Homework: stable theories

Introduction to Model Theory

Due March 10, 2022

Please justify your answers.

1. $(\mathbb{C}, +, \cdot)$ is an algebraically closed field. Show that the algebraic set $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$ is reducible, i.e., not a variety. *Hint:* first consider the similar set $x^2 - y^2 = 0$.

2. Consider the theory of dense linear orders (DLO). Let $\varphi(x, y)$ be the formula $x < y$. One can show that $\varphi(x, y)$ has the dichotomy property. Show by giving an example that D_3 is consistent.

3. In the structure $M = (\mathbb{R}, +, \cdot, 0, 1, \leq)$, let $\varphi(\bar{x}; \bar{y})$ be the formula $x_1 y_1 + x_2 y_2 = 1$. Thus $\varphi(\mathbb{R}^2; \bar{b})$ is a line for most $\bar{b} \in \mathbb{R}^2$. It turns out that the formula φ does not have the dichotomy property. Find the largest n such that D_n is consistent.

4. Let T be the complete theory of the structure $(\mathbb{Z}, +, -, 0)$. Show that T is not $\aleph_0$-stable. *Warning:* we are not including $\leq$ in the structure. Also, it turns out that $(\mathbb{Z}, +, -, 0)$ is $2^{\aleph_0}$-stable, so there are no formulas with the dichotomy property and all types are definable. *Hint:* show that there are too many 1-types over $(\mathbb{Z}, +)$. One way to do this uses least significant digits of base-10 expansions.