Generic Properties Of Groups

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1 Preliminaries

If $p(\bar{x})$ is a type over A, then we call the set of realizations of p in M

$$p(M^n) = \{\bar{a} \in M^n : (\forall \varphi(\bar{x}) \in p(\bar{x}))M \models \varphi(\bar{a})\} \models \bigcap_{\varphi(\bar{x}) \in p(\bar{x})} \varphi(M^n)$$

type definable over A. If V is a 0-type-definable subset of M^n, then we sometimes identify V with the set

$$[V] = \{\mathrm{tp}(\bar{a}) : \bar{a} \in V\} \subseteq S_n(\emptyset)$$

A first order structure M is κ-saturated if for any $A \subseteq M$ with $|A| < \kappa$, $n < \omega$ and $p \in S_n(A)$, p has a realization in M.

A group $(G, \cdot)$ is definable in a structure M if G is a definable subset of M^n for some $n < \omega$ and the group action $\cdot : G \times G \to G$ is a definable function in M. If $p(x)$ is a type over G and $g \in G$, then

We call a first order structure $(M, \cdot, \ldots)$ a group if $(M, \cdot)$ satisfies group axioms. We usually denote it by $(G, \cdot, \ldots)$. A structure of the form $(G, \cdot)$ is called a **pure** group.

$$g \cdot p(x) = \{g \cdot \varphi(x) : \varphi(x) \in p(x)\} = \{\varphi(g^{-1} \cdot x) : \varphi(x) \in p(x)\}$$

A group $(G, \cdot)$ is definable in a structure M if G is a definable

An infinite totally ordered first order structure $(M, <, \ldots)$ is **o-minimal** if every definable subset of M is a union of finitely many intervals and points.

Let $(M, <, \ldots)$ be an o-minimal structure. We usually say "ultimately" instead of "for all sufficiently large $a \in M$". We denote an open interval with endpoints a and b by (a, b) and a closed one by $[a, b]$. In contrast, $\langle a, b\rangle$ denotes the pair of elements a and b.

If $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$, $a < b$ and $f : (a, b) \to M$ is a definable function, then there are $a = a_1 < \cdots < a_n = b$ s.t. each interval (a_i, a_{i+1}) of f is either constant or strictly monotone and continuous in the order topology. In particular, every definable function $f : M \to M$ is ultimately continuous and monotone

2 Weak generic types

2.1 Introduction

Definition 2.1. A set $X \subseteq G$ is **(left) generic** if some finitely many left G-translates of X cover G. We say that a formula $\varphi(x)$ is **(left) generic** if the set $\varphi(G)$ of elements of G realizing φ is **(left) generic**. Finally, we say that a type $p(x)$ of elements of G is **(left) generic** if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is (left) generic

In the stable case left generic = right generic

and each partial generic type extends to a complete generic type (since type is definable)

Definition 2.2. A set $A \subseteq G$ is **weak generic**, if for some non-generic $B \subseteq G$ we have that $A \cup B$ is generic. A formula $\varphi(x)$ is **weak generic** if the set $\varphi(G)$ is weak generic. A type $p(x)$ of elements of G is weak generic if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is weak generic

2.2 Basic properties of weak generic sets and types

Lemma 2.3. *Assume that G is a group and X is a definable subset of G. TFAE*

1. *the set X is weak generic*
2. *for some finitely many elements $a_1, \dots, a_n \in G$ the set $G \setminus \bigcup_{i=1}^n a_i \cdot X$ is not generic*
3. *for some definable non-generic set $Y \subseteq G$ the set $X \cup Y$ is generic*

Proof. $1 \Rightarrow 2$: Assume X is weak generic, then there is non-generic set $Y \subseteq G$ s.t. $X \cup Y$ is generic. Then there are $a_1, \dots, a_n \in G$ s.t.

$$\bigcup_{i=1}^n a_i \cdot (X \cup Y) = \bigcup_{i=1}^n a_i \cdot X \cup \bigcup_{i=1}^n a_i \cdot Y = G$$

This means that

$$G \setminus \bigcup_{i=1}^n a_i \cdot X \subseteq \bigcup_{i=1}^n a_i \cdot Y$$

$2 \Rightarrow 3$: Let $Y = G \setminus \bigcup_{i=1}^n a_i \cdot X$. Then Y is definable and not generic so putting $a_{n+1} = e$. Then $G = \bigcup_{i=1}^{n+1} a_i \cdot (X \cup Y)$ □

Lemma 2.4. 1. *If $X, Y \subseteq G$ are not weak generic, then $X \cup Y$ is not weak generic*

2. *If $p(x)$ is a (partial) weak generic type over $A \subseteq G$, then $p(x)$ may be extended to a complete weak generic type over A*

Proof. 1. Let $Z \subseteq G$ be non-generic. Y is not weak generic so $Y \cup Z$ is not generic, so $Y \cup Z \cup X$ is not generic

2. non weak generics form an ideal

   Let $q(x) = \{\varphi(x) \in L(A) : p(x) \cup \{\neg\varphi(x)\} \text{ is not weak generic}\}$. Then $p \subseteq q$. We shall show that q is a consistent partial type over A. If not, then

$$G \models \neg\exists x \bigwedge_{k=1}^n \varphi_k(x)$$

   for some $n < \omega$ and $\varphi_1, \dots, \varphi_n \in q$. By compactness, for each $k \in \{1, \dots, n\}$ we can find a finite set of formulas $p_k \subseteq p$ s.t. the type $p_k(x) \cup \{\neg\varphi_k(x)\}$ is not weak generic. Let $\psi(x) = \bigwedge\{p_k(x) : 1 \leq k \leq n\}$ and note that for every $k \in \{1, \dots, n\}$ the set $\psi(G) \cap \neg\varphi_k(G)$ is not weak generic. By 1, neither is the union

$$\bigcup_{k=1}^n (\psi(G) \cap \neg\varphi_k(G)) = \psi(G) \cap \bigcup_{k=1}^n \neg\varphi_k(G) = \psi(G) \cap G = \psi(G)$$

contradicting the fact that $p(x) \vdash \psi(x)$. Finally we take any $r(x) \in S(A)$ with $r \supseteq q$ and the proof is complete

$\square$

We see that (complete) weak generic types exist. By Lemma 2.4, the set

$$WGEN(A) = \{p \in S(A) : p \text{ is weak generic}\}$$

is closed and non-empty in $S(A)$

Lemma 2.5. *Assume G is a group and $A \subseteq G$*

1. *If some weak generic type $p(x) \in S(G)$ is generic, then all weak generic types $q(x) \in S(A)$ are generic*

2. *If for every $p, q \in WGEN(G)$ there is $g \in G$ s.t. $g \cdot p = q$, then all weak generic types $q(x) \in S(A)$ are generic*

3. *If there is just one weak generic type in $S(A)$, then it is generic*

Proof. 1. Suppose that some weak generic type $q(x) \in S(A)$ is not generic. Then some definable generic set $X \subseteq G$ may be divided into two non-generic definable sets X_1, X_2. Since X is generic, some left G-translates X' of X belongs to $p(x)$. Then the corresponding translates X_1', X_2' of X_1, X_2 are also non-generic and one of them belongs to $p(x)$. Hence $p(x)$ is not generic, a contradiction

2. If not, then we can find a formula $\varphi(x) \in L(A)$ which is weak generic but not generic. Note that $\{\neg g \cdot \varphi(x) : g \in G\}$ is a partial weak generic type over G: for each $m < \omega$ and $g_1, \dots, g_m \in G$, the set $\bigcup_{i=1}^{m} g_i \cdot \varphi(G)$ is not generic, which implies that the set $\bigcap_{i=1}^{m}(G \setminus g_i \cdot \varphi(G))$ is weak generic. Extend the type $\{\neg g \cdot \varphi(x) : g \in G\}$ to some $q(x) \in WGEN(G)$. Next extend $\varphi(x)$ to $p(x) \in WGEN(G)$. Then $\forall g \in G\ g \cdot p \neq q$, a contradiction

3. by 2, immediately

$\square$

By Lemma 2.5 (1), in the stable case weak generic = generic

As an example note that if $G = (G, <, +, \dots)$ is o-minimal, then there are exactly two complete weak generic types, corresponding to $-\infty$ and $+\infty$, and they are not generic

Lemma 2.6. *Assume that $G \prec H$ and $\varphi(x) \in L(G)$*

1. *If $\varphi(G)$ is weak generic in G, then $\varphi(H)$ is weak generic in H*

2. *If G is $\aleph_0$-saturated and $\varphi(H)$ is weak generic in H, then $\varphi(G)$ is weak generic in G*

Proof. 1. There is a non-generic formula $\psi(x) \in L(G)$ s.t. $\varphi(G) \cup \psi(G)$ is generic in G, therefore $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. Thus $\varphi(H)$ is weak generic in H

2. There is a formula $\psi(x) \in L(H)$ s.t. $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. We have that $\psi(x) = \psi(x, b)$ where $b \subset H$. Let $A \subseteq G$ be a finite set containing all parameters of $\varphi(x)$. By $\aleph_0$-saturation of G, we are able to find in G a tuple $a \subset G$ s.t. $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Then $\psi(x, a) \in L(G)$ has properties needed to deduce the weak genericity of the set $\varphi(G)$ in G. Namely $\psi(G, a)$ is not generic in G and $\varphi(G) \cup \psi(G, a)$ is generic in G. If $\psi(G, a)$ is generic in G, then for some $0 < n < \omega$ we have that

$$G \models \exists x_1, \dots, x_n \forall y \exists z (\psi(z, a) \wedge \bigvee_{k=1}^{n} y = x_k \cdot z)$$

and the same holds in H since $G \prec H$, which would lead to a contradiction

□

All lemmas in this section remain true if we consider a group $(G, \cdot)$ definable in a first order structure M. Then G is a definable subset of M^n for some $n < \omega$ and for every $A \subseteq M$ we define the set $WGEN(A)$ of complete weak generic types over A as the set

$$\{p \in S_n(A) : \forall \varphi(x_1, \dots, x_n) \in p, G \cap \varphi(M^n) \text{ is weak generic in } G\}$$

2.3 Characterizations of weak genericity

Proposition 2.7. *Assume G is a definable group in an o-minimal structure M and X is a definable weak generic subset of G. Then* $\dim(X) = \dim(G)$

Proof. Suppose $\dim(X) < \dim(G)$. Take a generic set A and a non-generic set B s.t. $A = B \cup X$ (where A and B are definable subsets of G, apply Lemma 2.3) Choose a finite $S \subseteq G$ with $S \cdot A = G$. Then $G \setminus (S \cdot B) \subseteq S \cdot X$ and

$$\dim(G \setminus (S \cdot B)) \leq \dim(S \cdot X) = \dim(X) < \dim(G)$$

Hence the set $S \cdot B$ is large in the sense □

Assume G is a group and $X, Y \subseteq G$. We say that the set X is **translation disjoint** from the set Y if for some $a \in G$, $a \cdot X \cap Y = \emptyset$

Lemma 2.8. *Assume G is a group and X is a weak generic subset of G. Then for some finite $A \subseteq G$ there is no finite covering of X by sets that are translation disjoint from $A \cdot X$*

Proof. take $Y \supseteq X$ generic and $Y \setminus X$ not generic. We have that $G = A \cdot Y$ for some finite $A \subseteq G$. We shall prove that A meets conditions of the lemma.

Suppose for some $X_0, \dots, X_{n-1} \subseteq G$ and $a_0, \dots, a_{n-1} \in G$ we have that

$$X = \bigcup_{i<n} X_i \text{ and } \bigwedge_{i<n} (a_i \cdot X_i) \cap (A \cdot X) = \emptyset$$

Then for each $i < n$, $a_i \cdot X_i \subseteq G \setminus A \cdot X \subseteq A \cdot (Y \setminus X)$. So for each $i < n$, $X_i \subseteq a_i^{-1} \cdot A \cdot (Y \setminus X)$, which implies that $X \subseteq \{a_0^{-1}, \dots, a_{n-1}^{-1}\} \cdot A \cdot (Y \setminus X)$ and finally

$$G = A \cdot Y = A \cdot (Y \setminus X) \cup A \cdot X \subseteq (A \cup (A \cdot \{a_0^{-1}, \dots, a_{n-1}^{-1}\} \cdot A)) \cdot (Y \setminus X)$$

Then G is covered by finitely many things $\square$

Corollary 2.9. *Assume G is a group and X is a weak generic subset of G. Then the set $X \cdot X^{-1}$ is generic in G*

Proof. Take a finite $A \subseteq G$ as in Lemma 2.8. Then for each $a \in G$, $a \cdot X \cap A \cdot X \neq \emptyset$, which implies that $a \in A \cdot X \cdot X^{-1}$. So $G = A \cdot X \cdot X^{-1}$ $\square$

From now on, let $(G, <, +, \dots)$ be an o-minimal expansion of an ordered group $(G, <, +)$. Then the group G is commutative, divisible and torsion-free. By $(G^n, +)$ we mean the product of groups $(G, +) \times \dots \times (G, +)$ (n times). The ordering of G is dense since for every $a, b \in G$ with $a < b$ we have that $a < \frac{a+b}{2} < b$

Theorem 2.10. *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$, $0 < n < \omega$ and $\varphi(x_1, \dots, x_n) \in L(G)$. TFAE*

1. *$\varphi(x_1, \dots, x_n)$ is weak generic in $(G^n, +)$*
2. *$\neg\varphi(x_1, \dots, x_n)$ is not generic in $(G^n, +)$*
3. *the set $\varphi(G^n)$ contains arbitrarily large n-dimensional boxes*

$$(\forall R > 0)(\exists a_1, \dots, a_n \in G)[a_1, a_1 + R] \times \dots \times [a_n, a_n + R] \subseteq \varphi(G^n)$$

Proof. $3 \Rightarrow 2$: suppose there is $k < \omega$ and $\langle g_1^1, \dots, g_n^1 \rangle, \dots, \langle g_1^k, \dots, g_n^k \rangle \in G^n$ we have that

$$G^n = \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j \rangle + (G^n \setminus \varphi(G^n))$$

Put $M = \max\{\left|g_i^j\right| : 1 \leq i \leq n, 1 \leq j \leq k\}$. Using 3 we are able to find $a_1, \dots, a_n \in G$ s.t.

$$[a_1 - M, a_1 + M] \times \dots \times [a_n - M, a_n + M] \subseteq \varphi(G^m)$$

Then

$$\langle a_1, \dots, a_n \rangle \notin \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j \rangle + (G^n \setminus \varphi(G^n)))$$

a contradiction

$2 \Rightarrow 1$: since the set $G^n = \varphi(G^n) \cup (G^n \setminus \varphi(G^n))$ is generic in $(G^n, +)$ and the set $G^n \setminus \varphi(G^n)$ is not generic

$1 \Rightarrow 3$: W.L.O.G., $n \geq 2$. Using Lemma 2.4 (2) find $p(x_1, \dots, x_n) \in S_n(G)$ s.t. p is a weak generic type in $(G^n, +)$ and $\varphi \in p$. Extend G to a $|G|^+$-saturated group $H \succ G$. Take $\langle a_1, \dots, a_n \rangle \in H^n$ realizing p and fix a positive $R \in G$. We shall show that the follwing condition holds

$$(\forall a \in H)(a_n \leq a \leq a_n + R \Rightarrow \text{tp}(a/Ga_{<n}) = \text{tp}(a_n/Ga_{<n})) \qquad (\star)$$

For the sake of contradiction assume that for some $a \in [a_n, a_n + R]_H$ the types $\text{tp}(a/Ga_{<n})$ and $\text{tp}(a_n/Ga_{<n})$ are distinct. By the o-minimality of H, we can find $b \in [a_n, a_n + R]_H$ with $b \in \text{dcl}(Ga_{<n})$ (dense). Let $\psi(x_1, \dots, x_{n-1}, y) \in L(G)$ be s.t. $H \vDash \psi(a_{<n}, b) \wedge \exists! y \psi(a_{<n}, y)$. As $b - R \leq a_n \leq b$, we have that $\chi \in p$ where

$$\chi(x_1, \dots, x_n) = \exists! y \psi(x_{<n}, y) \wedge \forall y (\psi(y_{<n}, y) \rightarrow (y - R \leq x_n \leq y))$$

Since $\chi \in p$, the set $\chi(G^n)$ is weak generic in $(G^n, +)$

We define $f : G^{n-1} \rightarrow G$ as:

$$f(c_{<n}) = \begin{cases} c_n - R & G \vDash \chi(\bar{c}) \\ 0 & \text{otherwise} \end{cases}$$

Take $\langle c_1, \dots, c_{n-1} \rangle \in G^{n-1}$. If there is $c_n \in G$ s.t. $G \vDash \chi(c_1, \dots, c_n)$, then there exists just one $d \in G$ with $G \vDash \psi(c_1, \dots, c_{n-1}, d)$ and we put $f(c_1, \dots, c_{n-1}) =$

$d - R$. Otherwise we put $f(c_1, \dots, c_{n-1}) = 0$. Then the function f is definable over G and we consider the following formula over G:

$$\delta(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1}) \leq x_n \leq f(x_1, \dots, x_{n-1}) + R$$

Since $\chi(G^n) \subseteq \delta(G^n) \subseteq G^n$, the set $\delta(G^n)$ is weak generic in $(G^n, +)$. Let $A \subseteq G^n$ be a finite set chosen for $\delta(G^n)$ as in Lemma 2.8. Consider an arbitrary $\langle h_1, \dots, h_{n-1} \rangle \in H^{n-1}$. Choose $M_{h_{<n}} \in G$ s.t.

$$\{\langle h_1, \dots, h_n \rangle : f(h_{<n}) + M_{h_{<n}} \leq h_n \leq f(h_{<n}) + M_{h_{<n}} + R\} \cap (A + \delta(H^n)) = \emptyset$$

(exists since is bounded and A is finite) If $\mathrm{tp}(h_{<n}/G) = \mathrm{tp}(h'_{<n}/G)$, then $M_{h_{<n}}$ is good also for $h'_{<n}$. By compactness, for each $q(x_1, \dots, x_{n-1}) \in S_{n-1}(G)$ we can find a formula $\varphi_q(x_1, \dots, x_{n-1}) \in L(G)$ and $M_q \in G$ s.t. for every $h_{<n} \in H^{n-1}$ with $H \vDash \varphi_q(h_{<n})$ we have

$$\{\langle h_1, \dots, h_n \rangle : f(h_{<n}) + M_q \leq h_n \leq f(h_{<n}) + M_q + R\} \cap (A + \delta(H^n)) = \emptyset$$

Again by compactness, $S_{n-1}(G) = [\varphi_{q_1}] \cup \dots \cup [\varphi_{q_k}]$ for some $k < \omega$ and $q_1, \dots, q_k \in S_{n-1}(G)$. If not, then $\forall n \in \omega$, $G \vDash \bigwedge_{i=1}^{n} \neg\varphi_q i$, that is, $\{\neg\varphi_{q_i} : i \in \omega\}$ is consistent with G, then realized by H, which leads to a contradiction. For $i \in \{1, \dots, k\}$ put $X_i = (\varphi_{q_i}(G^{n-1}) \times G) \cap \delta(G^n)$ and $e_i = \langle 0, \dots, 0, M_{q_i} \rangle \in G^n$. Then $\delta(G^n) = X_1 \cup \dots \cup X_k$ and for every $i \in \{1, \dots, k\}$ we have that $(e_i + X_i) \cap (A + \delta(G^n)) = \emptyset$. This contradicts the choice of A and finishes the proof of $(\star)$

By $(\star)$, we have that

$$H \vDash \forall y((a_n \leq y \wedge y \leq a_n + R) \rightarrow \varphi(a_1, \dots, a_{n-1}, y))$$

Therefore the formula $\forall y((x_n \leq y \leq x_n + R \rightarrow \varphi(x_1, \dots, x_{n-1}, y)))$ belongs to p. In general, for each formula $\psi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$, $k \in \{1, \dots, n\}$ and positive $R \in G$ the formula

$$\forall y((x_k \leq y \leq x_k + R) \rightarrow \psi(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n))$$

belongs to p. We inductively create formulas $\varphi_k(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$, $k = \{1, \dots, n\}$,. Namely, provided that $\varphi_1(x_1, \dots, x_n), \dots, \varphi_{k-1}(x_1, \dots, x_n)$ have already been defined, let $\varphi_k(x_1, \dots, x_n)$ be the formula

$$\forall y((x_k \leq y \leq x_k + R) \rightarrow (\varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_{k-1}(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n)))$$

Finally, we take any $\bar{g} \in (\varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_n)(G^n)$ and see that

$$[g_1, g_1 + R] \times \dots \times [g_n, g_n + R] \subseteq \varphi(G^n)$$

$\square$

Corollary 2.11. *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$, $0 < n, k < \omega$ and $\varphi(x_1, \dots, x_n, y_1, \dots, y_k) \in L$*

1. *there is $\psi_1(y_1, \dots, y_k)$ s.t. for every $\langle a_1, \dots, a_k\rangle \in G^k$ we have that $G \models \psi_1(a)$ iff $\varphi(G^n, a)$ is weak generic in $(G^n, +)$*

2. *There is $\psi_2(y_1, \dots, y_k)$ s.t. for every $\langle a_1, \dots, a_k\rangle \in G^k$ we have that $G \models \psi_2(a)$ iff $\varphi(G^n, a)$ is generic in $(G^n, +)$*

3. *there is a natural number N s.t. for every φ-definable $X \subseteq G^n$ the set X is generic in $(G^n, +)$ iff G^n may be covered by at most N left translates of X*

Proof. 1. let $\psi_1(y_1, \dots, y_k)$ be

$$\forall r \exists z_1, \dots, z_n \forall x_1, \dots, x_n ((\bigwedge_{i=1}^{n} z_i \leq x_i \wedge x_i \leq z_i + r) \to \varphi(x_1, \dots, x_n, y_1, \dots, y_k))$$

3. Assume that $n = 1$. Let $\psi_2(y_1, \dots, y_k)$ be such as 2. Suppose for every $N < \omega$ we can find $\langle a_1, \dots, a_k\rangle \in G^k$ s.t. the set $\varphi(G, a_1, \dots, a_k)$ is generic in G but not N-generic. Then the set of formulas

$$\bigcup_{N<\omega} \{\psi_2(y_1, \dots, y_k) \wedge \forall z_1, \dots, z_N \exists t \forall x (\varphi(x, y_1, \dots, y_k) \to \bigwedge_{i=1}^{N} t \neq z_i \cdot x)\}$$

is a type in variables $y_1, \dots, y_k$ and has a realization $\langle b_1, \dots, b_k\rangle \in H^k$ in some $\aleph_0$-saturated elementary extension H of G. Then we reach a contradiction as the set $\varphi(H, b_1, \dots, b_k)$ is simultaneously generic and not generic in H

□

Corollary 2.12. *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$, $0 < n < \omega$, and $p(x_1, \dots, x_n) \in S_n(G)$. TFAE*

1. *$p(x_1, \dots, x_n)$ is weak generic in $(G^n, +)$*

2. *$\langle g_1, \dots, g_n\rangle + p(x_1, \dots, x_n) = p(x_1, \dots, x_n)$ for every $\langle g_1, \dots, g_n\rangle \in G^n$*

Proof. $1 \Rightarrow 2$: suppose

$$\langle g_1, \dots, g_n\rangle + p(x_1, \dots, x_n) \neq p(x_1, \dots, x_n)$$

for some $\langle g_1, \dots, g_n\rangle \in G^n$. Then for some $\varphi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$ we have that $(\langle g_1, \dots, g_n\rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$. $\varphi(G^n)$ is weak generic in

$(G^n,+)$ and hence contains arbitrarily large boxes. Take any $R > \max(|g_1|,\dots,|g_n|)$ and choose $a_1,\dots,a_n \in G$ s.t.

$$B = [a_1, a_1+R] \times \cdots \times [a_n, a_n+R] \subseteq \varphi(G^n)$$

we obtain

$$\emptyset \neq (\langle g_1,\dots,g_n\rangle + B) \cap B \subseteq (\langle g_1,\dots,g_n\rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$$

a contradiction

$2 \Rightarrow 1$: we shall prove a more general fact. Namely if G is a group and $p(x) \in S(G)$ is s.t. for every $g \in G$ we have that $g \cdot p = p$, then p is weak generic in G

If not, then we can find a formula $\varphi(x) \in p(x)$ which is not weak generic in G. Then $\neg\varphi(x)$ is generic in G so there are $m < \omega$ and $g_1,\dots,g_m \in G$ s.t $G = \bigcup_{i=1}^m g_i(G \setminus \varphi(G))$. Thus $\bigcap_{i=1}^m g_i \cdot \varphi(G) = \emptyset$, which contradicts the fact that the formulas $g_1 \cdot \varphi,\dots,g_m \cdot \varphi$ belong to the consistent type $p(x)$ $\square$

2.4 Stationary

In this section we assume that $(G,<,+,\dots)$ is an o-minimal expansion of an ordered group $(G,<,+)$

Recall that in stable group all weak generic types are generic. Moreover, all of them are stationary over any model M. This means that every (weak) generic type $p \in S(M)$ has a unique extension to a (weak) generic type $q \in S(A)$ for each $A \supseteq M$

Definition 2.13. We call a weak generic type p over a set A **stationary** if for every $B \supseteq A$ the type p has just one extension to a complete weak generic type over B

In general weak generic types do not need to be stationary

Example 2.1. we shall prove that the types $p_1(x) = \{x < a : a \in G\}$ and $p_2(x) = \{x > a : a \in G\}$ are the only two weak generic types in $(G,+)$ complete over G and that both of them are stationary

By the o-minimality of $(G,<,+,\dots)$, every definable subset of G is a union of finitely many points and intervals. For every $a,b \in G$ the interval (a,b) is not weak generic in $(G,+)$ by Lemma 2.3 (2). Thus no type in $S_1(G)$ but p_1 and p_2 is weak generic in $(G,+)$

On the other hand, all intervals of the form $(-\infty,a)$ or $(b,+\infty)$ are weak generic in $(G,+)$ since their complements in G are not generic in $(G,+)$. This gives us the weak genericity of the types p_1 and p_2

If H is any elementary extension of G, then there are also two complete (over H) weak generic types in $(H, +)$. This means that p_1 and p_2 are stationary

Definition 2.14. We call an o-minimal structure $(M, <, \dots)$ **stationary** if for every elementary extension N of M and N-definable function $g : N \to N$ there exists an M-definable function $f : N \to N$ s.t. $g(x) \leq f(x)$ for all sufficiently large $x \in N$

Theorem 2.15. *Assume $(M, <, \dots)$ is a stationary o-minimal structure and $N \succ M$. For every N-definable map $g : N \to N$ with $\lim_{x\to+\infty} g(x) = +\infty$ we can find an M-definable map $f : N \to N$ s.t. $\lim_{x\to+\infty} f(x) = +\infty$ and $f(x) \leq g(x)$ for all sufficiently large $x \in N$*

Proof. First of all, assume that g is a bijection. Then g^{-1} exists and by the stationary of $(M, <, \dots)$ we can find an M-definable function $f : N \to N$ s.t. ultimately $g^{-1} \leq f$. We have that $\lim_{x\to+\infty} g^{-1}(x) = +\infty$, which implies that $\lim_{x\to+\infty} f(x) = +\infty$. Since f is M-definable, we can choose $a \in M$ s.t. f is strictly increasing on $(a, +\infty)$ (monotonicity theorem). We define a function $f_1 : N \to N$ as follows

$$f_1(x) = \begin{cases} f(x) & x > a \\ f(a) + x - a & x \leq a \end{cases}$$

Then f_1 is an M-definable bijection, hence f_1^{-1} exists and also is M-definable. Moreover, $\lim_{x\to+\infty} f_1^{-1}(x) = +\infty$ and ultimately $f_1^{-1} \leq g$ so f_1^{-1} has the desired properties

If g is not a bijection, then proceeding as above we can find an N-definable bijection $g_1 : N \to N$ s.t. ultimately $g_1 = g$ $\square$

By the o-minimality of $(G, <, +, \dots)$, every definable subset of the set $G \times G$ is a union of finitely many cells of dimension 0,1,2. By Proposition 2.7, we are interested only in cells of dimension 2 (we are interested in weak generic subsets). They are of the form

$$C_{a,b}^{f,g} = \{\langle x, y\rangle \in G \times G : a < x < b \wedge f(x) < y < g(x)\}$$

where $\{-\infty\} \cup G \ni a < b \in G \cup \{\infty\}$ and $f, g : (a, b) \to G \cup \{-\infty, \infty\}$ are definable maps s.t. $f(x) < g(x)$ for each $x \in (a, b)$. If $a, b \in G$, then the cell $C_{a,b}^{f,g}$ is not weak generic in $(G, +) \times (G, +)$ by Theorem 2.10. Since we shall consider only weak generic types $p(x, y)$ in $(G, +) \times (G, +)$ s.t. $\{x > a : a \in G\} \subseteq p(x, y)$, we shall be interested only in weak generic cells of the form $C_{a,b}^{f,g}$ where $a \in G$ and $b = +\infty$

Definition 2.16. Assume that functions $f, g : G \to G$ are definable

1. $f \ll g$ if $f(x) < g(x)$ for all sufficiently large $x \in G$ and the set

$$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

   is weak generic in $(G, +) \times (G, +)$ $(C^{f,g}_{0,+\infty})$

2. $f \sim g$ if

$$\{\langle x, y\rangle \in G \times G : x > 0 \wedge f(x) < y \wedge y < g(x)\}$$

   is not weak generic in $(G, +) \times (G, +)$

$\sim$ is an equivalence relation on the set of all definable functions from G to G and that equivalence classes of $\sim$ are convex (i.e., if $f, g, h : G \to G$ are definable, $f \sim h$ and ultimately $f(x) \leq g(x) \leq h(x)$, then $f \sim g$ and $g \sim h$)

Definition 2.17. Let $f : G \to G$ be a definable function

1. Let $p^+_f(x, y)$ denote the only extension of the type

$$\{x > a : a \in G\} \cup \{y > f(x)\} \cup \{y < g(x) : g \gg f\}$$

   to a type which is complete over G and weak generic in $(G, +) \times (G, +)$

2. Let $p^-_f(x, y)$ denote the only extension of the type

$$\{x > a : a \in G\} \cup \{y < f(x)\} \cup \{y > g(x) : g \ll f\}$$

   to a type which is complete over G and weak generic in $(G, +) \times (G, +)$

3. Let $p_{+\infty}(x, y)$ denote the weak generic type

$$\{x > a : a \in G\} \cup \{y > g(x) : g : G \to G \text{ definable}\}$$

4. Let $p_{-\infty}(x, y)$ denote the weak generic type

$$\{x > a : a \in G\} \cup \{y < g(x) : g : G \to G \text{ definable}\}$$

Theorem 2.18. *Assume that $(G, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(G, <, +)$. TFAE*

1. *$p^+_f(x, y)$ and $p^-_f(x, y)$ are stationary for each definable function $f : G \to G$*
2. *$p_{+\infty}(x, y)$ and $p_{-\infty}(x, y)$ are stationary*

3. $(G, <, +, \dots)$ *are stationary*

Proof. $1 \Rightarrow 2$: Let $f : G \to G$ be a map constantly equal to 0. Then $p_{+\infty}(x, y) = p_f^+(y, x)$ and therefore $p_{+\infty}$ is stationary

$2 \Rightarrow 3$: Suppose the structure $(G, <, +, \dots)$ is not stationary. Then there exist an $H \succ G$ and a H-definable function $g : H \to H$ s.t. no G-definable map $f : H \to H$ dominates g

Consider the following partial types over H:

$$p_1(x, y) = p_{+\infty}(x, y) \cup \{y < g(x)\}$$
$$p_2(x, y) = p_{+\infty}(x, y) \cup \{y > g(x)\}$$

To reach a contradiction, it is enough to prove that both of them are weak generic in $(H, +) \times (H, +)$, and therefore $p_+(x, y)$ is not stationary. We begin with p_1.

Goal:

$$(\bigwedge_{i=1}^{m} x > a_i) \wedge (\bigwedge_{i=1}^{n} y > f_i(x)) \wedge y < g(x)$$

is weak generic in $(H, +) \times (H, +)$ where $a_1, \dots, a_m \in G$ and $f_1, \dots, f_n$ are functions from H to H definable over G.

Take $a = \max(a_1, \dots, a_n)$ and $f = \max(f_1, \dots, f_n)$ we can confine our attention to the sets X of the form

$$X = \{\langle x, y\rangle \in H \times H : x > a \wedge y > f(x) \wedge y < g(x)\}$$

where $a \in G$ and $f : H \to H$ is definable over G. W.L.O.G., we can assume that f is ultimately non-decreasing

Consider a map $h : H \to H$ defined as follows: $h(a) = f(2a) + a$ for each $a \in H$. Since h is G-definable, g dominates h . Actually, $\forall x \in N \exists x < y \in N$ s.t. $g(y) > h(y)$. Therefore we can define g' to be

$$g'(x) = \min\{g(y) : x < y \wedge g(y) > h(y)\}$$

Since h is non-decreasing, g' dominates h. Note that for each large enough $M \in H$ the area between the graphs of f and g in $H \times H$ contains the square whose vertices are

$$\langle M, f(2M)\rangle, \langle M, f(2M) + M\rangle, \langle 2M, f(2M)\rangle, \langle 2M, f(2M) + M\rangle$$

By Theorem 2.10, X is weak generic in $(H, +) \times (H, +)$. As a result, the type p_1 is weak generic in $(H, +) \times (H, +)$

$3 \Rightarrow 1$: Take any definable $f : G \to G$. We shall show that both p_f^+ and p_f^- are stationary weak generic types

By the o-minimality of G, f is ultimately non-negative or ultimately non-positive. It is easy to see that p_f^+ is stationary iff p_{-f}^- is stationary and p_f^- is stationary iff p_{-f}^+ is stationary. Therefore, W.L.O.G, we can assume that f is ultimately non-negative. Moreover, f is ultimately non-increasing or ultimately non-decreasing. If f is ultimately non-increasing, then $p_f^+ = p_z^+$ and $p_f^- = p_z^-$ where $z : G \to G$ is constantly equal to 0. So we can assume that f is ultimately non-decreasing (this includes the constant case)

Consider definable sets:

$$A = \{a \in G : (\exists b > a)(\forall c \in (a,b))f(c) - f(a) \leq c - a\}$$
$$B = \{a \in G : (\exists b > a)(\forall c \in (a,b))f(c) - f(a) > c - a\}$$

Note that by the o-minimality of G, we have that $G = A \cup B$ and for some $M \in G$ either $(M, +\infty) \subseteq A$ or $(M, +\infty) \subseteq B$. Enlarge M in order to ensure that f is continuous on $(M, +\infty)$

Case 1: $(M, +\infty) \subseteq A$. Then f grows "slowly" on $(M, +\infty)$:

$$(\forall a > M)(\exists b > 0)(\forall c \in (0,b))f(a+c) \leq f(a) + c \qquad (\star)$$

By $(\star)$ and the continuity of f

$$(\forall a > M)(\forall c > 0)f(a+c) \leq f(a) + c \qquad (\star\star)$$

Because if not, then the opposite holds: $(\exists a > M)(\exists c > 0)f(a+c) > f(a)+c$. Let $C = \{c > 0 : f(a+c) > f(a) + c\}$ and $c_0 = \inf(C)$. Assertion $(\star)$ implies that $c_0 > 0$. Since f is continuous at c_0, $c_0 \notin C$. Choose $d > c_0$ s.t. $(c_0, d) \subseteq C$. Since $c_0 \notin C$, $f(a + c_0) \leq f(a) + c_0$. On the other hand, by the continuity of f at $a + c_0$, we have that $f(a + c_0) \geq f(a) + c_0$. Thus $f(a + c_0) = f(a) + c_0$ and for every $e \in (0, d - c_0)$ we have that

$$f(a + c_0 + e) > f(a) + c_0 + e = f(a + c_0) + e$$

which implies that $a + c_0 \notin A$. But $a + c_0 \in (M, +\infty) \subseteq A$, a contradiction. So ($\star\star$ holds)

For the sake of contradiction assume that p_f^+ is not stationary. Then for some $H \succ G$ and H-definable $g : H \to H$ we have that $f \ll g$ and $g \ll h$ for each G-definable $h : H \to H$ with $f \ll h$. For any $q \supseteq p_f^+$, $\{x > a : a \in H\} \subseteq q$. Since $\lim_{x \to +\infty}(g(x) - f(x)) = +\infty$, there exists an increasing to $+\infty$ G-definable function $h : H \to H$ s.t. ultimately $h \leq g - f$ by 2.15. Enlarging M we can assume that h is increasing on $(M, \ +\infty)$.

Now fix any positive $R \in H$ and find $a > M$ with $h(a) \geq 2R$. By $(\star\star)$, we have that $f(a+R) \leq f(a) + R$. So the area between the graphs of f and $f + h$ contains the square whose vertices are

$$\langle a, f(a) + R\rangle, \langle a, f(a) + 2R\rangle, \langle a + R, f(a) + R\rangle, \langle a + R, f(a) + 2R\rangle$$

As R was arbitrary, we can use Theorem 2.10 to conclude that the area between the graphs of f and $f + h$ is weak generic in $(H, +) \times (H, +)$. So $f \ll f + h$ and therefore $g \ll f + h$, which contradicts the fact that ultimately $g \geq f + h$. So the type p_f^+ is stationary

Case 2: $(M, +\infty) \subseteq B$. Then f grows "quickly" on $(M, +\infty)$, which implies that $\lim_{x \to +\infty} f(x) = +\infty$. As in 2.15 find a definable bijection $f_1 : G \to G$ s.t. $f_1(af = f(a))$ for each $a \in (M, +\infty)$. If $g = f_1^{-1}$, then g grows "slowly" on $(M, +\infty)$ and from the previous case we know that the types p_g^+ and p_g^- are stationary. The proof is complete since $p_f^+(x, y) = p_{f_1}^+(x, y) = p_g^-(y, x)$ and $p_f^-(x, y) = p_{f_1}^-(x, y) = p_g^+(x, y)$ □

Example 2.2. If $(G, <, +)$ is an o-minimal ordered group, then every definable function $f : G \to G$ is ultimately equal to $f_q(x) + a$ for some $a \in G$ and $q \in \mathbb{Q}$ where $f_q(x) = q \cdot x$ ([**?**], Corollary 1.7.6) by considering G as a $\mathbb{Q}$-vector space. Below we list all weak generic types in $(G, +) \times (G, +)$ that are complete over G and contain the formula $x > 0$

1. $p_{-\infty}(x, y)$ and $p_{+\infty}(x, y)$
2. $p_{f_q}^-(x, y)$ and $p_{f_q}^+(x, y)$, $q \in \mathbb{Q}$
3. $\{x > a : a \in G\} \cup \{y > q \cdot x : q \in \mathbb{Q} \wedge q < r\} \cup \{y < q \cdot x : q \in \mathbb{Q} \wedge q > r\}$, $r \in \mathbb{R} \setminus \mathbb{Q}$

The structure $(G, <, +)$ is stationary since its elementary extensions are all linearly bounded. Thus by Theorem 2.18, weak generic types of the form (1) and (2) are stationary.

2.5 Expansions of real closed fields

In this section, $(R, <, +, \cdot, 0, 1, \dots)$ is an o-minimal expansion of an ordered ring $(R, <, +, \cdot, -, 1)$. Such a ring must be a real closed field. Since $(R, <, +, \cdot, 0, 1, \dots)$ is an o-minimal expansion of the ordered group $(R, <, +)$, all results obtained in the previous section apply

Definition 2.19. We call a structure $(R, <, +, \cdot, \dots)$ **polynomially bounded** if for every definable function $f : R \to R$ there is $n \in \mathbb{N}^+$ s.t. $|f(x)| \leq x^n$ for all sufficiently large $x \in R$

Remark. If a real closed field $(R, <, +, \cdot, \dots)$ is polynomially bounded and o-minimal, then for every definable $f : R \to R$ with $\lim_{x \to +\infty} f(x) = +\infty$ we can find $n \in \mathbb{N}_+$ s.t. $f(x) \geq \sqrt[n]{x}$ for all sufficiently large $x \in R$

Proof. We proceed as in the proof of 2.15. Since f is ultimately increasing, we are able to find a definable bijection $g : R \to R$ s.t. $f(x) = g(x)$ for all sufficiently large $x \in R$. We know that the inverse map g^{-1} is ultimately dominated by the polynomial function $x \mapsto x^n$ for some $n \in \mathbb{N}_+$. And this implies $f(x) = g(x) \geq \sqrt[n]{x}$ for sufficiently large x $\square$

Assume $(R, <, +, \cdot)$ is a pure real closed field. Since every definable map $f : R \to R$ is semi-algebraic, it follows from Proposition 2.6.1 in [**?**] that the structure $(R, <, +, \cdot)$ is polynomially bounded

Corollary 2.20. *Every pure real closed field $(R, <, +, \cdot)$ is stationary and so are the weak generic types $p_f^-(x, y)$ and $p_f^+(x, y)$ for each definable $f : R \to R$*

Proof. Consider an arbitrary elementary extension S of R and any definable map $f : S \to S$. Since the real closed field $(S, <, +, \cdot)$ is polynomially bounded, there exists $n \in \mathbb{N}_+$ s.t. ultimately $|f(x)| \leq x^n$. This gives us the stationary of the structure $(R, <, +, \cdot)$ $\square$

Definition 2.21. Assume $(R, +, \cdot, 0, 1)$ is a field, $f, g : R \to R$ and $g(x) \neq 0$ for all sufficiently large $x \in R$. We write $f \approx g$ iff

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$$

Lemma 2.22. *Assume $(R, <, +, \cdot)$ is a pure real closed field. If a function $f : R \to R$ is definable and ultimately non-zero, then for some $q \in \mathbb{Q}$ and $c \in R \setminus \{0\}$ we have that $f(x) \approx c \cdot x^q$*

Proof. Let S be an arbitrary $|R|^+$-saturated elementary extension of R. We can find $a \in S$ s.t. $a > r$ for every $r \in R$. Let

$$T = \{s \in S : |s| < r \text{ for some } r \in R\}$$

Then T is a convex subring of S,

$$T^* = \{s \in S : \frac{1}{r} < |s| < r \text{ for some } r \in R\}$$

and $(T^*, \cdot)$ is a subgroup of the multiplicative group $(S^*, \cdot)$. This definition is stronger than $|s| > 0$ since there may be infinitesimal The quotient group $(S^*/T^*, *, 1)$ may be ordered in the following way:

$$s_1/T^* \leq s_2/T^* \Leftrightarrow \frac{s_1}{s_2} \in T$$

We define a function $\nu : S \to S^*/T^* \cup \{-\infty\}$ (where for every $s \in S^*$, $-\infty < s/T^*$ and $(-\infty) * s/T^* = -\infty$) as follows:

$$\nu(s) = \begin{cases} -\infty & s = 0 \\ s/T^* & \text{otherwise} \end{cases}$$

Then ν is a valuation of the field S, i.e., $\forall x, y \in S$,

1. $\nu(x \cdot y) = \nu(x) * \nu(y)$
2. $\nu(x + y) \geq \min(\nu(x), \nu(y))$
3. $\nu(x) \neq \nu(y) \Rightarrow \nu(x + y) = \min(\nu(x), \nu(y))$
1. $\nu(x \cdot y) = \nu(x) * \nu(y)$
2. $\nu(x + y) \leq \max(\nu(x), \nu(y))$
3. $\nu(x) \neq \nu(y) \Rightarrow \nu(x + y) = \max(\nu(x), \nu(y))$

Since f is semi-algebraic, by Lemma 2.5.2 in [**?**], there exists a non-zero polynomial $P(X, Y) \in R[X, Y]$ s.t. $R \vDash \forall x(P(x, f(x)) = 0)$. So $S \vDash \forall x(P(x, f(x)) = 0)$ and, in particular, $P(a, f(a)) = 0$. The polynomial $P(X, Y)$ is of the form

$$P(X, Y) = \sum_{i=1}^{n} r_i \cdot X^{k_i} \cdot Y^{l_i}$$

for some $n \in \mathbb{N}_+$, $r_i \in R \setminus \{0\}$ and $k_i, l_i < \omega$ s.t. $\langle k_i, l_i \rangle \neq \langle k_j, l_j \rangle$ for every $i \neq j \in \{1, \dots, n\}$. Thus

$$0 = \sum_{i=1}^{n} r_i \cdot a^{k_i} \cdot f(a)^{l_i}$$

and for some $i \neq j \in \{1, \dots, n\}$ we have that

$$\nu(r_i \cdot a^{k_i} \cdot f(a)^{l_i}) = \nu(r_j \cdot a^{k_j} \cdot f(a)^{l_j}) \neq -\infty$$

since $f(a) \neq 0$ (if $f(a) = 0$, then $f : R \to R$ would be ultimately equal to 0)

This implies that $\nu(\frac{r_i}{f_j} \cdot a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = 1$ and $\nu(a^{k_i - k_j} \cdot f(a)^{l_i - l_j}) = 1$. So $a^{k_i - k_j} \cdot f(a)^{l_i - l_j} \in T^*$. If $l_i = l_j$, then $k_i \neq k_j$ and $a^{k_i - k_j} \in T^*$, which implies that $a \in T^*$ ($(T^*, \cdot, 1)$ is divisible), a contradiction.

So $l_i \neq l_j$. Let $q = -\frac{k_i - k_j}{l_i - l_j} \in \mathbb{Q}$ we obtain $\frac{f(a)}{a^q} \in T^*$. Therefore $\frac{1}{r} < \left|\frac{f(a)}{a^q}\right| < r$ for some $r \in R$. If $b \in S$ and $b > a$, then $\text{tp}(a/R) = \text{tp}(b/R)$. Hence for every $b > a$ we have that $\frac{1}{r} < \left|\frac{f(b)}{b^q}\right| < r$ and consequently

$$S \vDash \exists y \forall x (x > y \to \frac{1}{r} < \left|\frac{f(x)}{x^q}\right| < r)$$

As $R \prec S$, this implies that $\frac{1}{r} < \left|\frac{f(x)}{x^q}\right| < r$ for all sufficiently large $x \in R$. By the o-minimality of R, for some $c \in R$ with $\frac{1}{r} \leq |c| \leq r$ we have that $\lim_{x \to +\infty} \frac{f(x)}{x^q} = c$, which finishes the proof $\square$

Theorem 2.23. *Assume* $(R, <, +, \cdot)$ *is a pure real closed field. Let*

$$f(x) = \sum_{i=1}^{m} a_i \cdot x^{p_i} \quad \textit{and} \quad g(x) = \sum_{j=1}^{n} b_j \cdot x^{q_j}$$

where $m, n \in \mathbb{N}_+$, $a_1, \dots, a_m, b_1, \dots, b_n \in R$, $a_1, b_1 > 0$, $p_1 > \dots > p_m \in \mathbb{Q}$ *and* $q_1 > \dots > q_n \in \mathbb{Q}$. *TFAE*

1. $f \ll f + g$
2. $q_1 > \max(0, p_1 - 1)$

Proof. We define a rate of growth $gr(f)$ of a definable map $f : R \to R$ as follows: if $f(x) \approx c \cdot x^q$ for some $c \in R \setminus \{0\}$ and $q \in \mathbb{Q}$, then $gr(f) = q$ (Lemma 2.22 implies that $gr(f)$ is well defined for each ultimately non-zero definable function $f : R \to R$) $\square$

3 Problems

2.1 2.4 2.4 :done 2.4 2.4