Seminar on Topological Dynamics of Definable Group Actions

Section1

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1 Lookback

Definition 1.1.

1. V is a homogeneous definable G-set in M, if $\varphi_V(M) = V$, and $\forall v_1 v_2 \in V \exists g \in G(g(v_1) = v_2)$
2. G-flow X is point transitive if for some $x_0 \in X$, its G-orbit is dense, i.e., cl($o(x_0)$)=X.
3. E is a G-invariant relation on X, if $\forall x, y \in X$, xEy implies $g(x)Eg(y)$, for all $g \in G$.
4. E is a btde relation on $V^{\mathfrak{C}}$, if E is type definable over $M \supseteq V$ with bounded number of classes, i.e., there is (partial) type $p(x, y)$ over M such that $E = \{(x, y) : V^{\mathfrak{C}} \models \varphi(x, y), \varphi \in p\}$ and $|V^{\mathfrak{C}}/E| < |V^{\mathfrak{C}}|$.
5. $p \in X$ is almost periodic iff cl(Gp) is a minimal flow.

Proposition 1.2. If V is a homogeneous G-set, then the flow V_E is point transitive. $S_G(M)$ is a universal point transitive G-flow.

2 Generic and weak generic types

Assume $U \subseteq V$. For $A \subseteq G$, by AU we mean $\bigcup_{g \in A} gU$.

Definition 2.1.

1. U is generic for G on V, if for some finite $A \subseteq G$, $V = AU$
2. U is weak generic, if for some non-generic $U' \subseteq V$, $U \cup U'$ is generic.
3. formula $\varphi(x)$ over M is [weak] generic, if $\varphi(M) \cap V$ is [weak] generic.
4. type $p(x)$ is [weak] generic, if every formula $\varphi(x) \in p(x)$ is [weak] generic.

Remark 2.2.

1. U is weak generic iff for some finite $A \subseteq G$, $V - AU$ not generic.

2. if U definable, then U weak generic iff for some non-generic definable $U' \subseteq V$, $U \cup U'$ is generic.

Proof.

- (1)$\Rightarrow$: U is weak generic, so for some A, U', $V = A(U \cup U')$, so $V - AU = AU'$. U' not generic, so AU' not generic.

- (1)$\Leftarrow$: for some finite A, $V - AU$ not generic, claim $U \cup (V - AU)$ generic. let e be identity in G, then $A' = A \cup \{e\}$ is finite, and $A'(U \cup (V - AU)) \supseteq AU \cup (V - AU)$.

- (2) $V - AU$ is definable if U is definable.

□

Definition 2.3. Assume X is a point transitive G-flow, $p \in X$. Say p is [weak] generic if its every open neighbourhood is [weak] generic.

Remark 2.4. in the case $X = S_G(M)$, close sets on X are in the form $[\varphi] = \{p \in S_G(M) : \varphi \in p\}$. and $[\neg\varphi]$ is close, so its complement set is open. every $[\varphi]$ is clopen. For $p \in X$, its open neighbourhood is $[\varphi]$, $[\psi]$,... $(\varphi, \psi \in p)$. Every open neighbourhood of p is [weak] generic is the same as every formula in p is [weak] generic.

Definition 2.5. Given a point transitive G-flow X, let $Gen(X)$ and $WGen(X)$ denote the sets of generic and weak generic points in X. When $X = S_V(M)$, denote $Gen(X)$ and $WGen(X)$ by $Gen_V(M)$ and $WGen_V(M)$.

Remark 2.6.

1. $Gen(X) \subseteq WGen(X)$.

2. And if $Gen(X)$ not empty, then $WGen(X) = Gen(X)$.

Proof. (1) is obvious.

(2) if not, some definable generic X can be divided into two non-generic definable A and B. If p is a generic point, there is some $g \in G$ such that $p \in gX$. But either gA or gB witness p not a generic point, contradiction. □

Lemma 2.7. Assume $f : X \to Y$ is a surjective morphism of point transitive G-flows. Then $f[Gen(X)] \subseteq Gen(Y)$ and $f[WGen(X)] = WGen(Y)$.

Proof. For any $f(x) \in f[Gen(X)]$, any open neighbourhood containing $f(U)$, U is open neighbourhood of x in X, its finite translations cover X. So finite translations of $f(U)$ covers $f(X) = Y$. □

Corollary 2.8. Assume E is a G-invariant btde-relation on $V^{\mathfrak{C}}$ and $a \in V^{\mathfrak{C}}$.

1. a_E is weak generic in V_E iff for some $a' \in [a]_E$, type $\mathrm{tp}(a'/M)$ is weak generic in $S_V(M)$.

2. if type $\mathrm{tp}(a/M)$ is generic in $S_V(M)$, then a_E is generic in V_E.

3. a_E is generic in V_E iff for every formula $\varphi(x, y) \in E(x, y)$, $\varphi(\mathfrak{C}, a)$ is generic for action of G on $V^{\mathfrak{C}}$.

Proof. (1) and (2) follows from that there is surjective morphism form $S_V(M)$ to V_E.

(3) by definition, closed sets in V_E are in the form of $\pi[U]$, where U is a type definable subset of $V^{\mathfrak{C}}$, π maps elements in $V^{\mathfrak{C}}$ to its equivalence class. open sets in V_E are image of definable sets and image of complements of type definable sets. Claim a_E is generic in V_E iff for every $\varphi(x, y) \in E(x, y)$, the set $\bigcup\{[b]_E : [b]_E \subseteq \varphi(\mathfrak{C}, a)\}$ is generic in $V^{\mathfrak{C}}$.

$\Rightarrow$: clearly holds.

$\Leftarrow$: suppose$\varphi(x, y), \varphi'(x, y) \in E(x, y)$ and$\varphi'(x, y) \wedge \varphi'(y, z) \vdash \phi(x, z)$. $\varphi'(\mathfrak{C}, a)$ is generic in $V^{\mathfrak{C}}$. For any x, if $\mathfrak{C} \models \varphi'(x, a) \wedge E(x, y)$, there is $\mathfrak{C} \models \varphi(y, a)$. So $\varphi'(\mathfrak{C}, a) \subseteq \bigcup\{[b]_E : [b]_E \subseteq \varphi(\mathfrak{C}, a)\}$. □

Remark 2.9. $p \in X$ is almost periodic iff for every open $U \ni p$, the set $\mathrm{cl}(Gp)$ is covered by AU for some finite $A \subseteq G$.

Proof. $\Leftarrow$: prove minimality of $\mathrm{cl}(Gp)$. suppose $q \in \mathrm{cl}(Gp)$ and open $U \ni p$. So finite translations AU covers $\mathrm{cl}(Gp) \ni q$, thus there is $g \in A \subseteq G$ such that $q \in gU$. That is $g^{-1}q \in U$, U meets Gq. Thus we find any open neighbourhood meets Gp also meets Gq, so $\mathrm{cl}(Gp) \subseteq \mathrm{cl}(Gq)$.

$\Rightarrow$: Let open $U \ni p$, then GU is open, $\mathrm{cl}(Gp) - GU$ is closed and G-invariant, hence a G-flow. By minimality of $\mathrm{cl}(Gp)$, $\mathrm{cl}(Gp) - GU$ must be empty. By compactness(?), finite translations of U covers $\mathrm{cl}(Gp)$. □

Lemma 2.10. Assume X is a point transitive G-flow and $p \in X$.

1. open $U \subseteq X$ is generic iff U meets every minimal subflow in X.

2. p is generic iff every open $U \ni p$ meets every minimal subflow in X.

3. p is weak generic iff every open $U \ni p$ meets some minimal subflow in X.

Proof.

1. $\Rightarrow$: U is generic, so AU covers X for some finite $A \subseteq G$, also every minimal subflow in X. if gU meets some minimal subflow at p, then U meets the same subflow at $g^{-1}p$.

   $\Leftarrow$: U is open, so GU is open and $X - GU$ is closed and G-invariant, hence a G-flow. For U meets every minimal flow, so $X - GU$ cannot contain any minimal flow, so it can only be empty. GU covers X. (?)By compactness, finite translations of U covers X, so U is generic.

2. follows from 1.

3. $\Rightarrow$: If not ,some weak generic U disjoint from any minimal subflow in X. (?)By regularity of X, we can assume cl(U) is such. If cl(GU) meets some subflow at q, then cl(U) meets the same subflow at $g^{-1}q$. So cl(GU) is also disjoint from any minimal subflow, equivalently, X-cl(GU) meets every minimal subflow. By 1., X-cl(GU) is generic. But U is weak generic, so $X-GU \supseteq X-$cl(GU) is not generic. Contradiction. Hence weak generic U must meet some minimal flow.

   $\Leftarrow$: Let open U meets a minimal flow V, choose open U' meeting V with cl(U')$\subseteq U$. $V-$cl(GU') is a G-flow, by minimality of V, it must be empty. (?)By compactness, there is finite $A \subseteq G$ such that $V \subseteq AU'$. Thus $X-$cl(AU') does not meet minimal flow V, hence not generic, $X - AU \subseteq X-$cl(AU') is not generic, so U is weak generic.

∎

Corollary 2.11. $WGen(X)$ is the closure of the union of all minimal flows in X.

Corollary 2.12. Assume $Gen(X)$ is not empty. Then $Gen(X) = WGen(X)$ is the only minimal flow in X.

Proof. Any point from a minimal flow is generic, so its every neighbourhood meets every minimal flow. But distinct minimal flows disjoint, so there is only one minimal flow. ∎

Definition 2.13. Say a closed set $C \subseteq X$ is almost generic, if every open $U \supseteq C$ is generic. Let $MGen(X)$ denote the family of all minimal closed almost generic sets $C \subseteq X$.

Remark 2.14.

1. $WGen(X)$ =cl($\bigcup MGen(X)$).

2. Every subflow $Y \subseteq X$ meets every closed almost generic set $C \subseteq X$.

Proof.

1. $\supseteq$: Suppose not, then take $p \in C - WGen(X)$ for some minimal almost generic C, let $U \ni p$ open, with cl(U) disjoint from $WGen(X)$. So U is not weak generic. Let $C' = C - U$, then for every open$U' \supseteq C'$, $C \subseteq U \cup U'$. C is almost generic, so $U \cup U'$ is generic. U is not weak generic, so U' is generic, therefor C' almost generic, contradicting the minimality of C.

   $\subseteq$: For $p \in WGen(X)$, let $U \ni p$ open, $U' \ni p$ open with cl(U')$\subseteq U$, thus U' weak generic. Then there is finite $A \subseteq G$ such that $X - AU'$ not generic. So open set $X - A\cdot$ cl(U') is not generic, thus almost generic sets cannot be subset of $X - A\cdot$ cl(U'). So $A\cdot$ cl(U') meets every minimal almost generic set C, so does AU. Or, for some $a \in A$, U meets $a^{-1}C$. Note that $a^{-1}C$ is also a minimal almost generic set. If any superset of C is generic, than its a^{-1} translation is still generic, thus $a^{-1}C$ is minimal almost generic. So open neighbourhood $U \ni p$ must meet cl($\bigcup MGen(X)$). Hence $p \in$ cl($\bigcup MGen(X)$), $WGen(X) \subseteq$cl($\bigcup MGen(X)$).

2. Let C be any almost generic set, and $U \supseteq C$ open, then U is generic, for some finite $A \subseteq G$, AU covers X. There is some $a \in A$ such that aU meets subflow Y, or U meets $a^{-1}Y$. Y is a subflow, so $Y = a^{-1}Y$, Y meets U, so Y meets C. So every subflow meets every almost generic set.

□

In model theory setting, closed $C \subseteq S_V(M)$ corresponds to a (partial) type(closed sets are defined as image of type definable subsets), minimal closed almost generic sets $C \subseteq S_V(M)$ correspond to maximal generic types over M containing $V(x)$.

If $T = Th(M)$ is stable, then there are generic types in $S_V(M)$, G acts transitively on them, hence there is just one minimal $G - flow$ in $S_V(M)$, consisting of a single orbit. (?)I n o-minimal case, when G is S^1(a circle?) interpreted in the field of reals, the minimal G flow consists of two orbits.

When $G = (\mathbb{R}^n, +)$, there are no generic types in $S_G(M)$, but a rich structure of weak generic types.

Notions of a minimal flow in X and a minimal closed almost generic set $C \subseteq X$ are quite orthogonal: minimal G-flows in X are pairwise disjoint and lie densely in $WGen(X)$, but they need not be pairwise disjoint.

Definition 2.15.

- Say a subset U of M is Borel if $U = \bigcup\{p(M) : p \in B\}$ where B is Borel subset of $S(\emptyset)$.
- A Borel set in a topological space is any set that can be formed from open sets(or, equivalently, closed sets) through countable union, countable intersection, relative complement.

Theorem 2.16. Assume G is a 0-definable group in an $\aleph_0$ -saturated structure M, covered by countably many Borel sets $X_n, n < \omega$. Then for some finite $A \subseteq G$ and some $n < \omega$ we have $G = A \cdot X_n \cdot X_n^{-1}$.

Proof. Let $r : S_G(M) \to S(\emptyset)$ be the restriction map, and for $n < \omega$ let $Y_n = r^{-1}[B_n]$, where $B_n \subset S(\emptyset)$ is the Borel set determining X_n. Each Y_n is a Borel subset of $S_G(M)$. Let $S \subseteq S_G(M)$ be a minimal G-flow. Countable X_n covers G, so countable Y_n covers $S_G(M)$, $S \subseteq S_G(M)$, therefore we can find some Y_n not meager in S. Choose a $\varphi(x)$ such that U meets S and $Y_n \cap S$ is co-meager(complement is meager) in $U \cap S$, where $U = S_G(M) \cap [\varphi]$. S is minimal flow, so any neighbourhood of its point can cover S by finite translations. There is finite $A \subseteq G$ such that $S \subseteq AU$. Hence $A(Y_n \cap S)$ is co-meager in S. claim

$$G = A \cdot X_n \cdot X_n^{-1}$$

To show this, let $g \in G$. $Y_n \cap S$ is not meager in S, $g \cdot S = S$, so $g(Y_n \cap S)$ is not meager in S, hence it meets $A \cdot (Y_n \cap S)$. So there are $p, q \in (Y_n \cap S)$ such that $g \cdot p = a \cdot q$ for some $a \in A$. Hence $g \in a \cdot q(\mathfrak{C}) \cdot p(\mathfrak{C})^{-1}$. Let q', p' be their restrictions on $S(\emptyset)$, by saturation of M we have $g \in a \cdot q'(M) \cdot p'(M)^{-1}$. $q'(M), p'(M) \subseteq X_n$, hence $g \in a \cdot X_n \cdot X_n^{-1}$. □

Proposition 2.17. Assume G is a 0-definable group in an $\aleph_0$-saturated structure M, covered by countably many 0-type-definable sets X_n, $n < \omega$. Then there is a finite set $A \subset G$ and $n < \omega$ such that

$$G = \bigcup_{a \in A} (X_{<n} \cdot X_{<n}^{-1})^a$$

where $X_{<n} = \bigcup_{i<n} X_i$ and $X^a = aXa^{-1}$.