

Chapter 2

Stability

December 24, 2021

1 Historic remarks and motivations

- Can a first order theory T determine its models;
- Any theory T with an infinite model has models of arbitrary infinite cardinalities (L-S-T);
- For a fixed infinite cardinal κ , how many models of T has cardinal κ ?
- Consider the function $I_T(-) : \kappa \mapsto \#\{\text{models of } T \text{ of card. } \kappa\}$.
- Then $1 \leq I_T(\kappa) \leq 2^\kappa$;
- $\#\{L\text{-structures of card. } \kappa\} \leq 2^\kappa$;

Fact 1.1. *[Morley's Theorem] Let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable cardinal κ , then $I_T(\kappa) = 1$ for all uncountable cardinal κ . (Categoricity)*

Example 1.2. .

- The Theory of infinite sets;
- The theory of vector space over a fixed countable field;
- The theory of algebraically closed fields with fixed char;
- The theory of $(\mathbb{Z}, S, 0)$.

Shelah's stability theory intended to generalize Morley's Theory and classify the complete first order theories.

Conjecture 1.3. *[Morley] Let T be countable, then the function $I_T(\kappa)$ is non-decreasing on uncountable cardinals.*

Fact 1.4. [Shelah's Main gap theorem] Let T be a countable first order complete theory T . then one of these situations holds:

- $\forall \alpha, I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$
- $\forall \alpha, I_T(\aleph_\alpha) < \beth_{\aleph_1}(|\alpha|)$.

Here, $\beth_0(\kappa) = \kappa$, $\beth_\alpha(\kappa) = 2^{\beth_{\alpha+1}(\kappa)}$, and $\beth_\nu(\kappa) = \sup\{\beth_\alpha(\kappa) \mid \alpha < \nu\}$ for limit ordinals ν .

Remark 1.5. .

- The name “Main Gap” refers to the gap between $\beth_{\aleph_1}(|\alpha|)$ and 2^{\aleph_α} ($\alpha \geq \omega$)
- Depending on α this may be no gap at all;
- But in general $\beth_{\aleph_1}(|\alpha|)$ goes only moderately compared to 2^{\aleph_α} ;
- The case $I_T(\aleph_\alpha) = 2^{\aleph_\alpha}$ is called the “non-structure case”, we have a kind of chaos.
- The second case, namely, the case where there are relatively few non-isomorphic models, is called the “structure case”;
- In this case every model can be characterized up to isomorphism in terms of certain invariants;
- The most important “dividing lines” on the space of first-order theories is “stability”;
- Main gap theorem says that: “If T is a first-order theory and is stable and . . . , then the class of models looks like Otherwise, there's no hope”.

2 Counting types and stability

Definition 2.1. For a complete first order theory T , let $f_T : \text{Card} \rightarrow \text{Card}$ be defined by

$$f_T(\kappa) = \sup\{ |S_1 M| : M \models T, |M| = \kappa \},$$

for κ an infinite cardinal.

It is easy to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$.

Fact 2.2. Let T be an arbitrary complete theory in a first order language. The $f_T(\kappa)$ is one of the following functions

$$\kappa, \kappa + 2^{\aleph_0}, \text{ded } \kappa, (\text{ded } \kappa)^{\aleph_0}, 2^\kappa$$

Here

$$\text{ded } \kappa = \sup\{ |I| : I \text{ is a linear order with a dense subset of size } \kappa \}$$

$$\text{ded } \kappa = \sup\{ \lambda : \text{There is a linear order of size } \kappa \text{ with } \lambda \text{ cuts} \}$$

Lemma 2.3. $\kappa < \text{ded } \kappa \leq 2^\kappa$.

Proof. $\kappa < \text{ded } \kappa$:

- Let μ be minimal such that $2^\mu > \kappa$;
- Consider 2^μ as a set of 0 – 1 sequence of length μ ;
- then $2^{<\mu}$ is a dense subset of 2^μ ;
- $\mu \leq \kappa \implies 2^{<\mu} \leq \kappa$;
- so $\text{ded } \kappa \geq \mu > \kappa$.

$\text{ded } \kappa \leq 2^\kappa$:

- Every cut is determined by the subset of elements in its lower half.

□

Definition 2.4. Let $M \models T$.

- A formula $\phi(x, y)$, with its variables partitioned into two groups x, y , has the k -order property, $k \in \omega$, if there are some $a_i \in M_x, b_j \in M_y$ for $i, j < k$ such that

$$M \models \phi(a_i, b_j) \iff i < j$$

- $\phi(x, y)$ has the order property if it has the k -order property for all $k \in \omega$;
- We say that a formula $\phi(x, y) \in L$ is stable if there is some $k \in \omega$ such that it does not have the k -order property.
- A theory is stable if it implies that all formulas are stable (note that this is indeed a property of a theory).

Proposition 2.5. Assume that T is unstable, then $f_T(\kappa) \geq \text{ded } \kappa$ for all cardinals $\kappa \geq |T|$.

Proof. • Fix a cardinal κ . Let $\phi(x, y) \in L$ be a formula has the k -order property for all $k \in \omega$;

- Let $(I, <)$ be a dense linear order order of size κ ;
- Let $a_{i \in I}$ and $b_{i \in I}$ be two sequences of new constants;
- Then $\{\phi(a_i, b_j) \mid i < j\} \cup \{\neg \phi(a_i, b_j) \mid i \geq j\}$ is consistent with T ;
- By compactness, there is a model $\mathcal{M} \models T$ and $a_{i \in I}, b_{i \in I}$ from M such that

$$\mathcal{M} \models \phi(a_i, b_j) \iff i < j$$

- By L-S-T, we may assume that $|M| = \kappa$;
- For any cut $C = (A, B)$ in I

$$\Phi_C(x) = \{\phi(x, b_j) \mid i \in B\} \cup \{\neg\phi(x, b_j) \mid j \in A\}$$

is a partial type over M ;

- It is easy to see that $C_1 \neq C_2 \implies \Phi_{C_1} \cup \Phi_{C_2}$ is inconsistent;
- Let $p_C(x) \in S_x(M)$ be a complete extension of $\Phi_C(x)$;
- Then $|S_x(M)| \geq \text{number of cuts in } I$;
- As I is arbitrary,

$$f_T(\kappa) = \sup\{|S_x(M)| \mid M \models T, |M| = \kappa\} \geq \text{ded } \kappa$$

□

Recall

Fact 2.6 (Ramsey Theory). $\aleph_0 \rightarrow (\aleph_0)_k^n$ holds for all $n, k \in \omega$ (i.e. for any coloring of subsets of N of size n in k colors, there is some infinite subset I of N such that all n -element subsets of I have the same color).

Lemma 2.7. Let $\phi(x, y)$, $\psi(x, z)$ be stable formulas (where y, z are not necessarily disjoint tuples of variables). Then:

1. Let $\phi^*(y, x) = \phi(x, y)$, i.e. we switch the roles of the variables. Then $\phi^*(y, x)$ is stable.
2. $\neg\phi(y, x)$ is stable.
3. $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ and $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ are stable.
4. If $y = uv$ and $c \in M_v$ then $\theta(x, u) := \phi(x, uc)$ is stable.
5. If T is stable, then every L^{eq} -formula is stable as well.

Proof. .

(1) and (2) are trivial.

(3):

- Suppose that $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ is unstable;
- there are $(a_i, b_i, c_i \mid i \in \mathbb{N})$ such that

$$\phi(a_i, b_j) \vee \psi(a_i, c_j) \iff i < j$$

- Let $f : [\mathbb{N}]^2 \rightarrow \{0, 1\}$ defined by: for each $i < j \in \mathbb{N}$

$$f(i, j) = 1 \iff \models \psi(a_i, c_j) \text{ and } f(i, j) = 0 \iff \models \neg\psi(a_i, c_j)$$

- By Ramsey, there is a infinite subset $I \subseteq J$ such that
- f is constant on I ;
- If $f(I) = 1$, then $\forall i, j \in I (\psi(a_i, b_j) \iff i < j)$
- If $f(I) = 0$, then $\forall i, j \in I (\phi(a_i, b_j) \iff i < j)$
- So either ϕ or ψ is unstable.

(4): Trivial. □

Theorem 2.8 (Erdős-Makkai). *Let B be an infinite set, $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(c_{i < \omega}) \subseteq B$ and $(S_{i < \omega}) \subseteq \mathcal{F}$ such that one of the following holds:*

1. $c_i \in S_j \iff j < i (\forall i, j \in \omega)$,
2. $c_i \in S_j \iff i < j (\forall i, j \in \omega)$.

We need the following lemma:

Lemma 2.9. *Let X be a set and Y_1, \dots, Y_n are subsets of X . Define:*

$$E(x, y) := \bigwedge_{i=1}^n (x \in X_i \iff y \in X_i).$$

Then E is an equivalence relation on X and $Z \subseteq X$ is a boolean combination of X_i 's iff

$$E(x, y) \implies (x \in Z \iff y \in Z)$$

Proof. Exercise. □

We now proof the Theorem 2.8

Proof. • Choose $\mathcal{F}' \subseteq \mathcal{F}$ such that

- $|\mathcal{F}'| = |B|$;
- For any finite $B_0, B_1 \subseteq B$,

$$\exists S \in \mathcal{F} (B_1 \subseteq S \wedge B_2 \subseteq B \setminus S) \implies \exists S' \in \mathcal{F}' (B_1 \subseteq S' \wedge B_2 \subseteq B \setminus S').$$

- \mathcal{F}' exists as there are at most $|B|$ -many different pairs of finite subsets of B ;

- $|\mathcal{F}| > |\mathcal{F}'| \implies \exists S^* \in \mathcal{F}$ which is not a boolean combination of elements of \mathcal{F}' ;
 - Let $a_0 \in S^*$ and $b_0 \notin S^*$;
 - There is $S_0 \in \mathcal{F}'$ s.t. $a_0 \in S_0$ and $b_0 \notin S_0$;
 - Since S^* is NOT a boolean combination of $\{S_0\}$, there are a_1, b_1 s.t. :
 - $a_1 \in S_0 \iff b_1 \in S_0$, and ;
 - $a_1 \in S^*$ but $b_1 \notin S^*$.
 - Now $\{a_0, a_1\} \subseteq S^*$ and $\{b_0, b_1\} \subseteq B \setminus S^*$;
 - By the assumption of \mathcal{F}' , $\exists S_1 \in \mathcal{F}' (\{a_0, a_1\} \subseteq S_1 \wedge \{b_0, b_1\} \subseteq B \setminus S_1)$;
 - Since S^* is NOT a b. c. of $\{S_0, S_1\}$, there are a_2, b_2 s.t. :
 - $a_2 \in S_i \iff b_2 \in S_i$, for $i < 2$, and ;
 - $a_2 \in S^*$ but $b_2 \notin S^*$.
 - ...
 - Inductively, we have infinite sequences $(a_{i < \omega}) \subseteq S^*$ and $(b_{i < \omega}) \subseteq B \setminus S^*$ s.t.
 - $a_n \in S_i \iff b_n \in S_i$, for $i < n$;
 - $\{a_0, \dots, a_n\} \subseteq S_n$, $\{b_0, \dots, b_n\} \subseteq B \setminus S_n$
- By Ramsey, there is an infinite $I \subseteq \omega$ such that
- either $\forall n > j \in I (a_n \in S_j) \implies \forall i, j \in I (b_i \in S_j \iff i > j)$,
 - or $\forall n > j \in I (a_n \notin S_j) \implies \forall i, j \in I (a_i \in S_j \iff i \leq j)$
- In the first case we set $c_i = b_i$;
 - In the second case we set $c_i = a_{i+1}$;

□

Definition 2.10. Let $\phi(x, y)$ be a formula, by a complete ϕ -type over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x, b), \neg\phi(x, b)$ where b ranges over A . Let $S_\phi(A)$ be the space of all complete ϕ -types over A .

Proposition 2.11. Assume that $|S_\phi(B)| > |B|$ for some infinite set of parameters B . Then $\phi(x, y)$ is unstable.

Proof. .

- For $a \in \mathbb{M}_x$, $\text{tp}_\phi(a/B)$ is determined by $\phi(a, B) = \{b \in B \mid \models \phi(a, b)\}$;

- $|S_\phi(B)| > |B| \implies |\{\phi(a, B) \mid a \in \mathbb{M}_x\}| > |B|$;
- By Erdős-Makkai, there are sequences $(a_{i < \omega})$ and $(b_{i < \omega})$ s.t.
either $\models \phi(a_i, b_j) \iff i < j$, or $\models \phi(a_i, b_j) \iff j < i$.

□

3 Local ranks and definability of types

Definition 3.1. We define Shelah's local 2-rank taking values in $\{-\infty\} \cup \omega \cup \{+\infty\}$ by induction on $n \in \omega$. Let Δ be a set of L -formulas, and $\theta(x)$ a partial type over \mathbb{M} .

- $R_\Delta(\theta(x)) \geq 0 \iff \theta$ is consistent (and $-\infty$ otherwise);
- $R_\Delta(\theta(x)) \geq n + 1$ if $\exists \phi(x, y) \in \Delta$ and $a \in \mathbb{M}_y$ s.t.

$$R_\Delta(\theta(x) \wedge \phi(x, a)) \geq n \text{ and } R_\Delta(\theta(x) \wedge \neg\phi(x, a)) \geq n$$

- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and $R_\Delta(\theta(x)) \not\geq n + 1$
- $R_\Delta(\theta(x)) = +\infty$ if $R_\Delta(\theta(x)) \geq n$ for all $n \in \omega$.

If ϕ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$.

Proposition 3.2. $\phi(x, y)$ is stable iff $R_\phi(x = x)$ is finite.

Proof. Assume that $\phi(x, y)$ is unstable:

- By compactness, there is a sequence $(a_i b_i \mid i \in [0, 1])$ such that

$$\models \phi(a_i, b_j) \iff i < j$$

- Both $\phi(x, b_{\frac{1}{2}})$ and $\neg\phi(x, b_{\frac{1}{2}})$ contain dense subsequences of a_i 's.
- Each of these sets can be split again, by $\phi(x, b_{\frac{1}{4}})$ and $\phi(x, b_{\frac{3}{4}})$;
- ...

Conversely, assume that the rank is infinite:

- We can find a infinity tree of parameters

$$B = \{b_\eta \mid \eta \in 2^{<\omega}\}$$

such that

- for each $\eta \in 2^\omega$, let

$$\Phi_\eta = \{\phi^{\eta(n)}(x, b_{\eta|n}) \mid n \in \omega\},$$

where $\phi^1 = \phi$ and $\phi^0 = \neg\phi$;

- Then each Φ_η is consistent;
- Different Φ_η 's are inconsistent;
- $|S_\phi(B)| \geq 2^{|B|} \implies \phi(x, y)$ is unstable.

□

Definition 3.3.

- Let $\phi(x, y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is definable over B if there is some $L(B)$ -formula $\psi(y)$ such that for all $a \in A$

$$\phi(x, a) \in p \iff \models \psi(a)$$

- A type $p \in S_x(A)$ is definable over B if $p|_\phi$ is definable over B for all $\phi(x, y) \in L$.
- A type is definable if it is definable over its domain.
- We say that types in T are uniformly definable if for every $\phi(x, y)$ there is some $\psi(y, z)$ such that every type can be defined by an instance of $\psi(y, z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that

$$\phi(x, a) \in p \iff \models \psi(a, b),$$

for all $a \in A$.

Remark 3.4.

- Let $A \subseteq \mathbb{M}_x$, and $B \subseteq A$. We say that B is externally definable if there is some \mathbb{M} -definable set X such that $B = X \cap A$
- If $X = \phi(\mathbb{M}, c)$. Then $\text{tp}_\phi(c/A)$ is definable iff $B = X \cap A$ is in fact internally definable.
- A set is called stably embedded if for every externally definable subset of it is internally definable.

Example 3.5. Consider $(\mathbb{Q}, <) \models DLO$, and let $p = \text{tp}(\pi/\mathbb{Q})$. Then $x < y \in p(x) \iff x < \pi$. By QE, p is not definable.

Lemma 3.6.

1. The set $\{e \in \mathbb{M}^k \mid R_\phi(\theta(x, e)) \geq n\}$ is definable for all $n \in \omega$;
2. If $R_\phi(\theta(x)) = n$, then for any $a \in \mathbb{M}_y$, at most one of $\theta(x) \wedge \phi(x, a)$, $\theta(x) \wedge \neg\phi(x, a)$ has R_ϕ -rank n .

Proof. (1):

- Induction on n .
- $n = 0$: $R_\phi(\theta(x, e)) \geq 0 \iff \models \exists x(\theta(x, e))$;

- $n = k + 1$:

$$R_\phi(\theta(x, e)) \geq k+1 \iff \exists y \left(\left(R_\phi(\theta(x, e) \wedge \phi(x, y)) \geq k \right) \wedge \left(R_\phi(\theta(x, e) \wedge \neg \phi(x, y)) \geq k \right) \right)$$

(2): Trivial. □

Proposition 3.7. *Let $\phi(x, y)$ be a stable formula. Then all ϕ -types are uniformly definable.*

Proof. .

- Suppose that $R_\phi(x = x)$ is $n \in \omega$;
- Let $p \in S_\phi(A)$;
- Then there is $\chi(x) \in p$ such that $R_\phi(\chi(x)) = \min\{R_\phi(\varphi(x)) \mid \varphi \in p\}$;
- For each $b \in A_y$, either $\phi(x, b) \in p$ or $\neg \phi(x, b) \in p$;
- either $R_\phi(\chi(x) \wedge \phi(x, b)) < n$ or $R_\phi(\chi(x) \wedge \neg \phi(x, b)) < n$;
- $R_\phi(\chi(x))$ is minimal $\implies (\phi(x, b) \in p \iff R_\phi(\chi(x) \wedge \phi(x, b)) = n)$.

□

Theorem 3.8. *The following are equivalent for a formula $\phi(x, y)$.*

1. $\phi(x, y)$ is stable;
2. $R_\phi(x = x) < \omega$;
3. All ϕ -types are uniformly definable;
4. All ϕ -types over models are uniformly definable;
5. $S_\phi(M) \leq \kappa$ for all $\kappa \geq |L|$ and $M \models T$ with $|M| = \kappa$;
6. There is some κ such that $|S_\phi(M)| < \text{ded } \kappa$ for all $M \models T$ with $|M| = \kappa$.

Proof. .

- (1) \iff (2) by Proposition 3.2;
- (2) \implies (3) by Proposition 3.7;
- (3) \implies (4) is obvious;
- (4) \implies (5), each ϕ -type is determined by a $L(M)$ -formula, its own definition;
- (5) \implies (6) is obvious;

- (6) \implies (1) is by Proposition 2.5.

□

Global case:

Theorem 3.9. *Let T be a complete theory. Then the following are equivalent.*

1. T is stable;
2. There is NO sequence of tuples $(c_i \mid i \in \omega)$ from \mathbb{M} and formula $\phi(z_1, z_2) \in L(M)$ such that

$$\models \phi(c_i, c_j) \iff i < j;$$
3. $f_T(\kappa) \leq \kappa^{|T|}$ for all infinite cardinals κ ;
4. There is some κ such that $f_T(\kappa) \leq \kappa$;
5. There is some κ such that $f_T(\kappa) < \text{ded } \kappa$;
6. All formulas of the form $\phi(x, y)$ where x is a singleton variable, are stable;
7. All types over models are definable.

Proof. .

- (1) \implies (2) by definition;
- (2) \implies (1):
 - Let $\psi(x, y)$ be a formula with order property witnessed by sequence

$$\{(a_i, b_i) \mid i < \omega\};$$
 - Let $\phi(x_1 y_1; x_2 y_2) := \psi(x_1, y_2)$ and $c_i := a_i b_i$;
 - Then $\models \phi(c_i, c_j) \iff i < j$.
- (1) \implies (3) : $S_x(M) \rightarrow \prod_{\phi \in L} S_\phi(M)$ is injective;
- (3) \implies (4) is obvious;
- (4) \implies (5) is obvious;
- (5) \implies (1) is by Proposition 2.5.
- (6) \iff (1 – 5): Fix some κ , then $S_1(M) \leq \kappa$ for all M with $|M| = \kappa$ iff $S_n(M) \leq \kappa$ for all M with $|M| = \kappa$;
- (7) \iff (1 – 5) by Theorem 3.9

□

Example 3.10. • Stability \iff all types over all models are definable;

- Some unstable theories have certain special models over which all types are definable;
- $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$, all types over \mathbb{R} are uniformly definable;
- $\mathcal{M} = (\mathbb{Q}_p, +, \times, 0, 1)$, all types over \mathbb{Q}_p are uniformly definable.

4 Indiscernible sequences and stability

Definition 4.1. Given a linear order I , a sequence of tuples $(a_i \mid i \in I)$ with $a_i \in \mathbb{M}_x$ is indiscernible over a set of parameters A if $a_{i_0}, \dots, a_{i_n} \equiv_A a_{j_0}, \dots, a_{j_n}$ for all $i_0 < \dots < i_n$ and $j_0 < \dots < j_n$ from I and all $n \in \omega$.

Example 4.2. .

1. A constant sequence is indiscernible over any set;
2. A subsequence of a A -indiscernible sequence is A -indiscernible;
3. In the theory of equality, any sequence of singletons is indiscernible;
4. Any increasing (or decreasing) sequence of singletons in a dense linear order is indiscernible;
5. Any basis in a vector space is an indiscernible sequence.

Definition 4.3. For any sequence $\bar{a} = (a_i \mid i \in I)$ and a set of parameters B , we define $\text{EM}(\bar{a}/B)$, the Ehrenfeucht-Mostowski type of the sequence \bar{a} over B , as a partial type over B in countably many variables indexed by I and given by the following collection of formulas

$$\phi(x_0, \dots, x_n) \in L(B) \mid \forall i_0 < \dots < i_n, \models \phi(a_{i_0}, \dots, a_{i_n}, n \in \omega)$$

Exercise 4.4. For any sequence $\bar{a} = (a_i \mid i \in I)$ and a set of parameters B . If J is an infinite linear order, then there is a sequence $\bar{b} = (b_i \mid i \in J)$ which realizes the EM-type of \bar{a} over A , i.e.

$$\models \phi(b_{i_0}, \dots, b_{i_n}) \text{ for all } i_0 < \dots < i_n \in J, \phi \in \text{EM}(\bar{a}/A)$$

Exercise 4.5. If $\bar{a} = (a_i \mid i \in I)$ is an A -indiscernible sequence. Then $\text{EM}(\bar{a}/A)$ is a complete ω -type over A .

Let $\bar{a} = (a_i \mid i \in I)$ and $\bar{b} = (b_j \mid j \in J)$ be A -indiscernible sequences. We denote $\bar{a} \equiv_{\text{EM}, A} \bar{b}$ if $\text{EM}(\bar{a}/A) = \text{EM}(\bar{b}/A)$

Proposition 4.6. Let $\bar{a} = (a_i \mid i \in J)$ be an arbitrary sequence in \mathbb{M} , where J is an arbitrary linear order and A is a small set of parameters. Then for any small linear order I we can find (in \mathbb{M}) an A -indiscernible sequence $(b_i \mid i \in I)$ realize the EM-type of \bar{a} over A .

Proof. 1. Let $\{c_i \mid i \in I\}$ be a set of new constants;

2. Let $L' = L \cup \{c_i \mid i \in I\}$;

3. Let $T' \supseteq T$ be in L' containing the following axioms:

- $\phi(c_{i_0}, \dots, c_{i_n})$ for all $i_0 < \dots < i_n \in I$ and $\phi \in \text{EM}(\bar{a}/A)$;
- $\psi(c_{i_0}, \dots, c_{i_n}) \leftrightarrow \psi(c_{j_0}, \dots, c_{j_n})$ for all $i_0 < \dots < i_n, j_0 < \dots < j_n \in I$ and $\psi \in L(A)$

4. It is enough to show that T' is consistent;
5. By compactness, it is enough to show that every finite $T_0 \subseteq T'$ is consistent;
6. T_0 involves only finitely many formulas $\Delta \subseteq L(A)$ with at most n variables, and new constants $\{c_{k_0}, \dots, c_{k_m}\}$;
7. Let $(b_i \mid i \in I) \subseteq \mathbb{M}$ realize the EM-type of \bar{a} over A ;
8. By Ramsey, there is an infinite subset $I_0 \subseteq I$ such that:
 - either $\models \phi(b_{i_0}, \dots, b_{i_n})$ for all $\phi \in \Delta$ and $i_0 < \dots < i_n \in I$;
 - or $\models \neg\phi(b_{i_0}, \dots, b_{i_n})$ for all $\phi \in \Delta$ and $i_0 < \dots < i_n \in I$.
9. Let $i_0 < \dots < i_m \in I_0$ and interpret c_{k_0}, \dots, c_{k_m} as b_{i_0}, \dots, b_{i_m} ;
10. Then $\mathbb{M} \models T_0$.

□

Corollary 4.7. *If $\bar{a} = (a_i \mid i \in I)$ is an A -indiscernible sequence and $J \supseteq I$ is an arbitrary linear order, then (in \mathbb{M}) there is an A -indiscernible sequence $(b_j \mid j \in J)$ such that $b_i = a_i$ for all $i \in I$ (every thing involved is small).*

Proof. .

- Let $(c_j \mid j \in J) \subseteq \mathbb{M}$ realize the EM-type of \bar{a} over A ;
- Then the subsequence $(c_j \mid j \in I)$ realize the type of $(a_i \mid i \in I)$ over A ;
- Namely, $(c_j \mid j \in I) \equiv_A (a_i \mid i \in I)$;
- By homogeneity, there is $(b_j \mid j \in J) \supseteq (a_i \mid i \in I)$ such that $(c_j \mid j \in J) \equiv_A (b_j \mid j \in J)$.

□

Lemma 4.8. *If $\bar{a} = (a_i \mid i \in I)$ is an infinite A -indiscernible sequence, then for all $S \subseteq I$ and $a_i \notin \text{acl}(A, a_{j \in S})$*

- $a_i \in \text{acl}(A, a_{j \in S}) \iff \exists S_0 \subseteq_{\text{fin}} S (a_i \in \text{acl}(A, a_{j \in S_0}))$;
- Let $(b_i \mid i < \omega) \equiv_{\text{EM}, A} (a_i \mid i \in I)$;
- Then for any $i_0 < \dots < i_n \in I$ and $j_0 < \dots < j_n \in \mathbb{Q}$

$$a_{i_k} \in \text{acl}(A, \{a_{i_s} \mid s \neq k, s \leq n\}) \iff b_{j_k} \in \text{acl}(A, \{b_{j_s} \mid s \neq k, s \leq n\})$$

- WLG, we assume that $I = (\mathbb{Q}, <)$;

- Suppose that

$$a_{i_k} \in \text{acl}(A, \{a_{i_s} \mid s \neq k, s \leq n\})$$

- Suppose that the formula $\phi(x_0, \dots, x_k, \dots, x_n) \in L(A)$ witness the property;
- Namely, $\models \phi(a_{i_0}, \dots, a_{i_n})$ and $\phi(a_{i_0}, \dots, \mathbb{M}, \dots, a_{i_n})$ is finite;
- Then for any $q \in \mathbb{Q}$ realizing the same cut of a_{i_k} over $\{a_{i_s} \mid s \neq k, s \leq n\}$, we have

$$\models \phi(a_{i_0}, \dots, a_q, \dots, a_{i_n})$$

- So $\phi(a_{i_0}, \dots, \mathbb{M}, \dots, a_{i_n})$ is infinite, a contradiction.

Exercise 4.9. Start with the sequence $(1, 2, 3, \dots)$ in $(+, \times, 0, 1) \models \text{ACF}_0$. Give an explicit example of an indiscernible sequence based on it.

A more power result is :

Proposition 4.10. *Let A be a set of parameters. If $\kappa \geq |T| + |A|$, $\lambda = \beth_{(2^\kappa)^+}$, and $(a_i \mid i < \lambda)$ is a sequence of tuples a_i of the same length $\leq \kappa$, then there is an A -indiscernible sequence $(b_i \mid i < \omega)$ such that for each $n < \omega$ there are $i_0 < \dots < i_n < \lambda$ such that*

$$b_0, \dots, b_n \equiv_A a_{i_0}, \dots, a_{i_n}.$$

See Enrique Casanovas's "Simple theories and hyperimaginaries", Prop. 1.6 for a proof (Using Erdős-Rado theorem: $\beth_n(\kappa)^+ \rightarrow \kappa^{+n+1}_\kappa$, which means if f is a coloring of the $n+1$ -element subsets of a set of cardinality $\beth_n(\kappa)^+$, in κ many colors, then there is a homogeneous set of cardinality κ^+ , instead of Ramsey).

Definition 4.11. A sequence $(a_i \mid i \in I)$ is totally indiscernible over A if $a_{i_0} \dots a_{i_n} \equiv_A a_{j_0} \dots a_{j_n}$ for any $i_0 \neq \dots \neq i_n, j_0 \neq \dots \neq j_n$ from I (so the order of the indices doesn't matter any longer).

Theorem 4.12. *T is stable if and only if every indiscernible sequence is totally indiscernible.*

Proof. \Rightarrow

- Suppose that T is stable, $(a_i \mid i \in I)$ is indiscernible over A ;
- If $(a_i \mid i \in I)$ is NOT totally indiscernible;
- then there are $i_0 \neq \dots \neq i_n, j_0 \neq \dots \neq j_n$ from I such that $a_{i_0} \dots a_{i_n} \not\equiv_A a_{j_0} \dots a_{j_n}$;
- WLG, assume that $I = (\mathbb{Q}, <)$ and $i_0 = 0, \dots, i_n = n$;
- there is $\sigma \in S_{n+1}$, the permutation group of $\{0, \dots, n\}$, such that

$$a_{\sigma(0)} \dots a_{\sigma(n)} \equiv_A a_{j_0} \dots a_{j_n}$$

- $\sigma = \tau_1 \dots \tau_m$, a product of a sequence of transpositions of two consecutive elements;
- there is $0 < k < m$ such that $a_{\tau_k(0)} \dots a_{\tau_k(n)} \not\equiv_A a_0 \dots a_n$;
- Assume that $\tau_k = (s, s+1)$, then there is a $L(A)$ -formula $\psi(x_0, \dots, x_n)$ such that

$$\models \psi(a_0, \dots, a_s, a_{s+1}, \dots, s_n) \wedge \neg \psi(a_0, \dots, a_{s+1}, a_s, \dots, s_n);$$

- Let $\phi(x, y) := \psi(a_0, \dots, a_{s-1}, x, y, a_{s+2}, \dots, s_n)$;
- Then for all $s < q, r < s+1$, $\models \phi(a_q, a_r) \iff a_q < a_r$.

□