Generic Properties Of Groups

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1 Preliminaries

If $p(\bar{x})$ is a type over A, then we call the set of realizations of p in M

$$p(M^n) = \{\bar{a} \in M^n : (\forall \varphi(\bar{x}) \in p(\bar{x}))M \models \varphi(\bar{a})\} \models \bigcap_{\varphi(\bar{x}) \in p(\bar{x})} \varphi(M^n)$$

type definable over A. If V is a 0-type-definable subset of M^n, then we sometimes identify V with the set

$$[V] = \{\text{tp}(\bar{a}) : \bar{a} \in V\} \subseteq S_n(\emptyset)$$

A first order structure M is κ-saturated if for any $A \subseteq M$ with $|A| < \kappa$, $n < \omega$ and $p \in S_n(A)$, p has a realization in M.

A group $(G, \cdot)$ is definable in a structure M is G is a definable subset of M^n for some $n < \omega$ and the group action $\cdot : G \times G \to G$ is a definable function in M. If $p(x)$ is a type over G and $g \in G$, then

$$g \cdot p(x) = \{g \cdot \varphi(x) : \varphi(x) \in p(x)\} = \{\varphi(g^{-1} \cdot x) : \varphi(x) \in p(x)\}$$

A group $(G, \cdot)$ is definable in a structure M if G is a definable

An infinite totally ordered first order structure $(M, <, \dots)$ is **o-minimal** if every definable subset of M is a union of finitely many intervals and points.

Let $(M, <, \dots)$ be an o-minimal structure. We usually say "ultimately" instead of "for all sufficiently large $a \in M$". We denote an open interval with endpoints a and b by (a, b) and a closed one by $[a, b]$. In contrast, $\langle a, b\rangle$ denotes the pair of elements a and b.

If $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$, $a < b$ and $f : (a, b) \to M$ is a definable function, then there are $a = a_1 < \dots < a_n = b$ s.t. each interval (a_i, a_{i+1}) of f is either constant or strictly monotone and continuous in the order topology. In particular, every definable function $f : M \to M$ is ultimately continuous and monotone

2 Weak generic types

2.1 Introduction

Definition 2.1. A set $X \subseteq G$ is **(left) generic** if some finitely many left G-translates of X cover G. We say that a formula $\varphi(x)$ is **(left) generic** if the set $\varphi(G)$ of elements of G realizing φ is **(left) generic**. Finally, we say that a type $p(x)$ of elements of G is **(left) generic** if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is (left) generic

In the stable case left generic = right generic

and each partial generic type extends to a complete generic type (since type is definable)

Definition 2.2. A set $A \subseteq G$ is **weak generic**, if for some non-generic $B \subseteq G$ we have that $A \cup B$ is generic. A formula $\varphi(x)$ is **weak generic** if the set $\varphi(G)$ is weak generic. A type $p(x)$ of elements of G is weak generic if every formula $\varphi(x)$ with $p(x) \vdash \varphi(x)$ is weak generic

2.2 Basic properties of weak generic sets and types

Lemma 2.3. *Assume that G is a group and X is a definable subset of G. TFAE*

1. *the set X is weak generic*
2. *for some finitely many elements $a_1, \dots, a_n \in G$ the set $G \setminus \bigcup_{i=1}^{n} a_i \cdot X$ is not generic*
3. *for some definable non-generic set $Y \subseteq G$ the set $X \cup Y$ is generic*

Proof. $1 \Rightarrow 2$: Assume X is weak generic, then there is non-generic set $Y \subseteq G$ s.t. $X \cup Y$ is generic. Then there are $a_1, \dots, a_n \in G$ s.t.

$$\bigcup_{i=1}^{n} a_i \cdot (X \cup Y) = \bigcup_{i=1}^{n} a_i \cdot X \cup \bigcup_{i=1}^{n} a_i \cdot Y = G$$

This means that

$$G \setminus \bigcup_{i=1}^{n} a_i \cdot X \subseteq \bigcup_{i=1}^{n} a_i \cdot Y$$

$2 \Rightarrow 3$: Let $Y = G \setminus \bigcup_{i=1}^{n} a_i \cdot X$. Then Y is definable and not generic so putting $a_{n+1} = e$. Then $G = \bigcup_{i=1}^{n+1} a_i \cdot (X \cup Y)$ □

Lemma 2.4. *1. If $X, Y \subseteq G$ are not weak generic, then $X \cup Y$ is not weak generic*

2. If $p(x)$ is a (partial) weak generic type over $A \subseteq G$, then $p(x)$ may be extended to a complete weak generic type over A

Proof. 1. Let $Z \subseteq G$ be non-generic. Y is not weak generic so $Y \cup Z$ is not generic, so $Y \cup Z \cup X$ is not generic

2. non weak generics form an ideal

Let $q(x) = \{\varphi(x) \in L(A) : p(x) \cup \{\neg\varphi(x)\}$ is not weak generic$\}$. Then $p \subseteq q$. We shall show that q is a consistent partial type over A. If not, then

$$G \vDash \neg \exists x \bigwedge_{k=1}^{n} \varphi_k(x)$$

for some $n < \omega$ and $\varphi_1, \dots, \varphi_n \in q$. By compactness, for each $k \in \{1, \dots, n\}$ we can find a finite set of formulas $p_k \subseteq p$ s.t. the type $p_k(x) \cup \{\neg\varphi_k(x)\}$ is not weak generic. Let $\psi(x) = \bigwedge\{p_k(x) : 1 \leq k \leq n\}$ and note that for every $k \in \{1, \dots, n\}$ the set $\psi(G) \cap \neg\varphi_k(G)$ is not weak generic. By 1, neither is the union

$$\bigcup_{k=1}^{n} (\psi(G) \cap \neg\varphi_k(G)) = \psi(G) \cap \bigcup_{k=1}^{n} \neg\varphi_k(G) = \psi(G) \cap G = \psi(G)$$

contradicting the fact that $p(x) \vdash \psi(x)$. Finally we take any $r(x) \in S(A)$ with $r \supseteq q$ and the proof is complete

□

We see that (complete) weak generic types exist. By Lemma 2.4, the set

$$WGEN(A) = \{p \in S(A) : p \text{ is weak generic}\}$$

is closed and non-empty in $S(A)$

Lemma 2.5. *Assume G is a group and $A \subseteq G$*

1. *If some weak generic type $p(x) \in S(G)$ is generic, then all weak generic types $q(x) \in S(A)$ are generic*
2. *If for every $p, q \in WGEN(G)$ there is $g \in G$ s.t. $g \cdot p = q$, then all weak generic types $q(x) \in S(A)$ are generic*
3. *If there is just one weak generic type in $S(A)$, then it is generic*

Proof.

1. Suppose that some weak generic type $q(x) \in S(A)$ is not generic. Then some definable generic set $X \subseteq G$ may be divided into two non-generic definable sets X_1, X_2. Since X is generic, some left G-translates X' of X belongs to $p(x)$. Then the corresponding translates X_1', X_2' of X_1, X_2 are also non-generic and one of them belongs to $p(x)$. Hence $p(x)$ is not generic, a contradiction
2. If not, then we can find a formula $\varphi(x) \in L(A)$ which is weak generic but not generic. Note that $\{\neg g \cdot \varphi(x) : g \in G\}$ is a partial weak generic type over G: for each $m < \omega$ and $g_1, \dots, g_m \in G$, the set $\bigcup_{i=1}^{m} g_i \cdot \varphi(G)$ is not generic, which implies that the set $\bigcap_{i=1}^{m}(G \setminus g_i \cdot \varphi(G))$ is weak generic. Extend the type $\{\neg g \cdot \varphi(x) : g \in G\}$ to some $q(x) \in WGEN(G)$. Next extend $\varphi(x)$ to $p(x) \in WGEN(G)$. Then $\forall g \in G\ g \cdot p \neq q$, a contradiction
3. by 2, immediately

□

By Lemma 2.5 (1), in the stable case weak generic = generic

As an example note that if $G = (G, <, +, \dots)$ is o-minimal, then there are exactly two complete weak generic types, corresponding to $-\infty$ and $+\infty$, and they are not generic

Lemma 2.6. *Assume that $G \prec H$ and $\varphi(x) \in L(G)$*

1. *If $\varphi(G)$ is weak generic in G, then $\varphi(H)$ is weak generic in H*
2. *If G is $\aleph_0$-saturated and $\varphi(H)$ is weak generic in H, then $\varphi(G)$ is weak generic in G*

Proof. 1. There is a non-generic formula $\psi(x) \in L(G)$ s.t. $\varphi(G) \cup \psi(G)$ is generic in G, therefore $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. Thus $\varphi(H)$ is weak generic in H

2. There is a formula $\psi(x) \in L(H)$ s.t. $\psi(H)$ is not generic in H and $\varphi(H) \cup \psi(H)$ is generic in H. We have that $\psi(x) = \psi(x, b)$ where $b \subset H$. Let $A \subseteq G$ be a finite set containing all parameters of $\varphi(x)$. By $\aleph_0$-saturation of G, we are able to find in G a tuple $a \subset G$ s.t. tp$(a/A) =$ tp(b/A). Then $\psi(x, a) \in L(G)$ has properties needed to deduce the weak genericity of the set $\varphi(G)$ in G. Namely $\psi(G, a)$ is not generic in G and $\varphi(G) \cup \psi(G, a)$ is generic in G. If $\psi(G, a)$ is generic in G, then for some $0 < n < \omega$ we have that

$$G \models \exists x_1, \dots, x_n \forall y \exists z (\psi(z, a) \wedge \bigvee_{k=1}^{n} y = x_k \cdot z)$$

and the same holds in H since $G \prec H$, which would lead to a contradiction

□

All lemmas in this section remain true if we consider a group $(G, \cdot)$ definable in a first order structure M. Then G is a definable subset of M^n for some $n < \omega$ and for every $A \subseteq M$ we define the set $WGEN(A)$ of complete weak generic types over A as the set

$$\{p \in S_n(A) : \forall \varphi(x_1, \dots, x_n) \in p, G \cap \varphi(M^n) \text{ is weak generic in } G\}$$

2.3 Characterizations of weak genericity

Proposition 2.7. *Assume G is a definable group in an o-minimal structure M and X is a definable weak generic subset of G. Then* $\dim(X) = \dim(G)$

Proof. Suppose $\dim(X) < \dim(G)$. Take a generic set A and a non-generic set B s.t. $A = B \cup X$ (where A and B are definable subsets of G, apply Lemma 2.3) Choose a finite $S \subseteq G$ with $S \cdot A = G$. Then $G \setminus (S \cdot B) \subseteq S \cdot X$ and

$$\dim(G \setminus (S \cdot B)) \leq \dim(S \cdot X) = \dim(X) < \dim(G)$$

Hence the set $S \cdot B$ is large in the sense □

Assume G is a group and $X, Y \subseteq G$. We say that the set X is **translation disjoint** from the set Y if for some $a \in G$, $a \cdot X \cap Y = \emptyset$

Lemma 2.8. *Assume G is a group and X is a weak generic subset of G. Then for some finite $A \subseteq G$ there is no finite covering of X by sets that are translation disjoint from $A \cdot X$*

Proof. take $Y \supseteq X$ generic and $Y \setminus X$ not generic. We have that $G = A \cdot Y$ for some finite $A \subseteq G$. We shall prove that A meets conditions of the lemma.

Suppose for some $X_0, \dots, X_{n-1} \subseteq G$ and $a_0, \dots, a_{n-1} \in G$ we have that

$$X = \bigcup_{i<n} X_i \text{ and } \bigwedge_{i<n} (a_i \cdot X_i) \cap (A \cdot X) = \emptyset$$

Then for each $i < n$, $a_i \cdot X_i \subseteq G \setminus A \cdot X \subseteq A \cdot (Y \setminus X)$. So for each $i < n$, $X_i \subseteq a_i^{-1} \cdot A \cdot (Y \setminus X)$, which implies that $X \subseteq \{a_0^{-1}, \dots, a_{n-1}^{-1}\} \cdot A \cdot (Y \setminus X)$ and finally

$$G = A \cdot Y = A \cdot (Y \setminus X) \cup A \cdot X \subseteq (A \cup (A \cdot \{a_0^{-1}, \dots, a_{n-1}^{-1}\} \cdot A)) \cdot (Y \setminus X)$$

Then G is covered by finitely many things $\square$

Corollary 2.9. *Assume G is a group and X is a weak generic subset of G. Then the set $X \cdot X^{-1}$ is generic in G*

Proof. Take a finite $A \subseteq G$ as in Lemma 2.8. Then for each $a \in G$, $a \cdot X \cap A \cdot X \neq \emptyset$, which implies that $a \in A \cdot X \cdot X^{-1}$. So $G = A \cdot X \cdot X^{-1}$ $\square$

From now on, let $(G, <, +, \dots)$ be an o-minimal expansion of an ordered group $(G, <, +)$. Then the group G is commutative, divisible and torsion-free. By $(G^n, +)$ we mean the product of groups $(G, +) \times \dots \times (G, +)$ (n times). The ordering of G is dense since for every $a, b \in G$ with $a < b$ we have that $a < \frac{a+b}{2} < b$

Theorem 2.10. *Assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$, $0 < n < \omega$ and $\varphi(x_1, \dots, x_n) \in L(G)$. TFAE*

1. *$\varphi(x_1, \dots, x_n)$ is weak generic in $(G^n, +)$*
2. *$\neg\varphi(x_1, \dots, x_n)$ is not generic in $(G^n, +)$*
3. *the set $\varphi(G^n)$ contains arbitrarily large n-dimensional boxes*

$$(\forall R > 0)(\exists a_1, \dots, a_n \in G)[a_1, a_1 + R] \times \dots \times [a_n, a_n + R] \subseteq \varphi(G^n)$$

Proof. $3 \Rightarrow 2$: suppose there is $k < \omega$ and $\langle g_1^1, \dots, g_n^1\rangle, \dots, \langle g_1^k, \dots, g_n^k\rangle \in G^n$ we have that

$$G^n = \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j\rangle + (G^n \setminus \varphi(G^n))$$

Put $M = \max\{\left|g_i^j\right| : 1 \leq i \leq n, 1 \leq j \leq k\}$. Using 3 we are able to find $a_1, \dots, a_n \in G$ s.t.

$$[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M] \subseteq \varphi(G^m)$$

Then

$$\langle a_1, \dots, a_n\rangle \notin \bigcup_{j=1}^{k} (\langle g_1^j, \dots, g_n^j\rangle + (G^n \setminus \varphi(G^n)))$$

a contradiction

$2 \Rightarrow 1$: since the set $G^n = \varphi(G^n) \cup (G^n \setminus \varphi(G^n))$ is generic in $(G^n, +)$ and the set $G^n \setminus \varphi(G^n)$ is not generic

$1 \Rightarrow 3$: W.L.O.G., $n \geq 2$. Using Lemma 2.4 (2) find $p(x_1, \dots, x_n) \in S_n(G)$ s.t. p is a weak generic type in $(G^n, +)$ and $\varphi \in p$. Extend G to a $|G|^+$-saturated group $H \succ G$. Take $\langle a_1, \dots, a_n\rangle \in H^n$ realizing p and fix a positive $R \in G$. We shall show that the follwing condition holds

$$(\forall a \in H)(a_n \leq a \leq a_n + R \Rightarrow \mathrm{tp}(a/Ga_{<n}) = \mathrm{tp}(a_n/Ga_{<n})) \qquad (\star)$$

For the sake of contradiction assume that for some $a \in [a_n, a_n + R]_H$ the types $\mathrm{tp}(a/Ga_{<n})$ and $\mathrm{tp}(a_n/Ga_{<n})$ are distinct. By the o-minimality of H, we can find $b \in [a_n, a_n + R]_H$ with $b \in \mathrm{dcl}(Ga_{<n})$ (dense). Let $\psi(x_1, \dots, x_{n-1}, y) \in L(G)$ be s.t. $H \vDash \psi(a_{<n}, b) \wedge \exists! y \psi(a_{<n}, y)$. As $b - R \leq a_n \leq b$, we have that $\chi \in p$ where

$$\chi(x_1, \dots, x_n) = \exists! y \psi(x_{<n}, y) \wedge \forall y(\psi(y_{<n}, y) \to (y - R \leq x_n \leq y))$$

Since $\chi \in p$, the set $\chi(G^n)$ is weak generic in $(G^n, +)$

We define $f : G^{n-1} \to G$ as:

$$f(c_{<n}) = \begin{cases} c_n - R & G \vDash \chi(\bar{c}) \\ 0 & \text{otherwise} \end{cases}$$

Take $\langle c_1, \dots, c_{n-1}\rangle \in G^{n-1}$. If there is $c_n \in G$ s.t. $G \vDash \chi(c_1, \dots, c_n)$, then there exists just one $d \in G$ with $G \vDash \psi(c_1, \dots, c_{n-1}, d)$ and we put $f(c_1, \dots, c_{n-1}) =$

$d-R$. Otherwise we put $f(c_1,\dots,c_{n-1})=0$. Then the function f is definable over G and we consider the following formula over G:

$$\delta(x_1,\dots,x_n)=f(x_1,\dots,x_{n-1})\leq x_n\leq f(x_1,\dots,x_{n-1})+R$$

Since $\chi(G^n)\subseteq\delta(G^n)\subseteq G^n$, the set $\delta(G^n)$ is weak generic in $(G^n,+)$. Let $A\subseteq G^n$ be a finite set chosen for $\delta(G^n)$ as in Lemma 2.8. Consider an arbitrary $\langle h_1,\dots,h_{n-1}\rangle\in H^{n-1}$. Choose $M_{h_{<n}}\in G$ s.t.

$$\{\langle h_1,\dots,h_n\rangle : f(h_{<n})+M_{h_{<n}}\leq h_n\leq f(h_{<n})+M_{h_{<n}}+R\}\cap(A+\delta(H^n))=\emptyset$$

(exists since $\delta(H^n)$ is bounded and A is finite) If $\mathrm{tp}(h_{<n}/G)=\mathrm{tp}(h'_{<n}/G)$, then $M_{h_{<n}}$ is good also for $h'_{<n}$. By compactness, for each $q(x_1,\dots,x_{n-1})\in S_{n-1}(G)$ we can find a formula $\varphi_q(x_1,\dots,x_{n-1})\in L(G)$ and $M_q\in G$ s.t. for every $h_{<n}\in H^{n-1}$ with $H\vDash\varphi_q(h_{<n})$ we have

$$\{\langle h_1,\dots,h_n\rangle : f(h_{<n})+M_q\leq h_n\leq f(h_{<n})+M_q+R\}\cap(A+\delta(H^n))=\emptyset$$

Again by compactness, $S_{n-1}(G)=[\varphi_{q_1}]\cup\dots\cup[\varphi_{q_k}]$ for some $k<\omega$ and $q_1,\dots,q_k\in S_{n-1}(G)$. If not, then $\forall n\in\omega$, $G\vDash\bigwedge_{i=1}^n\neg\varphi_q i$, that is, $\{\neg\varphi_{q_i} : i\in\omega\}$ is consistent with G, then realized by H, which leads to a contradiction. For $i\in\{1,\dots,k\}$ put $X_i=(\varphi_{q_i}(G^{n-1})\times G)\cap\delta(G^n)$ and $e_i=\langle 0,\dots,0,M_{q_i}\rangle\in G^n$. Then $\delta(G^n)=X_1\cup\dots\cup X_k$ and for every $i\in\{1,\dots,k\}$ we have that $(e_i+X_i)\cap(A+\delta(G^n))=\emptyset$. This contradicts the choice of A and finishes the proof of $(\star)$ $\square$

Corollary 2.11. *Assume that $(G,<,+,\dots)$ is an o-minimal expansion of an ordered group $(G,<,+)$, $0<n,k<\omega$ and $\varphi(x_1,\dots,x_n,y_1,\dots,y_k)\in L$*

1. *there is $\psi_1(y_1,\dots,y_k)$ s.t. for every $\langle a_1,\dots,a_k\rangle\in G^k$ we have that $G\vDash\psi_1(a)$ iff $\varphi(G^n,a)$ is weak generic in $(G^n,+)$*
2. *There is $\psi_2(y_1,\dots,y_k)$ s.t. for every $\langle a_1,\dots,a_k\rangle\in G^k$ we have that $G\vDash\psi_2(a)$ iff $\varphi(G^n,a)$ is generic in $(G^n,+)$*
3. *there is a natural number N s.t. for every φ-definable $X\subseteq G^n$ the set X is generic in $(G^n,+)$ iff G^n may be covered by at most N left translates of X*

Proof. 1. let $\psi_1(y_1,\dots,y_k)$ be

$$\forall r\exists z_1,\dots,z_n\forall x_1,\dots,x_n((\bigwedge_{i=1}^n z_i\leq x_i\wedge x_i\leq z_i+r)\to\varphi(x_1,\dots,x_n,y_1,\dots,y_k))$$

3. Assume that $n = 1$. Let $\psi_2(y_1, \dots, y_k)$ be such as 2. Suppose for every $N < \omega$ we can find $\langle a_1, \dots, a_k \rangle \in G^k$ s.t. the set $\varphi(G, a_1, \dots, a_k)$ is generic in G but not N-generic. Then the set of formulas

$$\bigcup_{N<\omega} \{\psi_2(y_1, \dots, y_k) \wedge \forall z_1, \dots, z_N \exists t \forall x (\varphi(x, y_1, \dots, y_k) \to \bigwedge_{i=1}^{N} t \neq z_i \cdot x)\}$$

is a type in variables $y_1, \dots, y_k$ and has a realization $\langle b_1, \dots, b_k \rangle \in H^k$ in some $\aleph_0$-saturated elementary extension H of G. Then we reach a contradiction as the set $\varphi(H, b_1, \dots, b_k)$ is simultaneously generic and not generic in H

□

Corollary 2.12. *Assume that* $(G, <, +, \dots)$ *is an o-minimal expansion of an ordered group* $(G, <, +)$, $0 < n < \omega$, *and* $p(x_1, \dots, x_n) \in S_n(G)$. *TFAE*

1. $p(x_1, \dots, x_n)$ *is weak generic in* $(G^n, +)$
2. $\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) = p(x_1, \dots, x_n)$ *for every* $\langle g_1, \dots, g_n \rangle \in G^n$

Proof. $1 \Rightarrow 2$: suppose

$$\langle g_1, \dots, g_n \rangle + p(x_1, \dots, x_n) \neq p(x_1, \dots, x_n)$$

for some $\langle g_1, \dots, g_n \rangle \in G^n$. Then for some $\varphi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$ we have that $(\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$. $\varphi(G^n)$ is weak generic in $(G^n, +)$ and hence contains arbitrarily large boxes. Take any $R > \max(|g_1|, \dots, |g_n|)$ and choose $a_1, \dots, a_n \in G$ s.t.

$$B = [a_1, a_1 + R] \times \dots \times [a_n, a_n + R] \subseteq \varphi(G^n)$$

we obtain

$$\emptyset \neq (\langle g_1, \dots, g_n \rangle + B) \cap B \subseteq (\langle g_1, \dots, g_n \rangle + \varphi(G^n)) \cap \varphi(G^n) = \emptyset$$

a contradiction

$2 \Rightarrow 1$: we shall prove a more general fact. Namely if G is a group and $p(x) \in S(G)$ is s.t. for every $g \in G$ we have that $g \cdot p = p$, then p is weak generic in G

If not, then we can find a formula $\varphi(x) \in p(x)$ which is not weak generic in G. Then $\neg\varphi(x)$ is generic in G so there are $m < \omega$ and $g_1, \dots, g_m \in G$ s.t $G = \bigcup_{i=1}^{m} g_i(G \setminus \varphi(G))$. Thus $\bigcap_{i=1}^{m} g_i \cdot \varphi(G) = \emptyset$, which contradicts the fact that the formulas $g_1 \cdot \varphi, \dots, g_m \cdot \varphi$ belong to the consistent type $p(x)$ □

2.4 Stationary

In this section we assume that $(G, <, +, \dots)$ is an o-minimal expansion of an ordered group $(G, <, +)$

Recall that in stable group all weak generic types are generic. Moreover, all of them are stationary over any model M. This means that every (weak) generic type $p \in S(M)$ has a unique extension to a (weak) generic type $q \in S(A)$ for each $A \supseteq M$

Definition 2.13. We call a weak generic type p over a set A **stationary** if for every $B \supseteq A$ the type p has just one extension to a complete weak generic type over B

In general weak generic types do not need to be stationary

Example 2.1. we shall prove that the types $p_1(x) = \{x < a : a \in G\}$ and $p_2(x) = \{x > a : a \in G\}$ are the only two weak generic types in $(G, +)$ complete over G and that both of them are stationary

By the o-minimality of $(G, <, +, \dots)$, every definable subset of G is a union of finitely many points and intervals. For every $a, b \in G$ the interval (a, b) is not weak generic in $(G, +)$ by Lemma 2.3 (2). Thus no type in $S_1(G)$ but p_1 and p_2 is weak generic in $(G, +)$

On the other hand, all intervals of the form $(-\infty, a)$ or $(b, +\infty)$ are weak generic in $(G, +)$ since their complements in G are not generic in $(G, +)$. This gives us the weak genericity of the types p_1 and p_2

If H is any elementary extension of G, then there are also two complete (over H) weak generic types in $(H, +)$. This means that p_1 and p_2 are stationary

Definition 2.14. We call an o-minimal structure $(M, <, \dots)$ **stationary** if for every elementary extension N of M and N-definable function $g : N \to N$ there exists an M-definable function $f : N \to N$ s.t. $g(x) \leq f(x)$ for all sufficiently large $x \in N$

Theorem 2.15. *Assume $(M, <, \dots)$ is a stationary o-minimal structure and $N \succ M$. For every N-definable map $g : N \to N$ with $\lim_{x \to +\infty} g(x) = +\infty$ we can find an M-definable map $f : N \to N$ s.t. $\lim_{x \to +\infty} f(x) = +\infty$ and $f(x) \leq g(x)$ for all sufficiently large $x \in N$*

Proof. First of all, assume that g is a bijection. Then g^{-1} exists and by the stationary of $(M, <, \dots)$ we can find an M-definable function $f : N \to N$ s.t. ultimately $g^{-1} \leq f$. We have that $\lim_{x \to +\infty} g^{-1}(x) = +\infty$, which implies that $\lim_{x \to +\infty} f(x) = +\infty$. Since f is M-definable, we can choose $a \in M$

s.t. f is strictly increasing on $(a, +\infty)$ (monotonicity theorem). We define a function $f_1 : N \to N$ as follows

$$f_1(x) = \begin{cases} f(x) & x > a \\ f(a) + x - a & x \leq a \end{cases}$$

Then f_1 is an M-definable bijection, hence f_1^{-1} exists and also is M-definable. Moreover, $\lim_{x \to +\infty} f_1^{-1}(x) = +\infty$ and ultimately $f_1^{-1} \leq g$ so f_1^{-1} has the desired properties

If g is not a bijection, then proceeding as above we can find an N-definable bijection $g_1 : N \to N$ s.t. ultimately $g_1 = g$ □

By the o-minimality of $(G, <, +, \dots)$, every definable subset of the set $G \times G$ is a union of finitely many cells of dimension 0,1,2. By Proposition 2.7, we are interested only in cells of dimension 2. They are of the form

$$C_{a,b}^{f,g} = \{\langle x, y\rangle \in G \times G : a < x < b \wedge f(x) < y < g(x)\}$$

where $\{-\infty\} \cup G \ni a < b \in G \cup \{\infty\}$ and $f, g : (a, b \to G \cup \{-\infty, \infty\}$ are definable maps s.t. $f(x) < g(x)$ for each $x \in (a, b)$. If $a, b \in G$, then the cell $C_{a,b}^{f,g}$ is not weak generic in $(G, +) \times (G, +)$ by Theorem 2.10

3 Problems

2.1 2.4