Coheirs and invariant types

Advanced Model Theory

March 10, 2022

Reference in the book: Section 12.1–12.2 (loosely).

1 Conclusion of the last lecture

Let λ be an infinite cardinal. Recall that T is λ-*stable* if the following equivalent conditions hold:

1. For any $A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$.
2. For any $A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_n(A)| \leq \lambda$.

(These are equivalent by Lemma 11 in last week's notes.) If $\lambda \geq |L|$, then (1) and (2) are also equivalent to the following:

3. For any $M \preceq \mathbb{M}$, if $|M| \leq \lambda$, then $|S_1(M)| \leq \lambda$.
4. For any $M \preceq \mathbb{M}$, if $|M| \leq \lambda$, then $|S_n(M)| \leq \lambda$.

(1) $\implies$ (3) trivially. Conversely, assume (3). If $A \subseteq \mathbb{M}$ and $|A| \leq \lambda$, then downward Löwenheim-Skolem gives a small model $M \preceq \mathbb{M}$ with $M \supseteq A$ and $|M| \leq \lambda$. Every type over A extends to a type over M, so $|S_1(A)| \leq |S_1(M)| \leq \lambda$. Thus (1) $\iff$ (3). The equivalence (2) $\iff$ (4) is similar.

Lemma 1. *Suppose for every model M and every $p \in S_1(M)$, p is definable. Then M is λ-stable for some λ.*

Proof. Take $\lambda = 2^{|L|} > |L|$. It suffices to show that if $M \preceq \mathbb{M}$ and $|M| \leq \lambda$, then $|S_1(M)| \leq \lambda$. Every type in $S_1(M)$ is definable. A definable type is determined by the map $\varphi \mapsto d\varphi$. There are $|L|$-many possibilities for φ, and $|L(M)|$-many possibilities for $d\varphi$. So the number of (definable) types is at most $|L(M)|^{|L|} \leq \lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|^2} = 2^{|L|} = \lambda$. $\square$

Theorem 2. *The following are equivalent for a theory T with monster model $\mathbb{M}$:*

1. *All types over models are definable.*

2. *All 1-types over models are definable.*

3. *No formula $\varphi(\bar{x};\bar{y})$ has the dichotomy property.*

4. *No formula $\varphi(x;\bar{y})$ has the dichotomy property.*

5. *T is λ-stable for at least one λ.*

Proof. (5) $\implies$ (3) by Proposition 10 last week. (3) $\implies$ (4) and (1) $\implies$ (2) are trivial. (4) $\implies$ (2) and (3) $\implies$ (1) by Proposition 8 last week. (2) $\implies$ (5) by Lemma 1. $\square$

A theory is **stable** if the equivalent conditions of Theorem 2 hold.

2 Coheirs

Definition 3. Suppose $p \in S_n(M)$, $N \succeq M$, $q \in S_n(N)$, and $q \supseteq p$. Then q is a *coheir* of p if for any $L(N)$-formula $\varphi(\bar{x}) \in q(\bar{x})$, there is $\bar{a} \in M$ with $N \models \varphi(\bar{a})$.

Note if $\varphi_1, \ldots, \varphi_n \in q$, we can let $\psi = \bigwedge_{i=1}^n \varphi_i$ and then $\psi \in q$, so there is $\bar{a} \in M$ satisfying ψ, or equivalently, $\bar{a} \in M$ satisfying the finite subtype $\{\varphi_1, \ldots, \varphi_n\} \subseteq q$. Thus:

q is a coheir of p iff q is finitely satisfiable in M.

Example. p is a coheir of itself, because a type over M is finitely satisfiable in M.

Example. Suppose M is strongly minimal and $N \succeq M$. Let p and q be the transcendental 1-types over M and N, respectively. Then q is a coheir of p.

Proof. If $\varphi(x) \in q(x)$, then $\varphi(N)$ is finite or cofinite. If $\varphi(N)$ is finite, then $\varphi(x) \notin q(x)$. If $\varphi(N)$ is cofinite, it intersects M because M is infinite. $\square$

Lemma 4. *Suppose $M \preceq N$, and $\Sigma(\bar{x})$ is a partial type over N that is finitely satisfiable in M. Then there is $q \in S_n(N)$ extending $\Sigma(\bar{x})$ such that q is finitely satisfiable in M, i.e., q is a coheir of $q \restriction M$.*

Proof. Let $\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(N) : \forall \bar{a} \in M\ (N \models \psi(\bar{a}))\}$. Any tuple $\bar{a} \in M$ satisfies $\Psi(\bar{x})$. Because $\Sigma(\bar{x})$ is finitely satisfiable in M, so is $\Sigma(\bar{x}) \cup \Psi(\bar{x})$. (The tuple from M satisfying $\Sigma_0 \subseteq_f \Sigma$ will also satisfy Ψ.) Take a completion $q \in S_n(N)$ of $\Sigma \cup \Psi$. We claim q is finitely satisfiable in M. Take $\varphi(\bar{x}) \in q(\bar{x})$. If there is no $\bar{a} \in M$ satisfying φ, then $\neg\varphi \in \Psi$, so $\neg\varphi \in q$, a contradiction. $\square$

Here are two consequences of the lemma:

Theorem 5 (Coheirs exist)**.** *If $p \in S_n(M)$ and $N \succeq M$, then there is $q \in S_n(N)$ a coheir of p.*

Proof. Apply the Lemma with $\Sigma(\bar{x}) = p(\bar{x})$. $\square$

Theorem 6. *Suppose $M_1 \preceq M_2 \preceq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2$. If p_2 is a coheir of p_1, then there is $p_3 \in S_n(M_3)$ a coheir of p_2 and p_1.*

Proof. Take $p_3 \in S_n(M_3)$ finitely satisfiable in M_1, extending p_2. $\square$

Warning 7. A coheir of a coheir of p needn't be a coheir of p.

3 Invariant types

Work in a monster model $\mathbb{M}$. Let $A \subseteq \mathbb{M}$ be a small set. Recall

$$\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A) \iff \exists \sigma \in \mathrm{Aut}(\mathbb{M}/A)\ (\sigma(\bar{a}) = \bar{b}).$$

Lemma 8. *The following are equivalent for $X \subseteq \mathbb{M}^n$:*

1. $\sigma(X) = X$ *for* $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.
2. *If* $\bar{a}, \bar{b} \in \mathbb{M}^n$, *then*

$$\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A) \implies (\bar{a} \in X \iff \bar{b} \in X).$$

3. *There is a function* $f : S_n(A) \to \{0, 1\}$ *such that* $X = \{\bar{a} \in M^n : f(\mathrm{tp}(\bar{a}/A)) = 1\}$.

Proof. (2) $\iff$ (3) is clear.

(1) $\iff$ (2): we can rewrite (2) as follows:

If $\bar{a}, \bar{b} \in \mathbb{M}^n$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ and $\bar{b} = \sigma(\bar{a})$, then $\bar{a} \in X \iff \bar{b} \in X$.

Or equivalently,

If $\bar{a} \in \mathbb{M}^n$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, then $\bar{a} \in X \iff \sigma(\bar{a}) \in X$.

Or equivalently,

If $\bar{a} \in \mathbb{M}^n$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/a)$, then $\bar{a} \in X \iff \bar{a} \in \sigma^{-1}(X)$

or equivalently, $X = \sigma^{-1}(X)$, which is equivalent to $X = \sigma(X)$. $\square$

Definition 9. A set $X \subseteq \mathbb{M}^n$ is $\mathrm{Aut}(\mathbb{M}/A)$-*invariant*, or A-*invariant* for short, if the equivalent conditions of Lemma 8 hold.

Example. If $D \subseteq \mathbb{M}^n$ is A-definable (defined by an $L(A)$-formula), then D is A-invariant.

Lemma 10. *If $D \subseteq \mathbb{M}^n$ is definable and A-invariant, then D is A-definable.*

Proof. The usual compactness arguments…

Step 1: If $\bar{a} \in D$, then $\mathrm{tp}(\bar{a}/A) \vdash \bar{x} \in D$. By compactness/saturation, there is an $L(A)$-formula $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/A)$ such that $\varphi(\bar{x}) \vdash \bar{x} \in D$. That is, $\varphi(\mathbb{M}^n) \subseteq D$.

Step 2: D is covered by A-definable sets $\varphi(\mathbb{M}^n)$. By compactness/saturation, D is covered by finitely many $\{\varphi_i(\mathbb{M}^n) : 1 \leq i \leq m\}$. Then D is the A-definable set $\bigcup_{i=1}^m \varphi_i(\mathbb{M}^n)$. $\square$

Definition 11. A *global type* is a complete type over the monster.

Definition 12. A global type $p \in S_n(\mathbb{M})$ is *A-invariant* if $\sigma(p) = p$ for $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$. Equivalently, p is A-invariant if the sets

$$\{\bar{b} \in \mathbb{M}^n : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

are A-invariant for each $\varphi \in L$.

Explicitly, p is A-invariant if for any formula $\varphi(\bar{x}, \bar{y})$ and any $\bar{b}, \bar{c} \in \mathbb{M}$,

$$\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A) \implies (\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \varphi(\bar{x}, \bar{c}) \in p(\bar{x})).$$

Definition 13. A global type $p \in S_n(\mathbb{M})$ is *A-definable* if the sets

$$\{\bar{b} \in \mathbb{M}^n : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

are A-definable for each $\varphi \in L$.

Remark 14.

1. An A-definable type is A-invariant.
2. A definable type that is A-invariant is A-definable.
3. Every definable type is A-invariant for some small $A \subseteq \mathbb{M}$, since each $d\varphi$ uses finitely many parameters and there are only $|L|$-many $d\varphi$.

Proposition 15. *Suppose $M \preceq \mathbb{M}$ is a small model.*

1. *If p is a definable type over M and $p^{\mathbb{M}}$ is its heir over $\mathbb{M}$, then $p^{\mathbb{M}}$ is M-definable.*
2. *This gives a bijection between definable types over M, and M-definable types over $\mathbb{M}$.*

Proof.

1. $p^{\mathbb{M}}$ and p have the same $d\varphi$, which are $L(M)$-formulas, so $p^{\mathbb{M}}$ is M-definable.
2. If $q \in S_n(\mathbb{M})$ is M-definable, then $q \upharpoonright M$ is M-definable and $q = (q \upharpoonright M)^{\mathbb{M}}$. Therefore $q \mapsto q \upharpoonright M$ is the inverse of $p \mapsto p^{\mathbb{M}}$. □

Warning 16. An M-invariant type is *not* necessarily determined by its restriction to M.

If $A \subseteq \mathbb{M}$ isn't a model, an A-definable type is *not* necessarily determined by its restriction to A.

Theorem 17. *Suppose $M \preceq \mathbb{M}$ is a small model and $p \in S_n(M)$ is any type.*

1. *If $q \in S_n(\mathbb{M})$ is a coheir of p, then q is M-invariant.*
2. *In particular, there is an M-invariant global type $q \supseteq p$.*

Proof. If q isn't M-invariant, then there are $\varphi(\bar{x}, \bar{y})$, $\bar{b}$, and $\bar{c}$ such that

$$\mathrm{tp}(\bar{b}/M) = \mathrm{tp}(\bar{c}/M) \text{ but } \varphi(\bar{x}, \bar{b}) \in q(\bar{x}),\ \varphi(\bar{x}, \bar{c}) \notin q(\bar{x}).$$

Then $\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$, so there is $\bar{a} \in M$ such that $\mathbb{M} \models \varphi(\bar{a}, \bar{b}) \wedge \neg\varphi(\bar{a}, \bar{c})$. This contradicts $\mathrm{tp}(\bar{b}/M) = \mathrm{tp}(\bar{c}/M)$.

(2) follows because coheirs exist by Theorem 5. □

Remark 18. If $p \in S_n(M)$ and $N \succeq M$ and $q \in S_n(N)$ extends p, then Poizat calls q a *special son* of p if there is a global M-invariant type $q' \in S_n(\mathbb{M})$ extending q. Theorems 17 and 6 imply that coheirs are special sons.

4 Coheirs and invariant types in stable theories

Suppose T is stable, meaning all types over models are definable. Then any $p \in S_n(M)$ has a unique global heir.

Lemma 19. *If T is stable, then A-invariant global types are A-definable.*

Proof. If $p \in S_n(\mathbb{M})$ is A-invariant, then p is definable by stability, so p is A-definable by Remark 14(2). $\square$

Theorem 20. *Suppose T is stable, $M \preceq \mathbb{M}$ is a small model, and $p \in S_n(M)$. Let $p^{\mathbb{M}}$ be the unique heir of p over $\mathbb{M}$.*

1. *$p^{\mathbb{M}}$ is the unique M-invariant global type extending p.*
2. *$p^{\mathbb{M}}$ is the unique coheir of p over $\mathbb{M}$.*
3. *If $M \preceq N \preceq \mathbb{M}$ and q is the unique heir of p over N, then q is the unique coheir of p over N.*

Proof.

1. "M-invariant" is equivalent to "M-definable," and Proposition 15 shows that $p^{\mathbb{M}}$ is the unique M-definable global type extending M.
2. Coheirs are M-invariant (Theorem 17(1)).
3. Suppose q' is a coheir of p over N. Theorem 6 gives $r \in S_n(\mathbb{M})$ a coheir of p and q'. Then $r = p^{\mathbb{M}}$ by (2), so $q' = (p^{\mathbb{M}} \restriction N) = q$. $\square$

Corollary 21. *In a stable theory, heirs are the same thing as coheirs, and coheirs are unique.*

Corollary 22. *In a stable theory, a coheir of a coheir is a coheir.*

5 Products of invariant types

Work in a monster model $\mathbb{M}$, not assumed to be stable. If $A \subseteq \mathbb{M}$ is small and $p \in S_n(A)$, then "$\bar{b} \models p$" means $\bar{b}$ satisfies p, i.e., $\mathrm{tp}(\bar{b}/A) = p$. Also, $\bar{a} \equiv_C \bar{b}$ means $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$.

Lemma 23. *Suppose $C \subseteq \mathbb{M}$ is small and we have two C-invariant types $p \in S_n(\mathbb{M})$ and $q \in S_m(\mathbb{M})$. Then there is $r \in S_{n+m}(C)$ such that a tuple $(\bar{a}, \bar{b})$ realizes r if and only if*

$$\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a}. \tag{*}$$

Proof. If $(\bar{a}, \bar{b})$ satisfies (*) and $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$, then $\sigma(\bar{a}, \bar{b})$ satisfies (*) as well, because p, q are C-invariant. Therefore (*) depends only on $\mathrm{tp}(\bar{a}, \bar{b}/C)$.

Take some $(\bar{a}_0, \bar{b}_0)$ satisfying (*), and let $r = \mathrm{tp}(\bar{a}_0, \bar{b}_0/C)$. Realizations of r satisfy (*). Conversely, suppose $(\bar{a}, \bar{b})$ satisfies (*). Then $\mathrm{tp}(\bar{a}/C) = \bar{a} \restriction C = \mathrm{tp}(\bar{a}_0/C)$. Moving $(\bar{a}, \bar{b})$ by $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$, we may assume $\bar{a} = \bar{a}_0$. Then (*) shows $\bar{b} \models q \restriction C\bar{a}_0$, so $\bar{b} \equiv_{C\bar{a}_0} \bar{b}_0$. Therefore $(\bar{a}, \bar{b}) = (\bar{a}_0, \bar{b}) \equiv_C (\bar{a}_0, \bar{b}_0)$, and $(\bar{a}, \bar{b}) \models r$. $\square$

Proposition 24. *Suppose $C \subseteq \mathbb{M}$ is small and we have two C-invariant types $p \in S_n(\mathbb{M})$ and $q \in S_m(\mathbb{M})$. Then there is a C-invariant type $p \otimes q \in S_{n+m}(\mathbb{M})$ such that for any $C \subseteq C' \subseteq \mathbb{M}$,*

$$(\bar{a} \models p \restriction C' \text{ and } \bar{b} \models q \restriction C'\bar{a}) \iff (\bar{a}, \bar{b}) \models (p \otimes q) \restriction C'. \qquad (**)$$

Proof. Lemma 23 gives a type $r_{C'} \in S_{n+m}(C')$ for each small $C' \supseteq C$ such that

$$(\bar{a}, \bar{b}) \models r_{C'} \iff (\bar{a} \models p \restriction C' \text{ and } \bar{b} \models q \restriction C'\bar{a}).$$

Note that if $C'' \supseteq C' \supseteq C$ then $(\bar{a}, \bar{b}) \models r_{C''} \implies (\bar{a}, \bar{b}) \models r_{C'}$, and so $r_{C'} = r_{C''} \restriction C'$. Let $p \otimes q = \bigcup_{C' \supseteq C} r_{C'}$. Then $p \otimes q$ is a global type and $(p \otimes q) \restriction C' = r_{C'}$, proving (**).

Note (**) determines $p \otimes q$ uniquely, because for any $\varphi(\bar{x}, \bar{y}, \bar{c}) \in L(\mathbb{M})$, there is a small $C' \supseteq C$ with $\bar{c} \in C'$. If $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$, then $\sigma(p \otimes q) = \sigma(p) \otimes \sigma(q) = p \otimes q$. Thus $p \otimes q$ is C-invariant. □

The type $p \otimes q$ is called the *product* or *Morley product* of p and q. Sometimes it is defined in reverse order, so $(p \otimes q)(x, y)$ here is $(q \otimes p)(y, x)$ in some papers.

Example. If $A \subseteq \mathbb{M}$, then the *algebraic closure* of A, written $\mathrm{acl}(A)$, is the union of all finite A-definable sets. If $\mathbb{M}$ is strongly minimal and $p \in S_1(\mathbb{M})$ is the transcendental 1-type, one can show that $a \models p \restriction B \iff a \notin \mathrm{acl}(B)$. Therefore, $(a_1, a_2) \models (p \otimes p) \restriction B$ if and only if

$$a_1 \notin \mathrm{acl}(B) \text{ and } a_2 \notin \mathrm{acl}(Ba_1).$$

In ACF, $\mathrm{acl}(B)$ is the field-theoretic algebraic closure of B, and this condition says a_1 and a_2 are algebraically independent over A.

Example. Suppose $\mathbb{M}$ is a monster model of ACF. Let p_V be the generic type of an (irreducible) variety $V \subseteq \mathbb{M}^n$. If V, W are varieties, then $V \times W$ is a variety and $p_V \otimes p_W = p_{V \times W}$.

Proof. The product $p_V \otimes p_W$ must be p_Z for *some* variety Z. Take a small model M defining V, W, Z. Take $\bar{a} \models p_V \restriction M$, a small model $N \supseteq M\bar{a}$, and $\bar{b} \models p_W \restriction N$. Then $\bar{b} \models p_W \restriction M\bar{a}$, so $(\bar{a}, \bar{b}) \models (p_V \otimes p_W) \restriction M = p_Z \restriction M$.

Note $\bar{a} \in V$ and $\bar{b} \in W$, so $(\bar{a}, \bar{b}) \in V \times W$, implying $Z \subseteq V \times W$. Suppose $Z \subsetneq V \times W$. Take $(\bar{a}_0, \bar{b}_0) \in V \times W \setminus Z$. We may assume $(\bar{a}_0, \bar{b}_0) \in M$, as $M \preceq \mathbb{M}$. Let $Z_{\bar{a}} = \{\bar{y} \in \mathbb{M} : (\bar{a}, \bar{y}) \in Z\}$. Then $Z_{\bar{a}}$ is an N-definable algebraic set, and $\bar{b} \in Z_{\bar{a}}$ because $(\bar{a}, \bar{b}) \in Z$. The fact that $\bar{b} \models p_W \restriction N$ and $\bar{b} \in Z_{\bar{a}}$ implies $W \subseteq Z_{\bar{a}}$. Then $\bar{b}_0 \in W \in Z_{\bar{a}}$, so $(\bar{a}, \bar{b}_0) \in Z$. Let $Z^{\bar{b}_0} = \{\bar{x} \in \mathbb{M} : (\bar{x}, \bar{b}_0) \in Z\}$. Then $Z^{\bar{b}_0}$ is an M-definable algebraic set, and $\bar{a} \in Z^{\bar{b}_0}$. The fact that $\bar{a} \models p_V \restriction M$ and $\bar{a} \in Z^{\bar{b}_0}$ implies $V \subseteq Z^{\bar{b}_0}$. Then $\bar{a}_0 \in V \subseteq Z^{\bar{b}_0}$, meaning $(\bar{a}_0, \bar{b}_0) \in Z$, contradicting the choice of $\bar{a}_0, \bar{b}_0$. □

Remark 25. One says that two invariant types $p(\bar{x})$ and $q(\bar{y})$ *commute* if $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$. Concretely, this means

$$\bar{a} \models p \restriction C \text{ and } \bar{b} \models q \restriction C\bar{a} \implies \bar{a} \models p \restriction C\bar{b} \text{ (and } \bar{b} \models q \restriction C).$$

In ACF, all types commute:

$$(p_V \otimes p_W)(\bar{x}, \bar{y}) = p_{V\times W}(\bar{x}, \bar{y}) = p_{W\times V}(\bar{y}, \bar{x}) = (p_W \otimes p_V)(\bar{y}, \bar{x}).$$

We will see later that in stable theories, all types commute.

If $p(\bar{x})$ is a definable type and $\varphi(\bar{x}, \bar{y})$ is a formula, let $(d_p\bar{x})\varphi(\bar{x}, \bar{y})$ denote the formula $d\varphi(\bar{y})$. Note that $(d_p\bar{x})$ works like a quantifier—the free variables in $(d_p\bar{x})\varphi(\bar{x}, \bar{y})$ are $\bar{y}$.

Example. If p is the transcendentaly 1-type in a strongly minimal theory, then

$$(d_p x)\varphi(x, \bar{y}) = \exists^\infty x\ \varphi(x, \bar{y}).$$

Proposition 26. *If p, q are A-definable types, then $p \otimes q$ is A-definable and*

$$(d_{p\otimes q}(\bar{x}, \bar{y}))\varphi(\bar{x}, \bar{y}, \bar{z}) = (d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{z}).$$

Proof. Fix $\bar{c} \in \mathbb{M}$. Take a small model M such that p, q are M-definable and $\bar{c} \in M$. Take $\bar{a} \models p \restriction M$ and $\bar{b} \models q \restriction M\bar{a}$. Then $(\bar{a}, \bar{b}) \models (p \otimes q) \restriction M$, and so

$$\begin{gathered}
\varphi(\bar{x}, \bar{y}, \bar{c}) \in (p \otimes q)(\bar{x}, \bar{y}) \iff \\
\mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \iff \\
\varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \iff \\
\mathbb{M} \models (d_q\bar{y})\varphi(\bar{a}, \bar{y}, \bar{c}) \iff \\
(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \in p(\bar{x}) \iff \\
\mathbb{M} \models (d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}).
\end{gathered}$$

This holds for any $\bar{c}$, so $p \otimes q$ is definable and $(d_{p\otimes q}(\bar{x}, \bar{y}))\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{z})$.

The fact that $p \otimes q$ is A-invariant and definable implies that it is A-definable. $\square$

Example. In a strongly minimal theory, if p is the transcendental 1-type and $q = p \otimes p$, then $(d_q(x, y)) \cdots = \exists^\infty x\ \exists^\infty y \cdots$.

Example. Two definable types p, q commute iff

$$(d_p\bar{x})(d_q\bar{y})\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_q\bar{y})(d_p\bar{x})\varphi(\bar{x}, \bar{y}, \bar{z})$$

for any formula φ.