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Topological dynamics for groups definable in real closed field

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We study the definable topological dynamics of groups definable in an *o*-minimal expansion of an arbitrary real closed field M. For a definable group G which admits a compact-torsion-free decomposition $G = HK$, we give a description of the minimal subflow and Ellis group of the universal definable $G(M)$-flow $S_{G,ext}(M)$. This Ellis group is isomorphic to $N_G(H) \cap K(\mathbb{R})$, which extends the result of G. Jagiella from [7]. We also consider $SL(2, M)$ as an example, explaining the difference between the universal definable $SL(2, \mathbb{R})$-flow, $S_G(\mathbb{R})$ and the universal definable $G(M)$-flow, $S_{G,ext}(M)$ for an arbitrary model $M \succ \mathbb{R}$.

1. Introduction and preliminaries

The model theoretic approach to topological dynamics was introduced by Newelski in [10], where he studied the action a definable group $G(M)$ on its type space $S_G(M)$ and proved that $S_{G,ext}(M)$ is the universal definable $G(M)$-flow and hence an Ellis semigroup. Also in [10], Newelski conjectured that in the case of a theory with NIP, the Ellis groups of $S_{G,ext}(M)$ are isomorphic to G/G^{00}. The first counterexample of this conjecture is found in [5], where the authors showed that Newelski's conjecture fails in the case of $G = SL(2)$ and $M = \mathbb{R}$. Moreover, in [7] G. Jagiella provided a range of counterexamples by extending results in [5]. But in both papers, one works over the field of reals $\mathbb{R}$, where every externally definable subset of $\mathbb{R}$ is also a definable subset of $\mathbb{R}$. So $S_{G,ext}(\mathbb{R})$ equals to $S_G(\mathbb{R})$. This paper is inspired by the ideas of [5] as well as [7], and provides a broader range of such counterexamples by extending results in [7] to the context of an arbitrary real closed field M. Now we highlight our main result as follows:

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Theorem. *Let G be a group definable over $\mathbb{R}$ admitting a compact-torsion-free decomposition $G = HK$ with H torsion free and K definably compact, and M be an arbitrary elementary extension of $\mathbb{R}$. Then the Ellis group of $S_G(M^{ext})$ is algebraically isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

In the rest of this introduction we give a description of the key aspects of the model-theoretic and topological dynamics context, as well as their interaction. References are [7,10,11].

In Section 2, we will turn to a class of definable group which admits so-called "compact-torsion-free decomposition", where we give precise definitions and prove some basic but nontrivial results.

In Section 3, we describe the minimal subflow and calculate the Ellis group of $S_G(M^{ext})$ (the notation will be defined later) for G admitting the compact-torsion-free decomposition and M an arbitrary elementary extension of $\mathbb{R}$. We find that the structure of minimal subflow is not preserved by different models. But the Ellis group is independent of the models.

In Section 4, we present the example of $G = SL(2)$, which illustrates the difference between $S_G(\mathbb{R})$ and $S_G(M^{ext})$ of an arbitrary model $M \succ \mathbb{R}$.

1.1. G-flows and its enveloping semigroup

Assume G is a group. By a (point-transitive) *G-flow* we mean a compact Hausdorff space X together with a left action of G on X by homeomorphism that contains a dense orbit. By a *subflow* of X we mean a closed subspace of X which is closed under the action of G. Since X is compact, there exist minimal subflows of X. The minimal subflows of X are very important objects, the one which are "dynamically indecomposable" and considered to be the most fundamental G-flows.

Let X^X denote the collection of all maps from X to itself, provided with the product topology. Then X^X is also a compact Hausdorff space by Tychonoff's theorem. Now each $g \in G$ defines a homeomorphism $\pi_g : X \mapsto X$, which is an element in X^X. Let $E(X)$ denote the closure of the set $\{\pi_g : g \in G\}$ in X^X. Then $E(X)$ together with the operation $*$ of function composition defines a semigroup structure $(E(X), *)$. We call $E(X)$ the *Ellis semigroup* of the G-flow X. For every $x \in X$, the closure of its G-orbit is exactly $E(X)(x) = \{f(x) : f \in E(X)\}$. $E(X)$ itself is a G-flow with function composition as the G-action. Every minimal subflow of the G-flow $E(X)$ is a minimal left ideal of the semigroup $E(X)$, and homeomorphic to each other as G-flows, i.e., for any minimal subflows I_1 and I_2 of $E(X)$, there exists a homeomorphism $f : I_1 \to I_2$ such that $f(gx) = gf(x)$ for all $x \in I_1$. We sometimes use the phrase *"minimal subflow of $E(X)$"* to denote the homeomorphism class of minimal subflows of $E(X)$. Every minimal subflow I is the closure of the G-orbit of every $p \in I$, hence is $E(X) * p$. By an *idempotent* of I we mean some $u \in I$ such that $u * u = u$. We denote the collection of all idempotents of I by $J(I)$. For any $u \in J(I)$, we have that $(u * I, *)$ is a group with u as its identity. All those groups are isomorphic to each other, even for different minimal left ideals. We call these groups the *ideal groups* and call their isomorphism class the *Ellis group* of $E(X)$. For more details, readers need to see Refs. [1,3].

1.2. The universal definable $G(M)$-flow $S_{G,ext}(M)$

Now we consider the topological dynamics in the model-theoretic context. We will assume a basic knowledge of model theory. Good references are [13] and [15]. We sometimes work in a sufficiently saturated model $\bar{M}$ of a theory T, in which every type over a small model M is realized. Definability usually means with parameters, and by a formula (or a partial type) we mean one definable with parameters from A, for A a subset of $\bar{M}$. By x, y, z we mean arbitrary n-variables and $a, b, c \in \bar{M}$ denote n-tuples in $\bar{M}^n$ with $n \in \mathbb{N}$.

Definition 1.1. Let M be a model of a theory T and $\bar{M} \succ M$ be a saturated model.

1. By an *externally definable subset* X of M we mean a subset of M^n of the form $\{a \in M; \bar{M} \models \phi(a,b)\}$ with $\phi(x,y)$ an L-formula and parameters $b \in \bar{M}$. We also denote X by $\phi(M,b)$. By $X \subseteq_{ext} M$ we mean X is an externally definable subset of M.
2. By $S_{n,ext}(M)$ we mean the space of ultrafilters on the Boolean algebra of externally definable subsets of M^n.
3. By M^{ext} we mean the expansion of M obtained by adding a predicate for every externally definable subset of M. We let L^{ext} denote the expanded language by adding predicates P_X for every $X \subseteq_{ext} M$.
4. By $S_n(M^{ext})$ we mean the space of complete n-types over M^{ext}.

In the NIP case, Shelah has proved that:

Fact 1.2. *(See [16].) Let M be a model of an NIP theory T. Then $Th(M^{ext})$ eliminates quantifiers, and is NIP.*

So with the assumption of NIP, $p \in S_{n,ext}(M)$ iff $p \in S_n(M^{ext})$. Now we assume NIP throughout this paper and we use the notation $S_n(M^{ext})$ instead of $S_{n,ext}(M)$.

Remark 1.3. For any $N \succ M$, if N is $|M|^+$-saturated then there is a homeomorphism α between $S_n(M^{ext})$ and the closed subspace of $S_n(N)$ consisting of types finitely satisfiable in M. The homeomorphism α maps $p \in S_n(M^{ext})$ to $q = \{\phi(x,b) : \phi(M,b) \in p\}$. We denote this $\alpha(p)$ by p^N, and $\alpha^{-1}(q)$ by q_M. Note that if $N' \succ M$ is any other $|M|^+$-saturated model, we have ${p^N}_M = {p^{N'}}_M = p$ for any $p \in S_n(M^{ext})$.

In this paper, we do not make a distinction between p and p^N (i.e. we consider p^N as an element of $S_n(M^{ext})$).

An important property of M^{ext} is that every externally definable subset of M^{ext} is definable: for any formula $\phi(x,y) \in L^{ext}$ and $b \notin M^{ext}$, there exists a formula $\theta(x) \in L^{ext}(M)$ such that $\theta(M) = \phi(M,b)$. So every type p over M^{ext} has a unique global coheir. Moreover, for any $\phi(x,y) \in L^{ext}$, if p' is the unique global coheir of p, then $\phi(x,b) \in p'$ if and only if $\theta(x) \in p$ for some $\theta(x)$ with parameters in M and $\theta(M) = \phi(M,b)$.

In the model-theoretic setting we are concerned with the $G(M)$-flow $S_{G,ext}(M)$, where G is an M-definable group and $S_{G,ext}(M)$ is the set (space) of elements $p(x) \in S_{n,ext}(M)$ which contains the set $G(M)$. With the assumption of NIP, $S_{G,ext}(M)$ is also the space of complete types over M^{ext} concentrating on G, which we denote by $S_G(M^{ext})$. As Newelski showed in [10], $E(S_G(M))$ coincides with $S_{G,ext}(M)$, which is the so-called *universal definable* $G(M)$-flow (from [4] for example). So $S_G(M^{ext})$ is also the universal definable $G(M)$-flow, and thus an Ellis semigroup. Now we use the notation $S_G(M^{ext})$ instead of $S_{G,ext}(M)$. The product $*$ of $S_G(M^{ext})$ (or $S_{G,ext}(M)$) is defined as follows:

Definition 1.4. For any $p, q \in S_G(M^{ext})$,

$$p * q = \left\{U \subseteq_{ext} G(M) : \left\{g \in M : g^{-1}U \in q\right\} \in p\right\}.$$

If $p \in S(M)$ is definable over M, then for any $B \supseteq M$, p has a unique heir over B. We denote the unique heir of p over B by $p|B$. Note that for any $a \in \bar{M}$, we have $a \models p|B$ if and only if $\mathrm{tp}(B/M,a)$ is finitely satisfiable in M. Namely $\mathrm{tp}(B/M,a)$ is a coheir of $\mathrm{tp}(B/M)$.

Lemma 1.5. *(See [11].) For any $p, q \in S_G(M^{ext})$, $p * q = \mathrm{tp}(ab/M^{ext})$, where $a \models p$ and $b \models q|(M^{ext}, a)$.*

For $p \in S_G(M^{ext})$ and $U \subseteq_{ext} G(M)$ let

$$d_p U = \{g \in M : U \in gp\} = \{g \in M : g^{-1}U \in p\}.$$

Lemma 1.6. *(See [11].) For $U \subseteq_{ext} G(M)$ we have $d_p U \subseteq_{ext} G(M)$.*

It is easy to see that for $U \subseteq_{ext} G(M)$, $U \in p * q$ if and only if $d_q U \in p$.

1.3. Definable topological dynamics of definably compact groups

We assume again G is an $\emptyset$-definable group and T is a theory with NIP. The following results are contained in [6]. As above, by $S_G(M)$ we mean the space of complete types over M concentrating on G.

Definition 1.7. G has *fsg* (finitely satisfiable generics) if there is some global type $p \in S_G(\bar{M})$ such that for every small model M_0, every left G-translate $gp = \{\phi(g^{-1}x) : \phi(x) \in p\}$ of p, with $g \in G(\bar{M})$, is finitely satisfiable in M_0.

Let X be a G-flow. Recall that a subset Y of X is *generic* if finitely many G-translate of Y cover X. A point $x \in X$ is generic if every open neighborhood of x is generic. Newelski showed in [10] that

Fact 1.8. *If a G-flow X has a generic point. Then X has a unique minimal subflow, which is exactly the collection of all generic points.*

A definable subset Y of G is left (or right) *generic* if finitely many left (or right) G-translate of Y cover G. A (partial) type p is called left (or right) generic is every $X \in p$ is left (or right) generic. From Fact 1.8, we conclude easily:

Fact 1.9. *If the G-flow $S_G(M)$ (or $S_G(M^{ext})$) has a left (right) generic type. Then $S_G(M)$ (or $S_G(M^{ext})$) has a unique left (right) minimal subflow, which is exactly the collection of all left (right) generic types.*

The following is true (cf. [6] Corollary 4.3).

Fact 1.10. *Assuming G has fsg and M_0 is an arbitrary small model, then*

1. *there is global generic type $p \in S_G(\bar{M})$,*
2. *every left G-translate of p is generic and finitely satisfiable in M_0,*
3. *every right G-translate of p is generic and finitely satisfiable in M_0,*
4. *G^{00} exists, and is exactly the $stab(p) = \{g \in G : gp = p = pg\}$.*

It is easy to see from Fact 1.10 that the left generics coincide with right generics, so if G has *fsg* we simply say generic definable subset (or type).

Fact 1.11. *If T is an o-minimal theory, G is a definably compact group definable in $\bar{M}$. Then G has fsg.*

Proof. By Corollary 8.4 of [6]. □

The following is due to Pillay [14].

Theorem 1.12. *If G has fsg then we have:*

1. *The universal $G(M)$-flow $S_G(M^{ext})$ has a unique minimal subflow (i.e. minimal left ideal) I, which is exactly the collection of all generic types in $S_G(M^{ext})$.*
2. *If $u \in J(I)$, then $u \vdash G^{00}$ (i.e. $G^{00} \subseteq u$) and the group $u * I$ is isomorphic to the compact group G/G^{00}.*

We denote the collection of all generic type in $S_G(M^{ext})$ by $Gen(G(M))$. Note that $p \in S_G(M^{ext})$ is generic if and only if $p^{\bar{M}} \in S_G(\bar{M})$ is generic.

Remark 1.13. Assume G has *fsg* and $p \in S_G(\bar{M})$ is a global type. Then the following are equivalent:

1. p is generic.
2. For any small submodel M_0, $p_{M_0} \in S_G({M_0}^{ext})$ is also a generic type in $S_G({M_0}^{ext})$.
3. For some submodel M_0, $p_{M_0} \in S_G({M_0}^{ext})$ is also a generic type in $S_G({M_0}^{ext})$.

By the above remark, we see that $Gen(G(M))$ is not determined by M, namely $Gen(G(M))$ is homeomorphic to $Gen(G(M_0))$ via $p \mapsto (p^{\bar{M}})_{M_0}$, for any other model M_0. So we simply denote it by $Gen(G)$.

Recall that if G is a group definable in $\mathbb{R}$ and has *fsg*, then G/G^{00} is naturally isomorphic to $G(\mathbb{R})$ since $G^{00}(\mathbb{R}) = 1_G$. So from Facts 1.11 and 1.12 we conclude easily:

Corollary 1.14. *If G is a definably compact group definable in $\mathbb{R}$, T is the theory of the o-minimal expansion of $\mathbb{R}$, $M \succ \mathbb{R}$ as a model of T. Then:*

1. *The universal $G(M)$-flow $S_G(M^{ext})$ has a unique minimal subflow (i.e. minimal left ideal) I, which is $Gen(G)$.*
2. *If $u \in J(I)$, then $u \vdash G^{00}$ and $G(\mathbb{R})$ is isomorphic to $u * I$ via $k \mapsto u(k)$, where $k \in T(\mathbb{R})$ and $u(k) = uk = ku$.*

2. Groups admitting a compact-torsion-free decomposition

We assume in this section that the reader is familiar with the basics of *o*-minimality. Reference is [8]. From now on T will denote the theory of an *o*-minimal expansion of $\mathbb{R}$. We fix $M \succ \mathbb{R}$ as an arbitrary model of T, G a group definable over $\mathbb{R}$. We first recall the definition of *compact-torsion-free decomposition* from [7]:

Definition 2.1. We say that the definable group G has a definable compact-torsion-free decomposition if there exist definable subgroups H and K such that $G = HK$ with K definably compact, H torsion-free, and $H \cap K = \{1_G\}$.

Now we assume that the $\mathbb{R}$-definable group G has a compact-torsion-free decomposition $G = HK$ with subgroups H and K definable over $\mathbb{R}$.

A fundamental theorem of A. Pillay in [12] shows that if G is definable in the *o*-minimal structure M, then it has a topology τ making itself a topological group. The topology τ on G has the following properties: G can be covered by finitely many *tau*-open subsets, each definably homeomorphic to an open subset of M^k for some fixed k, which is called the (*o*-minimal) dimension of G. In particular, if G is definable in $\mathbb{R}$, then G is a real Lie group with this topology. Now G has a definable compact-torsion-free decomposition $G = HK$, we see that K is homeomorphic to the quotient space G/H via $k \mapsto k/H$, since $H \cap K = \{1_G\}$. The action of G on its quotient space G/H induces an action of G on K defined by $g \star k = k_1$ with $k_1/H = gk/H$.

Since for any $g \in G$, there is a unique $h_1 \in H$ and $k_1 \in K$ such that $gk = k_1 h_1$, we see that $g \star k$ is this k_1. Note that if $g \in K$ then $g \star k = gk$.

This action of $G(M)$ on $K(M)$ can be extended to the action of $G(M)$ on the type space $S_K(M^{ext})$ via $g \star \mathrm{tp}(k/M^{ext}) = \mathrm{tp}(g \star k/M^{ext})$ with $g \in G(M)$ and $\mathrm{tp}(k/M^{ext}) \in S_K(M^{ext})$. This action makes $S_K(M^{ext})$ a $G(M)$-flow. For any $g \in G(M)$, let π_g denote the mapping in $S_K(M^{ext})^{S_K(M^{ext})}$ induced by the action of g. For $p \in S_G(M^{ext})$, we define a mapping $\pi_p : S_K(M^{ext}) \to S_K(M^{ext})$ by

$$\pi_p(q) = \{U \subseteq_{ext} K(M) : \{g \in G(M) : g^{-1} \star U \in q\} \in p\}$$

for any $q \in S_K(M^{ext})$.

$\pi_p(q)$ is a complete type over M^{ext} since $g^{-1} \star U \in q$ if and only if $g^{-1} \star \neg U \notin q$. As with Lemma 1.6, we have $\{g \in G(M) : g^{-1} \star U \in q\} \subseteq_{ext} M$. Therefore, the mapping π_p is well-defined. We now describe the Ellis semigroup of the $G(M)$-flow $S_K(M^{ext})$.

Lemma 2.2. $E(S_K(M^{ext})) = \{\pi_p : p \in S_G(M^{ext})\}$.

Proof. For any $p \in S_G(M^{ext})$, there is a directed set D and a net $\{g_i\}_{i \in D}$ such that the net $\{g_i\}_{i \in D}$ converges to p. This means for any $X \in p$, there exists $j \in D$ such that $g_i \in X$ for any $i > j$. As $S_K(M^{ext})^{S_K(M^{ext})}$ is a compact Hausdorff space, the net $\{\pi_{g_i}\}_{i \in D}$ converges to some mapping $f \in S_K(M^{ext})^{S_K(M^{ext})}$. For any $q \in S_K(M^{ext})$, we see that $Y \in f(q)$ if and only if there exists some j such that $Y \in \pi_{g_i}(q)$ for all $i > j$. Now we claim that

$$Y \in f(q) \Longrightarrow \{g \in G(M) : g^{-1} \star Y \in q\} \in p.$$

Since $Y \subseteq_{ext} M$, we see that $\{g \in G(M) : g^{-1} \star Y \in q\} \subseteq_{ext} M$. Suppose $\{g \in G(M) : g^{-1} \star Y \in q\} \notin p$. Then for any $j \in D$, there is some $i > j$ such that $g_i \notin \{g \in G(M) : g^{-1} \star Y \in q\}$. This implies $Y \notin \pi_{g_i}(q)$. A contradiction. So we have $f(q) \subseteq \pi_p(q)$ by the claim. But as both $f(p)$ and $\pi_p(q)$ are complete types over M^{ext}, we have $f(q) = \pi_p(q)$ and hence $\{\pi_p : p \in S_G(M^{ext})\} \subseteq E(S_K(M^{ext}))$.

Conversely, if $f \in E(S_K(M^{ext}))$, then there is a directed set D and a net $\{\pi_{g_i}\}_{i \in D}$ such that the net $\{\pi_{g_i}\}_{i \in D}$ converges to f. Since $S_G(M^{ext})$ is a compact Hausdorff space, there is some $p \in S_G(M^{ext})$ which is an accumulation point of $\{g_i\}_{i \in D}$. A similar argument as above shows that $f = \pi_p$. This completes the proof. □

For $p \in S_G(M^{ext})$ and $q \in S_K(M^{ext})$, we denote $\pi_p(q)$ by $p \star q$. By Lemma 2.2, we could easily conclude that:

Theorem 2.3. *For $q \in S_K(M^{ext})$, the closure of its $G(M)$-orbit $G(M) \star q$ is $S_G(M^{ext}) \star q$.*

Proof. The closure of $G(M) \star q$ is $\{f(q) : f \in E(S_K(M^{ext}))\}$, which is $\{\pi_p(q) : p \in S_G(M^{ext})\}$ by Lemma 2.2. □

Lemma 2.4. *For any $p \in S_G(M^{ext})$ and $q \in S_K(M^{ext})$, $p \star q = \mathrm{tp}(a \star b/M^{ext})$, where $a \models p$ and $b \models q|(M^{ext}, a)$.*

Proof. Let $\phi(x) \in L^{ext}(M)$, we denote $g^{-1} \star \phi(x)$ by $\theta(x, g)$. Then

$$\phi(x) \in p \star q \quad \Longleftrightarrow \quad \{g \in G(M) : g^{-1} \star \phi(x) \in q\} \in p \quad \Longleftrightarrow \quad \theta(b, M) \in p.$$

Since $p' = \mathrm{tp}(a/M^{ext}, b)$ is the unique coheir of p over b, we see that $\theta(b, x) \in p'$ and hence $\models \theta(b, a)$. So we have $\phi(a \star b)$ as required. □

Remark 2.5. Note that $(S_K(M^{ext}), *)$ is a subsemigroup of $(S_G(M^{ext}), *)$. Moreover, for any $p, q \in S_K(M^{ext})$, we have that $p \star q = p * q$.

The natural projection $G(M) \to G(M)/H(M)$ induces a surjective continuous map π from $G(M)$ to $K(M)$ defined by $\pi(kh) = k$ with $k \in K(M)$ and $h \in H(M)$. It is also a homomorphism between the $G(M)$-flows. This π can be extended to a $G(M)$-flow homomorphism from $S_G(M^{ext})$ to $S_K(M^{ext})$, by mapping $\mathrm{tp}(kh/M^{ext}) \mapsto \mathrm{tp}(k/M^{ext})$, which we also call π.

Corollary 2.6. *Let π be the map from $S_G(M^{ext})$ to $S_K(M^{ext})$ given by $\pi(\mathrm{tp}(kh/M^{ext})) = \mathrm{tp}(k/M^{ext})$. Then $\pi(p' * p) = p' \star \pi(p)$ for any $p' \in S_G(M^{ext})$.*

Proof. Let g' realize p' and $g = kh$ realize $p|(M^{ext}, g')$. Then $p' * p = \mathrm{tp}(g'kh/M^{ext})$ by Lemma 1.5. But $g'kh = (g' \star k)h'$ for some $h' \in H$, and hence $\pi(p' * p) = \mathrm{tp}(g' \star k/M^{ext})$. Now $\mathrm{tp}(g'/M^{ext}, k)$ is finitely satisfiable in M and k realizes $\pi(p)$. So by Lemma 2.4, $p' \star \pi(p) = \mathrm{tp}(g' \star k/M^{ext})$ as required. $\square$

3. Minimal subflow and the Ellis group

In this section, we assume again we are working with the theory T of an o-minimal expansion of $\mathbb{R}$. $M \models T$ is an elementary extension of $\mathbb{R}$ and $\bar{M}$ is the saturated model of T. G is an $\mathbb{R}$-definable group which admits a compact-torsion-free decomposition $G = HK$, where K is an $\mathbb{R}$-definable definably compact group and H is an $\mathbb{R}$-definable torsion free subgroup. "Generic" always means left generic in the $G(M)$-flow $S_G(M^{ext})$ unless we use the phrase "right generic". We will describe the minimal subflow as well as the Ellis group of the definable universal $G(M)$-flow $S_G(M^{ext})$.

Lemma 3.1. *(See [7].) There is a left $H(\mathbb{R})$-invariant type $p_0 \in S_H(\mathbb{R})$.*

Lemma 3.2. *(See [2].) Assume that T has NIP and $M \models T$. If $p(x) \in S(M)$ is definable. Then $p(x)$ induces a unique type $p^*(x) \in S(M^{ext})$.*

It is well-known that all types over $\mathbb{R}$ are definable [9]. So we have:

Corollary 3.3. *Let $p_1 \in S_H(M)$ be the unique heir of p_0 over M. Then $p_1 \in S_H(M^{ext})$ and is $H(M)$-invariant.*

Proof. Clearly, the type p_1 is definable over $\mathbb{R}$. So p_1 induces a unique type ${p_1}^*(x) \in S(M^{ext})$ by Lemma 3.2. We can identify p_1 with ${p_1}^*$. $\square$

We fix this type p_1 in our following argument. For any n-tuple $a = (a_1, ..., a_n) \in \bar{M}$, we say that a is *finite* if there is some $0 < c \in \mathbb{R}$ such that $-c < a_i < c$ for each $1 \le i \le n$. For a finite n-tuple a, by $st(a)$ we mean the standard part of a in $\mathbb{R}$.

As K is a definably compact group definable over $\mathbb{R}$, we see that every element of $K(\bar{M})$ is finite.

Remark 3.4. For any $k_1, k_2 \in K(\bar{M})$ we have $st(k_1 k_2) = st(k_1)\, st(k_2)$.

Proof. Since $K^{00}(\mathbb{R}) = 1_K$, we see that $(kK^{00})(\mathbb{R}) = st(k)$. $\square$

Lemma 3.5. *If f is an $\mathbb{R}$-definable function and $f(a)$ and a in $\bar{M}$ are finite. Then $f(st(a)) = st(f(a))$.*

Proof. If $f(st(a)) \neq st(f(a))$ then there is an R-definable open set U such that $f(a) \in U$ and $f(st(a)) \notin U$. Now $V = f^{-1}U$ is also $\mathbb{R}$-definable with $a \in V$ and $st(a) \notin V$. Note that $st(a) \notin V$ if and only if $a \notin V$. A contradiction. □

Lemma 3.6.

$$\left\{st(k); k \in K(\bar{M}) \textit{ is a realization of some } q \in p_1 \star S_K\left(M^{ext}\right)\right\} = N_G(H) \cap K(\mathbb{R}).$$

Proof. If $k \in N_G(H) \cap K(\mathbb{R})$ we see that $\mathbb{R} \models \forall h \in H\ (h \star k = k)$. Hence $p_1 \star k = k$ and thus $k \models p_1 \star \mathrm{tp}(k/M^{ext})$ for any $k \in N_G(H) \cap K(\mathbb{R})$. So we have

$$N_G(H) \cap K(\mathbb{R}) \subseteq \left\{st(k); k \in K(\bar{M}) \text{ is a realization of some } q \in p_1 \star S_K\left(M^{ext}\right)\right\}.$$

Conversely, we claim that

Claim. *for any $h' \in H(\mathbb{R})$ and $k' \in K$ realizing some $q \in p_1 \star S_K(M^{ext})$, we have $h' \star st(k') = st(k')$.*

Proof. Since $k' = h \star k$ for some $h \in H(\bar{M})$ and $K \in K(\bar{M})$ such that $h \models p_1$ and $\mathrm{tp}(h/M^{ext}, k)$ is finitely satisfiable in M. Now the action of h' on K is an $\mathbb{R}$-definable homeomorphism. So $h' \star st(h \star k) = st((h'h) \star k)$ by Lemma 3.5. Since $\mathrm{tp}(h/M^{ext})$ is $H(M)$-invariant, we see that $\mathrm{tp}(h'h/M^{ext}) = \mathrm{tp}(h/M^{ext}) = p_1$. Moreover, $\mathrm{tp}(h'h/M^{ext}, k)$ is finitely satisfiable in M. So, by Lemma 2.4, we have

$$\mathrm{tp}\left(\left(h'h\right) \star k/M^{ext}\right) = \mathrm{tp}\left(h'h/M^{ext}\right) \star \mathrm{tp}\left(k/M^{ext}\right) = p_1 \star \mathrm{tp}\left(k/M^{ext}\right).$$

But $p_1 \star \mathrm{tp}(k/M^{ext})$ is $\mathrm{tp}(h \star k/M^{ext})$ and thus $\mathrm{tp}((h'h) \star k/\mathbb{R}) = \mathrm{tp}(h \star k/\mathbb{R})$. So we have $h' \star st(k') = h' \star st(h \star k) = st((h'h) \star k) = st(h \star k) = st(k')$. This proves the claim. □

Since $h' \star st(k') = st(k')$ for any $h' \in H(\mathbb{R})$, we see that $st(k') \in N_G(H)(\mathbb{R})$ as required. □

Recall that $Gen(K)$ is the collection of all generic type in $S_K(M^{ext})$. By Fact 1.11, K has *fsg*. So, by Remark 1.13, $Gen(K)$ is independent of the model M.

Lemma 3.7. *$Gen(K)$ is a two-sided ideal of the semigroup $(S_K(M^{ext}), *)$.*

Proof. As we have showed above, the definably compact group K has *fsg*. So the left generics coincide with right generics in $S_K(M^{ext})$ (or in $S_K(\bar{M})$). □

Corollary 3.8. *Let $u \in Gen(K)$ be an idempotent. Then the ideal group of $S_K(M^{ext})$ generated by u is $u * S_K(M^{ext})$.*

Proof. For any $q \in S_K(M^{ext})$, $u * q = u * (u * q) \in u * Gen(K)$. □

Proposition 3.9. *If $p \in S_H(M^{ext})$ is a left $H(M)$-invariant type, and I is a minimal subflow of the $G(M)$-flow $S_K(M^{ext})$, then $I * p$ is a minimal subflow of the $G(M)$-flow $S_G(M^{ext})$.*

Proof. We show that for any $q \in I$, the ideal $S_G(M) * q * p$ is $I * p$.

Let $r \in S_G(M^{ext})$. The type $r * q * p$ is realized by some $k'h'kh$ for $k', k \in K, h', h \in H$ with

$$k'h' \models r, \qquad k \models q\big|\left(M^{ext}, k', h'\right) \quad \text{and} \quad h \models p\big|\left(M^{ext}, k', h', k\right).$$

Let $h'k = k_1h_1$ (namely $h' \star k = k_1$). Then $h \models p|(M^{ext}, k', h', k.k_1, h_1)$, since (h', k) and (k_1, h_1) are interdefinable. Now p is $H(M)$-invariant, so for any $N \succ M$, $p|N$ is also $H(N)$-invariant. Hence $h_1h \models p|(M^{ext}, k', h', k.k_1, h_1)$. We see that

$$\mathrm{tp}\big(k'h'kh/M^{ext}\big) = \mathrm{tp}\big(k'k_1h_1h/M^{ext}\big) = \mathrm{tp}\big(k'k_1/M^{ext}\big) * p.$$

It is easy to see that $r \star q = \mathrm{tp}(k'k_1/M^{ext})$. So we have $(r \star q) * p = r * q * p$. Note that $I = S_G(M^{ext}) \star q$, hence $S_G(M) * q * p = I * p$. $\square$

Proposition 3.10. *The $G(M)$-flow $S_K(M^{ext})$ has a unique minimal subflow $\mathcal{I}$.*

Proof. Let $v_0 \in Gen(K)$. Then v_0 is generic in the $K(M)$-flow $(K(M), S_K(M^{ext}))$, so which is also generic in the $G(M)$-flow $(G(M), S_K(M^{ext}))$. So $\mathcal{I} = S_G(M^{ext}) \star v_0$ is the unique minimal subflow of $(G(M), S_K(M^{ext}))$. $\square$

Remark 3.11. Jagiella indicated in [7] that in the case of $M = \mathbb{R}$, $Gen(K)$, the minimal subflow of $(K(\mathbb{R}), S_K(\mathbb{R}))$, is left H-invariant. This implies $Gen(K)$ is also minimal as a $G(\mathbb{R})$-subflow of $(G(\mathbb{R}), S_K(\mathbb{R}))$. So $\mathcal{I}$ is precisely the $Gen(K)$. But this is not true for arbitrary model M. In the next section we will give a counterexample where $Gen(K)$ is a proper subset of $\mathcal{I}$.

We fix a generic type $v_0 \in Gen(K)$ ($v_0 \in S_K(M^{ext})$) with $v_0 \vdash K^{00}$ in our following argument. From Propositions 3.9 and 3.10 we conclude easily:

Corollary 3.12. *$\mathcal{J} = \mathcal{I} * p_1$ is a minimal subflow of $S_G(M^{ext})$.*

Remark 3.13. Let $v_0 \in S_K(M^{ext})$ be a generic type such that $v_0 \vdash G^{00}$. Then $v_0(k) = v_0k = kv_0$ with $k \in K(\mathbb{R})$ and $\{v_0(k), k \in K(\mathbb{R})\}$ is an ideal group (Ellis group) of $S_K(M^{ext})$ isomorphic to $K(\mathbb{R})$ via $v_0(k) \mapsto k$.

Note that $p_1 \star v_0 \vdash k_0K^{00}$ for some $k_0 \in N_G(H) \cap K(\mathbb{R})$ by Lemma 3.6. We fix this k_0 in our following argument.

Lemma 3.14. $p_1 \star v_0({k_0}^{-1}) \vdash K^{00}$.

Proof. $p_1 \star v_0({k_0}^{-1}) = p_1 \star (v_0{k_0}^{-1})$. This type is realized by $h \star (k{k_0}^{-1})$, with $h \models p_1$ and $k \models v_0|(M^{ext}, h)$. Now since $h \star k \in k_0K^{00}$, $h(k{k_0}^{-1}) = k_0k_1h'{k_0}^{-1}$ with some $k_1 \in K^{00}$ and $h' \in H$, and as ${k_0}^{-1} \in N_G(H) \cap K(\mathbb{R})$, we have $h'({k_0}^{-1}) = {k_0}^{-1}h''$ for some $h'' \in H$. This concludes $h \star (k{k_0}^{-1}) = k_0k_1{k_0}^{-1}$, which is in K^{00} since K^{00} is a normal subgroup of K. $\square$

Lemma 3.15. *For any $p \in S_H(M^{ext})$ and $q \in S_K(M^{ext})$, we have $p * q * p_1 = (p \star q) * p_1$.*

Proof. The type $p * q * p_1$ is realized by $hk'h'$, for some $k' \in K, h, h' \in H$ with

$$h \models p\big|\big(M^{ext}\big), \qquad k' \models q\big|\big(M^{ext}, h\big) \quad \text{and} \quad h' \models p_1\big|\big(M^{ext}, h, k'\big).$$

Let $hk' = k_1h_1$. Then $k_1 = h \star k'$. Since (h, k') is interdefinable with (k_1, h_1), we see that $h' \models p_1|(M^{ext}, k_1, h_1)$. Now p_1 is a left $H(M)$-invariant type, so for any model $N \succ M$, $p_1|N$ is also a left $H(N)$-invariant type. Hence $h_1h' \models p_1|(M^{ext}, k_1, h_1)$.

Now we see that $p \star q = \mathrm{tp}(k_1/M^{ext})$. So $(p \star q) * p_1 = \mathrm{tp}(k_1h_1h'/M^{ext}) = p * q * p_1$. $\square$

Proposition 3.16. *Let $r_0 = v_0({k_0}^{-1}) * p_1$. Then r_0 is an idempotent.*

Proof. Let $v' = (p_1 \star v_0({k_0}^{-1}))$. Then by Lemma 3.15, $r_0 * r_0 = v_0({k_0}^{-1}) * v' * p_1$. We also see that $v' \vdash K^{00}$ by Lemma 3.14.

Now it suffices to show that $v_0({k_0}^{-1}) * v' = v_0({k_0}^{-1})$. Let $k \models v_0({k_0}^{-1})$ and $k_1 \models v'|(M^{ext}, k)$. Then kk_1 realizes $v_0({k_0}^{-1}) * v'$. By Corollary 3.8 and Remark 3.13, we see that $v_0 * v' = v_0(k')$ with some $k' \in K(\mathbb{R})$. Now we have $v_0({k_0}^{-1}) * v' \vdash {k_0}^{-1}K^{00}$ since $v_0({k_0}^{-1}) \vdash {k_0}^{-1}K^{00}$ and $v' \vdash K^{00}$. So we conclude $k' = {k_0}^{-1}$ and $v_0({k_0}^{-1}) * v' = v_0({k_0}^{-1})$. $\square$

Theorem 3.17. *The Ellis group of $S_G(M^{ext})$ is algebraically isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

Proof. Let $E = r_0 * \mathcal{J}$ be the ideal group of $S_G(M^{ext})$ generated by r_0 where $\mathcal{J}$ is given by Corollary 3.12. We see that

$$E = \left\{v_0\left({k_0}^{-1}\right) * p_1 * q * p_1 : q \in \mathcal{I}\right\}.$$

Claim. $E = \{v_0(k) * p_1 : k \in N_G(H) \cap K(\mathbb{R})\}$.

Proof. Let $p_1 \star q = \operatorname{tp}(k_1/M^{ext})$ with some $q \in \mathcal{I}$. Then $st(k_1) \in N_G(H) \cap T(\mathbb{R})$ by Lemma 3.6. Now by Lemma 3.7, $v_0({k_0}^{-1}) * \operatorname{tp}(k_1/M^{ext})$ is a generic type. Let $a \models v_0({k_0}^{-1})$ such that $\operatorname{tp}(a/M^{ext}, k_1)$ is finitely satisfiable in M. Then the type $v_0({k_0}^{-1}) * \operatorname{tp}(k_1/M^{ext})$ is realized by ak_1. By Remark 3.4, $st(ak_1) = {k_0}^{-1}\, st(k_1)$ since $st(a) = {k_0}^{-1}$. By Corollary 3.8, $\operatorname{tp}(ak_1/M^{ext}) \in v_0 * Gen(K)$. So we see that $v_0({k_0}^{-1}) * \operatorname{tp}(k_1/M^{ext})$ is $v_0({k_0}^{-1}\, st(k_1))$. Now by Lemma 3.15, we conclude that $v_0 * p_1 * q * p_1$ is $v_0({k_0}^{-1}\, st(k_1)) * p_1$. So we have showed $E \subseteq \{v_0(k) * p_1 : k \in N_G(H) \cap K(\mathbb{R})\}$.

Conversely, for $k \in N_G(H) \cap K(\mathbb{R})$, let $q = v_0 k$. Let $h \models p_1$ and $a \models v_0|(M^{ext}, h)$. Then ak realizes q. Let $h \star a = a_1 h_1$, we see that $a_1 \in K^{00}$. Since $k \in N_G(H) \cap K(\mathbb{R})$, we have $h_1 k = k h_2$ with some $h_2 \in H$. So $h \star (ak) = a_1 k$, hence $p_1 \star q = \operatorname{tp}(a_1 k/M^{ext})$ with $a_1 \in K^{00}$. By Lemma 3.15, we see that $v_0({k_0}^{-1}) * p_1 * v_0(k) * p_1$ is $v_0({k_0}^{-1}) * v'k * p_1$, where $v' = \operatorname{tp}(a_1/M^{ext})$. Now the type $v_0({k_0}^{-1}) * v'k$ is $v_0({k_0}^{-1}k)$ since $v' \vdash K^{00}$. So we conclude $v_0({k_0}^{-1}) * p_1 * kv_0 * p_1 = v_0({k_0}^{-1}k) * p_1$. This completes the proof of the claim as k is arbitrary. $\square$

Now we show that the map $\beta : v_0({k_0}^{-1}a) * p_1 \mapsto a$ is a group isomorphism between E and $N_G(H) \cap K(\mathbb{R})$. It suffices to show that this β is a group morphism. For any $b \in N_G(H) \cap K(\mathbb{R})$, we see that $p_1 \star v_0({k_0}^{-1}b)$ is $p_1 \star (v_0({k_0}^{-1}))b$ since $v_0({k_0}^{-1}b) = v_0({k_0}^{-1})b$ and b is invariant under the action of $H(\bar{M})$. As $(p_1 \star v_0({k_0}^{-1})) \vdash K^{00}$, we have $v_0({k_0}^{-1}a) * (p_1 \star v_0({k_0}^{-1}b)) * p_1 = v_0({k_0}^{-1}ab) * p_1$, and hence $\beta(v_0({k_0}^{-1}a) * p_1 * v_0({k_0}^{-1}b) * p_1) = \beta(v_0({k_0}^{-1}ab) * p_1) = ab$. So β is a group isomorphism. $\square$

4. $SL(2)$

We assume again that $M \succ \mathbb{R}$ is a model of an o-minimal expansion of $\mathbb{R}$ and $\bar{M} \succ M$ is the saturated model.

We say that $a \in \bar{M}$ is *positive infinite* if $a > M$, and *negative infinite* if $a < M$. We say that a 1-type $p \in S_1(M)$ is positive (or negative) infinite type if its realizations are positive (or negative) infinite over M. We say that $a \in \bar{M}$ is *positive infinitesimally close to* $b \in M$ if $b < a < c$ for any $c \in M$ with $c > b$. Likewise for *a negative infinitesimally close to* b over M. A 1-type $p \in S(M)$ is said to be positive (negative) infinitesimally close to $b \in M$ if a realization of p is positive (negative) infinitesimally close to b.

It is well-known that the special linear group $G = SL(2)$ has a compact-torsion-free decomposition $G = HT$ with

$$H(b,c) = \left\{ \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} : b > 0 \right\} \quad \text{and} \quad T(x,y) = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}.$$

We denote the identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by I and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ by $-I$. For any $p \in S_G(M^{ext})$, the type $-p$ means $-Ip$ (or $p(-I)$).

There is a standard action of $G(M)$ on the projection space $\mathbb{P}^1(M)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is a representative of an element of $\mathbb{P}^1(M)$. This action is well-defined since $det \left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = 1 \neq 0$. We identify $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ with $x \in M$ and consider $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ as the "point at infinity" over M.

Lemma 4.1. *If $h \in H$, $t \in T$, and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{P}^1$ then $(h \star t)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = ht\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.*

Proof. Assume $ht = t_1 h_1$. Then $(h \star t)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = t_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. But $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right) = h_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. So we have $t_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = t_1 h_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = ht\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ as required. □

Let $p_0 = \mathrm{tp}(b, c/\mathbb{R})$ with $b > \mathbb{R}$ and $c > dcl(\mathbb{R}, b)$. It is easy to check that p_0 is a left $H(\mathbb{R})$-invariant. By Lemma 3.2, we see that the heir p_1 of p_0 over M is also in $S_H(M^{ext})$, and $H(M)$-invariant.

In fact, we could define p_1 as follows:

Let $\bar{M} \succ M$ be the saturated model. Then every cell in $\bar{M}^2$ is of form either $\{(x,y) \mid y = f_0(x), x \in I\}$ or $\{(x,y) \mid f_1(x) < y < f_2(x), x \in I\}$, where I is an interval and f_i's are continuous $\bar{M}$-definable functions.

Let $W = p(\bar{M})$, where $p(v) = \{v > d \mid d \in M\}$ is the positive infinite type over M. Let $\mathcal{F}$ be the collection of $\bar{M}$-definable functions f which satisfy:

(i) There exist $d \in M$ and $r \in W$ such that $(d, r) \subseteq dom(f)$.
(ii) For any $\alpha \in dom(f) \cap M$, there are $\beta, \gamma \in dom(f) \cap M$ such that $\beta > \alpha$ and $f(\beta) < \gamma$.

Let $p_1 = \{U_f \subseteq_{ext} M^2 | U_f = M^2 \cap \{(x,y) \mid y > f(x)\}, f \in \mathcal{F}\}$. It is easy to check that p_1 is an $H(M)$-invariant complete type over M^{ext}. Moreover, any (b,c) realizes p_1 satisfying $b > M$ and $c > dcl(M, b)$.

As $dim(T) = 1$, we see that $Gen(T)$ is the collection of all (unique) coheirs of non-algebraic types in $S_T(\mathbb{R})$. Now let $v = \mathrm{tp}(x, y/\mathbb{R})$ be the type in $S_T(\mathbb{R})$ with y negative infinitesimally close to 0 and x the positive square root of $1 - y^2$. This is the type infinitesimally close to the identity on the *"negative"* side and hence $v \vdash T^{00}$. This is a non-algebraic type over $\mathbb{R}$, so is a generic type. The unique global coheir $v^{\bar{M}}$ of v is also a generic type. Let v_0 be the type $v^{\bar{M}}{}_M$. We see that $v_0 \in S_T(M^{ext})$ and is generic.

Assume $t = (x,y) \in T$ with $x > 0$ and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{P}^1$, we see that y is positive (negative) infinitesimally close to 0 over M if and only if $t\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \begin{pmatrix} x/y \\ 1 \end{pmatrix}$ (or x/y) is positive (negative) infinite over M. Conversely, if $x < 0$ then y is positive (negative) infinitesimally close to 0 if and only if $t\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is negative (positive) infinite over M.

Now, for $h = (b,c)$ realizes p_1 and $t = (x,y)$ realizes $v_0 | (M^{ext}, h)$, we see that

$$(h \star t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = ht\begin{pmatrix} 1 \\ 0 \end{pmatrix} = b^2(x/y) + bc,$$

by Lemma 4.1.

If $M = \mathbb{R}$, we have:

Theorem 4.2. *The ideal group of $S_{SL(2)}(\mathbb{R})$ generated by $(v_0) * p_1$ is $\{v_0 * p_1, -v_0 * p_1\}$ with $v_0 * p_1$ as its identity.*

Proof.

Claim. $p_1 \star v_0 = v_0$. *(Note that $v_0 = v$ in the case of $M = \mathbb{R}$.)*

Proof. Let $h \star t = (x', y')$. Now h acts on $T(N)$ as a homeomorphism for any model $N \succ \mathbb{R}$ which contains h. Since t is infinitesimally close to I over $\mathbb{R}, h$, so we have (x', y') is also infinitesimally close to I over $\mathbb{R}, h$. This means $x' > 0$. Since x/y is negative infinite over R, b, c, so is $b^2(x/y) + bc$ as $b^2 > 0$. This implies y' is negative infinitesimally close to 0 over $\mathbb{R}$. So (x', y') realizes v_0 as required. $\square$

Clearly, $p_1 \star v_0 \vdash T^{00}$. By Lemma 3.16, we see that $v_0 * p_1$ is the idempotent of a minimal left ideal of the universal G-flow $S_G(M^{ext})$. As $N_G(H) \cap T(\mathbb{R})$ is $\{I, -I\}$, the ideal group generated by $(v_0) * p_1$ turns out to be $\{v_0 * p_1, -v_0 * p_1\}$ with $v_0 * p_1$ as its identity. $\square$

But if M is a proper elementary extension of $\mathbb{R}$, we have:

Theorem 4.3. *The ideal group of $S_{SL(2)}(M^{ext})$ generated by $(v_0) * p_1$ is $\{v_0 * p_1, -v_0 * p_1\}$ with $-v_0 * p_1$ as its identity.*

Proof.

Claim. $p_1 \star v_0 \vdash -IT^{00}$.

Proof. Let $t' = (x', y') = h \star t$ and $h' = (b', c') \in H$ with $ht = t'h'$. It is easy to see that $b^{-1}y = y'b'^{-1}$ by the product of matrices. As $b, b' > 0$ and $y < 0$, we see that $y' < 0$. Since v_0 contains the coheir of v over M (note that the coheir of v over M is a partial type over M^{ext}), we have $x/y < x_1/y_1$ for any $(x_1, y_1) \in T(M)$ realizing v. So $b^2(x/y) + bc$ is positive infinite over M as (x/y) is “finite” over M. Namely $t' \binom{1}{0} = x'/y'$ is positive infinite over M. As we have seen that $y' < 0$, so $x' < 0$ and hence infinitesimally close to $-I$. So $t' \in -IT^{00}$ as required. $\square$

By Lemma 3.16, we see that $(-v_0) * p_1$ is an idempotent of a minimal left ideal of the universal G-flow $S_G(M^{ext})$. The ideal group generated by $(-v_0) * p_1$ turns out to be $\{v_0 * p_1, -v_0 * p_1\}$ with $-v_0 * p_1$ as its identity. $\square$

Note that for any $p \in S_G(M^{ext})$, the types $p \star q$ and q have same dimension if $dim(q) = dim(T)$ since G acts on T as a homeomorphism. In the case of $M = \mathbb{R}$, any type $q \in S_T(\mathbb{R})$ with $dim(q) = dim(T)$ is a generic type. So $Gen(T)$ is invariant under the action $\star$ of $S_G(\mathbb{R})$. But if $M \neq \mathbb{R}$, we have:

Corollary 4.4. *$Gen(T)$ is not $H(M)$-invariant, and thus a proper subset of $\mathcal{I} = S_G(M^{ext}) \star v_0$.*

Proof. The above claim says that $p_1 \star v_0$ is infinitesimally close to $-I$ over M on the on the “negative” side. So it is the heir of the type infinitesimally close to $-I$ over $\mathbb{R}$ on the “negative” side, which is not generic since it is not finitely satisfiable in $\mathbb{R}$. So $Gen(T)$ is not invariant under the action of $S_H(M^{ext})$ and hence not invariant under the action of $S_G(M^{ext})$. $\square$

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