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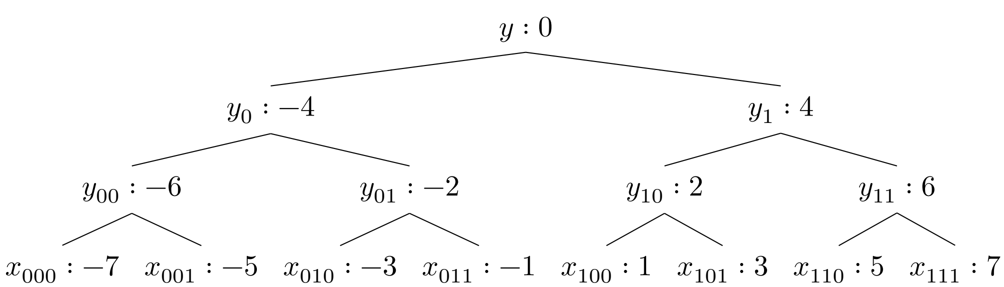
Exercise 1. $(\mathbb{C}, +, \cdot)$ is an algebraically closed field. Show that the algebraic set $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$ is reducible, i.e., not a variety

Proof. Since $x^2+y^2 = (x+yi)(x-yi)$, $\{(x, y) \in \mathbb{C}^2 : x^2+y^2 = 0\} = \{(x, y) \in \mathbb{C}^2 : x + yi = 0\} \cup \{(x, y) \in \mathbb{C}^2 : x - yi = 0\}$ □

Exercise 2. Consider the theory of dense linear orders. Let $\varphi(x, y)$ be the formula $x < y$. One can show that $\varphi(x, y)$ has dichotomy property. Show by giving an example that D_3 is consistent

Proof. Consider



□

Exercise 3. In the structure $M = (\mathbb{R}, +, \cdot, 0, 1, \leq)$, let $\varphi(\bar{x}, \bar{y})$ be the formula $x_1y_1 + x_2y_2 = 1$. Thus $\varphi(\mathbb{R}^2, \bar{b})$ is a line for most $\bar{b} \in \mathbb{R}^2$. It turns out that the formula φ does not have the dichotomy property. Find the largest n s.t. D_n is consistent

Proof. Largest n is 1. For a fixed $\bar{y} = (a, b)$ with $ab \neq 0$, we could take $\bar{x}_0$ on the line of $xy = 1 - ab$ and $\bar{x}_0$ outside the line.

Now for $n = 2$, suppose we have $\bar{y} = (a, b)$, $\bar{y}_0 = (a_0, b_0)$, $\bar{y}_1 = (a_1, b_1)$, $\bar{x}_{ij} = (a_{ij}, b_{ij})$ for $i, j = 0, 1$ and D_n is consistent. Then since $\varphi(\bar{x}_{00}, \bar{y})$ and $\varphi(\bar{x}_{01}, \bar{y})$.

Suppose $ab = 1$, then $a_{00}b_{00} = a_{01}b_{01} = 0$. Since $\varphi(\bar{x}_{00}, \bar{y}_0)$, $a_0 b_0 = 1$ and hence $\varphi(\bar{x}_{01}, \bar{y}_1)$, a contradiction.

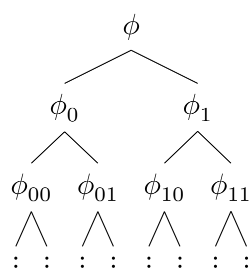
Now since $ab \neq 1$, $\bar{x}_{00}$ and $\bar{x}_{01}$ are on the same line $xy = 1 - ab$, and there is no such $\bar{y}_0$ to get a line $xy = 1 - a_0 b_0$ to isolate $\bar{x}_{00}$ and $\bar{x}_{01}$.

Thus D_2 is inconsistent $\square$

Exercise 4. Let T be a complete theory of the structure $(\mathbb{Z}, +, -, 0)$. Show that T is not $\aleph_0$-stable

Proof. Suppose we are working in base-2 system.

Given $\sigma \in 2^{<\omega}$, let $\phi_{\sigma 0}(x) = \exists y (x = y \cdot (10)^{\mathrm{lh}(\sigma)+2} + \sigma)$ and $\phi_{\sigma 1}(x) = \exists (x = y \cdot (10)^{\mathrm{lh}(\sigma)+2} + \sigma + 1 \cdot (10)^{\mathrm{lh}(\sigma)+1})$ where $\mathrm{lh}(\sigma)$ denotes the length of σ. Then $\phi_{\sigma i}(x) \Leftrightarrow x$ extends σi for $i = 0, 1$. Thus we have a tree



where ϕ is $x = x$.

Now note that for any $\sigma \in 2^{<\omega}$ $\phi_\sigma \leftrightarrow \phi_{\sigma 0} \vee \phi_{\sigma 1}$ and $\phi_{\sigma i} \models \neg \phi_{\sigma(1-i)}$ for $i = 0, 1$. For each $f : \omega \to 2$, $[\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \cdots$ and since $S_1(\mathbb{Z})$ is compact, there is $p_f \in \bigcap_{i \in \omega} [\phi_{f|i}]$. If $f, g \in 2^\omega$ and $f \neq g$, then there is n s.t. $f(n) \neq g(n)$ and $f \mid n = g \mid n$. Then since $\phi_{f|(n+1)} \models \neg \phi_{g|(n+1)}$, $[\phi_{f|n+1}] \cap [\phi_{g|n+1}] = \emptyset$ and hence $p_f \neq p_g$. Thus $|S_1(\mathbb{Z})| \geq 2^{\aleph_0}$ $\square$