Lattices

Introduction to Model Theory
(Third hour)

December 9, 2021

Section 1

Semilattices

Suprema and infima

Definition

Let $(P, \leq)$ be a poset and $S \subseteq P$ be a subset.

1. An *upper bound* of S is an element $a \in P$ such that $x \in S \implies x \leq a$.
2. A *lower bound* of S is an element $a \in P$ such that $x \in S \implies x \geq a$.
3. A *maximum* of S is an upper bound of S in S.
4. A *minimum* of S is a lower bound of S in S.
5. A *supremum* or *least upper bound* of S is a minimum in the set of upper bounds of S.
6. An *infimum* or *greatest lower bound* of S is a maximum in the set of lower bounds of S.

Suprema and infima

Suppse $S \subseteq (P, \leq)$.

1. S has at most one maximum.
2. S has at most one minimum.
3. S has at most one supremum.
4. S has at most one infimum.
5. If $a = \max(S)$, then $a = \sup(S)$.
6. If $a = \min(S)$, then $a = \inf(S)$.

Example

In $(\mathbb{R}, \leq)$, the set $S = [0, 1)$ has $0 = \min(S) = \inf(S)$, $1 = \sup(S)$, and no maximum.

Suprema and infima of $\varnothing$

Let $(P, \leq)$ be a poset.

Fact

$a = \inf(\varnothing) \iff a = \max(P)$. *[sic]*
$a = \sup(\varnothing) \iff a = \min(P)$. *[sic]*

$\varnothing$ is the one set where $\sup(S) < \inf(S)$ can happen.

Meet-semilattices as posets

Definition

A *meet-semilattice* is a poset $(S, \leq)$ such that $\inf\{x, y\}$ exists for any $x, y \in S$.

$\inf\{x, y\}$ is called the *meet* of x and y, and is written $x \wedge y$.

Example

Any linear order $(S, \leq)$ is a meet-semilattice, with $x \wedge y = \min\{x, y\}$.

Example

The powerset $(P(A), \subseteq)$ is a meet-semilattice with $x \wedge y = x \cap y$.

Meet-semilattices as algebraic structures

Definition

A *meet-semilattice* is a set S with an operation $\wedge$ satisfying the identities:

$$x \wedge y = y \wedge x$$
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$
$$x \wedge x = x.$$

To recover $\leq$:

$$x \leq y \iff x = x \wedge y.$$

Meet-semilattices as posets

Fact

A poset $(S, \leq)$ is a meet-semilattice iff $\inf A$ *exists for any finite non-empty $A \subseteq S$.*

Proof idea: $\inf\{x_1, \ldots, x_n\} = x_1 \wedge x_2 \wedge \cdots \wedge x_n$.

Bounded meet-semilattices

A *bounded meet-semilattice* is...

1. A poset $(S, \leq)$ such that $\inf A$ exists for every finite $A \subseteq S$.
2. A meet-semilattice $(S, \leq)$ such that $\inf \varnothing$ exists.
3. A meet-semilattice $(S, \leq)$ with a maximum element 1.
4. A meet-semilattice $(S, \wedge)$ with an element 1 such that $x \wedge 1 = x$ for all $x \in S$.

Posets as categories

We can regard a poset P as a category where

- Objects are elements of P.
- If $x \leq y$, there is a unique morphism from x to y.
- If $x \not\leq y$, there are no morphisms from x to y.

Fact

P has finite limits iff P is a bounded meet-semilattice.

Join-semilattices

Definition

A *join-semilattice* is a poset $(S, \leq)$ such that $\sup\{x, y\}$ exists for any $x, y \in S$.

$\sup\{x, y\}$ is called the *join* of x and y, and is written $x \vee y$.

Definition

A *bounded join-semilattice* is a join-semilattice with a least element 0.

Section 2

Lattices

Lattices

Definition

A *lattice* is a partial order $(L, \leq)$ that is a meet-semilattice and a join-semilattice.
That is, every finite non-empty subset of L has a supremum and infimum.
That is, every two-element subset $\{x, y\} \subseteq L$ has a supremum and infimum.

Examples:

- Linear orders.
- Powersets $(P(A), \subseteq)$.

Lattices

Definition

A *lattice* is a set L with two operations $\wedge, \vee$ satisfying the identities

$$
\begin{aligned}
x \wedge y &= y \wedge x & x \vee y &= y \vee x \\
x \wedge (y \wedge z) &= (x \wedge y) \wedge z & x \vee (y \vee z) &= (x \vee y) \vee z \\
x \wedge x &= x & x \vee x &= x \\
x \wedge (x \vee y) &= x & x \vee (x \wedge y) &= x.
\end{aligned}
$$

Absorption laws

The identities

$$x \wedge (x \vee y) = x$$
$$x \vee (x \wedge y) = x.$$

ensure that $\wedge$ and $\vee$ define the same partial order on L.

Bounded lattices

Definition

A *bounded lattice* is a partial order $(L, \leq)$ such that $\inf S$ and $\sup S$ exists for every finite $S \subseteq L$.

Definition

A *bounded lattice* is a lattice $(L, \leq)$ with a maximum 1 and a minimum 0.

Definition

A *bounded lattice* is a lattice $(L, \wedge, \vee)$ with elements $0, 1$ satisfying the identities

$$x \wedge 1 = x \vee 0 = x.$$

Sublattices

Let $(L, \leq)$ be a lattice.

Definition

A *sublattice* is a subset $S \subseteq L$ such that S is closed under $\wedge$ and $\vee$.

Now suppose $(L, \leq)$ is a bounded lattice.

Definition

A *bounded sublattice* is a sublattice $S \subseteq L$ containing the minimum 0 and the maximum 1.

Warning

$([1, 2], \leq)$ is a sublattice of $([0, 3], \leq)$, and $[1, 2]$ is a bounded lattice, but $[1, 2]$ is not a bounded sublattice.

The category of lattices

Definition

A *lattice homomorphism* from a lattice A to a lattice B is a function $f : A \to B$ preserving $\wedge, \vee$:

$$f(x \vee y) = f(x) \vee f(y)$$
$$f(x \wedge y) = f(x) \wedge f(y).$$

Definition

The *category of lattices* has lattices as objects, and lattice homomorphisms as morphisms.

The category of bounded lattices

Definition

A *bounded lattice homomorphism* from a bounded lattice A to a bounded lattice B is a function $f : A \to B$ preserving $\wedge, \vee, 0, 1$:

$$f(x \vee y) = f(x) \vee f(y)$$
$$f(x \wedge y) = f(x) \wedge f(y)$$
$$f(0) = 0$$
$$f(1) = 1.$$

This defines the *category of bounded lattices*.

Warning

The inclusion $[1, 2] \hookrightarrow [0, 3]$ is a lattice homomorphism between bounded lattices, but not a bounded lattice homomorphism.

The wrong category

Let C be the category where

- Objects are bounded lattices.
- Morphisms are lattice homomorphisms.

Fact

C does not have finite limits. C does not have finite colimits.

In contrast...

Fact

The category of bounded lattices has all small limits and small colimits.

Section 3

Complete lattices

Complete lattices

Definition

A *complete lattice* is a partial order $(L, \leq)$ such that $\inf S$ and $\sup S$ exist for every $S \subseteq L$.

- A complete lattice is a bounded lattice.
- Complete lattices are non-empty (take $S = \varnothing$).

Complete lattices

Example

$([0,1], \leq)$ is a complete lattice, by the completeness axiom of $\mathbb{R}$.

Example

For any set A, the powerset $(P(A), \subseteq)$ is a complete lattice, with $\sup \mathcal{F} = \bigcup \mathcal{F}$ and $\inf \mathcal{F} = \bigcap \mathcal{F}$.

Example

Any finite non-empty lattice is a complete lattice.

Complete semilattices. . . ?

Fact

Let $(P, \leq)$ be a poset. The following are equivalent:

- *Every $S \subseteq P$ has an infimum.*
- *Every $S \subseteq P$ has a supremum.*

Consequences:

- "complete lattice" = "complete meet-semilattice" = "complete join-semilattice"
- Any finite bounded semilattice is a complete lattice.

More complete lattices

These are complete lattices:

- The lattice of subrings of a ring R.
- The lattice of ideals in a ring R.
- The lattice of subgroups of a group G.
- The lattice of normal subgroups of a group G.
- The lattice of linear subspaces of a vector space V.
- The lattice of sublattices of a lattice L.
- The lattice of closed sets in a topological space T.

In each case, $\inf S$ is $\bigcap_{X \in s} X$.

Closure operations

Definition

A *closure operation* on a set S is a map $\mathrm{cl} : P(S) \to P(S)$ satisfying these properties:

- $X \subseteq \mathrm{cl}(X)$.
- If $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

Examples:

- Topological closure.
- $\mathrm{cl}(S) =$ the subring generated by $S \subseteq (R, +, \cdot)$.

Closure operations

Let $\mathrm{cl}(-)$ be a closure operation on S.

Definition

$X \subseteq S$ is *closed* if $\mathrm{cl}(X) = X$.

Fact

Let $\mathrm{cl}(-)$ *be a closure operation on* S*. Let* $\mathcal{C}$ *be the family of closed sets.*

1. $(\mathcal{C}, \subseteq)$ *is a complete lattice.*
2. $\bigwedge_{i \in I} X_i$ *is* $\bigcap_{i \in I} X_i$.
3. $\bigvee_{i \in I} X_i$ *is* $\mathrm{cl}\left(\bigcup_{i \in I} X_i\right)$.

Topological closure

Fact

A closure operation $\mathrm{cl}(-)$ *on* S *corresponds to a topology on* S *if and only if this identity holds:*

$$\mathrm{cl}(X \cup Y) = \mathrm{cl}(X) \cup \mathrm{cl}(Y).$$

Knaster-Tarski Theorem

Fact (Knaster-Tarski)

Let $(L, \leq)$ *be a complete lattice. Let* $f : L \to L$ *be monotone:*

$$x \leq y \implies f(x) \leq f(y).$$

Let $L' = \{x \in L : f(x) = x\}$.
THEN:

- L' *is non-empty.*
- $(L', \leq)$ *is a complete lattice.*

Warning

$(L', \leq)$ is usually not a sublattice of $(L, \leq)$.

Knaster-Tarski Theorem: example

Let T be a topological space. Define

$$f : P(T) \to P(T)$$
$$f(X) = \mathrm{int}(\overline{X}).$$

The fixed points are the *regular open sets*.

- $(0,1) \cup (2,3) \subseteq \mathbb{R}$ is a regular open set.
- $(0,1) \cup (1,3) \subseteq \mathbb{R}$ is a non-regular open set.

The poset of regular open sets is a complete lattice, by Knaster-Tarski.

Section 4

Distributive lattices

Distributive lattices

Theorem

Let $(L, \leq)$ be a lattice. The following are equivalent:

1. *L satisfies the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*
2. *L satisfies the identity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Proof.

$(1) \implies (2)$:

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= (x \wedge x) \vee (x \wedge z) \vee (y \wedge x) \vee (y \wedge z)\\ &= x \vee (x \wedge z) \vee (x \wedge y) \vee (y \wedge z)\\ &= x \vee (x \wedge y) \vee (y \wedge z)\\ &= x \vee (y \wedge z).\end{aligned}$$

$\square$

Distributive lattices

Definition

A *distributive lattice* is a lattice satisfying the equivalent identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Examples:

- Linear orders.
- Power sets.

Distributive lattices and power sets

Fact

$(L, \wedge, \vee)$ *is distributive iff* L *is isomorphic to a sublattice of* $(P(A), \cap, \cup)$ *for some set* A.

Distributive lattices from partial orders

Fact

Let $(P, \leq)$ be a partial order. Let $\mathcal{D}$ be the family of downward-closed sets, i.e., sets $A \subseteq P$ for which

$$x \leq y \in A \implies x \in A.$$

Then $(\mathcal{D}, \cap, \cup)$ is a distributive lattice.

Fact

All finite distributive bounded lattices arise this way.

Fact

Any finitely generated distributive lattice is finite.

(This doesn't hold for general lattices.)

Section 5

Boolean algebras

Complements

Let L be a bounded lattice.

Definition

x and y are *complements* if

$$x \wedge y = 0 \text{ and } x \vee y = 1.$$

Fact

In a bounded distributive lattice, complements are unique: if y, y' are complements of x, then $y = y'$.

Boolean algebras

Definition

A *Boolean algebra* is a distributive lattice in which every element has a complement.

We write the complement of x as $\neg x$.

Example

The powerset $(P(A), \subseteq)$ is a Boolean algebra, with $\neg X = A \setminus X$.

Finite Boolean algebras

Fact

Every finite Boolean algebra has the form $(P(A), \subseteq)$ for some finite set A.

Fact

The category of finite Boolean algebras is dual to the category of finite sets.

Categories C and D are *dual* if C is equivalent to D^{op}, or equivalently, C^{op} is equivalent to D.

Boolean algebras from topological spaces

Fact

Let T be a topological space. Let $\mathcal{B}$ be the family of clopen subsets of T. Then $\mathcal{B}$ is a Boolean algebra.

Fact

All Boolean algebras arise this way.

Another characterization of Boolean algebras

Fact

A structure $(B, \wedge, \vee, 0, 1, \neg)$ *is a Boolean algebra iff it is isomorphic to a substructure of* $(P(A), \cap, \cup, \varnothing, A, \neg)$ *for some set* A*.*

Stone spaces

Fact

If T is a compact topological space, then the following conditions are equivalent:

- *T is totally separated: for any $x, y \in T$, there is a clopen set U with $x \in U$ and $y \notin U$.*
- *T is totally disconnected: the connected components of T are points.*
- *T is Hausdorff and zero-dimensional: there is a basis of open sets consisting of clopen sets.*

Stone spaces

Definition

A *Stone space* is a compact totally disconnected space.

Examples:

- A finite discrete topological space.
- The Cantor set.
- The one-point compactification of a discrete topological space.
- $\{0, 1, 1/2, 1/3, 1/4, \ldots\}$.

Stone duality

Fact

The category of Stone spaces is dual to the category of Boolean algebras.

- A Stone space S corresponds to the Boolean algebra of clopen subsets.
- The other direction involves ultrafilters...

Ultrafilters

Let B be a Boolean algebra.

Definition

An *ultrafilter* in B is a subset $U \subseteq B$ such that

- $x, y \in U \implies x \wedge y \in U$.
- $x \in U$ and $x \leq y$ imply $y \in U$.
- $1 \in U$.
- $0 \notin U$.

Ultrafilters

Fact

An ultrafilter U determines a Boolean algebra homomorphism

$$B \to \{0,1\}$$
$$x \mapsto \begin{cases} 1 & x \in U \\ 0 & x \notin U. \end{cases}$$

Ultrafilters in B correspond bijectively to Boolean algebra homomorphisms $B \to \{0,1\}$.

The space of ultrafilters

Given a Boolean algebra B, let $S(B)$ be the set of ultrafilters in B.

- For $x \in B$, let $[x] = \{U \in S(B) : x \in U\}$.
- The family $\{[x] : x \in B\}$ is a basis for a topology on $S(B)$.
- This makes $S(B)$ into a Stone space
- This gives the functor from Boolean algebras to Stone spaces.

Remark

If M is a model and $A \subseteq M$, then the type space $S_n(A)$ is a Stone space, dual to the Boolean algebra of A-definable subsets $X \subseteq M^n$.

Boolean rings

Definition

A *Boolean ring* is a (unital) ring R satisfying the identity $x^2 = x$ for all $x \in R$.

Fact

Boolean rings are equivalent to Boolean algebras.

The correspondence works like so:

$$0 = 0 \qquad 1 = 1$$
$$x \cdot y = x \wedge y$$
$$x + y = (x \wedge \neg y) \vee (\neg x \wedge y)$$
$$x \vee y = x + y + xy$$
$$\neg x = 1 - x.$$

In a Boolean ring, maximal ideals = prime ideals, and these correspond bijectively to ultrafilters.

Heyting algebras

Definition

A bounded lattice $(L, \leq)$ is a *Heyting algebra* if for every a, b, the set $\{x \in L : x \wedge a \leq b\}$ has a maximum.

This maximum is denoted $a \to b$.

Fact

1. *Boolean algebras are Heyting algebras.*
   - *$a \to b$ is the logical implication operator $\neg a \vee b$.*
2. *The lattice of open sets in a topological space is a Heyting algebra.*
3. *Heyting algebras are distributive lattices.*

Heyting algebras and logic

1. Boolean algebras govern classical logic.
   - A propositional identity holds in classical logic iff it holds in all Boolean algebras.
2. Heyting algebras govern intuitionistic logic.
   - A propositional identity holds in intuitionistic logic iff it holds in all Heyting algebras.
3. If T is a topos and $A \in T$, then the subobject poset $\mathrm{Sub}(A)$ is always a Heyting algebra.
4. A poset $(P, \leq)$ is a Heyting algebra iff the category P is *cartesian closed*: it has finite limits, finite colimits, and exponential objects.
   - The exponential object b^a is $a \to b$.
5. More generally, the *Curry-Howard isomorphism* connects intuitionistic logic and cartesian closed categories.

Section 6

Modular lattices

Modular lattices: definition 1

Fact

Let $(L, \leq)$ *be a lattice. Suppose* $a \leq b$ *and* x *is arbitrary. Then*

$$(x \wedge b) \vee a \leq (x \vee a) \wedge b \tag{1}$$

Intuition: we are trying to “round” x into the interval $[a, b]$.

Definition

A lattice $(L, \leq)$ is *modular* if equality always holds in (1):

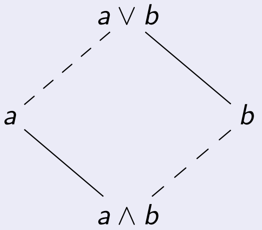
$$(x \wedge b) \vee a = (x \vee a) \wedge b.$$

Modular lattices: definition 2

Fact

Let $(M, \leq)$ be a modular lattice. For any a, b, there is an isomorphism

$$f : [a \wedge b, a] \to [b, a \vee b]$$
$$f(x) = x \vee b$$
$$f^{-1}(y) = y \wedge a$$



Fact

This property characterizes modular lattices.

Modular lattices: examples

Fact

These are modular lattices:

- *The lattice of ideals in a ring R.*
- *The lattice of normal subgroups in a group G.*
- *The lattice of subgroups in an abelian group A.*
- *The lattice of subspaces in a vector space V.*
- *Any distributive lattice.*

Three of these examples are instances of

- The lattice of submodules of an R-module M,

which is the inspiration for the name "modular lattice."

Length

Definition

If $a \leq b$, the *length* $\ell(b/a)$ is the maximum n such that there exist

$$a = x_0 < x_1 < \cdots < x_n = b,$$

or ∞ if there is no finite maximum.

- $\ell(b/a) = 0$ iff $b = a$
- $\ell(b/a) > 0$ iff $b > a$.
- $\ell(b/a) = 1$ iff b *"covers"* a, i.e., $b > a$ and there is no $b > x > a$.

Jordan-Hölder

Fact

If $a \leq b \leq c$ *in a modular lattice, then* $\ell(c/a) = \ell(c/b) + \ell(b/a)$*.*

Definition

A *composition series* from a to b is a maximal chain $a = x_0 < x_1 < \cdots < x_n = b$.

Then $\ell(b/a) = \sum_{i=1}^{n} \ell(x_i/x_{i-1}) = \sum_{i=1}^{n} 1$, so...

Theorem (Jordan-Hölder)

In a modular lattice, any composition series from a *to* b *has length* $\ell(b/a)$*.*

Length in vector spaces

Let M be the lattice of linear subspaces of $\mathbb{R}^n$.

Fact

If $W \subseteq V \subseteq \mathbb{R}^n$ are subspaces, then
$\ell(V/W) = \dim(V/W) = \dim(V) - \dim(W)$.

Finite-length modular lattices

Let M be a bounded modular lattice of finite length. Define $\rho(x) = \ell(x/0)$. Then

$$\rho(x \vee y) = \rho(x) + \rho(y) - \rho(x \wedge y) \tag{2}$$

$$x < y \implies \rho(x) < \rho(y) \tag{3}$$

$$\ell(y/x) = \rho(y) - \rho(x). \tag{4}$$

Fact

Let $(L, \leq)$ be a lattice and ρ be a function from L to an ordered abelian group satisfying (2) and (3). Then $(L, \leq)$ is a modular lattice.