Homework 1

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Exercise 1. Consider the structure $(\mathbb{Z}, +, \cdot, <)$. Show that there is a complete type $p \in S_1(\mathbb{Z})$ containing the formula $n < x$ for each $n \in \mathbb{Z}$

Proof. Let $\Gamma = \{n < x : n \in \mathbb{Z}\}$. Then Γ is finitely satisfiable and hence there is a complete type $q \in S_1(\mathbb{Z})$ s.t. $q(x) \supset \Gamma$ $\square$

Exercise 2. Let $p \in S_1(\mathbb{Z})$ be as in the previous problem, meaning that the formula $n < x$ is in $p(x)$ for all $n \in \mathbb{Z}$. Suppose $M \succeq \mathbb{Z}$ and $q \in S_1(M)$ is an heir of p. Show that $q(x)$ contains the formula $n < x$ for each $n \in M$

Proof. If for some $n \in M$, $\psi(x, n) := n < x \notin q(x)$. Then $\neg\psi(x, n) \in q(x)$ and hence there is $n' \in \mathbb{Z}$ s.t. $\neg\psi(x, n') \in p$, which is impossible $\square$

Exercise 3. Find a first-order formula $\varphi(x, y, z)$ equivalent to $\exists^\infty w(xw^2 + yw + z = 0)$ in the structure $\mathbb{C}$

Proof. Let $\psi(x) := \forall y(y \cdot x = x)$ and let $\varphi(x, y, z) := \psi(x) \wedge \psi(y) \wedge \psi(z)$ $\square$

Exercise 4. Let $M = \mathbb{R} \setminus [0, 2]$ and $N = \mathbb{R} \setminus [0, 1)$. From quantifier elimination in DLO, one can show that $(M, \leq) \preceq (N, \leq) \preceq (\mathbb{R}, \leq)$. It turns out that $\text{tp}(0/N)$ is an heir of $\text{tp}(0/M)$. Show that $\text{tp}(0/N)$ is not a strong heir of $\text{tp}(0/M)$

Proof. Let $p = \text{tp}(0/M)$ and $q = \text{tp}(0/N)$. Let $\varphi(x, y) := x > y$, then $(N, dq) \vDash \forall y(d\varphi(y) \leftrightarrow y < 1)$. But in M, for any $c \in M$, $M \vDash x < 0 \not\leftrightarrow x < c$. Thus $(M, dp) \vDash \neg\exists c(\forall y(d\varphi) \leftrightarrow y < c)$. Hence $\text{tp}(0/N)$ is not a strong heir of $\text{tp}(0/M)$ $\square$