2. More back-and-forth

Introductory Model Theory

September 23, 2021

The recommended reading for today is Chapter 1 of Poizat.

1 Review from last time

If $(M, R), (M', R')$ are binary relations, then $S_0(M, M')$ is the set of local isomorphisms from M to M'. (A local isomorphism is an isomorphism from a finite restriction of M to a finite restriction of M'.)

Definition 1. Let (M, R) and (M', R') be binary relations. A *Karpian family* for M and M' is a set $K \subseteq S_0(M, M')$ satisfying the following two conditions for any $f \in K$:

1. **(forth)** If $a \in M$ then there is $g \in K$ with $g \supseteq f$ and $a \in \mathrm{dom}(g)$.
2. **(back)** If $b \in M'$ then there is $g \in K$ with $g \supseteq f$ and $b \in \mathrm{im}(g)$.

M and M' are *∞-equivalent*, written $M \sim_\infty M'$, if there is a non-empty Karpian family.

Recall that $M \cong M'$ means that M and M' are isomorphic (i.e., there is an isomorphism between them). The relation $\cong$ is an equivalence relation.

Theorem 2. *Let (M, R) and (M', R') be binary relations.*

- *If $M \cong M'$, then $M \sim_\infty M'$.*
- *If M, M' are countable and $M \sim_\infty M'$, then $M \cong M'$.*

Definition 3. A *dense linear order without endpoints* (DLO) is a binary relation $(C, \leq)$ satisfying the conditions:

1. $\leq$ is reflexive: $\forall x \in C : x \leq x$.
2. $\leq$ is transitive: $\forall x, y, z \in C : (x \leq y \text{ and } y \leq z) \implies x \leq z$.
3. $\leq$ is anti-symmetric: $\forall x, y \in C : (x \leq y \text{ and } y \leq x) \implies x = y$.
4. $\leq$ is total: $\forall x, y \in C : (x \leq y \text{ or } y \leq x)$.

5. $C \neq \varnothing$.

6. $(C, \leq)$ is dense: $\forall x, y \in C : x < y \implies \exists z \in C : x < z < y$,
   where $x < y$ means $x \leq y$ and $x \neq y$.

7. $(C, \leq)$ has no endpoints: $\forall x \in C\ \exists y, z \in C : y < x < z$.

Conditions 1–3 say that $(C, \leq)$ is a partial order, and conditions 1–4 say that $(C, \leq)$ is a linear order. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are DLOs, but $(\mathbb{Z}, \leq)$ and $([0,1], \leq)$ and $(\mathbb{Q}, <)$ are not.

Theorem 4. *If $(C, \leq)$, $(C', \leq)$ are DLOs, then $S_0(C, C')$ is Karpian and $C \sim_\infty C'$.*

Definition 5. Let (M, R) and (M', R') be binary relations. For $p \leq \omega$, the set $S_p(M, M')$ of *p-isomorphisms* from M to M' is defined recursively as follows:

- For $p = 0$, $S_0(M, M')$ is the set of local isomorphisms from M to M'.

- For $0 < p < \omega$, $S_p(M, M')$ is the set of $f \in S_0(M, M')$ satisfying the following:

  1. **(forth)** For any $a \in M$, there is $g \in S_{p-1}(M, M')$ with $g \supseteq f$ and $a \in \mathrm{dom}(g)$.
  2. **(back)** For any $b \in M'$ there is $g \in S_{p-1}(M, M')$ with $g \supseteq f$ and $b \in \mathrm{im}(g)$.

- For $p = \omega$, $S_\omega(M, M') = \bigcap_{i=0}^{\infty} S_p(M, M')$.

For example, f is a 1-isomorphism if for any $a \in M$ there is a local isomorphism g extending f with $a \in \mathrm{dom}(g)$ and for any $b \in M$ there is a local isomorphism g extending f with $b \in \mathrm{im}(g)$. And f is an ω-isomorphism if it is a p-isomorphism for all $p = 0, 1, 2, 3, \ldots$.

Theorem 6. *If $g \in S_p(M, M')$ and $f \subseteq g$, then $f \in S_p(M, M')$.*

This says that restrictions of p-isomorphisms are p-isomorphisms.

Definition 7. M and M' are *p-equivalent* ($M \sim_p M'$) if the following equivalent conditions hold:

- $S_p(M, M') \neq \varnothing$ (there is at least one p-isomorphism).

- $\varnothing \in S_p(M, M')$ (the empty function is a p-isomorphism).

M and M' are ***elementarily equivalent*** ($M \equiv M'$) if they are ω-equivalent.

The symbol $\equiv$ means the same thing as $\sim_\omega$. We saw that $M \sim_\omega M'$ if and only if $\forall p < \omega : M \sim_p M'$.

2 Ehrenfeucht-Fraïssé games

Definition 8. Let $(M, R), (M', R')$ be binary relations. The Ehfrenfeucht-Fraïssé game of length n, denoted $\mathrm{EF}_n(M, M')$, is played as follows.

- There are two players, the Duplicator and Spoiler.
- There are n rounds.
- In the ith round, the Spoiler chooses either an $a_i \in M$ or a $b_i \in M'$.
- The Duplicator responds with a $b_i \in M'$ or an $a_i \in M$, respectively.
- At the end of the game, the Duplicator wins if
$$\{(a_1, b_1), \ldots, (a_n, b_n)\}$$
is a local isomorphism from R to R'.
- Otherwise, the Spoiler wins.

Lemma. *Suppose we are playing $\mathrm{EF}_n(M, M')$ and there have been q rounds so far, with $p = n - q$ rounds remaining. Suppose the moves so far are $(a_1, b_1), \ldots, (a_n, b_n)$. Let $f = \{(a_1, b_1), \ldots, (a_q, b_q)\}$. Then the following are equivalent:*

- *Duplicator has a winning strategy.*
- *f is a p-isomorphism.*

Proof. By induction on p.

- $p = 0$. Then the game is over, so Duplicator wins if and only if $f \in S_0(M, M')$.
- $p > 0$. If f isn't a local isomorphism, then Duplicator will definitely lose, and f isn't a p-isomorphism. So we may assume $f \in S_0(M, M')$. Then the following are equivalent:
  - Duplicator wins.
  - For any $a_{q+1} \in M$, there is a $b_{q+1} \in M'$ such that Duplicator wins in the position $(a_1, b_1, \ldots, a_{q+1}, b_{q+1})$, AND for any $b_{q+1} \in M'$ there is an $a_{q+1} \in M$ such that Duplicator wins in the position $(a_1, b_1, \ldots, a_{q+1}, b_{q+1})$.
  - For any $a_{q+1} \in M$ there is a $b_{q+1} \in M'$ such that $f \cup \{(a_{q+1}, b_{q+1})\} \in S_{p-1}(M, M')$, AND for any $b_{q+1} \in M'$ there is $a_{q+1} \in M$ such that $f \cup \{(a_{q+1}, b_{q+1})\} \in S_{p-1}(M, M')$.
  - For any $a_{q+1} \in M$ there is $g \in S_{p-1}(M, M')$ such that $g \supseteq f$ and $a_{q+1} \in \mathrm{dom}(g)$, AND for any $b_{q+1} \in M'$ there is $g \in S_{p-1}(M, M')$ such that $g \supseteq f$ and $b_{q+1} \in \mathrm{im}(g)$.
  - $f \in S_p(M, M')$. □

Theorem. *If M is p-equivalent to M', then $\mathrm{EF}_p(M, M')$ is a win for the Duplicator. Otherwise it is a win for the Spoiler.*

Proof. Take $q = 0$ and $n = p$ in the lemma. □

3 More about p-isomorphisms

Theorem. *Every $(p+1)$-isomorphism is a p-isomorphism.*

Proof. By induction on p.
$p = 0$: every 1-isomorphism is a 0-isomorphism. True by definition.
$p > 0$. Suppose s is a $p+1$-isomorphism. We claim s is a p-isomorphism.

(forth) Given $a \in M$, there is $t \in S_p(M, M')$ such that $t \supseteq s$ and $a \in \mathrm{dom}(t)$. By induction, $t \in S_{p-1}(M, M')$.

(back) Given $b \in M'$, there is $t \in S_p(M, M')$ such that $t \supseteq s$ and $b \in \mathrm{im}(t)$. By induction, $t \in S_{p-1}(M, M')$. □

So $S_0(M, M') \supseteq S_1(M, M') \supseteq S_2(M, M') \supseteq \cdots$. In terms of the Ehfrenfeuch-Fraïssé game, if we reduce the number of remaining rounds, it can only help the Duplicator.

Theorem. *Suppose $s \in S_p(M, M')$ and $t \in S_p(M', M'')$ and $\mathrm{dom}(t) = \mathrm{im}(s)$. Then $u := t \circ s \in S_p(M, M'')$.*

Proof. By induction on p. For $p = 0$, this says we can compose (local) isomorphisms; this is easy.
Suppose $p > 0$.

(forth) Given $a \in M$, there is $b \in M'$ such that $s' := s \cup \{(a, b)\} \in S_{p-1}(M, M')$. There is $c \in M''$ such that $t' := t \cup \{(b, c)\} \in S_{p-1}(M', M'')$. By induction, $u' := t' \circ s' = u \cup \{(a, c)\} \in S_{p-1}(M, M'')$.

(back) Similar. □

Corollary. *If $M \sim_p M'$ and $M' \sim_p M''$, then $M \sim_p M''$.*

Theorem. *Suppose $s \in S_p(M, M')$. Then $s^{-1} \in S_p(M', M)$.*

Proof. Exercise. □

Corollary. *If $M \sim_p M'$, then $M' \sim_p M$.*

Theorem. *Let K be a Karpian family for (M, R) and (M', R'). Then $K \subseteq S_p(M, M')$ for all p.*

Proof. By induction on p. For $p = 0$, we have $s \in K \subseteq S_0(M, M')$ by definition.
Suppose $p > 0$:

(forth) For any $a \in M$ there is $t \in K$ with $t \supseteq s$ and $a \in \mathrm{dom}(t)$. By induction $t \in S_{p-1}(M, M')$.

(back) Similar. □

Corollary. *If M, M' are DLOs, then $S_0(M, M') = S_p(M, M')$ for all p. $M \sim_\omega M'$.*

Corollary. $A \cong B \implies A \sim_\infty B \implies A \sim_\omega B \implies A \sim_p B$.

Corollary. *$\sim_p$ and $\sim_\omega$ are equivalence relations.*

4 More dense linear orders

Theorem. *Suppose* $(\mathbb{Q}, \leq) \sim_\omega (C, R)$. *Then* (C, R) *is a DLO.*

Proof. Suppose (C, R) is not a DLO and break into cases:

- R not reflexive. Spoiler chooses $b_1 \in C$ such that $(b_1, b_1) \notin R$. Then Duplicator must choose $a_1 \in \mathbb{Q}$ such that $a_1 \not\leq a_1$, impossible.

- R not antisymmetric. Spoiler chooses $b_1, b_2 \in C$ such that $b_1 \neq b_2$ but $b_1 R b_2$ and $b_2 R b_1$. Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ such that $a_1 \neq a_2$ and $a_1 \leq a_2$ and $a_2 \leq a_1$, impossible.

- R not transitive. Spoiler chooses $b_1, b_2, b_3 \in C$ such that $b_1 R b_2$ and $b_2 R b_3$ but $b_1 \not R\, b_3$. Duplicator must choose a_1, a_2, a_3 with $a_1 \leq a_2 \leq a_3$ and $a_1 \not\leq {}_3$, impossible.

- R not total. Spoiler chooses $b_1, b_2 \in C$ with $b_1 \not R\, b_2$ and $b_2 \not R\, b_1$. Again, Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ with $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, impossible.

- (C, R) has a maximum. Spoiler chooses $b_1 = \max(C)$. Duplicator chooses some $a_1 \in \mathbb{Q}$. Spoiler chooses $a_2 \in \mathbb{Q}$ greater than a_1. Spoiler must choose $b_2 \in C$ with $b_2 > b_1$, impossible.

- (C, R) has a minimum. Similar.

- (C, R) is not dense. Spoiler chooses $b_1, b_2 \in C$ with $b_1 < b_2$ with nothing between them. Duplicator must choose $a_1, a_2 \in \mathbb{Q}$ with $A_1 < a_2$. Spoiler then chooses $a_3 = (a_1 + a_2)/2$, so that $a_1 < a_3 < a_2$. Duplicator must choose $b_3 \in C$ with $b_1 < b_3 < b_2$, impossible. □

Corollary. *The class of DLOs is the* $\sim_\omega$*-equivalence class of* $(\mathbb{Q}, \leq)$

5 Discrete linear orders

Definition. A linear order $(C, \leq)$ is *discrete* without endpoints if $C \neq \varnothing$ and

$$\forall a \exists b : a \lhd b$$
$$\forall b \exists a : a \lhd b,$$

where $a \lhd b$ means $a < b$ and not $\exists c : a < c < b$.

Example. $(\mathbb{Z}, \leq)$ is a discrete linear order without endpoints. So is $(C, \leq)$, where

$$\begin{aligned} C =& \{\ldots, -3, -2, -1\} \\ & \cup \{-1/2, -1/3, -1/4, -1/5, \ldots\} \\ & \cup \{\ldots, 1/5, 1/4, 1/3, 1/2\} \\ & \cup \{1, 2, 3, \ldots\}. \end{aligned}$$

Definition. Let $(C, <)$ be discrete. If $a \leq b \in C$, then $d(a, b)$ is the size of $[a, b) = \{x \in C : a \leq x < b\}$, or ∞ if infinite. If $a > b$, then $d(a, b) = d(b, a)$.

Lemma. *Let $(C, <)$ and $(C', <)$ be discrete linear orders without endpoints. Suppose $a_1 < \cdots < a_n$ in C and $b_1 < \cdots < b_n$ in C'. Let f be the local isomorphism $f(a_i) = b_i$. Suppose that for every $1 \leq i < n$, we have*

$$d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or } d(a_i, a_{i+1}) \geq 2^p \leq d(b_i, b_{i+1}).$$

Then f is a p-isomorphism.

Here is the idea: a 0-isomorphism needs to respect the order. A 1-isomorphism needs to respect the order plus the relation $d(x, y) = 1$. A 2-isomorphism needs to respect the order plus the relations $d(x, y) = i$, for $i = 1, 2, 3$. A 3-isomorphism needs to respect the order plus the relations $d(x, y) = i$ for $i = 1, 2, 3, \ldots, 7$.

Proof. By induction on p. $p = 0$ is trivial.

Suppose $p > 0$. We verify the forth condition (back is similar). Let $a \in C$ be given. We must find $b \in C'$ such that $f \cup \{(a, b)\}$ is a $(p-1)$-isomorphism. Break into cases:

- If $a < a_1$ and $d(a, a_1) = q < \infty$, take $b < b_1$ such that $d(b, b_1) = q$.
- If $a < a_1$ and $d(a, a_1) = \infty$, take $b < b_1$ such that $d(b, b_1) = 2^{p-1}$.
- If $a > a_n$, do something similar.
- If $a_i < a < a_{i+1}$, then we need to choose b between b_i and b_{i+1}. If $d(a_i, a) < 2^{p-1}$ we need $d(b_i, b) = d(a_i, a)$. If $d(a_i, a) \geq 2^{p-1}$ then we need $d(b_i, b) \geq 2^{p-1}$. On the other side, if $d(a, a_{i+1}) < 2^{p-1}$ then we need $d(b, b_{i+1}) = d(a, a_{i+1})$. If $d(a, a_{i+1}) \geq 2^{p-1}$ then we need $d(b, b_{i+1}) \geq 2^{p-1}$. Here is how we proceed:
  - If $d(a_i, a_{i+1}) < 2^p$, then $d(b_i, b_{i+1}) = d(a_i, a_{i+1})$. Take b with $b_i < b < b_{i+1}$ and $d(b_i, b) = d(a_i, a)$.
  - If $d(a_i, a_{i+1}) \geq 2^p$, then $d(b_i, b_{i+1}) \geq 2^p$. There are three cases:
    * If $d(a_i, a) = q < 2^{p-1}$, then take $b > b_i$ with $d(b_i, b) = q$.
    * If $d(a, a_{i+1}) = q < 2^{p-1}$, take $b < b_{i+1}$ with $d(b, b_{i+1}) = q$.
    * If $d(a_i, a) \geq 2^{p-1} \leq d(a_{i+1}, a)$, take $b > b_i$ with $d(b_i, b) = 2^{p-1}$.
- If $a = a_i$, take $b = b_i$. □

In fact, Theorem 1.8 in Poizat shows that this condition exactly characterizes p-isomorphisms.

Theorem. *Let $(C, \leq)$ and $(C', \leq)$ be discrete linear orders without endpoints. Then $\varnothing$ is a p-equivalence from $(C, \leq)$ to $(C', \leq)$ for all p. Therefore $(C, \leq) \sim_\omega (C', \leq)$.*

Remark. Again, one can show that if $(\mathbb{Z}, \leq) \sim_\omega (C, R)$, then (C, R) is a discrete linear order without endpoints. So the discrete linear orders without endpoints are exactly the $\sim_\omega$-equivalence class of $(\mathbb{Z}, \leq)$.

Remark. Let C be the set

$$\begin{aligned} C = &\{\ldots, -3, -2, -1\} \\ &\cup \{-1/2, -1/3, -1/4, -1/5, \ldots\} \\ &\cup \{\ldots, 1/5, 1/4, 1/3, 1/2\} \\ &\cup \{1, 2, 3, \ldots\}. \end{aligned}$$

Then C is a discrete linear order without endpoints, so $(C, \leq) \sim_\omega (\mathbb{Z}, \leq)$. But $(C, \leq) \not\sim_\infty (\mathbb{Z}, \leq)$, since $(C, \leq) \ncong (\mathbb{Z}, \leq)$. So $\sim_\infty$ is stronger than $\sim_\omega$.

6 The infinite Ehrenfeucht-Fraïssé game

Definition 9. Let R, R' be binary relations with universes M, M'. The *infinite Ehrenfeucht-Fraïssé game*, denoted $\mathrm{EF}_\infty(M, M')$, is played as follows:

- There are two players, the Duplicator and Spoiler.
- There are infinitely many rounds (indexed by ω).
- In the nth round, the Spoiler chooses either an $a_n \in M$ or a $b_n \in M'$.
- The Duplicator responds with a $b_n \in M'$ or an $a_n \in M$, respectively.
- If $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ is *not* a local isomorphism, then the Spoiler immediately wins.
- The Duplicator wins if the Spoiler has not won by the end of the game.

Theorem. *The following are equivalent:*

1. $R \sim_\infty R'$*, i.e., there is a non-empty Karpian family* K*.*
2. *Duplicator has a winning strategy for* $\mathrm{EF}_\infty(M, M')$*.*
3. *Spoiler does not have a winning strategy for* $\mathrm{EF}_\infty(M, M')$*.*