Beth's theorem and saturated models

Introductory Model Theory

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Recommended reading: This material is covered in Sections 9.1–9.3 of the textbook, but using a different approach involving p-equivalence and resplendent models.

1 Expansions and reducts

Definition 1. Let $L \subseteq L'$ be languages.

1. If M is an L'-structure, then $M \upharpoonright L$ is the L-structure obtained by forgetting the symbols in $L' \setminus L$.

2. Let M be an L-structure and N be an L'-structure. Then M is a *reduct* of N, and N is an *expansion* of M, if $M = N \upharpoonright L$.

Lemma 2. *Let M be a κ-saturated L-structure. For $L_0 \subseteq L$, the reduct $M \upharpoonright L_0$ is κ-saturated.*

Proof. Let A be a subset of M with $|A| < \kappa$. Let p be a complete 1-type over A in $M \upharpoonright L_0$. Then p is a finitely satisfiable set of $L(A)$-formulas, so p is realized in M. □

Lemma 3. *Let M be an L-structure and κ be a cardinal. There is an L-structure $N \succeq M$ such that for every $L_0 \subseteq L$, the reduct $N \upharpoonright L_0$ is κ-saturated and κ-strongly homogeneous.*

Proof. In the proof of Lemma 14 last week, we showed

1. There is an elementary chain $\{M_\alpha\}_{\alpha<\kappa^+}$ such that $M_0 = M$, and $M_{\alpha+1}$ is $|M_\alpha|^+$-saturated for each $\alpha < \kappa^+$.

2. Given such a chain, the union $N = \bigcup_{\alpha<\kappa^+} M_\alpha$ is κ-saturated and strongly κ-homogeneous.

For any $L_0 \subseteq L$, the chain of reducts $\{M_\alpha \upharpoonright L_0\}_{\alpha<\kappa^+}$ has the same saturation properties (Lemma 2), so $N \upharpoonright L_0$ is strongly κ-homogeneous. □

2 Beth's implicit definability theorem

Let L be a language and let $L(R)$ be the language obtained by adding one new n-ary relation symbol R.

Definition 4. Let T be an $L(R)$-theory.

1. R is *implicitly defined* in T if for every L-structure M, there is at most one $R \subseteq M^n$ such that $(M, R) \models T$.

2. R is *explicitly defined* in T if there is an L-formula $\phi(x_1, \ldots, x_n)$ such that $T \vdash \forall \bar{x}\ (R(\bar{x}) \leftrightarrow \phi(\bar{x}))$.

If $\bar{a}$ is a tuple in an $L(R)$-structure M, let $\mathrm{tp}^L(\bar{a})$ denote the type in the reduct $M \restriction L$. In other words, $\mathrm{tp}^L(\bar{a})$ is the set of L-formulas satisfied by $\bar{a}$.

Lemma 5. *Suppose R is not explicitly defined in T. Then there are $M, N \models T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$, such that*

- $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$.

- $M \models R(\bar{a})$ *and* $N \models \neg R(\bar{b})$.

Proof. Suppose not. Let $S = \{\mathrm{tp}^L(\bar{a}) : M \models T, \bar{a} \in M^n\}$. For $p \in S$, one of two things happens:

1. Every realization of p satisfies R.

2. Every realization of p satisfies $\neg R$.

Otherwise, we can find a realization $\bar{a}$ satisfying R and a realization $\bar{b}$ satisfying $\neg R$, as desired.

By compactness, for each $p \in S$ there is an L-formula $\phi_p(\bar{x}) \in p(\bar{x})$ such that one of two things happens:

1. $T \cup \{\phi_p(\bar{x})\} \vdash R(\bar{x})$

2. $T \cup \{\phi_p(\bar{x})\} \vdash \neg R(\bar{x})$.

(p is closed under conjuction: if $\phi_1, \ldots, \phi_n \in p$, then $\phi_1 \wedge \cdots \wedge \phi_n \in p$. This is why we only need one formula from p.)

Let $\Sigma(\bar{x}) = T \cup \{\neg\phi_p(\bar{x}) : p \in S\}$. If $\Sigma(\bar{x})$ is consistent, there is $M \models T$ and $\bar{a} \in M^n$ satisfying $\Sigma(\bar{x})$. Let $p = \mathrm{tp}^L(\bar{a})$. Then $\bar{a}$ satisfies p, so it satisfies ϕ_p, but it also satisfies $\neg\phi_p$ (by choice of Σ), a contradiction.

Therefore $\Sigma(\bar{x})$ is inconsistent. By compactness there are $p_1, \ldots, p_n, q_1, \ldots, q_m \in S$ such that

$$\begin{aligned} T &\vdash \bigvee_{i=1}^{n} \phi_{p_i}(\bar{x}) \vee \bigvee_{i=1}^{m} \phi_{q_i}(\bar{x}) \\ T \cup \{\phi_{p_i}(\bar{x})\} &\vdash R(\bar{x}) \qquad \text{for } i = 1, \ldots, n \\ T \cup \{\phi_{q_i}(\bar{x})\} &\vdash \neg R(\bar{x}) \qquad \text{for } i = 1, \ldots, m. \end{aligned}$$

Then $T \vdash \forall \bar{x} \ \ (R(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \phi_{p_i}(\bar{x}))$. The $\leftarrow$ is by choice of the ϕ_{p_i}. The $\rightarrow$ is because if none of the ϕ_{p_i} hold, then one of the ϕ_{q_i} holds, and then $\neg R$ must hold.

Finally, $\bigvee_{i=1}^{n} \phi_{p_i}(\bar{x})$ is an explicit definition of R. □

Theorem 6 (Beth)**.** *If R is implicitly defined in T, then R is explicitly defined in T.*

Proof.

Case 1: T is complete. If R is not explicitly defined, we obtain $M, N \models T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$ with $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$, but $M \models R(\bar{a})$ and $N \models \neg R(\bar{a})$. Since T is complete, we have $M \equiv N$. By elementary amalgamation, we may find elementary embeddings $M \to N'$ and $N \to N'$. Replacing M and N by N' and N', we may assume $M = N$.

By Lemma 3, we may replace M with an elementary extension and assume M and $M \restriction L$ are $\aleph_0$-saturated and $\aleph_0$-strongly homogeneous. The fact that $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$ implies that there is an automorphism $\sigma \in \mathrm{Aut}(M \restriction L)$ with $\sigma(\bar{a}) = \bar{b}$. Let $R' = \sigma(R)$. Let $M' = (M \restriction L, R')$. Then σ is an isomorphism from M to M', so $M' \models T$. But $M' \restriction L = M \restriction L$. Because R is implicitly defined, $R = R'$. But then

$$\bar{a} \in R \iff \sigma(\bar{a}) \in \sigma(R) \iff \bar{b} \in R' \iff \bar{b} \in R,$$

contradicting the fact that $M \models R(\bar{a})$ and $M \models \neg R(\bar{b})$.

Case 2: T is not complete. Any completion of T implicitly defines R. By Case 1, any completion of T explicitly defines R. So in any model $M \models T$, there is an L-formula ϕ_M such that $M \models \forall \bar{x} \ (R(\bar{x}) \leftrightarrow \phi_M(\bar{x}))$.

Assume R is not explicitly defined. By Lemma 5 there are $M, N \models T$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$, with $\mathrm{tp}^L(\bar{a}) = \mathrm{tp}^L(\bar{b})$ and $M \models R(\bar{a})$ and $N \models \neg R(\bar{b})$. Let T' be the L-theory obtained from T by replacing every "R" with ϕ_M, the L-formula defining R in M. Then $M \models T'$. The type $\mathrm{tp}^L(\bar{a})$ contains the following:

- The formula $\phi_M(\bar{x})$.
- The sentences in T' (these do not involve $\bar{x}$).

These formulas are in $\mathrm{tp}^L(\bar{b}) = \mathrm{tp}^L(\bar{a})$, so $N \models \phi_M(\bar{b})$ and $N \models T'$.

Let $R' = \{\bar{c} \in N^n : N \models \phi_M(\bar{c})\}$. Then $(N \restriction L, R') \models T$ because $N \models T'$. Therefore $R' = R$ because R is implicitly defined. But $N \models \phi_M(\bar{b})$ and $N \models \neg R(\bar{b})$, a contradiction. □

3 Saturated models

Definition 7. A structure M is *saturated* if it is $|M|$-saturated. That is, for any $A \subseteq M$ with $|A| < |M|$ and any $p \in S_1(A)$, the type p is realized in M.

Example. A countably infinite model M is saturated iff it is ω-saturated.

Lemma 8. *Let M, N be saturated structures of cardinality κ. Let $A \subseteq M, B \subseteq N$ be sets and $f : A \to B$ be a partial elementary map from M to N. Suppose $|A| = |B| < \kappa$. Then we can extend f to an isomorphism $g : M \to N$.*

Proof. Let $\{a_\alpha : \alpha < \kappa\}$ and $\{b_\alpha : \alpha < \kappa\}$ be enumerations of M and N. Recursively define an increasing chain of small partial elementary maps $\{f_\alpha\}_{\alpha<\kappa}$ as follows:

1. $f_0 = f$.
2. Given f_α, take some $c \in N$ such that $f_\alpha \cup \{(a_\alpha, c)\}$ is a partial elementary map. Take some $d \in M$ such that $f_\alpha \cup \{(a_\alpha, c)\} \cup \{(d, b_\alpha)\}$ is a partial elementary map. Let $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, c)\} \cup \{(d, b_\alpha)\}$. Both steps are possible by Lemma 6 last week.
3. If α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Let $g = \bigcup_{\alpha<\kappa} f_\alpha$. Then $\mathrm{dom}(g)$ contains every a_α and $\mathrm{im}(g)$ contains every b_α. Thus $\mathrm{dom}(g) = M$ and $\mathrm{im}(g) = N$. It follows that g is an isomorphism from M to N. □

Theorem 9. *Let M, N be saturated models of cardinality κ. If $M \equiv N$, then $M \cong N$.*

Proof. Apply Lemma 8 to $f = \varnothing$. □

Theorem 10. *Let M be a saturated model. Let $f : A \to B$ be a partial elementary map from M to M. If $|A| < |M|$, then f extends to an automorphism $\sigma \in \mathrm{Aut}(M)$.*

In other words, M is strongly $|M|$-homogeneous.

Proof. Apply Lemma 8 to $N = M$. □

4 Countable saturated models

Lemma 11. *Let M be a structure and $A = \{a_1, \dots, a_m\}$ be a finite subset. Suppose $N, K \succeq M$. Suppose $\bar{b} \in N^n$ and $\bar{c} \in K^n$. Then*

$$\mathrm{tp}^N(\bar{b}/A) = \mathrm{tp}^K(\bar{c}/A) \iff \mathrm{tp}^N(\bar{b}, \bar{a}) = \mathrm{tp}^K(\bar{c}, \bar{a}).$$

Proof. Every $L(A)$-formula has the form $\varphi(\bar{x}, \bar{a})$ for some L-formula $\varphi(\bar{x}, y_1, \dots, y_m)$. So both sides say that for any L-formula $\varphi(\bar{x}, y_1, \dots, y_m)$,

$$M \models \varphi(\bar{b}, \bar{a}) \iff N \models \varphi(\bar{c}, \bar{a}).$$

□

If T is a complete theory, then $S_n(T)$ denotes $S_n^M(\varnothing)$ for some $M \models T$. (The choice of M doesn't matter.)

Definition 12. A complete theory T is *small* if $S_n(T)$ is countable for all n.

Lemma 13. *If M is a model of a small theory T, and $A \subseteq_f M$, then $S_n(A)$ is countable.*

Proof. Let $A = \{a_1, \ldots, a_m\}$. Define a map $f : S_n(A) \to S_{n+m}(\varnothing) = S_{n+m}(T)$ sending $\mathrm{tp}^N(\bar{b}/A)$ to $\mathrm{tp}^N(\bar{b}, \bar{a})$ for any $N \succeq M$ and $\bar{b} \in N^n$. This is well-defined and injective by Lemma 11. So $|S_n(A)| \leq |S_{n+m}(T)| \leq \aleph_0$. $\square$

Theorem 14. *Let T be a complete theory. Then T has a countable ω-saturated model if and only if T is small.*

Proof. First suppose there is a countable ω-saturated model M. Every type in $S_n(T) = S_n(\varnothing)$ is realized in M, so $S_n(T)$ is countable.

Conversely, suppose $S_n(T)$ is countable for any n. Take some ω-saturated model M^+. For each finite set $A \subseteq M^+$ and type $p \in S_1(A)$, take some element $c_{A,p} \in M$ realizing p. Define an increasing chain of countable subsets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots M^+$ as follows:

- $A_0 = \varnothing$.
- $A_{i+1} = A_i \cup \{c_{A,p} : A \subseteq_f A_i,\ p \in S_1(A)\}$.

This is countable because A_i has countably many finite subsets, and for each finite subset A, the type space $S_1(A)$ is countable by Lemma 13.

Define $M = \bigcup_{i=0}^{\infty} A_i$. This is a countable union of countable sets, so it is countable.

Claim. Work in M^+. If $A \subseteq_f M$ and $p \in S_1(A)$, then p is realized by an element of M.

Proof. Take i large enough that $A \subseteq A_i$. Then $c_{A,p} \in A_{i+1} \subseteq M$ and $c_{A,p}$ realizes p. $\square_{\text{Claim}}$

Claim. $M \preceq M^+$.

Proof. Use the Tarski-Vaught criterion. Suppose $M^+ \models \exists x\ \varphi(\bar{a}, y)$ for some L-formula $\phi(\bar{x}, y)$ and some tuple $\bar{a} \in M^n$. Take $b \in M^+$ such that $M^+ \models \phi(\bar{a}, b)$. By the previous claim, $\mathrm{tp}(b/\bar{a})$ is realized by some c in M. Then $M^+ \models \phi(\bar{a}, c)$, as needed. $\square_{\text{Claim}}$

Therefore $M \models T$. Finally, we show M is ω-saturated. Suppose $A \subseteq_f M$ and $p \in S_1^M(A)$. Because $M \preceq M^+$, we have $S_1^M(A) = S_1^{M^+}(A)$. By the first claim, there is $b \in M$ with $\mathrm{tp}^{M^+}(b/A) = p$. Then $\mathrm{tp}^M(b/A) = \mathrm{tp}^{M^+}(b/A)$ because $M \preceq M^+$, so b realizes p in M. $\square$

5 Appendix: how to think about implicit definitions

Let T be an L-theory. Let $L(R)$ be L plus a new n-ary relation symbol R.

- An *explicit definition* of R in terms of L-formulas is an expression of the form

$$\forall \bar{x}\ (R(\bar{x}) \leftrightarrow \varphi(\bar{x}))$$

  for some L-formula φ.

- An *implicit definition* of R in T is an $L(R)$-theory T' such that for every model $M \models T$, there is a unique $R \subseteq M^n$ such that $(M, R) \models T'$.

Here is a more conventional statement of Beth's theorem:

Theorem 15. *Let T' be an implicit definition of R in T. Then there is an explicit definition of R in terms of L-formulas.*

More precisely, there is an L-formula $\varphi(x_1, \ldots, x_n)$ such that for any model $M \models T$, the unique $R \subseteq M^n$ satisfying T' is defined by $\varphi(x_1, \ldots, x_n)$.

Proof. Apply Theorem 6 to the $L(R)$-theory $T \cup T'$. There is an L-formula $\varphi(\bar{x})$ such that $T \cup T' \vdash \forall \bar{x}\ (R(\bar{x}) \leftrightarrow \varphi(\bar{x}))$. Suppose $M \models T$. Let R be the unique n-ary relation such that $(M, R) \models T'$. Then $(M, R) \models T$ (since T is an L-theory and the reduct M satisfies T), and so $(M, R) \models T \cup T'$. Consequently $(M, R) \models \forall \bar{x}\ (R(\bar{x}) \leftrightarrow \varphi(\bar{x}))$, and R is defined by φ as claimed. $\square$

Example. Let L be the language of $(\mathbb{R}, +, \cdot, 0)$ and let T be the complete theory of $(\mathbb{R}, +, \cdot, 0)$. Let R be the 2-ary relation symbol $\leq$. Let T' be the set of sentences

$$\forall x, y, z : x \leq y \rightarrow x + z \leq y + z$$
$$\forall x, y, z : (x \leq y\ \wedge\ 0 \leq z) \rightarrow (xz \leq yz).$$

One of the homework problems is to show that this set of sentences is an implicit definition of $\leq$ in T: if $(M, +, \cdot, 0)$ is a model of T then there is a unique binary relation $\leq$ on M satisfying T'.

Beth's theorem then implies that there must be an explicit definition of $\leq$ in terms of L-formulas. Another homework problem is to find such a definition.

Remark 16. Above we have discussed implicit definitions of relation symbols. But we can also define function symbols implicitly, and Beth's theorem works. In fact, an n-ary function is a special kind of $(n+1)$-ary relation. Similarly, we can implicitly define constant symbols. Constant symbols are 0-ary function symbols.

Remark 17. The terms "implicit definition" and "explicit definition" are by analogy with definitions of functions in calculus/analysis. An *explicit definition* of a function is something like

$$f(x) = \frac{x}{x^2 + 1}$$

which tells you exactly how to calculate f. An *implicit definition* is something like

$$f(x)^5 + f(x) + x = 0,$$

a statement about f which uniquely determines f, but doesn't tell you how to calculate f. (In this case, $f(x)$ is the Bring radical of x.) The Implicit function theorem[1] gives a sufficient criterion for an implicit definition to work.

[1] Sometimes called the Hidden Function Theorem in Chinese.