Morley's Theorem

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References: $\underbrace{\S 10.2, 10.4}_{\text{prime models}}$, $\underbrace{18.6}_{\text{Morley's theorem}}$

§1 Prime models in t.t. theories

T complete.

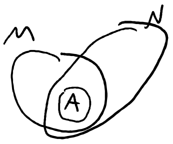
Def 1.1 $M \models T$ is prime if $\forall N \models T$, $\exists f : M \xrightarrow{\preccurlyeq} N$.

$M \preccurlyeq \mathbb{M}$ is prime if $\forall N \preccurlyeq \mathbb{M}$ $\exists \sigma \in Aut(\mathbb{M})$ $\sigma(M) \subseteq N$.

Fact If T is t.t., $\exists$ prime model, unique up to isomorphism.

Warning 1.2 For general T, existence or uniqueness can fail.

Def 1.3 $M \preccurlyeq \mathbb{M}$, M is prime over $A \subseteq M$ if
$\forall N \preccurlyeq \mathbb{M}, N \supseteq A$: $\exists \sigma \in Aut(\mathbb{M}/A)$ $\sigma(M) \subseteq N$.



$f : M \xrightarrow{\preccurlyeq} N$
$f \supseteq id_A$.

Example 1.4 If $acl(A) \preccurlyeq \mathbb{M}$, then $acl(A)$ is prime$/A$.
Let $M = acl(A)$. If $A \subseteq N \preccurlyeq \mathbb{M}$ then $M = acl(A) \subseteq N$.

This holds $\forall A \subseteq M \models ACF$.

Remark 1.5 Suppose $|L| = \aleph_0$, $M \models T \Rightarrow |M| = \infty$.
Then, if M is prime / $A \subseteq M$, then $|M| = |A| + \aleph_0 = \max(|A|, \aleph_0)$.

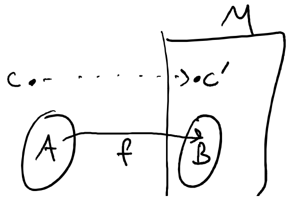
Proof: $|M| \geq |A|$, $|M| \geq \aleph_0$ $\boxed{|M| \geq |A| + \aleph_0.}$
Löwen...-Sk $\Rightarrow \exists N \supseteq A$, $N \preccurlyeq \mathbb{M}$, $|N| \leq |A| + |L| = |A| + \aleph_0$.
M is prime $\Rightarrow \exists M \xrightarrow{\preccurlyeq} N$, $\boxed{|M| \leq |N| \leq |A| + \aleph_0}$

Recall $p \in S_n(B)$ is isolated if $\{p\}$ is clopen
if $\{p\} = [\varphi]$ for some $\varphi \in L(B)$
if $a \models p \iff \mathbb{M} \models \varphi(a)$ $(\varphi(x) \in p(x))$.
φ "isolates" p in this case.

Def 1.6 B is constructible / A if $B = \{b_\alpha : \alpha < \beta\}$ where
$tp(b_\alpha / AB_{<\alpha})$ is isolated $\forall \alpha < \beta$.
$B_{<\alpha} = \{b_\gamma : \gamma < \alpha\}$

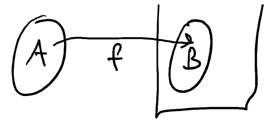
We call $(b_\alpha : \alpha < \beta)$ a construction of B over A.

Lemma 1.8 Suppose $tp(c/A)$ is isolated.
Suppose $f : A \to B \subseteq M \preccurlyeq \mathbb{M}$, f is a p.e.m.
Then $\exists c' \in M$ s.t. $f \cup \{(c, c')\}$ is a p.e.m.



Suppose $f: A \to B \subseteq M \preceq \mathbb{M}$, f is a p.e.m.

Then $\exists c' \in M$ s.t. $f \cup \{(c,c')\}$ is a p.e.m.



Proof $\exists \sigma \in Aut(\mathbb{M})$, $\sigma \supseteq f$. ~~Move~~

Replace A, c with $\sigma(A), \sigma(c)$, WMA $A = B$, $f = id_A$.

Take $\varphi(x) \in tp(c/A)$ with φ isolating $tp(c/A)$.

$\mathbb{M} \models \exists x\, \varphi(x)$ so $M \models \exists x\, \varphi(x)$

Take $c' \in M$ $M \models \varphi(c')$. $c' \models \varphi \Rightarrow c' \models tp(c/A)$

$c' \equiv_A c$ ~~f~~ $id_A \cup \{(c,c')\}$ is a p.e.m. $\square$

Theorem 1.9 If $A \subseteq M \preceq \mathbb{M}$ and M is constr. $/A$.

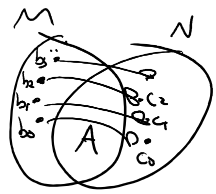
Then M is prime $/A$.

Proof ~~Let~~ If $A \subseteq N \preceq \mathbb{M} \ldots$

Take a construction $(b_\alpha : \alpha < \beta)$ for M $/A$.

By Lemma 1.8 and induction, $\exists\, c_0, c_1, c_2, \ldots \underset{\alpha<\beta}{c_\alpha}, \ldots \in N$

such that $id_A \cup \{(b_0, c_0), (b_1, c_1), \ldots (b_\alpha, c_\alpha)\}$ is a p.e.m



~~Then~~ Let $f = id_A \cup \{(b_\alpha, c_\alpha) : \alpha < \beta\}$ f is a p.e.m.

$f: M \xrightarrow{\preceq} N$ $\square$

Lemma 1.7 If $(B_\alpha : \alpha < \beta)$ is a seq. if

B_α is consr. $/ A \cup \bigcup_{\gamma<\alpha} B_\gamma$ $(\forall \alpha)$

then $\bigcup_{\alpha<\beta} B_\alpha$ is constr. $/A$.

Ex If B_0 is constr. $/A$, B_1 is constr. $/AB_0$ then B_0B_1 is constr. $/A$.

Proof Lem 1.7 Concatenate the constructions. $\square$

Fact 1.10 (T is t.t.) If $\emptyset \neq D \subseteq \mathbb{M}^n$ and D is A-definable.

then $\exists\, \bar{b} \in D$ s.t. $tp(\bar{b}/A)$ is isolated.

(Thm 9.2 in May 7 notes)

Lemma 1.11 (T is t.t.) Given $A \subseteq \mathbb{M}$ $\exists\, B \subseteq \mathbb{M}$ s.t

- B is constr. $/A$
- If $D \subseteq \mathbb{M}$, D is A-def., $D \neq \emptyset$, then $B \cap D \neq \emptyset$.

Proof Let $(D_\alpha : \alpha < \kappa)$ be all nonempty A-def sets.

By Fact 1.10 & recursion, choose $b_\alpha \in D_\alpha$ such that

$tp(b_\alpha / \underbrace{A \cup \{b_\gamma : \gamma < \alpha\}})$ is isolated.

D_α is def. over this.

Let $B = \{b_\alpha : \alpha < \kappa\}$, then B is constr. $/A$, $(b_\alpha : \alpha < \kappa)$ is a construction.

$\square$

Remark: In Lemma 1.11, $B \supseteq A$ because if $a \in A$, $\{a\}$ is A-definable, $\{a\} \cap B \neq \emptyset$, $a \in B$.

$A \subseteq B$.

Thm 1.12 (T is t.t.) If $A \subseteq \mathbb{M}$ then $\exists\, A \subseteq M \preceq \mathbb{M}$, M is constructible $/A$, M is prime $/A$.

Proof By Lem 1.11, $\exists$ $B_0, B_1, B_2, \dots$ (length ω) such that B_i is constr. $/ A B_0 B_1 \dots B_{i-1}$ and, if $\emptyset \neq D \subseteq \mathbb{M}$, D def $/ A B_0 \dots B_{i-1}$, then $D \cap B_i \neq \emptyset$.

By Remark, $A \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$

Let $M = \bigcup_{i<\omega} B_i$. $A \subseteq M$, M is constr. $/A$ by Lem 1.7.

$M \preceq \mathbb{M}$ by Tarski-Vaught test: if $\emptyset \neq D \subseteq \mathbb{M}$ and D is M-def., Want $D \cap M \neq \emptyset$. D is B_i-def. for $i \gg 0$. $D \cap B_{i+1} \neq \emptyset$, $D \cap M \neq \emptyset$.

$M \preceq \mathbb{M}$

M is prime over A. $\square$

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§2 Vaught pairs (Vaughtian pairs)

~~Def~~ $M \prec N$ means $M \preceq N$ and $M \neq N$.

Rem 2.2 If $\varphi \in L(M)$ and $|\varphi(M)| < \infty$ and $N \succeq M$, then $\varphi(N) = \varphi(M)$.

Proof: if $\varphi(M) = \{c_1, \dots c_n\}$, then $M \models \forall x \left[\varphi(x) \to \left(\bigvee_{i=1}^{n} x = c_i\right)\right]$

$N \models$ ——

$\varphi(M) \subseteq \varphi(N) \subseteq \{c_1 \dots c_n\} = \varphi(M)$.

Def 2.1 A Vaught pair is $M \prec N$ and $\varphi(x) \in L(M)$ s.t. $\varphi(M) = \varphi(N)$ and $|\varphi(M)| = \infty$.

Remark 2.3 If $\varphi(x, \bar{y}) \in L$, $\exists\, T_\varphi$ such that

$(M, P, \bar{b}) \models T_\varphi \iff$
$M \models T$
$P \prec M$
$\bar{b} \in P$
$\varphi(M, \bar{b}) = \varphi(P, \bar{b})$

Cor If T has a Vaught pair, then $\exists$ Vaught pair $M \prec N$, φ where $|N| \leq |L|$.

Proof Apply downward Lö-SK to T_φ from Rem 2.3.

Cor Suppose $\forall n<\omega$, $\exists$ $(M, P, \bar{b}) \models T_\varphi$ with $|\varphi(M,\bar{b})| \geq n$.
Then by compactness, $\exists$ $(M, P, \bar{b}) \models T_\varphi$ with $|\varphi(M,\bar{b})| = \infty$
so T has a Vaught pair.

Suppose $\varphi(x,\bar{y}) \in L$, $\forall n<\omega$, $\exists M \models T$, $\bar{b} \in M$, $n \leq |\varphi(M,\bar{b})| < \infty$
then take $N \succ M$, Rem 2.2 $\varphi(N,\bar{b}) = \varphi(M,\bar{b})$,
get $(N, M, \bar{b}) \models T_\varphi$ with $|\varphi(N,\bar{b})| \geq n$.
get a Vaught pair.

Cor If $\forall n<\omega$ $\exists$ $M \models T$, $\bar{b} \in M$, $n \leq |\varphi(M,\bar{b})| < \infty$
then T has a Vaught pair.

Lemma 2.5 If T does not have a Vaught pair, then
$\forall \varphi$ $\exists n_\varphi$ $\forall M \models T$ $\forall \bar{b} \in M$, either $|\varphi(M,\bar{b})| < n_\varphi$
or $|\varphi(M,\bar{b})| = \infty$.

$\Rightarrow$ T eliminates $\exists^\infty$. $\exists^\infty x\, \varphi(x,\bar{y}) \Leftrightarrow \exists^{>n_\varphi} x\, \varphi(x,\bar{y})$.

Resume ~~10:50.~~ 11:00
OR, whenever I come back from hesuan :(Sorry

Theorem 2.4 (Morley, Baldwin, Lachlan) Let T be complete L-theory
$|L| \leq \aleph_0$, $M \models T \Rightarrow |M| = \infty$.

TFAE:

1) $\exists \kappa > \aleph_0$: T is κ-cat.
2) T is t.t. & T has no Vaught pairs
3) $\forall \kappa > \aleph_0$: T is κ-categorical.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}$$

$1 \Leftrightarrow 3$: Morley's theorem
$3 \Leftrightarrow 2$: Baldwin, Lachlan

$3 \to 1$ trivial
$2 \to 3$ now
$1 \to 2$ later

(2)$\Rightarrow$(3). Assume T is t.t. ~~(= ω-stable)~~ T has no Vaught pairs.

By Lemma 2.5, T eliminates $\exists^\infty$.

Def 2.6 If $M \preceq \mathbb{M}$, D is M-definable, then.

1) D is minimal if $|D| = \infty$, $\forall$ M-def $D' \subseteq D$, $|D'| < \infty$ or $|D \setminus D'| < \infty$.
2) D is strongly minimal if ——— $\mathbb{M}$-def ———

Lemma 2.7 ~~Each~~ If $M \preceq \mathcal{M}$

1) $\exists$ M-def. minimal set $D \subseteq \mathcal{M}$

2) If $D \subseteq \mathcal{M}$ is minimal then D is strongly min.

Proof 1) Let $R(-)$ be CB rank on $S_1(M)$.

Easy: $R(D) > 0 \iff |D| = \infty$ for M-def. $D \subseteq \mathcal{M}$.

Let $E_\alpha = \{ p \in S_1(M) : R(p) \geq \alpha \}$. $\qquad E_{\alpha+1} = (E_\alpha)'$

$\underbrace{R(S_1(M))}_{[D]} = R(\underbrace{\mathcal{M}}_{D}) > 0 \qquad E_1 \neq \emptyset$.

If $E_1 = E_2 = E_1'$ then E_1 is perfect, non empty,

$S_1(M)$ is not scattered, (Lem 7.1 in May 5–7 notes)

$\Rightarrow T$ is not t.t. $\Rightarrow\Leftarrow$.

So $E_1 \supsetneq E_2$, take $p \in E_1 \setminus E_2$. $\quad R(p) = 1$.

$p \notin E_2 = E_1'$ means ~~R~~ p is isolated in E_1.

So $\exists [D] \subseteq S_1(M) \quad [D] \cap E_1 = \{p\}$.

$D \subseteq \mathcal{M}$ is M-definable. $\qquad D = D' \sqcup (D \setminus D')$.

Claim: D is minimal.

If $D' \subseteq D$, $\quad [D] = [D'] \sqcup [D \setminus D']$

If $p \in [D']$ then $E_1 \cap [D \setminus D'] = \emptyset$

$R([D \setminus D']) \leq 0$

$D \setminus D'$ is finite

If $p \in [D \setminus D']$ then D' is finite. $\Big\}$ so $D \setminus D'$ or D' is finite. ✓.

2) Suppose $\mathcal{M}$-definable $D' \subseteq D$, $|D'| = \infty$ $\quad |D \setminus D'| = \infty$.

Let $D = \varphi(\mathcal{M}, b) \quad b \in M$

$D' = \psi(\mathcal{M}, c) \quad c \in \mathcal{M}$.

$$\mathcal{M} \models \exists^\infty x \; [\varphi(x,b) \wedge \psi(x,c)]$$
$$\mathcal{M} \models \underbrace{\exists^\infty x \; (\varphi(x,b) \wedge \neg\psi(x,c))}_{\text{First order, by lemma 2.5}}]$$

$\Theta(b,c)$

$M \preceq \mathcal{M}$, so we may assume $c \in M$. $\qquad \mathcal{M} \models \exists z \; \Theta(b,z)$

$M \models \exists z \; \Theta(b,z)$.

$\square$

By Thm 1.12 $\Rightarrow$ $\exists$ prime model M_0

By Remark 1.5, ~~By~~ $|M_0| = \aleph_0$

By Lemma 2.7, $\exists$ M_0-def str. min. set $D \subseteq \mathcal{M}$.

~~There is~~ $\exists!$ $p \in S_1(M_0)$, $p \in [D]$, p is not constant.

If $a, b \in D$, $a, b \notin M_0$, then $tp(a/M_0) = tp(b/M_0)$.
(else $\exists$ M-def $D' \subseteq D$ $a \in D'$, $b \notin D'$
WLOG $|D'| < \infty$, $a \in acl(M_0) = M_0$).

Take $\hat{p} \in S_1(\mathbb{M})$ similarly.

$\hat{p}$ is M_0-invariant, M_0-def.

$\hat{p} \supseteq p$, p has an M_0-def extension, p is stationary, $p \sqsubseteq \hat{p}$.

<u>Remark 2.8</u> Let $I = (b_\alpha : \alpha < \beta)$ where $b_\alpha \models p$.

TFAE
1) I is independent $/ M_0$
2) I is a Morley seq. of $\hat{p}$ $/ M_0$: $b_\alpha \models \hat{p} | M_0 b_{<\alpha}$.
3) $b_\alpha \notin acl(M_0 b_{<\alpha})$.

Proof: $\sim$ to str. min. theories (§2 in May 5 notes)

For each κ, let I_κ be a Morley seq of $\hat{p}$ over M_0 of length κ.

$I_\kappa \models \hat{p}^{\otimes \kappa} | M_0$.

$\aleph_0$	$M_0, M_1, M_2, \ldots M_\omega$
$\aleph_1$	$M_{\aleph_1}$
$\aleph_2$	$M_{\aleph_2}$

Let M_κ be a prime model $/ M_0 I_\kappa$.

By Rem 1.5, $|M_\kappa| = |M_0| + |I_\kappa| + \aleph_0 = \aleph_0 + \kappa + \aleph_0 = \kappa + \aleph_0$.

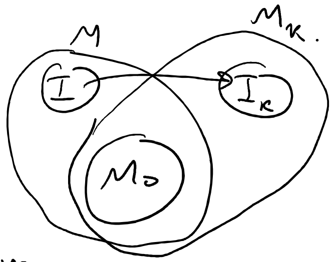
<u>Lemma 2.9</u> If $M \models T$, $M \cong M_\kappa$ for some κ.

<u>Proof</u> M_0 is prime, $M_0 \xrightarrow{\preceq} M$, move M by $\sigma \in Aut(\mathbb{M})$,
WLOG $M_0 \subseteq M$.

Let I be a maximal indep$/^{M_0}$ set of realizations of p in M.
subset of $\{a \in M : a \models p\} \subseteq D(M)$.

<u>Note</u> if $b \in D(M)$ and $b \notin acl(M_0 I)$ then $b \models \hat{p} | M_0 I$
$I \cup \{b\}$ is indep. $\Rightarrow\Leftarrow$ maximality of I.

$\rightarrow$ $\therefore$ if $b \in D(M)$ then $b \in acl(M_0 I)$.



Let $\kappa = |I|$. $I, I_\kappa \models \hat{p}^{\otimes \kappa} | M_0$.

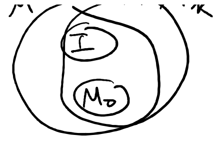
$I \equiv_{M_0} I_\kappa$

If $f : I \to I_\kappa$ is a bijection then $f \cup id_{M_0}$ is a p.e.m.
(by proof of Lem 2.4 in May 5 notes).

Move M, I by $\sigma \in Aut(\mathbb{M}/M_0)$, WLOG $I = I_\kappa$.

M_κ is prime $/ M_0 I$

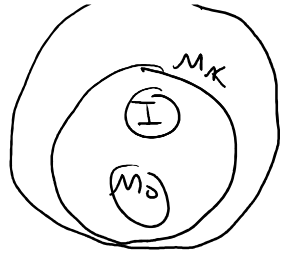
so $M_\kappa \xhookrightarrow{\preceq} M$ over $M_0 I$.



so $M_\kappa \xrightarrow{\sim} M$ over $M_0 I$.

Move M by $\sigma \in \mathrm{Aut}(\mathbb{M}/M_0 I)$

WLOG $M \supseteq M_\kappa$.



If $M_\kappa = M$, done.

If not $M_\kappa \prec M$. By No Vaught pairs, $D(M_\kappa) \subsetneq D(M)$.

Take $b \in D(M) \setminus D(M_\kappa)$ $\quad b \in \mathrm{acl}(M_0 I) \subseteq \mathrm{acl}(M_\kappa) = M_\kappa \Rightarrow\Leftarrow$. $\square$

<u>Cor 2.10</u> T is κ-cat $\forall \kappa > \aleph_0$.

<u>Proof</u> M_κ is the only model of size κ, for $\kappa > \aleph_0$.

Goal: If T is κ-cat for some $\kappa > \aleph_0$
then T is t.t. ($\Leftrightarrow \omega$-stable, by Theorem 7.2 on May 7)
and T has no vaught pairs.

T is κ-cat $\Longrightarrow$ T is ω-stable $\Longleftrightarrow$ T is t.t.

(Thm 32 on Mar 24) $\qquad$ (Thm 7.2, May 7).

Goal: If T is κ-cat, $\kappa > \aleph_0$, and T is t.t., then T has no Vaught pairs.

<u>§3</u> Constructible models are atomic

<u>Def 3.1</u> B is <u>atomic</u> over A if $tp(\bar{b}/A)$ is isolated $\forall \bar{b} \in B$.

<u>Rem 3.2</u> A is atomic over A: $tp(\bar{b}/A)$ is isolated by $\bar{x} = \bar{b}$ when $\bar{b} \in A$.

<u>Remark 3.3</u> If $tp(b/A)$ is isolated by $\varphi(y) \in L(A)$
then Ab is atomic $/A$ because if $\bar{c} \in A$
$tp(b\bar{c}/A)$ is isolated by $\varphi(y) \wedge (\bar{z} = \bar{c})$

<u>Lemma 3.4</u> If $A \subseteq B \subseteq C$, if B is atomic $/A$, C is atomic $/B$
then C is atomic $/A$.

<u>Proof</u> If $\bar{c} \in C$. Take $\varphi(\bar{x}, \bar{b}) \in tp(\bar{c}/B)$ isolating $tp(\bar{c}/B)$
Take $\psi(\bar{y}) \in tp(\bar{b}/A)$ ~~with~~ isolating $tp(\bar{b}/A)$.

<u>Claim</u> $\theta(\bar{x}) = (\exists \bar{y}\ \varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}))$ isolates $tp(\bar{c}/A)$.

<u>Proof</u> $\bar{c} \models \theta(\bar{x})$ (take $\bar{y} = \bar{b}$).

Suppose $\bar{c}_0 \models \theta$. Take $\bar{b}_0$ s.t. $\boxed{\varphi(\bar{c}_0, \bar{b}_0)} \wedge \boxed{\psi(\bar{b}_0)}$

$\psi(\bar{b}_0) \Longrightarrow \bar{b}_0 \equiv_A \bar{b}$.

Move $\bar{c}_0, \bar{b}_0$ by $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, WLOG $\bar{b}_0 = \bar{b}$.

Then $\cancel{\models}\ \varphi(\bar{c}_0, \bar{b}) \Rightarrow \bar{c}_0 \equiv_B \bar{c} \Rightarrow \bar{c}_0 \equiv_A \bar{c}$. $\square_{\text{claim}}$. $\square_{\text{lemma}}$.

<u>Prop 3.5</u> If $A \subseteq M \preceq \mathcal{M}$ and M is constructible $/A$ then M is atomic $/A$.

<u>Proof</u> Take $(b_\alpha : \alpha < \beta)$ ~~of~~ construction of M $/A$.

Show $A \cup \underbrace{\{b_\alpha : \alpha < \gamma\}}_{M_{<\gamma}}$ is atomic $/A$ for $\gamma \leq \beta$

by induction on γ.

$\gamma = 0$: Rem 3.2

γ is a limit: If $\bar{c} \in A \cup \{b_\alpha : \alpha < \gamma\}$ then $\exists \gamma' < \gamma$ $\bar{c} \in A \cup \{b_\alpha : \alpha < \gamma'\}$.

by induction $tp(\bar{c}/A)$ isolated.

$\gamma + 1$: by induction $A\underline{M_{<\gamma}}$ is atomic $/ \underline{A}$

$tp(b_\gamma / AM_{<\gamma})$ is isolated.

By Rem 3.3 $\underline{AM_{<\gamma+1}} = AM_{<\gamma} b_\gamma$ is atomic $/ \underline{AM_{<\gamma}}$.

By Lem 3.4, $AM_{<\gamma+1}$ atomic $/A$.

Take $\gamma = \beta$, $M = AM$ is atomic $/A$. $\square$.

<u>§4</u> Orthogonality

Assume T is t.t.

<u>Def 4.1</u> If $M \models T$, $p \in S_1(M)$, $\varphi(x) \in L(M)$, then

p is orthogonal to $\varphi(x)$ $(p \perp \varphi)$

if $\exists$ $N \succcurlyeq M$, p is realized in N but $\varphi(M) = \varphi(N)$.

<u>Rem 4.2</u> In Def 4.1, if $b \models p$, $b \in N$, we may assume N is ~~prime~~ constructible $/ Mb$.

<u>Lemma 4.3</u> If $p \in S_1(M)$, $\varphi(x) \in L(M)$, then $p \not\perp \varphi$ iff

$\exists$ $\psi(x, y, \bar{z}) \in L$, $\exists \bar{c} \in M$, such that

- $p(y) \vdash \exists x [\varphi(x) \wedge \psi(x, y, c)]$
- If $a \in M$, then $p(y) \vdash \neg\psi(a, y, c)$.

(If $b \models p$.
$\mathcal{M} \models \exists x\ \varphi(x) \wedge \psi(x, b, c)$
<u>but</u> if $a \in M$, then $\neg\psi(a, b, c)$).

<u>Proof</u> Suppose ψ, c exist

Suppose $N \succcurlyeq M$, $b \in N$, $b \models p$. There is $a_0 \in N$

$N \models \varphi(a_0) \wedge \psi(a_0, b, c)$

$a_0 \in \varphi(N)$. If $a_0 \in M$, then $\neg\psi(a_0, b, c) \Rightarrow\Leftarrow$

So $a_0 \notin M$, $a_0 \in \varphi(N) \setminus \varphi(M)$. $\varphi(N) \supsetneq \varphi(M)$.

Conversely, suppose $p \not\perp \varphi$. Take $b \models p$, take N constructible / Mb.

By non orthogonality $\varphi(N) \supsetneq \varphi(M)$. Take $a_0 \in \varphi(N) \setminus \varphi(M)$.

N is atomic over Mb, $tp(a_0/Mb)$ is isolated.

Take $\psi(x, b, \bar{c})$ isolating $tp(a_0/Mb)$, $\psi \in L$, $\bar{c} \in M$.

$N \models \varphi(a_0) \wedge \psi(a_0, b, \bar{c})$, so $N \models \exists x(\varphi(x) \wedge \psi(x, b, \bar{c}))$

If $a \in M$ and $\psi(a, b, \bar{c})$ then $a \underset{Mb}{\equiv} a_0$, so $a_0 \notin M \Rightarrow a \notin M \Rightarrow\!\Leftarrow$.

If $a \in M$ then $\neg\psi(a, b, \bar{c})$. $\square$

Lemma 4.4 If $M \preceq M' \preceq \mathbb{M}$, $p \in S_1(M)$, $p' \in S_1(M')$ if $p' \supseteq p$, and $\varphi(x) \in L(M)$, then

$$p \perp \varphi \iff p' \perp \varphi.$$

Proof The conditions of Lem 4.3 can be translated into first-order properties of (M, dp)

and $(M, dp) \preceq (M', dp')$. ($p'$ is a strong heir of p) $\square$

expansion of M by definable sets, p' has same def. as p.

Prop 4.5 (Stretching Vaught pairs in t.t. theories)

Suppose T is t.t. Suppose $\exists$ Vaught pair $M \prec N$, $\varphi(x) \in L(M)$

$\varphi(M) = \varphi(N)$, $|\varphi(M)| = \infty$.

Then:

1) $\forall M' \succeq M \ \exists N' \succ M'$ with $\varphi(N') = \varphi(M')$, $|N'| \leq |M'| + |L|$. (by Löwenheim Skolem)

2) For $\kappa \geq |M| + |L|$, $\exists N \succeq M$ with $|N| = \kappa$, $\varphi(N) = \varphi(M)$.

Proof 1) Take $b \in N \setminus M$. Let $p = tp(b/M)$. $p \perp \varphi$.

Let $p' \in S_1(M')$ be the heir. By Lem 4.4, $p' \perp \varphi$.

$\exists N' \succeq M'$ $\exists b' \models p'$ $b' \in N'$, $\varphi(N') = \varphi(M')$.

p not const $\Rightarrow p'$ not const, so $b' \notin M'$ so $N' \succ M'$.

2) Iterate (1) and build a chain $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ length κ.

$M_0 = M$

$M_{\alpha+1} \succ M_\alpha$ with $\varphi(M_{\alpha+1}) = \varphi(M_\alpha)$ (using (1) applied to $M' = M_\alpha \succeq M_0 = M$)

If β is a limit then $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$. $|M_{\alpha+1}| \leq |M_\alpha| + |L|$.

$N = \bigcup_{\alpha<\kappa} M_\alpha$.

So $N \succeq M$ $\varphi(N) = \varphi(M)$. $|N| \geq \kappa$. (added κ-many elements)

$|M_\alpha| \leq |\alpha| + |L| + |M| \quad \forall \alpha < \kappa$

by induction on α.

$|M_\alpha| \leq |\alpha| + |L| + |M| \quad \forall \alpha < \kappa$

by induction on α.

$|N| \leq \kappa + |L| + |M| = \kappa$ $\square$.

<u>Cor</u> Suppose $|L| = \aleph_0$, T is t.t., T has a Vaught Pair.

$M \prec N$, φ. WLOG $|M| = \aleph_0$. (Lö-Sk and T_φ).

By Prop 4.5, $\forall \kappa \geq \aleph_0$, $\exists N \succeq M$ $|N| = \kappa$.

$\varphi(N) = \varphi(M)$ $\quad |\varphi(N)| = \aleph_0$.

$\geq \aleph_0$ by def of V. pair

$|\varphi(N)| = |\varphi(M)| \leq |M| \leq \aleph_0$.

<u>So</u>: $\exists N \models T$

$|N| = \kappa$

$\exists$ def set $\varphi(N)$ $\quad |\varphi(N)| = \aleph_0$.

Goal: If T is κ-cat, $\kappa > \aleph_1$, T is t.t, T has a vaught pair

then ~~T has no Vaught Pair~~ $\exists$ contradiction.

<u>Def</u> $N \succ M$ "enlarges infinite sets" if $\varphi(N) \supsetneq \varphi(M)$

whenever $|\varphi(M)| = \infty$.

<u>Lemma 5.1</u> If $M \models T$ ~~and~~ $\exists N \succ M$ that enlarges inf sets,

$|N| \leq |M| + |L|$.

<u>Proof</u> Let $(D_\alpha : \alpha < \kappa)$ be all infinite M-def sets. $\kappa \leq |M| + |L|$.

$D_\alpha(N) \supsetneq D_\alpha(M)$. Take $b_\alpha \in D_\alpha$ $\quad b_\alpha \notin M$.

Let N = a model containing $M \cup \{b_\alpha : \alpha \in \kappa\}$.

Lö-Sk, WLOG $|N| \leq |M| + \kappa = |M| + |L|$. $\square$.

<u>Prop 5.2</u> If $(M \models T \Rightarrow |M| = \infty)$, if $\kappa \geq |L|$ then

$\exists M \models T$ $\quad |M| = \kappa$ s.t. $\forall$ M-def $D \subseteq M$,

$|D| < \infty$ or $|D(M)| = \kappa$.

<u>Proof</u> Build $(M_\alpha : \alpha < \kappa)$ elementary chain:

M_0 = model of size $\leq |L|$.

$M_{\alpha+1} \succ M_\alpha$ ~~&~~ enlarges infinite sets, $|M_{\alpha+1}| \leq |M_\alpha| + |L|$.

$M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for β a limit.

$N = \bigcup_{\alpha<\kappa} M_\alpha$. $\qquad$ As before, $|N| \leq \kappa$. $N \succeq M$.

If $\varphi \in L(N)$ then $\varphi \in L(M_\alpha)$ $\alpha < \kappa$.

$$\varphi(\underbrace{M_\beta) \subsetneq \varphi(}M_{\beta+1}) \quad \forall\ \beta \quad \alpha<\beta<\kappa$$

so $|\varphi(N)| \geq \kappa$ $\quad |\{\beta : \alpha<\beta<\kappa\}| = \kappa.$

$\square$.

So: if T is κ-cat, $\kappa > \aleph_0$, $|L| = \aleph_0$.

T is ω-stable by Thm 32 in Mar 24.

T is t.t. by Thm 7.2 in May 7 notes

~~$\exists$ M~~ Suppose T has a vaughtpar.

By Prop 4.5 (2) $\exists\ M \models T$ $|M| = \kappa$

$|\varphi(M)| = \aleph_0 < \kappa$

By Prop 5.2 $\exists\ N \models T$ $\forall \varphi \in L(N)$, $|\varphi(N)| < \aleph_0$ or $|\varphi(N)| = \kappa$.

$N \cong M$, T is not κ-cat $\Rightarrow \Leftarrow$,

T has no vaught pairs.

$\square$.