Stability

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1 Preface

A combination of various notes from [Pillay(2018)] [Chernikov(2019)] [Tent and Ziegler(2012)] [van den Dries(2019)]

A monster model $\mathfrak{C}$

[Pillay(2018)] has many typos😄

2 Preliminaries

2.1 Indiscernibles

Definition 2.1. Let I be a linear order and $\mathfrak{A}$ an L-structure. A family $(a_i)_{i\in I}$ of elements of A is called a **sequence of indiscernibles** if for all L-formulas $\varphi(x_1,\dots,x_n)$ and all $i_1<\dots<i_n$ and $j_1<\dots<j_n$ from I

$$\mathfrak{A}\vDash\varphi(a_{i_1},\dots,a_{i_n})\leftrightarrow\varphi(a_{j_1},\dots,a_{j_n})$$

or

$$\mathrm{tp}(a_{i_1},\dots,a_{i_n})=\mathrm{tp}(a_{j_1},\dots,a_{j_n})$$

Theorem 2.2. *Compactness let us "stretch" indiscernibles. Let $(a_i:i\in\omega)$ be indiscernibles in $\mathfrak{C}$, and $(I,<)$ an ordering. Then there exists an indiscernible $(b_i:i\in I)$ in $\mathfrak{C}$ s.t. $\forall i_1<\dots<i_n\in I$*

$$\mathrm{tp}(a_1,\dots,a_n)=\mathrm{tp}(b_{i_1},\dots,b_{i_n})$$

Indiscernible sequence are a fundamental tool of model theory, and there are many ways to obtain them.

Theorem 2.3 (Ramsey, extended)**.** *Let $n_1,\dots,n_r<\omega$. For each $i=1,\dots,r$, let $X_{i,1}$, $X_{i,2}$ be a partition of $[\omega]^{n_i}$. Then there is an infinite subset $Y\subseteq\omega$ which is homogeneous, i.e., $\forall i=1,\dots,r$, either $[Y]^{n_i}\subseteq X_{i,1}$ or $[Y]^{n_i}\subseteq Y_{i,2}$*

Proposition 2.4. *For each $n\in\omega$, let $\Sigma_n(x_1,\dots,x_n)$ be a collection of L-formulas in variables $x_1,\dots,x_n$. Suppose that there are $a_1,a_2,\dots\in\mathfrak{C}$ s.t.*

$$\vDash\Sigma_n(a_{i_1},\dots,a_{i_n}),\quad\forall i_1<\dots<i_n<\omega$$

Then there is an indiscernible $(b_i:i\in\omega)$ in $\mathfrak{C}$ s.t.

$$\vDash\Sigma_n(b_{i_1},\dots,b_{i_n}),\quad\forall i_1<\dots<i_n<\omega$$

we can expand $\bigcup_{n\in\omega}\Sigma_n$ and obtain the Ehrenfeucht-Mostowski type $\mathrm{EM}((a_i)_{i\in\omega})$. This is just the Standard Lemma in Tent

Example 2.1. Suppose $\Sigma_2 = \{x_1 \neq x_2\}$. Then the proposition yields the existence of infinite indiscernible sequences

Proof. Consider

$$\begin{aligned}\Gamma(x_1, x_2, \dots) = \{&\varphi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \varphi(x_{j_1}, \dots, x_{j_n}) : \\ &i_1 < \dots < i_n, j_1 < \dots < j_n \in \omega, \varphi \in L\} \\ \cup &\bigcup_n \Sigma(x_1, \dots, x_n)\end{aligned}$$

Let $\Gamma'(x_1, \dots, x_n) \subseteq_f \Gamma$. Let $\varphi_1, \dots, \varphi_r$ be the L-formulas appearing in Γ'. For $i = 1, \dots, r$, let

$$\begin{aligned}X_{i,1} &= \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \vDash \varphi_i(a_{j_1}, \dots, a_{j_n})\} \\ X_{i,2} &= \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \vDash \neg\varphi_i(a_{j_1}, \dots, a_{j_n})\}\end{aligned}$$

By Ramsey's theorem, there exists an infinite $Y \subseteq \mathbb{N}$ s.t. $\forall i = 1, \dots, r$, $[Y]^{n_i}$ is either contained in $X_{i,1}$ or in $X_{i,2}$. Write $Y = \{k_1 < k_2 < \dots\}$. Interpret each x_i as a_{k_i} to satisfy Γ' □

Definition 2.5. Let $M \prec N \prec \mathfrak{C}$ be models, and $p(\bar{x}) \in S_{\bar{x}}(N)$. We say p is finitely satisfiable in M, or $p(\bar{x})$ is a **coheir** of $p \restriction M \in S_{\bar{x}}(M)$, if every $\varphi(\bar{x}) \in p(\bar{x})$ is satisfied by some $\bar{a} \in M$

Remark. $p(\bar{x}) \in S_n(N)$ is finitely satisfiable (f.s.) in M iff $p(\bar{x})$ is in the topological closure of $\{\mathrm{tp}(\bar{a}/N) : \bar{a} \in M\} \subseteq S_n(N)$

Lemma 2.6. *Suppose $p(\bar{x}) \in S_{\bar{x}}(M)$ and $M \prec N$, then there is $p'(\bar{x}) \in S_{\bar{x}}(N)$ s.t. $p \subseteq p'$ and p' is f.s. in M*

Proof. Consider $\Gamma(\bar{x}) = p(\bar{x}) \cup \{\neg\varphi(\bar{x}) : \varphi(\bar{x}) \in L_N$ and not realized in $M\}$. Let $\Gamma \supseteq_f \Gamma' = \{\Psi(\bar{x}), \neg\varphi_1(\bar{x}), \dots, \neg\varphi_r(\bar{x})\} \in p$. Then any solution $\bar{a}$ of Ψ in M satisfies Γ' as $M \vDash \forall\bar{x}(\neg\varphi_i(\bar{x}))$ □

Remark. Let $i_M : M^{\bar{x}} \to S_{\bar{x}}(M)$ s.t. $m \mapsto \mathrm{tp}(m/M)$. Define $i_N : M^{\bar{x}} \to S_{\bar{x}}(N)$ similarly. Let $r : S_{\bar{x}}(N) \to S_{\bar{x}}(M)$. Note that $r \circ i_N = i_M$ and the set of types in $S_{\bar{x}}(N)$ that are f.s. in M is exactly the closure of $i_N(M^{\bar{x}})$ in $S_{\bar{x}}(N)$. Hence its image under r is closed. However the image must contain $i_M(M^{\bar{x}})$ which is dense in $S_{\bar{x}}(M)$. Therefore it must be onto, which proves the desired result

r is continuous and $r(\overline{i_N(M^n)}) \supseteq i_M(M^n)$ is closed. $\overline{i_M(M^n)} = S_n(M)$. Then r is onto? Then its preimage of p is what we want

Proposition 2.7. *Let $p(\bar{x}) \in S_{\bar{x}}(M)$, $N \succ M$ be $|M|^{+}$-saturated, and $p'(\bar{x}) \in S_{\bar{x}}(N)$ a coheir of p. Let $\bar{a}_1, \bar{a}_2, \dots \in N$ be defined as follows*

$$\begin{aligned}
&\bar{a}_1 \text{ realises } p(\bar{x}) \\
&\bar{a}_2 \text{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1) \\
&\bar{a}_3 \text{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1, \bar{a}_2) \\
&\dots
\end{aligned}$$

Then $(\bar{a}_i : i \in \omega)$ is indiscernible over M

Proof. We prove by induction on k that for any $n \leq k$ and $i_1 < \dots < i_n \leq k$ and $j_1 < \dots < j_n \leq k$, we have

$$\operatorname{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/M) = \operatorname{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/M)$$

Assume this is true for k and consider $k+1$. Let $i_1 < \dots < i_n \leq k$, $j_1 < \dots < j_n \leq k$. We need to show that

$$\operatorname{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}_{k+1}/M) = \operatorname{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}_{k+1}/M)$$

Consider a formula $\varphi(\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1}) \in L_M$. Assume by contradiction that

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}_{k+1}) \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}_{k+1})$$

But $\operatorname{tp}(\bar{a}_{k+1}/M, \bar{a}_1, \dots, \bar{a}_k)$ is f.s. in M, so there is $\bar{a}' \in M$ s.t.

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}') \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}')$$

contradicting IH ☐

2.2 Definability and Generalizations

Definition 2.8. $X \subseteq \mathfrak{C}^n$ is **definable almost over** A if there is an A-definable equivalence relation E on $\mathfrak{C}^n$ with finitely many classes and X is a union of some E-classes

Lemma 2.9. *Let $\mathbb{D}$ be a definable class and A a set of parameters. T.F.A.E.*

1. *$\mathbb{D}$ is definable over A*
2. *$\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix A pointwise*

$S \subseteq K^{\text{alg}} \Rightarrow M \setminus S \subseteq K^{\text{alg}}$

Proof. $\Rightarrow$ is easy as for any $F \in \text{Aut}(\mathfrak{C}/A)$ and $\mathbb{D} = \varphi(\mathfrak{C}, \bar{a})$, $\mathfrak{C} \vDash \varphi(\bar{s}, \bar{a})$ iff $\mathfrak{C} \vDash \varphi(F(\bar{s}), \bar{a})$. StackExchange

$x \in \mathbb{D} \Leftrightarrow \vDash \varphi(x, \bar{a}) \Leftrightarrow \varphi(F(x), F(\bar{a})) \leftrightarrow \varphi(F(x), \bar{a}) \Leftrightarrow F(x) \in \mathbb{D}$

$\Leftarrow$. Another proof from Chernikov. Assume that $\mathbb{D} = \varphi(\mathfrak{C}, b)$ where $b \in \mathfrak{C}$, and let $p(y) = \text{tp}(b/A)$

Claim 1. $p(y) \vdash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$, which says that for any realisations b', $\varphi(\mathfrak{C}, b) = \varphi(\mathfrak{C}, b')$

Indeed, let $b' \vDash p(y)$ be arbitrary. Then $\text{tp}(b/A) = \text{tp}(b'/A)$ so there is some $\sigma \in \text{Aut}(\mathfrak{C}/A)$ with $\sigma(b) = b'$. Then $\sigma(X) = \varphi(\mathfrak{C}, b')$ and by assumption $\sigma(X) = X$, thus $\varphi(\mathfrak{C}, b) = X = \varphi(\mathfrak{C}, b')$.

There is some $\psi(y) \in p$ (there is a finite subset of $p(y)$ that does the job and we take the conjunction) s.t.

$$\psi(y) \vDash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$$

Let $\theta(x)$ be the formula $\exists y(\psi(y) \wedge \varphi(x, y))$. Note that $\theta(x)$ is an $L(A)$-formula, as $\psi(y)$ is

Claim 2. $X = \theta(\mathfrak{C})$

If $a \in X$, then $\vDash \varphi(a, b)$, and as $\psi(y) \in \text{tp}(b/A)$ we have $\vDash \theta(a)$. Conversely, if $\vDash \theta(a)$, let b' be s.t. $\vDash \psi(b') \wedge \varphi(a, b')$. But by the choice of ψ this implies that $\vDash \varphi(a, b)$

$\Leftarrow$ Let $\mathbb{D}$ be defined by φ, defined over $B \supset A$. Consider the maps

$$\mathfrak{C} \xrightarrow{\tau} S(B) \xrightarrow{\pi} S(A)$$

where $\tau(c) = \text{tp}(c/B)$ and π is the restriction map. Let Y be the image of $\mathbb{D}$ in $S(A)$. Since $Y = \pi[\varphi]$. Y is closed. Note that $\tau(\mathbb{D}) = [\varphi]$. $\tau(\mathbb{D}) = \{\text{tp}(c/B) : \mathfrak{C} \vDash \varphi(c)\} \subseteq [\varphi]$. For any $q(x) \in [\varphi]$, as $\mathfrak{C}$ is saturated, $\mathfrak{C} \vDash q(d)$ and $d \in \mathbb{D}$. Thus $q \in \tau(\mathbb{D})$. π is continuous

Assume that $\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix A pointwise. Since elements which have the same type over A are conjugate by an automorphism of $\mathfrak{C}$, this means that $\mathbb{D}$-membership depends only on the type over A, i.e., $\mathbb{D} = (\pi\tau)^{-1}(Y)$. For any $\text{tp}(c/A) = \text{tp}(d/A)$ and $c \in \mathbb{D}$, as c and d are conjugate, $d \in \mathbb{D}$.

For any $c \notin \mathbb{D}$, $\pi\tau(c) \in Y$ iff $\text{tp}(c/A) \in \pi[\varphi]$ iff there is $d \in \mathbb{D}$ s.t. $\text{tp}(c/A) = \text{tp}(d/A)$ but then $c \in \mathbb{D}$.

This implies that $[\varphi] = \pi^{-1}(Y)$ $\tau(\mathbb{D}) = [\varphi] = \tau(\tau^{-1}\pi^{-1})(Y) = \pi^{-1}(Y)$, or $S(A) \setminus Y = \pi[\neg\varphi]$; hence $S(A) \setminus Y$ is also closed and we conclude that Y is clopen. By Lemma **??** $Y = [\psi]$ for some $L(A)$-formula ψ. This ψ defines $\mathbb{D}$. For any $d \in \mathfrak{C}$

$$\vDash \psi(d) \Leftrightarrow \text{tp}(d/A) \Leftrightarrow d \in \mathbb{D}$$

$\square$

A slight generalization of the previous lemma

Lemma 2.10. *Let $X \subseteq \mathfrak{C}^n$ be definable. TFAE*

1. *X is almost A-definable, i.e., there is an A-definable equivalence relation E on $\mathfrak{C}^n$ with finitely many classes, s.t. X is a union of E-classes*

2. *The set $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ is finite*

3. *The set $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ is small*

Proof. $1 \to 2$. Let $\varphi(x_1, x_2) \in L(A)$ be the A-definable equivalence relation E, and let $b_1, \dots, b_n \in M$ be representatives in each equivalence class so that each class can be written as $[b_i] = \varphi(\mathfrak{C}, b_i)$. Given $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$, since $\varphi(x_1, x_2) \leftrightarrow \varphi(\sigma(x_1), \sigma(x_2))$, the image of each $[b_i]$ under σ will be

$$\sigma([b_i]) = \{\sigma(x) : \varphi(x, b_i)\} = \{x' : \varphi(x', \sigma(b_i))\} = \{x : \varphi(x, b_{j_i})\} = [b_{j_1}]$$

for some $j_i \leq n$. Now X is a disjoint union of some $[b_i]$'s, so $\sigma(X)$ is a disjoint union of some $[b_j]$'s. Since there are only finitely many equivalence classes, there can only be finitely many possibilities for disjoint unions of these classes

$2 \to 1$. Let $X = \varphi(\mathfrak{C}, b)$ and $p(y) = \mathrm{tp}(b/A)$. Given $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$, we have $\sigma(X) = \varphi(\mathfrak{C}, \sigma(b))$. Then from assumption, there must be distinct $b_1, \dots, b_n$ s.t.

$$\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{C}/A)\} = \{\varphi(\mathfrak{C}, b_i) : i \leq n\}$$

Now if $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$, then strong homogeneity yields some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Then the above argument again shows that $\varphi(x, b')$ defines $\sigma(X)$ for some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$. Thus $\sigma(X) = \varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i \leq k$. Therefore $p(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. By compactness there is some $\psi(y) \in p$ s.t. $\psi(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. Now define $E(x_1, x_2)$ as

$$\forall y(\psi(y) \to (\phi(x_1, y) \leftrightarrow \phi(x_2, y)))$$

so it is A-definable. It is easy to check that E is an equivalence relation with finitely many classes, and that X is a union of E-classes ($a_1 E a_2$ iff they agree on $\phi(x, b_i)$ for all $i \leq k$, and so $X = \phi(\mathfrak{C}, b_0)$ is given by the union of all possible combinations intersected with it)

$3 \to 1$ Assume for contradiction that

$$|\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{C}/A)\}| = \lambda \geq \omega$$

we can find λ-many elements $(b_i : i < \lambda) \subset \mathfrak{C}$ to represent the distinct images under automorphisms. Then the set

$$q(y) = p(y) \cup \{\neg\forall x(\varphi(x, y) \leftrightarrow \varphi(x, b_i)) : i < \lambda\}$$

will be finitely satisfiable. Thus $q(y)$ is realised by some b'. But such b' has the same type as b over A and so strong homogeneity yields some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Applying such σ on X gives the image $\varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i < \lambda$, a contradiction □

Proposition 2.11. *We can identify definable sets with continuous functions in a certain settings*

1. *Formulas $\varphi(\bar{x}), \psi(\bar{x}) \in L_A$ are equivalent iff $[\varphi(\bar{x})] = [\psi(\bar{x})]$*
2. *The clopen subsets of $S_{\bar{x}}(A)$ are precisely the basic clopen sets*
3. *Clopen subsets X of $S_{\bar{x}}(A)$ correspond exactly to continuous functions $f : S_{\bar{x}}(A) \to 2$ (with discrete topology) where $f(p(\bar{x})) = 1$ if $p(\bar{x}) \in X$ and 0 otherwise*
4. *The definable subsets of $\mathfrak{C}^c$ are in one-to-one correspondence with continuous functions from $S_{\bar{x}}(A)$ to 2*

Proof. 3. If X is clopen, then $f^{-1}(2) = S_{\bar{x}}(A)$, $f^{-1}(0) = \emptyset$, $f^{-1}(\{1\}) = X$, $f^{-1}(\{0\}) = X^c$

4. By 1, definable sets are in one-to-one correspondence with basic clopen subsets. By 2, basic clopen sets are exactly all of the clopen subsets, so definable sets are in one-to-one correspondence with clopen sets. By 3, clopen sets are in one-to-one correspondence with continuous functions $f : S_{\bar{x}}(A) \to 2$

□

2.3 Imaginaries and T^{eq}

A **multi-sorted** structure is a family of sets $(M_s)_{s\in S}$ equipped with relations

$$R \subseteq M_{s_1} \times \cdots \times M_{s_m}, \quad (s_1, \ldots, s_m \in S)$$

A multi-sorted language L is a triple (S, L^r, L^f) and S are the sorts of L

M_s is the underlying set of sort s. Elements of M_s are also called "elements of $\mathcal{M}$" of sort s Given any tuple $\bar{s} = (s_i)_{i\in I}$ of sorts in S, we let $M_{\bar{s}} = \prod_{i\in I} M_{s_i}$

Given a variable $x = (x_i)_{i\in I}$ of L, with x_i of sorts s_i for $i \in I$, we define the x**-set of** $\mathcal{M}$ to be the product set

$$M_x := M_{\bar{s}} = \prod_i M_{s_i}, \quad \bar{s} = (s_i)_{i\in I}$$

$x = (x_i)_{i\in I}$ and $y = (y_j)_{j\in J}$ is **disjoint** if $x_i \neq y_j$ for all $i \in I$ and $j \in J$, and in that case we put $M_{x,y} = M_x \times M_y$. If in addition $I = J$ and x_i and y_i have the same sort for $i \in I$ (so that $M_x = M_y$), we call x and y **disjoint and similar**

Definition 2.12. The **definable closure** $\mathrm{dcl}(A)$ of A is the set of elements c for which there is an $L(A)$-formula $\varphi(x)$ s.t. c is the unique element satisfying φ. Elements or tuples a and b are said to be **interdefinable** if $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$.

Lemma 2.13. *Assume $A \subseteq \mathfrak{C}$ and $\bar{b} \in \mathfrak{C}$*

1. *$\bar{b} \in \mathrm{acl}(A)$ iff $\{f(\bar{b}) : f \in Aut(\mathfrak{C}/A)\}$ is finite*
2. *$\bar{b} \in \mathrm{dcl}(A)$ iff $f(\bar{b}) = \bar{b}$ for all $f \in Aut(\mathfrak{C}/A)$*

Proof.

1. Suppose $\bar{b} \in \mathrm{acl}(A)$ with witness $\exists^{\le k}\varphi(\bar{x})$. Then $\varphi(\mathfrak{C})$ is A-definable and hence is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant by Lemma 2.9

   Suppose the finiteness. Since the composition of automorphisms is an automorphism, this set is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant and therefore A-definable by some $\varphi(\bar{x})$.

2. $\{\bar{b}\}$ is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant

□

The first motivation to develop T^{eq} is dealing with quotient objects, without leaving the context of first order logic. That is, if E is some definable equivalence relation on some definable set X, we want to view X/E as a definable set

We work in the setting of multi-sorted languages. Let L be a 1-sorted language and let T be a (complete) L-theory. We shall build a many-sorted language L^{eq}-theory T^{eq}. We will ensure that in natural sense, L^{eq} contains L and T^{eq} contains T

First we define L^{eq}. Consider the set L-formula $\varphi(x, y)$, up to equivalence, such that T models that φ is an equivalence relation. For each φ, define s_φ to be a new sort in L^{eq}. Of particular importance is $s_=$, the sort given by the formula "$x = y$". = is an equivalence relation This sort $s_=$ will yield, in each model of T^{eq}, a model of T

Also define f_φ to be a function symbol with domain sort $s_=^n$ (where φ has n free variables) and codomain sort s_φ

For each m-place relation symbol $R \in L$, make R^{eq} an m-place relation symbol in L^{eq} on $s_=^m$. Likewise for all constant and function symbols in L. Finally, for the sake of formality, we put a unique equality symbol $=_\varphi$ on each sort

Remark. Let N be an L^{eq} structure. Then N has interpretations $s_\varphi(N)$ of each sort s_φ and $f_\varphi(N) : s_=(N)^{n_{f_\varphi}} \to s_\varphi(N)$ of each function symbol f_φ. Additionally, N will contain an L-structure consisting of $s_=$ and interpretations of the symbols of L inside of $s_=$

Definition 2.14. T^{eq} is the L^{eq}-theory which is axiomatised by the following

1. T, where the quantifiers in the formulas of T now range over the sort $s_=$
2. For each suitable L-formula $\varphi(x, y)$, the axiom $\forall_{s_=} \bar{x} \forall_{s_=} \bar{y} (\varphi(x, y) \leftrightarrow f_\varphi(\bar{x}) = f_\varphi(\bar{y}))$
3. For each L-formula φ, the axiom $\forall_{s_\varphi} y \exists_{s_=} \bar{x} (f_\varphi(\bar{x}) = y)$

Axioms 2 and 3 simply state that f_φ is the quotient function for the equivalence relation given by φ

Definition 2.15. Let $M \vDash T$. Then M^{eq} is the L^{eq} structure s.t. $s_=(M^{\mathrm{eq}}) = M$ and for each suitable L-formula $\varphi(x, y)$ of n variables, the sort $s_\varphi(M^{\mathrm{eq}})$ is equal to $M^{n_{f_\varphi}}/E$ where E is the equivalence relation defined by $\varphi(x, y)$ and $f_\varphi(M^{\mathrm{eq}})(b) = b/E$

Example 2.2 (Projective planes)**.** From Hodges.

Suppose A is a three-dimensional vector space over a finite field, and let L be the first-order language of A. Then we can write a formula $\theta(x, y)$ of L which expresses 'vectors x and y are non-zero and are linearly dependent on each other'. The formula θ is an equivalence formula of A, and the sort s_θ is the set of points of the projective plane P associated with A

Now $M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$. Moreover, passing from T to T^{eq} is a canonical operation, in the following sense

Lemma 2.16. 1. *For any $N \models T^{\mathrm{eq}}$, there is an $M \models T$ s.t. $N \cong M^{\mathrm{eq}}$*

2. *Suppose $M, N \models T$ are isomorphic, and let $h : M \cong N$. Then h extends uniquely to $h^{\mathrm{eq}} : M^{\mathrm{eq}} \cong N^{\mathrm{eq}}$*

3. *T^{eq} is a complete L^{eq}-theory*

4. *Suppose $M, N \models T$ and let $\bar{a} \in M$, $\bar{b} \in N$ with $\mathrm{tp}_M(\bar{a}) = \mathrm{tp}_N(\bar{b})$. Then $\mathrm{tp}_{M^{\mathrm{eq}}}(\bar{a}) = \mathrm{tp}_{N^{\mathrm{eq}}}(\bar{b})$*

Proof. 1. Take $M = s_=(N)$

2. Let $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to N^{\mathrm{eq}}$ be defined as $h^{\mathrm{eq}}(f_\varphi(M^{\mathrm{eq}})(b)) = f_\varphi(N^{\mathrm{eq}})(h(b))$ for each $\varphi \in L$. This defines a function on M^{eq}, because $f_\varphi(M^{\mathrm{eq}})$ is surjective by the T^{eq} axioms. Moreover h^{eq} is well-defined. Suppose $f_\varphi(M^{\mathrm{eq}})(b) = f_\varphi(M^{\mathrm{eq}})(b')$, then $\varphi(b, b')$ and hence $\varphi(h(b), h(b'))$, therefore $f_\varphi(N^{\mathrm{eq}})(h(b)) = f_\varphi(N^{\mathrm{eq}})(h(b'))$. Injectivity is the same since $\varphi(b, b') \leftrightarrow \varphi(h(b), h(b'))$.

$$\begin{aligned}
f_\varphi(N^{\mathrm{eq}})(h(b)) = f_\varphi(N^{\mathrm{eq}})(h(b')) &\Leftrightarrow h(b)/E_\varphi = h(b')/E_\varphi \\
&\Leftrightarrow \varphi(h(b), h(b')) \\
&\Leftrightarrow \varphi(b, b') \\
&\Leftrightarrow f_\varphi(M^{\mathrm{eq}})(b) = f_\varphi(M^{\mathrm{eq}})(b')
\end{aligned}$$

3. Let $M, N \models T^{\mathrm{eq}}$, we want to show that they are elementary equivalent. Assume the generalized continuum hypothesis. By GCH, there are $M', N' \models T^{\mathrm{eq}}$ which are λ saturated of size λ, for some large λ (strongly inaccessible), which $M \preceq M'$ and $N \preceq N'$. Since we want to show elementary equivalence, we can replace M, N with M' and N'. By 1, we have $M = M_0^{\mathrm{eq}}$, $N = N_0^{\mathrm{eq}}$ for some $M_0, N_0 \models T$. Furthermore, M_0, N_0 are λ-saturated of size λ. By assumption, T is complete, so $M_0 \equiv N_0$, and therefore $M_0 \cong N_0$. By 2, $M \cong N$, and therefore $M \equiv N$

We could simply prove that there is a back and forth system between M and N, using such a system between $M \supset M_0 \models T$ and $N \supset N_0 \models T$ $M_0 \equiv N_0$ iff $M_0 \sim_\omega N_0$. We want to show that $M \sim_\omega N$. For any $p \in \omega$,

- given $a \in s_=(M)$, choose according to M
- given $a \in s_\varphi(M)$, then there is $\bar{b}\bar{c} \in s_=(M)$ s.t. $f_\varphi(M^{\mathrm{eq}})(\bar{b}\bar{c}) = a$ and $\varphi(\bar{b}, \bar{c})$. If $\bar{b} \in s_=(M^{\mathrm{eq}})^n$, then there is a local isomorphism $\bar{b} \mapsto \bar{d}$ as $M \sim_\omega N$. Take $b = \bar{d}/E_\varphi$.

4. Let $M, N \models T$, they are elementary submodels of $\mathfrak{C}$. Since $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$, there exists an $\sigma \in \text{Aut}(\mathfrak{C}/A)$ with $\sigma(\bar{a}) = \bar{b}$. By 2, this automorphism extends to $\sigma^{\text{eq}} : \mathfrak{C}^{\text{eq}} \to \mathfrak{C}^{\text{eq}}$ with $\sigma^{\text{eq}}(a) = b$, hence $\text{tp}_{M^{\text{eq}}}(a) = \text{tp}_{\mathfrak{C}^{\text{eq}}}(a) = \text{tp}_{\mathfrak{C}^{\text{eq}}}(b) = \text{tp}_{N^{\text{eq}}}(b)$

$\square$

Corollary 2.17. *Consider the Strong space $S_{(s_=)^n}(T^{\text{eq}})$. The forgetful map $\pi : S_{(s_=)^n}(T^{\text{eq}}) \to S_n(T)$ is a homeomorphism*

Proof. Observe that it is continuous and surjective. By 4 of the previous lemma it is injective. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism $\square$

Proposition 2.18. *Let $\varphi(x_1, \dots, x_k)$ be an L^{eq} formula, where x_i is of sort S_{E_i}. There is an L-formula $\psi(\bar{y}_1, \dots, \bar{y}_k)$ s.t.*

$$T^{\text{eq}} \models \forall \bar{y}_1, \dots, \bar{y}_k(\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$$

Proof. Let n be the length of $\bar{y}_1, \dots, \bar{y}_k$. Consider the set $\pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$, it is a clopen subset of $S_n(T)$ by the previous lemma, hence equal to $\psi(\bar{y}_1, \dots, \bar{y}_k)$ for some formula ψ.

Guess the intuition is $[\varphi] = [\psi]$ iff $\models \varphi \leftrightarrow \psi$. Consider $\pi[\psi(\bar{y}_1, \dots, \bar{y}_k)] = \pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ and as π is homeomorphism, $[\psi(\bar{y}_1, \dots, \bar{y}_k)] = [\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$

$\square$

This proposition also shows that T^{eq} is complete since f_{E_i} is surjective
Also, for any $\bar{c} \in \mathfrak{C}$, $\bar{c} \in \text{dcl}^{\text{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \text{dcl}(\emptyset)$, $\bar{c} \in \text{acl}^{\text{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \text{acl}(\emptyset)$

Corollary 2.19. 1. *Let $M, N \models T$, and let $h : M \to N$ be an elementary embedding. Then $h^{\text{eq}} : M^{\text{eq}} \to N^{\text{eq}}$ is also an elementary embedding*

2. *$\mathfrak{C}^{\text{eq}}$ is also κ-saturated*

Proof. 1. $h : M \to \text{im}(h)$ is an isomorphism and can extend to $h^{\text{eq}} : M^{\text{eq}} \to (\text{im}(h))^{\text{eq}}$, and $(\text{im}(h))^{\text{eq}} \subseteq N^{\text{eq}}$

2. By Proposition 2.18

$\square$

Remark. For $M \models T$, a definable set $X \subseteq M^n$ can be viewed as an element of M^{eq}. Suppose X is defined in M by $\varphi(\bar{x}, \bar{a})$ where $\bar{a} \in M$. Consider the equivalence relation E_ψ defined by $\psi(\bar{y}_1, \bar{y}_2) = \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$ $\bar{y}_1 \sim \bar{y}_2$ iff this $\varphi(M, \bar{y}_1) = \varphi(M, \bar{y}_2)$, and consider $c = \bar{a}/E_\psi = f_\psi(\bar{a}) \in$

M^{eq}. Then X is defined in M^{eq} by $\chi(\bar{x}, c) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f_\psi(\bar{y}) = c)$. Moreover, if $c' \in S_\psi(M^{\text{eq}})$ and $\forall \bar{x}(\chi(\bar{x}, c) \leftrightarrow \chi(\bar{x}, c'))$, then $c = c'$. To see this, let $c' = f_\psi(\bar{a}')$, and let X' be defined in M by $\varphi(\bar{x}, \bar{a}')$. Then X' is defined in M^{eq} by $\chi(\bar{x}, c')$, so we have that $X = X'$ (in M^{eq}). And then $X = X'$ (in M) so $c = f_\psi(\bar{a}) = f_{\psi'}(\bar{a}') = c'$

Definition 2.20. With the above considerations in mind, given $M \vDash T$ and a definable set $X \subseteq M^n$, we call such a $c \in M^{\text{eq}}$ a **code** for X

Remark. Any automorphism of $\mathfrak{C}^{\text{eq}}$ fixes a definable set X set-wise iff it fixes a code for X. However, the choice of a code for X will depend on the formula φ used to define it

$$\begin{aligned}\sigma(X) = X &\Leftrightarrow \sigma(X) = \{\sigma(x) : \varphi(x, b)\} = \{x : \varphi(x, \sigma(b))\} = \{x : \varphi(x, b)\} = X \\ &\Leftrightarrow \forall x(\varphi(x, b) \leftrightarrow \varphi(x, \sigma(b))) \\ &\Leftrightarrow \psi(b, \sigma(b)) \Leftrightarrow f_\psi(b) = f_\psi(\sigma(b))\end{aligned}$$

We can think of $\mathfrak{C}^{\text{eq}}$ as adjoining codes for all definable equivalence relations (as c/E' codes $E'(x, c)$ for an arbitrary equivalence relation E)

Definition 2.21. Let $A \subseteq M \vDash T$. Then $\text{acl}^{\text{eq}}(A) = \{c \in M^{\text{eq}} : c \in \text{acl}_{M^{\text{eq}}}(A)\}$ and $\text{dcl}^{\text{eq}}(A)$ is defined similarly

Remark. Suppose $A \subseteq M \prec N$, then $\text{acl}_{N^{\text{eq}}}(A), \text{dcl}_{N^{\text{eq}}}(A) \subseteq M^{\text{eq}}$, so this notation is unambiguous

Lemma 2.22. *Let $M \vDash T$, a definable subset X of M^n, and $A \subseteq M$. Then X is almost A-definable iff X is definable in M^{eq} by a formula with parameters in* $\text{acl}^{\text{eq}}(A)$

Proof. We can work in $\mathfrak{C}$, since $M \prec \mathfrak{C}$. Let c be a code for X. From 2.10 X is almost A-definable iff $|\{\sigma(X) : \sigma \in \text{Aut}(\mathfrak{C}/A)\}| < \omega$ iff $|\{\sigma(c) : \sigma \in \text{Aut}(\mathfrak{C}^{\text{eq}}/A)\}| < \omega$ (note that σ extends uniquely in $\mathfrak{C}^{\text{eq}}$), that is, $c \in \text{acl}^{\text{eq}}(A)$.

$$\begin{aligned}\sigma(b)/E = \sigma'(b)/E &\Leftrightarrow \forall x(\varphi(x, \sigma(b)) \leftrightarrow \varphi(x, \sigma'(b))) \\ &\Leftrightarrow \sigma(X) = \sigma'(X)\end{aligned}$$

□

Definition 2.23. Let $\bar{a}, \bar{b} \in \mathfrak{C}$ have length n. Let $\bar{a}, \bar{b}$ have the same strong type over A (written as $\text{stp}_{\mathfrak{C}}(\bar{a}/A) = \text{stp}_{\mathfrak{C}}(\bar{a}/A)$) if $E(\bar{a}, \bar{b})$ for any finite equivalence relation (finitely many classes) defined over A

Remark. If $\varphi(\bar{x})$ is a formula over A, then it defines an equivalence with two classes $E(\bar{x}_1, \bar{x}_2)$ iff $(\varphi(\bar{x}_1) \wedge \varphi(\bar{x}_2)) \vee (\neg\varphi(\bar{x}_1) \wedge \neg\varphi(\bar{x}_2))$. Hence strong types are a refinement of types

Hence for any formula if $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/B)$, at least we have $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$

Lemma 2.24. *If $A = M \prec \mathfrak{C}$, then $\mathrm{tp}_{\mathfrak{C}}(a/M) \vDash \mathrm{stp}_{\mathfrak{C}}(a/M)$*

$$\mathrm{tp}_{\mathfrak{C}}(a/M) = \mathrm{tp}_{\mathfrak{C}}(b/M) \Rightarrow \mathrm{stp}_{\mathfrak{C}}(a/M) = \mathrm{stp}_{\mathfrak{C}}(b/M)$$

Proof. Let E be an equivalence relation with finitely many classes, defined over M, and $\bar{b}$ another realization of $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/M)$, we want to show $E(a,b)$. Since E has only finitely many classes, and M is a model, there are representants $e_1, \dots, e_n$ of each E-class in M. Hence we must have $E(a, e_i)$ for some i, and therefore $E(b, e_i)$, which yields $E(a, b)$ □

Lemma 2.25. *Let $A \subseteq M \vDash T$, and let $\bar{a}, \bar{b} \in M$. TFAE*

1. $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$
2. *$\bar{a}, \bar{b}$ satisfy the same formulas almost A-definable*
3. $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}_{\mathfrak{C}}(\bar{b}/\operatorname{acl}^{\mathrm{eq}}(A))$

Proof. $3 \to 2$. 2.22. Suppose $X = \varphi(\mathfrak{C}, \bar{d})$ is almost A-definable, then $\bar{a}, \bar{b} \in \varphi(\mathfrak{C}, \bar{d})$ iff $\bar{a}, \bar{b} \in \theta(\mathfrak{C}) := \exists \bar{y}(\varphi(\mathfrak{C}, \bar{y}) \wedge \bar{y}/E_\psi = \bar{c})$ where $\bar{c} = \bar{d}/E_\psi \in \operatorname{acl}^{\mathrm{eq}}(A)$.

$2 \to 3$

$1 \to 2$. Let X be almost definable over A. We want to show that $\bar{a} \in X$ iff $\bar{b} \in X$.

Since X is almost definable over A, there is an A-definable equivalence relation E with finitely many classes, and $\bar{c}_1, \dots, \bar{c}_n$ s.t. for all $\bar{x} \in M$, we have $\bar{x} \in X$ iff $M \vDash E(\bar{x}, \bar{c}_1) \vee \dots \vee E(\bar{x}, \bar{c}_n)$. Hence $E(\bar{a}, \bar{c}_i)$ for some i, so by assumption $E(\bar{b}, \bar{c}_i)$.

$2 \to 1$. Let E be an A-definable equivalence relation with finitely many classes, we want to show that $E(\bar{a}, \bar{b})$. The set $X = \{\bar{x} \in M : E(\bar{x}, \bar{a})\}$ is definable almost over A. But $\bar{a} \in X$, so $\bar{b} \in X$, hence $E(\bar{a}, \bar{b})$ □

Here is a note from scanlon

Definition 2.26. An **imaginary element** of $\mathfrak{A}$ is a class a/E where $a \in A^n$ and E is a definable equivalence relation on A^n

Definition 2.27. $\mathfrak{A}$ **eliminates imaginaries** if, for every definable equivalence relation E on A^n there exists definable function $f : A^n \to A^m$ s.t. for $x, y \in A^n$ we have

$$xEy \Leftrightarrow f(x) = f(y)$$

Remark. The definition give above is what Hodges calls **uniform elimination of imaginaries**

Remark. If $\mathfrak{A}$ eliminates imaginaries, then for any definable set X and definable equivalence relation E on X, there is a definable set Y and a definable bijection $f : X/E \to Y$. Of course this is not literally true, we should rather say that there is a definable map $f' : X \to Y$ s.t. f' is invariant on the equivalence classes defined by E

So elimination of imaginaries is saying that quotients exists in the category of definable sets

Remark. If $\mathfrak{A}$ eliminates imaginaries then for any imaginaries element $a/E = \tilde{a}$ there is some tuple $\hat{a} \in A^m$ s.t. $\tilde{a}$ and $\hat{a}$ are **interdefinable**, i.e. there is a formula $\varphi(x, y)$ s.t.

- $\mathfrak{A} \models \varphi(a, \tilde{a})$
- If $a'Ea$ then $\mathfrak{A} \models \varphi(a', \hat{a})$
- If $\varphi(b, \hat{a})$ then bEa
- If $\varphi(a, c)$ then $c = \hat{a}$

To get the formula φ we use the function f given by the definition of elimination of imaginaries and let $\varphi(x, y) := f(x) = y$

Almost conversely, if for every $\mathfrak{A}' \equiv \mathfrak{A}$ every imaginary in $\mathfrak{A}'$ is interdefinable with a **real** (non-imaginary) tuple then $\mathfrak{A}$ eliminates imaginaries

$$\{\forall xy(xEy \leftrightarrow f_E(x) = f_E(y)) \mid \forall E\}$$

Example 2.3. For any structure $\mathfrak{A}$, every imaginary in $\mathfrak{A}_A$ is interdefinable with a sequence of real elements

Example 2.4. Let $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$. Then $\mathfrak{A}$ eliminates imaginaries. For example, to eliminate the "odd/even" equivalence relation, E, we can define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(x) = y \Leftrightarrow xEy \wedge \forall z[xEz \to y < z \vee y = z]$$

Definition 2.28. $\mathfrak{A}$ has **definable choice functions** if for any formula $\theta(\bar{x}, \bar{y})$ there is a definable function $f(\bar{y})$ s.t.

$$\forall \bar{y} \exists \bar{x} [\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(f(\bar{y}), \bar{y})]$$

(i.e., f is a skolem function for θ) and s.t.

$$\forall \bar{y} \forall \bar{z} [\forall \bar{x} (\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(\bar{x}, \bar{z})) \rightarrow f(\bar{y}) = f(\bar{z})]$$

Proof. If $\mathfrak{A}$ has definable choice functions then $\mathfrak{A}$ eliminates imaginaries □

Proof. Given a definable equivalence relation E on A^n let f be a definable choice function for $E(\bar{x}, \bar{y})$. Since E is an equivalence relation we have $\forall \bar{y} E(f(\bar{y}), \bar{y})$ and

$$\forall \bar{y} \bar{z} [\bar{y}/E = \bar{z}/E \rightarrow f(\bar{y}) = f(\bar{z})]$$

Thus $f(\bar{y}) = f(\bar{z}) \Leftrightarrow \bar{y} E \bar{z}$ □

Example 2.5. We now see that $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$ eliminates imaginaries. Basically since $\mathfrak{A}$ is well ordered, we can find a least element to witness membership of definable sets, hence we have definable functions

Example 2.6. $\mathfrak{A} = (\mathbb{N}, \equiv \mod 2)$ does not eliminate imaginaries

First note that the only definable subsets of $\mathbb{N}$ are $\emptyset, \mathbb{N}, 2\mathbb{N}, (2n+1)\mathbb{N}$. This is because $\mathfrak{A}$ has an automorphisms which switches $(2n+1)\mathbb{N}$ and $2\mathbb{N}$

Now suppose $f : \mathbb{N} \rightarrow \mathbb{N}^m$ eliminates the equivalence relation $\equiv \mod 2$, i.e.,

$$f(x) = f(y) \Leftrightarrow y \equiv 2 \mod 2$$

The $\mathrm{im}(f)$ is definable and has cardinality 2. Since there are no definable subsets of $\mathbb{N}$ of cardinality 2, we must have $m > 1$. Now let $\pi : \mathbb{N}^m \rightarrow \mathbb{N}$ be a projection. Then $\pi(\mathrm{im}(f))$ is a finite nonempty definable subset of $\mathbb{N}$. But no such set exists

Proposition 2.29. *If $\mathfrak{A}$ eliminates imaginaries, then $\mathfrak{A}_A$ eliminates imaginaries*

Proof. The idea is that an equivalence relation with parameters can be obtained as a fiber of an equivalence relation in more variables. Let $E \subseteq A^n$ be an equivalence relation definable in $\mathfrak{A}_A$. Let $\varphi(x, y; z) \in L$ and $a \in A^l$ be s.t.

$$xEy \Leftrightarrow \mathfrak{A} \models \varphi(x, y; a)$$

We now define

$$\psi(x,u,y,v) = \begin{cases} u = v \wedge \text{"}\varphi \text{ defines an equivalence relation"} & \text{or} \\ u \neq v & \text{or} \\ \text{"}\varphi(x,y,v) \text{ does not define an equivalence relation"} \end{cases}$$

Now ψ defines an equivalence relation on A^{n+l}. Let $f : A^{n+l} \to A^m$ eliminate ψ, then $f(-,a)$ eliminates E □

Back to [Pillay(2018)]

Definition 2.30. 1. T has elimination of imaginaries (EI) if for any model $M \vDash T$ and $e \in M^{\mathrm{eq}}$, there is a $\bar{c} \in M$ s.t. $e \in \mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{c})$ and $\bar{c} \in \mathrm{dcl}_{M^{\mathrm{eq}}}(e)$

2. T has weak elimination of imaginaries if, as above, except $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$ (that is, $e \in \mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{c})$ and $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$)

3. T has geometric elimination of imaginaries if, as above, except $e \in \mathrm{acl}_{M^{\mathrm{eq}}(\bar{c})}$ and $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$

Note that in particular, elimination of imaginaries imply the existence of codes for definable sets

Proposition 2.31. *TFAE*

1. *T has EI*

2. *For some model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation E, there is a partition of M^n into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

3. *For any model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation E, there is a partition of M^n into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

4. *For any model $M \vDash T$, and any definable $X \subseteq M^n$ there is an L-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in M$ s.t. X is defined by $\varphi(\bar{x}, \bar{b})$ and for all $\bar{b}' \in M$ if X is defined by $\varphi(\bar{x}, \bar{b}')$ then $\bar{b} = \bar{b}'$. We call such a $\bar{b}$ a code for X.*

most typos i've ever seen in a proof

Proof. $2 \Leftrightarrow 3$. Since we concern only $\emptyset$-definable relations and functions, if it is true in some model, then it is true in any model

$1 \to 2$. Let $\pi_E : S^n_= \to S_E$ the canonical definable quotient map. Let $e \in S_E$. By assumption, there is $k \in \mathbb{N}$ and $\bar{c} \in \mathfrak{C}^k$ s.t. e and $\bar{c}$ are interdefinable. In other words, there is a formula $\varphi_e(x, \bar{y})$ over $\emptyset$ s.t. $\varphi_e(e, \bar{c})$. Moreover, $|\varphi_e(\mathfrak{C}, \bar{c})| = |\varphi_e(e, \mathfrak{C})| = 1$

Let

$$
\begin{aligned}
X_e = \{\bar{x} \in \mathfrak{C}, \models \exists! \bar{y} \varphi_e(\pi_E(\bar{x}), \bar{y}) \\
\wedge \forall z(E(\bar{x}, \bar{z}) \leftrightarrow \\
(\forall y(\varphi_e(\pi_E(\bar{x}), \bar{y}) \leftrightarrow \varphi_e(\pi_E(\bar{z}), \bar{y}))))\}
\end{aligned}
$$

This means that φ_e defines a function on X_e, and that this function separates E-classes.

Then $\pi^{-1}(\{e\}) \subset X_e$.

Since each X_e contains $\pi^{-1}(\{e\})$, we get $\mathfrak{C}^n = \bigcup_{e \in \pi_E(\mathfrak{C}^n)} X_e$, and by compactness, there are $e_1, \dots, e_l$ s.t. $\mathfrak{C}^n = \bigcup_{i=1}^{l} X_{e_i}$. As each X_e is $\emptyset$-definable. Let $\bar{x} \in X_e \Leftrightarrow \theta_e(\bar{x})$. Suppose there is no such l, then $\{x = x\} \cup \{\neg\theta_e(x)\}$ is satisfiable and realised since $\mathfrak{C}$ is saturated Naively, we can pick $f_i = \varphi_{e_i} \circ \pi_E$, but X_{e_i} are not disjoint

However we can consider $Y_1, \dots, Y_r$ to be the atoms of the boolean algebra generated by the X_i. These are disjoint, and we can pick, for each Y_j, appropriate f_i, to get the result

$3 \to 4$. Let $X = \varphi(\mathfrak{C}, \bar{a})$. Consider the $\emptyset$-definable equivalence relation $E(\bar{y}, \bar{z}) \Leftrightarrow \forall x(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{z}))$. Let Y_i and f_i be as in 3 and say $\bar{a} \in Y_1$, and let $\bar{b} = f_1(\bar{a})$. Then $\exists \bar{y}(f_1(\bar{y}) = \bar{b} \wedge \varphi(\bar{x}, \bar{y}))$ defines X, call this formula ψ

We have to show that $\bar{b}$ is unique. Let $\bar{b}'$ be s.t. $\exists \bar{y}(f_1(\bar{y}) = \bar{b}' \wedge \varphi(\bar{x}, \bar{y}))$ also defines X, and let $\bar{a}_0$ be as the $\bar{y}$ in the formula. Then $\varphi(x, \bar{a}_0)$ defines X, hence $\bar{a}_0 E \bar{a}$, which implies $\bar{b}' = f_1(\bar{a}_0) = f_1(\bar{a}) = \bar{b}$

$4 \to 1$. Let $e \in \mathfrak{C}^{\mathrm{eq}}$, then $e = \pi_E(\bar{a})$ for some $\bar{a} \in \mathfrak{C}^n$ and some $\emptyset$-definable equivalence relation E

The set $X = \{\bar{x} \in \mathfrak{C}^n \mid\models E(\bar{x}, \bar{a})\}$ has a code $\bar{b} \in \mathfrak{C}^k$, so that $X = \psi(\mathfrak{C}^n, \bar{b})$. We are going to prove interdefinability of e and $\bar{b}$ using automorphisms of $\mathfrak{C}$

First suppose that $\sigma \in \mathrm{Aut}(\mathfrak{C})$, and fixes e. We have $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b}))$. Applying σ, we get $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \sigma(\bar{b})))$. But $\bar{b}$ is a code for X, hence $\bar{b} = \sigma(\bar{b})$. This implies $\bar{b} \in \mathrm{dcl}(e)$

Now suppose $\sigma \in \mathrm{Aut}(\mathfrak{C})$ and fixes $\bar{b}$. Again $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow$

$\psi(\bar{x}, \bar{b}))$ and $\mathfrak{C}^{\text{eq}} \vDash \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\sigma(\bar{a})) \leftrightarrow \psi(\bar{x}, \bar{b}))$. But $\psi(\bar{a}, \bar{b})$, $e = \pi_E(\bar{a}) = \pi_E(\sigma(\bar{a})) = \sigma(e)$. Hence $e \in \text{dcl}(\bar{b})$ □

Note that condition 2 is somewhat unsatisfying, as we would like to have a quotient function for E, that is, $r = 1$

Proposition 2.32. *Suppose T eliminates imaginaries. We get $r = 1$ in condition 2 iff* $\text{dcl}(\emptyset)$ *has at least two elements*

Proof. First, suppose that $r = 1$. Consider the equivalence on $\mathfrak{C}^2$ given by $E((x, y), (x', y'))$ iff $x = y \leftrightarrow x' = y'$. In other words, the E classes are the diagonal and its complement (only two). Then $\pi_E(\mathfrak{C}^2)$ has two elements, and they belong to $\text{dcl}^{\text{eq}}(\emptyset)$. But because T eliminates imaginaries, this implies that there is also two elements in $\text{dcl}(\emptyset)$ by Proposition 2.18

Second, suppose that $\text{dcl}(\emptyset)$ contains two constants a and b. Let Y_i, f_i be as in condition 2. Using a and b,we can find some number k and functions $g_i : \mathfrak{C}^{k_i} \to \mathfrak{C}^k$ s.t. $g_i(\mathfrak{C}^{k_i})$ are pairwise disjoint. We can check that the $\emptyset$-definable function $f : \mathfrak{C}^n \to \mathfrak{C}^k$ sending $y \in Y_i$ to $g_i(f_i(y))$ has all the required properties □

Remark. Elimination of imaginaries also makes sense for many sorted theories

Proposition 2.33 (Assume T 1-sorted)**.** *T^{eq} has elimination of imaginaries*

Proof. Prove a strong version of 2 in Proposition 2.31 that is, we don't need to distinguish $Y_1, \ldots, Y_r$ and $f_1, \ldots, f_r$. Let E' be a $\emptyset$-definable equivalence relation on a sort s_E in some model M^{eq} of T^{eq}. By Proposition 2.18 there is an L-formula $\psi(\bar{y}_1, \bar{y}_2)$ ($\bar{y}_i$ the appropriate length) s.t. for all $\bar{a}_1, \bar{a}_2 \in M$, $M \vDash \psi(\bar{a}_1, \bar{a}_2)$ iff $M^{\text{eq}} \vDash E'(f_E(\bar{a}_1), f_E(\bar{a}_2))$. So $\psi(\bar{y}_1, \bar{y}_2)$ is an L-formula defining an equivalence relation on M^k for the suitable length k. Consider the map h, taking $e \in S_E$ to $f_\psi(\bar{a})$ for any $\bar{a} \in M^k$ s.t. $f_E(\bar{a}) = e$ for any $\bar{a} \in M^k$ s.t. $f_E(\bar{a}) = e$. Suppose $f_E(\bar{a}) = e = f_E(\bar{a}')$, we easily see that $f_\psi(\bar{a}) = f_\psi(\bar{a}')$, hence the map h is well-defined, and satisfies 2 of 2.31 □

2.4 Examples and counterexamples

Example 2.7. The theory of an infinite set has weak elimination of imaginaries but not full elimination of imaginaries

Proof. First, we show that T has weak elimination of imaginaries. Let M be an infinite set and let $e \in M^{\mathrm{eq}}$ be an imaginary element. Suppose that. Let $A \subset M$ be a finite set over which X is definable ??. Consider the set

$$\hat{A} := \bigcap_{\substack{\sigma \in \mathrm{Aut}(M) \\ \sigma(X) = X}} \sigma(A)$$

Since A is finite, there are $\sigma_1, \dots, \sigma_n$ s.t. $\hat{A} = \bigcap_i \sigma_i(A)$

To see that T does not have full elimination of imaginaries, observe that there is never a code for any finite set. Indeed, if M is an infinite set, $X \subset_f M$, and $\bar{a} \in M$, we can find a permutation of M which fixes X as a set but does not fix $\bar{a}$, meaning $\bar{a}$ could not be a code for X □

Example 2.8. Let $T = \mathrm{Th}(M, <, \dots)$ where $<$ is a total well-ordering. Then T has elimination of imaginaries

Proof. Every definable set has a least element. We verify (2) in 2.31. Let E be a $\emptyset$-definable equivalence relation on M^n. Let $f : M^n \to M^n$ s.t. for any $\bar{a}$, $f(\bar{a})$ is the least element of the E-class of $\bar{a}$. Notice that f is $\emptyset$-definable, and for all $\bar{a}, \bar{b}$, $f(\bar{a}) = f(\bar{b})$ iff $E(\bar{a}, \bar{b})$ □

Lemma 2.34. *Let T be strongly minimal and* $\mathrm{acl}(\emptyset)$ *be infinite (in some, any model). Then T has weak elimination of imaginaries*

Proof. Fix a model M. Let $e \in M^{\mathrm{eq}}$ Ok, now i think the convention for pillay is that $e \in M^{\mathrm{eq}}$ is automatically imaginary, so $e = \bar{a}/E$ for some $\bar{a}$ and E some $\emptyset$-definable equivalence relation. Let $A = \mathrm{acl}_{M^{\mathrm{eq}}}(e) \cap M$. A is infinite as it contains $\mathrm{acl}(\emptyset)$. A is infinite as it contains $\mathrm{acl}(\emptyset)$.

We first prove that there exists some $\bar{b} \subset A$ s.t. $E(\bar{a}, \bar{b})$. Let $X_1 = \{y_1 \in M : M \vDash \exists y_2 \dots y_n (\bar{y} E \bar{a})\}$. It is definable over e. If X_1 is finite, any $b_1 \in X_1$ then belongs to A. Otherwise, X_1 is cofinite, hence meets the infinite set A. Either way, $X_1 \cap A \neq \emptyset$ and we have $b_1 \in X_1 \cap A$

Now let $X_2 = \{y_2 \in M : M \vDash \exists y_3 \dots y_n (b_1 \bar{y} E \bar{a})\}$. We remark $X_2 \neq \emptyset$ since $b_1 \in X_1$. Now X_2 is either finite or cofinite since T is strongly minimal. By the same argument above, we may find $b_2 \in X_2 \cap A$. Then repeating this process, we may find $\bar{b} \subset A$. Therefore $\bar{b} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$.

Finally notice that $e \in \mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{b})$ since $\bar{a}/E = \bar{b}/E = e$ □

Example 2.9. The theory ACF_p has elimination of imaginaries, for any p

Proof. By Lemma 2.34, ACF_p has weak elimination of imaginaries. Therefore it suffices to show that every finite set can be coded. Let K be an algebraically closed field and let $X = \{c_1, \dots, c_n\} \subseteq K$. Consider the polynomial

$$\begin{aligned} P(x) &= \prod_{i=1}^{n}(x - c_i) \\ &= x^n + e_{n-1}x^{n-1} + \cdots + e_1 x + e_0 \end{aligned}$$

Then we may take the tuple $\bar{e} = (e_n, \dots, e_0)$ to be our code for X. $\square$

3 Stability

3.1 Historic remarks and motivations

Thoughout this chapter we will fix a complete theory T in some language L. Moreover, we will have no problem in working in T^{eq} (that is to say, to assume $T = T^{\mathrm{eq}}$)

For a given theory T, the spectrum functions is given as

$$I(T, -) : Card \to Card$$
$$I(T, \lambda) = \#\text{ of models of } T \text{ or cardinality } \lambda \text{ (up to isomorphism)}$$

Conjecture 3.1 (Morley)**.** *Let T be countable, then function $I_T(\kappa)$ is non-decreasing on uncountable cardinals*

One of such dividing lines is stability

3.2 Counting types and stability

Definition 3.2. For a complete first order theory T, let $f_T : Card \to Card$ be defined by $f_T(\kappa) = \sup\{|S_1(M)| : M \vDash T, |M| = \kappa\}$, for κ an infinite cardinal

Exercise 3.2.1. Show that

$$f_T(\kappa) = \sup\{|S_n(M)| : M \vDash T, |M| = \kappa, n \in \omega\}$$

gives an equivalent definition

It is easy to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$

Fact 3.3 (Keisler, Shelah [Keisler(1976)]). *Let T be an arbitrary complete theory in a countable language. Then $f_T(\kappa)$ is one of the following functions (and all of these options occur for some T):*

$$\kappa, \kappa + 2^{\aleph_0}, \kappa^{\aleph_0}, \operatorname{ded}\kappa, (\operatorname{ded}\kappa)^{\aleph_0}, 2^\kappa$$

Here, $\operatorname{ded}\kappa = \sup\{|I| : I \text{ is a linear order with a dense subset of size } \kappa\}$*, equivalently* $\sup\{\lambda : \text{ there is a linear order of size } \kappa \text{ with } \lambda \text{ cuts}\}$

ded is called the **Dedekind function**

Lemma 3.4. $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$

Proof. Let μ be minimal s.t. $2^\mu > \kappa$, and consider the tree $2^{<\mu}$. Take the lexicographic ordering I on it, then $|I| = 2^{<\mu} \leq \kappa$ by the minimality of μ, but there are at least $2^\mu > \kappa$ cuts

Every cut is **uniquely** determined by the subset of elements in its lower half $\square$

Definition 3.5. Let $M \vDash T$

1. A formula $\phi(x, y)$ with its variables partitioned into two groups x, y, has the **k-order property**, $k \in \omega$, if there are some $a_i \in M_x$, $b_i \in M_y$ for $i < k$ s.t. $M \vDash \phi(a_i, b_j) \Leftrightarrow i < j$
2. $\phi(x, y)$ has the **order property** if it has the k-order property for all $k \in \omega$
3. A formula $\phi(x, y)$ is **stable** if there is some $k \in \omega$ s.t. it does not have the k-order property
4. A theory is **stable** if it implies that all formulas are stable

Proposition 3.6. *Assume that T is unstable, then $f_T(\kappa) \geq \operatorname{ded}\kappa$ for all cardinals $\kappa \geq |T|$*

Proof. Fix a cardinal κ. Let $\phi(x, y) \in L$ be a formula that has the k-order property for all $k \in \omega$. Then by compactness we have:

Claim. Let I be an arbitrary linear order. Then we can find some $\mathcal{M} \vDash T$ and $a_i b_i : i \in I$ from M s.t. $\mathcal{M} \vDash \phi(a_i, b_j) \Leftrightarrow i < j$, for all $i, j \in I$

Consider

$$T' = T \cup \{\phi(a_i, b_j) : i < j\} \cup \{\neg\phi(a_i, b_j) : igej\}$$

Let I be an arbitrary dense linear order of size κ, and let $(a_i b_i : i \in I)$ in $\mathcal{M}$ be as given by the claim. By Löwenheim–Skolem Theorem, we can assume that $|\mathcal{M}| = \kappa$

Given a cut $C = (A, B)$ in I, consider the set of $L(M)$-formulas

$$\Phi_C = \{\phi(x, b_j) : j \in B\} \cup \{\neg\phi(x, b_j) : j \in A\}$$

Note that by compactness it is a partial type, let $p_C \in S_x(M)$ be a complete type over M extending $\Phi_C(x)$. Given two cuts C_1, C_2, we have $p_{C_1} \neq p_{C_2}$. As I was arbitrary, this shows that $\sup\{|S_x(M)| : M \vDash T, |M| = \kappa\} \geq \operatorname{ded} \kappa$. Note that we may have $|x| > 1$, however using Exercise **??** we get $f_T(\kappa) \geq \operatorname{ded} \kappa$ $\square$

Fact 3.7 (Ramsey). *$\aleph_0 \to (\aleph_0)^n_k$ holds for all $n, k \in \omega$ (i.e., for any coloring of subsets of $\mathbb{N}$ of size n in k colors, there is some infinite subset I of $\mathbb{N}$ s.t. all n-element subsets of I have the same color)*

Lemma 3.8. *Let $\phi(x, y)$, $\psi(x, z)$ be stable formulas (where y, z are not necessarily disjoint tuples of variables). Then*

1. *$\neg\phi(x, y)$ is stable*
2. *Let $\phi^*(y, x) := \phi(x, y)$, i.e., we switch the roles of the variables. Then $\phi^*(y, x)$ is stable*
3. *$\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ and $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ are stable*
4. *If $y = uv$ and $c \in M_v$, then $\theta(x, u) := \phi(x, uc)$ is stable*
5. *If T is stable, then every L^{eq}-formula is stable as well*
6. *The formula $\varphi(x, y)$ is stable for T iff there is $n < \omega$ s.t. $\varphi(x, y)$ is n-stable: it is not the case that there are a_i, b_i (in $\mathfrak{C}$, or in some/any $M \vDash T$), $i < n$, s.t. $\vDash \varphi(a_i, b_i)$ iff $i < j$ for all $i, j < n$*
7. *There are T, $M \vDash T$ and $\varphi(x, y)$ s.t. $\varphi(x, y)$ is stable in M but it is not stable for T*

Proof. 1. Suppose $\neg\phi(x, y)$ is unstable, then there is $I = (a_i, b_i)_{i \in \omega}$ s.t. $\vDash \neg\varphi(a_i, b_j) \Leftrightarrow i < j$, equivalently, $\vDash \varphi(a_i, b_j) \Leftrightarrow i \geq j$. Then add constants $(a_i, b_i)_{i \in \omega}$ and consider

$$\Gamma = T \cup \{\varphi(a_i, b_j) : i < j\} \cup \{\neg\varphi(a_i, b_j) : i \geq j\}$$

For any finite subset $\Gamma' \subset_f \Gamma$, we can reverse the order of I: suppose n is the maximum index and then let $i' = n - i$, $j' = n + 1 - j$. Then $i' < j' \Leftrightarrow n - i < n + 1 - j \Leftrightarrow i \geq j$. Hence I satisfies this, and hence $\varphi(x, y)$ is unstable

2. Suppose $\varphi^*(y, x)$ is not stable, then $\neg\varphi^*(y, x)$ is also unstable. Let a_i, b_i be witnesses in $\mathfrak{C}$ of the latter. Then $a_i' = b_i$ and $b_i' = a_{i+1}$, $i < \omega$, witness the instability of $\varphi(x, y)$ as $j + 1 > i$

3. Suppose that $\theta'(x, yz)$ is unstable, i.e., there are $(a_i, b_i b_i' : i \in \mathbb{N})$ s.t. $\vDash \phi(a_i, b_j) \vee \psi(a_i, b_j') \Leftrightarrow i < j$ for all $i, j \in \mathbb{N}$. Let

   $$P := \{(i, j) \in \mathbb{N}^2 : i < j, \vDash \phi(a_i, b_j)\}, Q := \{(i, j) \in \mathbb{N}^2 : i < j, \vDash \psi(a_i, b_j')\}$$

   then $P \cup Q = \{(i, j) \in \mathbb{N}^2 : i < j\}$. By Ramsey there is an infinite $I \subseteq \mathbb{N}$ s.t. either all increasing pairs from I belong to P, or all increasing pairs from I belong to Q

7. Consider the graph G, disjoint union of all finite graphs. Then the edge relation E is stable in G. Indeed, if it wasn't, we would have a vertex x_0 and infinitely many vertices $\{y_i : i \in \mathbb{N}\}$ s.t. $E(x_0, y_i)$ for all i, which is impossible

   But by 6, edge relation is not stable in $\mathrm{Th}(G)$

□

Lemma 3.9. *Let X be a set and $Y_1, \ldots, Y_n$ are subsets of X. Define*

$$E(x, y) := \bigwedge_{i=1}^{n} (x \in X_i \Leftrightarrow y \in X_i)$$

Then E is an equivalence relation on X and $Z \subseteq X$ is a boolean combination of X_i's iff

$$E(x, y) \Rightarrow (x \in Z \Leftrightarrow y \in Z)$$

Proof. E is an equivalence relation is obvious

$\Rightarrow$: obvious

$\Leftarrow$: Let U be the set of all boolean combination of X_i's. Let V be all the set Z satisfying $E(x, y) \Rightarrow (x \in Z \Leftrightarrow y \in Z)$. We want to show that $U \subseteq V$. First each X_i satisfies the condition. □

Theorem 3.10 (Erdős-Makkai)**.** *Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(b_i : i < \omega)$ of elements of B and $(S_i : i < \omega)$ of elements of $\mathcal{F}$ s.t. one of the following holds*

1. $b_i \in S_j \Leftrightarrow j < i (\forall i, j \in \omega)$
2. $b_i \in S_j \Leftrightarrow i < j (\forall i, j \in \omega)$

Proof. Choose $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = |B|$, s.t. any two finite subsets B_0, B_1 of B, if n$\exists S \in \mathcal{F}$ with $B_0 \subseteq S$, $B_1 \subseteq B \setminus S$, then there is some $S' \in \mathcal{F}'$ with $B_0 \subseteq S'$, $B_1 \subseteq B \setminus S'$ (possible as there are at most $|B|$-many pairs of finite subsets of B)

By assumption there is some $S^* \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$ (again there are at most $|B|$-many different Boolean combinations of sets from $\mathcal{F}'$)

We choose by induction sequences $(b'_i : i < \omega)$ in S^*, $(b''_i : i < \omega)$ in $B \setminus S^*$ and $(S_i : i < \omega)$ in $\mathcal{F}'$ s.t.

- $\{b'_0, \dots, b'_n\} \subseteq S_n$ and $\{b''_0, \dots, b''_n\} \subseteq B \setminus S_n$
- $\forall i < n (b'_n \in S_i \Leftrightarrow b''_n \in S_i)$

Assume $(b'_i : i < n)$, $(b''_i : i < n)$ and $(S_i : i < n)$ have already been constructed. Since S^* is not a Boolean combination of $S_0, \dots, S_{n-1}$, there are $b'_n \in S^*$, $b''_n \in B \setminus S^*$ s.t. for all $i < n$

$$b'_n \in S_i \Leftrightarrow b''_n \in S_i$$

by Lemma 3.9

By the choice of $\mathcal{F}'$, there is some $S_n \in \mathcal{F}'$ with $\{b'_0, \dots, b'_n\} \subseteq S_n$ and $\{b''_0, \dots, b''_n\} \subseteq B \setminus S_n$.

Now by Ramsey theorem we may assume that either $b'_n \in S_i$ for all $i < n < \omega$ or $b'_n \notin S_i$ for all $i < n < \omega$ (for $\{x, y\} \subset [\mathbb{N}]^2$ and assume $x < y$, color it according to whether $b'_y \in S_x$. Thus by Ramsey, there is an infinite $I \subseteq \omega$ s.t.

- either $\forall n > j \in I (b'_n \in S_j) \Rightarrow \forall i, j \in I (b''_i \in S_j \Leftrightarrow i > j)$
- or $\forall n > j \in I (b'_n \notin S_j) \Rightarrow \forall i, j \in I (a'_i \in S_j \Leftrightarrow i \leq j)$

Note that if $b''_i \in S_j$ and $i \leq j$, then as $\{b''_0, \dots, b''_i\} \subseteq B \setminus S_j$, $b''_i \notin S_j$

In the first case we set $b_i = b''_i$ and get 1, in the second case we set $b_i = b'_{i+1}$ and get 2. □

Definition 3.11. Fix $\varphi(x, y) \in L$. By a **complete φ-type over** $A \subseteq M_y$, we mean a maximal consistent collection of formulas of the form $\varphi(x, b)$, $\neg\varphi(x, b)$ where b ranges over A. Let $S_\varphi(A)$ be the space of all complete φ-types over A

Proposition 3.12. *Assume that $|S_\varphi(B)| > |B|$ for some infinite set of parameters B. Then $\varphi(x,y)$ is unstable*

Proof. For $a \in \mathbb{M}_x$, $\mathrm{tp}_\varphi(a/B)$ is determined by $\varphi(a,B) = \{b \in B \mid\models \phi(a,b)\}$. Then $|S_\varphi(B)| > |B| \Rightarrow |\{|\phi(a,B) \mid a \in \mathbb{M}_x\} > |B|$. By Erdős-Makkai, there are sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ s.t.

$$\text{either } \models \phi(a_i,b_j) \Leftrightarrow i<j, \text{ or } \models \phi(a_i,b_j) \Leftrightarrow j<i$$

□

Remark.

1. By a φ**-formula over** M we mean a Boolean combination of instances (over M) of φ and $\neg\varphi$. For example, $(\varphi(x,c) \wedge \varphi(x,b)) \vee \neg\varphi(x,d)$ is a φ-formula

2. Any type $p(x) \in S_\varphi(M)$ decides any φ-formula $\psi(x)$ over M, that is to say $p(x) \models \psi(x)$ or $p(x) \models \neg\psi(x)$, so in fact $p(x)$ extends to a unique maximal consistent set of φ-formulas over M

3. By defining the basic open sets of $S_\varphi(M)$ to be $\{p(x) \in S_\varphi(M) : \psi(x) \in p\}$ for ψ a φ-formula, $S_\varphi(M)$ becomes a compact totally disconnected space, where in addition the clopen sets are precisely given by φ-formulas, i.e., they are the basic clopen sets

4. Any $p(x) \in S_\varphi(M)$ extends to some $q(x) \in S_x(M)$ s.t. $p = q \restriction \varphi$, where $q \restriction \varphi$ is the set of φ-formulas in $q(x)$ (or instances of φ, $\neg\varphi$ in $q(x)$)

3.3 Local ranks and definability of types

Definition 3.13. We define **Shelah's local 2-rank** taking values in $\{-\infty\} \cup \omega \cup \{+\infty\}$ by induction on $n \in \omega$. Let Δ be a set of L-formulas, and $\theta(x)$ a partial type over $\mathfrak{C}$

- $R_\Delta(\theta(x)) \geq 0$ iff $\theta(x)$ is consistent ($-\infty$ otherwise)
- $R_\Delta(\theta(x)) \geq n+1$ if for some $\phi(x,y) \in \Delta$ and $a \in \mathfrak{C}_y$ we have both $R_\Delta(\theta(x) \wedge \phi(x,a)) \geq n$ and $R_\Delta(\theta(x) \wedge \neg\phi(x,a)) \geq n$
- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and $R_\Delta(\theta(x)) \not\geq n+1$, and $R_\Delta(\theta(x)) = \infty$ if for $n \in \omega$, $R_\Delta(\theta(x)) \geq n$

If $\phi(x,y)$ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$

Proposition 3.14. *$\phi(x,y)$ is stable iff $R_\phi(x=x)$ is finite (and so also $R_\phi(\theta(x))$ is finite for any partial type θ). Here $x=(x_i : i \in I)$ is a tuple of variables and $x=x$ is an abuse of notation for $\bigwedge_{i\in I} x_i = x_i$*

Proof. If $\phi(x,y)$ is unstable, i.e., it has the k-order property for all $k \in \omega$, by compactness, we find $(a_i b_i : i \in [0,1])$ s.t. $\vDash \phi(a_i, b_j) \Leftrightarrow i < j$. We know both $\phi(x, b_{1/2})$ and $\neg\phi(x, b_{1/2})$ contain dense subsequences of a_i's. Each of these sets can be split again

Conversely, suppose the rank is infinite, then we can find an infinite tree of parameters $B=(B_\eta : \eta \in 2^{<\omega})$ s.t. for every $\eta \in 2^\omega$ there set of formulas $\{\phi^{\eta(i)}(x, b_{\eta|i}) : i < \omega\}$ is consistent where $\phi^1 = \phi$ and $\phi^0 = \neg\phi$ (rank being $\geq k$ guarantees that we can find such a tree of height k, and then use compactness to find one of infinite height). This gives us that $\left|S_\phi(B)\right| > |B|$, which by Proposition 3.12 implies that $\phi(x,y)$ is unstable □

Definition 3.15.

1. Let $\phi(x,y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is **definable over** B if there is some $L(B)$-formula $\psi(y)$ s.t. $\forall a \in A$

$$\phi(x,a) \in p \Leftrightarrow \vDash \psi(a)$$

2. A type $p \in S_x(A)$ is definable over B if $p \mid \phi$ is definable over B for all $\phi(x,y) \in L$

   $\forall \phi(x,y) \in L,\ \exists \psi(y) \in L(B),\ \forall a \in A$ s.t.

$$\phi(x,a) \in p \Leftrightarrow \vDash \ \psi(a)$$

3. A type is **definable** if it is definable over its domain

4. types in T are **uniformly definable** if for every $\phi(x,y)$ there is some $\psi(y,z)$ s.t. every type can be defined by an instance of $\psi(y,z)$, i.e., for any A and $p \in S_\phi(A)$ there is some $b \in A$ s.t. $\phi(x,a) \in p \Leftrightarrow \vDash \psi(a,b)$ for all $a \in A$

Remark. Another way to think about it:

Given a set $A \subseteq \mathfrak{C}_x$, $B \subseteq A$ is **externally definable** (as a subset of A) if there is some definable (over $\mathfrak{C}$) set X s.t. $B = X \cap A$

Assume moreover that we have $X = \phi(c, \mathfrak{C})$ above. Then $\mathrm{tp}_\phi(c/A)$ is definable iff B is internally definable, i.e., $B = A \cap Y$ for some A-definable Y. A set is called **stably embedded** if every externally definable subset of it is internally definable. $\phi(x,a) \in \mathrm{tp}_\phi(c/A) \Leftrightarrow \vDash \phi(c,a) \Leftrightarrow a \in X \Leftrightarrow \vDash \psi(a)$. Thus $X = \phi(c,\mathfrak{C}) = \psi(\mathfrak{C})$

Example 3.1. Consider $(\mathbb{Q}, <) \vDash \mathsf{DLO}$ and let $p = \mathrm{tp}(\pi/\mathbb{Q})$. Then $x < y \in p(y) \Leftrightarrow x < \pi$. By QE, p is not definable

Lemma 3.16. *1. The set $\{e \in \mathbb{M}^k : R_\phi(\theta(x, e)) \geq n\}$ is definable, for all $n \in \omega$*

2. If $R_\phi(\theta(x)) = n$, then for any $a \in \mathbb{M}_y$, at most one of $\theta(x) \wedge \phi(x, a)$, $\theta(x) \wedge \neg\phi(x, a)$ has R_ϕ-rank n

Proof. 1. Let $S_n(\theta) = \{e : R_\phi(\theta(x, e)) \geq n\}$ and suppose it is defined by $\psi_{n,\theta}(x)$. Induction on n to show that $S_n(\theta)$ is definable for any θ. For $n = 0$, consider $\psi_{0,\theta}(x) := \exists y(\theta(y, x))$. Then $e \in R_0(\theta)$ iff $\theta(x, e)$ is consistent iff $\vDash \exists x(\theta(x, e))$ iff $e \in \psi_{0,\theta}(\mathfrak{C})$.

Now for n, $e \in S_n(\theta)$ iff $\exists a(R_\phi(\theta(x, e) \wedge \phi(x, a)) \geq n-1 \wedge R_\phi(\theta(x, e) \wedge \neg\phi(x, a)) \geq n-1)$

□

Proposition 3.17. *Let $\phi(x, y)$ be a stable formula. Then all ϕ-types are uniformly definable*

Proof. Suppose that $R_\phi(x = x) = n \in \omega$. Let $p \in S_\phi(A)$. Then there is $\chi(x) \in p$ s.t. $R_\phi(\chi(x)) = \min\{R_\phi(\varphi(x)) \mid \varphi \in p\}$. For each $b \in A_y$ either $\phi(x, b) \in p$ or $\neg\phi(x, b) \in p$. Either $R_\phi(\chi(x) \wedge \phi(x, b)) < n$ or $R_\phi(\chi(x) \wedge \neg\phi(x, b)) < n$.

$R_\phi(\chi(x))$ is minimal $\Rightarrow (\phi(x, b) \in p \Leftrightarrow R_\phi(\chi(x) \wedge \phi(x, b)) = n)$ □

Summary

Theorem 3.18. *TFAE*

1. *$\phi(x, y)$ is stable*
2. *$R_\phi(x = x) < \omega$*
3. *All ϕ-types are uniformly definable*
4. *All ϕ-types over models are definable*
5. *$|S_\phi(M)| \leq \kappa$ for all $\kappa \geq |L|$ and $M \vDash T$ with $|M| = \kappa$*
6. *There is some κ s.t. $|S_\phi(M)| < \mathrm{ded}\,\kappa$ for all $M \vDash T$ with $|M| = \kappa$*

Proof. $1 \leftrightarrow 2$ 3.14. $1 \to 3$ 3.17. $3 \to 4$ obvious.

$4 \to 5$. There are $|L| + \kappa = \kappa$ possible formulas defining $S_\phi(M)$ over M

$6 \to 1$ 3.6 □

Global case:

Theorem 3.19. *Let T be a complete theory. TFAE:*

1. *T is stable*
2. *There is NO sequence of tuples $(c_i)_{i\in\omega}$ from $\mathbb{M}$ and formula $\phi(z_1, z_2) \in L(M)$ s.t.*
$$\vDash \phi(c_i, c_j) \Leftrightarrow i < j$$
3. *$f_T(\kappa) \leq \kappa^{|T|}$ for all infinite cardinals κ*
4. *There is some κ s.t. $f_T(\kappa) \leq \kappa$*
5. *There is some κ s.t. $f_T(\kappa) < \operatorname{ded} \kappa$*
6. *All formulas of the form $\phi(x, y)$ where x is a singleton variable are stable*
7. *All types over models are definable*

Proof. $1 \to 2$: definition

$2 \to 1$: Let $\psi(x, y)$ be a formula with order property witnessed by sequence
$$\{(a_i, b_i) \mid i < \omega\}$$
Let $\phi(x_1y_1, x_2y_2) := \psi(x_1, y_2)$ and $c_i : a_ib_i$. Then $\vDash \phi(c_ic_j) \Leftrightarrow i < j$

$1 \to 3$: $S_x(M) \to \prod_{\phi\in L} S_\phi(M)$ is injective

$3 \to 4$, $4 \to 5$: obvious

$5 \to 1$: 3.6

$6 \leftrightarrow 1$: Fix some κ, then $S_1(M) \leq \kappa$ for all M with $|M| = \kappa$ iff $S_n(M) \leq \kappa$ for all M with $|M| = \kappa$

$1 \leftrightarrow 7$: 3.18 □

Example 3.2.

- stability $\Leftrightarrow$ all types over all models are definable
- some unstable theories have certain special models over which all types are definable
- $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$, all types over $\mathbb{R}$ are uniformly definable

As we will see later, a theory T iff all types over **all** models of T are definable.

Note that there are unstable theories for which all the types over a certain models are definable. For instance, in the case of dense linear orders, all types over $\mathbb{R}$ are definable

Indeed, by quantifier elimination, any non-realised 1-type over any model of DLO corresponds to a cut in its order. But in the case of $\mathbb{R}$, the order is complete, so for any cut, there will in fact exist a real number r s.t. the cut is of the form $(\{l \in \mathbb{R}, l < r\}, \{d \in \mathbb{R}, d > r\})$. Using this real number r, one can easily show definability of 1-types over $\mathbb{R}$

Proposition 3.20. *Fix a model $M \vDash T$ and an L-formula $\varphi(x, y)$. TFAE*

1. *$\varphi(x, y)$ is stable in M*
2. *Whenever $M^* \succ M$ is $|M|^+$-saturated and* $\text{tp}(a^*/M^*)$ *is finitely satisfiable in M, then* $\text{tp}_\varphi(a^*/M^*)$ *is definable over M and, moreover, it is defined by some φ-formula φ^*, i.e., a Boolean combination of $\varphi(a, y)$'s, $a \in M$*

1#+BEGIN$_{\text{proof}}$ $1 \to 2$. Fix some $p^*(x) = \text{tp}_\varphi(a^*/M^*)$ finitely satisfiable in M. We want to prove $\text{tp}_\varphi(a^*/M^*)$ is definable over M by a φ^*-formula. Note first that, as p^* is finitely satisfiable in M, whether or not some $\varphi(x, b)$, $b \in M^*$, is in p^* depends only on $\text{tp}(b/M)$; in fact, even only on $\text{tp}_{\varphi^*}(b/M) = q(y) \in S_{\varphi^*}(M)$

Suppose we had $b' \in M^*$ s.t. $\text{tp}_{\varphi^*}(b'/M) = \text{tp}_{\varphi^*}(b/M)$, but $\varphi(x, b) \in p^*$ and $\neg\varphi(x, b') \in p^*$. Then we would have $\vDash$ #+END$_{\text{proof}}$

3.4 Cantor-Bendixson Rank

Definition 3.21 (Cantor-Bendixson Rank)**.** Let X be a topological space. The **Cantor-Bendixson rank** is a function $CB_X : X \to On \cup \{\infty\}$. Let $p \in X$, then:

1. $CB_X(p) \geq 0$
2. $CB_X(p) = \alpha$ if $CB_X(p) \geq \alpha$ and p is isolated in the (closed) subspace $\{q \in X : CB_X(q) \geq \alpha\}$
3. $CB_X(p) = \infty$ if $CB_X(p) > \alpha$ for every ordinal α

For example, $CB_X(p) = 0$ if p is isolated, equivalently if $\{p\}$ is open. $CB_X(p) \geq 1$ otherwise

Note that 2 claims that the subspace $\{q \in X : CB_X(q) \geq \alpha\}$ is closed for all α. This is a consequence of the fact that the set of isolated points of any topological space form an open set, as a union of open sets

Proposition 3.22. *Suppose X is compact and $CB_X(p) < \infty$ for every p in X. Then there exists a maximal element α of $\{CB_X(p) : p \in X\}$ and $\{p \in X : CB_X(p) = \alpha\}$ is finite and non empty*

Proof. Assume there is no maximal element. Then, for each ordinal α there exists some p_α in X s.t. $CB_X(p_\alpha) > \alpha$. The set $\{p_\alpha : \alpha \in On\}$ must have a limit point p in the compact set X, which cannot be isolated in any of the $\{q \in X : CB_X(q) \geq \alpha\}$. Hence $CB_X(p) = \infty$, a contradiction

Let $\alpha = \sup\{CB_X(p) : p \in X\}$. We want to show that $X_\alpha = \{p \in X : CB_X(p) = \alpha\}$ is non-empty. We only need to consider the limit case. Assume it is empty and for each $\beta < \alpha$, $X_{<\beta} = \{p \in X : CB_X(p) < \beta\}$. Since $\mathcal{C} = \{X_\beta : \beta < \alpha\}$ is an open cover of X which clearly has no finite subcover as α is a limit ordinal, a contradiction

$\{p \in X : CB_X(p)\} \geq \alpha$ is closed, so compact. Since α is maximal, all points in $\{p \in X : CB_X(p) \geq \alpha\}$ are isolated. Therefore $\{p \in X : CB_X(p) \geq \alpha\}$ is finite $\square$

Lemma 3.23. *Suppose $\varphi(x, y)$ is stable in T. Let $M \vDash T$, $X = S_\varphi(M)$. Then $CB_X(p) < \infty$ for each $p \in X$*

Proof. $X_\alpha = \{p \in X : CB_X(p) \geq \alpha\}$. If $\exists q \in X$ s.t. $CB_X(q) = \infty$, then for some α, $X_\alpha \neq \emptyset$ and has no isolated points. If not, then each X_α has at least one isolation point and we could conclude that $CB_X(p) \leq |X|$ for any $p \in X$

Now fix an α. Since there are no isolated points in X_α, we can find $p_0, p_1 \in X_\alpha$ where $p_0 \neq p_1$. Since $S_\varphi(M)$ is Hausdorff, we can find $\psi_0(x)$ s.t. $\psi_0(x) \in p_0$ and $\neg\psi_0(x) \in p_1$. Notice that $\{p : p \in X_\alpha\} \cap [\psi_0(x)]$ and $\{p : p \in X_\alpha\} \cap [\psi_1(x)]$ have no isolated points. Thus we could build a tree and $|S_\varphi(M')| \geq 2^{\aleph_0}$ for some countable model M' by Löwenheim–Skolem Theorem since there is only countable many parameters $\square$

3.5 Indiscernible sequences and stability

Definition 3.24. Given a linear order I, a sequence of tuples $(a_i : i \in I)$ with $a_i \in \mathfrak{C}_x$ is **indiscernible** over a set of parameters A if $a_{i_0} \dots a_{i_n} \equiv_A a_{j_0} \dots a_{j_n}$ for all $i_0 < \dots < i_n$ and $j_0 < \dots < j_n$ from I and all $n \in \omega$

Example 3.3.

1. A constant sequence is indiscernible over any set
2. A subsequence of a A-indiscernible sequence is A-indiscernible
3. In the theory of equality, any sequence of singletons is indiscernible
4. Any increasing sequence of singletons in a dense linear order is indiscernible
5. Any basis in a vector space is an indiscernible sequence

Definition 3.25. For any sequence $\bar{a} = (a_i \mid i \in I)$ and a set of parameters B, we define EM$(\bar{a}/B)$, the Ehrenfeucht-Mostowski type of the sequence $\bar{a}$ over B, as a partial type over B in countably many variables indexed by ω and given by the following collection of formulas

$$\{\phi(x_0, \dots, x_n) \in L(B) \mid \forall i_0 < \dots < i_n, \models \phi(a_{i_0}, \dots, a_{i_n}), n \in \omega\}$$

Exercise 3.5.1. For any sequence $\bar{a} = (a_i \mid i \in I)$ and a set of parameters B. If J is an infinite linear order, then there is a sequence $\bar{b} = (b_i \mid i \in J)$ which realises EM$(\bar{a}/A)$

Exercise 3.5.2. If $\bar{a} = (a_i \mid i \in I)$ is an A-indiscernible sequence. Then EM$(\bar{a}/A)$ is a complete ω-type over A

Let $\bar{a} = (a_i \mid i \in I)$ and $\bar{b} = (b_j \mid j \in J)$ be A-indiscernible sequences. We denote $\bar{a} \equiv_{\text{EM},A} \bar{b}$ if EM$(\bar{a}/A) =$ EM$(\bar{b}/A) \in S_\omega(A)$

Proposition 3.26. *Let $\bar{a} = (a_i : i \in J)$ be an arbitrary sequence in $\mathfrak{C}$, where J is an arbitrary linear order and A is a small set of parameters. Then for any small linear order I we can find (in $\mathbb{M}$) an A-indiscernible sequence $(b_i : i \in I)$ realize the EM-type of $\bar{a}$ over A*

Corollary 3.27. *If $(a_i : i \in I)$ is an A-indiscernible sequence and $J \supseteq I$ is an arbitrary linear order, then there is an A-indiscernible sequence $(b_j : j \in J)$ s.t. $b_i = a_i$ for all $i \in I$ (everything involved is small)*

Proof. Let $(b_j : j \in J)$ be an arbitrary A-indiscernible sequence in $\mathfrak{C}$ based on I, obtained by 3.26. In particular

$$(b_j : j \in I) \equiv_A (a_j : i \in I)$$

which by strong homogeneity of $\mathfrak{C}$ implies that there is some $\sigma \in \text{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b_j) = a_j$. Then define $b'_j = \sigma(b_j)$ for all $j \in J$ $\square$

Lemma 3.28. *If $\bar{a} = (a_i \mid i \in I)$ is an infinite A-indiscernible sequence, then for all $S \subseteq I$ and $i \in I \setminus S$, $a_i \notin \text{acl}(A, a_{j \in S})$*

Proof. $a_i \in \text{acl}(A, a_{j\in S}) \Leftrightarrow \exists S_0 \subseteq_f S(a_i \in \text{acl}(A, a_{j \in S_0}))$. Let $(b_i \mid i \in \mathbb{Q}) \equiv_{\text{EM},A} (a_i \mid i \in I)$. Then for any $i_0 < \dots < i_n \in I$ and $j_0 < \dots < j_n \in \mathbb{Q}$

$$a_{i_k} \in \text{acl}(A, \{a_{i_s} \mid s \neq k, s \leq n\}) \Leftrightarrow b_{j_k} \in \text{acl}(A, \{b_{j_s} \mid s \neq k, s \leq n\})$$

WLOG, we assume that $I = (\mathbb{Q}, <)$.

Suppose that

$$a_{i_k} \in \text{acl}(A, \{a_{i_s} \mid s \neq k, s \leq n\})$$

and $\phi(x_0, \dots, x_k, \dots, x_n) \in L(A)$ witness the property. Then for any $q \in \mathbb{Q}$ realizing the same cut of a_{i_k} over $\{a_{i_s} \mid s \neq k, s \leq n\}$ we have

$$\vDash \phi(a_{i_0}, \dots, a_q, \dots, a_{i_n})$$

So $\phi(a_{i_0}, \dots, \mathbb{M}, \dots, a_{i_n})$ is infinite, a contradiction $\square$

Exercise 3.5.3. Start with the sequence $\bar{a} = (1, 2, 3, \dots)$ in $(\mathbb{C}, +, \times, 0, 1) \vDash \mathsf{ACF}_0$. Give an explicit example of an indiscernible sequence realizing $\mathrm{EM}(\bar{a})$

Proof. $x \in \mathbb{R}_{>0} \Leftrightarrow \exists y\ x = y^2 \wedge x \neq 0$. And in $\mathbb{R}_{>0}$ we can define an order $x > y \Leftrightarrow \exists z(x = y + z^2 \wedge z \neq 0)$. Note that $\mathrm{EM}_{\mathbb{R}_{>0}}(\bar{a}) \subseteq \mathrm{EM}_{\mathbb{C}}(\bar{a})$.

Thus $\bar{b}$ should be an increasing sequence of reals greater than or equal to 1. $\square$

Proposition 3.29. *Let κ, λ be small cardinals and let $(a_i)_{i \in \lambda}$ be a sequence of tuples with $|a_i| < \kappa$ and a set B be given. If $\lambda \geq \beth_{(2^{\kappa+|B|+|T|})^+}$ there is a B-indiscernible sequence $(a'_i)_{i \in \omega}$ s.t. for every $n \in \omega$ there are $i_0 < \dots < i_n \in \kappa$ s.t. $a'_0 \dots a'_n \equiv_B a_{i_0} \dots a_{i_n}$*

Let A be a set of parameters, and $\lambda \geq |S_\kappa(A)|$ (for example, $\lambda = 2^{|T|+|A|+\kappa}$). Set $\mu = \beth_{\lambda^+}$. Then for any sequence $(a_i : i < \mu)$ of κ-tuples there is an A-indiscernible sequence $(b_i : i < \omega)$ s.t. for all $n < \omega$ there are $i_0 < \dots < i_{n-1} < \mu$ for which $b_0 \dots b_{n-1} \equiv_A a_{i_0} \dots a_{i_{n-1}}$

Proof. We construct by induction a sequence of types p_n, each one a complete $n \times \kappa$-type over A, s.t. for all n

1. for any $i_0 < \dots < i_{m-1} < n$ we have $p_n(x_0, \dots, x_{n-1}) \vdash p_m(x_{i_0}, \dots, x_{i_{m-1}})$
2. For all $\eta < \mu$ there is $I \subseteq \mu$, $|I| = \eta$ s.t. every n elements in order from a_I satisfy p_n

For $n = 0$ there is nothing to do. Given p_n, consider the set of all $(n + 1) \times \kappa$-types over A that satisfy the first condition. If there is $q \in S$ that also satisfies the second,we are done. If not, then for each $q \in S$ there is an $\eta_q < \mu$ that witnesses it. As $|S| \leq \lambda < \mathrm{cf}(\mu) = \lambda^+$, we have that $\eta = \lambda + \sup\{\eta_q : q \in S\} < \mu$ is such that for all $q \in S$, for all $I \subseteq \mu$ with $|I| = \eta$, not all $(n + 1)$-sub-tuples in order from a_I satisfy q. As $\eta < \mu$, $\eta < \beth_\theta$ for some $\theta < \lambda^+$. Write $\nu = \beth_{\theta+n+1}$. Then on the one hand, $\nu < \mu$. On the other, $\nu \geq \beth_n(\eta)^+$. By the inductive hypothesis, there is $I \subseteq \mu$, $|I| = \nu$ s.t. all n-tuples in order in a_I satisfy p_n. As there are at most λ possible A-types for $(n + 1)$-tuples and $\lambda \leq \eta$, the Erdős-Rado theorem gives us $I' \subseteq I$ with $|I'| = \eta^+$ where all $(n + 1)$-tuples in order have the same type over A. This gives the wanted contradiction. Take p_ω as the limit of p_n $\square$

Definition 3.30. A sequence $(a_i \mid i \in I)$ is **totally indiscernible over** A if $a_{i_0} \dots a_{i_n} \equiv_A a_{j_0} \dots a_{j_n}$ for any $i_0 \neq \dots \neq i_n$, $j_0 \neq \dots \neq j_n$ from I

Theorem 3.31. *T is stable iff every indiscernible sequence is totally indiscernible*

Proof. $\Rightarrow$: Suppose T is stable and $(a_i \mid i \in I)$ is indiscernible over A. If $(a_i \mid i \in I)$ is not totally indiscernible, then there are $i_0 \neq \dots \neq i_n$, $j_0 \neq \dots \neq j_n$ from I s.t. $a_{i_0} \dots a_{i_n} \not\equiv_A a_{j_0} \dots a_{j_n}$ which implies they are in different orders. WLOG, assume that $I = (\mathbb{Q}, <)$ and $i_0 = 0, \dots, i_n = n$. Then there is $\sigma \in S_{n+1}$ s.t.

$$a_{\sigma(0)} \dots a_{\sigma(n)} \equiv_A a_{j_0} \dots a_{j_n}$$

$\sigma = \tau_m \dots \tau_1$, where $\tau_1, \dots, \tau_m$ are transpositions. Then there is $0 < k < m$ s.t. $a_{\tau_k(0),\dots,a_{\tau_k(n)}} \not\equiv_A a_0 \dots a_n$. Assume $\tau_k = (s, s+1)$, then there is an $L(A)$-formula $\psi(x_0, \dots, x_n)$ s.t.

$$\vDash \psi(a_0, \dots, a_s, a_{s+1}, \dots, s_n) \wedge \neg\psi(a_0, \dots, a_{s+1}, a_s, \dots, s_n)$$

Let $\phi(x, y) := \psi(a_0, \dots, a_{s-1}, x, y, a_{s+2}, \dots, a_n)$. Then for all $s < q$, $r < s+1$, $\vDash \phi(a_q, q_r) \Leftrightarrow q < r$, contradicting 3.19

$\Leftarrow$: Assume T is unstable. Then suppose that $\bar{c} = (c_i \mid i \in \omega)$ witnesses the order property of $\phi(x, y)$. Let $\bar{a} = (a_i \mid i \in \omega)$ be an indiscernible sequence based on $\bar{c}$. Then

$$\vDash \phi(a_i, a_j) \Leftrightarrow i < j$$

so $\bar{a}$ is not totally indiscernible $\square$

Proposition 3.32. *For any stable formula $\phi(x, y)$, in an arbitrary theory, there is some $k_\phi \in \omega$ depending just on ϕ s.t. for any indiscernible sequence $I \subseteq \mathbb{M}_x$ and any $b \in \mathbb{M}_y$, either $|\phi(I, b)| \leq k_\phi$ or $|\neg\phi(I, b)| \leq k_\phi$*

Proof. Suppose that $|\phi(I, b)| > k$ and $|\neg\phi(I, b)| > k$. By compactness, we assume that $I = \omega$. Then either $\phi(I, b)$ or $\neg\phi(I, b)$ is infinite. Assume that $\neg\phi(I, b)$ is infinite. Then there is a subsequence $J = \{n_0 < n_1 < \dots\} \subseteq \omega$ s.t.

$$\vDash \phi(a_{n_i}, b) \Leftrightarrow i \leq k$$

by $\vDash \neg\phi(a_{n_i}, b) \Leftrightarrow i > k$.

Let $c_i = a_{n_i}$ and $b_k = b$ we have

$$\vDash \bigwedge_{i \leq k} \phi(c_i, b_k) \wedge \bigwedge_{i=k+1}^{2k} \neg\phi(c_i, b_k)$$

Since $(c_i)_{i<\omega}$ is indiscernible, we have

$$\models \exists y \left(\bigwedge_{i \leq k} \phi(c_i, y) \wedge \bigwedge_{i=k+1}^{2k} \neg\phi(c_i, y) \right) \to \exists y \left(\bigwedge_{i \leq j} \phi(c_i, y) \wedge \bigwedge_{i=j+1}^{k} \neg\phi(c_i, y) \right)$$

for each $j < k$ (pick k elements from $2k$ and choose by indiscernibility)

Let

$$b_j \models \bigwedge_{i \leq j} \phi(c_i, y) \wedge \bigwedge_{i=j+1}^{k} \neg\phi(c_i, y)$$

Then $\models \phi(c_i, b_j) \Leftrightarrow i \leq j$, so ϕ has k-order property. Since ϕ is stable, k_ϕ exists $\square$

Corollary 3.33. *In a stable theory, we can define the average type of an indiscernible sequence $\bar{b} = (b_i)_{i \in I}$ over a set of parameters A as*

$$\mathrm{Av}(\bar{b}/A) = \{\phi(x) \in L(A) \mid \models \phi(b_i) \textit{ for all but finitely many } i \in I\}$$

By proposition 3.32 it is a complete consistent type over A

3.6 Stable=NIP∩NSOP and the classification picture

Definition 3.34 (NSOP).

- A (partitioned) formula $\phi(x, y) \in L$ has the **strict order property**, or **SOP**, if there is an infinite sequence $(b_i)_{i \in \omega}$ s.t. $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j \in \omega$
- A theory T has **SOP** if some formula does
- T is **NSOP** if it does not have the strict order property

Remark.

- SOP implies order property by picking an element in each $\phi(\mathbb{M}, b_{i+1}) \setminus \phi(\mathbb{M}, b_i)$
- If $\phi(x, y)$ has SOP, then by 3.26 we can choose an indiscernible sequence $(b_i)_{i \in \omega}$ satisfying the condition above
- DLO has SOP
- T is NSOP iff all formulas with parameters are NSOP iff all formulas $\phi(x, y)$ with x singleton are NSOP

Exercise 3.6.1. T has SOP iff there is a definable partial order with infinite chains

Proof. □

Definition 3.35 (NIP)**.** A (partitioned) formula $\phi(x, y)$ has the **independence property**, or **IP**, if (in $\mathbb{M}$) there are infinite sequences $(b_i)_{i\in\omega}$ and $(a_s)_{s\subseteq\omega}$ s.t.

$$\models \phi(a_s, b_i) \Leftrightarrow i \in s$$

Thus we can define any subset of $(b_i)_{i\in\omega}$ and there is no special subset

A theory T has **IP** if some formula does, otherwise T is **NIP**

Remark.

- If we have arbitrary long finite sequences $(b_i)_{i<n}$ satisfying the condition above for a fixed formula $\phi(x, y)$ then by compactness we can find an infinite sequence satisfying the condition above, hence $\phi(x, y)$ has IP
- If $\phi(x, y)$ has IP, then by Ramsey and compactness we can choose an indiscernible sequence $(b_i)_{i\in\omega}$ in the definition above

Lemma 3.36. *A formula $\phi(x, y)$ has IP iff there is an indiscernible sequence $\bar{b} = (b_n)_{n\in\omega}$ and a parameter c s.t.*

$$\models \phi(c, b_n) \Leftrightarrow n \textit{ is even}$$

Proof. $\Rightarrow$: Suppose $\phi(x, y)$ has IP. There are $\bar{b} = (b_n)_{n\in\omega}$ and $\bar{a} = (a_s)_{s\subseteq\omega}$ s.t. $\phi(a_s, b_n) \Leftrightarrow n \in s$. We may assume that $\bar{b}$ is indiscernible and let $s = \{0, 2, 4, \dots\}$. Let $c = a_s$, then $\models \phi(c, b_n) \Leftrightarrow n$ is even

$\Leftarrow$: □

Theorem 3.37 (Shelah)**.** *T is unstable iff*

3.7 Examples of stable theories

Example 3.4. The theory of a countable number of equivalence relations E_n for $n = 0, 1, 2, \dots,$

- Each equivalence relation has an infinite number of equivalence classes
- Each equivalence class of E_n is the union of an infinite number of different classes of E_{n+1}

This theory has QE by Back-and-Forth

So 1-types are determined by specifying the class w.r.t. each of the equivalence relation, which implies that over an set A, a type $p \in S_1(A)$ is determined by the function

$$f : \omega \to A \cup \{\infty\}$$

where $f(n) = a$ if $\exists a \in A$ s.t. $E_n(x, a) \in p$, otherwise $f(n) = \infty$

There are at most $|A|^{\aleph_0}$ many 1-types (3.19)

Example 3.5 (Modules are stable)**.**

Example 3.6. ACF_0 and ACF_p are stable

All strongly minimal theories are stable

3.8 Number of types and definability of types is NIP

Lemma 3.38. *If $F \subseteq 2^\lambda$ and $|F| > \operatorname{ded}\lambda$, then for each $n < \omega$ there is some $I \subseteq \lambda$ s.t. $|I| = n$ and $F \restriction I = 2^I$*

Proof. Consider each element of 2^λ as a $\{0,1\}$-sequence of length λ, then 2^λ is a dense linear order. For $f < g \in F$, there is $\alpha < \lambda$ s.t. $f \restriction \alpha = g \restriction \alpha$ and $f(\alpha) < g(\alpha)$. So each $f \in F$ realize a cut over $(\bigcup_{\alpha<\lambda} F \restriction \alpha) \subseteq 2^{<\lambda}$. $|F| > \operatorname{ded}\lambda \Rightarrow \left|\bigcup_{\alpha<\lambda} F \restriction \alpha\right| > \lambda \Rightarrow |F \restriction \alpha| > \lambda$ for some α

Let λ and F be a counterexample s.t. λ is minimal. By the minimality of λ, we have $|F \restriction \alpha| \leq \operatorname{ded}\lambda$ for each $\alpha < \lambda$

For each $f \in F \restriction \alpha$, let

$$
\begin{aligned}
\mathrm{Ext}_F(f) &:= \{g \in F : f \subseteq g\} \\
G_\alpha &:= \{f \in F \restriction \alpha : |\mathrm{Ext}_F(f)| > \operatorname{ded}\lambda\} \\
G &:= \{f \in F : \forall \alpha < \lambda (f \restriction \alpha \in G_\alpha)\}
\end{aligned}
$$

Then $F \setminus G = \bigcup_{\alpha<\lambda} \bigcup_{f \in F\restriction\alpha \setminus G_\alpha} \mathrm{Ext}_F(f)$. $|F \setminus G| \leq \lambda \times \operatorname{ded}\lambda \times \operatorname{ded}\lambda \leq \operatorname{ded}\lambda$, which implies $|G| = |F|$, we may assume that $F = G$. Namely, for each $f \in F$ and $\alpha < \lambda$, $|\mathrm{Ext}_F(f \restriction \alpha)| > \operatorname{ded}\lambda$. We now prove by induction on $n < \omega$ that:

$$
\begin{gathered}
\forall n < \omega, \forall \alpha < \lambda, \forall h \in F \restriction \alpha, \text{there is } I \subseteq \lambda \text{ with } |I| = n \text{ s.t.} \\
\mathrm{Ext}_F(h) \restriction I = 2^I
\end{gathered}
$$

It is for $n = 0$ since $\mathrm{Ext}_F(h) \neq \emptyset$.

We now consider the case of $n+1$. $|\mathrm{Ext}_F(h)| > \operatorname{ded}\lambda \Rightarrow |\mathrm{Ext}_F(h) \restriction \alpha| > \lambda$ for some $\alpha < \lambda$. For each $g \in \mathrm{Ext}_F(h) \restriction \alpha$ there is $I_g \subseteq \lambda$ with $\left|I_g\right| = n$ s.t. $\mathrm{Ext}_F(g) \restriction I_g = 2^{I_g}$. There are at most λ-many I_g's for $g \in \mathrm{Ext}_F(h)$, thus there are $f, g \in \mathrm{Ext}_F(h)$ s.t. $I_g = I_h$. Let $a \in f \triangle g$ ($f(a) \neq g(a)$) and $I = I_g \cup \{a\}$, then $\mathrm{Ext}_F(h) \restriction I = 2^I$ $\square$

Proposition 3.39. *1. If $\phi(x, y)$ has IP, then for each cardinal κ there is a set A of cardinality κ s.t. $\left|S_\phi(A)\right| = 2^\kappa$*

2. *If $\phi(x, y)$ has NIP, then for each cardinal κ and a set A of cardinality κ, we have $|S_\phi(A)| \le \operatorname{ded} \kappa$*

Proof. 1. If $\phi(x; y)$ has IP. Let $C = \{c_i : i < \kappa\}$ and $\{d_S \mid S \subseteq \kappa\}$ be two sets of new constants. By compactness

$$\{\phi(c_i, d_S) : i \in S\} \cup \{\neg\phi(c_j, d_S) : j \notin S\}$$

is consistent, then $S_1(C) = 2^{|C|}$

2. Suppose that $|S_\phi(A)| > \operatorname{ded} \kappa$. $S_\phi(A) = \{\mathrm{tp}_\phi(a/A) : a \in \mathbb{M}\}$ and $\mathrm{tp}_\phi(a/A)$ is determined by $\phi(a, A) \subseteq A$. Hence we are considering $T = \{\phi(a, A) \subseteq A : a \in \mathbb{M}\} \subseteq 2^A$. By Lemma 3.38, for each $n < \omega$, there is a finite subset $B \subseteq A$ with $|B| = n$ s.t.

$$\{\phi(a, B) : a \in \mathbb{M}\} = \mathcal{P}(B)$$

For each $S \subseteq B$, there is a_S s.t. $\vDash \phi(a_S, b) \Leftrightarrow b \in S$ for all $b \in B$. By compactness, ϕ has IP

□

Lemma 3.40. *A formula $\phi(x; y)$ is NIP iff there are some $d, c \in \omega$ s.t. for any finite set A with $|A| = n$ we have $|S_\phi(A)| \le cn^d$. In fact, d can be taken to be the maximal size of a set that can be shattered by instances of $\phi(x; y)$*

4 Forking Calculus

4.1 Keisler measures and generically prime ideals

Definition 4.1. 1. A **Keisler measure** (**over a set of parameters** A) is a finitely-additive probability measure on the Boolean algebra of A-definable subsets of $\mathbb{M}_x$. That is, a Keisler measure over A is a map $\mu : \mathrm{Def}_x(A) \to [0, 1]$ s.t.

   (a) $\mu(\mathbb{M}_x) = 1$

   (b) $\mu(P \cup Q) = \mu(P) + \mu(Q)$ for all disjoint $P, Q \in \mathrm{Def}_x(A)$

2. A Keisler measure μ is **invariant over** A if $a \equiv_A b$ implies $\mu(\phi(x, a)) = \mu(\phi(x, b))$

A type can be thought of as a $\{0, 1\}$-measure

Definition 4.2. A set $I \subseteq \mathrm{Def}_x(A)$ is an **ideal** if

1. $\emptyset \in I$

2. $\phi(x, a) \vdash \psi(x, b)$ and $\psi(x, b) \in I$ implies $\phi(x, a) \in I$

3. $\phi(x, a) \in I$ and $\psi(x, b) \in I$ implies $\phi(x, a) \vee \psi(x, b) \in I$

Lemma 4.3 (Extension of a type avoiding an ideal)**.** *If a partial type $\pi(x)$ over a set A doesn't imply a formula from an ideal $\mathcal{I}$, then for any set $B \supseteq A$ there is a complete type $p(x)$ over B not containing any formulas from $\mathcal{I}$*

Proof. We claim that the set of formulas

$$\tau(x) := \pi(x) \cup \{\neg\phi(x, b) : b \in B \text{ and } \phi(x, b) \in \mathcal{I}\}$$

is consistent. If not, then by compactness there are finitely many formulas $\phi_i(x, b_i) \in \mathcal{I}$ s.t. $\pi(x) \vdash \bigvee \phi_i(x, b_i)$. As $\mathcal{I}$ is an ideal, this is a contradiction

Hence any complete type $p(x)$ over B extending $\tau(x)$ satisfies the requirement □

An ideal I is **invariant over** A if $\phi(x, a) \in I$ and $a \equiv_A b$ implies $\phi(x, b) \in I$. As usual, an ideal I in Def($\mathbb{M}$) is **prime** if whenever $\phi(x, a) \wedge \psi(x, b) \in I$, then either $\phi(x, a) \in I$ or $\psi(x, b) \in I$. However, in the Boolean algebra Def$_x(\mathbb{M})$, prime ideals correspond to complete types in $S_x(\mathbb{M})$ (as for any $\phi(x, b)$, $\phi(x, b) \wedge \neg\phi(x, b) = \emptyset$, so either $\phi(x, b)$ or $\neg\phi(x, b)$ has to belong to I). We introduce weaker version

Definition 4.4. Given a cardinal κ, we say that an ideal $\mathcal{I}$ in Def$_x(A)$ is **κ-prime** if for any family $(S_i)_{i<\kappa}$ of A-definable sets with $S_i \notin \mathcal{I}$ for all $i < \kappa$, there are some $i < j < \kappa$ s.t. $S_i \cap S_j \notin \mathcal{I}$. We say that an ideal $\mathcal{I}$ is **generically prime** if it is κ-prime for some κ

Example 4.1.

1. An ideal is prime iff it is 2-prime

2. Let μ be an arbitrary finitely-additive probability measure on X, and let 0_μ be its 0-ideal containing all 0-measure elements. Then 0_μ is $\aleph_1$-prime. Indeed, take $J = \aleph_1$ and assume we are given a family $(S_i : i \in J)$ of sets of positive measure, say $\mu(S_i) > \frac{1}{n_i}$ for some $n_i \in \omega$. Then by pigeon-hole there is some $n \in \omega$ and some infinite $J' \subseteq J$ s.t. $\mu(S_i) > \frac{1}{n}$ for all $i \in J'$.

Proposition 4.5. *Let I be an A-invariant ideal in* Def$_x(\mathbb{M})$*. TFAE*

1. *I is S1, i.e., for any A-indiscernible sequence $(b_i)_{i\in\omega}$ and any formula $\phi(x, y)$, if $\phi(x, b_0) \notin I$ then $\phi(x, b_0) \wedge \phi(x, b_1) \notin I$*

2. I *is generically prime*

3. I *is* $(2^{|A|+|T|})^{+}$*-prime*

Proof. Assume that we have an A-indiscernible sequence $(a_i)_{i\in\omega}$ s.t. $\phi(x,a_0)\wedge\phi(x,a_1)\in I$ but $\phi(x,a_0)\notin I$. By compactness, indiscernibility and invariance of I, for any κ we can find a sequence $(a_i)_{i\in\kappa}$ s.t. $\phi(x,a_i)\notin I$ and $\phi(x,a_i)\wedge\phi(x,a_j)\in I$ for all $i\neq j\in\kappa$, thus I is not generically prime By indiscernibility, $\phi(x,a_i)\notin I$ for any $i\in\omega$. $\phi(x,a_i)\wedge\phi(x,a_j)\in I$ for all $i\neq j\in\omega$ by indiscernibility. And we can extend ω to κ by compactness

Conversely, assume that I is not generically prime. Then for any κ we can find $(\phi_i(x,a_i))_{i\in\kappa}$ with $\phi_i(x,a_i)\notin I$ and $\phi_i(x,a_i)\wedge\phi_j(x,a_j)\in I$. Taking κ large enough and applying 3.29 we find an A-indiscernible sequence starting with a_i,a_j for some $i\neq j$ and s.t. $\phi_i=\phi_j$ □

4.2 Dividing and forking

Definition 4.6. 1. A formula $\phi(x,a)$ **divides** over B if there is a sequence $(a_i)_{i\in\omega}$ and $k\in\omega$ s.t. $a_i\equiv_B a$ and $\{\phi(x,a_i)\}_{i\in\omega}$ is k-inconsistent. Equivalently, if there is a B-indiscernible sequence $(a_i)_{i\in\omega}$ starting with a and s.t. $\{\phi(x,a_i)_{i\in\omega}\}$ is inconsistent (by compactness and 3.29) Tent Lemma 7.1.4

2. A formula $\phi(x,a)$ **forks** over B if it belongs to the ideal generated by the formulas dividing over B, i.e., if there are $\psi_i(x,c_i)$ dividing over B for $i<n$ and s.t.
$$\phi(x,a)\vdash\bigvee_{i<n}\psi_i(x,c_i)$$

3. We denote by $\mathbf{F}(B)$ the ideal of formulas forking over B. It is invariant over B If $\phi(x,b)$ divides over B and given a $\sigma\in\mathrm{Aut}(\mathcal{U}/B)$, then $\sigma(b)\equiv_B b$ and hence $\phi(x,\sigma(b))$ divides over B

Example 4.2. Let T be DLO, then $a<x$ does not divide over $\emptyset$, but $a<x<b$ does

Example 4.3. In general there are formulas which fork, but don't divide. Consider the unit circle around the origin on the plane, and a ternary relation $R(x,y,z)$ on it which holds iff y is between x and z, ordered clock-wise. Let T be the theory of this relation. Check

1. This theory has QE

2. There is a unique 2-type $p(x, y)$ over $\emptyset$ consistent with "$x \neq y$". There is no constant and we can talk nothing:D

3. $R(a, y, c)$ divides over $\emptyset$ for any a, c

4. The formula "$x = x$" forks over $\emptyset$(but it does not divide - no formula can divide over its own parameters)

Definition 4.7. A (partial) type **does not divide** (fork) over B if it does not imply any formula which divides (resp. forks) over B

Note: if $a \notin \text{acl}(A)$ then $\text{tp}(a/Aa)$ divides over A (take $x = a$). Also, if $\pi(x)$ is consistent and defined over $\text{acl}(A)$, then it doesn't divide over A

Exercise 4.2.1. Let $p \in S_x(\mathbb{M})$ be a global type, and assume that it doesn't divide over a small set A. Then it doesn't fork over A

Proposition 4.8. *$\boldsymbol{F}(B)$ is contained in every generically prime B-invariant ideal*

Proof. It is enough to show that if $\varphi(x, a)$ divides over B and I is generically prime ideal, then $\varphi(x, a) \in I$. We use the equivalence from Proposition 4.5. Let $(a_i)_{i \in \omega}$ be indiscernible over B with $a_0 = a$ and $\{\varphi(x, a_i)_{i \in \omega}\}$ inconsistent. If $\varphi(x, a_0) \notin I$, then by induction using that I is generically prime (and that if $(a_i)_{i \in \omega}$ is indiscernible over B, then $(a_{2i}a_{2i+1})_{i \in \omega}$ is indiscernible over B), we see that $\bigwedge_{i<k} \varphi(x, a_i) \notin I$ for all $k \in \omega$. But as $\emptyset \in I$ this would imply that $\{\varphi(x, a_i)\}$ is consistent, a contradiction $\square$

Note that any intersection of B-invariant generically prime ideals is still B-invariant and generically prime

Definition 4.9. 1. Let $\mathbf{GP}(A)$ be the smallest generically prime ideal invariant over A

2. Let $\mathbf{0}(A)$ be the ideal of formulas which have measure 0 w.r.t. every A-invariant Keisler measure

Summing up the previous observations, we have

Proposition 4.10. *In any theory and for any set A, $\boldsymbol{F}(A) \subseteq \boldsymbol{GP}(A) \subseteq \mathbf{0}(A)$*

Example 4.4. There are theories with $\mathbf{F}(A) \subsetneq \mathbf{GP}(A)$, equivalently theories

4.3 Special extensions of types

- Let $A \subseteq B$ and $p \in S_x(A)$. Then there is some $q \in S_x(B)$ with $p \subseteq q$ (as p is a filter in $\text{Def}_x(B)$, so extends to an ultrafilter)
- We would like to be able to choose a "generic" extension q of p, s.t. it doesn't add any new conditions on q w.r.t. the new parameters from B which were not already present w.r.t. the parameters from A

Definition 4.11. A global type $p(x) \in S(\mathbb{M})$ is called **invariant** over C if it is invariant under all automorphisms of $\mathbb{M}$ fixing C.

Applying Proposition 4.8 to $\{0, 1\}$-measures, every global type invariant over A is non-forking over A

Definition 4.12. Let $A \subseteq B$, $p \in S_x(A)$ and $q \in S_x(B)$ extending p be given (so $p = q \upharpoonright \text{Def}_x(A)$, which also denote as $p = q|A$)

1. q is an **heir** of p (or "an heir over A") if for every formula $\phi(x, y) \in L(A)$, if $\phi(x, b) \in q$ for some $b \in B$, then $\phi(x, b') \in p$ for some $b' \in A$. Note that if q is an heir of p, then in fact A has to be a model of T
2. q is a **coheir** of p ("coheir over A", "finitely satisfiable in A") if for any $\phi(x, b) \in q$ there is some $a \in A$ s.t. $\vDash \phi(a, b)$

Exercise 4.3.1. $A \subseteq B$

1. If a type $q \in S(B)$ is definable over A or is finitely satisfiable in A, then it **does not split** over A, i.e., for all $a \equiv_A a'$ from B and $\phi(x, y) \in L(A)$ we have that $\phi(x, a) \in q \Leftrightarrow \phi(x, a') \in q$. In particular, if $B = \mathbb{M}$ then q is A-invariant
2. If A is a model of T and $q \in S(B)$ is definable over A, then it is an heir over A
3. If $B = \mathbb{M}$ and $q \in S(B)$ is A-invariant then it doesn't fork over A
4. tp(a/Mc) is an heir of tp(a/M) iff tp(c/Ma) is a coheir of tp(b/M)

Proof. 1. Obvious

2. $\phi(x, b) \in q \Leftarrow\vDash d\phi(b) \Rightarrow\vDash \exists x d\phi(x) \Rightarrow A \vDash d\phi(a)$ for some $a \in A$
3. $L(xB) \setminus p$ is an A-invariant prime ideal and $\mathbf{F}(B) \subseteq L(xB) \setminus p$ by 4.8

4. tp(a/Mc) is an heir of tp(a/M) iff $\forall\phi(x,y) \in L(M)$, $\phi(x,b) \in$ tp$(a/Mc) \Rightarrow \exists b'.\phi(x,b') \in$ tp(a/M) iff $\forall\phi(x,y) \in L(M)$, $\phi(x,c) \in$ tp$(a/Mc) \Rightarrow \exists b'\ \phi(x,b') \in$ tp(a/M) iff tp(c/Ma) is a coheir of tp(b/M)

□

Example 4.5. Let $M = (\mathbb{Q}, <)$ and consider the type $p \in S_x(M)$ given by $p = \{a < x : a \in M\}$. Now consider two global extensions $q_1, q_2 \in S_x(\mathbb{M})$ of p:

- $q_1(x) = \{a < x : a \in \mathbb{M}\}$
- $q_2(x) = p(x) \cup \{x < b : M < b \in \mathbb{M}\}$

q_1 is M-definable, so it is an heir of p, but not a heir of p. On the other hand, q_2 is a coheir of p, but it is not an heir over M

Remark. Note the space of A-invariant global types is a closed subset of $S(\mathbb{M})$ (as it equals $\bigcap_{\phi\in L, a\equiv_A b\in\mathbb{M}} \langle\phi(x,a) \leftrightarrow \phi(x,b)\rangle$), thus compact. Similarly, the space of types finitely satisfiable in A is a closed subset of A - it equals $\bigcap_{\phi(x,a)\in L(M), \phi(A,a)=\emptyset} \langle\neg\phi(x,a)\rangle$. It can also be described as the closure of the set of types realized in A, i.e., of $\{\text{tp}(a/\mathbb{M}) : a \in A\}$

Exercise 4.3.2.

1. If $\pi(x)$ is finitely satisfiable in A, then there exists a complete global type extending $\pi(x)$ and finitely satisfiable in A
2. Every global type finitely satisfiable in A is invariant over A

Proof.

1. Let $p = \pi(x) \cup \{\phi(x) : \pi(x) \cup \{\phi(x)\}$ is finitely satisfiable in $A\}$

□

Proposition 4.13. *Let $p \in S_x(M)$ be arbitrary, where $M \vDash T$ is a small model*

1. *There is a global coheir q of p*
2. *There is a global heir r of p*

Proof.

1. Let $A \subseteq \mathbb{M}_x$ be small and let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}(A)$. We can define a global type $q_{\mathcal{U}} \in S_x(\mathbb{M})$ in the following way. For a formula $\phi(x,b) \in L(\mathbb{M})$ we define $\phi(x,b) \in q_{\mathcal{U}} \Leftrightarrow \phi(A,b) \in \mathcal{U}$. Then $q_{\mathcal{U}}$ is finitely satisfiable in A.

   Conversely, every global type q finitely satisfiable in A is of the form $q_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on $\mathcal{P}(A)$

   Now any $p \in S_x(M)$ is finitely satisfiable in M since $M \prec \mathbb{M}$. It follows that $\{\phi(M) : \phi(x) \in p\}$ is a filter, so extends to some ultrafilter $\mathcal{U}$ on $\mathcal{P}(M)$. Then the global type $q_{\mathcal{U}}$ is a coheir of p

2. It is enough to show that the following set of formulas is consistent

   $s(x) := p(x) \cup \{\phi(x,c) : c \in \mathbb{M}, \phi(x,y) \in L(M), \phi(x,m) \in p \text{ for all } m \in M\}$

   As then any complete type $r(x) \in S_x(\mathbb{M})$ with $r \supseteq s$ is an heir of p. If for $\phi(x,b) \in r$, for all $b' \in M$, $\phi(x,b') \notin p$, then $\neg\phi(x,b) \in r$.

   Assume it is not consistent, then by compactness there are formulas $\phi(x,c) \in p$ and $\phi_i(x,c_i), i < n$ from $s(x)$ s.t. $\models \phi(x,c) \to \bigvee_{i<n} \neg\phi_i(x,c_i)$. As $\phi(x,c) \in L(M)$ and $M \prec \mathbb{M}$, it follows that there are m_i, $i < n$ s.t. $M \models \phi(x,c) \to \bigvee_{i<n} \neg\phi_i(x,m_i)$. But by definition of $s(x)$ we have $\phi_i(x,m_i) \in p$ for all $i < n$, as well as $\phi(x,c) \in p$ - thus their conjunction is consistent, a contradiction

   $\square$

Proposition 4.14. *Let $p \in S_x(M)$ be a definable type. Then it has a unique global heir $q \supseteq p$ which is definable over M*

Proof. First we show that p has a global M-definable extension. As $p(x)$ is definable, it follows that for every $\phi(x,y) \in L$ there is some $d\phi(y) \in L(M)$ s.t. $\phi(x,a) \in p \Leftrightarrow \models d\phi(a)$, for all $a \in M$. Consider the following set of formulas

$$q(x) := \{\phi(x,a) : \phi(x,y) \in L, a \in \mathbb{M}_y, \models d\phi(a)\}$$

By compactness, it is enough to show that for any $\phi_1(x,y_1), \phi_2(x,y_2)$

$$\models \forall y_1 y_2 \exists x(\phi_1(x,y_1) \wedge \phi_2(x,y_2))$$

As $M \prec \mathbb{M}$, this is equivalent to

$$M \models \forall y_1 y_2 \exists(\phi_1(x,y_1) \wedge \phi_2(x_2,y_2))$$

But for any $a_1, a_2 \in M$, $\phi_1(x,a_1) \wedge \phi_2(x,a_2) \Leftrightarrow \models d\phi(a_1) \wedge d\phi(a_2)$. Thus this holds.

Assume that q, r are two global types extending p which are both definable over M. This implies that for their corresponding defining schemas $(d_q(\phi))_{\phi(x,y)\in L}$ and $(d_r\phi(y))_{\phi(x,y)\in L}$ we must have $d_q\phi(M) = d_r\phi(M)$ and hence $M \models \forall y(\phi(x,y) \leftrightarrow d\phi(y))$. But again as $M \prec \mathbb{M}$, this implies that $d_q\phi(\mathbb{M}) = d_r\phi(\mathbb{M})$, and so $q = r$

By Exercise 4.3.1, $q(x)$ is an heir of $p(x)$. Now if $q \neq q'$ is another global type extending p, then for some $\phi(x,b) \in q'$ we have $\neg\phi(x,b) \in q$ and so $\not\models d\phi(b)$, and so $(\phi(x,b) \wedge \neg d\phi(b)) \in q'$. But as there can be no $m \in M$ with $\models \phi(x.m) \wedge \neg d\phi(m)$ and as $\phi(x,y) \wedge \neg d\phi(y) \in L$, it follows that q' is not a heir of p $\square$

Proposition 4.15. *Let $p \in S_x(\mathbb{M})$ be a global A-invariant type*

1. *If p is definable, then in fact it is definable over A*
2. *If p is finitely satisfiable in some small set, then in fact it is finitely satisfiable in any model $M \supseteq A$*

Proof. 1. As p is definable, for any formula $\phi(x,y) \in L$ there is some $d\phi(y) \in L(\mathbb{M})$ s.t. for any $b \in \mathbb{M}$ we have $\phi(x,b) \in p \Leftarrow b \in d\phi(\mathbb{M})$. As p is A-invariant, the definable set $d\phi(\mathbb{M})$ is also $\mathrm{Aut}(\mathbb{M}/A)$-invariant. But then the set $d\phi(\mathbb{M})$ is in fact A-definable by Lemma 2.9

2. Suppose p is finitely satisfiable in some small model N. Let M be an arbitrary small model containing A. Let $\phi(x,b) \in p$ be arbitrary. Consider the type $\mathrm{tp}(N/M)$. By Proposition 4.13, this type has a global coheir $r(x)$, let $N_1 \vDash r|Mb$. Then by invariance p is finitely satisfiable in N_1, in particular $\phi(N_1,b) \neq \emptyset$. But as the type $\mathrm{tp}(N_1/Mb)$ is finitely satisfiable in M, it follows that $\phi(M,b) \neq \emptyset$

□

4.4 Tensor product of invariant types and Morley sequences

Definition 4.16. Let $p \in S_x(\mathbb{M})$, $q \in S_y(\mathbb{M})$ be two global, A-invariant types. Then we define their tensor product $p \otimes q \in S_{xy}(\mathbb{M})$ as follows:

given a formula $\phi(x,y) \in L(B)$, $A \subseteq B \subseteq \mathbb{M}$, we set $\phi(x,y) \in p \otimes q \Leftrightarrow \phi(x,b) \in p$ for some (equivalently, any, by invariance of p) $b \in \mathbb{M}_y$ s.t. $b \vDash q|B$ since $b \vDash q|B \Rightarrow \mathrm{tp}(b/B) = q|B$

For any small $B \supseteq A$, $ab \vDash p \otimes q$ iff $b \vDash q|B$ and $a \vDash p|Bb$.

Remark. 1. Note that $p \otimes q$ is a complete type, as

$$p \otimes q = \bigcup_{A \subseteq B \subset_{small} \mathbb{M}} \{\mathrm{tp}(ab/B) : a \vDash p|Bb, b \vDash q|B\}$$

2. If both p and q are A-invariant, then so is $p \otimes q$. If $\phi(x,y,c) \in p \otimes q$, then there is $\phi(x,b,c) \in p$ and $b \vDash q|c$. Since p and q are A-invariant, for any $\sigma(\mathbb{M}/A)$, $\phi(x,\sigma(b),\sigma(c)) \in p$ and $\mathrm{tp}(\sigma(b)/\sigma(c)) = \sigma(q|c) = q|\sigma(c) \Rightarrow \sigma(b) \vDash q|\sigma(c)$. Hence $\phi(x,y,\sigma(c)) \in p \otimes q$

3. The operation $\otimes$ is associative, i.e., $p \otimes (q \otimes r) = (p \otimes q) \otimes r$. For any small B, both products restricted to B are equal to $\mathrm{tp}(abc/B)$ for $c \vDash r|B$, $b \vDash q|Bc$, $a \vDash q|Bbc$

4. $\otimes$ need not be commutative. Let T be DLO, and let $p = q$ be the type at $+\infty$, it is $\emptyset$-invariant. Then $p(x) \otimes q(y) \vdash x > y$, while $q(y) \otimes p(x) \vdash x < y$

5. In fact, in the definition of the tensor product, we have only used that p is invariant

Definition 4.17. Let $p \in S_x(\mathbb{M})$ be a global A-invariant type. Then for any $n \in \omega$ we define by induction $p^{(1)}(x_0) := p(x_0)$ and $p^{(n+1)}(x_0, \dots, x_n) := p(x_n) \otimes p^{(n)}(x_0, \dots, x_{n-1})$. We also let $p^{(\omega)} = (x_0, x_1, \dots) := \bigcup_{n \in \omega} p^{(n)}(x_0, \dots, x_{n-1})$. For any set $B \supseteq A$, a sequence $(a_i : i \in \omega) \vDash p^{(\omega)}|B$ is called a **Morley sequence** of p over B (indexed by ω)

Remark.

1. We can define $p^{(I)}$ for an arbitrary order type I in a natural way

2. Note that for any $(a_i : i < \omega), (b_i : i < \omega) \vDash p^{(\omega)}|B$

   $$(a_i : i < \omega) \equiv_B (b_i : i < \omega)$$

   as $\mathrm{tp}((a_i)_{i<\omega}/B) = \mathrm{tp}((b_i)_{i<\omega}/B)$. In particular, any Morley sequence of p over B is B-indiscernible, by the associativity of $\otimes$

   For any $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$, let $l_m = \max(i_m, j_m)$ for $1 \le m \le n$. Then

   $$i_1 \dots i_n \equiv_B l_1 \dots l_n \equiv_B j_1 \dots j_n$$

Lemma 4.18. *TFAE*

1. $\mathrm{tp}(a/Ab)$ *doesn't divide over* A

2. *For every infinite A-indiscernible sequence I s.t. $b \in I$, there is some $a' \equiv_{Ab} a$ s.t. I is Aa'-indiscernible*

3. *For every infinite A-indiscernible sequence I s.t. $b \in I$, there is some $J \equiv_{Ab} I$ s.t. J is Aa-indiscernible*

Proof. $2 \leftrightarrow 3$: by an A-automorphism

$1 \to 3$: □

Corollary 4.19. *If* $\mathrm{tp}(a/B)$ *does not divide over* $A \subseteq B$ *and* $\mathrm{tp}(b/Ba)$ *does not divide over* Aa*, then* $\mathrm{tp}(ab/B)$ *does not divide over* A

Proof. By Lemma 4.18. Let I be an arbitrary A-indiscernible sequence starting with B. Then we can find $I' \equiv_B I$ with I' Aa-indiscernible and $I'' \equiv_{Ba} I'$ with I'' abB-indiscernible. In particular $I'' \equiv_B I$ $\square$

Corollary 4.20. *If $\phi(x, a)$ k-divides over A and* tp(b/Aa) *does not divide over A, then $\phi(x, a)$ k-divides over Ab*

Proof. Let $I = (a_i : i \in \omega)$ be an infinite A-indiscernible sequence s.t. $a_0 = a$ and $\{\phi(x, a_i) : i \in \omega\}$ is k-inconsistent. By assumption and Lemma 4.18 there is $J \equiv_{Aa} I$ which is Ab-indiscernible. Then J witnesses that $\phi(x, a)$ k-divides over Ab $\square$

Proposition 4.21. *Let $p \in S_x(\mathbb{M})$ be a global type, and let M be a small model. TFAE*

1. *If p is definable over A, then p does not divide over A*
2. *If T is stable and p does not divide over M, then p is definable over M*

Proof. (1) is obvious

Assume that T is stable and that p does not divide over M. We will show that p is an heir of $p|M$, which is enough (as $p|M$ is a definable type by stability and Theorem 3.17, which using Proposition 4.14 implies that p is definable over M) So let $\phi(x, y) \in L(M)$ be given and assume that $\phi(x, b) \in p$. We want to show that $\phi(x, b') \in p$ for some $b' \in M$. Let $I = (b_i : i \in \omega)$ be a Morley sequence of a global coheir extension of tp(b/M) over M starting with $b_0 = b$ (exists by Proposition 4.13 and take the automorphism to shift b_0 to b) Let $a \models p|Mb$. Since tp(a/Mb) doesn't divide over M, by Lemma 4.18, we may assume that I is indiscernible over Ma. Condition of Morley sequence is in EM-type. So we have $\models \phi(a, b_i)$ for all $i \in \omega$. Again by stability and Theorem 3.17, the type $q = $ tp(a/MI) is definable. Let $n \in \omega$ be s.t. all of the parameters of $d\phi(y)$ are in $M \cup \{b_0, \dots, b_{n-1}\}$. Since tp$(b_n/b_{<n}M)$ is a coheir of tp(b/M) and $\models d\phi(b_n)$ (as $\models \phi(a, b_n)$), it follows that there is some $b' \in M$ with $\models d_q\phi(b')$. This implies that $\models \phi(a, b')$, and so $\phi(x, b') \in$ tp$(a/M) = p|M$, as wanted $\square$

4.5 Forking and dividing in simple theories

Definition 4.22. A theory T is **simple** if every type $p \in S_x(A)$ does not divide over some subset $A_0 \subseteq A$ of size $|A| \leq |T|$

Exercise 4.5.1. 1. Show that if T is stable then it is simple, and that if T is simple then it is NSOP

2. Show that the theory of a random graph is simple

Proof. 1. □

Note that, according to Definition 4.6, it is possible that a formula $\phi(x, a)$ divides over A, witnessed by a certain A-indiscernible sequence $I = (a_i)$, yet there is some other A-indiscernible sequence $J = (b_i)$ s.t. $b_0 = a$ but $\{\phi(x, b_i)\}$ is consistent. However, we can isolate a class of indiscernible sequences which always witness that a formula divides Consider the trivial indiscernible sequence $\bar{a} = aaaaaaa\ldots$

Lemma 4.23 (Kim's lemma for simple theories)**.** *Let T be simple. Assume that $\phi(x, a)$ divides over A and let $(a_i : i \in \omega)$ be an A-indiscernible sequence s.t. moreover* $\mathrm{tp}(a_i/a_{<i}A)$ *does not split over A, for all i (such a sequence is also called a Morley sequence in the type* $\mathrm{tp}(a/A)$*). Then $\{\phi(x, a_i) : i \in \omega\}$ is inconsistent*

Proof. WLOG, $A = \emptyset$. Assume $\phi(x, a)$ divides over A, but for some Morley sequence (a_i) in $\mathrm{tp}(a/\emptyset)$ we have $\{\phi(x, a_i)\}$ is consistent. □

5 TODO Problems

3.5.3	2.1	2.3	2.4	3.22
3.6.1	2		2	4.5.1

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7 References

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