Homework7

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November 17, 2021

Exercise 1. Let M and N be L-structures. Let T be the set of all L-sentences satisfied by M. Show that $M \equiv N$ iff $N \vDash T$

Proof. If $M \equiv N$, then for any L-sentence φ, if $M \vDash \varphi \Leftrightarrow N \vDash \varphi$. Hence $N \vDash T$.

If $N \vDash T$, then for any L-sentence φ, if $M \vDash \varphi$, then $N \vDash \varphi$. If there is a sentence ψ such that $N \vDash \psi$ and $M \vDash \neg\psi$. Then as $\neg\psi \in T$, we have $N \vDash \neg\psi$, a contradiction. Thus if $N \vDash \varphi$ then $M \vDash \varphi$. Hence $M \equiv N$ $\square$

Exercise 2. Show that if $(M, \leq)$ is a countable linear order, then there is an embedding $(M, \leq) \to (\mathbb{Q}, \leq)$

Proof. Take a element m from M and $q \in \mathbb{Q}$ and let $f_0 = \{(m, q)\}$. We build a chain of map

$$f_0 \subset f_1 \subset f_2 \subset \cdots$$

such that if $m_1, m_2 \in f_i$, then $m_1 \leq m_2$ iff $f(m_1) \leq f(m_2)$. Suppose f_i is defined and we can enumerate it $\{(m_0, q_0), \dots, (m_{i-1}, q_{i-1})\}$ such that $m_0 < m_1 < \cdots < m_{i-1}$.. Take an element m_i of $M \setminus \mathrm{dom}(f_i)$, there are three cases

1. If $m_i > m_{i-1}$, then take $q_i = \max\{q_0, \dots, q_{i-1}\} + 1 = q_{i-1} + 1$
2. If there is j such that $m_j < m_i < m_{j+1}$, then take a $q_i \in (q_j, q_{j+1})$
3. If $m_i < m_0$, then take $q_i = q_0 - 1$

Let $f_{i+1} = f_i \cup \{(m_i, q_i)\}$. Then for any $n \in \mathrm{dom}(f_i)$, $m_i < n$ if and only if $f_{i+1}(m_i) < f_{i+1}(n)$. Hence for any $m, n \in \mathrm{dom}(f_{i+1})$, we have $m < n$ if and only if $f_{i+1}(m) < f_i(n)$.

Let $f = \bigcup_{i \in |M|} f_i = \bigcup_{i \in \omega} f_0$. For any $a, b \in M$, there is $i \in \omega$ s.t. $a, b \in \mathrm{dom}(f_i)$ and $a \leq b$ if and only if $f(a) = f_i(a) \leq f_i(b) = f(b)$. Hence f is an embedding. $\square$

Exercise 3. Let L be a language and L' be a bigger language. Let M_1 be an L-structure and M_2 be an L'-structure. Suppose that $M_1 \equiv M_2 \restriction L$. Show that there is an L'-structure M_3 with an L'-elementary embedding $i_2 : M_2 \to M_3$ and an L-elementary embedding $i_1 : M_1 \to (M_3 \restriction L)$

Proof. Let $T_L(M_1) = \{\varphi(\overline{m}) \mid \varphi$ L-formula and $M_1 \vDash \varphi(\overline{m})\}$ and $T_{L'}(M_2) = \{\psi(\overline{m}') \mid \psi$ L'-formula and $M_2 \vDash \psi(\overline{m}')\}$. Let $L'' = L' \cup M_1 \cup M_2$ and $\Gamma = T_L(M_1) \cup T_{L'}(M_2)$ an L''-theory. For any $\phi(\overline{m}) \wedge \psi(\overline{m}')$ where $\phi(\overline{m}) \in T_L(M_1)$ and $\psi(\overline{m}') \in T_{L'}(M_1)$. We have $M_1 \vDash \phi(\overline{m})$ and hence $M_1 \vDash \exists \bar{x}\ \phi(\bar{x})$. As $M_1 \equiv M_2 \restriction L$, $M_2 \vDash \exists \bar{x}\ \phi(\bar{x})$ and there is $\bar{n} \in M_2^m$ such that $M_2 \vDash \phi(\bar{n})$. By interpreting $\overline{m}$ to $\bar{n}$ and $M_1 \setminus \overline{m}$ by arbitrary M_2 element, we have $M_2 \vDash \phi(\bar{n}) \wedge \psi(\overline{m}')$ and hence $\phi(\overline{m}) \wedge \psi(\overline{m}')$ is satisfiable.

Thus by compactness, Γ is satisfiable and there is a L''-structure N such that $N \vDash \Gamma$. Let $M_3 = N \restriction L'$,we have $M_3 \vDash T(M_2)$ and $M_3 \restriction L \vDash T(M_1)$. Hence M_3 is an elementary extension of M_2 and $M_3 \restriction L$ is an elementary extension of M_1 □

Exercise 4. Show that there is a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$

Proof. Consider a new language $L' = L \cup \mathbb{Q}$ and L'-theory $\Gamma = \mathrm{Th}(M, \leq, P) \cup \mathrm{Diag}(\mathbb{Q}, \leq) \cup \{P(q) : q \in \mathbb{Q}\}$. For any finite $\Delta_1 \cup \Delta_2 \cup \Delta_3 \subseteq_f \Gamma$ where $\Delta_1 \subseteq_f \mathrm{Th}(M, \leq, P)$, $\Delta_2 \subseteq_f \mathrm{Diag}(\mathbb{Q}, \leq)$ and $\Delta_3 \subseteq_f \{P(q) : q \in \mathbb{Q}\}$, let $A \subseteq \mathbb{Q}$ denote the constants of $\mathbb{Q}$ occurring in $\Delta_2 \cup \Delta_3$. As A is finite and M is an infinite linear order, we can find suitable interpretation such that for any $a_1, a_2 \in A$, $\mathbb{Q} \vDash a_1 \leq a_2$ if and only if $(M, \leq, P) \vDash a_1^M \leq a_2^M$. By interpreting $\mathbb{Q} \setminus A$ to arbitrary elements of M, we have $(M, \leq, P, \mathbb{Q}^M) \vDash \Delta_1 \cup \Delta_2 \cup \Delta_3$.

Hence Γ is satisfiable and suppose $(N', \leq, P^{N'}, \mathbb{Q}^{N'}) \vDash \Gamma$. Let $(N, \leq, P^N) = N' \restriction L$, which means $N = N'$ and $P^N = P^{N'}$. Then as $(N, \leq, P^N) \vDash \mathrm{Th}(M, \leq, P)$, we have $(N, \leq, P^N) \equiv (M, \leq, P)$. Also as for any $q \in \mathbb{Q}$, $q^{N'} \in N'$ and so $q^{N'} \in N$ and $P(q^{N'})$. Thus for any $q_1, q_2 \in \mathbb{Q}$, $\mathbb{Q} \vDash q_1 \leq q_2$ if and only if $N' \vDash q_1^{N'} \leq q_2^{N'}$ if and only if $N \vDash q_1^{N'} \leq q_2^{N'}$ if and only if $P^N \vDash q_1^{N'} \leq q_2^{N'}$ as $(P^N, \leq)$ is indeed a substructure of $(N, \leq)$. □

Exercise 5. Show that there is a countable structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$

Proof. From previous exercise we get a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$. Let $S = i(\mathbb{Q}) \subseteq N$, then Downward Löwenheim–Skolem Theorem we can get a countable elementary substructure $(N', \leq, P^{N'})$ of $(N, \leq, P^N)$ containing $i(\mathbb{Q})$. Hence $(N', \leq, P^{N'}) \equiv (M, \leq, P)$.

Now we prove that there is an embedding $j : (\mathbb{Q}, \leq) \to (P^{N'}, \leq)$. For any $q \in \mathbb{Q}$, as $N' \vDash P^{N'} i(q)$ if and only if $N \vDash P^N i(q)$, we have $i(\mathbb{Q}) \subseteq P^{N'}$. Hence we define $j(q) = i(q)$ for any $q \in \mathbb{Q}$. Then for any $q_1, q_2 \in \mathbb{Q}$, $\mathbb{Q} \vDash q_1 \leq q_2 \Leftrightarrow P^N \vDash i(q_1) \leq i(q_2) \Leftrightarrow P^{N'} \vDash j(q_1) \leq j(q_2)$ and j is indeed an embedding. $\square$

Exercise 6. Show that there is a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $f : (N, \leq) \to (P^N, \leq)$

Proof. Suppose we have a countable structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $i : (\mathbb{Q}, \leq) \to (P^N, \leq)$. Then N is a countable linear order and by Exercise 2 we have an embedding $j : (N, \leq) \to (\mathbb{Q}, \leq)$. Then $i \circ j$ is still an embedding as for any $a, b \in N$, $N \vDash a \leq b \Leftrightarrow \mathbb{Q} \vDash j(a) \leq j(b) \Leftrightarrow P^N \vDash ij(a) \leq ij(b)$ $\square$

Exercise 7. Show that there is an elementary extension $(N, \leq, P^N) \succeq (M, \leq, P)$ and an embedding $f : (N, \leq) \to (P^N, \leq)$

Proof. Consider a new language $L' = L \cup M \cup \{f\}$, let φ be $\forall x, y(x \leq y \leftrightarrow f(x) \leq f(y) \wedge P(f(x)) \wedge P(f(y)))$ and a theory $\Gamma = \text{Diag}_{\text{el}}(M, \leq, P) \cup \{\varphi\}$

From previous exercise, we have a structure $(N, \leq, P^N) \equiv (M, \leq, P)$ and an embedding $g : (N, \leq) \to (P^N, \leq)$. For any $\psi(\overline{m}) \wedge \varphi$ where $\psi(\overline{m}) \in \text{Diag}_{\text{el}}(M, \leq, P)$ is a L-formula, $M \vDash \exists \bar{x} \psi(\bar{x})$ and $N \vDash \exists \bar{x} \psi(\bar{x})$. So there is $\bar{n} \in N^n$ such that $N \vDash \psi(\bar{n})$. By interpreting $\overline{m}$ as $\bar{n}$ and $M \setminus \overline{m}$ as arbitrary elements of N, $(N, \leq, P^N, M^N, g) \vDash \psi(\overline{m}) \wedge \varphi$, hence $\psi \wedge \varphi$ is satisfiable and thus Γ is satisfiable.

Then there is a model $(O, \leq, P^O, M^O, f^O) \vDash \Gamma$ such that $(O, \leq, P^O) \succeq (M, \leq, P)$ and $f^O : (O, \leq) \to (P^O, \leq)$ is an embedding $\square$