Homework4

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Exercise 1. Let $\mathcal{L} = \{P\}$, a language with only one unary relation symbol. Classify complete theories with $\mathcal{L}$, i.e. determine all complete theories with only one unary symbol

Proof. Suppose $\mathfrak{M} = (M, R)$ and $\mathfrak{N} = (N, S)$.

1. If the universe of $\mathfrak{M}$ is finite. Then $|\mathfrak{M}| = a$ and $|P(\mathfrak{M})| = b$ for some $a, b \in \mathbb{N}$ and $b \le a$.

   For $n \in \mathbb{N}$, let

   $$\varphi_n = \exists x_1 \dots x_n (\bigwedge_{1 \le i < j \le n} x_i \ne x_j \wedge \forall x (\bigwedge_{i=1}^{n} x = x_i))$$
   $$\varphi_{n,P} = \exists x_1 \dots x_n (\bigwedge_{1 \le i < j \le n} x_i \ne x_j \wedge \forall x (\bigwedge_{i=1}^{n} x = x_i) \wedge \bigwedge_{i=1}^{n} P(x_i))$$

   Then φ_n says there are exactly n elements and $\varphi_{n,P}$ says there are exactly n elements satisfying P. Let $\theta_{a,b} = \varphi_a \wedge \varphi_{b,P}$, then $\mathfrak{M} \vDash \theta_{a,b}$.

   Claim $\mathfrak{M} \cong \mathfrak{N} \Leftrightarrow \mathfrak{M} \vDash \theta_{a,b}$ and $\mathfrak{N} \vDash \theta_{a,b}$

   Left to right is obvious. Now suppose both $\mathfrak{M}$ and $\mathfrak{N}$ both satisfy $\theta_{a,b}$. Then they have same number of elements and have same number of elements satisfying P. Let $f : P(\mathfrak{M}) \to P(\mathfrak{N})$ be a bijection as they have same cardinality and let $f' : \mathfrak{M} \to \mathfrak{N}$ be the bijection such that $f' \supseteq f$. Thus for any $a \in M$, $Ra \Leftrightarrow Sf'(a)$ and f' is an isomorphism.

   Thus $\mathfrak{M} \equiv \mathfrak{N}$ and any complete theory consisting of $\theta_{a,b}$ is unique.

2. If the universe of $\mathfrak{M}$ is infinite.

   For each $n \in \mathbb{N}$, let

$$\psi_n = \exists x_1 \ldots x_n (\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j)$$

   and $S = \{\psi_i : i \in \mathbb{N}\}$.

   (a) If $|P(\mathfrak{M})| = a$.
   Let $\Gamma_a = S \cup \{\varphi_{a,P}\}$. For any $p \in \mathbb{N}$, we show that Duplicator wins $\mathrm{EF}_p(R, S)$: if Spoiler chooses an element from $M \setminus P(\mathfrak{M})(N \setminus P(\mathfrak{N}))$, then Duplicator chooses a new element from $N \setminus P(\mathfrak{N})$ $(M \setminus P(\mathfrak{M}))$; if Spoiler chooses an element from $P(\mathfrak{M})$ $(P(\mathfrak{N}))$, then Duplicator chooses a element from $P(\mathfrak{N})$ $(P(\mathfrak{M}))$. Then we get a map s and $Ra \Leftrightarrow Ss(a)$ for $a \in \mathrm{dom}(s)$. Thus s is a local isomorphism and $\mathfrak{M} \equiv \mathfrak{N}$. Hence any complete theory that contains Γ_a is unique

   (b) if $|P(\mathfrak{M})|$ is infinite.
   Let $\Gamma_\omega = S \cup \{\varphi_{n,P} : n \in \mathbb{N}\}$. Similarly we can that prove any complete theory containing Γ_ω is unique

Thus we prove that there is three kinds of complete theory

1. For each $n \in \omega$ and $m \leq n$, there is a unique complete theory containing $\theta_{m,n}$
2. For each $n \in \omega$ there is a unique complete theory containing Γ_n
3. Unique complete theory containing Γ_ω

□

Exercise 2. Show that there is a structure $(M, +, \cdot, <, 0, 1)$ elementarily equivalent to $(\mathbb{R}, +, \cdot, <, 0, 1)$ s.t. the order on M is not complete: there is a bounded set with no supremum

Proof. Take $\mathbb{Q} \subseteq \mathbb{R}$, then by Löwenheim's theorem, we can find an elementarily restriction $\mathfrak{Q}$ of $(\mathbb{R}, +, \cdot, <, 0, 1)$ whose domain Q contains $\mathbb{Q}$ and $|\mathfrak{Q}| = \aleph_0$. Thus by Theorem 1.7 $\mathfrak{Q}$ is also a dense linear order. As every countable dense linear set is isomorphic to $\mathbb{Q}$, there is a f s.t. $f : \mathfrak{Q} \cong \mathbb{Q}$. Since $\mathbb{Q}$ is not complete, $\mathfrak{Q}$ is also not complete: if $\mathfrak{Q}$ is complete, then for any bounded subset $A \subset \mathbb{Q}$, $f(A)$ is also a bounded in $\mathfrak{Q}$ and thus has a supremum a in $\mathfrak{Q}$. Then $f^{-1}(a)$ is the supremum in $\mathbb{Q}$, a contradiction. □

Exercise 3. Show that the open interval $((0,1)<))$ is an elementary substructure of $(\mathbb{R},<)$

Proof. First $(\mathbb{R},<) \cong ((0,1),<)$ as we have the isomorphic function $f(x) = \arctan(x)$.

We first show that $((0,1),<) \sim_\omega (\mathbb{R},<)$. For any $p \in \mathbb{N}$ and game $\mathrm{EF}_p(((0,1),<),(\mathbb{R},<))$

- If Spoiler chooses $x \in \mathbb{R}$, then Duplicator chooses $y = f(x) \in (0,1)$
- If Spoiler chooses $y \in (0,1)$, then Duplicator chooses $x = f^{-1}(y)$

The induced map s is a local isomorphism as $s \subset f$. Thus Duplicator wins.

Thus $((0,1),<)$ is an elementary substructure of $(\mathbb{R},<)$ □

Exercise 4. Show that every formula is equivalent to a "nice" formula.

Proof. First we show that any formula φ of the form $y = t(\bar{x})$ can be transformed into a "nice" formula. We describe an algorithm for this transformation:

1. If t is a variable or constant, then return $y = t(\bar{x})$
2. If $t = f(t_1(\bar{x}), \dots, t_n(\bar{x}))$ and $t_{r_1}(\bar{x}), \dots, t_{r_m}(\bar{x})$ among $t_1(\bar{x}), \dots, t_n(\bar{x})$ are not "nice", let $\varphi_i(y_i, \bar{x})$ be $y_i = t_{r_i}(\bar{x})$ for $1 \le i \le m$ and we transform them into "nice" formula φ_i' by the algorithm
3. Let $\varphi'(y, \bar{x})$ be
$$y = \exists y_1 \dots y_m \left(f(t_1(\bar{x}), \dots, t_n(\bar{x}))^{t_{r_1}(\bar{x}), \dots, t_{r_m}(\bar{x})}_{y_1, \dots, y_n} \wedge \bigwedge_{i=1}^{m} \varphi_i'(y_i, \bar{x}) \right)$$
and return $\varphi'(y, \bar{x})$

As every formula φ is a finite string, this process will end and we will get a "nice" formula φ' s.t. $\vDash \varphi \leftrightarrow \varphi'$

We prove this by induction on the complexity of φ

1. If φ is atomic formula
   (a) If φ is of the form $t_1(\bar{x}) = t_2(\bar{x})$
   Let $\varphi_i := y_i = t_i(\bar{x})$ for $i = 1, 2$. We can transform φ_i into "nice" formula φ_i'. Hence we have nice formula
   $$\varphi'(\bar{x}) := \exists y_1 y_2 (y_1 = y_2 \wedge \varphi_1'(y_1, \bar{x}) \wedge \varphi_2'(y_2, \bar{x}))$$
   and $\vDash \varphi \leftrightarrow \varphi'$

(b) If φ is of the form $R(t_1(\bar{x}), \dots, t_m(\bar{x}))$
Let $\varphi_i := y_i = t_i(\bar{x})$ for $1 \leq i \leq m$. We can transform φ_i into "nice" formula φ_i' and let

$$\varphi'(\bar{x}) := \exists y_1 \dots y_n \left(R(y_1, \dots, y_m) \wedge \bigwedge_{i=1}^{m} \varphi_i'(y_i, \bar{x}) \right)$$

which is "nice" and $\vDash \varphi \leftrightarrow \varphi'$

2. If φ is of the form $\neg\psi$, $\psi \wedge \theta$, $\psi \vee \theta$, $\forall x \psi$ or $\exists x \psi$. As we can transform ψ and θ into nice formulas ψ' and θ' respectively

   (a) for $\neg\psi$, $\psi \wedge \theta$ or $\exists x \psi$, $\neg\psi'$, $\psi' \wedge \theta'$ and $\exists x \psi'$ are want we want

   (b) for $\psi \vee \theta$, let $\varphi' = \neg(\neg\psi' \wedge \theta')$

   (c) for $\forall x \psi$, let $\varphi' = \neg\exists x \neg\psi'$

$\square$

Exercise 5. Let T be the set of $\mathcal{L}_{\text{ring}}$-sentences true in $(\mathbb{R}, +, \cdot, 0, 1)$. Show that T is finitely satisfiable and complete, but does not have the witness property

Proof. Let $\mathfrak{M} = (\mathbb{R}, +, \cdot, 0, 1)$. Then $T = \text{Th}(\mathfrak{M})$. As for every $\mathcal{L}_{\text{ring}}$-sentence φ, either $\mathfrak{M} \vDash \varphi$ or $\mathfrak{M} \vDash \neg\varphi$, thus either $\varphi \in T$ or $\neg\varphi \in T$. Hence T is complete.

For every finite subset S of $\text{Th}(\mathfrak{M})$, we can enumerate them as $\varphi_1, \dots, \varphi_n$. Let $\psi = \bigwedge_{i=1}^{n} \varphi_i$. As $\mathfrak{M} \vDash \psi$, S is satisfiable and thus $\text{Th}(\mathfrak{M})$ is finitely satisfiable.

Sentence $\varphi = \exists x(x \cdot x = 1 + 1)$ doesn't have the witness property. $\square$