The Journal of Symbolic Logic
Volume 74, Number 1, March 2009

TOPOLOGICAL DYNAMICS OF DEFINABLE GROUP ACTIONS

LUDOMIR NEWELSKI

Abstract. We interpret the basic notions of topological dynamics in the model-theoretic setting, relating them to generic types of definable group actions and their generalizations.

Introduction. Assume G is a group. By a G-flow we mean a compact topological space X together with a left action of G on X by homeomorphisms. We say that a non-empty closed subset Y of X is a subflow of X if Y is closed under the action of G. We say that a flow is minimal if it has no proper subflow. We say that a flow is point-transitive, if it contains a dense G-orbit. Topological dynamics is concerned with the orbits of the actions of G on various G-flows, and particularly with minimal flows. For more background on topological dynamics the reader is referred to [E, A].

Assume X is a G-flow. For $x \in X$ let $o(x)$ denote the orbit of X. Then the closure $\mathrm{cl}(o(x))$ is a subflow of X and for any orbit $\mathscr{O} \subseteq X$ and every $x \in \mathrm{cl}(\mathscr{O})$ we have that $\mathrm{cl}(o(x)) \subseteq \mathrm{cl}(\mathscr{O})$. By Zorn lemma it follows that for any orbit $\mathscr{O} \subseteq X$, the closure $\mathrm{cl}(\mathscr{O})$ contains a minimal flow. If the flow $\mathrm{cl}(o(x))$ is minimal, then we call the point x almost periodic. In fact, every minimal flow is point transitive, any point in a minimal flow is almost periodic and any two distinct minimal subflows of X are disjoint.

There is a natural notion of a morphism of G-flows (called G-mapping), so that G-flows become a category. Point-transitive G-flows are a subcategory. There is a unique largest universal point-transitive G-flow, namely $X = \beta G$, the space of ultrafilters on G, where the action of G on βG is by left translation:

For $g \in G$ and $\mathscr{U} \in \beta G$ we let $g \cdot \mathscr{U} = \{g \cdot U : U \in \mathscr{U}\}$. The orbit consisting of the principal ultrafilters is dense in βG.

For every point-transitive G-flow X there is a surjective G-mapping $\beta G \to X$ and every minimal flow in X is an image of a minimal flow in βG under this mapping.

In model theory we often consider a definable counterpart of this situation. Namely, assume G is a group definable in some first-order structure M, with the underlying language L and $T = Th(M)$. By a definable G-set we mean a definable set V in M together with a definable group action on V. If this action is transitive, we call V a homogeneous G-set. Again, the largest homogeneous G-set is $V = G$,

Received June 28, 2006.
2000 *Mathematics Subject Classification.* Primary 03C45. Secondary 37B05.
Research supported by KBN grant 2 P03A 018 24.

0022-4812/09/7401-0004/$3.30

with the action on itself by left translation. Definable G-sets do not carry topology. However, in model theory we have various topological spaces related to a G-set V.

We use the standard model-theoretic notation. We consider M and all other models we investigate to be elementary submodels of a fixed, very saturated monster model $\mathfrak{C}$. For a definable subset V of M and $N \succ M$, by V^N we mean the set $\varphi(N)$, where φ is a formula defining V in M.

Assume V is a definable G-set in M. Then for any $N \succ M$ also G^N acts on V^N, in particular, $G^{\mathfrak{C}}$ acts on $V^{\mathfrak{C}}$. Also, G is a subgroup of $G^{\mathfrak{C}}$, hence G acts also on $V^{\mathfrak{C}}$.

Assume moreover that E is an equivalence relation on $V^{\mathfrak{C}}$, type-definable over M, with bounded number of classes (shortly: btde-relation). Then on the quotient set $V^{\mathfrak{C}}/E$ (which we denote also by V_E) there is a natural topology, with the closed sets of the form $\pi[U]$ for type-definable sets $U \subseteq V^{\mathfrak{C}}$. Here $\pi : V^{\mathfrak{C}} \to V^{\mathfrak{C}}/E$ is the quotient map. This topology is compact and Hausdorff [W], and is sometimes called the logic topology on V_E.

We say that a btde-relation E on $V^{\mathfrak{C}}$ is G-invariant if for $x, y \in V^{\mathfrak{C}}$, xEy implies $gxEgy$ for every $g \in G^{\mathfrak{C}}$. In this case the action of G on the G-set V induces the action of G on the space V_E by homeomorphisms. Hence V_E becomes a G-flow. We call any G-flow of this kind a definable G-flow.

Remark 0.1. *If V is a homogeneous G-set, then the flow V_E is point transitive.*

Proof. The orbit consisting of the elements $a_E, a \in V$, is dense in V_E. Indeed, suppose $b \in V^{\mathfrak{C}}$ and let W be a type-definable set in $\mathfrak{C}$ disjoint from $[b]_E$, so that $\pi[W]^c$ is an open neighbourhood of $b_E = \pi(b)$. Thus for some symmetric formula $\varphi(x, y) \in E(x, y)$ we have that $\varphi(\mathfrak{C}, b) \cap W = \emptyset$. Choose a symmetric $\varphi'(x, y) \in E(x, y)$ such that $\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z)$.

Since E is bounded, $\varphi'(\mathfrak{C}, b)$ meets M. Let $a \in \varphi'(\mathfrak{C}, b) \cap V$. It follows that $\varphi'(\mathfrak{C}, a) \cap W = \emptyset$, hence $[a]_E \cap W = \emptyset$ and $a_E \in \pi[W]^c$. ⊣

There is a finest bounded type-definable over M equivalence relation on $V^{\mathfrak{C}}$, namely the elementary equivalence $\equiv_M$ given by: $x \equiv_M y$ iff $tp(x/M) = tp(y/M)$. This equivalence relation is G-invariant. Hence the G-flow $V_{\equiv_M}$ is the largest one among the G-flows of the form V_E, where E is a G-invariant btde-relation on $V^{\mathfrak{C}}$. Since G, regarded as a definable G-set acted upon by G by left translation, is universal among all definable homogeneous G-sets, also the G-flow $G_{\equiv_M}$ is universal for all the point-transitive definable G-flows. Hence any point transitive G-flow of the form V_E is isomorphic to the G-flow G_E for some bounded G-invariant type-definable over M equivalence relation E on G, coarser than $\equiv_M$. This justifies our interest in the G-space $G_{\equiv_M}$, which we denote by $S_G(M)$.

One of the central notions in stable model theory is that of a generic type for a definable G-set V [Po, W]. In [NP1, NP2] we tried to generalize this notion to a broader, unstable context, introducing *weak generic* types. In this paper we relate these notions to the basic ideas of topological dynamics sketched above. We believe that topological dynamics provides a proper set-up here.

In Section 4 we interpret the Ellis semigroup as a model-theoretic object. This enables us to find a general counterpart of some notions introduced in [N1] in the stable case. Most importantly, it raises several questions relating some subgroups of the Ellis semigroup to the group G/G^{00}.

Assume G is a definable group in M and $N \prec \mathfrak{C}$ is an extension of M. Let $S_G(N)$ be the set of types $\{tp(g/N) : g \in G^{\mathfrak{C}}\}$ (this is consistent with the definition of $S_G(M)$). Then $S_G(N)$ is a point-transitive G^N-flow (and also a G-flow) and there is a natural continuous restriction function $r : S_G(N) \to S_G(M)$, which is a morphism of G-flows. Investigation of the relationship between the topological dynamics of $S_G(N)$ and $S_G(M)$ has a new, specifically model theoretic flavour.

In the stable context, generic types on $S_G(M)$ are thought of as "large" types, and then it is natural that the restriction of a generic type in $S_G(N)$ to M is again a generic type. Moreover, a type $q \in S_G(N)$ is generic iff $q|M$ is generic in $S_G(M)$ and the extension $q \supseteq q|M$ is non-forking. So in the stable context the notion of a generic type is closely related to forking independence.

In the general case (where the generic types may be missing) we discern within $S_G(M)$ the weak generic types as a surrogate for generic types. In this paper inside the set of weak generic types we distinguish an even smaller subset of almost periodic types, which have more regularity properties than weak generic types. This raises the question, which of these two notions is the correct general counterpart of the notion of generic type. We do not provide a definite answer here, however to shed some light on this problem we investigate the extension and restriction properties of weak generic and almost periodic types.

Now assume that both M and N are $\aleph_0$-saturated. It turns out that both weak generic and almost periodic types in $S_G(M)$ have weak generic and almost periodic extensions in $S_G(N)$, respectively. However, while the restriction of a weak generic type in $S_G(N)$ to M remains weak generic, it may happen that the restriction of an almost periodic type in $S_G(N)$ to M is not almost periodic. We give a rather complicated example of this phenomenon in Section 3. The theory in this example is simple, of SU-rank 1, so we get an example of a simple structure, where the notions of almost periodic and forking generic types differ. Still no example of a simple structure is known, where the notions of weak generic types and forking generic types differ.

I would like to thank T. Downarowicz and A. Ivanov for stimulating conversations.

§1. Generic and weak generic types. Assume G is a definable group in M and V is a homogeneous definable G-set in M. For $A \subseteq G$ and $U \subseteq V$ by AU we mean $\bigcup_{g \in A} gU$. First we recall the definitions of generic and weak generic sets and types in V.

DEFINITION 1.1. [NP2, Po, W] Assume $U \subseteq V$.

(1) We say that U is (left) generic for the action of G on V if for some finite $A \subseteq G$ we have that $V = AU$.
(2) We say that U is weak (left) generic if for some non-generic $U' \subseteq V$ we have that $U \cup U'$ is generic.
(3) We say that a formula $\varphi(x)$ over M is [weak] generic if the set $\varphi(M) \cap V$ is [weak] generic.
(4) We call a type $p(x) \in S(M) \cap [V(x)]$ [weak] generic if every formula $\varphi(x) \in p(x)$ is [weak] generic.

By a [weak] generic subset [type] of G we mean a [weak] generic subset [type] of $V = G$ acted upon by G by left translation.

REMARK 1.2. [NP2] *Assume $U \subseteq V$.*

(1) *U is weak generic iff for some finite $A \subseteq G$ we have that the set $V \setminus AU$ is not generic.*

(2) *Assume U is definable. Then U is weak generic iff for some non-generic definable $U' \subseteq V$ we have that $U \cup U'$ is generic.*

We can naturally extend Definition 1.1 to point transitive G-flows. Actually, in topological dynamics generic subsets of G are called syndetic.

DEFINITION 1.3. Assume X is a point-transitive G-flow and $p \in X$. We say that p is [weak] generic if every open neighbourhood U of p in X is [weak] generic.

In the case, where $X = S_V(M)$ is the G-flow of complete types of elements of $V^{\mathfrak{C}}$ over M, this definition agrees with Definition 1.1. Given a point-transitive G-flow X let $Gen(X)$ and $WGen(X)$ denote the sets of generic and weak generic points in X, respectively. Clearly, these sets are closed and $Gen(X) \subseteq WGen(X)$. By [NP2], if $Gen(X) \neq \emptyset$, then $Gen(X) = WGen(X)$. Also, $WGen(X) \neq \emptyset$, while often $Gen(X) = \emptyset$.

When $X = S_V(M)$ then we denote $Gen(X)$ and $WGen(X)$ by $Gen_V(M)$ and $WGen_V(M)$, respectively.

LEMMA 1.4. [NP2, Lemma 3.4] *Assume $f : X \to Y$ is a surjective morphism of point-transitive G-flows. Then $f[Gen(X)] \subseteq Gen(Y)$ and $f[WGen(X)] = WGen(Y)$.*

COROLLARY 1.5. *Assume E is a G-invariant btde-relation on $V^{\mathfrak{C}}$ and $a \in V^{\mathfrak{C}}$.*

(1) *a_E is weak generic in V_E iff for some $a' \in [a]_E$, the type $tp(a'/M)$ is weak generic in $S_V(M)$.*

(2) *If the type $tp(a/M)$ is generic in $S_V(M)$, then a_E is generic in V_E.*

(3) *a_E is generic in V_E iff for every formula $\varphi(x, y) \in E(x, y)$ we have that $\varphi(\mathfrak{C}, a)$ is generic for the action of G on $V^{\mathfrak{C}}$ (notice that this action is not transitive).*

PROOF. There is a natural surjective morphism $r : S_V(M) \to V_E$ of G-flows. So (1) and (2) follow from Lemma 1.4 and the description of [weak] generic types in $S_V(M)$.

(3) By the definition of the topology on V_E, a_E is generic in V_E iff

($*$) for every $\varphi(x, y) \in E(x, y)$, the set $\bigcup\{[b]_E : [b]_E \subseteq \varphi(\mathfrak{C}, a)\}$ is generic in $V^{\mathfrak{C}}$ (regarded as a G-set),

so clearly $\Rightarrow$ holds.

For $\Leftarrow$, let $\varphi(x, y) \in E(x, y)$. To show ($*$) choose $\varphi'(x, y) \in E(x, y)$ such that

($**$) $\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z)$.

By assumption we have that $\varphi'(\mathfrak{C}, a)$ is generic in $V^{\mathfrak{C}}$ (with respect to the action of G). However by ($**$),

$$\varphi'(\mathfrak{C}, a) \subseteq \bigcup\{[b]_E : [b]_E \subseteq \varphi(\mathfrak{C}, a)\},$$

hence we are done. ⊣

The notions of generic and weak generic points in a G-flow X are clarified by reference to almost periodic points. Recall that $p \in X$ is almost periodic iff $\mathrm{cl}(Gp)$ is a minimal flow.

REMARK 1.6. *$p \in X$ is almost periodic iff for every open $U \ni p$ (equivalently: every open $U \subseteq X$ meeting $\mathrm{cl}(Gp)$), the set $\mathrm{cl}(Gp)$ is covered by AU for some finite $A \subseteq G$.*

PROOF. $\Leftarrow$ Suppose $q \in \mathrm{cl}(Gp)$ and $U \subseteq X$ is an open set meeting $\mathrm{cl}(Gp)$. Then $\mathrm{cl}(Gp) \subseteq AU$ for some finite $A \subseteq G$, hence $q \in gU$ for some $g \in A$. Thus $g^{-1}q \in U$, showing that Gq meets U. So $\mathrm{cl}(Gq) = \mathrm{cl}(Gp)$.

$\Rightarrow$ Let $U \ni p$ be open. The set $\mathrm{cl}(Gp) \setminus GU$ is closed and G-invariant, so by the minimality of $\mathrm{cl}(Gp)$ it is empty. So we are done by compactness. ⊣

LEMMA 1.7. *Assume X is a point transitive G-flow and $p \in X$.*

(1) *An open set $U \subseteq X$ is generic iff U meets every minimal subflow in X.*
(2) *p is generic iff every open $U \ni p$ meets every minimal subflow in X.*
(3) *p is weak generic iff every open $U \ni p$ meets some minimal subflow in X.*

PROOF. (1) $\Rightarrow$ is immediate. $\Leftarrow$ The set $X \setminus GU$ is closed, G-invariant and by Remark 1.6 it is disjoint from any minimal suflow in X, hence it is empty. So we are done by compactness.

(2) follows from (1).

(3) $\Rightarrow$ Suppose some open $U \ni p$ is disjoint from any minimal flow in X. By regularity of X we can assume that also $\mathrm{cl}(U)$ is such. Then for every $g \in G$ the set $\mathrm{cl}(gU)$ is disjoint from any minimal flow in X.

Since p is weak generic, for some finite $A \subseteq G$ we have that $X \setminus AU$ is non-generic. On the other hand, all minimal flows in X are contained in $X \setminus \mathrm{cl}(AU)$, hence by (1) the open set $X \setminus \mathrm{cl}(AU)$ is generic, hence also the set $X \setminus AU$ is generic, a contradiction.

$\Leftarrow$ Let $U \ni p$ be open. Let $V \subseteq X$ be a minimal flow meeting U. Choose an open $U' \ni p$ meeting V with $\mathrm{cl}(U') \subseteq U$. By Remark 1.6 choose a finite $A \subseteq G$ such that $V \subseteq AU'$. Then by (1) the set $X \setminus \mathrm{cl}(AU')$ is not generic, hence also the set $X \setminus AU$ is not generic. ⊣

COROLLARY 1.8. *$WGen(X)$ is the closure of the union of all minimal flows in X.*

COROLLARY 1.9. *Assume $Gen(X) \neq \emptyset$. Then $Gen(X) = WGen(X)$ is the only minimal flow in X.*

Assume X is a point transitive G-flow. Inside X we have discerned the closed set $WGen(X)$, and inside $WGen(X)$ we have found still more perfect almost periodic points, belonging to the minimal flows in X. There is however still another way of refining the notion of a weak generic point in X. Namely, we say that a closed set $C \subseteq X$ is almost generic if every open $U \supseteq C$ is generic.

Since the intersection of a decreasing chain of closed almost generic sets $C \subseteq X$ is again a closed almost generic set, by Zorn lemma we have that every closed almost generic set contains a minimal closed almost generic set C'. Let $MGen(X)$ be the family of all minimal closed almost generic sets $C \subseteq X$.

REMARK 1.10. (1) *$WGen(X) = \mathrm{cl}(\bigcup MGen(X))$.*
(2) *Every subflow $Y \subseteq X$ meets every closed almost generic set $C \subseteq X$.*

PROOF. (1) $\supseteq$ Suppose $p \in C \setminus WGen(X)$ for some $C \in MGen(X)$. Let $U \ni p$ be open, with $\mathrm{cl}(U)$ disjoint from $WGen(X)$. So U is not weakly generic. Let $C' = C \setminus U$. Then for every open $U' \supseteq C'$ we have that $C \subseteq U \cup U'$, hence $U \cup U'$ is generic and U' is generic (as U is not weakly generic). So C' is almost generic, contradicting the minimality of C.

$\subseteq$ Suppose $p \in WGen(X)$. Let $U \ni p$ be open, and then let $U' \ni p$ be open, with $\mathrm{cl}(U') \subseteq U$. Choose a finite set $A \subseteq G$ such that the set $X \setminus AU'$ is not generic. It follows that also the open set $X \setminus A \cdot \mathrm{cl}(U')$ is not generic. So every set $C \in MGen(X)$ meets $A \cdot \mathrm{cl}(U')$, hence also AU. Thus for some $a \in A$ we have that $a^{-1}C \cap U \neq \emptyset$. Therefore U meets $\bigcup MGen(X)$.

(2) Let $U \supseteq C$ be open. Since U is generic, for some finite $A \subseteq G$ we have that $AU = X$. Thus for some $a \in A$ we have that $aU \cap Y \neq \emptyset$. Hence $U \cap a^{-1}Y \neq \emptyset$. But $a^{-1}Y = Y$, hence Y meets U. Since U was arbitrary, Y meets C. ⊣

Now we return to G and V definable in M. In the model-theoretic setting a closed set $C \subseteq S_V(M)$ corresponds to an (incomplete) type p, and C is almost generic iff p is generic (as a type). Hence the minimal closed almost generic sets $C \subseteq S_V(M)$ correspond to maximal (possibly incomplete) generic types over M containing the formula $V(x)$.

If $T = Th(M)$ is stable, then in $S_V(M)$ there are generic types and G acts transitively on them, hence there is just one minimal G-flow in $S_V(M)$, consisting of a single orbit. In general the situation may be more complicated . For example, if M is an o-minimal expansion of a real closed field, and G is definably compact, then by [PP] there are generic types in $S_G(M)$, so there is just one minimal flow there, but it need not be a single orbit. E.g. when G is S^1 interpreted in the field of reals, then the minimal G-flow consists of two orbits. Still in these two examples there is just one minimal flow in $S_G(M)$ and $MGen$ in $S_G(M)$ consists of singletons in this subflow.

When $G = (\mathbb{R}^n, +)$ (interpreted in the field of reals), then there are no generic types in $S_G(M)$, there is a rich structure of weak generic types there. Each weak generic type in $S_G(M)$ is G-invariant, hence it is a minimal G-flow itself. Hence in this example there is just one set in $MGen$, namely $WGen$, and this set consists of 1-point minimal G-flows. When $n = 1$ (i.e. $G = (\mathbb{R}, +)$), there are just two weak generic types in $S_G(M)$, while when $n > 1$, there are $\geq 2^{\aleph_0}$- many of them. See [Pe] for more details.

We see that the notions of a minimal flow in X and a minimal closed almost generic set $C \subseteq X$ are quite orthogonal: the minimal G-flows in X are pairwise disjoint and lie densely in $WGen(X)$. The sets $C \in MGen(X)$ also lie densely in $WGen(X)$, but they need not be pairwise disjoint. This happens for example in the group $\mathbb{R}^+ \times S^1$, and also in the group $\mathbb{R}^+ \times \{-1, +1\}$, considered in the ordered field of reals. Each such C meets every minimal G-flow in X.

Using minimal flows we can improve our results from [NP2]. Namely, consider the set-up, where M is $\aleph_0$-saturated. We say that a subset U of M is Borel if $U = \bigcup\{p(M) : p \in B\}$ for some Borel subset B of $S(\emptyset)$. The next theorem improves our result from [NP2], where we assume that the sets X_n are type-definable (hence correspond to closed sets of types).

THEOREM 1.11. *Assume G is a 0-definable group in an $\aleph_0$-saturated structure M, covered by countably many Borel sets $X_n, n < \omega$. Then for some finite set $A \subseteq G$ and some $n < \omega$ we have that $G = A \cdot X_n \cdot X_n^{-1}$.*

PROOF. Let $r : S_G(M) \to S(\emptyset)$ be the restriction map and for $n < \omega$ let $Y_n = r^{-1}[B_n]$, where $B_n \subseteq S(\emptyset)$ is the Borel set determining X_n. Each Y_n is a Borel subset of $S_G(M)$.

Let $S \subseteq S_G(M)$ be a minimal G-flow. Choose an $n < \omega$ such that $Y_n \cap S$ is not meager in S. Choose a formula $\varphi(x)$ such that $U \cap S \neq \emptyset$ and $Y_n \cap S$ is co-meager in $U \cap S$, where $U = S_G(M) \cap [\varphi]$. By Remark 1.6 there is a finite set $A \subseteq G$ such that $S \subseteq AU$. Hence the set $A(Y_n \cap S)$ is co-meager in S. We claim that

$$G = A \cdot X_n \cdot X_n^{-1}.$$

To show this, let $g \in G$. $g \cdot (Y_n \cap S)$ is not meager in S, hence it meets $A \cdot (Y_n \cap S)$. So there are $p, q \in Y_n \cap S$ such that $g \cdot p = a \cdot q$ for some $a \in A$. Hence $g \in a \cdot q(\mathfrak{C}) \cdot p(\mathfrak{C})^{-1}$.

Let $q' = q|\emptyset$, $p' = p|\emptyset$. Then by the saturation of M we have that $g \in a \cdot q'(M) \cdot p'(M)^{-1}$. Also, $q'(M), p'(M) \subseteq X_n$, hence $g \in a \cdot X_n \cdot X_n^{-1}$. ⊣

We can also give a new, short proof of [NP2, Theorem 2.4].

PROPOSITION 1.12. *Assume G is a 0-definable group in an $\aleph_0$-saturated structure M, covered by countably many 0-type-definable sets $X_n, n < \omega$. Then there is a finite set $A \subseteq G$ and an $n < \omega$ such that*

$$G = \bigcup_{a \in A} (X_{<n} \cdot X_{<n}^{-1})^a,$$

where $X_{<n} = \bigcup_{i<n} X_i$ and $X^a = aXa^{-1}$.

PROOF. Let $Y_n, n < \omega$, be as in the proof of Theorem 1.11 and let $S \subseteq S_G(M)$ be a minimal flow. The sets Y_n are closed now, so by the Baire category theorem, for some $k < \omega$ the set $Y_k \cap S$ has non-empty relative interior in S. This means that for some formula $\varphi(x)$ over M we have that $\emptyset \neq S \cap U \subseteq Y_k$, where $U = S_G(M) \cap [\varphi]$. Choose A as in the proof of Theorem 1.11. Now $S \subseteq AY_k$.

Fix some $p \in S$ and for $a \in A$ let $p_a = a^{-1}p$. Let $p'_a = p_a|\emptyset$. Choose $n > k$ so large that for each $a \in A$ we have $p'_a(M) \subseteq X_{<n}$. We claim that

$$G \subseteq \bigcup_{a \in A} (X_{<n} \cdot X_{<n}^{-1})^a.$$

To prove this, let $g \in G$. Since $gp \in S \subseteq A \cdot Y_k$, there are some $q \in Y_k$ and $a \in A$ such that $gp = aq_k$. Then also $g \cdot a \cdot p_a = a \cdot q$, hence

$$g \in a \cdot q(\mathfrak{C}) \cdot p_a(\mathfrak{C})^{-1} \cdot a^{-1}.$$

By the saturation of M we have that $g \in a \cdot q'(M) \cdot p'_a(M)^{-1} \cdot a^{-1}$, where $q' = q|\emptyset$. hence $g \in a \cdot X_{<n} \cdot X_{<n}^{-1} \cdot a^{-1}$. ⊣

The question remains if this proposition still holds when the sets X_n are just Borel.

§2. Extensions of types. Assume again that G and V are definable in M. Assume $N \prec \mathfrak{C}$ is an extension of M. Then G^N acts transitively on V^N, and we have a G-flow $S_V(M)$ and a G^N-flow $S_V(N)$. There is a natural restriction function $r : S_V(N) \to S_V(M)$. We can consider weak generic types and almost periodic types both in $S_V(M)$ and $S_V(N)$. Here we describe how these notions are related via the function r. Let $WGen_V(M)$, $WGen_V(N)$ be the sets of weak generic types in $S_V(M), S_V(N)$ respectively.

LEMMA 2.1. [NP2] *Assume $\varphi(x)$ is a formula over M.*

(1) *If φ is weak generic in V^M, then φ is weak generic in V^N.*
(2) *If M is $\aleph_0$-saturated, then in* (1) *also the converse holds.*
(3) *If $p \in S_V(M)$ is weak generic, then there is a weak generic type $q \in S_V(N)$ extending p, that is $WGen_V(M) \subseteq r[WGen_V(N)]$.*
(4) *If M is $\aleph_0$-saturated, then $WGen_V(M) = r[WGen_V(N)]$, that is for every weak generic $q \in S_V(N)$ we have that $q|M$ is weak generic in $S_V(M)$.*

Recall that a type $q \in S(N)$ is an heir of $p \in S(M)$ if $q \supseteq p$ and for every formula $\varphi(x, m, n) \in q$ (where $m \subseteq M, n \subseteq N$) there is some $n' \subseteq M$ with $\varphi(x, m, n') \in p$. We say that $q \in S(N)$ is a co-heir extension of p if $q \supseteq p$ and every formula in q is satisfied in M. In the stable case heirs and co-heirs coincide with the non-forking extensions. In general this is not true, although heir and co-heir extensions always exist. To deal with almost periodic types we need the following lemma.

LEMMA 2.2. (1) *Assume $p \in S_V(M)$ and $q \in S_V(N)$ is an heir of p. Then* $r[\mathrm{cl}(G^N \cdot q)] \subseteq \mathrm{cl}(G^M \cdot p)$.
(2) *Assume $q \in S_V(N)$. Then $r[\mathrm{cl}(G^N \cdot q)]$ is G^M-invariant in $S_V(M)$.*

PROOF. (1) Suppose not. Then for some consistent formula $\varphi(x)$ over M we have that

$$S_V(M) \cap [\varphi] \cap \mathrm{cl}(G^M \cdot p) = \emptyset$$

and for some $g \in G^N$ we have that $\varphi \in g \cdot q$. Hence $\varphi(g^{-1} \cdot x) \in q$. Since q is an heir of p, for some $g' \in G^M$ we have that $\varphi(g'^{-1} \cdot x) \in p$, hence $\varphi \in g' \cdot p$, a contradiction.

(2) G^M is a subgroup of G^N, acting by homeomorphisms on $S_V(N)$, hence the set $\mathrm{cl}(G^N \cdot q)$ is G^M-invariant. Since r is a morphism of G^M-flows $S_V(N)$ and $S_V(M)$, also $r[\mathrm{cl}(G^N \cdot q)]$ is G^M-invariant. ⊣

PROPOSITION 2.3. *Assume $p \in S_V(M)$ is almost periodic. Then there is an almost periodic $q \in S_V(N)$ extending p.*

PROOF. Let $q_0 \in S_V(N)$ be an heir of p. By Lemma 2.2(1), $r[\mathrm{cl}(G^N \cdot q_0)] \subseteq \mathrm{cl}(G^M \cdot p)$. Moreover, by Lemma 2.2(2), for every $q \in S_V(N)$ with $r[\mathrm{cl}(G^N \cdot q)] \subseteq \mathrm{cl}(G^M \cdot p)$ the set $r[\mathrm{cl}(G^N \cdot q)]$ is G^M-invariant, hence by the minimality of $\mathrm{cl}(G^M \cdot p)$ we have that $r[\mathrm{cl}(G^N \cdot q)] = \mathrm{cl}(G^M \cdot p)$.

Let $q_1 \in \mathrm{cl}(G^N \cdot q_0)$ be almost periodic. It follows that $\mathrm{cl}(G^N \cdot q_1) \subseteq \mathrm{cl}(G^N \cdot q_0)$, hence $r[\mathrm{cl}(G^N \cdot q_1)] \subseteq r[\mathrm{cl}(G^N \cdot q_0)]$ and $r[\mathrm{cl}(G^N \cdot q_1)] = \mathrm{cl}(G^M \cdot p)$.

Choose a type $q \in \mathrm{cl}(G^N \cdot q_1)$ with $r(q) = p$. Then q is an almost periodic extension of p, since any type in $\mathrm{cl}(G^N \cdot q_1)$ is almost periodic. ⊣

There is a question if every weak generic type in $S_V(M)$ has an heir extension to a weak generic type in $S_V(N)$. Lemma 2.1(4) shows that the notion of a weak generic type is local in the following sense.

COROLLARY 2.4. *Assume G is a 0-definable group in an $\aleph_0$-saturated model N and $p \in S_G(N)$. Then the following conditions are equivalent.*

(1) *p is weak generic.*
(2) *For every $\aleph_0$-saturated $M \preceq N$ we have that $p|M$ is weakly generic in $S_G(M)$.*

(3) *For every finite $A \subseteq N$ there is an $\aleph_0$-saturated $M \preceq N$ containing A, such that $p|M$ is weakly generic in $S_G(M)$.*

Assume $M \prec N$ are $\aleph_0$-saturated and $p \in S_G(N)$ is almost periodic. Then p is weakly generic, hence its restriction p' to M is also weakly generic. So no type in $S_G(M)$ that is not weakly generic can be extended to an almost periodic or even weakly generic type in $S_G(N)$. However, it can happen that a weakly generic and not almost periodic type in $S_G(M)$ extends to an almost periodic type in $S_G(N)$. We will show an example in the next section.

§3. Two examples. The starting point is an example from topological dynamics pointed to me by T. Downarowicz.

Example 1. Consider the group $(\mathbb{Z}, +)$ acting on $2^{\mathbb{Z}}$ by right shift. That is, for $k \in \mathbb{Z}$ and $f \in 2^{\mathbb{Z}}$ the function $k * f \in 2^{\mathbb{Z}}$ is defined by $(k * f)(n) = f(n-k)$. Here we consider $2^{\mathbb{Z}}$ as the topological product of countably many discrete spaces $2 = \{0, 1\}$.

There is an $\eta \in 2^{\mathbb{Z}}$ such that the orbit $\mathbb{Z} * \eta$ of η is dense in $2^{\mathbb{Z}}$, hence $2^{\mathbb{Z}}$ is a point-transitive $\mathbb{Z}$-flow. If $f \in 2^{\mathbb{Z}}$ is periodic, then the orbit of f is finite, hence f is almost periodic (as a point of the $\mathbb{Z}$-flow $2^{\mathbb{Z}}$).

Clearly, periodic functions are dense in $2^{\mathbb{Z}}$, hence every point in $2^{\mathbb{Z}}$ is weak generic, however not every point in $2^{\mathbb{Z}}$ is almost periodic, for example η is not. This example shows that, at least in topological dynamics, the notion of a weak generic point is weaker than that of an almost periodic point.

To transfer this example into a model-theoretic setting we must interpret $2^{\mathbb{Z}}$ as a space of types over some model. If M is a model, then M embeds into $S(M)$ as a dense discrete subspace of algebraic types. So $S(M)$ is a compactification of M (regarded as a discrete space). We can identify $\mathbb{Z}$ with the dense orbit $\mathbb{Z} * \eta \subseteq 2^{\mathbb{Z}}$, in this way $2^{\mathbb{Z}}$ becomes a compactification of $\mathbb{Z}$. We just need to expand the group structure $(\mathbb{Z}, +)$ by some unary predicates corresponding to the standard topological basis of $2^{\mathbb{Z}}$ to make $2^{\mathbb{Z}}$ a space of types over $\mathbb{Z}$.

For any set A let $2^{\subseteq A}$ [$2^{<A}$, respectively] denote the set of all [finite] functions $\sigma \subseteq A \times 2$. $\mathbb{Z}$ acts by right shift not only on $2^{\mathbb{Z}}$, but also on $2^{<\mathbb{Z}}$ and $2^{\subseteq \mathbb{Z}}$. The sets $[\sigma] = \{f \in 2^{\mathbb{Z}} : \sigma \subset f\}, \sigma \in 2^{<\mathbb{Z}}$, form a basis of the topology on $2^{\mathbb{Z}}$.

We expand the structure $(\mathbb{Z}, +)$ by unary predicates $P_\sigma(x), \sigma \in 2^{<\mathbb{Z}}$, defined by

$$P_\sigma(n) \Leftrightarrow n * \eta \in [\sigma].$$

Let $M = (\mathbb{Z}, +, P_\sigma)_{\sigma \in 2^{<\mathbb{Z}}}$ and let $T = Th(M)$.

REMARK 3.1. (1) *For all $n \in \mathbb{Z}$, $P_\sigma(x-n)$ is equivalent in M to $P_{n*\sigma}(x)$.*

(2) *$\neg P_\sigma(x)$ is equivalent to the disjunction $\bigvee\{P_\nu(x) : \nu \in 2^X \setminus \{\sigma\}\}$, where $X = Dom(\sigma)$.*

(3) *If $\sigma_1, \sigma_2 \in 2^{<\mathbb{Z}}$ are compatible, then $P_{\sigma_1}(x) \wedge P_{\sigma_2}(x)$ is equivalent in M to $P_{\sigma_1 \cup \sigma_2}(x)$, otherwise it is inconsistent.*

Let $\Delta = \{P_\sigma(x) : \sigma \in 2^{<\mathbb{Z}}\}$. By Remark 3.1 each function $f \in 2^{\mathbb{Z}}$ determines a complete Δ-type $p_f(x)$ in T, generated by $\{P_{f|X}(x) : X \subseteq \mathbb{Z}$ is finite$\}$; $tp_\Delta(0) = p_\eta(x)$. $\mathbb{Z}$ acts on $S_\Delta(\emptyset) = S_\Delta(M)$ by translation, so $S_\Delta(\emptyset)$ is a $\mathbb{Z}$-flow and the mapping $f \mapsto p_f$ is an isomorphsm of $\mathbb{Z}$-flows $2^{\mathbb{Z}}$ and $S_\Delta(M)$. It follows that every type in $S_\Delta(M)$ is weak generic, however (just as in $2^{\mathbb{Z}}$) not every type in $S_\Delta(M)$ is almost periodic.

REMARK 3.2. *In $S(M)$ there are weak generic types that are not almost periodic.*

PROOF. For $f \in 2^{\mathbb{Z}}$ let $P_f = \{p \in WGen(M) : p_f \subseteq p\}$. So each P_f is a closed, non-empty set. We claim that no type $r \in P_\eta$ is almost periodic.

Indeed, consider a type $r \in P_\eta$. Then for every $\sigma \in 2^{<\mathbb{Z}}$ we have that $\mathbb{Z}r$ meets $[P_\sigma(x)]$, hence for every $f \in 2^{\mathbb{Z}}$ we have that $\mathrm{cl}(\mathbb{Z}r) \cap P_f \neq \emptyset$.

Since for perodic f (with period k), the set $\bar{P}_f = \bigcup_{i<k} P_{i*f}$ is closed and $\mathbb{Z}$-invariant, also $\bar{P}_f \cap \mathrm{cl}(\mathbb{Z}r)$ is closed and $\mathbb{Z}$-invariant, and $\bar{P}_f \cap \mathrm{cl}(\mathbb{Z}r) \neq \mathrm{cl}(\mathbb{Z}r)$. Hence the $\mathbb{Z}$-flow $\mathrm{cl}(\mathbb{Z}r)$ is not minimal. ⊣

We will modify the last example to get a similar phenomenon in a simple structure.

Example 2. Consider the additive group of rationals $(\mathbb{Q}, +)$ acting on the space $2^{\mathbb{Q}}$ by right shift. We consider $2^{\mathbb{Q}}$ as the product of countably many discrete spaces $2 = \{0, 1\}$. Again, there is an $\eta \in 2^{\mathbb{Q}}$ such that the orbit $\mathbb{Q} * \eta$ of η is dense in $2^{\mathbb{Q}}$, hence $2^{\mathbb{Q}}$ is a point transitive $\mathbb{Q}$-flow. Since $(\mathbb{Q}, +)$ has no proper subgroup of finite index, there are no finite orbits in $2^{\mathbb{Q}}$, except for two 1-element orbits of two constant functions.

Again, $(\mathbb{Q}, +)$ acts also on $2^{<\mathbb{Q}}$ and $2^{\subseteq\mathbb{Q}}$. For $\sigma \in 2^{<\mathbb{Q}}$ let $[\sigma] = \{f \in 2^{\mathbb{Q}} : \sigma \subseteq f\}$. Clearly, if $\sigma \neq \emptyset$, then $[\sigma]$ is not generic. It follows that every point in $2^{\mathbb{Q}}$ is weak generic. If $f \in 2^{\mathbb{Q}}$ is periodic and piece-wise constant, then $\mathrm{cl}(\mathbb{Q} * f)$ is a proper subflow of $2^{\mathbb{Q}}$, and flows of this form are dense in $2^{\mathbb{Q}}$. Hence η is not almost periodic.

Again we identify $\mathbb{Q}$ with the orbit $\mathbb{Q} * \eta$ and expand the group structure $(\mathbb{Q}, +)$ with some unary predicates so that we get a model-theoretic isomorphic version of the $\mathbb{Q}$-flow $2^{\mathbb{Q}}$. So for $\sigma \in 2^{<\mathbb{Q}}$ we define a predicate $P_\sigma(x)$ on $\mathbb{Q}$ by:

$$P_\sigma(x) \Leftrightarrow x * \eta \in [\sigma].$$

Let M be the expansion of the $\mathbb{Q}$-vector space $(\mathbb{Q}, +, 0, q)_{q\in\mathbb{Q}}$ by unary predicates $P_\sigma(x), \sigma \in 2^{<\mathbb{Q}}$. Let $T_\eta = Th(M)$.

REMARK 3.3. *The counterpart of Remark* 3.1 *holds here.*

The assumption that $\mathbb{Q} * \eta$ is dense in $2^{\mathbb{Q}}$ does not determine T_η. In order for T_η to have some nice properties we need to assume additionally that η is "sufficiently generic" in $2^{\mathbb{Q}}$, that is η belongs to sufficiently many dense open subsets of $2^{\mathbb{Q}}$ (so we mean Cohen genericity here). We do not want to specify the dense open sets in question, but there will be countably many of them, so a sufficiently generic η exists. From now on we assume η is sufficiently generic.

PROPOSITION 3.4. *If η is sufficiently generic, then T_η admits elimination of quantifiers.*

PROOF. Atomic formulas in the language of T_η are of the form $t(\bar{x}) = 0$ or $P_\sigma(t(\bar{x}))$, where $t(\bar{x})$ is a $\mathbb{Q}$-linear combination of variables $\bar{x}$. It is enough to consider the formula

$$\varphi(\bar{x}, y) = \bigwedge_i \theta_i(\bar{x}, y),$$

where each θ_i is either atomic or negation of an atomic formula and in the term $t(\bar{x}, y)$ in θ_i, the variable y occurs non-trivially. Then we must show that the formula $\exists y \varphi(\bar{x}, y)$ is equivalent in M to a quantifier-free formula.

By Remark 3.3 we can assume that no θ_i is of the form $\neg P_\sigma(\dots)$. Also, we can assume that no θ_i is of the form $t(\bar{x}, y) = 0$ (since in this case y may be expressed as some $t'(\bar{x})$ and eliminated from $\varphi(\bar{x}, y)$).

To facilitate the proof we introduce some auxiliary predicates on M. For $q \in \mathbb{Q}^*$ and $\sigma \in 2^{<\mathbb{Q}}$ let

$$P^q_\sigma(x) \Leftrightarrow P_\sigma(q \cdot x).$$

After these simplifications we can write each θ_i in the form $P^{q_i}_{\sigma_i}(t_i(\bar{x}) + y)$ or $y \neq t_i(\bar{x})$ for some suitable q_i, σ_i and t_i. Hence now $\varphi(\bar{x}, y)$ equals

$$\bigwedge_{i \in I} P^{q_i}_{\sigma_i}(t_i(\bar{x}) + y) \wedge \bigwedge_{j \in J} y \neq t_j(\bar{x}),$$

for some finite disjoint sets of indices I and J. We will see that by genericity of η, for every $\bar{x} \subseteq M$, if there is an $y \in M$ satisfying

$$\bigwedge_{i \in I} P^{q_i}_{\sigma_i}(t_i(\bar{x}) + y), \tag{$*$}$$

then there are infinitely many such y. So we can assume that $\varphi(\bar{x}, y)$ equals $(*)$.

Let $q^0, \dots, q^{n-1}$ be all the distinct numbers among the q_i's. For $t < n$ let

$$\varphi_t(\bar{x}, y) = \bigwedge \{P^{q_i}_{\sigma_i}(t_i(\bar{x}) + y) : q_i = q^t\}.$$

So $\varphi(\bar{x}, y) = \bigwedge_{t<n} \varphi_t(\bar{x}, y)$. Now by genericity of η we have that for every $\bar{a} \subseteq M$:

(a) The fact that for some $y \in M$ we have that $M \models \varphi_t(\bar{a}, y)$ is equivalent to the functions from the set $\{(-t_i(\bar{a})) * \sigma_i : q_i = q^t\}$ being compatible, which is q.f. expressible of $\bar{a}$ in the structure $(\mathbb{Q}, +, 0, q)_{q \in \mathbb{Q}}$.

(b) If every formula $\exists y \varphi_t(\bar{a}, y)$ is true in M, then also $M \models \exists y \bigwedge_{t<n} \varphi_t(\bar{a}, y)$, and this is witnessed by infinitely many y. $\dashv$

Notice that the conjunction $\bigwedge_i P_{\sigma_i}(t_i(\bar{x}))$ is either inconsistent in M or non-algebraic. One can see that $T_\eta = Th(M)$ is simple, of SU-rank 1 (it is an expansion of the strongly minimal vector space over $\mathbb{Q}$ by some "semi-generic" predicates).

Let $\mathfrak{C}$ be a monster model extending M. Let $a \in \mathfrak{C} \setminus M$. Then the type $tp(a/M)$ is determined by the sequence of functions $\bar{f} = \langle f_q : q \in \mathbb{Q}^* \rangle \in \prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$, where

$$f_q = \bigcup \{\sigma \in 2^{<\mathbb{Q}} : \models P_\sigma(q \cdot a)\}.$$

Moreover, every seqence $\bar{f} = \langle f_q : q \in \mathbb{Q}^* \rangle \in \prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$ determines a unique non-algebraic type $p_{\bar{f}} \in S_1(M)$ given by

$$p_{\bar{f}}(x) \equiv \{x \neq a : a \in M\} \cup \{P_{f_q | X}(q \cdot x) : X \subseteq_{fin} \mathbb{Q} \wedge q \in \mathbb{Q}^*\}.$$

Of course the space $S_1(M)$ is a point transitive $\mathbb{Q}$-flow. We will see that all non-algebraic types in $S_1(M)$ are weak generic, so anticipating this fact let $WGen(M)$ be the set of non-algebraic types in $S_1(M)$. We see that $WGen(M)$ is homeomorphic to $\prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$, with the product topology. The homeomorphism $\bar{f} \mapsto p_{\bar{f}}$ between $\prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$ and $WGen(M)$ induces on $\prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$ the structure of a $\mathbb{Q}$-flow. We may describe the action of $(\mathbb{Q}, +)$ on this space explicitly.

REMARK 3.5. *The induced action of* $(\mathbb{Q}, +)$ *on* $\prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ *is given by*

$$\alpha * \langle f_q : q \in \mathbb{Q}^* \rangle = \langle (q\alpha) * f_q : q \in \mathbb{Q}^* \rangle,$$

where $(q\alpha) * f_q$ *is given by the shift action of* $(\mathbb{Q}, +)$ *on* $2^{\mathbb{Q}}$.

PROOF. Let $\psi(x) = P_\sigma(q \cdot x)$ and $\alpha \in \mathbb{Q}$. Then

$$\alpha * ([\psi(x)] \cap S_1(M)) = [\psi(x - \alpha)] \cap S_1(M).$$

We have that

$$\psi(x - \alpha) \Leftrightarrow P_\sigma(q \cdot (x - \alpha)) \Leftrightarrow \sigma \subseteq (q \cdot (x - \alpha)) * \eta \Leftrightarrow$$
$$(q \cdot \alpha) * \sigma \subseteq (q \cdot x) * \eta \Leftrightarrow P_{(q\cdot\alpha)*\sigma}(q \cdot x). \qquad \dashv$$

Since η is sufficiently generic, we have that $(\mathbb{Q}, +) * \bar{\eta}$ is dense in $\prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$, where $\bar{\eta} = \langle \eta_q : q \in \mathbb{Q}^* \rangle \in \prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ is the constant sequence qiven by $\eta_q = \eta, q \in \mathbb{Q}^*$. As in the case of $2^{\mathbb{Q}}$, the complement of any non-empty open set in $\prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ is not generic, hence every point in $\prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$ is weak generic. Hence the same is true of the non-algebraic types in $S_1(M)$, which justifies our notation $WGen(M)$. Also, the orbit $\mathbb{Q} * p_{\bar{\eta}}$ is dense in $WGen(M)$ (since $p_{\bar{\eta}}$ in $WGen(M)$ corresponds to $\bar{\eta}$ in $\prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$). Hence also $WGen(M)$ is a point transitive $\mathbb{Q}$-flow. Similarly as in $2^{\mathbb{Q}}$ we see that $p_{\bar{\eta}}$ is not almost periodic.

REMARK 3.6. *Every element of* M *is definable. So* M *is a prime minimal model of* T_η.

PROOF. Let $m \in M$. Then m is the unique $y \in M$ such that (for any non-empty $\sigma \in 2^{<\mathbb{Q}}$) $y + P_\sigma(M) = P_{m*\sigma}(M)$. $\dashv$

By Remark 3.6 we can assume that any model N of T_η we consider contains M as an elementary submodel.

Let N be any model of T_η. So N is a $\mathbb{Q}$-vector space. As it was the case with $(\mathbb{Q}, +)$, N acts by right shift on the product space 2^N and on the sets $2^{<N}$ and $2^{\subseteq N}$. Also N acts by homeomorphisms on the space $\prod_{q\in\mathbb{Q}^*} 2^N$, regarded with the product topology. Namely, for $n \in N$ and $\bar{f} = \langle f_q : q \in \mathbb{Q}^* \rangle \in \prod_{q\in\mathbb{Q}^*} 2^N$ we define $n * \bar{f}$ as $\langle qn * f_q \rangle_{q\in\mathbb{Q}^*}$, where $qn * f_q \in 2^N$ is given by

$$(qn * f_q)(k) = f_q(k - qn).$$

By a standard book-keeping argument one can find an $\bar{\eta}_N \in \prod_{q\in\mathbb{Q}^*} 2^N$ such that the N-orbit of $\bar{\eta}_N$ is dense in $\prod_{q\in\mathbb{Q}^*} 2^N$. So $\prod_{q\in\mathbb{Q}^*} 2^N$ is a point-transitive N-flow. We will see that $\prod_{q\in\mathbb{Q}^*} 2^N$ is naturally isomorphic (as an N-flow) to the space $WGen(N)$ of weak generic types in $S(N)$.

Let $N = M \oplus V$ for some subspace V of N.

LEMMA 3.7. *For every* $f \in 2^{\subseteq N}$ *there are unique* $f_v \in 2^{\subseteq M}, v \in V$, *such that* f *is the disjoint union of the functions* $v * f_v, v \in V$.

PROOF. Let $f_v = (-v) * f|(v + M)$. $\dashv$

LEMMA 3.8. (1) *Every non-algebraic atomic formula over* N, *in variable* x, *is equivalent to* $P_\sigma(q(x - v))$ *for some* $\sigma \in 2^{<\mathbb{Q}}, q \in \mathbb{Q}^*$ *and* $v \in V$.

(2) *The negation of each formula as in* (1) *is equivalent in* N *to a disjunction of formulas of this kind.*

(3) *If (v_i, q_i), $i < n$, are some finitely many pairwise distinct elements of $V \times \mathbb{Q}^*$ and for each $i < n$ we have some finitely many compatible functions $\sigma_{i,j} \in 2^{<\mathbb{Q}}$, $j < m_i$, then the formula*

$$\bigwedge_{i<n} \bigwedge_{j<m_i} P_{\sigma_{i,j}}(q_i(x - v_i))$$

is consistent in N.

Proof. (1) The non-algebraic atomic formulas in variable x over N are of the form $P_\sigma(q(x - n))$, where $\sigma \in 2^{<\mathbb{Q}}, q \in \mathbb{Q}^*$ and $n \in N$. Write n as $v + m$ for some unique $v \in V$ and $m \in M$. Then $P_\sigma(q(x - n))$ is equivalent to $P_\sigma(q(x - v - m))$, hence to $P_{qm*\sigma}(q(x - v))$.

(2),(3) are left as an exercise. In (3) use the genericity of η. ⊣

Let $a \in \mathfrak{C} \setminus N$. Then the type $tp(a/N)$ is determined by the sequence $\bar{f} = \langle f_q \rangle \in \prod_{q \in \mathbb{Q}^*} 2^N$, where $f_q = \bigcup_{v \in V} v * f_{q,v}$ and $f_{q,v} \in 2^M$ are defined by:

$$f_{q,v} = \bigcup \{\sigma \in 2^{<\mathbb{Q}} : \models P_\sigma(q(a - v))\}.$$

Moreover, every sequence $\bar{f} = \langle f_q \rangle \in \prod_{q \in \mathbb{Q}^*} 2^N$, where $f_q = \bigcup_{v \in V} v * f_{q,v}$, $f_{q,v} \in 2^M$, determines a unique non-algebraic type $p_{\bar{f}}(x) \in S_1(N)$ generated by the set of formulas

$$\{x \neq n : n \in N\} \cup \{P_\sigma(q(x - v)) : \sigma = f_{q,v}|X \text{ for some } X \subseteq_{fin} \mathbb{Q}\}.$$

Lemma 3.9. *The mapping $\bar{f} \mapsto p_{\bar{f}}$ is an isomorphism of N-flows $\prod_{q \in \mathbb{Q}^*} 2^N$ and $WGen(N)$.*

Proof. Similar to that of Remark 3.5. ⊣

Since $N * \bar{\eta}_N$ is dense in $\prod_{q \in \mathbb{Q}^*} 2^N$ we have that $N * p_{\bar{\eta}_N}$ is dense in $WGen(N)$. It is easy to see that $p_{\bar{\eta}_N}$ is not almost periodic, so also in $WGen(N)$ (just as in $WGen(M)$) not all types are almost periodic. We will consider some more specific examples of types in $WGen(N)$ that are not almost periodic.

Definition 3.10. (1) Assume $N' = N \oplus W$ for some $\mathbb{Q}$-vector space W and $\bar{f} = \langle f_q \rangle \in \prod_{q \in \mathbb{Q}^*} 2^N$. We define the W-prolongation $\bar{f}'$ of $\bar{f}$ as the sequence $\langle f'_q \rangle \in \prod_{q \in \mathbb{Q}^*} 2^{N'}$ given by $f'_q(n + w) = f_q(n)$, where $n \in N$ and $w \in W$.
(2) We say that $\bar{f} = \langle f_q \rangle \in \prod_{q \in \mathbb{Q}^*} 2^N$ has countable support if for some countable subspace N' of N, $\bar{f}$ is a W-prolongation of some $\bar{f}' \in \prod_{q \in \mathbb{Q}^*} 2^{N'}$ (for some W with $N = N' \oplus W$).
(3) We say that $p \in WGen(N)$ has countable support if $p = p_{\bar{f}}$ for some $\bar{f}$ with countable support.

We will prove that every type $p \in WGen(N)$ with countable support has an extension to an almost periodic type (over some $N' \succ N$). First notice the following proposition.

Proposition 3.11. *There are types $p \in WGen(N)$ that have countable support and are not almost periodic.*

Proof. Recall that $\eta \in 2^{\mathbb{Q}}$ is sufficiently generic in $2^{\mathbb{Q}}$, so that $\mathbb{Q} * \bar{\eta}$ is dense in $\prod_{q \in \mathbb{Q}^*} 2^{\mathbb{Q}}$. Let $\eta' = \bigcup_{v \in V} v * \eta$ and let $\bar{\eta}' = \langle \eta'_q \rangle_{q \in \mathbb{Q}^*}$, where $\eta'_q = \eta'$. Let $p = p_{\bar{\eta}'}$.

Clearly, p has countable support. To see that p is not almost periodic, by Lemma 3.9 it is enough to show that $\bar{\eta}'$ is not almost periodic in $\prod_{q\in\mathbb{Q}^*} 2^N$.

Consider any periodic piece-wise constant function $\xi \in 2^{\mathbb{Q}}$. Let $\bar{\xi} = \langle \xi_q \rangle \in \prod_{q\in\mathbb{Q}^*} 2^{\mathbb{Q}}$, where $\xi_q = \xi$. Let $\xi' = \bigcup_{v\in V} v * \xi$ and let $\bar{\xi}' = \langle \xi'_q \rangle_{q\in\mathbb{Q}^*}$, where $\xi'_q = \xi'$. Since $\bar{\xi} \in \mathrm{cl}(\mathbb{Q} * \bar{\eta})$ and $\mathrm{cl}(\mathbb{Q} * \bar{\xi})$ is a proper subset of $\mathrm{cl}(\mathbb{Q} * \bar{\eta})$, also $\bar{\xi}' \in \mathrm{cl}(N * \bar{\eta}')$ and $\mathrm{cl}(N * \bar{\xi}')$ is a proper subset of $\mathrm{cl}(N * \bar{\eta}')$. Hence $\bar{\eta}'$ is not almost periodic. ⊣

It will be more convenient to work with the N-flow $\prod_{q\in\mathbb{Q}^*} 2^N$ rather than with *WGen*(N). We introduce a technical property of $\bar{f} = \langle f_q \rangle \in \prod_{q\in\mathbb{Q}^*} 2^N$ implying that $\bar{f}$ is almost periodic.

Definition 3.12. (1) We say that a function $f \in 2^N$ is self-replicating if the following holds:

For every finite $X \subseteq N$ there is a finite $Y \subseteq N$ such that for every $n \in N$ there is an $m \in N$ with $m * (f|X) \subseteq f|(n + Y)$.

(2) We say that $\bar{f} = \langle f_q \rangle \in \prod_{q\in\mathbb{Q}^*} 2^N$ is self-replicating if the following holds.

For every finite sequence $q_i \in \mathbb{Q}^*, i \in I$, of distinct numbers, for every finite set $X \subseteq N$, there is a finite set $Y \subseteq N$ such that for every $n \in N$ there is an $m \in N$ such that for every $i \in I$ we have that $q_i m * (f_{q_i}|X) \subseteq f_{q_i}|(q_i n + Y)$

Lemma 3.13. (1) *Assume $f \in 2^N$ is self-replicating. Then f is almost periodic.*
(2) *Assume $\bar{f} = \langle f_q \rangle \in \prod_{q\in\mathbb{Q}^*} 2^N$ is self-replicating. Then $\bar{f}$ is almost periodic.*

Proof. (1) Suppose $g \in \mathrm{cl}(N * f)$. We must show that $f \in \mathrm{cl}(N * g)$. To see this, let $X \subseteq N$ be finite. Choose a finite $Y \subseteq N$ provided by the self-replication of f. Since $g \in \mathrm{cl}(N * f)$, there is some $n \in N$ such that $g|Y \subseteq n * f$. Choose an $m \in N$ with $m * (f|X) \subseteq f|(-n + Y)$. It follows that

$$((-n) * g)|(-n + Y) = f|(-n + Y)$$

contains $m * (f|X)$, hence $f|X \subseteq (-(n + m)) * g$.

(2) A similar proof. ⊣

Lemma 3.14. *Assume $N' = N \oplus W$ and $\bar{f} = \langle f_q \rangle \in \prod_{q\in\mathbb{Q}^*} 2^N$ is self-replicating. Then the W-prolongation of $\bar{f}$ is also self-replicating.*

Proof. Easy. ⊣

Lemma 3.15. (1) *The set of self-replicating functions $f \in 2^N$ is dense in 2^N.*
(2) *The set of self-replicating sequences $\bar{f} \in \prod_{q\in\mathbb{Q}^*} 2^N$ is dense in $\prod_{q\in\mathbb{Q}^*} 2^N$.*

Proof. We will prove only (1). The proof of (2) is a technical complication of the proof of (1). First consider the case, where $\dim_{\mathbb{Q}}(N)$ is finite. Assume $Z \subseteq N$ is finite and $\sigma \in 2^Z$. We want to find a self-replicating $f \in 2^N$ extending σ. Translating and possibly enlarging Z we can assume that for some finite basis $\mathscr{B} = \{b_i : i \in I\}$ of N and some $k \in \mathbb{N}$ we have that

$$Z = \left\{ \sum l_i b_i : l_i \in \mathbb{N} \text{ and } l_i < k \right\}.$$

We define $f \in 2^N$ as follows.

$$f\left(\sum (l_i + t_i k + r_i) b_i\right) = \sigma\left(\sum l_i b_i\right),$$

where $t_i, l_i \in \mathbb{Z}$, $0 \leq l_i < k$ and $r_i \in \mathbb{Q} \cap [0,1)$. We check that f is self-replicating. So consider a finite $X \subseteq \mathbb{N}$. Again, since f is periodic, enlarging and translating X we can assume that for some $l, t \in \mathbb{N}$,

$$X = \left\{ \sum \frac{l_i}{l} b_i : l_i \in \mathbb{N} \text{ and } 0 \leq \frac{l_i}{l} < tk \right\}.$$

One can see that the set

$$Y = \left\{ \sum \frac{l_i}{l} b_i : l_i \in \mathbb{N} \text{ and } 0 \leq \frac{l_i}{l} < (t+1)k \right\}$$

witnesses the self-replication of f.

To deal with the general case assume again that $Z \subseteq N$ is finite and $\sigma \in 2^Z$. Present N as $W_0 \oplus W_1$, where $\dim W_0$ is finite and $Z \subseteq W_0$. By the first part of the proof there is a self-replicating $f_0 \in 2^{W_0}$ extending σ. Let $f \in 2^N$ be the W_1-prolongation of f_0. By Lemma 3.14, f is self-replicating. ⊣

Proposition 3.16. *Assume N is countable and N' is a $\mathbb{Q}$-vector space properly extending N.*

(1) *Let $f \in 2^N$. Then there is a self-replicating $f' \in 2^{N'}$ extending f.*
(2) *Let $\bar{f} \in \prod_{q \in \mathbb{Q}^*} 2^N$. Then there is a self-replicating $\bar{f}' \in \prod_{q \in \mathbb{Q}^*} 2^{N'}$ extending $\bar{f}$.*

Proof. (1) By Lemma 3.14 it is enough to consider the case, where $N' = N \oplus W$ for some 1-dimensional W. To simplify notation we identify W with $\mathbb{Q}$. We present N as an increasing union of finite sets $X_l, l < \omega$. By Lemma 3.15, for each l we choose a self-replicating function $f_l \in 2^N$, extending $f|X_l$. Modifyng the construction in the proof of Lemma 3.15 we can further assume that the functions $f_l, l < \omega$, are self-replicating almost uniformly, meaning that for every $l^* < \omega$, for every finite $X \subseteq N$, there is a finite $Y \subseteq N$ such that for every $n \in N$ there is an $m \in N$ such that for every $l < l^*$ we have that

$$m * (f_l|X) \subseteq f_l|(n + Y).$$

We define a function $f' \in 2^{N'}$ as follows.

(a) For $n' = n + r$, where $n \in N$ and $r \in \mathbb{Q} \cap [0,1)$, let $f'(n') = f(n)$. So f' extends f.
(b) For $n' = n + k + r$, where $n \in N, k \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathbb{Q} \cap [0,1)$, let $f'(n') = f_t(k)$, where 2^t is the highest power of 2 dividing k.

We claim that f' is self-replicating. To check this, consider a finite $X \subseteq N$. Enlarging X we can assume that $X = X_t \times X^*$, where $X^* = \{\frac{i}{s} : i \in \mathbb{N}, -2^t \leq \frac{i}{s} \leq 2^t\}$, for some $s, t < \omega$.

Let $Y_0 \subseteq N$ be the finite set provided for X_t and $l^* = t + 1$ by almost uniform self-replication of the functions f_l. Let

$$Y^* = \left\{ \frac{i}{s} : i \in \mathbb{N},\ -2^{t+3} \leq \frac{i}{s} \leq 2^{t+3} \right\}.$$

Let $Y = Y_0 + Y^*$. One can see that Y witnesses self-replication of f' for X.

Hint: Assume $n \in N$ and $q \in \mathbb{Q}$. Then for some integer $k' = 2^{t+1}(2u+1)$ and some $r \in \mathbb{Q} \cap [0,1)$ we have that $(k' + r) + X^* \subseteq q + Y^*$. By the almost uniform self-replication of the functions f_l we have that there is an $m \in N$ such that

$m*(f'|((k'+r)+X)) \subseteq f'|((n+q)+Y)$. Since $(k'+r)*(f|X_t) = f'|(X_t+(k'+r))$, we have that $(m+k'+r)*(f'|X) \subseteq f'|((n+q)+Y)$.

(2) The proof is a technical complication of the proof of (1). ⊣

PROPOSITION 3.17. *Assume $p \in WGen(N)$ has countable support. Assume N' is a proper elementary extension of N. Then there is an almost periodic type $p' \in WGen(N')$ extending p.*

PROOF. There are subspaces N_0 and W of N such that $N = N_0 \oplus W$, N_0 is countable and

(a) $p|_{N_0} = p_{\bar{f}_0}$ for some $\bar{f}_0 \in \prod_{q\in\mathbb{Q}^*} 2^{N_0}$,
(b) $p = p_{\bar{f}}$ for the W-prolongation $\bar{f} \in \prod_{q\in\mathbb{Q}^*} 2^N$ of $\bar{f}_0$.

Choose $W' < N'$ with $N' = N \oplus W'$. Let $N_1 = N_0 \oplus W'$. By Proposition 3.16 there is a self-replicating $\bar{f}_1 \in \prod_{q\in\mathbb{Q}^*} 2^{N_1}$ extending $\bar{f}_0$. We have that $N' = N_1 \oplus W$. Let $\bar{f}'$ be the W-prolongation of $\bar{f}_1$. By Lemma 3.14, $\bar{f}'$ is self-replicating. Also, it extends $\bar{f}$. Hence by Lemmas 3.9 and 3.13, the type $p' = p_{\bar{f}'}$ is almost periodic and extends p. ⊣

So we have shown that in our example even when N is $\aleph_0$-saturated, many types in $WGen(N)$ that are not almost periodic, have almost periodic extensions in $WGen(N')$ for any proper elementary extension N' of N. This shows that the notion of almost periodic type is not down-absolute even between $\aleph_0$-saturated models. However, the notion of weak generic type is down absolute between $\aleph_0$-saturated models. So maybe, after all, the weak generic types are the correct generalization of generic types ?

In our example it remains unclear to me if every type $p \in WGen(N)$ has an almost periodic extension in $WGen(N')$ for some $N' \succ N$.

We have given an example of a simple unstable structure, where weak generic types differ from almost periodic types. We do not know any such o-minimal example.

§4. Ellis semigroup and co-heir extensions. Assume X is a point-transitive G-flow. Each $g \in G$ determines a homeomorphism $\pi_g : X \to X$ given by $\pi_g(x) = gx$. Each π_g belongs to the product X^X. Let $E(X)$ be the closure of the set $\{\pi_g : g \in G\} \subseteq X^X$, in the topolgy of pointwise convergence (which is the Tychonov product topology on X^X). $E(X)$ is a semi-group (the semi-group operation being the composition of functions), called the Ellis semigroup of the G-flow X. This is an important object in topological dynamics. Here we will interpret it model-theoretically. Now we sketch briefly some properties of the Ellis semigroup. The reader is referred to [E, A] for more information.

The semigroup operation in $E(X)$ is continuous in the first coordinate, but in general not in the second one. $E(X)$ is a G-flow itself: $g * f = \pi_g \circ f$. $E(X)$ is point-transitive: for the neutral element e of G we have $\pi_e = id_X$ and $G * id_X = \{\pi_g : g \in G\}$ is dense in $E(X)$. In a way, $E(X)$ is universal for X, since $E(E(X))$ is G-isomorphic to $E(X)$.

The minimal G-flows in X are closely related to the minimal left ideals in $E(X)$, and the idempotents in $E(X)$. More specifically, we have that if $I \subseteq E(X)$ is a minimal left ideal, then for every $x \in X$ we have that Ix is a minimal flow in X

and every minimal flow in X arises in this way. Also, $x \in X$ is almost periodic iff $x \in Ix$ iff $ux = x$ for some idempotent $u \in I$.

The minimal left ideals in $E(X)$ are precisely the minimal flows in $E(X)$, and they are all isomorphic (as G-flows). Let $J = \{u \in E(X) : u^2 = u\}$ be the set of idempotents of $E(X)$ and for a minimal left ideal $I \subseteq E(X)$ let $J(I) = J \cap I$. Then $J(I) \neq \emptyset$. Also, for $u \in J(I)$, the set uI is a subgroup of I with the neutral element u and I is the disjoint union of the subgroups $uI, u \in J(I)$. Also, $uI = uE(X)u$, so the idempotent u alone determines the group uI. In this paper we call the subgroups of $E(X)$ of the form uI the I-subgroups of $E(X)$.

Let $u, v \in J(I)$. Then the groups uI and vI are isomorphic: for every $p \in uI$ there is a unique $r \in vI$ with $pr = v$ and $rp = u$. The mapping $p \mapsto r$ is an isomorphism of the groups uI and vI.

If I_1, I_2 are two minimal left ideals in $E(X)$, then for $u \in J(I_1)$ and $v \in J(I_2)$ the groups uI_1 and vI_2 are also isomorphic, so all I-subgroups of $E(X)$ are isomorphic. The dynamical and algebraic properties of Ellis semigroup $E(X)$ express a lot of information about the original G-flow X.

Now we return to the model-theoretic setting. Again, G is a 0-definable group in a model M of a complete first-order theory T. Then $S_G(M)$ is a point-transitive G-flow, so we can consider its Ellis semigroup $E(S_G(M))$, which we will denote by $E_G(M)$. In this section we will interpret this object model-theoretically.

$S_G(M)$ is the Stone space of ultrafilters in the Boolean algebra of M-definable subsets of G. For technical reasons it is convenient to consider the space βG of ultrafilters in the Boolean algebra of all subsets of G and the action of G on βG given by left translation: For $g \in G$ and $\mathscr{U} \in \beta G$

$$g * \mathscr{U} = \{gX : X \in \mathscr{U}\}.$$

βG is a point transitive G-flow and the restriction function $r : \beta G \to S_G(M)$ is an epimorphism of G-flows.

In fact, βG is a universal point-transitive G-flow, hence it is naturally isomorphic to $E(\beta G)$. Namely, for $g \in G$ let $\pi_g^* : \beta G \to \beta G$ be the homeomorphism induced by the action of G on βG. Then for $\mathscr{U} \in \beta G$ let $\pi_{\mathscr{U}}^* : \beta G \to \beta G$ be given by $\pi_{\mathscr{U}}^* = \lim_{\mathscr{U}} \pi_g^*$ in the space $\beta G^{\beta G}$.

We can give an explicit definition of $\pi_{\mathscr{U}}^*$: For $\mathscr{V} \in \beta G$ let

$$\pi_{\mathscr{U}}^*(\mathscr{V}) = \{X \subseteq G : \{g \in G : g^{-1}X \in \mathscr{V}\} \in \mathscr{U}\}.$$

Clearly, $E(\beta G) = \{\pi_{\mathscr{U}}^* : \mathscr{U} \in \beta G\}$ and the mapping $\mathscr{U} \mapsto \pi_{\mathscr{U}}^*$ is an isomorphism of G-flows βG and $E(\beta G)$. The composition $\pi_{\mathscr{U}_1}^* \circ \pi_{\mathscr{U}_2}^*$ is given by the following formula.

$$X \in \pi_{\mathscr{U}_1}^* \pi_{\mathscr{U}_2}^*(\mathscr{V}) \Leftrightarrow \{g \in G : \{h \in G : h^{-1}g^{-1}X \in \mathscr{V}\} \in \mathscr{U}_2\} \in \mathscr{U}_1.$$

Hence $\pi_{\mathscr{U}_1}^* \pi_{\mathscr{U}_2}^* = \pi_{\mathscr{U}_1 * \mathscr{U}_2}^*$, where

$$\mathscr{U}_1 * \mathscr{U}_2 = \{Y \subseteq G : \{g \in G : \{h \in G : g \cdot h \in Y\} \in \mathscr{U}_2\} \in \mathscr{U}_1\}.$$

To describe $E_G(M)$ we need to consider some auxiliary G-flows consisting of co-heirs of types in $S_G(M)$. Recall the following "co-heir" type construction due to Lascar. Assume $\mathscr{U} \in \beta G$. For every set of parameters $A \supseteq M$ we define the type $p_{\mathscr{U}}|A \in S(A)$ by

$$\varphi(x) \in p_{\mathscr{U}}|A \Leftrightarrow \varphi(\mathfrak{C}) \cap M \in \mathscr{U}.$$

When $A = M$, we write $p_{\mathscr{U}}$ instead of $p_{\mathscr{U}}|M$. Clearly, $p_{\mathscr{U}} = r(\mathscr{U})$ and for any $A \supseteq M$, the type $p_{\mathscr{U}}|A$ is a co-heir of $p_{\mathscr{U}}$ and every co-heir of $p_{\mathscr{U}}$ in $S(A)$ arises in this way.

Let $S_{\beta G}(A) = \{p_{\mathscr{U}}|A : \mathscr{U} \in \beta G\}$. $S_{\beta G}(A)$ is a closed subset of $S(A)$. For any $\|M\|^{+}$-saturated $N \succ M$ the types in $S_{\beta G}(N)$ are stationary in the sense, that they have unique extensions in $S_{\beta G}(\mathfrak{C})$. Each type $p \in S_{\beta G}(N)$ has the only extension in $S(\mathfrak{C})$ that is a co-heir of $p|M$. In fact, $S_{\beta G}(\mathfrak{C})$ and $S_{\beta G}(N)$ are homeomorphic via the restriction function.

The natural restriction functions $\beta G \to S_{\beta G}(A)$ induce on $S_{\beta G}(A)$ the structure of a point-transitive G-flow. In the case of $S_{\beta G}(M) = S_G(M)$, this structure agrees with the old one.

For $g \in G$ let $\pi_g^A : S_{\beta G}(A) \to S_{\beta G}(A)$ be the homeomorphism induced by the action of G on $S_{\beta G}(A)$. For $\mathscr{U} \in \beta G$ let $\pi_{\mathscr{U}}^A : S_{\beta G}(A) \to S_{\beta G}(A)$ be given by $\pi_{\mathscr{U}}^A = \lim_{\mathscr{U}} \pi_g^A$ in the space $S_{\beta G}(A)^{S_{\beta G}(A)}$. Hence for $p \in S_{\beta G}(A)$,

$$\pi_{\mathscr{U}}^A(p) = \{\varphi(x) \in L(A) : \{g \in G : \varphi(g \cdot x) \in p\} \in \mathscr{U}\}.$$

For any $B \supseteq A \supseteq M$ and any $\mathscr{U} \in \beta G$ we have the following commuting diagram.

$$\begin{tikzcd}
\beta G \arrow[r] \arrow[d, "\pi_U^*"] & S_{\beta G}(B) \arrow[r] \arrow[d, "\pi_U^B"] & S_{\beta G}(A) \arrow[d, "\pi_U^A"] \\
\beta G \arrow[r] & S_{\beta G}(B) \arrow[r] & S_{\beta G}(A)
\end{tikzcd}$$

The horizontal arrows in this diagram are the restriction functions. Let $E_G(A)$ be the Ellis semigroup $E(S_{\beta G}(A))$. So

$$E_G(A) = \{\pi_{\mathscr{U}}^A : \mathscr{U} \in \beta G\}.$$

By the commuting of the diagram we have that $\pi_{\mathscr{U}_1}^A \circ \pi_{\mathscr{U}_2}^A = \pi_{\mathscr{U}_1 * \mathscr{U}_2}^A$. The mapping $\pi_{\mathscr{U}}^* \mapsto \pi_{\mathscr{U}}^A$ is a semi-group epimorphism of G-flows $E(\beta G) \cong \beta G$ and $E_G(A)$. The mapping $\pi_{\mathscr{U}}^B \mapsto \pi_{\mathscr{U}}^A$ is a semigroup epimorphism of G-flows $E_G(B)$ and $E_G(A)$.

Lemma 4.1. *Assume* $\mathscr{U}_1, \mathscr{U}_2 \in \beta G$, $b \models p_{\mathscr{U}_2}|A$ *and* $a \models p_{\mathscr{U}_1}|Ab$. *Then*

(1) $a \cdot b \models p_{\mathscr{U}_1 * \mathscr{U}_2}|A$ *and*
(2) $\pi_{\mathscr{U}_1}^A(p_{\mathscr{U}_2}|A) = p_{\mathscr{U}_1 * \mathscr{U}_2}|A$.

Proof. (1) We have that for $\varphi(x) \in L(A)$ and $g \in G$,

$$\models \varphi(g \cdot b) \Leftrightarrow \varphi(g \cdot x) \in p_{\mathscr{U}_2}|A \Leftrightarrow \{h \in G : \models \varphi(g \cdot h)\} \in \mathscr{U}_2.$$

Also,

$$\models \varphi(a \cdot b) \Leftrightarrow \varphi(x \cdot b) \in p_{\mathscr{U}_1}|Ab \Leftrightarrow \{g \in G : \models \varphi(g \cdot b)\} \in \mathscr{U}_1.$$

Hence

$$\varphi(x) \in p_{\mathscr{U}_1 * \mathscr{U}_2}|A \Leftrightarrow \{g \in G : \{h \in G : \models \varphi(g \cdot h)\} \in \mathscr{U}_2\} \in \mathscr{U}_1$$

$$\Leftrightarrow \{g \in G : \models \varphi(g \cdot b)\} \in \mathscr{U}_1 \Leftrightarrow \models \varphi(a \cdot b).$$

(2) follows from (1), since $p_{\mathscr{U}_2}|A = \pi_{\mathscr{U}_2}^A(tp(e/A))$, where e is the neutral element of G. ⊣

As we have seen above, among the flows $S_{\beta G}(A)$ there is the largest one: $S_{\beta G}(\mathfrak{C})$, which is isomorphic to $S_{\beta G}(N)$ for any $\|M\|^+$-saturated $N \succ M$. So let $N \succ M$ be $\|M\|^+$-saturated. We define a binary operation $*$ on $S_{\beta G}(N)$ as follows. Let $p, q \in S_{\beta G}(N)$. We define $p * q$ as $p_{\mathcal{U}_1 * \mathcal{U}_2}|N$, where $p = p_{\mathcal{U}_1}|N$ and $q = p_{\mathcal{U}_2}|N$. The next lemma shows that this definition is correct.

LEMMA 4.2. *If $p_{\mathcal{U}_1}|N = p_{\mathcal{U}_1'}|N$ and $p_{\mathcal{U}_2}|N = p_{\mathcal{U}_2'}|N$, then $p_{\mathcal{U}_1 * \mathcal{U}_2}|N = p_{\mathcal{U}_1' * \mathcal{U}_2'}|N$.*

PROOF. Let b realize the type $p_{\mathcal{U}_2}|N = p_{\mathcal{U}_2'}|N$. By stationarity, $p_{\mathcal{U}_1}|Nb = p_{\mathcal{U}_1'}|Nb$. Let a realize this type. By Lemma 4.1(1), $a \cdot b$ realizes both $p_{\mathcal{U}_1 * \mathcal{U}_2}|N$ and $p_{\mathcal{U}_1' * \mathcal{U}_2'}|N$. $\dashv$

We see that $p * q = tp(a \cdot b/N)$, where $a \models p, b \models q$ and the type $tp(a/Nb)$ is a co-heir of $tp(a/M)$. For $p \in S_{\beta G}(N)$ we define $\pi_p : S_{\beta G}(N) \to S_{\beta G}(N)$ by $\pi_p(q) = p * q$. By Lemma 4.1(2), $\pi_p \in E_G(N)$, in fact $\pi_{p_{\mathcal{U}}|N} = \pi^N_{\mathcal{U}}$. The next proposition follows directly from Lemma 4.1 and the definitions.

PROPOSITION 4.3. (1) *The function $p \to \pi_p$ is a semi-group isomorphism of G-flows $S_{\beta G}(N)$ and $E_G(N)$.*

(2) *For any $A \supseteq M$ there is a surjective morphism of G-flows $S_{\beta G}(N)$ and $E_G(A)$, that is also a homomorphism of semigroups.*

(3) *The semi-group $E_G(M)$ is a homomorphic image of $S_{\beta G}(N)$. The I-subgroups of $E_G(M)$ are also homomorphic images of the I-subgroups of $S_{\beta G}(N)$.*

Assume $tp(a/N), tp(b/N) \in S_{\beta G}(N)$ and $tp(a/Nb)$ is a co-heir of $tp(a/M)$. Then, as mentioned above, $tp(a \cdot b/N) = tp(a/N) * tp(b/N)$. So the mapping $a \mapsto tp(a/N)$ is "generically" a homomorphism between $G^{\mathfrak{C}}$ and the semi-group $S_{\beta G}(N)$.

It may happen that already the types in $S_G(M)$ are stationary, that is, the following condition holds:

$$\text{the types in } S_G(M) \text{ have unique co-heir extensions in } S(\mathfrak{C}). \qquad (S)$$

In this case in the above construction we can take $N = M$. Then $S_{\beta G}(N) = S_G(M)$ and since by Proposition 4.3(1), $S_{\beta G}(N)$ and $E_G(N)$ are isomorphic, we get that $S_G(M)$ has an induced semi-group structure, isomorphic to $E_G(M)$.

In the case where the theory is stable, the co-heir extensions are the non-forking extensions, and the types in $S_G(M)$ are stationary, so condition (S) holds. Then for $p, q \in S_G(M)$, $p * q$ is the type $tp(a \cdot b/M)$ for any independent realizations $a \models p$ and $b \models q$. In the semigroup $S_G(M)$ there is just one left minimal ideal $\mathcal{G}$ consisting of the generic types. There is just one idempotent in it, namely the generic type of the connected component G^0 of G. $\mathcal{G}$ is a group, and actually it is a topological group (i.e. the group operations are continuous). The semi-group structure on $S_G(M)$ was already considered in the stable case in [N1].

Now we consider the case, where M is o-minimal. Moreover, assume M is Dedekind complete. In this case by [MS] (S) holds . Hence, as above, $S_G(M)$ has a natural semi-group structure isomorphic to $E_G(M)$. However, unlike in the stable case, usually there is more than one idempotent there. For example, consider the additive group of reals $G = \mathbb{R}^+$ in an o-minimal expansion of the field of real numbers $M = \mathcal{R} = (\mathbb{R}, +, \cdot, <, \dots)$. Then in $S_G(M)$ there are two minimal left ideals, consisting of two idempotents: the two weak generic types. So the I-subgroups of $S_G(M)$ are trivial. For another example, consider $G = S^1$, definable in M. There is again a single minimal left ideal $Gen(M)$ in $S_G(M)$, consisting of the generic types. There are two idempotents in it, the generic types of G^{00}, and $Gen(M)$ is the disjoint union of two I-subgroups isomorphic to G/G^{00}.

More generally, $S_G(M)$ is naturally isomorphic to $E_G(M)$ under a weaker assumption than (S). Namely, it is enough to assume that

$$\text{for every } a, b \in G^{\mathfrak{C}} \text{ with } tp(a/Mb) \text{ being a co-heir of } tp(a/M) \text{, the type } tp(a \cdot b/M) \text{ depends only on the types } tp(a/M) \text{ and } tp(b/M). \tag{S'}$$

Then every type $p \in S_G(M)$ determines a function $\pi_p : S_G(M) \to S_G(M)$ given by $\pi_p(q) = tp(a \cdot b/M)$, where $a \models p$, $b \models q$ and $tp(a/Mb)$ is a co-heir of p. The mapping $p \mapsto \pi_p$ is an isomorphism of G-flows $S_G(M)$ and $E_G(M)$. It was so in the above examples.

However if M is o-minimal, but not Dedekind complete, condition (S') may fail. For example, let G be the additive group of reals considered in a countable elementary substructure M of $\mathscr{R}$. Let $b \in \mathbb{R} \setminus M$, $p = tp(b/M)$ and $q = tp(-b/M)$. q has two co-heir extensions $q_1, q_2 \in S(Nb)$. If $a_1 \models q_1$ and $a_2 \models q_2$, then $\models a_1 < -b < a_2$. Consequently $tp(a_1 + b/M) \neq tp(a_2 + b/M)$.

Now let us return to the general situation. I believe that the groups of the form uI, where I is a maximal left ideal in $E_G(N)$ or $E_G(M)$ and $u \in I$ is an idempotent, are closely related to G_M^{00}. I conjecture that in the case of a theory with NIP, these groups are in fact naturally isomorphic to G/G^{00}. In this case G^{00} exists by [Sh] (for a proof, see also [HPP]).

To be more precise, consider again a group G definable in M and let N be a sufficiently saturated elementary extension of M. Inside $G^{\mathfrak{C}}$ we define the group G_M^{00}, the smallest type-definable over M subgroup of bounded index in $G^{\mathfrak{C}}$. If there is the smallest type-definable subgroup of bounded index in $G^{\mathfrak{C}}$, it is denoted by G^{00}. If among the subgroups of bounded index $G^{\mathfrak{C}}$, invariant over a small subset of $\mathfrak{C}$ there is the smallest one, it is denoted by G^{∞}. Both G_M^{00} and G^{00} are normal subgroups of $G^{\mathfrak{C}}$ and $G^{\infty} < G^{00}$ (provided they exist). Notice that by [N2], if T is small and G^{∞} exists, then G^{00} exists and $G^{\infty} = G^{00}$.

We consider the G-flow $S_{\beta G}(N)$ with its semi-group structure.

In general, the idempotents in the minimal left ideals in $S_{\beta G}(N)$ seem to play the role of "generic types of connected component" of G.

Proposition 4.4. *Assume $H < G^{\mathfrak{C}}$ has bounded index and $u \in S_{\beta G}(N)$ is an idempotent.*

(1) *If H is invariant over M, then $u(\mathfrak{C}) \subseteq H$.*
(2) *If $u \in I$ for some minimal left ideal I in $S_{\beta G}(N)$ and H is type-definable over M and normal in $G^{\mathfrak{C}}$, then the group $G^{\mathfrak{C}}/H$ is a homomorphic image of the group uI. In particular, the group $G^{\mathfrak{C}}/G_M^{00}$ is a homomorphic image of uI.*

Proof. (1) We may assume that H is normal in $G^{\mathfrak{C}}$. Since H has bounded index in $G^{\mathfrak{C}}$, the equivalence relation E defined by $xEy \Leftrightarrow xH = yH$ is M-invariant and bounded, hence coarser that $\equiv_M$. Hence the co-sets of H are unions of sets that are type-definable over M. So $u(\mathfrak{C})$ is contained in a sigle coset X of H. Then $u * u(\mathfrak{C}) \subseteq X \cdot X$, hence $X \cdot X = X$ meaning that $X = H$.

(2) Since the index of H in $G^{\mathfrak{C}}$ is bounded, the H-cosets of elements of G are dense in $G^{\mathfrak{C}}/H$. Hence the set of cosets of H meeting some $g \cdot u(\mathfrak{C})$, $g \in G$, is

also dense in $G^{\mathfrak{C}}/H$. Since $Gu \subseteq I$ and I is closed, we get that every coset of H contains a realization of some type in I. Also, $u(\mathfrak{C}) \subseteq H$, hence every co-set of H contains a realization of some type in the group uI.

For $r \in uI$ define $f(r)$ as the unique co-set of H in $G^{\mathfrak{C}}$ containing $r(\mathfrak{C})$. Clearly, $f : uI \to G^{\mathfrak{C}}/H$ is an epimorphism. ⊣

PROPOSITION 4.5. *Assume in $S(\mathfrak{C})$ there is a bounded number of weak generic types of $G^{\mathfrak{C}}$. Then G^{∞} exists and it is the stabilizer of the weak generic types of $G^{\mathfrak{C}}$ in $S(\mathfrak{C})$.*

PROOF. $G^{\mathfrak{C}}$ acts on the set $WGen(\mathfrak{C})$. Let H be the subgroup of $G^{\mathfrak{C}}$ consisting of all these $g \in G^{\mathfrak{C}}$ that fix every weak generic type in $S(\mathfrak{C})$. Since the cardinality of $WGen(\mathfrak{C})$ is bounded, H has bounded index in G. Also, H is normal in G and invariant under $Aut(\mathfrak{C})$. We will prove that $H = G^{\infty}$.

Supopose not. Possibly extending the signature we can assume that there is an invariant normal subgroup S of bounded index in G such that S is a proper subgroup of H. Assume M is sufficiently saturated, so that $H^M \neq S^M$. We will prove that every $h \in H^M$ fixes $WGen(M)$ pointwise.

Indeed, suppose $p \in WGen(M)$. Choose $q \in WGen(\mathfrak{C})$ with $p = q|M$. Since $h * q = q$, also $h * p = p$.

Let N be a sufficiently saturated elementary extension of M, and let $u \in S_{\beta G}(N)$ be an idempotent and an almost periodic point. Let $u' = u|M$. By Proposition 4.4, $u'(\mathfrak{C}) \subseteq S(\mathfrak{C})$. Let $h \in H^M \setminus S^M$. Then $h * u' = u'$, hence $h \cdot u'(\mathfrak{C}) = u'(\mathfrak{C})$. On the other hand, $h \cdot u'(\mathfrak{C}) \subseteq H \setminus S$, a contradiction. ⊣

It is not hard to prove that under the assumptions of Proposition 4.5, $G^{\infty} = G^{00}$ and it is the stabilizer of any weak generic type of $G^{\mathfrak{C}}$. This was noticed independently also by M.Petrykowski. In Proposition 4.5 it is natural to expect additionally that $G^{\mathfrak{C}}/G^{00}$ is isomorphic to any I-subgroup of $S_{\beta G}(N)$. By Proposition 4.4 this is equivalent to the fact, that every almost periodic type $r \in S_{\beta G}(N)$ with $r(\mathfrak{C}) \subseteq G^{00}$, is an idempotent. I could not prove this.

As we indicated in the introduction, a new feature of topological dynamics in model theory is a possible relationship between the topological dynamics of G (definable in M) and G^N (in $N \succ M$). Let $M' \succ M$, $N' \succ N$ be sufficiently saturated. It would be interesting to compare the Ellis semigroups $E_G(M), E_{G^N}(N), S_{\beta G}(M')$ and $S_{\beta G^N}(N')$, and also their I-subgroups. I conjecture that if N is a sufficiently elementary extension of M, then the I-subgroups of $S_{\beta G}(M')$ are homomorphic images of the I-subgroups of $S_{\beta G^N}(N')$. The next remark shows that the semigroups $S_{\beta G}(M')$ and $S_{\beta G^N}(N')$ are related.

REMARK 4.6. *There is a continuous semi-group monomorphism $i : S_{\beta G}(M') \to S_{\beta G^N}(N')$. i is also a morphism of G-flows.*

PROOF. Without loss of generality we may assume that $M' = N'$. Then naturally $S_{\beta G}(M') \to S_{\beta G^N}(N')$ and $i = id$ works. ⊣

We conclude this section with a remark on existence of some interesting indiscernible sequences in $G^{\mathfrak{C}}$, following from existence of idempotents in $S_{\beta G}(N)$. Given a sequence $(a_n)_{n<\omega}$ of elements of $G^{\mathfrak{C}}$ and a finite non-empty set $I \subseteq \omega$ we let $a_I = a_{i_0} \cdot \dots \cdot a_{i_n}$, where $i_0 < \dots < i_n$ is the increasing enumeration of I. For finite non-empty sets $I, J \subseteq \omega$ we write $I < J$ if $\max I < \min J$.

REMARK 4.7. *There is an infinite M-indiscernible sequence $(a_n)_{n<\omega}$ of elements of $G^{\mathfrak{C}}$ such that for every sequence $I_k \subseteq \omega, k < \omega$, of finite non-empty sets with $I_k < I_{k+1}$ we have that*

$$tp((a_n)_{n<\omega}/M) = tp((a_{I_n})_{n<\omega}/M).$$

PROOF. Choose an almost periodic idempotent $p \in S_{\beta G}(N)$. There is a sequence $(a_n)_{n<\omega}$ of realizations of p such that for each n, the type $tp(a_n/N \cup \{a_k : k > n\})$ is a co-heir of $p|M$. Clearly, the sequence $(a_n)_{n<\omega}$ has the required properties. ⊣

§5. Open questions and problems. Here we list the open questions and problems scattered throughout the paper.

QUESTION 5.1. *Is Proposition* 1.12 *still valid if we assume that the sets $X_n, n < \omega$, are just Borel, and not necessarily* 0*-type-definable*?

Regarding extensions of almost periodic types and Proposition 2.3, we have the following question.

QUESTION 5.2. *Assume G is a group acting transitively on a set V, everything is definable in a model M. Assume that $N \succ M$. Does every weak generic type in $S_V(M)$ have an heir extension to a weakly generic type in $S_V(N)$?*

The combinatorial nature of Example 2 from Section 3 is not completely clear to me. So let N be a model of the theory considered there.

QUESTION 5.3. *Does every weak generic type in $S(N)$ have an almost periodic extension in $S(N')$ for some $N' \succ N$?*

PROBLEM 5.4. *Find an o-minimal example, where weak generic types differ from almost periodic types. Or even, find an example, where the theory has NIP.*

Regarding the Ellis semigroup and its model-theoretic interpretation, we ask the following question and problem. Again assume G is a group definable in M and N is a sufficiently saturated elementary extension of M. Then $S_{\beta G}(N)$ is a semi-group.

QUESTION 5.5. *Assume there is a bounded number of weak generic types in $S_G(\mathfrak{C})$.*
(1) *Is $G^{\mathfrak{C}}/G^{00}$ isomorphic to any I-subgroup of $S_{\beta G}(N)$?*
(2) *Is every almost periodic type $r \in S_{\beta G}(N)$ with $r(\mathfrak{C}) \subseteq G^{00}$ an idempotent*?

In virtue of Proposition 4.4 items (1) and (2) in this question are equivalent. This question is related to Question 5.2.

PROBLEM 5.6. *Clarify the relationship between G/G^{00} and the I-subgroups of $S_{\beta G}(N)$.*

REFERENCES

[A] J. AUSLANDER, ***Minimal flows and their extensions***, North Holland, Amsterdam, 1988.

[E] R. ELLIS, ***Lectures on topological dynamics***, Benjamin, New York, 1969.

[HPP] E. HRUSHOVSKI, Y. PETERZIL, and A. PILLAY, *Groups, measures, and the NIP*, ***Journal of the American Mathematical Society***, vol. 21 (2008), pp. 563–596.

[MS] D. MARKER and CH. STEINHORN, *Definable types in o-minimal theories*, this JOURNAL, vol. 59 (1994), pp. 185–198.

[N1] L. NEWELSKI, *On type-definable subgroups of a stable group*, ***Notre Dame Journal of Formal Logic***, vol. 32 (1991), pp. 173–187.

[N2] ———, *The diameter of a Lascar strong type*, ***Fundamenta Mathematicae***, vol. 176 (2003), pp. 157–170.

[NP1] L. Newelski and M. Petrykowski, *Coverings of groups and types*, ***Journal of London Mathematical Society. Second Series***, vol. 71 (2005), pp. 1–21.

[NP2] ———, *Weak generic types and coverings of groups I*, ***Fundamenta Mathematicae***, vol. 191 (2006), pp. 201–225.

[PP] Y. Peterzil and A. Pillay, *Generic sets in definably compact groups*, ***Fundamenta Mathematicae***, vol. 193 (2007), pp. 153–170.

[Pe] M. Petrykowski, ***Generic properties of groups***, Ph.D. thesis, Uniwersytet Wrocławski, 2006.

[Po] B. Poizat, *Groupes stables, avec types generiques reguliers*, this Journal, vol. 48 (1983), pp. 339–355.

[Sh] S. Shelah, *Maximal bounded index subgroup for dependent theories*, ***Proceedings of the American Mathematical Society***, vol. 136 (2008), pp. 1087–1091.

[W] F. Wagner, ***Simple theories***, Kluwer, Dodrecht, 2000.

INSTYTUT MATEMATYCZNY
UNIWERSYTET WROCŁAWSKI
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: newelski@math.uni.wroc.pl