Week6

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Exercise 1. Let $p \in S_1(M)$ be a non-constant type. Show that $[p]$ is not a minimal element in the fundamental order.

Proof. Suppose p is realized by a in $N \succeq M$ and let $q(x) = \text{tp}(a/N) \supseteq p$. Then $(x = a) \in q(x)$. But p is not a constant type, hence $(x = y) \in [q]$ and $(x = y) \notin [p]$. Therefore $[q] < [p]$. $\square$

Exercise 2. Let $p \in S_1(M)$ be a constant type and let $q \in S_1(N)$ be an extension of p. Show that $[q] = [p]$

Proof. If $p = \text{tp}(a/M)$ for some $a \in M$, then $q = \text{tp}(a/N)$. For any $\varphi(x, b) \in q(x)$ with $b \in N$, we have $\mathbb{M} \models \varphi(a, b)$ and therefore $\mathbb{M} \models \exists y \varphi(a, y)$. Hence $\exists y \varphi(x, y) \in p$ and so there is $b' \in M$ with $\varphi(x, b') \in p$. Thus $[p] \leq [q]$ and so $[p] = [q]$ $\square$

Exercise 3. Let $p \in S_1(M)$ be a constant type. Show that $[p]$ is a minimal element in the fundamental order.

Proof. For any $N \preceq \mathbb{M}, q \in S_1(N)$ and $[q] \leq [p]$, by Proposition 7, there is an ultrafilter $\mathcal{U}$ and an elementary embedding $M \to N^{\mathcal{U}}$ making $q^{\mathcal{U}}$ an extension of p. Then $[q^{\mathcal{U}}] = [p]$ as p is a constant type. But since $q \subseteq q^{\mathcal{U}}$, we have $[q^{\mathcal{U}}] \leq [q]$. Therefore $[q] = [p]$. $\square$

Exercise 4. Suppose the theory T is DLO. Let M, N be small models. Let a, b be elements of $\mathbb{M}$. Suppose $a \notin M$ and $\text{tp}(a/M)$ is not the type at $+\infty$ or $-\infty$. Suppose $b \notin N$, and $\text{tp}(b/N)$ is not the type at $+\infty$ or $-\infty$. Let $\varphi(x, \bar{c})$ be a formula in $\text{tp}(a/M)$. Show there is $\bar{c}' \in N$ s.t. $\varphi(x, \bar{c}')$ is a formula in $\text{tp}(b/N)$.

Proof. We may assume $\varphi(x, y_1, \dots, y_n)$ is a quantifier-free L-formula and it defines a linear order among x and $\bar{y}$ since otherwise we may find $\psi(x, y_1, \dots, y_n)$ which defines a linear order and implies φ. By rearranging variables, we may assume $y_1 \leq \dots \leq y_k \leq x \leq y_{k+1} \leq \dots y_n$ where $1 \leq k \leq n$. Then we can find $d_1 \leq \dots \leq d_n$ in N with $d_1 \leq \dots d_k \leq b \leq d_{k+1} \leq \dots \leq d_n$. Then $\varphi(x, \bar{d}) \in \mathrm{tp}(b/N)$ $\square$