Introduction To Model Theory

Will Johnson

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1 Back-and-forth Equivalence

Convention: Relations and functions are sets of pairs (x, y)

Definition 1.1. A **binary relation** is a pair (E, R) where E is a set and $R \subseteq E^2$. We call E the **universe** of the relation. For $a, b \in E$, write aEb if $(a, b) \in R$

We abbreviate (E, R) as R or E, if E or R is clear

Example 1.1. $(\mathbb{R}, <)$, $(\mathbb{R}, =)$, $(\mathbb{R}, \geq)$,$(\mathbb{Z}, <)$

Definition 1.2. A binary relation R is said to be

- **reflexive** if aRa $(\forall a \in E)$
- **symmetric** if $aRb \Rightarrow bRa$ $(\forall a, b \in E)$
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$ $(\forall a, b, c \in E)$
- **antisymmetric** if $aRb \wedge bRa \Rightarrow a = b$ $(\forall a, b \in E)$
- **total** if $aRb \vee bRa$ $(\forall a, b \in E)$
- an **equivalence relation** if it's reflexive, symmetric and transitive
- a **partial order** if it's reflexive, antisymmetric and transitive

- a **linear order** if it's a total partial order

Example 1.2. $=$ is an equivalence relation
$\subseteq$ is a partial order
$\leq$ is a linear order

Definition 1.3. An **isomorphism** from (E, R) to (E', R') is a bijection $f : E \to E'$ s.t. for any $a, b \in E$, $aRb \Leftrightarrow f(a)R'f(b)$. Two binary relations (E, R) and (E', R') are **isomorphic** ($\cong$) if there is an isomorphism between them

Example 1.3. $f : (\mathbb{Z}, <) \to (2\mathbb{Z}, >)$ and $f(x) = -2x$ is an isomorphism. $x < y \Leftrightarrow -2x > -2y$

$\cong$ is an equivalence relation

Definition 1.4. A **local isomorphism** from R to R' is an isomorphism from a finite restriction of R to a finite restriction of R'. The set of local isomorphisms from R to R' is denoted $S_0(R, R')$. For $f \in S_0(R, R')$, dom(f) and im(f) denote the domain and range of f

Example 1.4. $(\mathbb{Z}, <)$ is a restriction of $(\mathbb{R}, <)$

Example 1.5. Suppose $R = R' = (\mathbb{Z}, <)$, there is $f \in S_0(R, R')$ given by dom$(f) = \{1, 2, 3\}$ and im$(f) = \{10, 20, 30\}$ and $f(1) = 10$,$f(2) = 20$, $f(3) = 30$

Definition 1.5. Let f, g be local isomorphisms from R to R'. Then f is a **restriction** of g if $f \subseteq g$ and f is an **extension** of g if $f \supseteq g$.

Example 1.6. $g : \{0, 1, 2, 3\} \to \{5, 10, 20, 30\}$, g extends f in the previous example

Definition 1.6. Let R, R' be binary relations with universe E, E'. A **Karpian family** for (R, R') is a set $K \subseteq S_0(R, R')$ satisfying the following two conditions for any $f \in K$

1. (**forth**) if $a \in E$ then there is $g \in K$ with $g \supseteq f$ and $a \in$ dom(g)
2. (**back**) if $b \in E'$ then there is $g \in K$ with $g \supseteq f$ and $b \in$ im(g)

R and R' are ∞**-equivalent**, write $R \sim_\infty R'$, if there is a non-empty Karpian family

Proposition 1.7. *If $f : (E, R) \to (E', R')$ an isomorphism and $K = \{g \subseteq f : g$ is finite$\}$, then K is Karpian and $R \sim_\infty R'$*

Proof. Suppose $g \in K$

- (forth) Suppose $a \in E$, take $b = f(a)$ and let $h = g \cup \{(a, b)\}$. Then $h \subseteq f$, so $h \in K$, $h \supseteq g$, $a \in \text{dom}(h)$
- (back) similarly

$\square$

Proposition 1.8. *If (E, R) and (E', R') are countable and $R \sim_\infty R'$, then $R \cong R'$*

Proof. Let $K \subseteq S_0(R, R')$ be Karpian, $K \neq \emptyset$, $E = \{e_1, e_2, e_3, \dots\}$, $E' = \{e'_1, e'_2, e'_3, \dots\}$

Recursively build $f_1 \subseteq f_2 \subseteq \cdots$, $f_i \in K$
Let f_1 be anything in K as K is non-empty.
f_{2i} some extension of f_{2i-1} with $e_i \in \text{dom}(f_{2i})$
f_{2i+1} some extension of f_{2i} with $e'_i \in \text{im}(f_{2i+1})$
Now let $g = \bigcup_{i=1}^{\infty} f_i$, then g is an isomorphism $\square$

Definition 1.9. A **dense linear order without endpoints** (DLO) is a linear order $(C, \leq)$ satisfying

1. $C \neq \emptyset$
2. $\forall x, y \in C$, $x < y \Rightarrow \exists z \in C\ x < z < y$
3. $\forall x \in C$, $\exists y, z \in C\ y < x < z$

Example 1.7. $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$
non-example: $(\mathbb{Z}, \leq)$, $([0, 1], \leq)$

Proposition 1.10. *Let $(C, \leq)$ and $(C', \leq)$ be DLO's. Then $S_0(C, C')$ is Karpian. So $C \sim_\infty C'$*

Proof. Let $f \in S_0(C, C')$, $\text{dom}(f) = \{a_1, \dots, a_n\}$, $a_1 < \cdots < a_n$ and $\text{im}(f) = b_1, \dots, b_n$, $b_1 < \cdots < b_n$. Since f is a local isomorphism, $f(a_i) = b_i$

- (forth) Suppose $a \in C$. We want $b \in C'$ s.t. $f \cup \{(a, b)\} \in S_0(C, C')$.
  - if $a_i < a < a_{i+1}$. We take $b \in C'$ s.t. $b_i < b < b_{i+1}$ since dense
  - if $a < a_1$. We take $b \in C'$ s.t. $b < b_1$ since no endpoints
  - if $a > a_n$, take $b \in C'$ s.t. $b > b_n$
  - if $a = a_i$, take $b = b_i$

- (back) similar

∎

Proposition 1.11. *If $(C, \leq)$ and $(C', \leq)$ are countable DLOs, then $C \sim_\infty C'$, so $C \cong C'$*

Hence

$$\begin{aligned}(\mathbb{Q}, \leq) &\cong (\mathbb{Q} \setminus \{0\}, \leq) \\ &\cong (\mathbb{Q} \cup \{\sqrt{2}\}, \leq) \\ &\cong (\mathbb{Q} \cap (0,1), \leq)\end{aligned}$$

Definition 1.12. Let R, R' be binary relations with universe E, E'

- A **0-isomorphism** from R to R' is a local isomorphism from R to R'
- For $p > 0$, a p**-isomorphism** from R to R' is a local isomorphism f from R to R' satisfying the following two conditions
  1. (**forth**) For any $a \in E$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $a \in \text{dom}(g)$
  2. (**back**) For any $b \in E'$, there is a $(p-1)$-isomorphism $g \supseteq f$ with $b \in \text{im}(g)$
- An ω**-isomorphism** from R to R' is a local isomorphism f from R to R' s.t. f is a p-isomorphism for all $p < \omega$

The set of p-isomorphisms from R to R' is denoted $S_p(R, R')$

Example 1.8. Suppose $R = R' = (\mathbb{Z}, <)$, $f : \{2,4\} \to \{1,2\}$ is a local isomorphism with $f(2) = 1$ and $f(4) = 2$. Then $f \notin S_1(\mathbb{Z}, \mathbb{Z})$ (forth) fails. For $a = 3$, there is no b s.t. $1 < b < 2$

$g : \{2,4\} \to \{1,5\}$ is a 1-isomorphism but not a 2-isomorphism

Proposition 1.13. *If $f \in S_p(R, R')$ and $g \subseteq f$, then $g \in S_p(R, R')$*

Proof. if $p = 0$ easy

if $p > 0$ (forward), $\forall a \in E$, $\exists h \in S_{p-1}(R, R')$ has $a \in \text{dom}(h)$ and $h \supseteq f \supseteq g$ ∎

Proposition 1.14. *$S_p(R, R') \neq \emptyset$ iff $\emptyset \in S_p(R, R')$*

Proof. $\Leftarrow$ immediate

$\Rightarrow$. Suppose $f \in S_p(R, R')$. Then $\emptyset \subseteq f$. Hence $\emptyset \in S_p(R, R')$. ∎

Definition 1.15. R and R' are p**-equivalent**, written $R \sim_p R'$, if there is a p-isomorphism from $R \to R'$

R and R' are ω**-equivalent** or **elementarily equivalent**, written $R \sim_\omega R'$ or $R \equiv R'$, if there is an ω-isomorphism from R to R'

Note: $R \sim_\omega R'$ iff $S_\omega(R, R') \neq \emptyset$ iff $\emptyset \in S_\omega(R, R')$ iff $\forall p\ \emptyset \in S_p(R, R')$ iff $\forall p\ R \sim_p R'$

Definition 1.16. Let R, R' be binary relations with universe E, E'. The Ehfrenfeucht-Fraïssé game of length n, denoted $\mathrm{EF}_n(R, R')$ is played as follows

- There are two players, the Duplicator and Spoiler
- There are n rounds
- In the ith round, the Spoiler chooses either an $a_i \in E$ or a $b_i \in E'$
- The Duplicator responds with a $b_i \in E'$ or an $a_i \in E$ respectively
- At the ends of the game, the Duplicator wins

  $$\{(a_i, b_i), \dots, (a_n, b_n)\}$$

  is a local isomorphism from R to R'
- Otherwise, the Spoiler wins

Example 1.9. For $\mathrm{EF}_3(\mathbb{Q}, \mathbb{R})$

$\mathbb{Q}$	$\mathbb{R}$
S:$a_1 = 7$	D:$b_1 = 7$
D:$a_2 = 1.4$	S:$b_2 = \sqrt{2}$
D:$a_3 = -10$	S:$b_3 = 1.41$

So D wins

Example 1.10. $\mathrm{EF}_3(\mathbb{R}, \mathbb{Z})$

$\mathbb{R}$	$\mathbb{Z}$
D:$a_1 = 1$	S:$b_1 = 1$
D:$a_2 = 1.1$	S:$b_2 = 2$
S:$a_3 = 1.01$	

D fails

Proposition 1.17. *$EF_n(R, R')$ is a win for Duplicator iff $R \sim_n R'$*

Proposition 1.18. *In $EF_n(R, R')$ if moves so far are $a_1, b_1, \dots, a_i, b_i$, $p = n - 1$, $f = \{(a_1, b_1), \dots, (a_i, b_i)\}$. Then Duplicator wins iff $f \in S_p(R, R')$*

A Metric Spaces

$\mathbb{R}_{\geq 0}$ denotes $[0, +\infty] = \{x \in \mathbb{R} : x \geq 0\}$

Definition A.1. A **metric** on a set M is a function $d : M \times M \to \mathbb{R}_{\geq 0}$ satisfying the following properties

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Example A.1. $M = \mathbb{R}^2$, $d(x, y) =$(the distance from x to y)

$$d(x_1, x_2; y_1, y_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Example A.2. The **Manhattan metric** on $\mathbb{R}^2$ is given by

$$d(x_1, x_2; y_1, y_2) = |x_1 - y_1| + |x_2 - y_2|$$

measure distances in a city grid

Example A.3. Let M be the set of strings. The **edit distance** from x to y is the minimum number of intersections, deletions, and substitutions to go from x to y

$$d(drip, rope) = 3$$
$$drip \mapsto drop \mapsto rop \mapsto rope$$

Edit distance is a metric on M

Definition A.2. A **metric space** is a pair (M, d) where M is a set and d is a metric space

- $(\mathbb{R}^n, d_{Euclidean})$ where $d_{Euclidean}$ is the usual Euclidean distance
- $(\mathbb{R}^2, d_{Manhattan})$ where $d_{Manhattan}$ is the Manhattan distance

Often we abbreviate (M, d) as M, when d is clear
Fix a metric space (M, d)

Definition A.3. If $p \in M$ and $\epsilon > 0$, then

$$B_\epsilon(p) = \{x \in M : d(x,p) < \epsilon\}$$
$$\overline{B}_\epsilon(p) = \{x \in M : d(x,p) \leq \epsilon\}$$

$B_\epsilon(p)$ and $\overline{B}_\epsilon(p)$ are called the **open** and **closed** balls of radius ϵ around p

Example A.4. In $\mathbb{R}^2$ with the Euclidean metric, the open ball of radius 2 around $(0,0)$ the open disk

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 2^2\}$$

Example A.5. In $\mathbb{R}^2$ with the Manhattan metric, the open ball of radius 1 around $(0,0)$ the open disk

$$\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$$

Suppose $p \in M$ and $X \subseteq M$

Definition A.4. p is an **interior point** of X if X contains an open ball of positive radius around p

In particular, p must be an element of X

Example A.6. If $X = [-1,1] \times [-1,1]$, then $(0,0)$ is an interior point of X, but $(1,0)$ and $(0,2)$ are not

Definition A.5. The **interior** $\text{int}(X)$ is the set of interior points

Warning: There are metric spaces where the interior of $\overline{B}_\epsilon(p)$ isn't $B_\epsilon(p)$

Definition A.6. A set $X \subseteq M$ is **open** if $X = \text{int}(X)$, i.e., every point of X is an interior point of X

Example A.7 (in $\mathbb{R}$)**.** The set $(-1,2)$ is open. The sets $[-1,2]$ and $[-1,2)$ are not; they have interior $(-1,2)$

Fact: the interior $\text{int}(X)$ is the unique largest open set contained in X
Let $a_1, a_2, \ldots$ be a sequence in a metric space (M,d) and let p be a point

Definition A.7. "$\lim_{i\to\infty} a_i = p$" if for every $\epsilon > 0$, there is n s.t.

$$\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq B_\epsilon(p)$$

Example A.8. Work in $\mathbb{R}$ with the usual distance. Let $a_n = 1/n$. Then $\lim_{n\to\infty} a_n = 0$ but $\lim_{n\to\infty} a_n \neq 1$

Fact: For any sequence $a_1, a_2, a_3, \cdots$ in (M, d), there is at most one point p s.t. $\lim_{i\to\infty} a_i = p$

If such a p exists, it is called the **limit**, and written $\lim_{i\to\infty} a_i$

let X be a set and p be a point in a metric space (M, d)

Definition A.8. p is an **accumulation point** of X if $p = \lim_{n\to\infty} a_n$ for some sequence a_n in X

Equivalently

Definition A.9. p is an accumulation point of X if for every $\epsilon > 0$, we have $B_\epsilon(p) \cap X \neq \emptyset$

Definition A.10. The **closure** of X, written cl(X) or $\overline{X}$, is the set of accumulation points

Definition A.11. A set $X \subseteq M$ is **closed** if $X = \text{cl}(X)$

Fact: The closure cl(X) is the unique smallest closed set containing X

Example A.9. Work in $\mathbb{R}$ with the distance $d(x, y) = |x - y|$

$\mathbb{Q}$ is neither closed nor open

$\mathbb{R}$ is both closed and open, so is $emptyset$

Let X^c denote the completement $M \setminus X$

Fact: X is closed iff X^c is open

Fact: int$(X) = \text{cl}(X^c)^c$ and cl$(X) = \text{int}(X^c)^c$

Let (M, d) and (M', d) be metric spaces. Let $f : M \to M'$ be a function

Definition A.12. f is **continuous** if

$$\lim_{n\to\infty} a_n = p \Rightarrow \lim_{n\to\infty} f(a_n) = f(p)$$

for $a_1, a_2, a_3, \ldots, p \in M$

idea: f is continuous iff f preserves limits

Example A.10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Then $\lim_{n\to\infty} 1/n = 0$, but

$$\lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 1 = 1 \neq -1 = f(0)$$

Proposition A.13. *Fix $f : (M, d) \to (M', d)$. The following are equivalent*

1. *f is continuous*
2. *For every open set $U \subseteq M'$, the preimage $f^{-1}(U)$ is open*
3. *For every $p \in M$, for every $\epsilon > 0$, there is $\delta > 0$ s.t. for every $x \in M$,*

$$d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$$

Fact: The functions sin, cos, exp, $\sqrt[3]{-}$ and polynomials are continuous

Proposition A.14. *If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then $f + g, f \cdot g, f - g, f \circ g$ are continuous*

Proposition A.15. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all x, then $1/f(x)$ is continuous. If $f(x) \geq 0$ for all x, then $\sqrt{f(x)}$ is continuous*

Example A.11. This function is continuous

$$h(x) = \exp\left(\frac{1}{1 + x^2}\right) - \frac{1}{17 + \sin(\sqrt[3]{x})}$$

Definition A.16. A function $f : M \to M'$ is **Lipschitz continuous** if there is $c \in \mathbb{R}$ s.t. for any $x, y \in M$

$$d(f(x), f(y)) \leq c \cdot d(x, y)$$

Example A.12 (In $\mathbb{R}$)**.** The function $f(x) = |x| + |x - 1|$ is Lipschitz continuous with $c = 2$

Proposition A.17. *If f is Lipschitz continuous, then f is continuous*

Example A.13. The function $f(x) = x^2$ is continuous but not Lipschitz continuous

Definition A.18. Let (M, d)be a metric space and $S \subseteq M$ be a set. Then (S, d') is a metric space, where $d'(x, y) = d(x, y)$ for $x, y \in S$

- d' is the restriction of d to $S \times S$
- We say that (S, d') is a **subspace** of (M, d)

Let (M, d), (M', d) be metric spaces, $S \subseteq M$ and $f : S \to M'$ be a function

Definition A.19. f is **continuous** if f is continuous as a map from the subspace (S, d') to (M', d)

Example A.14 (in $\mathbb{R}$)**.** Let $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ be given by $f(x) = 1/x$. Then f is continuous

Definition A.20. An **isometry** or **isomorphism** from (M, d) to (M', d') is a bijection $f : M \to M'$ s.t. for any $x, y \in M$

$$d(x, y) = d'(f(x), f(y))$$

Example A.15 (in $\mathbb{R}^2$)**.** The map $(x, y) \mapsto (x + 1, y - 7)$ is an isometry

So is the map $(x, y) \mapsto (3/5x + 4/5y, -4/5x + 3/5y)$