Homework5

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Exercise 1. Let M be a structure and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Let $M^{\mathcal{U}}$ be the ultrapower $\prod_{i\in\mathbb{N}} M/\mathcal{U}$. Let $f : M \to M^{\mathcal{U}}$ be the map sending $a \in M$ to the class of $(a, a, a, \dots)$, i.e., the class of constant function $g_a(x) = a$. Show that f is an elementary embedding.

Proof. Let $M_i = M$ for all $i \in \mathbb{N}$, $\alpha_a = f(a)$

First we show that f is an embedding. f is injective from definition.

1. For any constant symbol c, $c^M \in M$. Then $c^M = a$ and $M \vDash c = a$. As $\{i \in \mathbb{N} : M_i \vDash c = a\} = \mathbb{N} \in \mathcal{U}$, we have $M^{\mathcal{U}} \vDash c = f(a)$ and $f(c^M) = f(a) = c^{M^{\mathcal{U}}}$.
2. For any function symbol h and suppose $M \vDash h(a_1, \dots, a_n) = a$. Then $\{i \in \mathbb{N} : M_i \vDash h(a_1, \dots, a_n)\} = \mathbb{N} \in \mathcal{U}$. Hence $M^{\mathcal{U}} \vDash h(f(a_1), \dots, f(a_n)) = f(a)$. Thus $f(h^M(a_1, \dots, a_n)) = f(a) = h^{M^{\mathcal{U}}}(f(a_1), \dots, f(a_n))$.
3. For any relation symbol R, $M \vDash R(a_1, \dots, a_n)$ iff $\{i \in \mathbb{N} : M_i \vDash R(a_1, \dots, a_n)\} = \mathbb{N} \in \mathcal{U}$ iff $M^{\mathcal{U}} \vDash R(f(a_1), \dots, f(a_n))$

Consequently f is an embedding. For arbitrary formula $\varphi(x_1, \dots, x_n)$, if $M \vDash \varphi(a_1, \dots, a_n)$ where $a_1, \dots, a_n \in M$. Then $\{i \in \mathbb{N} : M \vDash \varphi(a_1, \dots, a_n)\} = \mathbb{N} \in \mathcal{U}$. Thus $M^{\mathcal{U}} \vDash \varphi(f(a_1), \dots, f(a_n))$. If $M^{\mathcal{U}} \vDash \varphi(f(a_1), \dots, f(a_n))$, then $S = \{i \in \mathbb{N} : M \vDash \varphi(a_1, \dots, a_n)\} \in \mathcal{U}$. As $S \neq \emptyset$, $M \vDash \varphi(a_1, \dots, a_n)$ and $S = \mathcal{U}$. Thus we have

$$M \vDash \varphi(a_1, \dots, a_n) \Leftrightarrow M^{\mathcal{U}} \vDash \varphi(f(a_1), \dots, f(a_n))$$

and f is elementary □

Exercise 2 (2). Say that a set $S \subseteq \mathbb{N}$ is **cofinite** if $\mathbb{N} \setminus S$ is finite. Show that there is an ultrafilter $\mathcal{U}$ containing every cofinite set (and possibly other sets as well)

Proof. Let $A = \{S \subseteq \mathbb{N} : S \text{ is cofinite}\}$. For any S_1, S_2, $S_1 \cap S_2 = \mathbb{N} - ((\mathbb{N} - S_1) \cup (\mathbb{N} - S_2))$. As $\mathbb{N} - S_1$ and $\mathbb{N} - S_2$ is finite, $S_1 \cap S_2$ is still cofinite. Thus A has finite intersection property and there is a filter $\mathcal{F} \supseteq A$. Hence there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F} \supseteq A$ □

Exercise 3. For $i \in \mathbb{N}$, let $\mathcal{U}_i$ be the set $\{S \subseteq \mathbb{N} : i \in S\}$. Show that $\mathcal{U}_i$ is an ultrafilter on $\mathbb{N}$

Proof.

1. As $i \in \mathbb{N}$ and $i \notin \emptyset$, we have $\mathbb{N} \in \mathcal{U}_i$ and $\emptyset \notin \mathcal{U}_i$
2. If $A \in \mathcal{U}_i$ and $A \subseteq B$, then $i \in A \subseteq B$, hence $B \in \mathcal{U}_i$
3. If $A, B \in \mathcal{U}_i$, then $i \in A$ and $i \in B$, so $i \in A \cap B$. Thus $A \cap B \in \mathcal{U}$
4. For any $A \subseteq \mathbb{N}$, either $i \in A$ or $i \notin A$, whence either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$

Consequently, $\mathcal{U}$ is an ultrafilter □

Exercise 4. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Show that $\mathcal{U}$ contains every cofinite set.

Proof. If there is a cofinite set S that isn't belong to $\mathcal{U}$. Then $S' = \mathbb{N} \setminus S$ is finite and belongs to $\mathcal{U}$. For every element $s \in S'$, either $\{s\} \in \mathcal{U}$ or $\mathbb{N} \setminus \{s\} \in \mathcal{U}$. If $\{s\} \in \mathcal{U}$, then $\mathcal{U}$ is principal. So for all $s \in S'$, $\mathbb{N} \setminus \{s\} \in \mathcal{U}$. Then the finite intersection $S' \cap \bigcap_{s \in S'} (\mathbb{N} \setminus \{s\}) = \emptyset \in \mathcal{U}$, which is impossible. Thus $\mathcal{U}$ contains every cofinite set. □

Exercise 5. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing every cofinite set. Let $\mathbb{R}^{\mathcal{U}}$ be the ultrapower of the structure $(\mathbb{R}, +, \cdot, 0, 1)$. Show that $\mathbb{R}^{\mathcal{U}}$ is not isomorphic to $\mathbb{R}$

Proof. Let $a \in \mathbb{R}^{\mathbb{N}}$ be the tuple $(0, 1, 2, 3, 4, 5, \dots)$ s.t. a is a function $\mathbb{N} \to \mathbb{R}$ s.t. for all $i \in \mathbb{N}$, $a(i) = i$. For each $n \in \mathbb{N}$, let $\varphi_n(x, y)$ be

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} + y \cdot y = x$$

As. Let $b : \mathbb{N} \to \mathbb{R}$ be

$$b(i) = \begin{cases} 0 & i \leq n \\ \sqrt{i - n} & i > n \end{cases}$$

We claim that $\mathbb{R}^{\mathcal{U}} \vDash \varphi_n([a], [b])$. Let $S = \{i \in \mathbb{N} : \mathbb{R} \vDash \phi(a(i), b(i))\}$. Then $\mathbb{N} \setminus S = \{0, \dots, n-1\}$ is finite. Thus $S \in \mathcal{U}$. As $\mathbb{R}^{\mathcal{U}} \vDash \varphi_n([a], [b]) \Leftrightarrow S = \{i \in \mathbb{N} : \mathbb{R} \vDash \varphi_n(a(i), b(i))\} \in \mathcal{U}$, we have $\mathbb{R}^{\mathcal{U}} \vDash \varphi_n([a], [b])$ and $\mathbb{R}^{\mathcal{U}} \vDash \exists y \varphi_n([a], y)$. Let $\Gamma(x) = \{\exists y \varphi_n(x, y) : n \in \mathbb{N}\}$. Then $\Gamma(x)$ is realized in $\mathbb{R}^{\mathcal{U}}$ but not in $\mathbb{R}$

since there is no such x whenever $n > x$. If there is such a isomorphism f, then $f^{-1}([a])$ realizes $\Gamma(x)$, which is impossible. Thus $\mathbb{R}^{\mathcal{U}}$ is not isomorphic to $\mathbb{R}$. $\square$