Homework8

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Exercise 0.0.1. Show that T is $\aleph_0$-categorical and complete

Proof. For any models (M,E) and (N,E') of T s.t. $|M| = |N| = \aleph_0$, let M_1, M_2 and N_1, N_2 be the two equivalence classes of M and N respectively, then $|M_1| = |M_2| = |N_1| = |N_2|$. Then there are bijections $f : M_1 \to N_1$ and $g : M_2 \to N_2$. Let $h = f \cup g$. Then for any $a, b \in M$, $aEb \Leftrightarrow h(a)E'h(b)$, therefore h is a homomorphism and hence an isomorphism.

For any models (M,E) and (N,E') of T and suppose $(M,E) \vDash \varphi$, then by Löwenheim–Skolem Theorem, there is $(M_0,E_0) \preceq (M,E)$ and $(N_0,E_0') \preceq (N,E)$ s.t. $|M_0| = |N_0| = \aleph_0$. Then $(M,E) \vDash \varphi \Rightarrow (M_0,E_0) \vDash \varphi \Rightarrow (N_0,E_0') \vDash \varphi \Rightarrow (N,E') \vDash \varphi$. Therefore for any φ, either $T \vDash \varphi$ or $T \vDash \neg\varphi$ and T is complete □

Exercise 0.0.2. Show that T is $\aleph_0$-stable

Proof. For any countable $M \preceq \mathbb{M}$ and non-constant type $p(x) \in S_1(M)$, then $x \neq a \in p(x)$ for any $a \in M$. But for any $a \in M$, either $xEa \in p(x)$ or $\neg xEa \in p(x)$. Therefore there is only two non-constant types in $S_1(M)$ and thus $|S_1(M)| = |M| + 2 = \aleph_0$ and T is $\aleph_0$-stable □

Lemma 0.1. *T has quantifier elimination*

Proof. For any models M, N of T and A a common substructure of M and N and for all formula φ of the form $\exists y\psi(y)$ where ψ is a quantifier free $L(A)$-formula with parameter set D, then ψ is equivalent to

$$\bigwedge_{b\in B} yEb \wedge \bigwedge_{c\in C} \neg yEb \wedge \theta$$

where $B, C \subseteq D$ and θ is a quantifier free sentence in $L(D)$. If $M \vDash \varphi$, then $M \vDash \theta$ and so $A \vDash \theta$ and $N \vDash \theta$. Also there is $e \in M$ such that

$M \vDash \bigwedge_{b\in B} eEb \wedge \bigwedge_{c\in C} \neg eEb$, then B and C are in different equivalence class. Since each class is infinite, we can find such e' in N in the same equivalence class as B, therefore $N \vDash \psi(e')$ and so $N \vDash \varphi$. Therefore T has quantifier elimination □

Exercise 0.0.3. Show that $S_1(\emptyset)$ has a single point: if $a, b \in \mathbb{M}$, then $\mathrm{tp}(a) = \mathrm{tp}(b)$

Proof. Since T has quantifier elimination and there is no constant symbol, every formula is equivalent or $\top$ or $\bot$. Therefore there is only one type in $S_1(\emptyset)$. □

Exercise 0.0.4. $\mathrm{acl}(\emptyset) = \emptyset$

Proof. Since T has quantifier elimination, for any $\varphi(x)$, $\varphi(\mathbb{M})$ is either $\mathbb{M}$ or $\emptyset$, therefore $\mathrm{acl}(\emptyset) = \emptyset$ □

Exercise 0.0.5. Let X be one of the equivalence classes. Show that X is $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable but not $\mathrm{acl}(\emptyset)$-definable

Proof. Suppose $\mathbb{M}/E = \{e_1, e_2\}$. Then $\{e_1, e_2\} \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. If $e_1 = [a]_E$ for some $a \in X$, then X is e_1-definable and therefore $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable.

Since $\mathrm{acl}(\emptyset) = \emptyset$ and the only 0-definable sets are $\emptyset$ and $\mathbb{M}$ by quantifier elimination, X is not $\mathrm{acl}(\emptyset)$-definable □

Exercise 0.0.6. Let p be the unique 1-type over $\emptyset$. Let q be a global non-forking extension. Show that q is $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable but not $\mathrm{acl}(\emptyset)$-definable

Proof. By Proposition 5.6, $q \sqsupseteq p$ iff $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable.

If q is 0-definable, then for any φ there is $d\varphi(x) \in L$ such that $\varphi(x, b) \in q \Leftrightarrow \mathbb{M} \vDash d\varphi(b)$. But $d\varphi(\mathbb{M})$ is either $\mathbb{M}$ or $\emptyset$. Hence $\varphi(x, b) \in q \Leftrightarrow \mathbb{M} \vDash \forall y \varphi(x, y)$. However, $(\forall y \varphi(x, y)) \vee (\forall y \neg\varphi(x, y))$ is not always true and q is only a partial type □