# Stability

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# 1 Preface

A combination of various notes from [Pillay(2018)] [Chernikov(2019)] [Tent and Ziegler(2012)] [van den Dries(2019)]

A monster model $\mathfrak{C}$

[Pillay(2018)] has many typos😅

## 2 Preliminaries

### 2.1 Indiscernibles

**Definition 2.1.** Let $I$ be a linear order and $\mathfrak{A}$ an $L$-structure. A family $(a_i)_{i\in I}$ of elements of $A$ is called a **sequence of indiscernibles** if for all $L$-formulas $\varphi(x_1, \dots, x_n)$ and all $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ from $I$

$$\mathfrak{A} \vDash \varphi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n})$$

or

$$\mathrm{tp}(a_{i_1}, \dots, a_{i_n}) = \mathrm{tp}(a_{j_1}, \dots, a_{j_n})$$

**Theorem 2.2.** *Compactness let us "stretch" indiscernibles. Let $(a_i : i \in \omega)$ be indiscernibles in $\mathfrak{C}$, and $(I, <)$ an ordering. Then there exists an indiscernible $(b_i : i \in I)$ in $\mathfrak{C}$ s.t. $\forall i_1 < \dots < i_n \in I$*

$$\mathrm{tp}(a_1, \dots, a_n) = \mathrm{tp}(b_{i_1}, \dots, b_{i_n})$$

Indiscernible sequence are a fundamental tool of model theory, and there are many ways to obtain them.

**Theorem 2.3** (Ramsey, extended)**.** *Let $n_1, \dots, n_r < \omega$. For each $i = 1, \dots, r$, let $X_{i,1}$, $X_{i,2}$ be a partition of $[\omega]^{n_i}$. Then there is an infinite subset $Y \subseteq \omega$ which is homogeneous, i.e., $\forall i = 1, \dots, r$, either $[Y]^{n_i} \subseteq X_{i,1}$ or $[Y]^{n_i} \subseteq Y_{i,2}$*

**Proposition 2.4.** *For each $n \in \omega$, let $\Sigma_n(x_1, \dots, x_n)$ be a collection of $L$-formulas in variables $x_1, \dots, x_n$. Suppose that there are $a_1, a_2, \dots \in \mathfrak{C}$ s.t.*

$$\vDash \Sigma_n(a_{i_1}, \dots, a_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

*Then there is an indiscernible $(b_i : i \in \omega)$ in $\mathfrak{C}$ s.t.*

$$\vDash \Sigma_n(b_{i_1}, \dots, b_{i_n}), \quad \forall i_1 < \dots < i_n < \omega$$

we can expand $\bigcup_{n\in\omega} \Sigma_n$ and obtain the Ehrenfeucht-Mostowski type $\mathrm{EM}((a_i)_{i\in\omega})$. This is just the Standard Lemma in Tent

**Example 2.1.** Suppose $\Sigma_2 = \{x_1 \neq x_2\}$. Then the proposition yields the existence of infinite indiscernible sequences

*Proof.* Consider

$$\begin{aligned}\Gamma(x_1, x_2, \dots) = \{&\varphi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \varphi(x_{j_1}, \dots, x_{j_n}) : \\ &i_1 < \dots < i_n, j_1 < \dots < j_n \in \omega, \varphi \in L\} \\ \cup &\bigcup_n \Sigma(x_1, \dots, x_n)\end{aligned}$$

Let $\Gamma'(x_1, \dots, x_n) \subseteq_f \Gamma$. Let $\varphi_1, \dots, \varphi_r$ be the $L$-formulas appearing in $\Gamma'$. For $i = 1, \dots, r$, let

$$X_{i,1} = \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \models \varphi_i(a_{j_1}, \dots, a_{j_n})\}$$
$$X_{i,2} = \{(j_1, \dots, j_n) : j_1 < \dots < j_n \in \omega, \models \neg\varphi_i(a_{j_1}, \dots, a_{j_n})\}$$

By Ramsey's theorem, there exists an infinite $Y \subseteq \mathbb{N}$ s.t. $\forall i = 1, \dots, r$, $[Y]^{n_i}$ is either contained in $X_{i,1}$ or in $X_{i,2}$. Write $Y = \{k_1 < k_2 < \dots\}$. Interpret each $x_i$ as $a_{k_i}$ to satisfy $\Gamma'$ □

**Definition 2.5.** Let $M \prec N \prec \mathfrak{C}$ be models, and $p(\bar{x}) \in S_{\bar{x}}(N)$. We say $p$ is finitely satisfiable in $M$, or $p(\bar{x})$ is a **coheir** of $p \restriction M \in S_{\bar{x}}(M)$, if every $\varphi(\bar{x}) \in p(\bar{x})$ is satisfied by some $\bar{a} \in M$

*Remark.* $p(\bar{x}) \in S_n(N)$ is finitely satisfiable (f.s.) in $M$ iff $p(\bar{x})$ is in the topological closure of $\{\text{tp}(\bar{a}/N) : \bar{a} \in M\} \subseteq S_n(N)$

**Lemma 2.6.** *Suppose $p(\bar{x}) \in S_{\bar{x}}(M)$ and $M \prec N$, then there is $p'(\bar{x}) \in S_{\bar{x}}(N)$ s.t. $p \subseteq p'$ and $p'$ is f.s. in $M$*

*Proof.* Consider $\Gamma(\bar{x}) = p(\bar{x}) \cup \{\neg\varphi(\bar{x}) : \varphi(\bar{x}) \in L_N$ and not realized in $M\}$. Let $\Gamma \supseteq_f \Gamma' = \{\Psi(\bar{x}), \neg\varphi_1(\bar{x}), \dots, \neg\varphi_r(\bar{x})\} \in p$. Then any solution $\bar{a}$ of $\Psi$ in $M$ satisfies $\Gamma'$ as $M \models \forall\bar{x}(\neg\varphi_i(\bar{x}))$ □

*Remark.* Let $i_M : M^{\bar{x}} \to S_{\bar{x}}(M)$ s.t. $m \mapsto \text{tp}(m/M)$. Define $i_N : M^{\bar{x}} \to S_{\bar{x}}(N)$ similarly. Let $r : S_{\bar{x}}(N) \to S_{\bar{x}}(M)$. Note that $r \circ i_N = i_M$ and the set of types in $S_{\bar{x}}(N)$ that are f.s. in $M$ is exactly the closure of $i_N(M^{\bar{x}})$ in $S_{\bar{x}}(N)$. Hence its image under $r$ is closed. However the image must contain $i_M(M^{\bar{x}})$ which is dense in $S_{\bar{x}}(M)$. Therefore it must be onto, which proves the desired result

$r$ is continuous and $r(\overline{i_N(M^n)}) \supseteq i_M(M^n)$ is closed. $\overline{i_M(M^n)} = S_n(M)$. Then $r$ is onto? Then its preimage of $p$ is what we want

**Proposition 2.7.** *Let $p(\bar{x}) \in S_{\bar{x}}(M)$, $N \succ M$ be $|M|^+$-saturated, and $p'(\bar{x}) \in S_{\bar{x}}(N)$ a coheir of $p$. Let $\bar{a}_1, \bar{a}_2, \dots \in N$ be defined as follows*

$$\bar{a}_1 \text{ realises } p(\bar{x})$$
$$\bar{a}_2 \text{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1)$$
$$\bar{a}_3 \text{ realises } p'(\bar{x}) \restriction (M, \bar{a}_1, \bar{a}_2)$$
$$\dots$$

*Then $(\bar{a}_i : i \in \omega)$ is indiscernible over $M$*

*Proof.* We prove by induction on $k$ that for any $n \leq k$ and $i_1 < \cdots < i_n \leq k$ and $j_1 < \cdots < j_n \leq k$, we have

$$\text{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/M) = \text{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/M)$$

Assume this is true for $k$ and consider $k+1$. Let $i_1 < \cdots < i_n \leq k$, $j_1 < \cdots < j_n \leq k$. We need to show that

$$\text{tp}_M(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}, \bar{a}_{k+1}/M) = \text{tp}_M(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}, \bar{a}_{k+1}/M)$$

Consider a formula $\varphi(\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1}) \in L_M$. Assume by contradiction that

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}_{k+1}) \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}_{k+1})$$

But $\text{tp}(\bar{a}_{k+1}/M, \bar{a}_1, \dots, \bar{a}_k)$ is f.s. in $M$, so there is $\bar{a}' \in M$ s.t.

$$M \vDash \varphi(\bar{a}_{i_1}, \cdots, \bar{a}_{i_n}, \bar{a}') \wedge \neg\varphi(\bar{a}_{j_1}, \cdots, \bar{a}_{j_n}, \bar{a}')$$

contradicting IH $\square$

## 2.2 Definability and Generalizations

**Definition 2.8.** $X \subseteq \mathfrak{C}^n$ is **definable almost over** $A$ if there is an $A$-definable equivalence relation $E$ on $\mathfrak{C}^n$ with finitely many classes and $X$ is a union of some $E$-classes

**Lemma 2.9.** *Let $\mathbb{D}$ be a definable class and $A$ a set of parameters. T.F.A.E.*

1. *$\mathbb{D}$ is definable over $A$*
2. *$\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix $A$ pointwise*

$S \subseteq K^{\text{alg}} \Rightarrow M \setminus S \subseteq K^{\text{alg}}$

*Proof.* $\Rightarrow$ is easy as for any $F \in \text{Aut}(\mathfrak{C}/A)$ and $\mathbb{D} = \varphi(\mathfrak{C}, \bar{a})$, $\mathfrak{C} \vDash \varphi(\bar{s}, \bar{a})$ iff $\mathfrak{C} \vDash \varphi(F(\bar{s}), \bar{a})$. StackExchange

$x \in \mathbb{D} \Leftrightarrow \vDash \varphi(x, \bar{a}) \Leftrightarrow \varphi(F(x), F(\bar{a})) \leftrightarrow \varphi(F(x), \bar{a}) \Leftrightarrow F(x) \in \mathbb{D}$

$\Leftarrow$. Another proof from Chernikov. Assume that $\mathbb{D} = \varphi(\mathfrak{C}, b)$ where $b \in \mathfrak{C}$, and let $p(y) = \text{tp}(b/A)$

**Claim 1**. $p(y) \vdash \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b))$, which says that for any realisations $b'$, $\varphi(\mathfrak{C}, b) = \varphi(\mathfrak{C}, b')$

Indeed, let $b' \vDash p(y)$ be arbitrary. Then $\text{tp}(b/A) = \text{tp}(b'/A)$ so there is some $\sigma \in \text{Aut}(\mathfrak{C}/A)$ with $\sigma(b) = b'$. Then $\sigma(X) = \varphi(\mathfrak{C}, b')$ and by assumption $\sigma(X) = X$, thus $\varphi(\mathfrak{C}, b) = X = \varphi(\mathfrak{C}, b')$.

There is some $\psi(y) \in p$ (there is a finite subset of $p(y)$ that does the job and we take the conjunction) s.t.

$$\psi(y) \models \forall x(\varphi(x,y) \leftrightarrow \varphi(x,b))$$

Let $\theta(x)$ be the formula $\exists y(\psi(y) \wedge \varphi(x,y))$. Note that $\theta(x)$ is an $L(A)$-formula, as $\psi(y)$ is

**Claim 2.** $X = \theta(\mathfrak{C})$

If $a \in X$, then $\models \varphi(a,b)$, and as $\psi(y) \in \text{tp}(b/A)$ we have $\models \theta(a)$. Conversely, if $\models \theta(a)$, let $b'$ be s.t. $\models \psi(b') \wedge \varphi(a,b')$. But by the choice of $\psi$ this implies that $\models \varphi(a,b)$

$\Leftarrow$ Let $\mathbb{D}$ be defined by $\varphi$, defined over $B \supset A$. Consider the maps

$$\mathfrak{C} \xrightarrow{\tau} S(B) \xrightarrow{\pi} S(A)$$

where $\tau(c) = \text{tp}(c/B)$ and $\pi$ is the restriction map. Let $Y$ be the image of $\mathbb{D}$ in $S(A)$. Since $Y = \pi[\varphi]$. $Y$ is closed. Note that $\tau(\mathbb{D}) = [\varphi]$. $\tau(\mathbb{D}) = \{\text{tp}(c/B) : \mathfrak{C} \models \varphi(c)\} \subseteq [\varphi]$. For any $q(x) \in [\varphi]$, as $\mathfrak{C}$ is saturated, $\mathfrak{C} \models q(d)$ and $d \in \mathbb{D}$. Thus $q \in \tau(\mathbb{D})$. $\pi$ is continuous

Assume that $\mathbb{D}$ is invariant under all automorphisms of $\mathfrak{C}$ which fix $A$ pointwise. Since elements which have the same type over $A$ are conjugate by an automorphism of $\mathfrak{C}$, this means that $\mathbb{D}$-membership depends only on the type over $A$, i.e., $\mathbb{D} = (\pi\tau)^{-1}(Y)$. For any $\text{tp}(c/A) = \text{tp}(d/A)$ and $c \in \mathbb{D}$, as $c$ and $d$ are conjugate, $d \in \mathbb{D}$.

For any $c \notin \mathbb{D}$, $\pi\tau(c) \in Y$ iff $\text{tp}(c/A) \in \pi[\varphi]$ iff there is $d \in \mathbb{D}$ s.t. $\text{tp}(c/A) = \text{tp}(d/A)$ but then $c \in \mathbb{D}$.

This implies that $[\varphi] = \pi^{-1}(Y)$ $\tau(\mathbb{D}) = [\varphi] = \tau(\tau^{-1}\pi^{-1})(Y) = \pi^{-1}(Y)$ , or $S(A) \setminus Y = \pi[\neg\varphi]$; hence $S(A) \setminus Y$ is also closed and we conclude that $Y$ is clopen. By Lemma **??** $Y = [\psi]$ for some $L(A)$-formula $\psi$. This $\psi$ defines $\mathbb{D}$. For any $d \in \mathfrak{C}$

$$\models \psi(d) \Leftrightarrow \text{tp}(d/A) \Leftrightarrow d \in \mathbb{D}$$

$\square$

A slight generalization of the previous lemma

**Lemma 2.10.** *Let $X \subseteq \mathfrak{C}^n$ be definable. TFAE*

1. *$X$ is almost $A$-definable, i.e., there is an $A$-definable equivalence relation $E$ on $\mathfrak{C}^n$ with finitely many classes, s.t. $X$ is a union of $E$-classes*

2. *The set $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ is finite*

3. *The set* $\{\sigma(X) : \sigma \in Aut(\mathfrak{C}/A)\}$ *is small*

*Proof.* $1 \to 2$. Let $\varphi(x_1, x_2) \in L(A)$ be the $A$-definable equivalence relation $E$, and let $b_1, \ldots, b_n \in M$ be representatives in each equivalence class so that each class can be written as $[b_i] = \varphi(\mathfrak{C}, b_i)$. Given $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$, since $\varphi(x_1, x_2) \leftrightarrow \varphi(\sigma(x_1), \sigma(x_2))$, the image of each $[b_i]$ under $\sigma$ will be

$$\sigma([b_i]) = \{\sigma(x) : \varphi(x, b_i)\} = \{x' : \varphi(x', \sigma(b_i))\} = \{x : \varphi(x, b_{j_i})\} = [b_{j_1}]$$

for some $j_i \leq n$. Now $X$ is a disjoint union of some $[b_i]$'s, so $\sigma(X)$ is a disjoint union of some $[b_j]$'s. Since there are only finitely many equivalence classes, there can only be finitely many possibilities for disjoint unions of these classes

$2 \to 1$. Let $X = \varphi(\mathfrak{C}, b)$ and $p(y) = \mathrm{tp}(b/A)$. Given $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$, we have $\sigma(X) = \varphi(\mathfrak{C}, \sigma(b))$. Then from assumption, there must be distinct $b_1, \ldots, b_n$ s.t.

$$\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{C}/A)\} = \{\varphi(\mathfrak{C}, b_i) : i \leq n\}$$

Now if $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$, then strong homogeneity yields some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Then the above argument again shows that $\varphi(x, b')$ defines $\sigma(X)$ for some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$. Thus $\sigma(X) = \varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i \leq k$. Therefore $p(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. By compactness there is some $\psi(y) \in p$ s.t. $\psi(y) \vdash \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. Now define $E(x_1, x_2)$ as

$$\forall y(\psi(y) \to (\phi(x_1, y) \leftrightarrow \phi(x_2, y)))$$

so it is $A$-definable. It is easy to check that $E$ is an equivalence relation with finitely many classes, and that $X$ is a union of $E$-classes ($a_1 E a_2$ iff they agree on $\phi(x, b_i)$ for all $i \leq k$, and so $X = \phi(\mathfrak{C}, b_0)$ is given by the union of all possible combinations intersected with it)

$3 \to 1$ Assume for contradiction that

$$|\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{C}/A)\}| = \lambda \geq \omega$$

we can find $\lambda$-many elements $(b_i : i < \lambda) \subset \mathfrak{C}$ to represent the distinct images under automorphisms. Then the set

$$q(y) = p(y) \cup \{\neg \forall x(\varphi(x, y) \leftrightarrow \varphi(x, b_i)) : i < \lambda\}$$

will be finitely satisfiable. Thus $q(y)$ is realised by some $b'$. But such $b'$ has the same type as $b$ over $A$ and so strong homogeneity yields some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ s.t. $\sigma(b) = b'$. Applying such $\sigma$ on $X$ gives the image $\varphi(\mathfrak{C}, b') = \varphi(\mathfrak{C}, b_i)$ for some $i < \lambda$, a contradiction $\square$

**Proposition 2.11.** *We can identify definable sets with continuous functions in a certain settings*

1. *Formulas $\varphi(\bar{x}), \psi(\bar{x}) \in L_A$ are equivalent iff $[\varphi(\bar{x})] = [\psi(\bar{x})]$*
2. *The clopen subsets of $S_{\bar{x}}(A)$ are precisely the basic clopen sets*
3. *Clopen subsets $X$ of $S_{\bar{x}}(A)$ correspond exactly to continuous functions $f : S_{\bar{x}}(A) \to 2$ (with discrete topology) where $f(p(\bar{x})) = 1$ if $p(\bar{x}) \in X$ and 0 otherwise*
4. *The definable subsets of $\mathfrak{C}^c$ are in one-to-one correspondence with continuous functions from $S_{\bar{x}}(A)$ to 2*

*Proof.*

3. If $X$ is clopen, then $f^{-1}(2) = S_{\bar{x}}(A)$, $f^{-1}(0) = \emptyset$, $f^{-1}(\{1\}) = X$, $f^{-1}(\{0\}) = X^c$
4. By 1, definable sets are in one-to-one correspondence with basic clopen subsets. By 2, basic clopen sets are exactly all of the clopen subsets, so definable sets are in one-to-one correspondence with clopen sets. By 3, clopen sets are in one-to-one correspondence with continuous functions $f : S_{\bar{x}}(A) \to 2$

□

## 2.3 Imaginaries and $T^{\text{eq}}$

A **multi-sorted** structure is a family of sets $(M_s)_{s\in S}$ equipped with relations

$$R \subseteq M_{s_1} \times \cdots \times M_{s_m}, \quad (s_1, \dots, s_m \in S)$$

A multi-sorted language $L$ is a triple $(S, L^r, L^f)$ and $S$ are the sorts of $L$

$M_s$ is the underlying set of sort $s$. Elements of $M_s$ are also called "elements of $\mathcal{M}$" of sort $s$ Given any tuple $\bar{s} = (s_i)_{i\in I}$ of sorts in $S$, we let $M_{\bar{s}} = \prod_{i\in I} M_{s_i}$

Given a variable $x = (x_i)_{i\in I}$ of $L$, with $x_i$ of sorts $s_i$ for $i \in I$, we define the $x$**-set of** $\mathcal{M}$ to be the product set

$$M_x := M_{\bar{s}} = \prod_i M_{s_i}, \quad \bar{s} = (s_i)_{i\in I}$$

$x = (x_i)_{i\in I}$ and $y = (y_j)_{j\in J}$ is **disjoint** if $x_i \neq y_j$ for all $i \in I$ and $j \in J$, and in that case we put $M_{x,y} = M_x \times M_y$. If in addition $I = J$ and $x_i$ and $y_i$ have the same sort for $i \in I$ (so that $M_x = M_y$), we call $x$ and $y$ **disjoint and similar**

**Definition 2.12.** The **definable closure** $\mathrm{dcl}(A)$ of $A$ is the set of elements $c$ for which there is an $L(A)$-formula $\varphi(x)$ s.t. $c$ is the unique element satisfying $\varphi$. Elements or tuples $a$ and $b$ are said to be **interdefinable** if $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$.

**Lemma 2.13.** *Assume $A \subseteq \mathfrak{C}$ and $\bar{b} \in \mathfrak{C}$*

1. *$\bar{b} \in \mathrm{acl}(A)$ iff $\{f(\bar{b}) : f \in Aut(\mathfrak{C}/A)\}$ is finite*
2. *$\bar{b} \in \mathrm{dcl}(A)$ iff $f(\bar{b}) = \bar{b}$ for all $f \in Aut(\mathfrak{C}/A)$*

*Proof.*

1. Suppose $\bar{b} \in \mathrm{acl}(A)$ with witness $\exists^{\leq k}\varphi(\bar{x})$. Then $\varphi(\mathfrak{C})$ is $A$-definable and hence is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant by Lemma 2.9

   Suppose the finiteness. Since the composition of automorphisms is an automorphism, this set is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant and therefore $A$-definable by some $\varphi(\bar{x})$.

2. $\{\bar{b}\}$ is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant

□

The first motivation to develop $T^{\mathrm{eq}}$ is dealing with quotient objects, without leaving the context of first order logic. That is, if $E$ is some definable equivalence relation on some definable set $X$, we want to view $X/E$ as a definable set

We work in the setting of multi-sorted languages. Let $L$ be a 1-sorted language and let $T$ be a (complete) $L$-theory. We shall build a many-sorted language $L^{\mathrm{eq}}$-theory $T^{\mathrm{eq}}$. We will ensure that in natural sense, $L^{\mathrm{eq}}$ contains $L$ and $T^{\mathrm{eq}}$ contains $T$

First we define $L^{\mathrm{eq}}$. Consider the set $L$-formula $\varphi(x, y)$, up to equivalence, such that $T$ models that $\varphi$ is an equivalence relation. For each $\varphi$, define $s_\varphi$ to be a new sort in $L^{\mathrm{eq}}$. Of particular importance is $s_=$, the sort given by the formula "$x = y$". = is an equivalence relation This sort $s_=$ will yield, in each model of $T^{\mathrm{eq}}$, a model of $T$

Also define $f_\varphi$ to be a function symbol with domain sort $s_=^n$ (where $\varphi$ has $n$ free variables) and codomain sort $s_\varphi$

For each $m$-place relation symbol $R \in L$, make $R^{\mathrm{eq}}$ an $m$-place relation symbol in $L^{\mathrm{eq}}$ on $s_=^m$. Likewise for all constant and function symbols in $L$. Finally, for the sake of formality, we put a unique equality symbol $=_\varphi$ on each sort

*Remark.* Let $N$ be an $L^{\mathrm{eq}}$ structure. Then $N$ has interpretations $s_\varphi(N)$ of each sort $s_\varphi$ and $f_\varphi(N) : s_=(N)^{n_{f_\varphi}} \to s_\varphi(N)$ of each function symbol $f_\varphi$. Additionally, $N$ will contain an $L$-structure consisting of $s_=$ and interpretations of the symbols of $L$ inside of $s_=$

**Definition 2.14.** $T^{\text{eq}}$ is the $L^{\text{eq}}$-theory which is axiomatised by the following

1. $T$, where the quantifiers in the formulas of $T$ now range over the sort $s_=$
2. For each suitable $L$-formula $\varphi(x,y)$, the axiom $\forall_{s_=}\bar{x}\forall_{s_=}\bar{y}(\varphi(x,y) \leftrightarrow f_\varphi(\bar{x}) = f_\varphi(\bar{y}))$
3. For each $L$-formula $\varphi$, the axiom $\forall_{s_\varphi} y \exists_{s_=}\bar{x}(f_\varphi(\bar{x}) = y)$

Axioms 2 and 3 simply state that $f_\varphi$ is the quotient function for the equivalence relation given by $\varphi$

**Definition 2.15.** Let $M \models T$. Then $M^{\text{eq}}$ is the $L^{\text{eq}}$ structure s.t. $s_=(M^{\text{eq}}) = M$ and for each suitable $L$-formula $\varphi(x,y)$ of $n$ variables, the sort $s_\varphi(M^{\text{eq}})$ is equal to $M^{n_{f_\varphi}}/E$ where $E$ is the equivalence relation defined by $\varphi(x,y)$ and $f_\varphi(M^{\text{eq}})(b) = b/E$

**Example 2.2** (Projective planes)**.** From Hodges.

Suppose $A$ is a three-dimensional vector space over a finite field, and let $L$ be the first-order language of $A$. Then we can write a formula $\theta(x,y)$ of $L$ which expresses 'vectors $x$ and $y$ are non-zero and are linearly dependent on each other'. The formula $\theta$ is an equivalence formula of $A$, and the sort $s_\theta$ is the set of points of the projective plane $P$ associated with $A$

Now $M^{\text{eq}} \models T^{\text{eq}}$. Moreover, passing from $T$ to $T^{\text{eq}}$ is a canonical operation, in the following sense

**Lemma 2.16.**
1. *For any $N \models T^{\text{eq}}$, there is an $M \models T$ s.t. $N \cong M^{\text{eq}}$*
2. *Suppose $M, N \models T$ are isomorphic, and let $h : M \cong N$. Then $h$ extends uniquely to $h^{\text{eq}} : M^{\text{eq}} \cong N^{\text{eq}}$*
3. *$T^{\text{eq}}$ is a complete $L^{\text{eq}}$-theory*
4. *Suppose $M, N \models T$ and let $\bar{a} \in M$, $\bar{b} \in N$ with $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$. Then $\text{tp}_{M^{\text{eq}}}(\bar{a}) = \text{tp}_{N^{\text{eq}}}(\bar{b})$*

*Proof.*
1. Take $M = s_=(N)$
2. Let $h^{\text{eq}} : M^{\text{eq}} \to N^{\text{eq}}$ be defined as $h^{\text{eq}}(f_\varphi(M^{\text{eq}})(b)) = f_\varphi(N^{\text{eq}})(h(b))$ for each $\varphi \in L$. This defines a function on $M^{\text{eq}}$, because $f_\varphi(M^{\text{eq}})$ is surjective by the $T^{\text{eq}}$ axioms. Moreover $h^{\text{eq}}$ is well-defined. Suppose $f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b')$, then $\varphi(b,b')$ and hence $\varphi(h(b),h(b'))$,

therefore $f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b'))$. Injectivity is the same since $\varphi(b, b') \leftrightarrow \varphi(h(b), h(b'))$.

$$\begin{aligned} f_\varphi(N^{\text{eq}})(h(b)) = f_\varphi(N^{\text{eq}})(h(b')) &\Leftrightarrow h(b)/E_\varphi = h(b')/E_\varphi \\ &\Leftrightarrow \varphi(h(b), h(b')) \\ &\Leftrightarrow \varphi(b, b') \\ &\Leftrightarrow f_\varphi(M^{\text{eq}})(b) = f_\varphi(M^{\text{eq}})(b') \end{aligned}$$

3. Let $M, N \vDash T^{\text{eq}}$, we want to show that they are elementary equivalent. Assume the generalized continuum hypothesis. By GCH, there are $M', N' \vDash T^{\text{eq}}$ which are $\lambda$ saturated of size $\lambda$, for some large $\lambda$ (strongly inaccessible), which $M \preceq M'$ and $N \preceq N'$. Since we want to show elementary equivalence, we can replace $M, N$ with $M'$ and $N'$. By 1, we have $M = M_0^{\text{eq}}$, $N = N_0^{\text{eq}}$ for some $M_0, N_0 \vDash T$. Furthermore, $M_0, N_0$ are $\lambda$-saturated of size $\lambda$. By assumption, $T$ is complete, so $M_0 \equiv N_0$, and therefore $M_0 \cong N_0$. By 2, $M \cong N$, and therefore $M \equiv N$

   We could simply prove that there is a back and forth system between $M$ and $N$, using such a system between $M \supset M_0 \vDash T$ and $N \supset N_0 \vDash T$ $M_0 \equiv N_0$ iff $M_0 \sim_\omega N_0$. We want to show that $M \sim_\omega N$. For any $p \in \omega$,

   - given $a \in s_=(M)$, choose according to $M$
   - given $a \in s_\varphi(M)$, then there is $\bar{b}\bar{c} \in s_=(M)$ s.t. $f_\varphi(M^{\text{eq}})(\bar{b}\bar{c}) = a$ and $\varphi(\bar{b}, \bar{c})$. If $\bar{b} \in s_=(M^{\text{eq}})^n$, then there is a local isomorphism $\bar{b} \mapsto \bar{d}$ as $M \sim_\omega N$. Take $b = \bar{d}/E_\varphi$.

4. Let $M, N \vDash T$, they are elementary submodels of $\mathfrak{C}$. Since $\text{tp}_M(\bar{a}) = \text{tp}_N(\bar{b})$, there exists an $\sigma \in \text{Aut}(\mathfrak{C}/A)$ with $\sigma(\bar{a}) = \bar{b}$. By 2, this automorphism extends to $\sigma^{\text{eq}} : \mathfrak{C}^{\text{eq}} \to \mathfrak{C}^{\text{eq}}$ with $\sigma^{\text{eq}}(a) = b$, hence $\text{tp}_{M^{\text{eq}}}(a) = \text{tp}_{\mathfrak{C}^{\text{eq}}}(a) = \text{tp}_{\mathfrak{C}^{\text{eq}}}(b) = \text{tp}_{N^{\text{eq}}}(b)$

$\square$

**Corollary 2.17.** *Consider the Strong space $S_{(s_=)^n}(T^{\text{eq}})$. The forgetful map $\pi : S_{(s_=)^n}(T^{\text{eq}}) \to S_n(T)$ is a homeomorphism*

*Proof.* Observe that it is continuous and surjective. By 4 of the previous lemma it is injective. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism $\square$

**Proposition 2.18.** *Let $\varphi(x_1, \dots, x_k)$ be an $L^{\mathrm{eq}}$ formula, where $x_i$ is of sort $S_{E_i}$. There is an $L$-formula $\psi(\bar{y}_1, \dots, \bar{y}_k)$ s.t.*

$$T^{\mathrm{eq}} \vDash \forall \bar{y}_1, \dots, \bar{y}_k(\psi(\bar{y}_1, \dots, \bar{y}_k) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k)))$$

*Proof.* Let $n$ be the length of $\bar{y}_1, \dots, \bar{y}_k$. Consider the set $\pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$, it is a clopen subset of $S_n(T)$ by the previous lemma, hence equal to $\psi(\bar{y}_1, \dots, \bar{y}_k)$ for some formula $\psi$.

Guess the intuition is $[\varphi] = [\psi]$ iff $\vDash \varphi \leftrightarrow \psi$. Consider $\pi[\psi(\bar{y}_1, \dots, \bar{y}_k)] = \pi[\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ and as $\pi$ is homeomorphism, $[\psi(\bar{y}_1, \dots, \bar{y}_k)] = [\varphi(f_{E_1}(\bar{y}_1), \dots, f_{E_k}(\bar{y}_k))]$ □

This proposition also shows that $T^{\mathrm{eq}}$ is complete since $f_{E_i}$ is surjective
Also, for any $\bar{c} \in \mathfrak{C}$, $\bar{c} \in \mathrm{dcl}^{\mathrm{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \mathrm{dcl}(\emptyset)$ , $\bar{c} \in \mathrm{acl}^{\mathrm{eq}}(\emptyset) \Leftrightarrow \bar{c} \in \mathrm{acl}(\emptyset)$

**Corollary 2.19.**
1. *Let $M, N \vDash T$, and let $h : M \to N$ be an elementary embedding. Then $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to N^{\mathrm{eq}}$ is also an elementary embedding*
2. *$\mathfrak{C}^{\mathrm{eq}}$ is also $\kappa$-saturated*

*Proof.*
1. $h : M \to \mathrm{im}(h)$ is an isomorphism and can extend to $h^{\mathrm{eq}} : M^{\mathrm{eq}} \to (\mathrm{im}(h))^{\mathrm{eq}}$, and $(\mathrm{im}(h))^{\mathrm{eq}} \subseteq N^{\mathrm{eq}}$
2. By Proposition 2.18

□

*Remark.* For $M \vDash T$, a definable set $X \subseteq M^n$ can be viewed as an element of $M^{\mathrm{eq}}$. Suppose $X$ is defined in $M$ by $\varphi(\bar{x}, \bar{a})$ where $\bar{a} \in M$. Consider the equivalence relation $E_\psi$ defined by $\psi(\bar{y}_1, \bar{y}_2) = \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$ $\bar{y}_1 \sim \bar{y}_2$ iff this $\varphi(M, \bar{y}_1) = \varphi(M, \bar{y}_2)$, and consider $c = \bar{a}/E_\psi = f_\psi(\bar{a}) \in M^{\mathrm{eq}}$. Then $X$ is defined in $M^{\mathrm{eq}}$ by $\chi(\bar{x}, c) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f_\psi(\bar{y}) = c)$. Moreover, if $c' \in S_\psi(M^{\mathrm{eq}})$ and $\forall \bar{x}(\chi(\bar{x}, c) \leftrightarrow \chi(\bar{x}, c'))$, then $c = c'$. To see this, let $c' = f_\psi(\bar{a}')$, and let $X'$ be defined in $M$ by $\varphi(\bar{x}, \bar{a}')$. Then $X'$ is defined in $M^{\mathrm{eq}}$ by $\chi(\bar{x}, c')$, so we have that $X = X'$ (in $M^{\mathrm{eq}}$). And then $X = X'$ (in $M$) so $c = f_\psi(\bar{a}) = f_{\psi'}(\bar{a}') = c'$

**Definition 2.20.** With the above considerations in mind, given $M \vDash T$ and a definable set $X \subseteq M^n$, we call such a $c \in M^{\mathrm{eq}}$ a **code** for $X$

*Remark.* Any automorphism of $\mathfrak{C}^{\mathrm{eq}}$ fixes a definable set $X$ set-wise iff it fixes a code for $X$. However, the choice of a code for $X$ will depend on the formula $\varphi$ used to define it

$$\begin{aligned}\sigma(X) = X \Leftrightarrow \sigma(X) = \{\sigma(x) : \varphi(x,b)\} = \{x : \varphi(x,\sigma(b))\} = \{x : \varphi(x,b)\} = X \\ \Leftrightarrow \forall x(\varphi(x,b) \leftrightarrow \varphi(x,\sigma(b))) \\ \Leftrightarrow \psi(b,\sigma(b)) \Leftrightarrow f_\psi(b) = f_\psi(\sigma(b))\end{aligned}$$

We can think of $\mathfrak{C}^{\mathrm{eq}}$ as adjoining codes for all definable equivalence relations (as $c/E'$ codes $E'(x,c)$ for an arbitrary equivalence relation $E$)

**Definition 2.21.** Let $A \subseteq M \vDash T$. Then $\mathrm{acl}^{\mathrm{eq}}(A) = \{c \in M^{\mathrm{eq}} : c \in \mathrm{acl}_{M^{\mathrm{eq}}}(A)\}$ and $\mathrm{dcl}^{\mathrm{eq}}(A)$ is defined similarly

*Remark.* Suppose $A \subseteq M \prec N$, then $\mathrm{acl}_{N^{\mathrm{eq}}}(A), \mathrm{dcl}_{N^{\mathrm{eq}}}(A) \subseteq M^{\mathrm{eq}}$, so this notation is unambiguous

**Lemma 2.22.** *Let $M \vDash T$, a definable subset $X$ of $M^n$, and $A \subseteq M$. Then $X$ is almost $A$-definable iff $X$ is definable in $M^{\mathrm{eq}}$ by a formula with parameters in* $\mathrm{acl}^{\mathrm{eq}}(A)$

*Proof.* We can work in $\mathfrak{C}$, since $M \prec \mathfrak{C}$. Let $c$ be a code for $X$. From 2.10 $X$ is almost $A$-definable iff $|\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{C}/A)\}| < \omega$ iff $|\{\sigma(c) : \sigma \in \mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}}/A)\}| < \omega$ (note that $\sigma$ extends uniquely in $\mathfrak{C}^{\mathrm{eq}}$), that is, $c \in \mathrm{acl}^{\mathrm{eq}}(A)$.

$$\begin{aligned}\sigma(b)/E = \sigma'(b)/E \Leftrightarrow \forall x(\varphi(x,\sigma(b)) \leftrightarrow \varphi(x,\sigma'(b))) \\ \Leftrightarrow \sigma(X) = \sigma'(X)\end{aligned}$$

$\square$

**Definition 2.23.** Let $\bar{a}, \bar{b} \in \mathfrak{C}$ have length $n$. Let $\bar{a}, \bar{b}$ have the same strong type over $A$ (written as $\mathrm{stp}_{\mathfrak{C}}(\bar{a}/A) = \mathrm{stp}_{\mathfrak{C}}(\bar{a}/A)$) if $E(\bar{a}, \bar{b})$ for any finite equivalence relation (finitely many classes) defined over $A$

*Remark.* If $\varphi(\bar{x})$ is a formula over $A$, then it defines an equivalence with two classes $E(\bar{x}_1, \bar{x}_2)$ iff $(\varphi(\bar{x}_1) \wedge \varphi(\bar{x}_2)) \vee (\neg\varphi(\bar{x}_1) \wedge \neg\varphi(\bar{x}_2))$. Hence strong types are a refinement of types

Hence for any formula if $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/B)$, at least we have $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$

**Lemma 2.24.** *If $A = M \prec \mathfrak{C}$, then* $\mathrm{tp}_{\mathfrak{C}}(a/M) \vDash \mathrm{stp}_{\mathfrak{C}}(a/M)$

$\mathrm{tp}_{\mathfrak{C}}(a/M) = \mathrm{tp}_{\mathfrak{C}}(b/M) \Rightarrow \mathrm{stp}_{\mathfrak{C}}(a/M) = \mathrm{stp}_{\mathfrak{C}}(b/M)$

*Proof.* Let $E$ be an equivalence relation with finitely many classes, defined over $M$, and $\bar{b}$ another realization of $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/M)$, we want to show $E(a,b)$. Since $E$ has only finitely many classes, and $M$ is a model, there are representants $e_1, \dots, e_n$ of each $E$-class in $M$. Hence we must have $E(a, e_i)$ for some $i$, and therefore $E(b, e_i)$, which yields $E(a,b)$ □

**Lemma 2.25.** *Let $A \subseteq M \models T$, and let $\bar{a}, \bar{b} \in M$. TFAE*

1. $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$
2. $\bar{a}, \bar{b}$ *satisfy the same formulas almost $A$-definable*
3. $\mathrm{tp}_{\mathfrak{C}}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}_{\mathfrak{C}}(\bar{b}/\operatorname{acl}^{\mathrm{eq}}(A))$

*Proof.* $3 \to 2$. 2.22. Suppose $X = \varphi(\mathfrak{C}, \bar{d})$ is almost $A$-definable, then $\bar{a}, \bar{b} \in \varphi(\mathfrak{C}, \bar{d})$ iff $\bar{a}, \bar{b} \in \theta(\mathfrak{C}) := \exists \bar{y}(\varphi(\mathfrak{C}, \bar{y}) \wedge \bar{y}/E_\psi = \bar{c})$ where $\bar{c} = \bar{d}/E_\psi \in \operatorname{acl}^{\mathrm{eq}}(A)$.

$2 \to 3$

$1 \to 2$. Let $X$ be almost definable over $A$. We want to show that $\bar{a} \in X$ iff $\bar{b} \in X$.

Since $X$ is almost definable over $A$, there is an $A$-definable equivalence relation $E$ with finitely many classes, and $\bar{c}_1, \dots, \bar{c}_n$ s.t. for all $\bar{x} \in M$, we have $\bar{x} \in X$ iff $M \models E(\bar{x}, \bar{c}_1) \vee \dots \vee E(\bar{x}, \bar{c}_n)$. Hence $E(\bar{a}, \bar{c}_i)$ for some $i$, so by assumption $E(\bar{b}, \bar{c}_i)$.

$2 \to 1$. Let $E$ be an $A$-definable equivalence relation with finitely many classes, we want to show that $E(\bar{a}, \bar{b})$. The set $X = \{\bar{x} \in M : E(\bar{x}, \bar{a})\}$ is definable almost over $A$. But $\bar{a} \in X$, so $\bar{b} \in X$, hence $E(\bar{a}, \bar{b})$ □

Here is a note from scanlon

**Definition 2.26.** An **imaginary element** of $\mathfrak{A}$ is a class $a/E$ where $a \in A^n$ and $E$ is a definable equivalence relation on $A^n$

**Definition 2.27.** $\mathfrak{A}$ **eliminates imaginaries** if, for every definable equivalence relation $E$ on $A^n$ there exists definable function $f : A^n \to A^m$ s.t. for $x, y \in A^n$ we have

$$xEy \Leftrightarrow f(x) = f(y)$$

*Remark.* The definition give above is what Hodges calls **uniform elimination of imaginaries**

*Remark.* If $\mathfrak{A}$ eliminates imaginaries, then for any definable set $X$ and definable equivalence relation $E$ on $X$, there is a definable set $Y$ and a definable bijection $f : X/E \to Y$. Of course this is not literally true, we should rather say that there is a definable map $f' : X \to Y$ s.t. $f'$ is invariant on the equivalence classes defined by $E$

So elimination of imaginaries is saying that quotients exists in the category of definable sets

*Remark.* If $\mathfrak{A}$ eliminates imaginaries then for any imaginaries element $a/E = \tilde{a}$ there is some tuple $\hat{a} \in A^m$ s.t. $\tilde{a}$ and $\hat{a}$ are **interdefinable**, i.e. there is a formula $\varphi(x, y)$ s.t.

- $\mathfrak{A} \models \varphi(a, \tilde{a})$
- If $a'Ea$ then $\mathfrak{A} \models \varphi(a', \hat{a})$
- If $\varphi(b, \hat{a})$ then $bEa$
- If $\varphi(a, c)$ then $c = \hat{a}$

To get the formula $\varphi$ we use the function $f$ given by the definition of elimination of imaginaries and let $\varphi(x, y) := f(x) = y$

Almost conversely, if for every $\mathfrak{A}' \equiv \mathfrak{A}$ every imaginary in $\mathfrak{A}'$ is interdefinable with a **real** (non-imaginary) tuple then $\mathfrak{A}$ eliminates imaginaries

$$\{\forall xy(xEy \leftrightarrow f_E(x) = f_E(y)) \mid \forall E\}$$

**Example 2.3.** For any structure $\mathfrak{A}$, every imaginary in $\mathfrak{A}_A$ is interdefinable with a sequence of real elements

**Example 2.4.** Let $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$. Then $\mathfrak{A}$ eliminates imaginaries. For example, to eliminate the "odd/even" equivalence relation, $E$, we can define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(x) = y \Leftrightarrow xEy \wedge \forall z[xEz \to y < z \vee y = z]$$

**Definition 2.28.** $\mathfrak{A}$ has **definable choice functions** if for any formula $\theta(\bar{x}, \bar{y})$ there is a definable function $f(\bar{y})$ s.t.

$$\forall \bar{y} \exists \bar{x}[\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(f(\bar{y}), \bar{y})]$$

(i.e., $f$ is a skolem function for $\theta$) and s.t.

$$\forall \bar{y} \forall \bar{z}[\forall \bar{x}(\theta(\bar{x}, \bar{y}) \leftrightarrow \theta(\bar{x}, \bar{z})) \to f(\bar{y}) = f(\bar{z})]$$

*Proof.* If $\mathfrak{A}$ has definable choice functions then $\mathfrak{A}$ eliminates imaginaries $\square$

*Proof.* Given a definable equivalence relation $E$ on $A^n$ let $f$ be a definable choice function for $E(\bar{x}, \bar{y})$. Since $E$ is an equivalence relation we have $\forall \bar{y} E(f(\bar{y}), \bar{y})$ and

$$\forall \bar{y} \bar{z}[\bar{y}/E = \bar{z}/E \to f(\bar{y}) = f(\bar{z})]$$

Thus $f(\bar{y}) = f(\bar{z}) \Leftrightarrow \bar{y} E \bar{z}$ $\square$

**Example 2.5.** We now see that $\mathfrak{A} = (\mathbb{N}, <, \equiv \mod 2)$ eliminates imaginaries. Basically since $\mathfrak{A}$ is well ordered, we can find a least element to witness membership of definable sets, hence we have definable functions

**Example 2.6.** $\mathfrak{A} = (\mathbb{N}, \equiv \mod 2)$ does not eliminate imaginaries

First note that the only definable subsets of $\mathbb{N}$ are $\emptyset, \mathbb{N}, 2\mathbb{N}, (2n+1)\mathbb{N}$. This is because $\mathfrak{A}$ has an automorphisms which switches $(2n+1)\mathbb{N}$ and $2\mathbb{N}$

Now suppose $f : \mathbb{N} \to \mathbb{N}^m$ eliminates the equivalence relation $\equiv \mod 2$, i.e.,

$$f(x) = f(y) \Leftrightarrow y \equiv 2 \mod 2$$

The $\mathrm{im}(f)$ is definable and has cardinality 2. Since there are no definable subsets of $\mathbb{N}$ of cardinality 2, we must have $m > 1$. Now let $\pi : \mathbb{N}^m \to \mathbb{N}$ be a projection. Then $\pi(\mathrm{im}(f))$ is a finite nonempty definable subset of $\mathbb{N}$. But no such set exists

**Proposition 2.29.** *If $\mathfrak{A}$ eliminates imaginaries, then $\mathfrak{A}_A$ eliminates imaginaries*

*Proof.* The idea is that an equivalence relation with parameters can be obtained as a fiber of an equivalence relation in more variables. Let $E \subseteq A^n$ be an equivalence relation definable in $\mathfrak{A}_A$. Let $\varphi(x, y; z) \in L$ and $a \in A^l$ be s.t.

$$xEy \Leftrightarrow \mathfrak{A} \models \varphi(x, y; a)$$

We now define

$$\psi(x, u, y, v) = \begin{cases} u = v \wedge \text{"}\varphi \text{ defines an equivalence relation"} & \text{or} \\ u \neq v & \text{or} \\ \text{"}\varphi(x, y, v) \text{ does not define an equivalence relation"} & \end{cases}$$

Now $\psi$ defines an equivalence relation on $A^{n+l}$. Let $f : A^{n+l} \to A^m$ eliminate $\psi$, then $f(-, a)$ eliminates $E$ ☐

Back to [Pillay(2018)]

**Definition 2.30.** 1. $T$ has elimination of imaginaries (EI) if for any model $M \models T$ and $e \in M^{\mathrm{eq}}$, there is a $\bar{c} \in M$ s.t. $e \in \mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{c})$ and $\bar{c} \in \mathrm{dcl}_{M^{\mathrm{eq}}}(e)$

2. $T$ has weak elimination of imaginaries if, as above, except $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$ (that is, $e \in \mathrm{dcl}_{M^{\mathrm{eq}}}(\bar{c})$ and $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$)

3. $T$ has geometric elimination of imaginaries if, as above, except $e \in \mathrm{acl}_{M^{\mathrm{eq}}(\bar{c})}$ and $\bar{c} \in \mathrm{acl}_{M^{\mathrm{eq}}}(e)$

Note that in particular, elimination of imaginaries imply the existence of codes for definable sets

**Proposition 2.31.** *TFAE*

1. *$T$ has EI*

2. *For some model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation $E$, there is a partition of $M^n$ into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

3. *For any model $M \vDash T$, we have that for any $\emptyset$-definable equivalence relation $E$, there is a partition of $M^n$ into $\emptyset$-definable sets $Y_1, \dots, Y_r$ and for each $i = 1, \dots, r$ a $\emptyset$-definable $f_i : Y_i \to M^{k_i}$ where $k_i \geq 1$ s.t. for each $i = 1, \dots, r$, for all $\bar{b}_1, \bar{b}_2 \in Y_i$, we have $E(\bar{b}_1, \bar{b}_2)$ iff $f_i(\bar{b}_1) = f_i(\bar{b}_2)$*

4. *For any model $M \vDash T$, and any definable $X \subseteq M^n$ there is an $L$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in M$ s.t. $X$ is defined by $\varphi(\bar{x}, \bar{b})$ and for all $\bar{b}' \in M$ if $X$ is defined by $\varphi(\bar{x}, \bar{b}')$ then $\bar{b} = \bar{b}'$. We call such a $\bar{b}$ a code for $X$.*

most typos i've ever seen in a proof

*Proof.* $2 \Leftrightarrow 3$. Since we concern only $\emptyset$-definable relations and functions, if it is true in some model, then it is true in any model

$1 \to 2$. Let $\pi_E : S^n_= \to S_E$ the canonical definable quotient map. Let $e \in S_E$. By assumption, there is $k \in \mathbb{N}$ and $\bar{c} \in \mathfrak{C}^k$ s.t. $e$ and $\bar{c}$ are interdefinable. In other words, there is a formula $\varphi_e(x, \bar{y})$ over $\emptyset$ s.t. $\varphi_e(e, \bar{c})$. Moreover, $|\varphi_e(\mathfrak{C}, \bar{c})| = |\varphi_e(e, \mathfrak{C})| = 1$

Let

$$
\begin{aligned}
X_e = \{\bar{x} \in \mathfrak{C}, \vDash \exists! \bar{y} \varphi_e(\pi_E(\bar{x}), \bar{y}) \\
\wedge \forall z (E(\bar{x}, \bar{z}) \leftrightarrow \\
(\forall y (\varphi_e(\pi_E(\bar{x}), \bar{y}) \leftrightarrow \varphi_e(\pi_E(\bar{z}), \bar{y}))))\}
\end{aligned}
$$

This means that $\varphi_e$ defines a function on $X_e$, and that this function separates $E$-classes.

Then $\pi^{-1}(\{e\}) \subset X_e$.

Since each $X_e$ contains $\pi^{-1}(\{e\})$, we get $\mathfrak{C}^n = \bigcup_{e \in \pi_E(\mathfrak{C}^n)} X_e$, and by compactness, there are $e_1, \dots, e_l$ s.t. $\mathfrak{C}^n = \bigcup_{i=1}^{l} X_{e_i}$. As each $X_e$ is $\emptyset$-definable. Let $\bar{x} \in X_e \Leftrightarrow \theta_e(\bar{x})$. Suppose there is no such $l$, then $\{x = x\} \cup \{\neg\theta_e(x)\}$

is satisfiable and realised since $\mathfrak{C}$ is saturated Naively, we can pick $f_i = \varphi_{e_i} \circ \pi_E$, but $X_{e_i}$ are not disjoint

However we can consider $Y_1, \dots, Y_r$ to be the atoms of the boolean algebra generated by the $X_i$. These are disjoint, and we can pick, for each $Y_j$, appropriate $f_i$, to get the result

$3 \to 4$. Let $X = \varphi(\mathfrak{C}, \bar{a})$. Consider the $\emptyset$-definable equivalence relation $E(\bar{y}, \bar{z}) \Leftrightarrow \forall x(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{z}))$. Let $Y_i$ and $f_i$ be as in 3 and say $\bar{a} \in Y_1$, and let $\bar{b} = f_1(\bar{a})$. Then $\exists \bar{y}(f_1(\bar{y}) = \bar{b} \wedge \varphi(\bar{x}, \bar{y}))$ defines $X$, call this formula $\psi$

We have to show that $\bar{b}$ is unique. Let $\bar{b}'$ be s.t. $\exists \bar{y}(f_1(\bar{y}) = \bar{b}' \wedge \varphi(\bar{x}, \bar{y}))$ also defines $X$, and let $\bar{a}_0$ be as the $\bar{y}$ in the formula. Then $\varphi(x, \bar{a}_0)$ defines $X$, hence $\bar{a}_0 E \bar{a}$, which implies $\bar{b}' = f_1(\bar{a}_0) = f_1(\bar{a}) = \bar{b}$

$4 \to 1$. Let $e \in \mathfrak{C}^{\mathrm{eq}}$, then $e = \pi_E(\bar{a})$ for some $\bar{a} \in \mathfrak{C}^n$ and some $\emptyset$-definable equivalence relation $E$

The set $X = \{\bar{x} \in \mathfrak{C}^n \mid \models E(\bar{x}, \bar{a})\}$ has a code $\bar{b} \in \mathfrak{C}^k$, so that $X = \psi(\mathfrak{C}^n, \bar{b})$. We are going to prove interdefinability of $e$ and $\bar{b}$ using automorphisms of $\mathfrak{C}$

First suppose that $\sigma \in \mathrm{Aut}(\mathfrak{C})$, and fixes $e$. We have $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b}))$. Applying $\sigma$, we get $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \sigma(\bar{b})))$. But $\bar{b}$ is a code for $X$, hence $\bar{b} = \sigma(\bar{b})$. This implies $\bar{b} \in \mathrm{dcl}(e)$

Now suppose $\sigma \in \mathrm{Aut}(\mathfrak{C})$ and fixes $\bar{b}$. Again $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\bar{a}) \leftrightarrow \psi(\bar{x}, \bar{b}))$ and $\mathfrak{C}^{\mathrm{eq}} \models \forall \bar{x}(\pi_E(\bar{x}) = \pi_E(\sigma(\bar{a})) \leftrightarrow \psi(\bar{x}, \bar{b}))$. But $\psi(\bar{a}, \bar{b})$, $e = \pi_E(\bar{a}) = \pi_E(\sigma(\bar{a})) = \sigma(e)$. Hence $e \in \mathrm{dcl}(\bar{b})$ □

Note that condition 2 is somewhat unsatisfying, as we would like to have a quotient function for $E$, that is, $r = 1$

**Proposition 2.32.** *Suppose $T$ eliminates imaginaries. We get $r = 1$ in condition 2 iff* $\mathrm{dcl}(\emptyset)$ *has at least two elements*

*Proof.* First, suppose that $r = 1$. Consider the equivalence on $\mathfrak{C}^2$ given by $E((x, y), (x', y'))$ iff $x = y \leftrightarrow x' = y'$. In other words, the $E$ classes are the diagonal and its complement (only two). Then $\pi_E(\mathfrak{C}^2)$ has two elements, and they belong to $\mathrm{dcl}^{\mathrm{eq}}(\emptyset)$. But because $T$ eliminates imaginaries, this implies that there is also two elements in $\mathrm{dcl}(\emptyset)$ by Proposition 2.18

Second, suppose that $\mathrm{dcl}(\emptyset)$ contains two constants $a$ and $b$. Let $Y_i, f_i$ be as in condition 2. Using $a$ and $b$,we can find some number $k$ and functions $g_i : \mathfrak{C}^{k_i} \to \mathfrak{C}^k$ s.t. $g_i(\mathfrak{C}^{k_i})$ are pairwise disjoint. We can check that the $\emptyset$-definable function $f : \mathfrak{C}^n \to \mathfrak{C}^k$ sending $y \in Y_i$ to $g_i(f_i(y))$ has all the required properties □

*Remark.* Elimination of imaginaries also makes sense for many sorted theories

**Proposition 2.33** (Assume $T$ 1-sorted)**.** *$T^{\text{eq}}$ has elimination of imaginaries*

*Proof.* Prove a strong version of 2 in Proposition 2.31 that is, we don't need to distinguish $Y_1, \dots, Y_r$ and $f_1, \dots, f_r$. Let $E'$ be a $\emptyset$-definable equivalence relation on a sort $s_E$ in some model $M^{\text{eq}}$ of $T^{\text{eq}}$. By Proposition 2.18 there is an $L$-formula $\psi(\bar{y}_1, \bar{y}_2)$ ($\bar{y}_i$ the appropriate length) s.t. for all $\bar{a}_1, \bar{a}_2 \in M$, $M \vDash \psi(\bar{a}_1, \bar{a}_2)$ iff $M^{\text{eq}} \vDash E'(f_E(\bar{a}_1), f_E(\bar{a}_2))$. So $\psi(\bar{y}_1, \bar{y}_2)$ is an $L$-formula defining an equivalence relation on $M^k$ for the suitable length $k$. Consider the map $h$, taking $e \in S_E$ to $f_\psi(\bar{a})$ for any $\bar{a} \in M^k$ s.t. $f_E(\bar{a}) = e$ for any $\bar{a} \in M^k$ s.t. $f_E(\bar{a}) = e$. Suppose $f_E(\bar{a}) = e = f_E(\bar{a}')$, we easily see that $f_\psi(\bar{a}) = f_\psi(\bar{a}')$, hence the map $h$ is well-defined, and satisfies 2 of 2.31 $\square$

### 2.4 Examples and counterexamples

**Example 2.7.** The theory of an infinite set has weak elimination of imaginaries but not full elimination of imaginaries

*Proof.* First, we show that $T$ has weak elimination of imaginaries. Let $M$ be an infinite set and let $e \in M^{\text{eq}}$ be an imaginary element. Suppose that. Let $A \subset M$ be a finite set over which $X$ is definable ??. Consider the set

$$\hat{A} := \bigcap_{\substack{\sigma \in \text{Aut}(M) \\ \sigma(X) = X}} \sigma(A)$$

Since $A$ is finite, there are $\sigma_1, \dots, \sigma_n$ s.t. $\hat{A} = \bigcap_i \sigma_i(A)$

To see that $T$ does not have full elimination of imaginaries, observe that there is never a code for any finite set. Indeed, if $M$ is an infinite set, $X \subset_f M$, and $\bar{a} \in M$, we can find a permutation of $M$ which fixes $X$ as a set but does not fix $\bar{a}$, meaning $\bar{a}$ could not be a code for $X$ $\square$

**Example 2.8.** Let $T = \text{Th}(M, <, \dots)$ where $<$ is a total well-ordering. Then $T$ has elimination of imaginaries

*Proof.* Every definable set has a least element. We verify (2) in 2.31. Let $E$ be a $\emptyset$-definable equivalence relation on $M^n$. Let $f : M^n \to M^n$ s.t. for any $\bar{a}$, $f(\bar{a})$ is the least element of the $E$-class of $\bar{a}$. Notice that $f$ is $\emptyset$-definable, and for all $\bar{a}, \bar{b}$, $f(\bar{a}) = f(\bar{b})$ iff $E(\bar{a}, \bar{b})$ $\square$

**Lemma 2.34.** *Let $T$ be strongly minimal and* $\text{acl}(\emptyset)$ *be infinite (in some, any model). Then $T$ has weak elimination of imaginaries*

*Proof.* Fix a model $M$. Let $e \in M^{\text{eq}}$ Ok, now i think the convention for pillay is that $e \in M^{\text{eq}}$ is automatically imaginary, so $e = \bar{a}/E$ for some $\bar{a}$ and $E$ some $\emptyset$-definable equivalence relation. Let $A = \text{acl}_{M^{\text{eq}}}(e) \cap M$. $A$ is infinite as it contains $\text{acl}(\emptyset)$. $A$ is infinite as it contains $\text{acl}(\emptyset)$.

We first prove that there exists some $\bar{b} \subset A$ s.t. $E(\bar{a}, \bar{b})$. Let $X_1 = \{y_1 \in M : M \vDash \exists y_2 \dots y_n(\bar{y}E\bar{a})\}$. It is definable over $e$. If $X_1$ is finite, any $b_1 \in X_1$ then belongs to $A$. Otherwise, $X_1$ is cofinite, hence meets the infinite set $A$. Either way, $X_1 \cap A \neq \emptyset$ and we have $b_1 \in X_1 \cap A$

Now let $X_2 = \{y_2 \in M : M \vDash \exists y_3 \dots y_n(b_1\bar{y}E\bar{a})\}$. We remark $X_2 \neq \emptyset$ since $b_1 \in X_1$. Now $X_2$ is either finite or cofinite since $T$ is strongly minimal. By the same argument above, we may find $b_2 \in X_2 \cap A$. Then repeating this process, we may find $\bar{b} \subset A$. Therefore $\bar{b} \in \text{acl}_{M^{\text{eq}}}(e)$.

Finally notice that $e \in \text{dcl}_{M^{\text{eq}}}(\bar{b})$ since $\bar{a}/E = \bar{b}/E = e$ $\square$

**Example 2.9.** The theory $\mathsf{ACF}_p$ has elimination of imaginaries, for any $p$

*Proof.* By Lemma 2.34, $\mathsf{ACF}_p$ has weak elimination of imaginaries. Therefore it suffices to show that every finite set can be coded. Let $K$ be an algebraically closed field and let $X = \{c_1, \dots, c_n\} \subseteq K$. Consider the polynomial

$$\begin{aligned} P(x) &= \prod_{i=1}^{n}(x - c_i) \\ &= x^n + e_{n-1}x^{n-1} + \dots + e_1 x + e_0 \end{aligned}$$

Then we may take the tuple $\bar{e} = (e_n, \dots, e_0)$ to be our code for $X$. $\square$

# 3 Stability

## 3.1 Historic remarks and motivations

Thoughout this chapter we will fix a complete theory $T$ in some language $L$. Moreover, we will have no problem in working in $T^{\text{eq}}$ (that is to say, to assume $T = T^{\text{eq}}$)

For a given theory $T$, the spectrum functions is given as

$$I(T, -) : Card \to Card$$
$$I(T, \lambda) = \# \text{ of models of } T \text{ or cardinality } \lambda \text{ (up to isomorphism)}$$

**Conjecture 3.1** (Morley). *Let $T$ be countable, then function $I_T(\kappa)$ is non-decreasing on uncountable cardinals*

One of such dividing lines is stability

## 3.2 Counting types and stability

**Definition 3.2.** For a complete first order theory $T$, let $f_T : Card \to Card$ be defined by $f_T(\kappa) = \sup\{|S_1(M)| : M \vDash T, |M| = \kappa\}$, for $\kappa$ an infinite cardinal

*Exercise* 3.2.1. Show that

$$f_T(\kappa) = \sup\{|S_n(M)| : M \vDash T, |M| = \kappa, n \in \omega\}$$

gives an equivalent definition

It is easy to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$

**Fact 3.3** (Keisler, Shelah [Keisler(1976)]). *Let $T$ be an arbitrary complete theory in a countable language. Then $f_T(\kappa)$ is one of the following functions (and all of these options occur for some $T$):*

$$\kappa, \kappa + 2^{\aleph_0}, \kappa^{\aleph_0}, \operatorname{ded}\kappa, (\operatorname{ded}\kappa)^{\aleph_0}, 2^{\kappa}$$

*Here,* $\operatorname{ded}\kappa = \sup\{|I| :$ *$I$ is a linear order with a dense subset of size $\kappa$*$\}$*, equivalently* $\sup\{\lambda :$ *there is a linear order of size $\kappa$ with $\lambda$ cuts*$\}$

ded is called the **Dedekind function**

**Lemma 3.4.** $\kappa < \operatorname{ded}\kappa \leq 2^{\kappa}$

*Proof.* Let $\mu$ be minimal s.t. $2^{\mu} > \kappa$, and consider the tree $2^{<\mu}$. Take the lexicographic ordering $I$ on it, then $|I| \leq \kappa$ by the minimality of $\mu$, but there are at least $2^{\mu} > \kappa$ cuts

Every cut is **uniquely** determined by the subset of elements in its lower half □

**Definition 3.5.** Let $M \vDash T$

1. A formula $\phi(x, y)$ with its variables partitioned into two groups $x, y$, has the $k$**-order property**, $k \in \omega$, if there are some $a_i \in M_x$, $b_i \in M_y$ for $i < k$ s.t. $M \vDash \phi(a_i, b_j) \Leftrightarrow i < j$

2. $\phi(x, y)$ has the **order property** if it has the $k$-order property for all $k \in \omega$

3. A formula $\phi(x, y)$ is **stable** if there is some $k \in \omega$ s.t. it does not have the $k$-order property

4. A theory is **stable** if it implies that all formulas are stable

**Proposition 3.6.** *Assume that $T$ is unstable, then $f_T(\kappa) \geq \operatorname{ded} \kappa$ for all cardinals $\kappa \geq |T|$*

*Proof.* Fix a cardinal $\kappa$ □

**Fact 3.7** (Ramsey)**.** *$\aleph_0 \to (\aleph_0)^n_k$ holds for all $n, k \in \omega$ (i.e., for any coloring of subsets of $\mathbb{N}$ of size $n$ in $k$ colors, there is some infinite subset $I$ of $\mathbb{N}$ s.t. all $n$-element subsets of $I$ have the same color)*

**Lemma 3.8.** *Let $\phi(x, y)$, $\psi(x, z)$ be stable formulas (where $y, z$ are not necessarily disjoint tuples of variables). Then*

1. *$\neg\phi(x, y)$ is stable*
2. *Let $\phi^*(y, x) := \phi(x, y)$, i.e., we switch the roles of the variables. Then $\phi^*(y, x)$ is stable*
3. *$\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ and $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ are stable*
4. *If $y = uv$ and $c \in M_v$, then $\theta(x, u) := \phi(x, uc)$ is stable*
5. *If $T$ is stable, then every $L^{\text{eq}}$-formula is stable as well*
6. *The formula $\varphi(x, y)$ is stable for $T$ iff there is $n < \omega$ s.t. $\varphi(x, y)$ is $n$-stable: it is not the case that there are $a_i, b_i$ (in $\mathfrak{C}$, or in some/any $M \vDash T$), $i < n$, s.t. $\vDash \varphi(a_i, b_i)$ iff $i < j$ for all $i, j < n$*
7. *There are $T$, $M \vDash T$ and $\varphi(x, y)$ s.t. $\varphi(x, y)$ is stable in $M$ but it is not stable for $T$*

*Proof.*

1. Suppose $\neg\phi(x, y)$ is unstable, then there is $I = (a_i, b_i)_{i\in\omega}$ s.t. $\vDash \neg\varphi(a_i, b_i) \Leftrightarrow i < j$, equivalently, $\vDash \varphi(a_i, b_i) \Leftrightarrow i \geq j$. Then add constants $(a_i, b_i)_{i\in\omega}$ and consider

$$\Gamma = T \cup \{\varphi(a_i, b_i) : i < j\} \cup \{\neg\varphi(a_i, b_i) : i \geq j\}$$

   For any finite subset $\Gamma' \subset_f \Gamma$, we can reverse the order of $I$: suppose $n$ is the maximum index and then let $i' = n - i$, $j' = n + 1 - j$. Then $i' < j' \Leftrightarrow n - i < n + 1 - j \Leftrightarrow i \geq j$. Hence $I$ satisfies this, and hence $\varphi(x, y)$ is unstable

2. Suppose $\varphi^*(y, x)$ is not stable, then $\neg\varphi^*(y, x)$ is also unstable. Let $a_i, b_i$ be witnesses in $\mathfrak{C}$ of the latter. Then $a'_i = b_i$ and $b'_i = a_{i+1}$, $i < \omega$, witness the instability of $\varphi(x, y)$ as $j + 1 > i$

3. Suppose that $\theta'(x, yz)$ is unstable, i.e., there are $(a_i, b_i b'_i : i \in \mathbb{N})$ s.t. $\models \phi(a_i, b_j) \vee \psi(a_i, b'_j) \Leftrightarrow i < j$ for all $i, j \in \mathbb{N}$. Let

   $P := \{(i,j) \in \mathbb{N}^2 : i < j, \models \phi(a_i, b_j)\}, Q := \{(i,j) \in \mathbb{N}^2 : i < j, \models \psi(a_i, b'_j)\}$

   then $P \cup Q = \{(i,j) \in \mathbb{N}^2 : i < j\}$. By Ramsey there is an infinite $I \subseteq \mathbb{N}$ s.t. either all increasing pairs from $I$ belong to $P$, or all increasing pairs from $I$ belong to $Q$

7. Consider the graph $G$, disjoint union of all finite graphs. Then the edge relation $E$ is stable in $G$. Indeed, if it wasn't, we would have a vertex $x_0$ and infinitely many vertices $\{y_i : i \in \mathbb{N}\}$ s.t. $E(x_0, y_i)$ for all $i$, which is impossible

   But by 6, edge relation is not stable in $\mathrm{Th}(G)$

□

**Definition 3.9.** Fix $\varphi(x, y) \in L$. By a **complete $\varphi$-type over** $M$, $M \models T$, we mean a maximal consistent set of instances of $\varphi$ and $\neg\varphi$ over $M$, namely $L_M$-formulas of the form $\varphi(x, b)$, $\neg\varphi(x, b)$ for $b \in M$. We write $S_\varphi(M)$ for the set of such complete $\varphi$-types over $M$

*Remark.*

1. By a **$\varphi$-formula over** $M$ we mean a Boolean combination of instances (over $M$) of $\varphi$ and $\neg\varphi$. For example, $(\varphi(x, c) \wedge \varphi(x, b)) \vee \neg\varphi(x, d)$ is a $\varphi$-formula

2. Any type $p(x) \in S_\varphi(M)$ decides any $\varphi$-formula $\psi(x)$ over $M$, that is to say $p(x) \models \psi(x)$ or $p(x) \models \neg\psi(x)$, so in fact $p(x)$ extends to a unique maximal consistent set of $\varphi$-formulas over $M$

3. By defining the basic open sets of $S_\varphi(M)$ to be $\{p(x) \in S_\varphi(M) : \psi(x) \in p\}$ for $\psi$ a $\varphi$-formula, $S_\varphi(M)$ becomes a compact totally disconnected space, where in addition the clopen sets are precisely given by $\varphi$-formulas, i.e., they are the basic clopen sets

4. Any $p(x) \in S_\varphi(M)$ extends to some $q(x) \in S_x(M)$ s.t. $p = q \restriction \varphi$, where $q \restriction \varphi$ is the set of $\varphi$-formulas in $q(x)$ (or instances of $\varphi$, $\neg\varphi$ in $q(x)$)

**Definition 3.10.**

1. Let $p(x) \in S_x(M)$ be a complete type over $M$. We say that $p(x)$ is **definable** if, for each $\varphi(x, y)$ in $L$, there is an $L_M$-formula $\psi(y)$ s.t. for all $m \in M$, we have $M \models \psi(m)$ iff $\varphi(x, m) \in p$ (note that such $\psi(y)$ is unique up to equivalence). We say the type $p(x)$ is **definable over** $A \subseteq M$ is each such $\psi(y)$ is over $A$

2. Likewise, we speak of the $\varphi$-type $p(x) \in S_\varphi(M)$ being **definable** when $\{b \in M : \varphi(x, b) \in p(x)\}$ is defined by a formula $\psi(y)$ of $L_M$ (Note that in this case $\psi(y)$ determines $p(x)$)

As we will see later, a theory $T$ iff all types over **all** models of $T$ are definable.

Note that there are unstable theories for which all the types over a certain models are definable. For instance, in the case of dense linear orders, all types over $\mathbb{R}$ are definable

Indeed, by quantifier elimination, any non-realised 1-type over any model of DLO corresponds to a cut in its order. But in the case of $\mathbb{R}$, the order is complete, so for any cut, there will in fact exist a real number $r$ s.t. the cut is of the form $(\{l \in \mathbb{R}, l < r\}, \{d \in \mathbb{R}, d > r\})$. Using this real number $r$, one can easily show definability of 1-types over $\mathbb{R}$

**Proposition 3.11.** *Fix a model $M \vDash T$ and an $L$-formula $\varphi(x, y)$. TFAE*

1. *$\varphi(x, y)$ is stable in $M$*
2. *Whenever $M^* \succ M$ is $|M|^+$-saturated and* tp$(a^*/M^*)$ *is finitely satisfiable in $M$, then* tp$_\varphi(a^*/M^*)$ *is definable over $M$ and, moreover, it is defined by some $\varphi^*$-formula, i.e., a Boolean combination of $\varphi(a, y)$'s, $a \in M$*

1#+BEGIN$_{\text{proof}}$ $1 \to 2$. Fix some $p^*(x) = \text{tp}(a^*/M^*)$ finitely satisfiable in $M$. We want to prove tp$_\varphi(a^*/M^*)$ is definable over $M$ by a $\varphi^*$-formula. Note first that, as $p^*$ is finitely satisfiable in $M$, whether or not some $\varphi(x, b)$, $b \in M^*$ is in $p^*$ depends only on tp$(b/M)$ #+END$_{\text{proof}}$

### 3.3 Local Stability

**Definition 3.12.** 1. Let $M \vDash T$. We say $\varphi(\bar{x}, \bar{y})$ is **stable in** $M$ if it is **not** the case that there are $\bar{a}_i, \bar{b}_i \in M$, for $i < \omega$, s.t. **either** for all $i, j < \omega$, $M \vDash \varphi(\bar{a}_i, \bar{b}_j)$ iff $i \leq j$, **or**, for all $i, j < \omega$, $M \vDash \neg\varphi(\bar{a}_i, \bar{b}_j)$ iff $i \leq j$ I think, this is from **??**

2. $\varphi(\bar{x}, \bar{y})$ is **stable** (for $T$) if it is stable in $M$ for all $M \vDash T$
3. $T$ is **stable** if every $L$-formula $\varphi(\bar{x}, \bar{y})$ is stable (for $T$)

*Remark.* A formula $\varphi(\bar{x}, \bar{y})$ is stable for $T$ iff it is not the case that there are $\bar{a}_i, \bar{b}_i \in \mathfrak{C}$, for $i < \omega$, s.t. $\mathfrak{C} \vDash \varphi(\bar{a}_i, \bar{b}_i)$ iff $i \leq j$ for all $i, j < \omega$

For simplicity, from now on we will write tuples simply as $x$ and $a$.

# 4 TODO Problems

2.1 2.3 2.4

# 5 Index

This is a functional link that will open a buffer of clickable index entries:

# 6 References

## References


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