Morley Sequences and the Order Property

Thursday, March 17, 2022 9:35 AM

Слава Україні
Героям слава

References: Sections 12.3 and 12.8 (sort of)

Morley product if $p \in S_n(\mathcal{M})$ $q \in S_m(\mathcal{M})$ are both A-invariant (ie Aut($\mathcal{M}/A$)-invariant)

(if $b_1 \equiv_A b_2$ then $\varphi(x,b_1) \in p(x) \iff \varphi(x,b_2) \in p(x)$)

($\{b : \varphi(x,b) \in p(x)\}$ is A-invariant)

$\boxed{\text{In a stable theory, A-invariant types are A-definable}}$

$p \otimes q \in S_{n+m}(\mathcal{M})$ is an A-invariant type such that for any $A' \supseteq A$, for $\bar{b}, \bar{c} \in \mathcal{M}$

→ $(\bar{b},\bar{c}) \models (p\otimes q)\restriction A' \iff \bar{b} \models p \restriction A'$ and $\bar{c} \models q \restriction A'\bar{b}$ ← determines $(p\otimes q)\restriction A' \in S_{n+m}(A')$

Example in a strongly minimal theory, $\exists\, p \in S_1(\mathcal{M})$

p is ~~generated by~~ $\{\varphi(x) \in L(\mathcal{M})$ such that $\varphi(\mathcal{M})$ is cofinite.$\}$ $\emptyset$-definable, $\emptyset$-invariant

~~$p \restriction A$~~ $a \models p \restriction B \iff a \notin acl(B)$

In ACF: $acl(B) = \bigcup \{\varphi(\mathcal{M}) : \varphi(x) \in L(B), |\varphi(\mathcal{M})| < \infty\}$.

$a \in acl(B) \iff a$ is a root of a polynomial over B.

So $(a,b) \models p\otimes p \restriction C$

$\iff a \notin acl(C)$ and $b \notin acl(Ca)$

Example In ACF, we know that any variety $V \subseteq \mathcal{M}^n$ determines a complete $p_V \in S_n(\mathcal{M})$, and $S_n(\mathcal{M}) = \{p_V : V \subseteq \mathcal{M}^n \text{ a variety}\}$.

If V, W are varieties, then $p_V \otimes p_W = p_{V\times W}$. (and $V \times W$ is a variety by the way)

Remark 25 If p, q are invariant types, we say p and q commute if

$(p\otimes q)(\bar{x},\bar{y}) = (q\otimes p)(\bar{y},\bar{x})$. That is, for small C,

if $(\bar{a},\bar{b}) \models (p\otimes q)\restriction C$ then $(\bar{b},\bar{a}) \models (q\otimes p)\restriction C$

$\boxed{\text{if } \bar{a} \models p\restriction C \text{ and } \bar{b} \models q \restriction C\bar{a} \text{ then } \underbrace{\bar{b} \models q\restriction C}_{\text{automatic}} \text{ and } \bar{a} \models p \restriction C\bar{b}}$

In ACF, all types commute:

$(p_V \otimes p_W)(\bar{x},\bar{y}) = p_{V\times W}(\bar{x},\bar{y}) = p_{W\times V}(\bar{y},\bar{x}) = (p_W \otimes p_V)(\bar{y},\bar{x})$.

$\otimes$ for definable types $\otimes$ \otimes

If ~~p~~ $p(\bar{x})$ is a definable type and $\varphi(\bar{x},\bar{y})$ is a formula

then $(d_p \bar{x})\, \varphi(\bar{x},\bar{y})$ means $d\varphi(\bar{y})$, the formula defining $\{\bar{b} \in \mathcal{M} : \varphi(\bar{x},\bar{b}) \in p(\bar{x})\}$

$d_p\bar{x}$ works like a quantifier: free vars. in $(d_p\bar{x})\,\varphi(\bar{x},\bar{y})$ are $\bar{y}$.

Example Suppose $\mathbb{M} \models T$ strongly minimal, let $p =$ transcendental 1-type

$\varphi(x,\bar{b}) \in p(x) \iff \varphi(\mathbb{M},\bar{b})$ is cofinite $\iff \varphi(\mathbb{M},\bar{b})$ is infinite.

$d\varphi(\bar{y})$ is $\exists^\infty x\ \varphi(x,\bar{y})$

$(d_p x)\ \varphi(x,\bar{y})$ is $(\exists^\infty x)\ \varphi(x,\bar{y})$. So "$d_p x$" is "$\exists^\infty x$" for this p.

Proposition 26 If p, q are A-definable global types, then $p \otimes q$ is A-definable and $\boxed{(d_{p\otimes q}(\bar{x}\bar{y}))\ \varphi(\bar{x},\bar{y},\bar{z}) \equiv (d_p\bar{x})(d_q\bar{y})\ \varphi(\bar{x},\bar{y},\bar{z}).}$

Example: in a strongly minimal theory, if $p \in S_1(\mathbb{M})$ is transcendental and $q = p \otimes p$ then $(d_q(x,y))\ \varphi(x,y,\bar{z})$ is $\exists^\infty x\ \exists^\infty y\ \varphi(x,y,\bar{z})$.

$(\neq \exists^\infty(x,y))$

$\exists^\infty x \exists^\infty y \iff \exists^\infty y \exists^\infty x$

Proof of Prop 26 Fix $\bar{c} \in \mathbb{M}$.

Take $M \preceq \mathbb{M}$ s.t. $\bar{c} \in M$ and $M \supseteq A$

So p, q are M-definable. Take $\bar{a} \models p\restriction M$, $\bar{b} \models q \restriction M\bar{a}$

so $(\bar{a},\bar{b}) \models (p\otimes q)\restriction M$. So...

$$\begin{aligned}
\varphi(\bar{x}\bar{y},\bar{c}) \in p\otimes q &\iff \varphi(\bar{x},\bar{y},\bar{c}) \in p\otimes q\restriction M && (\bar{c}\in M)\\
&\iff \mathbb{M} \models \varphi(\bar{a},\bar{b},\bar{c}) && (\bar{a},\bar{b}) \models p\otimes q\restriction M\\
&\iff \varphi(\bar{a},\bar{y},\bar{c}) \in q(\bar{y})\restriction M\bar{a} && \bar{b}\models q\restriction M\bar{a}\\
&\iff \varphi(\bar{a},\bar{y},\bar{c}) \in q(\bar{y})\\
&\iff \mathbb{M} \models (d_q\bar{y})\,\varphi(\bar{a},\bar{y},\bar{c})\\
&\iff (d_q\bar{y})\,\varphi(\bar{x},\bar{y},\bar{c}) \in p(\bar{x}) && \bar{a}\models p\restriction M\\
&\iff \mathbb{M} \models (d_p\bar{x})(d_q\bar{y})\,\varphi(\bar{x},\bar{y},\bar{c})
\end{aligned}$$

$\mathbb{M} \models (d_{p\otimes q}(\bar{x}\bar{y}))\,\varphi(\bar{x}\bar{y},\bar{c})$

$\forall \bar{c}$

So $(d_{p\otimes q}(\bar{x},\bar{y}))\ \varphi(\bar{x},\bar{y},\bar{z}) \equiv (d_p\bar{x})(d_q\bar{y})\ \varphi(\bar{x},\bar{y},\bar{z})$. $\square$.

Two definable types p, q commute iff $(d_p\bar{x})(d_q\bar{y})\ \varphi(\bar{x},\bar{y},\bar{z}) \equiv (d_q\bar{y})(d_p\bar{x})\ \varphi(\bar{x},\bar{y},\bar{z})$.

Morley sequences $p\otimes p\otimes p\otimes p\otimes \cdots$

Let $p \in S_n(\mathbb{M})$ be A-invariant.

Definition 1 A Morley sequence of p over A is a sequence $\bar{b}_1, \bar{b}_2, \bar{b}_3, \ldots \in \mathbb{M}^n$ such that

$$\bar{b}_1 \models p\restriction A,\ \bar{b}_2 \models p\restriction A\bar{b}_1,\ \bar{b}_3 \models p\restriction A\bar{b}_1\bar{b}_2,\ \ldots,\ \bar{b}_i \models p\restriction A\bar{b}_1 b_2\cdots b_{i-1},\ \ldots$$

So $(\bar{b}_1, \dots, \bar{b}_n) \models \underbrace{p \otimes p \otimes \cdots \otimes p}_{n \text{ times}} \restriction A$

<u>Example</u> If T is strongly min., p is transcendental 1-type, a Morley seq over A is $b_1, b_2, b_3, \dots$ such that $b_1 \notin \operatorname{acl}(A)$, $b_2 \notin \operatorname{acl}(Ab_1)$, $b_3 \notin \operatorname{acl}(Ab_1b_2)$, $\dots$ $b_i \notin \operatorname{acl}(Ab_1 \dots b_{i-1})$.

<u>Definition 2</u> Let $(I, \leq)$ be infinite. Let $(\bar{b}_i : i \in I)$ be a sequence in M^n. Then $(\bar{b}_i : i \in I)$ is <u>A-indiscernible</u> $(A \subseteq M)$ if $\forall m$ $\forall\, i_1 < \dots < i_m$ in I $\forall\, j_1 < \dots < j_m$ in I

$(\bar{b}_{i_1}, \bar{b}_{i_2}, \dots \bar{b}_{i_m}) \equiv_A (\bar{b}_{j_1}, \dots \bar{b}_{j_m})$.

<u>Example</u> If $b_1, b_2, b_3, \dots$ is A-indiscernible then

$b_1 \equiv_A b_2 \equiv_A b_3 \equiv_A \cdots$ $\quad b_1 b_2 \equiv_A b_1 b_3 \equiv_A b_2 b_3 \equiv_A b_1 b_4 \equiv_A b_2 b_4 \equiv_A b_{10} b_{3000} \equiv_A \cdots$

$b_1 b_2 b_3 \equiv_A b_1 b_4 b_7 \equiv_A b_2 b_3 b_4 \equiv_A \cdots$

$b_1 b_2 \equiv_A b_1 b_3 \not\equiv_A b_2 b_1$

<u>Example</u> In DLO, in $(\mathbb{R}, \leq)$

$1, 2, 3, 4, 5, \dots$ is indiscernible $\quad b_i = i$

if $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ then $\quad (1,2,3) \equiv (1,4,7) \equiv (2,8,10)$

$(b_{i_1}, \dots b_{i_n}) \equiv (b_{j_1}, \dots b_{j_n})$ because $b_{i_1} < b_{i_2} < \dots < b_{i_n}$ and $b_{j_1} < b_{j_2} < \dots < b_{j_n}$

by Q.E. in DLO

Any increasing sequence is indiscernible in DLO.

In $(\mathbb{R}, \leq, +, \cdot)$, the sequence $1, 2, 3, 4, 5, \dots$ is not indiscernible

$1 \not\equiv 2 \not\equiv 3 \not\equiv 1$ etc..

<u>Theorem 4</u> If $p \in S_n(M)$ is A-invariant and $(\bar{b}_i : i < \omega)$ is a Morley seq. of p over A then $(\bar{b}_i : i < \omega)$ is A-indiscernible

<u>Proof</u> If $i_1 < i_2 < \dots < i_n$ then $\bar{b}_{i_1} \models p \restriction A$, $\bar{b}_{i_2} \models p \restriction A \bar{b}_{i_1}$, $\dots$ $\bar{b}_{i_n} \models p \restriction A \bar{b}_{i_1} \dots \bar{b}_{i_{n-1}}$

so $tp(\bar{b}_{i_1}, \dots \bar{b}_{i_n} / A)$ is $(\underbrace{p \otimes p \otimes \cdots \otimes p}_{n \text{ times}}) \restriction A$.

Doesn't depend on $i_1, \dots i_m$. $\square$

Order property $\qquad$ Fix T, M

<u>Def 5</u> Let $\varphi(\bar{x}; \bar{y})$ be a formula. φ has the <u>order property</u> if there are $\bar{a}_i, \bar{b}_i$ for $i \in \mathbb{Z}$ such that

$M \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j$

<u>Example</u> In DLO $\varphi(x, y) \equiv (x < y)$ has the O.P.

take $a_i = b_i = i$ $\qquad a_i < b_j \iff i < j$.

<u>Remark 6</u> If φ has O.P., then $\neg\varphi$ has O.P. if $\varphi^T(\bar y, \bar x) = \varphi(\bar x, \bar y)$ then φ^T has O.P.

Suppose $\varphi(\bar a_i, \bar b_j) \iff i < j$

then $\neg\varphi(\bar a_{-i}, \bar b_{1-j}) \iff -i \not< 1-j \iff i < j$ $\qquad \bar a'_i = \bar a_{-i}$, $\bar b'_j = \bar b_{1-j}$ then $\neg\varphi(\bar a'_i, \bar b'_j) \iff i<j$.

See notes for φ^T

OP $\Rightarrow$ unstable

<u>Lemma 7</u> For $\lambda \geq \aleph_0$ there is a linear order $(I, \leq)$ and $S \subseteq I$ such that $|I| > \lambda$, $|S| \leq \lambda$, S is dense in I

$\forall x, y \in I \quad x < y \Rightarrow \exists z \in S \quad x \leq z \leq y$

<u>Example</u> if $\lambda = \aleph_0$: $I = \mathbb{R}$, $S = \mathbb{Q}$

<u>Proof</u> (Lemma 9 Mar 3) $\Rightarrow \exists \mu \quad |2^\mu| > \lambda$, $|2^{<\mu}| \leq \lambda$

2^μ = binary strings of len. μ
$2^{<\mu}$ = strings of len $\alpha < \mu$.

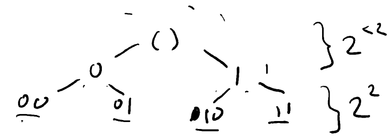
Let $I = 2^\mu \cup 2^{<\mu}$ $\qquad |I| > \lambda$
$S = 2^{<\mu}$ $\qquad |S| \leq \lambda$

$2^{<\mu} = \bigcup_{\alpha<\mu} 2^\alpha$.

if $\sigma \in 2^{<\mu}$, pad on the right with u. $\qquad \{0,1\}$, $\{0,u,1\}$ $\qquad 0 < u < 1$

$010 \rightsquigarrow \underbrace{010uuu\ldots}_{\text{length } \mu}$

if $\mu = 2$



Take lexicographic order on the padded strings

lex. order: $00 < 0 < 01 < () < 10 < 1 < 11$
$[\,00 < 0u < 01 < uu < 10 < 1u < 11$

S is dense in I

if $x, y \in 2^\mu \quad x < y$

need $z \in 2^{<\mu} \quad x < z < y$

$x = \tau 0 \ldots$
$y = \tau 1 \ldots$ $\qquad$ for some $\tau \in 2^{<\mu}$

then $x < \tau < y$. $\qquad \tau \in S$. $\qquad \square$.

<u>Theorem 8</u> If $\varphi(\bar x, \bar y)$ has O.P. then T is not λ-stable for any λ (T is unstable)

<u>Proof</u> Take $I \supseteq S$ as in Lemma 7 $\qquad S$ dense in $(I, \leq)$, $|S| \leq \lambda$, $|I| > \lambda$

$\bar a_i \; \bar b_j \quad i, j \in \mathbb{Z} \quad \varphi(\bar a_i, \bar b_j) \iff i < j$.

By compactness, $\exists \; \bar a_i, \bar b_j$ for $i, j \in I$ such that

$\mathcal{M} \models \varphi(\bar a_i, \bar b_j) \iff i < j$.

Let $C = \{\bar b_j : j \in S\}$ $\qquad |C| \leq \lambda$

<u>Claim</u> $I \setminus S \to S_n(C)$
$i \mapsto tp(\bar a_i / C)$

is injective.

Proof If $i_1 < i_2$ then $\exists\, j \in S$ $i_1 < j < i_2$ $\quad i_1 < j$, $i_2 \not< j$.
then $\varphi(\bar{a}_{i_1}, \bar{b}_j) \wedge \neg \varphi(\bar{a}_{i_2}, \bar{b}_j)$ $\quad \bar{b}_j \in C$
so $\bar{a}_{i_1} \not\equiv_C \bar{a}_{i_2}$ $\quad tp(\bar{a}_{i_1}/C) \neq tp(\bar{a}_{i_2}/C)$. $\square_{\text{claim}}$.

$|S_n(C)| \geq |I \setminus S| > \lambda$ $\quad |I| > |S|$ so $|I \setminus S| = |I| > \lambda$
λ-stability fails. $\square$.

---

§4 Lemma 9 If $\varphi(\bar{x}, \bar{y})$ doesn't have O.P. then $\exists\, n_\varphi < \omega$ s.t. there are no $a_0, a_1, \ldots a_{n_\varphi - 1}, b_0, b_1, \ldots b_{n_\varphi - 1}$ s.t. $\mathcal{M} \models \varphi(a_i, b_j) \iff i < j$.

Proof Compactness. If n_φ doesn't exist, then
$\{\varphi(a_i, b_j) : i < j \in \mathbb{Z}\} \cup \{\neg\varphi(a_i, b_j) : i \geq j \in \mathbb{Z}\}$
is consistent, φ has O.P. $\square$.

Lemma 10 Suppose $\varphi(\bar{x}, \bar{y})$ doesn't have O.P. Let n_φ be from Lemma 9.
Let $\bar{b}_1, \bar{b}_2, \bar{b}_3, \ldots$ be indiscernible (over $\varnothing$) Then there is no $\bar{a}$ such that
$\mathcal{M} \models \varphi(\bar{a}, \bar{b}_i)$ for $0 \leq i < n_\varphi$
$\mathcal{M} \models \neg\varphi(\bar{a}, b_i)$ for $n_\varphi \leq i < 2n_\varphi$. $\qquad \varphi(\bar{a}, \mathcal{M})$

$\underbrace{\top\,\top\,\top \ldots \top}_{n}\ \underbrace{\bot \ldots \bot}_{n}$

Proof $n = n_\varphi$. Suppose $\bar{a}$ exists. For $0 \leq j < n$
$\bar{b}_{n-j}\, \bar{b}_{n-j+1} \cdots \bar{b}_{(n-j)+(n-1)} \equiv \bar{b}_0\, \bar{b}_1 \cdots \bar{b}_{n-1}$
$\exists\, \sigma_j \in \mathrm{Aut}(\mathcal{M})$
$\sigma_j(\bar{b}_{n-j} \cdots \bar{b}_{n-j+(n-1)}) = (\bar{b}_0 \ldots \bar{b}_{n-1})$

$$\begin{array}{ll} \top\top\top\top & a_0 \\ \top\top\top\bot & a_1 \\ \top\top\bot\bot & a_2 \\ \top\bot\bot\bot & a_3 \\ \bot\bot\bot\bot & a_4 \end{array}$$

Let $\bar{a}_j = \sigma_j(\bar{a})$. For $i, j < n$
$\mathcal{M} \models \varphi(\bar{a}_j, \bar{b}_i) \iff \mathcal{M} \models \varphi(\sigma_j(\bar{a}), \sigma_j(\bar{b}_{i+n-j}))$
$\iff \mathcal{M} \models \varphi(\bar{a}, \bar{b}_{i+n-j}) \iff i+n-j < n \iff i < j$.

Contradicts choice of n_φ (should have been n_{φ^T} oops).

Lemma 11 Suppose $\varphi(x_1, \ldots x_n; \bar{y})$ doesn't have O.P. Take $N > \max(n_{\varphi^T}, n_{\neg\varphi^T})$.
Let p be an A-invariant type$/\mathcal{M}$. Let $a_1, a_2, a_3, \ldots$ be a Morley sequence of p over A.
1) If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$ then $\mathcal{M} \models \varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$.
2) If $\varphi(x, \bar{b}) \notin p(\bar{x})$ then $\mathcal{M} \models \neg\varphi(\bar{a}_i, \bar{b})$ for most of $i < 2N$.
(>50%)

Proof (1) Suppose $\neg\varphi(x, \bar{b}) \in p(\bar{x})$ but $\underbrace{\varphi(\bar{a}_i, \bar{b}) \text{ for most}}_{\text{at least } N}$ $i < 2N$ $\quad i \in \{0, 1, \ldots 2N-1\}$
then $\exists\, j_1 < j_2 < \cdots < j_N < 2N$ such that $\varphi(\bar{a}_{j_i}, \bar{b})$ for $1 \leq i \leq N$.
Take $\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3, \ldots$ a Morley seq. of p over $A\, a_1\, a_2\, a_3 \ldots \bar{b}$.

1) $\bar{c}_i \models p \restriction \bar{b}$ so $\mathfrak{M} \models \neg\varphi(\bar{c}_i, \bar{b})$ because $\neg\varphi(x, \bar{b}) \in p \restriction \bar{b}$.

2) $a_0 a_1 a_2 \ldots c_0 c_1 c_2 \ldots$ is a Morley seq. of length ~~$\omega \cdot 2$~~ $\omega + \omega$.

in particular

$a_{j_1}, a_{j_2}, \ldots, a_{j_N}, c_0, c_1, c_2, c_3, \ldots$ is a Morley seq. of p over A. ✓

~~$a_{j_1} \models p \restriction A$~~

$c_0 \models p \restriction A a_{j_1} a_{j_2} \ldots a_{j_N}$

$c_1 \models p \restriction A a_{j_1} \ldots a_{j_N} c_0$

$\Rightarrow$ $a_{j_1}, a_{j_2}, \ldots a_{j_N}, c_0, c_1, c_2, \ldots$ is indiscernible over A.

is ——— $\emptyset$.

$\mathfrak{M} \models \varphi(a_{j_i}, b)$ for $1 \leq i \leq N$

$\mathfrak{M} \models \neg\varphi(c_i, b)$ for all i. This contradicts Lemma 10.

(2) similar. $\square$

Prop 12 Suppose $\varphi(x_1, \ldots x_n; \bar{y})$ doesn't have the O.P.

If $M \preceq \mathfrak{M}$, $p \in S_n(M)$, then $d_p\varphi(\bar{y})$ exists. (is definable)

$\{\bar{b} \in M : \varphi(x, \bar{b}) \in p(\bar{x})\}$ is definable in M.

Proof Take $q \in S_n(\mathfrak{M})$ coheir of p. (Theorem 5, 3/10 notes)

q is M-invariant (Theorem 17, 3/10 notes)

Let $(\bar{a}_i : i < \omega)$ be a Morley seq. of q over M.

By Lemma 11, $d_q\varphi(\bar{y})$ is defined by

$$\boxed{\varphi(\bar{x}, \bar{b}) \in q(\bar{x}) \iff \mathfrak{M} \models \bigvee_{\substack{S \subseteq 2N \\ |S| \geq N}} \bigwedge_{i \in S} \varphi(\bar{a}_i, \bar{b}).}$$

is Lemma 11.

$\{\bar{b} \in \mathfrak{M} : \varphi(\bar{x}, \bar{b}) \in q(\bar{x})\}$ is definable

is M-invariant

is M-definable (Lemma 10, 3/10).

$\{\bar{b} \in M : \varphi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is $(M$-$)$ definable. $\square$.

Theorem 13 Fix $n < \omega$. Suppose no fmla $\varphi(x_1, x_2, \ldots x_n; \bar{y})$ has the O.P.

Then $\forall M \models T$ $\forall p \in S_n(M)$, p is definable.

Cor 14 The following are equivalent

1.) All $p \in S_n(M)$ are definable

2.) All $p \in S_1(M)$ are definable

3.) No $\varphi(\bar{x}, \bar{y})$ has O.P

4.) No $\varphi(x, \bar{y})$ has O.P.

5.) T is λ-stable for some λ

OP is a property of a "partitioned fmla." Whether $\varphi(x_1, \ldots x_n; y_1, \ldots y_m)$ has O.P. depends on n. $\varphi(x, y; z)$ could have OP but $\varphi(x; y, z)$ doesn't.

Proof $\sim$ (Thm 2, 3/10). using Thm 8

Thm 13 today.

Thm 8

(5)

$$\begin{tikzcd}
 & & (5) \arrow[dll, bend right, "\text{Thm 8}"'] \\
(3) \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\text{Thm 13}"] & (4) \arrow[d, Rightarrow, "\text{Thm 13}"] & \\
(1) \arrow[r, Rightarrow] & (2) \arrow[uur, bend right, "\text{See 3/10}"'] &
\end{tikzcd}$$

$\square$.

Fact 15 $\varphi(\bar{x},\bar{y})$ has the O.P. $\Longleftrightarrow$ $\varphi(\bar{x},\bar{y})$ has the dichotomy property.

Example if T is strongly min. then T is stable b/c if $\varphi(x,\bar{y})$ has the O.P.

then $\exists$ $a_i, \bar{b}_i \in M$ $\quad M \models \varphi(a_i, \bar{b}_j) \Longleftrightarrow i<j$ for $i,j \in \mathbb{Z}$. $\quad a_i \neq a_j$ for $i \neq j$.

Hmm, so $\varphi(M, \bar{b}_0)$ contains a_i for $i<0$ but not a_i for $i \geq 0$ so it's neither finite nor cofinite. $\Rightarrow\Leftarrow$

---

§5 Theorem 16 If T is stable and p, q are global types then $(p \otimes q)(\bar{x},\bar{y}) = (q \otimes p)(\bar{y},\bar{x})$.

Proof Suppose not. Take $\varphi(\bar{x},\bar{y}) \in L(M)$

$$\varphi(\bar{x},\bar{y}) \in (p \otimes q)(\bar{x},\bar{y})$$
$$\varphi(\bar{x},\bar{y}) \notin (q \otimes p)(\bar{y},\bar{x}).$$

Take A s.t. p, q are A-definable and $\varphi(\bar{x},\bar{y}) \in L(A)$.

Take $p \otimes q \otimes p \otimes q \otimes p \otimes q \otimes \cdots$

$((b_i, c_i) : i<\omega)$ a Morley seq. of $p \otimes q$ over A.

$b_0 \models p \restriction A \qquad c_0 \models q \restriction Ab_0$

$b_1 \models p \restriction Ab_0c_0 \qquad b_2 \models q \restriction Ab_0c_0b_1$

etc ...

If $i \leq j$ $\quad b_i \models p \restriction A \quad c_j \models q \restriction Ab_i \quad (b_i, c_j) \models p \otimes q \restriction A \Rightarrow M \models \varphi(b_i, c_j)$

If $i > j$ $\quad c_j \models q \restriction A \quad b_i \models p \restriction Ac_j \quad (c_j, b_i) \models q \otimes p \restriction A \Rightarrow M \models \neg\varphi(b_i, c_j)$

So $M \models \varphi(b_i, c_j) \Longleftrightarrow i \leq j$ $\quad \varphi$ has the O.P. contra stability $\Rightarrow\Leftarrow$. $\square$.

Example Suppose T is strongly minimal, $p \in S_1(M)$ transcendental.

$(p \otimes p)(x,y) = (p \otimes p)(y,x)$ $\quad$ (p commutes with p)

So, if $A \subseteq M$, $b, c \in M$,

$b \models p \restriction A \quad c \models p \restriction Ab \implies c \models p \restriction A \quad b \models p \restriction Ac$

$b \notin \operatorname{acl}(A)$, $c \notin \operatorname{acl}(Ab) \implies c \notin \operatorname{acl}(A)$, $b \notin \operatorname{acl}(Ac)$

$b \notin \operatorname{acl}(A)$ and $c \notin \operatorname{acl}(Ab) \implies b \notin \operatorname{acl}(Ac)$

$b \notin \operatorname{acl}(A) \quad b \in \operatorname{acl}(Ac) \implies c \in \operatorname{acl}(Ab)$

Steinitz exchange property: if $b, c \notin \mathrm{acl}(A)$
$b \in \mathrm{acl}(Ac) \Rightarrow c \in \mathrm{acl}(Ab)$.

a ( ) is a pregeometry / matroids.

(Geometric Stability Theory).

T is any complete theory, $\mathbb{M}$ monster.

<u>Theorem</u> If $\mathbb{M}$ is infinite, then $\exists (b_i : i < \omega)$ indiscernible with $b_i \neq b_j$ for $i \neq j$.

<u>Proof</u> Take small $M \preceq \mathbb{M}$. $\mathbb{M}$ infinite $\Rightarrow$ $\mathbb{M}$ is not small.
$M \subsetneq \mathbb{M}$

$p = tp(a/M)$ $a \notin M$. $p \in S_1(M)$. Let $q \in S_1(\mathbb{M})$ be a coheir of p.
(q is f. sat. in M) q is M-invariant type.

<u>Claim</u> $q(x) \not\vdash x = c$ for any $c \in \mathbb{M}$.

<u>Proof</u> q is f.sat in M, if $(x=c) \in q(x)$ then $\exists a_0 \in M$ $a_0 = c$.
~~$c \in \mathbb{M}$~~ $c = a_0 \in M$. $x = c \in q \restriction M = p$.
$a \models p$ so $a = c = a_0 \in M$ $\Rightarrow\Leftarrow$ $a \notin M$. $\square_{\text{claim}}$.

If $b \models q \restriction A$ then $b \notin A$ (else $(x=b) \in q \restriction A \subseteq q \Rightarrow\Leftarrow$ claim)

Take $b_1, b_2, b_3, \ldots$ a Morley seq. of q over M.
Then $(b_i : i < \omega)$ is (M-)indiscernible, and ~~$b_i \models q$~~ $b_i \models q \restriction M b_j$ so $b_i \neq b_j$ for $j < i$. $\square$.