Homework2

Qi'ao Chen
21210160025

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Exercise 1. Given two chains $(C, \leq)$ and $(D, \leq)$, define the lexicographic product of C and D to be the chain defined on the Cartesian product of their universes s.t. $(a, b) < (c, d)$ in the sense of $C \times D$ if $b < d$ in the sense of D, or $b = d$and $a < c$ in the sense of C

1. Show that a discrete chains without endpoints are those that be written in the form $\mathbb{Z} \times C$, where C is a linear order
2. Show that if $C \sim_\omega C'$ and $D \sim_\omega D'$ then so is $C \times D \sim_\omega C' \times D'$

Proof. 1. Given a discrete chains without endpoints $(D, <)$, for any $a, b \in D$, let $a \sim b$ if $d(a, b) \neq \infty$. Suppose $a < b$ and $d(a, b) = \infty$, then for any $a' \in [a]$, as $d(a, a')$ is finite, $d(a', b) = \infty$. Hence $a' < b$. Similarly, for any $b' \in [b]$, $a < b'$. Thus we define $[a] \prec [b]$ if $a < b$. As $<$ is a strict linear order, $\prec$ is also a strict linear order. We choose an element from each equivalence class, denoted by b_i for $i \in I$. For each $a \in [b_i]$, define $f : D \to \mathbb{Z}$ as

$$f(a) = \begin{cases} d(a, b_i) & \text{if } b_i < a \\ -d(a, b_i) & \text{otherwise} \end{cases}$$

and we have an isomorphism $f : [a] \cong \mathbb{Z}$ for any $a \in D$. Then we define the $\phi : D \to \mathbb{Z} \times D/\sim$ with $a \mapsto (f(a), [a])$. ϕ is an isomorphism as f is an isomorphism and $-/\sim : D \to D/\sim$ is surjective.

Given a lexicographic product $(\mathbb{Z} \times C, <)$ where C is a linear order, for any $(n, c) \in \mathbb{Z} \times C$, we have $(n-1, c)$ and $(n+1, c)$ such that $(n-1, c) < (n, c) < (n+1, c)$. Also there is no $(m, c) \in \mathbb{Z} \times C$ such that $(n-1, c) < (m, c) < (n, c)$ or $(n, c) < (m, c) < (n+1, c)$. Hence $(\mathbb{Z} \times C, <)$ is a discrete order.

2. If $C \sim_\omega C'$ and $D \sim_\omega D'$, then $(C', \leq)$ and $(D', \leq)$ are also chains (here we abuse the symbols of relations on C' and D').

   We prove $C \times D \sim_\omega C' \times D'$ by induction on p to show $S_p(C \times D, C' \times D')$ is not empty.

   For $p = 0$, $S_0(C \times D, C' \times D') \neq \emptyset$ as $\emptyset \in S_0(C \times D, C' \times D')$.

   For $p = n+1$, as $S_n(C \times D, C' \times D') \neq \emptyset$, we can build a local isomorphism $s = \{((c_i, d_i), (c'_i, d'_i)) | 0 < i \leq n\}$ where i respect the order of $C \times D$.

   Now for any $(c, d) \in C \times D$

   - if $(c, d) > (c_n, d_n)$, then either $d > d_n$ or $d = d_n$ and $c > c_n$. In both cases, as $C \sim_\omega C'$ and $D \sim_\omega D'$, we can find $(c', d') \in C' \times D'$ such that either $d' > d'_n$ or $d' = d'_n$ and $c' > c'_n$. Hence $(c', d') > (c'_n, d'_n)$
   - if $(c_i, d_i) < (c, d) < (c_{i+1}, d_{i+1})$, similarly we can find $(c', d') \in C' \times D'$ such that $(c'_i, d'_i) < (c', d') < (c_{i+1}, d_{i+1})$
   - if $(c, d) < (c_1, d_1)$, similarly we can find $(c', d') \in C' \times D'$ with $(c', d') < (c'_1, d'_1)$

Hence $t = s \cup \{((c, d), (c', d'))\}$ is a local isomorphism. Backward condition is similar and thus $\emptyset \in S_n(C \times D, C' \times D')$.

Consequently, $C \times D \sim_\omega C' \times D'$ □

Exercise 2. Given two chains $(C, \leq)$ and $(D, \leq)$, by $C + D$ we mean the chain

$$C \times \{0\} \cup D \times \{1\}$$

s.t. $C \times \{0\}$ is a copy of C, $D \times \{1\}$ is a copy of D, and each element of $C \times \{0\}$ is smaller than each element of $D \times \{1\}$

1. Show that the linear orders $\mathbb{R}$ and $\mathbb{R} + \mathbb{Q}$ are not isomorphic

2. Construct two discrete linear orders such that they are ∞-equivalent but not isomorphic

Proof. 1. Suppose there is an isomorphism $f : \mathbb{R} \to \mathbb{R} + \mathbb{Q}$. Then

$$\mathbb{R} \cong [(f^{-1}(0,1), 0), (f^{-1}(1,1), 0)] \cong [(0,1), (1,1)] \cong \mathbb{Q}$$

that is impossible.

2. $\mathbb{R}$ and $\mathbb{R}+\mathbb{Q}$ are ∞-equivalent. We prove $\mathbb{R} \sim_\infty \mathbb{R}+\mathbb{Q}$ by induction on α.

   Clearly $\mathbb{R} \sim_0 \mathbb{R}+\mathbb{Q}$.

   If $\alpha = \beta + 1$ and $\mathbb{R} \sim_\beta \mathbb{R}+\mathbb{Q}$, then $\emptyset \in S_\beta(\mathbb{R}, \mathbb{R}+\mathbb{Q})$ and we get a local isomorphism $s = \{(a_i, b_i)_{0 \le i < \beta}\}$ and for each ordinal $\gamma < \lambda < \beta$, $a_\gamma < a_\lambda$ and $b_\gamma < b_\lambda$.

   - if $a < a_0$, then we choose $b = (\inf(\text{dom}\, s) - 1, 0)$
   - if $a_\gamma < a < a_\lambda$ and there is no a_i such that $a_\gamma < a_i < a_\lambda$ for all $0 \le i < \beta$, then we choose a b such that $b_\gamma < b < b_\lambda$ as $\mathbb{R}$ and $\mathbb{Q}$ are both dense
   - if $a > \sup(\text{dom}\, s)$, then we choose $b = (\sup(\text{im}\, s) + 1, 1)$

Let $t = s \cup \{(a, b)\}$ and we get a new local isomorphism preserving all order relations. And the backward case is similar. Hence $\mathbb{R} \sim_\alpha \mathbb{R}+\mathbb{Q}$

If α is a limit ordinal, then clearly $\mathbb{R} \sim_\alpha \mathbb{R}+\mathbb{Q}$.

Hence $\mathbb{R} \sim_\infty \mathbb{R}+\mathbb{Q}$. □

Exercise 3. 1. Show that $\mathbb{Z}+\mathbb{Z}$ and $\mathbb{Z}$ are $\omega+1$-equivalent but not $\omega+2$-equivalent

2. Construct two discrete linear orders such that they are $\omega+n$-equivalent but not $\omega+n+1$-equivalent for each $n \in \mathbb{N}$

Proof. 1. We first prove that $\mathbb{Z}+\mathbb{Z} \sim_{\omega+1} \mathbb{Z}$

   For any $a \in \mathbb{Z}$, we choose $(a, 0) \in \mathbb{Z}+\mathbb{Z}$. Now we prove that $s = \{(a, (a, 0))\}$ is an ω-isomorphism, that is, for any $p \in \omega$, s is a p-isomorphism. But by Theorem 1.8, s is indeed a p-isomorphism. Hence s is an ω-isomorphism and forward condition is satisfied.

   Then backward is similar and $\mathbb{Z}+\mathbb{Z} \sim_{\omega+1} \mathbb{Z}$.

   But for $\omega+2$, if we choose $a, b \in \mathbb{Z}+\mathbb{Z}$ as $d(a, b) = \infty$, and suppose $s = \{(a, c), (b, d)\}$. To show s is an ω-isomorphism, by Theorem 1.8, for any $p \in \omega$, $d(c, d)$ should be greater than or equal to $2^p - 1$, which is impossible in $\mathbb{Z}$. Thus $\mathbb{Z}+\mathbb{Z} \nsim_{\omega+2} \mathbb{Z}$

2. We claim that

$$\sum_{i=1}^{2^n-1} \mathbb{Z} \sim_{\omega+n} \sum_{i=1}^{2^n} \mathbb{Z} \quad \text{and} \quad \sum_{i=1}^{2^n-1} \mathbb{Z} \nsim_{\omega+n+1} \sum_{i=1}^{2^n} \mathbb{Z}$$

   Let $S_n = \{1, 2, \dots, n\}$, first we prove that

If $m, n \geq 2^{p-1}$, then $S_m \sim_p S_n$ for $m, n, p \in \mathbb{N}^+$

Let $C = C' = \mathbb{Z}$ and $a_1 = b_1 = 0$, $a_2 = m+1$ and $b_2 = n+1$. Let $s = \{(a_1, b_1), (a_2, b_2)\}$, then by Theorem 1.8, s is a p-isomorphism. So s is still a p-isomorphism if we restriction domain and image to $S_m \cup \{0, m+1\}$ and $S_n \cup \{0, n+1\}$ respectively. Thus $S_m \sim_p S_n$.

So $S_{2^p-1} \sim_p S_{2^p}$ and $S_{2^p-1} \nsim_{p+1} S_{2^p}$.

First note that $\sum_{i=1}^{n} \mathbb{Z} \cong \mathbb{Z} \times S_n$. So we present a winning strategy for Duplicator in $\mathrm{EF}_{\omega+n}(\mathbb{Z} \times S_{2^n-1}, \mathbb{Z} \times S_{2^n})$. Suppose Spoiler and Duplicator have already chosen $\{((a_1, p_1), (b_1, q_1)), \dots, ((a_r, p_r), (b_r, q_r))\}$ in round r, let $C_{2^n-1}(r) = \{p_1, \dots, p_r\}$ and $C_{2^n}(r) = \{q_1, \dots, q_r\}$. Let $f = \emptyset$.

In first n rounds:

- If Spoiler chooses (a, p) from $\mathbb{Z} \times S_{2^n-1}$ in round r and $p \notin C_{2^n-1}(r-1)$. Then Duplicator chooses a new (a, q) where the choice of $q \in S_{2^n}$ is according to $\mathrm{EF}_n(S_{2^n-1}, S_{2^n})$ and let $f = f \cup \{(p, q)\}$.
- If Spoiler chooses (a, q) from $\mathbb{Z} \times S_{2^n}$ in round r and $p \notin C_{2^n}(r-1)$, then Duplicator chooses (a, p) similarly and let $f = f \cup \{(p, q)\}$.
- If Spoiler chooses (a, p) from $\mathbb{Z} \times S_{2^n-1}$ in round r and $p \in C_{2^n-1}(r-1)$. Then Duplicator chooses $(a, f(p))$.
- If Spoiler chooses (a, q) from $\mathbb{Z} \times S_{2^n-1}$ in round r and $q \in C_{2^n-1}(r-1)$. Then Duplicator chooses $(a, f^{-1}(q))$ as f is injective by $\mathrm{EF}_n(S_{2^n-1}, S_{2^n})$.

Then $s = \{((a_1, p_1), (a_1, q_1)), \dots, ((a_n, p_n), (a_n, q_n))\}$ is an ω-isomorphism. For each $i \in S_{2^n-1}$, $s|_{\mathbb{Z} \times S_i}$ is an ω-isomorphism by Theorem 1.8. Then Duplicator only needs to choose the right integer from S_j according to Spoiler's choice of S_i.

Thus $\sum_{i=1}^{2^n-1} \mathbb{Z} \sim_{\omega+n} \sum_{i=1}^{2^n} \mathbb{Z}$. Also as $S_{2^n-1} \nsim_{n+1} S_{2^n}$, $\sum_{i=1}^{2^n-1} \mathbb{Z} \nsim_{\omega+n+1} \sum_{i=1}^{2^n} \mathbb{Z}$.

$\square$