Homework 2 Solutions

Introductory to Model Theory

Autumn 2021

1.1. *Assume L is a discrete linear order without end points, define*

$$d(a,b) = \begin{cases} n & a \leq b \text{ and } |\{x : a \leq x < b\}| = n < \infty \\ \infty & a \leq b \text{ and } |\{x : a \leq x < b\}| = \infty \\ d(b,a) & a > b \end{cases}$$

define $a \sim b$ iff $d(a,b) < \infty$. It's easy to verify that $\sim$ is an equivalence relation and $[x]_\sim = \bigcup_{i<\infty}\{a : d(x,a) = i\}$. For any $x \in L$, $[x]_\sim \cong \mathbb{Z}$ by $f_x : [x]_\sim \to \mathbb{Z}$ as

$$f(a) = \begin{cases} n & a \geq x \text{ and } d(x,a) = n \\ -n & a < x \text{ and } d(x,a) = n \end{cases}$$

Define $[a]_\sim <_\sim [b]_\sim$ iff for any $x \in [a]_\sim$ and $y \in [b]_\sim$, $x < y$. Then $C = \{[x]_\sim | x \in L\}$ is a linear order under $<_\sim$. We claim $L \cong \mathbb{Z} \times C$.

We pick an element in each equivalence class by axiom of choice, which gives an injection $\tau : C \to L$. We will construct the isomorphism $\sigma : \mathbb{Z} \times C \to L$. For $(m, [x]_\sim) \in \mathbb{Z} \times C$, we may assume $\tau([x]_\sim) = x$, let $\sigma(m, [x]_\sim) = f_x^{-1}(m)$.

□

1.2. *Suppose $\langle (a_1, c_1), ..., (a_n, c_n)\rangle$ and $\langle (b_1, d_1), ..., (b_n, d_n)\rangle$ satisfy $a_i \in C, b_i \in D, c_i \in C', d_i \in D'$ for any $1 \leq i \leq n$, $s = \{(a_1, c_1), ..., (a_n, c_n)\}$ and $t = \{(b_1, d_1), ..., (b_n, d_n)\}$ are both p-isomorphism. We claim $s \times t = \{((a_1, b_1), (c_1, d_1)), ..., ((a_n, b_n), (c_n, d_n))\}$ is a p-isomorphism.*

The proof is based on induction on p. The case of $p = 0$ is to check $s \times t$ is a local isomorphism. Assume $(a_i, b_i) < (a_j, b_j)$, then $b_i < b_j$ or $b_i = b_j$ and $a_i < a_j$. By s and t, we have $d_i < d_j$ or $d_i = d_j$ and $c_i < c_j$, so $(c_i, d_i) < (c_j, d_j)$.

Suppose the case of p is correct, we need to check forth and back condition. For any $(a_{n+1}, b_{n+1}) \in C \times D$, we need to find $(c_{n+1}, d_{n+1}) \in C' \times D'$ s.t. $s \times t \cup \{((a_{n+1}, b_{n+1}), (c_{n+1}, d_{n+1}))\}$ is a p-isomorphism. Since s and t are both p-isomorphism, we have c_{n+1} and d_{n+1} to make $s \cup \{(a_{n+1}, c_{n+1})\}$ and $t \cup \{(b_{n+1}, d_{n+1})\}$ p-isomorphism. Then by hypothesis induction, $s \times t \cup \{((a_{n+1}, b_{n+1}), (c_{n+1}, d_{n+1}))\}$ is a p-isomorphism, which shows the forth condition works and the back direction is similar. $\square$

2.1. *Otherwise, suppose $f : \mathbb{R} \to \mathbb{R} + \mathbb{Q}$ is an isomorphism, let $x = (0, 1) \in \mathbb{R} + \mathbb{Q}$, then there are only countable elements bigger than it but uncountable elements bigger than $f(x)$. A contradiction!* $\square$

2.2. *Let $L_1 = \mathbb{Z} \times \mathbb{R}$ and $L_2 = \mathbb{Z} \times (\mathbb{R} + \mathbb{Q})$, they are not isomorphic for a similar reason as proof of the previous problem.*

$L_1 \sim_\infty L_2$ because Duplicator has a winning strategy in $EF_\infty(L_1, L_2)$. Assume $s_n = \{((a_1, b_1), (c_1, d_1)), ..., ((a_n, b_n), (c_n, d_n))\}$ is the local isomorphism after n rounds. $s_0 = \emptyset$ is a local isomorphism obviously. If Spoiler choose $(a_{n+1}, b_{n+1}) \in L_1$, we need to find corresponded (c_{n+1}, d_{n+1}) to make $s_{n+1} = s_n \cup \{((a_{n+1}, b_{n+1}), (c_{n+1}, d_{n+1}))\}$. There are two cases.

- $b_{n+1} \in \{b_1, ..., b_n\}$.

  Assume $b_{n+1} = b_i$, we choose $d_{n+1} = s_n(b_i)$ and $c_{n+1} = s_n(a_i) + a_{n+1} - a_i$.

- $b_{n+1} \notin \{b_1, ..., b_n\}$.

  Assume b_i is the greatest element less than b_{n+1} and b_j is the least element greater than b_{n+1} where $1 \leq i, j \leq n$. Since $\mathbb{R} + \mathbb{Q}$ is dense, there is $s_n(b_i) < d_{n+1} < s_n(b_j)$. c_{n+1} is chosen in $\mathbb{Z}$ arbitarily. The case b_{n+1} is bigger or smaller than any b_i for $1 \leq i \leq n$ is similar but we need to use the fact that $\mathbb{R} + \mathbb{Q}$ doesn't have end points.

Arguments above show the forth direction, and back direction is similar for the sake that $\mathbb{R}$ is also a dense linear order without end points.

$\square$

3.1. *Denote $L_1 = \mathbb{Z}+\mathbb{Z}$, $L_2 = \mathbb{Z}$, for any $a \in L_1$, $b \in L_2$, $\{(a,b)\} \in S_\omega(L_1, L_2)$. This is because for any $p \in \mathbb{N}$, $\{(a,b)\} \in S_p(L_1, L_2)$ since L_1 and L_2 are both discreate linear order without end points and we can apply Theorem 1.8 in Poizat. So $L_1 \sim_{\omega+1} L_2$.*

We now show $L_1 \not\sim_{\omega+2} L_2$. If $\emptyset \in S_{\omega+2}(L_1, L_2)$, let $(a,0),(b,1) \in L_1$, then there are m, n s.t. $\{((a,0),m),((b,1),n)\} \in S_\omega(L_1, L_2)$. We choose $p = n-m > 0$, then we have $d_{L_1}((a,0),(b,1)) = \infty > 2^p$ but $d_{L_2}(m,n) = p < 2^p$, so $\{((a,0),m),((b,1),n)\} \notin S_p(L_1, L_2)$ by Theorem 1.8 in Poizat, a contradiction!

$\square$

3.2. *Let $L_1 = \mathbb{Z} \times 2^n$ and $L_2 = \mathbb{Z} \times 2^{n+1}$.*

Claim 1. *L_1 and L_2 are $(\omega+n)$-isomorphic but not $(\omega+n+1)$-isomorphic.*

Denote $[n] = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. The key claim is that:

Claim 2. *$[2^n]$ and $[2^{n+1}]$ are n-isomorphic but not $(n+1)$-isomorphic.*

Proof of Claim 2. *The direct proof is similar to the proof of Theorem 1.8 in Poizat. We can also use that theorem by choosing the discrete linear order without end-points to be $\mathbb{Z}$ and $a_1 = b_1 = 0$,$a_2 = 2^n$, $b_2 = 2^{n+1}$. Then that s is an n-isomorphism gives $[2^n]$ and $[2^{n+1}]$ are n-isomorphic. Proof of no $(n+1)$-isomorphism is still by induction on n the same as part of the proof of Theorem 1.8 in Poizat.* $\square_{\textit{Claim 2.}}$

Proof of Claim 1. *Suppose $s_m = \{(a_1,a_1),...,(a_m,a_m)\}$, $t = \{(b_1,d_1),...,(b_m, d_m)\}$ and t_m is an $n-m$-isomorphism between $[2^n]$ and $[2^{n+1}]$, then $s_m \times t_m = \{((a_1,b_1),(a_1,d_1)),...,((a_m,b_m),(a_m,d_m))\}$ is an ω-isomorphism because for any $p \in \mathbb{N}$, we can apply Theorem 1.8 in Poizat. By induction on $m' = n-m$, we prove $s_m \times t_m$ is an $(\omega+m')$-isomorphism. The case of $m' = 0$ is trivial. If the case* *of* $m' - 1 \geq 0$ *is true, for any* $(a_{m+1}, b_{m+1}) \in L_1$*, since* t_m *is an* m'*-isomorphism between* $[2^n]$ *and* $[2^{n+1}]$*, we have* d_{m+1} *to make* $t_{m+1} = t_m \cup \{(b_{m+1}, d_{m+1})\}$ *an* $m' - 1$*-isomorphism between* $[2^n]$ *and* $[2^{n+1}]$*. Let* $s_{m+1} = s_m \cup \{(a_{m+1}, a_{m+1})\}$*, by induction hypothesis,* $s_{m+1} \times t_{m+1}$ *is an* $(\omega + m' - 1)$*-isomorphism between* L_1 *and* L_2*. The back direction is similar and thus* $L_1 \sim_{\omega+n} L_2$*.*

Suppose $b_1 \leq b_2 \leq ... \leq b_m$*,* $d_1 \leq d_2 \leq ... \leq d_m$*, for any* $1 \leq i \leq m$*,* $b_i \in [2^n], d_i \in [2^{n+1}]$*. If there is* $1 \leq j < m$ *s.t.* $b_j = b_{j+1}$*,* $d_j < d_{j+1}$*, then no matter the choices of* a_i *and* c_i*,* $\{((a_1, b_1), (c_1, d_1)), ..., ((a_m, b_m), (c_m, d_m))\}$ *can't be an* ω*-isomorphism because it fails to be a* p*-isomorphism when* $p = |a_{j+1} - a_j| > 0$ *by Theorem 1.8 in Poizat. So if* $L_1 \sim_{\omega+n+1} L_2$*, it would tell* $[2^n] \sim_{n+1} [2^{n+1}]$*, a contradiction!* $\square_{Claim\ 1.}$

$\square$