Algebraic closure and imaginaries

Advanced model theory

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Reference in the book: Sections 6.1 [sic], 16.4, 16.5.

Recall a set $D \subseteq \mathbb{M}^n$ is *A-invariant* if it satisfies the equivalent conditions:

- $\sigma(D) = D$ for $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.
- If $\bar{b} \equiv_A \bar{c}$, then $\bar{b} \in D \iff \bar{c} \in D$.

We'll repeatedly use the following fact:

Fact 1 (Lemma 10 in the 2022-3-10 notes)**.** *If $A \subseteq \mathbb{M}$ is small and $D \subseteq \mathbb{M}^n$ is definable, then D is A-invariant iff D is A-definable.*

Definition 2. A set $D \subseteq M^n$ is *A-definable* if $D = \varphi(M^n) = \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}$ for some $L(A)$-formula φ, i.e., a formula with parameters from A. If $D_1, \ldots, D_{n+1}$ are A-definable and f is a function $f : \prod_{i=1}^n D_i \to D_{n+1}$, then f is *A-definable* if the graph $\Gamma(f) := \{(\bar{a}, b) \in \prod_{i=1}^{n+1} D_i : f(\bar{a}) = b\}$ is an A-definable set. Without a prefix, *definable* means "M-definable". Also, *0-definable* is short for "$\varnothing$-definable".

1 Definable closure

Definition 3. The *definable closure* $\mathrm{dcl}(A)$ of $A \subseteq \mathbb{M}$ is $\{b \in \mathbb{M} : \{b\} \text{ is } A\text{-definable}\}$.

Example 4. In a field $(K, +, \cdot)$, $a \div b$ is in $\mathrm{dcl}(a, b)$ because $\{a \div b\}$ is defined by the formula $x \cdot b = a$.

If $\bar{b}$ is a tuple, note that $\bar{b} \in \mathrm{dcl}(A)$ iff $\{\bar{b}\} \subseteq \mathbb{M}^n$ is A-definable.

Proposition 5. *The following are equivalent for $\bar{b} \in \mathbb{M}^n$ and small $A \subseteq \mathbb{M}$:*

1. $\bar{b} \in \mathrm{dcl}(A)$*, i.e., $\{\bar{b}\}$ is A-definable.*
2. $\mathrm{Aut}(\mathbb{M}/A)$ *fixes $\bar{b}$, i.e., $\{\bar{b}\}$ is A-invariant.*
3. $\mathrm{tp}(\bar{b}/A)$ *has only one realization.*

Proof. (1) $\iff$ (2): Fact 1.

(2) $\iff$ (3): Let $S = \{\bar{c} \in \mathbb{M} : \bar{c} \models \mathrm{tp}(\bar{b}/A)\} = \{\sigma(\bar{b}) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$. (2) and (3) both say $S = \{\bar{b}\}$. ☐

Proposition 6.

1. *$A \subseteq \mathrm{dcl}(A)$.*
2. *$A \subseteq B \implies \mathrm{dcl}(A) \subseteq \mathrm{dcl}(B)$.*
3. *$\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$.*
4. *$D \subseteq M^n$ is A-definable iff D is $\mathrm{dcl}(A)$-definable.*

Conditions (1)–(3) say that $\mathrm{dcl}(-)$ is an abstract "closure operator."

Proof. (1): If $b \in A$ then the formula $x = b$ defines $\{b\}$.

(2): If $A \subseteq B$, then A-definable sets are B-definable, so if $\{c\}$ is A-definable then $\{c\}$ is B-definable.

(4): $A \subseteq \mathrm{dcl}(A)$ so A-definable sets are $\mathrm{dcl}(A)$-definable. Conversely, suppose D is $\mathrm{dcl}(A)$-definable. If $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, then $\sigma(b) = b$ for $b \in \mathrm{dcl}(A)$, so $\sigma \in \mathrm{Aut}(\mathbb{M}/\mathrm{dcl}(A))$ and $\sigma(D) = D$. Then D is A-invariant, and therefore A-definable by Fact 1.

(3): $A \subseteq \mathrm{dcl}(A)$ by (1), so $\mathrm{dcl}(A) \subseteq \mathrm{dcl}(\mathrm{dcl}(A))$ by (2). If $b \in \mathrm{dcl}(\mathrm{dcl}(A))$ then $\{b\}$ is $\mathrm{dcl}(A)$-definable, hence A-definable by (4), and so $b \in \mathrm{dcl}(A)$. This shows $\mathrm{dcl}(\mathrm{dcl}(A)) \subseteq \mathrm{dcl}(A)$. ☐

Definition 7. A is *definably closed* if $\mathrm{dcl}(A) = A$.

Proposition 8. $\mathrm{dcl}(A)$ *is the smallest definably closed set containing A.*

Proof. $\mathrm{dcl}(A)$ is definably closed and contains A by (3) and (1) in Proposition 6.

Suppose $\mathrm{dcl}(B) = B$ and $B \supseteq A$. Then $\mathrm{dcl}(B) \supseteq \mathrm{dcl}(A)$ by Proposition 6(2), so $B = \mathrm{dcl}(B) \supseteq \mathrm{dcl}(A)$. ☐

Here is a characterizatio of definable closure in ACF_0:

Fact 9. *If $A \subseteq M \models \mathrm{ACF}_0$, then A is definably closed iff A is a subfield of M.*

The easy part is that if $A = \mathrm{dcl}(A)$, then A is a subfield. (Compare with Example 4). The hard part is showing that subfields are definably closed. For comparison, in ACF_p, not all subfields are definably closed.[1]

Definition 10. $\bar{a}, \bar{b}$ are *interdefinable* if $\mathrm{dcl}(\bar{a}) = \mathrm{dcl}(\bar{b})$.

Equivalently, $\bar{a}, \bar{b}$ are interdefinable if $\bar{a} \in \mathrm{dcl}(\bar{b})$ and $\bar{b} \in \mathrm{dcl}(\bar{a})$.

Lemma 11. *$\bar{a}$ is interdefinable with $\bar{b}$ iff $\mathrm{Aut}(\mathbb{M}/\bar{a}) = \mathrm{Aut}(\mathbb{M}/\bar{b})$.*

[1] In ACF_p, it turns out that a subfield K is definably closed only if K is closed under pth roots.

Proof. $\mathrm{dcl}(\bar{a}) \subseteq \mathrm{dcl}(\bar{b}) \iff \bar{a} \in \mathrm{dcl}(\bar{b}) \iff \mathrm{Aut}(\mathbb{M}/\bar{b}) \subseteq \mathrm{Aut}(\mathbb{M}/\bar{a})$. □

Lemma 12. *If $\bar{a}$ is interdefinable with $\bar{b}$, then there is a 0-definable bijection $f : X \to Y$ with $f(\bar{a}) = \bar{b}$.*

Proof. Take φ_1, φ_2 so $\varphi_1(\bar{a}, \mathbb{M}^m) = \{\bar{b}\}$ and $\varphi_2(\mathbb{M}^n, \bar{b}) = \{\bar{a}\}$. Replacing φ_1, φ_2 both with $\varphi_1 \wedge \varphi_2$, we may assume $\varphi_1 = \varphi_2 =: \varphi$. Let $\psi(\bar{x}, \bar{y})$ be

$$\varphi(\bar{x}, \bar{y}) \ \wedge \ (\exists! \bar{z}\ \varphi(\bar{x}, \bar{z})) \ \wedge \ (\exists! \bar{w}\ \varphi(\bar{w}, \bar{y})).$$

Then $\mathbb{M} \models \psi(\bar{a}, \bar{b})$, and ψ defines a bijection. □

2 Algebraic closure

Definition 13. The algebraic closure $\mathrm{acl}(A)$ of $A \subseteq \mathbb{M}$ is the union of all finite A-definable sets $D \subseteq \mathbb{M}$.

Note that $\bar{b} \in \mathrm{acl}(A)$ iff $\bar{b} \in D$ for some finite A-definable $D \subseteq \mathbb{M}^n$. (If $\bar{b} \in D$, then each coordinate b_i is in the finite A-definable set $\pi_i(D)$, where π_i is the ith coordinate projection. Conversely, if b_i is in a finite A-definable set D_i for all i, then $\bar{b}$ is in the finite A-definable set $\prod_{i=1}^n D_i$.)

Proposition 14. *Suppose $\bar{b} \in \mathbb{M}^n$ and $A \subseteq \mathbb{M}$. Let $S = \{\bar{c} \in \mathbb{M}^n : \bar{c} \equiv_A \bar{b}\} = \{\sigma(\bar{b}) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$. ($S$ is the set of realizations of $\mathrm{tp}(\bar{b}/A)$.)*

1. *If $\bar{b} \in \mathrm{acl}(A)$ then S is finite and A-definable.*
2. *If $\bar{b} \notin \mathrm{acl}(A)$ then S is large.*

Proof. (1): Take a finite A-definable set $D \ni \bar{b}$. Then $S \subseteq D$ so $|S| < \infty$. The set S is definable and A-invariant, hence A-definable.

(2): Suppose S is small. Let $\Sigma(\bar{x}) = \mathrm{tp}(\bar{b}/A) \cup \{\bar{x} \neq \bar{c} : \bar{c} \in S\}$. $\Sigma(\bar{x})$ is small and has no realizations, so $\Sigma(\bar{x})$ is inconsistent. By compactness there are $\psi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$ and $\bar{c}_1, \ldots, \bar{c}_n \in S$ such that

$$\left\{ \psi(\bar{x}) \wedge \bigwedge_{i=1}^{n} \bar{x} \neq \bar{c}_i \right\} \vdash \bot.$$

Then $\psi(\mathbb{M}^n) \subseteq \{\bar{c}_i : 1 \leq i \leq n\}$, so $\psi(\mathbb{M}^n)$ is finite. Then $\bar{b}$ is in the finite A-definable set $\psi(\mathbb{M}^n)$, showing that $\bar{b} \in \mathrm{acl}(A)$. □

Proposition 15.

1. *$A \subseteq \mathrm{acl}(A)$.*
2. *$A \subseteq B \implies \mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.*
3. *$\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.*

Proof. Like Proposition 6, except for the proof of

$$\mathrm{acl}(\mathrm{acl}(A)) \subseteq \mathrm{acl}(A).$$

Take $b \in \mathrm{acl}(\mathrm{acl}(A))$. Then b is in some finite $\mathrm{acl}(A)$-definable set D. Write D as $\varphi(\mathbb{M}, \bar{c})$ with $\bar{c} \in \mathrm{acl}(A)$. The family

$$\{\sigma(D) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\} = \{\varphi(\mathbb{M}, \sigma(\bar{c})) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$$

is finite by Proposition 14, as $\bar{c} \in \mathrm{acl}(A)$. Each set in the family is the image of the finite set D under an automorphism, so each set in the family is finite. Therefore, the union $\bigcup_{\sigma \in \mathrm{Aut}(\mathbb{M}/A)}$ is finite. This union is A-invariant ($=$ A-definable), and contains b, so $b \in \mathrm{acl}(A)$. □

Definition 16. A is *algebraically closed* if $\mathrm{acl}(A) = A$.

Proposition 17. $\mathrm{acl}(A)$ *is the smallest algebraically closed set containing* A.

Proof. Like Proposition 8. □

Proposition 18. *If* $M \preceq \mathbb{M}$ *then* $\mathrm{acl}(M) = M$.

Proof. If $b \in \mathrm{acl}(M) \setminus M$ and $S = \{\sigma(b) : b \in \mathrm{Aut}(\mathbb{M}/M)\}$, then S is non-empty, and S is M-definable by Proposition 14. But $S \cap M = \varnothing$, contradicting the Tarski-Vaught criterion. (The easy direction of Tarski-Vaught. If $\varphi(\bar{x}, \bar{b})$ is an $L(M)$-formula defining S, then $\mathbb{M} \models \exists \bar{x}\ \varphi(\bar{x}, \bar{b})$ but $M \not\models \exists \bar{x}\ \varphi(\bar{x}, \bar{b})$, contradicting $M \preceq \mathbb{M}$.) □

Proposition 19. *If* $M \models \mathrm{ACF}$ *and* K *is a subfield of* M*, the following are equivalent:*

1. $K = \mathrm{acl}(K)$.
2. $K \models \mathrm{ACF}$*: every polynomial* $P(x) \in K[x]$ *factors into linear polynomials.*
3. $K \preceq M$.

In particular, K is model-theoretically algebraically closed (1) iff K is field-theoretically algebraically closed (2).

Proof. (1) $\Longrightarrow$ (2): if $P(x) \in K[x]$ then $P(x) = c \cdot \prod_{i=1}^{n} (x - r_i)$ for $c \in K$ and $r_1, \ldots, r_n \in M$, because $M \models \mathrm{ACF}$. But $\{r_1, \ldots, r_n\}$ is finite and K-definable, defined by "$P(x) = 0$", so each r_i is in $\mathrm{acl}(K) = K$.

(2) $\Longrightarrow$ (3): The embedding $K \hookrightarrow M$ is elementary by quantifier elimination. Therefore $K \preceq M$.

(3) $\Longrightarrow$ (1): Proposition 18. □

3 $\mathbb{M}^{\text{eq}}$

Definition 20. An *A-interpretable set* is a quotient D/E, where D is an A-definable set and $E \subseteq D \times D$ is an A-definable equivalence relation. "0-interpretable" is short for $\varnothing$-interpretable.

If $\bar{a} \in D$, then $[\bar{a}]_E$ denotes the E-equivalence class in D/E. Some authors write $[\bar{a}]_E$ as $\bar{a}/E$.

Definition 21. M^{eq} is the expansion of M by the following, for each 0-interpretable set D/E:

- A new sort D/E.
- A relation symbol for the graph of $D \to D/E$.

In other words, M^{eq} is the expansion of M obtained by adding each 0-interpretable set as a new sort, with enough data to connect the new sorts to the old sorts.

The structure M^{eq} is closely connected to M, as explained in Fact 22 below. Here is a brief summary. If $\mathbb{M}$ is a monster model, then $\mathbb{M}^{\text{eq}}$ is a monster model (7) with the "same" automorphism group (6) and the "same" small models (5). If we restrict our attention to the original sorts from $\mathbb{M}$, then $\mathbb{M}^{\text{eq}}$ and $\mathbb{M}$ have the same definable sets (1), (4), and the same partial elementary maps (2). However, $\mathbb{M}^{\text{eq}}$ has some new elements, and the definable sets in $\mathbb{M}^{\text{eq}}$ correspond exactly to the interpretable sets in the original structure $\mathbb{M}$ (8) and (9). On the other hand, the new elements of $\mathbb{M}^{\text{eq}}$ are definable from the old elements (3). So $\mathbb{M}^{\text{eq}}$ is a way of converting interpretable sets into definable sets while preserving most other things.

We omit the proof of Fact 22. Most of the proof is straightforward, though (7) requires a little cleverness.

Fact 22.

1. *If $X \subseteq M^n$, then X is 0-definable in M iff X is 0-definable in M^{eq}.*[2] *In other words, M^{eq} doesn't define any new sets on the original sorts of M.*

2. *Consequently, if $A, B \subseteq M$ and $f : A \to B$ is bijection, then f is a partial elementary map in M iff f is a partial elementary map in M^{eq}.*

3. *In M^{eq}, $\operatorname{dcl}(M) = M^{\text{eq}}$.*[3]

4. *Consequently, any M^{eq}-definable set $X \subseteq M^n$ is M-definable in M^{eq}, and therefore M-definable in M.*[4].

[2] More generally, one can show that $X \subseteq M^n \times \prod_{j=1}^{m}(D_j/E_j)$ is 0-definable in M^{eq} iff $\tilde{X} \subseteq M^n \times \prod_{j=1}^{m} D_j$ is 0-definable in M, where $\tilde{X} = \{(a_1, \ldots, a_n, b_1, \ldots, b_m) : (a_1, \ldots, a_n, [b_1]_{E_1}, \ldots, [b_m]_{E_m}) \in X\}$. One proves this fact by induction on the complexity of formulas, and then deduces Fact 22(1) from it.

[3] This is obvious, using the definable functions $D \to D/E$.

[4] Note Fact 22(1) means that if $\bar{x}$ is a tuple of variables in the old sorts of M, then any L^{eq}-formula $\phi(x)$ is equivalent to an L-formula.

5. *If $N \preceq M$, then $\mathrm{dcl}_{M^{\mathrm{eq}}}(N)$ is an elementary substructure of M^{eq}, isomorphic to N^{eq}. Moreover, all elementary substructures of M arise this way. This yields an order-preserving bijection between the elementary substructures of M and the elementary substructures of M^{eq}. In particular, all elementary substructures of M^{eq} arise this way.*[5]

6. *If $\sigma \in \mathrm{Aut}(M)$, then σ induces an automorphism $\hat{\sigma} \in \mathrm{Aut}(M^{\mathrm{eq}})$. This yields an isomorphism $\mathrm{Aut}(M) \cong \mathrm{Aut}(M^{\mathrm{eq}})$ between the automorphism groups.*

7. *If M^{eq} is κ-saturated and strongly κ-homogeneous if M is. Consequently, if $\mathbb{M}$ is a monster model, then $\mathbb{M}^{\mathrm{eq}}$ is a monster model.*[6]

8. *Every 0-interpretable set D/E in M is a 0-definable set in M^{eq}, by construction.*

9. *Conversely, if X is 0-definable in M^{eq}, then there is a 0-interpretable set D/E in M and a 0-definable bijection $X \to D/E$ in M^{eq}.*[7]

We write $\mathrm{acl}^{\mathrm{eq}}$ and $\mathrm{dcl}^{\mathrm{eq}}$ to mean acl and dcl in the structure $\mathbb{M}^{\mathrm{eq}}$. From now on, we use the word "interpretable" to mean "definable in $\mathbb{M}^{\mathrm{eq}}$," and "definable" to mean "definable in $\mathbb{M}$." An *imaginary* (or *imaginary element*) is an element of $\mathbb{M}^{\mathrm{eq}}$. (Elements of $\mathbb{M}$ are sometimes called *reals*.)

[5]Behind the scenes, there is a theory T^{eq}, and $M \models T \implies M^{\mathrm{eq}} \models T^{\mathrm{eq}}$. Moreover, all models of T^{eq} have the form M^{eq} up to isomorphism. Finally, elementary embeddings $M \to N$ correspond bijectively to elementary embeddings $M^{\mathrm{eq}} \to N^{\mathrm{eq}}$. If you know category theory, this means there is an equivalence of categories between models of T and models of T^{eq}. (In these categories, the morphisms are elementary embeddings.)

[6]This fact is slightly harder to prove than the others on this list. The easier direction is that if M^{eq} is κ-saturated and strongly κ-homogeneous, then M is also, essentially by Fact 22(2). If you just want a monster model $\mathbb{M}$ such that $\mathbb{M}^{\mathrm{eq}}$ is a monster model, you can do the following: take $M \models T$, construct M^{eq}, take some monster elementary extension $U \succeq M^{\mathrm{eq}}$, then check that U is $\mathbb{M}^{\mathrm{eq}}$ for some $\mathbb{M} \succeq M$. By the "easy" direction, $\mathbb{M}$ is a monster model.

Nevertheless, it's nice to know that the "hard" direction holds: if M is κ-saturated and strongly κ-homogeneous, then M^{eq} is too. (In terms of monster models, this means there's no need to further enlarge the monster model to make sure $\mathbb{M}^{\mathrm{eq}}$ is a monster.) It's not that hard to prove that κ-saturation transfers from M to M^{eq}, especially if you think of κ-saturation in terms of a compactness-like property: if $|A| < \kappa$ and a collection of A-definable sets has FIP, then it has non-empty intersection. To transfer this from M to M^{eq}, one takes the A-definable sets in M^{eq} and lifts them to A-definable sets in M using the maps $D \to D/E$. The proof of strong κ-homogeneity is a little more complicated, and uses κ-saturation.

[7]This comes down to the following things. First, if D/E and D'/E' are two interpretable sets, then $(D/E) \times (D'/E')$ "is" an interpretable set, namely $(D \times D')/E''$, where $(a,b)E''(c,d) \iff aEc \wedge bE'd$. Secondly, if X is a definable subset of D/E, then X "is" an interpretable set D'/E', where $D' = \{a \in D : [a]_E \in X\}$, and E' is the restriction of E to D'.

4 Elimination of imaginaries

Definition 23. T has *elimination of imaginaries* if every $e \in \mathbb{M}^{\mathrm{eq}}$ is interdefinable with a tuple $\bar{b} \in \mathbb{M}$.

Definition 24. T has *uniform elimination of imaginaries* if the following equivalent conditions hold:

1. For every 0-interpretable set D/E, there is a 0-definable set Y and 0-interpretable bijection $f : D/E \to Y$.

2. For every 0-interpretable set D/E, there is a 0-definable set Y and a 0-definable surjection $g : D \to Y$ such that for $x, y \in D$,

$$g(x) = g(y) \iff [x]_E = [y]_E \iff E(x, y)$$

Uniform elimination of imaginaries implies elimination of imaginaries. (If $e \in D/E$ and $f : D/E \to Y$ is as in Definition 24(1), then e is interdefinable with $f(e)$.) In Theorem 26, we will see that the converse often holds.

Lemma 25. *If T has elimination of imaginaries and D/E is 0-interpretable, then we can partition D/E into 0-interpretable subsets $X_1, \ldots, X_n$ with 0-interpretable bijections $f_i : X_i \to Y_i$ to 0-definable sets Y_i.*

Proof. Say a 0-interpretable $X \subseteq D/E$ is "good" if there is 0-definable Y and a 0-definable bijection $f : X \to Y$. If X is good and $X_0 \subseteq X$ is 0-interpretable, then X_0 is good. (To see this, replace f with its restriction $f \restriction X_0$.)

Claim. Good sets cover D/E.

Proof. If $e \in D/E$, then e is interdefinable with some $\bar{b} \in \mathbb{M}^n$, by elimination of imaginaries. By Lemma 12, there is a 0-interpretable $X \subseteq D/E$ and a 0-definable $Y \subseteq \mathbb{M}^n$ and a 0-definable bijection $f : X \to Y$ with $e \in X$ and $f(e) = \bar{b}$. Then X is good. $\square_{\text{Claim}}$

There are at most $|L|$-many good sets. By saturation, D/E is a finite union of good sets $D/E = \bigcup_{i=1}^n X_i$. Replacing X_i with $X_i' := X_i \setminus \bigcup_{j<i} X_j$, we may assume the X_i are pairwise disjoint. $\square$

Theorem 26. *Suppose T is single-sorted and at least two elements are definable ($|\operatorname{dcl}(\varnothing)| > 1$). Then T has uniform elimination of imaginaries iff T has elimination of imaginaries.*

Proof. If T has uniform elimination of imaginaries, then it has elimination of imaginaries, as noted above. Conversely suppose T has elimination of imaginaries. Let D/E be 0-interpretable. Take f_i, X_i, Y_i for $1 \leq i \leq n$ as in Lemma 25. Fix two distinct definable elements $a, b \in \operatorname{dcl}(\varnothing) \subseteq \mathbb{M}$. Replacing Y_i with $Y_i \times \{(a, a, \ldots, a)\}$ we may assume all the Y_i are definable subsets of $\mathbb{M}^m$ for some fixed m not depending on i. Take N so large that $2^N > n$, and take distinct tuples $\bar{c}_1, \ldots, \bar{c}_n \in \{a, b\}^N$. Replacing Y_i with $Y_i \times \{\bar{c}_i\}$, we may assume $Y_i \cap Y_j = \varnothing$ for $i \neq j$. Then $\bigcup_i f_i$ is a 0-interpretable bijection from D/E to $\bigcup_i Y_i \subseteq \mathbb{M}^m$. $\square$

The assumption "two elements are definable" holds in many natural theories. For example, in theories of fields like ACF and RCF, we can take the two elements 0 and 1.

DLO is an example of a theory with elimination of imaginaries but not uniform elimination of imaginaries.[8]

Remark 27. $\mathbb{M}^{\mathrm{eq}}$ has uniform elimination of imaginaries: if D/E is 0-interpretable and $E' \subseteq (D/E) \times (D/E)$ is a 0-interpretable equivalence relation on D/E, then $(D/E)/E'$ is also 0-interpretable. In fact, it's D/E'' where

$$E''(\bar{a}, \bar{b}) \iff E'([\bar{a}]_E, [\bar{b}]_E).$$

5 Codes and elimination of imaginaries

Definition 28. A real tuple or imaginary e is a *code* for a definable set D if

$$\{\sigma \in \mathrm{Aut}(\mathbb{M}) : \sigma(D) = D\} = \mathrm{Aut}(\mathbb{M}/e).$$

We also say that e *codes* D if e is a code for D.

Remark 29. If e and e' are both codes for D, then e and e' are interdefinable by Lemma 11.

Remark 30. Suppose e codes D, and $A \subseteq \mathbb{M}^{\mathrm{eq}}$. Then D is A-definable iff $e \in \mathrm{dcl}^{\mathrm{eq}}(A)$.

Proof. The following are equivalent, using Fact 1 and Proposition 5:

$$\begin{gathered} D \text{ is } A\text{-definable} \\ D \text{ is } A\text{-invariant} \\ \forall \sigma \in \mathrm{Aut}(\mathbb{M}/A) : \sigma(D) = D \\ \forall \sigma \in \mathrm{Aut}(\mathbb{M}/A) : \sigma(e) = e \\ e \in \mathrm{dcl}^{\mathrm{eq}}(A). \end{gathered}$$

□

Example 31. Suppose $T = \mathrm{ACF}$ and $S = \{r_1, \ldots, r_n\} \subseteq \mathbb{M}^1$. Let $P(x) = \prod_{i=1}^n (x - r_i)$. Write $P(x)$ as $x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Then $(c_0, c_1, \ldots, c_{n-1})$ is a code for S. Indeed, $\sigma(\bar{c}) = \bar{c} \iff \sigma(P(x)) \equiv P(x) \iff \prod_{i=1}^n (x - \sigma(r_i)) \equiv \prod_{i=1}^n (x - r_i) \iff \{\sigma(r_1), \ldots, \sigma(r_n)\} = \{r_1, \ldots, r_n\} \iff \sigma(S) = S$. For example, $(r_1 r_2, -r_1 - r_2)$ is a code for $\{r_1, r_2\}$.

[8] Here is an example where uniform elimination of imaginaries fails. Let E be the equivalence relation on $\mathbb{M}^2$ with two classes, one of which is the line $y = x$ and the other is its complement. If there was a 0-interpretable bijection from $\mathbb{M}^2/E$ to $Y \subseteq \mathbb{M}^n$, then Y would contain two elements, both of which are in $\mathrm{dcl}(\varnothing)$. But $\mathrm{dcl}(\varnothing) = \varnothing$, so Y cannot have any elements unless $n = 0$, and when $n = 0$ the set Y can only have one element. The proof that DLO has elimination of imaginaries is a little harder. One approach uses the machinery of Section 5 below, specifically Proposition 37. The hard part is to prove that if $D \subseteq \mathbb{M}^n$ is definable, then there is a unique smallest set $A \subseteq \mathbb{M}$ such that D is A-definable. If $a_1 < a_2 < \cdots < a_m$ are the elements of A listed in ascending order, then $(a_1, a_2, \ldots, a_m)$ is a *code* for D in the sense of Definition 28. Then every definable set has a code, so elimination of imaginaries holds by Proposition 37.

Example 32. If D/E is 0-interpretable and $e \in D/E$, then e is an E-equivalence class $X = E(\mathbb{M}, \bar{a})$, and $\sigma(e) = \sigma(X)$ for all σ. Therefore e codes X.

Lemma 33. *Let $\varphi(\bar{x}, \bar{y})$ be a formula. Let $f(\bar{y})$ be a 0-definable function such that*

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \iff f(\bar{b}) = f(\bar{c}).$$

Then $f(\bar{b})$ is a code for $\varphi(\mathbb{M}, \bar{b})$, for each $\bar{b}$.

Proof. $\sigma(f(\bar{b})) = f(\bar{b}) \iff f(\sigma(\bar{b})) = f(\bar{b}) \iff \varphi(\mathbb{M}, \sigma(\bar{b})) = \varphi(\mathbb{M}, \bar{b}) \iff \sigma(\varphi(\mathbb{M}, \bar{b})) = \varphi(\mathbb{M}, \bar{b})$. □

Proposition 34. *The following are equivalent:*

1. *T has uniform elimination of imaginaries.*
2. *For any formula $\varphi(\bar{x}; \bar{y})$, there is a 0-definable function $f_\varphi(\bar{y})$ such that*

$$\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}) \iff f_\varphi(\bar{b}) = f_\varphi(\bar{c}).$$

Proof. (1) $\implies$ (2): apply uniform elimination of imaginaries to $\mathbb{M}^n/E$, where $E(\bar{b}, \bar{c}) \iff (\varphi(\mathbb{M}, \bar{b}) = \varphi(\mathbb{M}, \bar{c}))$.

(2) $\implies$ (1): given a 0-interpretable set D/E, note that

$$E(\bar{b}, \bar{c}) \iff E(\mathbb{M}, \bar{b}) = E(\mathbb{M}, \bar{c}) \iff f_E(\bar{b}) = f_E(\bar{c})$$

for $\bar{b}, \bar{c} \in D$. So we have a 0-definable function on D satisfying condition (2) of Definition 24. □

Corollary 35. *If T has uniform elimination of imaginaries, then every definable set has a code in $\mathbb{M}$.*

Proof. Combine Lemma 33 and Proposition 34. □

Corollary 36. *Every definable set has a code in $\mathbb{M}^{\mathrm{eq}}$.*

Proof. $\mathbb{M}^{\mathrm{eq}}$ has uniform elimination of imaginaries (Remark 27). □

Proposition 37. *The following are equivalent:*

1. *T has elimination of imaginaries.*
2. *Every definable $D \subseteq \mathbb{M}^n$ has a code in $\mathbb{M}$.*

Proof. (1) $\implies$ (2): given D take a code $e \in \mathbb{M}^{\mathrm{eq}}$, then take $\bar{b} \in \mathbb{M}^m$ interdefinable with e. Then

$$\mathrm{Aut}(\mathbb{M}/\bar{b}) = \mathrm{Aut}(\mathbb{M}/e) = \{\sigma \in \mathrm{Aut}(\mathbb{M}) : \sigma(D) = D\}.$$

so $\bar{b}$ is a code for D.

(2) $\implies$ (1): if $e \in D/E \subseteq \mathbb{M}^{\mathrm{eq}}$, then e codes a definable set X (namely $X = e$, see Example 32). By (2), some $\bar{b} \in \mathbb{M}^m$ codes X. By uniqueness of codes (Remark 29), e is interdefinable with $\bar{b}$. □

We write $\ulcorner D \urcorner$ for "the" code of D, which is unique up to interdefinability. If elimination of imaginaries holds, we can take $\ulcorner D \urcorner$ in $\mathbb{M}$; otherwise it's in $\mathbb{M}^{\text{eq}}$. By Remark 29, $\operatorname{dcl}^{\text{eq}}(\ulcorner D \urcorner)$ is uniquely determined, and by Remark 30, $\operatorname{dcl}^{\text{eq}}(\ulcorner D \urcorner)$ is the smallest definably closed $A \subseteq \mathbb{M}^{\text{eq}}$ defining D.

6 Elimination of imaginaries and naming parameters

Proposition 38. *Uniform elimination of imaginaries is preserved by naming parameters.*

Proof. We claim that condition (2) in Proposition 34 is preserved by naming a set of parameters $A \subseteq \mathbb{M}$.

Suppose the condition holds in the original L-structure $\mathbb{M}$. Let $\psi(\bar{x}; \bar{y})$ be an $L(A)$-formula. Write $\psi(\bar{x}, \bar{y})$ as $\varphi(\bar{x}; \bar{y}, \bar{a})$ for some L-formula φ and some tuple $\bar{a} \in A$. Take a 0-definable function $f(\bar{y}, \bar{z})$ such that

$$\varphi(\mathbb{M}; \bar{b}, \bar{a}) = \varphi(\mathbb{M}; \bar{b}', \bar{a}') \iff f(\bar{b}, \bar{a}) = f(\bar{b}', \bar{a}').$$

Let $g(\bar{y})$ be the A-definable function $g(\bar{y}) = f(\bar{y}, \bar{a})$. Then

$$\psi(\mathbb{M}; \bar{b}) = \psi(\mathbb{M}; \bar{b}') \iff \varphi(\mathbb{M}; \bar{b}, \bar{a}) = \varphi(\mathbb{M}; \bar{b}', \bar{a}) \iff f(\bar{b}, \bar{a}) = f(\bar{b}', \bar{a}) \iff g(\bar{b}) = g(\bar{b}').$$

Therefore, condition (2) in Proposition 34 holds in the $L(A)$-structure. $\square$

Proposition 39. *Elimination of imaginaries is preserved by naming parameters.*

Proof. Condition (2) in Proposition 37 is preserved. Let $\mathbb{M}_A$ be $\mathbb{M}$ as an $L(A)$-structure. Then $\mathbb{M}$ and $\mathbb{M}_A$ have the same definable sets, and

$$(\bar{b} \text{ codes } D \text{ in } \mathbb{M}) \implies (\bar{b} \text{ codes } D \text{ in } \mathbb{M}_A)$$

because $\operatorname{Aut}(\mathbb{M}_A) \subseteq \operatorname{Aut}(\mathbb{M})$. $\square$

Corollary 40. *If T has elimination of imaginaries and D/E is $\mathbb{M}$-interpretable, then there is an $\mathbb{M}$-interpretable bijection $D/E \to X$ where X is $\mathbb{M}$-definable.*

Proof. If $|\mathbb{M}| \leq 1$, then $D \to D/E$ is a bijection. Otherwise, take $A \subseteq \mathbb{M}$ such that D, E are A-definable and $|A| \geq 2$. After naming A, D/E is 0-interpretable in $\mathbb{M}_A$. Elimination of imaginaries is preserved, and $\operatorname{dcl}(\varnothing) \supseteq A$, so uniform elimination of imaginaries holds in $\mathbb{M}_A$ by Theorem 26. Thus there is an A-definable X and an A-interpretable bijection $f : D/E \to X$. $\square$

7 Elimination of imaginaries in Peano Arithmetic and ACF

Theorem 41. *If T is a completion of Peano Arithmetic, then T has uniform elimination of imaginaries.*

Proof. Fix 0-interpretable D/E where $D \subseteq \mathbb{M}^n$. Take lexicographic order on $\mathbb{M}^n$. The induction axiom implies $\min(X)$ exists for any non-empty definable $X \subseteq \mathbb{M}^n$. Let $f : D/E \to \mathbb{M}^n$ be $f(X) = \min(X)$. Then f is a 0-interpretable injection. $\square$

Next consider ACF_0. Fix a monster model $\mathbb{M}$.

Fact 42. *If $S \subseteq \mathbb{M}^n$ is finite, then S has a code.*

The $n = 1$ case was Example 31. Here is a proof of $n = 2$. (The general case is similar.)

Proof. For $q \in \mathbb{Q}$ let $\pi_q : \mathbb{M}^2 \to \mathbb{M}$ be $\pi_q(x, y) = y - qx$. Each $\pi_q(S)$ is a finite subset of $\mathbb{M}$, so has a code in $\mathbb{M}$ by Example 31. Let $A = \{\ulcorner \pi_q(S) \urcorner : q \in \mathbb{Q}\} \subseteq \mathbb{M}$.
Claim. If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $\sigma(S) = S \iff \sigma \in \mathrm{Aut}(\mathbb{M}/A)$.

Proof. $\Rightarrow$: Easy: if $\sigma(S) = S$ then $\sigma(\pi_q(S)) = \pi_q(\sigma(S)) = \pi_q(S)$, and so $\sigma(\ulcorner \pi_q(S) \urcorner) = \ulcorner \pi_q(S) \urcorner$. Therefore σ fixes A pointwise.

$\Leftarrow$: Suppose $S' = \sigma(S) \neq S$. Then $S' \not\subset S$ and $S \not\subset S'$ (since $|S'| = |S|$). Therefore $S' \cup S \supsetneq S$. The map $\pi_q : S' \cup S \to \mathbb{M}$ is injective for all but finitely many $q \in \mathbb{Q}$. (Consider the finite set of lines through two points in $S \cup S'$, and take q not equal to the slope of any of these lines.) Fix a $q \in \mathbb{Q}$ such that π_q is injective on $S' \cup S$. Then

$$|\pi_q(S) \cup \pi_q(S')| = |\pi_q(S \cup S')| = |S \cup S'| > |S| \geq |\pi_q(S)|.$$

Therefore $\pi_q(S) \neq \pi_q(S') = \sigma(\pi_q(S))$, and so σ doesn't fix $\ulcorner \pi_q(S) \urcorner \in A$. $\square_{\text{Claim}}$

Then S is A-invariant hence A-definable. Take $\bar{b} \in A$ defining S. For any σ,

$$\sigma(\bar{b}) = \bar{b} \implies \sigma(S) = S \implies \sigma \in \mathrm{Aut}(\mathbb{M}/A) \implies \sigma(\bar{b}) = \bar{b}.$$

Therefore $\bar{b}$ codes S. $\square$

Lemma 43. *If $A \subseteq \mathbb{M}^{\mathrm{eq}}$ and $D \subseteq \mathbb{M}^n$ is non-empty and A-definable then there is $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$ with $\bar{b} \in D$.*

Proof. By induction on n.

- $n = 1$
  - D is finite. Then $D \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. Take any $b \in D$.
  - D is cofinite. Then $D \cap \mathbb{Q} \neq \varnothing$. Take any $b \in D \cap \mathbb{Q}$.
- $n > 1$: Let $D' = \{\bar{b} \in \mathbb{M}^{n-1} : \exists c \in \mathbb{M}\ (\bar{b}, c) \in D\}$. Then D' is non-empty and A-definable. By induction there is $\bar{b} \in D'$, $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$. Let $D'' = \{c \in \mathbb{M} : (\bar{b}, c) \in D\}$. Then D'' is non-empty and $A\bar{b}$-definable. By induction there is $c \in D''$ with

  $$c \in \mathrm{acl}^{\mathrm{eq}}(A\bar{b}) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{acl}^{\mathrm{eq}}(A).$$

  Then $(\bar{b}, c) \in \mathrm{acl}^{\mathrm{eq}}(A)$ and $(\bar{b}, c) \in D$. $\square$

Theorem 44. ACF_0 *has uniform elimination of imaginaries.*

Proof. $\mathrm{ACF}_0 \vdash 0 \neq 1$, so it suffices to show elimination of imaginaries by Theorem 26. Take e in a 0-interpretable set D/E. By Example 32, e is the code of an E-equivalence class X (namely $X = e$). By Lemma 43 there is $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(e)$ with $\bar{a} \in X$. Let $S = \{\sigma(\bar{a}) : \sigma \in \mathrm{Aut}(\mathbb{M}/e)\}$. Note $S \subseteq X$ because $\bar{a} \in X$ and X is e-invariant. By Proposition 14, S is finite and e-definable. Take $\ulcorner S \urcorner \in \mathbb{M}^m$ by Fact 42. Then $\ulcorner S \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(e)$. On the other hand, X is the unique E-equivalence class containing S, so X and e are $\ulcorner S \urcorner$-definable, and $e \in \mathrm{dcl}^{\mathrm{eq}}(\ulcorner S \urcorner)$.[9] Thus e is interdefinable with $\ulcorner S \urcorner \in \mathbb{M}^m$. $\square$

With some modifications to the proof, one can also handle algebraically closed fields of positive characteristic:

Fact 45. ACF *has uniform elimination of imaginaries.*

[9]More precisely, if $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner$, then $\sigma(S) = S$. As D, E are 0-definable, $\sigma(X)$ is some E-equivalence class. But $S \subseteq X \implies \sigma(S) \subseteq \sigma(X)$. So $S \subseteq \sigma(X)$. Therefore $\sigma(X)$ must be the same E-equivalence class as X. Then $\sigma(X) = X$. This argument shows $\sigma(\ulcorner S \urcorner) = \ulcorner S \urcorner \implies \sigma(X) = X$, which means X is $\ulcorner S \urcorner$-definable.