Week 7

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Exercise 0.0.1. Suppose $\sqrt{a}$ and $\sqrt{b}$ both exist (in $\mathbb{M}$). Show that $\sqrt{a}+\sqrt{b} \in \mathrm{acl}(\{a,b\})$

Proof. Let $\varphi(x)$ be $\exists y, z(y \cdot y = a \wedge z \cdot z = b \wedge x = y + z)$. Then since $\sqrt{a}$ and $\sqrt{b}$ are unique, $|\varphi(\mathbb{M})| = 1$ and $\mathbb{M} \vDash \varphi(\sqrt{a}+\sqrt{b})$. Therefore $\sqrt{a}+\sqrt{b} \in \mathrm{acl}(\{a,b\})$ □

Exercise 0.0.2. Suppose $c \neq 0$. Let $D = \{(x,y,z) \in \mathbb{M}^3 : ax + by + cz = 0\}$. Show that $(a/c, b/c)$ is a "code" for D

Proof. If $\sigma \in \mathrm{Aut}(\mathbb{M})$ fixes a/c and b/c, then for any $(x,y,z) \in D$, then

$$\begin{aligned}(a/c)\sigma(x) + (b/c)\sigma(x) + \sigma(z) &= \sigma(ax/c) + \sigma(by/c) + \sigma(z) \\ &= \sigma(ax/c + by/c + z) = \sigma(0) = 0\end{aligned}$$

Therefore $a\sigma(x) + b\sigma(y) + c\sigma(z) = 0 \cdot c = 0$ and $(\sigma(x), \sigma(y), \sigma(z)) \in D$

If there is $\sigma \in \mathrm{Aut}(\mathbb{M})$ that fixes D and not fixes b/c, then since $b/c - \sigma(b)/\sigma(c) \neq 0$, $a\sigma(x)+b\sigma(y)+c\sigma(z) = 0$ and $\sigma(a)\sigma(x)+\sigma(b)\sigma(y)+\sigma(c)\sigma(z) = 0$, we have

$$\begin{aligned}\sigma(y) &= \frac{a\sigma(c) - c\sigma(a)}{b\sigma(c) - c\sigma(b)}\sigma(x) \\ y &= \frac{c\sigma^{-1}(a) - a\sigma^{-1}(c)}{c\sigma^{-1}(b) - b\sigma^{-1}(c)}x\end{aligned}$$

let $k = \frac{c\sigma^{-1}(a) - a\sigma^{-1}(c)}{c\sigma^{-1}(b) - b\sigma^{-1}(c)}$, then

$$z = -\frac{a+bk}{c}x$$

But $D = \{(x, kx, -\frac{a+bk}{c}x) : x \in \mathbb{M}\} \subsetneq \{(x,y,z) \in \mathbb{M}^3 : ax + by + c = 0\}$, therefore we have a contradiction. Thus for any $\sigma \in \mathrm{Aut}(\mathbb{M})$ fixing D, σ fixes a/c and b/c □

Exercise 0.0.3. Let $D = \mathbb{M}^3 \setminus \{(0,0,0)\}$. Let E be the equivalence relation on D where $(a_1, a_2, a_3)E(b_1, b_2, b_3)$ iff the two vectors are parallel. Find a definable function $f : D \to \mathbb{M}^n$ s.t.

$$f(a_1, a_2, a_3) = f(b_1, b_2, b_3) \Leftrightarrow (a_1, a_2, a_3)E(b_1, b_2, b_3)$$

Proof. Let $f : D \to \mathbb{M}^3$ be

$$f(a_1, a_2, a_3) = \begin{cases} (1, a_2/a_1, a_3/a_1) & a_1 \neq 0 \\ (0, 1, a_3/a_2) & a_1 = 0 \wedge a_2 \neq 0 \\ (0,0,1) & a_1 = 0 \wedge a_2 = 0 \wedge a_3 \neq 0 \\ (0,0,0) & \text{otherwise} \end{cases}$$

Then if $a_1 \neq 0$, $b_1 \neq 0$ for otherwise $\lambda = 0$ and $b_1 = b_2 = b_3 = 0$.

$$\begin{aligned} (a_1, a_2, a_3)E(b_1, b_2, b_3) &\Leftrightarrow (1, a_2/a_1, a_3/a_1)E(1, b_2/b_1, b_3/b_1) \\ &\Leftrightarrow (1, a_2/a_1, a_3/a_1) = (1, b_2/b_1, b_3/b_1) \\ &\Leftrightarrow f(a_1, a_2, a_3) = f(b_1, b_2, b_3) \end{aligned}$$

Other cases are similar $\square$

Exercise 0.0.4. Suppose $\leq$ is a definable linear order on $\mathbb{M}$. Show that $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq \mathbb{M}$

Proof. If $a \in \mathrm{acl}(A)$, then there is a $L(A)$-formula $\varphi(x)$ s.t. $\mathbb{M} \vDash \varphi(a)$ and $|\varphi(\mathbb{M})|$ is finite. Suppose $\varphi(\mathbb{M}) = \{a_1, \dots, a_i, a, a_{i+1}, \dots, a_m\}$ with $a_1 < a_2 < \dots < a_i < a < a_{i+1} < \dots < a_m$ for some $i, m \in \mathbb{N}$. Then if there is $b \in A$ with $b < a$ and a is the nth elements greater than b in $\varphi(\mathbb{M})$, then we can take $\psi(x)$ as

$$\begin{aligned} \exists x_1, \dots, x_{n-1} \Big(\varphi(x) \wedge \bigwedge_{i=1}^{n-1} \varphi(x_i) \\ &\wedge (a < x_1 < \dots < x_{n-1} < x) \\ &\wedge \bigwedge_{1 \leq i < j \leq n-1} x_i \neq x_j \\ &\wedge \neg \exists y (a < y < x \wedge \varphi(y) \wedge \bigwedge_{i=1}^{n-1} x \neq x_i) \Big) \end{aligned}$$

and $\psi(\mathbb{M}) = \{a\}$.

If there is no such $b \in A$, then there is some $c \in A$ and a is the nth elements lesser than b in $\varphi(\mathbb{M})$, then we can take $\psi(x)$ as

$$\begin{aligned}
\exists x_1, \dots, x_{n-1} \Big(\varphi(x) \wedge \bigwedge_{i=1}^{n-1} \varphi(x_i) \\
&\wedge (x < x_1 < \dots < x_{n-1} < c) \\
&\wedge \bigwedge_{1 \leq i < j \leq n-1} x_i \neq x_j \\
&\wedge \neg \exists y (x < y < c \wedge \varphi(y) \wedge \bigwedge_{i=1}^{n-1} x \neq x_i) \Big)
\end{aligned}$$

and $\psi(\mathbb{M}) = \{a\}$ $\square$