

2.) If X is B-def, then $RM(X) = \max_{\bar{a}\in X} RM(\bar{a}/B)$.

<u>Proof</u> 1.) Prop 4.7:

$$RM(\bar{a}/B) = RM(tp(\bar{a}/B)) = \min_{\Sigma_0\subseteq_f tp(\bar{a}/B)} RM(\Sigma_0) = \min_{\varphi\in tp(\bar{a}/B)} RM(\varphi)$$

2.) If $\bar{a}\in X$, $[tp(\bar{a}/B)]\subseteq [X]\subseteq S_n(M)$.

If $X=\varphi(M^n)$, $\varphi\in L(B)$, then $\varphi\in tp(\bar{a}/B)$.

so $[tp(\bar{a}/B)]\subseteq[\varphi]$.

So $RM(tp(\bar{a}/B))\leq RM(X)$.

$RM(X)\geq \max_{\bar{a}\in X} RM(\bar{a}/B)$.

$RM(X) = \max_{p\in S_n(M)} RM(p)$.

Take $p\in S_n(M)$ $p\in[X]$, with $RM(p)=RM(X)$.

Let $q = p\restriction B$. $\quad p(\bar{x}) \vdash q(\bar{x})$ $\quad$ $p(\bar{x})\vdash \bar{x}\in X$

$p(\bar{x})\vdash\varphi(\bar{x})$.

where φ defines X.

$q(\bar{x})\vdash\varphi(\bar{x})$

$RM(p)\leq RM(q)\leq RM(\varphi) = RM(X) = RM(p)$

So $RM(q) = RM(X)$. Take $\bar{a}\models q$, $q = tp(\bar{a}/B)$,

$M\models\varphi(\bar{a})$, so $\bar{a}\in X$, $RM(\bar{a}/B) = RM(q) = RM(X)$. $\square$

Example If T is s.min, $a \in \mathbb{M}$, $B \subseteq \mathbb{M}$,

$$RM(a/B) = \begin{cases} 0 & \text{if } a \in acl(B) \\ 1 & \text{if } a \notin acl(B) \end{cases}$$

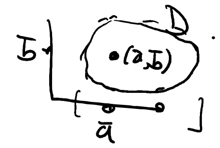
Why? $RM(a/B) = \min\{RM(X) : X \text{ is } B\text{-def}, X \ni a\}. = \begin{cases} 0 & \text{if } \exists B\text{-def } X \ni a \text{ with } |X| < \infty \\ 1 & \text{otherwise} \end{cases}$

1 if $|X| = \infty$ $\quad RM(X) \leq RM(\mathbb{M}) = 1$
0 if $|X| < \infty$.

Lemma 6.10 If $\bar{b} \in acl(\bar{a}C)$ then $RM(\bar{b}/C) \leq RM(\bar{a}/C)$.

Proof Let $\pi_1(\bar{x}, \bar{y}) = \bar{x}$ $\quad \pi_2(\bar{x}, \bar{y}) = \bar{y}$ $\quad |\bar{x}| = |\bar{a}|$ $\quad |\bar{y}| = |\bar{b}|$.

If D is C-def, and $D \ni (\bar{a}, \bar{b})$
and D is small enough, then.



— 1) $RM(\pi_1(D)) = RM(\bar{a}/C)$ $\quad$ (idea: Prop 6.9, $\exists$ C-def D_1 $\quad D_1 \ni \bar{a}$, $RM(D_1) = RM(\bar{a}/C)$, replace D with $D \cap \pi_1^{-1}(D_1)$ now $\pi_1(D) \subseteq D_1$, so $RM(\pi_1(D)) \leq RM(D_1) \leq RM(\bar{a}/C)$. $(\bar{a}\bar{b}) \in D$, $\bar{a} \in \pi_1(D)$ $\quad RM(\pi_1(D)) \geq RM(\bar{a}/C)$.)

— 2) $RM(\pi_2(D)) = RM(\bar{b}/C)$

→ 3) $D \to \pi_1(D)$ has finite fibers $\quad \to \forall \bar{a}' \, \exists^{<\infty} \bar{b}'$ with $(\bar{a}', \bar{b}') \in D$.

because $\bar{b} \in acl(C\bar{a})$.

$\exists \varphi(\bar{x}, \bar{y}) \in L(C)$, $(\bar{a}, \bar{b}) \in \varphi(\mathbb{M})$

$\bar{b} \in \varphi(\bar{a}, \mathbb{M})$, $|\varphi(\bar{a}, \mathbb{M})|$ finite k.

Shrink φ, make $|\varphi(\bar{a}', \mathbb{M})| \leq k$ $\forall \bar{a}'$.

take $D \subseteq \varphi(\mathbb{M})$, then $D \to \pi_1(D)$ has finite fibers

Then $RM(\bar{b}/C) = RM(\pi_2(D)) \leq RM(D) \leq RM(\pi_1(D)) = RM(\bar{a}/C)$. $\square$

Theorem 6.11 If T is strongly minimal,
let $p \in S_1(\mathbb{M})$ be transcendental. Let $p^{\otimes n} = \underbrace{p \otimes p \otimes \cdots \otimes p}_{n \text{ times}} \in S_n(\mathbb{M})$.

Let $C \subseteq \mathbb{M}$, $\bar{a} \in \mathbb{M}^n$.

1) If $\bar{a} \not\models p^{\otimes n} \restriction C$, then $RM(\bar{a}/C) < n$

2) $RM(\mathbb{M}^n) = n$

3) If $\bar{a} \models p^{\otimes n} \restriction C$, then $RM(\bar{a}/C) = n$.

Proof Prove (1–3) jointly, by induction on n.

1) If $a_i \notin \mathrm{acl}(Ca_1,\dots a_{i-1})$ $\forall i$, then $a_i \models p\restriction Ca_1\dots a_{i-1}$

$\bar a \models p^{\otimes n}\restriction C \Rightarrow\Leftarrow$.

So $\exists i$ such that $a_i \in \mathrm{acl}(Ca_1\dots a_{i-1})$.

$\bar a \in \mathrm{acl}(C\underbrace{a_1 a_2 \dots a_{i-1}\, a_{i+1}\, a_{i+2} \dots a_n}_{\bar b \in \mathbb{M}^{n-1}})$

$RM(\bar a/C) \le RM(\bar b/C) \le RM(\mathbb{M}^{n-1}) \le n-1$ by induction.

2) Example 6.5 $\Rightarrow$ $\underline{RM(\mathbb{M}^n) \ge n}$, so we need $RM(\mathbb{M}^n) \le n$.

Take $\aleph_0$-sat $M \preceq \mathbb{M}$. (CB-rank/M = Morley Rank by Lem 6.2)

If $q \in S_n(M)$ and $q \ne p^{\otimes n}\restriction M$, then $RM(q) < n$ by part (1).

If $\alpha = n$, $E_\alpha = \{q \in S_n(M) : R(q) \ge n\} = \{p^{\otimes n}\restriction M\}$.

$E_n \ne \emptyset$ so $R(p^{\otimes n}\restriction M) = n$.

$p^{\otimes n}\restriction M$ is the only point in E_n, so it's isolated

So $E_n' = \emptyset = E_{n+1}$, $\nexists q \in S_n(M)$, $R(q) \ge n+1$.

So $R(S_n(M)) = n$. $RM(\mathbb{M}^n) = n$.

3) Prop 6.9 $\Rightarrow$ $\underset{\overset{\|}{n}}{RM(\mathbb{M}^n)} = \max\limits_{\bar a \in \mathbb{M}^n} RM(\bar a/C)$.

$\exists q \in S_n(C)$, $RM(q) = n$. Part 1 $\Rightarrow$ if $q \ne p^{\otimes n}\restriction C$ then $RM(q) < n$

$\Rightarrow q = p^{\otimes n}\restriction C$.

$RM(p^{\otimes n}\restriction C) = n$. $\square$

__Theorem 6.12__ If T is strongly min., $M \models T$

if $B_1, B_2 \subseteq M$ are bases of M, then $|B_1| = |B_2|$.

__Def__ $\dim(M) = |B|$ for any basis $B \subseteq M$.

$\dim(M)$ is a complete invariant for M.

__Proof__ We WMA $|B_1| \le |B_2|$.

(Resume 11:40.)

Case 1: $|B_2| < \infty$. Let $\{a_1,\dots a_n\} = B_1$

$\{b_1,\dots b_m\} = B_2$

$\mathrm{acl}(B_1) = M$

$\mathrm{acl}(B_2) = M$

$\bar a \models p^{\otimes n}\restriction \emptyset$

$\bar b \models p^{\otimes m}\restriction \emptyset$

Use Thm 6.11 with $C = \emptyset$.

$\cancel{\bar a \in \mathrm{acl}(\bar b)}$

$\bar b \in \mathrm{acl}(\bar a)$

$n = \cancel{RM(\bar a/\emptyset) \le}$

$m = RM(\bar b/\emptyset) \le RM(\bar a/\emptyset) = n = |B_1|$

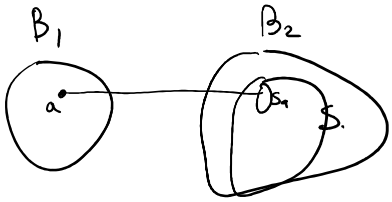
$|B_2| =$

So $|B_1| = |B_2|$.

Case 2: $|B_2| = \kappa \geq \aleph_0$. Suppose $|B_1| < |B_2|$. ~~($|B_1|$ may or)~~

If $a \in B_1$, $a \in M = \mathrm{acl}(B_2)$, $\exists S_a \subseteq_{fin} B_2$ $a \in \mathrm{acl}(S_a)$

Let $S = \bigcup_{a \in B_1} S_a$. $S \subseteq B_2$ $|S| < |B_2|$.



either

$|B_1| < \aleph_0$ $\Rightarrow |S| < \aleph_0 \leq |B_2|$

$|B_1| \geq \aleph_0$, then $|S| \leq |B_1| < |B_2|$.

$B_1 \subseteq \mathrm{acl}(S)$ (if $a \in B_1$, $a \in \mathrm{acl}(S_a) \subseteq \mathrm{acl}(S)$).

$M = \mathrm{acl}(B_1) \subseteq \mathrm{acl}(\mathrm{acl}(S)) = \mathrm{acl}(S)$.

Take $c \in B_2 \setminus S$. $c \in \mathrm{acl}(S)$. $S \subseteq B_2 \setminus \{c\}$.

$c \in \mathrm{acl}(B_2 \setminus \{c\})$, so B_2 not indep. $\Rightarrow\Leftarrow$. $\square$

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§7 Totally transcendental theories

t.t.

If $L_0 \subseteq L$, let $S_n^{L_0}(A) = \{$ n-types / A in $M \restriction$ $\}$,

Recall, $S_n(A)$ is scattered $\iff R(S_n(A)) < \infty$.

no ~~nonperfect~~ perfect non-empty sets

$R(S_n(A))$ is an ordinal (not "is finite").

Lemma 7.1 Fix $n < \infty$. TFAE.

1) $R(S_n(M)) < \infty$. $(RM(M^n) < \infty$

2) $R(S_n(A)) < \infty$ $\forall A \subseteq M$

3) $R(S_n(A)) < \infty$ $\forall A \subseteq M$ with $|A| \leq \aleph_0$

4) $R(S_n^{L_0}(A)) < \infty$ $\forall$ countable $A \subseteq M$ and countable $L_0 \subseteq L$.

Proof By Lemma 5.3,

$\neg(1)$ says $\exists \{D_\sigma\}_{\sigma \in 2^{<\omega}}$ $D_\sigma \subseteq M^n$ definable,

$D_\sigma \neq \emptyset$

$D_\sigma = D_{\sigma 0} \sqcup D_{\sigma 1}$ $\quad D_\sigma = D_{\sigma 0} \cup D_{\sigma 1}$

$\emptyset = D_{\sigma 0} \cap D_{\sigma 1}$

$\neg(3)$ says $\exists A \subseteq M$, $|A| \leq \aleph_0$,

$\exists \{D_\sigma\}_{\sigma \in 2^{<\omega}}$, $D_\sigma \neq \emptyset$ $D_\sigma \subseteq M^n$

$D_\sigma = D_{\sigma 0} \sqcup D_{\sigma 1}$

$\ldots \sigma \in 2^{<\omega}$, $D_\sigma \vdash p$ $\ldots = |M|$

$D_\sigma = D_{\sigma 0} \sqcup D_{\sigma 1}$

and D_σ is A-definable.

<u>Theorem 7.2</u> Suppose $|L| = \aleph_0$. TFAE.

1) $RM(\Sigma(\bar{x})) < \infty$ for any $\Sigma(x_1, \ldots x_n)$ partial type $\quad (R(S_n(M)) < \infty \ \forall n.)$

2) $RM(x=x) < \infty$, i.e. $RM(M') < \infty$. $\quad (R(S_1(M)) < \infty)$

3) T is ω-stable $\aleph_0$-stable

<u>Proof</u> $1 \Rightarrow 2$: clear.

$2 \Rightarrow 3$: $(2) \Rightarrow R(S_1(M)) < \infty$. Lem 7.1 $\Rightarrow$ if $A \subseteq M$ and $|A| \leq \aleph_0$, then $R(S_1(A)) < \infty$.

Lemma 5.3(3) $\Rightarrow$ $|S_1(A)| \leq |L| + |A| \leq \aleph_0$.

$|A| \leq \aleph_0 \Rightarrow |S_1(A)| \leq \aleph_0$. T is $\aleph_0$-stable.

$3 \Rightarrow 1$: By ω-stability, If $|A| \leq \aleph_0$, then $|S_n(A)| \leq \aleph_0 < 2^{\aleph_0}$

Lemma 5.3(2), $R(S_n(A)) < \infty$.

Lemma 7.3 $\Rightarrow$ $R(S_n(M)) < \infty$. $\square$.

<u>Theorem 7.3</u> For any T, TFAE

1) $RM(\Sigma(x_1, \ldots x_n)) < \infty$

2) $RM(M') = RM(x=x) < \infty$

3) $T \upharpoonright L_0$ is ω-stable for any countable $L_0 \subseteq L$.

<u>Proof</u> Like Theorem 7.2

<u>Def</u> T is <u>totally transcendental</u> if (1)–(3) hold.

<u>Ex</u> Strongly min. $\Rightarrow$ t.t. because $RM(M') = 1 < \infty$.

<u>Theorem 7.6</u> If T is t.t., then T is λ-stable $\forall \lambda \geq |L|$.

<u>Proof</u> If $\lambda \geq |L|$, $A \subseteq M$ $|A| \leq \lambda$

then $S_1(A)$ is scattered by Lemma 7.1.

$R(S_1(A)) < \infty$

Lemma 5.3(3), $|S_1(A)| \leq |L| + |A| \leq \lambda$. $\square$.

<u>Cor</u> t.t. $\Rightarrow$ superstable.

Cor t.t. $\Rightarrow$ superstable.

§8. RM and forking.

Assume T is t.t.

Lemma 8.1 Suppose $p \in S_n(A)$.

1) If $q \in S_n(\mathbb{M})$, $q \supseteq p$, then $RM(q) \le RM(p)$
2) $\exists$ $r \in S_n(\mathbb{M})$, $r \supseteq p$ with $RM(r) = RM(p)$
3) If $q \in S_n(\mathbb{M})$, $q \supseteq p$, and $RM(q) = RM(p)$, then $q \sqsupseteq p$
4) If $q \in S_n(\mathbb{M})$, $q \supseteq p$ and $q \sqsupseteq p$, then $RM(q) = RM(p)$.

Proof 1-2: by definition:
$$RM(p) = \max\{RM(q) : q \in S_n(\mathbb{M}), q \supseteq p\}$$

3) Prop 4.8 $\Rightarrow$ $\{q \in S_n(\mathbb{M}) : q \supseteq p, RM(q) = RM(p)\}$ is finite.

(If $C \subseteq S_n(A)$ is closed, and $R(C) = \alpha < \infty$, then $C \cap E_\alpha$ is finite).

If $\sigma \in Aut(\mathbb{M}/A)$, $RM(\sigma(q)) = RM(q) = RM(p)$.
$\sigma(q) \in \{$that set$\}$

$\{\sigma(q) : \sigma \in Aut(\mathbb{M}/A)\}$ is finite, q is almost A-def.

So $q \sqsupseteq p$. (by Prop 5.6 in 4-21).

4) Fix r as in (2).
By (3), $r \sqsupseteq p$.
Suppose $q \sqsupseteq p$. $\exists \sigma \in Aut(\mathbb{M}/A)$, $\sigma(r) = q$. (Prop 4.5, 4-21).
$RM(q) = RM(r) = RM(p)$. $\square$

If $q \in S_n(\mathbb{M})$, $q \supseteq p$, then
$$RM(q) = RM(p) \iff q \sqsupseteq p$$

Prop 8.2 If $p \in S_n(A)$, $A \subseteq B$, $q \in S_n(B)$ $q \supseteq p$.
then
1) $RM(q) \le RM(p)$
2) $RM(q) = RM(p) \iff q \sqsupseteq p$.

Proof 1) "Clear"... $\max\{RM(r) : r \in S_n(\mathbb{M}), r \supseteq \underset{q}{\overset{\text{or}}{p}}\}$. $[q] \subseteq [p] \subseteq S_n(\mathbb{M})$.

2) Take $r \in S_n(\mathbb{M})$, $r \sqsupseteq q$. $RM(r) = RM(q)$. by Lemma 8.1.

So
$$q \sqsupseteq p \iff r \sqsupseteq p \iff RM(r) = RM(p) \iff RM(q) = RM(p). \quad \square$$

So

($r \supseteq q$ and $\cancel{r \exists q}$) $\quad q \supseteq p \iff r \supseteq p \overset{}{\underset{8.1}{\iff}} RM(r) = RM(p) \iff RM(q) = RM(p)$. $\square$

(Gives another proof of superstability)

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§9 Recall $p \in S_n(A)$ is isolated if $\{p\}$ is clopen

$\{p\} = [\varphi]$. $\quad \varphi \in L(A)$

$a \models p \iff \models \varphi(a)$.

Lemma 9.1 If $S_n(A)$ is scattered, if $U \subseteq S_n(A)$ clopen, nonempty, then $\exists\, p \in S_n(A)$ isolated, $p \in U$.

Proof U is not perfect, take $p \in U$ isolated in U.

$\exists$ clopen V, $\quad V \cap U = \{p\}$.

$V \cap U$ is clopen. so p is isolated in $S_n(A)$.

Theorem 9.2 If T is t.t., if $D \subseteq M^n$ A-definable, $D \neq \emptyset$, then $\exists\, \bar{b} \in D$, $tp(\bar{b}/A)$ is isolated in $S_n(A)$.

Proof T is t.t. $\Rightarrow R(S_n(M)) < \infty \overset{7.1}{\Longrightarrow} R(S_n(A)) < \infty$.

$\Rightarrow S_n(A)$ scattered, $\overset{9.1}{\Longrightarrow} \exists\, p \in S_n(A)$, p is isolated, $p \in [D]$.

take $\bar{b} \models p$, $\quad tp(\bar{b}/A) = p$ is isolated, $\bar{b} \in D$ because $p \in [D]$.