Topological spaces I

Introduction to Model Theory
(Third hour)

September 23, 2021

Section 1

Review

Open sets and interior

Let (M, d) be a metric space.

- An *open ball* of radius r around $p \in M$ is the set

$$B_r(p) = \{x \in M : d(x, p) < r\}.$$

- p is an *interior point* of $X \subseteq M$ if there is $r > 0$ such that $B_r(p) \subseteq X$.
- The set of interior points of X is the *interior* of X.
- X is *open* if it equals its interior—every point in X is an interior point of X.

Fact

The interior of X is the largest open subset of X.

Closed sets and closure

Let (M, d) be a metric space.

- p is an *accumulation point* of X if for every $r > 0$, $B_r(p) \cap X \neq \varnothing$.
- The *closure* of X is the set of accumulation points.
- X is *closed* if it equals its closure—X contains every accumulation point of X.

WARNING: accumulation point can mean different things for different authors.

Fact

The closure of X is the smallest closed superset of X.

Section 2

Topologies and topological spaces

Nearness relations

Definition

A "nearness relation" on a set S is a relation $\leftarrow$ between elements of S and subsets of S, satisfying the following axioms:

- If $a \in X$, then $a \leftarrow X$.
- If $a \leftarrow X$ and $X \subseteq Y$, the $a \leftarrow Y$.
- If $a \leftarrow (X \cup Y)$, then $a \leftarrow X$ or $a \leftarrow Y$.
- Suppose that $a \leftarrow X$. Suppose that for every $b \in X$, $b \leftarrow Y$. Then $a \leftarrow Y$.

Intuition: $a \leftarrow X$ if a is a limit of points in X.

Example

In a metric space M, define $a \leftarrow X$ to mean that there is $b_1, b_2, b_3, \ldots \in M$ with $\lim_{i \to \infty} b_i = a$.

The Sorgenfrey line

On $\mathbb{R}$, define $a \leftarrow X$ if there are $b_1, b_2, \ldots \in X$ such that $\lim_{i \to \infty} b_i = a$, and $b_i > a$ for all i.

Fact

This defines a nearness relation.
No metric on $\mathbb{R}$ induces this nearness relation.

Topologies

Definition

Let S be a set. A *topology* on S is a set $\mathcal{T}$ satisfying the following:

- If $U \in \mathcal{T}$, then $U \subseteq S$.
- $\varnothing, S \in \mathcal{T}$.
- Suppose I is a set and $U_i \in \mathcal{T}$ for all $i \in I$. Then $\bigcup_{i \in I} U_i \in \mathcal{T}$.
- Suppose I is a *finite* set and $U_i \in \mathcal{T}$ for all $i \in I$. Then $\bigcap_{i \in I} U_i \in \mathcal{T}$.

Intuition: $\mathcal{T}$ is the collection of "open sets."

Fact

In a metric space M, let $\mathcal{T}$ be the collection of open sets. Then $\mathcal{T}$ is a topology.

Topologies and Nearness relations

Fact

Let S be a set. There is a one-to-one correspondence between topologies on S and nearness relations on S.

Let $(\leftarrow)$ be a nearness relation on S.

- Say $X \subseteq S$ is *closed* if $a \leftarrow X \implies a \in X$.
- Say $X \subseteq S$ is *open* if the complement $S \setminus X$ is closed.
- Let $\mathcal{T}$ be the collection of open sets.
- Then $\mathcal{T}$ is a topology.

Topologies and Nearness relations

Fact

Let S be a set. There is a one-to-one correspondence between topologies on S and nearness relations on S.

Let $\mathcal{T}$ be a topology on S.

- Say that $U \subseteq S$ is *open* if $U \in \mathcal{T}$.
- Say that $C \subseteq S$ is *closed* if the complement $S \setminus C$ is open.
- Define the *closure* of X to be the intersection of all closed supersets of X.
- Define $a \leftarrow X$ to mean that a is in the closure of X.
- Then $\leftarrow$ is a nearness relation.

Topologies and metric spaces

Let S be a set.

- Any metric on S induces a topology.
- Different metrics can induce the same topology.
- A topology is *metrizable* if it is induced by at least one metric.

Example

The Manhattan metric $d(\bar{x}, \bar{y}) = \sum_{i=1}^{n} |x_i - y_i|$ and the Euclidean metric $d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ induce the same topology on $\mathbb{R}^n$.

Example

The Sorgenfrey topology on $\mathbb{R}$ is not metrizable.

Topological spaces

Definition

A *topological space* is a pair $(S, \mathcal{T})$, where S is a set and $\mathcal{T}$ is a topology.

Section 3

Bases

Bases

Let $(S, \mathcal{T})$ be a topological space. A *basis* for the topology is a subset $\mathcal{B} \subseteq \mathcal{T}$ satisfying the following equivalent conditions:

1. For every open set U and $p \in U$, there is $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
2. Every open set has the form $\bigcup_{i \in I} B_i$ for some set I (possibly infinite) and some $B_i \in \mathcal{B}$.

Example

In a metric space, the collection of open balls is a basis for the standard topology.

Bases

Suppose $\mathcal{B}$ is a basis for $\mathcal{T}$. Then $\mathcal{T}$ is determined by $\mathcal{B}$:

Fact

$U \in \mathcal{T}$ if and only if there is a set I and a family $\{B_i\}_{i \in I}$ of sets in B such that $U = \bigcup_{i \in I} B_i$.

Fact

Let S be a set and $\mathcal{B}$ be a collection of subsets. Then $\mathcal{B}$ is a basis for a topology if and only if the following hold:

- $\bigcup \mathcal{B} = S$.
- *For any $B_1, B_2 \in \mathcal{B}$, the intersection $B_1 \cap B_2$ is a union of sets in $\mathcal{B}$.*

So we can specify a topology by specifying a basis.

Topologies defined by bases

Example

The sets $[a, b) \subseteq \mathbb{R}$ are a basis for a topology on $\mathbb{R}$. This is the Sorgenfrey line.

Example

Let $(C, \leq)$ be any linear order. The sets (a, b) with $a, b \in C \cup \{\pm\infty\}$ are a basis for a topology on C, the *order topology*.

Section 4

Basic topological notions

Topology in a topological space

Let $(S, \mathcal{T})$ be a topological space.

- A set $X \subseteq S$ is *open* if $X \in \mathcal{T}$.
- A set $X \subseteq S$ is *closed* if the complement $S \setminus X$ is open.
- The *interior* of $X \subseteq S$ is the largest open subset of X.
- The *closure* of $X \subseteq S$ is the smallest closed superset of X.
- The *boundary* of X is the closure minus the interior.

Topology in a topological space

Let S be a topological space.

- A set N is a *neighborhood* of a if a is in the interior of N.
- A point a is an *interior point* of X if a is in the interior of X.
- A point a is an *accumulation point* of X if a is in the closure of X.

Important facts

- X is open if and only if X equals its interior if and only if X is contained in its interior.
- X is closed if and only if $X = \overline{X}$ if and only if $X \supseteq \overline{X}$.
- $\text{int}(X^c)^c = \overline{X}$, and $\overline{X} = \text{int}(X^c)^c$.
- a is an interior point of X if and only if X contains an open neighborhood of a if and only if X contains a neighborhood of X.
- a is an accumulation point of X if and only if every neighborhood of a intersects X if and only if every open neighborhood of a intersects X.
- If N_1, N_2 are neighborhoods of a, then $N_1 \cap N_2$ is a neighborhood of a.

More topological notions

- A point $p \in X$ is *isolated* if there is a neighborhood N of p such that $N \cap X = \{p\}$. Equivalently, p is isolated if $p \notin \overline{X \setminus \{p\}}$.
- X is *discrete* if every point is isolated.
- X is *dense* if $\overline{X}$ is the whole topological space.
- X is *perfect* if X is closed and no point is isolated.
- The *derived set* of X is the set of non-isolated accumulation points of X.
- The *boundary* of X is $\overline{X} \setminus \mathrm{int}(X)$.
- The *frontier* of X is $\overline{X} \setminus X$.

Section 5

Continuous maps and homeomorphisms

Continuity

Let S_1, S_2 be topological spaces. Let $f : S_1 \to S_2$ be a function.

Definition

f is *continuous* if for every open set $U \subseteq S_2$, the preimage $f^{-1}(U)$ is open in S_1.

In terms of $(\leftarrow)$:

Definition

f is *continuous* if

$$a \leftarrow X \implies f(a) \leftarrow f(X),$$

where $f(X) = \{f(b) : b \in X\}$.

These definitions generalize the case of metric spaces.

Continuity at a point

$f : S_1 \to S_2$ is *continuous* at $p \in S_1$ if for any neighborhood $E \ni f(p)$, there is a neighborhood $D \ni p$ such that for any $x \in D$, $f(x) \in E$.

- This is like the ϵ-δ definition in metric spaces.
- $f : S_1 \to S_2$ is continuous iff f is continuous at every point in S_1.

Homeomorphisms

Definition

A *homeomorphism* from $f : S_1 \to S_2$ is a bijection such that f and f^{-1} are continuous.

- It is not enough to require that f is a continuous bijection!
- Homeomorphisms are "isomorphisms of topological spaces."
    - A homeomorphism is a bijection $f : S_1 \to S_2$ such that for every $U \subseteq S_1$,
    $$U \text{ is open} \iff f(U) \text{ is open.}$$

Definition

Two topological spaces S_1, S_2 are *homeomorphic* if there is a homeomorphism $S_1 \to S_2$.

Homeomorphisms

TODO: picture of a tea cup or something.

Section 6

Limits and Hausdorffness

Limits of a sequence

Let S be a topological space. Suppose $a_1, a_2, a_3, \ldots \in S$.

Definition

"$\lim_{n\to\infty} a_n = b$" means that for any neighborhood $U \ni b$, there is an integer n such that

$$\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq U.$$

The Hausdorff condition

Definition

A topological space or topology is *Hausdorff* if the following holds: for any two distinct points $p \neq q$, there are neighborhoods $U \ni p$ and $V \ni q$ with $U \cap V = \varnothing$.

Example

Metric spaces are Hausdorff: take

$$\begin{aligned} \epsilon &= d(p,q)/3 \\ U &= B_\epsilon(p) \\ V &= B_\epsilon(q). \end{aligned}$$

The Sorgenfrey line is also Hausdorff.

Limits in Hausdorff spaces

Fact

Let $a_1, a_2, a_3, \ldots$ be a sequence in a Hausdorff topological space. Suppose

$$\lim_{n\to\infty} a_n = b$$
$$\lim_{n\to\infty} a_n = b'.$$

Then $b = b'$.

Warning

This can fail in non-Hausdorff topological spaces. We should say that b is "a limit" rather than "the limit", and maybe avoid notation like $\lim_{n\to\infty} a_n$.

Non-Hausdorff spaces

Let S be any set.

Definition

The *trivial* or *indiscrete* topology on S is $\{\varnothing, S\}$. (The only open sets are $\varnothing$ and S.)

If $|S| > 1$, then the trivial topology isn't Hausdorff.

Definition

The *cofinite* topology on S is the topology where the closed sets are S and its finite subsets.

If $|S| \geq \aleph_0$, then the cofinite topology isn't Hausdorff.

Limits and closure

Work in a topological space S.

Fact

Suppose $X \subseteq S$ is a set with closure $\overline{X}$. Suppose $a_1, a_2, \ldots \in X$. If $\lim_{n\to\infty} a_n = b$, then $b \in \overline{X}$.

Warning

In metric spaces, there is a converse:

> *Every point of $\overline{X}$ is a limit of points in X.*

This no longer holds in topological spaces.

But if we replace sequences with "nets", then things work.

Limits and continuity

Let $f : S_1 \to S_2$ be a function between topological spaces.

Fact

If f is continuous then f preserves limits of sequences:

$$\lim_{n\to\infty} a_n = b \implies \lim_{n\to\infty} f(a_n) = f(b).$$

Warning

In metric spaces, there is a converse: if f preserves limits of sequences, then f is continuous.
This no longer holds in topological spaces.
But if we replace sequences with "nets", then it's true.