Morley's theorem

Advanced model theory

May 12, 2022

References: Sections 10.2, 10.4, 18.6 in Poizat's *Course in Model Theory.*

1 Prime models in totally transcendental theories

Definition 1.1. $M \models T$ is *prime* if for every model $N \models T$, there is an elementary embedding $M \to N$.

Warning 1.2. There are theories without prime models. There are theories with multiple non-isomorphic prime models.

Definition 1.3. A model $M \preceq \mathbb{M}$ is *prime over* a subset $A \subseteq M$ if for any model $N \preceq \mathbb{M}$ with $N \supseteq A$, there is an elementary embedding $M \to N$ extending id_A, or equivalently, an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(M) \subseteq N$.

Equivalently, M is prime over $A \subseteq M$ if M is prime as an $L(A)$-structure.

Example 1.4. If $\mathrm{acl}(A) \preceq M$, then $\mathrm{acl}(A)$ is a prime model over A. This happens for any A in ACF.

Remark 1.5. Suppose $|L| = \aleph_0$ and models of T are infinite. If M is prime over A, then $|M| = |A| + \aleph_0 = \max(|A|, \aleph_0)$. First, $|M| \geq \aleph_0$ by assumption and $|M| \geq |A|$ because $M \supseteq A$. In the other direction, take $N \preceq \mathbb{M}$ with $N \supseteq A$. By Löwenheim-Skolem, we may assume $|N| \leq |A| + \aleph_0$. There is an elementary embedding $M \to N$, so $|M| \leq |N| \leq |A| + \aleph_0$.

Definition 1.6. B is *constructible over* A if there is an enumeration $(b_\alpha : \alpha < \beta)$ of B such that for each $\alpha < \beta$, $\mathrm{tp}(b_\alpha/B_{<\alpha}A)$ is isolated, where $B_{<\alpha} = \{b_\gamma : \gamma < \alpha\}$. The enumeration $(b_\alpha : \alpha < \beta)$ is called a *construction* of B over A.

Lemma 1.7. *If B_0 is constructible over A, and B_1 is constructible over AB_0, then B_0B_1 is constructible over A. More generally, if $(B_\alpha : \alpha < \beta)$ is a sequence of sets and B_α is constructible over $A \cup \bigcup_{\gamma<\alpha} B_\gamma$ for each α, then $\bigcup_{\alpha<\beta} B_\alpha$ is constructible over A.*

Proof. Concatenate the constructions. $\square$

Lemma 1.8. *Suppose* $\mathrm{tp}(c/A)$ *is isolated. If* $M \preceq \mathbb{M}$ *and* $f : A \to B \subseteq M$ *is a partial elementary map, then there is* $c' \in M$ *such that* $f \cup \{(c, c')\}$ *is a partial elementary map.*

Proof. Moving A, c by an automorphism, we may assume $A = B$ and $f = \mathrm{id}_A$. Take $\varphi(x) \in L(A)$ isolating $\mathrm{tp}(c/A)$. Then

$$\mathbb{M} \models \varphi(c) \implies \mathbb{M} \models \exists x\, \varphi(x) \implies M \models \exists x\, \varphi(x)$$

since $A \subseteq M$. Take $c' \in M$ with $M \models \varphi(c')$. Then $c' \equiv_A c$ because $\varphi(x)$ isolates $\mathrm{tp}(c/A)$. The fact that $c' \equiv_A c$ means $\mathrm{id}_A \cup \{(c, c')\}$ is a partial elementary map. $\square$

Theorem 1.9. *If* $A \subseteq M \preceq \mathbb{M}$ *and* M *is constructible over* A*, then* M *is prime over* A*.*

Proof. Let $N \supseteq A$ be another model. Let $(b_\alpha : \alpha < \beta)$ be a construction of M over A. By induction using Lemma 1.8, we can find $c_\alpha \in N$ such that $f = \mathrm{id}_A \cup \{(b_0, c_0), (b_1, c_1), \ldots\}$ is a partial elementary map. Then f is a partial elementary embedding of M into N over A. $\square$

Fact 1.10. *Suppose* T *is totally trascendental. If* $D \subseteq \mathbb{M}^n$ *is non-empty and* A*-definable, then there is* $\bar{b} \in D$ *such that* $\mathrm{tp}(\bar{b}/A)$ *is isolated.*

Fact 1.10 was Theorem 9.2 in the May 7 notes.

Lemma 1.11. *Suppose* T *is totally transcendental. Given* $A \subseteq \mathbb{M}$*, there is* $B \subseteq \mathbb{M}$ *constructible over* A *such that* $D \cap B \neq \varnothing$ *for every non-empty* A*-definable* $D \subseteq \mathbb{M}$*.*

Proof. Let $\{D_\alpha : \alpha < \kappa\}$ enumerate the non-empty A-definable subsets of $\mathbb{M}$. By Fact 1.10, recursively choose b_α for $\alpha < \kappa$ such that $b_\alpha \in D_\alpha$ and

$$\mathrm{tp}(b_\alpha/A \cup \{b_\beta : \beta < \alpha\})$$

is isolated. Then $B = \{b_\alpha : \alpha < \kappa\}$ is constructible over A and has the desired property. $\square$

Theorem 1.12. *If* T *is totally transcendental and* $A \subseteq \mathbb{M}$*, then there is a model* $M \preceq \mathbb{M}$ *that is constructible and prime over* A*.*

Proof. By Lemma 1.11, build a sequence $B_0, B_1, B_2, \ldots$ of length ω such that

- B_i is constructible over $AB_{<i}$, where $B_{<i} = \bigcup_{j<i} B_j$.
- $B_i \cap D \neq \varnothing$ for every D definable over $AB_{<i}$.

(Note that $\{a\}$ is A-definable for $a \in A$, so $A \subseteq B_i$ for each i.) Let $M = \bigcup_{i<\omega} B_i \supseteq A$. Then M is constructible over A by Lemma 1.7, and $M \preceq \mathbb{M}$ by the Tarski-Vaught criterion. $\square$

Fact 1.13. *The prime model in Theorem 1.12 is unique up to isomorphism.*

We won't use Fact 1.13, which is proved in Section 10.4 and 18.1 of the textbook.

2 Vaught pairs

Definition 2.1. A *Vaught pair* is a proper elementary extension $M \prec N$ (with $N \neq M$) and an $L(M)$-formula such that

- $\varphi(M) = \varphi(N)$ (even though $M \subsetneq N$)
- $\varphi(M)$ is infinite.

Remark 2.2. If $\varphi(M)$ is finite, then $\varphi(M) = \varphi(N)$ always holds for any $N \succeq M$. Indeed, if $\varphi(M) = \{c_1, \ldots, c_n\}$, then $M \models \forall x \ (\varphi(x) \to \bigvee_{i=1}^{n}(x = c_i))$ so the same holds in N.

Remark 2.3. For any formula $\varphi(\bar{x}; \bar{y})$, there is a theory T_φ such that a model of T_φ is a triple $(M, P, \bar{b})$ where $M \models T$, $P \preceq M$, $P \neq M$, $\bar{b} \in P$, and $\varphi(P; \bar{b}) = \varphi(M; \bar{b})$.

Theorem 2.4 (Morley, Baldwin, Lachlan)**.** *Let T be a complete theory in a countable language, with infinite models. The following are equivalent:*

1. *T is κ-categorical for some $\kappa > \aleph_0$.*
2. *T is totally transcendental and has no Vaught pairs.*
3. *T is κ-categorical for all $\kappa > \aleph_0$.*

In the rest of these notes, we will prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). (3) $\implies$ (1) is obvious. We prove (2) $\implies$ (3) in this section, and (1) $\implies$ (2) in later sections.

Assume in this section that the language is countable, models are infinite, and T is totally transcendental with no Vaugtian pairs.

Lemma 2.5. $\exists^\infty$ *is eliminated.*

Proof. Fix a formula $\varphi(\bar{x}; \bar{y})$. It suffices to prove the claim below, which implies

$$\exists^\infty \bar{x} \ \varphi(\bar{x}; \bar{y}) \iff \exists^{>N_\varphi} \bar{x} \ \varphi(\bar{x}; \bar{y}).$$

Claim. There is an integer N_φ such that for any $\bar{b} \in \mathbb{M}$,

$$|\varphi(\mathbb{M}; \bar{b})| = \infty \text{ or } |\varphi(\mathbb{M}; \bar{b})| \leq N_\varphi.$$

Suppose the claim fails. Then for every n there is $\bar{b}_n$ such that

$$n < |\varphi(\mathbb{M}; \bar{b}_n)| < \infty.$$

Let M_n be a small model containing $\bar{b}_n$. Then $\varphi(M_n; \bar{b}) = \varphi(\mathbb{M}; \bar{b}_n)$ (see Remark 2.2). With T_φ as in Remark 2.3 we get a model $(\mathbb{M}, M_n, \bar{b}_n) \models T_\varphi$ with $n < |\varphi(\mathbb{M}; \bar{b}_n)|$ for each n. By compactness, there is a model $(M, P, \bar{b}) \models T_\varphi$ with $|\varphi(M; \bar{b})| = \infty$, which is a Vaught pair. $\square$

Definition 2.6. An M-definable set $D \subseteq \mathbb{M}^n$ is *minimal* (with respect to M) if D is infinite and for every M-definable set $D' \subseteq D$, either D' is finite or $D \setminus D'$ is finite. D is *strongly minimal* if it is minimal with respect to $\mathbb{M}$.

Lemma 2.7. *Let M be a model.*

1. *There is an M-definable minimal set $D \subseteq \mathbb{M}$.*
2. *If D is minimal then D is strongly minimal.*

Proof. 1. Let $\mathrm{R}(-)$ denote Cantor-Bendixson rank over M. It's easy to see that for M-definable D, $\mathrm{R}(D) > 0$ iff D is infinite. Let $E_\alpha = \{p \in S_1(M) : \mathrm{R}(p) \geq \alpha\}$. Then $\mathrm{R}(S_1(M)) = \mathrm{R}([\mathbb{M}]) > 0$ because $\mathbb{M}$ is infinite, so $E_1 \neq \varnothing$. If $E_1 = E_2$, then E_1 is perfect, contradicting the fact that $S_1(M)$ is scattered because T is totally transcendental (see Lemma 7.1 in the May 5–7 notes). Therefore $E_1 \supsetneq E_2$. Take $p \in E_1 \setminus E_2$. Then p is an isolated point of E_1. Take a clopen set $[D] \subseteq S_1(M)$ with $[D] \cap E_1 = \{p\}$. Then $\mathrm{R}([D]) = 1 > 0$, so D is infinite. For M-definable $D' \subseteq D$, we have $[D] = [D'] \sqcup [D \setminus D']$. If $p \in [D']$ then $[D \setminus D'] \cap E_1 = \varnothing$ so $\mathrm{R}(D \setminus D') \leq 0$ and $|D \setminus D'| < \infty$. Similarly, if $p \in [D \setminus D']$, then $|D'| < \infty$. Thus D is minimal.

2. Suppose D is not strongly minimal, witnessed by $\mathbb{M}$-definable $D' \subsetneq D$ such that D' and $D \setminus D'$ are infinite. Write D and D' as $\varphi(\mathbb{M}; \bar{b})$ and $\psi(\mathbb{M}; \bar{c})$ for $\bar{b} \in M$ and $\bar{c} \in \mathbb{M}$. Then

$$\mathbb{M} \models \exists^\infty \bar{x}\ (\varphi(\bar{x}; \bar{b})\ \wedge\ \psi(\bar{x}; \bar{c}))$$
$$\mathbb{M} \models \exists^\infty \bar{x}\ (\varphi(\bar{x}; \bar{b})\ \wedge\ \neg\psi(\bar{x}; \bar{c})).$$

By Lemma 2.5 the right hand sides are first-order. The fact that $M \preceq \mathbb{M}$ implies we can change $\bar{c}$ and take $\bar{c} \in M$. Then D' is M-definable, and D isn't minimal. $\square$

By Theorem 1.12, T has a prime model M_0. By Lemma 2.7 there is an M_0-definable strongly minimal set $D \subseteq \mathbb{M}$. Let $p \in S_1(M_0)$ be the transcendental type of D, the type generated by

$$\{x \in D\} \cup \{x \notin D' : D' \subseteq D,\ D' \text{ is } M_0\text{-definable},\ |D'| < \infty\}.$$

As in Section 2 of the May 5 notes, this is a complete type. Let $\hat{p} \in S_1(\mathbb{M})$ be the analogous global transcendental type in D. Then $\hat{p}$ is M_0-invariant hence M_0-definable, and $\hat{p} \supseteq p$, implying that p is stationary and $\hat{p}$ is its global non-forking extension.

Remark 2.8. Let $I = (b_\alpha : \alpha < \kappa)$ be a sequence of realizations of p. The following are equivalent:

1. I is independent over M_0.
2. I is a Morley sequence of $\hat{p}$ over M_0, meaning $b_\alpha \models \hat{p} \restriction (M_0\{b_\beta : \beta < \alpha\})$ for each $\alpha < \kappa$.

3. $b_\alpha \notin \operatorname{acl}(M_0 \cup \{b_\beta : \beta < \alpha\})$ for each $\alpha < \kappa$.

The proof is similar to the case of strongly minimal theories (see §2 in the May 5 notes).

For each small cardinal κ, let I_κ be a Morley sequence of $\hat{p}$ over M_0 of length κ. Let $M_\kappa \preceq \mathbb{M}$ be prime over $M_0 \cup I_\kappa$. Note $|M_\kappa| = \kappa + \aleph_0$ by Remark 1.5.

Lemma 2.9. *If M is a small model, then $M \cong M_\kappa$ for some κ.*

Proof. Because M_0 is prime, there is an elementary embedding $M_0 \to M$. Moving M by an automorphism, we may assume $M_0 \subseteq M$. Let I be a maximal independent set of realizations of p in M. Let $\kappa = |I|$. Let $f : I \to I_\kappa$ be any bijection. Then f is a partial elementary map over M_0, by the proof of Lemma 2.4 in the May 5 notes. So $f \cup \mathrm{id}_{M_0} : (M_0 \cup I) \to (M_0 \cup I_\kappa)$ is a partial elementary map. Moving M and I by an automorphism, we may assume $I = I_\kappa$. As M_κ is prime over $M_0 \cup I_\kappa$, there is an elementary embedding $M_\kappa \to M$ over $M_0 \cup I_\kappa$. Moving M by an automorphism over $M_0 \cup I_\kappa$, we may assume $M_\kappa \subseteq M$. We claim $M_\kappa = M$. Suppose $M_\kappa \subsetneq M$. Then $D(M_\kappa) \subsetneq D(M)$, or else (M_κ, M, D) is a Vaught pair. Take $b \in D(M) \setminus D(M_\kappa)$. Note $\operatorname{acl}(M_0 I_\kappa) \subseteq \operatorname{acl}(M_\kappa) = M_\kappa$. ($\operatorname{acl}(M_\kappa) = M_\kappa$ by Proposition 18 in the April 7 notes.) Therefore $b \notin \operatorname{acl}(M_0 I_\kappa)$. Then $b \models \hat{p} \restriction M_0 I_\kappa$, and $I_\kappa \cup \{b\}$ is a larger independent set of realizations of p in M, contradicting the maximality of I. $\square$

Since the cardinality of M_κ is $\max(\aleph_0, \kappa)$, we see that for $\kappa > \aleph_0$, the only model of size κ is M_κ.

Corollary 2.10. *T is κ-categorical for $\kappa > \aleph_0$.*

This completes the proof of (2) $\implies$ (3) in Theorem 2.4.

3 Constructible models are atomic

Definition 3.1. B is *atomic* over A if for every finite tuple $\bar{b} \in B$, $\operatorname{tp}(\bar{b}/A)$ is isolated.

Remark 3.2. A is atomic over A: if $\bar{b} \in A$, then $\operatorname{tp}(\bar{b}/A)$ is isolated by the formula $\bar{x} = \bar{b}$.

Remark 3.3. If $\operatorname{tp}(b/A)$ is isolated, then Ab is atomic over A. Indeed, if $\varphi(y) \in L(A)$ isolates $\operatorname{tp}(b/A)$, and if $\bar{a}$ is a tuple in A, then $\operatorname{tp}(\bar{a}, b/A)$ is isolated by the formula $(\bar{x} = \bar{a}) \ \wedge \ \varphi(y)$.

Lemma 3.4. *Suppose $A \subseteq B \subseteq C$. If C is atomic over B and B is atomic over A, then C is atomic over A.*

Proof. For $\bar{c} \in C$, take a formula $\varphi(\bar{x}; \bar{b})$ isolating $\operatorname{tp}(\bar{c}/B)$, with $\bar{b} \in B$. Take an $L(A)$-formula $\psi(\bar{y})$ isolating $\operatorname{tp}(\bar{b}/A)$. Let $\theta(\bar{x}) \in L(A)$ be

$$\exists \bar{y} \ (\varphi(\bar{x}; \bar{y}) \ \wedge \ \psi(\bar{y})).$$

Then $\mathbb{M} \models \theta(\bar{c})$ because we can take $\bar{y} = \bar{b}$. We claim $\theta(\bar{x})$ isolates $\operatorname{tp}(\bar{c}/A)$. Suppose $\mathbb{M} \models \theta(\bar{c}_0)$. Take $\bar{b}_0$ such that $\mathbb{M} \models \varphi(\bar{c}_0; \bar{b}_0) \ \wedge \ \psi(\bar{b}_0)$. Then $\psi(\bar{b}_0)$ ensures $\bar{b}_0$ realizes $\operatorname{tp}(\bar{b}/A)$. Moving $(\bar{b}_0, \bar{c}_0)$ by an automorphism over A, we may assume $\bar{b}_0 = \bar{b}$. Then $\varphi(\bar{c}_0; \bar{b})$ ensures $\bar{c}_0$ realizes $\operatorname{tp}(\bar{c}/B)$. In particular, it realizes $\operatorname{tp}(\bar{c}/A)$. $\square$

Proposition 3.5. *If $M \preceq \mathbb{M}$ is constructible over $A \subseteq M$, then M is atomic over A.*

Proof. Take $(b_\alpha : \alpha < \beta)$ a construction of M over A. For $\alpha \le \beta$, let $B_{<\alpha} = \{b_\gamma : \gamma < \alpha\}$. We prove by induction on α that $AB_{<\alpha}$ is atomic over A.

- $\alpha = 0$ is handled by Remark 3.2
- If α is a limit ordinal, then any finite tuple in $AB_{<\alpha}$ comes from $AB_{<\gamma}$ for some $\gamma < \alpha$, so things work by induction.
- Consider $\alpha + 1$. By induction, $AB_{<\alpha}$ is atomic over A. We know $\operatorname{tp}(b_\alpha/AB_{<\alpha})$ is isolated. By Remark 3.3, $AB_{<\alpha+1} = AB_{<\alpha}b_\alpha$ is atomic over $AB_{<\alpha}$. By Lemma 3.4, $AB_{<\alpha+1}$ is atomic over A.

This completes the induction. Taking $\alpha = \beta$, we see that AM is atomic over A, which implies M is atomic over A. $\square$

4 Orthogonality

Assume T is totally transcendental.

Definition 4.1. Let M be a model. Suppose $p \in S_n(M)$ and $\varphi(\bar{x}) \in L(M)$. Say that p is *orthogonal* to $\varphi(\bar{x})$ if there is a model $N \succeq M$ containing a realization $b \models p$, and $\varphi(N) = \varphi(M)$.

Remark 4.2. In Definition 4.1, N contains a copy of a prime model over $M \cup \{b\}$. Without loss of generality, N is a prime model over $M \cup \{b\}$.

Lemma 4.3. *If $p \in S_1(M)$ and $\varphi(\bar{x}) \in L(M)$, then p is non-orthogonal to $\varphi(\bar{x})$ if and only if there is an L-formula $\psi(\bar{x}; y, \bar{z})$ and a tuple $\bar{c} \in M$ such that*

- *The formula $(\exists \bar{x})(\varphi(\bar{x}) \ \wedge \ \psi(\bar{x}; y; \bar{c}))$ is in $p(y)$.*
- *For each $\bar{a} \in M$, the formula $\neg\psi(\bar{a}; y; \bar{c})$ is in $p(y)$.*

Proof. First suppose $\psi, \bar{c}$ exist. Suppose $N \succeq M$ contains a realization $b \models p$. Then

$$N \models (\exists \bar{x})(\varphi(\bar{x}) \ \wedge \ \psi(\bar{x}; b; \bar{c}))$$

but for each $\bar{a} \in M$,

$$N \models \neg\psi(\bar{a}; b; \bar{c}). \qquad (*)$$

Take $\bar{a}_0 \in N$ satisfying $\varphi(\bar{x}) \ \wedge \ \psi(\bar{x}; b, \bar{c})$. By $(*)$, $\bar{a}_0 \notin M$. Then $\bar{a}_0 \in \varphi(N) \setminus \varphi(M)$, showing non-orthogonality.

Conversely, suppose p is non-orthogonal to $\varphi(\bar{x})$. Take $b \in \mathbb{M}$ realizing p and take $N \preceq \mathbb{M}$ constructible over Mb. By non-orthogonality, $\varphi(N) \supsetneq \varphi(M)$. Take $\bar{a}_0 \in \varphi(N) \setminus \varphi(M)$. By

Proposition 3.5, N is atomic over Mb. Then $\mathrm{tp}(\bar{a}_0/Mb)$ is isolated by some formula $\psi(\bar{x}; b; \bar{c})$ with $\bar{c} \in M$. The element $\bar{a}_0$ shows

$$\mathbb{M} \models (\exists \bar{x})(\varphi(\bar{x}) \ \wedge \ \psi(\bar{x}; b; \bar{c})). \tag{1}$$

Note that if $\psi(\bar{a}; b; \bar{c})$ holds, then $\bar{a} \equiv_{Mb} \bar{a}_0$, and so $\bar{a} \notin M$. Taking contrapositives, we see that

$$\mathbb{M} \models \neg\psi(\bar{a}; b; \bar{c}) \qquad \text{for any } \bar{a} \in M. \tag{2}$$

Using the fact that $p = \mathrm{tp}(b/M)$, equations (1)–(2) turn into the two desired conditions. $\square$

Lemma 4.4. *Suppose $M \preceq M' \preceq \mathbb{M}$, $p \in S_1(M)$, $p' \in S_1(M)$, p' is the heir of p, and $\varphi(\bar{x}) \in M$. Then p is orthogonal to $\varphi(\bar{x})$ iff p' is orthogonal to $\varphi(\bar{x})$.*

Proof. The conditions of Lemma 4.3 can be expressed as first-order sentences in the structures (M, dp) and (M', dp'). From the proof of Propositions 15–16 in the February 24 notes, one can see that $(M, dp) \preceq (M', dp')$, so things transfer between (M, dp) and (M', dp'). $\square$

In the following, $M \prec N$ and $N \succ M$ mean $M \preceq N$ and $M \neq N$. (This notation is slightly non-standard; many authors use $M \prec N$ to mean $M \preceq N$.)

Proposition 4.5 (Stretching Vaught pairs)**.** *Suppose there is a Vaught pair: suppose that $M \prec N$ and suppose $\varphi(M) = \varphi(N)$ for some $L(M)$-formula $\varphi(\bar{x})$.*

1. *For every $M' \succeq M$, there is $N' \succ M'$ with $\varphi(N') = \varphi(M')$. We may take $|N'| \leq |M'| + |L|$.*
2. *For every $\kappa \geq |M| + |L|$, there is an elementary extension $N \succeq M$ with $|N| = \kappa$ and $\varphi(N) = \varphi(M)$.*

Proof. 1. Take $b \in N \setminus M$. Then $p := \mathrm{tp}(b/M)$ is orthogonal to $\varphi(\bar{x})$. Let $p' \in S_1(M')$ be the heir of p. By Lemma 4.4, p' is orthogonal to $\varphi(\bar{x})$. So there is $N' \succeq M'$ containing a realization $b' \models p'$, with $\varphi(N') = \varphi(M')$. The fact that p is non-constant implies that its heir p' is non-constant, so $b' \notin M'$ and $N' \succ M'$. By downward Löwenheim-Skolem we may assume $|N'| \leq |M'| + |L|$.

2. Iterating part (1), build an increasing elementary chain $(M_\alpha : \alpha < \kappa)$ where...
   - $M_0 = M$.
   - $M_{\alpha+1} \succ M_\alpha$ with $\varphi(M_{\alpha+1}) = \varphi(M_\alpha)$ and $|M_{\alpha+1}| \leq |M_\alpha| + |L|$.
   - If β is a limit ordinal, then $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$.

   Take $N = \bigcup_\alpha M_\alpha$. Then $N \succeq M$ with $\varphi(N) = \varphi(M)$. Note $|M_\alpha| \leq |\alpha| + |L| + |M|$ by induction on α, so $|N| \leq \kappa + |L| + |M| = \kappa$. On the other hand, $\kappa \leq |N|$ because there were at least κ strict inclusions in the chain. $\square$

5 The end of the proof

Say that an elementary extension $N \succeq M$ *enlarges infinite sets* if $D(N) \supsetneq D(M)$ for every infinite M-definable set $D \subseteq \mathbb{M}^n$.

Lemma 5.1. *If $M \preceq \mathbb{M}$, then there is $N \succeq M$ such that $|N| \le |M| + |L|$ and the extension N enlarges infinite sets.*

Proof. Let $\kappa = |L(M)| = |L| + |M|$. Let $\{D_\alpha : \alpha < \kappa\}$ enumerate all infinite M-definable sets. For each α, the set $D_\alpha = D_\alpha(\mathbb{M})$ is large(r than $D_\alpha(M)$), by saturation. Take $\bar{b}_\alpha \in D_\alpha(\mathbb{M}) \setminus D_\alpha(M)$. By downward Löwenheim-Skolem there is a small model N containing $M \cup \{\bar{b}_\alpha : \alpha < \kappa\}$ with $|N| \le |M| + \kappa = |M| + |L|$. □

Proposition 5.2. *Suppose models of T are infinite. If $\kappa \ge |L|$, then there is a model $M \preceq \mathbb{M}$ such that $|M| = \kappa$ and $|D(M)| = \kappa$ for every infinite M-definable set.*

Proof. Recursively define an increasing elementary chain $(M_\alpha : \alpha < \kappa)$ as follows:

- M_0 is a model of size $|L|$, which exists by Löwenheim-Skolem.
- If β is a limit ordinal, then $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$.
- $M_{\alpha+1} \succeq M_\alpha$ is an extension which enlarges infinite sets, with $|M_{\alpha+1}| \le |L| + |M_\alpha|$.

Let $N = \bigcup_{\alpha<\kappa} M_\alpha$. As in the proof of Proposition 4.5(2), $|N| \le \kappa$. It remains to show that $|D(N)| \ge \kappa$ for infinite N-definable sets D. The set D is M_α-definable for some $\alpha < \kappa$. For all β with $\alpha < \beta < \kappa$, we have $D(M_{\beta+1}) \supsetneq D(M_\beta)$. The number of such β is κ, so $|D(N)| \ge \kappa$. □

Theorem 5.3. *Let T be a complete theory with infinite models in a countable language. If T is κ-categorical for some $\kappa > \aleph_0$, then T is totally transcendental and T has no Vaught pairs.*

Proof. T is ω-stable by Theorem 32 in the March 24th notes. By Theorem 7.2 in the May 7 notes, "ω-stable" is equivalent to "totally transcendental" since the language is countable.

Assume for the sake of contradiction that T has a Vaught pair $(M, N, \varphi(\bar{x}; \bar{b}))$. Then $(N, M, \bar{b})$ is a model of the theory T_φ of Remark 2.3. Applying downward Löwenheim-Skolem, we may assume N, M are countable. By Proposition 4.5(2), there is an elementary extension $N_1 \succeq M$ with $|N_1| = \kappa$, such that $\varphi(N_1; \bar{b}) = \varphi(M; \bar{b})$. In particular, $|\varphi(N_1; \bar{b})| = \aleph_0 < \kappa$.

By Proposition 5.2, there is also a model N_2 of cardinality κ such that $D(N_2) = \kappa$ for every infinite N_2-definable set. There is an infinite N_1-definable set D such that $D(N_1) \neq \kappa$, namely $D = \varphi(\mathbb{M}; \bar{b})$. Therefore $N_1 \not\cong N_2$, and κ-categoricity fails. □

This completes the proof of Morley's theorem plus the Baldin-Lachlan characterization of uncountable categoricity (which is: totally transcendental plus no Vaught pairs).