Topological spaces II

Introduction to Model Theory
(Third hour)

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Section 1

Subspaces

The case of metric spaces

Let (M, d) be a metric space. Let (M_0, d_0) be a subspace, meaning

- $M_0 \subseteq M$.
- d_0 is the restriction of d to M_0.

Fact

Suppose $X \subseteq M_0$.

- *U is open in (M_0, d_0) iff there is an open set U' in (M, d) with $U = U' \cap M_0$.*
- *C is closed in (M_0, d_0) iff there is a closed set C' in (M, d) with $C = C' \cap M_0$.*

Subspaces

Let $(S, \mathcal{T})$ and $(S_0, \mathcal{T}_0)$ be topological spaces.

Definition

$(S_0, \mathcal{T}_0)$ is a *subspace* of $(S, \mathcal{T})$ if $S_0 \subseteq S$ and the following equivalent conditions hold:

- U is open in $(S_0, \mathcal{T}_0)$ iff there is an open set U' in $(S, \mathcal{T})$ with $U = U' \cap S_0$.
- C is closed in $(S_0, \mathcal{T}_0)$ iff there is a closed set C' in $(S, \mathcal{T})$ with $C = C' \cap S_0$.

The subspace topology

Fact

Suppose $(S, \mathcal{T})$ is a topological space and $S_0 \subseteq S$. There is a unique topology $\mathcal{T}_0$ on S_0 making $(S_0, \mathcal{T}_0)$ be a subspace of $(S, \mathcal{T})$.

$\mathcal{T}_0$ is called the *subspace topology* on S_0. It is simply

$$\mathcal{T}_0 := \{U \cap S_0 : U \in \mathcal{T}\}.$$

Corollary

Subspaces of $(S, \mathcal{T})$ correspond bijectively with subsets of S.

Open and closed subspaces

Definition

A subspace $(S_0, \mathcal{T}_0) \subseteq (S, \mathcal{T})$ is *open* (resp. *closed*) if S_0 is an open (resp. closed) subset of S.

Fact

Let S_0 be an open subspace of S and suppose $X \subseteq S_0$. Then X is open in S_0 iff X is open in S.

Fact

Let S_0 be a closed subspace of S and suppose $X \subseteq S_0$. Then X is closed in S_0 iff X is closed in S.

The discrete topology

Let S be a set. The *discrete topology* is the topology where all subsets of S are open.

Fact

A topological space $(S, \mathcal{T})$ is discrete if and only if every singleton $\{p\}$ is open.

Discrete sets

Let S be a topological space:

Definition

A set $X \subseteq S$ is *discrete* if the subspace topology on X is discrete.

Definition

A point $p \in X$ is *isolated* if there is a neighborhood $N \ni p$ with $N \cap X = \{p\}$.

Fact

X is discrete if and only if every point is isolated.

Section 2

Product spaces

The product of two topological spaces

Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be two topological spaces.

Definition

The *product topological space* is $(S_1 \times S_2, \mathcal{T}_\times)$, where the *product topology* $\mathcal{T}_\times$ has a basis $\{U_1 \times U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$.

Example

The product topology on $\mathbb{R} \times \mathbb{R}$ has basic open sets $(a, b) \times (c, d)$ with $a < b$ and $c < d$.

This is the usual topology on $\mathbb{R}^2$.

The product topology on metric spaces

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. We can define several metrics on $M_1 \times M_2$:

$$d((x_1, x_2); (y_1, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$
$$d'((x_1, x_2); (y_1, y_2)) := d(x_1, y_1) + d(x_2, y_2)$$
$$d''((x_1, x_2); (y_1, y_2)) := \max(d(x_1, y_1), d(x_2, y_2))$$

Fact

Each of these is a metric on $M_1 \times M_2$. They all define the same topology, which is the product topology on $M_1 \times M_2$.

The product topology and limits

Fact

Let S_1, S_2 be two topological spaces. Let $a_1, a_2, \ldots$ be a sequence in S_1 and $b_1, b_2, \ldots$ be a sequence in S_2. Then $\lim_{i\to\infty}(a_i, b_i) = (c, d)$ if and only if

$$\lim_{i\to\infty} a_i = c$$
$$\lim_{i\to\infty} b_i = d.$$

This almost characterizes the product topology.

The product topology and continuity

Let S_1 and S_2 be topological spaces and $S_1 \times S_2$ be the product.

1. The projection maps $S_1 \times S_2 \to S_1$ and $S_1 \times S_2 \to S_2$ are continuous.
2. Let S_0 be a topological space and let $f_i : S \to S_i$ be a function for $i = 1, 2$. Let $(f_1, f_2) : S_0 \to S_1 \times S_2$ be the function $(f_1, f_2)(x) = (f_1(x), f_2(x))$. Then

$$(f_1, f_2) \text{ is continuous } \iff f_1 \text{ and } f_2 \text{ are continuous.}$$

3. Let $f : S_1 \times S_2 \to S_0$ be a function. Then f is continuous at $(p, q) \in S_1 \times S_2$ iff the following holds: for any neighborhood $E \ni f(p, q)$, there are neighborhoods $U_1 \ni p$ and $U_2 \ni q$ such that if $p' \in U_1$ and $q' \in U_2$, then $f(p', q') \in E$.

Section 3

Connectedness

Connectedness for topological spaces

Definition

A topological space S is *disconnnected* if there exists a clopen set X other than $\varnothing$ and S. Otherwise, S is *connected*.

Equivalently:

Definition

A topological space S is *disconnected* if there is a non-constant continuous function $f : S \to \{0, 1\}$ where $\{0, 1\}$ has the discrete topology. S is *connected* if every continuous function $f : S \to \{0, 1\}$ is constant.

Connectedness for sets

Let S be a topological space.

Definition

A subset $X \subseteq S$ is *connected* (resp. *disconected*) if the subspace X is connected (resp. disconnected).

Fact

Let X be open. Then X is disconnected if and only if $X = X_1 \cup X_2$ where X_1, X_2 are non-empty open sets and $X_1 \cap X_2 = \varnothing$.

Fact

Let X be closed. Then X is disconnected if and only if $X = X_1 \cup X_2$ where X_1, X_2 are non-empty closed sets and $X_1 \cap X_2 = \varnothing$.

Connectedness and continuity

Fact

Let $f : S \to S'$ be continuous. If $X \subseteq S$ is connected, then $f(X) \subseteq S'$ is connected.

Connected components

Definition

A *connected component* of X is a maximal connected subset of X.

Fact

The connected components form a partition of X. There is an equivalence relation $\sim$ on X such that $a \sim b$ if and only if a and b are in the same connected component.

X is connected if there is just one connected component.

Definition

X is *totally disconnected* if every connected component is a single point.

Warning (Cantor's Leaky Tent)

There is a connected set $X \subseteq \mathbb{R}^2$ and a point $p \in X$ such that $X \setminus \{p\}$ is totally disconnected.

Path-connectedness

Let X be a set.

Definition

For $p, q \in X$, a "path" in X is a continuous function $f : [0,1] \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition

X is *path-connected* if for any $p, q \in X$, there is a path from p to q.

Fact

If X is path-connected, then X is connected.

Warning (Topologist's sine curve)

There is a subset $X \subseteq \mathbb{R}^2$ that is connected but not path connected.

Section 4

Compactness

Compactness

Let $(S, \mathcal{T})$ be a topological space.

Definition

An *open cover* of S is a set $\mathcal{C} \subseteq \mathcal{T}$ with $\bigcup \mathcal{C} = S$.
A *subcover* of $\mathcal{C}$ is another cover $\mathcal{C}_0$ with $\mathcal{C}_0 \subseteq \mathcal{C}$.

Definition

S is *compact* if every cover has a finite subcover.
$X \subseteq S$ is *compact* if it is compact as a subspace.

Remark

Let $\mathcal{B}$ be a basis for $\mathcal{T}$. In both definitions, it suffices to only consider covers $\mathcal{C} \subseteq \mathcal{B}$.

Compactness via FIP

Definition

A family of sets $\mathcal{F}$ has the *finite intersection property* (FIP) if for any $X_1, \dots, X_n \in \mathcal{F}$, the intersection $\bigcap_{i=1}^{n} \mathcal{F}$ is non-empty.

Fact

A topological space S is compact if and only if the following holds: let $\mathcal{F}$ be a family of closed sets with FIP. Then $\bigcap \mathcal{F} \neq \varnothing$.

Compactness: important facts

Fact

Let $f : S_1 \to S_2$ be continuous. If $X \subseteq S_1$ is compact, then $f(X) \subseteq S_2$ is compact.

Fact

If S is Hausdorff and $X \subseteq S$ is compact, then X is closed.

Fact

Finite sets are compact.

Fact

A finite union of compact sets is compact.

Fact

A product of compact topological spaces is compact, even infinitely many.

Compactness and cluster points

Definition

b is a *cluster point* of $a_1, a_2, \ldots$ if for any neighborhood $N \ni b$, there are infinitely many i with $a_i \in N$.

Warning

In metric spaces, b is a cluster point of $a_1, a_2, \ldots$ iff some subsequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ converges to b. This does not hold in general topological spaces.

Fact

In a compact topological space, any sequence has a cluster point.

Warning

In metric spaces, this characterizes compactness. This does not hold in general topological spaces.

Compactness and quasi-compactness

Warning

English	French
Compact	Quasi-compact
Compact and Hausdorff	Compact

Algebraic geometry often follows the French convention.

Completeness

Fact

A metric space (M, d) is compact iff M is complete and M is totally bounded.

Neither "complete" nor "totally bounded" makes sense in topological spaces:

- $\mathbb{R}$ is homeomorphic to $(0, 1) \subseteq \mathbb{R}$.
- $\mathbb{R}$ is complete, but $(0, 1)$ is not.
- $(0, 1)$ is totally bounded, but $\mathbb{R}$ is not.

Section 5

Metrizability

Metrizability

Definition

A topology $\mathcal{T}$ on a set S is *metrizable* if there is a metric d on S inducing $\mathcal{T}$.

Not all topologies are metrizable!

- For example, metrizable topologies are always Hausdorff, and non-Hausdorff topologies exist.
- The Sorgenfrey line is not metrizable. The cofinite and trivial topologies are usually not metrizable.
- The order topology on ω_1 is not metrizable.

The discrete topology

Theorem

The discrete topology on a set S is metrizable.

Proof.

Use the "discrete metric"

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

□

Three important properties

Let $(S, \mathcal{T})$ be a topological space.

Definition

S is *second countable* if there is a countable basis $\mathcal{B} \subseteq \mathcal{T}$.

Definition

S is *separable* if there is a countable set $X \subseteq S$ with $\overline{X} = S$ (i.e., X is dense).

Definition

S is *Lindelöf* if every cover has a countable subcover.

Three important properties

Fact

In metric spaces,

$$(\textit{compact}) \Longrightarrow (\textit{Lindelöf}) \Longleftrightarrow (\textit{separable}) \Longleftrightarrow (\textit{second countable}).$$

Fact

In topological spaces,

$$(\textit{compact}) \Rightarrow (\textit{Lindelöf}) \Leftarrow (\textit{second countable}) \Rightarrow (\textit{separable}).$$

and no other logical relations hold between these notions.

Metrization theorems

Fact

Let $(S, \mathcal{T})$ be a compact topological space. Then S is metrizable if and only if S is Hausdorff and second-countable.

There are other more complicated theorems, like Urysohn's metrization theorem and the Bing-Nagata-Smirnov metrization theorem.

Section 6

Polish spaces

Polish spaces

Definition

A topological space $(S, \mathcal{T})$ is *Polish* if

- It is separable.
- It is metrizable, by a complete metric d on S.

Example

$\mathbb{R}$ is a Polish space.

Polish spaces are important in set theory and computability theory, especially in *descriptive set theory*.

Polish spaces

Fact

Let S be a Polish space.

- *Any closed subspace of S is Polish.*
- *Any open subspace of S is Polish.*
- *If $X \subseteq S$ is countable, then $S \setminus X$ is a Polish subspace.*

Example

The Cantor set is a Polish space.

Example

$\mathbb{R} \setminus \mathbb{Q}$ is a Polish space.

A typical theorem

Fact

Let S be a Polish space. Then $|S| \leq \aleph_0$ or $|S| = 2^{\aleph_0}$. More generally, if X is an open or closed subset of S, then $|X| \leq \aleph_0$ or $|X| = 2^{\aleph_0}$.

Definition

The collection of *Borel sets* is the smallest $\mathcal{B} \subseteq Pow(S)$ containing the open and closed sets, closed under complements and countable unions and countable intersections.

Fact

Let S be Polish and $X \subseteq S$ be Borel. Then $|X| \leq \aleph_0$ or $|X| = 2^{\aleph_0}$.

Intuition: there are no easy counterexamples to the continuum hypothesis.

Section 7

Beyond point-set topology

An overview

There are many subjects within topology:

1. Point-set topology
2. Algebraic topology
3. Differential topology
4. Knot theory
5. Low-dimensional topology
6. Symplectic topology
7. ...

We have only been talking about point-set topology.
Technically, it is the foundation for other branches of topology.
But it is nothing like the rest of topology.

Tameness

- "Wild" topological spaces like $\mathbb{Q}$ or the Cantor set are important in point-set topology
- But most topologists study much "tamer" sets, like $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
  - Low-dimensional topologists focus on manifolds.
  - Algebraic topologists focus on CW complexes or simplicial complexes.
  - ...

Remark

In model theory, *o-minimality* is a way to avoid "wild" topological spaces and ensure automatic tameness.

Manifolds

Definition

A topological space M is an *n-dimensional manifold* if the following conditions hold:

- M is Hausdorff.
- For every point $p \in M$, there is an open neighborhood $U \ni p$ homeomorphic to a ball in $\mathbb{R}^n$ (or equivalently, homeomorphic to $\mathbb{R}^n$).
- *M is second countable or paracompact or something.* (Conventions vary.)

Example

The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a 1-dimensional manifold.

Manifolds

Fact

2-dimensional compact connected manifolds are classified up to homeomorphism.

For more information, see one of the following:

- *A Guide to the Classification Theorem for Compact Surfaces* by Jean Gallier and Dianna Xu
  - `https://www.cis.upenn.edu/~jean/surfclass-n.pdf`
- *Fantastic Topological Surfaces and How to Classify Them* by Khorben Boyer
  - `https://digitalcommons.wou.edu/cgi/viewcontent.cgi?article=1102&context=aes`
- *An Introduction to Topology: The Classification Theorem for Surfaces* by E. C. Zeeman
  - `https://www.maths.ed.ac.uk/~v1ranick/surgery/zeeman.pdf`.

Smooth manifolds

- So far we have discussed *topological manifolds*.
- A *smooth manifold* is a topological manifold with additional information allowing one to talk about derivatives of functions.
- Smooth manifolds are much easier to work with than topological manifolds.
- Smooth manifolds are studied in *differential topology*.