Coheir independence

Advanced Model Theory

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This document is about the relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$ of *coheir independence* (Definition 1), which provides new insights into the material we have covered in class.

Using $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$, we see how the two notions "coheir" and "heir" are dual (Remark 3). As an application, the existence of coheirs is logically equivalent to the existence of heirs (Corollary 5). The relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$ is also connected to ultrafilters (Proposition 6), and this explains why heirs extend to ultrapower types (Remark 10) and why coheirs extend to further coheirs (Corollary 9). In stable theories, the relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$ is symmetric, and this is connected to the equivalence of coheirs and heirs (Theorem 11). In stable theories, the relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$ is generalized by the relation $\mathop{\smile\!\!\!\!\!\!\downarrow}$ of *non-forking independence* (Section 5).

1 Coheir independence

Work in a monster model $\mathbb{M}$ of a complete L-theory T. Recall that a "small model" is a small elementary substructure $M \preceq \mathbb{M}$.

Definition 1. Let M be a small model and $\bar{a}, \bar{b}$ be small tuples (possibly infinite). Then $\bar{a}$ is *coheir independent* from $\bar{b}$ over M, written

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u \bar{b},$$

if $\operatorname{tp}(\bar{a}/M\bar{b})$ is finitely satisfiable in M. In other words, for any $L(M)$ formula $\varphi(\bar{x}, \bar{y})$, if $\mathbb{M} \models \varphi(\bar{a}, \bar{b})$, then there is $\bar{a}' \in M$ such that $\mathbb{M} \models \varphi(\bar{a}', \bar{b})$.

If A, B are small sets and M is a small model, then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u B$ means $\bar{a} \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u \bar{b}$ where $\bar{a}, \bar{b}$ are tuples enumerating A and B, respectively. (The choice of the enumerations doesn't matter.)

Note that $\mathop{\smile\!\!\!\!\!\!\downarrow}^u$ is a ternary (3-ary) relation.

Remark 2. The relation $A \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u B$ is finitary with respect to the arguments A and B, in the following sense. $A \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u B$ holds iff the following does:

For any finite tuple $\bar{a} \in A$ and any finite tuple $\bar{b} \in B$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\downarrow}_M^u \bar{b}$.

The reason is that a formula $\varphi(\bar{x}, \bar{y})$ can only refer to finitely many variables in the tuple $\bar{x}$, and finitely many variables in the tuple $\bar{y}$.

Remark 3. The relation $\mathop{\smile\!\!\!\!\!\mid}^u$ can be used to define heirs and coheirs, as follows. Suppose M, N are small models with $M \preceq N$. Suppose $p \in S_n(M)$ and $q \in S_n(N)$ with $q \supseteq p$. Take $\bar{a} \in \mathbb{M}^n$ realizing q and therefore realizing p too.

1. $q = \operatorname{tp}(\bar{a}/N)$ is a coheir of $p = \operatorname{tp}(\bar{a}/M)$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^u_M N$.
2. $q = \operatorname{tp}(\bar{a}/N)$ is an heir of $p = \operatorname{tp}(\bar{a}/M)$ if and only if $N \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{a}$.

Part (1) is the definition of "coheir." Part (2) is easy to prove by unwinding the definitions: the condition $N \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{a}$ means that for any $L(M)$-formula $\varphi(\bar{x}, \bar{y})$,

$$\left[\exists \bar{b} \in N : \mathbb{M} \models \varphi(\bar{a}, \bar{b})\right] \implies \left[\exists \bar{b}' \in M : \mathbb{M} \models \varphi(\bar{a}, \bar{b}')\right],$$

or equivalently

$$\left[\exists \bar{b} \in N : \varphi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{a}/N)\right] \implies \left[\exists \bar{b}' \in M : \varphi(\bar{x}, \bar{b}') \in \operatorname{tp}(\bar{a}/M)\right]$$

which is the definition of "heir."

Remark 3 shows that heirs and coheirs are dual in some sense. This is the reason for the name "coheir."

2 Existence

Lemma 4. *Let M be a small model and $\bar{a}, \bar{b}$ be tuples, possibly infinite.*

1. *There is $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$ such that $\sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{b}$.*
2. *There is $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^u_M \sigma(\bar{b})$.*

Proof. 1. Let α be the length of $\bar{a}$ and let $\bar{x}$ be an α-tuple of variables. Let

$$\Psi(\bar{x}) = \{\psi(\bar{x}) \in L(M\bar{b}) : \psi(\bar{x}) \text{ is satisfied by every } \bar{a}' \in M^\alpha\}.$$

Any $\bar{a}' \in M^\alpha$ satisfies $\Psi(\bar{x})$. If $\varphi(\bar{x}) \in \operatorname{tp}(\bar{a}/M)$, then there is $\bar{a}' \in M^\alpha$ satisfying $\varphi(\bar{x})$ because $\operatorname{tp}(\bar{a}/M)$ is finitely satisfiable in M. Then $\bar{a}'$ satisfies $\{\varphi(\bar{x})\} \cup \Psi(\bar{x})$. This shows $\operatorname{tp}(\bar{a}/M) \cup \Psi(\bar{x})$ is finitely satisfiable, hence realized by some $\bar{a}' \in \mathbb{M}^\alpha$.

Then $\bar{a}'$ realizes $\operatorname{tp}(\bar{a}/M)$, so $\operatorname{tp}(\bar{a}'/M) = \operatorname{tp}(\bar{a}/M)$, and there is $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$ such that $\sigma(\bar{a}) = \bar{a}'$. Finally, $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{b}$ by choice of $\Psi(\bar{x})$: if $\varphi(\bar{x}) \in \operatorname{tp}(\bar{a}'/M\bar{b})$ and $\varphi(\bar{x})$ isn't satisfiable in M, then $\neg\varphi(\bar{x}) \in \Psi(\bar{x})$ and $\bar{a}'$ doesn't satisfy $\varphi(\bar{x})$, a contradiction.

2. By (1), there is $\tau \in \operatorname{Aut}(\mathbb{M}/M)$ such that $\tau(\bar{a}) \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{b}$. Let $\sigma = \tau^{-1}$. Then $\sigma(\tau(\bar{a})) \mathop{\smile\!\!\!\!\!\mid}^u_{\sigma(M)} \sigma(\bar{b})$, or equivalently, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^u_M \sigma(\bar{b})$. $\square$

Corollary 5. *Suppose $p \in S_n(M)$ and $N \succeq M$.*

1. *There is $q \in S_n(M)$ such that q is a coheir of p.*
2. *There is $q \in S_n(M)$ such that q is an heir of p.*

Proof. 1. Take $\bar{a} \in \mathbb{M}^n$ realizing p. Let $\bar{b}$ enumerate N. By Lemma 4(1), there is $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ such that $\sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\mid}^u_M \bar{b}$, i.e., $\sigma(\bar{a}) \mathop{\smile\!\!\!\!\!\mid}^u_M N$. Thus $\mathrm{tp}(\sigma(\bar{a})/N)$ is a coheir of $\mathrm{tp}(\sigma(\bar{a})/M) = \mathrm{tp}(\bar{a}/M) = p$.

2. Similar, using Lemma 4(2) to get $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ such that $N \mathop{\smile\!\!\!\!\!\mid}^u_M \sigma(\bar{a})$. $\square$

Both halves of Corollary 5 were proved previously.

- The existence of coheirs was proved in Theorem 5 of the March 10th notes, via the exact proof given above.
- The existence of heirs was proved in Proposition 14 of the February 24th notes. The proof there looks different, but I believe the two proofs are secretly equivalent on some level.

3 "u" for "ultrafilter"

Proposition 6. *Let $\bar{a}$ be an α-tuple in $\mathbb{M}$. Let M be a small model and B be a small set. The following are equivalent:*

1. $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^u_M B$
2. *There is an ultrafilter $\mathcal{U}$ on the set M^α such that for any $L(MB)$-formula $\varphi(\bar{x})$,*

$$\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB) \iff \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}. \qquad (*)$$

Proof. (1) $\implies$ (2): for $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB)$, let $\varphi(M^\alpha) = \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\}$. Let $I = M^\alpha$ and $\mathcal{F} = \{\varphi(M^\alpha) : \varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB)\}$. We claim $\mathcal{F}$ has the FIP. Indeed, if $\varphi_1, \ldots, \varphi_n \in \mathrm{tp}(\bar{a}/MB)$, then there is $\bar{a}' \in M^\alpha$ satisfying each φ_i, meaning $\bar{a}' \in \bigcap_{i=1}^n \varphi_i(M^\alpha)$.

Let $\mathcal{U}$ be an ultrafilter on M^α extending $\mathcal{F}$. Then for any $L(MB)$-formula,

$$\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB) \implies \varphi(M^\alpha) \in \mathcal{F} \implies \varphi(M^\alpha) \in \mathcal{U} \iff \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\} \in \mathcal{U}.$$

Then,

$$\begin{aligned} \varphi(\bar{x}) \notin \mathrm{tp}(\bar{a}/MB) &\implies \neg\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB) \implies \{\bar{a}' \in M^\alpha : \mathbb{M} \models \neg\varphi(\bar{a}')\} \in \mathcal{U} \\ &\implies \{\bar{a}' \in M^\alpha : \mathbb{M} \models \varphi(\bar{a}')\} \notin \mathcal{U}. \end{aligned}$$

So both directions of $(*)$ hold.

(2) $\implies$ (1): suppose $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/MB)$. By (2), $\varphi(M^\alpha) \in \mathcal{U}$, which implies $\varphi(M^\alpha) \neq \varnothing$, which means $\varphi(\bar{x})$ is satisfiable in M. $\square$

Proposition 6 explains why there's a "u" in $\mathop{\smile\hskip-0.9em\raise0.2em\hbox{$\vert$}}^{u}$; it stands for "ultrafilter".

Remark 7. Topologically, condition (2) in Proposition 6 says that $\text{tp}(\bar{a}/MB)$ is an *ultralimit* of $\text{tp}(\bar{a}'/MB)$ for various $\bar{a}' \in M^\alpha$. Proposition 6 is related to the fact that the set $X = \{p \in S_\alpha(MB) : p \text{ is finitely satisfiable in } M\}$ is the topological closure of $X_0 = \{\text{tp}(\bar{a}'/MB) : \bar{a}' \in M^\alpha\}$, and so every type in X is a "limit" of types in X_0.

Proposition 8. *Suppose $p \in S_n(M)$ and $N \succeq M$.*

1. *If $q \in S_n(N)$ is a coheir of p, then there is an ultrafilter $\mathcal{U}$ on M^n such that*
$$q(\bar{x}) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\}. \tag{**}$$
2. *Conversely, if $\mathcal{U}$ is an ultrafilter on M^n and we define $q(\bar{x})$ according to (**), then $q(\bar{x}) \in S_n(N)$ and q is a coheir of p.*

Proof. 1. Take $\bar{a}$ realizing q and p. Then $\bar{a} \mathop{\smile\hskip-0.9em\raise0.2em\hbox{$\vert$}}^{u}_{M} N$. Apply Proposition 6.

2. It suffices to show that q is finitely satisfiable in M and complete:

- q is finitely satisfiable in M: if $\varphi_1, \ldots, \varphi_n \in q(\bar{x})$, then $\varphi_i(M^n) \in \mathcal{U}$ for each i, so $\bigcap_{i=1}^n \varphi_i(M^n) \neq \varnothing$.
- q is complete: if $\psi = \neg\varphi$, then $\psi(M^n)$ and $\varphi(M^n)$ are complements, so one is in $\mathcal{U}$, and then $\psi \in q$ or $\varphi \in q$. □

Corollary 9 (Coheirs extend)**.** *Suppose $M \preceq N \preceq N'$ and $p \in S_n(M)$ and $q \in S_n(N)$ is a coheir of p. Then there is $q' \in S_n(N')$ with $q' \supseteq q$ and q' a coheir of p.*

(This was Theorem 6 in the March 10th notes.)

Proof. By Proposition 8 there is an ultrafilter $\mathcal{U}$ on M^n such that
$$q(x) = \{\varphi(\bar{x}) \in L(N) : \varphi(M^n) \in \mathcal{U}\}.$$
Take $q'(x) = \{\varphi(\bar{x}) \in L(N') : \varphi(M^n) \in \mathcal{U}\}$. □

Remark 10. Suppose $q \in S_n(N)$ is an heir of $p \in S_n(M)$. Then $N \mathop{\smile\hskip-0.9em\raise0.2em\hbox{$\vert$}}^{u}_{M} \bar{a}$ for a realization $\bar{a}$. Proposition 6 gives an ultrafilter $\mathcal{U}$ and tells us *something*. The details are confusing to work through, but the ultimate conclusion is

> There is an ultrapower $M^{\mathcal{U}} \succeq N$ such that $p^{\mathcal{U}} \supseteq q$.

This was Proposition 2 in the March 3rd notes.

4 Symmetry

Suppose $q \in S_n(N)$ is an extension of $p \in S_n(M)$. In Corollary 21 of the March 10th notes, we proved

> q is a coheir of p if and only if q is an heir of p.

Taking a realization $\bar{a} \in \mathbb{M}^n$ of q and p, this means

$$\bar{a} \mathop{\downarrow}_{M}^{u} N \iff N \mathop{\downarrow}_{M}^{u} \bar{a}. \tag{1}$$

More generally, the relation $\downarrow^u$ is symmetric in stable theories:

Theorem 11. *If T is stable, then*

$$\bar{a} \mathop{\downarrow}_{M}^{u} \bar{b} \implies \bar{b} \mathop{\downarrow}_{M}^{u} \bar{a}.$$

Proof. It suffices to prove $\Rightarrow$. Let α be the length of $\bar{a}$. Take a small model N containing M and $\bar{b}$. By the method of Corollary 9, one can find a type $q \in S_\alpha(N)$ extending $\mathrm{tp}(\bar{a}/M\bar{b})$, finitely satisfiable in M. Take $\bar{a}'$ realizing q. Then $\bar{a}' \downarrow_M^u N$. Also $\mathrm{tp}(\bar{a}'/M\bar{b}) = q \restriction (M\bar{b}) = \mathrm{tp}(\bar{a}/M\bar{b})$, so there is $\sigma \in \mathrm{Aut}(\mathbb{M}/M\bar{b})$ such that $\sigma(\bar{a}') = \bar{a}$. Then

$$\bar{a}' \mathop{\downarrow}_{M}^{u} N \implies \sigma(\bar{a}') \mathop{\downarrow}_{\sigma(M)}^{u} \sigma(N) \iff \bar{a} \mathop{\downarrow}_{M}^{u} \sigma(N).$$

Replacing N with $\sigma(N)$, we may assume $\bar{a} \downarrow_M^u N$. By equation (1) above, $N \downarrow_M^u \bar{a}$. As $\bar{b} \in N$, this implies $\bar{b} \downarrow_M^u \bar{a}$. $\square$

5 Onwards

Let A, B, C be small sets. Shelah defines a ternary relation

$$A \mathop{\downarrow}_{C} B$$

called *(non-)forking independence*. In stable theories, Shelah's $\downarrow$ satisfies many nice properties. Here are two:

- $A \downarrow_C B \iff B \downarrow_C A$, analogous to Theorem 11.
- If M is a model, then $A \downarrow_M B \iff A \downarrow_M^u B$.

The relation $\downarrow$ is very important in the study of stable theories, and we will learn more about it in the weeks ahead.