Note 03

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1 First-order languages

Definition 1.1. The alphabet of a first-order language consists of the following groups of symbols:

- Parenthesis: (and)
- Connectives: [$\neg$,negation,not], [$\wedge$, conjunction,and], and [$\vee$, disjunction,or]];
- Quantifiers: [$\forall$,for all] and [$\exists$, there exists];
- A denumerably infinite list of variables: $v_0, v_1, ..., v_n, ...$;
- $=$;
- A set of constant symbols $\mathcal{C}$;
- A set of function symbols $\mathcal{F}$, and positive integers n_f for each $f \in \mathcal{F}$, which is referred to as the arity of the function;
- A set of relation symbols $\mathcal{R}$, and positive integers n_R for each $R \in \mathcal{R}$, which is referred to as the arity of the relation.

In this chapter, we only consider the language has a unique m-ary relation r.

Definition 1.2. We define the sets $F_0, ..., F_n$ by induction on n as follows:

- F_0, called the set of atomic formulas or formulas of complexity 0, consists of all the words of the form $x_1 = x_2$ and $r(x_1, ..., x_m)$, where $x_1, ..., x_m$ are variables, not necessarily distinct.
- F_{n+1}, called the set of formulas of complexity $n+1$, consists of all words of the form

$$\neg(f),\ (f) \wedge (g),\ (f) \vee (g),\ (\exists x)(f),\ \text{or}\ (\forall x)(f),$$

where x is a variable and $f, g \in F_0 \cup ... \cup F_n$.

The union of all the F_n is called the set F of formulas.

Definition 1.3. Let f be a formula, then the set $S(f)$ of subformulas of f is defined by induction on the complexity of f:

- If f is atomic, then $S(f) = \{f\}$;
- If f is $\neg(g)$, or $(\exists x)(g)$, or $(\forall x)(g)$, then $S(f) = S(g) \cup \{f\}$;
- If f is $(g) \wedge (h)$, or $(g) \vee (h)$, then $S(f) = S(g) \cup S(h) \cup \{f\}$.

Definition 1.4. Let f be a formula, then the quantifier rank of f, denoted by $QR(f)$, is defined by induction on the complexity of f:

- If f is atomic, then $QR(f) = 0$;
- If f is $\neg(g)$, then $QR(f) = QR(g)$;
- If f is $(g) \wedge (h)$, or $(g) \vee (h)$, then $QR(f) = \max\{QF(g),\ QF(h)\}$;
- If f is $(\exists x)(g)$, or $(\forall x)(g)$, then $QR(f) = QR(g) + 1$.

The formulas of quantifier rank 0 are call quantifier-free formulas, which are exactly the Boolean combinations of the atomic formulas.

Definition 1.5. Let f be a formula, then we define the set $FV(f)$ of the free variables of f, as follows:

- If f is atomic, then $FV(f) =$ all variables occurring in f;
- If f is $\neg(g)$, then $FV(f) = FV(g)$;
- If f is $(g) \wedge (h)$, or $(g) \vee (h)$, then $FV(f) = FV(g) \cup QF(h)$;
- If f is $(\exists x)(g)$, or $(\forall x)(g)$, then $FV(f) = FV(g) \backslash \{x\}$.

If $FV(f) = \emptyset$, then we call f a closed formula or sentence.

Definition 1.6.

- When we write a formula $f(\bar{x})$, where $\bar{x}$ is an n-tuple of variables $(x_1, ..., x_n)$, we understand that all free variables of f are contained among $x_1, ..., x_n$. Namely,
$$\{x_1, ..., x_n\} \subseteq FV(f)$$
- Let (M, R) be an m-ary relation, and $\bar{a} = (a_1, ..., a_n) \in M^n$;
- We will define, by induction on the complexity of f, what it means for R to satisfy $f(\bar{a})$, or equivalently for $f(\bar{a})$ to be true for R. We write
$$(M, R) \models f(\bar{a})$$
to mean (M, R) satisfies $f(\bar{a})$, where $f(a)$ is not a formula in our language, but rather what we get from the formula $f(\bar{x})$ by replacing free occurrences of $x_1, ..., x_n$ by $a_1, ..., a_n$, respectively.

- If f is of the form $x = y$, then $(M, R) \models a = b$ iff a and b are identical;
- If f is of the form $r(x_1, ..., x_n)$, then $(M, R) \models r(a_1, ..., a_n)$ iff $(a_1, ..., a_n) \in R$;
- $(M, R) \models \neg(f)(\bar{a})$ iff (M, R) does not satisfy $f(\bar{a})$;
- $(M, R) \models (f) \vee (g)(\bar{a})$ iff (M, R) satisfies $f(\bar{a})$ or (M, R) satisfies $g(\bar{a})$;
- $(M, R) \models (f) \wedge (g)(\bar{a})$ iff (M, R) satisfies $f(\bar{a})$ and (M, R) satisfies $g(\bar{a})$;
- $(M, R) \models (\exists x)(f)(\bar{a}, x)$ iff there exists $b \in M$ such that R satisfies $f(\bar{a}, b)$;
- $(M, R) \models (\forall x)(f)(\bar{a}, x)$ iff for all $b \in M$, R satisfies $f(\bar{a}, b)$.

We will assume that the universe of every relation is not empty.

Definition 1.7. Let $f(\bar{x})$ and $g(\bar{x})$, we say that f and g are equivalent if for any n-tuple $\bar{a}$ and any relation (M, R),

$$(M, R) \models f(\bar{a}) \iff (M, R) \models g(\bar{x}).$$

$(\forall x)(f)$ is equivalent to $\neg(\exists x)\neg(f)$

Definition 1.8. A formula is said to be in *prenex form* if all its quantifiers occur at the beginning.

Lemma 1.9. *Every formula has an equivalent prenex form.*

2 Connections to Back-and-Forth Technique

Theorem 2.1. *Let (M, R) and (N, S) be m-ary relations, let $\bar{a} \in M^n$ and $\bar{b} \in N^n$. Then $\bar{a}$ and $\bar{b}$ are p-equivalent iff*

$$(M, R) \models f(\bar{a}) \iff (N, R) \models f(\bar{b})$$

for any formula $f(\bar{x})$ with quantifier rank at most p.

Proof. $\Rightarrow$: By induction on p.

- If $\bar{a} \sim_0 \bar{b}$;
- Then, by definition, they satisfy the same atomic formulas;
- Therefore, they satisfy the same quantifier-free formulas.
- Suppose that $a \sim_{p+1} b$;
- The formula $f := (\exists y)g(\bar{x}, y)$ has quantifier rank at most $p + 1$;

- So $g(\bar{x}, y)$ is a formula of quantifier rank at most p;
- $(M, R) \models f(\bar{a})$ iff there is $c \in M$ such that $(M, R) \models g(\bar{a}, c)$;
- there is $d \in N$ such that $\bar{c} \sim_p \bar{b}d$;
- by induction hypothesis, $(N, S) \models g(\bar{b}, d)$, and thus $\models (\exists y) g(\bar{b}, y)$;
- Similarly, $(N, S) \models (\exists y) g(\bar{b}, y) \implies (M, R) \models (\exists y) g(\bar{a}, y)$.

To prove the converse, we need the following lemma: $\square$

Lemma 2.2. *If the arity m of a relation, and the integers n and p, are fixed, there is only finite number $C(n, p)$ of p-equivalence classes of n-typles.*

Proof. Induction on p. If $p = 0$, then

- Consider a set of symbols $X = \{x_1, ..., x_n\}$;
- There are at most finitely many m-ary relations defined on X;
- Also, there are at most finitely many ways to interpret the relation "$=$" on X;
- Let (M, R) and (N, S) be m-ary relations, $\bar{a} \in M^n$ and $\bar{b} \in N^n$;
- Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$;
- Let $R_A = R \cap A^m$, the restriction of R on A;
- Let $S_B = S \cap B^m$, the restriction of S on B;
- If $p = 0$, $\bar{a} \sim_0 \bar{b}$ iff R_A is isomorphic to R_B via $a_i \mapsto b_i$, $i = 1, ..., n$;
- So there are at most finitely many 0-equivalence classes of n-tuples;

From p to $p + 1$:

- by induction hypothesis,
  - there exists relations $\{(M_k, R_k) |\ k \leq C(n + 1, p)\}$, and
  - $\{\bar{d}_k \in M_k^{n+1} |\ k \leq C(n + 1, p)\}$
- such that each $n + 1$-tuple is p-equivalent to some $\bar{d}_k$;
- Now consider an arbitrary relation (M, R) and an n-tuple $\bar{a}$;
- We define $[\bar{a}] = \{k |\ \exists c \in M(\bar{a}c \sim_p \bar{d}_k)\}$
- For any relation (N, S) and $\bar{b} \in N^n$;

- It is easy to see that $\bar{a} \sim_{p+1} \bar{b} \iff [\bar{a}] = [\bar{b}]$;
- So $C(n, p+1)$ is bounded by $2^{C(n+1,p)}$.

$\square$

Proof of Theorem 2.1, Part 2: We now show that if $\bar{a}$ and $\bar{b}$ satisfy the same formulas of QR at most p, then $\bar{a} \sim_p \bar{a}$.

We Claim that for each p-equivalence class C, there is a formula f_C of QR p such that the tuples in C are exactly those satisfy f_C.

- Induction on p.
- If $p = 0$:
- Given an n-tuple $\bar{a}$;
- There are only finitely many atomic formulas with variables $x_1, ..., x_n$;
- Let f_C be the conjunction of those satisfied by $\bar{a}$ and negation of those not satisfied by $\bar{a}$.
- Then f_C characterizes the 0-equivalence class of $\bar{a}$.

From p to $p+1$:

- Let $\bar{a}$ be an n-tuple of (M, R);
- Let $f_1(\bar{x}, y), ..., f_k(\bar{x}, y)$ characterize the p-equivalence classes C_1,...,C_k, on $n+1$-tuples, respectively;
- Let $\langle \bar{a} \rangle = \{i \leq k |\ (M, R) \models (\exists y) f_i(\bar{a}, y)\}$;
- it is easy to see that $\langle \bar{a} \rangle = [a]$ if we list $C_1, ..., C_k$ as in the previous lemma.
- Let $f_C(\bar{x}) = \bigwedge_{i \in \langle \bar{a} \rangle} (\exists y) f_i(\bar{x}, y) \wedge \bigwedge_{i \notin \langle \bar{a} \rangle} \neg (\exists y) f_i(\bar{x}, y)$;
- It is easy to see that $\bar{b} \sim_{p+1} \bar{a}$ iff $[\bar{a}] = [\bar{b}]$ iff $\langle \bar{a} \rangle = \langle \bar{b} \rangle$ iff $f_C(\bar{b})$ holds.

$\square$