Essential Stability Theory

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1 Preliminaries and Notations

Exercise 1.0.1. Let $\mathcal{M}$ be a finite model in a language L. Show that

$$\mathcal{N} \equiv \mathcal{M} \Rightarrow \mathcal{N} \cong \mathcal{M}$$

2 Constructing Models with Special Properties

2.1 Prime and Atomic Models

Proposition 2.1. *T complete countable theory*

1. *A countable $\mathcal{M} \models T$ is prime iff $\mathcal{M}$ is atomic*
2. *If $\mathcal{M}$ and $\mathcal{N}$ are both countable atomic models of T, then $\mathcal{M} \cong \mathcal{N}$*

Then our question is: **does every complete theory have a prime model, or can we find a meaningful characterization of those which do?**

Example 2.1 (A countable complete theory with no atomic model)**.** Let $L = \{P_i : i < \omega\}$ where each P_i is a unary relation symbol. Let $X = 2^{<\omega}$. The theory T is defined so that for any model $\mathcal{M} \models T$ and $s \in X$, the intersection of the family of sets $\{P_i(\mathcal{M}) : s(i) = 0\} \cup \{M \setminus P_i(\mathcal{M}) : s(i) = 1\}$ is nonempty. Let $P_i^0(v)$ denote the formula $P_i(v)$, and $P_i^1(v)$ the formula $\neg P_i(v)$.

For $s \in X$, let $\varphi_s(v) := \bigwedge_{i<lh(s)} P_i^{s(i)}(v)$ where lh is the length function, $\sigma_s := \exists v \varphi_s(v)$ and $T = \{\sigma_s : s \in X\}$. T is a complete quantifier-eliminable theory.

Thus, if $\mathcal{M} \models T$ and $a \in M$, tp(a) is implied by $\{P_i^j(v) : \mathcal{M} \models P_i^j(a), i < \omega, j = 0, 1\}$. We claim that every complete 1-type in T is nonisolated. If, to the contrary, p is an isolated 1-type, then by the characterization of types just mentioned p is isolated by some $\varphi_s \in p$. However, if $j = lh(s)$, both $\exists v(\varphi_s(v) \wedge P_j(v))$ and $\exists v(\varphi_s(v) \wedge \neg P_j(v))$ are in T, proving that φ_s does not isolate a complete type in T. Since T has no isolated 1-types over $\emptyset$, no model of T can be atomic

Proposition 2.2. *Let T be a countable complete theory. Then T has a countable atomic model iff the isolated types of T are dense*

Lemma 2.3. *If T is a complete theory with $|S(\emptyset)| < 2^{\aleph_0}$ then the isolated types of T are dense*

Thus, for a countable complete theory, having fewer than continuum many complete types is sufficient to guarantee the existence of a prime model.

But this condition is not necessary. Consider Th$(\mathbb{Z}, +, 1)$. $|S_1(\emptyset)| = 2^{\aleph_0}$. However, since every element of the model $(\mathbb{Z}, +, 1)$ interprets a term of the language, it is an elementary submodel of any model of T

Remark. An algebraic formula is contained in only finitely many complete types in T, each of which is isolated

Proof. φ algebraic and $\varphi \in p$. Then $p(\mathcal{M})$ is finite
$p = q \Leftrightarrow p(\mathcal{M}) = q(\mathcal{M})$ □

If $\mathcal{M}$ is a model and $A \subset M$, $\mathcal{M}$ is called a **prime model over** A if $\mathcal{M}_A$ is a prime model over Th$(\mathcal{M}_A)$. Note that $\mathcal{N} \models$ Th$(\mathcal{M}_A)$ iff $\mathcal{N} \equiv \mathcal{M}$ and there is a elementary map $f : A \to \mathcal{N}$

Exercise 2.1.1. Let T be a complete theory and φ a formula in n variables which is contained in only finitely many complete n-types of T. Show that every complete n-types containing φ is isolated

Proof. If there are $p_1, \dots, p_n$, then there is $\phi_1, \dots, \phi_n$ s.t. for any $\varphi \in q$, $\phi_i \in q \Leftrightarrow q = p_i$. Thus $[\phi_i] = \{p_i\}$. Thus for any φ, either $[\phi_i \wedge \varphi]$ or $[\phi_i \vee \neg\varphi]$ is empty. Hence ϕ_i is complete □

Exercise 2.1.2. Suppose $\bar{a}$ and $\bar{b}$ are sequences from a model $\mathcal{M}$ which have the same complete types in $\mathcal{M}$ and $\varphi(v, \bar{a})$ isolates a complete type over $\bar{a}$. Show that $\varphi(v, \bar{b})$ isolates a complete type over $\bar{b}$

Proof. If $\varphi(v, \bar{a})$ isolates $p(v) = \{\varphi(v, \bar{a})\}$. First, $q(v) = \{\varphi(v, \bar{b})\}$ is a complete type.

Then $\varphi(v, \bar{b})$ isolates it □

Exercise 2.1.3. Suppose that $\bar{a}$ and $\bar{b}$ be finite sequences from the universe of the model $\mathcal{M}$. Prove that $\text{tp}_{\mathcal{M}}(\bar{a}\bar{b})$ is isolated iff $\text{tp}_{\mathcal{M}}(\bar{a}/\bar{b})$ and $\text{tp}(\bar{b})$ are both isolated. Using this fact show that when $\mathcal{M}$ is an atomic model and $\bar{a}$ is a finite sequence from M, then $\mathcal{M}$ is atomic over $\bar{a}$. Conversely, if $\mathcal{M}$ is atomic over $\bar{a}$ and $\text{tp}_{\mathcal{M}}(\bar{a})$ is isolated, then $\mathcal{M}$ is atomic

Proof. If $\varphi(\bar{x}, \bar{y})$ isolates $\text{tp}_{\mathcal{M}}(\bar{a}\bar{b})$, then $\varphi(\bar{x}, \bar{b})$ isolates $\text{tp}_{\mathcal{M}}(\bar{a}/\bar{b})$ and $\exists\bar{x}\varphi(\bar{x}, \bar{y})$ isolates $\text{tp}_{\mathcal{M}}(\bar{b})$

If $\varphi(\bar{x}, \bar{b})$ isolates $\text{tp}_{\mathcal{M}}(\bar{a}/\bar{b})$ and $\psi(\bar{y})$ isolates $\text{tp}(\bar{b})$. Then $\psi(\bar{y}) \wedge \varphi(\bar{x}, \bar{y})$ isolates $\text{tp}(\bar{a}\bar{b})$.

For any $\theta(\bar{x}, \bar{y}) \in \text{tp}(\bar{a}\bar{b})$. $\mathcal{M} \vDash \forall\bar{x}(\varphi(\bar{x}, \bar{b}) \to \theta(\bar{x}, \bar{b}))$. Hence $\mathcal{M} \vDash \forall\bar{y}(\psi(\bar{y}) \to \forall\bar{x}(\varphi(\bar{x}, \bar{y}) \to \theta(\bar{x}, \bar{y})))$ □

Exercise 2.1.4. Show that the complete type realized by 1 in $(\mathbb{Z}, +)$ is non-isolated

Proof. $\text{tp}(1/2)$ is isolated by $x + x = 2$. □

Exercise 2.1.5. Show that $\text{Th}(\mathbb{Z}, +e)$ has continuum many complete 1-types over $\emptyset$

Exercise 2.1.6. Given an example of a model $\mathcal{M}$ containing an element a which is the only realization of $\text{tp}_{\mathcal{M}}(a)$ in $\mathcal{M}$, although this type is not even isolated

Proof. Not isolated means there is no minimum element under $\subseteq$ in $\{\varphi(\mathcal{M}) : \varphi \in \text{tp}(a)\}$ □

Exercise 2.1.7. Let $\mathcal{M}$ be a model s.t. the type in $\mathcal{M}$ of each tuple from M is algebraic. Prove that $\mathcal{M}$ is a prime and minimal model of its theory

Proof. □

2.2 Saturated and Homogeneous Models

Proposition 2.4. *A countable complete theory T has a saturated countable model iff it is small*

Let T be a countable complete theory. We proved that T has a countable atomic model when $|S(\emptyset)| < 2^{\aleph_0}$ and T has a countable saturated model when when $S(\emptyset)$ is countable. It is natural to ask **if there is a countable complete theory with** $|S(\emptyset)|$ **strictly between** $\aleph_0$ **and** $2^{\aleph_0}$

The Cantor-Bendixson Theorem from point-set topology quickly gives a negative answer: $S_n(\emptyset)$ is strictly between $\aleph_0$ and $2^{\aleph_0}$

First, we prove Cantor-Bendixson theorem first from here

Definition 2.5. $a \in X$ is **isolaed in** X iff $\{a\}$ is open. Otherwise a is a limit point

Definition 2.6. X is a **perfect set** iff X is closed and has no isolated points

Cantor set is perfect since each point of it is a limit point

Lemma 2.7. *If P is a perfect set and I is an open interval on $\mathbb{R}$ s.t. $I \cap P \neq \emptyset$, then there exist disjoint closed intervals $J_0, J_1 \subset I$ s.t. $int(J_0) \cap P \neq \emptyset$ and $int(J_1) \cap P \neq \emptyset$. Moreover, we can pick J_0 and J_1 s.t. their lengths are both less than any $\epsilon > 0$*

Proof. Since P has no isolated points, there must be at least two points $a_0, a_1 \in I \cap P$. Then pick $J_0 \ni a_0$ and $J_1 \ni a_1$ to be small enough □

Lemma 2.8. *If P is a nonempty perfect set, then $|P| = \mathfrak{c}$*

Proof. We exhibit a one-to-one mapping $G : 2^\omega \to P$

We build a binary tree. For each $s \in 2^{<\omega}$, we associate an interval I_s s.t.

- I_s is closed
- $I_s \cap P \neq \emptyset$
- $I_{s,b} \subset I_s$
- $I_{s,0} \cap I_{s,1} = \emptyset$
- $|I_s| < 1/(|s| + 1)$

where $|I|$ denotes the length of interval I and $|s|$ denotes the length of sequence s

Let $\langle\rangle$ denotes the emptyset sequence, let $I_{\langle\rangle}$ be the closure of $I \cap P$ for some open interval I with length at most 1 whose intersection with P is nonempty. Then by 2.7 choose appropriate $I_{s,0}$ and $I_{s,1}$

Now for all $f \in 2^{\omega}$, define

$$G(f) = \bigcap_{i \in \omega} I_{f \upharpoonright i}$$

If we pick elements from each $I_{f \upharpoonright i}$, then $G(f)$ is their limit, which is contained in P since P is closed

Suppose $f, f' \in 2^{\omega}$ and $f \neq f'$. Let $n \in \omega$ be the smallest index s.t. $f(n) \neq f'(n)$. Then $I_{f \upharpoonright n} \cap I_{f' \upharpoonright n} = \emptyset$ □

Theorem 2.9 (Cantor-Bendixson). *If $C \subseteq \mathbb{R}$ is closed and uncountable, then there exists some perfect, nonempty $P \subseteq C$.*

Proof. Let $C \subseteq \mathbb{R}$ be closed. Define the **Cantor-Bendixson derivative**

$$C' = \{a \in C \mid a \text{ is a limit point of } C\}$$

This operation maps closed sets to closed sets, since closed sets in $\mathbb{R}$ are those which contain all their limit points, and the derivative is monotone and retains all limit points. Then define

$$\begin{aligned} C_0 &= C \\ C_{\alpha+1} &= (C_\alpha)' \\ C'_\lambda &= \bigcap_{\beta < \lambda} C_\beta \end{aligned}$$

Note that C_β is closed for all β by induction

Claim: $C_\gamma = C_{\gamma+1}$ for some γ. For if not, $C_\alpha \neq C_\beta$ for any $\alpha \neq \beta$, since C is monotone, then $C\_$ would be an injection $Ord \to \mathcal{P}(C)$, which is absurd

Note that C_γ is perfect, since it consists solely of limit points and is closed. If $C_\gamma \neq \emptyset$, we are done

We claim that C_γ cannot be $\emptyset$ since this would imply that C is countable. Consider $C_\beta - C_{\beta+1}$, which contains all the isolated points in C_β. That is, if $a \in C_\beta - C_{\beta+1}$, there exists an open interval $I_a \ni a$ s.t. $C_\beta \cap I_a = \{a\}$. In particular, we may choose I_a to be an open interval with rational endpoints

Note that I_a is distinct; otherwise, at the earliest stage when I_a arose, it would have contained more than one point. Therefore we have an injection from C into the set of intervals with rational endpoints, which is countable

□

Remark. The above proof shows that every closed set can be decomposed into a perfect subset and a countable subset.

Definition 2.10. The smallest γ in the above proof for which $C_\gamma = C_{\gamma+1}$ is called the **Cantor-Bendixson rank** of C, and the above proofs shows that $\gamma < \aleph_1$

It can be shown that for every $\gamma < \aleph_1$, there exists a closed $C \subseteq \mathbb{R}$ with Cantor-Bendixson rank γ

Lemma 2.11. *There are $2^{\aleph_0}$ perfect sets*

Proof. There is an injection from $\mathcal{P}(\mathbb{N})$ to the set of all perfect sets: for each set of naturals, identify each natural with a small closed interval containing it, and take the union. There are at most $2^{\aleph_0}$ perfect sets since there are $2^{\aleph_0}$ closed sets □

Theorem 2.12. *There exists a set X with $|X| = 2^{\aleph_0} = |\mathbb{R} - X|$ s.t. for every perfect set P, $P \nsubseteq X$ and $P \nsubseteq \mathbb{R} - X$*

Proof. Let $(P_\alpha : \alpha < 2^{\aleph_0})$ be an enumeration of the perfects sets. Also let x_α be an enumeration of $\mathbb{R}$. Now define r_γ □

Now we come back to book

Definition 2.13. Let T be a complete theory. φ a formula in n variables

1. $CB(\varphi) = -1$ if φ is inconsistent
2. Let $\Psi_\alpha = \{\psi : CB(\psi) = \beta < \alpha\}$
   $CB(\varphi) = \alpha$ if $\{p \in S_n(\emptyset) : \varphi \in p \wedge \forall \psi \in \Psi_\alpha(\neg\psi \in p)\}$ is nonempty and finite

For $p \in S_n(T)$, $CB(p)$ is

$$\inf\{CB(\varphi) : p \vDash \varphi\}$$

When $CB(p) = \alpha$ we say that the **Cantor-Bendixson rank** of p is α. If there is no such α, $CB(p) = \infty$ and say that the Cantor-Bendixson rank of p does not exist