Applications of compactness

Introductory Model Theory

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Recommended reading: Poizat's *Course in Model Theory*, Sections 4.2 and 5.1, and Section 5.2 up to Theorem 5.1.

1 Proof of compactness via ultraproducts

Recall from last time...

Fact 1. *Let I be a set. Let $\mathcal{P}(I)$ be the power set of I. Suppose $\mathcal{S} \subseteq I$. If $\mathcal{S}$ has the finite intersection property (FIP), then there is an ultrafilter $\mathcal{U}$ on I with $\mathcal{U} \supseteq \mathcal{S}$.*

Fact 2 (part of Łoś's theorem)**.** *Let M_i be an L-structure for each $i \in I$. Let $\mathcal{U}$ be an ultrafilter on I and let M be the ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$. Then for any L-sentence σ,*

$$M \models \sigma \iff \{i \in I : M_i \models \sigma\} \in \mathcal{U}.$$

We can use ultraproducts and Łoś's theorem to give another proof of compactness.

Theorem 3 (Compactness theorem)**.** *If T is a finitely satisfiable L-theory, then T is satisfiable.*

Proof. Let $\{M_i : i \in I\}$ be a collection of L-structures containing at least one representative from every elementary equivalence class. For ϕ an L-sentence, let $[\phi] = \{i \in I : M_i \models \phi\}$. Let $\mathcal{S} = \{[\phi] : \phi \in T\}$. We claim $\mathcal{S}$ has FIP. Otherwise, there are $\phi_1, \ldots, \phi_n \in T$ such that $\varnothing = \bigcap_{i=1}^n [\phi_i] = [\bigwedge_{i=1}^n \phi_i]$. But T is finitely satisfiable, so there is some $N \models \bigwedge_{i=1}^n \phi_i$. There is some $M_j \equiv N$, and then $j \in [\bigwedge_{i=1}^n \phi_i] = \varnothing$, a contradiction.

So there is an ultrafilter $\mathcal{U}$ on I containing $\mathcal{S}$. Let $M = \prod_{i \in I} M_i/\mathcal{U}$. Then for $\phi \in T$, we have

$$\{i \in I : M_i \models \phi\} = [\phi] \in \mathcal{S} \subseteq \mathcal{U},$$

so $M \models \phi$ by Łoś's theorem. Thus $M \models T$. □

2 The Löwenheim-Skolem theorem

Theorem 4 (Löwenheim-Skolem). *Let T be an L-theory. Suppose T has an infinite model, or that for every $n < \omega$, T has a model of size $> n$. Then for any $\kappa \geq |L|$, T has a model of size κ.*

Proof. Let L' be L plus new constant symbols c_α for $\alpha < \kappa$. Let T' be T plus the sentences $c_\alpha \neq c_\beta$ for $\alpha < \beta < \kappa$.

Claim. T' is finitely satisfiable.

Proof. Let $T_0 \subseteq_f T'$. Then there is $S \subseteq_f \kappa$ such that

$$T_0 \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in S,\ \alpha < \beta\}.$$

Take $M \models T$ with $|M| \geq |S|$. Expand M to an L'-structure by interpreting c_α for $\alpha \in S$ as distinct elements of M (and define c_α randomly for $\alpha \notin S$). Then $M \models T_0$. $\square_{\text{Claim}}$

By compactness, T' has a model M. Then the c_α^M are pairwise distinct, so $|M| \geq \kappa$. By Downward Löwenheim-Skolem (also called Löwenheim's theorem), we can find an elementary substructure $N \preceq M$ with $|N| = \kappa$. Then $N \equiv M$, so $N \models T$. $\square$

3 Elementary amalgamation

Given an L-structure M, and $A \subseteq M$, let $L(A)$ be L plus a new constant symbol for each element of A. Then M is naturally an $L(A)$-structure.

Definition 5. $T(A)$ is the set of $L(A)$-sentences true in M.

Taking $A = M$, the set $T(M)$ is called the *elementary diagram* of M, and sometimes written $\mathrm{eldiag}(M)$. Poizat calls this the *diagram* of M, but some authors use "diagram" to mean the quantifier-free part of $T(M)$.

Remark 6. Suppose $N \models T(M)$. Define $f : M \to N$ to be the map sending $c \in M$ to its interpretation $c^N \in N$. Then

$$M \models \phi(a_1, \ldots, a_n) \iff \phi(a_1, \ldots, a_n) \in T(M) \iff N \models \phi(f(a_1), \ldots, f(a_n)).$$

So $f : M \to N$ is an elementary embedding. Conversely, if $f : M \to N$ is an elementary embedding then N is naturally a model of $T(M)$.

Theorem 7. *If $M_1 \equiv M_2$, then there is a structure N and elementary embeddings $M_1 \to N$ and $M_2 \to N$.*

Proof. Note $T(M_i)$ is closed under conjunction for $i = 1, 2$.

Claim. $T(M_1) \cup T(M_2)$ is finitely satisfiable.

Proof. Otherwise, there is $\phi \in T(M_1)$ and $\psi \in T(M_2)$ with $\phi \wedge \psi$ being unsatisfiable. We can write ϕ as $\phi(\bar{a})$ for some L-formula ϕ and $\bar{a}$ in M_1. Similarly, we can write ψ as $\psi(\bar{b})$ for some L-formula ψ and $\bar{b}$ in M_2.

Then $M_2 \models \psi(\bar{b})$, so $M_2 \models \exists \bar{x}\ \psi(\bar{x})$, so $M_1 \models \exists \bar{x}\ \psi(\bar{x})$. Take $\bar{c}$ in M_1 with $M_1 \models \psi(\bar{c})$. Expand M_1 to an $(L(M_1) \cup L(M_2))$-structure by interpreting b_i as c_i. Then $M_1 \models \phi(\bar{a}) \wedge \psi(\bar{b})$, a contradiction. $\square_{\text{Claim}}$

By compactness, there is $N \models T(M_1) \cup T(M_2)$. Then there are elementary embeddings $M_1 \to N$ and $M_2 \to N$. $\square$

4 Types

Let M be an L-structure and $A \subseteq M$. Recall $L(A)$ and $T(A)$ from above.

Definition 8. Suppose $N \succeq M$ and $\bar{b} \in N^n$. The *type of $\bar{b}$ over A*, written $\text{tp}(\bar{b}/A)$ or $\text{tp}^N(\bar{b}/A)$ is the set of $L(A)$-formulas satisfied by $\bar{b}$:

$$\{\phi(x_1, \ldots, x_n) \in L(A) : N \models \phi(\bar{b})\}.$$

The *space of n-types over A* is

$$S_n(A) = \{\text{tp}^N(\bar{b}/A) : N \succeq M,\ \bar{b} \in N^n\}.$$

Definition 9. Let $\Sigma(\bar{x})$ be a set of $L(A)$-formulas.

1. If $N \succeq M$ and $\bar{b} \in N^n$, then $\bar{b}$ *satisfies* $\Sigma(\bar{x})$, written $N \models \Sigma(\bar{b})$, if for every $\phi(\bar{x}) \in \Sigma(\bar{x})$ we have $N \models \phi(\bar{b})$. We also say that $\bar{b}$ *realizes* Σ, or $\bar{b}$ *is a realization* of Σ.

2. Σ is *satisfiable in M* if there is a realization in M.

3. Σ is *finitely satisfiable in M* if every finite $\Sigma_0 \subseteq_f \Sigma$ is satisfiable in M.

Exercise 10. Suppose $p \in S_n(A)$ and $\bar{b} \in N^n$ for some $N \succeq M$. Then $\bar{b}$ realizes p iff $\text{tp}(\bar{b}/A) = p$.

Theorem 11. *Fix a structure M and subset $A \subseteq M$.*

1. *If $p(\bar{x})$ is an n-type, then p is finitely satisfiable in M.*

2. *If $\Sigma(\bar{x})$ is a set of $L(A)$-formulas that is finitely satisfiable in M, then there is $p \in S_n(A)$ with $p \supseteq \Sigma$.*

Proof. 1. Take $N \succeq M$ and $\bar{b} \in N^n$ with $\text{tp}(\bar{b}/A) = p$. Suppose $\Sigma_0 \subseteq_f p$. Then $N \models \bigwedge \Sigma_0(\bar{b})$ so $N \models \exists \bar{x} \bigwedge \Sigma_0(\bar{x})$ so $M \models \exists \bar{x} \bigwedge \Sigma_0(\bar{x})$ so $\Sigma_0(\bar{x})$ is realized in M.

2. Let $c_1, \ldots, c_n$ be new constant symbols not in $L(A)$. For any finite $\Sigma_0(\bar{x}) \subseteq \Sigma(\bar{x})$, there is a model of $T(M) \cup \Sigma_0(\bar{c})$, namely M with $\bar{c}$ interpreted as a realization of $\Sigma_0(\bar{x})$.

   By compactness, $T(M) \cup \Sigma(\bar{c})$ has a model N. There is an elementary embedding $M \to N$ because $N \models T(M)$; without loss of generality $M \preceq N$. Then $\bar{c}$ satisfies $\Sigma(\bar{x})$, so $\text{tp}(\bar{c}/A) \supseteq \Sigma(\bar{x})$. $\square$

5 ω-saturated structures

Definition 12. A structure M is *ω-saturated* if for every finite $A \subseteq_f M$, every $p \in S_1(A)$ is realized in M.

Lemma 13. *Suppose $M_1 \preceq M_2 \preceq M_3 \preceq \cdots$. Then there is an L-structure on $M = \bigcup_i M_i$ such that $M_i \preceq M$ for all i.*

Proof. For each i, $T(M_i)$ is finitely satisfiable, complete, and has the witness property. Therefore the union $\bigcup_i T(M_i)$ is finitely satisfiable, complete, and has the witness property. Take the canonical model. (Or see Theorem 2.6 in Poizat's textbook for a more elementary proof.) $\square$

Lemma 14. *Let M be a structure. There is $M' \succeq M$ such that if $A \subseteq_f M$ and $p \in S_1(A)$, then p is realized in M'.*

Proof. Add a new constant symbol c_p for each $p \in \bigcup_{A \subseteq_f M} S_1(A)$. Then $T(M) \cup \bigcup_{p \in S_1(A)} p(c_p)$ is finitely satisfiable, because each p is finitely satisfiable in M. Therefore there is a model N. There is an elementary embedding $M \to N$ because $N \models T(M)$. We may assume $M \preceq N$. Then c_p^N (the interpretation of c_p in N) realizes p. $\square$

Theorem 15. *Let M be any structure. Then there is an ω-saturated $M' \succeq M$.*

Proof. Build a chain $M = M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ such that every type over a finite set in M_i is realized in M_{i+1}. Let $M' = \bigcup_i M_i$. Then $M' \succeq M_0 = M$. If $A \subseteq_f M'$, then $A \subseteq M_i$ for $i \gg 0$, so every type over A is realized in M_{i+1}, hence in M' (as $M_{i+1} \preceq M'$). $\square$

Remark 16. Let M be ω-saturated. Let A be a finite subset. Let $\Sigma(x)$ be a set of $L(A)$-formulas in x. If $\Sigma(x)$ is finitely satisfiable in M, then $\Sigma(x)$ extends to a type $p(x)$, so there is some $a \in M$ realizing $p(x)$ and therefore realizing $\Sigma(x)$.

Example. $(\mathbb{R}, +, \cdot, 0, 1)$ is *not* ω-saturated. The following set of formulas is finitely satisfiable in $\mathbb{R}$, but not realized in $\mathbb{R}$:

$$\Sigma(x) = \{\exists y \ (y \cdot y + \underbrace{1 + \cdots + 1}_{n \text{ times}} = x) : n \in \mathbb{N}\}.$$

On the other hand, one can show that $(\mathbb{C}, +, \cdot, 0, 1)$ *is* ω-saturated.