Introductory model theory with elements of universal algebra (version 0)

Will Johnson

Autumn 2022

Preface

These are a provisional draft of the course notes for Introductory Model Theory at Fudan University in Autumn 2022. They aim to present the basic topics of model theory without assuming much background in abstract algebra. Please send any comments or corrections to willjohnson@fudan.edu.cn.

The current draft has no examples, motivations, explanations, or introduction. I hope to remedy this soon.

Chapter 1

Introduction

Under construction

Contents

Chapter 2

Algebras and equational classes

2.1 Monoids and groups

Let A be a set. For $k \geq 0$, a *k-ary operation* or *k-ary function* on A is a function $f : A^k \to A$. *Binary*, *unary*, and *nullary* mean 2-ary, 1-ary, and 0-ary, respectively.

Note that A^0 has one element (). Therefore, we can identify nullary fuctions $f : A^0 \to A$ with elements of A via the correspondence

$$f \mapsto f().$$

A *magma* is a pair $(A, \star)$ where A is a set and $\star$ is a binary operation on A. A magma $(A, \star)$ is a semigroup if $\star$ is associative, meaning that:

$$x \star (y \star z) = (x \star y) \star z$$

for all $x, y, z \in A$.

Let $(A, \star)$ be a semigroup. An element $e \in A$ is is an *identity element* if

$$x \star e = x = e \star x$$

holds for all $x \in A$.

Theorem 2.1.1. *If $(A, \star)$ is a semigroup, there is at most one identity element.*

Proof. Suppose e_1, e_2 are identity elements. Then $e_1 \star e_2 = e_1$ (because e_2 is an identity element), and $e_1 \star e_2 = e_2$ (because e_1 is an identity element. Therefore $e_1 = e_2$. □

Definition 2.1.2. A *monoid* is a triple $(A, \star, e)$ where $(A, \star)$ is a semigroup and e is an identity element.

Let $(A, \star, e)$ be a monoid. If $x \in A$, then an *inverse* of x is an element $x' \in A$ such that $x \star x' = x' \star x = e$.

Theorem 2.1.3. *Let $(A, \star, e)$ be a monoid. If $x \in A$, then x has at most one inverse.*

Proof. Suppose y, z are both inverses of x. Then

$$y = y \star e = y \star (x \star z) = (y \star x) \star z = e \star z = z. \qquad \square$$

Definition 2.1.4. A *group* is a 4-tuple $(A, \star, e, (-)')$ where $(A, \star, e)$ is a monoid and $(-)'$ is a unary function $A \to A$ such that x' is an inverse of x for every $x \in A$.

Equivalently, a group is a 4-tuple $(A, \star, e, (-)')$ where A is a set, $\star$ is a binary operation on A, $e \in A$ is a nullary operation (an element), and $(-)'$ is a unary operation such that the following equations hold for all $x, y, z \in A$.

$$\begin{gathered} x \star (y \star z) = (x \star y) \star z \\ x \star e = e \star x = x \\ x \star x' = x' \star x = e. \end{gathered}$$

A semigroup, monoid, or group is *commutative* if the equation

$$x \star y = y \star x$$

holds for any x, y. Commutative groups are usually called *abelian groups*.

Remark 2.1.5. Groups are usually written using multiplicative or additive notation.

	Multiplicative notation	Additive notation
$x \star y$	$x \cdot y$	$x + y$
e	1	0
x'	x^{-1}	$-x$

Additive notation is traditionally reserved for abelian groups.

2.2 Rings and fields

Definition 2.2.1. A *ring* is a 6-tuple $(A, +, \cdot, -, 0, 1)$ such that

1. $(A, +, 0, -)$ is an abelian group.
2. $(A, \cdot, 1)$ is a commutative monoid.
3. The distributive law holds for $x, y, z \in A$:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

Definition 2.2.2. A *field* is a ring $(K, +, \cdot, -, 0, 1)$ such that $0 \neq 1$ and any $x \neq 0$ has an inverse x^{-1} in the monoid $(K, \cdot, 1)$.

2.3 Languages, algebras, and equational classes

Definition 2.3.1. A *(functional) language* $\mathcal{L}$ is a set of *function symbols* and a map assigning to each function symbol f an integer $n_f \in \mathbb{N}$ called the *arity* of f. An *n-ary function symbol* is a function symbol of arity n. Nullary function symbols are usually called *constant symbols*.

In Chapters 2–3, all "languages" will be functional languages. Later, we wil see a more general notion of "language" (Definition 4.1.1).

Example 2.3.2. The *language of abelian groups* has one binary function symbol $+$, one constant symbol 0, and one unary function symbol $-$.

Definition 2.3.3. Given a language $\mathcal{L}$, an *$\mathcal{L}$-algebra* $\mathcal{A}$ is a set A and a map assigning to each n-ary function symbol f in $\mathcal{L}$ a corresponding n-ary function $f^{\mathcal{A}} : A^n \to A$. The set A is called the *underlying set* of $\mathcal{A}$, and $f^{\mathcal{A}}$ is called the *interpretation of f in $\mathcal{A}$*.

Usually we don't distinguish between an algebra $\mathcal{A}$ and its underlying set A, writing both as A.

Example 2.3.4. If $\mathcal{L}$ is the language of abelian groups, then an $\mathcal{L}$-algebra is essentially a 4-tuple $(A, +^A, 0^A, -^A)$ where A is a set, $+^A$ is a binary operation on A, $0^A \in A$, and $-^A$ is a unary operation on A.

Fix some infinite set $\mathcal{V} = \{x, y, z, \ldots\}$ of "variable symbols."

Definition 2.3.5. An $\mathcal{L}$-*term* is a string generated by the following rules:

- If x is a variable symbol, then x is a term.
- If f is an n-ary function symbol in $\mathcal{L}$, and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.

Example 2.3.6. These are terms in the language of abelian groups:

$$x + (-y),\ 0 + (x + 0),\ z,\ -(-(0 + x)),\ 0.$$

When we say "let $t(x_1, \ldots, x_n)$ be a term," we mean that $t(x_1, \ldots, x_n)$ is a term and the variables occurring in $t(x_1, \ldots, x_n)$ are contained in $\{x_1, \ldots, x_n\}$. If $s_1, \ldots, s_n$ are terms and $t(x_1, \ldots, x_n)$ is a term, then $t(s_1, \ldots, s_n)$ denotes the result of replacing x_i with s_i in $t(x_1, \ldots, x_n)$.

A *closed term* is a term with no variables. If t is a closed term and A is an $\mathcal{L}$-algebra, we define the *interpretation of* t *in* A, written t^A, recursively as follows:

$$f(t_1, \ldots, t_k)^A = f^A(t_1^A, \ldots, t_k^A).$$

The language $\mathcal{L}(A)$ consists of $\mathcal{L}$ with each element of A added as a new constant symbol. We regard A as an $\mathcal{L}(A)$-algebra by interpreting each new constant symbol as the corresponding element of A, so that $c^A = c$ for $c \in A$. If $t(x_1, \ldots, x_n)$ is an $\mathcal{L}$-term and $a_1, \ldots, a_n \in A$, then $t(a_1, \ldots, a_n)$ is a closed $\mathcal{L}(A)$-term. The *interpretation of* t *in* A, written t^A, is the function $t^A : A^n \to A$ defined by

$$t^A(a_1, \ldots, a_n) = (t(a_1, \ldots, a_n))^A.$$

Example 2.3.7. If $\mathcal{L}$ is the language of groups and $t(x, y) = (-x) + (0 + y)$, then the interpretation of t in an $\mathcal{L}$-algebra $(A, +^A, 0^A, -^A)$ is the function

$$t^A(a, b) = ((-a) + (0 + b))^A = (-^A a) +^A (0^A +^A b).$$

Definition 2.3.8. An $\mathcal{L}$-*equation* is a formal expression of the form

$$t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$$

for two $\mathcal{L}$-terms $t(\bar{x})$ and $s(\bar{x})$. An $\mathcal{L}$-algebra A *satisfies* an equation $(t(\bar{x}) = s(\bar{x}))$ if for any $\bar{a} \in A^n$,

$$t^A(\bar{a}) = s^A(\bar{a}).$$

The notation $A \models \varphi$ means that A satisfies φ.

Definition 2.3.9. An *equational $\mathcal{L}$-theory* is a set Σ of $\mathcal{L}$-equations. Elements of Σ are called *axioms* of Σ. If Σ is an equational $\mathcal{L}$-theory, and A is an $\mathcal{L}$-algebra, then A is a *model* of Σ, written $A \models \Sigma$, if $A \models \varphi$ for every $\varphi \in \Sigma$. The class of models of Σ is written $\mathrm{Mod}(\Sigma)$. An *equational class* is a class of the form $\mathrm{Mod}(\Sigma)$ for some Σ.

Example 2.3.10. Let $\mathcal{L}$ be the language of abelian groups. The *theory of abelian groups* consists of the equations

$$\begin{gathered} x + (y + z) = (x + y) + z \\ x + 0 = x \\ x + (-x) = 0 \\ x + y = y + x. \end{gathered}$$

Models are abelian groups.

Example 2.3.11. The classes of rings, groups, monoids, semigroups, and magmas are equational classes.

Example 2.3.12. An *idempotent monoid* is a monoid $(A, \star, e)$ in which the equation $x \star x = x$ holds. Idempotent monoids form an equational class.

Example 2.3.13. A *boolean algebra* is a 6-tuple $(B, \wedge, \vee, \neg, 0, 1)$ where $(B, \vee, 0)$ is an idempotent commutative monoid, $(B, \wedge, 1)$ is an idempotent commutative monoid, and $\neg : B \to B$ is a unary operation such that the following equations hold:

$$\begin{gathered} x \wedge \neg x = 0 \\ x \vee \neg x = 1 \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \end{gathered}$$

Boolean algebras form an equational class. If S is a set, then the powerset $\mathfrak{P}(S)$ is a boolean algebra with

$$\begin{gathered} x \wedge y = x \cap y \\ x \vee y = x \cup y \\ 0 = \varnothing \\ 1 = S \\ \neg x = S \setminus x \end{gathered}$$

2.4 Homomorphisms and isomorphisms

Fix a language $\mathcal{L}$.

Definition 2.4.1. Let A, B be $\mathcal{L}$-algebras. A *homomorphism* from A to B is a function $\alpha : A \to B$ such that for any k-ary function symbol $f \in \mathcal{L}$, and any $a_1, \ldots, a_k \in A$,

$$\alpha(f^A(a_1, \ldots, a_k)) = f^B(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_k)). \qquad (*)$$

An *isomorphism* is a bijective homomorphism. Two $\mathcal{L}$-algebras A, B are *isomorphic*, written $A \cong B$, if there is an isomorphism $\alpha : A \to B$.

In what follows, we extend functions to tuples componentwise, so that $\alpha(a_1, \ldots, a_k) := (\alpha(a_1), \ldots, \alpha(a_k))$. Then $(*)$ can be rewritten as $\alpha(f^A(\bar{a})) = f^B(\alpha(\bar{a}))$.

Theorem 2.4.2. *Let $\mathcal{L}$ be a language and A, B, C be $\mathcal{L}$-algebras.*

1. *Let $\alpha : A \to B$ and $\beta : B \to C$ be homomorphisms. Then $\beta \circ \alpha : A \to C$ is a homomorphism.*
2. *The identity map $\mathrm{id}_A : A \to A$ is an isomorphism.*
3. *If $\alpha : A \to B$ is an isomorphism, then $\alpha^{-1} : B \to A$ is an isomorphism.*

Proof. 1. If $f \in \mathcal{L}$ is a k-ary function symbol, then

$$\beta(\alpha(f^A(\bar{a}))) = \beta(f^B(\alpha(\bar{a}))) = f^C(\beta(\alpha(\bar{a})))$$

because β and α are homomorphisms.

2. If $f \in \mathcal{L}$ is a k-ary relation symbol, then

$$\mathrm{id}(f(\bar{a})) = f(\bar{a}) = f(\mathrm{id}(\bar{a})).$$

3. Suppose $f \in \mathcal{L}$ is a k-ary function symbol and $b_1, \ldots, b_k \in B$. Let $a_i = \alpha^{-1}(b_i) \in A$. Then $b_i = \alpha(a_i)$. As α is a homomorphism,

$$\alpha(f^A(\bar{a})) = f^B(\alpha(\bar{a})) = f^B(\bar{b}).$$

Applying α^{-1} to both sides, we see that

$$\alpha^{-1}(f^B(\bar{b})) = f^A(\bar{a}) = f^A(\alpha^{-1}(\bar{b})).$$

□

Corollary 2.4.3. *The relation of isomorphism is an equivalence relation on $\mathcal{L}$-algebras:*

1. *$A \cong A$ for any $\mathcal{L}$-algebra A.*
2. *If $A \cong B$, then $B \cong A$.*
3. *If $A \cong B$ and $B \cong C$, then $A \cong C$.*

Proof.

1. $\mathrm{id}_A : A \to A$ is an isomorphism.
2. If $\alpha : A \to B$ is an isomorphism, then $\alpha^{-1} : B \to A$ is an isomorphism.
3. If $\alpha : A \to B$ and $\beta : B \to C$ are isomorphisms, then $\beta \circ \alpha : A \to C$ are isomorphisms. □

Theorem 2.4.4. *Let $\alpha : A \to B$ be a homomorphism of $\mathcal{L}$-algebras. Let $t(x_1, \ldots, x_n)$ be an $\mathcal{L}$-term. Then for any $a_1, \ldots, a_n \in A$, we have*

$$\alpha(t^A(\bar{a})) = t^B(\alpha(\bar{a})). \tag{†}$$

Proof. Proceed by induction on the complexity of t.

- If $t(\bar{x}) = x_i$, then both sides of (†) are $\alpha(a_i)$.
- Suppose $t(\bar{x}) = f(s_1(\bar{x}), \ldots, s_k(\bar{x}))$ for some k-ary function symbol f and some simpler $\mathcal{L}$-terms $s_1, \ldots, s_k$. By definition of t^A,

  $$\alpha(t^A(\bar{a})) = \alpha(f^A(s_1^A(\bar{a}), \ldots, s_k^A(\bar{a}))).$$

  As α is a homomorphism,

  $$\alpha(f^A(s_1^A(\bar{a}), \ldots, s_k^A(\bar{a}))) = f^B(\alpha(s_1^A(\bar{a})), \ldots, \alpha(s_k^A(\bar{a})))).$$

  By induction,

  $$\alpha(s_i^A(\bar{a})) = s_i^B(\alpha(\bar{a})).$$

  for each i. Therefore,

  $$f^B(\alpha(s_1^A(\bar{a})), \ldots, \alpha(s_k^A(\bar{a})))) = f^B(s_1^B(\alpha(\bar{a})), \ldots, s_k^B(\alpha(\bar{a}))).$$

  By definition, the right hand side is $t^B(\alpha(\bar{a}))$. □

Lemma 2.4.5. *Let $\alpha : A \to B$ be a surjective homomorphism of $\mathcal{L}$-algebras, and let φ be an $\mathcal{L}$-equation. Then $A \models \varphi \implies B \models \varphi$.*

Proof. Suppose φ is $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$. Suppose $b_1, \ldots, b_n \in B$. By surjectivity, we can write b_i as $\alpha(a_i)$ for some $a_i \in A$. Then

$$t^B(\bar{b}) = t^B(\alpha(\bar{a})) = \alpha(t^A(\bar{a}))$$
$$s^B(\bar{b}) = s^B(\alpha(\bar{a})) = \alpha(s^A(\bar{a}))$$

by Theorem 2.4.4, as α is a homomorphism. Because $A \models \varphi$, the right hand sides are equal. Therefore the left hand sides are equal, meaning

$$t^B(\bar{b}) = s^B(\bar{b}).$$ □

Theorem 2.4.6. *Suppose $A \cong B$.*

1. $A \models s = t \iff B \models s = t$.
2. $A \models \Sigma \iff B \models \Sigma$
3. $A \in \mathrm{Mod}(\Sigma) \iff B \in \mathrm{Mod}(\Sigma)$.
4. *If $\mathcal{K}$ is an equational class, then $A \in \mathcal{K} \iff B \in \mathcal{K}$.*

Group homomorphisms

Lemma 2.4.7. *Let G be a group. If $ax = ay$, then $x = y$. Similarly, if $xa = ya$, then $x = y$.*

Proof. If $ax = ay$, then

$$x = 1x = (a^{-1}a)x = a^{-1}(ax) = a^{-1}(ay) = (a^{-1}a)y = 1y = y.$$

The other case is similar. □

Theorem 2.4.8. *Let G, H be groups. If $f : G \to H$ is a semigroup homomorphism, meaning that f preserves multiplication*

$$f(xy) = f(x)f(y),$$

then f also preserves 1 and $(-)^{-1}$:

$$f(1) = 1$$
$$f(x^{-1}) = f(x)^{-1},$$

and so f is a group homomorphism.

Proof. First note that

$$f(1) \cdot f(1) = f(1 \cdot 1) = f(1) = f(1) \cdot 1.$$

Canceling $f(1)$ from both sides, we see $f(1) = 1$. Next,

$$f(x)f(x^{-1}) = f(xx^{-1}) = f(1) = 1 = f(x)f(x)^{-1}.$$

Canceling $f(x)$ from both sides, we see $f(x^{-1}) = f(x)^{-1}$. □

Corollary 2.4.9. *Let R be a ring. Then the following equations hold, for $a, x \in R$:*

$$\begin{aligned} 0a &= 0 \\ (-x)a &= -(xa) \\ (-1)a &= -a \end{aligned}$$

Proof. Let $\mu_a(x) = ax$. Then μ_a is a semigroup homomorphism $(R, +) \to (R, +)$, by the distributive law:

$$a(x + y) = ax + ay.$$

Thus μ_a preserves zero and negation:

$$\begin{aligned} \mu_a(0) &= 0 \\ \mu_a(-x) &= -\mu_a(x), \end{aligned}$$

which are the first two equations. The third equation holds by taking $x = 1$ in the second equation. □

Lemma 2.4.10. *The following holds in any field K:*

$$xy = xz \implies y = z \text{ when } x \neq 0.$$

Proof. Multiply both sides of $xy = xz$ by x^{-1}, as in Lemma 2.4.7. □

Theorem 2.4.11. *If R is a field and $x, y \in R$, then*

$$xy = 0 \iff (x = 0 \text{ or } y = 0)$$

Proof. If $x = 0$ or $y = 0$, then $xy = 0$ by the zero law (Corollary 2.4.9). Conversely, suppose $xy = 0$, but $x \neq 0$ and $y \neq 0$. Then

$$xy = 0 = x0$$

by the zero law. Cancelling x from both sides, $y = 0$, a contradiction. □

Chapter 3

New algebras from old

3.1 Subalgebras and generators

Definition 3.1.1. Let A be an $\mathcal{L}$-algebra. A *subalgebra* is a subset $B \subseteq A$ such that for any k-ary relation symbol $f \in \mathcal{L}$,

$$b_1, \ldots, b_k \in B \implies f^A(b_1, \ldots, b_k) \in B.$$

If B is a subalgebra of an $\mathcal{L}$-algebra A, then we can make B into an $\mathcal{L}$-algebra by defining f^B to be the restriction of f^A to B, for each function symbol f:

$$f^B(b_1, \ldots, b_k) := f^A(b_1, \ldots, b_k) \in B.$$

In this way, we regard subalgebras as algebras, not just sets.

Theorem 3.1.2. *Suppose B is a subalgebra of A.*

1. *The inclusion $B \to A$ is a homomorphism.*
2. *If $A \models s = t$, then $B \models s = t$.*
3. *If $\mathcal{K} = \mathrm{Mod}(\Sigma)$ is an equational class, then $A \in \mathcal{K} \implies B \in \mathcal{K}$.*

Proof. (1) holds by choice of the structure on B. For (2), note that

$$t^B(\bar{b}) = t^A(\bar{b})$$
$$s^B(\bar{b}) = s^A(\bar{b})$$

for $\bar{b}$ in B by Theorem 2.4.4 and (1). If $t^A = s^A$, then $t^B = s^B$. Finally, (3) follows directly from (2). □

Example 3.1.3. The ring $\mathbb{R}$ is a field, but the subring $\mathbb{Z}$ is not. Therefore, fields are not an equational class.

Definition 3.1.4. If S is a subset of an algebra A, then $\langle S \rangle$ or $\langle S \rangle_A$ denotes the smallest subalgebra of A containing S, which is

$$\{t^A(\bar{b}) : t(x_1, \ldots, x_n) \text{ is an } \mathcal{L}\text{-term and } \bar{b} \in S^n\}$$

The subalgebra $\langle S \rangle$ is called the subalgebra *generated* by S. We say that A is *finitely generated* if $A = \langle S \rangle$ for some finite $S \subseteq A$.

We often omit brackets $\{,\}$ inside $\langle , \rangle$, using abbreviations like

$$\langle a_1, \ldots, a_n \rangle = \langle \{a_1, \ldots, a_n\} \rangle$$
$$\langle A, b \rangle = \langle A \cup \{b\} \rangle.$$

3.2 Products

Binary products

Definition 3.2.1. Let A, B be two $\mathcal{L}$-algebras. The *product algebra* $A \times B$ is the $\mathcal{L}$-algebra with underlying set $A \times B$ and

$$f^{A \times B}((a_1, b_1), \ldots, (a_k, b_k)) := (f^A(a_1, \ldots, a_k), f^B(b_1, \ldots, b_k)).$$

for each k-ary function symbol $f \in \mathcal{L}$.

Remark 3.2.2. Let A_1, A_2 be two $\mathcal{L}$-algebras. For $i = 1, 2$, let $\pi_i : A_1 \times A_2 \to A_i$ be the projection map $\pi_i(x_1, x_2) = x_i$. Then each π_i is a homomorphism. For example,

$$\begin{aligned} \pi_1(f^{A_1 \times A_2}((a_1, b_1), \ldots, (a_k, b_k))) &= f^{A_1}(a_1, \ldots, a_k) \\ &= f^{A_1}(\pi_1(a_1, b_1), \ldots, \pi_1(a_k, b_k)). \end{aligned}$$

Theorem 3.2.3. *Let A_1, A_2 be $\mathcal{L}$-algebras.*

1. *If φ is an equation and $A_i \models \varphi$ for $i = 1, 2$, then $A_1 \times A_2 \models \varphi$.*
2. *If $\mathcal{K}$ is an equational class and $A_1, A_2 \in \mathcal{K}$, then $A_1 \times A_2 \in \mathcal{K}$.*

Proof. Part (2) follows formally from (1). For (1), suppose $A_i \models t = s$ for $i = 1, 2$. Let $P = A_1 \times A_2$. We claim $P \models t = s$. Otherwise there are $a_1, \ldots, a_k \in P$ such that $t^P(\bar{a}) \neq s^P(\bar{a})$. Then there is $i \in \{1, 2\}$ such that

$$\pi_i(t^P(\bar{a})) \neq \pi_i(s^P(\bar{a})).$$

As π_i is a homomorphism, we can rewrite the two sides as follows:

$$t^{A_i}(\pi_i(\bar{a})) \neq s^{A_i}(\pi_i(\bar{a})).$$

Then $A_i \not\models t = s$, a contradiction. $\square$

Example 3.2.4. The ring $\mathbb{R}$ is a field, but $\mathbb{R} \times \mathbb{R}$ is not, because the zero law (Theorem 2.4.11) fails:

$$(0, 1) \neq (0, 0)$$
$$(1, 0) \neq (0, 0)$$
$$(0, 1) \cdot (1, 0) = (0, 0).$$

Therefore fields are not an equational class.

Infinite products

Let I be a set.

Definition 3.2.5. An I*-tuple* is a function with domain I. If a is an I-tuple, we write $a(i)$ as $\pi_i(a)$. The notation $(a_i : i \in I)$ means the function $i \mapsto a_i$.

When $I = \{1, 2, \ldots, n\}$, we identify n-tuples and I-tuples, so that

$$(a_1, a_2, \ldots, a_n) = (a_i : i \in \{1, \ldots, n\}).$$

Likewise, we think of ω-tuples as tuples of length ω, i.e., sequences, so that

$$(a_1, a_2, a_3, \ldots) = (a_i : i \in \omega).$$

Definition 3.2.6. Let $\{A_i\}_{i \in I}$ be a family of sets. The *direct product* $\prod_{i \in I} A_i$ is the set of I-tuples $(a_i : i \in I)$ such that $a_i \in A_i$ for each $i \in I$.

Remark 3.2.7. Suppose A_i doesn't depend on i, so that $A_i = A$ for some fixed set A as i varies. Then $\prod_{i \in I} A$ is the set of functions from I to A. This set is also written A^I, and is called a *power* of A.

Definition 3.2.8. Let I be a set and let A_i be an $\mathcal{L}$-algebra for each $i \in I$. We make $\prod_{i \in I} A_i$ into an $\mathcal{L}$-algebra by interpreting each k-ary function symbol f as

$$f(a_1, \ldots, a_k) = (f^{A_i}(\pi_i(a_1), \ldots, \pi_i(a_k)) : i \in I).$$

Remark 3.2.9. The structure on $\prod_{i \in I} A_i$ is chosen to make π_i a homomorphism:

$$\pi_i(f(a_1, \ldots, a_k)) = f^{A_i}(\pi_i(a_1), \ldots, \pi_i(a_k)).$$

Theorem 3.2.10. *Let A_i be an $\mathcal{L}$-algebra for each $i \in I$.*

1. *If φ is an equation and $A_i \models \varphi$ for each $i \in I$, then $\prod_{i \in I} A_i \models \varphi$.*
2. *If $\mathcal{K}$ is an equational class and $A_i \in \mathcal{K}$ for all $i \in I$, then $\prod_{i \in I} A_i \in \mathcal{K}$.*

Proof. The proof of Theorem 3.2.3 works here. □

Example 3.2.11 (Powers)**.** If I is a set and A is an $\mathcal{L}$-algebra, the power $A^I := \prod_{i \in I} A$ is the set of functions $I \to A$, with all the operations defined pointwise.

3.3 Descending functions along surjections

Theorem 3.3.1. *Let $\pi : A \to A'$ be a surjection. Let $f : A \to B$ be a function such that*

$$\pi(x) = \pi(y) \implies f(x) = f(y). \qquad (*)$$

Then there is a unique function $f' : A' \to B$ such that $f'(\pi(x)) = f(x)$, i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ A' \arrow[r, dashed, "f'"'] & B. \end{tikzcd}$$

Proof. For $z \in A'$, let $S_z = \{f(x) : x \in A,\ \pi(x) = z\}$. There is at least one $x \in A$ with $\pi(x) = z$ because π is surjective, so S_z is non-empty. If $x_1, x_2 \in A$ satisfy $\pi(x_i) = z$ for $i = 1, 2$, then $f(x_1) = f(x_2)$ by $(*)$. Therefore S_z has a unique element. Let $f'(z)$ be the unique element of S_z.

If $x \in A$ and $z = \pi(x)$, then $f(x) \in S_z = \{f'(z)\}$, so $f(x) = f'(z) = f'(\pi(x))$. This proves existence of f'. For uniqueness, suppose $f'' : A' \to B$ is another function satisfying $f''(\pi(x)) = f(x)$. For any $z \in A'$, there is $x \in A$ with $\pi(x) = z$, and then

$$f''(z) = f''(\pi(x)) = f(x) = f'(\pi(x)) = f'(z),$$

so $f'' = f'$. $\square$

Example 3.3.2. Let $\mathcal{K}$ be the class of $\mathcal{L}$-algebras. If $X, Y \in \mathcal{K}$ are isomorphic, then $|X| = |Y|$. Therefore, there is a map from the class $\mathcal{K}/{\cong}$ of isomorphism classes to the class Card of cardinals sending the isomorphism class of X to the cardinality $|X|$. Ignoring the difference between sets and classes, this is an instance of Theorem 3.3.1:

$$\begin{tikzcd} \mathcal{K} \arrow[d] \arrow[dr] & \\ \mathcal{K}/{\cong} \arrow[r, dashed] & \text{Card}. \end{tikzcd}$$

The surjectivity of $\mathcal{K} \to \mathcal{K}/{\cong}$ holds because every isomorphism class is the isomorphism class of some algebra. The condition $(*)$ in Theorem 3.3.1 holds because

$$[X]_{\cong} = [Y]_{\cong} \iff X \cong Y \implies |X| = |Y|.$$

The conclusion of Theorem 3.3.1 says that the map $f' : \mathcal{K}/{\cong} \to \text{Card}$ satisfies

$$f'([X]_{\cong}) = |X|,$$

so that f' sends the isomorphism class of X to the cardinality of X.

3.4 Congruences and quotients

Definition 3.4.1. Let A be an $\mathcal{L}$-algebra. A *congruence* on A is an equivalence relation E on A that is also a subalgebra of $A \times A$.

Unwinding the definition, an equivalence relation $\sim$ on A is a congruence iff

$$(a_1 \sim b_1 \text{ and } a_2 \sim b_2 \text{ and} \dots \text{and } a_k \sim b_k) \implies f(a_1, \dots, a_k) \sim f(b_1, \dots, b_k)$$

for any k-ary function symbol $f \in \mathcal{L}$.

Theorem 3.4.2. *Let $\sim$ be an equivalence relation on an $\mathcal{L}$-algebra A. Then $\sim$ is a congruence iff the following holds: for any k-ary function symbol $f \in \mathcal{L}$ and any $1 \leq i \leq k$,*

$$a_i \sim a_i' \implies f(a_1, \dots, a_k) \sim f(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_k). \qquad (*)$$

Proof. First suppose $\sim$ is a congruence, and $a_i \sim a_i'$. For $j \neq i$ define $a_j' := a_j$. Then $a_j \sim a_j'$ because $\sim$ is reflexive. As $a_j \sim a_j'$ for *all* $j \leq k$, we see that

$$f(a_1, \dots, a_k) \sim f(a_1', \dots, a_k') = f(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_k).$$

Conversely, suppose $(*)$ holds. Suppose $a_i \sim b_i$ for $i = 1, \dots, k$. Then $(*)$ gives

$$\begin{aligned} & f(a_1, a_2, a_3, \dots, a_k) \\ \sim & f(b_1, a_2, a_3, \dots, a_k) \\ \sim & f(b_1, b_2, a_3, \dots, a_k) \\ \sim & \cdots \\ \sim & f(b_1, b_2, \dots, b_k). \end{aligned}$$

□

Definition 3.4.3. Let R be a ring. An *ideal* is a subset $I \subseteq R$ such that

1. $0 \in I$.
2. $x, y \in I \implies x + y \in I$.
3. $(x \in R \text{ and } y \in I) \implies xy \in I$.

Remark 3.4.4. If R is a ring and $a \in R$, then the set $aR := \{ax : x \in R\}$ is an ideal. Such ideals are called *principal ideals*.

Theorem 3.4.5. *Let R be a ring.*

1. *If I is an ideal, define $x \equiv_I y$ to mean $x - y \in I$. Then $\equiv_I$ is a congruence.*

2. *This gives a bijection between congruences on R and ideals on R.*

Proof. First, note that if $\sim$ is a congruence, then

$$x \sim y \iff x - y \sim 0$$

for any $x, y \in R$. Indeed,

$$\begin{aligned} x \sim y &\implies x - y \sim y - y = 0 \\ x - y \sim 0 &\implies x = (x - y) + y \sim 0 + y = y. \end{aligned}$$

Therefore $\sim$ must have the form

$$x \sim y \iff x - y \in I$$

for some set $I \subseteq R$, namely $I = \{z \in R : z \sim 0\}$. It remains to characterize which sets I yield congruences.

1. Reflexivity says that $x - x \in I$ for any x. This holds iff $0 \in I$.

2. Symmetry says that $x - y \in I \iff y - x \in I$. This holds iff I is closed under negation.

3. Transitivity says that if $x - y \in I$ and $y - z \in I$, then $x - z \in I$. This holds iff I is closed under addition.

4. Compatibility with $+$ says that if $x - y \in I$, then $(x + a) - (y + a) \in I$. This condition holds for any I.

5. Compatibility with $\cdot$ says that if $x - y \in I$, then $(ax) - (ay) \in I$. This condition holds iff I is closed under multiplication by R.

In summary, I yields a congruence if and only if the following four properties hold:

- $0 \in I$
- $x \in I \implies -x \in I$
- $x, y \in I \implies x + y \in I$

- $a \in R,\ x \in I \implies ax \in I$.

The second condition is an instance of the fourth (take $a = -1$), so it can be removed. The remaining three conditions are the definition of "ideal." □

Remark 3.4.6. The relation $x \equiv_I y$ is called "congruence modulo I", and is usually written like $x \equiv y \pmod{I}$. When I is a principal ideal aR, it is usually written $x \equiv y \pmod{a}$.

If X is a set and E is an equivalence relation on X, let X/E denote the quotient, the set $\{[a]_E : a \in X\}$, where $[a]_E$ is the E-equivalence class $[a]_E = \{b \in X : a\ E\ b\}$. Recall that $[a]_E = [b]_E \iff a\ E\ b$. We omit the subscript when E is clear from context.

Theorem 3.4.7 (Quotients)**.** *Let A be an $\mathcal{L}$-algebra and E be a congruence on A. Then there is a unique $\mathcal{L}$-algebra with underlying set A/E such that $A \to A/E$ is a homomorphism, meaning that*

$$f^{A/E}([a_1], \ldots, [a_k]) = [f^A(a_1, \ldots, a_k)] \tag{†}$$

for any k-ary function symbol in $\mathcal{L}$.

Proof. By Theorem 3.3.1, $f^{A/E} : (A/E)^k \to A/E$ is uniquely determined by (†), as long as

$$([a_1], \ldots, [a_k]) = ([b_1], \ldots, [b_k]) \implies [f^A(\bar{a})] = [f^A(\bar{b})],$$

or equivalently,

$$(a_i\ E\ b_i \text{ for } i = 1, \ldots, k) \implies (f^A(\bar{a})\ E\ f^A(\bar{b})).$$

This holds because E is a congruence. □

Definition 3.4.8. If A is an $\mathcal{L}$-algebra and E is a congruence, then A/E is called the *quotient algebra* of A by E.

Theorem 3.4.9. *Let A be an $\mathcal{L}$-algebra and E be a congruence on A.*

1. *If A satisfies an equation φ, then A/E satisfies φ.*
2. *If $\mathcal{K} = \mathrm{Mod}(\Sigma)$ is an equational class, then $A \in \mathcal{K} \implies A/E \in \mathcal{K}$.*

Proof. By construction, there is a surjective homomorphism $A \to A/E$. Then $A \models \varphi \implies A/E \models \varphi$ (Lemma 2.4.5). □

If E is an equivalence relation on A, a *set of representatives* for E is a set $S \subseteq A$ containing exactly one element from each E-equivalence class.

Theorem 3.4.10. *Let A be an $\mathcal{L}$-algebra and E be a congruence. Let S be a set of representatives for E. Let $\rho : A \to S$ be the map sending $x \in A$ to the unique $y \in S \cap [x]_E$. For each k-ary function symbol $f \in \mathcal{L}$, define*

$$f^S(x_1, \dots, x_k) = \rho(f^A(x_1, \dots, x_k)).$$

This makes S into an $\mathcal{L}$-algebra isomorphic to A/E.

Proof. The construction certainly makes S into an $\mathcal{L}$-algebra. Define $\alpha : S \to A/E$ by $\alpha(x) = [x]$. We claim that α is an isomorphism.

1. α is a homomorphism: if f is a k-ary function symbol, then

   $$\alpha(f^S(x_1, \dots, x_k)) = [\rho(f^A(x_1, \dots, x_k))]$$

   But $\rho(y)\ E\ y$ for any $y \in A$, so $[\rho(y)] = [y]$. Therefore

   $$[\rho(f^A(x_1, \dots, x_k))] = [f^A(x_1, \dots, x_k)].$$

   By construction of the quotient,

   $$[f^A(x_1, \dots, x_k)] = f^{A/E}([x_1], \dots, [x_k]) = f^{A/E}(\alpha(x_1), \dots, \alpha(x_k)).$$

   Putting everything together,

   $$\alpha(f^S(x_1, \dots, x_k)) = f^{A/E}(\alpha(x_1), \dots, \alpha(x_k)).$$

2. α is a bijection: clear by choice of S. □

3.5 Application: finite fields

Work in the ring of integers $\mathbb{Z}$.

Lemma 3.5.1. *If $n > 0$, then $\{0, 1, \dots, n-1\}$ is a system of representatives for $\equiv_n$. That is, for every $x \in \mathbb{Z}$ there is a unique $y \in \{0, \dots, n-1\}$ with $x \equiv y \pmod{n}$.*

Proof. An exercise, by induction on x. □

Lemma 3.5.2. *If $0 \neq n \in \mathbb{Z}$, and $x \in \mathbb{Z}$, there is $y \in \mathbb{Z}$ with*

$$x \equiv y \pmod{n}$$
$$|y| < |n|.$$

Proof. Since $n\mathbb{Z} = (-n)\mathbb{Z}$, we may assume $n > 0$ by replacing n with $-n$ if necessary. Then there is some $y \in \{0, 1, 2, \ldots, n-1\}$ with $x \equiv y \pmod{n}$ by Lemma 3.5.1. □

Theorem 3.5.3. *Every ideal $I \subseteq \mathbb{Z}$ is a principal ideal $I = n\mathbb{Z}$ for some $n \geq 0$.*

Proof. Note that $\{0\} \subseteq I$. If $I = \{0\}$, take $n = 0$. Otherwise, take $n \in I \setminus \{0\}$ minimizing $|n|$. Replacing n with $-n$ if necessary, we may assume $n \geq 0$. Then $n\mathbb{Z} \subseteq I$ because I is an ideal. We claim $n\mathbb{Z} = I$. Otherwise, take $a \in I \setminus n\mathbb{Z}$. By Lemma 3.5.2 there is $b \in \mathbb{Z}$ with

$$b \equiv a \pmod{n}$$
$$|b| < |n|.$$

Then $b - a \in n\mathbb{Z} \subseteq I$, and $a \in I$, so $b \in I$. If $b \neq 0$ then b contradicts the choice of n. If $b = 0$, then $a \equiv b = 0 \pmod{n}$, so $a \in n\mathbb{Z}$, contradicting the choice of a. □

Theorem 3.5.4. *If $n > 0$, then $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $R = (\{0, 1, \ldots, n-1\}, +^R, \cdot^R)$, where $x +^R y$ is the unique $z \in R$ with $x + y \equiv z \pmod{n}$, and $x \cdot^R y$ is the unique $w \in R$ with $xy \equiv w \pmod{n}$. This follows by Theorem 3.4.10 applied to the system of representatives in Lemma 3.5.1. In particular, $\mathbb{Z}/n\mathbb{Z}$ is finite, of size n.*

Lemma 3.5.5. *Let I, J be ideals in a ring R. Let $I + J := \{x + y : x \in I,\ y \in J\}$. Then $I + J$ is an ideal containing I and J.*

Proof. To show that $I + J$ is an ideal, there are three things to check:

1. $0 \in I + J$: take $x = 0 \in I$ and $y = 0 \in J$.

2. $I + J$ is closed under addition: if $x, x' \in I$ and $y, y' \in J$, then $(x + y) + (x' + y') = (x + x') + (y + y') \in I + J$.

3. $I + J$ is closed under multiplication by R: if $x \in I, y \in J$, and $a \in R$, then $a(x + y) = (ax) + (ay) \in I + J$.

Finally, if $x \in I$ then $x + 0 \in I + J$ because $0 \in J$. This shows $I \subseteq I + J$, and $J \subseteq I + J$ follows similarly. $\square$

An ideal $I \subseteq R$ is *proper* if $1 \notin I$. A *maximal ideal* is a maximal proper ideal.

Theorem 3.5.6. *If I is a maximal ideal, then R/I is a field.*

Proof. Let $[a] \in R/I$ denote the image of $a \in R$. First we show that $1 \neq 0$ in R/I. The fact that $1 \notin I$ means that $1 \not\equiv 0 \pmod{I}$, so $[1] \neq [0]$. Next we show that any $[a] \neq 0$ has a multiplicative inverse. By Lemma 3.5.5, $aR + I$ is an ideal containing aR and I. The fact that $[a] \neq 0$ means that $a = a - 0 \notin I$. Therefore $aR + I \supsetneq I$, as $a \in aR \subseteq aR + I$. By maximality of I, $aR + I$ is improper, so $1 \in aR + I$. Therefore there are $x \in R$ and $y \in I$ with $1 = ax + y$. Then $ax - 1 = y \in I$, so that $ax \equiv 1 \pmod{I}$ and $[a][x] = [ax] = [1]$. Then $[x]$ is the multiplicative inverse of $[a]$. $\square$

Theorem 3.5.7. *If p is prime, then $\mathbb{Z}/p\mathbb{Z}$ is a field.*

Proof. It suffices to show that the ideal $p\mathbb{Z}$ is maximal. If not, take a larger proper ideal $I \supsetneq p\mathbb{Z}$. By Theorem 3.5.3, $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. Then $p \in p\mathbb{Z} \subseteq n\mathbb{Z}$, so p is a multiple of n. In other words, $p = nm$ for some $m \in \mathbb{Z}$. As p is prime, one of n or m is ± 1.

- If $n = \pm 1$, then $n\mathbb{Z}$ is improper, a contradiction.
- If $m = \pm 1$, then $n = \pm p$, and $n\mathbb{Z} = p\mathbb{Z}$, a contradiction. $\square$

3.6 The fundamental theorem on homomorphisms

Theorem 3.6.1 (Images)**.** *Let $\alpha : A \to B$ be a homomorphism of $\mathcal{L}$-algebras. Then the image* $\mathrm{im}(\alpha) = \alpha(A) = \{\alpha(x) : x \in A\}$ *is a subalgebra of B.*

Proof. Suppose $f \in \mathcal{L}$ is a k-ary function symbol, and $b_1, \ldots, b_k \in \mathrm{im}(\alpha)$. Each b_i can be written as $\alpha(a_i)$ for some $a_i \in A$. Then

$$f(b_1, \ldots, b_k) = f(\alpha(a_1), \ldots, \alpha(a_k)) = \alpha(f(a_1, \ldots, a_k)) \in \mathrm{im}(\alpha). \qquad \square$$

Definition 3.6.2. Let $\alpha : A \to B$ be a homomorphism of $\mathcal{L}$-algebras. The *kernel* of α is the equivalence relation

$$a \sim b \iff \alpha(a) = \alpha(b).$$

We write the kernel as $\ker(\alpha)$.

Theorem 3.6.3. *If $\alpha : A \to B$ is a homomorphism of $\mathcal{L}$-algebras, then the kernel is a congruence on A.*

Proof. Let $E = \ker(\alpha)$. Let $f \in \mathcal{L}$ be a k-ary function symbol. If $a_i \; E \; b_i$ for $i = 1, \ldots, k$, then $\alpha(a_i) = \alpha(b_i)$ for each i. As α is a homomorphism,

$$\alpha(f(\bar{a})) = f(\alpha(\bar{a})) = f(\alpha(\bar{b})) = \alpha(f(\bar{b})).$$

Therefore $f(\bar{a}) \; E \; f(\bar{b})$. $\square$

Remark 3.6.4. In ring theory, the kernel of a homomorphism $f : R \to S$ is the *ideal* $\{x \in R : f(x) = 0\}$. This is the ideal corresponding to the kernel in the sense of Definition 3.6.2.

Lemma 3.6.5. *Let A, B, C be algebras. Let $\alpha : A \to B$ be a surjective homomorphism and $\beta : B \to C$ be a function such that $\beta \circ \alpha : A \to C$ is a homomorphism. Then β is a homomorphism.*

Proof. Let f be a k-ary function symbol. If $\bar{b} \in B^k$, then $\bar{b} = \alpha(\bar{a})$ for some $\bar{a} \in A^k$. Then

$$\beta(f(\bar{b})) = \beta(f(\alpha(\bar{a}))) = \beta(\alpha(f(\bar{a})) = f(\beta(\alpha(\bar{a})) = f(\beta(\bar{b}))$$

because α and $\beta \circ \alpha$ are homomorphisms. Therefore β is a homomorphism. $\square$

Theorem 3.6.6 (Universal property of quotients)**.** *Let A be an algebra and E be a congruence on A. If $\alpha : A \to B$ is a homomorphism and $E \subseteq \ker(\alpha)$, then there is a unique homomorphism $\beta : A/E \to B$ such that $\alpha(x) = \beta([x])$, or equivalently, the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, "\alpha"] & \\
A/E \arrow[r, dashed, "\beta"'] & B
\end{tikzcd}$$

Proof. The condition $E \subseteq \ker(\alpha)$ means that

$$[x]_E = [y]_E \implies \alpha(x) = \alpha(y).$$

By Theorem 3.3.1, there is a unique *function* $\beta : A/E \to B$ such that $\alpha(x) = \beta([x])$. We need β to be a homomorphism. The composition $A \to A/E \to B$ is the homomorphism α, and $A \to A/E$ is a surjective homomorphism, so $\beta : A/E \to B$ is a homomorphism by Lemma 3.6.5. □

Theorem 3.6.7 (Fundamental theorem on homomorphisms)**.** *Let* $\alpha : A \to B$ *be a homomorphism of* $\mathcal{L}$*-algebras. Let* E *be the kernel (a congruence on* A*) and let* $\operatorname{im}(\alpha)$ *be the image (a subalgebra of* B*). There is an isomorphism* $\beta : A/E \to \operatorname{im}(\alpha)$*, and* α *is the composition of the following three homomorphisms:*

$$A \to A/E \xrightarrow{\beta} \operatorname{im}(\alpha) \xrightarrow{\subseteq} B$$

Proof. We can regard $\alpha : A \to B$ as a surjective homomorphism $\alpha' : A \to \operatorname{im}(\alpha)$. Then $\ker(\alpha') = \ker(\alpha) \supseteq E$, so Theorem 3.6.6 gives a homomorphism $\beta : A/E \to \operatorname{im}(\alpha)$ making the diagram commute:

$$\begin{tikzcd}
A \arrow[rrr, "\alpha"] \arrow[dr] \arrow[drr, "\alpha'"] & & & B. \\
& A/E \arrow[r, dashed, "\beta"'] & \operatorname{im}(\alpha) \arrow[ur, hook] &
\end{tikzcd}$$

Then β is an isomorphism:

- β is surjective because any element of $\operatorname{im}(\alpha)$ has the form $\alpha(a) = \beta([a])$ for some $a \in A$.

- β is injective because for $a, b \in A$,

$$\beta([a]) = \beta([b]) \iff \alpha(a) = \alpha(b) \iff a \; E \; b \iff [a] = [b]. \qquad \square$$

In the case of surjective homomorphisms, Theorem 3.6.7 says the following:

Corollary 3.6.8. *If* $\alpha : A \to B$ *is a surjective homomorphism, then there is an isomorphism* $A/\ker(\alpha) \to B$*.*

3.7 Application: characteristic of fields

If R is a ring and $n \in \mathbb{Z}$, let n^R be the interpretation of n in R, that is,

$$n^R = \begin{cases} (\underbrace{1 + \cdots + 1}_{n \text{ times}})^R & \text{if } n > 0 \\ 0^R & \text{if } n = 0 \\ (-(\underbrace{1 + \cdots + 1}_{n \text{ times}}))^R & \text{if } n < 0 \end{cases}$$

For example, $n^{\mathbb{Z}} = n$.

Lemma 3.7.1. *The map $\alpha : n \mapsto n^R$ is the unique homomorphism from $\mathbb{Z}$ to R.*

Proof. The fact that α is a homomorphism is an exercise in induction and the ring axioms. If $\beta : \mathbb{Z} \to R$ is another homomorphism, then $\beta(n) = \alpha(n)$ by Theorem 2.4.4. For example,

$$\beta(3) = \beta(1 + 1 + 1) = (1 + 1 + 1)^R = \alpha(3). \qquad \square$$

The subring of R generated by the empty set, written $\langle \varnothing \rangle_R$, can be described abstractly as the smallest subring of R, or explicitly as the set of things of the form t^R for t a closed term.

Lemma 3.7.2. *The image* $\operatorname{im}(\alpha)$ *equals* $\langle \varnothing \rangle_R$.

Proof. The image $\operatorname{im}(\alpha)$ is a subring, so $\operatorname{im}(\alpha) \supseteq \langle \varnothing \rangle_R$. On the other hand, $\alpha(n) = n^R$ is clearly t^R for some closed term t, so $\alpha(n) \in \langle \varnothing \rangle_R$. Thus $\operatorname{im}(\alpha) \subseteq \langle \varnothing \rangle_R$. $\square$

We summarize the situation below:

Theorem 3.7.3. *If R is a ring, then there is a unique homomorphism $\alpha : \mathbb{Z} \to R$ given by $\alpha(n) = n^R$, and the image* $\operatorname{im}(\alpha)$ *is the minimal subring $\langle \varnothing \rangle_R$.*

Definition 3.7.4. The *characteristic* of R, written $\operatorname{char}(R)$, is the unique $n \in \mathbb{N}$ such that the kernel of $\mathbb{Z} \to R$ is the principal ideal $n\mathbb{Z}$.

Theorem 3.7.5. *If R has characteristic n, then the minimal subring $\langle \varnothing \rangle_R$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Proof. By Theorem 3.7.3, $\langle\varnothing\rangle_R$ is the image of the unique homomorphism $\alpha : \mathbb{Z} \to R$. By the fundamental theorem on homomorphisms, $\langle\varnothing\rangle_R = \operatorname{im}(\alpha) \cong \mathbb{Z}/\ker(\alpha) = \mathbb{Z}/n\mathbb{Z}$, where $n = \operatorname{char}(R)$. □

Theorem 3.7.6. *If K is a field, then* $\operatorname{char}(K) \in \{0, 2, 3, 5, 7, 11, \ldots\}$.

Proof. If $n = \operatorname{char}(K)$, and $\alpha : \mathbb{Z} \to R$ is the unique homomorphism, then $\ker(\alpha) = n\mathbb{Z}$. We must rule out the following cases:

- $n = 1$. Then $1 \in 1\mathbb{Z} = \ker(\alpha)$, so $1^K = \alpha(1) = 0^K$, and K is not a field.
- n is a composite number ab, for some integers $a, b > 1$. Then $a, b \notin n\mathbb{Z}$ and $n \in n\mathbb{Z}$, so $a, b \notin \ker(\alpha)$ but $ab = n \in \ker(\alpha)$. This means that
$$\begin{gathered}\alpha(a) \neq 0\\ \alpha(b) \neq 0\\ \alpha(a)\alpha(b) = \alpha(ab) = 0,\end{gathered}$$
contradicting the zero law (Theorem 2.4.11). □

Theorem 3.7.7. *If $p \in \{0, 2, 3, 5, 7, \ldots\}$, then there is a field of characteristic p.*

Proof. The field $\mathbb{R}$ has characteristic 0. If $p > 0$, the kernel of $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is $p\mathbb{Z}$, so the field $\mathbb{Z}/p\mathbb{Z}$ has characteristic p. □

Lemma 3.7.8. *Let R be a ring. Then $R/0R \cong R$.*

Proof. By the fundamental theorem on homomorphisms (Corollary 3.6.8) applied to the surjective homomorphism $\operatorname{id}_R : R \to R$, we have $R \cong R/I$, where $I = \ker(\operatorname{id}_R) = \{x \in R : \operatorname{id}(x) = 0\} = \{0\} = 0R$. □

Theorem 3.7.9. *If K is a field, then the minimal subring $\langle\varnothing\rangle_K$ is isomorphic to $\mathbb{Z}$ if* $\operatorname{char}(K) = 0$, *and $\mathbb{Z}/p\mathbb{Z}$ if* $\operatorname{char}(K) = p > 0$.

Theorem 3.7.10. *Let $L \supseteq K$ be an extension of fields. Then* $\operatorname{char}(L) = \operatorname{char}(K)$.

Proof. Let $\alpha : \mathbb{Z} \to K$ be the unique homomorphism from $\mathbb{Z}$ to K. Then α is also the unique homomorphism from $\mathbb{Z}$ to L. The kernel of α is the same whether we regard α as a homomorphism to K or to L. □

3.8 Term algebras and free algebras

Let $\bar{x}$ be a tuple of variables, possibly infinite. Let $T(\bar{x})$ be the set of terms in the variables $\bar{x}$. We make $T(\bar{x})$ into an algebra by setting

$$f^{T(\bar{x})}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k).$$

Theorem 3.8.1. *If A is an algebra and $\bar{a}$ is a tuple in A of the same length as $\bar{x}$, then the map*

$$\begin{aligned} \eta : T(\bar{x}) &\to A \\ \eta(t(\bar{x})) &= t^A(\bar{a}) \end{aligned}$$

is a homomorphism $T(\bar{x}) \to A$.

Proof. Suppose f is a k-ary function symbol, and $t_1(\bar{x}), \ldots, t_k(\bar{x}) \in T(\bar{x})$. Then

$$\begin{aligned} &\eta(f^{T(\bar{x})}(t_1(\bar{x}), \ldots, t_k(\bar{x}))) = \eta(f(t_1(\bar{x}), \ldots, t_k(\bar{x}))) \\ &= f^A(t_1^A(\bar{a}), \ldots, t_k^A(\bar{a})) = f^A(\eta(t_1(\bar{x})), \ldots, \eta(t_k(\bar{x}))). \end{aligned} \qquad \square$$

Fix an equational class $\mathcal{K}$. Let $\mathrm{Eq}(\mathcal{K})$ denote the set of equations φ such that every $A \in \mathcal{K}$ satisfies φ.

Definition 3.8.2. If t, s are two terms, then $t \equiv_{\mathcal{K}} s$ if the equation $t = s$ holds for all $A \in \mathcal{K}$. In other words, $t \equiv_{\mathcal{K}} s$ means that $(t = s) \in \mathrm{Eq}(\mathcal{K})$.

Theorem 3.8.3. *If $\bar{x}$ is a tuple of variables, then $\equiv_{\mathcal{K}}$ is a congruence on the term algebra $T(\bar{x})$.*

Proof. It is easy to see that $\equiv_{\mathcal{K}}$ is an equivalence relation. Suppose $t_1, \ldots, t_k$ and $s_1, \ldots, s_k$ are terms in $T(\bar{x})$ with $t_i \equiv_{\mathcal{K}} s_i$ for each i, and f is a k-ary function symbol. Then for any $A \in \mathcal{K}$ and tuple $\bar{a}$ in A,

$$t_i^A(\bar{a}) = s_i^A(\bar{a}) \text{ for } i = 1, \ldots, k,$$

and so

$$f(t_1(\bar{a}), \ldots, t_k(\bar{a}))^A = f(s_1(\bar{a}), \ldots, s_k(\bar{a}))^A.$$

It follows that

$$f(t_1(\bar{x}), \ldots, t_k(\bar{x})) \equiv_{\mathcal{K}} f(s_1(\bar{x}), \ldots, s_k(\bar{x})). \qquad \square$$

Definition 3.8.4. The *($\mathcal{K}$-)free algebra* on the variables $\bar{x}$, written $F_{\mathcal{K}}(\bar{x})$ or $F(\bar{x})$, is the quotient $T(\bar{x})/{\equiv_{\mathcal{K}}}$.

Theorem 3.8.5. *$F_{\mathcal{K}}(\bar{x}) \in \mathcal{K}$.*

Proof. Let $t(y_1, \dots, y_k) = s(y_1, \dots, y_k)$ be any of the axioms defining $\mathcal{K}$. For any terms $u_1, \dots, u_k \in T(\bar{x})$, any $A \in \mathcal{K}$, and any $\bar{a}$ in A, we have

$$t(u_1(\bar{a}), \dots, u_k(\bar{a})) = s(u_1(\bar{a}), \dots, u_k(\bar{a}))$$

because $A \models t = s$. Therefore

$$\begin{aligned} t(u_1(\bar{x}), \dots, u_k(\bar{x})) &\equiv_{\mathcal{K}} s(u_1(\bar{x}), \dots, u_k(\bar{x})) \\ [t(u_1(\bar{x}), \dots, u_k(\bar{x}))] &= [s(u_1(\bar{x}), \dots, u_k(\bar{x}))] \text{ in } F(\bar{x}) \\ t([u_1(\bar{x})], \dots, [u_k(\bar{x})]) &= s([u_1(\bar{x})], \dots, [u_k(\bar{x})]) \text{ in } F(\bar{x}) \end{aligned}$$

Thus $F(\bar{x}) \models t = s$. □

Theorem 3.8.6 (Universal mapping property)**.** *For any $A \in \mathcal{K}$ and tuple $\bar{a}$ in A (of the same length as $\bar{x}$), there is a homomorphism $\alpha : F(\bar{x}) \to A$ sending $[t(\bar{x})]$ to $t(\bar{a})$.*

Proof. Let $\beta : T(\bar{x}) \to A$ be the evaluation map $t(\bar{x}) \mapsto t(\bar{a})$. Note that

$$t \equiv_{\mathcal{K}} s \implies t(\bar{a}) = s(\bar{a}) \iff \beta(t) = \beta(s).$$

Therefore $(\equiv_{\mathcal{K}}) \subseteq \ker(\beta)$. By the universal property of quotients (Theorem 3.6.6), there is a homomorphism $\alpha : F(\bar{x}) \to A$ making the diagram commute

$$\begin{tikzcd} T(\bar{x}) \arrow[d] \arrow[dr, "\beta"] & \\ F(\bar{x}) \arrow[r, dashed, "\alpha"'] & A. \end{tikzcd}$$

Thus $\alpha([t(\bar{x})]) = \beta(t(\bar{x})) = t(\bar{a})$.

□

Corollary 3.8.7. *If $A \in \mathcal{K}$, then there is a surjective homomorphism $\alpha : F(\bar{x}) \to A$ for some $\bar{x}$.*

Proof. Let $\bar{a} = (a_i : i \in I)$ be a tuple (probably infinite) enumerating all of A. Let $\bar{x} = (x_i : i \in I)$ be a tuple of variables of the same length. Let $\alpha : F(\bar{x}) \to A$ be the homomorphism sending $[t(\bar{x})]$ to $t^A(\bar{a})$. Letting $t(\bar{x}) = x_i$, we see that $\alpha([x_i]) = a_i$, so α is surjective. □

3.9 Birkhoff's HSP theorem

Lemma 3.9.1. *Let $\alpha_i : A \to B_i$ be a homomorphism of $\mathcal{L}$-algebras for $i \in I$. Let $\alpha : A \to \prod_{i \in I} B_i$ be the map $\alpha(a) = (\alpha_i(a) : i \in I)$. Then α is a homomorphism.*

Proof. Let π_j be the jth coordinate projection from $\prod_{i \in I} B_i$ to B_j. If f is a k-ary function symbol and $a_1, \ldots, a_k \in A$, we must show

$$\alpha(f(\bar{a})) \stackrel{?}{=} f(\alpha(\bar{a})).$$

Both sides are in $\prod_{i \in I} B_i$. If the two sides disagree, then there is $i \in I$ such that

$$\pi_i(\alpha(f(\bar{a}))) \neq \pi_i(f(\alpha(\bar{a}))).$$

As π_i is a homomorphism (Remark 3.2.9), we can change the right hand side:

$$\pi_i(\alpha(f(\bar{a}))) \neq f(\pi_i(\alpha(\bar{a}))).$$

Now $\pi_i \circ \alpha = \alpha_i$ by definition of α, so

$$\alpha_i(f(\bar{a})) \neq f(\alpha_i(\bar{a})).$$

This contradicts the fact that α_i is a homomorphism. $\square$

Theorem 3.9.2 (Birkhoff's HSP theorem)**.** *Let $\mathcal{K}$ be a class of algebras. Then $\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is closed under isomorphisms, subalgebras, products, and quotients.*

Proof. Equational classes are closed under isomorphisms (Theorem 2.4.6), subalgebras (Theorem 3.1.2), products (Theorem 3.2.10), and quotients (Theorem 3.4.9).

Conversely, suppose $\mathcal{K}$ is closed under isomorphisms, subalgebras, products, and quotients. Let Σ be $\mathrm{Eq}(\mathcal{K})$, the set of equations holding on $\mathcal{K}$. If $A \in \mathcal{K}$, then $A \models \Sigma$, and so $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma) =: \overline{\mathcal{K}}$.

Claim. If $\bar{x}$ is a tuple of variables, then the $\overline{\mathcal{K}}$-free algebra $F_{\overline{\mathcal{K}}}(\bar{x})$ is in $\mathcal{K}$.

Proof. Let $\{(a_i, b_i)\}_{i \in I}$ enumerate all the pairs of distinct elements of $F = F_{\overline{\mathcal{K}}}(\bar{x})$. For each i, we can write $a_i = [t_i(\bar{x})]$ and $b_i = [s_i(\bar{x})]$ for terms $t_i, s_i \in T(\bar{x})$. The fact that $a_i \neq b_i$ means that $t_i(\bar{x}) \not\equiv_{\overline{\mathcal{K}}} s_i(\bar{x})$. Thus $(t_i = s_i) \notin \Sigma = \mathrm{Eq}(\mathcal{K})$, so there is $A_i \in \mathcal{K}$ with $A_i \not\models t_i = s_i$. Then there is $\bar{a}_i \in A_i$ with $t_i^A(\bar{a}_i) \neq s_i^A(\bar{a}_i)$. As $A_i \in \mathcal{K} \subseteq \overline{\mathcal{K}}$, there is a homomorphism $\beta_i : F(\bar{x}) \to A_i$ sending $[t(\bar{x})]$ to $t^{A_i}(\bar{a}_i)$ (see Theorem 3.8.6). Then

$$\beta_i(a_i) = \beta_i([t_i(\bar{x})]) = t_i^{A_i}(\bar{a}_i) \neq s_i^{A_i}(\bar{a}_i) = \beta_i([s_i(\bar{x})]) = \beta_i(b_i).$$

Thus, for every $i \in I$,

$$\beta_i(a_i) \neq \beta_i(b_i). \tag{$*$}$$

By Lemma 3.9.1, there is a homomorphism $\beta : F(\bar{x}) \to \prod_{i \in I} A_i$ with $\beta(x) = (\beta_i(x) : i \in I)$. Note that $\prod_{i \in I} A_i \in \mathcal{K}$ because $\mathcal{K}$ is closed under products. We claim that β is injective. Indeed, if $a, b \in F(\bar{x})$ and $a \neq b$, then $(a, b) = (a_i, b_i)$ for some $i \in I$. Then $\beta_i(a) = \beta_i(a_i) \neq \beta_i(b_i) = \beta_i(b)$, so $\beta(a)$ and $\beta(b)$ differ at the ith coordinate.

Then β is an isomorphism from $F(\bar{x})$ to $\mathrm{im}(\beta)$. As $\mathrm{im}(\beta)$ is a subalgebra of $\prod_{i \in I} A_i \in \mathcal{K}$, we have $\mathrm{im}(\beta) \in \mathcal{K}$ and then $F(\bar{x}) \in \mathcal{K}$. $\square_{\text{Claim}}$

Now if $A \in \overline{\mathcal{K}}$ then there is a surjective homomorphism $F(\bar{x}) \to A$ for some $\overline{\mathcal{K}}$-free algebra $F(\bar{x})$ (Corollary 3.8.7). By the fundamental theorem on homomorphisms (Corollary 3.6.8), A is isomorphic to a quotient of $F(\bar{x})$. By the claim, $F(\bar{x}) \in \mathcal{K}$, and thus $A \in \mathcal{K}$.

This shows that $\overline{\mathcal{K}} \subseteq \mathcal{K}$. As $\mathcal{K} \subseteq \overline{\mathcal{K}}$, the class $\mathcal{K}$ equals the equational class $\overline{\mathcal{K}}$. $\square$

Chapter 4

First-order logic

4.1 Languages, structures, formulas, and satisfaction

Definition 4.1.1. A *language* $\mathcal{L}$ consists of a set of *function symbols*, a set of *relation symbols*, and a function assigning to each function or relation symbol X an integer $n_X \in \mathbb{N}$ called the *arity* of X. A symbol X is said to be *k-ary* if $n_X = k$. Nullary function symbols are called *constant symbols*.

Example 4.1.2. The *language of posets* contains one binary relation symbol $\leq$. The *language of monoids* contains one binary function symbol $\cdot$ and one constant symbol 1.

Let M be a set. For $n \geq 0$, an *n-ary relation* on M is a subset $R \subseteq M^n$. If $a_1, \ldots, a_n \in M$, then "$R(a_1, \ldots, a_n)$" means "$(a_1, \ldots, a_n) \in R$". We think of R as an n-ary function $M^n \to \{\textsc{False}, \textsc{True}\}$.

Definition 4.1.3. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-structure* $\mathcal{M}$ consists of the following:

1. A set M, called the *underlying set* of $\mathcal{M}$.
2. A map assigning an n-ary operation $f^{\mathcal{M}} : M^n \to M$ to each n-ary function symbol f.
3. A map assigning an n-ary relation $R^{\mathcal{M}} \subseteq M^n$ to each n-ary relation symbol R.

Usually we don't distinguish between a structure $\mathcal{M}$ and its underlying set M, writing both as M.

Example 4.1.4. If $\mathcal{L}$ is the language of posets, then an $\mathcal{L}$-structure is essentially a pair $(M, \leq^M)$ where M is a set and $\leq^M$ is a binary relation on M.

Definition 4.1.5. Let $\mathcal{L}^+$ be a language and $\mathcal{L}^-$ be a sublanguage. For any $\mathcal{L}^+$-structure M, we can form an $\mathcal{L}^-$-structure from M by forgetting about the symbols in $\mathcal{L}^+ \setminus \mathcal{L}^-$. The resulting structure $M \restriction \mathcal{L}^-$ is called a *reduct* of M. Conversely, M is an *expansion* of $M \restriction \mathcal{L}^-$.

Fix a language $\mathcal{L}$ and a class $\mathcal{V} = \{x, y, z, \ldots\}$ of *variable symbols* disjoint from the symbols in $\mathcal{L}$.

Definition 4.1.6. An $\mathcal{L}$-*term* is a string generated by the following rules:

- If x is a variable symbol, then x is a term.
- If f is an n-ary function symbol in $\mathcal{L}$, and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.

Definition 4.1.7. An $\mathcal{L}$-*formula* is a string generated by the following rules:

1. If R is a k-ary relation symbol in $\mathcal{L}$ and $t_1, \ldots, t_k$ are $\mathcal{L}$-terms, then $R(t_1, \ldots, t_k)$ is an $\mathcal{L}$-formula.
2. If t and s are $\mathcal{L}$-terms, then $(t = s)$ is an $\mathcal{L}$-formula.
3. If φ and ψ are $\mathcal{L}$-formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\neg\varphi$.
4. $\top$ and $\bot$ are $\mathcal{L}$-formulas.
5. If φ is an $\mathcal{L}$-formula and x is a variable symbol, then the following are $\mathcal{L}$-formulas:

$$\exists x \ (\varphi) \in \mathcal{F}$$
$$\forall x \ (\varphi) \in \mathcal{F}.$$

The set of *atomic* $\mathcal{L}$-*formulas* is the subset generated by (1)–(2), while the set of *quantifier-free* $\mathcal{L}$-*formulas* is generated by (1)–(4). If x is a variable symbol and α is a term or formula, an occurrence of x in α is *bound* if it occurs inside a quantifier $\exists x\ (\dots)$ or $\forall x\ (\dots)$, and *free* otherwise. The *free variables* of α are the variables with free occurrences in α. A *closed term* is a term with no free variables (i.e., no variables). A *sentence* is a formula with no free variables.

We write "let $\varphi(x_1, \dots, x_n)$ be a formula" to mean that $\varphi(x_1, \dots, x_n)$ is a formula whose free variables are contained in $\{x_1, \dots, x_n\}$. We use the same convention for terms. If $\varphi(x_1, \dots, x_n)$ is a formula and $t_1, \dots, t_n$ are terms, then $\varphi(t_1, \dots, t_n)$ denotes the result of replacing the free occurrences of x_i with t_i in φ.

Let M be an $\mathcal{L}$-structure and A be a subset of M. The language $\mathcal{L}(A)$ is obtained by adding each element of A as a new constant symbol. We regard M as an $\mathcal{L}(A)$-structure by interpreting each new constant symbol as the corresponding element of A:

$$a^M = a.$$

If t is a closed $\mathcal{L}(M)$-term, define the interpretation t^M recursively as follows:

$$f(t_1, \dots, t_k)^M = f^M(t_1^M, \dots, t_k^M),$$

If $t(x_1, \dots, x_n)$ is an $\mathcal{L}$-term or $\mathcal{L}(M)$-term, then the interpretation t^M is the function $M^n \to M$ defined by

$$t^M(a_1, \dots, a_n) = (t(a_1, \dots, a_n))^M.$$

If φ is an $\mathcal{L}(M)$-sentence, we define $M \models \varphi$ recursively:

$$\begin{gathered}
M \models t = s \iff t^M = s^M \\
M \models R(t_1, \dots, t_k) \iff (t_1^M, \dots, t_k^M) \in R^M \\
M \models \varphi \vee \psi \iff (M \models \varphi \text{ or } M \models \psi) \\
M \models \varphi \wedge \psi \iff (M \models \varphi \text{ and } M \models \psi) \\
M \models \neg\varphi \iff M \not\models \varphi \\
M \models \top \text{ is always true} \\
M \models \bot \text{ is always false} \\
M \models \exists x\ \varphi(x) \iff \exists a \in M : M \models \varphi(a) \\
M \models \forall x\ \varphi(x) \iff \forall a \in M : M \models \varphi(a).
\end{gathered}$$

If $M \models \varphi$, we say that M *satisfies* φ, or φ is *true* in M.

4.2 Theories and elementary classes

Fix a language $\mathcal{L}$. An *$\mathcal{L}$-theory* is a set T of $\mathcal{L}$-sentences. Elements of T are called *axioms*. An $\mathcal{L}$-structure M *satisfies* T, or is a *model* of T, written $M \models T$, if

$$M \models \varphi \text{ for every } \varphi \in T.$$

The set of models of T is written $\mathrm{Mod}(T)$. An *elementary class* is a class of the form $\mathrm{Mod}(T)$.

Example 4.2.1. Let $\mathcal{L}$ be the language of rings. The *theory of rings* T_{rings} has the following axioms:

$$\forall x \; \forall y \; (x + y = y + x)$$
$$\forall x \; \forall y \; \forall z \; (x + (y + z) = (x + y) + z)$$
$$\cdots$$

Models of T_{rings} are exactly rings. The *theory of fields* T_{fields} is T_{rings} plus the following two axioms:

$$\neg(0 = 1)$$
$$\forall x \; (x = 0 \vee \exists y \; (x \cdot y = 1)).$$

Models of T_{fields} are fields.

4.3 Common abbreviations

The following abbreviations are standard:

Abbreviation	Meaning
$\varphi \to \psi$	$\neg\varphi \vee \psi$
$\varphi \leftarrow \psi$	$\psi \to \varphi$
$\varphi \leftrightarrow \psi$	$(\varphi \to \psi) \wedge (\psi \to \varphi)$
$t \neq s$	$\neg t = s$
$\bigwedge_{i=1}^n \varphi_i$	$\varphi_1 \wedge \cdots \wedge \varphi_n$ [or $\top$ if $n = 0$]
$\bigvee_{i=1}^n \varphi_i$	$\varphi_i \vee \cdots \vee \varphi_n$ [or $\perp$ if $n = 0$]
$\exists^{\geq n} \varphi(x)$	$\exists y_1, \ldots, y_n \left(\bigwedge_{i=1}^n \varphi(y_i) \wedge \bigwedge_{1 \leq i < j \leq n} (y_i \neq y_j) \right)$
$\exists^{=n} \varphi(x)$	$\exists^{\geq n} x \; \varphi(x) \wedge \neg \exists^{\geq n+1} x \; \varphi(x)$
$\exists! x \; \varphi(x)$	$\exists^{=1} x \; \varphi(x)$.

Additionally, if $\mathcal{L}$ contains a symbol $\leq$, then

$$t \geq s \text{ means } s \leq t$$
$$t < s \text{ means } t \leq s \wedge t \neq s$$
$$t > s \text{ means } s < t.$$

Example 4.3.1. Let $\mathcal{L} = \{\leq\}$ be the language of posets. The *theory of posets*, whose models are posets, has the following axioms:

$$\forall x(x \leq x)$$
$$\forall x, y, z(x \leq y \wedge y \leq z \to x \leq z)$$
$$\forall x, y(x \leq y \wedge y \leq x \to x = y).$$

The *theory of linear orders* adds one more axiom:

$$\forall x, y(x \leq y \vee y \leq x).$$

4.4 Elementary equivalence and embeddings

The *complete theory* of an $\mathcal{L}$-structure M is the set

$$\mathrm{Th}(M) := \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence and } M \models \varphi\}.$$

Two $\mathcal{L}$-structures M and N are *elementarily equivalent*, written $M \equiv N$, if $\mathrm{Th}(M) = \mathrm{Th}(N)$, meaning that $M \models \varphi \iff N \models \varphi$ for every $\mathcal{L}$-sentence φ.

Theorem 4.4.1. *A structure N is a model of* $\mathrm{Th}(M)$ *if and only if $N \equiv M$.*

Proof. Note that $N \models \mathrm{Th}(M)$ iff $\mathrm{Th}(N) \supseteq \mathrm{Th}(M)$. (More generally, $N \models T$ iff $\mathrm{Th}(N) \supseteq T$.) So we must rule out the case that $\mathrm{Th}(N) \supsetneq \mathrm{Th}(M)$. Suppose $\varphi \in \mathrm{Th}(N) \setminus \mathrm{Th}(M)$. Then $N \models \varphi$ and $M \not\models \varphi$, implying that $N \not\models \neg\varphi$ and $M \models \neg\varphi$. Then $(\neg\varphi) \in \mathrm{Th}(M) \setminus \mathrm{Th}(N)$, contradicting the fact that $\mathrm{Th}(N) \supseteq \mathrm{Th}(M)$. $\square$

Definition 4.4.2. Let M, N be $\mathcal{L}$-structures. A function $\alpha : M \to N$ is an *embedding* if α is injective, and α strictly preserves the function and relation symbols:

$$\alpha(f^M(b_1, \ldots, b_n)) = f^N(\alpha(b_1), \ldots, \alpha(b_n))$$
$$R^M(b_1, \ldots, b_n) \iff R^N(\alpha(b_1), \ldots, \alpha(b_n)).$$

An *isomorphism* is a bijective embedding. Two structures M and N are *isomorphic*, written $M \cong N$, if there is an isomorphism from M to N.

The analogue of Theorem 2.4.2 holds, and so $\cong$ is an equivalence relation.

Theorem 4.4.3. *Let $f : M \to N$ be an embedding and $a_1, \ldots, a_n$ be in M.*

1. *If $t(x_1, \ldots, x_n)$ is a term, then*

$$f(t^M(a_1, \ldots, a_n)) = t^N(f(a_1), \ldots, f(a_n)).$$

2. *If $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula, then*

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n)).$$

3. *If f is an isomorphism and $\varphi(x_1, \ldots, x_n)$ is any formula, then*

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n)).$$

Proof. By induction on the complexity of t or φ, like in Theorem 2.4.4. (In fact, part (1) is an instance of Theorem 2.4.4.) $\square$

Definition 4.4.4. A function $\alpha : M \to N$ is an *elementary embedding* if α preserves all $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$, in the sense that

$$M \models \varphi(b_1, \ldots, b_n) \iff N \models \varphi(\alpha(b_1), \ldots, \alpha(b_n)). \qquad (*)$$

Remark 4.4.5. 1. Any elementary embedding is an embedding: apply $(*)$ to the following formulas:

$$x_1 = x_2$$
$$R(x_1, \ldots, x_k)$$
$$f(x_1, \ldots, x_k) = x_{k+1}$$

2. Theorem 4.4.3(3) says that isomorphisms are elementary embeddings.

3. If $\alpha : M \to N$ is an elementary embedding, then $M \equiv N$. To see this, take $n = 0$ in $(*)$, so that φ is a sentence. Then $(*)$ says that $M \models \varphi \iff N \models \varphi$.

Corollary 4.4.6. *If $M \cong N$, then $M \equiv N$.*

4.5 Substructures and extensions

Let M be an $\mathcal{L}$-structure.

Definition 4.5.1. A *substructure* of M is a subset $A \subseteq M$ such that for any k-ary function symbol f in $\mathcal{L}$,

$$a_1, \ldots, a_k \in A \implies f^M(a_1, \ldots, a_k) \in A.$$

If A is a substructure of M, we regard A as an $\mathcal{L}$-structure by defining f^A and R^A to be the restrictions of f^M and R^M to A. In this way, substructures are structures, not just sets.

If M is an $\mathcal{L}$-structure, an *extension* of M is an $\mathcal{L}$-structure N such that M is a substructure of N.

Definition 4.5.2. If A is a subset of M, then $\langle A \rangle_M$ denotes the smallest substructure of M containing A, which is

$$\{t^M(\bar{b}) : t(x_1, \ldots, x_n) \text{ is an } \mathcal{L}\text{-term and } \bar{b} \in A^n\}$$

The substructure $\langle A \rangle_M$ is called the substructure *generated* by A. We say that M is *finitely generated* if $M = \langle A \rangle$ for some finite $A \subseteq M$.

Remark 4.5.3. If the language $\mathcal{L}$ has no function symbols or constant symbols, then every subset of M is a substructure, and so $\langle A \rangle_M = A$.

Remark 4.5.4. If A is a substructure of M, then the inclusion $A \hookrightarrow M$ is an embedding. Therefore

$$A \models \varphi(\bar{a}) \iff M \models \varphi(\bar{a})$$

for *quantifier-free* formulas φ and tuples $\bar{a}$ in A.

Definition 4.5.5. Let M and N be $\mathcal{L}$-structures. We say that M is an *elementary substructure* of N, written $M \preceq N$, or that N is an *elementary extension* of M, written $N \succeq M$, if M is a substructure of N and the inclusion $M \to N$ is an elementary embedding, meaning that

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(a_1, \ldots, a_n) \qquad (*)$$

for $\mathcal{L}$-formulas φ and $\bar{a} \in M^n$.

Equation $(*)$ says that M and N satisfy the same $\mathcal{L}(M)$-sentences.

Theorem 4.5.6. *Let f be an embedding from M to N.*

1. *The image* $\mathrm{im}(f)$ *is a substructure of N, and $M \to \mathrm{im}(f)$ is an isomorphism.*

2. *If f is an elementary embedding, then* $\mathrm{im}(f) \preceq N$.

Proof. 1. Straightforward. The fact that im() is a substructure is like the analogous fact for homomorphisms (Theorem 3.6.1).

2. If $M' = \mathrm{im}(f)$, then the inclusion $M' \hookrightarrow N$ is the composition of two elementary maps:

$$M' \xrightarrow{\cong} M \xrightarrow{f} N.$$

□

4.6 Complete theories

We write $T \vdash \varphi$ if φ is provable from T, and $T \models \varphi$ if every model of T satisfies φ.

Fact 4.6.1 (Soundness and completeness theorem)**.** *If T is a theory and φ is a sentence, then*

$$T \vdash \varphi \iff T \models \varphi.$$

A theory T is *inconsistent* if the following equivalent conditions hold:

1. $T \vdash \bot$.
2. $T \vdash \varphi$ for all φ.
3. $T \vdash \varphi$ and $T \vdash \neg\varphi$ for some φ.

Otherwise, T is *consistent*. Note that $T \models \bot$ if and only if T has no models, because no structure M satisfies $\bot$. Therefore the completeness theorem implies the following:

Theorem 4.6.2. *T is consistent if and only if T has a model.*

A theory T is *complete* if T is consistent and for any sentence φ,

$$T \vdash \varphi \text{ or } T \vdash \neg\varphi.$$

Theorem 4.6.3. *If T is complete and $M \models T$, then for any sentence φ,*

$$M \models \varphi \iff T \vdash \varphi.$$

Proof. As $M \models T$ and T is complete,

$$T \vdash \varphi \implies M \models \varphi$$
$$T \nvdash \varphi \implies T \vdash \neg\varphi \implies M \models \neg\varphi \implies M \not\models \varphi. \qquad \square$$

Theorem 4.6.4. *Let T be a theory. Then T is complete if and only if any two models $M_1, M_2 \models T$ are elementarily equivalent.*

Proof. First suppose T is complete. If M_1, M_2 are two models, then $\mathrm{Th}(M_1) = \{\varphi : T \vdash \varphi\} = \mathrm{Th}(M_2)$ by Theorem 4.6.3, and so $M_1 \equiv M_2$.

Conversely suppose T is incomplete, with $T \nvdash \varphi$ and $T \nvdash \neg\varphi$. By the completeness theorem, there are models $M_1, M_2 \models T$ with $M_1 \models \neg\varphi$ and $M_2 \models \varphi$. Then $M_1 \not\equiv M_2$. $\square$

Fact 4.6.5. *Let $\mathcal{L}$ be a countable language. If T is finite or more generally if T is computably enumerable, then the set $\overline{T} = \{\varphi : T \vdash \varphi\}$ is computably enumerable.*

Theorem 4.6.6. *If $\mathcal{L}$ is countable, T is complete and computably enumerable, and M is a model of T, then $\mathrm{Th}(M)$ is computable—there is an algorithm which takes an $\mathcal{L}$-sentence φ and determines whether $M \models \varphi$.*

Proof. By Theorem 4.6.3, $\mathrm{Th}(M) = \overline{T}$, which is computably enumerable by Fact 4.6.5. However,

$$\varphi \notin \mathrm{Th}(M) \iff \neg\varphi \in \mathrm{Th}(M),$$

and so $\mathrm{Th}(M)$ is also co-c.e. As $\mathrm{Th}(M)$ is both c.e. and co-c.e., it is computable. $\square$

4.7 Completeness via back-and-forth systems

Let M, N be two $\mathcal{L}$-structures.

Definition 4.7.1. A *partial elementary map* from M to N is a bijection $f : A \to B$ where A and B are subsets of M and N, respectively, and f preserves all $\mathcal{L}$-formulas, in the sense that

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n))$$

for $n \geq 0$, $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula, and $a_1, \ldots, a_n \in A$.

Remark 4.7.2. Taking $n = 0$, we see that if f is a partial elementary map from M to N, then $M \equiv N$.

Conversely, $\varnothing : \varnothing \to \varnothing$ is a partial elementary map from M to N *if and only if* $M \equiv N$.

Remark 4.7.3. An elementary embedding from M to N is the same thing as a partial elementary map f from M to N with $\mathrm{dom}(f) = M$.

Definition 4.7.4. A *partial isomorphism* from M to N is an isomorphism $f : A \to B$ where A is a substructure of M and B is a substructure of N.

By Theorem 4.4.3, partial isomorphisms preserve quantifier-free formulas: if $f : A \to B$ is a partial isomorphism and $\bar{a} \in A^n$ and $\varphi(\bar{x})$ is quantifier-free, then

$$M \models \varphi(\bar{a}) \iff A \models \varphi(\bar{a}) \iff B \models \varphi(f(\bar{a})) \iff N \models \varphi(f(\bar{a})).$$

Definition 4.7.5. A *back-and-forth system* between M and N is a family $\mathcal{F}$ of partial isomorphisms satisfying the following conditions:

1. Forward: if $f : A \to B$ is in $\mathcal{F}$, and $a' \in M$, then there is $f' : A' \to B'$ in $\mathcal{F}$ such that f' extends f and $a' \in A'$.

2. Backward: if $f : A \to B$ is in $\mathcal{F}$, and $b' \in N$, then there is $f' : A' \to B'$ in $\mathcal{F}$ such that f' extends f and $b' \in B'$.

Theorem 4.7.6. *If $\mathcal{F}$ is a back-and-forth system, then every $f \in \mathcal{F}$ is a partial elementary map from M to N.*

Proof. We show by induction on the complexity of $\varphi(x_1, \ldots, x_n)$ that if $f : A \to B$ is in $\mathcal{F}$ and $a_1, \ldots, a_n \in A$, then

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n)). \qquad (\dagger)$$

The case of atomic formulas holds because f is a partial isomorphism. The case of boolean operations is straightforward. It remains to consider the case where $\varphi(\bar{x})$ is $\exists y : \psi(\bar{x}, y)$. By symmetry, we only need to prove the $(\Rightarrow)$ direction of (†). Suppose $M \models \exists y : \psi(\bar{a}, y)$. Take some $a' \in M$ such that $M \models \psi(\bar{a}, a')$. By the "forward" condition, there is $g \in \mathcal{F}$ extending f with $a' \in \mathrm{dom}(g)$. By induction, $M \models \psi(\bar{a}, a') \implies N \models \psi(g(\bar{a}), g(a'))$, and then $N \models \varphi(g(\bar{a}))$. But $g(\bar{a}) = f(\bar{a})$, so $N \models \varphi(f(\bar{a}))$ as desired. □

Corollary 4.7.7. *Let M, N be two structures. If there is a non-empty back-and-forth system between M and N, then $M \equiv N$.*

Definition 4.7.8. A linear order $(M, \leq)$ is *dense* if for any $x < y$ there is z with $x < z < y$.

Definition 4.7.9. DLO is the theory of non-empty dense linear orders without endpoints. That is, $(M, \leq) \models \mathrm{DLO}$ if $(M, \leq)$ is a non-empty dense linear order without a greatest or least element.

Theorem 4.7.10. *Let M and N be models of* DLO *and let $\mathcal{F}$ be the class of finite partial isomorphisms $f : A \to B$ between M and N. Then $\mathcal{F}$ is a back-and-forth system.*

Proof. We prove the forward condition; the backward condition follows by symmetry. Fix a finite partial isomorphism $f : A \to B$ and an element $a' \in M$. We must extend f to a larger finite partial isomorphism f' containing a'. Enumerate the elements of A:

$$A = \{a_1, \ldots, a_n\}$$
$$a_1 < a_2 < \cdots < a_n$$

Let $b_i = f(a_i)$. Then

$$B = \{b_1, \ldots, b_n\}$$
$$b_1 < b_2 < \cdots < b_n.$$

There are five cases.

1. $n = 0$, so that A, B are empty. Take any $b' \in N$. (N is non-empty.) Then $f' : \{a'\} \to \{b'\}$ extends f and is defined at a'.

2. $a' = a_i \in A$. Then we can take $f' = f$.

3. $a' < a_1$. Take $b' \in N$ with $b' < b_1$. (N has no minimum.) Then there is a partial isomorphism

$$f' : \{a', a_1, \ldots, a_n\} \to \{b', b_1, \ldots, b_n\}$$

extending f and sending a' to b'.

4. $a' > a_n$. This case is similar to the previous case.

5. $a_i < a' < a_{i+1}$ for some i. Take $b' \in N$ with $b_i < b' < b_{i+1}$. (N is densely ordered.) Then there is a partial isomorphism

$$f' : \{a_1, \ldots, a_i, a', a_{i+1}, \ldots, a_n\} \to \{b_1, \ldots, b_i, b', b_{i+1}, \ldots, b_n\}$$

extending f and sending a' to b'. □

Corollary 4.7.11. *1. The theory* DLO *is complete—any two models M, N are elementarily equivalent.*

2. If $M \models$ DLO*, then* $\mathrm{Th}(M)$ *is decidable.*

For example, the complete theory of $(\mathbb{Q}, \leq)$ is decidable.

4.8 Definable sets and functions

Fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure M. If $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula, then $\varphi(M^n)$ or $\varphi(M)$ denotes the set *defined by* φ:

$$\varphi(M^n) := \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}.$$

More generally, if $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ is an $\mathcal{L}$-formula and $\bar{b} \in M^m$, then $\varphi(M^n, \bar{b})$ or $\varphi(M, \bar{b})$ denotes the set defined by $\varphi(\bar{x}, \bar{b})$:

$$\varphi(M^n, \bar{b}) := \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}.$$

Sets of this form are called *definable sets*. A set $D \subseteq M^n$ is *A-definable* or *definable over A* if $D = \varphi(M^n, \bar{b})$ with $\bar{b} \in A^m$. That is, an A-definable set is a set defined by an $\mathcal{L}(A)$-formula. Definable sets are the same thing as M-definable sets. A *0-definable set* is a $\varnothing$-definable set. If D, E are definable sets, a function $f : D \to E$ is *definable* if the graph $\Gamma(f) = \{(x, y) \in D \times E : f(x) = y\}$ is a definable set. A-definable functions are defined similarly.

Remark 4.8.1. If $t(x_1, \ldots, x_n)$ is an $\mathcal{L}(M)$-term, then the function $t^M : M^n \to M$ is definable, defined by the formula $t(\bar{x}) = y$.

Theorem 4.8.2. *If $f : X \to Y$ and $g : Y \to Z$ are definable, then $g \circ f : X \to Z$ is definable.*

Proof. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}(M)$-formula defining f, meaning that

$$f(\bar{a}) = \bar{b} \iff M \models \varphi(\bar{a}, \bar{b}).$$

Similarly, let $\psi(\bar{y}, \bar{z})$ define g. Let $\theta(\bar{x}, \bar{z})$ be

$$\exists \bar{y}\ (\varphi(\bar{x}, \bar{y})\ \wedge\ \psi(\bar{y}, \bar{z})).$$

Then θ defines $g \circ f$:

$$\begin{aligned} M \models \theta(\bar{a}, \bar{c}) &\iff \exists \bar{b}\ \big(M \models \varphi(\bar{a}, \bar{b}) \text{ and } M \models \psi(\bar{b}, \bar{c})\big) \\ &\iff \exists \bar{b}\ (f(\bar{a}) = \bar{b} \text{ and } g(\bar{b}) = \bar{c}) \\ &\iff g(f(\bar{a})) = \bar{c}. \end{aligned}$$ □

4.9 Incompleteness of Peano Arithmetic

Consider $\mathbb{N}$ as a structure $(\mathbb{N}, +, \cdot, 0, 1)$.

Remark 4.9.1. The relation $x \leq y$ is definable, defined by

$$\exists z : z + x = y.$$

Remark 4.9.2. The function $|x - y|$ is definable.

Proof. $|x - y| = z$ if and only if $x = y + z \vee y = x + z$. □

Definition 4.9.3. If $n > 0$ and $a \in \mathbb{N}$, then $a \bmod n$ denotes the unique $x \in \{0, 1, \ldots, n-1\}$ such that $x \equiv a \pmod n$.

Lemma 4.9.4. *The function $x \bmod y$ is definable.*

Proof. $(x \bmod y) = z$ if and only if

$$z + 1 \leq y \wedge (\exists k : k \cdot y + z = x).$$ □

Definition 4.9.5. For $a, b, x \in \mathbb{N}$, $\gamma(a, b, x) = (a \bmod ((x+1)b+1))$.

Lemma 4.9.6. *For any finite sequence $c_0, c_1, \ldots, c_n \in \mathbb{N}$, there are $a, b \in \mathbb{N}$ such that $c_i = \gamma(a, b, i)$ for $0 \leq i \leq n$.*

Proof. Take $m > \max(c_0, \ldots, c_n, n)$, and let $b = m! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots m$.
Claim. The numbers $\{b+1, 2b+1, 3b+1, \ldots, (n+1)b+1\}$ are pairwise coprime.

Proof. Suppose a prime number p divides both $ib+1$ and $jb+1$ for some $1 \leq i < j \leq n+1$. Then p divides the difference $(j-i)b$, and one of two things happens:

- p divides b.
- p divides $j - i$. Then $p \leq j - i \leq n \leq m$, so p divides $m! = b$.

Either way, p divides b, and so

$$0 \equiv ib + 1 \equiv i0 + 1 = 1 \pmod{p},$$

a contradiction. $\square_{\text{Claim}}$

By the Chinese remainder theorem, there is an a such that

$$a \equiv c_i \pmod{(i+1)b+1}$$

for each $0 \leq i \leq n$. But $c_i \leq m \leq b < (i+1)b+1$, and so

$$c_i = (a \bmod ((i+1)b+1)) = \gamma(a, b, i).$$ $\square$

Theorem 4.9.7. *Let $f(n)$ be the nth Fibonacci number* $(1, 1, 2, 3, 5, 8, 13, \ldots)$

$$f(0) = f(1) = 1$$
$$f(n+2) = f(n) + f(n+1)$$

Then the function $f : \mathbb{N} \to \mathbb{N}$ is definable.

Proof. For fixed n and k, the following are equivalent:

1. $f(n) = k$.

2. There are $c_0, \ldots, c_n$ such that the following hold:

$$\begin{aligned} c_0 &= 1 \\ c_1 &= 1 \\ c_i &= c_{i-1} + c_{i-2} \text{ for all } 2 \leq i \leq n. \end{aligned}$$

3. There are a and b such that the following hold:

$$\begin{aligned} \gamma(a, b, 0) &= 1 \\ \gamma(a, b, 1) &= 1 \\ \gamma(a, b, i) &= \gamma(a, b, i-1) + \gamma(a, b, i-2) \text{ for all } 2 \leq i \leq n. \end{aligned}$$

Indeed, (2) and (3) are equivalent by Lemma 4.9.6. Condition (3) is easily expressed by a formula. □

Theorem 4.9.8. *If $f : \mathbb{N}^k \to \mathbb{N}$ is computable, then f is definable.*

Proof sketch. The only hard thing to check is primitive recursion, which is handled by the method of Theorem 4.9.7. □

Corollary 4.9.9. *If $S \subseteq \mathbb{N}^k$ is computable, then S is definable.*

Theorem 4.9.10. $\mathrm{Th}(\mathbb{N})$ *is incomputable.*

Proof. Fix a reasonable enumeration of Turing machines. Let $S \subseteq \mathbb{N}^2$ be the set of pairs (n, k) such that the nth Turing machine halts within k steps. By Corollary 4.9.9, S is definable, defined by an $\mathcal{L}(\mathbb{N})$-formula $\varphi(x, y, n_1, n_2, \ldots)$. Replacing n_i with $\underbrace{1 + 1 + \cdots + 1}_{n_i \text{ times}}$, we see that S is $\varnothing$-definable, defined by an $\mathcal{L}$-formula $\varphi(x, y)$. Then for any $n \in \mathbb{N}$,

$$\mathbb{N} \models \exists y\ \varphi(\underbrace{1 + 1 + \cdots +}_{n \text{ times}}, y) \iff (\text{the } n\text{th Turing machine halts}).$$

Therefore the halting problem reduces to $\mathrm{Th}(\mathbb{N})$. □

Let $\mathcal{L}$ be the language $\{+, \cdot, 0, 1\}$. *Peano Arithmetic* (PA) consists of the following axioms:

$$\begin{gathered}
\forall x : x + 1 \neq 0 \\
\forall x : (x = 0 \vee (\exists y : x = y + 1)) \\
\forall x, y : (x + 1 = y + 1 \rightarrow x = y) \\
\forall x : x + 0 = x \\
\forall x, y : x + (y + 1) = (x + y) + 1 \\
\forall x : x \cdot 0 = 0 \\
\forall x, y : x \cdot (y + 1) = (x \cdot y) + x
\end{gathered}$$

as well as the infinite set of axioms $\{Ind_\varphi : \varphi(x, y_1, \ldots, y_n) \text{ an } \mathcal{L}\text{-formula}\}$, where Ind_φ is the induction axiom

$$\forall \bar{y} \, (\varphi(0, \bar{y}) \wedge (\forall x \ \varphi(x, \bar{y}) \rightarrow \varphi(x + 1, \bar{y})) \rightarrow \forall x \ \varphi(x, \bar{y}))\,.$$

For example, $(\mathbb{N}, +, \cdot, 0, 1) \models \mathrm{PA}$.

Remark 4.9.11. The induction schema says that if $D \subseteq M^1$ is a definable set such that $0 \in D$ and $x \in D \implies x + 1 \in D$, then D contains every $x \in M$.

PA is computably axiomatized, so Theorems 4.6.6 and 4.9.10 yield the following:

Corollary 4.9.12. *Peano arithmetic is incomplete.*

Then Theorem 4.6.4 gives the following:

Corollary 4.9.13. *There is a model* $M \models \mathrm{PA}$ *with* $M \not\equiv \mathbb{N}$.

Chapter 5

The compactness theorem

5.1 A temporary convention

When we defined formulas (Definition 4.1.7), we chose the following logical operators to be fundamental:

$$\wedge, \vee, \top, \bot, \exists, \forall, \neg.$$

In contrast, notions like $\leftrightarrow$ and $\exists!$ are defined in terms the fundamental operators (see Section 4.3). For example, $\varphi \to \psi$ is an abbreviation for $\neg\varphi \vee \psi$.

In this chapter, we temporarily switch conventions, regarding

$$\wedge, \top, \exists, \neg,$$

as fundamental logical operators, and $\vee, \forall, \bot$ as derived notions like $\leftrightarrow$ or $\exists!$. Specifically:

$$\begin{aligned} \varphi \vee \psi &:= \neg(\neg\varphi \wedge \neg\psi) \\ \bot &:= \neg\top \\ \exists x\ \varphi(x) &:= \neg\forall x\ \neg\varphi(x). \end{aligned}$$

This convention simplifies the proofs of the compactness and completeness theorems, without impacting the conclusions of these theorems.

5.2 Premodels

A $\mathcal{L}$-*prestructure* is a pair $(M, \approx)$ where M is an $\mathcal{L}$-structure and $\approx$ is an equivalence relation on M such that the following hold:

1. If f is a k-ary function symbol and $a_i \approx b_i$ for $i = 1, \ldots, k$, then
$$f(a_1, \ldots, a_k) \approx f(b_1, \ldots, b_k).$$
2. If R is a k-ary relation symbol and $a_i \approx b_i$ for $i = 1, \ldots, k$, then
$$R(a_1, \ldots, a_k) \iff R(b_1, \ldots, b_k).$$

The first condition says that $\approx$ is a congruence with respect to the function symbols.

Definition 5.2.1. If $(M, \approx)$ is an $\mathcal{L}$-prestructure, then $M/{\approx}$ is the following $\mathcal{L}$-structure:

1. The underlying set is $M/{\approx} = \{[a] : a \in M\}$, where $[a] = \{b \in M : b \approx a\}$.
2. If f is a k-ary function symbol, then
$$f([a_1], \ldots, [a_k]) = [f(a_1, \ldots, a_k)]$$
3. If R is a k-ary relation symbol, then
$$R([a_1], \ldots, [a_n]) \iff R(a_1, \ldots, a_n).$$

This is well-defined by definition of prestructure (using Theorem 3.3.1).

If M is an $\mathcal{L}$-prestructure, if $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-sentence, and $a_1, \ldots, a_n \in M$, then we define $M \models \varphi(a_1, \ldots, a_n)$ exactly as for ordinary structures, except that
$$M \models t(\bar{a}) = s(\bar{a}) \iff t^M(\bar{a}) \approx s^M(\bar{a}).$$
In other words, we treat the symbol "=" as a non-logical symbol whose interpretation is $\approx$.

Definition 5.2.2. A *premodel* of T is an $\mathcal{L}$-prestructure satisfying T.

Theorem 5.2.3. *If $(M, \approx)$ is an $\mathcal{L}$-prestructure, then*

$$M \models \varphi(a_1, \ldots, a_n) \iff (M/\approx) \models \varphi([a_1], \ldots, [a_n]).$$

Proof. By induction on the complexity of φ. For example, if $\varphi(x_1, \ldots, x_n)$ is $\exists y : \psi(\bar{x}, y)$, then

$$\begin{aligned} M \models \varphi(a_1, \ldots, a_n) &\iff \exists b \in M : M \models \psi(\bar{a}, b) \\ &\iff \exists b \in M : (M/\approx) \models \psi([a_1], \ldots, [a_n], [b]) \\ &\iff \exists c \in (M/\approx) : (M/\approx) \models \psi([a_1], \ldots, [a_n], c) \\ &\iff (M/\approx) \models \varphi([a_1], \ldots, [a_n]), \end{aligned}$$

as every element of $M/\approx$ has the form $[b]$ for some $b \in M/\approx$. □

Corollary 5.2.4. *If $(M, \approx)$ is a premodel of T, then $M/\approx$ is a model of T.*

5.3 The witness property

Definition 5.3.1. Let T be an $\mathcal{L}$-theory.

- T is *satisfiable* if T has a model.
- T is *finitely satisfiable* if every $T_0 \subseteq_f T$ is satisfiable.
- T is *complete* if for every $\mathcal{L}$-sentence φ, $\varphi \in T$ or $\neg\varphi \in T$.
- T has the *witness property* if whenever $(\exists x\ \varphi(x)) \in T$, there is a closed term t such that $\varphi(t) \in T$.

Recall from §4.6 that the notation $T \models \varphi$ means that every model of T satisfies φ.

Lemma 5.3.2. *Suppose an $\mathcal{L}$-theory T is finitely satisfiable and complete, $T_0 \subseteq_f T$, and $T_0 \models \varphi$. Then $\varphi \in T$.*

Proof. Otherwise, $\neg\varphi \in T$. Then $T_0 \cup \{\neg\varphi\}$ shows T is *not* finitely satisfiable. □

Lemma 5.3.3. *Suppose an $\mathcal{L}$-theory T is finitely satisfiable and complete and has the witness property. Let φ, ψ be sentences and $\theta(x)$ be a formula.*

1. $\neg\varphi \in T \iff \varphi \notin T$.

2. $\varphi \wedge \psi \in T \iff (\varphi \in T \textit{ and } \psi \in T)$

3. $\top \in T$.

4. $(\exists x\ \theta(x)) \in T$ *if and only if there is a closed term* t *such that* $\theta(t) \in T$.

Proof. 1. Either $\varphi \in T$ or $\neg\varphi \in T$ by completeness. If both φ and $\neg\varphi$ are in T, then T is not finitely satisfiable.

2. Both directions hold by Lemma 5.3.2, because

$$\{\varphi, \psi\} \models \varphi \wedge \psi$$
$$\varphi \wedge \psi \models \varphi \text{ and } \varphi \wedge \psi \models \psi.$$

3. Lemma 5.3.2 again.

4. The $\Leftarrow$ direction holds by Lemma 5.3.2, and the $\Rightarrow$ direction holds because T has the witness property. $\square$

Theorem 5.3.4. *Let* T *be finitely satisfiable and complete, with the witness property. Then* T *has a model.*

Proof. By Corollary 5.2.4, it suffices to build a premodel $(M, \approx) \models T$. The underlying set of M will be the set of closed $\mathcal{L}$-terms. If f is a k-ary function symbol and $t_1, \ldots, t_k \in M$, let $f^M(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$. If R is a k-ary function symbol and $t_1, \ldots, t_k \in M$, let $R^M(t_1, \ldots, t_k)$ hold iff $R(t_1, \ldots, t_k) \in T$. Finally, let $t \approx s$ hold iff $(t = s) \in T$.

Claim. The relation $\approx$ is transitive.

Proof. Suppose $\{s = t, t = u\} \subseteq T$. Note that $\{s = t, t = u\} \models \{s = u\}$. (Any structure which satisfies $s = t$ and $t = u$ must also satisfy $s = u$.) By Lemma 5.3.2, $(s = u) \in T$. $\square_{\text{Claim}}$

Similar arguments show that $\approx$ is symmetric, reflexive, and respects the function and relation symbols. Thus M is an $\mathcal{L}$-prestructure.

Claim. If φ is an $\mathcal{L}$-sentence, then $M \models \varphi \iff \varphi \in T$.

Proof. Proceed by induction on φ:

1. If φ is an atomic formula, then $M \models \varphi \iff \varphi \in T$ by choice of the $\mathcal{L}$-structure:
   (a) $M \models R(t_1, \ldots, t_k) \iff R(t_1, \ldots, t_k) \in T$ by choice of R^M.
   (b) $M \models t = s \iff (t = s) \in T$ by choice of $\approx$.
   (c) $M \models f(t_1, \ldots, t_k) = s \iff (f(t_1, \ldots, t_k) = s) \in T$ by choice of f^M.
2. The logical operators $\neg, \top, \wedge, \exists$ work correctly by Lemma 5.3.3 and induction. For example,

$$\begin{aligned} M \models \exists x\ \varphi(x) &\iff \exists t \in M : M \models \varphi(t) \\ &\iff \exists t \in M : \varphi(t) \in T \qquad \text{by induction} \\ &\iff (\exists x\ \varphi(x)) \in T \qquad \text{by Lemma 5.3.3(4).} \qquad \square_{\text{Claim}} \end{aligned}$$

By the Claim, M is a premodel of T. By Corollary 5.2.4, the quotient $M/{\approx}$ is a model of T. $\square$

Definition 5.3.5. Suppose T is finitely satisfiable and complete and has the witness property. The *canonical model* of T is the model constructed in the proof of Theorem 5.3.4.

The canonical model of T is characterized up to isomorphism by the fact that every element $a \in M$ is named by a closed term t, in the sense that $a = t^M$.

Definition 5.3.6. Let $\mathcal{C}$ be a collection of closed $\mathcal{L}$-terms. An $\mathcal{L}$-theory *has the witness property over* $\mathcal{C}$ if whenever $(\exists x\ \varphi(x)) \in T$, there is a term $t \in \mathcal{C}$ such that $\varphi(t) \in T$.

Theorem 5.3.7. *Let T be a finitely satisfiable, complete theory with the witness property over $\mathcal{C}$, and let M be the canonical model of T. Then every element of M is named by a term in $\mathcal{C}$.*

Proof. Fix $a \in M$. Then $a = t^M$ for some term t. The $\mathcal{L}$-sentence

$$\exists x\ x = t$$

holds in any $\mathcal{L}$-structure, so it must be in T by Lemma 5.3.2. By the witness property over $\mathcal{C}$, there is $t' \in \mathcal{C}$ such that $(t' = t) \in T$. Then $M \models t' = t$, so $a = t^M = (t')^M$, and a is named by a term in $\mathcal{C}$. $\square$

5.4 Compactness via Henkin's method

Theorem 5.4.1. *Let I be a linear order and $\{T_i\}_{i\in I}$ be a chain of finitely satisfiable theories, meaning that $i < i' \implies T_i \subseteq T_{i'}$. Then the union $T = \bigcup_{i\in I} T_i$ is finitely satisfiable.*

Proof. Suppose $T_0 \subseteq_f T$. Let $T_0 = \{\varphi_1, \ldots, \varphi_n\}$. For each $j \leq n$, we have $\varphi_j \in T = \bigcup_i T_i$, so there is some $i_j \in I$ with $\varphi_j \in T_{i_j}$. Let $\ell = \max(i_1, \ldots, i_n) \in I$. Then for each $j \leq n$, we have $\varphi_j \in T_{i_j} \subseteq T_\ell$. Thus $T_0 \subseteq_f T_\ell$, and T_0 is satisfiable because T_ℓ is finitely satisfiable. □

Lemma 5.4.2. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a complete, finitely satisfiable $\mathcal{L}$-theory $T' \supseteq T$.*

Proof. By Zorn's lemma and Theorem 5.4.1 there is a $T' \supseteq T$ which is maximal among finitely satisfiable $\mathcal{L}$-theories. We claim T' is complete. Otherwise there is a sentence φ with $\varphi \notin T'$ and $\neg\varphi \notin T'$. By maximality, $T' \cup \{\varphi\}$ is not finitely satisfiable, so there is $T_1 \subseteq_f T$ with $T_1 \models \neg\varphi$. Similarly, $\neg\varphi \notin T'$ implies that there is $T_2 \subseteq_f T$ with $T_2 \models \varphi$. Then $T_1 \cup T_2$ is not satisfiable, a contradiction. □

Lemma 5.4.3. *Let T be a finitely satisfiable $\mathcal{L}$-theory containing a sentence $\exists x\ \varphi(x)$. Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$ where c is a new constant symbol. Then $T \cup \{\varphi(c)\}$ is a finitely satisfiable $\mathcal{L}'$-theory.*

Proof. Otherwise, $T_0 \cup \{\varphi(c)\}$ is unsatisfiable for some $T_0 \subseteq_f T$. Take $M \models T_0 \cup \{\exists x\ \varphi(x)\}$ and take $b \in M$ such that $M \models \varphi(b)$. Expand the $\mathcal{L}$-structure M to an $\mathcal{L}'$-structure by interpreting c as b. Then $M \models T_0 \cup \{\varphi(c)\}$, contradicting the choice of T_0. □

Lemma 5.4.4. *Let T be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}' \supseteq \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supseteq T$ that is finitely satisfiable and complete, and has the witness property.*

Proof. Build increasing chains

$$\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots$$
$$T_0 \subseteq T_1 \subseteq \cdots$$

where T_i is a finitely satisfiable $\mathcal{L}_i$-theory as follows:

1. $\mathcal{L}_0 = \mathcal{L}$ and $T_0 = T$.

2. If $n > 0$ and n is odd, then $\mathcal{L}_n = \mathcal{L}_{n-1}$ and T_n is a completion of T_{n-1} from Lemma 5.4.2.

3. If $n > 0$ and n is even, let $\{\varphi_i(x) : i \in I\}$ enumerate the $\mathcal{L}_{n-1}$-formulas such that $(\exists x\ \varphi(x)) \in T_{n-1}$. Let $\mathcal{L}_n = \mathcal{L}_{n-1} \cup \{c_i : i \in I\}$ where the c_i are new constant symbols. Let $T_n = T_{n-1} \cup \{\varphi_i(c_i) : i \in I\}$. Then T_n is finitely satisfiable by Lemma 5.4.3.

Finally, take $\mathcal{L}' = \bigcup_n \mathcal{L}_n$ and $T' = \bigcup_n T_n$. Then T' is finitely satisfiable because each T_i is, T' is complete because of the odd-numbered steps, and T' has the witness property because of the even-numbered steps. $\square$

Theorem 5.4.5 (Compactness)**.** *If T is finitely satisfiable, then T has a model.*

Proof. By Lemma 5.4.4, there is a language $\mathcal{L}' \supseteq \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supseteq T$ such that T' is finitely satisfiable and complete and has the witness property. By Theorem 5.3.4 there is an $\mathcal{L}'$-structure M satisfying T', and therefore T. $\square$

Chapter 6

Categoricity

6.1 A useful fact around finite satisfiability

Let T be a set of sentences. Suppose T is closed under finite conjunctions, in the sense that

$$\varphi_1, \ldots, \varphi_n \in T \implies \bigwedge_{i=1}^{n} \varphi_i \in T.$$

Then T is finitely satisfiable if and only if every $\varphi \in T$ is satisfiable, because any finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq T$ is equivalent to a single formula $\bigwedge_{i=1}^{n} \varphi_i$ in T.

More generally, suppose T_1, T_2 are two sets of sentences, and T_1 is closed under finite conjunctions. Then the following are equivalent:

- $T_1 \cup T_2$ is finitely satisfiable.
- For every $\varphi \in T_1$ and finite subset $T_2' \subseteq_f T_2$, the theory $\{\varphi\} \cup \{T_2'\}$ is satisfiable.

We will use this fact implicitly in what follows.

6.2 Diagrams and elementary amalgamation

Let M be an $\mathcal{L}$-structure and A be a subset of M. The language $\mathcal{L}(A)$ is obtained by adding each element of A as a new constant symbol. We regard

M as an $\mathcal{L}(A)$-structure by interpreting each new constant symbol as the corresponding element of A:

$$a^M = a.$$

Definition 6.2.1. The *elementary diagram* of M, written $\text{eldiag}(M)$, is the set of $\mathcal{L}(M)$-sentences φ such that $M \models \varphi$. The *diagram* of M is the set of atomic and negated atomic formulas in $\text{eldiag}(M)$.

A model of $\text{diag}(M)$ consists of an $\mathcal{L}$-structure N and a map

$$\begin{aligned} M &\to N \\ a &\mapsto a^N \end{aligned}$$

such that for any atomic $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and any $\bar{a} \in M^n$,

$$\begin{aligned} M \models \varphi(a_1, \ldots, a_n) &\implies N \models \varphi(a_1^N, \ldots, a_n^N) \\ M \models \neg\varphi(a_1, \ldots, a_n) &\implies N \models \neg\varphi(a_1^N, \ldots, a_n^N). \end{aligned}$$

This means that $a \mapsto a^N$ is an embedding.

Therefore, a model of $\text{diag}(M)$ is the same thing as an $\mathcal{L}$-structure N with an embedding $f : M \to N$. In particular, if N is an extension of M, then N is a model of $\text{diag}(M)$ in a natural way, interpreting $a \in M$ as $a \in N$. Every model of $\text{diag}(M)$ is isomorphic as an $\mathcal{L}(M)$-structure to an extension of M.

Similarly, a model of $\text{eldiag}(M)$ is the same thing as an $\mathcal{L}$-structure N with an elementary embedding $f : M \to N$. Any elementary extension of M is a model of $\text{eldiag}(M)$, and every model of $\text{eldiag}(M)$ is isomorphic as an $\mathcal{L}(M)$-structure to an elementary extension of M.

Theorem 6.2.2 (Elementary amalgamation)**.** *If M_1 and M_2 are elementarily equivalent, then there is a structure N and elementary embeddings $M_1 \to N$ and $M_2 \to N$.*

We first prove a lemma. Recall the notion of *expansion* from Definition 4.1.5.

Lemma 6.2.3. *If $M_1 \equiv M_2$ and $\varphi(\bar{c}) \in \text{eldiag}(M_2)$, then there is an $\mathcal{L}(M_2)$-structure N expanding the $\mathcal{L}$-structure M_1 such that $N \models \varphi$.*

Proof. The fact that $M_2 \models \varphi(\bar{c})$ implies that $M_2 \models \exists \bar{x}\ \varphi(\bar{x})$ and then $M_1 \models \exists \bar{x}\ \varphi(\bar{x})$. Take $\bar{a}$ in M_1 such that $M_1 \models \varphi(\bar{a})$. Expand M_1 to an $\mathcal{L}(M_2)$-structure N by interpreting the symbols $c_1, \ldots, c_n$ as $a_1, \ldots, a_n$, respectively. Then $N \models \varphi(\bar{c})$ because $\bar{c}^N = \bar{a}$ and $M_1 \models \varphi(\bar{a})$. □

Now we prove Theorem 6.2.2.

Proof. It suffices to show that $\operatorname{eldiag}(M_1) \cup \operatorname{eldiag}(M_2)$ is consistent. Otherwise, by compactness, there is a tuple $\bar{c}$ in M_2 and formula $\varphi(\bar{c}) \in \operatorname{eldiag}(M_2)$ such that $\operatorname{eldiag}(M_1) \cup \{\varphi(\bar{c})\}$ is inconsistent. But $M_1 \models \operatorname{eldiag}(M_1)$, and Lemma 6.2.3 gives an expansion of M_1 satisfying $\{\varphi(\bar{c})\}$, and so $\operatorname{eldiag}(M_1) \cup \{\varphi(\bar{c})\}$ is consistent. □

6.3 Tarski-Vaught

Theorem 6.3.1. *Let M be a structure and A be a subset. Then $A \preceq M$ iff the following holds: for every non-empty A-definable $D \subseteq M$, we have $D \cap A \neq \varnothing$.*

This condition is called the *Tarski-Vaught criterion*.

Proof. First suppose $A \preceq M$. Suppose D is A-definable and non-empty. Write D as $\varphi(M, \bar{a})$ for some $\mathcal{L}(A)$-formula $\varphi(x, \bar{a})$. Then

$$M \models \exists x : \varphi(x, \bar{a}) \implies A \models \exists x : \varphi(x, \bar{a})$$

so there is $b \in A$ with $A \models \varphi(b, \bar{a})$, which implies $M \models \varphi(b, \bar{a})$ as $A \preceq M$. Then $b \in A$ and $b \in D = \varphi(M, \bar{a})$.

Conversely, suppose the Tarski-Vaught criterion holds. First we show that A is a substructure. Suppose f is a k-ary function symbol and $a_1, \ldots, a_k \in A$. The set $\{f(\bar{a})\}$ is non-empty and A-definable, so A intersects it, meaning that $f(\bar{a}) \in A$.

Next we show that for any $\mathcal{L}$-formula $\varphi(\bar{x})$ and any $\bar{a} \in A$,

$$M \models \varphi(\bar{a}) \iff A \models \varphi(\bar{a}). \tag{$*$}$$

We may assume φ makes no use of $\wedge$, $\forall$, $\top$, $\perp$. Proceed by induction on the complexity of φ. If φ is atomic or more generally quantifier-free, then $(*)$ holds because A is a substructure.

If φ is $\psi \vee \theta$, then

$$\begin{aligned} M \models \varphi(\bar{a}) &\iff M \models \psi(\bar{a}) \text{ or } M \models \theta(\bar{a}) \\ &\iff A \models \psi(\bar{a}) \text{ or } A \models \theta(\bar{a}) \\ &\iff A \models \varphi(\bar{a}) \end{aligned}$$

by induction. Finally, if $\varphi(\bar{x})$ is $\exists y : \psi(\bar{x}, y)$, then

$$\begin{aligned} M \models \varphi(\bar{a}) &\iff \exists b \in M : M \models \psi(\bar{a}, b) \\ &\overset{\text{TV}}{\iff} \exists b \in A : M \models \psi(\bar{a}, b) \\ &\overset{\text{ind}}{\iff} \exists b \in A : A \models \psi(\bar{a}, b) \\ &\iff A \models \varphi(\bar{a}). \end{aligned}$$

The second line uses the Tarski-Vaught criterion, and the third line uses induction. □

6.4 The Löwenheim-Skolem theorem

Definition 6.4.1. If $\mathcal{L}$ is a language, the *size* of $\mathcal{L}$, written $|\mathcal{L}|$, is $\aleph_0$ plus the number of symbols in $\mathcal{L}$.

If $\bar{x}$ is a finite tuple of variables, then $|\mathcal{L}|$ is the number of $\mathcal{L}$-formulas $\varphi(\bar{x})$.

Remark 6.4.2. If M is a $\mathcal{L}$-structure and $A \subseteq M$, then the number of A-definable sets in M is at most $|A| + |\mathcal{L}|$.

Theorem 6.4.3 (Downward Löwenheim-Skoelm theorem)**.** *Let M be an $\mathcal{L}$-structure.*

1. *If $A \subseteq M$, there is an elementary substructure $N \preceq M$ with $N \supseteq A$ and $|N| \leq |A| + |\mathcal{L}|$.*

2. *For any κ with $|\mathcal{L}| \leq \kappa \leq |M|$, there is $N \preceq M$ with $|N| = \kappa$.*

Proof. 1. Let $F : \mathfrak{P}(M) \setminus \{\varnothing\} \to M$ be a function such that $F(X) \in X$. Recursively define $A_0 \subseteq A_1 \subseteq \cdots$ by letting $A_0 = A$ and

$$A_{n+1} = A_n \cup \{F(X) : X \subseteq M \text{ is } A_n\text{-definable and non-empty}\}.$$

Let $N = \bigcup_{n=0}^{\infty} A_n$. Then $A = A_0 \subseteq N$. By induction on n, each A_n has size at most $|A| + |\mathcal{L}|$, and therefore $|N| \leq |A| + |\mathcal{L}|$. If X is N-definable and non-empty, then X is A_n-definable for some n, so $F(X) \in A_{n+1} \subseteq N$. Thus N intersects every N-definable set, and $N \preceq M$ by Tarski-Vaught.

2. Take a subset $A \subseteq M$ with $|A| = \kappa$, and take $N \preceq M$ as in part (1). Then $|N| \leq |A| + |\mathcal{L}| = \kappa + |\mathcal{L}| = \kappa$. On the other hand, $|N| \geq \kappa$ because $N \supseteq A$. $\square$

Theorem 6.4.4 (Löwenheim-Skolem theorem)**.** *Let T be an $\mathcal{L}$-theory. Suppose T has an infinite model, or more generally that for every $n < \omega$, T has a model of size at least n. Then for any $\kappa \geq |\mathcal{L}|$, T has a model of size κ.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where the c_α are new constant symbols. Let $\Sigma = \{(c_\alpha \neq c_\beta) : \alpha < \beta < \kappa\}$.

Claim. $T \cup \Sigma$ is finitely satisfiable.

Proof. Suppose $\Sigma_0 \subseteq_f \Sigma$. Let S be the finite set of $\alpha \in \kappa$ such that c_α appears in Σ_0. Take a model $M \models T$ with $|M| > |S|$. Expand M to an $\mathcal{L}'$-structure by interpreting the c_α for $\alpha \in S$ as distinct elements of M, and interpreting c_α for $\alpha \in \kappa \setminus S$ arbitrarily. Then $M \models T \cup \Sigma_0$. $\square_{\text{Claim}}$

By compactness, $T \cup \Sigma$ has a model M. Then the c_α^M are pairwise distinct, so $|M| \geq \kappa$. By downward Löwenheim-Skolem (Theorem 6.4.3), there is an elementary substructure $N \preceq M$ with $|N| = \kappa$. Then $N \equiv M$ and $M \models T$, so $N \models T$. $\square$

Corollary 6.4.5 (Löwenheim-Skolem theorem, second version)**.** *Let M be an infinite $\mathcal{L}$-structure. For any $\kappa \geq |\mathcal{L}|$, there is $N \equiv M$ with $|N| = \kappa$.*

Proof. Apply Theorem 6.4.4 to the $\mathcal{L}$-theory $\text{Th}(M)$. Models of $\text{Th}(M)$ are elementarily equivalent to M (Theorem 4.4.1). $\square$

Example 6.4.6. Earlier we saw that Peano Arithmetic has models which are not elementarily equivalent to $\mathbb{N}$ (Corollary 4.9.13). Using the Löwenheim-Skolem theorem, we can also produce models that are elementarily equivalent to $\mathbb{N}$ but not isomorphic to it. For example, Corollary 6.4.5 gives $M \equiv \mathbb{N}$ with $|M| > \aleph_0$. Then $M \models \text{PA}$ but $M \not\cong \mathbb{N}$.

Theorem 6.4.7 (Upward Löwenheim-Skolem theorem)**.** *If M is an infinite $\mathcal{L}$-structure and $\kappa \geq |M|+|\mathcal{L}|$, then there is an elementary extension $N \succeq M$ with $|N| = \kappa$.*

Proof. As $\kappa \geq |M| + |\mathcal{L}| = |\mathcal{L}(M)|$, we can apply the Löwenheim-Skolem theorem (Theorem 6.4.4) to the $\mathcal{L}(M)$-theory $\mathrm{eldiag}(M)$ to get a model $N \models \mathrm{eldiag}(M)$ with $|N| = \kappa$. Moving N by an isomorphism, we may assume $N \succeq M$. □

6.5 Absolute categoricity

Definition 6.5.1. A theory T is *absolutely categorical* if there is a unique model of T, up to isomorphism.

Theorem 6.5.2. *If T is absolutely categorical, then the unique model M is finite.*

Proof. Take $\kappa > |M| + |\mathcal{L}|$. If M is infinite, then there is a model $N \models T$ with $|N| = \kappa$ by Theorem 6.4.4. Then M and N are two non-isomorphic models, contradicting categoricity. □

Conversely, if M is finite, then $\mathrm{Th}(M)$ is absolutely categorical. Recall that models of $\mathrm{Th}(M)$ are the same thing as structures elementarily equivalent to M (Theorem 4.4.1).

Theorem 6.5.3. *If M is finite and $M \equiv N$, then $M \cong N$.*

Proof. By elementary amalgamation (Theorem 6.2.2) there are elementary embeddings $M \to M'$ and $N \to M'$ for some structure M'. Moving M and N by isomorphisms, we may assume $M \preceq M'$ and $N \preceq M'$. Let n be the size of M. By Lemma 6.5.4 below, $|M| = |M'| = |N| = n$. As M and N are subsets of M', we must have $M = M' = N$. □

Lemma 6.5.4. *Suppose $M \equiv N$. If $|M| = n < \infty$, then $|N| = n$. Consequently, M is finite iff N is finite.*

Proof. Suppose $|M| = n$. Let φ be the sentence $\exists^{=n} x\ \top$ (see Section 4.3). Then φ says that there are exactly n elements, so $M \models \varphi$. Therefore $N \models \varphi$ and $|N| = n$. □

6.6 κ-categoricity

Definition 6.6.1. Let κ be an infinite cardinal. A theory T is *κ-categorical* if there is a unique model of cardinality κ.

Theorem 6.6.2 (Łoś-Vaught criterion)**.** *Suppose T is κ-categorical for some $\kappa \geq |\mathcal{L}|$.*

1. *Any two infinite models of T are elementarily equivalent.*

2. *If all models of T are infinite, then T is complete.*

Proof. 1. Given infinite models M_1, M_2, use Löwenheim-Skolem (Corollary 6.4.5) to get $N_i \equiv M_i$ with $|N_i| = \kappa$. By κ-categoricity, $N_1 \cong N_2$. Then $M_1 \equiv N_1 \equiv N_2 \equiv M_2$.

2. Clear. □

Lemma 6.6.3. *Let M, N be countable structures and let $\mathcal{F}$ be a back-and-forth system between M and N. If $f_0 \in \mathcal{F}$, then there is an isomorphism $f : M \to N$ extending f_0.*

Proof. Let $a_1, a_2, \ldots$ be an enumeration of M and $b_1, b_2, \ldots$ be an enumeration of N. Recursively build an increasing sequence of partial isomorphisms $f_i : A_i \to B_i$ in $\mathcal{F}$ as follows:

- For $i = 0$, let $f_0 : A_0 \to B_0$ be the given partial isomorphism.

- For $i = 2k - 1$, take f_i to be an extension of f_{i-1} with $a_k \in \mathrm{dom}(f_i)$. Such an f_i exists by the "forward" condition.

- For $i = 2k$, take f_i to be an extension of f_{i-1} with $b_k \in \mathrm{im}(f_i)$. Such an f_i exists by the "backward" condition.

Take $f = \bigcup_{i=0}^{\infty} f_i$. Then f is an isomorphism from M to N. □

Theorem 6.6.4. *Let M, N be countable structures. If there is a non-empty back-and-forth system between M and N, then $M \cong N$.*

Recall the theory DLO from Definition 4.7.9. Theorem 4.7.10 gives a non-empty back-and-forth system between any two models of DLO.

Corollary 6.6.5 (Cantor)**.** *DLO is $\aleph_0$-categorical: any two countable models of DLO are isomorphic.*

Any model of DLO is infinite, so the Łoś-Vaught criterion shows that DLO is complete, as we saw earlier (Corollary 4.7.11).

Theorem 6.6.6. *DLO is not κ-categorical for $\kappa > \aleph_0$.*

Proof. If A, B are linear orders, let $A \times B$ denote the lexicographic product, the set $A \times B$ with the lexicographic order

$$(a, b) < (a', b') \iff a < a' \text{ for } a \neq a'$$
$$(a, b) < (a, b') \iff b < b'.$$

Also, let A^* denote A with the reversed order.

Let $M = \kappa \times \mathbb{Q}$. It is straightforward to see that M and M^* are models of DLO, of size κ. We claim that $M \ncong M^*$. Note that for any $a \in M$,

$$|\{x \in M : x < a\}| < \kappa$$
$$|\{x \in M : x > a\}| = \kappa.$$

In M^*, the reverse properties hold, so $M \ncong M^*$, and DLO is not κ-categorical. $\square$

Chapter 7

Ultraproducts

7.1 Ultrafilters

Let I be a set.

Definition 7.1.1. A *filter* on I is a set $\mathcal{F} \subseteq \mathfrak{P}(I)$ satisfying the following:

1. If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
2. If $X \subseteq Y \subseteq I$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.
3. $I \in \mathcal{F}$.

A filter is *proper* if $\varnothing \notin \mathcal{F}$.

Definition 7.1.2. A family of sets $\mathcal{S} \subseteq \mathfrak{P}(I)$ has the *finite intersection property* (FIP) if for any $n \geq 0$ and $X_1, \ldots, X_n \in \mathcal{S}$, we have $\bigcap_{i=1}^n X_i \neq \varnothing$.

(When $n = 0$, $\bigcap_{i=1}^n X_i$ is defined to be I.)

Remark 7.1.3. If $\mathcal{F}$ is a proper filter, then $\mathcal{F}$ has the FIP, because $\varnothing \notin \mathcal{F}$, and $\mathcal{F}$ is closed under intersection.

Lemma 7.1.4. *If $\mathcal{S} \subseteq \mathfrak{P}(I)$ has the FIP, then there is a proper filter $\mathcal{F} \supseteq \mathcal{S}$.*

Proof. Let $\mathcal{F}$ be the set of $X \subseteq I$ such that there are $n \geq 0$ and $Y_1, \ldots, Y_n \in \mathcal{S}$ with $X \supseteq \bigcap_{i=1}^n Y_i$. Then $\mathcal{F}$ is a proper filter containing $\mathcal{S}$. □

Definition 7.1.5. An *ultrafilter* on I is a proper filter $\mathcal{U}$ such that for any $X \subseteq I$,

$$X \in \mathcal{U} \text{ or } I \setminus X \in \mathcal{U}.$$

Lemma 7.1.6. *If $\mathcal{F}$ is a proper filter on I, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.*

Proof. By Zorn's lemma, there is a maximal proper filter $\mathcal{U} \supseteq \mathcal{F}$. We claim that $\mathcal{U}$ is an ultrafilter. Otherwise, there is $X \subseteq I$ with $X \notin \mathcal{U}$ and $I \setminus X \notin \mathcal{U}$. By maximality, $\mathcal{U} \cup \{X\}$ is not contained in a filter, and does not have FIP. Therefore there are $Y_1, \ldots, Y_n \in \mathcal{U}$ such that $X \cap \bigcap_{i=1}^n Y_i = \varnothing$. As $\mathcal{U}$ is a filter, $\mathcal{U}$ contains $Y := \bigcap_{i=1}^n Y_i$. Then $X \cap Y = \varnothing$, so $Y \subseteq I \setminus X$.

Applying the same argument to the complement of X, we get $Z \in \mathcal{U}$ with $Z \subseteq X$. Then $Y \cap Z \in \mathcal{U}$ because $\mathcal{U}$ is a filter. However, $Y \cap Z \subseteq (I \setminus X) \cap X = \varnothing$, so $\varnothing \in \mathcal{U}$ contradicting the fact that $\mathcal{U}$ is proper. □

Theorem 7.1.7. *If $\mathcal{S} \subseteq \mathfrak{P}(I)$ has the FIP, then $\mathcal{S}$ is contained in an ultrafilter $\mathcal{U}$ on I.*

Fact 7.1.8. *Let $\mathcal{U}$ be an ultrafilter on I. Let $\chi : \mathfrak{P}(I) \to \mathcal{U}$ be the characteristic function of $\mathcal{U}$:*

$$\chi(X) = \begin{cases} 1 & X \in \mathcal{U} \\ 0 & X \notin \mathcal{U}. \end{cases}$$

Then χ is a homomorphism of boolean algebras. In fact, ultrafilters on I correspond bijectively to homomorphisms $\mathfrak{P}(I) \to \{0, 1\}$.

7.2 Ultraproducts

Let I be a set and let $\mathcal{U}$ be an ultrafilter on I. Let M_i be a *non-empty* $\mathcal{L}$-structure for each $i \in I$. Let $P = \prod_{i \in I} M_i$ and let $\pi_i : P \to M_i$ be the ith coordinate projection.

Let $\mathcal{L}(P)$ be $\mathcal{L}$ with a new constant symbol added for each $a \in P$. Regard M_i as an $\mathcal{L}(P)$-structure by interpreting $a \in P$ as $\pi_i(a) \in M_i$.

For any $\mathcal{L}(P)$-sentence φ, let $[\![\varphi]\!] = \{i \in I : M_i \models \varphi\}$. Note the following:

$$\begin{aligned} [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\neg\varphi]\!] &= I \setminus [\![\varphi]\!] \end{aligned}$$

Lemma 7.2.1. *Let T be the set of $\mathcal{L}(P)$-sentences φ such that $[\![\varphi]\!] \in \mathcal{U}$. Then T is finitely satisfiable and complete and has the witness property over P in the sense of Definition 5.3.6.*

Proof. For finite satisfiability, suppose $\varphi_1, \ldots, \varphi_n \in T$. If $S = [\![\bigwedge_{i=1}^n \varphi_i]\!] = \bigcap_{i=1}^n [\![\varphi_i]\!]$, then $S \in \mathcal{U}$ so $S \neq \varnothing$. Taking $i \in S$, we have $M_i \models \bigwedge_{i=1}^n \varphi_i$.

For completeness, note that $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ are complementary, so one of them is in $\mathcal{U}$, and therefore one of φ and $\neg\varphi$ is in T.

For the witness property, suppose $\exists x\ \varphi(x)$ is in T, meaning that $S := [\![\exists x\ \varphi(x)]\!] \in \mathcal{U}$. Define $c = (c_i : i \in I) \in P$ as follows:

1. If $i \in S$, then $M_i \models \exists x \varphi(x)$. Choose $c_i \in M_i$ so that $M_i \models \varphi(c_i)$.
2. If $i \notin S$, take any $c_i \in M_i$.

Then $i \in S \implies M_i \models \varphi(c)$, so $[\![\varphi(c)]\!] \supseteq S$. It follows that $[\![\varphi(c)]\!] \in \mathcal{U}$ and $\varphi(c) \in T$. $\square$

By Definition 5.3.5 and Theorem 5.3.7, we can talk about the canonical model of T—the unique model in which every element is named by an element of P.

Definition 7.2.2. The *ultraproduct* $\prod_{i\in I}^{/\mathcal{U}} M_i$ is the canonical model of T.

Theorem 7.2.3. *Let $P = \prod_{i\in I} M_i$, let $M = \prod_{i\in I}^{/\mathcal{U}} M_i$, and let $[c]$ be the element of M named by $c \in P$.*

1. *Every element of M has the form $[c]$ for some $c \in P$.*
2. *$[c] = [d]$ if and only if $\{i \in I : \pi_i(c) = \pi_i(d)\} \in \mathcal{U}$.*
3. *Let R be a k-ary relation symbol. Then*
$$R^M([c_1], \ldots, [c_k]) \iff \{i \in I : M_i \models R(\pi_i(c_1), \ldots, \pi_i(c_k))\} \in \mathcal{U}.$$
4. *If f is a k-ary function symbol, then*
$$f^M([c_1], \ldots, [c_k]) = [f^P(c_1, \ldots, c_k)],$$
where P is given the product structure so that
$$f^P(c_1, \ldots, c_k) = (f^{M_i}(\pi_i(c_1), \ldots, \pi_i(c_k)) : i \in I).$$
5. *If $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula, then*
$$M \models \varphi([c_1], \ldots, [c_k]) \iff \{i \in I : M_i \models \varphi(\pi_i(c_1), \ldots, \pi_i(c_k))\} \in \mathcal{U}.$$

6. *If φ is an $\mathcal{L}$-sentence, then*

$$M \models \varphi \iff \{i \in I : M_i \models \varphi\} \in \mathcal{U}.$$

Proof. Part (1) is true by construction, as M is the canonical model of T. Part (5) is also true by construction:

$$\begin{aligned} M \models \varphi([c_1], \dots, [c_k]) &\iff M \models \varphi(c_1, \dots, c_k) \iff \varphi(c_1, \dots, c_k) \in T \\ &\iff [\![\varphi(c_1, \dots, c_k)]\!] \in \mathcal{U} \\ &\iff \{i \in I : M_i \models \varphi(\pi_i(c_1), \dots, \pi_i(c_k))\} \in \mathcal{U}. \end{aligned}$$

Remember that a constant symbol $c \in P$ is interpreted as $[c]$ in M, and $\pi_i(c)$ in M_i.

Parts (2) and (3) follow by specializing part (5) to atomic formulas $x = y$ and $R(x_1, \dots, x_k)$. Part (6) is the $n = 0$ case of part (5). It remains to prove part (4). Fix $c_1, \dots, c_k \in P$ and let $d = f^P(c_1, \dots, c_k)$. Then $\pi_i(d) = f^{M_i}(\pi_i(c_1), \dots, \pi_i(c_k))$ for each $i \in I$. Then

$$\{i \in I : M_i \models \pi_i(d) = f(\pi_i(c_1), \dots, \pi_i(c_k))\} = I \in \mathcal{U},$$

so by part (5), we have

$$M \models [d] = f([c_1], \dots, [c_k]),$$

which means that $f^M([c_1], \dots, [c_k]) = [d] = [f^P(c_1, \dots, c_k)]$. $\square$

Remark 7.2.4. Parts (1)—(4) determine the structure of M up to isomorphism, and are usually taken as the *definition* of the ultraproduct. Then part (5) is called *Łoś's Theorem.*

Theorem 7.2.5. *Let I be a set, $\mathcal{U}$ be an ultrafilter on I, M_i be an $\mathcal{L}$-structure for $i \in I$, and M be the ultraproduct $\prod_{i \in I}^{/\mathcal{U}} M_i$.*

1. *Let T be a theory. If $M_i \models T$ for all i, then $M \models T$.*

2. *Let $\mathcal{K}$ be an elementary class. If $M_i \in \mathcal{K}$ for all i, then $M \in \mathcal{K}$.*

Proof. (1) follows from Łoś's theorem: if $\varphi \in T$, then

$$\{i \in I : M_i \models \varphi\} = I \in \mathcal{U},$$

so $M \models \varphi$. (2) is a restatement of (1). $\square$

If $\mathcal{K}$ is a class of $\mathcal{L}$-structures and T is an $\mathcal{L}$-theory, say that T is *finitely satisfiable in* $\mathcal{K}$ if for any finite subtheory $T_0 \subseteq_f T$ there is $M \in \mathcal{K}$ satisfying T_0.

Theorem 7.2.6. *If T is finitely satisfiable in $\mathcal{K}$, then there is an ultraproduct M of structures in $\mathcal{K}$ such that $M \models T$.*

Proof. Let $\{M_i\}_{i\in I}$ be a small collection of structures in $\mathcal{K}$ containing at least one representative from every elementary equivalence class. If φ is an $\mathcal{L}$-sentence, let $[\![\varphi]\!] = \{i \in I : M_i \models \varphi\}$. Let $\mathcal{F} = \{[\![\varphi]\!] : \varphi \in T\}$. We claim that $\mathcal{F}$ has the FIP. Indeed, if $\varphi_1, \ldots, \varphi_n \in T$ then there is some M_i satisfying $\bigwedge_{j=1}^n \varphi_j$, and then $i \in \bigcap_{j=1}^n [\![\varphi_j]\!]$.

Because $\mathcal{F}$ has the FIP, it is contained in an ultrafilter $\mathcal{U}$ on I (Theorem 7.1.7). Let $M = \prod_{i\in I}^{/\mathcal{U}} M_i$. Then for $\varphi \in T$ we have

$$\{i \in I : M_i \models \varphi\} = [\![\varphi]\!] \in \mathcal{F} \subseteq \mathcal{U},$$

and so $M \models \varphi$ by Łoś's theorem. $\square$

Note that this gives another proof of the compactness theorem (Theorem 5.4.5).

Definition 7.2.7. If M is an $\mathcal{L}$-structure and $\mathcal{U}$ is an ultrafilter on a set I, then the *ultrapower* $M^{\mathcal{U}}$ is the ultraproduct $\prod_{i\in I}^{/\mathcal{U}} M_i$.

Theorem 7.2.8. *Let $\Delta : M \to M^{\mathcal{U}}$ be the diagonal map sending $a \in M$ to the class of the tuple $(a : i \in I) \in M^I = \prod_{i\in I} M$. Then Δ is an elementary embedding.*

Proof. Fix a formula $\varphi(x_1, \ldots, x_n)$ and a tuple $\bar{a} = (a_1, \ldots, a_n) \in M^n$. Let $S = \{i \in I : M \models \varphi(a_1, \ldots, a_n)\}$. By Łoś's theorem,

$$M^{\mathcal{U}} \models \varphi(\Delta(a_1), \ldots, \Delta(a_n)) \iff S \in \mathcal{U}.$$

But

$$S = \begin{cases} I & \text{if } M \models \varphi(\bar{a}) \\ \varnothing & \text{if } M \not\models \varphi(\bar{a}). \end{cases}$$

Therefore $S \in \mathcal{U} \iff M \models \varphi(\bar{a})$, and

$$M^{\mathcal{U}} \models \varphi(\Delta(\bar{a})) \iff M \models \varphi(\bar{a}). \qquad \square$$

7.3 Characterizations of elementary classes

Lemma 7.3.1. *If M_1 and M_2 are elementarily equivalent, then there is an elementary embedding from M_2 to an ultrapower $M_1^{\mathcal{U}}$.*

Proof. In the proof of elementary amalgamation, we saw that $\mathrm{eldiag}(M_2)$ is finitely satisfiable in expansions of M_1 to $\mathcal{L}(M_2)$-structures (Lemma 6.2.3). By Theorem 7.2.6, there is an ultraproduct $N = \prod_{i\in I}^{/\mathcal{U}} N_i$ satisfying $\mathrm{eldiag}(M_2)$, where each N_i is an expansion of M_1 to an $\mathcal{L}(M_2)$-structure. The fact that $N \models \mathrm{eldiag}(M_2)$ gives an elementary embedding $M_2 \to N$. On the other hand, $N \restriction \mathcal{L}$ is

$$\prod_{i\in I}^{/\mathcal{U}} (N_i \restriction \mathcal{L}) = \prod_{i\in I}^{/\mathcal{U}} M_1 = M_1^{\mathcal{U}},$$

and we have the desired elementary embedding $M_2 \to M_1^{\mathcal{U}}$. □

Theorem 7.3.2. *A class $\mathcal{K}$ of $\mathcal{L}$-structures is elementary iff $\mathcal{K}$ is closed under ultraproducts, isomorphisms, and elementary substructures.*

Proof. Suppose $\mathcal{K}$ is closed under ultraproducts, isomorphisms, and elementary substructures.

Claim. $\mathcal{K}$ is closed under elementary equivalence: if $M \equiv N$ and $M \in \mathcal{K}$ then $N \in \mathcal{K}$.

Proof. By Lemma 7.3.1, there is an ultrapower $M^{\mathcal{U}}$ and an elementary embedding $f : N \to M^{\mathcal{U}}$. Then $N \cong \mathrm{im}(f) \preceq M^{\mathcal{U}}$, so

$$M \in \mathcal{K} \implies M^{\mathcal{U}} \in \mathcal{K} \implies \mathrm{im}(f) \in \mathcal{K} \implies N \in \mathcal{K}. \qquad \square_{\mathrm{Claim}}$$

Let T be the set of all sentences φ such that every $N \in \mathcal{K}$ satisfies φ. Then $\mathcal{K} \subseteq \mathrm{Mod}(T)$. We claim that $\mathcal{K} = \mathrm{Mod}(T)$. Fix $M \in \mathrm{Mod}(T)$; we claim that $M \in \mathcal{K}$. Break into two cases:

1. $\mathrm{Th}(M)$ is finitely satisfiable in $\mathcal{K}$. By Theorem 7.2.6, there is a model $N \models \mathrm{Th}(M)$ such that N is an ultraproduct of structures in $\mathcal{K}$. By assumption, $N \in \mathcal{K}$. The fact that $N \models \mathrm{Th}(M)$ means that $N \equiv M$ (Theorem 4.4.1), and so $M \in \mathcal{K}$.

2. $\mathrm{Th}(M)$ is not finitely satisfiable in $\mathcal{K}$. Then there is $\varphi \in \mathrm{Th}(M)$ such that no $N \in \mathcal{K}$ satisfies φ. Then every $N \in \mathcal{K}$ satisfies $\neg\varphi$, so $\neg\varphi \in T$ and $M \models \neg\varphi$, contradicting the fact that $\varphi \in \mathrm{Th}(M)$. □

Fact 7.3.3 (Keisler-Shelah)**.** *If $M \equiv N$, then there is a set I and an ultrafilter $\mathcal{U}$ on I such that the ultraproducts $M^{\mathcal{U}}$ and $N^{\mathcal{U}}$ are isomorphic.*

Definition 7.3.4. A structure M is an *ultraroot* of a structure N if N is an ultrapower of M. A class $\mathcal{K}$ is *closed under ultraroots* if $M^{\mathcal{U}} \in \mathcal{K} \implies M \in \mathcal{K}$.

If M is an ultraroot of N, then $N \succeq M$, so $N \equiv M$. Therefore, elementary classes are closed under ultraroots.

Theorem 7.3.5. *A class $\mathcal{K}$ is elementary iff $\mathcal{K}$ is closed under ultraproducts, ultraroots, and isomorphisms.*

Proof. Suppose $\mathcal{K}$ is closed under ultraproducts, ultraroots, and isomorphisms. By the Keisler-Shelah theorem, $\mathcal{K}$ is closed under elementary equivalence. Then the proof of Theorem 7.3.2 applies. $\square$

7.4 Universal classes

Lemma 7.4.1 (Lemma on constants)**.** *Let $\bar{c} = (c_1, \ldots, c_n)$ be a tuple of constant symbols not appearing in T. If $T \vdash \varphi(\bar{c})$, then $T \vdash \forall \bar{x}\ \varphi(\bar{x})$.*

Proof. Otherwise, there is $M \models T$ with $M \not\models \forall \bar{x}\ \varphi(\bar{x})$. Take $\bar{a} \in M$ with $M \models \neg\varphi(\bar{a})$. Interpreting $\bar{c}$ as $\bar{a}$, we get a model of $T \cup \{\neg\varphi(\bar{c})\}$, contradicting $T \vdash \varphi(\bar{c})$. $\square$

A *universal formula* or *$\forall$-formula* is one of the form $\forall \bar{y} : \varphi(\bar{x}, \bar{y})$ where φ is quantifier-free. A *universal theory* is a set of universal sentences.

Lemma 7.4.2. *If T is a universal theory, and $M \models T$, then any substructure of M is a model of T.*

Proof. Let ψ be a sentence in T. We can write ψ as $\forall \bar{x} : \varphi(\bar{x})$ where φ is quantifier-free. Let N be a substructure of M. If $\bar{a} \in N$, then

$$M \models \forall \bar{x} : \varphi(\bar{x}) \implies M \models \varphi(\bar{a}) \implies N \models \varphi(\bar{a}),$$

where the second implication holds because $\varphi(\bar{x})$ is quantifier-free (see Remark 4.5.4). As this holds for any $\bar{a} \in N$, we have $N \models \forall \bar{x} : \varphi(\bar{x})$. $\square$

If T is a theory, then $T_\forall$ denotes the set of universal sentences φ such that $T \vdash \varphi$.

Theorem 7.4.3. *$M \models T_\forall$ if and only if M is a substructure of a model of T.*

Proof. First suppose M is a substructure of $N \models T$. Then $N \models T_\forall$, so $M \models T_\forall$ as $T_\forall$ is a universal theory.

Conversely, suppose $M \models T_\forall$. Break into two cases:

1. The $\mathcal{L}(M)$-theory $\mathrm{diag}(M) \cup T$ is consistent. Take a model N. Moving N by an isomorphism, we may assume that N is an extension of M, and $N \models T$. Then we are done.

2. $\mathrm{diag}(M) \cup T$ is inconsistent. By compactness, there is a finite tuple $\bar{a} \in M$ and a quantifier-free formula $\varphi(\bar{a}) = \bigwedge_{i=1}^n \varphi_i(\bar{a})$ with $\varphi_i(\bar{a}) \in \mathrm{diag}(M)$ such that $\{\varphi(\bar{a})\} \cup T$ is inconsistent. Then $T \vdash \neg\varphi(\bar{a})$, so $T \vdash \forall \bar{x}\, \neg\varphi(\bar{x})$ by the lemma on constants (Lemma 7.4.1). But then
$$(\forall \bar{x}\, \neg\varphi(\bar{x})) \in T_\forall,$$
so $M \models \forall \bar{x}\, \neg\varphi(\bar{x})$, contradicting the fact that $M \models \varphi(\bar{a})$. □

Theorem 7.4.4. *Let T be a theory. Then T is axiomatizable by universal sentences if and only if $\mathrm{Mod}(T)$ is closed under substructures.*

Proof. If T is axiomatizable by universal sentences, then $\mathrm{Mod}(T)$ is closed under substructures by Lemma 7.4.2.

Next suppose that $\mathrm{Mod}(T)$ is closed under substructures. We claim that $T_\forall$ axiomatizes T. Certainly $M \models T \implies M \models T_\forall$. Conversely, suppose $M \models T_\forall$. By Theorem 7.4.3 there is an extension $N \supseteq M$ with $N \models T$. By the assumption on T, $M \models T$. □

Theorem 7.4.5. *Let $\mathcal{K}$ be a class of structures. Then the following are equivalent:*

1. *$\mathcal{K}$ is axiomatized by a universal theory.*

2. *$\mathcal{K}$ is closed under isomorphism, substructures, and ultraproducts.*

Proof. (1) $\implies$ (2) is clear.

(2) $\implies$ (1): by Theorem 7.3.2, $\mathcal{K}$ is an elementary class. Then use Theorem 7.4.4. □

Chapter 8

Topologies

8.1 Topological spaces

Fix a set S.

Definition 8.1.1. A family of sets $\mathcal{F} \subseteq \mathfrak{P}(S)$ is *closed under infinite unions* if whenever I is a set and $X_i \in \mathcal{F}$ for all $i \in I$, we have $\bigcup_{i \in I} X_i \in \mathcal{F}$. We say that $\mathcal{F} \subseteq \mathfrak{P}(S)$ is *closed under finite unions* if this holds for finite I.

Remark 8.1.2. The definition of "*closed under (in)finite intersections*" is similar, with fine print. We understand $\bigcap_{i \in I} X_i$ to mean

$$\{x \in S : \forall i \in I \ x \in X_i\}$$

Consequently, when $I = \varnothing$, the intersection $\bigcap_{i \in I} X_i$ is defined to be S, and not the set-theoretic universe, for example.

Definition 8.1.3. A *metric space* is a set S and function $d : S^2 \to \mathbb{R}$ satisfying the axioms:

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) = 0 \iff x = y$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Example 8.1.4. $\mathbb{R}^2$ is a metric space with respect to the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Fix a metric space (S, d). An *open ball* is a set of the form

$$B_\epsilon(p) = \{x \in S : d(x, p) < \epsilon\}$$

for some $p \in S$ and $\epsilon > 0$. A set $X \subseteq S$ is *open* if for every $p \in X$, there is $\epsilon > 0$ such that $B_\epsilon(p) \subseteq X$.

Fact 8.1.5. *Let (S, d) be a metric space.*

1. *Every open ball is open.*
2. *The collection of open sets is closed under infinite unions and finite intersections.*

Definition 8.1.6. A *topology* on S is a family of sets τ closed under finite intersections and infinite unions. A *topological space* is a set S with a topology τ.

Fix a topological space (S, τ).

Definition 8.1.7. A set $X \subseteq S$ is *open* if $X \in \tau$ and *closed* if $S \setminus X$ is open.

Fact 8.1.8. *In a metric space, a set X is closed if and only if it "closed under limits" in the sense that:*

$$\left(b_1, b_2, \ldots \in X \text{ and } \lim_{i \to \infty} b_i = a\right) \implies a \in X.$$

Definition 8.1.9. A topological space S is *Hausdorff* if for any $a_1 \neq a_2$ in S, there are open sets $U_i \ni a_i$ with $U_1 \cap U_2 = \varnothing$.

Remark 8.1.10. Metric spaces are Hausdorff: if $a_1 \neq a_2$, then $B_\epsilon(a_1) \cap B_\epsilon(a_2) = \varnothing$ for $\epsilon = d(a_1, a_2)/3$.

Theorem 8.1.11. *If S is Hausdorff and $p \in S$, then $\{p\}$ is closed.*

Proof. For every $q \neq p$, take open sets U_q, V_q with $q \in U_q$, $p \in V_q$, and $U_q \cap V_q = \varnothing$. In particular, $q \in U_q$ and $p \notin U_q$ for each $q \neq p$. Then $S \setminus \{p\}$ is the open set $\bigcup_{q \neq p} U_q$. □

Definition 8.1.12. An *open cover* is a collection $\mathcal{C}$ of open sets with $\bigcup \mathcal{C} = S$. A *subcover* of $\mathcal{C}$ is another open cover $\mathcal{C}' \subseteq \mathcal{C}$.

Definition 8.1.13. A topological space S is *compact* if every open cover has a finite subcover.

Fact 8.1.14. *A metric space (S, d) is compact if and only if every sequence has a convergent subsequence: for any $a_1, a_2, a_3, \ldots \in S$, there is a subsequence*

$$a_{i_1}, a_{i_2}, a_{i_3}, \ldots$$

(with $i_1 < i_2 < \cdots$) such that $\lim_{n\to\infty} a_{i_n}$ exists.

Theorem 8.1.15. *If a topological space S is compact and $\mathcal{F}$ is a family of closed sets with the FIP, then $\bigcap \mathcal{F} \neq \varnothing$. Conversely, this property characterizes compactness.*

Proof. Let X_i be open for $i \in I$, and let Y_i be the complementary closed sets. We claim that the following are equivalent:

1. If $\bigcup_{i\in I} X_i = S$, then there is $I_0 \subseteq_f I$ with $\bigcup_{i\in I_0} X_i = S$.
2. If $\bigcap_{i\in I} Y_i = \varnothing$, then there is $I_0 \subseteq_f I$ with $\bigcap_{i\in I_0} Y_i = \varnothing$.
3. If $\bigcap_{i\in I_0} Y_i \neq \varnothing$ for every $I_0 \subseteq_f I$, then $\bigcap_{i\in I} Y_i \neq \varnothing$.

Indeed, (1) and (2) are equivalent by de Morgan's laws, and (2) and (3) are contrapositives. Finally, observe that (1) is the definition of compactness, and (3) says that if $\{Y_i\}_{i\in I}$ has FIP then $\bigcap_i Y_i \neq \varnothing$. □

Definition 8.1.16. Let S_1, S_2 be two topological spaces and $f : S_1 \to S_2$ be a function. Then f is *continuous* if for every open set $U \subseteq S_2$, the preimage $f^{-1}(U)$ is open in S_1.

Fact 8.1.17. *If S_1, S_2 are metric spaces, a function $f : S_1 \to S_2$ is continuous iff f "preserves limits", in the sense that*

$$\lim_{i\to\infty} b_i = a \implies \lim_{i\to\infty} f(b_i) = f(a).$$

Definition 8.1.18. A function $f : S_1 \to S_2$ is a *homeomorphism* if f is continuous, f is a bijection, and $f^{-1} : S_2 \to S_1$ is continuous. Two topological spaces S_1, S_2 are *homeomorphic* if there is a homeomorphism from S_1 to S_2.

8.2 Ultralimits

Definition 8.2.1. If I is a set, $a_i \in S$ for $i \in I$, $\mathcal{U}$ is an ultrafilter on I, and $b \in S$, then b is an *ultralimit* of the a_i, written

$$b = \lim_{i \to \mathcal{U}} a_i$$

if for every open set $N \ni b$,

$$\{i \in I : a_i \in N\} \in \mathcal{U}.$$

Fact 8.2.2. *If S is Hausdorff then ultralimits are unique: for any $I, \mathcal{U}, \{a_i\}_{i \in I}$, there is at most one b with $b = \lim_{i \to \mathcal{U}} a_i$. In fact, this property holds if and only if S is Hausdorff.*

Half-proof. Suppose S is Hausdorff, and

$$b = \lim_{i \to \mathcal{U}} a_i$$
$$c = \lim_{i \to \mathcal{U}} a_i.$$

If $b \neq c$, by Hausdorffness there are open sets $N_1 \ni b$ and $N_2 \ni c$ with $N_1 \cap N_2 = \varnothing$. By definition of ultralimit, the sets

$$\{i \in I : a_i \in N_1\}$$
$$\{i \in I : a_i \in N_2\}$$

are in the ultrafilter $\mathcal{U}$. But their intersection is empty, and $\varnothing \notin \mathcal{U}$, a contradiction. □

Fact 8.2.3. *If S is compact then ultralimits exist: for any $I, \mathcal{U}, \{a_i\}_{i \in I}$, there is at least one b with $b = \lim_{i \to \mathcal{U}} a_i$. In fact, this property holds if and only if S is compact.*

Half-proof. Suppose S is compact. Say an open set $N \subseteq S$ is *good* if $\{i \in I : a_i \in N\} \in \mathcal{U}$, and *bad* otherwise, i.e., if $\{i \in I : a_i \notin N\} \in \mathcal{U}$. A finite union of bad sets is bad, because $\mathcal{U}$ is closed under finite intersections. The set S is good. There are two cases:

- S is covered by bad open sets. Then compactness gives a finite subcover, so S is a finite union of bad sets and S is bad, a contradiction.

- S is not covered by bad open sets. Take $p \in S$ such that p is in no bad set. Then every open set $N \ni p$ is good, which means $p = \lim_{i\to\mathcal{U}} a_i$. □

Theorem 8.2.4. *A set $C \subseteq S$ is closed if and only if C is closed under ultralimits, in the sense that*

$$\left(a_i \in C \text{ for all } i \in I \text{ and } b = \lim_{i\to\mathcal{U}} a_i\right) \implies b \in C.$$

Proof. First suppose C is closed, and $a_i \in C$ for all $i \in I$. If $b \notin C$, then the complement $S \setminus C$ is an open set containing b, but $\{i \in I : a_i \in S \setminus C\} = \varnothing \notin \mathcal{U}$, contradicting the definition of ultralimits. Thus $b \in C$, and C is closed under ultralimits.

Conversely, suppose C is closed under ultralimits. Let U_0 be the union of open sets disjoint from C. It suffices to show that $C \cup U_0 = S$, as then C is the complement of the open set U_0. Fix $p \notin U_0$; we claim $p \in C$. Let

$$\mathcal{F} = \{C \cap U : U \text{ is open and } U \ni p\}.$$

Then $\mathcal{F}$ is closed under finite intersections, because a finite intersection of open sets is open. Moreover, $\varnothing \notin \mathcal{F}$, or else $C \cap U = \varnothing$ and U shows $p \in U_0$, a contradiction. Thus $\mathcal{F}$ has the FIP and is contained in an ultrafilter $\mathcal{U}$ on C. Then

$$p = \lim_{x\to\mathcal{U}} x$$

because for any open set U containing p,

$$\{x \in C : x \in U\} = C \cap U \in \mathcal{F} \subseteq \mathcal{U}.$$

By assumption on C, $p \in C$. □

Fact 8.2.5. *If S_1, S_2 are topological spaces and $f : S_1 \to S_2$ is continuous, then f preserves ultralimits, in the sense that*

$$b = \lim_{i\to\mathcal{U}} a_i \implies f(b) = \lim_{i\to\mathcal{U}} f(a_i).$$

In fact, this property holds if and only if f is continuous.

Half-proof. Suppose f is continuous and $b = \lim_{i\to\mathcal{U}} a_i$. If N is an open set containing $f(b)$, then $f^{-1}(N)$ is an open set containing b, and so

$$\{i \in I : f(a_i) \in N\} = \{i \in I : a_i \in f^{-1}(N)\} \in \mathcal{U}.$$ □

Theorem 8.2.6. *Let S_1, S_2 be compact Hausdorff spaces and let $f : S_1 \to S_2$ be continuous.*

1. *If $X \subseteq S_1$ is closed, then the image $f(X)$ is closed.*
2. *If f is a bijection, then f is a homeomorphism.*

Proof. 1. By Theorem 8.2.4, it suffices to show that $f(X)$ is closed under ultralimits. Suppose $a_i \in f(X)$ and $\lim_{i \to \mathcal{U}} a_i = b$. We must show $b \in f(X)$. Write a_i as $f(\alpha_i)$ for some $\alpha_i \in X$. Let

$$\beta = \lim_{i \to \mathcal{U}} \alpha_i.$$

The ultralimit exists as S_1 is compact (Fact 8.2.3). The ultralimit β is in X because X is closed (Theorem 8.2.4). Then

$$f(\beta) = \lim_{i \to \mathcal{U}} f(\alpha_i) = \lim_{i \to \mathcal{U}} a_i$$

because the continuous function f preserves ultralimits (Fact 8.2.5). As S_2 is Hausdorff, ultralimits are unique (Fact 8.2.2), and so $f(\beta) = b$. Then $\beta \in X \implies b = f(\beta) \in f(X)$.

2. Let X be a subset of S_1. Part (1) shows that

$$X \text{ is closed in } S_1 \implies f(X) \text{ is closed in } S_2.$$

The reverse direction holds by continuity. Thus f is a homeomorphism.

□

8.3 The space of complete theories

Fix a language $\mathcal{L}$. Let $S = \{\mathrm{Th}(M) : M \text{ is an } \mathcal{L}\text{-structure}\}$. For each $\mathcal{L}$-sentence φ, let

$$[\![\varphi]\!] = \{T \in S : \varphi \in T\} = \{\mathrm{Th}(M) : M \models \varphi\}.$$

Note that $M \models \varphi \wedge \psi \iff (M \models \varphi \text{ and } M \models \psi)$, and so

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!].$$

Similarly,

$$\begin{aligned} [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\neg\varphi]\!] &= S \setminus [\![\varphi]\!] \\ [\![\top]\!] &= S \\ [\![\bot]\!] &= \varnothing. \end{aligned}$$

Definition 8.3.1. A set $X \subseteq S$ is *clopen* if $S = [\![\varphi]\!]$ for some sentence φ, and *open* if $X = \bigcup_{i \in I} Y_i$ where each Y_i is clopen.

Theorem 8.3.2. *The collection of open sets is closed under infinite unions and finite intersections, so it defines a topology on S. Moreover, the topology is Hausdorff.*

Proof. Infinite unions are clear: an infinite union of infinite unions of clopen sets is an infinite union of clopen sets. For finite intersections, note that

$$\left(\bigcup_{i \in I} [\![\varphi_i]\!]\right) \cap \left(\bigcup_{j \in J} [\![\psi_j]\!]\right) = \bigcup_{(i,j) \in I \times J} ([\![\varphi_i]\!] \cap [\![\psi_j]\!]) = \bigcup_{(i,j) \in I \times J} [\![\varphi_i \wedge \psi_j]\!].$$

For Hausdorffness, suppose $T_1, T_2 \in S$ and $T_1 \neq T_2$. Then there is a sentence φ such that, say, $\varphi \in T_1$ and $\varphi \notin T_2$. This means $T_1 \in [\![\varphi]\!]$ and $T_2 \notin [\![\varphi]\!]$. Then $T_2 \in S \setminus [\![\varphi]\!] = [\![\neg\varphi]\!]$. The two open sets $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ separate T_1 from T_2. ☐

Lemma 8.3.3. *Let X_i be clopen for each $i \in I$.*

1. *Suppose $\{X_i : i \in I\}$ has the FIP: for any $I_0 \subseteq_f I$, we have $\bigcap_{i \in I_0} X_i \neq \varnothing$. Then $\bigcap_{i \in I} X_i \neq \varnothing$.*

2. *If $S = \bigcup_{i \in I} X_i$, then there is finite $I_0 \subseteq_f I$ such that $S = \bigcup_{i \in I_0} X_i$.*

Proof.

1. Let $X_i = [\![\varphi_i]\!]$. For any $I_0 \subseteq I$, there is M with $\mathrm{Th}(M) \in \bigcap_{i \in I_0} [\![\varphi_i]\!]$, meaning that $M \models \{\varphi_i : i \in I_0\}$. By the compactness theorem, there is M satisfying $\{\varphi_i : i \in I\}$, and then $\mathrm{Th}(M) \in \bigcap_{i \in I} [\![\varphi_i]\!]$.

2. Let $Y_i = S \setminus X_i$. Apply part (1) to the family of clopen sets $\{Y_i\}_{i \in I}$. By assumption, $\bigcap_{i \in I} Y_i = \varnothing$, so there must be $I_0 \subseteq_f I$ such that $\bigcap_{i \in I_0} Y_i = \varnothing$, or equivalently, $\bigcup_{i \in I_0} X_i = S$. ☐

Theorem 8.3.4. *The topological space S is compact.*

Proof. Suppose $S = \bigcup_{i\in I} U_i$ for some open sets U_i. Let $\mathcal{F}$ be the family of clopen sets X such that $X \subseteq U_i$ for some i. Every open set is the union of its clopen subsets, so $U_i \subseteq \bigcup \mathcal{F}$ for each i. Then $S = \bigcup_{i\in I} U_i \subseteq \bigcup \mathcal{F}$. Applying Lemma 8.3.3(2) to the clopen cover $\mathcal{F}$, there is a finite subcover $S = \bigcup_{j=1}^n X_j$ with $X_j \in \mathcal{F}$. For each j, choose a $U_{i_j} \supseteq X_j$. Then $S = \bigcup_{j=1}^n X_j \subseteq \bigcup_{j=1}^n U_{i_j}$, and $\{U_{i_1}, \ldots, U_{i_n}\}$ is an open subcover of the original cover. □

If S is compact, then ultralimits should exist (Fact 8.2.3). In fact, ultralimits correspond exactly to ultraproducts:

Theorem 8.3.5. *Let M be an ultraproduct $\prod_{i\in I}^{/\mathcal{U}} M_i$. Then $\mathrm{Th}(M)$ is the ultralimit $\lim_{i\to\mathcal{U}} \mathrm{Th}(M_i)$ in the topological space S.*

Proof. Let N be an open set containing $\mathrm{Th}(M)$. Then there is a clopen set $[\![\varphi]\!]$ with $\mathrm{Th}(M) \in [\![\varphi]\!] \subseteq N_0$, because N_0 is a union of clopen sets. Then

$$\mathrm{Th}(M) \in [\![\varphi]\!] \implies M \models \varphi \implies \{i \in I : M_i \models \varphi\} \in \mathcal{U}$$

by Łoś's theorem (Theorem 7.2.3(5)). If $M_i \models \varphi$, then $\mathrm{Th}(M_i) \in [\![\varphi]\!] \subseteq N$. Therefore

$$\{i \in I : \mathrm{Th}(M_i) \in N\} \in \mathcal{U}.$$ □

Recall that X is closed iff $S \setminus X$ is open.

Theorem 8.3.6. *A set $X \subseteq S$ is clopen if and only if X is both closed and open.*

Proof. If X is clopen, then X is open. The complement $S \setminus X$ is clopen, hence open, and so X is also closed.

Conversely, suppose X is closed and open. Then $X = \bigcup_{i\in I} Y_i$ and $S \setminus X = \bigcup_{j\in J} Z_j$ where the Y_i and Z_i are clopen sets. Note that $S = \bigcup_{i\in I} Y_i \cup \bigcup_{j\in J} Z_j$. By Lemma 8.3.3(2), there are finite $I_0 \subseteq_f I$ and $J_0 \subseteq_f J$ such that $S = \bigcup_{i\in I_0} Y_i \cup \bigcup_{j\in J_0} Z_j$. Then X is the clopen set $\bigcup_{i\in I_0} Y_i$. □

8.4 Stone spaces

Definition 8.4.1. In any topological space, a *clopen set* is a set that is both closed and open.

Definition 8.4.2. A *Stone space* is a compact, Hausdorff topological space in which every open set is a union of clopen sets.

Theorem 8.4.3. *Let S be a set. Let $\mathcal{B}$ be a boolean subalgebra of $\mathfrak{P}(S)$. Suppose the following two conditions hold:*

1. *If $a, b \in S$ are distinct, then there is $X \in \mathcal{B}$ with $a \in X$ and $b \notin X$, or $b \in X$ and $a \notin X$.*

2. *If $\{X_i\}_{i \in I}$ is a family of sets in $\mathcal{B}$ with the FIP, then $\bigcap_{i \in I} X_i \neq \varnothing$.*

Then there is a Stone space topology on S in which the clopen sets are exactly the elements of $\mathcal{B}$.

Proof. This follows by the arguments of Section 8.3. □

Theorem 8.4.4. *Let S_1, S_2 be Stone spaces and $f : S_1 \to S_2$ be a map. Suppose that for any clopen set X in S_2, the preimage $f^{-1}(X)$ is clopen in S_1. Then f is continuous.*

Proof. If U is open in S_2, then $U = \bigcup_{i \in I} X_i$ for some clopen sets X_i. Then $f^{-1}(U) = \bigcup_{i \in I} f^{-1}(X_i)$ which is open in S_1. □

Chapter 9

Types and quantifier elimination

9.1 Types

Definition 9.1.1. Let M be an $\mathcal{L}$-structure, let A be a subset, and let $\bar{b}$ be an n-tuple. The *type* of $\bar{b}$ over A, written $\mathrm{tp}(\bar{b}/A)$, is the set of $\mathcal{L}(A)$-formulas $\varphi(x_1, \ldots, x_n)$ such that $M \models \varphi(\bar{b})$. When $A = \varnothing$, we write $\mathrm{tp}(\bar{b}/A)$ as $\mathrm{tp}(\bar{b})$. We write $\mathrm{tp}(-)$ as $\mathrm{tp}^M(-)$ when we need to specify M.

Remark 9.1.2. Partial elementary maps preserve types: if f is a partial elementary map from M to N and $\bar{a}$ is a tuple in $\mathrm{dom}(f)$, then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(f(\bar{a}))$. Indeed, for any formula $\varphi(\bar{x})$,

$$\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}) \iff M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})) \iff \varphi(\bar{x}) \in \mathrm{tp}(f(\bar{a})).$$

Remark 9.1.3. If $\bar{b}, \bar{c}$ are two n-tuples in the same structure M, then the following are equivalent:

1. $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$.

2. For every A-definable set D, $\bar{b} \in D \iff \bar{c} \in D$.

Indeed, (1) and (2) are equivalent to

3. For every $\mathcal{L}(A)$-formula $\varphi(\bar{x})$, $M \models \varphi(\bar{b}) \iff M \models \varphi(\bar{c})$.

Let M be an $\mathcal{L}$-structure and A be a set.

Definition 9.1.4. A *complete n-type* over A is something of the form $\mathrm{tp}^N(\bar{b}/A)$ for some $N \succeq M$ and n-tuple $\bar{b} \in N^n$. The set of complete n-types is written $S_n(A)$.

Lemma 9.1.5.
1. *If $p \in S_n(A)$ and $\varphi(\bar{x})$ is an $\mathcal{L}(A)$-formula, then $\neg\varphi \in p \iff \varphi \notin p$.*
2. *If $p, q \in S_n(A)$ and $p \subseteq q$, then $p = q$.*

Proof.
1. If $p = \mathrm{tp}^N(\bar{b}/A)$ for some elementary extension $N \succeq M$ and $\bar{b} \in N^n$, then

$$\neg\varphi \in p \iff N \models \neg\varphi(\bar{b}) \iff N \not\models \varphi(\bar{b}) \iff \varphi \notin p.$$

2. Otherwise take $\varphi \in q \setminus p$. Then $\varphi \in q \implies \neg\varphi \notin q$, and $\varphi \notin p \implies \neg\varphi \in p$. Then $\neg\varphi \in p \setminus q$, contradicting $p \subseteq q$. □

Definition 9.1.6. Let $\Sigma(x_1, \ldots, x_n)$ be a set of $\mathcal{L}(A)$-formulas in the variables $x_1, \ldots, x_n$. Then $\bar{b} \in M^n$ *realizes* Σ if $\Sigma \subseteq \mathrm{tp}(\bar{b}/A)$. We say that $\Sigma(\bar{x})$ is *realized* in M if some $\bar{a} \in M^n$ realizes Σ, and *omitted* in M otherwise.

Remark 9.1.7. If $p \in S_n(A)$ is a complete type, then a tuple $\bar{b}$ realizes p if and only if $\mathrm{tp}(\bar{b}/A) = p$, by Lemma 9.1.5(2) applied to the complete types p and $\mathrm{tp}(\bar{b}/A)$.

Theorem 9.1.8. *Let $\Sigma(\bar{x})$ be a set of $\mathcal{L}(A)$-formulas in the variables $x_1, \ldots, x_n$. The following are equivalent:*

1. *Every finite subset $\Sigma_0 \subseteq_f \Sigma$ is realized in M.*
2. *Σ is realized in an elementary extension of M.*

Proof. Consider a third condition:

3. Every finite subset $\Sigma_0 \subseteq_f \Sigma$ is realized in an elementary extension of M.

We claim $(3) \implies (1) \implies (2) \implies (3)$. The direction $(2) \implies (3)$ is clear.

$(3) \implies (1)$: if $\Sigma_0 = \{\varphi_1, \ldots, \varphi_n\}$, and Σ_0 is realized in $N \succeq M$, then

$$N \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}).$$

As $M \preceq N$, the same sentence holds in M, which means that Σ_0 is realized in M.

(1) $\Longrightarrow$ (2): Let $\mathcal{L}' = \mathcal{L}(A) \cup \{c_1, \ldots, c_n\}$ where the c_i are new constant symbols. If $\Sigma_0 \subseteq_f \Sigma$, then the $\mathcal{L}'$-structure

$$\text{eldiag}(M) \cup \{\varphi(\bar{c}) : \varphi \in \Sigma_0\}$$

is satisfied by the structure M with $\bar{c}$ interpreted as a realization of Σ_0. By compactness,

$$\text{eldiag}(M) \cup \{\varphi(\bar{c}) : \varphi \in \Sigma\}$$

is satisfied by some $\mathcal{L}'$-structure N. The fact that $N \models \text{eldiag}(M)$ means that, up to isomorphism, $N \succeq M$. Then the interpretation $\bar{c}^N$ of $\bar{c}$ in N is an n-tuple realizing $\Sigma(\bar{x})$, and so (2) holds. □

Definition 9.1.9. A *partial n-type* over A is a set $\Sigma(\bar{x})$ of $\mathcal{L}(A)$-formulas in the variables $x_1, \ldots, x_n$ satisfying the equivalent conditions of Theorem 9.1.8.

Remark 9.1.10. Condition (2) of Theorem 9.1.8 says that $\Sigma(\bar{x})$ is a partial type if and only if $\Sigma(\bar{x}) \subseteq \text{tp}^N(\bar{b}/A)$ for some tuple $\bar{b}$ in an elementary extension $N \succeq M$. Equivalently, $\Sigma(\bar{x})$ is a partial type if and only if Σ is a subset of a complete type. In particular, complete types are partial types.

Theorem 9.1.11. *Let $p(\bar{x})$ be a set of $\mathcal{L}(A)$-formulas in $\bar{x}$. Then $p(\bar{x})$ is a complete type over A if and only if $p(\bar{x})$ is a maximal partial type over A.*

Proof. By Remark 9.1.10, every partial type is contained in a complete type. Therefore, any maximal partial type is a complete type.

Conversely, suppose p is a complete type, but not maximal. Take a larger partial type $\Sigma \supsetneq p$. Then $\Sigma \subseteq q$ for some complete type q, and we have $p \subsetneq \Sigma \subseteq q$, contradicting the incomparability of complete types (Lemma 9.1.5(2)). □

Types over a theory

Let T be an $\mathcal{L}$-theory.

Definition 9.1.12. A *complete n-type* over T is something of the form $\text{tp}^M(\bar{a})$ for some $M \models T$ and n-tuple $\bar{a} \in M^n$. The set of complete n-types is written $S_n(T)$.

Lemma 9.1.13. *If $p, q \in S_n(T)$ and $p \subseteq q$, then $p = q$.*

Proof. Like Lemma 9.1.5. □

Theorem 9.1.14. *If $M \models T$ and $\bar{a} \in M^n$ realizes $p \in S_n(T)$, then $\operatorname{tp}(\bar{a}) = p$.*

Proof. Like Remark 9.1.7. □

Theorem 9.1.15. *Let $\Sigma(\bar{x})$ be a set of $\mathcal{L}$-formulas in the variables $x_1, \ldots, x_n$. The following are equivalent:*

1. *Every finite subset $\Sigma_0 \subseteq \Sigma$ is realized in a model of T.*

2. *Σ is realized in a model of T.*

Proof. Like Theorem 9.1.8, but more straightforward. □

Definition 9.1.16. A *partial n-type* over T is a set $\Sigma(\bar{x})$ of $\mathcal{L}$-formulas in the variables $x_1, \ldots, x_n$ satisfying the equivalent conditions of Theorem 9.1.15.

Remark 9.1.17. As in Remark 9.1.10, $\Sigma(\bar{x})$ is a partial type if and only if $\Sigma(\bar{x})$ is a subset of a complete type. In particular, complete types are partial types.

Theorem 9.1.18. *Let $p(\bar{x})$ be a set of $\mathcal{L}$-formulas in $\bar{x}$. Then $p(\bar{x})$ is a complete type over T if and only if $p(\bar{x})$ is a maximal partial type over T.*

Proof. Like Theorem 9.1.11 □

Lemma 9.1.19. *Let T be a* complete *theory, M be a model, and $\Sigma(\bar{x})$ be a set of $\mathcal{L}$-formulas in the variables $\bar{x}$.*

1. *If $\Sigma(\bar{x})$ is finite, then $\Sigma(\bar{x})$ is realized in a model of T if and only if $\Sigma(\bar{x})$ is realized in M.*

2. *$\Sigma(\bar{x})$ is a partial type over T if and only if $\Sigma(\bar{x})$ is a partial type over $\varnothing \subseteq M$.*

3. *$\Sigma(\bar{x})$ is a complete type over T if and only if $\Sigma(\bar{x})$ is a complete type over $\varnothing \subseteq M$.*

Proof.
1. If $N \models \exists \bar{x} \bigwedge_{\varphi \in \Sigma} \varphi(\bar{x})$ and $N \models T$ then $M \models \exists \bar{x} \bigwedge_{\varphi \in \Sigma} \varphi(\bar{x})$ because $M \equiv N$.

2. By part (1), $\Sigma(\bar{x})$ is finitely realized in models of T if and only if $\Sigma(\bar{x})$ is finitely realized in M.

3. The complete types over T are the maximal partial types over T, and similarly for types over $A = \varnothing$. Since the sets of partial types agree by part (2), the sets of maximal partial types agree. □

Corollary 9.1.20. *If T is a complete theory and M is a model, then $S_n(T) = S_n^M(\varnothing)$.*

Fact 9.1.21. *If $A \subseteq M$ and $T_{A,M}$ is the complete $\mathcal{L}(A)$-theory of M, then $S_n^M(A) = S_n(T_{A,M})$.*

9.2 The topology on type space

Let M be an $\mathcal{L}$-structure and A be a subset of M.

Theorem 9.2.1. *For each $\mathcal{L}(A)$-formula $\varphi(x_1, \ldots, x_n)$, let $[\![\varphi]\!] = \{p \in S_n(A) : \varphi \in p\}$. Then*

$$
\begin{gathered}
[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\neg\varphi]\!] = S_n(A) \setminus [\![\varphi]\!] \\
[\![\top]\!] = S_n(A) \qquad [\![\bot]\!] = \varnothing
\end{gathered}
$$

Proof. This is a matter of unwinding definitions. For example, if $p \in S_n(A)$ is $\mathrm{tp}^N(\bar{b}/A)$, then

$$
\begin{gathered}
p \in [\![\varphi \wedge \psi]\!] \iff \varphi \wedge \psi \in p \iff N \models \varphi(\bar{b}) \wedge \psi(\bar{b}) \\
\iff N \left(\models \varphi(\bar{b}) \text{ and } N \models \psi(\bar{b})\right) \iff \cdots \iff (p \in [\![\varphi]\!] \text{ and } p \in [\![\psi]\!]),
\end{gathered}
$$

showing that $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$. □

Theorem 9.2.2. *If φ, ψ are $\mathcal{L}(A)$-formulas, then*

$$
\begin{aligned}
[\![\varphi]\!] \subseteq [\![\psi]\!] &\iff \varphi(M) \subseteq \psi(M) \\
[\![\varphi]\!] = [\![\psi]\!] &\iff \varphi(M) = \psi(M).
\end{aligned}
$$

Proof. We prove the first line, which directly implies the second line. Unwinding the definitions, $[\![\varphi]\!] \subseteq [\![\psi]\!]$ means that for every $N \succeq M$ and $\bar{b} \in N$, if $N \models \varphi(\bar{b})$ then $N \models \psi(\bar{b})$. Equivalently, every $N \succeq M$ satisfies

$$N \models \forall \bar{x}\ (\varphi(\bar{x}) \to \psi(\bar{x})).$$

Similarly, $\varphi(M) \subseteq \psi(M)$ means

$$M \models \forall \bar{x}\ (\varphi(\bar{x}) \to \psi(\bar{x})).$$

The two conditions are equivalent because $N \succeq M$. □

Theorem 9.2.3. *The collection of sets $[\![\varphi]\!]$ form a basis for a topology making $S_n(A)$ into a Stone space, and the sets $[\![\varphi]\!]$ are exactly the clopen sets in this topology.*

Proof. We use Theorem 8.4.3. There are two things to check:

1. If $p, q \in S_n(A)$ are distinct, then there is a set $[\![\varphi]\!]$ distinguishing p and q. Indeed, take φ in $p \setminus q$.
2. If $\{[\![\varphi_i]\!]\}_{i \in I}$ has the FIP, then $\bigcap_i [\![\varphi_i]\!] \neq \varnothing$. Equivalently, if every finite subset of $\{\varphi_i : i \in I\}$ is contained in a complete type, then $\{\varphi_i : i \in I\}$ is contained in a complete type. This is clear by Theorem 9.1.8. □

Using Theorem 9.2.2 we see the following:

Corollary 9.2.4. *The boolean algebra of clopen sets in $S_n(A)$ is isomorphic to the boolean algebra of A-definable subsets of M^n via the isomorphism $[\![\varphi]\!] \mapsto \varphi(M^n)$.*

There is an analogous picture for type spaces of a theory. Let T be an $\mathcal{L}$-theory:

Theorem 9.2.5. *For each $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, let $[\![\varphi]\!] = \{p \in S_n(T) : \varphi \in p\}$. Then*

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$$
$$[\![\neg\varphi]\!] = S_n(T) \setminus [\![\varphi]\!]$$
$$[\![\top]\!] = S_n(T) \qquad [\![\bot]\!] = \varnothing$$

Moreover, $[\![\varphi]\!] = [\![\psi]\!]$ if and only if φ and ψ are equivalent modulo T, in the sense that

$$T \vdash \forall \bar{x}\ (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

or equivalently $\varphi(M) = \psi(M)$ for all $M \models T$.

Theorem 9.2.6. *The collection of sets $[\![\varphi]\!]$ is a basis for a topology making $S_n(T)$ a Stone space, and the clopen sets are exactly the sets $[\![\varphi]\!]$.*

9.3 Quantifier elimination

Definition 9.3.1. A theory T has *quantifier elimination* if for every formula $\varphi(\bar{x})$, there is a quantifier-free formula $\psi(\bar{x})$ that is equivalent to φ in models of T:

$$T \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Definition 9.3.2. If $(a_1, \ldots, a_n)$ is a tuple in a structure M, then the *quantifier-free type* $\mathrm{qftp}(\bar{a})$ is the set of quantifier-free formulas $\varphi(x_1, \ldots, x_n)$ with $M \models \varphi(\bar{a})$.

For any theory T, let $S_n^{\mathrm{qfree}}(T)$ be the space of quantifier-free n-types, analogous to the space of complete n-types $S_n(T)$. Each quantifier-free formula $\varphi(x_1, \ldots, x_n)$ defines a set $[\![\varphi]\!]_{\mathrm{qfree}} \subseteq S_n^{\mathrm{qfree}}(T)$, and these sets form a basis for a topology making $S_n^{\mathrm{qfree}}(T)$ into a Stone space. Moreover, every clopen set in $S_n^{\mathrm{qfree}}(T)$ has the form $[\![\varphi]\!]_{\mathrm{qfree}}$ for some quantifier-free φ. The proofs are analogous to the case of $S_n(T)$ (Section 9.2). There is a restriction map

$$S_n(T) \to S_n^{\mathrm{qfree}}(T)$$

sending $\mathrm{tp}(\bar{a})$ to $\mathrm{qftp}(\bar{a})$. Note that

$$\mathrm{qftp}^M(\bar{a}) \in [\![\varphi]\!]_{\mathrm{qfree}} \iff M \models \varphi(\bar{a}) \iff \mathrm{tp}^M(\bar{a}) \in [\![\varphi]\!]$$

so the preimage of $[\![\varphi]\!]_{\mathrm{qfree}}$ is $[\![\varphi]\!]$. Because preimages of clopen sets are clopen, the restriction map is continuous (Theorem 8.4.4).

Theorem 9.3.3. *The following are equivalent:*

1. *T has quantifier elimination.*

2. *For any $M, N \models T$ and $\bar{a} \in M^n$ and $\bar{b} \in N^n$,*

$$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b}) \implies \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b}).$$

Proof. Consider a third condition:

3. For every n, the restriction map $S_n(T) \to S_n^{\mathrm{qfree}}(T)$ is a homeomorphism.

We claim (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2): Suppose that T has quantifier elimination and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. For any formula $\varphi(\bar{x})$, there is an equivalent quantifier free-formula $\psi(\bar{x})$, and then

$$M \models \varphi(\bar{a}) \iff M \models \psi(\bar{a}) \iff N \models \psi(\bar{b}) \iff N \models \varphi(\bar{b}),$$

so that $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$.

(2) $\implies$ (3): The restriction map is surjective, since $\mathrm{qftp}^M(\bar{a})$ lifts to $\mathrm{tp}^M(\bar{a})$. The restriction map is injective by (2). Then the restriction map is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism (Theorem 8.2.6).

(3) $\implies$ (1): Let $\varphi(x_1, \ldots, x_n)$ be a formula. Then $[\![\varphi]\!] \subseteq S_n(T)$ is clopen. The image under the homeomorphism $S_n(T) \to S_n^{\mathrm{qfree}}(T)$ is a clopen set, which must be $[\![\psi]\!]_{\mathrm{qfree}} \subseteq S_n^{\mathrm{qfree}}(T)$ for some quantifier-free formula ψ. Then

$$M \models \varphi(\bar{a}) \iff \mathrm{tp}(\bar{a}) \in [\![\varphi]\!] \iff \mathrm{qftp}(\bar{a}) \in [\![\psi]\!]_{\mathrm{qfree}} \iff M \models \psi(\bar{a}),$$

so $T \vdash \varphi \leftrightarrow \psi$. □

Theorem 9.3.4. *Suppose M, N are $\mathcal{L}$-structures and $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then the following are equivalent:*

1. $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$.

2. *There is an isomorphism $f : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ with $f(\bar{a}) = \bar{b}$.*

Proof. (1) $\implies$ (2): Let $p = \mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. Note that every element of $\langle \bar{a} \rangle_M$ has the form $t^M(\bar{a})$ for some term t. Similarly, $\langle \bar{b} \rangle_M = \{t^N(\bar{b}) : t(\bar{x}) \text{ a term}\}$. Moreover,

$$t^M(\bar{a}) = s^M(\bar{a}) \iff (t(\bar{x}) = s(\bar{x})) \in p \iff t^N(\bar{b}) = s^N(\bar{b}),$$

Therefore there is a bijection $f : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ sending $t^M(\bar{a})$ to $t^N(\bar{b})$. If R is a k-ary relation symbol and $t_1, \ldots, t_k$ are terms in $\bar{x}$, then

$$R^M(t_1^M(\bar{a}), \ldots, t_k^M(\bar{a})) \iff R(t_1, \ldots, t_k) \in p \iff R^N(t_1^N(\bar{a}), \ldots, t_k^N(\bar{a})),$$

and so f preserves relation symbols. A similar argument shows f preserves function symbols.

(2) $\implies$ (1): Let $A = \langle \bar{a} \rangle_M$ and $B = \langle \bar{b} \rangle_N$. Then

$$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^A(\bar{a}) = \mathrm{qftp}^B(\bar{b}) = \mathrm{qftp}^N(\bar{b})$$

by Theorem 4.4.3 applied to the embeddings $A \to M$, $B \to N$, and $A \xrightarrow{\cong} B$. $\square$

Theorem 9.3.5. *Suppose T has quantifier elimination. If $M, N \models T$, then*

$$M \equiv N \iff \langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N.$$

Proof. The left hand side says that $\mathrm{tp}^M() = \mathrm{tp}^N()$. Indeed, $\mathrm{tp}^M()$ is the set of formulas in 0 free variables (i.e., sentences) satisfied by the empty tuple (i.e., true in M), which is just $\mathrm{Th}(M)$.

Meanwhile, the right hand side of $(*)$ is equivalent to $\mathrm{qftp}^M() = \mathrm{qftp}^N()$ by Theorem 9.3.4. We don't have to worry about how the isomorphism acts on the generators, because there are no generators to check.

Finally, quantifier elimination gives

$$\mathrm{tp}^M() = \mathrm{tp}^N() \iff \mathrm{qftp}^M() = \mathrm{qftp}^N(). \qquad \square$$

Theorem 9.3.6. *Suppose T has quantifier elimination and $M, N \models T$.*

1. *If $f : M \to N$ is an embedding, then f is an elementary embedding.*

2. *If M is a substructure of N, then M is an elementary substructure of N.*

Proof. Embeddings preserve quantifier-free formulas (Theorem 4.4.3(3)), and quantifier elimination allows us to replace any formula with an equivalent quantifier-free formula. $\square$

9.4 Quantifier elimination in DLO

Theorem 9.4.1. *DLO has quantifier elimination.*

Proof. We use the criterion of Theorem 9.3.3. Suppose $M, N \models \mathrm{DLO}$, $\bar{a} \in M^n$, $\bar{b} \in N^n$, and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. We must show $\mathrm{tp}^M(\bar{a}) \stackrel{?}{=} \mathrm{tp}^N(\bar{b})$.

By Theorem 9.3.4, there is a partial isomorphism $f : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ with $f(\bar{a}) = \bar{b}$. As the language of orders is relational, all terms are trivial, and so $\mathrm{dom}(f) = \langle \bar{a} \rangle_M = \{a_1, \ldots, a_n\}$. Then f is a finite partial isomorphism. By Theorem 4.7.10, the class of finite partial isomorphisms is a back-and-forth system, and so f is a partial elementary map by Theorem 4.7.6. Therefore $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(f(\bar{a})) = \mathrm{tp}^N(\bar{b})$. $\square$

If M, N are models of DLO, then $\langle \varnothing \rangle_M$ and $\langle \varnothing \rangle_N$ are both the empty order $\varnothing$, so $M \equiv N$. This gives another proof of the completeness of DLO.

Corollary 9.4.2. $\mathbb{Q}$ *and the open interval* $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ *are elementary substructures of* $\mathbb{R}$*.*

Definition 9.4.3. If $(M, \leq) \models \mathrm{DLO}$ and $a < b$ are two points in M, then we define

$$\begin{aligned}
(a, b) &:= \{x \in M : a < x < b\} \\
(a, +\infty) &:= \{x \in M : a < x\} \\
(-\infty, a) &:= \{x \in M : x < a\} \\
(-\infty, +\infty) &:= M
\end{aligned}$$

Sets of these forms are called *open intervals.*

Corollary 9.4.4. *If* $M \models \mathrm{DLO}$ *and* $D \subseteq M^1$ *is definable, then* D *is a finite union of points and open intervals.*

Proof. Let $\mathcal{F}$ be the collection of sets $D \subseteq M$ such that D is a finite union of points and open intervals. It is an exercise to see that $\mathcal{F}$ is closed under boolean operations (it is a boolean subalgebra of $\mathfrak{P}(M)$). We must show that $D \in \mathcal{F}$. Write D as $\varphi(M, \bar{b})$ for some $\mathcal{L}$-formula $\varphi(x, \bar{y})$ and tuple of parameters $\bar{b}$. By quantifier-elimination, we may assume φ is quantifier-free. Then φ is a boolean combination of atomic formulas. As $\mathcal{F}$ is closed under boolean combinations, we may assume φ is atomic. Then $\varphi(x, \bar{b})$ has one of the following forms:

$$x \leq x,\ x \leq b_i,\ b_i \leq x,\ b_i \leq b_j,\ x = x,\ x = b_i,\ b_i = x,\ b_i = b_j.$$

Each of these formulas defines a set in $\mathcal{F}$. $\square$

Chapter 10

Algebraically closed fields

10.1 Polynomial rings

Fact 10.1.1. *Let R be a ring and x be a symbol. There is a ring $R[x]$ extending R, generated by $R \cup \{x\}$, with the following properties:*

1. *Every element of $R[x]$ has the form $\sum_{i=0}^{n} a_i x^i$ for some $n \geq 0$ and $a_0, \ldots, a_n \in R$.*

2. *Two elements $\sum_{i=0}^{n} a_i x^i$ and $\sum_{i=0}^{n} b_i x^i$ are equal if and only if the tuples $\bar{a}$ and $\bar{b}$ are equal.*

3. *The sum of two elements is given by*

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i.$$

4. *The product of two elements is given by*

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} c_k x^k, \text{ where } c_k = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m \\ i+j=k}} a_i b_j.$$

Elements of $R[x]$ are called polynomials*.*

If $P(x)$ is a non-zero polynomial, then we can write

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $n \geq 0$ and $a_n \neq 0$. Then $a_n x^n$ is called the *leading term*, a_n is called the *leading coefficient*, and n is called the *degree* of $P(x)$. We write the degree of P as $\deg P$. The degree of the zero polynomial is defined to be $-\infty$. A nonzero polynomial is *monic* if the leading coefficient is 1.

Definition 10.1.2 (Evaluating polynomials)**.** If $P(x) \in R[x]$ has the form

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

and if $b \in R$, then

$$P(b) := \sum_{i=0}^{n} a_i b^i.$$

Fact 10.1.3. *For fixed b, the map*

$$\begin{aligned} R[x] &\to R \\ P(x) &\to P(b) \end{aligned}$$

is a ring homomorphism, meaning among other things that

$$\begin{aligned} (P+Q)(b) &= P(b) + Q(b) \\ (PQ)(b) &= P(b)Q(b) \end{aligned}$$

Lemma 10.1.4. *Let K be a field, and let $A(x), B(x) \in K[x]$ be polynomials with $B(x)$ non-zero. then there is $R(x) \in K[x]$ such that*

$$\begin{aligned} A(x) &\equiv R(x) \pmod{B(x)} \\ \deg R(x) &< \deg B(x). \end{aligned}$$

Proof. If bx^m is the leading term of $B(x)$, replace $B(x)$ with $b^{-1}B(x)$. Then we can assume $B(x)$ is monic. Proceed by induction on $\deg A(x)$. If $\deg A(x) < \deg B(x)$, take $R(x) = A(x)$. Otherwise, let

$$\begin{aligned} A(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots \\ B(x) &= x^m + b_{m-1}x^{m-1} + \cdots \end{aligned}$$

where $m = \deg(B) \leq n = \deg(A)$. Let

$$\begin{aligned} A'(x) &= A(x) - a_n x^{n-m} B(x) \\ &= (a_n x^n + a_{n-1} x^{n-1} + \cdots) - (a_n x^n + a_n b_{m-1} x^{n-1} + \cdots + a_n b_0 x^{n-m}). \end{aligned}$$

Then $\deg A'(x) < n = \deg A(x)$, but $A(x) \equiv A'(x) \pmod{B(x)}$. By induction, $A'(x) \equiv R(x) \pmod{B(x)}$ for some suitable $R(x)$. □

Let K be a field.

Theorem 10.1.5. *Suppose $P(x) \in K[x]$ and $P(a) = 0$ for some $a \in K$. Then $P(x) = (x - a)Q(x)$ for some $Q(x) \in K[x]$.*

Proof. Apply the division lemma to $P(x)$ and $(x - a)$, to get $R(x) \in K[x]$ with

$$\begin{aligned} P(x) &\equiv R(x) \pmod{x - a} \\ \deg R(x) &< \deg(x - a) = 1. \end{aligned}$$

The first line means there is $Q(x) \in K[x]$ with

$$P(x) = (x - a)Q(x) + R(x). \tag{$*$}$$

The second line means $R(x) = c$ for some constant c. Substituting $x = a$ into $(*)$, we see

$$\begin{aligned} P(a) &= (a - a)Q(a) + c \\ 0 &= 0 + c. \end{aligned}$$

Then $c = 0$, so $R(x) = 0$ and $P(x) = (x - a)Q(x)$. □

A *root* of $P(x) \in K[x]$ is an element $a \in K$ with $P(a) = 0$. Note that a is a root of $P(x)Q(x)$ if and only if a is a root of $P(x)$ or a is a root of $Q(x)$, by the zero law (Theorem 2.4.11).

Theorem 10.1.6. *If $P(x)$ is a non-zero polynomial, then the number of roots of $P(x)$ in K is at most* $\deg P(x)$.

Proof. Let $d = \deg P(x)$. If $P(x)$ has no roots, then the claim holds. Otherwise, take some root a. Then $P(x) = (x - a)Q(x)$ for some polynomial $Q(x)$ of degree $d - 1$. By induction on d, $Q(x)$ has at most $d - 1$ roots. As $(x - a)$ has one root, $P(x)$ has at most $(d - 1) + 1 = d$ roots. □

Theorem 10.1.7. *If K is a field, every ideal $I \subseteq K[x]$ is a principal ideal $P(x)K[x]$ for some polynomial $P(x)$.*

Proof. Like the proof of the same fact in $\mathbb{Z}$ (Theorem 3.5.3), but using Lemma 10.1.4 instead of Lemma 3.5.2 and using $\deg P$ instead of $|n|$. □

Remark 10.1.8. In Theorem 10.1.7, scaling $P(x)$ by a non-zero constant, we may assume $P(x)$ is zero or monic. Then $P(x)$ is uniquely determined.

Definition 10.1.9. A non-constant polynomial $P(x)$ is *reducible* if $P(x)$ is a product of two non-constant polynomials, and *irreducible* otherwise.

Theorem 10.1.10. *If $P(x)$ is an irreducible polynomial in $K[x]$, then the quotient $K[x]/P(x)K[x]$ is a field.*

Proof. Like the proof that $\mathbb{Z}/p\mathbb{Z}$ is a field when p is prime (Theorem 3.5.7). □

10.2 The theory ACF

Definition 10.2.1. A field K is *algebraically closed* if every polynomial $P(x) \in K[x]$ with $\deg P > 0$ has a root. That is, for every $n > 0$ and $a_0, a_1, \ldots, a_n \in K$ with $a_n \neq 0$, there is $x \in K$ with

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Note that the class of algebraically closed fields is elementary, defined by a theory consisting of the field axioms plus an axiom schema

$$\forall a_1, \ldots, a_n \left(a_n \neq 0 \to \exists x \sum_{i=0}^{n} a_i x^i = 0 \right) \text{ for } n > 0.$$

This theory is usually called *ACF*.

Fact 10.2.2 (Fundamental theorem of algebra)**.** *The field $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ is algebraically closed.*

Theorem 10.2.3. *If K is algebraically closed, then K is infinite.*

Proof. If $K = \{a_1, \ldots, a_n\}$, then $P(x) = 1 + \prod_{i=1}^{n}(x - a_i)$ has no root in K, so $K \not\models$ ACF. □

10.3 Algebraic and transcendental elements

Let L be a field and K be a subfield.

Definition 10.3.1. An element $a \in L$ is *algebraic* over K if $P(a) = 0$ for some non-zero polynomial $P(x) \in K[x]$. Otherwise, a is *transcendental* over K.

If $a \in L$, then $K[a]$ denotes the subring of L generated by $K \cup \{a\}$.

Theorem 10.3.2. *Fix $a \in L$. Let $I_{a/K} = \{P(x) \in K[x] : P(a) = 0\}$.*

1. *$I_{a/K}$ is an ideal in $K[x]$.*
2. *$K[x]/I_{a/K}$ is isomorphic to $K[a]$ via the isomorphism sending $P(x)$ to $P(a)$.*
3. *If a is transcendental over K, then $I_{a/K}$ is the zero ideal $0 \cdot K[x] = \{0\}$.*
4. *If a is algebraic over K, then $I_{a/K} = P(x) \cdot K[x]$ for some irreducible monic polynomial $P(x)$.*

Proof. Let $f : K[x] \to K[a]$ be the ring homomorphism $P(x) \mapsto P(a)$. Then $I_{a/K}$ is the kernel. The image $\mathrm{im}(f)$ is a subring of $K[a]$ containing $f(K) = K$ and $f(x) = a$, so it must be all of $K[a]$. By the fundamental theorem on homomorphisms, $K[x]/I_{a/K}$ is isomorphic to the image $K[a]$ via the map $P(x) \mapsto P(a)$.

By Theorem 10.1.7 and Remark 10.1.8, the ideal $I_{a/K}$ is $P(x) \cdot K[x]$, where $P(x)$ is zero or a monic polynomial. In the first case, $I_{a/K} = \{0\}$, which means precisely that a is transcendental. In the second case, $P(x) \in I_{a/K}$ implies that $P(a) = 0$, and so a is algebraic. Suppose for the sake of contradiction that $P(x)$ is reducible as $P(x) = Q_1(x)Q_2(x)$. Note that $\deg P > \deg Q_i$ for $i = 1, 2$, so $Q_i \notin P(x) \cdot K[x] = I_{a/K}$, and therefore $Q_i(a) \neq 0$ for $i = 1, 2$. But then $0 = P(a) = Q_1(a)Q_2(a) \neq 0$, a contradiction. $\square$

Definition 10.3.3. If $a \in L$ is algebraic over K, then the *minimal polynomial* of a over K is the monic irreducible polynomial $P(x)$ appearing in Theorem 10.3.2(4).

Theorem 10.3.4. *If $P(x) \in K[x]$ is a monic irreducible polynomial and $a \in L$ is a root of $P(x)$, then $P(x)$ is the minimal polynomial of a.*

Proof. Let $P_0(x)$ be the actual minimal polynomial of a. Then

$$P(a) = 0 \implies P(x) \in I_{a/K} = P_0(x) \cdot K[x].$$

Thus $P(x)$ is a multiple of $P_0(x)$:

$$P(x) = P_0(x)Q(x).$$

As $P(x)$ is irreducible, this forces $Q(x) = 1$ and $P(x) = P_0(x)$. $\square$

10.4 Quantifier elimination in ACF

Lemma 10.4.1. *Let M_1, M_2 be two fields. Let $f : R_1 \to R_2$ be a partial isomorphism. Then there is a larger partial isomorphism $g : K_1 \to K_2$ such that K_1 and K_2 are fields.*

Proof. Let $K_i = \{a/b : a, b \in R_i,\ b \neq 0\}$. Then K_i is a subfield of M_i. For example, K_i is closed under addition because

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Let $g : K_1 \to K_2$ be defined by $g(a/b) = f(a)/f(b)$. This is well-defined because

$$\frac{a}{b} = \frac{a'}{b'} \iff ab' = a'b \implies f(ab') = f(a'b)$$
$$\iff f(a)f(b') = f(a')f(b) \iff \frac{f(a)}{f(b)} = \frac{f(a')}{f(b')}.$$

It is an exercise in algebra to see that g is an isomorphism. For example, g preserves addition because

$$g\left(\frac{a}{b} + \frac{c}{d}\right) = g\left(\frac{ad + bc}{bd}\right) = \frac{f(ad + bc)}{f(bd)} = \frac{f(a)f(d) + f(b)f(c)}{f(b)f(d)}$$
$$= \frac{f(a)}{f(b)} + \frac{f(c)}{f(d)} = g\left(\frac{a}{b}\right) + g\left(\frac{c}{d}\right).$$ $\square$

Lemma 10.4.2. *Let M_1, M_2 be two uncountable algebraically closed fields extending a countable subfield K. For any $a \in M_1$, there is $b \in M_2$ and an isomorphism $f : K[a] \to K[b]$ sending a to b and fixing K.*

Proof. Recall the notation

$$I_{a/K} = \{P(x) \in K[x] : P(a) = 0\}$$

from Theorem 10.3.2.

Claim. There is $b \in M_2$ with $I_{a/K} = I_{b/K}$.

Proof. First suppose a is transcendental. There are countably many non-zero polynomials in $K[x]$, and each has finitely many roots (Theorem 10.1.6). Therefore only countable many $b \in M_2$ are algebraic over K. Take $b \in M_2$ transcendental over K. Then $I_{a/K} = \{0\} = I_{b/K}$ by Theorem 10.3.2.

Next suppose a is algebraic with minimal polynomial $P(x) \in K[x]$. As M_2 is algebraically closed, there is $b \in M_2$ with $P(b) = 0$. By Theorem 10.3.4, $P(x)$ is the minimal polynomial of b, and then

$$I_{a/K} = P(x) \cdot K[x] = I_{b/K}. \qquad \square_{\text{Claim}}$$

Fix $b \in M_2$ as in the claim, and let $I = I_{a/K} = I_{b/K}$. By Theorem 10.3.2, we have isomorphisms

$$K[x]/I \to K[a]$$
$$K[x]/I \to K[b]$$

sending $P(x)$ to $P(a)$ and $P(b)$, respectively. The composition

$$K[a] \to K[x]/I \to K[b]$$

is the desired isomorphism. $\square$

Lemma 10.4.3. *Let M_1, M_2 be two uncountable algebraically closed fields. Let $f : R_1 \to R_2$ be a finitely-generated partial isomorphism. For any $a \in M_1$, there is $b \in M_2$ and an isomorphism $R_1[a] \to R_2[b]$ extending f.*

Proof. By Lemma 10.4.1, the isomorphism $f : R_1 \to R_2$ extends to an isomorphism $f' : K_1 \to K_2$ where the K_i are fields. Moving M_2, we may assume $K_1 = K_2$ and f' is the identity map. Applying Lemma 10.4.2, there is an element $b \in M_2$ and an isomorphism $K_1[b] \to K_2[b]$ extending f'. This isomorphism restricts to an isomorphism $R_1[b] \to R_2[b]$ extending f. $\square$

Theorem 10.4.4. *The theory ACF has quantifier elimination.*

Proof. We use the criterion of Theorem 9.3.3. Suppose $M, N \models \mathrm{ACF}$, $\bar{a} \in M^n$, $\bar{b} \in N^n$, and $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. We must show $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$. Recall that M and N are infinite (Theorem 10.2.3). Replacing M, N with elementary extensions, we may assume M and N are uncountable. By Theorem 9.3.4, there is a partial isomorphism $f : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ with $f(\bar{a}) = \bar{b}$. Let $\mathcal{F}$ be the collection of finitely generated partial isomorphisms from M to N. Then $\mathcal{F}$ is a back-and-forth system: Lemma 10.4.3 gives the forward condition, and the backward condition holds by symmetry. By Theorem 4.7.6, the fact that $f \in \mathcal{F}$ implies that f is a partial elementary map, and so $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$. $\square$

Corollary 10.4.5. *If* $M, N \models \mathrm{ACF}$*, then* $M \equiv N \iff \mathrm{char}(M) = \mathrm{char}(N)$.

Proof. By quantifier elimination in ACF and Theorem 9.3.5, we have

$$M \equiv N \iff \langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N.$$

By Theorem 3.7.5, the right hand side is equivalent to

$$\mathbb{Z}/n_M\mathbb{Z} \stackrel{?}{\cong} \mathbb{Z}/n_N\mathbb{Z}, \qquad (*)$$

where $n_M = \mathrm{char}(M)$ and $n_N = \mathrm{char}(N)$. Clearly, $(*)$ holds if and only if $n_M = n_N$. $\square$

Definition 10.4.6. For $p \in \{0, 2, 3, 5, 7, 11, 13, \ldots\}$, ACF_p is the theory of algebraically closed fields of characteristic p:

$$\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} \neq 0 : p = 2, 3, 5, 7, \ldots\}$$
$$\mathrm{ACF}_p = \mathrm{ACF} \cup \{\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0\} \text{ for } p > 0$$

For example, $\mathbb{C} \models \mathrm{ACF}_0$. We will see later (Corollary 11.3.5) that ACF_p is consistent for each p.

Corollary 10.4.7. *Each theory* ACF_p *is complete.*

Corollary 10.4.8. $\mathrm{Th}(\mathbb{C})$ *is decidable.*

Chapter 11

Existentially closed models

11.1 Unions of chains of structures

Fix a language $\mathcal{L}$.

Definition 11.1.1. A *chain of $\mathcal{L}$-structures* is a family $\{M_i\}_{i \in I}$ where $(I, \leq)$ is a linear order, M_i is an $\mathcal{L}$-structure for $i \in I$, and M_i is a substructure of M_j for $i \leq j$.

If $\{M_i\}_{i \in I}$ is a chain of $\mathcal{L}$-structures, we can make $M = \bigcup_{i \in I} M_i$ into a $\mathcal{L}$-structure by defining

$$f^M(a_1, \ldots, a_n) = f^{M_i}(a_1, \ldots, a_n)$$
$$R^M(a_1, \ldots, a_n) \iff R^{M_i}(a_1, \ldots, a_n)$$

for any $i \in I$ large enough that $\{a_1, \ldots, a_n\} \subseteq M_i$. The choice of i doesn't matter—if j is another choice, then

$$f^{M_i}(\bar{a}) = f^{M_j}(\bar{a})$$
$$R^{M_i}(\bar{a}) \iff R^{M_j}(\bar{a})$$

because M_i is a substructure or extension of M_j.

Definition 11.1.2. A theory T is *inductive* if whenever $\{M_i\}_{i \in I}$ is a chain of models of T, the union $\bigcup_{i \in I} M_i$ is also a model of T.

Definition 11.1.3. An $\forall\exists$-*sentence* is one of the form $\forall \bar{x}\ \exists \bar{y}\ \varphi(\bar{x}, \bar{y})$, where φ is quantifier-free. An $\forall\exists$-*theory* is a set of $\forall\exists$-sentences.

Theorem 11.1.4. *If T is an $\forall\exists$-theory, then T is inductive.*

Proof. Suppose $\{M_i\}_{i\in I}$ is a chain of structures, and

$$M_i \models \forall \bar{x}\ \exists \bar{y}\ \varphi(\bar{x}, \bar{y}) \tag{$*$}$$

for all i, where φ is quantifier-free. Let $M = \bigcup_i M_i$. If $\bar{a} \in M^n$, then $\bar{a} \in M_i^n$ for large enough i. By $(*)$, there is $\bar{b} \in M_i^m$ with $M_i \models \varphi(\bar{a}, \bar{b})$. Then $M \models \varphi(\bar{a}, \bar{b})$ because φ is quantifier-free. We have shown

$$M \models \forall \bar{x}\ \exists \bar{y}\ \varphi(\bar{x}, \bar{y}). \qquad \square$$

Example 11.1.5. The theory of fields is an $\forall\exists$-theory, so it is inductive.

11.2 Existentially closed models

Definition 11.2.1. Let $M \subseteq N$ be structures. Then M is *existentially closed* in N, written $M \preceq_1 N$, if for any quantifier-free $\mathcal{L}(M)$-formula $\varphi(\bar{x})$,

$$N \models \exists \bar{x}\ \varphi(\bar{x}) \implies M \models \exists \bar{x}\ \varphi \bar{x}.$$

Remark 11.2.2. $M \preceq N \implies M \preceq_1 N$.

Fix a theory T. **"Model" will always mean model of T.**

Definition 11.2.3. A model $M \models T$ is *existentially closed* if for any larger model $N \models T$ with $N \supseteq M$, we have $M \preceq_1 N$.

Theorem 11.2.4. *If T has quantifier-elimination, then every model of T is existentially closed.*

Proof. Suppose $M \subseteq N$ are models of T. Then $M \preceq N$ by quantifier elimination (Theorem 9.3.6), and so $M \preceq_1 N$. $\square$

Theorem 11.2.5. *If T is inductive, then any model M embeds into an existentially closed model $N \supseteq M$.*

Proof. If $M \models T$ and $\varphi(\bar{x})$ is a quantifier-free $\mathcal{L}(M)$-formula, say that $\varphi(\bar{x})$ is *realizable beyond* M if it is realized in some larger model $N \supseteq M$. We make two observations:

1. A model M is existentially closed iff every quantifier-free $\mathcal{L}(M)$-formula that is realizable beyond M is already realized in M.

2. If N extends M and a quantifier-free $\mathcal{L}(M)$-formula φ is realizable beyond N, then it is realizable beyond M—the extension of N where φ is realized is an extension of M.

Claim. For any model M, there is a larger model $M^* \supseteq M$ such that every quantifier-free $\mathcal{L}(M)$-formula that is realizable beyond M^* is realized in M^*.

Proof. Let $\{\varphi_\alpha\}_{\alpha<\kappa}$ enumerate all quantifier-free $\mathcal{L}(M)$-formulas. Build an increasing chain of models $\{M_\alpha\}_{\alpha<\kappa}$ as follows:

1. $M_0 = M$.

2. If φ_α is realizable beyond M_α, then $M_{\alpha+1}$ is an extension of M_α with a realization of φ_α. Otherwise take $M_{\alpha+1} = M_\alpha$.

3. If α is a limit ordinal, take $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. This is a model because the theory is inductive.

Let $M^* = \bigcup_{\alpha<\kappa} M_\alpha$. Again, this is a model because T is inductive. Suppose φ is a quantifier-free $\mathcal{L}(M)$-formula that is realizable beyond M^*. Then $\varphi = \varphi_\alpha$ for some α. By observation (2) above, φ_α is realizable beyond $M_\alpha \subseteq M^*$. Then φ_α is realized in $M_{\alpha+1}$ by construction. That is, $M_{\alpha+1} \models \varphi_\alpha(\bar{c})$ for some tuple $\bar{c}$ in $M_{\alpha+1}$. As M^* extends $M_{\alpha+1}$ and φ_α is quantifier-free, $M^* \models \varphi_\alpha(\bar{c})$. $\square_{\text{Claim}}$

Using the claim, build an increasing chain of length ω:

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

where $M_{i+1} = M_i^*$. Let $N = \bigcup_{n=0}^{\infty} M_n$. Again, $N \models T$ because T is inductive. We claim N is existentially closed. Take a quantifier-free $\mathcal{L}(N)$-formula φ that is realizable beyond N. Then φ is an $\mathcal{L}(M_i)$-formula for some $i < \omega$. By observation (2), φ is realizable beyond $M_{i+1} = M_i^*$, so φ is realized in M_{i+1}. Then φ is realized in N, and N is existentially closed by observation (1). $\square$

11.3 Existentially closed fields

Lemma 11.3.1. *Let K, L be fields, and $\alpha : K \to L$ be a ring homomorphism. Then α is injective.*

Proof. Let $I = \ker(\alpha)$. We claim $I = \{0\}$. Otherwise take $a \in I \setminus \{0\}$. Then $1 = a^{-1}a \in I$, so $1^L = \alpha(1^K) = 0^L$, contradicting the definition of "field."

As $\ker(\alpha) = \{0\}$, it follows that

$$\alpha(x) = \alpha(y) \iff x - y \in \ker(\alpha) \iff x - y = 0 \iff x = y.$$

This means that α is injective. □

Lemma 11.3.2. *If K is a field and $P(x)$ is a polynomial of positive degree, then there is a larger field $L \supseteq K$ in which $P(x)$ has a root.*

Proof. Write $P(x)$ as a product $\prod_{i=1}^n Q_i(x)$ of irreducible factors. Replacing $P(x)$ with one of its irreducible factors, we may assume $P(x)$ is irreducible. Then $K[x]/P(x)K[x]$ is a field (Theorem 10.1.10). The composition

$$K \stackrel{\subseteq}{\to} K[x] \to K[x]/P(x)K[x]$$

is a homomorphism of fields, hence an embedding (Lemma 11.3.1). Up to isomorphism, $L := K[x]/P(x)K[x]$ is a field extending K. The fact that

$$P(x) \equiv 0 \pmod{P(x)} \text{ in } K[x]$$

means that

$$P(x) = 0 \text{ in } L$$

and so the element x is a root of P in L. □

An *existentially closed field* is a field that is existentially closed among the class of all fields.

Theorem 11.3.3. *If K is an existentially closed field, then K is algebraically closed.*

Proof. Let $P(x) \in K[x]$ be a polynomial of positive degree. By Lemma 11.3.2, the quantifier-free $\mathcal{L}(K)$-formula $P(x) = 0$ is realized in a field extending K. As K is existentially closed, it is realized in K, meaning that $P(x)$ has a root. □

Corollary 11.3.4. *If K is a field, then there is an algebraically closed field L extending K.*

Proof. The class of fields is inductive, so Theorem 11.2.5 applies. □

Corollary 11.3.5. *For every $p \in \{0, 2, 3, 5, 7, \ldots\}$, there are algebraically closed fields of characteristic p.*

Proof. Take a field K of characteristic p (Theorem 3.7.7) and an algebraically closed field $M \supseteq K$ by Corollary 11.3.4. Then $\mathrm{char}(M) = \mathrm{char}(K) = p$ because characteristic doesn't change in field extensions (Theorem 3.7.10). □

Theorem 11.3.6. *A field K is existentially closed if and only if it is algebraically closed.*

Proof. If K is existentially closed, then K is algebraically closed by Theorem 11.3.3. Conversely, suppose K is algebraically closed. We claim K is existentially closed. Let $L \supseteq K$ be an extension. By Corollary 11.3.4 there is an algebraically closed field $M \supseteq L \supseteq K$. Because ACF has quantifier elimination, algebraically closed fields are existentially closed *among algebraically closed fields*, and so $K \preceq_1 M$. But this implies $K \preceq_1 L$: if a quantifier-free $\mathcal{L}(K)$-formula φ is realized in L, then it is realized in M, hence realized in K. □

Chapter 12

Monster models

12.1 Pushing types along partial elementary maps

Theorem 12.1.1. *Let M, N be $\mathcal{L}$-structures and $f : A \to B$ be a partial elementary map from M to N. If $\Sigma(\bar{x})$ is a set of $\mathcal{L}(A)$-formulas, let $f_*\Sigma(\bar{x})$ be*

$$\{\varphi(\bar{x}, f(\bar{c})) : \bar{c} \in A,\ \varphi(\bar{x}, \bar{c}) \in \Sigma(\bar{x})\}.$$

1. *If $\Sigma(\bar{x})$ is finite, then $\Sigma(\bar{x})$ is realized in M if and only if $f_*\Sigma(\bar{x})$ is realized in N.*

2. *$\Sigma(\bar{x})$ is finitely realized in M if and only if $f_*\Sigma(\bar{x})$ is finitely realized in N.*

3. *f_* gives a bijection between partial types over A and partial types over B.*

4. *f_* gives a bijection between $S_n(A)$ and $S_n(B)$.*

Proof. 1. Write $\Sigma(\bar{x})$ as $\{\varphi_i(\bar{x}, \bar{c}_i) : 1 \leq i \leq n\}$. Then

$$M \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{c}_i) \iff N \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, f(\bar{c}_i)),$$

because f is a partial elementary map.

2. Clear from (1).

3. By (2), f_* is a map from partial types over A to partial types over B. The inverse map is $(f^{-1})_*$.

4. Clear from (3), since $S_n(A)$ and $S_n(B)$ are exactly the maximal partial types over A and B. □

Corollary 12.1.2. *If $A \subseteq M \preceq N$, then $S_n^M(A) = S_n^N(A)$.*

Proof. The identity map $\mathrm{id}_A : A \to A$ is a partial elementary map from M to N. □

12.2 κ-saturated models

Let κ be an infinite cardinal.

Definition 12.2.1. A structure M is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$ and every $p \in S_1(A)$, p is realized in M.

Fix $\mathcal{L}$-structures N, M, where M is κ-saturated.

Lemma 12.2.2. *Let $f : A \to B$ be a partial elementary map from N to M, with $|A| < \kappa$. For any $\alpha \in N$ there is $\beta \in M$ such that $f \cup \{(\alpha, \beta)\}$ is a partial elementary map.*

Proof. Let $p = \mathrm{tp}(\alpha/A) \in S_1(A)$. Let $f_*p = \{\varphi(x, f(\bar{a})) : \varphi(x, \bar{a}) \in p\}$. By Theorem 12.1.1, $f_*p \in S_1(B)$. As $|B| = |A| < \kappa$, there is $\beta \in M$ realizing f_*p. Then

$$N \models \varphi(\alpha, \bar{c}) \iff \varphi(x, \bar{c}) \in p(x)$$
$$\iff \varphi(x, f(\bar{c})) \in f_*p(x) \iff M \models \varphi(\beta, f(\bar{c}))$$

for any $\mathcal{L}$-formula α and tuple $\bar{c}$ in A, so $f \cup \{(\alpha, \beta)\}$ is a partial elementary map. □

Lemma 12.2.3. *Let $f : A \to B$ be a partial elementary map from N to M. Suppose $A \subseteq A' \subseteq N$, $|A| < \kappa$, and $|A'| \leq \kappa$. Then f can be extended to a partial elementary map $f : A' \to B'$.*

Proof. Write A' as $\{a_\alpha : \alpha < \kappa\}$. Recursively choose partial elementary maps f_α for $\alpha < \kappa$ as follows:

- $f_0 = f$.
- $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ for some $b \in M$.
- $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$ if β is a limit ordinal.

The successor step $\alpha + 1$ works because $\mathrm{dom}(f_\alpha) \leq |A| + |\alpha| < \kappa$, so Lemma 12.2.2 applies.

Let $g = \bigcup_{\alpha<\kappa} f_\alpha$. Then g is a partial elementary map with domain A'. $\square$

Theorem 12.2.4 (κ-universality)**.** *If M is κ-saturated and $N \equiv M$ with $|N| \leq \kappa$, then there is an elementary embedding $g : N \to M$. In particular, N is isomorphic to an elementary substructure of M.*

Proof. Note that $\varnothing : \varnothing \to \varnothing$ is a partial elementary map from N to M. Use Lemma 12.2.3 to extend to a partial elementary map g with $\mathrm{dom}(g) = N$. Then g is an elementary embedding $N \to M$. $\square$

Theorem 12.2.5. *Suppose M is κ-saturated and $A \subseteq M$ with $|A| < \kappa$. For any finite $n < \omega$ and $p \in S_n(A)$, p is realized in M.*

Proof. Take $N \succeq M$ containing a realization $\bar{b} = p$. Note that $\mathrm{id}_A : A \to A$ is a partial elementary map from N to M. By Lemma 12.2.3, we can extend it to a partial elementary map f with $\mathrm{dom}(f) = A \cup \{a_1, \ldots, a_n\}$. Then

$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \implies N \models \varphi(\bar{a}, \bar{b}) \iff M \models \varphi(f(\bar{a}), \bar{b})$$

for any formula φ and tuple $\bar{b}$ in A, and so $f(\bar{a}) \in M^n$ realizes p. $\square$

Theorem 12.2.6. *Suppose M is κ-saturated, $A \subseteq M$ satisfies $|A| < \kappa$, and $\Sigma(\bar{x})$ is a partial type over A in at most κ variables. Then $\Sigma(\bar{x})$ is realized in M.*

Proof. Similar to Theorem 12.2.5. $\square$

Theorem 12.2.7 (κ-compactness)**.** *Let M be κ-saturated.*

1. *Let $\Sigma(\bar{x})$ be a partial n-type over M. If $|\Sigma| < \kappa$, then $\Sigma(\bar{x})$ is realized in M.*
2. *Suppose $|I| < \kappa$ and X_i is a definable subset of M^n for each $i \in I$. If $\{X_i : i \in I\}$ has FIP, then $\bigcap_i X_i \neq \varnothing$.*

3. *Suppose $X \subseteq M^n$ is definable, $|I| < \kappa$, $Y_i \subseteq M^n$ is definable for $i \in I$, and $X \subseteq \bigcup_{i \in I} Y_i$. Then there is a finite $I_0 \subseteq_f I$ such that $X \subseteq \bigcup_{i \in I_0} Y_i$.*

Proof. 1. Let A be the set of parameters used in $\Sigma(\bar{x})$. Then $|A| < \kappa$, and $\Sigma(\bar{x})$ is a partial n-type over A. Take a complete n-type $p \in S_n(A)$ with $p \supseteq \Sigma(\bar{x})$. Then p is realized in M by Theorem 12.2.5.

2. Write X_i as $\varphi_i(M^n)$ for some $\mathcal{L}(M)$-formula φ_i. Let $\Sigma = \{\varphi_i : i \in I\}$. The FIP means that Σ is finitely realized in M, i.e., a partial type. Apply (1) to find a point realizing Σ, i.e., a point in $\bigcap_{i \in I} X_i$.

3. If there is no finite subcover, then the family

$$\{X\} \cup \{M^n \setminus Y_i : i \in I\}$$

has the FIP. By (2), there is a point $\bar{a} \in X \cap \bigcap_i (M^n \setminus Y_i)$. Then $\bar{a} \in X$ but $\bar{a} \notin \bigcup_i Y_i$, a contradiction. □

Corollary 12.2.8. *If M is κ-saturated and $D \subseteq M^n$ is definable, then one of two things happens:*

1. *D is finite.*

2. *$|D| \geq \kappa$.*

Proof. Otherwise, $\{\{p\} : p \in D\}$ is a small cover of D without a finite subcover, contradicting Theorem 12.2.7(3). □

12.3 Strongly κ-homogeneous models

If M is a structure, then $\mathrm{Aut}(M)$ denotes the set of automorphisms of M, i.e., isomorphisms from M to M. If $A \subseteq M$, then

$$\mathrm{Aut}(M/A) := \{\sigma \in \mathrm{Aut}(M) : \forall x \in A \ \sigma(x) = x\}$$

Definition 12.3.1. M is *strongly κ-homogeneous* if any partial elementary map f from M to M with $|\operatorname{dom}(f)| < \kappa$ can be extended to an automorphism $\sigma \in \mathrm{Aut}(M)$.

Theorem 12.3.2. *Suppose M is strongly κ-homogeneous. Suppose $\bar{a}, \bar{b} \in M^n$, $C \subseteq M$, and $|C| < \kappa$. Then the following are equivalent:*

1. $\operatorname{tp}(\bar{a}/C) = \operatorname{tp}(\bar{b}/C)$.

2. *There is* $\sigma \in \operatorname{Aut}(M/C)$ *with* $\sigma(\bar{a}) = \bar{b}$.

Proof. (1) $\implies$ (2): if $\operatorname{tp}(\bar{a}/C) = \operatorname{tp}(\bar{b}/C)$, then there is a partial elementary map

$$f : C \cup \{a_1, \ldots, a_n\} \to C \cup \{b_1, \ldots, b_n\}$$
$$f(x) = \begin{cases} x & \text{if } x \in C \\ b_i & \text{if } x = a_i \end{cases}$$

Then f extends to an automorphism $\sigma \in \operatorname{Aut}(M)$. Note that $\sigma \supseteq f \supseteq \mathrm{id}_C$, so $\sigma \in \operatorname{Aut}(M/C)$, and $\sigma(\bar{a}) = f(\bar{a}) = \bar{b}$.

(2) $\implies$ (1): isomorphisms preserve all formulas. $\square$

Definition 12.3.3. $\bar{a} \equiv_C \bar{b}$ means $\operatorname{tp}(\bar{a}/C) = \operatorname{tp}(\bar{b}/C)$.

Informal Definition 12.3.4. A *monster model* is a structure that is κ-saturated and strongly κ-homogeneous for some cardinal κ bigger than any cardinals we care about. A set X is "small" or "large" depending on whether $|X| < \kappa$ or $|X| \geq \kappa$.

Work in a monster model $\mathbb{M}$. Fix a small set $A \subseteq \mathbb{M}$.

Definition 12.3.5. A set $X \subseteq \mathbb{M}^n$ is *A-invariant* if $\sigma(X) = X$ for all $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$.

Theorem 12.3.6. *The following are equivalent for* $X \subseteq \mathbb{M}^n$*:*

1. *X is A-invariant.*

2. *If* $\bar{b}, \bar{c} \in \mathbb{M}^n$*, then*

$$\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A) \implies (\bar{b} \in A \iff \bar{c} \in A).$$

3. *There is a subset* $X' \subseteq S_n(A)$ *such that*

$$X = \{\bar{b} \in \mathbb{M}^n : \operatorname{tp}(\bar{b}/A) \in X'\}.$$

Proof. Note that (1) means the following: if $\bar{a} \in \mathbb{M}^n$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$, then

$$\bar{a} \in X \iff \sigma(\bar{a}) \in X.$$

This is equivalent to (2) by Theorem 12.3.2. The equivalence of (2) and (3) is clear. $\square$

Remark 12.3.7. An A-definable set $\varphi(\mathbb{M}^n)$ corresponds to the clopen set $[\![\varphi]\!] \subseteq S_n(A)$, so A-definable sets are A-invariant.

Theorem 12.3.8. *If $D \subseteq \mathbb{M}^n$ is definable and A-invariant, then D is A-definable.*

Proof. Note that D is B-definable for some small $B \supseteq A$. Let $f : S_n(B) \to S_n(A)$ be the restriction map sending $\mathrm{tp}(\bar{c}/B)$ to $\mathrm{tp}(\bar{c}/A)$. This map is surjective because every complete type over A extends to a complete type over B. Additionally, it is continuous by Theorem 8.4.4, because the preimage of the clopen set $[\![\varphi]\!] \subseteq S_n(A)$ is the clopen set $[\![\varphi]\!] \subseteq S_n(B)$. As X is A-invariant and B-invariant, there are sets $X_A \subseteq S_n(A)$ and $X_B \subseteq S_n(B)$ such that

$$\bar{c} \in X \iff \mathrm{tp}(\bar{c}/A) \in X_A$$
$$\bar{c} \in X \iff \mathrm{tp}(\bar{c}/B) \in X_B.$$

Then $X_B = f^{-1}(X_A)$. As f is surjective

$$X_A = f(X_B)$$
$$S_n(A) \setminus X_A = f(S_n(B) \setminus X_B).$$

Because X is B-definable the set $X_B \subseteq S_n(B)$ is clopen, and so X_B and $S_n(B) \setminus X_B$ are closed. The image of a closed set is closed (Theorem 8.2.6), so X_A and its complement are both closed. Then X_A is a clopen set $[\![\psi]\!]$, and X is the A-definable set $\psi(\mathbb{M}^n)$. $\square$

12.4 Construction of monster models

A cardinal κ is *regular* if whenever $|I| < \kappa$ and $|X_i| < \kappa$ for every $i \in I$, we have $\left|\bigcup_{i \in I} X_i\right| < \kappa$. The cardinal $\aleph_0$ is regular, because a finite union of finite sets is finite. For any cardinal κ, the successor κ^+ is regular, because a union of at most κ-many sets of size at most κ has size at most $\kappa^2 = \kappa < \kappa^+$. Consequently, for any cardinal κ we can find a larger cardinal that is regular.

If κ is a regular cardinal and $A \subseteq \kappa$ with $|A| < \kappa$, then $\sup(A) < \kappa$. Otherwise, $\sup(A) = \kappa$, and then κ is a union $\bigcup_{\alpha \in A} \alpha$ where $|A| < \kappa$ and $|\alpha| < \kappa$ for each $\alpha \in \kappa$, contradicting regularity.

Lemma 12.4.1. *Let κ be a regular cardinal. Let $\{S_\alpha\}_{\alpha<\kappa}$ be an increasing chain of sets, indexed by κ. If $A \subseteq \bigcup_{\alpha<\kappa} S_\alpha$ and $|A| < \kappa$, then $A \subseteq S_\alpha$ for some $\alpha < \kappa$.*

Proof. Define $f : A \to \kappa$ by $f(x) = \min\{\alpha < \kappa : x \in S_\alpha\}$. Then $|f(A)| \leq |A| < \kappa$, so $\alpha := \sup f(A) < \kappa$. For any $x \in A$, we have $f(x) \in f(A)$ and so $f(x) \leq \alpha$. Then $x \in S_{f(x)} \subseteq S_\alpha$ for any $x \in A$, so $A \subseteq S_\alpha$. □

Definition 12.4.2. A chain of structures $\{M_i\}_{i\in I}$ is *elementary* if $M_i \preceq M_j$ for $i \leq j$.

Theorem 12.4.3 (Tarski-Vaught)**.** *Let $\{M_i\}_{i\in I}$ be an elementary chain of $\mathcal{L}$-structures. Let $M = \bigcup_{i\in I} M_i$. Then $M_i \preceq M$ for all $i \in I$.*

Proof sketch. For each i, note that $\mathrm{eldiag}(M_i)$ is finitely satisfiable, complete, and has the witness property. Therefore the union $\bigcup_i \mathrm{eldiag}(M_i)$ is finitely satisfiable, complete, and has the witness property. The canonical model is M. Then $M \succeq M_i$ because $M \models \mathrm{eldiag}(M_i)$. □

Lemma 12.4.4. *If M is a structure, there is $N \succeq M$ such that every type in $S_1(M)$ is realized in N.*

Proof. Let $\{p_i(x) : i \in I\}$ enumerate $S_1(M)$. Let $\bar{x} = (x_i : i \in I)$ be a tuple of variables, one for each $i \in I$. Let $\Sigma(\bar{x}) = \{p_i(x_i) : i \in i\}$. Then $\Sigma(\bar{x})$ is finitely satisfiable in M because each p_i is finitely satisfiable in M. Therefore, $\Sigma(\bar{x})$ is realized by some tuple $\bar{a} = (a_i : i \in I)$ in an elementary extension $N \succeq M$. The element a_i realizes p_i. □

Theorem 12.4.5. *If M is a structure and κ is a cardinal, there is a κ-saturated $N \succeq M$.*

Proof. Replacing κ with κ^+ if necessary, we may assume κ is regular. Build an elementary chain $\{M_\alpha\}_{\alpha<\kappa}$ by recursion on α:

- $M_0 = M$.
- $M_{\alpha+1}$ is an elementary extension of M_α realizing every complete 1-type over M_α (using Lemma 12.4.4).

- If α is a limit ordinal, take $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, using the Tarski-Vaught theorem on chains (Theorem 12.4.3).

Let $N = \bigcup_{\alpha<\kappa} M_\alpha$. Then $N \succeq M_0 = M$ by Tarski-Vaught again. If $A \subseteq N$ and $|A| < \kappa$, then $A \subseteq M_\alpha$ for some $\alpha < \kappa$ by Lemma 12.4.1. If $p \in S_1(A)$, then p is a partial type over M_α, so it extends to a complete type $p' \in S_1(M_\alpha)$, which is then realized by some $b \in M_{\alpha+1} \subseteq N$. Therefore N is κ-saturated. □

Lemma 12.4.6. *Let M be a structure and N be an $|M|^+$-saturated elementary extension. Let f be a partial elementary map from M to M.*

1. *There is a partial elementary map g from N to N extending f, with* $\mathrm{dom}(g) = M$.

2. *There is a partial elementary map h from N to N extending f, with* $\mathrm{im}(h) = M$.

Proof. 1. Let $\kappa = |M|^+$. Then N is κ-saturated and $|M| < \kappa$. Note that f is a partial elementary map from M to N. By Lemma 12.2.3, we can extend f to a partial elementary map g from M to N with $\mathrm{dom}(g) = M$.

2. This follows from (1) by symmetry. More precisely, apply part (1) to f^{-1} to get a partial elementary map g extending f^{-1} with $\mathrm{dom}(g) = M$. Then set $h = g^{-1}$, so that $\mathrm{im}(h) = \mathrm{dom}(g) = M$. □

Lemma 12.4.7. *For any M there is an elementary extension $N \succeq M$ with the following properties:*

1. *Every complete type over M is realized in N.*

2. *If f is a partial elementary map from M to M, then there is $\sigma \in$* $\mathrm{Aut}(N)$ *extending f.*

Proof. Build an elementary chain $\{M_i\}_{i<\omega}$ by recursion on i:

- $M_0 = M$.

- M_{i+1} is an $|M_i|^+$-saturated elementary extension of M_i. This is possible by Theorem 12.4.5.

Let $N = \bigcup_{i=0}^{\infty} M_i$. Then $M = M_0 \preceq N$ by the Tarski-Vaught theorem on chains (Theorem 12.4.3). Every complete type over M is already realized in M_1, hence in N.

Let $f : A \to B$ be a partial elementary map from M to M. Recursively build an increasing chain of partial elementary maps $\{f_i\}_{i<\omega}$ with $\mathrm{dom}(f_i), \mathrm{im}(f_i) \subseteq M_i$ as follows:

- $f_0 = f$.
- If $n > 0$, then $f_n : M_n \dashrightarrow M_n$ is a partial elementary map extending $f_{n-1} : M_{n-1} \dashrightarrow M_{n-1}$ with

$$\mathrm{dom}(f_n) = M_{n-1} \text{ if } n \text{ is odd}$$
$$\mathrm{im}(f_n) = M_{n-1} \text{ if } n \text{ is even.}$$

Take $\sigma = \bigcup_{n=0}^{\infty} f_n$. Then σ is a partial elementary map from N to N. The odd steps ensure $\mathrm{dom}(\sigma) \supseteq \bigcup_n M_n = N$, and the even steps ensure $\mathrm{im}(\sigma) \supseteq \bigcup_n M_n = N$. Thus $\mathrm{dom}(\sigma) = \mathrm{im}(\sigma) = N$, and $\sigma \in \mathrm{Aut}(N)$. $\square$

Theorem 12.4.8. *If M is a structure and κ is a cardinal, then there is a κ-saturated, strongly κ-homogeneous elementary extension $N \succeq M$.*

Proof. Replacing κ with κ^+ if necessary, we may assume κ is regular. Build an elementary chain $\{M_\alpha\}_{\alpha<\kappa}$ by recursion on α as follows:

1. $M_0 = M$.
2. $M_{\alpha+1}$ is an elementary extension of M_α as in Lemma 12.4.7. In particular, every complete 1-type over M_α is realized in $M_{\alpha+1}$.
3. If α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. This works by the Tarski-Vaught theorem on chains (Theorem 12.4.3).

Let $N = \bigcup_{\alpha<\kappa} M_\alpha$. As in the proof of Theorem 12.4.5, N is κ-saturated. We prove strong κ-homogeneity. Suppose f is a partial elementary map from N to N with $|\mathrm{dom}(f)| = |\mathrm{im}(f)| < \kappa$. By Lemma 12.4.1, there is $\alpha < \kappa$ with $\mathrm{dom}(f) \cup \mathrm{im}(f) \subseteq M_\alpha$. Then f is a partial elementary map from M_α to M_α. By Lemma 12.4.7, we can extend f to an automorphism $\sigma_{\alpha+1} \in \mathrm{Aut}(M_{\alpha+1})$. Build an increasing chain $\{\sigma_\beta\}_{\alpha<\beta<\kappa}$ with $\sigma_\beta \in \mathrm{Aut}(M_\beta)$ by repeatedly applying Lemma 12.4.7. Take $\sigma = \bigcup_{\alpha<\beta<\kappa} \sigma_\beta$. Then $\sigma \in \mathrm{Aut}(N)$, and $\sigma \supseteq \sigma_{\alpha+1} \supseteq f$. Therefore N is strongly κ-homogeneous. $\square$

12.5 Saturation and back-and-forth

Theorem 12.5.1. *Let M and N be ω-saturated. For any $\bar{a} \in M^n$ and $\bar{b} \in N^n$ with $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, let $g_{\bar{a},\bar{b}}$ be the isomorphism from $\langle \bar{a} \rangle_M$ to $\langle \bar{b} \rangle_N$ sending $\bar{a}$ to $\bar{b}$ as in Theorem 9.3.4. Then the family $\mathcal{F} = \{g_{\bar{a},\bar{b}} : n \in \mathbb{N},\ \bar{a} \in M,\ \bar{b} \in N,\ \ \mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})\}$ is a back-and-forth system.*

Proof. We verify the "forward" condition; "backward" is similar. Fix some $g_{\bar{a},\bar{b}}$. Let a' be an element of M. By Lemma 12.2.2 we can extend the partial elementary map $\bar{a} \mapsto \bar{b}$ to a partial elementary map $(\bar{a}, a') \mapsto (\bar{b}, b')$. Then $\mathrm{tp}(\bar{a}, a') = \mathrm{tp}(\bar{b}, b')$, and $g_{\bar{a}a',\bar{b}b'}$ is the desired partial isomorphism in $\mathcal{F}$ extending $g_{\bar{a},\bar{b}}$ and containing a' in its domain. $\square$

Corollary 12.5.2. *If M, N are countable and ω-saturated, and $M \equiv N$, then $M \cong N$.*

Proof. Let $\mathcal{F}$ be as in Theorem 12.5.1. By Theorem 6.6.4, it suffices to show that $\mathcal{F}$ is non-empty. We only need to find some $n \in \mathbb{N}$, some $\bar{a} \in M^n$, and $\bar{b} \in N^n$ with $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Take $n = 0$, $\bar{a} = () \in M^0$, and $\bar{b} = () \in N^0$. Then $\mathrm{tp}(\bar{a}) = \mathrm{Th}(M) = \mathrm{Th}(N) = \mathrm{tp}(\bar{b})$. $\square$

Corollary 12.5.3. *If M is countable and ω-saturated, then M is strongly ω-homogeneous.*

Proof. Let $f : A \to B$ be a partial elementary map from M to M, with A and B finite. Let $\bar{a}$ be a tuple enumerating A. Then $\bar{b} := f(\bar{a})$ enumerates B, and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. We must find $\sigma \in \mathrm{Aut}(M)$ with $\sigma(\bar{a}) = \bar{b}$. If $\mathcal{F}$ is the back-and-forth system between M and M given by Theorem 12.5.1, then $g_{\bar{a},\bar{b}} \in \mathcal{F}$. By Lemma 6.6.3, there is an isomorphism $\sigma : M \to M$ extending $g_{\bar{a},\bar{b}}$. Then $\sigma(\bar{a}) = \bar{b}$. $\square$

Theorem 12.5.4. *Let κ be an infinite cardinal. The following are equivalent:*

1. *T has quantifier elimination.*

2. *If M, N are κ-saturated structures, then the family $\mathcal{F}_0$ of* all *finitely generated partial isomorphisms from M to N is a back-and-forth system.*

Proof. (1) $\implies$ (2): Note that M, N are ω-saturated. If $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$, let $g_{\bar{a},\bar{b}} : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$ be the partial isomorphism sending $\bar{a}$ to $\bar{b}$ from Theorem 9.3.4. Then

$$\mathcal{F}_0 = \{g_{\bar{a},\bar{b}} : \mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})\}.$$

By quantifier elimination, this is

$$\mathcal{F}_0 = \{g_{\bar{a},\bar{b}} : \mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})\},$$

which is a back-and-forth system by Theorem 12.5.1.

(2) $\implies$ (1): By Theorem 9.3.3, it suffices to show that if $M, N \models T$, if $\bar{a} \in M^n$ and $\bar{b} \in N^n$, then

$$\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b}) \implies \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b}).$$

Replacing M and N by elementary extensions, we may assume M and N are κ-saturated. By (2), the family $\mathcal{F}_0$ of finitely generated partial isomorphisms from M to N is a back-and-forth system. Suppose $\mathrm{qftp}^M(\bar{a}) = \mathrm{qftp}^N(\bar{b})$. Then $\mathcal{F}$ contains the isomorphism $g_{\bar{a},\bar{b}} : \langle \bar{a} \rangle_M \to \langle \bar{b} \rangle_N$. By Theorem 4.7.6, $g_{\bar{a},\bar{b}}$ is a partial elementary map, so $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. $\square$

12.6 Application: discrete linear orders with endpoints

Definition 12.6.1. Let x, y be elements in a linear order $(M, \leq)$. Then y is a *successor* of x, and x is a *predecessor* of y, written $x \lhd y$, if $x < y$ but there is no $z \in M$ with $x < z < y$.

Let T be the theory of linear orders $(M, \leq)$ such that the following hold:

1. $\min(M)$ and $\max(M)$ exist. In particular, M is non-empty.

2. Every $x \in M$ other than $\max(M)$ has a successor $y \rhd x$.

3. Every $x \in M$ other than $\min(M)$ has a predecessor $y \lhd x$.

Example 12.6.2. Any finite non-empty linear order is a model of T.

Definition 12.6.3. Suppose $M \models T$ and $x, y \in M$. The *distance* between x and y, written $d(x, y)$, is the element of $\mathbb{N} \cup \{\infty\}$ defined as follows:

1. If $x = y$, then $d(x, y) = 0$.
2. If $x < y$, then $d(x, y)$ is $1 + |\{z \in M : x < z < y\}|$.
3. If $x > y$, then $d(y, x) = d(x, y)$.

Note that we define $d(x, y)$ to be the symbol "∞" when $[x, y]$ is infinite, rather than distinguishing cardinalities.

Let $\mathcal{L}'$ be the base language of orders $\mathcal{L} = \{\leq\}$ plus two new constant symbols $\min, \max$ and a binary relation R_n for each $n < \infty$. Expand any model $M \models T$ to an $\mathcal{L}'$-structure by interpreting the new symbols as follows:

1. $\min^M = \min(M)$.
2. $\max^M = \max(M)$.
3. $R_n^M(x, y) \iff d(x, y) = n$.

The resulting $\mathcal{L}'$-structures are the models of some $\mathcal{L}'$-theory T'.

Remark 12.6.4. Let M, N be models of T'. Suppose

$$\begin{gathered} \min(M) = a_1 < a_2 < \cdots < a_n = \max(M) \\ \min(N) = b_1 < b_2 < \cdots < b_n = \max(N) \\ d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ for } 1 \leq i < n. \end{gathered}$$

Then there is a partial isomorphism

$$\begin{gathered} f : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\} \\ f(a_i) = b_i. \end{gathered}$$

Moreover, all finite partial isomorphisms from M to N arise in this way.

Lemma 12.6.5. *Let M, N be $\aleph_1$-saturated models of T'. Let $\mathcal{F}$ be the collection of finite partial isomorphisms $f : A \to B$. Then $\mathcal{F}$ is a back-and-forth system.*

Proof. By symmetry we only need to prove the forward condition. Let $f : A \to B$ be a finite partial isomorphism and α be an element of M. We must find $f' \in \mathcal{F}$ extending f with $\alpha \in \mathrm{dom}(f')$. We may assume $\alpha \notin \mathrm{dom}(f)$; otherwise take $f' = f$. By Remark 12.6.4, f has the form

$$f : \{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$$
$$f(a_i) = b_i$$

where

$$\min(M) = a_1 < a_2 < \cdots < a_n = \max(M)$$
$$\min(N) = b_1 < b_2 < \cdots < b_n = \max(N)$$
$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ for } 1 \leq i < n.$$

There must be some i such that $a_i < \alpha < a_{i+1}$. (The cases $\alpha < a_1 = \min(M)$ and $\alpha > a_n = \max(M)$ are impossible.) By Remark 12.6.4, we must find β such that

$$b_i < \beta < b_{i+1}$$
$$d(b_i, \beta) = d(a_i, \alpha) =: x$$
$$d(\beta, b_{i+1}) = d(\alpha, a_{i+1}) =: y.$$

Note that $d(b_i, b_{i+1}) = d(a_i, a_{i+1}) = x + y$. There are four cases:

1. $x, y < \infty$. Take β between b_i and b_{i+1} with $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = (x + y) - x = y$ as desired.

2. $x < \infty = y$. Then $d(b_i, b_{i+1}) = \infty$. Take β between b_i and b_{i+1} with $d(b_i, \beta) = x$. Then $d(\beta, b_{i+1}) = \infty$.

3. $y < \infty = x$. Similar.

4. $x = y = \infty$. As $d(b_i, b_{i+1}) = \infty$, there are

$$b_i \lhd c_1 \lhd c_2 \lhd \cdots \leq \cdots \lhd d_2 \lhd d_1 \lhd b_{i+1}.$$

   The partial type $\{x > c_i : i < \omega\} \cup \{x < d_i : i < \omega\}$ is realized in M, by $\aleph_1$-saturation. Let β be a realization. Then $b_i < \beta < b_{i+1}$, and $d(b_i, \beta) = d(\beta, b_{i+1}) = \infty$. $\square$

Theorem 12.6.6. *The theory T' has quantifier elimination.*

Corollary 12.6.7. *Let M be an infinite model of T. Let M_0 be the set of $x \in M$ such that $d(x, \min(M)) < \infty$ or $d(x, \max(M)) < \infty$. Then $M_0 \preceq M$.*

Proof. Working in the expanded language $\mathcal{L}'$, it is easy to see that M_0 is a substructure of M and $M_0 \models T'$. Then $M_0 \preceq M$ because submodels are elementary substructures in theories with quantifier elimination (Theorem 9.3.6(2)). $\square$

Definition 12.6.8. If M is a model of T, the *length* of M, written $\ell(M)$, is $d(\min(M), \max(M)) \in \mathbb{N} \cup \{\infty\}$.

Corollary 12.6.9. *Two models $M, N \models T$ are elementarily equivalent if and only if $\ell(M) = \ell(N)$.*

Proof. If $M \equiv N$, then certainly $\ell(M) = \ell(N)$, essentially because the relations R_n are definable, or simply because $\ell(M)$ is $|M| - 1$.

Conversely, suppose $\ell(M) = \ell(N)$. Let M' and N' be the expansions of M and N to models of T. Note that the minimal substructure $\langle \varnothing \rangle_{M'}$ contains only the two points $\min(M)$ and $\max(M)$. This substructure is determined up to isomorphism by $\ell(M)$. For example, $R_n(\min(M), \max(M))$ holds iff $\ell(M) = n$. Because $\ell(M) = \ell(N)$, we have $\langle \varnothing \rangle_{M'} \cong \langle \varnothing \rangle_{N'}$, and so $M' \equiv N'$ by quantifier elimination (because of Theorem 9.3.5). Restricting to $\mathcal{L}$-sentences, we see $M \equiv N$. $\square$

Corollary 12.6.10. *The theory of infinite models of T is complete and decidable.*

Corollary 12.6.11. *The class $\mathcal{K}$ of models of T is the elementary class generated by finite non-empty linear orders.*

Proof. Certainly $\mathcal{K}$ is an elementary class containing the finite non-empty linear orders. If $\mathcal{K}'$ is a smaller such elementary class, take $M \in \mathcal{K} \setminus \mathcal{K}'$. Then M is infinite or else $M \in \mathcal{K}'$. Because $\mathcal{K}'$ contains models of size $> n$ for each n, it must contain an infinite model N (Theorem 6.4.4). Then M, N are infinite models of T, so Corollary 12.6.10 shows $M \equiv N \in \mathcal{K}'$. Then $M \in \mathcal{K}'$, contradicting the choice of M. $\square$

Corollary 12.6.12. *If φ is a sentence in the language of orders, then the following are equivalent:*

- *$T \vdash \varphi$.*
- *Every finite non-empty linear order satisfies φ.*

Chapter 13

Countable categoricity

13.1 Baire category theorem

Let S be a non-empty Stone space (Definition 8.4.2).

Definition 13.1.1. $X \subseteq S$ is *dense* if X intersects any non-empty clopen set $Y \subseteq S$.

Definition 13.1.2. A set $X \subseteq S$ is *comeager* if X contains a countable intersection of dense open sets.

Remark 13.1.3. Dense open sets are comeager. A countable intersection of comeager sets is comeager.

Theorem 13.1.4 (Baire category theorem)**.** *If $X \subseteq S$ is comeager, then X is dense, hence non-empty.*

Proof. Because S is a Stone space, every open set is a union of clopen sets. Therefore every non-empty open set contains a non-empty clopen set.

Suppose $X \supseteq \bigcup_{i=1}^{\infty} U_i$ where each U_i is open and dense. Let V_0 be a non-empty clopen set. Then $V_0 \cap U_1$ is a non-empty open set. It contains a non-empty clopen set V_1. Similarly, $V_1 \cap U_2$ contains a non-empty clopen set V_2. Continuing, we can build a descending chain of clopen sets

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

with $V_i \subseteq U_i$. The family $\{V_0, V_1, V_2, \ldots\}$ has the FIP, so the intersection is non-empty by compactness (Theorem 8.1.15). Take $p \in \bigcap_{i=0}^{\infty} V_i \subseteq V_0 \cap \bigcap_{i=1}^{\infty} U_i \subseteq V_0 \cap X$. Then X intersects V_0 as desired. $\square$

13.2 The omitting types theorem

Fix an $\mathcal{L}$-structure M, subset $A \subseteq M$, and $n < \omega$.

Definition 13.2.1. A complete type $p \in S_n(A)$ is *isolated* if $\{p\}$ is clopen.

Theorem 13.2.2. *If p is isolated, then p is realized in M.*

Proof. As $\{p\}$ is clopen, $\{p\} = [\![\varphi]\!] \subseteq S_n(A)$ for some $\mathcal{L}(A)$-formula $\varphi(\bar{x}) \in p(\bar{x})$. Then $\varphi(\bar{x})$ is satisfied by $b \in M$, as p is finitely satisfiable. If $p' = \mathrm{tp}(b/A) \in S_n(A)$, then $p' \in [\![\varphi]\!] = \{p\}$, so $p' = p$ and b realizes p. □

Fix a complete theory T in a countable language $\mathcal{L}$.

Definition 13.2.3. If $p \in S_n(T)$ and $M \models T$, then M *omits* p if p isn't realized in M.

Let $S_\omega(T)$ be the space of complete ω-types $\mathrm{tp}(a_0, a_1, a_2, \ldots)$, i.e., complete types in the variables $\bar{x} = (x_0, x_1, x_2, \ldots)$. Work in a monster model $\mathbb{M} \models T$.

Lemma 13.2.4. *There is a comeager set $W \subseteq S_\omega(T)$ such that if $\bar{c} \in \mathbb{M}^\omega$ and $\mathrm{tp}(\bar{c}) \in W$, then $\{c_i : i \in \omega\} \preceq \mathbb{M}$.*

Proof.

Claim. For any formula $\varphi(\bar{x}, y)$, the following open set is dense:

$$U_\varphi := [\![\neg\exists y\ \varphi(\bar{x}, y)]\!] \cup \bigcup_{i=0}^{\infty} [\![\varphi(\bar{x}, x_i)]\!].$$

Proof. Take non-empty $[\![\psi]\!] \subseteq S_\omega(T)$. Suppose for the sake of contradiction that $[\![\psi]\!] \cap U_\varphi = \varnothing$. Then $[\![\psi]\!]$ doesn't intersect the sets in the union, which means the following:

$$\psi(\bar{x}) \wedge (\neg\exists y\ \varphi(\bar{x}, y)) \text{ is inconsistent}$$
$$\psi(\bar{x}) \wedge \varphi(\bar{x}, x_i) \text{ is inconsistent, for each } i < \omega.$$

Take $\bar{c} \in \mathbb{M}^\omega$ realizing $\psi(\bar{x})$. By the first line, $\mathbb{M} \models \exists y\ \varphi(\bar{c}, y)$. Thus $\mathbb{M} \models \varphi(\bar{c}, e)$ for some $e \in \mathbb{M}$. Take $i \gg 0$ so that x_i doesn't occur in $\psi(\bar{x})$ or $\varphi(\bar{x}, y)$. Changing c_i to e, we may assume $\mathbb{M} \models \varphi(\bar{c}, c_i)$, contradicting the second line. $\square_{\text{Claim}}$

Now let W be the comeager set $\bigcap_\varphi U_\varphi$. Suppose $\bar{c} \in \mathbb{M}^\omega$ and $\mathrm{tp}(\bar{c}) \in W$. We claim that $M := \{c_i : i \in \omega\} \preceq \mathbb{M}$ by the Tarski-Vaught criterion (Theorem 6.3.1). Suppose $D \subseteq \mathbb{M}^1$ is M-definable, defined as $\varphi(\bar{c}, \mathbb{M})$ for some $\varphi(\bar{x}, y)$. If $D \neq \varnothing$, then $\mathbb{M} \models \exists y\ \varphi(\bar{c}, y)$, so $\mathrm{tp}(\bar{c}) \notin [\![\neg\exists y\ \varphi(\bar{x}, y)]\!]$. But $\mathrm{tp}(\bar{c}) \in U_\varphi$, so $\mathrm{tp}(\bar{c}) \in [\![\varphi(\bar{x}, x_i)]\!]$ for some i, which means $\mathbb{M} \models \varphi(\bar{c}, c_i)$, or equivalently, $c_i \in D$. Thus $M = \{c_i : i \in \omega\}$ intersects every non-empty M-definable set $D \subseteq \mathbb{M}^1$. $\square$

Lemma 13.2.5. *For any $j_1, \ldots, j_n < \omega$, let $f_{\bar{j}} : S_\omega(T) \to S_n(T)$ be the restriction map $\mathrm{tp}(\bar{c}) \mapsto \mathrm{tp}(c_{j_1}, \ldots, c_{j_n})$.*

1. *f is continuous.*

2. *If $X \subseteq S_\omega(T)$ is clopen, then $f(X) \subseteq S_n(T)$ is clopen.*

Proof. 1. The preimage of $[\![\varphi(y_1, \ldots, y_n)]\!] \subseteq S_n(T)$ is $[\![\varphi(x_{j_1}, \ldots, x_{j_n})]\!] \subseteq S_\omega(T)$.

2. If $X = [\![\varphi]\!]$, then

$$X = \{\mathrm{tp}(\bar{c}) : \bar{c} \in \mathbb{M}^\omega,\ \mathbb{M} \models \varphi(\bar{c})\}$$
$$f(X) = \{\mathrm{tp}(c_{j_1}, \ldots, c_{j_n}) : \bar{c} \in \mathbb{M}^\omega,\ \mathbb{M} \models \varphi(\bar{c})\}.$$

(The first line holds by saturation of $\mathbb{M}$.) Now the set

$$\{(c_{j_1}, \ldots, c_{j_n}) : \bar{c} \in \mathbb{M}^\omega,\ \mathbb{M} \models \varphi(\bar{c})\}$$

is definable, defined by the formula

$$\psi(y_1, \ldots, y_n) := \exists x_0, \ldots, x_N \ \left(\varphi(\bar{x}) \wedge \bigwedge_{i=1}^{n} y_i = x_{j_i}\right),$$

where N is large enough to quantify away all x_i's appearing in $\varphi(\bar{x})$. Then

$$f(X) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in \psi(\mathbb{M}^n)\} = [\![\psi]\!].$$ $\square$

Lemma 13.2.6. *Let $p \in S_n(T)$ be non-isolated. There is a comeager set $V_p \subseteq S_\omega(T)$ such that if $\bar{c} \in \mathbb{M}^\omega$ and $\mathrm{tp}(\bar{c}) \in V_p$, then p is not realized by any tuple in $C = \{c_i : i < \omega\}$.*

Proof.

Claim. For any $j_1, \ldots, j_n \in \omega$, there is a dense open set $U_{\bar{j}} \subseteq S_\omega(T)$ such that if $\mathrm{tp}(\bar{c}) \in U_{\bar{j}}$, then $(c_{j_1}, \ldots, c_{j_n})$ doesn't realize p.

Proof. Let $f = f_{\bar{j}} : S_\omega(T) \to S_n(T)$ be the restriction map from Lemma 13.2.5. Points are closed (Theorem 8.1.11) so $S_n(T) \setminus \{p\}$ is open and the preimage $U := f^{-1}(S_n(T) \setminus \{p\})$ is open. For density, suppose $X \subseteq S_\omega(T)$ is clopen and non-empty, but $X \cap U = \varnothing$. Then $f(X) \subseteq \{p\}$. But $f(X)$ is clopen by Lemma 13.2.5(2), and non-empty, so $f(X) = \{p\}$ and p is isolated, a contradiction. Thus U is dense. Finally, if $\mathrm{tp}(\bar{c}) \in U$, then $\mathrm{tp}(c_{j_1}, \ldots, c_{j_n}) \in S_n(T) \setminus \{p\}$, meaning that $(c_{j_1}, \ldots, c_{j_n})$ doesn't realize p. $\square_{\text{Claim}}$

Take $V_p = \bigcap_{\bar{j} \in \mathbb{N}^n} U_{\bar{j}}$. If $\mathrm{tp}(\bar{c}) \in V_p$, then $(c_{j_1}, \ldots, c_{j_n})$ doesn't realize p for any $\bar{j} \in \mathbb{N}^\omega$. $\square$

Theorem 13.2.7 (Omitting types theorem). *Let $p_1, p_2, \ldots$ be a countable list of non-isolated complete types $p_i \in S_{n_i}(T)$. Then there is a countable model $M \models T$ omitting every p_i.*

Proof. Let W and V_p be the comeager sets from Lemmas 13.2.4 and 13.2.6. The set $Q = W \cap \bigcap_{i=1}^\infty V_{p_i}$ is comeager, hence non-empty by the Baire Category Theorem. Take $\bar{c} \in \mathbb{M}^\omega$ with $\mathrm{tp}(\bar{c}) \in Q$, and let $M = \{c_i : i < \omega\}$. Then $M \preceq \mathbb{M}$ because $\mathrm{tp}(\bar{c}) \in W$, and M omits p_i because $\mathrm{tp}(\bar{c}) \in V_{p_i}$. $\square$

13.3 Countably categorical theories

Work in a monster model $\mathbb{M} \models T$.

Lemma 13.3.1. *$S_n(A)$ is finite iff all types in $S_n(A)$ are isolated.*

Proof. Suppose $S_n(A)$ is finite. Every point is closed (Theorem 8.1.11), and a finite union of closed sets is closed, so every subset of $S_n(A)$ is closed, every subset is open, and every subset is clopen, implying every point is isolated.

Conversely, suppose every point is isolated. Then $S_n(A) = \bigcup_{p \in S_n(A)} \{p\}$ is an open cover. By compactness of $S_n(A)$, there is a finite subcover, and so $S_n(A)$ is finite. $\square$

Lemma 13.3.2. *$S_n(A)$ is finite iff there are only finitely many A-definable sets $D \subseteq M^n$.*

Proof. The boolean algebra of A-definable sets is isomorphic to the boolean algebra of clopen sets in $S_n(A)$. If $S_n(A)$ is finite, then there are only finitely many clopen sets. Conversely, if there the number of clopen sets is $k < \infty$, then there are at most 2^k open sets, because every open set is a union of clopen sets. Then there are at most 2^k closed sets, hence 2^k points (as every point is closed by Theorem 8.1.11). $\square$

Lemma 13.3.3. *If $\bar{b}, \bar{c} \in \mathbb{M}^n$ and $A = \{a_1, \ldots, a_m\} \subseteq \mathbb{M}$, then*

$$\bar{b} \equiv_A \bar{c} \iff \bar{a}\bar{b} \equiv_\varnothing \bar{a}\bar{c}.$$

Proof. Every $\mathcal{L}(A)$-formula has the form $\varphi(\bar{a}, \bar{x})$ for some $\mathcal{L}$-formula $\varphi(\bar{w}, \bar{x})$. Therefore both sides say that for any $\mathcal{L}$-formula $\varphi(\bar{w}, \bar{x})$,

$$\mathbb{M} \models \varphi(\bar{a}, \bar{b}) \iff \mathbb{M} \models \varphi(\bar{a}, \bar{c}). \qquad \square$$

Lemma 13.3.4. *If $|S_n(\varnothing)| < \infty$ for all n, then $|S_n(A)| < \infty$ for all n and all finite $A \subseteq \mathbb{M}$.*

Proof. Fix some $n < \omega$ and $A = \{a_1, \ldots, a_m\} \subseteq M$. Then the map

$$\begin{aligned} S_n(A) &\to S_{m+n}(\varnothing) \\ \operatorname{tp}(\bar{b}/A) &\mapsto \operatorname{tp}(\bar{a}\bar{b}/\varnothing) \end{aligned}$$

is well-defined and injective by Lemma 13.3.3. Thus $|S_n(A)| \leq |S_{m+n}(\varnothing)| < \infty$. $\square$

Recall that $S_n(\varnothing) = S_n(T)$ (Corollary 9.1.20).

Theorem 13.3.5 (Engeler, Ryll-Nardzewski, Svenonius)**.** *Let T be a complete theory in a countable language and let $\mathbb{M}$ be a monster model. The following are equivalent:*

1. *T has a unique countable model.*

2. *$S_n(T)$ is finite for all $n < \omega$.*

3. *$S_n(A)$ is finite for all $n < \omega$ and finite $A \subseteq \mathbb{M}$.*

4. *Every countable model of T is ω-saturated.*

Proof. (1) $\implies$ (2): Suppose $S_n(T)$ is infinite for some n. By Lemma 13.3.1, some $p \in S_n(T)$ is non-isolated. Then p is realized in some model M. By the downward Löwenheim-Skolem theorem (Theorem 6.4.3), we may assume M is countable. By the omitting types theorem (Theorem 13.2.7), there is a countable model N omitting p. Then $M \not\cong N$, and (1) fails.

(2) $\implies$ (3): Lemma 13.3.4.

(3) $\implies$ (4): Assume (3). Let M be a countable model. If A is a finite subset of M, then $S_n(A)$ is finite by (3), so every $p \in S_n(A)$ is isolated by Lemma 13.3.1, and then every $p \in S_n(A)$ is realized by Theorem 13.2.2. This shows that M is ω-saturated.

(4) $\implies$ (1): There is at most one countable ω-saturated model by Corollary 12.5.2. □

Theorem 13.3.6. *Let M be a countable structure in a countable language. The following are equivalent:*

1. $\mathrm{Th}(M)$ *is countably categorical.*

2. *For every n, the action of* $\mathrm{Aut}(M)$ *on M^n has finitely many orbits.*

Proof. (1) $\implies$ (2): by Theorem 13.3.5(4), M is ω-saturated, hence strongly ω-homogeneous by Corollary 12.5.3. Therefore $\bar{a}, \bar{b} \in M^n$ are in the same orbit of $\mathrm{Aut}(M)$ iff $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. By Theorem 13.3.5(2), there are only finitely many complete n-types.

(2) $\implies$ (1): if $D \subseteq M^n$ is 0-definable, then D is $\mathrm{Aut}(M)$-invariant so D is a union of orbits. There are only finitely many orbits, hence finitely many possibilities for D. Then the boolean algebra of 0-definable sets $D \subseteq M^n$ is finite. By Lemma 13.3.2, $S_n(\varnothing) = S_n(T)$ is finite. By Theorem 13.3.5, $\mathrm{Th}(M)$ is countably categorical. □

Chapter 14

Closure operators and pregeometries

14.1 Closure operators

Definition 14.1.1. A *closure operator* on a set S is an operation $\mathrm{cl}(-) : \mathfrak{P}(S) \to \mathfrak{P}(S)$ with the following properties:

- Idempotent: $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.
- Monotone: $X \subseteq Y \implies \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.
- Increasing: $X \subseteq \mathrm{cl}(X)$.

The set $\mathrm{cl}(X)$ is called the *closure* of X.

Fix a set S and a closure operator $\mathrm{cl}(-)$ on S. We say that $X \subseteq S$ is *closed* if $X = \mathrm{cl}(X)$. By idempotence, the closed sets are exactly the sets of the form $\mathrm{cl}(X)$.

Theorem 14.1.2. *If $X \subseteq S$, then $\mathrm{cl}(X)$ is the smallest closed set containing X.*

Proof. The set $\mathrm{cl}(X)$ contains X because $\mathrm{cl}(-)$ is increasing, and $\mathrm{cl}(X)$ is closed because $\mathrm{cl}(-)$ is idempotent. Suppose Y is a closed set containing X. Then $Y = \mathrm{cl}(Y) \supseteq \mathrm{cl}(X)$, because $\mathrm{cl}(-)$ is monotone. $\square$

Definition 14.1.3. A *closure system* on a set S is a family $\mathcal{C} \subseteq \mathfrak{P}(S)$ of sets, called *closed sets*, such that for any $X \subseteq S$ there is a smallest closed set containing X.

Remark 14.1.4. If $\mathcal{C}$ is a closure system, define $\mathrm{cl}(X)$ to be the smallest closed set containing X. It is easy to see that $\mathrm{cl}(-)$ is a closure operator on S. This gives a map from closure systems to closure operators. Conversely, any closure operator $\mathrm{cl}(-)$ defines a closure system $\{X \subseteq S : \mathrm{cl}(X) = X\}$ by Theorem 14.1.2. It is easy to see that these two maps are inverses. Thus closure operators correspond bijectively with closure systems.

Theorem 14.1.5. *A family $\mathcal{C} \subseteq \mathfrak{P}(S)$ is a closure system iff $\mathcal{C}$ is closed uder infinite intersections.*

Proof. If $\mathcal{C}$ is closed under infinite intersections, then the smallest closed set containing X exists—it is $\bigcap\{Y \in \mathcal{C} : Y \supseteq X\}$. Therefore $\mathcal{C}$ is a closure system.

Conversely, suppose $\mathcal{C}$ is a closure system. Let $\{X_i\}_{i \in I}$ be a family of closed sets and let $Y = \bigcap_{i \in I} X_i$. We claim that Y is closed. For each i, we have $\mathrm{cl}(Y) \subseteq \mathrm{cl}(X_i) = X_i$ by monotonicity. Thus $\mathrm{cl}(Y) \subseteq \bigcap_{i \in I} X_i = Y$. Conversely $Y \subseteq \mathrm{cl}(Y)$ because $\mathrm{cl}(-)$ is increasing. $\square$

Example 14.1.6. If S is a topological space, the family of closed sets is a closure system. The closure $\mathrm{cl}(X)$ is the smallest closed set containing X.

Example 14.1.7. Let M be a structure. Recall that $\langle A \rangle_M$ is the smallest substructure of M containing A. Therefore the class of substructures of M is a closure system, and $\langle - \rangle_M$ is a closure operation.

Definition 14.1.8. A closure operator $\mathrm{cl}(-)$ on a set S is *finitary* if for any $a \in S$, $X \subseteq S$ with $a \in \mathrm{cl}(X)$, there is a finite subset $X_0 \subseteq_f X$ with $a \in \mathrm{cl}(X_0)$.

More generally, we say that an operation $F : \mathfrak{P}(S) \to \mathfrak{P}(S)$ is *finitary* if $F(X) \subseteq \bigcup\{F(X_0) : X_0 \subseteq_f X\}$.

Example 14.1.9. If M is a structure, the closure operation $\langle - \rangle_M$ is finitary: if $b \in \langle A \rangle$ then $b = t(\bar{a})$ for some term t and finite tuple $\bar{a}$. If $A_0 = \{a_1, \ldots, a_n\}$, then $b \in \langle A_0 \rangle_M$.

Lemma 14.1.10. *Suppose* $\mathrm{cl}(-) : \mathfrak{P}(S) \to \mathfrak{P}(S)$ *is monotone, finitary, increasing, and satisfies the property*

$$a \in \mathrm{cl}(X) \text{ and } b \in \mathrm{cl}(X \cup \{a\}) \implies b \in \mathrm{cl}(X). \qquad (*)$$

Then $\mathrm{cl}(-)$ *is idempotent, and therefore a finitary closure operator.*

Proof. Note that $(*)$ implies

$$\mathrm{cl}(X \cup \{a\}) = \mathrm{cl}(X) \text{ for } a \in \mathrm{cl}(X). \tag{$\dagger$}$$

Fix X. Then $\mathrm{cl}(\mathrm{cl}(X)) \supseteq \mathrm{cl}(X)$ because $\mathrm{cl}(-)$ is increasing. For the reverse inclusion, suppose $b \in \mathrm{cl}(\mathrm{cl}(X))$. As $\mathrm{cl}(-)$ is finitary, $b \in \mathrm{cl}(\{a_1, \ldots, a_n\})$ for some $a_1, \ldots, a_n \in \mathrm{cl}(X)$. By n applications of $(\dagger)$,

$$\mathrm{cl}(X) = \mathrm{cl}(X \cup \{a_1\}) = \mathrm{cl}(X \cup \{a_1, a_2\}) = \cdots = \mathrm{cl}(X \cup \{a_1, \ldots, a_n\}).$$

Then

$$b \in \mathrm{cl}(\{a_1, \ldots, a_n\}) \subseteq \mathrm{cl}(X \cup \{a_1, \ldots, a_n\}) = \mathrm{cl}(X).$$

□

14.2 Algebraic and definable closure

Definition 14.2.1. Let M be a structure and $A \subseteq M$ be a subset.

1. $b \in M$ is *definable* over A if $\{b\}$ is A-definable.
2. $b \in M$ is *algebraic* over A if $b \in D$ for some finite A-definable $D \subseteq M$.
3. The *definable closure* of A, written $\mathrm{dcl}(A)$ or $\mathrm{dcl}^M(A)$, is the set of $b \in M$ definable over A.
4. The *algebraic closure* of A, written $\mathrm{acl}(A)$ or $\mathrm{acl}^M(A)$, is the set of $b \in M$ algebraic over A.

Lemma 14.2.2. *If $b \in \mathrm{acl}(A)$, then there is a finite A-definable set $X \ni b$ such that $\mathrm{tp}(c/A) = \mathrm{tp}(b/A)$ for every $c \in X$.*

Proof. Take a minimal finite A-definable set $X \ni b$. If X does not have the desired property, take $c \in x$ with $\mathrm{tp}(c/A) \neq \mathrm{tp}(b/A)$. Then there is an A-definable set D with $b \in D$ and $c \notin D$ (Remark 9.1.3). Then $X \cap D$ is a strictly smaller A-definable set containing b, a contradiction. □

Lemma 14.2.3. *Fix a structure M.*

1. *If $A \subseteq B$ then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.*
2. *If $b \in \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A_0)$ for some finite subset $A_0 \subseteq A$.*
3. *$A \subseteq \mathrm{acl}(A)$.*

4. *If* $b \in \mathrm{acl}(A)$ *and* $c \in \mathrm{acl}(A \cup \{b\})$, *then* $c \in \mathrm{acl}(A)$.

Proof. 1. Any finite A-definable set is a finite B-definable set.

2. Any finite A-definable set is A_0-definable for some finite $A_0 \subseteq A$.

3. If $b \in A$, then the set $\{b\}$ is finite and A-definable, so $b \in \mathrm{acl}(A)$.

4. As $b \in \mathrm{acl}(A)$ and $c \in \mathrm{acl}(A \cup \{b\})$, there is a finite A-definable set $X \ni b$ and a finite Ab-definable set $Y \ni c$. By Lemma 14.2.2, we may assume all elements of X have the same type over A. Write Y as $\varphi(M, b)$ for some $\mathcal{L}(A)$-formula $\varphi(x, y)$. Let $n = |Y| = |\varphi(M, b)|$. Then $n = |\varphi(M, b')|$ for every $b' \in X$, since this is expressed by a formula in $\mathrm{tp}(b'/A)$. Let $Z = \bigcup_{b' \in X} \varphi(M, b')$. Then Z is a finite A-definable set containing c. □

Theorem 14.2.4. *Algebraic closure* $\mathrm{acl}(-)$ *is a finitary closure operator.*

Proof. Lemma 14.2.3 plus Lemma 14.1.10. □

A set $A \subseteq M$ is *algebraically closed* if $\mathrm{acl}(A) = A$. The algebraic closure of A is the smallest algebraically closed set containing A.

Theorem 14.2.5. *Definable closure* $\mathrm{dcl}(-)$ *is a finitary closure operator.*

Proof. Like Theorem 14.2.4 but slightly easier. □

A set $A \subseteq M$ is *definably closed* if $\mathrm{dcl}(A) = A$. The definable closure of A is the smallest definably closed set containing A.

14.3 Definable closure, algebraic closure, and substructures

Theorem 14.3.1. *Let M be a structure. If $A \subseteq M$ and $A = \mathrm{dcl}(A)$, then A is a substructure of M.*

Proof. Let f be a k-ary function symbol, and let $a_1, \ldots, a_k$ be elements of A. Then $f(a_1, \ldots, a_k)$ is in $\mathrm{dcl}(A)$, as it is defined by the $\mathcal{L}(A)$-formula $x = f(a_1, \ldots, a_k)$. □

Theorem 14.3.2. *Suppose $M \preceq N$ and φ is an $\mathcal{L}(M)$-formula.*

1. *$\varphi(M) \subseteq \varphi(N)$.*

2. *If $\varphi(M)$ is finite, then $\varphi(M) = \varphi(N)$.*

Proof. 1. If $\bar{b} \in \varphi(M)$, then $M \models \varphi(\bar{b})$, so $N \models \varphi(\bar{b})$, and $\bar{b} \in \varphi(N))$.

2. Let $n = |\varphi(M)|$. Then $M \models \exists^{=n}\bar{x}\ \varphi(\bar{x})$, so the same sentence holds in N, and so $|\varphi(N)| = n$. As $\varphi(M) \subseteq \varphi(N)$, the two sets must be equal. □

Theorem 14.3.3. *If $A \subseteq M \preceq N$, then $\mathrm{acl}^M(A) = \mathrm{acl}^N(A)$.*

Proof. If $b \in \mathrm{acl}^M(A)$, then b is in a finite set of the form $\varphi(M)$ for some $\mathcal{L}(A)$-formula $\varphi(x)$. By Theorem 14.3.2, $\varphi(N) = \varphi(M)$. Then $b \in \varphi(N) \implies b \in \mathrm{acl}^N(A)$. Thus $\mathrm{acl}^M(A) \subseteq \mathrm{acl}^N(A)$.

The proof that $\mathrm{acl}^N(A) \subseteq \mathrm{acl}^M(A)$ is identical, exchanging M and N (even though there is no symmetry between M and N). □

Theorem 14.3.4. *If $M \preceq N$, then M is algebraically closed as a subset of N.*

Proof. $\mathrm{acl}^N(M) = \mathrm{acl}^M(M) = M$. □

Theorem 14.3.5. *Let $\mathbb{M}$ be a monster model, let A be a small subset, and let b be an element. Let S be the set of realizations of $\mathrm{tp}(b/A)$.*

1. *If $b \in \mathrm{acl}(A)$, then S is finite.*

2. *If $b \notin \mathrm{acl}(A)$, then S is large.*

Proof. If $b \in \mathrm{acl}(A)$, then there is a finite A-definable set $X \ni b$. Every realization of $\mathrm{tp}(b/A)$ is in X, so $S \subseteq X$.

Conversely, suppose S is small. Let

$$\Sigma(x) = \mathrm{tp}(b/A) \cup \{x \neq c : c \in S\}.$$

Then $\Sigma(x)$ is a set of formulas over the small set $A \cup S$ with no realizations in $\mathbb{M}$. By saturation, $\Sigma(x)$ isn't finitely realizable. Therefore there is $\varphi(x) \in \mathrm{tp}(b/A)$ and $c_1, \ldots, c_n \in S$ such that

$$\varphi(x) \cup \{x \neq c_i : 1 \leq i \leq n\}$$

isn't realized in $\mathbb{M}$. This means that $\varphi(\mathbb{M}) \subseteq \{c_1, \ldots, c_n\}$. Then b is in the finite A-definable set $\varphi(\mathbb{M})$. □

Remark 14.3.6. By strong homogeneity, the set S in Theorem 14.3.5 is the set $\{\sigma(b) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$.

14.4 Countably categorical fields

Theorem 14.4.1. *Let M be a field and A be a definably closed subset. Then A is a subfield.*

Proof. By Theorem 14.3.1, A is a substructure, i.e., a subring. It remains to show that A is closed under multiplicative inverses. If $a \in A$ is non-zero, then the $\mathcal{L}(A)$-formula $xa = 1$ defines the set $\{a^{-1}\}$, so $a^{-1} \in \operatorname{dcl}(A) = A$. $\square$

Lemma 14.4.2. *Let T be $\aleph_0$-categorical, let M be a model of T, and let A be a finite subset of M. Then $\operatorname{dcl}(A)$ is finite.*

Proof. By Theorem 13.3.5(3), $S_1(A)$ is finite. Therefore there are only finitely many A-definable sets $D \subseteq M^1$. By definition, $b \in \operatorname{dcl}(A)$ if and only if $\{b\}$ is definable. Thus $\operatorname{dcl}(A)$ is finite. $\square$

Theorem 14.4.3. *There is no $\aleph_0$-categorical theory of fields.*

Proof. Let T be an $\aleph_0$-categorical theory of fields and let K be an $\aleph_1$-saturated model. For any $n > 0$, the set $\{x \in K : x^n - 1 = 0\}$ is finite, by Theorem 10.1.6. Then the following partial type is finitely satisfiable in K:

$$\Sigma(x) = \{x \neq 0, x \neq 1, x^2 \neq 1, x^3 \neq 1, x^4 \neq 1, \dots\}.$$

Then $\Sigma(x)$ is realized in K. Take $b \in K$ realizing $\Sigma(x)$. Consider the group homomorphism

$$f : \mathbb{Z} \to K^\times$$
$$f(n) = b^n.$$

As b realizes Σ, we have $f(b) = b^n \neq 1$ for $n \neq 0$. Therefore $\ker(f) = \{0\}$, and the image $f(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/\{0\} \cong \mathbb{Z}$. In particular $f(\mathbb{Z})$ is infinite. However, $f(n) = b^n \in \operatorname{dcl}(\{b\})$ for each n, and $\operatorname{dcl}(\{b\})$ is finite. $\square$

14.5 Algebraic closure in ACF

If $M \models \mathrm{ACF}$ and K is a subfield, then K^{alg} denotes the set of $a \in M$ that are algebraic over K, meaning that $P(a) = 0$ for some non-zero polynomial $P(x) \in K[x]$ (Definition 10.3.1).

Lemma 14.5.1. *Suppose $M \models$ ACF and K is a subfield. If $D \subseteq M^1$ is K-definable, then there is a finite subset $S \subseteq K^{\text{alg}}$ such that $D = S$ or $D = M \setminus S$.*

Proof. Let $\mathcal{F}$ be the class of sets of the form S or $M \setminus S$ for finite $S \subseteq K^{\text{alg}}$. We must show $D \in \mathcal{F}$. Note that $\mathcal{F}$ is closed under boolean combinations.

By quantifier-elimination, $D = \varphi(K)$ for some quantifier-free $\mathcal{L}(K)$-formula φ, which is a boolean combination of atomic formulas. Because $\mathcal{F}$ is closed under boolean combinations, we may assume that φ is atomic.

Then φ has the form $P(x) = Q(x)$ for some polynomials $P, Q \in K[x]$. If $P - Q$ is identically zero, then $\varphi(M) = M$. Otherwise, $\varphi(M)$ is the set of roots of $P - Q$, which is a subset of K^{alg} by definition of K^{alg}, and finite by Theorem 10.1.6. Either way, $\varphi(M) \in \mathcal{F}$. □

Fix $M \models$ ACF.

Theorem 14.5.2. *Let K be a subfield of M. Then $\operatorname{acl}(K) = K^{\text{alg}}$.*

Proof. If $a \in K^{\text{alg}}$, then a is in a finite K-definable set of the form $\{x \in M : P(x) = 0\}$ for some $P(x) \in K[x]$, and so $a \in \operatorname{acl}(K)$.

Conversely, suppose $a \in \operatorname{acl}(K)$. Then $a \in D$ for some finite K-definable set D. By Lemma 14.5.1, $D \subseteq K^{\text{alg}}$, so $a \in K^{\text{alg}}$. □

Theorem 14.5.3. *Let K be a subset of M. The following are equivalent:*

1. *K is a subfield and $K = K^{\text{alg}}$.*
2. *K is a substructure and $K \models$ ACF.*
3. *$K \preceq M$.*
4. *$K = \operatorname{acl}(K)$.*

Proof. We show (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

Assume (1). If $P(x) \in K[x]$ is non-constant, then there is $c \in M$ with $P(c) = 0$, because $M \models$ ACF. The element c is in K^{alg} by definition, so $c \in K$. Thus $K \models$ ACF.

Assume (2). Then $K \preceq M$ because ACF has quantifier elimination and both K and M are models of ACF (see Theorem 9.3.6).

Assume (3). Then $K = \operatorname{acl}(K)$ by Theorem 14.3.4.

Assume (4). Then K is a subfield by Theorem 14.3.1 (definably closed sets are subfields) and $K = \operatorname{acl}(K) = K^{\text{alg}}$ by Theorem 14.5.2. □

Corollary 14.5.4. *If K is a subfield of M, then K^{alg} is also a subfield, and $K^{\mathrm{alg}} \preceq M$.*

Proof. $\mathrm{acl}(K^{\mathrm{alg}}) = \mathrm{acl}(\mathrm{acl}(K)) = \mathrm{acl}(K) = K^{\mathrm{alg}}$, so K^{alg} satisfies condition (4) of Theorem 14.5.3. Then K^{alg} is a subfield and an elementary substructure by (1) and (3) of Theorem 14.5.3. □

Example 14.5.5. The set of algebraic numbers $\mathbb{Q}^{\mathrm{alg}} \subseteq \mathbb{C}$ is an algebraically closed field, and an elementary substructure of $\mathbb{C}$.

14.6 Pregeometries and vector spaces

Definition 14.6.1. Let K be a field. A *K-vector space* is an abelian group $(V, +)$ with a function

$$\cdot : K \times V \to V$$

satisfying the axioms

$$\begin{aligned} a \cdot (v + w) &= (a \cdot v) + (a \cdot w) \\ 1 \cdot v &= v \\ (a + b) \cdot v &= a \cdot v + b \cdot v \qquad (\dagger) \\ (a \cdot b) \cdot v &= a \cdot (b \cdot v). \end{aligned}$$

For fixed K, we can regard the class of K-vector spaces as an equational class by thinking of the map $\cdot : K \times V \to V$ as a family of unary maps

$$\mu_a : V \to V$$
$$\mu_a(v) = a \cdot v,$$

one for each $a \in K$. The axioms in Definition 14.6.1 become axiom schemas. For example, axiom ($\dagger$) becomes the axiom schema

$$\mu_{a+b}(x) = \mu_a(x) + \mu_b(x) \text{ for each } a, b \in K.$$

Subalgebras of vector spaces are called *linear subspaces*.

Remark 14.6.2. In a vector space, the following equations hold:

$$\begin{aligned} (-1) \cdot v &= -v \\ 0 \cdot v &= 0 \\ a \cdot (-v) &= -(a \cdot v) \\ a \cdot 0 &= 0. \end{aligned}$$

Fix a field K and vector space V.

Definition 14.6.3. The *span* of a set $S \subseteq V$, written $\mathrm{span}(S)$, is the collection of elements of the form

$$a_1 v_1 + \cdots + a_n v_n$$

where $n \geq 0$, $a_1, \ldots, a_n \in K$, and $v_1, \ldots, v_n \in S$.

Theorem 14.6.4. *The span of S is the linear subspace of V generated by S:*

$$\mathrm{span}(S) = \langle S \rangle_V.$$

Proof. It is an exercise in algebra (using Remark 14.6.2) to see that $\mathrm{span}(S)$ is a linear subspace. It contains S because if $v \in S$, then $v = 1 \cdot v \in \mathrm{span}(S)$. Therefore $\mathrm{span}(S) \supseteq \langle S \rangle_V$. On the other hand, every element of $\mathrm{span}(S)$ has the form $t(v_1, \ldots, v_n)$ for some term t and tuple $\bar{v}$ in S, and so $\mathrm{span}(S) \subseteq \langle S \rangle_V$. □

Corollary 14.6.5. *Span is a finitary closure operator on V.*

Definition 14.6.6. A *pregeometry* is a pair (X, cl) where X is a set, cl is a finitary closure operator on X, and the following *exchange property* holds for $a, b \in X$ and $C \subseteq X$:

$$a \in \mathrm{cl}(C \cup \{b\}) \setminus \mathrm{cl}(C) \implies b \in \mathrm{cl}(C \cup \{a\}).$$

Theorem 14.6.7. *In a vector space,* $\mathrm{span}(-)$ *has the exchange property, and therefore defines a pregeometry.*

Proof. Suppose $v \in \mathrm{span}(S \cup \{w\})$ but $v \notin \mathrm{span}(S)$. Then

$$v = a_1 v_1 + \cdots + a_n v_n + bw \tag{$*$}$$

for some $a_1, \ldots, a_n, b \in K$ and $v_1, \ldots, v_n \in S$. If $b = 0$, then $v \in \mathrm{span}(S)$, a contradiction. Thus $b \neq 0$, and b^{-1} exists. Rearranging $(*)$, we see

$$\begin{aligned} -bw &= a_1 v_1 + \cdots + a_n v_n - v \\ w &= (-b^{-1}a_1)v_1 + \cdots + (-b^{-1}a_n)v_n + b^{-1}v \\ w &\in \mathrm{span}(S \cup \{v\}). \end{aligned}$$

□

Rank

Fix a pregeometry (X, cl).

Definition 14.6.8. If $\bar{a} \in X^n$ and $B \subseteq X$, the *rank* of $\bar{a}$ over B, written $\mathrm{rk}(\bar{a}/B)$, is defined to be the number of $i \in \{1, \ldots, n\}$ such that $a_i \notin \mathrm{cl}(B \cup \{a_1, \ldots, a_{i-1}\})$. We write $\mathrm{rk}(\bar{a}/\varnothing)$ as $\mathrm{rk}(\bar{a})$.

Theorem 14.6.9. $\mathrm{rk}(\bar{a}, \bar{b}/C) = \mathrm{rk}(\bar{a}/C) + \mathrm{rk}(\bar{b}/C\bar{a})$.

Proof. Clear from the definition. □

Theorem 14.6.10. $\mathrm{rk}(\bar{a}/B) = 0 \iff \{a_1, \ldots, a_n\} \subseteq \mathrm{cl}(B)$.

Proof. If $a_i \in \mathrm{cl}(B)$ for each i, then $a_i \in \mathrm{cl}(B \cup \{a_1, \ldots, a_{i-1}\})$ for each i, so $\mathrm{rk}(\bar{a}/B) = 0$. Conversely, if $\mathrm{rk}(\bar{a}/B) = 0$, then $a_i \in \mathrm{cl}(B \cup \{a_1, \ldots, a_{i-1}\})$ for each i. If S is a closed set containing B, then

$$\{a_1, \ldots, a_{i-1}\} \subseteq S \implies a_i \in S$$

and so $\{a_1, \ldots, a_n\} \subseteq S$ by induction. Taking $S = \mathrm{cl}(B)$, we see $\{a_1, \ldots, a_n\} \subseteq \mathrm{cl}(B)$. □

Lemma 14.6.11. $\mathrm{rk}(a, b/C) = \mathrm{rk}(b, a/C)$.

Proof. Both sides are in $\{0, 1, 2\}$. It suffices to show

$$\mathrm{rk}(a, b/C) = 0 \iff \mathrm{rk}(b, a/C) = 0 \tag{14.1}$$
$$\mathrm{rk}(a, b/C) = 2 \iff \mathrm{rk}(b, a/C) = 2 \tag{14.2}$$

By symmetry it suffices to show the $\Rightarrow$ directions.

(14.1): by Theorem 14.6.10, both sides say $\{a, b\} \subseteq C$.

(14.2): suppose $\mathrm{rk}(a, b/C) = 2$. Then $a \notin \mathrm{cl}(C)$ and $b \notin \mathrm{cl}(Ca)$. By monotonicity, $b \notin \mathrm{cl}(C)$. If $a \in \mathrm{cl}(Cb)$, then

$$a \in \mathrm{cl}(Cb) \setminus \mathrm{cl}(C) \text{ but } b \notin \mathrm{cl}(Ca),$$

contradicting the exchange property. Thus $a \notin \mathrm{cl}(Cb)$. With $b \notin \mathrm{cl}(C)$, this implies $\mathrm{rk}(b, a/C) = 2$. □

Lemma 14.6.12. *If $\bar{a}, \bar{d}$ are tuples and b, c are elements and A is a set, then*

$$\mathrm{rk}(\bar{a}, b, c, \bar{d}/A) = \mathrm{rk}(\bar{a}, c, b, \bar{d}/A).$$

Proof. Using Theorem 14.6.9 and Lemma 14.6.11,

$$\begin{aligned}\mathrm{rk}(\bar{a}, b, c, \bar{d}/A) &= \mathrm{rk}(\bar{a}/A) + \mathrm{rk}(b, c/A, \bar{a}) + \mathrm{rk}(\bar{d}/A, \bar{a}, b, c)\\ &= \mathrm{rk}(\bar{a}/A) + \mathrm{rk}(c, b/A, \bar{a}) + \mathrm{rk}(\bar{d}/A, \bar{a}, c, b)\\ &= \mathrm{rk}(\bar{a}, c, b, \bar{d}/A).\end{aligned}$$ □

Theorem 14.6.13. *If π is a permutation of $\{1, \dots, n\}$, then*

$$\mathrm{rk}(a_1, \dots, a_n/B) = \mathrm{rk}(a_{\pi(1)}, \dots, a_{\pi(n)}/B).$$

Proof. Repeated applications of Lemma 14.6.12. □

Theorem 14.6.14. *If $\bar{a} \subseteq \mathrm{cl}(C\bar{b})$, then $\mathrm{rk}(\bar{a}/C) \leq \mathrm{rk}(\bar{b}/C)$.*

Proof.

$$\begin{aligned}\mathrm{rk}(\bar{a}/C) &\leq \mathrm{rk}(\bar{a}/C) + \mathrm{rk}(\bar{b}/C\bar{a}) = \mathrm{rk}(\bar{a}, \bar{b}/C)\\ &= \mathrm{rk}(\bar{b}, \bar{a}/C) = \mathrm{rk}(\bar{b}/C) + \mathrm{rk}(\bar{a}/C\bar{b}) = \mathrm{rk}(\bar{b}/C).\end{aligned}$$ □

Theorem 14.6.15. *If $C \supseteq B$, then $\mathrm{rk}(\bar{a}/C) \leq \mathrm{rk}(\bar{a}/B)$.*

Proof. For each i, monotonicity of $\mathrm{cl}(-)$ shows

$$a_i \notin \mathrm{cl}(C \cup \{a_1, \dots, a_{i-1}\}) \implies a_i \notin \mathrm{cl}(B \cup \{a_i, \dots, a_{i-1}\}).$$ □

Independence and bases

Definition 14.6.16. A set $I \subseteq X$ is *independent* if $a \notin \mathrm{cl}(I \setminus \{a\})$ for each $a \in I$.

Theorem 14.6.17. *In a vector space V, a set $I \subseteq V$ is independent if and only if the following condition holds: for any distinct $v_1, \dots, v_n \in V$ and $a_1, \dots, a_n \in K$,*

$$a_1 v_1 + \cdots + a_n v_n = 0 \implies v_1 = v_2 = \cdots = v_n = 0. \qquad (*)$$

Proof. If I is not independent, then there is some $v_1 \in \mathrm{span}(I \setminus \{v_1\})$. Thus there are distinct $v_2, \dots, v_n \in I \setminus \{v_1\}$ with

$$v_1 = a_2 v_2 + \cdots + a_n v_n.$$

Setting $a_1 = -1 \in K$, we get $\sum_{i=1}^n a_i v_i = 0$, contradicting $(*)$.

Conversely, suppose $(*)$ fails. Permuting the v_i, we may assume $a_1 \neq 0$. Then

$$a_1 v_1 + \cdots + a_n v_n = 0$$
$$v_1 = (-a_1^{-1} a_2) v_2 + \cdots + (-a_1^{-1} a_n) v_n \in \mathrm{span}(v_2, \ldots, v_n),$$

and I is not independent. □

Now return to a general pregeometry.

Definition 14.6.18. A *basis* is a maximal independent set.

Remark 14.6.19.

1. If I is independent and $J \subseteq I$, then J is independent, by monotonicity of $\mathrm{cl}(-)$.

2. A set $I \subseteq X$ is independent if and only if every finite subset is independent, because $\mathrm{cl}(-)$ is finitary.

Theorem 14.6.20. *There is at least one basis.*

Proof. Zorn's lemma. □

Lemma 14.6.21. *If $a_1, \ldots, a_n \in X$ are distinct, then $\mathrm{rk}(\bar{a}) = n$ if and only if the set $I = \{a_1, \ldots, a_n\}$ is independent.*

Proof. Note that $\mathrm{rk}(\bar{a}) = n$ if and only if

$$a_i \notin \mathrm{cl}(a_1, \ldots, a_{i-1}) \text{ for all } 1 \leq i \leq n, \tag{$*$}$$

by definition of rank. Moreover, the condition $\mathrm{rk}(\bar{a}) = n$ is invariant under permuting $\bar{a}$, by Theorem 14.6.13.

Independence clearly implies $(*)$, as $\mathrm{cl}(-)$ is monotone. For the converse, suppose $\mathrm{rk}(\bar{a}) = n$ and $i \leq n$. We must show $a_i \notin \mathrm{cl}(I \setminus \{a_i\})$. Permuting $\bar{a}$, we reduce to the case $i = n$. Then we must show $a_n \notin \mathrm{cl}(a_1, \ldots, a_{n-1})$, which is a case of $(*)$. □

Lemma 14.6.22. *If I is independent and $a \notin \mathrm{cl}(I)$, then $I \cup \{a\}$ is independent.*

Proof. By Remark 14.6.19(2), we may assume I is a finite set $\{b_1, \ldots, b_n\}$. By Lemma 14.6.21 and Theorem 14.6.9

$$\mathrm{rk}(\bar{b}, a) = \mathrm{rk}(\bar{b}) + \mathrm{rk}(a/\bar{b}) = n + 1,$$

and so $\{b_1, \ldots, b_n, a\}$ is independent. $\square$

Theorem 14.6.23. *If B is a basis, then $\mathrm{cl}(B) = X$.*

Proof. Otherwise, take $a \in X \setminus \mathrm{cl}(B)$, and $B \cup \{a\}$ is independent, contradicting the fact that B is a maximal independent set. $\square$

Remark 14.6.24. Conversely, if B is independent and $\mathrm{cl}(B) = X$, then B is a basis. Otherwise, if $I \supsetneq B$ is a larger independent set, then $I \subseteq X = \mathrm{cl}(B)$, contradicting Lemma 14.6.25 below. Thus B is a basis if and only if B is independent and $\mathrm{cl}(B) = X$.

Lemma 14.6.25. *If I is independent and J is a proper subset, then $I \not\subseteq \mathrm{cl}(J)$.*

Proof. Otherwise, take $a \in I \setminus J$. Then $J \subseteq I \setminus \{a\}$, and so

$$a \in I \subseteq \mathrm{cl}(J) \subseteq \mathrm{cl}(I \setminus \{a\}),$$

contradicting independence. $\square$

Theorem 14.6.26. *If B and C are two bases, then $|B| = |C|$.*

Proof. Suppose not. Without loss of generality, $|B| < |C|$. There are two cases:

1. B and C are finite. Let $\bar{b}$ and $\bar{c}$ enumerate B and C. Then

   $$\mathrm{rk}(\bar{b}) = |B| < |C| = \mathrm{rk}(\bar{c})$$

   by Lemma 14.6.21. However, $C \subseteq X = \mathrm{cl}(B)$ by Theorem 14.6.23, and so $\mathrm{rk}(\bar{c}) \leq \mathrm{rk}(\bar{b})$ by Theorem 14.6.14.

2. C is infinite. By Theorem 14.6.23, $B \subseteq \mathrm{cl}(C)$ and $C \subseteq \mathrm{cl}(B)$. As $\mathrm{cl}(-)$ is finitary, we can find a finite subset $C_b \subseteq C$ for each $b \in B$ such that $b \in \mathrm{cl}(C_b)$. Let $C' = \bigcup_{b \in B} C_b$. Then $B \subseteq \mathrm{cl}(C')$, implying $C \subseteq \mathrm{cl}(B) \subseteq \mathrm{cl}(\mathrm{cl}(C')) = \mathrm{cl}(C')$. This contradicts independence of C (see Lemma 14.6.25) unless $C' = C$.

   However, $|C'| < |C|$. Indeed, if B is finite then C' is finite, and so $|C'| < |C|$. If B is infinite then $|C'| \leq |B| < |C|$. Thus C' is a proper subset of C, a contradiction. $\square$

Definition 14.6.27. The *rank* of a pregeometry is the cardinality of any basis.

14.7 The classification of vector spaces

Fix a field K. We can regard K as a vector space by defining

$$a \cdot v = a \cdot v$$
$$v + w = v + w$$

where the left-hand $\cdot$ and $+$ are the vector space operations, and the right-hand $\cdot$ and $+$ are the field operations.

If λ is a cardinal, let K^λ be the power vector space, i.e., the set of functions $f : \lambda \to K$ with the vector space operations defined pointwise:

$$(f + g)(x) = f(x) + g(x)$$
$$(a \cdot f)(x) = a \cdot (f(x)).$$

Say that $f \in K^\lambda$ has *finite support* if

$$\text{supp}(f) := \{x \in \lambda : f(x) \neq 0\}$$

is finite. Note that

$$\text{supp}(f + g) \subseteq \text{supp}(f) + \text{supp}(g)$$
$$\text{supp}(af) \subseteq \text{supp}(f)$$
$$\text{supp}(0) = \varnothing.$$

Therefore, $F_\lambda := \{f \in K^\lambda : \text{supp}(f) \text{ is finite}\}$ is a linear subspace of K^λ.

Lemma 14.7.1. *Let V be a K-vector space. Suppose $v_i \in V$ for each $i \in \lambda$.*

1. *There is a homomorphism $\alpha : F_\lambda \to V$ defined by*

$$\alpha(f) = \sum_{i \in \text{supp}(f)} f(i) \cdot v_i.$$

2. *α is surjective if and only if $V = \text{span}\{v_i : i \in \lambda\}$.*

3. *α is injective if and only if $\{v_i : i \in \lambda\}$ is independent (and the v_i are distinct).*

4. *α is an isomorphism if and only if $\{v_i : i \in \lambda\}$ is a basis (and the v_i are distinct).*

5. *All homomorphisms $F_\lambda \to V$ arise as in part (1).*

Proof.

1. An exercise in algebra. Essentially this works because $\alpha(f) = \sum_{i\in\lambda} f(i) \cdot v_i$, which depends linearly on f. The sum makes sense as almost all the terms in it are zero.

2. True by definition of span.

3. α is injective if and only if $\ker(\alpha) = \{0\}$. This condition means that if $f \in K^\lambda$ has finite support and
$$\sum_{i\in\lambda} f(i)v_i = 0,$$
then f must vanish. This is a rephrasing of Theorem 14.6.17.

4. α is an isomorphism if and only if it is injective and surjective. By the previous two points, this means that the set $B = \{v_i : i \in \lambda\}$ is independent and has $\mathrm{span}(B) = V$. By Remark 14.6.24, this means that B is a basis.

5. Let $\beta : F_\lambda \to V$ be a homomorphism. Let $e_i : \lambda \to K$ be the function
$$e_i(x) = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{if } x \neq i. \end{cases}$$
Then $\mathrm{supp}(e_i) = \{i\}$, so $e_i \in F_\lambda$. Note that for any $f \in F_\lambda$,
$$f = \sum_{i\in\mathrm{supp}(f)} f(i) \cdot e_i.$$
Therefore
$$\beta(f) = \sum_{i\in\mathrm{supp}(f)} f(i) \cdot \beta(e_i).$$
Taking $v_i = \beta(e_i)$, we see that β has the desired form. $\square$

Lemma 14.7.1 immediately yields the following:

Theorem 14.7.2. *Let V be a K-vector space. Then $V \cong F_\lambda$ if and only if V has a basis of cardinality λ.*

By Theorems 14.6.20 and 14.6.26, there is a unique cardinal λ such that V has a basis of cardinality λ.

Theorem 14.7.3. *Let V be a K-vector space. Then $V \cong F_\lambda$ for a unique cardinal λ, called the* dimension *of V.*

Lemma 14.7.4. *Let V be a K-vector space of dimension λ.*

1. *If λ is finite, then $|V| = |K|^\lambda$.*

2. *If λ or $|K|$ is infinite, then $|V| = |K| + \lambda$.*

Proof. Without loss of generality, $V = F_\lambda$.

1. $F_\lambda = K^\lambda$, whose cardinality if $|K|^\lambda$.

2. If $B \subseteq V$ is a basis of size λ, then $B \subseteq V \implies \lambda \leq |V|$. If $v \in V$ is non-zero, then there is an injection

$$\begin{aligned} K &\to V \\ a &\mapsto av, \end{aligned}$$

and so $|K| \leq |V|$. Thus $|V| \geq \max(|K|, |\lambda|) = |K| + |\lambda|$. On the other hand, V is generated by a set of size λ, and the language has size $\aleph_0 + |K|$, so that $|V| \leq \lambda + |K| + \aleph_0 = |K| + \lambda$. □

Theorem 14.7.5. *If V is an infinite K-vector space and $|V| > |K|$, then* $\dim(V) = |V|$.

Proof. Let $\lambda = \dim(V)$. If K and λ are both finite then $|V| = |K|^\lambda < \aleph_0$, a contradiction. Therefore K or λ is infinite, and so $|V| = \max(|K|, \lambda)$. As $|V| > |K|$, this implies $|V| = \lambda$. □

Corollary 14.7.6. *Let K be a field. The theory of infinite K-vector spaces is complete and λ-categorical for every infinite $\lambda > |K|$.*

Definition 14.7.7. A theory T in a countable language is *totally categorical* if T is λ-categorical for any λ.

Corollary 14.7.8. *If K is a finite field, then the theory of infinite K-vector spaces is totally categorical.*

Chapter 15

Strongly minimal and geometric theories

15.1 Strong minimality

Definition 15.1.1. Let S be a set. A subset $X \subseteq S$ is *cofinite* (in S) if $S \setminus X$ is finite.

Definition 15.1.2. A structure M is *minimal* if M is infinite, and every definable set $X \subseteq M$ is finite or cofinite.

Note this is only a statement about definable sets in one variable; it says nothing about definable subsets of M^n for $n > 1$.

Definition 15.1.3. A theory T is *strongly minimal* if all its models are minimal. A structure M is *strongly minimal* if its complete theory is strongly minimal.

Note that models of a strongly minimal theory are strongly minimal.

Example 15.1.4. ACF is strongly minimal by Lemma 14.5.1.

Theorem 15.1.5. *Suppose M is strongly minimal and $A \subseteq M$. Then $A \preceq M$ if and only if $A = \mathrm{acl}(A)$ and A is infinite.*

Proof. First suppose $A \preceq M$. Then A is infinite because M is (Lemma 6.5.4), and $A = \mathrm{acl}(A)$ because A is an elementary substructure (Theorem 14.3.4).

Conversely, suppose $A = \mathrm{acl}(A)$ and A is infinite. Then $A \preceq M$ by the Tarski-Vaught criterion (Theorem 6.3.1). Indeed, suppose $D \subseteq M$ is non-empty and A-definable. By strong minimality, D is finite or cofinite. If D is finite then $D \subseteq \mathrm{acl}(A) = A$, so $A \cap D = D \neq \varnothing$. If D is cofinite, it intersects A because A is infinite. $\square$

Theorem 15.1.6. *If M is strongly minimal, then* $\mathrm{acl}(-)$ *satisfies the exchange property:*

$$a \in \mathrm{acl}(Cb) \setminus \mathrm{acl}(C) \implies b \in \mathrm{acl}(Ca).$$

Consequently, $\mathrm{acl}(-)$ *defines a pregeometry.*

Proof. Because $a \in \mathrm{acl}(Cb)$, there is a finite Cb-definable set X containing a. Write X as $\varphi(M, b)$ for some $\mathcal{L}(C)$-formula $\varphi(x, y)$. Let $n = |X|$. Replacing $\varphi(x, y)$ with

$$\varphi(x, y) \wedge \neg \exists^{\geq n+1} z \; \varphi(z, y),$$

we may assume that $|\varphi(M, b')| \leq n$ for any $b' \in M$. As $\varphi(a, b)$ holds, $b \in \varphi(a, M)$. If $\varphi(a, M)$ is finite then $b \in \mathrm{acl}(Ca)$ as desired.

Otherwise, $\varphi(a, M)$ is cofinite, and its complement has cardinality $k < \infty$. Let D be the C-definable set of $a' \in M$ such that $|M \setminus \varphi(a', M)| = k$. Then $a \in D$, so D is infinite because $a \notin \mathrm{acl}(C)$. Take distinct $a_1, \ldots, a_{n+1} \in D$. Then $\varphi(a_i, M)$ is cofinite for each i. An intersection of cofinite sets is cofinite, hence non-empty, so there is some

$$b' \in \bigcap_{i=1}^{n+1} \varphi(a_i, M).$$

This means that $\varphi(a_i, b')$ holds for all i, and so $|\varphi(M, b')| \geq n + 1$, contradicting the choice of φ. $\square$

15.2 Uncountable categoricity

Let $\mathbb{M}$ be a monster model of a *complete* strongly minimal theory in a countable language.

Theorem 15.2.1. *For any small set $A \subseteq \mathbb{M}$, there is a unique 1-type $p \in S_1(A)$ whose realizations are the elements of $\mathbb{M} \setminus \mathrm{acl}(A)$.*

Proof. In other words, we must show $b, c \notin \operatorname{acl}(A)$ implies $b \equiv_A c$. Otherwise there is an A-definable set D with $b \in D$ and $c \notin D$ (Remark 9.1.3). If D is finite, then $b \in \operatorname{acl}(A)$ and if D is cofinite, then $c \in \operatorname{acl}(A)$. Either way, we get a contradiction. □

The pregeometry rank $\operatorname{rk}(\bar{a}/B)$ (Definition 14.6.8) is usually written as $\dim(\bar{a}/B)$. Say that a finite tuple $\bar{a} \in \mathbb{M}^n$ is *independent* if $\dim(\bar{a}) = n$. By definition, this means $a_i \notin \operatorname{acl}(a_1, \ldots, a_{i-1})$ for each i. By Lemma 14.6.21, $\bar{a}$ is independent if and only if the a_i are pairwise distinct and $\{a_1, \ldots, a_n\}$ is independent as a set.

Lemma 15.2.2. *Let $\bar{a}, \bar{b}$ be two finite tuples of length n in $\mathbb{M}$. If $\dim(\bar{a}) = n$ and $\dim(\bar{b}) = n$, then $\bar{a} \equiv_\varnothing \bar{b}$.*

Proof. Proceed by induction on n. The case $n = 0$ is trivial. Note that the subtuples $(a_1, \ldots, a_{n-1})$ and $(b_1, \ldots, b_{n-1})$ are independent. By induction, $(a_1, \ldots, a_{n-1}) \equiv_\varnothing (b_1, \ldots, b_{n-1})$. Take $\sigma \in \operatorname{Aut}(\mathbb{M})$ moving $(a_1, \ldots, a_{n-1})$ to $(b_1, \ldots, b_{n-1})$. Replacing $\bar{a}$ with $\sigma(\bar{a})$, we may assume $a_i = b_i$ for $i < n$. By independence,

$$\begin{aligned} a_n &\notin \operatorname{acl}(a_1, \ldots, a_{n-1}) \\ b_n &\notin \operatorname{acl}(b_1, \ldots, b_{n-1}) = \operatorname{acl}(a_1, \ldots, a_{n-1}). \end{aligned}$$

By Theorem 15.2.1, a_n and b_n have the same type over $(a_1, \ldots, a_{n-1})$. Therefore $(a_1, \ldots, a_n) \equiv_\varnothing (a_1, \ldots, a_{n-1}, b_n) = \bar{b}$. □

Theorem 15.2.3. *Let M be a model of T. Let $f : I_1 \to I_2$ be a bijection between two independent sets. Then f is a partial elementary map.*

Proof. If $a_1, \ldots, a_n$ are distinct elements of I_1, then $f(a_1), \ldots, f(a_n)$ are distinct elements of I_2, and the two tuples $\bar{a}$ and $f(\bar{a})$ are both independent n-tuples. By Lemma 15.2.2, they have the same type. □

Definition 15.2.4. If $M \models T$, the *rank* of M, written $\operatorname{rk}(M)$, is the rank of the pregeometry $(M, \operatorname{acl}(-))$, i.e., the cardinality of a basis.

Remark 15.2.5. If $M \models T$, and B is a basis, then $M = \operatorname{acl}(B)$ (Theorem 14.6.23), and so

$$|M| = |\operatorname{acl}(B)| = |B| + \aleph_0 = \operatorname{rk}(M) + \aleph_0.$$

Theorem 15.2.6. *Two models M_1, M_2 are isomorphic if and only if* $\mathrm{rk}(M_1) = \mathrm{rk}(M_2)$.

Proof. If $M_1 \cong M_2$, it is easy to see that $\mathrm{rk}(M_1) = \mathrm{rk}(M_2)$. Conversely, suppose the ranks are equal. Embed M_1 and M_2 into a monster model $\mathbb{M}$. Let B_i be a basis of M_i for $i = 1, 2$. Then $|B_1| = |B_2|$. Take a bijection $f : B_1 \to B_2$. By Theorem 15.2.3, f is a partial elementary map from $\mathbb{M}$ to $\mathbb{M}$. By strong homogeneity, f extends to an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$. Then $\sigma(M_1) = \sigma(\mathrm{acl}(B_1)) = \mathrm{acl}(B_2) = M_2$. The automorphism σ restricts to an isomorphism from M_1 to M_2. □

Corollary 15.2.7. *T is κ-categorical for all $\kappa > \aleph_0$.*

Theorem 15.2.8. *Suppose that in models of T, algebraically closed sets are infinite. Then there is a unique model of rank κ for every cardinal κ, finite or infinite.*

Proof. Take a model M_0 of size greater than $\kappa + \aleph_0$, and let B_0 be a basis of M_0. Then $|B_0| > \kappa$. Take a subset $I \subseteq B_0$ with $|I| = \kappa$, and let $M = \mathrm{acl}(I)$. By assumption, M is infinite. Then $M \preceq M_0$ by Theorem 15.1.5. By Remark 14.6.24, the independent set I is a basis of M, so $\mathrm{rk}(M) = |I| = \kappa$. □

The classification of algebraically closed fields

Definition 15.2.9. The *transcendence degree* of an algebraically closed field K is its rank in the sense of Definition 15.2.4.

Theorem 15.2.10. *For each $p \in \{0, 2, 3, 5, 7, \ldots\}$ and each cardinal κ, there is a unique algebraically closed field of characteristic p and transcendence degree κ.*

Proof. The theory ACF_p is complete and strongly minimal. Algebraically closed sets are infinite by Theorem 14.5.3(2) and Theorem 10.2.3. Therefore Theorem 15.2.8 applies, giving a unique model $M \models \mathrm{ACF}_p$ with transcendence degree κ. □

15.3 Uniform finiteness

The notation $\exists^\infty x\ P(x)$ means that there are infinitely many x such that $P(x)$ holds. In general, $\exists^\infty x$ cannot be expressed in first-order logic.

Definition 15.3.1. A structure M *eliminates* $\exists^\infty$ if for any first-order formula $\varphi(x, \bar{y})$, there is a formula $\psi(\bar{y})$ such that for any $\bar{b}$,

$$M \models \exists^\infty x\ \varphi(x, \bar{b}) \iff M \models \psi(\bar{b}).$$

Here, the left-hand side really means $\exists^\infty a \in M : M \models \varphi(a, \bar{b})$, or equivalently, $\varphi(M, \bar{b})$ is infinite.

Recall from Section 4.3 that the notation $\exists^{\geq n} x\ P(x)$ means that there are at least n values of x such that $P(x)$ holds. Unlike $\exists^\infty$, this can be expressed in first-order logic.

Definition 15.3.2. A structure M has *uniform finiteness* if for any formula $\varphi(x, \bar{y})$, there is a number n_φ such that for any $\bar{b}$ in M,

$$|\varphi(M, \bar{b})| < \infty \iff |\varphi(M, \bar{b})| < n_\varphi.$$

Theorem 15.3.3. *Let M be a structure.*

1. *If M has uniform finiteness, then M eliminates $\exists^\infty$.*

2. *If M has uniform finiteness and $N \equiv M$, then N has uniform finiteness.*

3. *M has uniform finiteness if and only if every $N \equiv M$ eliminates $\exists^\infty$.*

Proof.

1. Uniform finiteness says $(\exists^\infty x)\varphi(x, \bar{y}) \iff (\exists^{\geq n_\varphi} x)\varphi(x, \bar{y})$.

2. Given φ, if n_φ works for M then it works for N. Otherwise there is $\bar{b}$ in N such that $n_\varphi \leq |\varphi(N, \bar{b})| < \infty$. Let $k = |\varphi(N, \bar{b})|$. Then

$$N \models \exists \bar{y}\ \exists^{=k} x\ \varphi(x, \bar{y}).$$

   This says that there is a $\bar{b}$ in N such that $\varphi(N, \bar{b})$ has size exactly k. As $M \equiv N$, the same holds in M, so there is a $\bar{b}$ in M such that $\varphi(M, \bar{b})$ has size exactly $k \geq n_\varphi$, contradicting the choice of n_φ.

3. If M has uniform finiteness and $N \equiv M$, then N has uniform finiteness and therefore eliminates $\exists^\infty$. Conversely, suppose every $N \equiv M$ eliminates $\exists^\infty$. Fix an $\aleph_1$-saturated elementary extension $N \succeq M$. By

part (2), it suffices to show that N has uniform finiteness. Fix $\varphi(x, \bar{y})$. Let

$$D_\infty = \{\bar{b} : |\varphi(N, \bar{b})| < \infty\}$$
$$D_k = \{\bar{b} : |\varphi(N, \bar{b})| < k\} \text{ for } k < \omega.$$

Then D_∞ is definable by elimination of $\exists^\infty$, and D_k is definable easily. Moreover,

$$D_0 \subseteq D_1 \subseteq \cdots \subseteq D_\infty$$

and $D_\infty = \bigcup_{i=0}^\infty D_i$. By $\aleph_0$-compactness (Theorem 12.2.7(3)), $D_\infty = D_{n_\varphi}$ for some n_φ, which means

$$|\varphi(N, \bar{b})| < \infty \implies |\varphi(N, \bar{b})| < n_\varphi. \qquad \square$$

Theorem 15.3.4. *If M is strongly minimal, then M has uniform finiteness.*

Proof. Without loss of generality, M is $\aleph_1$-saturated. Let $\varphi(x; y_1, \ldots, y_n)$ be a formula. Let D_k be the set of $\bar{b}$ such that $\varphi(M, \bar{b})$ or $M \setminus \varphi(M, \bar{b})$ has size less than k. Then $D_0 \subseteq D_1 \subseteq \cdots \subseteq M^n$. Strong minimality means that $M^n = \bigcup_{k=0}^\infty D_k$. By $\aleph_0$-compactness (Theorem 12.2.7(3)), $D_k = M^n$ for some k. This means that for any $\bar{b}$,

$$|\varphi(M, \bar{b})| < k \text{ or } |M \setminus \varphi(M, \bar{b})| < k.$$

Therefore,

$$|\varphi(M, \bar{b})| < k \text{ or } |\varphi(M, \bar{b})| = \infty,$$

which is uniform finiteness. $\square$

15.4 Pregeometric theories

Definition 15.4.1. A theory T is *pregeometric* if $\mathrm{acl}(-)$ satisfies exchange in models of T.

Definition 15.4.2. A complete theory T is *geometric* if it is pregeometric and has uniform finiteness.

For example, strongly minimal theories are geometric.

Let $\mathbb{M}$ be a monster model of a pregeometric theory. Let $\dim(\bar{a}/B)$ denote the rank of $\bar{a}$ over B with respect to $\mathrm{acl}(-)$, i.e., the number of values $i \in \{1, \ldots, n\}$ such that $a_i \notin \mathrm{acl}(Ba_1a_2 \cdots a_{i-1})$.

Theorem 15.4.3. *1.* $\dim(\bar{a}/B) = \dim(\sigma(\bar{a})/\sigma(B))$ *for any* $\sigma \in \mathrm{Aut}(\mathbb{M})$.

2. If $\bar{a} \equiv_C \bar{b}$, *then* $\dim(\bar{a}/C) = \dim(\bar{b}/C)$.

Proof. 1. The definition of $\dim(-/-)$ is clearly automorphism-invariant.

2. Take $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ with $\sigma(\bar{a}) = \bar{b}$ and apply part (1). □

In particular, $\mathrm{tp}(\bar{a}/C)$ determines $\dim(\bar{a}/C)$. If $p \in S_n(C)$, we let $\dim(p)$ denote $\dim(\bar{a}/C)$ for any $\bar{a}$ realizing p.

Theorem 15.4.4 (Extension)**.** *Let* $B \subseteq C$ *be small sets in* $\mathbb{M}$.

1. If $p \in S_n(B)$, *then there is an extension* $q \in S_n(C)$ *with* $q \supseteq p$.

2. If $\bar{a} \in \mathbb{M}^n$, *then there is* $\bar{a}' \equiv_B \bar{a}$ *with* $\dim(\bar{a}'/C) = \dim(\bar{a}/B)$.

Proof. Note that (1) $\iff$ (2), by taking $p = \mathrm{tp}(\bar{a}/B)$ and $q = \mathrm{tp}(\bar{a}'/B)$. We prove (2) by induction on n. First suppose $n = 1$. If $a \notin \mathrm{acl}(B)$, then the set X of realizations of $\mathrm{tp}(a/B)$ is large by Theorem 14.3.5, so there is $a' \in X \setminus \mathrm{acl}(C)$. Then $a' \equiv_B a$ and $\dim(a'/C) = 1 = \dim(a/B)$. On the other hand, if $a \in \mathrm{acl}(B) \subseteq \mathrm{acl}(C)$, then $\dim(a/C) = 0 = \dim(a/B)$, and we can take $a' = a$.

Next suppose $n > 1$. Write $\bar{a}$ as $(\bar{a}_1, \bar{a}_2)$, where $\bar{a}_1$ and $\bar{a}_2$ are tuples of shorter length. By induction there is $\bar{a}_1' \equiv_B \bar{a}_1$ with $\dim(\bar{a}_1'/C) = \dim(\bar{a}_1/B)$. Moving $\bar{a}$ by an automorphism over B sending $\bar{a}_1$ to $\bar{a}_1'$, we may assume $\bar{a}_1 = \bar{a}_1'$. Then

$$\dim(\bar{a}_1/C) = \dim(\bar{a}_1/B). \tag{$*$}$$

By induction applied to $\bar{a}_2$ and the inclusion $B\bar{a}_1 \subseteq C\bar{a}_1$, there is $\bar{a}_2' \equiv_{B\bar{a}_1} \bar{a}_2$ with $\dim(\bar{a}_2'/C\bar{a}_1) = \dim(\bar{a}_2/B\bar{a}_1)$. Moving $\bar{a}$ by an automorphism over $B\bar{a}_1$ sending $\bar{a}_2$ to $\bar{a}_2'$, we may assume $\bar{a}_2' = \bar{a}_2$. Then

$$\dim(\bar{a}_2/C\bar{a}_1) = \dim(\bar{a}_2/B\bar{a}_1). \tag{$\dagger$}$$

Adding equations ($*$) and (†) and using additivity,

$$\dim(\bar{a}_1\bar{a}_2/C) = \dim(\bar{a}_1\bar{a}_2/B).$$ □

15.5 Dimension theory in the monster

Work in a monster model $\mathbb{M}$ of a pregeometric theory.

Definition 15.5.1. If B is small and $X \subseteq \mathbb{M}^n$ is B-definable, then $\dim_B(X) = \max_{\bar{a} \in X} \dim(\bar{a}/B)$. If X is empty, we set $\dim(X) = -\infty$.

Theorem 15.5.2. *If X is B-definable and C-definable, then* $\dim_B(X) = \dim_C(X)$.

Proof. First suppose $B \subseteq C$. Note that $\dim(\bar{a}/B) \geq \dim(\bar{a}/C)$ for any $\bar{b}$ (Theorem 14.6.15), so

$$\dim_B(X) = \max_{\bar{a} \in X} \dim(\bar{a}/B) \geq \max_{\bar{a} \in X} \dim(\bar{a}/C) = \dim_C(X).$$

Conversely, take $\bar{a} \in X$ with $\dim(\bar{a}/B) = \dim_B(X)$. By Theorem 15.4.4, there is $\bar{a}' \equiv_B \bar{a}$ with $\dim(\bar{a}'/C) = \dim(\bar{a}/B)$. Then $\bar{a} \in X \implies \bar{a}' \in X$ because X is B-definable (Remark 9.1.3), and so

$$\dim_C(X) \geq \dim(\bar{a}'/C) = \dim(\bar{a}/B) = \dim_B(X).$$

This completes the case where $B \subseteq C$. The general case then follows:

$$\dim_B(X) = \dim_{B \cup C}(X) = \dim_C(X). \qquad \square$$

By Theorem 15.5.2, $\dim_B(X)$ doesn't depend on B, so we just write $\dim(X)$. The following two facts are a restatement of the definition:

- If X is B-definable and $\bar{a} \in X$, then $\dim(\bar{a}/B) \leq \dim(X)$.
- If X is B-definable and non-empty, then there is $\bar{a} \in X$ with $\dim(\bar{a}/B) = \dim(X)$.

We will use these repeatedly in what follows.

Theorem 15.5.3 (Basic properties of dimension)**.** *Let X, Y be definable sets.*

1. $\dim(X) \leq 0$ *if and only if* $|X| < \infty$.
2. *If* $X \subseteq Y$, *then* $\dim(X) \leq \dim(Y)$.
3. *If* $X, Y \subseteq \mathbb{M}^n$, *then* $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$.

4. *$\dim(X \times Y) = \dim(X) + \dim(Y)$.*

5. *$\dim(\mathbb{M}^n) = n$, assuming $\mathbb{M}$ is infinite.*

6. *If $f : X \to Y$ is a definable surjection, then $\dim(X) \geq \dim(Y)$.*

7. *If $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.*

8. *If $f : X \to Y$ is a definable injection, then $\dim(X) \leq \dim(Y)$.*

Proof. Take a small set C over which all the sets and functions are defined.

1. If X is finite, then every $\bar{a} \in X$ is algebraic over C, and so $\dim(\bar{a}/C) \leq 0$ (Theorem 14.6.10). Thus $\dim(X) \leq 0$.

   Conversely, if X is infinite, then X is large (Corollary 12.2.8), so there is $\bar{a} \in X$ with $\bar{a} \notin \operatorname{acl}(C)$. Then $\dim(X) \geq \dim(\bar{a}/C) > 0$.

2. Clear.

3. Clear.

4. If $(\bar{a}, \bar{b}) \in X \times Y$, then

   $$\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) + \dim(\bar{b}/C\bar{a}) \leq \dim(X) + \dim(Y)$$

   because $\bar{a}$ is in the C-definable set X and $\bar{b}$ is in the $C\bar{a}$-definable set Y. As this holds for all $(\bar{a}, \bar{b}) \in X \times Y$, we have $\dim(X \times Y) \leq \dim(X) + \dim(Y)$.

   Conversely, take $\bar{a} \in X$ with $\dim(\bar{a}/C) = \dim(X)$. The set Y is $C\bar{a}$-definable, so there is $\bar{b} \in Y$ with $\dim(\bar{b}/C\bar{a}) = \dim(Y)$. Then

   $$\dim(X \times Y) \geq \dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) + \dim(\bar{b}/C\bar{a}) = \dim(X) + \dim(Y).$$

5. By part (4), it suffices to show $\dim(\mathbb{M}^1) = 1$. If $a \in \mathbb{M}^1$, then $\dim(a/C) \leq 1$, as a has length 1. Thus $\dim(\mathbb{M}^1) \leq 1$. On the other hand, $\dim(\mathbb{M}) \geq 1$ by part (1).

6. Take $\bar{b} \in Y$ with $\dim(\bar{b}/C) = \dim(Y)$. Since f is surjective, there is $\bar{a} \in X$ with $f(\bar{a}) = \bar{b}$. Then $\bar{b} \in \operatorname{dcl}(C\bar{a}) \subseteq \operatorname{acl}(C\bar{a})$, so by Theorem 14.6.14,

   $$\dim(X) \geq \dim(\bar{a}/C) \geq \dim(\bar{b}/C) = \dim(Y).$$

7. Apply part (6) to f and f^{-1}.

8. By parts (2) and (7), $\dim(X) = \dim(\mathrm{im}(f)) \leq \dim(Y)$. □

Theorem 15.5.4 (Fiber dimension theorem)**.** *Let $f : X \to Y$ be a definable function. For every $b \in Y$, let $X_b = f^{-1}(b) = \{x \in X : f(x) = b\}$. If $\dim(X_b) = k$ for all $b \in Y$, then $\dim(X) = k + \dim(Y)$.*

Proof. Take a small set C defining f, X, and Y. Note that $\bar{a}$ and $(\bar{a}, f(\bar{a}))$ are interalgebraic over C for any $\bar{a} \in X$, so

$$\dim(\bar{a}/C) = \dim(\bar{a}, f(\bar{a})/C) = \dim(\bar{a}/Cf(\bar{a})) + \dim(f(\bar{a})/C)$$

by Theorem 14.6.14 and additivity (Theorem 14.6.9). If $\bar{b} = f(\bar{a})$, then

$$\dim(\bar{a}/C) = \dim(\bar{a}/C\bar{b}) + \dim(\bar{b}/C) \leq \dim(X_b) + \dim(Y) = k + \dim(Y).$$

because $\bar{a}$ is in the $C\bar{b}$-definable set X_b, and $\bar{b}$ is in the C-definable set Y. As this holds for any $\bar{a} \in X$, we see

$$\dim(X) \leq k + \dim(Y).$$

For the converse, take $\bar{b} \in Y$ with $\dim(\bar{b}/C) = \dim(Y)$. The set X_b is $C\bar{b}$-definable, so there is $\bar{a} \in X_{\bar{b}}$ with $\dim(\bar{a}/C\bar{b}) = \dim(X_b) = k$. Then $\bar{b} = f(\bar{a})$, so

$$\dim(X) \geq \dim(\bar{a}/C) = \dim(\bar{a}/C\bar{b}) + \dim(\bar{b}/C) = k + \dim(Y). \qquad \square$$

Proof. Let $Z_k = f^{-1}(Y_k)$. Then $\dim(Z_k) = k + \dim(Y_k)$, and $X = \bigcup_k Z_k$. □

Now suppose the the theory is geometric (Definition 15.4.2), meaning that uniform finiteness holds.

Lemma 15.5.5. *Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}(\mathbb{M})$-formula and let $\psi(x_1, \ldots, x_{n-1})$ be an $\mathcal{L}(\mathbb{M})$-formula equivalent to $\exists^\infty x_n\ \varphi$. Then $\varphi(\mathbb{M}^n)$ has dimension n if and only if $\psi(\mathbb{M}^{n-1})$ has dimension $n - 1$.*

Proof. Take a finite set B such that φ and ψ are $\mathcal{L}(B)$-formulas. If $\psi(\mathbb{M}^{n-1})$ has dimension $n-1$, take $\bar{a} \in \psi(\mathbb{M}^{n-1})$ with $\dim(\bar{a}/B) = n-1$. Then $\varphi(\bar{a}, \mathbb{M})$ is infinite, hence large. Take $a_n \in \varphi(\bar{a}, \mathbb{M}) \setminus \mathrm{acl}(B\bar{a})$. Then $(\bar{a}, a_n) \in \varphi(\mathbb{M}^n)$ and

$$\dim(\bar{a}, a_n/B) = \dim(\bar{a}/B) + \dim(a_n/B\bar{a}) = (n-1) + 1 = n,$$

and $\varphi(\mathbb{M}^n)$ has dimension n.

Conversely, suppose $\varphi(\mathbb{M}^n)$ has dimension n. Take $(\bar{a}, a_n) \in \varphi(\mathbb{M}^n)$ with $\dim(\bar{a}, a_n/B) = n$. Then $a_n \notin \operatorname{acl}(B\bar{a})$ and $\dim(\bar{a}/B) = n-1$. As $\varphi(\bar{a}, \mathbb{M})$ is $B\bar{a}$-definable and contains $a_n \notin \operatorname{acl}(B\bar{a})$, it must be infinite, meaning that $\bar{a} \in \psi(\mathbb{M}^{n-1})$, and then $\dim(\psi(\mathbb{M}^{n-1})) \geq \dim(\bar{a}/B) = n-1$. □

Lemma 15.5.6. *Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be an $\mathcal{L}$-formula. The set*

$$\{\bar{b} \in \mathbb{M}^m : \dim(\varphi(\mathbb{M}^n, \bar{b})) = n\}$$

is definable.

Proof. The case $n = 0$ is easy. If $n > 0$, let $\psi(x_1, \ldots, x_{n-1}; \bar{y})$ be the $\mathcal{L}$-formula equivalent to $\exists^\infty x_n\, \varphi$. Then for any $\bar{b} \in \mathbb{M}^m$,

$$\dim(\varphi(\mathbb{M}^n, \bar{b})) = n \iff \dim(\psi(\mathbb{M}^{n-1}, \bar{b})) = n-1,$$

by Lemma 15.5.5, and the right hand side is definable by induction on n. □

Lemma 15.5.7. *Let $D \subseteq \mathbb{M}^n$ be definable and let k be in $\{0, 1, \ldots, n\}$. Then $\dim(D) \geq k$ if and only if there is a coordinate projection $\pi : \mathbb{M}^n \to \mathbb{M}^k$ such that $\pi(D) \subseteq \mathbb{M}^k$ has dimension k.*

Proof. If $\dim(\pi(D)) = k$, then $\dim(D) \geq k$ by Theorem 15.5.3(6). Conversely, suppose $\dim(D) \geq k$. Take a small set B over which D is defined, and take $\bar{a} \in D$ with $\dim(\bar{a}/B) = \dim(D) \geq k$. There are at least k values of i such that $a_i \notin \operatorname{acl}(Ba_1, \ldots, a_{i-1})$. Let $i_1 < i_2 < \cdots < i_k$ be some such values. Then

$$a_{i_j} \notin \operatorname{acl}(Ba_{i_1}a_{i_2} \cdots a_{i_{j-1}})$$

for each j, and so

$$\dim(a_{i_1}a_{i_2} \cdots a_{i_k}/B) = k.$$

Let $\pi : \mathbb{M}^n \to \mathbb{M}^k$ be the coordinate projection $\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k})$. Then $\pi(D)$ is B-definable, $\pi(\bar{a}) \in \pi(D)$, and $\dim(\pi(\bar{a})/B) = k$. We conclude that $\dim(\pi(D)) \geq k$. □

Theorem 15.5.8 (Definability of dimension)**.** *Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula. For each $k \leq n$, the set*

$$\{\bar{b} \in \mathbb{M}^m : \dim(\varphi(\mathbb{M}^n; \bar{b})) = k\}$$

is definable.

Proof. Let Π_k^n be the finite set of coordinate projections $\mathbb{M}^n \to \mathbb{M}^k$. For each $\pi \in \Pi_k^n$, let $\theta_\pi(z_1, \ldots, z_k, \bar{y})$ be the formula

$$\exists \bar{x}\ \varphi(\bar{x}; \bar{y}) \wedge (\pi(\bar{x}) = \bar{z}).$$

Then $\theta_\pi(\mathbb{M}^k, \bar{b})$ is the image of $\varphi(\mathbb{M}^n, \bar{b})$ under π. By Lemma 15.5.7,

$$D_k := \{\bar{b} \in \mathbb{M}^m : \dim(\varphi(\mathbb{M}^n; \bar{b})) \geq k\} = \bigcup_{\pi \in \Pi_k^n} \{\bar{b} \in \mathbb{M}^m : \dim(\theta_\pi(\mathbb{M}^k; \bar{b})) = k\},$$

and the sets on the right-hand side are definable by Lemma 15.5.6. Thus each D_k is definable. The set we want is $D_k \setminus D_{k-1}$. □

Theorem 15.5.9. *Let $f : X \to Y$ be a definable function. For each $b \in Y$, let $X_b = f^{-1}(b)$. For each $k \leq \dim(X)$, let $Y_k = \{b \in Y : \dim(X_b) = k\}$. Then each set Y_k is definable, and*

$$\dim(X) = \max_k (k + \dim(Y_k)).$$

Proof. Theorems 15.5.8 and 15.5.4. □

15.6 Dimension theory in small models

If D is a definable set in a model M, and N is an elementary extension of M, then $D(N)$ denotes $\varphi(N)$, where $\varphi(\bar{x})$ is the $\mathcal{L}(M)$-formula defining D in M. The choice of φ doesn't matter, because

$$\begin{aligned} \varphi(M) = \psi(M) &\iff M \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) \\ \iff N \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) &\iff \varphi(N) = \psi(N). \end{aligned}$$

Note that $D(M) = D$.

Lemma 15.6.1. *Let $M \preceq N$ be monster models of a pregeometric theory.*

1. *If $\bar{a} \in M^n$ and $B \subseteq M$ is small, then* $\dim^M(\bar{a}/B) = \dim^N(\bar{a}/B)$.
2. *If D is M-definable, then* $\dim(D(M)) = \dim(D(N))$.

Proof. 1. By Theorem 14.3.3, $\operatorname{acl}(Ba_1, \ldots, a_{i-1})$ is the same whether calculated in M or N.

2. Take a small set $B \subseteq M$ defining D. Take $\bar{a} \in D(M)$ with $\dim(\bar{a}/B) = \dim(D(M))$. Then $\bar{a} \in D(N)$ and $\dim(\bar{a}/B) = \dim(D(M))$, so
$$\dim(D(M)) \leq \dim(D(N)).$$
Conversely, take $\bar{a} \in D(N)$ with $\dim(\bar{a}/B) = \dim(D(N))$. Then $\mathrm{tp}(\bar{a}/B)$ is realized in M by some $\bar{c}$. The formula defining $D(N)$ is in $\mathrm{tp}(\bar{a}/B)$, so $\bar{c} \in D(N) \cap M^n = D(M)$. Also, $\dim(\bar{c}/B) = \dim(\bar{a}/B)$ by automorphism invariance. The fact that $\bar{c} \in D(M)$ then shows
$$\dim(D(M)) \geq \dim(\bar{c}/B) = \dim(\bar{a}/B) = \dim(D(N)). \qquad \square$$

Definition 15.6.2. Let M be a small model of a pregeometric theory. If $D \subseteq M^n$ is definable, then $\dim(D)$ is defined to be $\dim(D(\mathbb{M}))$ where $\mathbb{M}$ is a monster model extending M.

Theorem 15.6.3. *The choice of* $\mathbb{M}$ *doesn't matter in Definition 15.6.2.*

Proof. Let $\mathbb{M}_1, \mathbb{M}_2$ be two monster models extending M. Then $\mathbb{M}_1$ and $\mathbb{M}_2$ are elementarily equivalent as $\mathcal{L}(M)$-structures. By elementary amalgamation there is a third monster model $\mathbb{M}_3$ extending $\mathbb{M}_1$ and $\mathbb{M}_2$, up to isomorphism. Then
$$\dim(D(\mathbb{M}_1)) = \dim(D(\mathbb{M}_3)) = \dim(D(\mathbb{M}_2))$$
by two applications of Lemma 15.6.1. $\square$

Corollary 15.6.4. *Dimension is invariant in elementary extensions: if* $M \preceq N$ *and* D *is a definable set in* M*, then* $\dim(D(M)) = \dim(D(N))$.

Proof. Take a monster model $\mathbb{M}$ extending N (and M). Then $\dim(D(M)) = \dim(D(\mathbb{M})) = \dim(D(N))$ by two applications of Definition 15.6.2. $\square$

Theorem 15.6.5. *Let* M *be a model of a pregeometric theory* T*. Define* $\dim(-)$ *as in Definition 15.6.2.*

1. *The basic properties of dimension (Theorem 15.5.3) hold in* M*.*

2. *If* T *is geometric, then the fiber dimension theorem (Theorem 15.5.4) and definability of dimension (Theorem 15.5.8) hold in* M*.*

Proof. 1. For example, we show $\dim(D_1 \times D_2) = \dim(D_1) + \dim(D_2)$. If $\varphi(\bar{x})$ defines D_1 and $\psi(\bar{y})$ defines D_2, then the formula $\theta(\bar{x}, \bar{y}) :\equiv \varphi(\bar{x}) \wedge \psi(\bar{y})$ defines $D_1 \times D_2$. Therefore

$$(D_1 \times D_2)(\mathbb{M}) = \theta(\mathbb{M}) = D_1(\mathbb{M}) \times D_2(\mathbb{M}).$$

Thus

$$\begin{aligned} \dim(D_1 \times D_2) &= \dim((D_1 \times D_2)(\mathbb{M})) \\ = \dim(D_1(\mathbb{M}) \times D_2(\mathbb{M})) &= \dim(D_1(\mathbb{M})) + \dim(D_2(\mathbb{M})) \\ &= \dim(D_1) + \dim(D_2). \end{aligned}$$

2. Suppose we have the configuration $f : X \to Y$ of Theorem 15.5.4, where $X_b = f^{-1}(b)$ has dimension k for every $b \in Y$. In order to reduce from M to the known case of $\mathbb{M}$, we need to show that the property

   $$\dim(X_b) = k \text{ for every } b \in Y$$

   transfers from M to $\mathbb{M}$. Let $f_{\mathbb{M}} : X(\mathbb{M}) \to Y(\mathbb{M})$ be the function defined by the same formula as f, and for $b \in Y(\mathbb{M})$ let $X_b(\mathbb{M})$ denote the fiber of $f_{\mathbb{M}}^{-1}(b)$. Note that when $b \in Y = Y(M)$, the set $X_b(\mathbb{M})$ really is the extension of X_b. By the definability of dimension (Theorem 15.5.8), the set

   $$Z = \{b \in Y(\mathbb{M}) : \dim(X_b(\mathbb{M})) \neq k\}$$

   is definable. It is $\mathrm{Aut}(\mathbb{M}/M)$-invariant, hence M-definable (Theorem 12.3.8). If $b \in Y(M)$, then $\dim(X_b(\mathbb{M})) = \dim(X_b) = k$, so $b \notin Z$. Therefore there are no M-points in Z. By the Tarski-Vaught criterion, $Z = \varnothing$. Thus $Z = \varnothing$, and $\dim(X_b(\mathbb{M})) = k$ for every $b \in Y(\mathbb{M})$. Then we can apply Theorem 15.5.4 over $\mathbb{M}$ to get what we want.

   For Theorem 15.5.8, fix a formula $\varphi(\bar{x}, \bar{y})$. The set

   $$Z_k = \{\bar{b} \in \mathbb{M} : \dim(\varphi(\mathbb{M}; \bar{b})) = k\}$$

   is definable by Theorem 15.5.8 applied to $\mathbb{M}$, and clearly $\mathrm{Aut}(\mathbb{M}/\varnothing)$-invariant. By Theorem 12.3.8, it is 0-definable, defined by some formula $\psi_k(\bar{y})$. Then for any $\bar{b}$ in M,

   $$\begin{aligned} \dim(\varphi(M, \bar{b})) = k &\iff \dim(\varphi(\mathbb{M}, \bar{b})) = k \\ \iff \mathbb{M} \models \psi_k(\bar{b}) &\iff M \models \psi_k(\bar{b}). \end{aligned}$$

The first equivalence holds by Definition 15.6.2. The second holds by choice of ψ_k. The third holds as $M \preceq \mathbb{M}$.

Therefore $\{\bar{b} \in M^m : \dim(\varphi(M^n; \bar{b})) = k\}$ is the definable set $\psi_k(M^m)$.

$\square$