Homework10

Qi'ao Chen
21210160025

December 13, 2021

Exercise 1. Let M be a κ-saturated structure for some $\kappa > \aleph_0$. Let X_i be a definable subset of M^n for $i = 1, 2, \dots$. Suppose that $X_0 \subseteq \bigcup_{i=1}^{\infty} X_i$. Show that there is an n s.t. $X_0 \subseteq \bigcup_{i=0}^{n} X_i$

Proof. Suppose for each X_i , $X_i = \varphi_i(M^n)$.

Suppose there is no such n, then for any $n \in \mathbb{N}$, there is $c_n \in M$ s.t. $\varphi_0(c_n) \wedge \neg \bigvee_{i=1}^{n} \varphi_i(c_n)$. Hence consider

$$\Gamma(x) = \{\varphi_0(x)\} \cup \{\neg\varphi_i(x) \mid i = 1, 2, \dots\}$$

This is finitely satisfiable by our discussion and hence realised by $a \in M$ since M is κ-saturated. Thus $X_0 \nsubseteq \bigcup_{i=1}^{\infty} X_i$, a contradiction □

Exercise 2. Consider the structure $(\mathbb{C}, +, \cdot)$. Let $\mathcal{F}$ be a family of definable subsets of $\mathbb{C}^1$. Suppose $\mathcal{F}$ has the finite intersection property. If $|\mathcal{F}| < |\mathbb{C}|$, show that $\bigcap \mathcal{F} \neq \emptyset$

Proof. Suppose $|\mathcal{F}| = \lambda$, then $\mathcal{F} = \{\varphi_\alpha(\mathbb{C}) \mid \alpha < \lambda\}$.

If all of $|\varphi_\alpha(\mathbb{C})|$ are cofinite, then each $\varphi_\alpha(x) \leftrightarrow \bigwedge_{i=1}^{n_\alpha} x \neq c_{\alpha,i}$ where $|\neg\varphi_\alpha(\mathbb{C})| = n_\alpha$ and $\mathbb{C} \vDash \neg\varphi(c_{\alpha,i})$. But as $\omega \cdot \lambda = \max\{\omega, \lambda\} < |\mathbb{C}|$, there is $c \in \bigcap \mathcal{F}$.

Otherwise, since $\mathcal{F}$ has the finite intersection property, $F = \{\varphi_\alpha(x) \mid \alpha < \lambda\}$ is finitely satisfiable. Add a new constant distinct constant c to $L(\mathbb{C})$ and then

$$\Gamma = \mathrm{Diag}_{\mathrm{el}}(\mathbb{C}) \cup \{\varphi_\alpha(c) \mid \alpha < \lambda\}$$

is satisfiable and let $\mathfrak{M} \vDash \Gamma$. Note that $|\varphi_\alpha(\mathfrak{M})| = |\varphi_\alpha(\mathbb{C})|$ for all $\alpha < \lambda$ by the fact as $|\varphi(\mathbb{C})| = n$ is definable. By the assumption, there is $\alpha < \lambda$ s.t. $|\varphi_\alpha(\mathbb{C})| = n < \omega$, then $\mathfrak{M} \vDash \bigvee_{i=1}^{n} c = c_i$ where $c_i \neq c_j$ for $i \neq j$ and $c_i, c_j \in \varphi_\alpha(\mathbb{C})$. Hence $c \in \mathbb{C}$. □

Exercise 3. Show that $\mathbb{C}$ is $|\mathbb{C}|$-saturated

Proof. For any $A \subseteq \mathbb{C}$ with $|A| < |\mathbb{C}|$ and $p \in S_n(A)$. Then $|p| = \max\{|A|, \aleph_0\} < |\mathbb{C}|$. Let $\mathcal{F} = \{\varphi(\mathbb{C}) \mid \varphi \in p\}$. Then $\mathcal{F}$ has the finite intersection property since p is finitely satisfiable. Then $\bigcap \mathcal{F} \neq \emptyset$ and hence p is realised by $c \in \bigcap \mathcal{F}$. Thus $\mathbb{C}$ is $|\mathbb{C}|$-saturated $\square$

Exercise 4. Show that $\mathbb{R}$ is not $|\mathbb{R}|$-saturated

Proof. Consider

$$\Gamma(x) = \{x > q : q \in \mathbb{Q}\}$$

then $\Gamma(x)$ is finitely satisfiable but it is not realised in $\mathbb{R}$ since there is no such element in $\mathbb{R}$. Thus $\mathbb{R}$ is not $\aleph_1$-saturated $\square$

Exercise 5. Let $f : \mathbb{C} \to \mathbb{C}$ be the complex conjugation map

$$f(x + iy) = x - iy \text{ for } x, y \in \mathbb{R}$$

Show that the structure $(\mathbb{C}, +, \cdot, f)$ is not $|\mathbb{C}|$-saturated

Proof. Let $\varphi(x) := f(x) = x$. Then $\mathbb{R} = \varphi(\mathbb{C})$. Thus consider

$$\Gamma(x) = \{x > q \wedge \varphi(x) : q \in \mathbb{Q}\}$$

This is finitely satisfiable in $\mathbb{C}$ but not realised in $\mathbb{C}$ $\square$

Exercise 6. Let M be an infinite structure. Show that M is not $|M|^+$-saturated.

Proof. Consider

$$\Gamma(x) = \{x \neq a : a \in M\}$$

Since it is finitely satisfiable, we can extend it to $p(x) \in S(M)$, which is obviously not realised in M. Thus M is not $|M|^+$-saturated $\square$