Homework9

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Exercise 1. *Show that* $A \mathop{\smile\!\!\!\!\!\mid}_C B \Leftrightarrow A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$ *for any* A, B, C

Proof. For any A, B, C:

Suppose $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then by Problem 2, $A \mathop{\smile\!\!\!\!\!\mid}_{BC} \mathrm{acl}(BC)$, therefore by transitivity, $A \mathop{\smile\!\!\!\!\!\mid}_C B\,\mathrm{acl}(BC)$, $A \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{acl}(BC)$. As $C \subseteq \mathrm{acl}(C) \subseteq \mathrm{acl}(BC)$, we have $A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$.

Suppose $A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$, by Problem 2, $A \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{acl}(C)$, then by transitivity, $A \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{acl}(BC)$, then by monotonicity, $A \mathop{\smile\!\!\!\!\!\mid}_C B$ as $B \subseteq acl(BC)$ □

Exercise 2. *Show there is a model* $M_1 \supseteq B_1$ *s.t.* $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_2$ *and a model* $M_2 \supseteq M_1 \cup B_2$ *s.t.* $M_2 \mathop{\smile\!\!\!\!\!\mid}_{M_1 B_2} A$

Proof. For any model $M_1' \supseteq B_1$, by Extension, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/B_1)$ such that $\sigma(M_1') \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_2$, but $\sigma(M_1')$ is still a model containing B_1, thus we can take $M_1 = \sigma(M_1')$ and $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_2$

Then by the first part of the problem, there is a model $M_2 \supseteq M_1 \cup B_2$ s.t. $M_2 \mathop{\smile\!\!\!\!\!\mid}_{M_1 B_2} AM_1B_2$. By Base monotonicity, $M_2 \mathop{\smile\!\!\!\!\!\mid}_{M_1 B_2} A$ □

Exercise 3. *Show that* $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} M_1$ *and* $A \mathop{\smile\!\!\!\!\!\mid}_{B_2} M_2$

Proof. As $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_2$, by Monotonicity, $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_1$, then by Base monotonicity, $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} A$, and by Symmetry, $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} M_1$

As $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_1} AB_2$, base monotonicity gives $M_1 \mathop{\smile\!\!\!\!\!\mid}_{B_2} A$, and symmetry gives $A \mathop{\smile\!\!\!\!\!\mid}_{B_2} M_1$, as $A \mathop{\smile\!\!\!\!\!\mid}_{B_2 M_1} M_2$, we have $A \mathop{\smile\!\!\!\!\!\mid}_{B_2} M_1M_2$ by transitivity, and this is equivalent to $A \mathop{\smile\!\!\!\!\!\mid}_{B_2} M_2$ □

Exercise 4. *Show that* $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} B_2 \Leftrightarrow A \mathop{\smile\!\!\!\!\!\mid}_{M_1} M_2$

Proof. $\Rightarrow$: Use transitivity on $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} B_2$ and $A \mathop{\smile\!\!\!\!\!\mid}_{B_2} M_2$ we get $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} M_2$, then by base monotonicity, $A \mathop{\smile\!\!\!\!\!\mid}_{M_1} M_2$

$\Leftarrow$: Use transitivity on $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} M_1$ and $A \mathop{\smile\!\!\!\!\!\mid}_{M_1} M_2$, we get $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} M_2$, then by monotonicity, $A \mathop{\smile\!\!\!\!\!\mid}_{B_1} B_2$ $\square$