Category Theory In Context

Emily Riehl

October 25, 2021

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1 Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Definition 1.1. A **category** consists of

- a collection of **objects** $X, Y, Z, \ldots$
- a collection of **morphisms** $f, g, h, \ldots$

so that

- Each morphism has specified **domain** and **codomain** objects; the notation $f : X \to Y$ signifies that f is a morphism with domain X and codomain Y
- Each object has a designated **identity morphism** $1_X : X \to X$
- For any pair of morphisms f, g with the codomain of f equal to the domain of g, there exists a specified **composite morphism** gf whose domain is equal to the domain of f and whose codomain is equal to the codomain of g, i.e., :

$$f : X \to Y, \quad g : Y \to Z \qquad \rightsquigarrow \qquad gf : X \to Z$$

This data is subject to the following two axioms

- For any $f : X \to Y$, the composites $1_Y f$ and $f 1_X$ are both equal to f
- For any composable triple of morphisms f, g, h, the composites $h(gf)$ and $(hg)f$ are equal and hence denoted by hgf.

$$f : X \to Y, \quad g : Y \to Z, \quad h : Z \to W \qquad \rightsquigarrow \qquad hgf : X \to W$$

Example 1.1. 1. For any language $\mathcal{L}$ and any theory T of $\mathcal{L}$, there is a category $\mathbf{MODEL}_T$ whose objects are models of T. Morphisms is just homomorphisms

Concrete categories are those whose objects have underlying sets and whose morphisms are functions between underlying sets

Definition 1.2. A category is **small** if it has only a set's worth of arrows Both $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ are sets

Thus it has only a set's worth of objects

Definition 1.3. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms

The set of arrows between a pair of fixed objects in a locally small category is typically called a **hom-set**

Definition 1.4. An **isomorphism** in a category is a morphism $f : X \to Y$ for which there exists a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_X$, denoted by $X \cong Y$

An **endomorphism** is a morphism whose domain equals its codomain

Definition 1.5. A **groupoid** is a category in which every morphism is an isomorphism

Lemma 1.6. *Any category $\mathcal{C}$ contains a **maximal groupoid**, the subcategory containing all of the objects and only those morphisms that are isomorphisms*

Exercise 1.1.1. 1. Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ s.t. $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism

2. Show that a morphism can have at most one inverse isomorphism

Proof. 1. $g = 1_x g = (hf)g = h(fg) = h1_y = h$

2. From 1

$\square$

Exercise 1.1.2. For any category $\mathcal{C}$ and any object $c \in \mathcal{C}$, show that

1. There is a category $c/\mathcal{C}$ whose objects are morphisms $f : c \to x$ with domain c in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes.

2. There is a category $\mathcal{C}/c$ whose objects are morphisms $f : x \to c$ with codomain c in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes

The category $c/\mathcal{C}$ and $\mathcal{C}/c$ are called **slice categories** of $\mathcal{C}$ **under** and **over** c, respectively

1.2 Duality

Definition 1.7. Let $\mathcal{C}$ be any category. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ has

- the same objects as in $\mathcal{C}$
- a morphism f^{op} in $\mathcal{C}^{\mathrm{op}}$ for each a morphism f in $\mathcal{C}$ so that the domain of f^{op} is defined to be the codomain of f and the codomain of f^{op} is defined to be the domain of f
- For each object X, the arrow 1_X^{op} serves as its identity in $\mathcal{C}^{\mathrm{op}}$

- A pair of morphisms $f^{\mathrm{op}}, g^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ is composable precisely when the pair g, f is composable in $\mathcal{C}$. We then define $g^{\mathrm{op}} \circ f^{\mathrm{op}}$ to be $(f \circ g)^{\mathrm{op}}$: i.e.
$$\mathrm{dom}(f^{\mathrm{op}}) = \mathrm{cod}(f) = \mathrm{dom}(g) = \mathrm{cod}(g^{\mathrm{op}})$$

Lemma 1.8. *T.F.A.E.*

1. *$f : x \to y$ is an isomorphism*

2. *For all objects $c \in \mathcal{C}$, post-composition with f defines a bijection*
$$f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$$

3. *For all objects $c \in \mathcal{C}$, pre-composition with f defines a bijection*
$$f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$$

Lemma 1.8 asserts that isomorphisms in a locally small category are defined *representably* in terms of isomorphisms in the category of sets.

Proof. $2 \to 1$. Let $c = y$, since f_* in an bijection, there must be an element $g \in \mathrm{Hom}(y, x)$ s.t. $f_*(g) = 1_y$. Hence $fg = 1_y$. Thus $gf, 1_x$ have common image under f_*, thus $gf = 1_x$. Whence f and g are inverse isomorphisms $\square$

Definition 1.9. A morphism $f : x \to y$ in a category is

1. a **monomorphism** if for any parallel morphisms $h, k : w \rightrightarrows x$, $fg = fk$ implies that $h = k$

2. an **epimorphism** if for any parallel morphisms $h, k : w \rightrightarrows x$, $hf = kf$ implies that $h = k$

Also, we can re-express it

1. $f : x \to y$ is a monomorphism in $\mathcal{C}$ iff for all objects $c \in \mathcal{C}$, $f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$ is injective

2. $f : x \to y$ is an epimorphism in $\mathcal{C}$ iff for all $c \in \mathcal{C}$, $f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$ is injective

Example 1.2. Suppose that $x \xrightarrow{s} y \xrightarrow{r} x$ are morphisms s.t. $rs = 1_x$. The map s is a **section** or **right inverse** to r, while the map r defines a **retraction** or **left inverse** to s. The maps s and r express the object x as a **retract** of the object y

In this case, s is always a monomorphism and, dually, r is always an epimorphism. To ackowledge the presence of these one-sided inverses, s is said to be a **split monomorphism** and r is said to be a **split epimorphism**

Example 1.3. By the previous example, an isomorphism is necessarily both monic and epic, but the converse need not hold in general. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic in the category **Rng**, but this map is not an isomorphism: there are no ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$

Lemma 1.10. *1. If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monomorphisms, then so is $gf : x \rightarrowtail z$*

2. If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is monic, then f is monic

Dually

1. If $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epimorphisms, then so is $gf : x \twoheadrightarrow z$

2. If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is epic, then g is epic

Exercise 1.2.1. 1. Show that a morphism $f : x \to y$ is a split epimorphism in a category $\mathcal{C}$ iff for all $c \in \mathcal{C}$, the post-composition function $f_* : \mathrm{Hom}(c, x) \to \mathrm{Hom}(c, y)$ is surjective

2. Show that a morphism $f : x \to y$ is a split monomorphism in a category $\mathcal{C}$ iff for all $c \in \mathcal{C}$, the post-composition function $f^* : \mathrm{Hom}(y, c) \to \mathrm{Hom}(x, c)$ is surjective

Exercise 1.2.2. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism

Proof. Suppose $y \xrightarrow{g} x \xrightarrow{f} y$ and $fg = 1_y$, then $fgf = f = f \circ 1_x$. Since f is mono, $gf = 1_x$ □

1.3 Functoriality

Definition 1.11. A **functor** $F : \mathcal{C} \to \mathcal{D}$, between categories $\mathcal{C}$ and $\mathcal{D}$, consists of the following data:

- An object $Fc \in \mathcal{D}$, for each objects $c \in \mathcal{C}$
- A morphism $Ff : Fc \to Fc' \in \mathcal{D}$, for each morphism $f : c \to c' \in \mathcal{C}$

Functoriality axioms

- For any composable pair $f, g \in \mathcal{C}$, $Fg \circ Ff = F(g \circ f)$
- For each object $c \in \mathcal{C}$, $F(1_c) = 1_{Fc}$

Definition 1.12. A **contravariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$

- A morphism $Ff : Fc' \to Fc \in \mathcal{D}$ for each morphism $f : c \to c' \in \mathcal{C}$
- For any composable pair $f, g \in \mathcal{C}$, $Ff \circ Fg = F(g \circ f)$

$$\mathcal{C}^{\text{op}} \xrightarrow{F} \mathcal{D}$$

$$\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \\
{} \arrow[r, mapsto] & {} \\
c' \arrow[r, mapsto] & Fc' \arrow[uu, "Ff"']
\end{tikzcd}$$

Lemma 1.13. *Functors preserve isomorphisms*

Proof. Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and an isomorphism $f : x \to y$ in $\mathcal{C}$ with inverse $g : y \to x$. Then

$$F(g)F(f) = F(gf) = F(1_x) = 1_{Fx}$$

Thus $Fg : Fy \to Fx$ is a left inverse to $Ff : Fx \to Fy$ $\square$

Definition 1.14. If $\mathcal{C}$ is locally small, then for any object $c \in \mathcal{C}$ we may define a pair of covariant and contravariant **functors represented by** c:

$$\mathcal{C} \xrightarrow{\text{Hom}(c,-)} \textbf{Sets} \qquad \mathcal{C}^{\text{op}} \xrightarrow{\text{Hom}(-,c)} \textbf{Sets}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, "f"'] & \text{Hom}(c,x) \arrow[dd, "f_*"] & x \arrow[r, mapsto] \arrow[dd, "f"'] & \text{Hom}(x,c) \\
{} \arrow[r, mapsto] & {} & {} \arrow[r, mapsto] & {} \\
y \arrow[r, mapsto] & \text{Hom}(c,y) & y \arrow[r, mapsto] & \text{Hom}(y,c) \arrow[uu, "f^*"]
\end{tikzcd}$$

Post-composition defines a **covariant** action on hom-sets

Definition 1.15. For any categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathcal{C} \times \mathcal{D}$, their **product**, whose

- objects are ordered pairs (c, d), where c is an object of $\mathcal{C}$ and d is an object of $\mathcal{D}$
- morphisms are ordered pairs $(f, g) : (c, d) \to (c', d')$, where $f : c \to c' \in \mathcal{C}$ and $g : d \to d' \in \mathcal{D}$ and
- in which composition and identities are defined componentwise

Definition 1.16. If $\mathcal{C}$ is locally small, then there is a **two-sided represented functor**

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Sets}$$

A pair of objects (x, y) is mapped to the hom-set $\mathrm{Hom}(x, y)$. A pair of morphisms $f : w \to x$ and $h : y \to z$ is sent to the function

$$\mathrm{Hom}(x, y) \xrightarrow{(f^*, h_*)} \mathrm{Hom}(w, z)$$

$$g \longmapsto hgf$$

An **isomorphism of categories** is given by a pair of inverse functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ s.t. the composites Gf and FG, respectively, equal the identity functors on $\mathcal{C}$ and $\mathcal{D}$

1.4 Naturality

Definition 1.17. Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of

- an arrow $\alpha_c : Fc \to Gc$ in $\mathcal{D}$ for each object $c \in \mathcal{C}$, the collection of which define the **components** of the natural transformation s.t. for any morphism $f : c \to c'$ in $\mathcal{C}$, the following square of morphisms in $\mathcal{D}$

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

commutes

A **natural isomorphism** is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_c is an isomorphism. In this case, the natural isomorphism may be depicted as $\alpha : F \cong G$

$$
\begin{tikzcd}
A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
$$