

Introduction To Model Theory

Will Johnson

May 3, 2022

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1 Back-and-forth Equivalence I

Convention: Relations and functions are sets of pairs (x, y)

Definition 1.1. A **binary relation** is a pair (E, R) where E is a set and $R \subseteq E^2$. We call E the **universe** of the relation. For $a, b \in E$, write aEb if $(a, b) \in R$

We abbreviate (E, R) as R or E , if E or R is clear

Example 1.1. $(\mathbb{R}, <)$, $(\mathbb{R}, =)$, (\mathbb{R}, \geq) , $(\mathbb{Z}, <)$

Definition 1.2. A binary relation R is said to be

- **reflexive** if aRa ($\forall a \in E$)
- **symmetric** if $aRb \Rightarrow bRa$ ($\forall a, b \in E$)
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$ ($\forall a, b, c \in E$)
- **antisymmetric** if $aRb \wedge bRa \Rightarrow a = b$ ($\forall a, b \in E$)

- **total** if $aRb \vee bRa$ ($\forall a, b \in E$)
- an **equivalence relation** if it's reflexive, symmetric and transitive
- a **partial order** if it's reflexive, antisymmetric and transitive
- a **linear order** if it's a total partial order

Example 1.2. $=$ is an equivalence relation

\subseteq is a partial order

\leq is a linear order

Definition 1.3. An **isomorphism** from (E, R) to (E', R') is a bijection $f : E \rightarrow E'$ s.t. for any $a, b \in E$, $aRb \Leftrightarrow f(a)R'f(b)$. Two binary relations (E, R) and (E', R') are **isomorphic** (\cong) if there is an isomorphism between them

Example 1.3. $f : (\mathbb{Z}, <) \rightarrow (2\mathbb{Z}, >)$ and $f(x) = -2x$ is an isomorphism.
 $x < y \Leftrightarrow -2x > -2y$

\cong is an equivalence relation

Definition 1.4. A **local isomorphism** from R to R' is an isomorphism from a finite restriction of R to a finite restriction of R' . The set of local isomorphisms from R to R' is denoted $S_0(R, R')$. For $f \in S_0(R, R')$, $\text{dom}(f)$ and $\text{im}(f)$ denote the domain and range of f

Example 1.4. $(\mathbb{Z}, <)$ is a restriction of $(\mathbb{R}, <)$

Example 1.5. Suppose $R = R' = (\mathbb{Z}, <)$, there is $f \in S_0(R, R')$ given by $\text{dom}(f) = \{1, 2, 3\}$ and $\text{im}(f) = \{10, 20, 30\}$ and $f(1) = 10, f(2) = 20, f(3) = 30$

Definition 1.5. Let f, g be local isomorphisms from R to R' . Then f is a **restriction** of g if $f \subseteq g$ and f is an **extension** of g if $f \supseteq g$.

Example 1.6. $g : \{0, 1, 2, 3\} \rightarrow \{5, 10, 20, 30\}$, g extends f in the previous example

Definition 1.6. Let R, R' be binary relations with universe E, E' . A **Karpian family** for (R, R') is a set $K \subseteq S_0(R, R')$ satisfying the following two conditions for any $f \in K$

1. (**forth**) if $a \in E$ then there is $g \in K$ with $g \supseteq f$ and $a \in \text{dom}(g)$
2. (**back**) if $b \in E'$ then there is $g \in K$ with $g \supseteq f$ and $b \in \text{im}(g)$

R and R' are ∞ -**equivalent**, write $R \sim_\infty R'$, if there is a non-empty Karpian family

Proposition 1.7. *If $f : (E, R) \rightarrow (E', R')$ an isomorphism and $K = \{g \subseteq f : g \text{ is finite}\}$, then K is Karpian and $R \sim_\infty R'$*

Proof. Suppose $g \in K$

- (forth) Suppose $a \in E$, take $b = f(a)$ and let $h = g \cup \{(a, b)\}$. Then $h \subseteq f$, so $h \in K$, $h \supseteq g$, $a \in \text{dom}(h)$
- (back) similarly

□

Proposition 1.8. *If (E, R) and (E', R') are countable and $R \sim_\infty R'$, then $R \cong R'$*

Proof. Let $K \subseteq S_0(R, R')$ be Karpian, $K \neq \emptyset$, $E = \{e_1, e_2, e_3, \dots\}$, $E' = \{e'_1, e'_2, e'_3, \dots\}$

Recursively build $f_1 \subseteq f_2 \subseteq \dots$, $f_i \in K$

Let f_1 be anything in K as K is non-empty.

f_{2i} some extension of f_{2i-1} with $e_i \in \text{dom}(f_{2i})$

f_{2i+1} some extension of f_{2i} with $e'_i \in \text{im}(f_{2i+1})$

Now let $g = \bigcup_{i=1}^\infty f_i$, then g is an isomorphism

□

Definition 1.9. A **dense linear order without endpoints** (DLO) is a linear order (C, \leq) satisfying

1. $C \neq \emptyset$
2. $\forall x, y \in C, x < y \Rightarrow \exists z \in C, x < z < y$
3. $\forall x \in C, \exists y, z \in C, y < x < z$

Example 1.7. (\mathbb{Q}, \leq) , (\mathbb{R}, \leq)

non-example: (\mathbb{Z}, \leq) , $([0, 1], \leq)$

Proposition 1.10. *Let (C, \leq) and (C', \leq) be DLO's. Then $S_0(C, C')$ is Karpian. So $C \sim_\infty C'$*

Proof. Let $f \in S_0(C, C')$, $\text{dom}(f) = \{a_1, \dots, a_n\}$, $a_1 < \dots < a_n$ and $\text{im}(f) = \{b_1, \dots, b_n\}$, $b_1 < \dots < b_n$. Since f is a local isomorphism, $f(a_i) = b_i$

- (forth) Suppose $a \in C$. We want $b \in C'$ s.t. $f \cup \{(a, b)\} \in S_0(C, C')$.

- if $a_i < a < a_{i+1}$. We take $b \in C'$ s.t. $b_i < b < b_{i+1}$ since dense
 - if $a < a_1$. We take $b \in C'$ s.t. $b < b_1$ since no endpoints
 - if $a > a_n$, take $b \in C'$ s.t. $b > b_n$
 - if $a = a_i$, take $b = b_i$
- (back) similar

□

Proposition 1.11. *If (C, \leq) and (C', \leq) are countable DLOs, then $C \sim_\infty C'$, so $C \cong C'$*

Hence

$$\begin{aligned}
 (\mathbb{Q}, \leq) &\cong (\mathbb{Q} \setminus \{0\}, \leq) \\
 &\cong (\mathbb{Q} \cup \{\sqrt{2}\}, \leq) \\
 &\cong (\mathbb{Q} \cap (0, 1), \leq)
 \end{aligned}$$

Definition 1.12. Let R, R' be binary relations with universe E, E'

- A **0-isomorphism** from R to R' is a local isomorphism from R to R'
- For $p > 0$, a **p -isomorphism** from R to R' is a local isomorphism f from R to R' satisfying the following two conditions
 1. (**forth**) For any $a \in E$, there is a $(p-1)$ -isomorphism $g \supseteq f$ with $a \in \text{dom}(g)$
 2. (**back**) For any $b \in E'$, there is a $(p-1)$ -isomorphism $g \supseteq f$ with $b \in \text{im}(g)$
- An **ω -isomorphism** from R to R' is a local isomorphism f from R to R' s.t. f is a p -isomorphism for all $p < \omega$

The set of p -isomorphisms from R to R' is denoted $S_p(R, R')$

Example 1.8. Suppose $R = R' = (\mathbb{Z}, <)$, $f : \{2, 4\} \rightarrow \{1, 2\}$ is a local isomorphism with $f(2) = 1$ and $f(4) = 2$. Then $f \notin S_1(\mathbb{Z}, \mathbb{Z})$ (forth) fails. For $a = 3$, there is no b s.t. $1 < b < 2$

$g : \{2, 4\} \rightarrow \{1, 5\}$ is a 1-isomorphism but not a 2-isomorphism

Proposition 1.13. *If $f \in S_p(R, R')$ and $g \subseteq f$, then $g \in S_p(R, R')$*

Proof. if $p = 0$ easy

if $p > 0$ (forward), $\forall a \in E, \exists h \in S_{p-1}(R, R')$ has $a \in \text{dom}(h)$ and $h \supseteq f \supseteq g$ \square

Proposition 1.14. $S_p(R, R') \neq \emptyset$ iff $\emptyset \in S_p(R, R')$

Proof. \Leftarrow immediate

\Rightarrow . Suppose $f \in S_p(R, R')$. Then $\emptyset \subseteq f$. Hence $\emptyset \in S_p(R, R')$. \square

Definition 1.15. R and R' are p -**equivalent**, written $R \sim_p R'$, if there is a p -isomorphism from $R \rightarrow R'$

R and R' are ω -**equivalent** or **elementarily equivalent**, written $R \sim_\omega R'$ or $R \equiv R'$, if there is an ω -isomorphism from R to R'

Note: $R \sim_\omega R'$ iff $S_\omega(R, R') \neq \emptyset$ iff $\emptyset \in S_\omega(R, R')$ iff $\forall p \emptyset \in S_p(R, R')$ iff $\forall p R \sim_p R'$

Definition 1.16. Let R, R' be binary relations with universe E, E' . The Ehrenfeucht-Fraïssé game of length n , denoted $\text{EF}_n(R, R')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are n rounds
- In the i th round, the Spoiler chooses either an $a_i \in E$ or a $b_i \in E'$
- The Duplicator responds with a $b_i \in E'$ or an $a_i \in E$ respectively
- At the ends of the game, the Duplicator wins

$$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

is a local isomorphism from R to R'

- Otherwise, the Spoiler wins

Example 1.9. For $\text{EF}_3(\mathbb{Q}, \mathbb{R})$

\mathbb{Q}	\mathbb{R}
S: $a_1 = 7$	D: $b_1 = 7$
D: $a_2 = 1.4$	S: $b_2 = \sqrt{2}$
D: $a_3 = -10$	S: $b_3 = 1.41$

So D wins

Example 1.10. $EF_3(\mathbb{R}, \mathbb{Z})$

\mathbb{R}	\mathbb{Z}
D: $a_1 = 1$	S: $b_1 = 1$
D: $a_2 = 1.1$	S: $b_2 = 2$
S: $a_3 = 1.01$	

D fails

Proposition 1.17. $EF_n(R, R')$ is a win for Duplicator iff $R \sim_n R'$

Proposition 1.18. In $EF_n(R, R')$ if moves so far are $a_1, b_1, \dots, a_i, b_i$, $p = n - 1$, $f = \{(a_1, b_1), \dots, (a_i, b_i)\}$. Then Duplicator wins iff $f \in S_p(R, R')$

2 Back-and-forth Equivalence II

Definition 2.1. Let $(M, R), (M', R')$ be binary relations.. The Ehrenfeucht-Fraïssé game of length n , denoted $EF_n(M, M')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are n rounds
- In the i th round, the Spoiler chooses either an $a_i \in M$ or a $b_i \in M'$
- The Duplicator responds with a $b_i \in M'$ or an $a_i \in M$ respectively
- At the ends of the game, the Duplicator wins

$$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

is a local isomorphism from R to R'

- Otherwise, the Spoiler wins

Lemma 2.2. Suppose we are playing $EF_n(M, M')$ and there have been q rounds so far, with $p = n - q$ rounds remaining. Suppose the moves so far are $(a_1, b_1), \dots, (a_n, b_n)$. Let $f = \{(a_1, b_1), \dots, (a_q, b_q)\}$. Then the following are equivalent

- Duplicator has a winning strategy
- f is a p -isomorphism

Proof. By induction on p .

if $p = 0$, then the game is over, so Duplicator wins iff $f \in S_0(M, M')$

$p > 0$. If f isn't a local isomorphism, then Duplicator will definitely lose, and f isn't a p -isomorphism. So we may assume $f \in S_0(M, M')$. Then the following are equivalent

- Duplicator wins
- For any $a_{q+1} \in M$, there is a $b_{q+1} \in M'$ s.t. Duplicator wins in the position $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$, AND for any $b_{q+1} \in M'$, there is a $a_{q+1} \in M$ s.t. Duplicator wins in the position $(a_1, b_1, \dots, a_{q+1}, b_{q+1})$,
- For any $a_{q+1} \in M$ there is a $b_{q+1} \in M'$ s.t. $f \cup \{(a_{q+1}, b_{q+1})\} \in S_{p-1}(M, M')$ (by induction) , AND ...
- For any $a_{q+1} \in M$, there is $g \in S_{p-1}(M, M')$ s.t. $g \supseteq f$ and $a_{q+1} \in \text{dom}(g)$, AND
- $f \in S_p(M, M')$

□

Theorem 2.3. *If M is p -equivalent to M' , then $EF_p(M, M')$ is a win for the Duplicator. Otherwise it is a win for the Spoiler*

Proof. We need to prove $\emptyset \in EF_p(M, M')$

□

Theorem 2.4. *Every $(p + 1)$ -isomorphism is a p -isomorphism*

Proof. By induction on p .

$p = 0$: every 1-isomorphism is a 0-isomorphism.

□

So $S_0(M, M') \supseteq S_1(M, M') \supseteq S_2(M, M') \supseteq \dots$ In terms of the Ehrenfeucht-Fraïssé game

Theorem 2.5. *Suppose $s \in S_p(M, M')$ and $t \in S_p(M', M'')$ and $\text{dom}(t) = \text{im}(s)$. Then $u := t \circ s \in S_p(M, M'')$*

Corollary 2.6. *If $M \sim_p M'$ and $M' \sim_p M''$, then $M \sim_p M''$*

Proof. $\emptyset \in S_p(M, M')$ and $\emptyset \in S_p(M', M'')$, hence $\emptyset \in S_p(M, M'')$

□

Theorem 2.7. *Suppose $s \in S_p(M, M')$. Then $s^{-1} \in S_p(M, M')$*

Proof. Since $s \in S_p(M, M')$, s is a local isomorphism from M onto M' . As s is a bijection, s^{-1} is also a bijection.

□

Corollary 2.8. *If $M \sim_p M'$, then $M' \sim_p M$*

\sim_p is an equivalence relation

Theorem 2.9. *Let K be a Karpian family for (M, R) and (M', R') . Then $K \subseteq S_p(M, M')$ for all p . (also for all α)*

Corollary 2.10. *If M, M' are DLOs, then $S_0(M, M') = S_p(M, M')$ for all p . $M \sim_\omega M'$*

Corollary 2.11. $A \cong B \implies A \sim_\infty B \implies A \sim_\omega B \implies A \sim_p B$

Corollary 2.12. \sim_p and \sim_ω are equivalence relations

Theorem 2.13. *Suppose $(\mathbb{Q}, \leq) \sim_\omega (C, R)$. Then (C, R) is a DLO*

Proof. Suppose (C, R) is not a DLO and break into cases

- R is not reflexive. As $\emptyset \in S_1(\mathbb{Q}, C)$. Spoiler chooses $b_1 \in C$ s.t. $(b_1, b_1) \notin R$. Then duplicator must choose $a_1 \in \mathbb{Q}$ s.t. $a_1 \not\leq a_1$, impossible
- R is antisymmetric. $\emptyset \in S_2(\mathbb{Q}, C)$. Let $b_1, b_2 \in C$ s.t. $b_1 R b_2$ and $b_2 R b_1$. We want to show that $b_1 = b_2$. Since $\emptyset \in S_2(\mathbb{Q}, C)$, we have a local isomorphism $\{(a_1, b_1), (a_2, b_2)\} \in S_0(\mathbb{Q}, C)$. Hence $a_1 \leq a_2$ and $a_2 \leq a_1$. As so $a_1 = a_2$. As this is a bijection, $b_1 = b_2$.
- R is transitive. $\emptyset \in S_3(\mathbb{Q}, C)$. Let $b_1, b_2, b_3 \in C$ s.t. $b_1 R b_2$ and $b_2 R b_3$. $\square\square\square \square a_1, a_2, a_3 \in \mathbb{Q}$ s.t. $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \in S_0(\mathbb{Q}, C)$.
- R is total. $\square\square\square S_2(\mathbb{Q}, C)$.
- (C, R) has no maximum. $\forall b_1 \in C$
- (C, R) has no minimum
- (C, R) is dense. For any $b_1 \neq b_2 \in C$ s.t. $b_1 R b_2$. $S_3(\mathbb{Q}, C)$

□

Corollary 2.14. *The class of DLOs is the \sim_ω -equivalence class of (\mathbb{Q}, \leq)*

Definition 2.15. A linear order (C, \leq) is **discrete** without endpoints if $C \neq \emptyset$ and

$$\forall a \exists b : a \triangleleft b$$

$$\forall b \exists a : a \triangleleft b$$

where $a \triangleleft b$ means $a < b$ and not $\exists c : a < c < b$

Example 2.1. (\mathbb{Z}, \leq) . So is (C, \leq) , where

$$C = \{\dots, -3, -2, -1\} \cup \\ \{-1/2, -1/3, -1/4, -1/5, \dots\} \cup \\ \{\dots, 1/5, 1/4, 1/3, 1/2\} \cup \\ \{1, 2, 3, \dots\}$$

Definition 2.16. Let $(C, <)$ be discrete. If $a \leq b \in C$, then $d(a, b)$ is the size of $[a, b) = \{x \in C : a \leq x < b\}$ or ∞ if infinite. If $a > b$, then $d(a, b) = d(b, a)$ (definition)

$$d(a, b) = 0 \Leftrightarrow a = b$$

Lemma 2.17. Let $(C, <)$ and $(C', <)$ be discrete linear orders without endpoints. Suppose $a_1 < \dots < a_n$ in C and $b_1 < \dots < b_n$ in C' . Let f be the local isomorphism $f(a_i) = b_i$. Suppose that for every $1 \leq i < n$, we have

$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or } d(a_i, a_{i+1}) \geq 2^p \leq d(b_i, b_{i+1})$$

Then f is a p -isomorphism

IDEA: a 0-isomorphism needs to respect the order. A 1-isomorphism needs to respect the order plus the relation $d(x, y) = 1$ (to make sure we can find the point). A 2-isomorphism needs to respect the order plus the relation $d(x, y) = i$ for $i = 1, 2, 3$. A 3-isomorphism needs to respect the order plus the relations $d(x, y) = i$ for $i = 1, 2, 3, \dots, 7$

this is like binary search algorithm:D

Proof. • $a_i < a < a_{i+1}$

– if $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$
which means they are finite

□

Theorem 2.18. Let (C, \leq) and (C', \leq') be discrete linear orders without points. Then \emptyset is a p -equivalence from (C, \leq) to (C', \leq') for all p . Therefore $(C, \leq) \sim \omega(C', \leq')$.

Remark. If $(\mathbb{Z}, \leq) \sim_\omega (C, R)$, then (C, R) is a dense linear order

Definition 2.19. Let $(M, R), (M', R')$ be binary relations.. The **infinite Ehrenfeucht-Fraïssé game**, denoted $\text{EF}_\infty(M, M')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are infinitely many rounds (indexed by ω)
- In the i th round, the Spoiler chooses either an $a_i \in M$ or a $b_i \in M'$
- The Duplicator responds with a $b_i \in M'$ or an $a_i \in M$ respectively
- if $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is not a local isomorphism, then the Spoiler immediately wins
- The Duplicator wins if the Spoiler has not won by the end of the game

Theorem 2.20. *TFAE*

1. $R \sim_\infty R'$, i.e., there is a non-empty Karpian family K
2. Duplicator has a winning strategy for $EF_\infty(M, M')$
3. Spoiler does not have a winning strategy for $EF_\infty(M, M')$

Proof. $1 \rightarrow 2$. Karpian family is the winning strategy □

3 Connections to Back-and-Forth Technique

Theorem 3.1 (Fraïssé's Theorem). *Let (M, R) and (N, S) be m -ary relations, let $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then \bar{a} and \bar{b} are p -equivalent iff*

$$(M, R) \models f(\bar{a}) \iff (N, S) \models f(\bar{b})$$

for any formula $f(\bar{x})$ with quantifier rank at most p

Proof. \Rightarrow . Induction on p . If $\bar{a} \sim_0 \bar{b}$, then by definition, they satisfy the same atomic formulas. Therefore they satisfy the same quantifier-free formulas.

Suppose that $\bar{a} \sim_{p+1} \bar{b}$. The formula $f := (\exists y)g(\bar{x}, y)$ has quantifier rank at most $p + 1$. So $g(\bar{x}, y)$ is a formula of quantifier rank at most p . $(M, R) \models f(\bar{a})$ iff there is a $c \in M$ s.t. $(M, R) \models g(\bar{a}, c)$. Then there is a $d \in N$ s.t. $\bar{a}c \sim_p \bar{b}d$. By IH, $(N, S) \models g(\bar{b}, d)$ and thus $(N, S) \models (\exists y)g(\bar{b}, y)$. Another direction is similar □

To prove the converse we need the following lemma

Lemma 3.2. *If the arity m of a relation, and the integers n and p are fixed, there is only finite number $C(n, p)$ of p -equivalence classes of n -tuples*

$(M, R_1, \bar{a}_1), \dots, (M, R_n, \bar{a}_n)$. For any (M, R) and $\bar{a} \in M$, $\exists 1 \leq i \leq n$ s.t. $\bar{a} \sim_p \bar{a}_i$

Proof. Induction on p . If $p = 0$, then consider a set of symbols $X = \{x_1, \dots, x_n\}$. There are at most finitely many m -ary relations defined on X . Also there are at most finitely many ways to interpret the relation "=" on X . Let (M, R) and (N, S) be m -ary relations, $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. Let $R_A = R \cap A^m$ and $S_B = S \cap B^m$. If $p = 0$, $\bar{a} \sim_0 \bar{b}$ iff R_A is isomorphic to R_B via $a_i \mapsto b_i, i = 1, \dots, n$. So there are at most finitely many 0-equivalence classes of n -tuples

By IH, there exists relations $\{(M_k, R_k) \mid k \leq C(n+1, p)\}$ and $\{\bar{d}_k \in M_k^{n+1} \mid k \leq C(n+1, p)\}$ s.t. each $n+1$ -tuple is p -equivalent to some \bar{d}_k . Now consider an arbitrary relation (M, R) and an n -tuple \bar{a} , we define $[\bar{a}] = \{k \mid \exists c \in M(\bar{a}c \sim_p \bar{d}_k)\}$. For any relation (N, S) and $\bar{b} \in N^n$, $\bar{a} \sim_{p+1} \bar{b} \Leftrightarrow [\bar{a}] = [\bar{b}]$ \square

Proof (continued). We now show that if \bar{a} and \bar{b} satisfy the same formulas of QR at most p , then $\bar{a} \sim_p \bar{b}$.

Claim: For each p -equivalence class C , there is a formula f_C of QR p s.t. the tuples in C are exactly those satisfy f_C . $(M, R, \bar{a}) \in C \Leftrightarrow R \models f_C(\bar{a})$.

Induction on p . If $p = 0$, given an n -tuple \bar{a} , there are finitely many atomic formulas with variables x_1, \dots, x_n . $n^2 + n^m$. $\{x_i = x_j \mid i, j \leq n\}$ and $\{r(x_{i_1}, \dots, x_{i_m}) \mid i_j \leq n\}$.

Let f_C be the conjunction of those satisfied by \bar{a} and negation of the others. Then f_C characterizes the 0-equivalence class of \bar{a} . (characterizes $R|_{\{a_1, \dots, a_n\}}$)

Now prove $p+1$. Let \bar{a} be an n -tuple of (M, R) . Let $f_1(\bar{x}, y), \dots, f_k(\bar{x}, y)$ characterize all the p -equivalence classes C_1, \dots, C_k on $n+1$ -tuples. Let $\langle \bar{a} \rangle = \{i \leq k \mid (M, R) \models (\exists y) f_i(\bar{a}, y)\}$. $\langle \bar{a} \rangle = [\bar{a}]$

Let $f_C(\bar{x}) = \bigwedge_{i \in \langle \bar{a} \rangle} (\exists y) f_i(\bar{x}, y) \wedge \bigwedge_{i \notin \langle \bar{a} \rangle} \neg(\exists y) f_i(\bar{x}, y)$. $\bar{b} \sim_{p+1} \bar{a}$ iff $[\bar{a}] = [\bar{b}]$ iff $\langle \bar{a} \rangle = \langle \bar{b} \rangle$ iff $f_C(\bar{b})$ holds \square

bracket system

4 Compactness

4.1 Ultraproducts

If I is a nonempty set, a **filter** is a set F of subsets of I s.t.

- $I \in F, \emptyset \in F$
- if $X, Y \in F$, then $X \cap Y \in F$
- if $X \in F$ and $X \subset Y$, then $Y \in F$

A **filter prebase** B is a set of subsets of I contained in a filter; this means that the intersection of a finite number of elements of B is never empty. The filter F_B consisting of subsets of I containing a finite intersection of elements of B is the smallest filter containing B ; we call it the filter **generated** by B . If, in addition, the intersection of two elements of B is always in B , we call B a **filter base**.

Example 4.1. Let J be a set and I the set of finite subsets of J ; for every $i \in I$, let $I_i = \{j : j \in I, j \supset i\}$, and let B be the set of all the I_i . Then $I_i \cap I_j = I_{i \cup j}$; B is closed under finite intersections and does contain \emptyset ; It is therefore a filter base.

Theorem 4.1. A filter F of subsets of I is an ultrafilter iff for every subset A of I , either A or its complement $I - A$ is in F .

Theorem 4.2. Let U be an ultrafilter of subsets of I . If I is covered by finitely many subsets A_1, \dots, A_n , then one of the A_i is in U ; moreover, if the A_i are pairwise disjoint, exactly one of the A_i is in U .

Ultrafilter and Compactness

A topological space X is compact if and only if every ultrafilter in X is convergent.

4.2 Applications of Compactness

Lemma 4.3. If M and N are elementarily equivalent structures, then M can be embedded into an ultraproduct of N .

Proof. Let I be the set of injections from finite subset of M to N . If $f(\bar{a})$ is a formula with parameters \bar{a} in M , $M \models f(\bar{a})$, let $I_{f(\bar{a})}$ denote the set of such injections s whose universe contains \bar{a} and s.t. $N \models f(s(\bar{a}))$. The set $I_{f(\bar{a})}$ is never empty, as $M \models f(\bar{a})$, so $M \models \exists \bar{x}(f(\bar{x}) \wedge D(\bar{x}))$, where D is the conjunction of the formulas $x_i = x_j$ if $a_i = a_j$, and $x_i \neq x_j$ otherwise, and N also satisfies this formula. On the other hand, $I_{f(\bar{a})} \cap I_{g(\bar{b})} = I_{f(\bar{a}) \wedge g(\bar{b})}$, so the $I_{f(\bar{a})}$ form a filter base, which can be extended to an ultrafilter.

Define a function S from M to N^U as follows: If $a \in M$, the i th coordinate of Sa is ia if i is defined at a , and any element of N otherwise.

(We are excluding the case of empty universes, which is trivial.) Note that $\{i : i \text{ is defined at } a\} = I_{a=a}$, and that changing the coordinates outside of $I_{a=a}$ will not change Sa modulo U , so S is well-defined. **If $a = b$, then $S(a) = S(b)$ iff $\{i : N \models i(a) = i(b)\} = I_{a=b} \in U$. If $a \neq b$, then $I_{a \neq b} \in U$, hence S is an injection.**

$N^U \models \phi(S(\bar{a}))$ iff $\{i : N \models \phi(i(\bar{a}))\} \in U$. If $M \models \phi(\bar{a})$, then $\{i : N \models \phi(i(\bar{a}))\} = I_{\phi(\bar{a})}$. \square

5 Quantifier elimination

Theorem 5.1. *If two structures M and N are elementarily equivalent and ω -saturated, they are ∞ -equivalent: More precisely, two tuples of the same type (over \emptyset), one in M and the other in N , can be matched up by an infinite back-and-forth construction*

If M is ω -saturated, then for every \bar{a} of M and every p of $S_n(\bar{a})$, p is realised in M

An ω -saturated model therefore realises all absolute n -types for all n . This condition, however, is not sufficient for a model to be ω -saturated. Example: let T be the theory of discrete order without endpoints; M is ω -saturated iff it has the form $\mathbb{Z} \times \mathbb{C}$ where \mathbb{C} is a dense chain without endpoints, while it realizes all pure n -types iff it has the form $\mathbb{Z} \times \mathbb{C}$ where \mathbb{C} is an infinite chain

If T is a complete theory and M is an ω -saturated model of T , then every denumerable model N of T can be elementarily embedded in M . In fact, if $N = \{a_0, a_1, \dots, a_n, \dots\}$, we can successively realize, in M , the type of a_0 , then the type of a_1 over a_0, \dots , the type of a_{n+1} over $(a_0, \dots, a_n), \dots$

As two denumerable, elementarily equivalent, ω -saturated structures are isomorphic. Under what conditions does a complete theory T have a (unique) ω -saturated denumerable model? That happens iff for every n , $S_n(T)$ is (finite or) denumerable. (Here, we do not assume that T is denumerable)

In fact, this condition further implies that for every $\bar{a} \in M$, $S_1(\bar{a})$ is denumerable (because to say that b and c have the same type over \bar{a} is to say that $\bar{a}b$ and $\bar{a}c$ have the same type over \emptyset). It is clearly necessary, because a denumerable model can realize only denumerable many n -types. To see that it is sufficient: Let A_1 be a denumerable subset of M that realizes all 1-types over \emptyset ; then let A_2 be a denumerable subset of M that realises all 1-types over finite subsets of A_1 ; etc. Let $A = \bigcup A_n$. A satisfies Tarski's test so it is an elementary submodel of M

Theorem 5.2. *Let T be a theory, not necessarily complete, and let F be a nonempty set of formulas $f(\bar{x})$ in the language L of T , having for free variables only $\bar{x} = (x_1, \dots, x_n)$, s.t. two n -tuples from models of T have the same type whenever they satisfy the same formulas of F . Then for every formula $g(\bar{x})$ of L in these variables, there is some $f(\bar{x})$ that is a Boolean combination of elements of F s.t. $T \models \forall \bar{x} (f(\bar{x}) \leftrightarrow g(\bar{x}))$*

Proof. Consider the clopen set $[g(\bar{x})]$ in $S_n(T)$. If $[g] = \emptyset$, then $[g] = [f \wedge \neg f]$, and if $[g] = S_n(T)$, then $[g] = [f \vee \neg f]$, where f is an arbitrary element of F , which is nonempty. Consider $p \in [g]$ and $q \notin [g]$. There is $f_{p,q} \in F$ s.t. $p \models f_{p,q}(\bar{x})$ and $q \models \neg f_{p,q}(\bar{x})$. **If p and q are different, then they are realised by two tuples satisfying different formulas of F . Here we consider the model amalgamated by the model realising p and the model realising q . Thus such $f_{p,q}$ exists**

Keeping p fixed and varying q , all the $[f_{p,q}]$ and $\neg[g]$ form a family of closed sets whose intersection is empty; $\bigcup [\neg f_{p,q}] \supset [\neg g]$. by compactness, one of its finite subfamilies must have empty intersection, meaning that for some $h_p = f_{p,q} \wedge \dots \wedge f_{p,q_n} \in [h_p] \subset [g]$

Now when we vary p , $[g]$ is a compact set that is covered by the open sets $[h_p]$, so a finite number of them are enough to cover it; the disjunction of these h_p , module T , is equivalent to g \square

Note that if we want that every sentence be equivalent module T to a quantifier-free sentence; that requires, naturally, that the set of sentences without quantifiers be nonempty, meaning that the language **involves** constant symbols, or else nullary relation symbols.

A theory T is **model complete** if it has the following property: If $M, N \models T$ and if $N \subseteq M$, then $N \leq M$

Two theories T_1 and T_2 in the same language L , are **companions** if every model of one can be embedded into a model of the other

Theorem 5.3. *Two theories are companions of each other iff they have the same universal consequences (a sentence being called **universal** if it is of the form $\forall x_1, \dots, x_n f(x_1, \dots, x_n)$ with f quantifier-free)*

Proof. A universal sentence f that is true in a structure is always true in its substructure; if $T_1 \models f$ and if there is a model of T_2 that doesn't satisfy f , it cannot be extended to a model of T_1

Conversely, suppose that T_1 and T_2 have the same universal consequences, and let $M_1 \models T_1$. We name each element of M_1 by a new constant, and let $D(M_1)$ be the set of all *quantifier-free* sentences in the new language that are

true in M_1 . If $D(M_1) \models f(a_1, \dots, a_n)$, then $M \models \exists \bar{x} f(\bar{x})$, so $\forall \bar{x} \neg f(\bar{x})$ is not a consequence of T_1 , and therefore not of T_2 . There is therefore some model $M_2 \models T_2$ with $\bar{b} \in M_2$ s.t. $M_2 \models f(\bar{b})$. By compactness, this means that $D(M_1) \cup T_2$ is consistent, in other words, that M_1 embeds into a model of T_2 \square

A theory T therefore has a minimal companion, which we shall denote by T_\forall , which is axiomatized by the universal consequences of T .

A theory T' is a **model companion** of T if it is a companion of T that is model complete

Theorem 5.4. *A theory has at most one model companion*

Proof. Let T_1 and T_2 be model companions of T . Therefore T_1 and T_2 are companions. Let $M_1 \models T_1$; it embeds into a $N_1 \models T_2$, which embeds into a $M_2 \models T_1$. We get a chain $M_1 \subset N_1 \subset M_2 \subset N_2 \subset \dots \subset M_n \subset N_n \subset \dots$, whose limit we call P . As T_1 is model complete, the chain of M_n is elementary, and P is an elementary extension of M_1 ; similarly $N_1 \leq P$. Therefore M_1 is also a model of T_2 ; by symmetry T_1 and T_2 have the same models, meaning $T_1 = T_2$ \square

We say that T' is a **model completion** of T if it is a model companion of T and also the following condition is satisfied: if $M \models T$, embeds into a model $M_1 \models T'$ and into a model $M_2 \models T'$, then a tuple \bar{a} of M satisfies the same formulas in M_1 and in M_2

Naturally a model complete theory is its own model completion, and it is clear that a theory that admits quantifier elimination is the model completion of every one of its companions. A theory is the model completion of every one of its companions iff it is the model completion of the weakest of them all, T_\forall

In the particular case where for every $n > 0$ we can take for F the quantifier-free formulas, we say that the theory T **eliminates quantifiers** or **admits quantifier elimination**.

Theorem 5.5. *The model completion of a universal theory (i.e., one that is axiomatized by universal sentences) admits quantifier elimination*

Proof. Let \bar{a} and \bar{b} satisfying the same quantifier-free formulas, be in two models M_1 and M_2 of this theory T' , and let $N_1 \subseteq M_1$, $N_2 \subseteq M_2$ generated by \bar{a} and \bar{b} respectively. \square

DLO has quantifier elimination

Facts. In DLO, any 0-isomorphism is an ω -isomorphism.

Suppose $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$, want $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$

$\exists f : \langle \bar{a} \rangle_{\mathfrak{M}} \rightarrow \langle \bar{b} \rangle_{\mathfrak{N}}$ an isomorphism by Theorem 6, $f \in S_0(\mathfrak{M}, \mathfrak{N}) = S_\omega(\mathfrak{M}, \mathfrak{N})$. Then by Fraïssé's theorem, $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$

$M \equiv N \Leftrightarrow \langle \emptyset \rangle_M \cong \langle \emptyset \rangle_N \Leftrightarrow \text{char}(M) = \text{char}(N)$

same characteristic determine same minimal subring

$M^n / \text{Aut}(M/A) \cong S_n(A)$

Algebraically closed fields are axiomatized by the field axioms plus the axiom schema

$$\forall y_0, \dots, y_n \left(y_n \neq 0 \rightarrow \exists x \sum_{i=0}^n y_i x^i = 0 \right)$$

Lemma 5.6. *If $K \models \text{ACF}$, then K is infinite*

Proof. If $K = \{a_1, \dots, a_n\}$, then $P(x) = 1 + \prod_{i=1}^n (x - a_i)$ has no root in K \square

If $M \models \text{ACF}$ and K is a subfield, then K^{alg} denotes the set of $a \in M$ algebraic over K

Lemma 5.7. *Given uncountable $M, N \models \text{ACF}$, suppose $\bar{a} \in M^n$ and $\bar{b} \in N^n$ and $\text{qftp}^M(\bar{a}) = \text{qftp}^N(\bar{b})$. Suppose $\alpha \in M$. Then there is $\beta \in N$ s.t. $\text{qftp}^M(\bar{a}, \alpha) = \text{qftp}^N(\bar{b}, \beta)$*

Proof. Let $A = \langle \bar{a} \rangle_M$ and $B = \langle \bar{b} \rangle_N$. There is an isomorphism $f : A \rightarrow B$ and we can extend f to an isomorphism $f : \text{Frac}(A) \rightarrow \text{Frac}(B)$ (Note that A and B are subrings since they are only closed under multiplication and addition). Moving N by an isomorphism we may assume $\text{Frac}(A) = \text{Frac}(B)$ and $f = \text{id}_{\text{Frac}(A)}$. (In particular, $\bar{a} = \bar{b}$). let $K = \text{Frac}(A)$. Let $K = \text{Frac}(A)$

Claim. There is $\beta \in N$ with $I(\alpha) = I(\beta)$ in K

Suppose α is algebraic over K with minimal polynomial $P(x)$. Take $\beta \in N$ with $P(\beta) = 0$. Let $Q(x)$ be the minimal polynomial over β over K . Then $P(x) \in Q(x) \cdot K[x]$. But $P(x)$ is irreducible, so $P(x) = Q(x)$. Then $I(\alpha) = I(\beta)$

suppose α is transcendental, since there are only countable many solutions, there is transcendental $\beta \in N$. Then $I(\alpha) = I(\beta) = 0$

Take such β , let $I = I(\alpha) = I(\beta)$

- If $P(x) \in K[x]$, $P(\alpha) = 0 \Leftrightarrow P(x) \in I \Leftrightarrow P(\beta) = 0$

- If $P(x), Q(x) \in K[x]$, then $P(\alpha) = Q(\alpha) \Leftrightarrow (P - Q)(\alpha) = 0 \Leftrightarrow (P - Q)(\beta) = 0 \Leftrightarrow P(\beta) = Q(\beta)$
- Hence if $\varphi(x)$ is an atomic $\mathcal{L}(K)$ -formula, then $M \models \varphi(\alpha) \Leftrightarrow N \models \varphi(\beta)$
- so is quantifier-free $\varphi(x) \in \mathcal{L}(K)$

□

Lemma 5.8. *Lemma 5.7 holds if we replace “uncountable” with “ ω -saturated”*

Proof. Take uncountable $M' \geq M$ and $N' \geq N$, this is possible since models of ACF are infinite. By Lemma 5.7, there is $\beta_0 \in N'$ s.t. $\text{qftp}(\bar{a}, \alpha) = \text{qftp}(\bar{b}, \beta_0)$. By ω -saturation, we can find $\beta \in N$ s.t. $\text{tp}(\beta/\bar{b}) = \text{tp}(\beta_0/\bar{b})$. Then $\text{tp}(\bar{b}, \beta) = \text{tp}(\bar{b}, \beta_0)$ □

Theorem 5.9. *ACF has quantifier elimination*

Theorem 5.10. *Suppose $M, N \models \text{ACF}$, then $M \equiv N \Leftrightarrow \text{char}(M) = \text{char}(N)$*

Proof. TFAE

- $M \equiv N$
- for every sentence φ , $M \models \varphi \Leftrightarrow N \models \varphi$
- for every quantifier-free sentence φ , $M \models \varphi \Leftrightarrow N \models \varphi$
- for every atomic sentence φ , $M \models \varphi \Leftrightarrow N \models \varphi$
- for any terms t_1, t_2 , $M \models t_1 = t_2 \Leftrightarrow N \models t_1 = t_2$
- for any term t , $M \models t = 0 \Leftrightarrow N \models t = 0$
- for any $n \in \mathbb{Z}$, $M \models n = 0 \Leftrightarrow N \models n = 0$
- $\{n \in \mathbb{Z} : n^M = 0\} = \{n \in \mathbb{Z} : n^N = 0\}$
- $\text{char}(M) = \text{char}(N)$

□

Corollary 5.11. *ACF_p is complete for each p*

Corollary 5.12. *\mathbb{C} is completely axiomatized by ACF_0*

Lemma 5.13. *Let M be algebraically closed. Let K be a field. Let $\varphi(x)$ be an $\mathcal{L}(K)$ -formula in one variable. Let $D = \varphi(M)$. Then there is a finite subset $S \subseteq K^{\text{alg}}$ s.t. $D = S$ or $D = M \setminus S$, that is, either $D \subseteq K^{\text{alg}}$ or $M \setminus K \subseteq K^{\text{alg}}$*

Proof. By Q.E., we may assume φ is quantifier-free. Then φ is a boolean combination of atomic formulas

Let $\mathcal{F} = \{S : S \subseteq_f K^{\text{alg}}\} \cup \{M \setminus S : S \subseteq_f K^{\text{alg}}\}$. Note that \mathcal{F} is closed under boolean combinations. So we may assume φ is an atomic formula

Then $\varphi(x)$ is $(P(x) = 0)$ for some $P(x) \in K[x]$. If $P(x) \equiv 0$, then $\varphi(M) = M \in \mathcal{F}$. Otherwise $\varphi(M) \subseteq_f K^{\text{alg}}$, so $\varphi(M) \in \mathcal{F}$ \square

Lemma 5.14. *Suppose $M \leq N \models \text{ACF}$ and K is a subfield of M . Suppose $c \in N$ is algebraic over K . Then $c \in M$*

Proof. Let $P(x)$ be the minimal polynomial of c over K . Let b_1, \dots, b_n be the roots of $P(x)$ in M . Then

$$M \models \forall x \left(P(x) = 0 \rightarrow \bigvee_{i=1}^n x = b_i \right)$$

so the same holds in N . Then $P(c) = 0 \Rightarrow c \in \{b_1, \dots, b_n\} \subseteq M$ \square

Theorem 5.15. *If $M \models \text{ACF}$ and K is a subfield, then K^{alg} is a subfield of M and $(K^{\text{alg}})^{\text{alg}} = K^{\text{alg}}$*

Proof. Suppose $a, b \in K^{\text{alg}}$. We claim $a + b \in K^{\text{alg}}$. Let $P(x)$ and $Q(y)$ be the minimal polynomials of a, b over K . Let $\varphi(z)$ be the $\mathcal{L}(K)$ -formula

$$\exists x, y (P(x) = 0 \wedge Q(y) = 0 \wedge x + y = z)$$

Then $M \models \varphi(a + b)$ and $\varphi(M) = \{x + y : P(x) = 0 = Q(y)\}$ is finite. Thus $a + b \in \varphi(M) \subseteq K^{\text{alg}}$

A similar argument shows K^{alg} is closed under the field operations, so K^{alg} is a subfield of M \square

Theorem 5.16. *Suppose $M \models \text{ACF}$ and K is a subfield. TFAE*

1. $K = K^{\text{alg}}$
2. $K \models \text{ACF}$
3. $K \leq M$

Proof. $1 \rightarrow 2$: suppose $P(x) \in K[x]$ has degree > 0 . Then there is $c \in M$ s.t. $P(c) = 0$. By definition, $c \in K^{\text{alg}} = K$

$2 \rightarrow 3$: quantifier elimination

$3 \rightarrow 1$. 5.14 □

Corollary 5.17. *If $M \models \text{ACF}$ and K is a subfield, then $K^{\text{alg}} \models \text{ACF}$*

K^{alg} is called the **algebraic closure** of K . It is independent of M :

Theorem 5.18. *Let M, N be two algebraically closed fields extending K . Let $(K^{\text{alg}})_M$ and $(K^{\text{alg}})_N$ be K^{alg} in M and N , respectively. Then $(K^{\text{alg}})_M \cong (K^{\text{alg}})_N$*

6 Saturated Models

Lemma 6.1. *Let $S_0 \subseteq S_1 \subseteq \dots \subseteq S_\alpha \subseteq \dots$ be an increasing chain of sets indexed by $\alpha < \kappa$ for some regular cardinal κ . If $A \subseteq \bigcup_{\alpha < \kappa} S_\alpha$ and $|A| < \kappa$, then $A \subseteq S_\alpha$ for some $\alpha < \kappa$*

Proof. define $f : A \rightarrow \kappa$ by $f(x) = \min\{\alpha : x \in S_\alpha\}$. Then $|f(A)| \leq |A| < \kappa$, so $\alpha := \sup f(A) < \kappa$. For any $x \in A$, we have $f(x) \leq \alpha$ and so $x \in S_{f(x)} \subseteq S_\alpha$ □

Theorem 6.2. *If M is a structure and κ is a cardinal, there is a κ -saturated $N \geq M$*

Proof. Build an elementary chain

$$M_0 \leq M_1 \leq \dots \leq M_\alpha \leq \dots$$

of length κ^+ , where

1. $M_0 = M$
2. $M_{\alpha+1}$ is an elementary extension of M_α realizing every type in $S_1(M_\alpha)$
3. If α is a limit ordinal, then $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$

Let $N = \bigcup_{\alpha < \kappa^+} M_\alpha$. If $A \subseteq N$ and $|A| < \kappa$, then $A \subseteq M_\alpha$ for some $\alpha < \kappa^+$ □

Theorem 6.3. *Suppose M is κ -saturated. If $A \subseteq M$ and $|A| < \kappa$, then every $p \in S_n(A)$ is realized in M*

Proof. Take $N \geq M$ containing a realization \bar{a} of p . We can extend the partial elementary map $\text{id}_A : A \rightarrow A$ to $f : A \cup \{a_1, \dots, a_n\} \rightarrow B$ where $B \subseteq M$. Then $\text{tp}^M(f(\bar{a})/A) = \text{tp}^N(\bar{a}/A) = p$, so $f(\bar{a})$ realizes p in M □

Lemma 6.4. *For any M there is an elementary extension $N \geq M$ with the following properties:*

- *Every type over M is realized in N*
- *If $A, B \subseteq M$ and $f : A \rightarrow B$ is a partial elementary map, then there is $\sigma \in \text{Aut}(N)$ with $\sigma \supseteq f$*

Proof. Build an elementary chain

$$M = M_0 \leq M_1 \leq \dots$$

of length ω , where M_{i+1} is $|M_i|^+$ -saturated. Every $p \in S_n(M)$ is realized in M_1

For the second point, let $f : A \rightarrow B$ be given. Recursively build an increasing chain of partial elementary maps f_n with $\text{dom}(f_n), \text{im}(f_n) \subseteq M_n$ as follows:

- $f_0 = f$
- If $n > 0$ is odd, then f_n is a partial elementary map extending f_{n-1} with $\text{dom}(f_n) = M_{n-1}$ and $\text{im}(f_n) \subseteq M_n$
- If $n > 0$ is even, then f_n is a partial elementary map extending f_{n-1} with $\text{dom}(f_n) \subseteq M_n$ and $\text{im}(f_n) = M_{n-1}$

□

Theorem 6.5. *If M is a structure and κ is a cardinal, there is a strongly κ -homogeneous κ -saturated $N \geq M$*

Proof. Build an elementary chain

$$M_0 \leq M_1 \leq \dots \leq M_\alpha \leq \dots$$

of length κ^+ .

□

Lemma 6.6. *Let M be a κ -saturated L -structure. For $L_0 \subseteq L$, the reduct $M \upharpoonright L_0$ is κ -saturated*

Lemma 6.7. *Let M be an L -structure and κ be a cardinal. There is an L -structure $N \geq M$ s.t. for every $L_0 \subseteq L$, the reduct $N \upharpoonright L_0$ is κ -saturated and κ -strongly homogeneous*

Definition 6.8. Let T be an $L(R)$ -theory

1. R is **implicitly defined** in T if for every L -structure M , there is at most one $R \subseteq M^n$ s.t. $(M, R) \models T$
2. R is **explicitly defined** in T if there is an L -formula $\phi(x_1, \dots, x_n)$ s.t. $T \vdash \forall \bar{x} (R(\bar{x}) \leftrightarrow \phi(\bar{x}))$

Lemma 6.9. Suppose R is not explicitly defined in T . Then there are $M, N \models T$ and $\bar{a} \in M^n, \bar{b} \in N^n$ s.t.

- $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$
- $M \models R(\bar{a})$ and $N \models \neg R(\bar{b})$

Proof. Suppose not. Let $S = \{\text{tp}^L(\bar{a}) : M \models T, \bar{a} \in M^n\}$. For $p \in S$, one of two things happens

1. Every realization of p satisfies R
2. Every realization of p satisfies $\neg R$

Otherwise we can find a realization \bar{a} satisfying R and a realization \bar{b} satisfying $\neg R$, as desired.

By compactness, for each $p \in S$ there is an L -formula $\phi_p(\bar{x}) \in p(\bar{x})$ s.t. one of two things happens

1. $T \cup \{\phi_p(\bar{x})\} \vdash R(\bar{x})$
2. $T \cup \{\phi_p(\bar{x})\} \vdash \neg R(\bar{x})$

Let $\Sigma(\bar{x}) = T \cup \{\neg \phi_p(\bar{x}) : p \in S\}$. If $\Sigma(\bar{x})$ is consistent, there is $M \models T$ and $\bar{a} \in M^n$ satisfying $\Sigma(\bar{x})$. Let $p = \text{tp}^L(\bar{a})$, so it satisfies ϕ_p but it also satisfies $\neg \phi_p$, a contradiction

Therefore $\Sigma(\bar{x})$ is inconsistent. By compactness there are $p_1, \dots, p_n, q_1, \dots, q_m \in S$ s.t.

$$\begin{aligned}
 T &\vdash \bigvee_{i=1}^n \phi_{p_i}(\bar{x}) \vee \bigvee_{i=1}^m \phi_{q_i}(\bar{x}) \\
 T \cup \{\phi_{p_i}(\bar{x})\} &\vdash R(\bar{x}) \quad \text{for } i = 1, \dots, n \\
 T \cup \{\phi_{q_i}(\bar{x})\} &\vdash \neg R(\bar{x}) \quad \text{for } i = 1, \dots, m
 \end{aligned}$$

Then $T \vdash \forall \bar{x} (R(\bar{x}) \leftrightarrow \bigvee_{i=1}^n \phi_{p_i}(\bar{x}))$. The \leftarrow is by the choice of the ϕ_{p_i} . The \rightarrow is because if none of the ϕ_{p_i} hold, then one of the ϕ_{q_i} holds, and then $\neg R$ must hold.

Finally $\bigvee_{i=1}^n \phi_{p_i}(\bar{x})$ is an explicit definition of R

If $m = 0$, then $T \vdash R(\bar{x})$, if $n = 0$, then $T \vdash \neg R(\bar{x})$ □

Theorem 6.10 (beth). *If R is implicitly defined in T , then R is explicitly defined in T*

Proof. **Case 1:** T is complete.

If R is not explicitly defined, we obtain $M, N \models T$ and $\bar{a} \in M^n, \bar{b} \in N^n$ with $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$ but $M \models R(\bar{a})$ and $N \models \neg R(\bar{a})$. Since T is complete, we have $M \equiv N$. By elementary amalgamation, we may find elementary embeddings $M \rightarrow N', N \rightarrow N'$. Replacing M and N by N' and N' , we may choose $M = N$. By Lemma 6.7, we may replace M with an elementary extension and assume M and $M \upharpoonright L$ are \aleph_0 -saturated and \aleph_0 -strongly homogeneous. The fact that $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$ implies that there is an automorphism $\sigma \in \text{Aut}(M \upharpoonright L)$ with $\sigma(\bar{a}) = \bar{b}$. Let $R' = \sigma(R)$. Let $M' = (M \upharpoonright L, R')$. Then σ is an isomorphism from M to M' , so $M' \models T$. But $M' \upharpoonright L = M \upharpoonright L$. Because R is implicitly defined, $R = R'$. But then

$$\bar{a} \in R \Leftrightarrow \sigma(\bar{a}) \in \sigma(R) \Leftrightarrow \bar{b} \in R' \Leftrightarrow \bar{b} \in R$$

contradicting the fact that $M \models R(\bar{a})$ and $M \models \neg R(\bar{b})$

Case 2: T is not complete. Any completion of T implicitly defines R . By Case 1, any completion of T explicitly defines R . So in any model $M \models T$, there is an L -formula ϕ_M s.t. $M \models \forall \bar{x} (R(\bar{x}) \leftrightarrow \phi_M(\bar{x}))$

Assume R is not explicitly defined, there are $M, N \models T$ and $\bar{a} \in M^n, \bar{b} \in N^n$, with $\text{tp}^L(\bar{a}) = \text{tp}^L(\bar{b})$ and $M \models R(\bar{a})$ and $N \models \neg R(\bar{a})$. Let T' be the L -theory obtained from T by replacing every R with ϕ_M . Then $M \models T'$. The type $\text{tp}^L(\bar{a})$ contains the following

- $\phi_M(\bar{x})$
- sentences in T'

So $N \models \phi_M(\bar{b})$ and $N \models T'$.

Let $R' = \{\bar{c} \in N^n : N \models \phi_M(\bar{c})\}$. Then $(N \upharpoonright L, R') \models T$ because $N \models T'$. Therefore $R' = R$ because R is implicitly defined. But $N \models \phi_M(\bar{b})$ and $N \models \neg R(\bar{b})$, a contradiction \square

Theorem 6.11. *Let T be a complete theory. Then T has a countable ω -saturated model iff T is small*

Proof. \Rightarrow : trivial

\Leftarrow : Suppose $S_n(T)$ is countable for any n . Take some ω -saturated model M^+ . For each finite set $A \subseteq M^+$ and type $p \in S_1(A)$, take some element $c_{A,p} \in M$ realizing p . Define an increasing chain of countable subsets $A_0 \subseteq A_1 \subseteq \dots \subseteq M^+$ as follows

- $A_0 = \emptyset$
- $A_{i+1} = A_i \cup \{c_{A,p} : A \subseteq_f A_i, p \in S_1(A)\}$

each A_i is countable, and define $M = \bigcup_{i=0}^{\infty} A_i$, which is countable

Now we only need to prove that M is ω -saturated and $M \leq M^+$ \square

7 Prime models

7.1 Omitting types theorem

Theorem 7.1 (Baire Category Theorem for $S_n(A)$). *Let U_1, U_2, \dots be dense open sets. Then $\bigcap_{i=1}^{\infty} U_i$ is dense*

Lemma 7.2. *$S_n(A)$ is finite iff all types in $S_n(A)$ are isolated*

Proof. If each $p \in S_n(A)$ is isolated. The family $\{\{p\} : p \in S_n(A)\}$ covers $S_n(A)$, so there is a finite cover. This is impossible unless $S_n(A)$ is finite \square

Definition 7.3. A set $X \subseteq S_n(A)$ is **comeager** if $X \supseteq \bigcap_{i=1}^{\infty} U_i$ for some dense open sets U_i

Work in $S_{\omega}(T)$.

Lemma 7.4. *If X_1, X_2, \dots are comeager, then $\bigcap_{i=1}^{\infty} X_i$ is comeager*

Lemma 7.5. *For any formula $\phi(x_0, \dots, x_n, y)$, there is a dense open set Z_{ϕ} s.t. if $M \models T$, $\bar{c} \in M^{\omega}$, $\text{tp}^M(\bar{c}) \in Z_{\phi}$ and $M \models \exists y \phi(c_0, \dots, c_n, y)$, then there is $i < \omega$ s.t. $M \models \phi(c_0, \dots, c_n, c_i)$*

Proof. Take $A = [\neg \exists y \phi(x_0, \dots, x_n, y)]$ and $B_i = [\phi(x_0, \dots, x_n, x_i)]$ for $i < \omega$. Let $Z_{\phi} = A \cup \bigcup_{i=0}^{\infty} B_i$, which is open. If $p = \text{tp}^M(\bar{c}) \in Z_{\phi}$ and $M \models \exists y \phi(c_0, \dots, c_n, y)$ then $p \notin A$, so there is $i < \omega$ s.t. $p \in B_i$ meaning $M \models \phi(c_0, \dots, c_n, c_i)$

It remains to show that Z_{ϕ} is dense. Take non-empty $[\psi] \subseteq S_{\omega}(T)$; we claim $Z_{\phi} \cap [\psi] \neq \emptyset$. Take $p = \text{tp}^M(\bar{e}) \in [\psi]$. We may assume $p \notin Z_{\phi}$, or we are done. Then $p \notin A$, so $M \models \exists y \phi(e_0, \dots, e_n, y)$. Take $b \in M$ s.t. $M \models \phi(e_0, \dots, e_n, b)$. Take $i > n$ large enough that x_i doesn't appear in ϕ . Let $\bar{c} = (e_0, \dots, e_{i-1}, b, e_{i+1}, e_{i+2}, \dots)$. We have $M \models \psi(\bar{e})$ because $\text{tp}(\bar{e}) \in [\psi]$ and therefore $M \models \psi(\bar{c})$, so $\text{tp}(\bar{c}) \in [\psi]$. Also $M \models \phi(c_0, \dots, c_n, c_i)$ \square

Proposition 7.6. *There is a comeager set $W \subseteq S_{\omega}(T)$ s.t. if $\text{tp}^M(\bar{c}) \in W$, then $\{c_i : i < \omega\} \leq M$*

Proof. Let $W = \bigcap_{\phi} Z_{\phi}$. Suppose $\text{tp}^M(\bar{c}) \in W$. Then for any $\phi(x_0, \dots, x_n, y)$, if $M \models \exists y \phi(c_0, \dots, c_n, y)$, then there is $i < \omega$ s.t. $M \models \phi(c_0, \dots, c_n, c_i)$. By Tarski-Vaught, $\{c_i : i < \omega\} \leq M$. \square

Lemma 7.7. *Let $p \in S_n(T)$ be non-isolated. For any $(j_1, \dots, j_n) \in \mathbb{N}^n$, there is a dense open set $V_{p, \bar{j}} \subseteq S_{\omega}(T)$ s.t. $\text{tp}^M(\bar{c}) \in V_{p, \bar{j}} \Leftrightarrow \text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$*

Proof. Let $V_{p, \bar{j}} = V = \bigcup_{\phi \in p} [\neg \phi(x_{j_1}, \dots, x_{j_n})]$. If $\text{tp}^M(\bar{c}) \in V$, then there is some $\phi \in p$ s.t. $M \models \neg \phi(c_{j_1}, \dots, c_{j_n})$, and so $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$. Conversely, if $\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$, there is $\phi \in p$ s.t. $M \models \neg \phi(c_{j_1}, \dots, c_{j_n})$, and then $\text{tp}^M(\bar{c}) \in V$.

It remains to show that V is dense. Suppose $[\psi] \subseteq S_{\omega}(T)$ is non-empty. Take $q = \text{tp}^M(\bar{e}) \in [\psi]$. We may assume $q \notin V$. By choice of V , $\text{tp}^M(e_{j_1}, \dots, e_{j_n}) = p$. Take m large enough so that $m \geq \max(j_1, \dots, j_n)$ and ψ is a formula in x_0, \dots, x_m . Let $\phi(y_1, \dots, y_n)$ be

$$\exists x_0, \dots, x_m \psi(x_0, \dots, x_m) \wedge \bigwedge_{i=1}^n (y_i = x_{j_i})$$

Then $(e_{j_1}, \dots, e_{j_n})$ satisfies ϕ , and so $\phi \in p$. As p is non isolated, there is $N \models \phi(d_1, \dots, d_n)$ with $\text{tp}^N(d_1, \dots, d_n) \neq p$. By definition of ϕ there are $c_0, \dots, c_m \in N$ with $N \models \psi(c_0, \dots, c_m)$ and $(d_1, \dots, d_n) = (c_{j_1}, \dots, c_{j_n})$. Choose $c_{m+1}, c_{m+2}, \dots \in N$ arbitrarily. Then $\bar{c} = (c_i : i < \omega) \in N^{\omega}$ and $\text{tp}(\bar{c}) \in [\psi]$, and $\text{tp}(c_{j_1}, \dots, c_{j_n}) = \text{tp}(d_1, \dots, d_n) \neq p$, so $\text{tp}(\bar{c}) \in V$, showing $V \cap [\psi] \neq \emptyset$. \square

Proposition 7.8. *Let $p \in S_n(T)$ be non-isolated. There is a comeager set $V_p \subseteq S_{\omega}(T)$ s.t. if $\text{tp}^M(\bar{c}) \in V_p$, then p is not realized by a tuple in $\{c_i : i < \omega\}$*

Proof. Let $V_p = \bigcap_{\bar{j} \in \mathbb{N}^n} V_{p, \bar{j}}$. If $\text{tp}^M(\bar{c}) \in V_p$, then for any $j_1, \dots, j_n \in \mathbb{N}$

$$\text{tp}^M(c_{j_1}, \dots, c_{j_n}) \neq p$$

\square

Theorem 7.9 (Omitting types theorem). *Let Π be a countable set of pairs (p, n) , where $n < \omega$ and p is a non-isolated type in $S_n(T)$. There is a countable model $M \models T$ omitting p for every $(p, n) \in \Pi$*

Proof. The set $Q = W \cap \bigcap_{(p, n) \in \Pi} V_p$ is comeager, hence non-empty. Take $\text{tp}^N(\bar{c}) \in Q$. Then $M := \{c_i : i < \omega\} \leq N$ because $\text{tp}^N(\bar{c}) \in W$. For $(p, n) \in \Pi$, M omits p because $\text{tp}(\bar{c}) \in V_p$. \square

Theorem 7.10 (Ryll-Nardzewski). *Let T be a complete theory in a countable language. Then T is ω -categorical iff $S_n(T)$ is finite for every $n < \omega$*

Proof. Suppose $S_n(T)$ is infinite for some n . By 7.2 there is a non-isolated $p \in S_n(T)$. By 7.9 there is a countable model $M_0 \models T$ omitting p . Take an elementary extension $M_1 \geq M_0$ where p is realized by $\bar{a} \in M_1^n$. By Löwenheim–Skolem Theorem we may assume M_1 is countable. Then $M_1 \not\equiv M_0$ \square

8 Heirs and definable types

8.1 Definable types

Definition 8.1. $p(\bar{x})$ is a **definable type** if for every formula $\varphi(\bar{x}; \bar{y})$ the set

$$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is definable, defined by some $L(M)$ -formula $d\varphi(\bar{y})$

Proposition 8.2. *If T is strongly minimal and $M \models T$, there is a 1-type $p(x) \in S_1(M)$ s.t.*

$$\varphi(x, \bar{b}) \in p(x) \Leftrightarrow \exists^\infty a \in M : M \models \varphi(a, \bar{b})$$

Moreover, $p = \text{tp}(c/M)$ for any $N \geq M$ and $c \in N \setminus M$

Proof. Take $N \succ M$ and $c \in N \setminus M$; let $p(x) = \text{tp}(c/M)$. We must show that

$$N \models \varphi(c, \bar{b}) \Leftrightarrow \exists^\infty a \in M : M \models \varphi(a, \bar{b})$$

\Rightarrow : if not, $N \models \varphi(c, \bar{b})$ but $\varphi(M, \bar{b})$ is finite, then $c \in M$

\Leftarrow : if $N \models \neg\varphi(c, \bar{b})$, then $\neg\varphi(M, \bar{b})$ is infinite and so $\varphi(M, \bar{b})$ is finite \square

$p(x)$ is called the **transcendental 1-type**

Proposition 8.3. *If T is strongly minimal*

1. T eliminates the \exists^∞ quantifier
2. If $M \models T$, the transcendental 1-type $p \in S_1(M)$ is definable

Proof. 1. For any $\varphi(x, y)$, there is $n_\varphi < \omega$ s.t. for every $M \models T$ and $\bar{b} \in M$

$$|\varphi(M, \bar{b})| < n_\varphi \text{ or } |\neg\varphi(M, \bar{b})| < n_\varphi$$

2. For each $\varphi(x, \bar{y})$, $d\varphi(\bar{y})$ is the formula $\exists^\infty x \varphi(x, \bar{y})$

\square

Corollary 8.4. *If $p \in S_1(M)$ and M is strongly minimal, then p is definable*

Definition 8.5. A theory T is **stable** if all n -types over models are definable

8.2 Heirs and strong heirs

Suppose $M \leq N$ and $p \in S_n(M)$. An **extension** or **son** of p is $q \in S_n(N)$ with $q \supseteq p$, i.e., $p = q \upharpoonright M$

Definition 8.6 (Heirs). $q \in S_n(N)$ is an **heir** of p , written $p \sqsubseteq q$, if for any $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$ with $\bar{b} \in M$ and $\bar{c} \in N$, there is $\bar{c}' \in M$ with $\varphi(\bar{x}, \bar{b}, \bar{c}') \in p(\bar{x})$

Lemma 8.7. Suppose $M_1 \leq M_2 \leq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$, with $p_1 \subseteq p_2 \subseteq p_3$

1. If $p_1 \sqsubseteq p_2 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_3$
2. If $p_1 \sqsubseteq p_3$, then $p_1 \sqsubseteq p_2$

Definition 8.8. If $p \in S_n(M)$, then (M, dp) is the expansion of M by relation symbols $d\varphi(\bar{y})$ for each $\varphi(\bar{x}, \bar{y})$, interpreted as follows:

$$(M, dp) \models d\varphi(\bar{b}) \Leftrightarrow \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$$

Remark. p is definable iff the new relations in (M, dp) are definable in the old structure M

Remark. The class of structures of the form (M, dp) with $M \models T$ and $p \in S_n(M)$ is an elementary class, axiomatized by T plus the following:

$$\forall \bar{y}_1 \dots \bar{y}_m \left(\bigwedge_{i=1}^m d\varphi_i(\bar{y}) \rightarrow \exists \bar{x} \bigwedge_{i=1}^m \varphi_i(\bar{x}, \bar{y}_i) \right) \text{ for formulas } \varphi_1(\bar{x}, \bar{y}_1), \dots, \varphi_n(\bar{x}, \bar{y}_n)$$

$$\forall \bar{y} (d\varphi(\bar{y}) \vee d\neg\varphi(\bar{y})) \text{ for each formula } \varphi(\bar{x}, \bar{y})$$

Any model of such theory has an underlying p

Lemma 8.9. If $(M, dp) \leq (N, dq)$, then $M \leq N$ and $p \sqsubseteq q$

Proof. $(N, dq) \geq (M, dp)$ implies $N \geq M$. Then:

- $q \supseteq p$: if $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ (with $\bar{b} \in M$), then $(M, dp) \models d\varphi(\bar{b})$, so $(N, dq) \models d\varphi(\bar{b})$, and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$
- $q \sqsupseteq p$: suppose $\varphi(\bar{x}, \bar{b}, \bar{c}) \in q(\bar{x})$, with $\bar{b} \in M$ and $\bar{c} \in N$. Then $(N, dq) \models d\varphi(\bar{b}, \bar{c})$, and $(N, dq) \models \exists \bar{z} d\varphi(\bar{b}, \bar{z})$. Then $(M, dp) \models \exists \bar{z} d\varphi(\bar{b}, \bar{z})$

□

Corollary 8.10. If $p \in S_n(M)$, then there is $M_0 \leq M$ with $|M_0| \leq |T|$, s.t. $p \sqsupseteq (p \upharpoonright M_0)$

Proof. Apply downward Löwenheim–Skolem theorem to (M, dp) to find $(M_0, dq) \preceq (M, dp)$ with $|M_0| \leq |T|$. Then $q = p \upharpoonright M_0$ and $p \sqsupseteq q$ \square

Definition 8.11. If $M \preceq N$ and $p \in S_n(M)$ and $q \in S_n(N)$, then q is a **strong heir** of p if $(N, dq) \succeq (M, dp)$

Proposition 8.12 (Types have heirs). *Suppose $M \preceq N$ and $p \in S_n(M)$*

1. *There is $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p*
2. *There is $q \in S_n(N)$ an heir of p*

Proof. 1. Let \bar{c} be an infinite tuple enumerating N . Then $\text{tp}^L(\bar{c}/M)$ is finitely satisfiable in M , hence finitely satisfiable in the expansion (M, dp) . Therefore it is satisfied in some $(N', dq) \succeq (M, dp)$. So there is \bar{e} in N' with $\text{tp}^L(\bar{e}/M) = \text{tp}^L(\bar{c}/M)$. Then the map $f(c_i) = e_i$ is an L -elementary embeddings of N into N' extending $\text{id}_M : M \rightarrow M$. Moving N' by an isomorphism, we may assume $N' \succeq N$

2. Take $N' \succeq N$ and $q' \in S_n(N')$ a strong heir of p . Let $q = q' \upharpoonright N$. Then $q' \supseteq q \supseteq p$ and $q' \sqsupseteq p$, so $q \sqsupseteq p$. \square

8.3 Heirs and definable types

Proposition 8.13. *Let $p \in S_n(M)$ be definable and $N \succeq M$*

1. *p has a unique heir $q \in S_n(N)$*
2. *For $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N$*

$$\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Leftrightarrow N \models d_p \varphi(\bar{b}) \quad (*)$$

3. *In particular, q is definable with $d_q \varphi = d_p \varphi$ for all φ*

Proof. Claim. If $q \in S_n(N)$ and $q \sqsupseteq p$, then q satisfies $(*)$
Take $\bar{a} \in N' \succeq N$ realizing q . If $(*)$ fails then

$$\begin{aligned} (\varphi(\bar{x}, \bar{b})) \in q(\bar{x}) &\not\Leftrightarrow N \models d_p \varphi(\bar{b}) \\ N' \models \neg(\varphi(\bar{a}, \bar{b}) \leftrightarrow d_p \varphi(\bar{b})) \\ \neg(\varphi(\bar{x}, \bar{b}) \leftrightarrow d_p \varphi(\bar{b})) &\in q(\bar{x}) \end{aligned}$$

As $q \sqsupseteq p$, there is $b' \in M$ s.t.

$$\begin{aligned} \neg(\varphi(\bar{x}, \bar{b}') \leftrightarrow d_p \varphi(\bar{b}')) &\in p(\bar{x}) \\ N' \models \neg(\varphi(\bar{a}, \bar{b}') \leftrightarrow d_p \varphi(\bar{b}')) \\ \varphi(\bar{x}, \bar{b}') \in p(\bar{x}) &\not\leftrightarrow M \models d_p \varphi(\bar{b}') \end{aligned}$$

a contradiction

There is at least one heir, and at most one heir satisfying (*) □

Example 8.1. Suppose T is strongly minimal and $M \leq N$ are models of T . Let p and q be the transcendental 1-types over M and N . For any $\varphi(x, \bar{y})$

$$d_p \varphi(\bar{y}) \equiv (\exists^\infty x \varphi(x, \bar{y})) \equiv d_q \varphi(\bar{y})$$

so q is the unique heir of p

Proposition 8.14. *TFAE for $p \in S_n(M)$*

1. p is definable
2. For every $N \geq M$, p has a unique heir over N

Proof. Suppose p has unique heirs. Then for any $N \geq M$, p has at most one strong heir over N . Therefore there is at most one way to expand N to an elementary extension of (M, dp) . Then the elementary diagram (M, dp) implicitly defines the relations $d\varphi$. By Beth's implicit definability theorem, (M, dp) is a expansion of M by definable relations, meaning p is definable □

Proposition 8.15. *Suppose $M_1 \leq M_2 \leq M_3$ and $p_i \in S_n(M_i)$ for $i = 1, 2, 3$ with $p_1 \sqsubseteq p_2 \sqsubseteq p_3$. Suppose p_1 is definable. Then $p_1 \sqsubseteq p_2 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_3$*

Proof. We only need to show the implication $p_1 \sqsubseteq p_3 \Rightarrow p_2 \sqsubseteq p_3$. Suppose $p_1 \sqsubseteq p_3$. Take $p'_2 \sqsupseteq p_1$ and $p'_3 \sqsupseteq p'_2$. By the uniqueness of heirs of definable types, $p'_2 = p_2$ and p_2 is definable. Then $p'_3 = p_3$ □

8.4 Types in ACF

A **positive quantifier free formula** is a quantifier-free formula that doesn't use \neg

Fix a model $M \models \text{ACF}$

Definition 8.16. A set $V \subseteq M^n$ is an **algebraic set** if

$$V = \varphi(M^n; \bar{b}) = \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}$$

where φ is positive quantifier free.

Remark. V is an algebraic set iff V is defined by finitely many polynomial equations

$$V = \{\bar{a} \in M^n : P_1(\bar{a}) = \dots = P_m(\bar{a}) = 0\}$$

Lemma 8.17. 1. M^n and \emptyset are algebraic sets

2. If $V, W \subseteq M^n$ are algebraic sets, then $V \cap W$ and $V \cup W$ are algebraic sets

3. Any finite subset of M^n is an algebraic set

Fact 8.18 (Quantifier elimination). Every definable set $D \subseteq M^n$ is a finite boolean combination of algebraic sets

Fact 8.19 (Consequence of Hilbert's basis theorem). The class of algebraic sets has the descending chain condition (DCC): there is no infinite chain of algebraic sets $V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$

Corollary 8.20. If \mathcal{S} is a non-empty collection of algebraic sets, then \mathcal{S} contains at least one minimal element

Corollary 8.21. An infinite intersection $\bigcap_{i \in I} V_i$ of algebraic sets is an algebraic set

Corollary 8.22. If $S \subseteq K[\bar{x}]$ is any set of polynomials, possibly infinite, then the subset of M^n defined by S is an algebraic set. All algebraic sets arise this way

Corollary 8.23 (Noetherian induction). Let \mathcal{S} be a class of algebraic sets. Suppose the following holds

If X is an algebraic set, and every algebraic set $Y \subsetneq X$ is in \mathcal{S} , then $X \in \mathcal{S}$

Then every algebraic set is in \mathcal{S}

Definition 8.24. An algebraic set V is **reducible** if $V = W_1 \cup W_2$ for algebraic sets $W_1, W_2 \subsetneq V$. A **variety** is a non-empty irreducible algebraic set

Remark. If V is an algebraic variety, then the set of algebraic proper subsets of V is closed under finite unions

Proposition 8.25. If V is an algebraic set, then V is a finite union of varieties

Proof. • $V = \emptyset$: V is a union of zero varieties

- V is irreducible: V is a union of one variety
- V is reducible: $V = X \cup Y$ where $X, Y \subsetneq V$. By Noetherian induction! \square

Definition 8.26. The **generic type** of V is the type generated by the following formulas

1. $x \in V$
2. $x \notin W$ for each algebraic proper subset $W \subsetneq V$

We will write this type as $p_V(\bar{x})$

Note that $x \in V$ and $x \notin W$ is all definable

Proposition 8.27. Let V be a variety

1. $p_V(\bar{x})$ is a consistent complete type
2. If W is an algebraic set, then $p_V(\bar{x}) \vdash \bar{x} \in W \Leftrightarrow W \supseteq V$

Proof. Finite satisfiability: given finitely many proper algebraic subsets $W_1, \dots, W_m \subsetneq V$, we have $V \not\supseteq \bigcup_{i=1}^m W_i$, so there is $\bar{a} \in V$ and $\bar{a} \notin W_i$ for $1 \leq i \leq m$

1. If $W \supseteq V$, then $p_V(\bar{x}) \vdash \bar{x} \in V \vdash \bar{x} \in W$. If $W \not\supseteq V$, then $(W \cap V) \subsetneq V$, so $p_V(\bar{x}) \vdash \bar{x} \notin W \cap V$. But $p_V(\bar{x}) \vdash \bar{x} \in V$ so $p_V(\bar{x}) \vdash \bar{x} \notin W$

Completeness: by 2, for any positive quantifier-free formula $\varphi(\bar{x})$

$$p_V(\bar{x}) \vdash \varphi(\bar{x}) \text{ or } p_V(\bar{x}) \vdash \neg\varphi(\bar{x})$$

\square

Theorem 8.28. The map $V \mapsto p_V$ is a bijection from the set of varieties $V \subseteq M^n$ to $S_n(M)$

Proof. Injectivity: suppose V, W are varieties and $V \neq W$. WLOG, $V \not\subseteq W$. Then $p_W(\bar{x}) \vdash \bar{x} \in W$ but $p_V(\bar{x}) \not\vdash \bar{x} \in W$, so $p_V \neq p_W$

Surjectivity: fix $p \in S_n(M)$. Take V a minimal algebraic set s.t. $p(\bar{x}) \vdash \bar{x} \in V$. (There is at least one such V , namely M^n). V is non-empty because p is consistent. If V is reducible as $V = X \cup Y$ for smaller algebraic sets X, Y , then $p(\bar{x}) \vdash \bar{x} \in X$ or $p(\bar{x}) \vdash \bar{x} \in Y$ by completeness, contradicting the choice of V . Thus V is a variety. By choice of V , $p(\bar{x}) \vdash \bar{x} \in V$. \square

Proposition 8.29. $N \geq M$, let $V \subseteq M^n$ be a variety, defined by a formula φ

1. φ defines a variety $V_N \subseteq N^n$
2. V_N depends only on V , not on the choice of φ

Proof. Take ψ a positive quantifier-free formula defining V . Then $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is satisfied by M , and therefore by N . Let $V_N = \psi(N)$. As ψ is positive quantifier free, V_N is an algebraic set. As $M \models \exists \bar{x} \psi(\bar{x})$, V_N is non-empty. If $V_N = W_1 \cup W_2$ where W_1, W_2 are algebraic proper subsets of V_N defined by $\theta_i(\bar{x}, \bar{b}_i)$ for some positive quantifier-free L -formula θ_i and tuple of parameters $\bar{b}_i \in N$. Then

$$N \models \exists \bar{y}_1 \bar{y}_2 \left(\forall \bar{x} \left(\psi(\bar{x}) \leftrightarrow \bigvee_{i=1}^2 \theta_i(\bar{x}, \bar{y}_i) \right) \wedge \bigwedge_{i=1}^2 \exists \bar{x} (\psi(\bar{x}) \wedge \neg \theta_i(\bar{x}, \bar{y}_i)) \right)$$

which implies V is reducible □

Theorem 8.30. Let $M \leq N$ be models of ACF. Let $V \subseteq M^n$ be a variety, and let $V_N \subseteq N^n$ be its extension. Then $p_{V_N} \in S_n(N)$ is the unique heir of $p_V \in S_n(M)$

Proof. Let $q \in S_n(N)$ be an heir of p_V . Let φ be an $L(M)$ -formula defining V and V_N . Then $\varphi(\bar{x}) \in p_V(\bar{x}) \subseteq q(\bar{x})$, so $q(\bar{x}) \vdash \bar{x} \in V_N$. Suppose $q(\bar{x}) \not\vdash \bar{x} \in W$ for some algebraic $W \subsetneq V_N$, $q(\bar{x}) \vdash \bar{x} \in W$. Let $\psi(\bar{x}, \bar{b})$ be a positive quantifier-free formula defining W . Let $\theta(\bar{b})$ be the $L(M)$ -formula

$$\forall \bar{x} (\psi(\bar{x}, \bar{b}) \rightarrow \varphi(\bar{x})) \wedge \exists \bar{x} (\varphi(\bar{x}) \wedge \neg \psi(\bar{x}, \bar{b}))$$

which says $\psi(M^n, \bar{b}) \subsetneq \varphi(M^n)$. $N \models \theta(\bar{b})$ since $W \subsetneq V$. Then $q(\bar{x}) \vdash \psi(\bar{x}, \bar{b}) \wedge \theta(\bar{b})$. Because $q \sqsupseteq p_V$, there is $\bar{b}' \in M$ s.t.

$$p_V(\bar{x}) \vdash \psi(\bar{x}, \bar{b}') \wedge \theta(\bar{b}')$$

Thus we find an algebraic proper subset of V □

General fact: If $q \sqsupseteq p$, suppose $\forall \bar{b} (\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in p(\bar{x}))$, then $\forall \bar{b} \in N$, $\varphi(\bar{b}) \Rightarrow \psi(\bar{x}, \bar{b}) \in q(\bar{x})$

8.5 1-types in DLO

9 Stable Theories

9.1 Strong heirs from ultrapowers

Definition 9.1. If $p \in S_n(M)$, I set, \mathcal{U} ultrafilter on I , $M^\mathcal{U} = M^I/\mathcal{U}$. The **ultrapower type** $p^\mathcal{U} \in S_n(M^\mathcal{U})$ is the strong heir of p s.t. $(M^\mathcal{U}, dp^\mathcal{U}) = (M, dp)^\mathcal{U}$

$p^\mathcal{U}$ is a strong heir of p
 If $\varphi(\bar{x}, \bar{y}) \in L$, $\bar{b} \in M^\mathcal{U}$ represented by $(\bar{b} : i \in I) \in M^I$,
 $\varphi(\bar{x}, \bar{b}) \in p^\mathcal{U} \Leftrightarrow (M, dp)^\mathcal{U} \models d\varphi(\bar{b}) \Leftrightarrow \{i \in I \mid (M, dp) \models d\varphi(\bar{b}_i)\} \in \mathcal{U} \Leftrightarrow \{i \in I \mid \varphi(\bar{x}, \bar{b}_i) \in p(\bar{x})\} \in \mathcal{U}$

Proposition 9.2. Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $q \supseteq p$. Then there is I , ultrafilter \mathcal{U} on I s.t. (for some copy of $M^\mathcal{U}$, moved by isomorphism), $M \preceq N \preceq M^\mathcal{U}$, $p \subseteq q \subseteq p^\mathcal{U}$

Proof. Let $I = \{f : N \rightarrow M \mid f \supseteq \text{id}_M\}$.

Note that if $\phi(\bar{x}, \bar{b}) \in q(\bar{x})$, $\bar{b} \in N$, there is $f \in I$, $\phi(\bar{x}, f(\bar{b})) \in p(\bar{x})$. (has some duplicate variable problem, if $b_1 = b_2$, but $c_1 \neq c_2$, but maybe we could take some equivalent formulas)

For each $\phi(\bar{x}, \bar{b})$, $\bar{b} \in N$, let $S_{\phi, \bar{b}} = \{f \in I \mid \phi(\bar{x}, f(\bar{b})) \in p(\bar{x})\}$. Let $\mathcal{F} = \{S_{\phi, \bar{b}} \mid \phi(\bar{x}, \bar{b}) \in q(\bar{x})\}$

Claim \mathcal{F} has F.I.P

Suppose $\phi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$, $1 \leq i \leq m$. So $\bigwedge_{i=1}^m \phi_i(\bar{x}, \bar{b}_i) \in q(\bar{x})$, then there is $f \in I$ s.t. $\bigwedge_{i=1}^m \phi_i(\bar{x}, f(\bar{b}_i)) \in p(\bar{x})$. Then $f \in S_{\phi_i, \bar{b}_i}$, so $\bigcap_{i=1}^m S_{\phi_i, \bar{b}_i} \neq \emptyset$

Thus there is $\mathcal{U} \supseteq \mathcal{F}$. Form $M^\mathcal{U}$, $p^\mathcal{U}$. Let $g : N \rightarrow M^\mathcal{U}$ as follows. If $c \in N$, $g(c) = [(f(c) : f \in I)]$. Note if $c \in M$, then $f(c) = c$ for all f , and so $g \upharpoonright M = \text{id}_M$

For any $\phi(\bar{x}, \bar{y})$, $\bar{b} \in N$, $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow S_{\phi, \bar{b}} \in \mathcal{F} \Rightarrow S_{\phi, \bar{b}} \in \mathcal{U} \Rightarrow \{f \in I \mid \phi(\bar{x}, f(\bar{b})) \in p(\bar{x})\} \in \mathcal{U} \Leftrightarrow \phi(\bar{x}, g(\bar{b})) \in p^\mathcal{U}$

So $g : N \rightarrow M^\mathcal{U}$, $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \phi(\bar{x}, g(\bar{b})) \in p^\mathcal{U}$. $N \models \phi(\bar{b}) \Rightarrow M^\mathcal{U} \models \phi(g(\bar{b}))$. WLOG, $N \preceq M^\mathcal{U}$ and $g \supseteq \text{id}_N$. $\phi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \phi(\bar{x}, \bar{b}) \in p^\mathcal{U}$. $q \subseteq p^\mathcal{U}$. \square

Since we can prove compactness by ultrapower. Everything we get from compactness can be got by some ultrapower

Corollary 9.3. Every heir of p extends to a strong heir of p

9.2 Stability

Definition 9.4. If α is an ordinal, then $2^\alpha =$ strings of length α in alphabet $\{0, 1\}$

Definition 9.5. $\varphi(\bar{x}, \bar{y})$ be a formula. For α an ordinal, take variables \bar{x}_σ for $\sigma \in 2^\alpha$, \bar{y}_τ for $\tau \in 2^{<\alpha}$.

$D_\alpha = \{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 0\} \cup \{\neg \varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \text{ extends } \tau 1\}$
 $\varphi(\bar{x}, \bar{y})$ has the **dichotomy property** if

1. D_ω is consistent
2. D_n is consistent for all $n \in \omega$
3. D_α is consistent for all α

1-3 are equivalent

Example 9.1. D_2 is $\varphi(x_{00}, y), \varphi(x_{00}, y_0), \varphi(x_{01}, y), \neg\varphi(x_{01}, y_0)$ and so on

$$y \text{ / } \backslash y_0 y_1 \text{ / } \backslash \text{ / } \backslash x_{00} x_{01} x_{10} x_{11}$$

Proposition 9.6. Fix T, \mathbb{M} , and an integer $n < \omega$. Suppose there is a small model $M \leq \mathbb{M}$ and a type $p \in S_n(M)$ that is not definable, then some formula $\varphi(x_1, \dots, x_n, \bar{y})$ has the dichotomy property

Proof. Because p is not definable, there is an $N \geq M, q_1, q_2 \in S_n(N), q_1, q_2 \supseteq p$ and $q_1 \neq q_2$. There is $\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x}) \setminus q_2(\bar{x}), \bar{b} \in N$.

Claim If $M' \geq N, p' \in S_n(M'), p' \supseteq p$, then there is some $N' \geq M', q'_1, q'_2 \in S_n(N'), q'_1, q'_2 \supseteq p', q'_1, q'_2 \supseteq p$. and there is $\bar{b}' \in N', \varphi(\bar{x}, \bar{b}') \in q'_1, \neg\varphi(\bar{x}, \bar{b}') \in q'_2$

There is $M^\mathcal{U}$ s.t. $M \leq M' \leq M^\mathcal{U}, p \subseteq p' \subseteq p^\mathcal{U}$. Then $M' \leq M^\mathcal{U} \leq N^\mathcal{U}$ and $p \subseteq p^\mathcal{U} \subseteq q_i^\mathcal{U}$ for $i = 1, 2$. Take $N' = N^\mathcal{U}, q'_i = q_i^\mathcal{U}$, and \bar{b}' to be the image of \bar{b} under the elementary embedding $N \rightarrow N^\mathcal{U}$

Recursively build a tree of $(M, p) \text{ / } \backslash (M_0, p_0) (M_1, p_1)$

build $(M_\tau, p_\tau, \varphi(\bar{x}, \bar{b}_\tau))$ for $\tau \in 2^{<\omega}$

$M_\emptyset = M, p_\tau \supseteq p. M_{\tau_0} = M_{\tau_1} \geq M_\tau, \bar{b}_\tau \in M_{\tau_0}, \varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau_0}(\bar{x}), \neg\varphi(\bar{x}, \bar{b}_\tau) \in p_{\tau_1}(\bar{x}).$

Then φ has dichotomy □

working in \mathbb{M}

Proposition 9.7. If some $\varphi(x_1, \dots, x_n, \bar{y})$ has dichotomy property, then for every cardinal $\lambda \geq \aleph_0$, there is $A \subseteq \mathbb{M}, |A| \leq \lambda, |S_n(A)| > \lambda$

Proof. take smallest cardinal μ s.t. $2^\mu > \lambda, \mu \leq \lambda$. note that $|2^{<\mu}| = |\bigcup_{\alpha < \mu} 2^\alpha| \leq \lambda$.

φ has dichotomy proposition, so D_μ is consistent. In the monster, there are \bar{a}_σ for $\sigma \in 2^\mu, \bar{b}_\tau$ for $\tau \in 2^{<\mu}$ s.t. if σ extends τ_0 then $\mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau)$ and if σ extends τ_1 then $\mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$. Let $A = \{\bar{b}_\tau : \tau \in 2^{<\mu}\}$. Then $|A| \leq \lambda$ but $\text{tp}(a_\sigma/A) \neq \text{tp}(a_{\sigma'}/A)$ for $\sigma \neq \sigma'$. Thus $|S_n(A)| \geq 2^\mu > \lambda$. □

Lemma 9.8. for λ infinite, TFAE

1. $\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $\forall n$, $|S_n(A)| \leq \lambda$

2. $\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$

Proof. $2 \rightarrow 1$: By induction on n , $|S_{n-1}(A)| \leq \lambda$. Then we can find $\bar{b}_\alpha \in \mathbb{M}^{n-1}$ for $\alpha < \lambda$ s.t.

$$S_{n-1}(A) = \{\text{tp}(\bar{b}_\alpha/A) : \alpha < \lambda\}$$

For each α , $|A\bar{b}_\alpha| \leq \lambda \Rightarrow |S_1(A\bar{b}_\alpha)| \leq \lambda$. So we can find $c_{\alpha,\beta} \in \mathbb{M}$ for $\beta < \lambda$ s.t.

$$S_1(A\bar{b}_\alpha) = \{\text{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha) : \beta < \lambda\} \text{ (for } \alpha < \lambda \text{)}$$

Claim: if $p \in S_n(A)$ then $p = \text{tp}(\bar{b}_\alpha c_{\alpha,\beta}/A)$ for some $\alpha, \beta < \lambda$

Take $(\bar{b}', c') \in \mathbb{M}^n$ realizing p . Then $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}_\alpha/A)$ for some $\alpha < \lambda$. Moving (\bar{b}', c') by an automorphism in $\text{Aut}(\mathbb{M}/A)$, we may assume $\bar{b}' = \bar{b}_\alpha$. Then $\text{tp}(c'/A\bar{b}_\alpha) = \text{tp}(c_{\alpha,\beta}/A\bar{b}_\alpha)$ for some $\beta < \lambda$. Moving c' by an automorphism in $\text{Aut}(\mathbb{M}/A\bar{b}_\alpha)$, we may assume $c' = c_{\alpha,\beta}$

By the claim, $|S_n(A)| \leq \lambda^2 = \lambda$ □

Definition 9.9. T is λ -stable if $|A| \leq \lambda \Rightarrow |S_1(A)| \leq \lambda$

Proposition 9.10. If $\lambda \geq |L|$, TFAE

1. $\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $\forall n$, $|S_n(A)| \leq \lambda$

2. $\forall A \subseteq \mathbb{M}$, if $|A| \leq \lambda$, then $|S_1(A)| \leq \lambda$

3. If $M \leq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_1(M)| \leq \lambda$

4. If $M \leq \mathbb{M}$, $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$

Proof. $3 \rightarrow 1$: Let $A \subseteq \mathbb{M}$, $|A| \leq \lambda$, using downward Löwenheim-Skolem Theorem to get a model $A \subseteq M \leq \mathbb{M}$ and $|A| + |L| = |M|$

$4 \rightarrow 2$: similar □

Example 9.2. strongly minimal theory is λ -stable for $\lambda \geq |L|$

Given $A \subseteq \mathbb{M}$, $\exists M \leq \mathbb{M}$, $|M| \leq \lambda$. $S_1(M) = \text{const types} + \text{transcendental types}$, so $|S_1(M)| = |M| + 1$

λ -stable \Rightarrow no φ has D.P \Rightarrow all types are definable

Lemma 9.11. Suppose $\forall M \leq \mathbb{M}$, $\forall p \in S_1(M)$ is definable. Then T is λ -stable for some λ

Proof. Take $\lambda = 2^{|L|} > |L|$. Suppose $M \leq \mathbb{M}$ and $|M| \leq \lambda$. $p \in S_1(M)$ is determined by $\varphi \in L \mapsto d_p \varphi \in L(M)$, $|S_1(M)| \leq |L(M)|^{|L|} \leq \lambda^{|L|} = 2^{|L|}$ □

Theorem 9.12. *TFAE*

1. T is λ -stable for some λ
2. no formula $\varphi(\bar{x}, \bar{y})$ has D.P.
3. no $\varphi(x, \bar{y})$ has D.P.
4. $M \models T, p \in S_1(M) \Rightarrow p$ is definable
5. $M \models T, p \in S_n(M) \Rightarrow p$ is definable

Proof. $5 \rightarrow 1$: Let $\lambda = 2^{|L|}$. Note that $\lambda^{|L|} = (2^{|L|})^{|L|} = 2^{|L|} = \lambda$. Take $A \subseteq \mathbb{M}$ with $|A| \leq \lambda$. By downward Löwenheim–Skolem Theorem, there is a small model $M \leq \mathbb{M}$ with $A \subseteq M$ and $|M| \leq \lambda$. Every n -type over A extends to an n -type over M , so $|S_n(A)| \leq |S_n(M)|$. It remains to show $|M| \leq \lambda \Rightarrow |S_n(M)| \leq \lambda$. (That is, we may assume A is a small model M). By (5), every n -type over M is definable. A definable type is determined by the map $\varphi \mapsto d\varphi$, which is a function from L -formulas to $L(M)$ -formulas. So the number of (definable) types over M is at most $|L(M)|^{|L|} \leq \lambda^{|L|} = \lambda$ \square

9.3 The dichotomy property and definability of types

We will prove

If no formula has the dichotomy property, then all types over *arbitrary sets* are definable

9.3.1 The dichotomy property

Fix a complete theory T and monster model \mathbb{M} . Fix a formula $\varphi(\bar{x}; \bar{y})$

Definition 9.13. “ D_α is consistent” if there are $(\bar{a}_\sigma : \sigma \in 2^\alpha)$ and $(\bar{b}_\tau : \tau \in 2^{<\alpha})$ s.t.

$$\begin{aligned} \sigma \sqsupseteq \tau 0 &\Rightarrow \mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau) \\ \sigma \sqsupseteq \tau 1 &\Rightarrow \mathbb{M} \models \neg \varphi(\bar{a}_\sigma, \bar{b}_\tau) \end{aligned}$$

Remark. “ D_α ” is the name for the set of formulas $\{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 0\} \cup \{\neg \varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 1\}$

Lemma 9.14. *If D_n is consistent for all $n < \omega$, then D_ω is consistent*

Proof. Let F be a finite fragment of D_ω . By compactness, it suffices to show that F is consistent. Take n bigger than the length of τ for any \bar{y}_τ appearing in F . Because D_n is consistent, there are $(\bar{a}^0_\sigma : \sigma \in 2^n)$ and $(\bar{b}^0_\tau : \tau \in 2^{<n})$. Define \square

9.3.2 φ -types

Continue to fix $T, \mathbb{M}, \varphi(\bar{x}; \bar{y})$. Let n be the length of the variable tuple \bar{x}

Definition 9.15. If $B \subseteq \mathbb{M}$ is a set and $\bar{a} \in \mathbb{M}^n$, then $\text{tp}^\varphi(\bar{a}/B)$ is the partial type

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \varphi(\bar{a}, \bar{b})\} \cup \{\neg\varphi(\bar{x}; \bar{b}) : \bar{b} \in B, \mathbb{M} \models \neg\varphi(\bar{a}, \bar{b})\}$$

Definition 9.16. $S_\varphi(B) = \{\text{tp}^\varphi(\bar{a}/B) : \bar{a} \in \mathbb{M}^n\}$

Definition 9.17. A φ -type $p \in S_\varphi(B)$ is **definable** if there is an $L(B)$ -formula $\psi(\bar{y})$ s.t.

$$\forall \bar{b} \in B, \quad \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

Theorem 9.18. Suppose φ does not have the dichotomy property. Then every φ -type over any set is definable

9.3.3 Proof of Theorem 9.18

Definition 9.19. Let $\Sigma(\bar{x})$ be a small set of $L(\mathbb{M})$ -formulas. Define “ $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ ” by recursion on n :

- $R_{\varphi,2}(\Sigma(\bar{x})) \geq 0$ iff $\Sigma(\bar{x})$ is consistent
- $R_{\varphi,2}(\Sigma(\bar{x})) \geq n + 1$ iff there is $\bar{b} \in \mathbb{M}$ s.t.

$$\begin{aligned} R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) &\geq n \\ R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) &\geq n \end{aligned}$$

Lemma 9.20. $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ iff there are $(\bar{a}_\sigma : \sigma \in 2^n)$ and $(\bar{b}_\tau : \tau \in 2^{<n})$ s.t.

- If σ extends $\tau 0$, then $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds
- If σ extends $\tau 1$, then $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds
- Each \bar{a}_σ satisfies $\Sigma(\bar{x})$

Definition 9.21. $R_{\varphi,2}(\Sigma(\bar{x}))$ is the largest n s.t. $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$, or $-\infty$ if there is no such n , or $+\infty$ if $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ for all n

$R_{\varphi,2}(\Sigma(\bar{x}))$ is called the “ φ -2-rank” or “ $R_{\varphi,2}$ -rank” of Σ

Remark (Monotonicity). If $\Sigma(x) \vdash \Sigma'(x)$, then $R_{\varphi,2}(\Sigma(\bar{x})) \leq R_{\varphi,2}(\Sigma'(\bar{x}))$

Remark. From Lemma 9.20 we see that $R_{\varphi,2}(\{\bar{x} = \bar{x}\}) \geq n$ iff “ D_n ” is consistent. By Lemma 9.14,

$$R_{\varphi,2}(\{\bar{x} = \bar{x}\}) = +\infty \text{ iff } \varphi \text{ has the dichotomy property}$$

In particular, if φ does not have the dichotomy property, then $R_{\varphi,2}(\{\bar{x} = \bar{x}\})$ is finite. By Monotonicity, $R_{\varphi,2}(\Sigma(\bar{x}))$ is finite for any $\Sigma(\bar{x})$

Remark (Definability). Suppose $\Sigma(\bar{x})$ is a finite partial type over $A \subseteq \mathbb{M}$ and suppose $n < \omega$. Then the set

$$\{\bar{b} \in \mathbb{M} : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n\}$$

is A -definable

Now we can prove Theorem 9.18. Suppose $p \in S_{\varphi}(B)$. Take a finite subtype $\Sigma(\bar{x}) \subseteq_f p(\bar{x})$ minimizing $R_{\varphi,2}(\Sigma(\bar{x}))$. Then $\Sigma(\bar{x})$ is a partial type over B . Let $k = R_{\varphi,2}(\Sigma(\bar{x}))$

Claim. If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) = k$$

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) < k$$

Proof. Monotonicity gives

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \leq k$$

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) \leq k$$

If the first inequality is sharp, then it contradicts the minimality of $\Sigma(\bar{x})$. If the second inequality is not sharp, then

$$R_{\varphi,2}(\Sigma(\bar{x})) \geq k + 1$$

□

Claim. If $\neg\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) < k$$

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) = k$$

Combining the two claims, we see that the set

$$\{\bar{b} \in B : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

is exactly

$$\{\bar{b} \in B : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k\}$$

By Definability, there is an $L(B)$ -formula $\psi(\bar{x})$ s.t.

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq k \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

Therefore p is a definable φ -type

9.3.4 Remarks on the proof

1. In the proof of theorem 9.18, the finite subtype $\Sigma(\bar{x}) \subseteq p(\bar{x})$ chosen to minimize $R_{\varphi,2}(\Sigma(\bar{x}))$ actually has

$$R_{\varphi,2}(\Sigma(\bar{x})) = R_{\varphi,2}(p(\bar{x}))$$

This comes from the following

- Fact 9.22.** (a) If $n < \omega$ and if $R_{\varphi,2}(\Sigma_0(\bar{x})) \geq n$ for every finite subtype $\Sigma_0(\bar{x})$, then $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$
- (b) $R_{\varphi,2}(\Sigma(\bar{x}))$ is the minimum of $R_{\varphi,2}(\Sigma_0(\bar{x}))$ as Σ_0 ranges over finite subtypes of Σ .

2. We have discussed $R_{\varphi,2}$ for partial types, but we can also define it for definable sets. If D is a definable set, defined by a finite type $\Sigma(\bar{x})$, then $R_{\varphi,2}(D) := R_{\varphi,2}(\Sigma(\bar{x}))$. The

9.3.5 Consequences of Theorem 9.18

Theorem 9.23. Suppose that no formula $\varphi(x_1, \dots, x_n; \bar{y})$ has the dichotomy property. For any model $M \models T$ and any $p \in S_n(M)$, p is definable

Proof. Take $\bar{a} \in \mathbb{M}$ realizing p . Let $\varphi(\bar{x}; \bar{y})$ be a formula. By Theorem 9.18, $\text{tp}^\varphi(\bar{a}/M)$ is definable, therefore p is definable \square

Remark. When $A \subseteq \mathbb{M}$ is arbitrary, one says that $p \in S_n(A)$ is **definable** if for any $\varphi(\bar{x}; \bar{y})$ there is an $L(A)$ -formula $\psi(\bar{y})$ s.t. for $\bar{b} \in A$

$$\varphi(\bar{x}; \bar{b}) \in p(\bar{x}) \Leftrightarrow \mathbb{M} \models \psi(\bar{b})$$

The proof of Theorem 9.23 shows more generally that

If no formula has the dichotomy property, then any $p \in S_n(A)$ is definable for any A

Therefore in a stable theory, any type over *any* set is definable

Warning 9.24. Definable types over arbitrary sets are not as well-behaved as definable types over models. For example

1. A definable type over $A \subseteq \mathbb{M}$ can have more than one A -definable extension to \mathbb{M} : In DLO, there is only one 1-type over \emptyset . It has two different \emptyset -definable extensions to \mathbb{M} : the types at $+\infty$ and $-\infty$

2. A definable type over $A \subseteq \mathbb{M}$ can have no A -definable extensions to \mathbb{M} : In ACF , $\text{tp}(\sqrt{2}/\mathbb{Q})$ is definable (because ACF is stable), but there is no \mathbb{Q} -definable extension to the monster model \mathbb{M} . Indeed, there are exactly two extensions to \mathbb{M} , namely $\text{tp}(\sqrt{2}/\mathbb{M})$ and $\text{tp}(-\sqrt{2}/\mathbb{M})$. These are exchanged by some automorphisms in $\text{Aut}(\mathbb{M}/\mathbb{Q})$ so neither one can be \mathbb{Q} -definable

Theorem 9.25. Assume T is stable and $M \models T$. Let $D \subseteq M^n$ be \emptyset -definable. If $X \subseteq D$ is definable (with parameters from M), then X is definable over parameters from D

Proof. Let $\psi(\bar{y})$ be a formula defining D and let $\varphi(\bar{a}; \bar{y})$ be a formula defining X . Then $\text{tp}^\varphi(\bar{a}/D)$ is definable by Theorem 9.18. Therefore there is a formula $\theta(\bar{y})$ with parameters from D s.t. if $\bar{b} \in D$, then

$$(M \models \varphi(\bar{a}; \bar{b})) \Leftrightarrow \varphi(\bar{a}; \bar{b}) \in \text{tp}^\varphi(\bar{a}/D) \Leftrightarrow (M \models \theta(\bar{b}))$$

In other words, if $\psi(\bar{y})$ holds, then $\varphi(\bar{a}; \bar{y})$ is equivalent to $\theta(\bar{y})$. Therefore X is defined by $\psi(\bar{y}) \wedge \theta(\bar{y})$, a formula with parameters from D \square

Fact 9.26. Let M be a stable structure and $D \subseteq M^n$ be A -definable. Let N be the structure whose domain is D , with an m -ary relation symbol for each A -definable $X \subseteq D^m$. Then the definable subsets of D^m in M agree with the definable subsets of D^m in N

9.4 Coheirs

Definition 9.27. If $M \leq N$, if $p \in S_n(M)$, if $q \in S_n(N)$, then q is a **coheir** of p if $q \supseteq p$ and q is finitely satisfiable in M (for any $\phi(x) \in q(x)$, there is $a \in M$ s.t. $N \models \phi(a)$)

Example 9.3. $\mathbb{Q}^{\text{alg}} \leq \mathbb{C}$, $q = \text{tp}(\pi/\mathbb{C})$, $p = \text{tp}(\pi/\mathbb{Q}^{\text{alg}})$. $q \supseteq p$, but q isn't a coheir since $x = \pi \in q(x)$

Example 9.4. If $M \leq N$ strongly minimal, $q(x) \in S_1(N)$, $p(x) \in S_1(M)$ is the transcendental 1-type, $p \subseteq q$, then q is a coheir of p ,

If $\varphi(x) \in q(x)$, then $\varphi(N)$ is cofinite and M is infinite, so $\varphi(N) \cap M \neq \emptyset$

Lemma 9.28. If $M \leq N$, $\Sigma(\bar{x})$ partial type over N , $\Sigma(\bar{x})$ is f.sat. in M , then $\exists q(\bar{x}) \in S_n(N)$, $q(\bar{x})$ is fsat. in M

Proof. Let $\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(N) : \forall \bar{a} \in M, N \models \psi(\bar{a})\}$

If $\bar{a} \in M$, then \bar{a} satisfies Ψ

Claim $\Sigma(\bar{x})$ fsat in $M \Rightarrow \Sigma \cup \Psi$ is fsat $\Rightarrow q \in S_n(N)$, $q \supseteq \Sigma \cup \Psi$

If q isn't fast. in M then $\varphi(\bar{x}) \in q(\bar{x})$, $\varphi(\bar{x})$ not sat. in M \square

Theorem 9.29. *If $p \in S_n(M)$, $N \geq M$, then $\exists q \in S_n(N)$, q is a coheir of p*

Theorem 9.30. *Suppose $M_1 \leq M_2 \leq M_3$, $p_1 \in S_n(M_1)$, $p_2 \in S_n(M_2)$, p_2 is a coheir of p_1 . Then $\exists p_3 \in S_n(M_3)$, p_3 is a coheir of p_1 and p_2*

9.5 Coheir Independence

9.5.1 Coheir independence

Definition 9.31. Let M be a small model, \bar{a}, \bar{b} small tuples (possibly infinite). Then \bar{a} is **coheir independent** from \bar{b} over M , written

$$\bar{a} \downarrow_M^u \bar{b}$$

if $\text{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M

Remark. The relation $A \downarrow_M^u B$ is finitary w.r.t. the arguments A and B , in the following sense. $A \downarrow_M^u B$ holds iff the following does:

For any finite tuple $\bar{a} \in A$ and any finite tuple $\bar{b} \in B$, we have $\bar{a} \downarrow_M^u \bar{b}$

Since a formula $\varphi(\bar{x}, \bar{y})$ can only refer to finitely many variables

Remark. The relation \downarrow^u can be used to define heirs and coheirs, as follows. Suppose M, N are small models with $M \leq N$. Suppose $p \in S_n(M)$ and $q \in S_n(N)$ with $q \supseteq p$. Take $\bar{a} \in \mathbb{M}^n$ realizing q

1. $q = \text{tp}(\bar{a}/N)$ is a coheir of $p = \text{tp}(\bar{a}/M)$ iff $\bar{a} \downarrow_M^u N$
2. $q = \text{tp}(\bar{a}/N)$ is an heir of $p = \text{tp}(\bar{a}/M)$ iff $N \downarrow_M^u \bar{a}$

9.5.2 Existence

Lemma 9.32. *Let M be a small model and \bar{a}, \bar{b} be tuples, possibly infinite*

1. *There is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) \downarrow_M^u \bar{b}$*
2. *There is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\bar{a} \downarrow_M^u \sigma(\bar{b})$*

Proof. 1. Let α be the length of \bar{a} and \bar{x} be an α -tuple of variables. Let

$$\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(M\bar{b}) : \psi(\bar{x}) \text{ is satisfied by every } \bar{a}' \in M^\alpha\}$$

If $\varphi(\bar{x}) \in \text{tp}(\bar{a}/M)$, then there is $\bar{a}' \in M^\alpha$ satisfying $\varphi(\bar{x})$ because $\text{tp}(\bar{a}/M)$ is finitely satisfiable in M . Then \bar{a}' satisfies $\{\varphi(\bar{x})\} \cup \Psi(\bar{x})$.

This shows $\text{tp}(\bar{a}/M) \cup \Psi(\bar{x})$ is finitely satisfiable, hence realized by some $\bar{a}' \in \mathbb{M}^\alpha$

Then \bar{a}' realizes $\text{tp}(\bar{a}/M)$, so $\text{tp}(\bar{a}'/M) = \text{tp}(\bar{a}/M)$, and there is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) = \bar{a}'$. Finally $\bar{a}' \downarrow_M^u \bar{b}$ by choice of $\Psi(\bar{x})$: if $\varphi(\bar{x}) \in \text{tp}(\bar{a}'/M\bar{b})$ and $\varphi(\bar{x})$ isn't satisfiable in M , then $M \models \neg \exists \bar{x} \varphi(\bar{x})$ and $M \models \forall \bar{x} \neg \varphi(\bar{x})$, hence $\neg \varphi(\bar{x}) \in \Psi(\bar{x})$ and \bar{a} doesn't satisfy $\varphi(\bar{x})$, a contradiction

2. By 1, there is $\tau \in \text{Aut}(\mathbb{M}/M)$ s.t. $\tau(\bar{a}) \downarrow_M^u \bar{b}$. Let $\sigma = \tau^{-1}$. Then $\sigma(\tau(\bar{a})) \downarrow_{\sigma(M)}^u \sigma(\bar{b})$, or equivalently, $\bar{a} \downarrow_M^u \sigma(\bar{b})$

□

Corollary 9.33. Suppose $p \in S_n(M)$ and $N \geq M$

1. There is $q \in S_n(M)$ s.t. q is a coheir of p
2. There is $q \in S_n(M)$ s.t. q is an heir of p

Proof. 1. Take $\bar{a} \in \mathbb{M}^n$ realizing p . Let \bar{b} enumerate N . By Lemma, there is $\sigma \in \text{Aut}(\mathbb{M}/M)$ s.t. $\sigma(\bar{a}) \downarrow_M^u \bar{b}$, i.e., $\sigma(\bar{a}) \downarrow_M^u N$. Thus $\text{tp}(\sigma(\bar{a})/N)$ is a coheir of $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M) = p$

2. Similarly we have $N \downarrow_M^u \sigma(\bar{a})$, and thus $\text{tp}(\sigma(\bar{a})/N)$ is an heir of $\text{tp}(\sigma(\bar{a})/M) = \text{tp}(\bar{a}/M)$

□

9.5.3 “u” for “ultrafilter”

Proposition 9.34. Let \bar{a} be an α -tuple in \mathbb{M} . Let M be a small model and B a small set. TFAE

1. $\bar{a} \downarrow_M^u B$
2. There is an ultrafilter \mathcal{U} on the set M^α s.t. for any $L(MB)$ -formula $\varphi(\bar{x})$

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Leftrightarrow \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}$$

Proof. \Rightarrow : For $\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)$, let $I = M^\alpha$ and $\mathcal{F} = \{\varphi(M^\alpha) : \varphi(\bar{x}) \in \text{tp}(\bar{a}/MB)\}$. We claim that \mathcal{F} has FIP. Let \mathcal{U} be an ultrafilter on M^α extending \mathcal{F} . Then for any $L(MB)$ -formula

$$\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \in \mathcal{F} \Rightarrow \varphi(M^\alpha) \in \mathcal{U} \Leftrightarrow \{\bar{a}' \in M : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}$$

Then

$$\varphi(\bar{x}) \notin \text{tp}(\bar{a}/MB) \Rightarrow \neg\varphi(\bar{x}) \in \text{tp}(\bar{a}/MB) \Rightarrow \varphi(M^\alpha) \notin \mathcal{U}$$

\Leftarrow :

□

Proposition 9.35. Suppose $p \in S_n(M)$ and $N \succeq M$

1. If $q \in S_n(N)$ is a coheir of p , then there is an ultrafilter \mathcal{U} on M^n s.t.

$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\} \quad (\star)$$

2. Conversely, if \mathcal{U} is an ultrafilter on M^n and we define $q(\bar{x})$ according to (\star) , then $q(\bar{x}) \in S_n(N)$ and q is a coheir of p

Proof. 1. Take \bar{a} realizing q and p , then $\bar{a} \downarrow_M^u N$. Apply proposition 9.34

2. It suffices to show that q is finitely satisfiable in M and complete

□

Corollary 9.36 (Coheirs extend). Suppose $M \preceq N \preceq N'$ and $p \in S_n(M)$ and $q \in S_n(N)$ is a coheir of p , then is $q' \in S_n(N')$ with $q' \supseteq q$ and q' is a coheir of p

Proof. By proposition 9.35 there is an ultrafilter \mathcal{U} on M^n s.t.

$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\}$$

Take $q'(\bar{x}) = \{\varphi(\bar{x}) \in L(N') : \varphi(M^n) \in \mathcal{U}\}$

□

Remark. Suppose $q \in S_n(N)$ is an heir of $p \in S_n(M)$. Then $N \downarrow_M^u \bar{a}$ for a realization \bar{a} . Proposition 9.34 gives an ultrafilter \mathcal{U} and tells us something, ultimate conclusion is

There is an ultrapower $M^\mathcal{U} \succeq N$ s.t. $p^\mathcal{U} \supseteq q$

9.5.4 Symmetry

Suppose $q \in S_n(N)$ is an extension of $p \in S_n(M)$.

In stable theory, coheir and heir are the same thing, so for any $q \in S_n(N)$ and $p \in S_n(M)$, $M \preceq N$

$$\bar{a} \downarrow_M^u N \Leftrightarrow N \downarrow_M^u \bar{a}$$

Theorem 9.37. If T is stable, then

$$\bar{a} \downarrow_M^u \bar{b} \Leftrightarrow \bar{b} \downarrow_M^u \bar{a}$$

Proof. It suffices to prove \Rightarrow . Let α be the length of \bar{a} . Take a small model N containing M and \bar{b} . By the method of 9.36, one can find a type $q \in S_\alpha(N)$ extending $\text{tp}(\bar{a}/M\bar{b})$ finitely satisfiable in M . Take \bar{a}' realizing q . Then $\bar{a}' \downarrow_M^u N$. Also $\text{tp}(\bar{a}'/M\bar{b}) = q \upharpoonright (M\bar{b}) = \text{tp}(\bar{a}/M\bar{b})$, so there is $\sigma \in \text{Aut}(\mathbb{M}/M\bar{b})$ s.t. $\sigma(\bar{a}') = \bar{a}$. Then

$$\bar{a}' \downarrow_M^u N \Rightarrow \sigma(\bar{a}') \downarrow_{\sigma(M)}^u \sigma(N) \Leftrightarrow \bar{a} \downarrow_M^u \sigma(N)$$

Replacing N with $\sigma(N)$, we may assume $\bar{a} \downarrow_M^u N$. Therefore we have $N \downarrow_M^u \bar{a}$. As $\bar{b} \in N$, this implies $\bar{b} \downarrow_M^u \bar{a}$ \square

9.5.5 Finitely satisfiable types commute with definable types

Recall that if $M \leq N \leq \mathbb{M}$, then

$$N \downarrow_M^u \bar{a} \Leftrightarrow \text{tp}(\bar{a}/N) \supseteq \text{tp}(\bar{a}/M)$$

Therefore the following lemma generalizes the fact that definable types have unique types

Lemma 9.38. *Let M be a small model. Suppose $\text{tp}(\bar{a}/M)$ is definable and $\bar{b} \downarrow_M^u \bar{a}$. Then $\text{tp}(\bar{a}/M\bar{b})$ is $p \upharpoonright M\bar{b}$, where p is the M -definable global type extending $\text{tp}(\bar{a}/M)$*

Proof. We must show that for any L -formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and any $\bar{c} \in M$,

$$\varphi(\bar{x}, \bar{b}, \bar{c}) \in \text{tp}(\bar{a}/M\bar{b}) \Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c})$$

Otherwise, these things are true

$$\begin{aligned} \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c}) \\ \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{b}, \bar{c}) \\ (\varphi(\bar{a}, \bar{y}, \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{y}, \bar{c})) \in \text{tp}(\bar{b}/M\bar{a}) \end{aligned}$$

As $\bar{b} \downarrow_M^u$, there is $\bar{b}' \in M$ s.t.

$$\begin{aligned} \mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) &\Leftrightarrow (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \\ \mathbb{M} \models \varphi(\bar{a}, \bar{b}', \bar{c}) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \\ \varphi(\bar{x}, \bar{b}', \bar{c}) \in \text{tp}(\bar{a}/M) &\Leftrightarrow \mathbb{M} \models (d_p \bar{x})\varphi(\bar{x}, \bar{b}', \bar{c}) \end{aligned}$$

A contradiction \square

Lemma 9.39. *Let $p \in S_n(\mathbb{M})$ be finitely satisfiable in a small model M . If $\bar{a} \models p \upharpoonright M\bar{b}$, then $\bar{a} \downarrow_M^u \bar{b}$*

Theorem 9.40. *Let p, q be global types. Suppose p is definable over some small set A . (p is A -invariant) Suppose q is finitely satisfiable in some small set B (q is B -invariant by 9.49). Then p and q commute*

Proof. Otherwise, there is an $L(\mathbb{M})$ -formula $\varphi(\bar{x}, \bar{y})$ s.t.

$$\begin{aligned} (p \otimes q)(\bar{x}, \bar{y}) &\vdash \varphi(\bar{x}, \bar{y}) \\ (q \otimes p)(\bar{y}, \bar{x}) &\vdash \neg \varphi(\bar{x}, \bar{y}) \end{aligned}$$

The formula φ uses only finitely many parameters \bar{c} from \mathbb{M} . By Löwenheim–Skolem Theorem there is a small model M containing $AB\bar{c}$. Then $\varphi(\bar{x}, \bar{y})$ is an $L(M)$ -formula. Also, p is M -definable and q is finitely satisfiable in M . Note that p, q and $p \otimes q, q \otimes p$ are M -invariant types. Take $(\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright M$ and $\bar{a} \models p \upharpoonright M, \bar{b} \models q \upharpoonright M\bar{a}$. By Lemma 9.39, $\bar{b} \downarrow_M^u \bar{a}$

Now $\text{tp}(\bar{a}/M)$ is the definable type $p \upharpoonright M$, so by Lemma 9.39

$$\bar{a} \models p \upharpoonright M\bar{b}$$

Thus $(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$

It follows that $(q \otimes p)(\bar{y}, \bar{x})$ and $(p \otimes q)(\bar{x}, \bar{y})$ have the same restriction to M . Then φ leads to a contradiction \square

9.5.6 Types commute in stable theories

Assume the theory T is stable

Proposition 9.41 (Assuming stability). *Let $p \in S_n(\mathbb{M})$ be a global type and M be a small model. TFAE*

1. p is finitely satisfiable in M
2. p is M -invariant
3. p is M -definable

Proof. $1 \rightarrow 2$: 9.49

$2 \rightarrow 3$: 9.51 \square

Theorem 9.42 (Assuming stability). *Let $p(\bar{x}), q(\bar{y})$ be two invariant global types. Then p and q commute*

Proof. The types p and q are invariant over small sets A and B respectively. Take a small model M containing $A \cup B$. Then p and q are M -invariant. By Proposition 9.41, p is M -definable and p is finitely satisfiable in M . Therefore p and q commute by Theorem 9.40 \square

9.5.7 Morley products and \downarrow^u

Let M be a small model. If p and q are M -definable types, then the Morley product $p \otimes q$ is also M -definable by 9.63. Since M -definable global types corresponds to (M) -definable types over M (Proposition 9.48), we can regard \otimes as an operation on definable types over M

If T is stable, then all types over M are definable, and we get an operation

$$\begin{aligned} S_n(M) \times S_n(M) &\rightarrow S_{m+n}(M) \\ (p, q) &\mapsto p \otimes q \end{aligned}$$

The following theorem shows that, at least in stable theories, there is a very close connection between the Morley product $p \otimes q$ and the coheir independence relation $\bar{a} \downarrow_M^u \bar{b}$

Theorem 9.43. *Assume T is stable. Let $M \preceq \mathbb{M}$ be a small model and \bar{a}, \bar{b} be tuples in \mathbb{M} . Then*

$$\bar{a} \downarrow_M^u \bar{b} \Leftrightarrow \text{tp}(\bar{b}, \bar{a}/M) = \text{tp}(\bar{b}/M) \otimes \text{tp}(\bar{a}/M)$$

Proof. First suppose $\bar{a} \downarrow_M^u \bar{b}$. Then $\text{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M . By Lemma 9.28, there is a global type p which is finitely satisfiable in M and extends $\text{tp}(\bar{a}/M\bar{b})$. By Proposition 9.41, p is M -definable. Then p is the unique M -definable global extension of the definable type $\text{tp}(\bar{a}/M)$. Let q be the unique M -definable global extension of the definable type $\text{tp}(\bar{b}/M)$. Then

$$\bar{b} \models q \upharpoonright M \quad \text{and} \quad \bar{a} \models p \upharpoonright M\bar{b}$$

because p extends $\text{tp}(\bar{a}/M\bar{b})$. Therefore

$$(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$$

or equivalently, $\text{tp}(\bar{b}, \bar{a}/M) = (q \otimes p) \upharpoonright M$.

Conversely, suppose $\text{tp}(\bar{b}, \bar{a}/M) = \text{tp}(\bar{b}/M) \otimes \text{tp}(\bar{a}/M)$. Let q be the unique M -definable global extension of the definable type $\text{tp}(\bar{b}/M)$ and let p be the unique M -definable global extension of the definable type $\text{tp}(\bar{a}/M)$ by 9.48. Then

$$(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright M$$

or equivalently

$$\bar{b} \models q \upharpoonright M \quad \text{and} \quad \bar{a} \models p \upharpoonright M\bar{b}$$

By Proposition 9.41 p is finitely satisfiable in M , and so

$$\bar{a} \models p \upharpoonright M\bar{b} \Rightarrow \bar{a} \underset{M}{\overset{u}{\downarrow}} \bar{b}$$

by Lemma 9.39 □

9.6 Invariant types

Lemma 9.44. *If $X \subseteq \mathbb{M}^n$, TFAE*

1. $\sigma(X) = X$ if $\sigma \in \text{Aut}(\mathbb{M}/A)$
2. If $\bar{a}, \bar{b} \in \mathbb{M}^n$, $\bar{a} \equiv_A \bar{b} \Rightarrow (\bar{a} \in X \Leftrightarrow \bar{b} \in X)$
3. There is $f : S_n(A) \rightarrow \{0, 1\}$ s.t. $\bar{a} \in X \Leftrightarrow f(\text{tp}(\bar{a}/A)) = 1$

Proof. rewrite (2) as

- If $\bar{a}, \bar{b} \in \mathbb{M}^n$, $\sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(\bar{a}) = \sigma(\bar{b})$, then $\bar{a} \in X \Leftrightarrow \bar{b} \in X$
 - If $\bar{a} \in M$, $\sigma \in \text{Aut}(\mathbb{M}/A)$, $\bar{a} \in X \Leftrightarrow \sigma(\bar{a}) \in X$
-

Definition 9.45. $X \subseteq \mathbb{M}^n$ is **A -invariant** if $\forall \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(X) = X$

Example 9.5. If X is A -definable, then X is A -invariant

Lemma 9.46. *If $D \subseteq \mathbb{M}^n$ is definable and A -invariant, then D is A -definable*

Proof. Step 1: If $\bar{b} \in D$ then $\text{tp}(\bar{b}/A) \vdash \bar{x} \in D$, by compactness, there is $\varphi(\bar{x}) \in \text{tp}(\bar{b}/A)$ s.t. $\varphi(\bar{x}) \vdash \bar{x} \in D$, $\varphi(\mathbb{M}^n) \subseteq D$

Step 2: So then D is covered by A -definable subsets of D . By compactness, \bar{D} is covered by finitely many of them, which implies D is A -definable

Suppose $D = \psi$, then $[\psi] = \bigcup [\varphi_i]$ □

Definition 9.47. p is **A -definable** if $\forall \varphi$, $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is A -definable

Remark. 1. p is A -definable $\Rightarrow p$ is A -invariant

2. If p is definable, then p is A -invariant $\Leftrightarrow p$ is A -definable

3. If p is definable thne p is A -definable for some small A

Each $d_p\varphi$ uses only finitely many parameters

Proposition 9.48. *Suppose $M \leq \mathbb{M}$, small*

1. *If $p \in S_n(M)$ definable and $p^{\mathbb{M}}$ is its heir over \mathbb{M} , then $p^{\mathbb{M}} \in S_n(\mathbb{M})$ is M -definable*
2. *$p \mapsto p^{\mathbb{M}}$ is a bijection from definable types over M to M -definable types over \mathbb{M}*

Proof. 1. $p^{\mathbb{M}}$ has the same definition as p , so it's M -definable

2. $q \mapsto q \upharpoonright M$ is an inverse to $p \mapsto p^{\mathbb{M}}$

□

Warning: an M -invariant type p is not determined by $p \upharpoonright M$. If $A \subseteq \mathbb{M}$, A -definable type p is not determined by $p \upharpoonright A$. Only works for models
CHECK

Theorem 9.49. *Suppose $M \leq \mathbb{M}$ and $p \in S_n(M)$*

1. *If $q \in S_n(\mathbb{M})$ and q is a coheir of p , then q is M -invariant*
2. *$\exists q \in S_n(\mathbb{M}), p \subseteq q$ is M -invariant*

Proof. If q is a coheir of p , but q is not M -invariant, then $\exists \bar{b}, \bar{c}, \bar{b} \equiv_M \bar{c}$, $\varphi(\bar{x}, \bar{b}) \in q$, $\varphi(\bar{x}, \bar{c}) \notin q$. Then $\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$. Because q is fsat. in M , $\exists \bar{a} \in M$, $M \models \varphi(\bar{a}, \bar{b}) \wedge \neg\varphi(\bar{a}, \bar{c})$, so $\bar{b} \not\equiv_M \bar{c}$ □

In stable theories:

Lemma 9.50. *If T is stable and p is A -invariant, then p is A -definable*

Theorem 9.51. *Suppose T stable, $M \leq \mathbb{M}$ small, $p \in S_n(M)$. Let $p^{\mathbb{M}}$ the global heir.*

1. *$p^{\mathbb{M}}$ is the only M -invariant global type extending p*
2. *$p^{\mathbb{M}}$ is the only global coheir of p*
3. *If $M \leq N \leq \mathbb{M}$ and q is the heir of p over N , then q is the unique coheir of p over N*

Proof. 1. M -invariant $\Leftrightarrow M$ -definable

2. there is some coheir of p . Any coheir is M -invariant, so p^M is the only coheir

□

Corollary 9.52. *In a stable theory, coheirs are unique and coheir=heir*

Corollary 9.53. *In a stable theory, “coheir” is transitive*

9.7 Morley sequences and the order property

9.7.1 Morley sequence

Lemma 9.54. *If p, q are A -invariant global types, $p \in S_n(\mathbb{M})$, $q \in S_m(\mathbb{M})$, then there is $r \in S_{n+m}(A)$ s.t. $(\bar{b}, \bar{c}) \models r$ iff*

$$\bar{b} \models p \upharpoonright A \quad \text{and} \quad \bar{c} \models q \upharpoonright (A\bar{b}) \quad (\star)$$

Proof. Let $X = \{(\bar{b}, \bar{c}) : \bar{b} \models p \upharpoonright A \text{ and } \bar{c} \models q \upharpoonright A\bar{b}\}$. If $(\bar{b}, \bar{c}) \in X$ and $\sigma \in \text{Aut}(\mathbb{M}/A)$, then $\sigma(\bar{b}) \models \sigma(p \upharpoonright A) = p \upharpoonright A$ and $\sigma(\bar{c}) \models q \upharpoonright A\sigma(\bar{b})$. So $\sigma(\bar{b}, \bar{c}) \in X$, X is A -invariant

Fix $\bar{b}_0 \models p \upharpoonright A$, $\bar{c}_0 \models q \upharpoonright A\bar{b}_0$, so $(\bar{b}_0, \bar{c}_0) \in X$. Let $r = \text{tp}(\bar{b}_0, \bar{c}_0/A)$. If $(\bar{b}, \bar{c}) \models r$, then $(\bar{b}, \bar{c}) \in X$

Conversely, if $(\bar{b}, \bar{c}) \in X$, want $(\bar{b}, \bar{c}) \models r$, i.e., $(\bar{b}, \bar{c}) \equiv_A (\bar{b}_0, \bar{c}_0)$

$\bar{b} \models p \upharpoonright A = \text{tp}(\bar{b}_0/A)$ so $\bar{b} \equiv_A \bar{b}_0$, $\exists \sigma \in \text{Aut}(A)$, $\sigma(\bar{b}) = \bar{b}_0$. Replace (\bar{b}, \bar{c}) with $(\sigma(\bar{b}), \sigma(\bar{c})) = (\bar{b}_0, \sigma(\bar{c}))$.

WMA $\bar{b} = \bar{b}_0$. Then \bar{c} and \bar{c}_0 both satisfy $q \upharpoonright A\bar{b}_0$. Move \bar{c} by $\tau \in \text{Aut}(\mathbb{M}/A\bar{b}_0)$, we may assume $\bar{c} = \bar{c}_0$. Then $\bar{c} \equiv_{A\bar{b}_0} \bar{c}_0 \Rightarrow \bar{b}\bar{c} \equiv_A \bar{b}_0\bar{c}_0$ □

Proposition 9.55. *If $p \in S_n(\mathbb{M})$, $q \in S_m(\mathbb{M})$ and both are A -invariant, then there is A -invariant $p \otimes q \in S_{n+m}(\mathbb{M})$ s.t. for any small $A' \supseteq A$,*

$$(\bar{b}, \bar{c}) \models (p \otimes q) \upharpoonright A' \Leftrightarrow \bar{b} \models p \upharpoonright A' \text{ and } \bar{c} \models q \upharpoonright A'\bar{b}$$

Proof. Note p, q are A' -invariant for any A' -invariant, so lemma gives $r_{A'} \in S_{n+m}(A')$ for each $A' \supseteq A$ s.t. $(\bar{b}, \bar{c}) \models r_{A'} \Leftrightarrow$ the condition

If $A'' \supseteq A' \supseteq A$, if $(\bar{b}, \bar{c}) \models r_{A''}$ then $(\bar{b}, \bar{c}) \models r_{A'}$ so $r_{A'} \models r_{A'} \upharpoonright A'$.

Let $p \otimes q = \bigcup_{A'} r_{A'}$, then $p \otimes q \in S_{n+m}(\mathbb{M})$ and $r_{A'} = p \otimes q \upharpoonright A'$ □

If $\sigma \in \text{Aut}(\mathbb{M}/A)$, then $\sigma(p \otimes q) = \sigma(p) \otimes \sigma(q) = p \otimes q$, so $p \otimes q$ is A -invariant

Fact 9.56. *If $p \in S_n(M)$ A -invariant where M is $|A|^+$ -saturated and $N \geq M$, then p has a unique A -invariant extension over N*

Fact 9.57. If $p, q \in S_{n+m}(\mathbb{M})$ A -invariant, take $\bar{b} \models p, \bar{b} \in \mathbb{M}_1 \geq \mathbb{M}$, take $\bar{c} \models q \upharpoonright \mathbb{M}_1$ then $\text{tp}(\bar{b}, \bar{c}/\mathbb{M}) = p \otimes q$

Definition 9.58. The **(Morley) product** of invariant types p, q is $p \otimes q$

If p, q are A -invariant, then $(\bar{b}, \bar{c}) \models (p \otimes q) \upharpoonright A \Leftrightarrow \bar{b} \models p \upharpoonright A$ and $\bar{c} \models q \upharpoonright A\bar{b}$

Definition 9.59. $\text{acl}(A) = \bigcup \{ \varphi(\mathbb{M}) : \varphi(x) \in L(A), |\varphi(\mathbb{M})| < \infty \}$

Fact 9.60. In ACF, if K a subfield of \mathbb{M} , then $\text{acl}(K)$ is K^{alg}

Fact 9.61. In any theory T , $\text{acl}(-)$ is a finitary closure operation

Example 9.6. If T is strongly minimal and $p \in S_1(\mathbb{M})$ transcendental 1-type, what is $p \otimes p$

$b \models p \upharpoonright A \Leftrightarrow b \notin \text{acl}(A)$

Therefore $(b, c) \models (p \otimes p) \upharpoonright A$ iff $b \models p \upharpoonright A$ and $c \models p \upharpoonright A\bar{b}$ iff $b \notin \text{acl}(A)$ and $c \notin \text{acl}(A\bar{b})$

idea: b, c are algebraically independent over A

In stable theories, $(p \otimes q)(x, y)$ is the “most free” completion of $p(\bar{x}) \cup q(\bar{y})$

Example 9.7. Suppose $\mathbb{M} \models \text{ACF}$. let p_V denote generic type of a variety $V \subseteq \mathbb{M} \{x \in V\} \cup \{x \notin W : W \subsetneq V, W \text{ algebraic}\}$

If $V \subseteq \mathbb{M}^n, W \subseteq \mathbb{M}^m$ varieties, then $V \times W$ is a variety, and $p_V \otimes p_W = p_{V \times W}$

Proof. $p_V \otimes p_W = p_Z$ for some variety $Z \subseteq \mathbb{M}^{n+m}$. Take small $M \preceq \mathbb{M}$ s.t. V, W, Z are M -definable. Take $\bar{a} \models p_V \upharpoonright M$, take small $N \preceq \mathbb{M}, N \supseteq M\bar{a}$. Take $\bar{b} \models p_W \upharpoonright N$, so $(\bar{a}, \bar{b}) \models p_V \otimes p_W \upharpoonright M = p_Z \upharpoonright M$.

“ $x \in V \in p_V \upharpoonright M$ ”, $\bar{a} \in V, \bar{b} \in W$, so $(\bar{a}, \bar{b}) \in V \times W$.

Fact: $p_Z(\bar{x}) \upharpoonright \bar{x} \in U \Leftrightarrow Z \subseteq U$ for U algebraic

So $(\bar{a}, \bar{b}) \in V \otimes W \Leftrightarrow Z \subseteq V \times W$

Suppose $Z \subsetneq V \times W$. Take $(\bar{a}_0, \bar{b}_0) \in V \times W \setminus Z$. Let $Z_{\bar{a}} = \{\bar{y} \in M : (\bar{a}, \bar{y}) \in Z\}$, then $Z_{\bar{a}}$ is an algebraic set over $N \supseteq M_{\bar{a}}$

L

□

Definition 9.62. invariant types p, q “commute” if $p \otimes q(\bar{x}, \bar{y}) = q \otimes p(\bar{y}, \bar{x})$

Example 9.8. In ACF, any two types commutes

$p_V \otimes p_W = p_{V \times W} = p_W \otimes p_V$

If p is a definable type and $\varphi(\bar{x}, \bar{y})$ is a formula, then $(d_p \bar{x})\varphi(\bar{x}, \bar{y})$ means $d\varphi(\bar{y})$, the formula defining $\{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$

$d_p \bar{x}$ works like quantifier, free variables in $(d_p \bar{x})\varphi(\bar{x}, \bar{y})$ are \bar{y}

Example 9.9. Suppose $\mathbb{M} \models T$ strongly minimal, let $p =$ transcendental 1-type, $\varphi()$

Proposition 9.63. *If p, q are A -definable global types, then $p \otimes q$ is A -definable and $(d_{p \otimes q}(\bar{x}, \bar{y}))\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{z})$*

Proof. Fix $\bar{c} \in \mathbb{M}$, take $M \preceq \mathbb{M}$ s.t. $\bar{c} \in M$ and $M \supseteq A$, so p, q are M -definable. Take $\bar{a} \models p \upharpoonright M$ and $\bar{b} \models q \upharpoonright M\bar{a}$, so $(\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright M$. So

$$\begin{aligned} \varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q &\Leftrightarrow \varphi(\bar{x}, \bar{y}, \bar{c}) \in p \otimes q \upharpoonright M \\ &\Leftrightarrow \mathbb{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \\ &\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \upharpoonright M\bar{a} \\ &\Leftrightarrow \varphi(\bar{a}, \bar{y}, \bar{c}) \in q(\bar{y}) \\ &\Leftrightarrow \mathbb{M} \models (d_q \bar{y})\varphi(\bar{a}, \bar{y}, \bar{c}) \\ &\Leftrightarrow (d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \in p(\bar{x}) \\ &\Leftrightarrow (d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{c}) \end{aligned}$$

□

Example 9.10. in a strongly minimal theory, if $p \in S_1(\mathbb{M})$ is transcendental and $q = p \otimes p$ then $(d_q(x, y))\varphi(x, y, \bar{z})$ is $\exists^\infty x \exists^\infty y \varphi(x, y, \bar{z})$

Two definable types p, q commute iff $(d_p \bar{x})(d_q \bar{y})\varphi(\bar{x}, \bar{y}, \bar{z}) \equiv (d_q \bar{y})(d_p \bar{x})\varphi(\bar{x}, \bar{y}, \bar{z})$
Let A -invariant $p \in S_n(\mathbb{M})$

Definition 9.64. A **Morley sequence** of p over A is a sequence $\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots \in \mathbb{M}^n$ s.t.

$$\bar{b}_1 \models p \upharpoonright A, \bar{b}_2 \models p \upharpoonright A\bar{b}_1, \dots, \bar{b}_i \models p \upharpoonright A\bar{b}_1 \dots \bar{b}_{i-1} \dots$$

$$\text{So } (\bar{b}_1, \dots, \bar{b}_n) \models \underbrace{p \otimes \dots \otimes p}_{n \text{ times}}$$

Example 9.11. If T is strongly minimal, p is transcendental 1-type, a Morley sequence over A is b_1, b_2, \dots s.t. $b_1 \notin \text{acl}(A), b_2 \notin \text{acl}(Ab_1), \dots$

Example 9.12. In DLO, in (\mathbb{R}, \leq) , $1, 2, 3, 4, \dots$ is indiscernible

An increasing sequence is indiscernible in DLO

Theorem 9.65. *If $p \in S_n(\mathbb{M})$ A -invariant and $(\bar{b}_i : i < \omega)$ is a Morley sequence of p over A , then it is A -indiscernible*

9.7.2 Order Property

Remark. If φ has O.P., then $\neg \varphi$

9.7.3 Instability from the order property

Lemma 9.66. For any infinite $\lambda \geq \aleph_0$ there is a linear order (I, \leq) and $S \subseteq I$ s.t. $|I| > \lambda$, $|S| \leq \lambda$, S is dense in I

Proof. there is μ s.t. $|2^\mu| > \lambda$ and $|2^{<\mu}| \leq \lambda$.

Let $I = 2^\mu \cup 2^{<\mu}$ and $S = 2^{<\mu}$ □

Theorem 9.67. If $\varphi(\bar{x}, \bar{y})$ has O.P., then T is not λ -stable for any λ

Proof. Take $I \supseteq S$ s.t. S dense in I , $|S| \leq \lambda$, $|I| > \lambda$

$\bar{a}_i, \bar{b}_j, i, j \in \mathbb{Z}$, $\varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$. By compactness, we can take any linear order. There is \bar{a}_i, \bar{b}_j for $i, j \in I$ s.t. $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$

Let $C = \{\bar{b}_j : j \in S\}$, $|C| \leq \lambda$.

Claim $I \setminus S \rightarrow S_n(C)$, $i \mapsto \text{tp}(\bar{a}_i/C)$ is an injection

If $i_1 < i_2$, then there is $j \in S$, $i_1 < j < i_2$ then $\varphi(\bar{a}_{i_1}, \bar{b}_j) \wedge \neg \varphi(\bar{a}_{i_2}, \bar{b}_j)$, $\bar{b}_j \in C$, so $\bar{a}_{i_1} \not\equiv_C \bar{a}_{i_2}$
 $|S_n(C)| \geq |I \setminus S| > \lambda$ □

9.7.4 The order property from instability

Lemma 9.68. Suppose $\varphi(\bar{x}, \bar{y})$ doesn't have O.P. Let n_φ be from Lemma 9. Let $\bar{b}_1, \bar{b}_2, \dots$ be indiscernible (over \emptyset). Then there is no \bar{a} s.t. $\mathbb{M} \models \varphi(\bar{a}, \bar{b}_i)$ for $0 \leq i < n_\varphi$ s.t.

Proof. $n = n_\varphi$. Suppose \bar{a} exists, for $0 \leq$ □

Lemma 9.69. Suppose $\varphi(x_1, \dots, x_n; \bar{y})$ doesn't have O.P.. Take $N > \max(n_\varphi, n_{\neg\varphi})$. let p be an A -invariant type over \mathbb{M} . Let a_1, a_2, \dots be a Morley sequence of p over A

1. If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$
2. If $\varphi(\bar{x}, \bar{b}) \notin p(\bar{x})$, then $\mathbb{M} \models \neg \varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$

Example 9.13. If T is strongly minimal then T is stable if $\varphi(x, \bar{y})$ has the O.P., then there is $a_i, \bar{b}_i \in \mathbb{M}$ $\mathbb{M} \models \varphi(a_i, \bar{b}_j) \Leftrightarrow i < j$ for $i, j \in \mathbb{Z}$

So $\varphi(\mathbb{M}, \bar{b}_0)$ is neither finite or cofinite

9.7.5 Commuting types

Theorem 9.70. If T is stable and p and q are global types (all types are definable and hence invariant for some A), then $(p \otimes q)(\bar{x}, \bar{y}) = (q \otimes p)(\bar{y}, \bar{x})$

Proof. Suppose not. Take $\varphi(\bar{x}, \bar{y}) \in L(\mathbb{M})$. $\varphi(\bar{x}, \bar{y}) \in (p \otimes q)(\bar{x}, \bar{y})$, $\varphi(\bar{x}, \bar{y}) \notin (q \otimes p)(\bar{y}, \bar{x})$.

Take A s.t. p, q are A -definable and $\varphi(\bar{x}, \bar{y}) \in L(A)$

Take $p \otimes q \otimes p \otimes q \otimes \dots$

$((b_i, c_i) : i \in \omega)$ a Morley sequence of $p \otimes q$ over A

If $i \leq j$, $(b_i, c_j) \models p \otimes q \upharpoonright A$, $\mathbb{M} \models \varphi(b_i, c_j)$

If $i > j$, $(c_j, b_i) \models q \otimes p \upharpoonright A$, $\mathbb{M} \models \neg \varphi(b_i, c_j)$ □

9.8 Ramsey's theorem and indiscernible sequences

Definition 9.71. X set, C a set of "colors", then $f : [X]^\kappa \rightarrow C$ is a coloring of κ -elements subsets of X

Definition 9.72. $Y \subseteq X$ is **homogeneous** if $f \upharpoonright [Y]^\kappa$ is constant

Definition 9.73. If N, m, n, k are cardinals, $N \rightarrow (m)_k^n$ means that if $|X| = N$, $|C| = k$, $f : [X]^n \rightarrow C$, then there is $Y \subseteq X$, Y is homogeneous and has size m

Fact 9.74 (Friends and strangers theorem). $|X| = 6$, $|C| = 2$ and $f : [X]^2 \rightarrow C$, then there is $Y \subseteq X$ homogeneous and size 3

Theorem 9.75 (Finite Ramsey's theorem). If $n, m, k \in \omega$ then there is $N < \omega$ s.t. $N \rightarrow (m)_k^n$

Proof. Let $L = \{R_1, \dots, R_k\}$, R_i is an n -ary predicate (relation) symbol. T is the L -theory that says:

- If $R_i(\bar{x})$ then \bar{x} is distinct
- If \bar{x} is distinct then $R_i(\bar{x})$ holds for exactly one i
- If \bar{y} is a permutation of \bar{x} , $R_i(\bar{x}) \leftrightarrow R_i(\bar{y})$

A model of T is a set M and a coloring of $[M]^n$

Let φ be the formula s.t. $M \models \varphi \Leftrightarrow$ there is a homogeneous $Y \subseteq M$, $|Y| = m$

$$\exists y_1, \dots, y_m \bigwedge_{1 \leq i_1 < \dots < i_n \leq m} \bigwedge_{1 \leq j_1 < \dots < j_n \leq m} \text{same color}$$

Suppose $N \not\rightarrow (m)_k^n$, then $\exists M \models T$ $|M| = N$ and $M \not\models \varphi$. Suppose $N \not\rightarrow (m)_k^n$ for any $N < \omega$, then by compactness, $T \cup \{\neg \varphi\}$ has infinite models. By theorem 17 last week, there is $M \models T \cup \{\neg \varphi\}$, indiscernible sequence $a_1, a_2, \dots \in M$ not constant, but indiscernibility $\Rightarrow \{a_1, a_2, \dots\}$ is homogeneous, $\{a_1, \dots, a_m\}$ is homogeneous □

Fact 9.76 (Infinite Ramsey's theorem). $\aleph_0 \rightarrow (\aleph_0)_k^n$ for $n, k \in \omega$

extracting indiscernibles

Working $\mathbb{M} \models T$. If (I, \leq) is a linear order and $(\bar{a}_i : i \in I)$ is a sequence in \mathbb{M} and if $B \subseteq \mathbb{M}$

Definition 9.77. $\text{tp}^{\text{EM}}(\bar{a}/B) = \{\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B) : \forall i_1 < \dots < i_n \in I, \mathbb{M} \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n})\}$, the **Ehrenfeucht-Mostowski type** over B

Remark. tp^{EM} is really a sequence of partial types over $B, \Sigma_1, \Sigma_2, \dots$

Example 9.14. In (\mathbb{R}, \leq) , $1, 1, 2, 2, 3, 3, 4, 4, \dots$

$(x_1 \leq x_2) \in \text{tp}^{\text{EM}}(\dots)$

$x_1 < x_2 \notin \text{tp}^{\text{EM}}$

Remark. If $(\bar{a}_i : i \in I)$ is a sequence, $I_0 \subseteq I$, then $\text{tp}^{\text{EM}}((\bar{a}_i : i \in I)/B) \subseteq \text{tp}^{\text{EM}}((\bar{a}_i : i \in I_0)/B)$

Definition 9.78. If $\varphi(\bar{x}_1, \dots, \bar{x}_n) \in L(B)$, $(\bar{a}_i : i \in I)$ is “ φ -indiscernible” if $\forall i_1 < \dots < i_n, \forall j_1 < \dots < j_n$,

$$\mathbb{M} \models \varphi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) \leftrightarrow \varphi(\bar{a}_{j_1}, \dots, \bar{a}_{j_n})$$

Remark. $(\bar{a}_i : i \in I)$ is B -indiscernible iff it is φ -indiscernible for all $\varphi \in L(B)$

Definition 9.79. If Δ is a set of formulas, \bar{a} is Δ -indiscernible if it is φ -indiscernible for all $\varphi \in \Delta$

Lemma 9.80. Let $(\bar{a}_i : i \in I)$ be infinite

1. If $m < \omega$, Δ is a finite set of L -formulas, then there is Δ -indiscernible subsequence of length m
2. If (J, \leq) is a linear order, Δ a set of formulas, then there is $(\bar{b}_j : j \in J) \in \mathbb{M}$ s.t. \bar{b} is Δ -indiscernible and $\text{tp}^{\text{EM}}(\bar{b}) \supseteq \text{tp}^{\text{EM}}(\bar{a})$

Proof. 1. By induction on $|\Delta|$.

$|\Delta| = 0$, take any subsequence of length m

$|\Delta| > 0$, $\Delta = \Delta_0 \cup \{\varphi\}$, $\varphi(x_1, \dots, x_n)$. Ramsey: there is $N \rightarrow (m)_2^n$, by induction there is subsequence $(\bar{b}_i : i < N)$ Δ_0 -indiscernible. Define $f : [N]^n \rightarrow \{0, 1\}$ by

$$f(\{i_1, \dots, i_n\}) = \begin{cases} 1 & \mathbb{M} \models \varphi(b_{i_1}, \dots, b_{i_n}) \\ 0 & \text{otherwise} \end{cases}$$

there is subsequence $(\bar{c}_i : i < m)$ that is homogeneous, φ -indiscernible

2. By compactness, we may assume J is finite, Δ is finite. By part 1

□

Theorem 9.81. *If $(\bar{a}_i : i \in I)$ an infinite sequence, B is a set of parameters, (J, \leq) infinite linear order, then there is B -indiscernible sequence $(\bar{b}_j : j \in J)$ with $\text{tp}^{\text{EM}}(\bar{b}/B) \supseteq \text{tp}^{\text{EM}}(\bar{a}/B)$*

Proof. Apply Lemma 9.80 with $\Delta = \{\text{all the } L(B)\text{-formulas}\}$

□

“Extracting indiscernible sequences”

Example 9.15 (=Theorem 17 last week). If $|\mathbb{M}| = \infty$, take distinct $a_0, a_1, a_2, \dots \in \mathbb{M}$, $x_1 \neq x_2 \in \text{tp}^{\text{EM}}(\bar{a})$. Take b_0, b_1, \dots indiscernible, extracted from \bar{a} , then $(x_1 \neq x_2) \in \text{tp}^{\text{EM}}(\bar{a}) \subseteq \text{tp}^{\text{EM}}(\bar{b})$, so $b_i \neq b_j$ for $i < j$. So \bar{b} is a non-constant indiscernible sequence

Example 9.16. Suppose $\mathbb{M} \geq (\mathbb{R}, +, \cdot, \leq, 0, 1, -)$. Suppose b_1, b_2, b_3, \dots is indiscernible, extracted from $1, 2, 3, \dots$

$$\begin{aligned} x_1 > 0 &\in \text{tp}^{\text{EM}}(\bar{a}) \subseteq \text{tp}^{\text{EM}}(\bar{b}) \\ x_2 - x_1 &\geq 1 \in \text{tp}^{\text{EM}}(\bar{b}) \end{aligned}$$

Remark. $(\bar{a}_i : i \in I)$ is B -indiscernible iff $\text{tp}^{\text{EM}}(\bar{a}/B)$ is “complete”, i.e., $\forall \varphi(x_1, \dots, x_n) \in L(B)$, $\varphi \in \text{tp}^{\text{EM}}$ or $\neg \varphi \in \text{tp}^{\text{EM}}$

Theorem 9.82. *If $(\bar{a}_i : i \in I)$ is B -indiscernible, if (J, \leq) is a linear order, then there is B -indiscernible $(\bar{b}_j : j \in J)$ with $\text{tp}^{\text{EM}}(\bar{b}/B) = \text{tp}^{\text{EM}}(\bar{a}/B)$*

Remark. If $(\bar{a}_i : i \in I)$ is B -indiscernible, then $\text{tp}(\bar{a}/B)$ is determined by $\text{tp}^{\text{EM}}(\bar{a}/B)$ and (I, \leq)

$$\mathbb{M} \models \varphi(a_{i_1}, \dots, a_{i_n}) \Leftrightarrow \varphi \in \text{tp}^{\text{EM}}(\bar{a}/B)$$

So if $(\bar{a}_i : i \in I)$, $(\bar{b}_i : i \in I)$ both B -indiscernible and $\text{tp}^{\text{EM}}(\bar{a}/B) = \text{tp}^{\text{EM}}(\bar{b}/B)$, then $\text{tp}(\bar{a}/B) = \text{tp}(\bar{b}/B)$

Theorem 9.83 (extending indiscernibles). *If $(\bar{a}_i : i \in I)$ is B -indiscernible, if (J, \leq) extends (I, \leq) , then $\exists \bar{a}_j$ for $j \in J \setminus I$ s.t. $(\bar{a}_j : j \in J)$ is B -indiscernible*

Proof. extract B -indiscernible $(\bar{c}_j : j \in J)$ from $(\bar{a}_i : i \in I)$, $\text{tp}^{\text{EM}}(\bar{c}/B) = \text{tp}^{\text{EM}}(\bar{a}/B)$

the subsequence $(\bar{c}_i : i \in I)$ has same EM-type as

there is $\sigma \in \text{Aut}(\mathbb{M}/B)$ s.t. $\sigma(\bar{c}_i) = \bar{a}_i$ for $i \in I$. Define $\bar{a}_j := \sigma(\bar{c}_j)$ for $j \in J \setminus I$

□

Theorem 9.84. *If $\varphi(\bar{x}, \bar{y}) \in L$, TFAE*

1. φ has O.P., $\bar{a}_i, \bar{b}_i, i \in \mathbb{Z}, \mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$
2. same as (1) but $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$ is indiscernible
3. There is an indiscernible $(\bar{a}_i : i \in \mathbb{Z})$ some \bar{b} s.t. $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$

Proof. 1 \rightarrow 2: extract an indiscernible sequence from

2 \rightarrow 3: take $\bar{b} = \bar{b}_0$

3 \rightarrow 1: For any $j \in \mathbb{Z}, (\bar{a}_i : i \in \mathbb{Z}) \equiv_B (\bar{a}_{i+j} : i \in \mathbb{Z})$, there is $\sigma_j \in \text{Aut}(\mathbb{M})$, $\sigma_j(\bar{a}_i) = \bar{a}_{i+j}$. Let $\bar{b}_j = \sigma_j(\bar{b})$. Then $\bar{a}_i \bar{b}_j = \sigma(\bar{a}_{i-j} \bar{b})$
 $\mathbb{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow \mathbb{M} \models \varphi(\bar{a}_{i-j}, \bar{b}) \Leftrightarrow i - j < 0 \Leftrightarrow i < j$ \square

Corollary 9.85. T is unstable \Leftrightarrow there is $\varphi(\bar{x}, \bar{y})$ with O.P. $\Leftrightarrow (\bar{a}_i : i \in \mathbb{Z})$, $\varphi(\bar{x}, \bar{y}), \bar{b}$ s.t. $\varphi(\bar{a}_i, \bar{b}) \Leftrightarrow i < 0$

Total indiscernibility

Example 9.17. In DLO, 1,2,3,4,... is indiscernible but not totally indiscernible

In a totally

Proposition 9.86. If T is unstable, then \exists indiscernible sequence that isn't totally indiscernible

Proof. Take φ with O.P., take $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$ witnessing O.P., then $\varphi(a_1, b_2) \wedge \neg \varphi(a_2, b_1)$, so $(\bar{a}_i \bar{b}_i : i \in \mathbb{Z})$ isn't totally indiscernible \square

Definition 9.87. $\text{tp}(a_1, \dots, a_n/B)$ is **symmetric** if \forall permutation $\sigma \in S(n)$ $\bar{a}_1, \dots, \bar{a}_n \equiv_B \bar{a}_{\sigma(1)}, \dots, \bar{a}_{\sigma(n)}$

Remark. Let σ_i be the permutation swapping i and $i+1$ and fixing everything else.

$\text{tp}(\bar{a}_1, \dots, \bar{b}_n/B)$ is symmetric iff it holds for each σ_i

Remark. Let $(\bar{a}_i : i \in I)$ be B -indiscernible. Let $p_n = \text{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/B)$ for any $i_1 < \dots < i_n$. Then $(\bar{a}_i : i \in I)$ is totally B -indiscernible iff each p_n is symmetric

Remark. If $(\bar{a}_i : i \in I)$ is B -indiscernible, then $\text{tp}^{\text{EM}}(\bar{a}/B)$ determines whether \bar{a} is totally B -indiscernible

tp^{EM} is p_1, p_2, \dots

Lemma 9.88. Let $(\bar{a}_i : i \in \mathbb{Z})$ be B -indiscernible. Let $C = \{\bar{a}_i : i \notin \{0, 1\}\}$. If $\bar{a}_0 \bar{a}_1 \equiv_{BC} \bar{a}_1 \bar{a}_0$. Then $(\bar{a}_i : i \in \mathbb{Z})$ is totally B -indiscernible

Proof. there is $\sigma_0 \in \text{Aut}(\mathbb{M}/BC)$, $\sigma_0(\bar{a}_0) = \bar{a}_1$, $\sigma(\bar{a}_1) = \bar{b}_0$

By indiscernibility, there is $\alpha_i \in \text{Aut}(\mathbb{M}/B)$ s.t. α_i swaps \bar{a}_i, \bar{a}_{i+1} fixes \bar{a}_j for $j \notin \{i, i+1\}$. This means $\bar{a}_1 \dots \bar{a}_n \equiv_B \bar{a}_{\sigma_i(1)} \dots \bar{a}_{\sigma_i(n)}$ so $\text{tp}(\bar{a}_1, \dots, \bar{a}_n/B)$ is symmetric \square

Proposition 9.89. *If \mathbb{M} is stable and $A \subseteq \mathbb{M}$ small, then \mathbb{M} is stable as an $L(A)$ -structure*

Proof. Otherwise, there is $L(A)$ -formula $\varphi(\bar{x}, \bar{y})$ with the O.P. $\varphi(\bar{x}, \bar{y}, \bar{c})$ for some $\bar{c} \in A$, $\bar{b}_i \bar{c}$ is the new \bar{b} \square

Theorem 9.90. *TFAE*

1. T is stable
2. every indiscernible sequence is totally indiscernible
3. B -indiscernible \Rightarrow totally B -indiscernible

Proof. $3 \rightarrow 2$: trivial

$1 \rightarrow 3$: Suppose T stable but $(\bar{a}_i : i \in I)$ B -indiscernible not totally B -indiscernible

Extract $(\bar{a}'_i : i \in I)$ from $(\bar{a}_i : i \in I)$ some \square

Corollary 9.91. *If T is stable, if $(\bar{a}_i : i \in I)$ is indiscernible, if D is definable, $\{i \in I : \bar{a}_i \in D\}$ is finite or cofinite in I*

Proof. Suppose not. Take $i_1, i_2, \dots \in I$ s.t. $a_{i_1}, a_{i_2}, \dots \notin D$, \square

10 Fundamental Order and Forking

10.1 The fundamental order

Fix $n < \omega$

Definition 10.1. If $M \preceq \mathbb{M}$, $p \in S_n(M)$, $\varphi(x_1, \dots, x_n; \bar{y})$. p **represents** φ if $\exists \bar{b} \in M \varphi(\bar{x}, \bar{b}) \in p(\bar{x})$. p **omits** φ otherwise

The **class** of p is $[p] = \{\varphi : p \text{ represents } \varphi\}$

$[p] \leq [q]$ if $[p] \supseteq [q]$

The **fundamental order** is $\{[p] : M \preceq \mathbb{M}, p \in S_n(M)\}$, with \leq (depends on n). $p \leq q$ means $[p] \leq [q]$

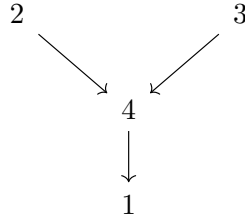
Remark. \leq is a partial order on the fundamental order but a preorder on the class $\{p : M \models T, p \in S_n(M)\}$

$[p]$ is not a standard notation

Example 10.1. $n = 1$, $\varphi(x, y) := x = y$. $p \in S_1(M)$ represents p iff $\exists b \in M$, $x = b \in p(x)$ iff p is a constant type

Example 10.2. $n = 1$, $T = \text{DLO}$, there are four classes:

1. constant types
2. types at $+\infty$
3. types at $-\infty$
4. others



$x = y$ is represented in 1

$x < y$ is represented in 1,3,4 $\text{tp}(2/\mathbb{R})$ has $x < 3$, $\text{tp}(-\infty/\mathbb{R})$ has $x < 0$, $\text{tp}(\sqrt{2}/\mathbb{Q})$ has $x < 2$, $\text{tp}(+\infty/\mathbb{R})$ doesn't have $x < b$

$x > y$ is represented in 1,2,4

$\text{tp}(\sqrt{2}/\mathbb{Q})$ and $\text{tp}(0^+/\mathbb{R})$ have the same class

Goal: in a stable theory: if q is an extension of p , then if $q \sqsupseteq p$, then $[q] = [p]$, if $q \sqsupset p$, then $[q] < [p]$

Proposition 10.2. Suppose $M \preceq N$, $p \in S_n(M)$, $q \in S_n(N)$, $p \subseteq q$

1. $[q] \leq [p]$
2. $[q] = [p]$ iff for any L -formula $\varphi(\bar{x}, \bar{y})$, if $\bar{b} \in N$ and $\varphi(\bar{x}, \bar{b}) \in q(\bar{x})$, then $\exists \bar{b}' \in M$ $\varphi(\bar{x}, \bar{b}') \in p$
3. if $q \sqsupset p$, then $[q] = [p]$

Proof. 1. every formula φ represented by p is represented by q

2. $[q] = [p] \Leftrightarrow [q] \geq [p] \Leftrightarrow [q] \subseteq [p] \Leftrightarrow$ this condition

- 3.

□

Remark. Suppose $M \leq N$, $p \in S_n(M)$, $q \in S_n(N)$, $p \subseteq q$

1. $[q] = [p]$ means that $\forall \varphi(\bar{x}, \bar{y}) \in L$, $\exists \bar{b} \in N$, $\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \Rightarrow \exists \bar{b} \in M$ $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$
 2. but $q \supseteq p$ considers $L(M)$ -formulas
- $q \supseteq p$ iff $[q] = [p]$ in $L(M)$

Proposition 10.3. $M, N \leq \mathbb{M}$, $p \in S_n(M)$, $q \in S_n(N)$, then $[p] \geq [q]$ iff \exists ultrafilter \mathcal{U} and elementary embedding $M \rightarrow N^{\mathcal{U}}$ making $q^{\mathcal{U}} \supseteq p$

Proof. \Rightarrow similar to 9.2

\Leftarrow : $[q^{\mathcal{U}}] = [q]$ because $q^{\mathcal{U}} \supseteq q$, $[q^{\mathcal{U}}] \leq [p]$ because $q^{\mathcal{U}} \supseteq p$ □

10.2 The fundamental order in stable theory

Assume T is stable

Lemma 10.4. Suppose $M \leq N \leq \mathbb{M}$, $p \in S_n(M)$, $q_1, q_2 \in S_n(N)$, $q_1, q_2 \supseteq p$ and $[q_1] = [p] = [q_2]$. Then $q_1 = q_2$.

In other words, there is at most one extension of p to N with the same class as p

Proof. similar to 9.6

Suppose $q_1 \neq q_2$, $\exists \varphi(\bar{x}, \bar{b})$ s.t. $\varphi \in q_1$, $\neg \varphi \in q_2$

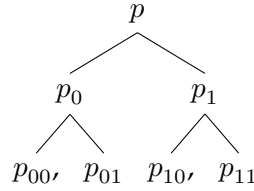
Let $\beta = [p]$

Claim: If $M' \leq \mathbb{M}$, $p' \in S_n(M')$, $[p'] = \beta$, then $\exists N' \geq M'$, $\exists q'_1, q'_2 \in S_n(N')$, $q'_1, q'_2 \supseteq p'$, $[q'_1] = [q'_2] = \beta$ and $\exists \bar{b}' \in N'$, $\varphi(\bar{x}, \bar{b}') \in q'_1$ and $\neg \varphi \in q'_2$.
 $[p'] \geq [p]$, so there \mathcal{U} , elementary embedding $M' \rightarrow M^{\mathcal{U}}$ s.t. $p^{\mathcal{U}} \supseteq p'$.

Then we have $M' \rightarrow M^{\mathcal{U}} \rightarrow N^{\mathcal{U}}$

$[q'_1] = [q_1] = \beta = [q_2] = [q'_2]$. Let $q'_i = q_i^{\mathcal{U}}$, $N' = N^{\mathcal{U}}$

Using the claim, we can build a tree of types



where $p_{\sigma 0}$ and $p_{\sigma 1}$ are extensions of p_{σ} differing by a formula $\varphi(\bar{x}, \bar{b}_{\sigma})$. Then φ has the dichotomy property □

Proposition 10.5. If $M \leq N$, $p \in S_n(M)$, $q \in S_n(N)$, $q \supseteq p$

$$1. q \sqsupseteq p \Leftrightarrow [q] = [p]$$

$$2. q \not\sqsupseteq p \Leftrightarrow [q] < [p]$$

Proof. Let q' be the heir of p , $q' \in S_n(N)$

If $q \sqsupseteq p$, then $q = q'$

If $[q] = [p]$, then $[q] = [q'] = [p]$ so Lemma 10.4 shows $q = q'$ \square

10.3 bounds

T is stable

Fix $A \subseteq \mathbb{M}$, $p \in S_n(A)$

Definition 10.6. If $M \leq \mathbb{M}$, $M \supseteq A$, then $\text{Ex}_M(p) = \{[q] : q \in S_n(M), q \sqsupseteq p\}$

Lemma 10.7. Every chain in $\text{Ex}_M(p)$ has an upper bound

Proof. Let $F = \{q \in S_n(M) : q \sqsupseteq p\}$. Suppose $\{[q_i] : i \in I\}$ is a chain, $q_i \in F$, (I, \leq) a linear order, $[q_i] \leq [q_j]$ for $i \leq j$

If $i \leq j$, q_i omits φ , then q_j omits φ

Let $\Sigma(\bar{x}) = \{\neg\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \text{ omitted by some } q, \bar{b} \in M\}$

Claim: $p(\bar{x}) \cup \Sigma(\bar{x})$ is consistent

suppose $\varphi_1, \dots, \varphi_m$, φ_j is omitted by q_{i_j} , $i_j \in I$, $\bar{b}_1, \dots, \bar{b}_m \in M$. Want $p \cup \{\neg\varphi_j(\bar{x}, \bar{b}_j) : 1 \leq j \leq m\}$ consistent. WLOG, $i_1 \leq \dots \leq i_m$, then $\varphi_j(\bar{x}, \bar{y})$ is omitted by q_{i_m} for all j . Then $q_{i_m}(\bar{x})$ extends $p(\bar{x}) \cup \{\neg\varphi_j(\bar{x}, \bar{b}_j) : 1 \leq j \leq m\}$

Take $q(\bar{x}) \in S_n(M)$ a completion of $p(\bar{x}) \cup \Sigma(\bar{x})$. Then $q \in F$, so $[q] \in \text{Ex}_M(p)$. \square

Definition 10.8. $\text{Bd}_M(p) = \{\text{maximal } \beta \in \text{Ex}_M(p)\}$

Elements of $\text{Bd}_M(p)$ are called **bounds** of p

Corollary 10.9. $\forall \beta \in \text{Ex}_M(p)$, $\exists \beta' \in \text{Bd}_M(p)$, $\beta' \geq \beta$, and $\text{Bd}_M(p)$ is not empty

Example 10.3. Suppose $A \leq \mathbb{M}$, $p \in S_n(A)$, A is a model

Claim: $[p] = \max \text{Ex}_M(p)$, so $\text{Bd}_M(p) = \{[p]\}$

Take $q \in S_n(M)$, $q \sqsupseteq p$, then $[q] = [p]$, $[q] \in \text{Ex}_M(p)$. If $r \in S_n(M)$, $r \sqsupseteq p$, then $[r] \leq [p]$, so if $p \in \text{Ex}_M(p)$ then $\beta \leq [p]$

Lemma 10.10. Suppose $M, N \leq \mathbb{M}$, $M, N \supseteq A$, $p \in S_n(A)$

$$1. \forall \beta \in \text{Ex}_M(p), \exists \beta' \in \text{Ex}_N(p), \beta' \geq \beta$$

$$2. \text{Bd}_M(p) = \text{Bd}_N(p)$$

Proof. 1. Take $M' \leq \mathbb{M}$, $M' \supseteq M \cup N$, $\beta \in \text{Ex}_M(p)$ means $\exists q \in S_n(M)$, $q \supseteq p$, $[q] = \beta$

Let $q' \in S_n(M')$ be $q' \supseteq q$

Let $r = q' \upharpoonright N$. Then $r \supseteq p$, so $[r] \in \text{Ex}_N(p)$. $[r] \geq [q'] = [q] = \beta$

2. suppose $\beta \in \text{Bd}_M(p)$

- by 1, there is $\beta' \in \text{Ex}_N(p)$ with $\beta \leq \beta'$
- by Corollary 10.9, there is $\beta'' \in \text{Bd}_N(p)$ with $\beta' \leq \beta''$
- By 1, there is $\beta''' \in \text{Ex}_M(p)$ with $\beta'' \leq \beta'''$

Then $\beta \leq \beta' \leq \beta'' \leq \beta''' \in \text{Ex}_M(p)$. Therefore

$$\beta = \beta' = \beta'' = \beta'''$$

This shows $\text{Bd}_M(p) \subseteq \text{Bd}_N(p)$

□

Since $\text{Bd}_M(p)$ doesn't depend on M , we write it as $\text{Bd}(p)$

10.4 Theorem of the bound

T is stable

Definition 10.11. $p \in S_n(\mathbb{M})$ is **Lascar A -invariant** if p is M -invariant for every $A \subseteq M \leq \mathbb{M}$

weaker than being A -invariant in stable theory

Lemma 10.12. If $A \subseteq M \leq \mathbb{M}$, $p \in S_n(A)$, $q \in S_n(M)$, $q \supseteq p$, $[q] \in \text{Bd}(p)$. Let $q^{\mathbb{M}}$ be the global heir of q . Then $q^{\mathbb{M}}$ is Lascar A -invariant

Proof. By 10.2, $[q^{\mathbb{M}}] = [q] \in \text{Bd}(p)$. If $q^{\mathbb{M}}$ isn't Lascar A -invariant, there is small $N \supseteq A$ $q^{\mathbb{M}}$ isn't N -invariant, not N -definable. Then $q^{\mathbb{M}} \not\sqsubseteq q^{\mathbb{M}} \upharpoonright N$ (or else $q^{\mathbb{M}}$ would be N -definable 9.48). By Proposition 10.5, $[q^{\mathbb{M}} \upharpoonright N] > [q^{\mathbb{M}}] = [q]$

Let $r = q^{\mathbb{M}} \upharpoonright N$, $r \supseteq p$, so $[r] \in \text{Ex}_N(p)$, $[q] \in \text{Bd}(p) = \text{Bd}_N(p)$ is maximal in $\text{Ex}_N(p)$, but $[r] > [q]$, $[r] \in \text{Ex}_N(p)$ □

Lemma 10.13. Fix \bar{b} and A , then $\exists M \supseteq A$, $M \leq \mathbb{M}$, the global heir of $\text{tp}(\bar{b}/M)$ is Lascar A -invariant. Also given $\beta \in \text{Bd}(\text{tp}(\bar{b}/A))$, can make $\text{tp}(\bar{b}/M)$ and it's heir have class β

Proof. Take $\beta \in \text{Bd}(p)$, $p = \text{tp}(\bar{b}/A)$. Take $M \supseteq A$, $M \leq \mathbb{M}$. Take $q \in S_n(M)$, $[q] = \beta$. Take $\bar{b}_0 \models q$, $\text{tp}(\bar{b}_0/A) = \text{tp}(\bar{b}/A)$. There is $\sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(\bar{b}_0) = \bar{b}$.

Move M, q, b_0 by σ . We may assume $\bar{b}_0 = \bar{b}$, so $\text{tp}(\bar{b}/M) = q$, $[q] = \beta$.

By 10.12, $q^{\mathbb{M}}$ is Lascar A -invariant \square

Lemma 10.14. Fix \bar{b}, A . Suppose $M_1, M_2 \leq \mathbb{M}$, $M_1, M_2 \supseteq A$. Let $p_i \in S_n(\mathbb{M})$ be the heir of $\text{tp}(\bar{b}/M_i)$. Suppose p_1, p_2 are Lascar A -invariant, then $p_1 = p_2$

Proof. Suppose $p_1 \neq p_2$. Take $\varphi(\bar{x}, \bar{c}) \in p_1(\bar{x})$, $\neg\varphi(\bar{x}, \bar{c}) \in p_2$.

Lemma 10.13 shows there is $M_3 \leq \mathbb{M}$, $M_3 \supseteq A$ s.t. $\text{tp}(\bar{c}/M_3) \sqsubseteq r \in S_n(\mathbb{M})$ and r is Lascar A -invariant.

Take $\bar{e} \models r \upharpoonright M_1 M_2 M_3 \bar{b}$. Note $\bar{b} \models p_1 \upharpoonright M_1$ and $\bar{e} \models r \upharpoonright M_1 \bar{b}$. Then $(\bar{b}, \bar{e}) \models (p_1 \otimes r) \upharpoonright M_1$ since p_1, r are M_1 -invariant. In stable theory, product commutes. Therefore $(\bar{e}, \bar{b}) \models (r \otimes p_1) \upharpoonright M_1$. Then $\bar{b} \models p_1 \upharpoonright M_1 e$.

$\bar{e} \models r \upharpoonright M_3 = \text{tp}(\bar{c}/M_3)$, $\bar{e} \equiv_{M_3} \bar{c}$, p_1 is M_3 -invariant. Hence $\varphi(\bar{x}, \bar{e}) \in p_1$. So $\mathbb{M} \models \varphi(\bar{c}, \bar{e})$

Same argument with p_2 , get $\mathbb{M} \models \neg\varphi(\bar{c}, \bar{e})$, a contradiction \square

Theorem 10.15. If $p \in S_n(A)$, $|\text{Bd}(p)| = 1$

Proof. Take $\bar{b} \models p$, $\beta_1, \beta_2 \in \text{Bd}(p)$. Lemma 10.13, there is $A \subseteq M_1, M_2 \leq \mathbb{M}$ s.t. $[\text{tp}(\bar{b}/M_i)] = \beta_i$ if $p_i = \text{tp}(\bar{b}/M_i)$, $p_i^{\mathbb{M}}$ is Lascar A -invariant.

Lemma 10.14 $p_1^{\mathbb{M}} = p_2^{\mathbb{M}}$ \square

Definition 10.16. $\text{bd}(p)$ = the bound of p

example

10.5 Non-forking extensions

Assume stability

Proposition 10.17. If $A \subseteq B$, $p \in S_n(A)$, $q \in S_n(B)$, $p \subseteq q$, then $\text{bd}(q) \leq \text{bd}(p)$

Proof. Take $M \supseteq B$, $M \leq \mathbb{M}$, $r \in S_n(M)$ extending q with $[r] = \text{bd}(q)$. Then r extends p , so $[r] \in \text{Ex}_M(p)$. As $\text{bd}(p)$ is the maximum of $\text{Ex}_M(p)$ we must have $[r] \leq \text{bd}(p)$ \square

Definition 10.18. If $A \subseteq B$, $p \in S_n(A)$, $q \in S_n(B)$, $q \supseteq p$, q is a **nonforking extension** of p iff $\text{bd}(q) = \text{bd}(p)$

Proposition 10.19. If $M \leq N$ and $q \in S_n(N)$ extends $p \in S_n(M)$, then q is a non-forking extension of p iff q is an heir of p

Proposition 10.19 ensures the notation $q \sqsupseteq p$ is unambiguous

Proof. $\text{bd}(p) = [p]$ and $\text{bd}(q) = [q]$ □

Proposition 10.20 (Full transitivity). *Suppose $A_1 \subseteq A_2 \subseteq A_3$ and $p_i \in S_n(A_i)$ for $i = 1, 2, 3$ with $p_1 \subseteq p_2 \subseteq p_3$. Then $p_1 \sqsubseteq p_3$ iff $p_1 \sqsubseteq p_2$ and $p_2 \sqsubseteq p_3$*

Proposition 10.21 (Extension). *If $p \in S_n(A)$ and $B \supseteq A$, then there is at least one $q \in S_n(B)$ with $q \sqsupseteq p$*

Proof. Take a small model $M \supseteq B$. Then $\text{bd}(p) \in \text{Bd}(p) \subseteq \text{Ex}_M(p)$, so there is $r \in S_n(M)$ extending p with $[r] = \text{bd}(p)$. Let $q = r \upharpoonright B$. Then $\text{bd}(r) = \text{bd}(p)$, so $r \sqsupseteq p$. By full transitivity, $q \sqsupseteq p$ □

10.6 Forking formulas and Lascar invariance

Lemma 10.22. *If $A \subseteq M \preceq \mathbb{M}$ and if the global heir of $\text{tp}(\bar{b}/M)$ is Lascar A -invariant, then $\text{tp}(\bar{b}/M) \sqsupseteq \text{tp}(\bar{b}/A)$*

Proof. Let β be the bound of $\text{tp}(\bar{b}/A)$. By Lemma 10.13 there is a small model $M' \supseteq A$ s.t. the global heir of $\text{tp}(\bar{b}/M')$ is Lascar A -invariant and has class β . By Lemma 10.14 $\text{tp}(\bar{b}/M')$ and $\text{tp}(\bar{b}/M)$ have the same global heir. By Proposition 10.2 they have the same class. Then the class of $\text{tp}(\bar{b}/M)$ is $\beta = \text{bd}(\text{tp}(\bar{b}/A))$, implying $\text{tp}(\bar{b}/M) \sqsupseteq \text{tp}(\bar{b}/A)$ □

Proposition 10.23 (Forking and Lascar A -invariance). *If p is a global type and $A \subseteq \mathbb{M}$, then $p \sqsupseteq (p \upharpoonright A)$ iff p is Lascar A -invariant*

Proof. First suppose $p \sqsupseteq (p \upharpoonright A)$. For any small model $M \supseteq A$, we have $p \sqsupseteq (p \upharpoonright M)$ by Full transitivity, which then means p is the heir of $p \upharpoonright M$ by Proposition 10.19. Then p is M -definable, so p is Lascar A -invariant

Conversely, suppose p is Lascar A -invariant. Take a small model $M \supseteq A$ and take $\bar{b} \models p \upharpoonright M$. Then p is M -definable, so p is the global heir of $p \upharpoonright M = \text{tp}(\bar{b}/M)$. By Lemma 10.22, $\text{tp}(\bar{b}/M) \sqsupseteq \text{tp}(\bar{b}/A) = p \upharpoonright A$. But p is the heir of $\text{tp}(\bar{b}/M)$, so $p \sqsupseteq \text{tp}(\bar{b}/M) \sqsupseteq p \upharpoonright A$. By transitivity we have $p \sqsupseteq (p \upharpoonright A)$ □

Corollary 10.24. *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \sqsupseteq p$ iff some global extension of q is Lascar A -invariant*

Definition 10.25. An $L(\mathbb{M})$ -formula $\varphi(\bar{x})$ **forks over** A if every global type containing it fails to be Lascar A -invariant

Proposition 10.26 (Finite Character). *If $A \subseteq B$ and $q \in S_n(B)$ extends $p \in S_n(A)$, then $q \not\sqsupseteq p$ (q is a forking extension of p) iff some formula in q forks over A*

Proof. For any model M , let $\Sigma_M(\bar{x})$ be the global partial type

$$\{\varphi(\bar{x}; \bar{b}) \leftrightarrow \varphi(\bar{x}; \bar{c}) : \varphi \in L, \bar{b} \equiv_M \bar{c}\}$$

A global type $p \in S_n(\mathbb{M})$ extends Σ_M iff it is M -invariant, iff it is M -definable. Define $\Sigma_A(\bar{x})$ to be the union of $\Sigma_M(\bar{x})$ for M ranging over small models containing A . Then $p \in S_n(\mathbb{M})$ extends $\Sigma_A(\bar{x})$ iff it is Lascar A -invariant. Therefore an $L(\mathbb{M})$ -formula $\psi(\bar{x})$ forks over A iff $\Sigma_A(\bar{x}) \cup \{\psi(\bar{x})\}$ is inconsistent. By Corollary 10.24, $q \not\sqsubseteq p$ iff $\Sigma_A(\bar{x}) \cup q(\bar{x})$ is inconsistent. Then the result follows by compactness \square

Intuition if φ forks over A , then $\varphi(\mathbb{M})$ is “small”, and $\{\varphi(\mathbb{M}) : \varphi \text{ forks over } A\}$ is an ideal

10.7 The dichotomy property and the fundamental order

Lemma 10.27. *Assume stability. Suppose $M \leq N \leq \mathbb{M}$, $p \in S_n(M)$, and $q_1, q_2 \in S_n(N)$ are exteqqeq*

11 Algebraic closure and imaginaries

11.1 Many-sorted logic

11.1.1 First approximation: many-sorted structures

Definition 11.1. A **(single sorted) structure** consists of a set M and a collection of **functions**, **relations** and **constants**. Each function is a function $f : M^{n_f} \rightarrow M$ for some number n_f called the **arity** of f . Each relation is a relation $R \subseteq M^{n_R}$ for some n_R called the **arity** of R . Each constant is an element of M

Definition 11.2. A **many-sorted structure** consists of a collection of **sorts**, **functions**, **relations** and **constants**. Each sort is a set. Each function is a function $f : X_1 \times X_2 \times \cdots \times X_n \rightarrow Y$ where X_1, X_2, \dots, X_n, Y are sorts. Each relation is a relation $R \subseteq X_1 \times \cdots \times X_n$ where X_1, \dots, X_n are sorts. Each constant is an element of a sort

This approach works if we only need to consider definable sets and formulas within a fixed structure. If we want to talk about theories or elementary equivalence, we need to define many-sorted languages before we can properly define many-sorted structures.

11.1.2 Many-sorted languages

Definition 11.3. A many-sorted language consists of the following data:

1. A set \mathcal{S} of **sorts**
2. A set \mathcal{F} of **function symbols**: for each $f \in \mathcal{F}$, a finite non-empty list of sorts (X_1, \dots, X_n, Y) , called the **signature** of f
3. A set \mathcal{R} of **relation symbols**: for each $R \in \mathcal{R}$, a finite list of sorts (X_1, \dots, X_n) , called the **signature** of R
4. A set \mathcal{C} of **constant symbols**: for each $c \in \mathcal{C}$, a sort X , called the **signature** of c

11.2 Definable closure

Work in a monster model \mathbb{M}

Fact 11.4. If \mathcal{F} is a small family of definable $D \subseteq \mathbb{M}^n$, suppose $X \subseteq \mathbb{M}^n$ definable. Suppose X is “infinite boolean combination” of \mathcal{F} , i.e., if $\bar{a}, \bar{a} \in \mathbb{M}^n$ and $\forall D \in \mathcal{F}, \bar{a} \in D \Leftrightarrow \bar{b} \in D$, then $\bar{a} \in X \Leftrightarrow \bar{b} \in X$. Then X is a (finite) boolean combination of sets in \mathcal{F}

Proof. WLOG, \mathcal{F} is closed under finite boolean combination

- If $\bar{a} \in X, \exists D \in \mathcal{F}, \bar{a} \in D$
- X is a finite union of things in \mathcal{F}

□

Fact 11.5 (9.46). If $D \subseteq \mathbb{M}^n$ definable, $A \subseteq \mathbb{M}$ small, then D is A -definable iff D is A -invariant

Definition 11.6. If D_1, \dots, D_{n+1} are A -definable, and $f : D_1 \times \dots \times D_n \rightarrow D_{n+1}$, then f is **A -definable** if $\Gamma(f) = \{(\bar{a}, b) : b = f(\bar{a})\}$ is definable

Definition 11.7. If $A \subseteq \mathbb{M}$, $\text{dcl}(A) = \{b \in \mathbb{M} : \{b\} \text{ is } A\text{-definable}\}$

Example 11.1. In a field, $a \div b \in \text{dcl}(\{a, b\})$ because $\{a \div b\}$ is $\varphi(\mathbb{M}), \varphi(x) := bx = a$

Note: if $\bar{b} \in \mathbb{M}^n, \bar{b} \in \text{dcl}(A)^n \Leftrightarrow \{\bar{b}\} \subseteq \mathbb{M}^n$ is A -definable

Proposition 11.8. If $\bar{b} \in \mathbb{M}^n, A \subseteq \mathbb{M}$, TFAE

1. $\bar{b} \in \text{dcl}(A)$, i.e., $\{\bar{b}\}$ is A -definable
2. $\forall \sigma \in \text{Aut}(\mathbb{M}/A), \sigma(\bar{b}) = \bar{b}$, i.e. $\{\bar{b}\}$ is A -invariant
3. \bar{b} is the only realization of $\text{tp}(\bar{b}/A)$

Proof. 1 \Leftrightarrow 2 by fact since $\{\bar{b}\}$ is definable

2 \Leftrightarrow 3: let $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$. □

Proposition 11.9. 1. $A \subseteq \text{dcl}(A)$

2. $A \subseteq B \Rightarrow \text{dcl}(A) \subseteq \text{dcl}(B)$

3. $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$

4. D is A -definable $\Leftrightarrow D$ is $\text{dcl}(A)$ -definable

Conditions 1-3 say that $\text{dcl}(-)$ is an abstract “closure operator”

Proof. 4. If D is $\text{dcl}(A)$ -definable, $\sigma \in \text{Aut}(\mathbb{M}/A)$, $b \in \text{dcl}(A)$, $\sigma(b) = b$ by Proposition 11.8, $\sigma \in \text{Aut}(\mathbb{M}/\text{dcl}(A))$, $\sigma(D) = D$, so D is A -invariant

3. take $b \in \text{dcl}(\text{dcl}(A))$, $\{b\}$ is $\text{dcl}(A)$ -definable, $\{b\}$ is A -definable, $b \in \text{dcl}(A)$ □

Definition 11.10. A is **definably closed** if $\text{dcl}(A) = A$

Proposition 11.11. $\text{dcl}(A)$ is the smallest definably closed set containing A

Proof. $\text{dcl}(A)$ is definably closed

$\text{dcl}(A) \supseteq A$

Now suppose $B = \text{dcl}(B)$ and $B \supseteq A$, by “monotonicity”, $\text{dcl}(B) \supseteq \text{dcl}(A)$, so $B \supseteq \text{dcl}(A)$ □

Fact 11.12. if $M \models \text{ACF}_0$, if $A \subseteq M$, then $A = \text{dcl}(A) \Leftrightarrow A$ is a subfield of M

\Rightarrow : easy \Leftarrow : harder

It fails in $\text{ACF}_{p'}$, in ACF_p , $p > 0$, $K \subseteq M$ is definably closed $\Leftrightarrow \forall x \in K, \sqrt[p]{x} \in K$

Definition 11.13. \bar{a}, \bar{b} are **interdefinable** iff $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b})$ iff $\bar{a} \in \text{dcl}(\bar{b})$ and $\bar{b} \in \text{dcl}(\bar{a})$

Lemma 11.14. $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b}) \Leftrightarrow \text{Aut}(\mathbb{M}/\bar{a}) = \text{Aut}(\mathbb{M}/\bar{b})$

Proof. By Proposition 11.8, $\text{dcl}(\bar{a}) \subseteq \text{dcl}(\bar{b}) \Leftrightarrow \bar{a} \in \text{dcl}(\bar{b}) \Leftrightarrow \text{Aut}(\mathbb{M}/\bar{b}) \subseteq \text{Aut}(\mathbb{M}/\bar{a})$ □

Lemma 11.15. *If $\text{dcl}(\bar{a}) = \text{dcl}(\bar{b})$, then $\exists \emptyset$ -definable bijection $f : X \rightarrow Y$ s.t. $f(\bar{a}) = \bar{b}$*

Proof. $\bar{a} \in \text{dcl}(\bar{b})$, so there is L -formula $\varphi_1(\bar{x}, \bar{y})$ s.t. $\{\bar{a}\} = \varphi_1(\mathbb{M}, \bar{b})$, also there is $\{\bar{b}\} = \varphi_2(\bar{a}, \mathbb{M})$

Let $\varphi = \varphi_1 \wedge \varphi_2$, can replace φ_1, φ_2 with φ .

Let $\psi(\bar{x}, \bar{y})$ be $\varphi(\bar{x}, \bar{y}) \wedge \exists! \bar{w} \varphi(\bar{w}, \bar{y}) \wedge \exists! \bar{z} \varphi(\bar{x}, \bar{z})$. Then $\psi(\bar{x}, \bar{y})$ defines a bijection and $f(\bar{a}) = \bar{b}$

We can extract the definition of $f(\bar{a}) = \bar{b}$ and fill f with garbage \square

11.3 Algebraic closure

Definition 11.16. $\text{acl}(A) = \bigcup \{D \subseteq \mathbb{M}^1 : D \text{ is } A\text{-definable}, |D| < \infty\}$

Example 11.2. In fields, $\sqrt{a} \in \text{acl}(a)$ because $\{\sqrt{a}, -\sqrt{a}\}$ is $\{a\}$ -definable

Note: $\bar{b} \in \text{acl}(A) \Leftrightarrow$ there is A -definable $D \subseteq \mathbb{M}^n, \bar{b} \in D, |D| < \infty$

If $\bar{b} \in D \subseteq \mathbb{M}^n$, then let $D_i = \pi_i(D), \pi_i(\bar{x}) = x_i$,

Proposition 11.17. *If $\bar{b} \in \mathbb{M}^n, A \subseteq \mathbb{M}$ small, let $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$*

1. *If $\bar{b} \in \text{acl}(A)$, then S is finite and A -definable*
2. *If $\bar{b} \notin \text{acl}(A)$, then S is large (S is not small)*

Proof. 1. $\bar{b} \in \text{acl}(A)$, there is D finite, A -definable, $\bar{b} \in D$. $\sigma(\bar{b}) \in \sigma(D) = D$ for $\sigma \in \text{Aut}(\mathbb{M}/A)$, so $S \subseteq D$, S is finite, then S is definable. Also S is A -invariant

for each $a \in D \setminus S$, we have a $L(A)$ -formula φ_a s.t. $\mathbb{M} \models \neg \varphi_a(a) \wedge \varphi_a(b)$. Then $\bigwedge_{a \in D \setminus S} \varphi_a$ extract S from D

2. If $\bar{b} \notin \text{acl}(A)$ but S is small. Let $\Sigma(\bar{x}) = \text{tp}(\bar{b}/A) \cup \{\bar{x} \neq \bar{c} : \bar{c} \in S\}$. $\Sigma(\bar{x})$ is inconsistent. By Compactness, there is $\psi(\bar{x}) \in \text{tp}(\bar{b}/A)$, $\bar{c}_1, \dots, \bar{c}_m \in S$, $\{\psi(\bar{x}), \bar{x} \neq \bar{c}_1, \dots, \bar{x} \neq \bar{c}_m\}$ is inconsistent. Thus $D = \psi(\mathbb{M}) \subseteq \{\bar{c}_1, \dots, \bar{c}_m\}$. D is A -definable, $\bar{b} \in D$, D is finite, so $\bar{b} \in \text{acl}(A)$ \square

Proposition 11.18. 1. $A \subseteq \text{acl}(A)$

2. $A \subseteq B \Rightarrow \text{acl}(A) \subseteq \text{acl}(B)$

3. $\text{acl}(\text{acl}(A)) = \text{acl}(A)$

Proof. 3. Take $b \in \text{acl}(\text{acl}(A))$, then $b \in \varphi(\mathbb{M}, \bar{c}) = D$, D is finite, D is $\text{acl}(A)$ -definable, $\bar{c} \in \text{acl}(A)$

$\mathcal{F} = \{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\} = \{\varphi(\mathbb{M}, \sigma(\bar{c})) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is finite by Proposition 11.17, $\sigma(D)$ is finite as D is finite. Therefore $D' = \bigcup \mathcal{F}$ is finite, A -invariant, $b \in D'$ as $b \in D \in \mathcal{F}$. Therefore $b \in \text{acl}(A)$ \square

Definition 11.19. A is **algebraically closed** if $A = \text{acl}(A)$

Proposition 11.20. $\text{acl}(A)$ is the smallest algebraically closed set containing A

Proposition 11.21. If $M \leq \mathbb{M}$, then $\text{acl}(M) = M$

Proof. Otherwise, $\text{acl}(M) \supsetneq M$, take $b \in \text{acl}(M) \setminus M$. $S = \{\sigma(b) : \sigma \in \text{Aut}(\mathbb{M}/M)\}$. By Proposition 11.17, S is finite, M -definable. M is M -invariant so $S \cap M = \emptyset$, contradicting to Tarski-Vaught criterion \square

Proposition 11.22. If $M \models \text{ACF}$ and K is a subfield. TFAE

1. $K = \text{acl}(K)$
2. $K \models \text{ACF}$
3. $K \leq M$

Idea: in ACF, field theoretic algebraic closure = model theoretic algebraic closure

Proof. 1 \rightarrow 2: Take $P(x)$, $P \neq 0$, $M \models \text{ACF}$, $P(x) = c \cdot (x - r_1) \dots (x - r_n)$, $c \in K$, $r_1, \dots, r_n \in M$. $D = \{r_1, \dots, r_n\}$ is K -definable. $K = \text{acl}(K)$ implies $D \subseteq \text{acl}(K) = K$

2 \rightarrow 3: q.e. \square

Fact 11.23. If T is strongly minimal, $A \subseteq M \models T$

$$A \leq M \Leftrightarrow |A| = \infty \text{ and } A = \text{acl}(A)$$

11.4 Imaginaries

Definition 11.24. An A -**interpretable set** is X/E where X is A -definable and $E \subseteq X^2$ is A -definable equivalence relation on X

Interpretable = \mathbb{M} -interpretable. 0-interpretable = \emptyset -interpretable

Definition 11.25. If M is a structure, M^{eq} is the expansion of M by

- A new sort for every 0-interpretable D/E
- Relation symbols for $D \rightarrow D/E$ (i.e., if $D \subseteq M^n$, add $R \subseteq M^n \times (D/E)$ where $R(a_1, \dots, a_n, b) \Leftrightarrow \bar{a} \in D, [a]_E = b$) If $D \subsetneq M^n$ can't add a function symbol

M is called the **home sort** (when M is one-sorted)

M^{eq} is the expansion of M obtained by adding each 0-interpretable set as a new sort, with enough data to connect the new sorts to the old sorts

If \mathbb{M} is a monster model, then \mathbb{M}^{eq} is a monster model (7) with the “same” automorphism group (6) and the “same” small models (5). If we restrict our attention to the original sorts from M , then \mathbb{M}^{eq} and \mathbb{M} have the same definable sets (1) (4) and the same partial elementary maps (2). However, \mathbb{M}^{eq} have some new elements, and the definable sets in \mathbb{M}^{eq} correspond exactly to the interpretable sets in the original structure \mathbb{M} (8) (9). On the other hand, the new elements of \mathbb{M}^{eq} are definable from the old elements (3). So \mathbb{M}^{eq} is a **way of covering interdefinable sets into definable sets while preserving most other things**

Fact 11.26. 1. If $X \subseteq M^n$, X is 0-definable in $M \Leftrightarrow X$ is 0-definable in M^{eq} .
In other words, M^{eq} doesn't define any new sets on the original sorts of M

2. If $A, B \subset M$, $f : A \rightarrow B$ bijection, then f is a partial elementary map in $M \Leftrightarrow f$ is a partial elementary map in M^{eq}

3. In M^{eq} , $\text{dcl}(M) = M^{\text{eq}}$

4. Consequently, any M^{eq} -definable set $X \subseteq M^n$ is M -definable in M^{eq} , and therefore M -definable in M

5. If $N \leq M$ then $N^{\text{eq}} \leq M^{\text{eq}}$ (more precisely, $N^{\text{eq}} \cong \text{dcl}_{M^{\text{eq}}}(N) \leq M^{\text{eq}}$)
This gives an \leq -preserving bijection.

Moreover, all elementary substructures of M arise this way. This yields an order-preserving bijection between the elementary substructures of M and the elementary substructures of M^{eq} . In particular, all elementary substructures of M^{eq} arise this way

6. If $\sigma \in \text{Aut}(M)$, σ acts on M^{eq} in a natural way. σ induces $\hat{\sigma} \in \text{Aut}(M^{\text{eq}})$.
This gives an isomorphism $\text{Aut}(M) \cong \text{Aut}(M^{\text{eq}})$

5 and 6 come from equivalence of categories $\text{Mod } T \rightarrow \text{Mod } T^{\text{eq}}$

7. M is κ -saturated and strongly κ -homogeneous $\Leftrightarrow M^{\text{eq}}$ is κ -saturated and strongly κ -homogeneous

8. If D/E is 0-interpretable in M , then D/E is 0-definable in M^{eq}
9. If X is 0-definable in M^{eq} , then there is 0-interpretable D/E in M and a 0-definable bijection $f : X \rightarrow D/E$ in M^{eq}

Ideas (8-9): interpretable in $M \Leftrightarrow$ definable in M^{eq}

Proof. Only some remarks

1. More generally, one can show that $X \subseteq M^n \times \prod_{j=1}^m (D_j/E_j)$ is 0-definable in M^{eq} iff $\tilde{X} \subseteq M^n \times \prod_{j=1}^m D_j$ is 0-definable in M where $\tilde{X} = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n, [b_1]_E, \dots, [b_m]_{E_M}) \in X\}$.
3. using the definable functions $D \rightarrow D/E$
4. Note (1) means that if \bar{x} is a tuple of variables in the old sorts of M , then any L^{eq} -formula $\phi(x)$ is equivalent to an L -formula
5. Behind the scenes, there is a theory T^{eq} and $M \models T \Rightarrow M^{\text{eq}} \models T^{\text{eq}}$. Moreover, all models of T^{eq} have the form M^{eq} up to isomorphism. Finally, elementary embeddings $M \rightarrow N$ correspond bijectively to elementary embeddings $M^{\text{eq}} \rightarrow N^{\text{eq}}$
7. \Leftarrow is easier by (2). If you just want a monster model \mathbb{M} s.t. \mathbb{M}^{eq} is a monster model, you can do the following: take $M \models T$, construct M^{eq} , take some monster elementary extension $U \geq M^{\text{eq}}$, then check that U is \mathbb{M}^{eq} for some $\mathbb{M} \geq M$

If M is κ -saturated and strongly κ -homogeneous. κ -saturation is not hard in terms of a compactness-like property: if $|A| < \kappa$ and a collection of A -definable sets has FIP, then it has non-empty intersection. To transfer this from M to M^{eq} , one takes the A -definable sets in M^{eq} and lifts them to A -definable sets in M using the maps $D \rightarrow D/E$

9. This comes down to the following things. First, if D/E and D'/E' are two interpretable sets, then $(D/E) \times (D'/E')$ "is" an interpretable set, namely $(D \times D')/E''$ where $(a, b)E''(c, d) \Leftrightarrow aEc \wedge bE'd$. Secondly if X is a definable subset of D/E , then X "is" an interpretable set D'/E' , where $D' = \{a \in D : [a]_E \in X\}$ and E' is the restriction of E to D'

□

From now on, we use the word "interpretable" to mean "definable in \mathbb{M}^{eq} " and "definable" to mean "definable" in \mathbb{M}

An **imaginary** is an element of \mathbb{M}^{eq}

11.5 Elimination of imaginaries

Definition 11.27. T has **elimination of imaginaries** (EI) if $\forall a \in \mathbb{M}^{\text{eq}}, \exists \bar{b} \in \mathbb{M}, \text{dcl}(a) = \text{dcl}(\bar{b})$

Definition 11.28. T has **uniform EI** if

1. If D/E is 0-interpretable, then there is 0-definable Y , there is bijection $f : D/E \rightarrow Y$, 0-interpretable (= 0-definable in M^{eq})
2. If D/E is 0-interpretable, then there is Y 0-definable, 0-definable surjection $g : D \rightarrow Y$ s.t. $g(x) = g(y)$ iff $E(x, y)$

$$1 \Leftrightarrow 2$$

Note: uniform EI implies EI

If $e \in D/E$, there is Y 0-definable, bijection $f : D/E \rightarrow Y$, e interpretable with $f(e) \in \mathbb{M}^n$

Lemma 11.29. If T has EI, if D/E is 0-interpretable, then there is 0-interpretable $X_i \subseteq D/E$, $X_i \cap X_j = \emptyset$, $D/E = \bigcup_{i=1}^n X_i$, each X_i has 0-interpretable bijection to a 0-definable set $f_i : X_i \rightarrow Y_i$

Proof. Say 0-interpretable $X \subseteq D/E$ is “good” if there is 0-definable Y , 0-interpretable bijection $f : X \rightarrow Y$.

If $X' \subseteq X$, X is “good”, X' is 0-interpretable, then X' is good

Claim D/E is covered by good sets

If $e \in D/E$, E.I. implies there is $\bar{b} \in \mathbb{M}^m$, $\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(\bar{b})$. Lemma 11.15 implies there is 0-interpretable bijection $f : X \rightarrow Y$, $f(e) = \bar{b}$, X is good

There are at most $|L|$ -many good sets. By saturation, $D/E = \bigcup_{i=1}^n X_i$ (class of good sets is small)

Replace X_i with $X_i \setminus (X_1 \cup \dots \cup X_{i-1})$, we may assume the X_i are pairwise disjoint □

Theorem 11.30. Suppose T has one-sort and $|\text{dcl}(\emptyset)| \geq 2$, then T has E.I. $\Leftrightarrow T$ has uniform E.I.

Proof. \Rightarrow : Take D/E 0-interpretable. Lemma 11.29 gives $D/E = \coprod_{i=1}^n X_i$, $f_i : X_i \rightarrow Y_i$, Y_i 0-definable

Fix $a, b \in \text{dcl}(\emptyset)$

By replacing y_i with $y_i \times \{(a, a, \dots, a)\}$. WMA there is m s.t. $Y_i \subseteq \mathbb{M}^m$ $\forall i$. Take $N \gg 0$, $2^N > n$, take distinct $\bar{c}_1, \dots, \bar{c}_n \in \{a, b\}^N$

Replacing Y_i with $Y_i \times \{\bar{c}_i\}$, now Y_i s are disjoint □

Example 11.3. DLO has E.I., doesn't have uniform E.I.

$D = M^2$, E two class $\{(x, y) : x = y\}$ and $\{(x, y) : x \neq y\}$

uniform E.I. would imply $D/E \leftrightarrow Y$, Y is 0-definable. But there is no 0-definable Y with two elements

Remark. M^{eq} has uniform E.I.

If D/E is 0-interpretable and $E' \subseteq (D/E) \times (D/E)$ is a 0-interpretable equivalence relation on D/E , then $(D/E)/E'$ is also 0-interpretable. In fact, it's D/E'' where

$$E''(\bar{a}, \bar{b}) \Leftrightarrow E'([\bar{a}]_E, [\bar{b}]_E)$$

Therefore $M^{\text{eq}} \approx (M^{\text{eq}})^{\text{eq}}$

Example 11.4. DLO is an example of a theory with elimination of imaginaries but not uniform elimination of imaginaries

Let E be the equivalence relation on M^2 with two classes, one of which is the line $y = x$ and the other is its complement. If there was a 0-interpretable bijection from M^2/E to $Y \subseteq M^n$, then Y would contain two elements, both of which are in $\text{dcl}(\emptyset)$. But $\text{dcl}(\emptyset) = \emptyset$, so Y cannot have any elements unless $n = 0$ and when $n = 0$ the set Y can only have one element.

11.6 Codes

Definition 11.31. A real tuple or imaginary e is a **code** for D (e **codes** D) if $\{\sigma \in \text{Aut}(M) : \sigma(D) = D\} = \text{Aut}(M^{\text{eq}}/e) = \text{Aut}(M/e)$

Remark. If e, e' code D , then $\text{Aut}(M/e) = \text{Aut}(M/e')$, so $\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(e')$ by Lemma 11.14

Remark. If e codes D , then D is A -definable $\Leftrightarrow e \in \text{dcl}^{\text{eq}}(A)$

Proof. TFAE

- D is A -definable
- D is A -invariant
- $\forall \sigma \in \text{Aut}(M/A), \sigma(D) = D$
- $\forall \sigma \in \text{Aut}(M/A), \sigma(e) = e$
- $e \in \text{dcl}^{\text{eq}}(A)$

□

Example 11.5. Suppose $T = \text{ACF}$ and $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$. Let $P(x) = \prod_{i=1}^n (x - r_i)$. Write $P(x)$ as $x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$. Then (c_0, \dots, c_{n-1}) is a code for S . Indeed

$$\begin{aligned} \sigma(\bar{c}) = \bar{c} &\Leftrightarrow \sigma(P(x)) \equiv P(x) \\ &\Leftrightarrow \prod_{i=1}^n (x - \sigma(r_i)) \equiv \prod_{i=1}^n (x - r_i) \\ &\Leftrightarrow \{\sigma(r_1), \dots, \sigma(r_n)\} = \{r_1, \dots, r_n\} \\ &\Leftrightarrow \sigma(S) = S \end{aligned}$$

Example 11.6. If D/E is 0-interpretable and $e \in D/E$, then e is an E -equivalence class $X = E(\mathbb{M}, \bar{a})$, and $\sigma(e) = \sigma(X)$ for all σ . Therefore σ codes X

Lemma 11.32. Let $\varphi(\bar{x}, \bar{y})$ be a formula. Let $f(\bar{y})$ be a 0-definable function s.t.

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f(\bar{b}) = f(\bar{c})$$

Then $f(\bar{b})$ is a code for $\varphi(\mathbb{M}, \bar{b})$, for each \bar{b}

Proposition 11.33. TFAE

1. T has uniform elimination of imaginaries
2. For any formula $\varphi(\bar{x}; \bar{y})$, there is a 0-definable function $f_\varphi(\bar{y})$ s.t.

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \Leftrightarrow f_\varphi(\bar{b}) = f_\varphi(\bar{c})$$

Proof. 1 \rightarrow 2: apply uniform E.I. to \mathbb{M}^n/E , where $E(\bar{b}, \bar{c}) \Leftrightarrow (\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}))$

2 \rightarrow 1: given a 0-interpretable set D/E ,

$$E(\bar{b}, \bar{c}) \Leftrightarrow E(\mathbb{M}, \bar{b}) = E(\mathbb{M}, \bar{c}) \Leftrightarrow f_E(\bar{b}) = f_E(\bar{c})$$

for $\bar{b}, \bar{c} \in D$. So we have a 0-definable function on D satisfying condition 2 of Definition □

Corollary 11.34. If T has uniform elimination of imaginaries, then every definable set has a code in \mathbb{M}

Corollary 11.35. Every definable set has a code in \mathbb{M}^{eq}

Proposition 11.36. TFAE

1. T has elimination of imaginaries
2. Every definable $D \subseteq \mathbb{M}^n$ has a code in \mathbb{M}

Proof. $1 \rightarrow 2$: given D take a code $e \in \mathbb{M}^{\text{eq}}$, then take $\bar{b} \in \mathbb{M}^m$ interdefinable with e . Then

$$\text{Aut}(\mathbb{M}/\bar{b}) = \text{Aut}(\mathbb{M}/e) = \{\sigma \in \text{Aut}(\mathbb{M}) : \sigma(D) = D\}$$

so \bar{b} is a code for D

$2 \rightarrow 1$: if $e \in D/E \subseteq \mathbb{M}^{\text{eq}}$, then e codes a definable set X □

Corollary 11.37. $\text{dcl}^{\text{eq}}(e)$ is the smallest definably closed set defining D

11.7 Elimination of imaginaries and naming parameters

Proposition 11.38. Uniform elimination of imaginaries is preserved by naming parameters

Proof. Fix D/E , $D \subseteq \mathbb{M}^n$ □

12 Forking and stability spectra

12.1 EI in PA and ACF

\mathbb{Q} is 0-definable

Theorem 12.1. If complete $T \supseteq \text{PA}$ (e.g., $T = \text{Th}(\mathbb{N})$), then T has uniform E.I.

Proof. Fix interpretable D/E . Want $D/E \rightarrow Y$ definable bijection, or $D/E \rightarrow \mathbb{M}^n$ definable injection

Take $f : D/E \rightarrow \mathbb{M}^n$, $f(X) = \min(X)$, \min is w.r.t. lexicographic order on \mathbb{M}^n . $\text{PA} \Rightarrow \mathbb{M}, \mathbb{M}^n$ are definably well-ordered □

Consider $T = \text{ACF}_0$

Fact 12.2. If $S \subseteq_f \mathbb{M}^n$, then $\exists^\top S^\top \in \mathbb{M}$

Proof. $n = 1$, if $S = \{r_1, \dots, r_n\} \subseteq \mathbb{M}$ form $P(x) = \prod_{i=1}^m (x - r_i)$. Then $P(x) = x^m + c_{m-1}x^{m-1} + \dots + c_1x + c_0$, \bar{c} is a code for S

$n = 2$, for $q \in \mathbb{Q}$, let $\pi_q : \mathbb{M}^2 \rightarrow \mathbb{M}$, $(x, y) \mapsto y - qx$. Let $A = \{\ulcorner \pi_q(S)^\top : q \in \mathbb{Q} \urcorner\} \subseteq \mathbb{M}$

Claim: If $\sigma \in \text{Aut}(\mathbb{M})$, $\sigma(S) = S \Leftrightarrow \sigma \in \text{Aut}(\mathbb{M}/A)$

$\Rightarrow: \forall q \in \mathbb{Q}, \sigma(\pi_q(S)) = \pi_q(\sigma(S)) = \pi_q(S)$ (π_q is 0-definable since $\mathbb{Q} \subseteq \text{dcl}(\emptyset)$), then $\sigma(\ulcorner \pi_q(S) \urcorner) = \ulcorner \pi_q(S) \urcorner, \sigma \in \text{Aut}(\mathbb{M}/A)$
 \Leftarrow : Suppose $\sigma \in \text{Aut}(\mathbb{M}/A)$ but $S' = \sigma(S) \neq S$, then $|S \cup S'| > |S| = |S'|$. $\exists q \in \mathbb{Q}$ s.t. $S \cup S' \rightarrow \pi_q(S \cup S')$ is injective (take any $q \notin \{\frac{y_1 - y_0}{x_1 - x_0} : (x_0, y_0), (x_1, y_1) \in S \cup S'\}$). Then $|\pi_q(S) \cup \pi_q(S')| = |\pi_q(S \cup S')| = |S \cup S'| > |S| \geq |\pi_q(S)|$. Therefore $\pi_q(S) \neq \pi_q(S') = \sigma(\pi_q(S))$. Then $\sigma(\ulcorner \pi_q(S) \urcorner) \neq \ulcorner \pi_q(S) \urcorner, \sigma \notin \text{Aut}(\mathbb{M}/A)$

S is A -invariant, so S is A -definable, so $\exists \bar{b} \in A, S$ is \bar{b} -definable
If $\sigma \in \text{Aut}(\mathbb{M})$, then

$$\sigma(\bar{b}) = \bar{b} \Rightarrow \sigma(S) = S \Rightarrow \sigma \in \text{Aut}(\mathbb{M}/A) \Rightarrow \sigma(\bar{b}) = \bar{b}$$

□

Lemma 12.3. *If $A \subseteq \mathbb{M}^{\text{eq}}$, if $D \subseteq \mathbb{M}^n$ is A -definable and $D \neq \emptyset$, then $\exists \bar{b} \in D, \bar{b} \in \text{acl}^{\text{eq}}(A)$*

Proof. Need $|\text{acl}(\emptyset)| = \infty$ and strongly minimality

Induction on n

$n = 1$: if $|D| < \infty, D \subseteq \text{acl}^{\text{eq}}(A)$, take any $b \in D$

If $|\mathbb{M} \setminus D| < \infty, D \cap \mathbb{Q} \neq \emptyset$, take $b \in D \cap \mathbb{Q}$

$n > 1$: $D' = \{\bar{b} \in \mathbb{M}^{n-1} : \exists c \in \mathbb{M}, (\bar{b}, c) \in D\}, D' \neq \emptyset, D'$ is A -definable, there is $\bar{b} \in D', \bar{b} \in \text{acl}^{\text{eq}}(A)$. $D'' = \{c \in \mathbb{M} : (\bar{b}, c) \in D\}, D'' \neq \emptyset, D''$ is \bar{b} -definable, there is $c \in D'', c \in \text{acl}^{\text{eq}}(A\bar{b}) \subseteq \text{acl}^{\text{eq}}(\text{acl}^{\text{eq}}(A)) = \text{acl}^{\text{eq}}(A)$ □

Theorem 12.4. *ACF_0 has uniform E.I.*

Proof. $\text{ACF}_0 \vdash 0 \neq 1$ so uniform EI \Leftrightarrow EI. Just need EI (11.30)

Take $e \in D/E, e = \ulcorner X \urcorner$ for some E -equivalence class X

By Lemma 12.3, there is $\bar{a} \in X, \bar{a} \in \text{acl}^{\text{eq}}(e)$. Let $S = \{\sigma(\bar{a}) : \sigma \in \text{Aut}(\mathbb{M}/e)\} = \text{realizations of } \text{tp}(\bar{a}/e)$. By Proposition 11.17, $|S| < \infty, S$ is e -definable. By the fact, take $\ulcorner S \urcorner \in \mathbb{M}^m$. $\ulcorner S \urcorner \in \text{dcl}^{\text{eq}}(e)$ because S is e -definable. Given S, X is the unique E -equivalence class containing S . So $e = \ulcorner X \urcorner \in \text{dcl}^{\text{eq}}(\ulcorner S \urcorner)$

$$\text{dcl}^{\text{eq}}(e) = \text{dcl}^{\text{eq}}(\ulcorner S \urcorner)$$

If $\sigma \in \text{Aut}(\mathbb{M})$ and $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner$, then $\sigma(S) = S$. As D, E are 0-definable, $\sigma(X)$ is some E -equivalence class. But $S \subseteq X \Rightarrow \sigma(S) \subseteq \sigma(X)$, so $S \subseteq \sigma(X)$. Therefore $\sigma(X)$ must be the same E -equivalence class as X , then $\sigma(X) = X$. This argument shows $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner \Rightarrow \sigma(X) = X$, which means X is $\ulcorner S \urcorner$ -definable □

Fact 12.5. *ACF_p has uniform E.I.*

12.2 Stability and \mathbb{M}^{eq}

Theorem 12.6. \mathbb{M} is λ -stable $\Leftrightarrow \mathbb{M}^{\text{eq}}$ is λ -stable

Proof. \Leftarrow : is easy.

\Rightarrow : Suppose $A \subseteq \mathbb{M}^{\text{eq}}$, $|A| \leq \lambda$, want $|S_{\bar{x}}(A)| \leq \lambda$

Take $B \subseteq \mathbb{M}$, $A \subseteq \text{dcl}^{\text{eq}}(B)$, $|B| \leq \lambda$. If $e \in D/E$, $e = [\bar{a}]_E$, then $e \in \text{dcl}^{\text{eq}}(\bar{a})$. Then $\text{tp}(e/B)$ determines $\text{tp}(e/A)$, so $|S_{\bar{x}}(B)| \geq |S_{\bar{x}}(A)|$, suffice to show $|S_{\bar{x}}(B)| \leq \lambda$. If $\bar{x} \in D/E$, let \bar{y} live in D , let $\pi : D \rightarrow D/E$. If $\bar{c} \in D$, $\text{tp}(\bar{c}/B)$ determines $\text{tp}(\pi(\bar{c})/B)$, so $|S_D(B)| \geq |S_{D/E}(B)|$, but $|S_D(B)| \leq \lambda$ \square

12.3 Almost A -definability

Proposition 12.7. If $A \subseteq \mathbb{M}$, $\text{acl}(A) = \bigcap_{M \leq \mathbb{M}, A \subseteq M} M$

Proof. If $M \supseteq A$, $M \leq \mathbb{M}$, $\text{acl}(A) \subseteq \text{acl}(M) = M$ 11.21

If $b \notin \text{acl}(A)$, then $\{\sigma(b) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is large. Take some $M_0 \supseteq A$, $M_0 \leq \mathbb{M}$. Take $\sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(b) \notin M_0$ since that is large, $b \notin \sigma^{-1}(M_0) := M$, $M \supseteq A$, $M \leq \mathbb{M}$, $b \notin \bigcap_{M \supseteq A} M$ \square

Proposition 12.8. If $D \subseteq \mathbb{M}^n$ is definable, $A \subseteq \mathbb{M}^{\text{eq}}$ small. TFAE

1. D is $\text{acl}^{\text{eq}}(A)$ -definable
2. $\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ finite
3. $\{\sigma(D) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is small
4. D is M -definable $\forall M \leq \mathbb{M}, M \supseteq A$ (forking)

Proof. By remark 11.6, the proposition is equivalent to

1. $\ulcorner D \urcorner \in \text{acl}^{\text{eq}}(A)$
2. $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ finite
3. $\{\sigma(\ulcorner D \urcorner) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ small
4. $\ulcorner D \urcorner \in M^{\text{eq}}, \forall M \supseteq A$

1 \leftrightarrow 2 \leftrightarrow 3 by Proposition 11.17

1 \leftrightarrow 4 Proposition 12.7 \square

D is **almost A -definable** if 1-4 hold

Proposition 12.9. *If $p \in S_n(\mathbb{M})$ is definable and $A \subseteq \mathbb{M}^{\text{eq}}$, TFAE*

1. p is $\text{acl}^{\text{eq}}(A)$ -definable
2. $\{\sigma(p) : \sigma \in \text{Aut}(\mathbb{M}/A)\}$ is small
3. p is M -definable for every $M \supseteq A$, $M \preceq \mathbb{M}$ (Lascar A -invariant)

Proof. Apply Proposition 12.8 to

$$D_\varphi = \{\bar{b} \in \mathbb{M} : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$$

□

12.4 Theorem of the bound

Assume T is stable. Recall 10.5, 10.9, 10.10

Fact 12.10. *If $p \in S_n(M)$, $q \in S_n(N)$, $p \subseteq q$, then $[q] \leq [p]$ and $[q] = [p]$ iff $q \supseteq p$*

If $p \in S_n(A)$, $A \subseteq M$, $\text{Ex}_M(p) = \{[q] : q \in S_n(M), q \supseteq p\}$, $\text{Bd}_M(p)$ = maximal elements of $\text{Ex}_M(p)$. Zorn's lemma \Rightarrow if $\beta \in \text{Ex}_M(p) \exists \beta' \in \text{Bd}_M(p), \beta' \supseteq \beta$

Fact 12.11. *If $A \subseteq M, N$, then $\text{Bd}_M(p) = \text{Bd}_N(p)$, and we can just write $\text{Bd}(p)$*

A “bound” of p is an element of $\text{Bd}(p)$

Lemma 12.12. *If $p \in S_n(A)$, $q \in S_n(\mathbb{M})$, $q \supseteq p$, if $[q] \in \text{Bd}(p)$, then*

1. *If $A \subseteq M \preceq \mathbb{M}$, then q is M -definable*
2. *q is $\text{acl}^{\text{eq}}(A)$ -definable*

Proof. 1. Let $r = q \upharpoonright M$. If $r \sqsubseteq q$ then q is M -definable. If $r \not\sqsubseteq q$, then $[q] < [r]$, contradicting $[q] \in \text{Bd}(p)$

2. Proposition 12.9

□

Corollary 12.13. *If $p \in S_n(A)$, $\exists q \in S_n(\mathbb{M})$, $q \supseteq p$, q is $\text{acl}^{\text{eq}}(A)$ -definable*

Proof. Take $\beta \in \text{Bd}(p) = \text{Bd}_{\mathbb{M}}(p) \subseteq \text{Ex}_{\mathbb{M}}(p)$, $\beta = [q]$ for some $q \in S_n(\mathbb{M})$, then by 12.12 □

Proposition 12.14. *If $A = \text{acl}^{\text{eq}}(A)$, if $p \in S_n(A)$, if $q_1, q_2 \in S_n(\mathbb{M})$, $q_1, q_2 \supseteq p$, q_1, q_2 A -definable, then $q_1 = q_2$*

Proof. If not, take $\varphi(\bar{x}, \bar{b}), q_1(\bar{x}) \vdash \varphi(\bar{x}, \bar{b}), q_2(\bar{x}) \vdash \neg\varphi(\bar{x}, \bar{b})$. $\text{tp}(\bar{b}/A)$ has a global A -definable extension r by Corollary 12.13. Take $\bar{c} \models q_1 \upharpoonright A\bar{b}$, then $\varphi(\bar{x}, \bar{b}) \in q_1(\bar{x}) \upharpoonright A\bar{b}$. $\bar{b} \models r \upharpoonright A$ and $\bar{c} \models q_1 \upharpoonright A\bar{b}$. $(\bar{b}, \bar{c}) \models (r \otimes q_1) \upharpoonright A$, $(\bar{c}, \bar{b}) \models (q_1 \otimes r) \upharpoonright A$, $\bar{c} \models q_1 \upharpoonright A$ and $\bar{b} \models r \upharpoonright A\bar{c}$, that is, $\bar{c} \models q_2 \upharpoonright A$, therefore $(\bar{c}, \bar{b}) \models (q_2 \otimes r) \upharpoonright A$, $(\bar{b}, \bar{c}) \models (r \otimes q_2) \upharpoonright A$, $\bar{c} \models q_2 \upharpoonright A\bar{b}$, a contradiction \square

Proposition 12.15. *If $A = \text{acl}^{\text{eq}}(A)$, $p \in S_n(A)$*

1. $\text{Bd}(p) = \{\beta\}$
2. *If $q \in S_n(\mathbb{M})$, $q \supseteq p$ then $[q] = \beta \Leftrightarrow q$ is A -definable*
3. *$\exists! q \in S_n(\mathbb{M})$, $q \supseteq p$, $[q] = \beta$, q is A -definable*

Proof. 1. If $\beta_1, \beta_2 \in \text{Bd}(p)$, take $q_1, q_2 \in S_n(\mathbb{M})$, $q_1, q_2 \supseteq p$, $[q_i] = \beta_i$. Lemma 12.12 shows that q_1, q_2 is A -definable, Proposition 12.14 says $q_1 = q_2$, then $\beta_1 = \beta_2$

2. If $[q] = \beta$, then q is A -definable by Lemma 12.12.

If q is A -definable, take some $q' \in S_n(\mathbb{M})$, $q' \supseteq p$, $[q'] = \beta$. Then Lemma 12.12 shows that q' is A -definable. By Proposition 12.14, $q = q'$

3. Existence by 12.13, uniqueness by 12.14

\square

Theorem 12.16 (Theorem of the bound). *If $A \subseteq \mathbb{M}^{\text{eq}}$, (small), $p \in S_n(A)$, then*

1. $\text{Bd}(p) = \{\beta\}$
2. *If $q \in S_n(\mathbb{M})$, $q \supseteq p$, then q is $\text{acl}^{\text{eq}}(A)$ -definable $\Leftrightarrow [q] = \beta$*
3. *If $X = \{q \in S_n(\mathbb{M}) : q \supseteq p, q \text{ is } \text{acl}^{\text{eq}}(A)\text{-definable}\}$, $Y = \{q \in S_n(\text{acl}^{\text{eq}}(A)) : q \supseteq p\}$, there is a bijection $X \rightarrow Y$, $q \mapsto q \upharpoonright \text{acl}^{\text{eq}}(A)$*
4. *If $q_1, q_2 \in S_n(\mathbb{M})$, $q_1, q_2 \supseteq p$, q_1, q_2 $\text{acl}^{\text{eq}}(A)$ -definable, then $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(q_1) = q_2$*

Proof. 3. Proposition 12.15 (3)

4. Take $c_i \models q_i \upharpoonright \text{acl}^{\text{eq}}(A)$ for $i = 1, 2$, $\text{tp}(c_1/A) = p = \text{tp}(c_2/A)$. $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(c_1) = c_2$. $\sigma(\text{tp}(c_1/\text{acl}^{\text{eq}}(A))) = \text{tp}(c_2/\text{acl}^{\text{eq}}(A))$, (if $a \in \text{acl}^{\text{eq}}(A)$, then $\sigma(a) \in \text{acl}^{\text{eq}}(A)$) $\sigma(q_1 \upharpoonright \text{acl}^{\text{eq}}(A)) = q_2 \upharpoonright \text{acl}^{\text{eq}}(A)$, $\sigma(q_1) \upharpoonright \text{acl}^{\text{eq}}(A) = q_2 \upharpoonright \text{acl}^{\text{eq}}(A)$, $\sigma(q_1), q_2$ are both almost A -definable, both $\supseteq p$, (3) shows that $\sigma(q_1) = q_2$

1. If $\beta_1, \beta_2 \in \text{Bd}(p) \subseteq \text{Ex}_{\mathbb{M}}(p)$, take $q_i \in S_n(\mathbb{M})$, $q_i \supseteq p$, $[q_i] = \beta_i$ for $i = 1, 2$. Lemma 12.12 shows that q_1, q_2 are $\text{acl}^{\text{eq}}(A)$ -definable, (4) shows that $\exists \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(q_1) = q_2$, $\beta_1 = [q_1] = [\sigma(q_1)] = [q_2] = \beta_2$
2. If $[q] = \beta \in \text{Bd}(p)$, then q is $\text{acl}^{\text{eq}}(A)$ -definable by Lemma 12.12. Take $q' \in S_n(\mathbb{M})$, $q' \supseteq p$, $[q'] = \beta$, Lemma 12.12 \Rightarrow q' is $\text{acl}^{\text{eq}}(A)$ -definable. (4) $\Rightarrow \exists \sigma \in \text{Aut}(\mathbb{M}/A)$, $\sigma(q) = q'$, $[q] = [\sigma(q)] = \beta$

□

Remark. $\text{stp}(\bar{b}/A) := \text{tp}(\bar{b}/\text{acl}^{\text{eq}}(A))$, the **strong type** of \bar{b} over A . A set X is $\text{acl}^{\text{eq}}(A)$ -invariant iff

$$\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A) \Rightarrow (\bar{b} \in X \Leftrightarrow \bar{c} \in X)$$

Proposition ?? (3) says that there is a bijection between strong types over A and $\text{acl}^{\text{eq}}(A)$ -invariant global types

One can define strong types without using \mathbb{M}^{eq} . It turns out that $\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A)$ iff $\bar{b}E\bar{c}$ for every A -definable “finite” equivalence relation E on \mathbb{M}^n
 $\text{stp}(\bar{b}/A) = \text{stp}(\bar{c}/A) \Leftrightarrow \forall X \text{ almost } A\text{-definable}, \bar{b} \in X \Leftrightarrow \bar{c} \in X$

12.5 Forking

Let $\text{Bd}(p) = \{\text{bd}(p)\}$. If $p \in S_n(M)$, $\text{bd}(p) = \max \text{Ex}_M(p) = [p]$

Remark. If $p \in S_n(A)$, $q \in S_n(B)$, $q \supseteq p$, then $\text{Ex}_{\mathbb{M}}(q) \subseteq \text{Ex}_{\mathbb{M}}(p)$, $\text{bd}(q) \leq \text{bd}(p)$

Definition 12.17. $q \in S_n(B)$, $p \in S_n(A)$, if $q \supseteq p$, q is a **nonforking extension**, written $q \sqsupseteq p$ if $\text{bd}(q) = \text{bd}(p)$,

Remark. If $p \in S_n(M)$ and $M \leq \mathbb{M}$ then $\text{bd}(p) = \max \text{Ex}_M(p) = \max\{[p]\} = [p]$

Proposition 12.18 (Full transitivity). *If $p_1 \subseteq p_2 \subseteq p_3$, then*

$$p_1 \sqsubseteq p_2 \wedge p_2 \sqsubseteq p_3 \Leftrightarrow p_1 \sqsubseteq p_3$$

Proposition 12.19 (Extension). *If $p \in S_n(A)$, $B \supseteq A$, then there is $q \in S_n(B)$, $q \sqsupseteq p$*

Proof. Take $M \supseteq B$, $M \leq \mathbb{M}$, take $r \in S_n(M)$, $r \supseteq p$ with $[r] = \text{bd}(p) \in \text{Ex}_M(p)$, then $\text{bd}(r) = \text{bd}(p)$, $r \sqsupseteq p$. □

Proposition 12.20. *If $p \in S_n(A)$, $q \in S_n(\mathbb{M})$, $q \supseteq p$, then $q \sqsupseteq p \Leftrightarrow [q] = \text{bd}(p) \Leftrightarrow q$ is $\text{acl}^{\text{eq}}(A)$ -definable*

In light of Proposition 12.20, Proposition 12.16 is really a statement about global non-forking extensions. In particular, global non-forking extensions of p correspond to extensions of p to $\text{acl}^{\text{eq}}(A)$, and any two extensions are conjugate over A

Proof. q is almost A -definable by Proposition 12.16 \square

Proposition 12.21. *If $q \in S_n(B)$, $p \in S_n(A)$, $q \supseteq p$, then $q \sqsupseteq p \Leftrightarrow \exists r \in S_n(\mathbb{M})$, $r \supseteq q$, r is $\text{acl}^{\text{eq}}(A)$ -definable*

Proof. If $q \sqsupseteq p$, by extension there is $r \in S_n(\mathbb{M})$ with $r \sqsupseteq q$ and then $r \sqsupseteq p$ by transitivity and r is $\text{acl}^{\text{eq}}(A)$ -definable by 12.20

If $r \supseteq q$ and r is $\text{acl}^{\text{eq}}(A)$ -definable, then $r \sqsupseteq p$ by 12.20, and so $q \sqsupseteq p$ by transitivity \square

Proposition 12.22. *If $p \in S_n(A)$ if $q \in S_n(\text{acl}^{\text{eq}}(A))$, $q \supseteq p$, then $q \sqsupseteq p$*

Proof. 12.13, q has a global $\text{acl}^{\text{eq}}(A)$ -definable extension \square

Definition 12.23. $\varphi(x) \in L(\mathbb{M})$ **forks** over A if there is no $p \in S_n(\mathbb{M})$ s.t. $p(\bar{x}) \supseteq \varphi(\bar{x})$ and $p(\bar{x})$ is $\text{acl}^{\text{eq}}(A)$ -definable

Proposition 12.24. *If $q \supseteq p \in S_n(A)$, then $q \not\sqsupseteq p$ iff $\exists \varphi(x) \in q(x)$, φ forks over A*

Proof. Let $\Sigma_A(\bar{x}) = \{\varphi(\bar{x}, \bar{b}) \leftrightarrow \varphi(\bar{x}, \bar{c}) : \bar{b} \equiv_{\text{acl}^{\text{eq}}(A)} \bar{c}\}$. If $r(\bar{x}) \in S_n(\mathbb{M})$, then $r \supseteq \Sigma_A \Leftrightarrow r$ is $\text{acl}^{\text{eq}}(A)$ -invariant $\Leftrightarrow r$ is $\text{acl}^{\text{eq}}(A)$ -definable. Therefore

1. An $L(\mathbb{M})$ -formula $\varphi(\bar{x})$ forks over A iff $\Sigma_A(\bar{x}) \cup \{\varphi(\bar{x})\}$ is inconsistent
2. $q \sqsupseteq p$ iff $q(\bar{x}) \cup \Sigma_A(\bar{x})$ is consistent by 12.21

Then $\varphi(\bar{x})$ forks over $A \Leftrightarrow \{\varphi(\bar{x})\} \cup \Sigma_A(\bar{x}) \vdash \perp$ \square

We say that $q \in S_n(B)$ **forks over** A if q contains a formula which forks over A

12.6 Stationary types

Lemma 12.25. *TFAE for $p \in S_n(A)$*

1. p has a unique non-forking extension over \mathbb{M}
2. p has a unique non-forking extension over any $B \supseteq A$

3. p has a unique extension to $\text{acl}^{\text{eq}}(A)$
4. p has an A -definable extensions over \mathbb{M}

Proof. $1 \rightarrow 2$: obvious

$2 \rightarrow 3$: take $B = \text{acl}^{\text{eq}}(A)$, use Proposition 12.22

$3 \leftrightarrow 1$: Proposition 12.16

$1 \rightarrow 4$: If q is the unique non-forking extension, then $\text{Aut}(\mathbb{M}/A)$ fixes q by symmetry, so q is A -invariant and A -definable

$4 \rightarrow 1$: Let q be the A -definable extension. Then q is $\text{acl}^{\text{eq}}(A)$ -definable (= non-forking). If q' is any other non-forking extension, then $q' = \sigma(q) = q$ for some $\sigma \in \text{Aut}(\mathbb{M}/A)$ by Proposition 12.16 \square

Example 12.1. 1. If $p \in S_n(M)$ and $M \leq \mathbb{M}$, then p is stationary

2. If $A = \text{acl}^{\text{eq}}(A)$ and $p \in S_n(A)$, then p is stationary. (It has an A -definable extension by Corollary 12.13) In the language of Remark 12.4, strong types are stationary
3. If $p \in S_n(A)$ is stationary and $q \in S_n(B)$ is a non-forking extension, then q is stationary (By transitivity, every global non-forking extension of q is a global non-forking extension of p . So q has no more global non-forking extension than p)
4. In a strongly minimal theory, the transcendental type over A is stationary (The global transcendental type is \emptyset -definable and extends it)
5. In $\mathbb{C} \models \text{ACF}$, $\text{tp}(i/\mathbb{Q}) = \text{tp}(-i/\mathbb{Q})$ is *not* stationary, because it has two extensions to $\text{acl}(\mathbb{Q}) = \mathbb{Q}^{\text{alg}}$, namely $\text{tp}(i/\mathbb{Q}^{\text{alg}})$ and $\text{tp}(-i/\mathbb{Q}^{\text{alg}})$

Lemma 12.26. Let $p \in S_n(A)$ be stationary, let $q \in S_n(B)$ be an extension of p , and let $p^{\mathbb{M}} \in S_n(\mathbb{M})$ be the unique A -definable global extension of p . Then $q \sqsupseteq p$ iff $q = p^{\mathbb{M}} \upharpoonright B$

Proof. Note $p^{\mathbb{M}} \sqsupseteq p$. If $q = p^{\mathbb{M}} \upharpoonright B$, then $q \sqsupseteq p$ by full transitivity. Conversely if $q \sqsupseteq p$, take some $r' \in S_n(\mathbb{M})$ with $r' \sqsupseteq q \sqsupseteq p$. Then $r' = p^{\mathbb{M}}$ by stationary, so $q \subseteq r' = p^{\mathbb{M}}$ and $q = p^{\mathbb{M}} \upharpoonright B$ \square

12.7 Local Character

Definition 12.27. $\kappa_n(T)$ is the smallest infinite cardinal s.t. there is no descending chain $(\beta_\alpha : \alpha < \kappa)$ of length κ in the fundamental order for n -types.

Remark. $\kappa_n(T) \leq |L|^+$. Otherwise, there is a descending chain $(\beta_\alpha : \alpha < |L|^+)$. If $\beta_\alpha = [p_\alpha]$, then $[p_\alpha] \subsetneq [p_{\alpha+1}]$, and we can take $\varphi_\alpha \in [p_{\alpha+1}] \setminus [p_\alpha]$. Then φ_α are pairwise distinct

Proposition 12.28 (Local character). *If $p \in S_n(A)$, then there is $B \subseteq A$ with $|B| < \kappa_n(T)$ s.t. $p \supseteq (p \upharpoonright B)$*

Proof. Suppose not ($p \not\supseteq p \upharpoonright B$ for $B \subseteq A$ with $|B| < \kappa_n(T)$)

Let $\kappa = \kappa_n(T)$. Build $(\bar{b}_\alpha : \alpha < \kappa)$ in A as follows:

Step α : $p \not\supseteq p \upharpoonright \{\bar{b}_\gamma : \gamma < \alpha\}$, $B_\alpha = \{\bar{b}_\gamma : \gamma < \alpha\}$, $\exists \varphi(x, \bar{b}_\alpha) \in p(x)$, $\varphi(x, \bar{b}_\alpha)$ forks over B_α

$(B_\alpha : \alpha < \kappa)$. For $\alpha < \gamma$, $p \upharpoonright B_\gamma \supseteq p \upharpoonright B_\alpha$, $\text{bd}(p \upharpoonright B_\alpha) \leq \text{bd}(p \upharpoonright B_\gamma)$, $p \upharpoonright B_{\alpha+1} \ni \varphi(x, \bar{b}_\alpha)$ forks over B_α , so $p \upharpoonright B_{\alpha+1} \not\supseteq p \upharpoonright B_\alpha$, then $\text{bd}(p \upharpoonright B_\gamma) < \text{bd}(p \upharpoonright B_\alpha)$ \square

12.8 Stability spectra

$\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$

Lemma 12.29. *If $M \models T$, λ cardinal, β, β' in fundamental order, $\beta > \beta'$, if $N \geq M$, N very saturated, strongly homogeneous, if $p \in S_n(M)$, $[p] = \beta$, then*

$$|\{q \in S_n(N) : q \supseteq p, [q] = \beta'\}| > \lambda$$

Proof. Since $\beta' \leq [p]$, $\exists N_0 \geq M$, $q_0 \in S_n(N_0)$, $q_0 \supseteq p$, $[q_0] = \beta'$ (10.3). WLOG, $|N_0| \leq |M| + |L|$. Embed $N_0 \hookrightarrow N$. WLOG, $M \leq N_0 \leq N$. Take $q \in S_n(N)$ the heir of q_0 , $[q] = \beta' < \beta = [p]$, $q \supseteq p$, $q \not\supseteq p$. (Treat N as monster, q is not $\text{acl}^{\text{eq}}(M)$ -definable). Then $\{\sigma(q) : \sigma \in \text{Aut}(N/M)\}$ is big \square

Proposition 12.30. *If $\lambda^\mu > \lambda$ for some $\mu < \kappa(T)$, then T is not λ -stable ($\lambda \geq \aleph_0$)*

Proof. Take μ minimal, $\lambda^{<\mu} \leq \lambda$

Take $(\beta_\alpha : \alpha < \mu)$ in descending fundamental order

Take $p \in S_1(M)$, $[p] = \beta_0$, build $M_0 \leq M_1 \leq \dots$ of length μ s.t. $M_{\alpha+1}$ is $(|M_\alpha| + \lambda)^+$ -saturated and -strongly homogeneous

Lemma 12.29 gives distinct p_i , $i < \lambda$, $p_i \in S_1(M_1)$, $p_i \supseteq p$, $[p_i] = \beta_1$

$\forall i < \lambda$, Lemma 12.29 gives distinct $p_{ij} \in S_1(M_2)$, $[p_{ij}] = \beta_2$, $p_{ij} \supseteq p_i$, etc.

Get p_σ for $\sigma \in \lambda^{<\mu}$, if $\sigma \in \lambda^\alpha$, $p_\sigma \in S_1(M_\alpha)$. If τ extends σ , then $p_\tau \supseteq p_\sigma$, $[p_\alpha] = \beta_\alpha$, $p_{\sigma i} \neq p_{\sigma j}$ for $i \neq j$.

For each σ , i, j , $i \neq j$, $i, j < \lambda$, take $\varphi(x, b) \in p_{\sigma i}$, $\varphi(x, b) \notin p_{\sigma j}$. Collect all the b in a set $B \subseteq \bigcup_{\alpha < \mu} M_\alpha$, $|B| = \lambda^{<\mu} \cdot \lambda \cdot \lambda \leq \lambda$

Claim: $\lambda^\mu \rightarrow S_1(B), \sigma \mapsto p_\sigma \upharpoonright B$ is injective

Since we have chosen the parameters to distinguish all these types

If $\sigma, \tau \in \lambda^\mu, \sigma \neq \tau$, there is $\alpha < \mu, \sigma \upharpoonright \alpha = \tau \upharpoonright \alpha, \sigma \upharpoonright (\alpha + 1) \neq \tau \upharpoonright (\alpha + 1)$ \square

Corollary 12.31. *If T is λ -stable, then $\lambda \geq \kappa(T)$*

Proof. Take $\mu = \lambda$ \square

Lemma 12.32. *If T is λ -stable, $A \subseteq \mathbb{M}, |A| \leq \lambda$, then $S_1(\text{acl}^{\text{eq}}(A)) \leq \lambda$*

Proof. Take $A = A_0 \subseteq A_1 \subseteq \dots$ of length $\omega, A_{i+1} = A_i \cup \{\text{a realization of each type over } A_i\}$. If $|A_i| \leq \lambda$, then $|S_1(A_1)| \leq \lambda$, so $|A_i| \leq \lambda$. Let $M = \bigcup_{i=0}^{\infty} A_i, |M| \leq \lambda$

Claim $M \preceq \mathbb{M}$

Use Tarski-Vaught criterion. If $D \subseteq \mathbb{M}, D$ is M -definable, $D \neq \emptyset$, then we want $D \cap M \neq \emptyset$. D is A_i -definable, $i < \omega$, take $b_0 \in D, \text{tp}(b_0/A_i)$ is realized by $b \in A_{i+1}$, therefore $b \in D, D \cap M \neq \emptyset$

$\text{dcl}^{\text{eq}}(M) = M^{\text{eq}} \supseteq \text{acl}^{\text{eq}}(A), \lambda \geq |S_1(M)|$ \square

Definition 12.33. $\lambda_0(T) = \min\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$

By Corollary 12.31, $\lambda_0(T) \geq \kappa(T)$

Theorem 12.34. T is λ -stable $\Leftrightarrow \lambda \geq \lambda_0(T)$ and $\forall \mu < \kappa(T), \lambda^\mu \leq \lambda$

Proof. \Leftarrow : Fix $A \subseteq \mathbb{M}, |A| \leq \lambda$.

Goal: $|S_1(A)| \leq \lambda$

If $p \in S_1(A)$, by local character, there is $B \subseteq A, p \sqsupseteq p \upharpoonright B$ and $\mu = |B| < \kappa(T)$. Number of choices of B is $\leq |A|^\mu \leq \lambda$

If $c \models p, p = \text{tp}(c/A)$, determined by $\text{tp}(c/\text{acl}^{\text{eq}}(A))$, determined by $\text{tp}(c/\text{acl}^{\text{eq}}(B))$, and the number of choices for $\text{tp}(c/\text{acl}^{\text{eq}}(B))$ is $\leq \lambda$ by lemma 12.32 \square

Remark. Theorem 12.34 shows that $\kappa(T)$ and $\lambda_0(T)$ determine the **stability spectrum** of T , the set $S = \{\lambda : T \text{ is } \lambda\text{-stable}\}$

12.9 Superstability

Definition 12.35. T is **superstable** if T is stable and $\kappa(T) = \aleph_0$ (fundamental order has DCC)

If T is superstable and $p \in S_n(A)$, then $\exists A_0 \subseteq_f A, p \sqsupseteq (p \upharpoonright A_0)$ by local character

Proposition 12.36. *T is superstable iff T is λ -stable for all sufficiently large λ*

Proof. If $\kappa(T) = \aleph_0$. Then T is λ -stable $\Leftrightarrow \lambda \geq \lambda_0(T)$ and $\forall \mu < \aleph_0$ for all $\mu < \aleph_0$

If $\kappa > \aleph_0$, $\lambda = \aleph_{\alpha+\omega}$ for any ordinal α , then $\text{cf}(\lambda) = \omega$. $\lambda^{\aleph_0} > \lambda$ by König's lemma (if $\forall i \in I, \lambda_i > \kappa_i$, then $\prod_{i \in I} \lambda_i > \sum_{i \in I} \kappa_i$), $\lambda^{\aleph_0} = \prod_{i \in \aleph_0} \lambda > \sum_{i \in \aleph_0} \aleph_{\alpha+i} = \lambda$

Let $\mu = \aleph_0$, $\mu < \kappa(T)$, $\lambda^\mu > \lambda$ so not λ -stable

If T is not superstable, then $\forall \alpha$, T is not $\aleph_{\alpha+\omega}$ -stable □

Now suppose $|L| = \aleph_0$, $\aleph_0 \leq \lambda_0(T)$, $\aleph \leq \kappa(T)$

Corollary 12.31: $\lambda_0(T) \geq \kappa(T)$

?: $\kappa(T) \leq |L|^+ = \aleph_1$

T is 2^{\aleph_0} -stable

So $\lambda_0 \leq 2^{\aleph_0}$

Fact 12.37. *If $\lambda_0 > \aleph_0$ then $\lambda_0 \geq 2^{\aleph_0}$*

So there are three possibilities

name	$\kappa(T)$	$\lambda_0(T)$	Spectrum
ω -stable	\aleph_0	\aleph_0	$\{\lambda : \lambda \geq \aleph_0\}$
Superstable but not ω -stable	\aleph_0	2^{\aleph_0}	$\{\lambda : \lambda \geq 2^{\aleph_0}\}$
Stable but not superstable	\aleph_1	2^{\aleph_0}	$\{\lambda : \lambda^{\aleph_0} = \lambda\} = \{\lambda^{\aleph_0} : \lambda\}$

Fact 12.38. • *Strongly minimal theory is ω -stable*

- $(\mathbb{Z}, +)$ is superstable but not ω -stable
- separably closed fields (other than ACF), free groups (other than $(\mathbb{Z}, +)$) are stable, but not superstable

12.10 Forking calculus

Definition 12.39. $\bar{a} \downarrow_C \bar{b}$ if $\text{tp}(\bar{a}/C\bar{b}) \sqsubseteq \text{tp}(\bar{a}/C)$

\bar{a} and \bar{b} are **independent over C**

Lemma 12.40. *Suppose $C = \text{acl}^{\text{eq}}(C)$, \bar{a}, \bar{b} are tuples. Let p, q be the unique global C -definable types (they are stationary) extend $\text{tp}(\bar{a}/C)$, $\text{tp}(\bar{b}/C)$. Then*

1. $\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C$
2. $\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{b} \downarrow_C \bar{a})$

Proof.

$$\begin{aligned}
& (\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C \\
& \Leftrightarrow \bar{b} \models q \upharpoonright C \text{ and } \bar{a} \models p \upharpoonright C\bar{b} \\
& \Leftrightarrow \bar{a} \models p \upharpoonright C\bar{b} \\
& \Leftrightarrow \text{tp}(\bar{a}/C\bar{b}) \subseteq p \\
& \Leftrightarrow \text{tp}(\bar{a}/C\bar{b}) \supseteq \text{tp}(\bar{a}/C) \quad (\text{Lemma 12.26}) \\
& \Leftrightarrow \bar{a} \downarrow_C \bar{b}
\end{aligned}$$

$\text{tp}(\bar{a}/C)$ and $\text{tp}(\bar{b}/C)$ are stationary by 12.25 (4)

Types commute in stable theory (9.70) □

Lemma 12.41. $\forall C, \bar{a}, \bar{b}$

1. $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a} \downarrow_{\text{acl}^{\text{eq}}(C)} \bar{b}$
2. $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{b} \downarrow_C \bar{a}$

Proof. $1 \rightarrow 2$ by 12.40

$$\begin{array}{ccc}
& \text{tp}(\bar{a}/C\bar{b}) & \sqsubseteq \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C\bar{b})) \\
\subseteq & & \sqsubseteq \sqcup \\
\text{tp}(\bar{a}/C) & & \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C)\bar{b}) \\
\sqsubseteq & & \subseteq \\
& \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(\bar{c})) &
\end{array}$$

Therefore $\text{tp}(\bar{a}/C) \sqsubseteq \text{tp}(\bar{a}/C\bar{b}) \Leftrightarrow \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(\bar{c})) \sqsubseteq \text{tp}(\bar{a}/\text{acl}^{\text{eq}}(C)\bar{b})$ □

Lemma 12.42. If \bar{a}, \bar{a}' enumerate A , \bar{b}, \bar{b}' enumerate B , then $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a}' \downarrow_C \bar{b}'$

Proof. $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \text{tp}(\bar{a}/CB) \supseteq \text{tp}(\bar{a}/C) \Leftrightarrow \bar{a} \downarrow_C \bar{b}' \Leftrightarrow \bar{b}' \downarrow_C \bar{a} \Leftrightarrow \bar{b}' \downarrow_C \bar{a}'$ □

Definition 12.43. $A \downarrow_C B$ if $\bar{a} \downarrow_C \bar{b}$ for tuples \bar{a}, \bar{b} enumerating A, B

Proposition 12.44. 1. (Symmetry) $A \downarrow_C B \Leftrightarrow B \downarrow_C A$

2. (Monotonicity) If $A' \subseteq A, B' \subseteq B$ then $A \downarrow_C B \Rightarrow A' \downarrow_C B'$

3. (*right Transitivity*) $A \downarrow_C B, A \downarrow_{CB} B' \Rightarrow A \downarrow_C BB'$
4. (*left Transitivity*) $A \downarrow_C B, A' \downarrow_{CA} \Rightarrow AA' \downarrow_C B$
5. (*Base monotonicity*) $A \downarrow_C BB' \Rightarrow A \downarrow_{CB} B'$
6. (*Normality*) $A \downarrow_C B \Rightarrow A \downarrow_C BC$
7. (*Invariance*) If $\sigma \in \text{Aut}(\mathbb{M})$ then $A \downarrow_C B \Rightarrow \sigma(A) \downarrow_{\sigma(C)} \sigma(B)$
8. (*Extension*) Given $A, B, C, \exists A' \equiv_C A$ with $A' \downarrow_C B$
9. (*Finite character*) If $A_0 \downarrow_C B_0 \forall A_0 \subseteq_f A, B_0 \subseteq_f B$, then $A \downarrow_C B$

Proof. $C \subseteq CB \subseteq CBB'$, so $\text{tp}(A/CBB') \supseteq \text{tp}(A/C) \Leftrightarrow \text{tp}(A/CBB') \supseteq \text{tp}(A/CB)$ and $\text{tp}(A/CB) \supseteq \text{tp}(A/C)$ this gives transitivity, base monotonicity and monotonicity

Extension: Proposition ?? $\text{tp}(A/C)$ has a non-forking extension q to BC , take $A' \models q$, then $A \equiv_C A'$

Finite character: only need to show $\forall B_0 \subseteq_f B (A \downarrow_C B_0) \Rightarrow A \downarrow_C B$

If $\text{tp}(A/BC) \not\supseteq \text{tp}(A/C)$, then $\varphi \in \text{tp}(A/BC)$, φ forks over C , $\varphi \in L(B_0C)$, $A \not\downarrow_C B_0$ \square

12.11 Examples

Proposition 12.45. If p, q are C -definable types and $\bar{a} \models p \upharpoonright C, \bar{b} \models q \upharpoonright C$, then

$$\bar{a} \downarrow_C \bar{b} \Leftrightarrow (\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright C$$

Proof. $\text{tp}(\bar{a}/C), \text{tp}(\bar{b}/C)$ stationary, like Lemma 12.40 \square

In ACF, $\bar{a} \downarrow_C \bar{b} \Leftrightarrow \bar{a} \downarrow_{\text{acl}(C)} \bar{b}$, $\text{acl}(M)$ is a model, $\text{tp}(\bar{a}/M) = p_V, \text{tp}(\bar{b}/M) = p_W$, then $\bar{a} \downarrow_M \bar{b} \Leftrightarrow (\bar{a}, \bar{b})$ is generic on $V \times W$

Proposition 12.46. $\bar{a} \downarrow_B \bar{a} \Leftrightarrow \bar{a} \in \text{acl}^{\text{eq}}(B)$

Proof. \Rightarrow : $S_n(\mathbb{M}) \ni p \supseteq \text{tp}(\bar{a}/B\bar{a}) \supseteq \text{tp}(\bar{a}/B)$, $p = \text{tp}(\bar{a}/\mathbb{M})$, p is $\text{acl}^{\text{eq}}(B)$ -definable, $\bar{a} \in \text{acl}^{\text{eq}}(B)$

\Leftarrow : By “extension”, $\exists \sigma \in \text{Aut}(\mathbb{M}/B), \sigma(\text{acl}^{\text{eq}}(B)) \downarrow_B \text{acl}^{\text{eq}}(B), \sigma(\text{acl}^{\text{eq}}(B)) = \text{acl}^{\text{eq}}(\sigma(B)) = \text{acl}^{\text{eq}}(B)$

Actually $\text{acl}^{\text{eq}}(B) \downarrow_B \text{acl}^{\text{eq}}(B)$, by monotonicity, $\bar{a} \downarrow_B \bar{a}$ \square

Proposition 12.47. $A \downarrow_C B \Rightarrow \text{acl}^{\text{eq}}(AC) \cap \text{acl}^{\text{eq}}(BC) = \text{acl}^{\text{eq}}(C)$

if $e \in \text{acl}(AC)$ and $e \in \text{acl}(BC)$ then $e \in \text{acl}(C)$

Proposition 12.48. Let $T = \text{theory of } \mathbb{R}\text{-vector spaces}$. If $A \subseteq M$

1. T is complete, has q.e.
2. T is strongly minimal, stable
3. $A \downarrow_{\emptyset} B \Leftarrow \text{span}(A) \cap \text{span}(B) = \{0\}$
4. $A \downarrow_C B \Leftrightarrow \text{span}(AC) \cap \text{span}(BC) = \text{span}(C)$

Proposition 12.49. $\kappa_n(T)$ doesn't depend on n

Proof. **Claim:** $\kappa_n(T)$ is smallest $\kappa \geq \aleph_0$ s.t. $\nexists \bar{a} \in \mathbb{M}^n$, increasing $(C_\alpha : \alpha < \kappa)$, $C_\alpha \subseteq \mathbb{M}$ s.t. $\text{tp}(\bar{a}/C_{\alpha+1}) \supseteq \text{tp}(\bar{a}/C_\alpha)$ for all α

Given a forking chain, $\text{bd}(\text{tp}(\bar{a}/C_{\alpha+1})) < \text{bd}(\text{tp}(\bar{a}/C_\alpha))$ so descending chain of length κ in fundamental order

Conversely, given a descending chain □

A Metric Spaces

$\mathbb{R}_{\geq 0}$ denotes $[0, +\infty] = \{x \in \mathbb{R} : x \geq 0\}$

Definition A.1. A **metric** on a set M is a function $d : M \times M \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Example A.1. $M = \mathbb{R}^2$, $d(x, y) = (\text{the distance from } x \text{ to } y)$

$$d(x_1, x_2; y_1, y_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Example A.2. The **Manhattan metric** on \mathbb{R}^2 is given by

$$d(x_1, x_2; y_1, y_2) = |x_1 - y_1| + |x_2 - y_2|$$

measure distances in a city grid

Example A.3. Let M be the set of strings. The **edit distance** from x to y is the minimum number of intersections, deletions, and substitutions to go from x to y

$$d(\text{drip}, \text{rope}) = 3$$

$$\text{drip} \mapsto \text{drop} \mapsto \text{rop} \mapsto \text{rope}$$

Edit distance is a metric on M

Definition A.2. A **metric space** is a pair (M, d) where M is a set and d is a metric space

- $(\mathbb{R}^n, d_{\text{Euclidean}})$ where $d_{\text{Euclidean}}$ is the usual Euclidean distance
- $(\mathbb{R}^2, d_{\text{Manhattan}})$ where $d_{\text{Manhattan}}$ is the Manhattan distance

Often we abbreviate (M, d) as M , when d is clear
Fix a metric space (M, d)

Definition A.3. If $p \in M$ and $\epsilon > 0$, then

$$B_\epsilon(p) = \{x \in M : d(x, p) < \epsilon\}$$

$$\bar{B}_\epsilon(p) = \{x \in M : d(x, p) \leq \epsilon\}$$

$B_\epsilon(p)$ and $\bar{B}_\epsilon(p)$ are called the **open** and **closed** balls of radius ϵ around p

Example A.4. In \mathbb{R}^2 with the Euclidean metric, the open ball of radius 2 around $(0, 0)$ the open disk

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2^2\}$$

Example A.5. In \mathbb{R}^2 with the Manhattan metric, the open ball of radius 1 around $(0, 0)$ the open disk

$$\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$$

Suppose $p \in M$ and $X \subseteq M$

Definition A.4. p is an **interior point** of X if X contains an open ball of positive radius around p

In particular, p must be an element of X

Example A.6. If $X = [-1, 1] \times [-1, 1]$, then $(0, 0)$ is an interior point of X , but $(1, 0)$ and $(0, 2)$ are not

Definition A.5. The **interior** $\text{int}(X)$ is the set of interior points

Warning: There are metric spaces where the interior of $\overline{B}_\epsilon(p)$ isn't $B_\epsilon(p)$

Definition A.6. A set $X \subseteq M$ is **open** if $X = \text{int}(X)$, i.e., every point of X is an interior point of X

Example A.7 (in \mathbb{R}). The set $(-1, 2)$ is open. The sets $[-1, 2]$ and $[-1, 2)$ are not; they have interior $(-1, 2)$

Fact: the interior $\text{int}(X)$ is the unique largest open set contained in X

Let a_1, a_2, \dots be a sequence in a metric space (M, d) and let p be a point

Definition A.7. " $\lim_{i \rightarrow \infty} a_i = p$ " if for every $\epsilon > 0$, there is n s.t.

$$\{a_n, a_{n+1}, a_{n+2}, \dots\} \subseteq B_\epsilon(p)$$

Example A.8. Work in \mathbb{R} with the usual distance. Let $a_n = 1/n$. Then $\lim_{n \rightarrow \infty} a_n = 0$ but $\lim_{n \rightarrow \infty} a_n \neq 1$

Fact: For any sequence a_1, a_2, a_3, \dots in (M, d) , there is at most one point p s.t. $\lim_{i \rightarrow \infty} a_i = p$

If such a p exists, it is called the **limit**, and written $\lim_{i \rightarrow \infty} a_i$

let X be a set and p be a point in a metric space (M, d)

Definition A.8. p is an **accumulation point** of X if $p = \lim_{n \rightarrow \infty} a_n$ for some sequence a_n in X

Equivalently

Definition A.9. p is an accumulation point of X if for every $\epsilon > 0$, we have $B_\epsilon(p) \cap X \neq \emptyset$

Definition A.10. The **closure** of X , written $\text{cl}(X)$ or \overline{X} , is the set of accumulation points

Definition A.11. A set $X \subseteq M$ is **closed** if $X = \text{cl}(X)$

Fact: The closure $\text{cl}(X)$ is the unique smallest closed set containing X

Example A.9. Work in \mathbb{R} with the distance $d(x, y) = |x - y|$

\mathbb{Q} is neither closed nor open

\mathbb{R} is both closed and open, so is *emptyset*

Let X^c denote the complement $M \setminus X$

Fact: X is closed iff X^c is open

Fact: $\text{int}(X) = \text{cl}(X^c)^c$ and $\text{cl}(X) = \text{int}(X^c)^c$

Let (M, d) and (M', d) be metric spaces. Let $f : M \rightarrow M'$ be a function

Definition A.12. f is **continuous** if

$$\lim_{n \rightarrow \infty} a_n = p \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(p)$$

for $a_1, a_2, a_3, \dots, p \in M$

idea: f is continuous iff f preserves limits

Example A.10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Then $\lim_{n \rightarrow \infty} 1/n = 0$, but

$$\lim_{n \rightarrow \infty} f(1/n) = \lim_{n \rightarrow \infty} 1 = 1 \neq -1 = f(0)$$

Proposition A.13. Fix $f : (M, d) \rightarrow (M', d)$. The following are equivalent

1. f is continuous
2. For every open set $U \subseteq M'$, the preimage $f^{-1}(U)$ is open
3. For every $p \in M$, for every $\epsilon > 0$, there is $\delta > 0$ s.t. for every $x \in M$,

$$d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$$

Fact: The functions \sin , \cos , \exp , $\sqrt[3]{}$ and polynomials are continuous

Proposition A.14. If $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, then $f + g$, $f \cdot g$, $f - g$, $f \circ g$ are continuous

Proposition A.15. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x) \neq 0$ for all x , then $1/f(x)$ is continuous. If $f(x) \geq 0$ for all x , then $\sqrt{f(x)}$ is continuous

Example A.11. This function is continuous

$$h(x) = \exp\left(\frac{1}{1+x^2}\right) - \frac{1}{17 + \sin(\sqrt[3]{x})}$$

Definition A.16. A function $f : M \rightarrow M'$ is **Lipschitz continuous** if there is $c \in \mathbb{R}$ s.t. for any $x, y \in M$

$$d(f(x), f(y)) \leq c \cdot d(x, y)$$

Example A.12 (In \mathbb{R}). The function $f(x) = |x| + |x - 1|$ is Lipschitz continuous with $c = 2$

Proposition A.17. If f is Lipschitz continuous, then f is continuous

Example A.13. The function $f(x) = x^2$ is continuous but not Lipschitz continuous

Definition A.18. Let (M, d) be a metric space and $S \subseteq M$ be a set. Then (S, d') is a metric space, where $d'(x, y) = d(x, y)$ for $x, y \in S$

- d' is the restriction of d to $S \times S$
- We say that (S, d') is a **subspace** of (M, d)

Let $(M, d), (M', d)$ be metric spaces, $S \subseteq M$ and $f : S \rightarrow M'$ be a function

Definition A.19. f is **continuous** if f is continuous as a map from the subspace (S, d') to (M', d)

Example A.14 (in \mathbb{R}). Let $f : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = 1/x$. Then f is continuous

Definition A.20. An **isometry** or **isomorphism** from (M, d) to (M', d') is a bijection $f : M \rightarrow M'$ s.t. for any $x, y \in M$

$$d(x, y) = d'(f(x), f(y))$$

Example A.15 (in \mathbb{R}^2). The map $(x, y) \mapsto (x + 1, y - 7)$ is an isometry

So is the map $(x, y) \mapsto (3/5x + 4/5y, -4/5x + 3/5y)$

These two metric spaces are isometric via the isometry $x \mapsto (x, 0)$

- \mathbb{R} with the usual distance
- The subspace $\mathbb{R} \times \{0\}$ inside \mathbb{R}^2 with the usual distance

Proposition A.21. The isometries of \mathbb{R}^2 are exactly the rotations, translations, reflections and glide reflections

Let X be a non-empty set in a metric space

Definition A.22. The **diameter** of X , written $\text{diam}(X)$, is

$$\sup\{d(p, q) : p, q \in X\}$$

(Possibly $\text{diam}(X) = +\infty$)

Example A.16. In \mathbb{R}^2 with the usual metric, the diameter of $B_r(p)$ is $2r$

Work in a metric space M

Definition A.23. A **Cauchy sequence** is a sequence a_1, a_2, a_3, \dots s.t.

$$\lim_{n \rightarrow \infty} \text{diam}(\{a_n, a_{n+1}, a_{n+2}, \dots\}) = 0$$

Proposition A.24. Every sequence which converges to a point in M is a Cauchy sequence

Proposition A.25. Let a_1, a_2, a_3, \dots be a sequence in a metric space (M, d) . The following are equivalent

- The sequence is a Cauchy sequence
- There is some metric space M' s.t. M is a subspace of M' , and $\lim_{n \rightarrow \infty} a_n$ converges in M'

Proposition A.26. In \mathbb{R} , every Cauchy sequence converges

This fails in the subspace \mathbb{Q}

Definition A.27. A metric space (M, d) is **complete** if every Cauchy sequence in M converges (to a point in M)

Example A.17. \mathbb{R} is complete. The subspace \mathbb{Q} and $(-1, 1)$ are not complete

Let (M, d) be a metric space

Definition A.28. The **completion** of M is a new metric space \overline{M} . Objects of \overline{M} are equivalence classes of Cauchy sequences in M . Two Cauchy sequences $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ are equivalent if $\lim_{i \rightarrow \infty} d(a_i, b_i) = 0$. The distance in \overline{M} between two Cauchy sequences $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ is $\lim_{i \rightarrow \infty} d(a_i, b_i)$

Proposition A.29. This is well-defined, and \overline{M} is complete

Proposition A.30. If we identify $c \in M$ with the constant sequence c, c, c, c, \dots then M is a dense subspace of \overline{M} . If M is complete, then $\overline{M} = M$

Example A.18. \mathbb{R} is the completion of \mathbb{Q} w.r.t. its usual metric

Example A.19. The p -**adic norm** on \mathbb{Q} is defined by

$$|0|_p = 0$$

$$|p^k a/b|_p = p^{-k} \text{ if } a, b \text{ are integers not divisible by } p$$

For example, $|1.3|_5 = |5^{-1} \cdot 13/2|_5 = 5^1$

The p -**adic metric** on \mathbb{Q} is given by $d(x, y) = |x - y|_p$. This is an incomplete metric. The completion is called \mathbb{Q}_p , the set of p -**adic numbers**

Definition A.31. $C([0, 1])$ is the space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$

Proposition A.32. *There is a metric on $C([0, 1])$ where $d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}$. This makes $C([0, 1])$ into a complete metric space.*

Definition A.33. A metric space (M, d) is **connected** if the only clopen sets are M and \emptyset . Otherwise M is disconnected

Definition A.34. A set $X \subseteq M$ is **connected** (resp. **disconnected**) if the subspace (X, d) is connected or disconnected as a metric space.

Proposition A.35. *X is disconnected iff there is a non-constant continuous function $f : X \rightarrow \{0, 1\}$*

Example A.20. The set $[-10, -1] \cup [1, 10]$ is disconnected, as witnessed by

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Example A.21. The set $[-10, 10] \setminus \{0\}$ is disconnected

Example A.22. The set \mathbb{Q} is disconnected, witnessed by

$$f(x) = \begin{cases} 0 & x < \sqrt{2} \\ 1 & x > \sqrt{2} \end{cases}$$

The set $\mathbb{R} \setminus \mathbb{Q}$ is disconnected by a similar argument

Proposition A.36. *If $X \subseteq \mathbb{R}$ is non-empty, then the following are equivalent*

- X is connected
- X is convex: if $a, b \in X$, then $[a, b] \subseteq X$

- X is an interval, a set of the form

$$[a, b], (a, b), (a, b], [a, b) \\ (-\infty, a), (-\infty, a], [a, +\infty), (a, +\infty), (-\infty, \infty)$$

Proposition A.37. *Let $f : M \rightarrow M'$ be continuous. If $X \subseteq M$ is connected, then $f(X) \subseteq M'$ is connected*

Corollary A.38 (Intermediate Value Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) < y < f(b)$, then there is $x \in [a, b]$ with $f(x) = y$*

Proof. $f([a, b])$ is connected, hence convex, so it contains $y \in [f(a), f(b)]$. Therefore there is $x \in [a, b]$ with $f(x) = y$ \square

There are discontinuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the IVT
classify infinite set with only 1 unary predicate

B Midterm review [5/7]

- ☒ 2.24 : definable types, heirs
- ☒ 3.3 : dichotomy property, λ -stability, stability
- ☒ 3.10 : stability, coheirs, invariant types, Morley products
- ☒ 3.17 : Morley sequences, the order property, indiscernible sequences
- ☐ 3.24 : Ramsey's theorem, indiscernible sequence, EM-types, total indiscernibility
- ☒ 4.7 : definable closure, algebraic closure, elimination of imaginaries
- ☐ Homework
 - ☐ 1
 - ☐ 2
 - ☐ 3
 - ☒ 4
 - ☐ 5
 - ☐ 6
 - ☐ 7

C Problems want to ask

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