Homework12

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Exercise 1. Show that if $p \in S_n(T)$ then $S_n(T) \setminus \{p\}$ is open

Proof. $S_n(T) \setminus \{p\} = \bigcup_{\varphi \in p} [\neg\varphi]$ □

Exercise 2. Suppose $X \subseteq S_n(T)$ is open and the complement $S_n(T) \setminus X$ is also open. Show that X is clopen

Proof. Since X and $S_n(T) \setminus X$ are open, $X = \bigcup_{\varphi \in E} [\varphi]$ and $S_n(T) \setminus X = \bigcup_{\psi \in D} [\psi]$ for some sets E, D of formulas. If one of E or D is finite, then X is clopen. Now suppose E and D are infinite, then since $S_n(T) = \bigcup_{\varphi \in E} [\varphi] \cup \bigcup_{\psi \in D} [\psi]$, by Lemma 5, there are finite subsets $E' \subseteq E$ and $D' \subseteq D$ s.t. $S_n(T) = \bigcup_{\varphi \in E'} [\varphi'] \cup \bigcup_{\psi \in D'} [\psi']$. Thus $X = \bigcup_{\varphi \in E'} [\varphi'] = [\bigvee_{\varphi \in E'} \varphi']$ and hence it is clopen. □

Exercise 3. Suppose I is a set and $U_i \subseteq S_n(T)$ is open for each $i \in I$. Suppose $\bigcup_{i \in I} U_i = S_n(T)$. Show that there is a finite set $I_0 \subseteq I$ s.t. $\bigcup_{i \in I_0} U_i = S_n(T)$

Proof. For each $i \in I$, since U_i is open, it is a union of clopen sets, that is, $U_i = \bigcup_{\varphi_i \in E_i} [\varphi_i]$ for some E_i. Hence $S_n(T) = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{\varphi_i \in E_i} [\varphi_i]$. Thus by Lemma 5, there is a finite subset E of $\bigcup_{i \in I} E_i$ s.t. $S_n(T) = \bigcup_{\varphi \in E} [\varphi]$. Since for each $\varphi \in E$, $[\varphi] \subseteq U_i$ for some $i \in I$, there is a finite set $I_0 \subseteq I$ s.t. $\bigcup_{i \in I_0} U_i = S_n(T)$ □

Exercise 4. Let $S_3(DLO)$ be the space of 3-types in DLO. What is the cardinality of $S_3(DLO)$?

Proof. Since DLO has quantifier elimination and has no constant, for variables x, y, z, the basic formulas are of the form

$$o = o, \quad o < o'$$

where o and o' is one of x, y, z. Thus there is 13 kinds of relation between x, y and z:

$$\begin{array}{lll} x = y = z & x = y < z & x < y < z \\ & z < x = y & x < z < y \\ & y < x = z & y < x < z \\ & x = z < y & y < z < x \\ & x < y = z & z < x < y \\ & y = z < x & z < y < x \end{array}$$

And thus $|(S_3(DLO))| = 13$ □

Exercise 5. Let K be an infinite field and $t \in K$ be a non-zero element. Suppose the type-space $S_1(\{t\})$ is finite. Show that there is a positive integer n s.t. $t^n = 1$

Proof. If there is no such n, then for each different $i, j \in \mathbb{N}^+$, $t^i \neq t^j$. Thus for each $i \in \mathbb{N}^+$, $x = t^i$ determines a unique type in $S_1(\{t\})$ and hence $S_1(\{t\})$ is infinite, a contradiction □

Exercise 6. Let T be a complete theory of infinite fields. Show that T is not ω-categorical

Proof. Suppose $M \vDash T$ and T is ω-categorical, then by Ryll-Nardzewski theorem $S_2(T)$ is finite. Then for each $a \in M$, $|S_1(\{a\})| \leq |S_2(T)|$ and hence there is a least positive n_a s.t. $a^{n_a} = 1$.

Let $d = \sup\{n_a : a \in M\}$. If $d = \omega$, then consider

$$\Gamma(x) = \{x^n \neq 1 : n \in \omega\}$$

is finitely satisfiable and hence there is a countable model $N \vDash T$ and a element $t \in N$ s.t. $S_1(\{t\})$ is infinite, a contradiction. Thus $d < \omega$.

But $x^d - 1 = 0$ has only finitely many solutions, contradicting to the infinity of M □