Homework9

Qi'ao Chen
21210160025

December 1, 2021

Exercise 1. Show that the following is not a field

Proof. As $a = 1 + 1$, $b = a + 1 = 1 + 1 + 1$, we have $a \cdot a = (1 + 1) \cdot (1 + 1) = 1 + 1 + 1 + 1 = 0 = b$. Then $a \cdot b = 1 = 0$. Hence this is not a field. $\square$

Proof. Let $\varphi(x)$ be the formula $\exists y(x \cdot y + y = 1)$. Find a quantifier-free formula $\psi(x)$ equivalent to $\varphi(x)$ in all algebraically closed fields $\square$

Proof. $x + 1 \neq 0$ $\square$

Exercise 2. Let $\varphi(x, y, z)$ be the formula $\exists w(x \cdot w^2 + y \cdot w + z = 0)$. Find a quantifier-free formula $\psi(x, y, z)$ equivalent to $\varphi(x, y, z)$ in all algebraically closed fields

Proof. $x \neq 0$ $\square$

Exercise 3. If M is a structure and $\varphi(x)$ is a formula in one variable, then $\varphi(M)$ denotes the set $\{a \in M : M \vDash \varphi(a)\}$. Show that if $M \preceq N$ and $\varphi(M)$ is finite, then $\varphi(M) = \varphi(N)$

Proof. Suppose $|\varphi(M)| = n$, then let ψ_n be

$$\exists x_1 \dots x_n \Big(\bigwedge_{\substack{i \neq j \\ 1 \leq i \leq n \\ 1 \leq j \leq n}} x_i \neq x_j \wedge \bigwedge_{i=1}^{n} \varphi(x_i) \Big)$$

and let $\psi := \psi_n \wedge \neg\psi_{n+1}$. Apparently $\neg\psi_{n+1} \vDash \neg\psi_{n+m}$ for all $m \geq 1$. Thus ψ states that there is exactly n solutions for $\varphi(x)$ and we have $M \vDash \psi$. As $M \preceq N$, we have $N \vDash \psi$ and N has exactly n solutions for $\varphi(x)$. But for any $m \in M$, $M \vDash \varphi(m) \Leftrightarrow N \vDash \varphi(m)$. Hence $\varphi(M) = \varphi(N)$ $\square$

Exercise 4. Let T be a theory with quantifier elimination. Let M be a structure and N be an extension. Suppose that M and N are both models of T. Let $\varphi(\bar{x})$ be a quantifier-free $L(M)$-formula in several variables. Suppose that $N \vDash \exists \bar{x} \varphi(\bar{x})$. Show that $M \vDash \exists \bar{x} \varphi(\bar{x})$

Proof. Given any formula $\psi(x, \bar{a})$ where $\bar{a} \in M^n$ and let $\chi(\bar{y}) := \exists \bar{x}\ \psi(\bar{x}, \bar{y})$, which is equivalent to a quantifier-free formula $\theta(\bar{y})$. As $N \vDash \chi(\bar{a})$ we have $N \vDash \theta(\bar{a})$. As M is a submodel of N, $N \vDash \theta(\bar{a}) \Leftrightarrow M \vDash \theta(\bar{a})$. Hence we have $M \vDash \exists \bar{x}\ \psi(\bar{x}, \bar{y})$, that is $M \vDash \exists \bar{x} \varphi(\bar{x})$ $\square$

Exercise 5. Let K be an algebraically closed field. Let $L \supseteq K$ be an extension field. Let $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ be polynomials over K. Suppose that the system of equations

$$\begin{aligned} P(x, y, z) &= 0 \\ Q(x, y, z) &= 0 \\ R(x, y, z) &= 0 \end{aligned}$$

has a solution in L. Show that it has a solution in K

Proof. By the fact, there is a model $M \supseteq L \supseteq K$ s.t. M and K are both algebraically closed field. By Theorem 36, $K \preceq M$. Let $\psi(x, y, z)$, an $\mathcal{L}_K$-formula, be

$$P(x, y, z) = 0 \wedge Q(x, y, z) = 0 \wedge R(x, y, z) = 0$$

As $L \vDash \exists xyz\ \psi(x, y, z)$, $M \vDash \exists xyz\ \psi(x, y, z)$ and hence $K \vDash \exists xyz\ \psi(x, y, z)$ $\square$