Stability spectra and the forking calculus

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Prop 7.3 (Local character)

If $p \in S_n(A)$, $\exists B \subseteq A$ $|B| < \kappa_n(T)$

$p \sqsupseteq (p \restriction B)$ (p doesn't fork over B)

Proof Suppose not ($p \not\sqsupseteq p \restriction B$ for $B \subseteq A$ with $|B| < \kappa_n(T)$)

Let $\kappa = \kappa_n(T)$.

Build $(\bar{b}_\alpha : \alpha < \kappa)$ in A as follows

Step α: $p \not\sqsupseteq p \restriction \underbrace{\{\bar{b}_\gamma : \gamma < \alpha\}}_{B_\alpha}$ $|B_\alpha| < \kappa$.

$\exists\, \varphi(x, \bar{b}_\alpha) \in p(x)$, $\varphi(x, \bar{b}_\alpha)$ forks over B_α.

there's $\bar{b}_\alpha$

$(B_\alpha : \alpha < \kappa)$ ~~inc~~ $\underset{\alpha < \gamma}{B_\alpha \subseteq B_\gamma} \subseteq A$

For $\alpha < \gamma$, $p \restriction B_\gamma \supseteq p \restriction B_\alpha$

$bd(p \restriction B_\gamma) \leq bd(p \restriction B_\alpha)$

$\bar{b}_\alpha \in B_{\alpha+1}$

If $\gamma > \alpha$ ~~$\gamma > \gamma$~~ $\gamma \geq \alpha + 1$ $p \restriction B_{\alpha+1} \ni \varphi(x, \bar{b}_\alpha)$ forks over B_α

so $p \restriction B_{\alpha+1} \not\sqsupseteq p \restriction B_\alpha$.

then $p \restriction B_\gamma \not\sqsupseteq p \restriction B_\alpha$

$bd(p \restriction B_\gamma) < bd(p \restriction B_\alpha)$, get desc. chain of length κ in fund. order $\Rightarrow\Leftarrow$. $\square$.

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§8 Stability Spectra

$\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$.

Lemma 8.1 If $M \models T$, λ cardinal, $\beta, \beta' \in$ Fund. order

$\beta > \beta'$, if $N \succcurlyeq M$ N very saturated, str. homog.

if $p \in S_n(M)$ $[p] = \beta$ then

$|\{q \in S_n(N) : q \supseteq p,\ [q] = \beta'\}| > \lambda$.

Proof Since $\beta' \leq [p]$, $\exists$ $N_0 \succcurlyeq M$ $q_0 \in S_n(N_0)$ $q_0 \supseteq p$

$[q_0] = \beta'$. (use ultrapowers)

Embed $N_0 \hookrightarrow N$. WLOG $|N_0| \leq |M| + |L|$

WLOG $M \preccurlyeq N_0 \preccurlyeq N$

Take $q \in S_n(N)$ the heir of q_0.

$[q] = \beta' < \beta = [p]$ $q \supseteq p$

$q \not\sqsupseteq p$.

(Treat N as monster, q is not $acl^{eq}(M)$-definable

(else $q \sqsupseteq p$)

$\{\sigma(q) : \sigma \in Aut(N/M)\}$ is big (Prop 3.3)

$|\{\sigma(q) : \sigma \in Aut(N/M)\}| > \lambda$

$[\sigma(q)] = [q] = \beta'$. $\sigma(q) \supseteq p$ $\square$.

Prop 8.2 If $\lambda^\mu > \lambda$ for some $\mu < \kappa(T)$, then T is not λ-stable. $(\lambda \geq \aleph_0)$

Proof Take μ minimal s.t. $\lambda^\mu > \lambda$.

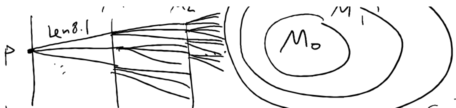
Take $(\beta_\alpha : \alpha < \mu)$ in fundamental order descending.

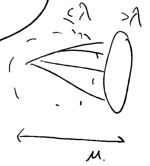
(WMA $\mu \geq \aleph_0$ since $\lambda^\mu \leq \lambda$ for $\mu < \aleph_0$.)

Take $p \in S_1(M_0)$ $[p] = \beta_0$.

Build $M_0 \preccurlyeq M_1 \preccurlyeq \dots$ length μ.

$M_{\alpha+1}$ is $(|M_\alpha| + \lambda)^+$-saturated, -strongly homogeneous.





Lem 8.1 gives distinct p_i $\quad i<\lambda \quad p_i \in S_1(M_1)$
$p_i \supseteq p$
$[p_i] = \beta_1$

$\forall i<\lambda$, Lem 8.1 gives distinct $p_{ij} \in S_1(M_2)$ $\quad [p_{ij}] = \beta_2$
$p_{ij} \supseteq p_i$

etc. . . .

Get p_σ for $\sigma \in \lambda^{\leq\mu}$, if $\sigma \in \lambda^\alpha$ $\quad p_\sigma \in S_1(M_\alpha)$ $\quad [p_\sigma] = \beta_\alpha$.

If τ extends σ then $p_\tau \supseteq p_\sigma$

$p_{\sigma i} \neq p_{\sigma j}$ for $i \neq j$

For each σ, i, j $\quad i \neq j$ $\quad i,j<\lambda$ $\quad$ ~~$\sigma \in \lambda^\alpha$, $\alpha \leq \mu$~~

Take $\varphi(x,b) \in p_{\sigma i}$
$\varphi(x,b) \notin p_{\sigma j}$

$|\lambda^{<\mu}| \leq \lambda$ by choice of μ.

Collect all the b in a set $B \subseteq \bigcup_{\alpha<\mu} M_\alpha$.

$|B| \leq |\lambda^{<\mu}| \lambda \lambda \leq \lambda$

Claim $\quad \lambda^\mu \to S_1(B)$
$\sigma \mapsto p_\sigma \restriction B$

is injective. (So then $|S_1(B)| \geq \lambda^\mu > \lambda$, so λ-stab. fails).

Proof If $\sigma, \tau \in \lambda^\mu$ $\quad \sigma \neq \tau$.
$\Rightarrow$ there is $\alpha<\mu$
~~$\sigma\restriction\alpha \neq \tau\restriction\alpha$~~
$\sigma\restriction\alpha = \tau\restriction\alpha$
$\sigma\restriction(\alpha+1) \neq \tau\restriction(\alpha+1)$.

σ:
τ:

α

$p_\sigma \restriction M_{\alpha+1} = p_{\sigma\restriction\alpha+1} \neq p_{\tau\restriction\alpha+1} = p_\tau \restriction M_{\alpha+1}$

By choice of B, $\exists b \in B$, $\varphi(x,b)$
$\varphi \in p_\sigma$
$\varphi \notin p_\tau$. $\quad$ So $p_\sigma \restriction B \neq p_\tau \restriction B$. $\quad \square$

Cor 8.3 If T is λ-stable, $\lambda \geq \kappa(T)$.

Proof Otherwise take $\mu = \lambda$. $\quad \mu < \kappa(T)$.
$\lambda^\mu > \lambda$ $\quad (\lambda^\mu = \lambda^\lambda \geq 2^\lambda > \lambda)$.
So Prop 8.2, T not λ-stable.

Lemma 8.4 If T is λ-stable, $A \subseteq \mathbb{M}$ $\quad |A| \leq \lambda$
then $|S_1(\mathrm{acl}^{eq}(A))| \leq \lambda$.

Proof Take $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ length ω,
$A_{i+1} = A_i \cup \{$a realization of each type over $A_i\}$.
If $|A_i| \leq \lambda$ then $|S_1(A_i)| \leq \lambda$ so $|A_{i+1}| \leq \lambda$.
Induction $\Rightarrow$ $|A_i| \leq \lambda$

Let $M = \bigcup_{i=0}^{\infty} A_i$. $\quad |M| \leq \lambda$.

Claim $M \preccurlyeq \mathbb{M}$.

Proof Use Tarski-Vaught criterion. If $D \subseteq \mathbb{M}$, D is M-definable, $D \neq \emptyset$, then we want $D \cap M \neq \emptyset$.
D is A_i-definable, $i<\omega$. Take $b_0 \in D$. $tp(b_0/A_i)$
$\exists\, b \in A_{i+1}$ $\quad b \models tp(b_0/A_i)$ $\quad b \in D$.
$b \in D \cap M \neq \emptyset$. $\quad \square_{\text{claim}}$

$M^{eq} \supseteq \mathrm{acl}^{eq}(A)$.

$\underset{\lambda\text{-stab}}{\lambda \geq} |S_1(M)| = |S_1(M^{eq})| \geq |S_1(\mathrm{acl}^{eq}(A))|$. $\quad \square$

Def 8.5 $\lambda_0(T) = \min\{\lambda \geq \aleph_0 : T \text{ is } \lambda\text{-stable}\}$.

Cor 8.3 $\Rightarrow$ $\lambda_0(T) \geq \kappa(T)$.

Theorem 8.6 T is λ-stable $\Leftrightarrow [\lambda \geq \lambda_0(T) \text{ and } \forall \mu < \kappa(T),\ \lambda^\mu \leq \lambda]$.

Proof: $(\Rightarrow)$ $\lambda \geq \lambda_0(T)$ by def, $\lambda^\mu \leq \lambda$ by Prop 8.2.

$(\Leftarrow)$ Fix $A \subseteq M$ $|A| \leq \lambda$

Goal: $|S_1(A)| \leq \lambda$.

Assuming $\lambda \geq \lambda_0(T) \geq \kappa(T)$.
$\lambda^\mu \leq \lambda$ $\forall \mu < \kappa(T)$.

If $p \in S_1(A)$, by local character (Prop 7.3)

$\exists B \subseteq A$ $p \sqsupseteq p\restriction B$ $|B| < \kappa(T) \leq \lambda_0(T)$

Let $\mu = |B|$.

\# choices for μ is $\leq \kappa(T)$.

Given μ, \# choices for B is $\leq |A|^\mu \leq \lambda^\mu \leq \lambda$

If $c \models p$ $p = tp(c/A)$, determined by $tp(c/acl^{eq}(A))$

determined by $tp(c/acl^{eq}(B))$.

$$\begin{array}{ccc} \overset{p}{tp(c/A)} & \sqsupseteq & \overset{p\restriction B}{tp(c/B)} \\ \sqcup\!\!\!\sqcup & & \sqcup\!\!\!\sqcup \\ tp(c/acl^{eq}(A)) & \sqsupseteq & \underbrace{tp(c/acl^{eq}(B))}_{\text{stationary}} \end{array}$$

\# choices for $tp(c/acl^{eq}(B))$ is $\leq \lambda_0$ by Lem 8.4. $|B| \leq \lambda_0$

All together \# choices for p is

$\leq \kappa(T) \times \lambda \times \lambda_0(T) = \lambda$.

$\kappa \leq \lambda_0 \leq \lambda$.

$|S_1(A)| \leq \lambda$. ∎

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§9 Superstability

Def 9.1 T is superstable if T is stable and $\underbrace{\kappa(T) = \aleph_0}_{\text{fnd. order has DCC. (no } \infty \text{ descend. chains)}}$.

If T is superstable & $p \in S_n(A)$ then $\exists A_0 \subseteq_f A$

$p \sqsupseteq (p\restriction A_0)$ (By local character)

Prop 9.2 T is superstable iff T is λ-stable for all suff. large λ.

Proof: If $\kappa = \aleph_0$. Then T is λ-stable

$\Leftrightarrow \lambda \geq \lambda_0$ and $(\underbrace{\lambda^\mu \leq \lambda \text{ for all } \mu < \aleph_0}_{\text{always true}})$

$\Leftrightarrow \lambda \geq \lambda_0$.

If $\kappa > \aleph_0$. If $\lambda = \aleph_{\alpha+\omega}$ for any ordinal α

then $cof(\lambda) = \omega$. $\lambda = \underbrace{\aleph_\alpha + \aleph_{\alpha+1} + \aleph_{\alpha+2} + \cdots}_{\aleph_0\text{-many}}$

$\lambda^{\aleph_0} > \lambda$ by König's lemma.

(if $\lambda_i > \kappa_i$ $\forall i \in I$ then $\prod_{i \in I} \lambda_i > \sum_{i \in I} \kappa_i$)

$$\lambda^{\aleph_0} = \prod_{i \in \aleph_0} \lambda > \sum_{i \in \aleph_0} \aleph_{\alpha+i} = \lambda.$$

Let $\mu = \aleph_0$, $\mu < \kappa(T)$, $\lambda^\mu > \lambda$, so not λ-stable

If T is not superstable, then T is not $(\aleph_{\alpha+\omega})$-stable $\forall \alpha$. ∎

Suppose $|L| = \aleph_0$

$\boxed{\aleph_0 \leq \lambda_0(T) \\ \aleph_0 \leq \kappa(T)}$ Cor 8.3: $\boxed{\lambda_0(T) \geq \kappa(T)}$

Rem 7.2: $\boxed{\kappa(T) \leq |L|^+ = \aleph_1}$

We know T is $2^{\aleph_0}$-stable (March 10, Lemma 1)

($2^{|L|}$-stable)

So $\boxed{\lambda_0 \leq 2^{\aleph_0}}$ $\longrightarrow$ λ ... $\aleph_0$ ... $2^{\aleph_0}$.

λ_0 is $\aleph_0$ or $2^{\aleph_0}$.

~~If $\kappa(T) = \aleph_0$ then~~

Fact $\boxed{\text{If } \lambda_0 > \aleph_0 \text{ then } \lambda_0 \geq 2^{\aleph_0}}$

If T is not $\aleph_0$-stable then $\exists$ countable A
$|S_1(A)| = 2^{\aleph_0}$.
so, not λ-stable for $\lambda < 2^{\aleph_0}$.

Cases:

	$\kappa(T)$	$\lambda_0(T)$	Stability Spectrum
superstable: $\aleph_0$-stable (ω-stable)	$\aleph_0$	$\aleph_0$	$\{\lambda : \lambda \geq \aleph_0\}$
superstable: Superstable but not $\aleph_0$-stable	$\aleph_0$	$2^{\aleph_0}$	$\{\lambda : \lambda \geq 2^{\aleph_0}\}$
Not superstable	$\aleph_1$	$2^{\aleph_0}$	$\{\lambda : \lambda^{\aleph_0} = \lambda\} = \{\lambda^{\aleph_0} : \lambda\}$
	~~$\aleph_1$~~	~~$\aleph_0$~~	

$\lambda^{\mu} = \lambda$ ~~for $\mu < \aleph_1$~~ for $\mu = \aleph_0$

Fact
- Strongly minimal are ω-stable
- $(\mathbb{Z}, +)$ is superstable but not ω-stable
- Separably closed fields (other than ACF), Free groups (other $(\mathbb{Z}, +)$) are stable, but not superstable

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Forking calculus $A \underset{C}{\perp} B$

Def 10.1 $\bar{a} \underset{C}{\perp} \bar{b}$ if $tp(\bar{a}/C\bar{b}) \sqsupseteq tp(\bar{a}/C)$

$\bar{a}, \bar{b}$ tuples (possibly ∞ long), C set

"$\bar{a}$ and $\bar{b}$ are independent over C"

Lemma 10.2 Suppose $C = acl^{eq}(C)$
$\bar{a}, \bar{b}$ are tuples. Let p, q be the unique global C-definable types extending $tp(\bar{a}/C)$, $tp(\bar{b}/C)$.

Then
1) $\bar{a} \underset{C}{\perp} \bar{b} \iff (\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C$
2) $\bar{a} \underset{C}{\perp} \bar{b} \iff \bar{b} \underset{C}{\perp} \bar{a}$

Proof 1) $(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C$
$\iff \bar{b} \models q \upharpoonright C$ and $\bar{a} \models p \upharpoonright C\bar{b}$
$\iff \bar{a} \models p \upharpoonright C\bar{b}$
$\iff tp(\bar{a}/C\bar{b}) \subseteq p$ $\qquad p \sqsupseteq tp(\bar{a}/C)$
$\iff tp(\bar{a}/C\bar{b}) \sqsupseteq tp(\bar{a}/C)$ (by Lemma 6-3)
$\iff \bar{a} \underset{C}{\perp} \bar{b}$.

2) $(\bar{b}, \bar{a}) \models (q \otimes p) \upharpoonright C \iff (\bar{a}, \bar{b}) \models (p \otimes q) \upharpoonright C$
(types commute in stable theories). $\square$

Lemma 10.3 $\forall C, \bar{a}, \bar{b}$,
1) $\bar{a} \underset{C}{\perp} \bar{b} \iff \bar{a} \underset{acl^{eq}(C)}{\perp} \bar{b}$
2) $\bar{a} \underset{C}{\perp} \bar{b} \iff \bar{b} \underset{C}{\perp} \bar{a}$.

Resume 11:40.

Proof (1) $\Rightarrow$ (2) by Lemma 10.2
Just need (1).

$tp(\bar{a}/C) \sqsubseteq tp(\bar{a}/C\bar{b}) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b}) \sqsubseteq tp(\bar{a}/acl^{eq}(C\bar{b}))$

$tp(\bar{a}/C) \sqsubseteq tp(\bar{a}/acl^{eq}(C)) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b})$

($\sqsubseteq$ by Prop 5.8)

$tp(\bar{a}/C) \sqsubseteq tp(\bar{a}/C\bar{b})$ $\qquad \bar{a} \underset{C}{\perp} \bar{b} \iff \bar{a} \underset{acl^{eq}(C)}{\perp} \bar{b}$
$\iff tp(\bar{a}/C) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b})$
$\iff tp(\bar{a}/acl^{eq}(C)) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b})$

$\iff tp(\bar{a}/C) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b})$

$\iff tp(\bar{a}/acl^{eq}(C)) \sqsubseteq tp(\bar{a}/acl^{eq}(C)\bar{b})$ $\square$.

Lemma 10.4 If $\bar{a}, \bar{a}'$ enumerate A, $\bar{b}, \bar{b}'$ enumerate B

then $\bar{a} \underset{C}{\perp} \bar{b} \iff \bar{a}' \underset{C}{\perp} \bar{b}'$

Proof $\bar{a} \underset{C}{\perp} \bar{b} \iff tp(\bar{a}/CB) \sqsupseteq tp(\bar{a}/C) \iff \bar{a} \underset{C}{\perp} \bar{b}'$.

Similarly $\bar{b}' \underset{C}{\perp} \bar{a} \iff \bar{b}' \underset{C}{\perp} \bar{a}'$

$\bar{a} \underset{C}{\perp} \bar{b} \iff \bar{a} \underset{C}{\perp} \bar{b}'$ $\quad \bar{a}' \underset{C}{\perp} \bar{b}'$. $\square$.

Def 10.5 $A \underset{C}{\perp} B$ if $\bar{a} \underset{C}{\perp} \bar{b}$ for tuples $\bar{a}, \bar{b}$ enumerating A, B.

Prop 10.6

(Symmetry) $A \underset{C}{\perp} B \iff B \underset{C}{\perp} A$

(Monotonicity) If $A' \subseteq A$, $B' \subseteq B$ then $A \underset{C}{\perp} B \Rightarrow A' \underset{C}{\perp} B'$.

(Transitivity) [right] $(A \underset{C}{\perp} B$ and $A \underset{CB}{\perp} B') \iff A \underset{C}{\perp} BB'$ $\quad$ CB means $C \cup B$.

[(left Transitivity) If $A \underset{C}{\perp} B$ and $A' \underset{CA}{\perp} B$ then $AA' \underset{C}{\perp} B$]

(Base monotonicity) $A \underset{C}{\perp} BB' \Rightarrow A \underset{CB}{\perp} B'$

(Normality) $A \underset{C}{\perp} B \Rightarrow A \underset{C}{\perp} BC$

(Invariance) If $\sigma \in Aut(M)$ then $A \underset{C}{\perp} B \Rightarrow \sigma(A) \underset{\sigma(C)}{\perp} \sigma(B)$

(Extension) Given A, B, C, $\exists A' \equiv_C A$ with $A' \underset{C}{\perp} B$

$\exists \sigma \in Aut(M/C)$, $\sigma(A) = A'$

(Finite character) If $[A_0 \underset{C}{\perp} B_0 \quad \forall A_0 \subseteq_f A \quad \forall B_0 \subseteq_f B]$ Then $A \underset{C}{\perp} B$.

Proof Symmetry was Lemma 10.3(2).

$C \subseteq CB \subseteq CBB'$ so...

(full transitivity)

$tp(A/CBB') \sqsupseteq tp(A/C) \iff$

$\left(\begin{array}{c} tp(A/CBB') \sqsupseteq tp(A/CB) \text{ and} \\ tp(A/CB) \sqsupseteq tp(A/C) \end{array} \right)$

$A \underset{C}{\perp} BB' \iff \left(A \underset{CB}{\perp} B' \text{ and } A \underset{C}{\perp} B \right)$.

Gives:
- Transitivity
- Base monotonicity
- Monotonicity on right (symmetry gives mon. on left).

Normality: by def.

Invariance: by def.

Extension: given A, B, C. Prop 5.5

$tp(A/C)$ has a non-forking extension to BC $\hat{q}$

take $A' \models q$

$tp(A'/BC) = q \sqsupseteq tp(A'/C) = tp(A/C)$

$tp(A'/C) = q \restriction C = tp(A/C)$ $\quad A \equiv_C A'$.

$A' \underset{C}{\perp} B$.

Finite character ~~$tp(A/BC) \not\sqsupseteq tp(A/C)$~~

Only need: $(A \underset{C}{\perp} B_0, \forall B_0 \subseteq_f B) \Rightarrow A \underset{C}{\perp} B$

$A \not\underset{C}{\downarrow} B \overset{??}{\Longrightarrow} \exists B_0 \subseteq_{fin} B \quad A \not\underset{C}{\downarrow} B_0$.

$tp(A/BC) \not\supseteq tp(A/C)$

$\varphi \in tp(A/BC)$ φ forks over C

~~φ only use~~ $\varphi \in L(B_0 C)$ $B_0 \subseteq_{fin} B$.

$\varphi \in tp(A/B_0C)$, φ forks over C

so... $A \not\underset{C}{\downarrow} B_0$. $\square$.

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§11 **Prop 11.1** If p, q are C-definable types and $\bar{a} \models p \restriction C$, $\bar{b} \models q \restriction C$ then...

$\bar{a} \underset{C}{\downarrow} \bar{b} \iff (\bar{a}, \bar{b}) \models (p \otimes q) \restriction C$.

In ACF, $V, W \subseteq M^n$, $P_V \otimes P_W = P_{V \times W}$.

Proof $tp(\bar{a}/C)$, $tp(\bar{b}/C)$ stationary

Proof is like Lemma 10.2. $\square$.

In ACF, $\bar{a} \underset{C}{\downarrow} \bar{b} \iff \bar{a} \underset{acl(C)}{\downarrow} \bar{b}$

$acl(C)$ is a model M

$\bar{a} \underset{M}{\downarrow} \bar{b}$ $tp(\bar{a}/M) = P_V$

$tp(\bar{b}/M) = P_W$.

$\Leftrightarrow$ $(\bar{a}, \bar{b})$ is generic on $V \times W$.

Prop 11.2 $\bar{a} \underset{B}{\downarrow} \bar{a} \iff \bar{a} \in acl^{eq}(B)$

Proof

($\Rightarrow$) $tp(\bar{a}/B\bar{a}) \supseteq tp(\bar{a}/B)$

$P \in S_n(M)$. $\bar{x} = \bar{a} \in p(\bar{x})$.

$p = tp(\bar{a}/M)$.

P is $acl^{eq}(B)$-definable

$\bar{a} \in acl^{eq}(B)$.

($\Leftarrow$) By "extension", $\exists \sigma \in Aut(M/B)$, $\sigma(acl^{eq}(B)) \underset{B}{\downarrow} acl^{eq}(B)$.

$\sigma(acl^{eq}(B)) = acl^{eq}(\sigma(B)) = acl^{eq}(B)$.

Actually $acl^{eq}(B) \underset{B}{\downarrow} acl^{eq}(B)$.

assuming $\bar{a} \in acl^{eq}(B)$

monotonicity $\Rightarrow$ $\bar{a} \underset{B}{\downarrow} \bar{a}$. $\square$.

Fact 11.3 $A \underset{C}{\downarrow} B \Rightarrow acl^{eq}(AC) \cap acl^{eq}(BC) = acl^{eq}(C)$.

if $e \in acl(AC)$ and $e \in acl(BC)$ then $e \in acl(C)$.

$A \cap B \subseteq acl^{eq}(C)$

Fact 11.4 Let T = theory of $\mathbb{R}$-vectorspaces.

If $A \subseteq M$, let $span(A) = \{ \sum_{i=1}^{n} x_i a_i \mid n \geq 0,\ x_i \in \mathbb{R},\ a_i \in A \}$.

1) T is complete, has Q.E.
2) T is strongly minimal, stable.
3) $A \underset{\emptyset}{\downarrow} B \iff span(A) \cap span(B) = \{0\}$.
4) $A \underset{C}{\downarrow} B \iff span(AC) \cap span(BC) = span(C)$.

$\Rightarrow$ easier
$\Leftarrow$ harder

(*) this condition determines $tp(A,B/C)$ from $tp(A/C)$ and $tp(B/C)$.

If (*) holds,

"Extension" $\Rightarrow \exists A' \underset{C}{\equiv} A$ $A' \underset{C}{\downarrow} B$ A', B, C satisfy (*)

$tp(A', B/C) = tp(A, B/C)$.

$A' \underset{C}{\downarrow} B \Rightarrow A \underset{C}{\downarrow} B$.

Prop 12.1 $\kappa_n(T)$ does not depend on n (for $1 \leq n < \omega$).

Proof sketch $\kappa_1(T) = \kappa_2(T)$.

Claim $\kappa_n(T)$ is smallest $\kappa \geq \aleph_0$ such that $\not\exists$ $\bar{a} \in M^n$, increasing $(C_\alpha : \alpha < \kappa)$ $C_\alpha \subseteq M$ such that $tp(\bar{a}/C_{\alpha+1}) \not\supseteq tp(\bar{a}/C_\alpha)$ $\forall \alpha$.

Proof Given a forking chain,

$$bd(tp(\bar{a}/C_{\alpha+1})) < bd(tp(\bar{a}/C_\alpha))$$

So get desc. chain of length κ in fnd. order.

Conversely, given a descending chain, get a forking chain as in Prop 8.2

$\square_{\text{claim}}$

$\kappa_1(T) = \kappa_2(T)$

$\kappa_1(T) \le \kappa_2(T)$ because if

$(C_\alpha : \alpha < \kappa)$ increasing

$a \not\perp_{C_\alpha} C_{\alpha+1}$

then $(a,a) \not\perp_{C_\alpha} C_{\alpha+1}$ so get a chain for 2-types.

$\kappa_2(T) \le \kappa_1(T)$.

Suppose have forking chain

$(a,b) \in M^2$, $(C_\alpha : \alpha < \kappa)$ increasing

$a,b \not\perp_{C_\alpha} C_{\alpha+1}$ $\forall \alpha$.

By "left transitivity"

$a \perp_{C_\alpha} C_{\alpha+1}$ and $b \perp_{aC_\alpha} C_{\alpha+1} \Rightarrow a,b \perp_{C_\alpha} C_{\alpha+1}$ (no)

$\forall \alpha$: $\left[a \not\perp_{C_\alpha} C_{\alpha+1} \text{ or } b \not\perp_{aC_\alpha} aC_{\alpha+1} \right]$

$(tp(a/C_\alpha) : \alpha < \kappa)$, $(tp(b/C_\alpha a) : \alpha < \kappa)$

one of these chains forks at least κ times.

So get a ~~cha~~ desc. chain of length κ in fnd. order for 1-types

$\kappa < \kappa_2 \Rightarrow \kappa < \kappa_1$. $\square$

§13 Independence

Def $\{A_i : i \in I\}$ is indep. over B if

$$A_i \perp_B \{A_j : j \neq i\}.$$

Ex ~~A, B are indep $/C$~~

A_1, A_2 are indep$/C$ if $A_1 \perp_C A_2$ and $A_2 \perp_C A_1$

if $A_1 \perp_C A_2$.

Ex A_1, A_2, A_3 are indep$/C$ if

$\underbrace{A_1 \perp_C A_2A_3 \text{ and } A_2 \perp_C A_1A_3 \text{ and } A_3 \perp_C A_1A_2}$.

Fact $\left(A_1 \perp_C A_2 \text{ and } A_1A_2 \perp_C A_3 \right) \Leftrightarrow$ (this).

$A_1 \perp_{A_2C} A_3$ ← base mon. on left.

$\xrightarrow{\text{trans.}} A_1 \perp_C A_2A_3$

$A_3 \perp_C A_1A_2$

similarly $A_2 \perp_C A_1A_3$

Fact $A_1, \ldots A_n$ indep$/C$ $\Leftrightarrow$ $A_i \perp_C A_1, \ldots A_{i-1}$.