Lascar rank

Advanced model theory

May 26–30, 2022

References: Poizat's *Course in Model Theory* §17.1, §19.2. The main result of the last three sections is proved in Poizat's *Stable Groups*, Corollary 2.14 (see eLearning).

1 Foundation rank

Let $(P, \leq)$ be a poset.

Definition 1.1. If $x \in P$ and α is an ordinal, then "$\mathrm{RF}(x) \geq \alpha$" is defined recursively as follows:

- $\mathrm{RF}(x) \geq 0$ is always true.
- $\mathrm{RF}(x) \geq \alpha + 1$ if there is $x' < x$ with $\mathrm{RF}(x') \geq \alpha$.
- If α is a limit ordinal, then $\mathrm{RF}(x) \geq \alpha$ if for all $\beta < \alpha$, $\mathrm{RF}(x) \geq \beta$.

For $x \in P$, we define $\mathrm{RF}(x)$ to be the greatest α such that $\mathrm{RF}(x) \geq \alpha$, or ∞ if $\mathrm{RF}(x) \geq \alpha$ for all α. $\mathrm{RF}(x)$ is called the *foundation rank* of x.

Lemma 1.2. *If* $\mathrm{RF}(x) = \infty$*, then there is* $y < x$ *such that* $\mathrm{RF}(y) = \infty$*.*

Proof. Suppose not. Then $\mathrm{RF}(y)$ is an ordinal for all $y < x$. Let $\alpha = \sup\{\mathrm{RF}(y) : y < x\}$. As $\mathrm{RF}(x) = \infty \geq \alpha + 2$, there is some $y < x$ such that $\mathrm{RF}(y) \geq \alpha + 1 > \alpha \geq \mathrm{RF}(y)$, a contradiction. □

Proposition 1.3. $\mathrm{RF}(x) = \infty$ *iff there is a descending chain* $x = x_0 > x_1 > x_2 > x_3 > \cdots$*.*

Proof. If $\mathrm{RF}(x) = \infty$, set $x_0 = x$, and recursively choose x_i such that $x_{i+1} < x_i$ and $\mathrm{RF}(x_{i+1}) = \infty$ by Lemma 1.2.

Conversely, suppose $x = x_0 > x_1 > x_2 > \cdots$. We prove by induction on α that for all $i < \omega$, $\mathrm{RF}(x_i) \geq \alpha$. The zero and limit cases are easy. Suppose $\mathrm{RF}(x_i) \geq \alpha$ for all i. Then $x_i > x_{i+1}$ and $\mathrm{RF}(x_{i+1}) \geq \alpha$ shows $\mathrm{RF}(x_i) \geq \alpha + 1$. This completes the inductive proof. So then $\mathrm{RF}(x_i) = \infty$ for all i, including $i = 0$. □

So $\mathrm{RF}(-)$ is ordinal-valued on P iff P satisfies the descending chain condition (DCC), meaning that there are no descending chains of length ω.

Remark 1.4. If $n < \omega$, then $\mathrm{RF}(x) \geq n$ iff there are $x = x_n > x_{n-1} > \cdots > x_0$. This is easy to prove by induction on n. On the other hand, it can happen that $\mathrm{RF}(x) \geq \omega$ without their being any infinite chains below x.

Remark 1.5.

1. If $x \geq y$ then $\mathrm{RF}(x) \geq \mathrm{RF}(y)$. (You show by induction on α that $x \geq y$ and $\mathrm{RF}(y) \geq \alpha$ imply $\mathrm{RF}(x) \geq \alpha$.)
2. If $x > y$ then $\mathrm{RF}(x) > \mathrm{RF}(y)$ unless $\mathrm{RF}(x) = \mathrm{RF}(y) = \infty$. (Certainly $\mathrm{RF}(x) \geq \mathrm{RF}(y)$ by the previous point. If equality holds, and $\mathrm{RF}(x) = \mathrm{RF}(y) < \infty$, let $\alpha = \mathrm{RF}(x) = \mathrm{RF}(y)$. Then $x > y$ and $\mathrm{RF}(y) = \alpha$ implies $\alpha = \mathrm{RF}(x) \geq \alpha + 1$, a contradiction.)
3. If P satisfies the DCC, then $x > y \implies \mathrm{RF}(x) > \mathrm{RF}(y)$, because no point has rank ∞.

Lemma 1.6. *If $\mathrm{RF}(x) = \alpha < \infty$ and $\beta \leq \alpha$, then there is $y \leq x$ with $\mathrm{RF}(y) = \beta$.*

Proof. Let $S = \{y \leq x : \mathrm{RF}(y) \geq \beta\}$. S is non-empty because $x \in S$. Take $y \in S$ minimizing $\mathrm{RF}(y)$. Then $\mathrm{RF}(y) \geq \beta$. If $\mathrm{RF}(y) = \beta$ we're done. Otherwise $\mathrm{RF}(y) \geq \beta + 1$, so there is $z < y$ with $\mathrm{RF}(z) \geq \beta$. Then $z \in S$. By Remark 1.5(2), $\mathrm{RF}(z) < \mathrm{RF}(y)$, contradicting the choice of y. $\square$

Therefore, if $(P, \leq)$ has the DCC, then $\{\mathrm{RF}(x) : x \in P\}$ is downwards closed (it's an initial segment of the ordinals).

2 Lascar rank

Assume T is stable.

Lemma 2.1. *If $p \in S_n(A)$ and $\beta = \mathrm{bd}(p)$ and $\beta' < \beta$, then there is an extension $p' \supseteq p$ with $\mathrm{bd}(p') = \beta$.*

Proof. If M is a model extending A, then some extension $q \in S_n(M)$ has $[q] = \beta$ by definition of "bound." By (April 21–28, Lemma 8.1), there is a further extension $p' \in S_n(N)$ with $\beta' = [p'] = \mathrm{bd}(p')$. $\square$

Definition 2.2. Let p be a (complete) type over some set A. The *Lascar U-rank* $\mathrm{U}(p)$ is the foundation rank of $\mathrm{bd}(p)$ (the bound of p) in the fundamental order.

Proposition 2.3. *Let p be a type.*

1. *$\mathrm{U}(p) \geq 0$ always holds.*
2. *If α is a limit ordinal, then $\mathrm{U}(p) \geq \alpha$ iff for all $\beta < \alpha$, $\mathrm{U}(p) \geq \beta$.*
3. *$\mathrm{U}(p) \geq \alpha + 1$ iff there is a forking extension $q \supseteq p$ with $\mathrm{U}(q) \geq \alpha$.*

Proof. (1) and (2) are easy. For (3), first suppose $\mathrm{U}(p) \geq \alpha + 1$. By definition of foundation rank, there is β in the fundamental order with $\mathrm{RF}(\beta) \geq \alpha$ and $\beta < \mathrm{bd}(p)$. By Lemma 2.1 there is an extension $q \in S_n(B)$ with $q \supseteq p$ and $\mathrm{bd}(q) = \beta < \mathrm{bd}(p)$. Then q is a forking extension of p and $\mathrm{U}(q) = \mathrm{RF}(\beta) \geq \alpha$.

Conversely suppose there is a forking extension $q \supseteq p$ with $\mathrm{U}(q) \geq \alpha$. Then $\mathrm{bd}(p) > \mathrm{bd}(q)$ and $\mathrm{RF}(\mathrm{bd}(q)) \geq \alpha$, so $\mathrm{U}(p) = \mathrm{RF}(\mathrm{bd}(p)) \geq \alpha + 1$. ☐

Proposition 2.3 can be used as an alternate definition of Lascar rank if you don't like the fundamental order (but then Proposition 2.4 below requires some work).

If T is superstable (April 21–28, Definition 9.1), then the fundamental order satisfies the DCC, and so $\mathrm{U}(p) < \infty$ for all types p. **Assume from now on that T is superstable.** This includes the case where T is totally transcendental (May 5–7, Theorem 7.6).

Proposition 2.4. *Suppose $p \in S_n(A)$ and $A \subseteq B$ and $q \in S_n(B)$ is an extension.*

1. $\mathrm{U}(q) \leq \mathrm{U}(p)$.
2. $\mathrm{U}(q) = \mathrm{U}(p) \iff q \sqsupseteq p$.

Proof. We know that $\mathrm{bd}(q) \leq \mathrm{bd}(p)$, so $\mathrm{U}(q) \leq \mathrm{U}(p)$ by Remark 1.5(1). By Remark 1.5(3), $\mathrm{U}(q) < \mathrm{U}(p) \iff \mathrm{bd}(q) < \mathrm{bd}(p) \iff q \not\sqsupseteq p$. ☐

Remark 2.5. Proposition 2.4 is analogous to what happens with Morley rank in totally transcendental theories. Recall from (May 5–7, Proposition 8.2) that if T is totally transcendental and $q \supseteq p$, then

1. $\mathrm{RM}(q) \leq \mathrm{RM}(p)$.
2. $\mathrm{RM}(q) = \mathrm{RM}(p) \iff q \sqsupseteq p$.

Proposition 2.6. *Suppose T is totally transcendental. Then $\mathrm{U}(p) \leq \mathrm{RM}(p)$ for any p.*

Proof. We prove by induction on α that

$$\mathrm{U}(p) \geq \alpha \implies \mathrm{RM}(p) \geq \alpha \qquad (*_\alpha)$$

The zero and limit cases are easy. Suppose $(*_\alpha)$ is known; we prove $(*_{\alpha+1})$. Suppose $\mathrm{U}(p) \geq \alpha + 1$. By Proposition 2.3 there is a forking extension q with $\mathrm{U}(q) \geq \alpha$. By $(*_\alpha)$, $\mathrm{RM}(q) \geq \alpha$. Then $\mathrm{RM}(p) > \mathrm{RM}(q) \geq \alpha$ because q is a forking extension, so $\mathrm{RM}(p) \geq \alpha + 1$. ☐

Remark 2.7 (U is jump-free)**.** If $\mathrm{U}(p) = \alpha$ and $\beta \leq \alpha$, there is q with $\mathrm{U}(q) = \beta$. This holds by Lemma 1.6.

Definition 2.8. $\mathrm{U}(\bar{a}/B) := \mathrm{U}(\mathrm{tp}(\bar{a}/B))$.

Proposition 2.4 says that if $B \subseteq C$, then

$$\mathrm{U}(\bar{a}/C) \leq \mathrm{U}(\bar{a}/B)$$
$$\mathrm{U}(\bar{a}/C) = \mathrm{U}(\bar{a}/B) \iff \bar{a} \underset{B}{\perp} C.$$

Remark 2.9. Proposition 2.3 says that $\mathrm{U}(\bar{a}/B) \geq \alpha + 1$ iff there is a forking extension $\mathrm{tp}(\bar{a}'/BC) \not\sqsupseteq \mathrm{tp}(\bar{a}/B)$ with $\mathrm{U}(\bar{a}'/BC) \geq \alpha$. But then $\mathrm{tp}(\bar{a}'/B) = \mathrm{tp}(\bar{a}/B)$, so moving $\bar{a}'C$ by $\sigma \in \mathrm{Aut}(\mathbb{M}/B)$ we may assume $\bar{a}' = \bar{a}$. So we see

$$\mathrm{U}(\bar{a}/B) \geq \alpha + 1 \iff \exists C \left(\mathrm{U}(\bar{a}/BC) \geq \alpha \text{ and } \bar{a} \underset{B}{\not\perp} C \right)$$

3 Lascar inequalities

Continue to assume superstability.

Proposition 3.1. $\mathrm{U}(\bar{a}/B) = 0 \iff \bar{a} \in \mathrm{acl}(B)$.

Proof. Equivalently, we want $\mathrm{U}(\bar{a}/B) > 0 \overset{?}{\iff} \bar{a} \notin \mathrm{acl}(B)$. By Remark 2.9, $\mathrm{U}(\bar{a}/B) > 0$ iff there is C with $\bar{a} \not\perp_B C$. If $\bar{a} \notin \mathrm{acl}(B)$, then $\bar{a} \not\perp_B \bar{a}$ by (April 21–28, Proposition 11.2). Conversely, if $\bar{a} \in \mathrm{acl}(B)$, then for any C we have $C \perp_B \mathrm{acl}(B)$ by (Homework 9, Problem 2), so $C \perp_B \bar{a}$ by monotonicity and $\bar{a} \perp_B C$ by symmetry. $\square$

Lemma 3.2. *If* $\mathrm{U}(\bar{a}/\bar{b}C) \geq \alpha$, *then* $\mathrm{U}(\bar{a}\bar{b}/C) \geq \alpha$.

Proof. By induction on α. The zero and limit cases are easy. Suppose $\mathrm{U}(\bar{a}/\bar{b}C) \geq \alpha + 1$. By Remark 2.9 there is C' such that $\bar{a} \not\perp_{\bar{b}C} C'$ and $\mathrm{U}(\bar{a}/\bar{b}CC') \geq \alpha$. By induction,

$$\mathrm{U}(\bar{a}/\bar{b}CC') \geq \alpha \implies \mathrm{U}(\bar{a}\bar{b}/CC') \geq \alpha.$$

If $\bar{a}\bar{b} \perp_C C'$ then base monotonicity on the left gives $\bar{a} \perp_{\bar{b}C} C'$, a contradiction. Therefore $\bar{a}\bar{b} \not\perp_C C'$. By Remark 2.9,

$$\mathrm{U}(\bar{a}\bar{b}/CC') \geq \alpha \implies \mathrm{U}(\bar{a}\bar{b}/C) \geq \alpha + 1. \qquad \square$$

For ordinals, there are two kinds of addition, $+$ and $\oplus$. (See pages 376–378 in the textbook for more about this.) Here are some examples highlighting the difference:

$$\begin{aligned} \omega + 1 &= \omega + 1 \\ 1 + \omega &= \omega \\ \omega \oplus 1 &= \omega + 1 \\ 1 \oplus \omega &= \omega + 1. \end{aligned}$$

The operation $\oplus$ is commutative, but $+$ is not. When $n, m < \omega$, $n + m$ and $n \oplus m$ are equal and are both the usual addition. If you like, you can only consider the case of finite ordinals in what follows.

Proposition 3.3. $\mathrm{U}(\bar{a}\bar{b}/C) \geq \mathrm{U}(\bar{a}/\bar{b}C) + \mathrm{U}(\bar{b}/C)$.

Proof. It suffices to show that

$$(\mathrm{U}(\bar{a}/\bar{b}C) \geq \alpha \text{ and } \mathrm{U}(\bar{b}/C) \geq \beta) \implies \mathrm{U}(\bar{a}\bar{b}/C) \geq \alpha + \beta.$$

We prove this by induction on β, holding α fixed. The $\beta = 0$ case is Lemma 3.2. The limit case is easy.[1] Suppose β works and consider $\beta + 1$. Suppose

$$\mathrm{U}(\bar{a}/\bar{b}C) \geq \alpha \text{ and } \mathrm{U}(\bar{b}/C) \geq \beta + 1.$$

By Remark 2.9, there is C' with $\mathrm{U}(\bar{b}/CC') \geq \beta$ and $\bar{b} \not\perp_C C'$. Moving C' by $\sigma \in \mathrm{Aut}(\mathbb{M}/\bar{b}C)$, we may assume $C' \perp_{\bar{b}C} \bar{a}$, by the extension/existence property of forking. Then $\mathrm{U}(\bar{a}/\bar{b}CC') = \mathrm{U}(\bar{a}/\bar{b}C) \geq \alpha$ by Proposition 2.4(2). By induction,

$$\mathrm{U}(\bar{a}/\bar{b}CC') \geq \alpha \text{ and } \mathrm{U}(\bar{b}/CC') \geq \beta \text{ imply } \mathrm{U}(\bar{a}\bar{b}/CC') \geq \alpha + \beta.$$

But $\bar{a}\bar{b} \not\perp_C C'$ (or else $\bar{b} \perp_C C'$ by monotonicity). By Remark 2.9,

$$\mathrm{U}(\bar{a}\bar{b}/CC') \geq \alpha + \beta \implies \mathrm{U}(\bar{a}\bar{b}/C) \geq (\alpha + \beta) + 1 = \alpha + (\beta + 1). \qquad \square$$

Proposition 3.4. $\mathrm{U}(\bar{a}\bar{b}/C) \leq \mathrm{U}(\bar{a}/\bar{b}C) \oplus \mathrm{U}(\bar{b}/C)$.

Proof. We prove by induction on α that $\mathrm{U}(\bar{a}\bar{b}/C) \geq \alpha \implies \mathrm{U}(\bar{a}/\bar{b}C) \oplus \mathrm{U}(\bar{b}/C) \geq \alpha$. The zero and limit cases are trivial. Suppose

$$\mathrm{U}(\bar{a}\bar{b}/C) \geq \alpha + 1.$$

By Remark 2.9 there is C' with $\bar{a}\bar{b} \not\perp_C C'$ and $\mathrm{U}(\bar{a}\bar{b}/CC') \geq \alpha$. By induction,

$$\mathrm{U}(\bar{a}/\bar{b}CC') \oplus \mathrm{U}(\bar{b}/CC') \geq \alpha. \tag{$\dagger$}$$

By left transitivity,

$$\left(\bar{a} \underset{\bar{b}C}{\perp} C' \text{ and } \bar{b} \underset{C}{\perp} C'\right) \implies \bar{a}\bar{b} \underset{C}{\perp} C',$$

so either $\bar{a} \not\perp_{\bar{b}C} C'$ or $\bar{b} \not\perp_C C'$. Then by Proposition 2.4,

$$\mathrm{U}(\bar{a}/\bar{b}C) \geq \mathrm{U}(\bar{a}/\bar{b}CC')$$
$$\mathrm{U}(\bar{b}/C) \geq \mathrm{U}(\bar{b}/CC'),$$

and *at least one inequality is strict.* Therefore

$$\mathrm{U}(\bar{a}/\bar{b}C) \oplus \mathrm{U}(\bar{b}/C) > \mathrm{U}(\bar{a}/\bar{b}CC') \oplus \mathrm{U}(\bar{b}/CC') \geq \alpha. \qquad \square$$

[1] When β is a limit ordinal, $\alpha + \beta$ is defined to be $\sup\{\alpha + \gamma : \gamma < \beta\}$, so things somehow work out.

Putting Propositions 3.3 and 3.4 together, we get the *Lascar inequalities*

$$\mathrm{U}(\bar{a}/\bar{b}C) + \mathrm{U}(\bar{b}/C) \leq \mathrm{U}(\bar{a}\bar{b}/C) \leq \mathrm{U}(\bar{a}/\bar{b}C) \oplus \mathrm{U}(\bar{b}/C).$$

When the ranks are finite, $+$ is equivalent to $\oplus$, so we get an equality

$$\mathrm{U}(\bar{a}\bar{b}/C) = \mathrm{U}(\bar{a}/\bar{b}C) + \mathrm{U}(\bar{b}/C).$$

Proposition 3.5. *Suppose* $\mathrm{U}(\bar{a}/C)$ *and* $\mathrm{U}(\bar{b}/C)$ *are finite. Then*

$$\mathrm{U}(\bar{a}\bar{b}/C) \leq \mathrm{U}(\bar{a}/C) + \mathrm{U}(\bar{b}/C),$$

with equality iff $\bar{a} \mathop{\smile\hspace{-0.9em}\raise0.1em\vert}_C \bar{b}$.

Proof. By the Lascar inequalities, the listed inequality is equivalent to

$$\mathrm{U}(\bar{a}/\bar{b}C) + \mathrm{U}(\bar{b}/C) \leq \mathrm{U}(\bar{a}/C) + \mathrm{U}(\bar{b}/C)$$

or equivalently

$$\mathrm{U}(\bar{a}/\bar{b}C) \leq \mathrm{U}(\bar{a}/C).$$

Now use Proposition 2.4. □

Remark 3.6. In general, $\mathrm{U}(\bar{a}\bar{b}/C) \leq \mathrm{U}(\bar{a}/C) \oplus \mathrm{U}(\bar{b}/C)$ (by the Lascar inequalities) and

$$\bar{a} \mathop{\smile\hspace{-0.9em}\raise0.1em\vert}_{C} \bar{b} \implies \mathrm{U}(\bar{a}\bar{b}/C) = \mathrm{U}(\bar{a}/C) \oplus \mathrm{U}(\bar{b}/C)$$

(see Theorem 19.5 in the textbook). But the reverse implication $\Leftarrow$ needn't hold, if I recall correctly.

Proposition 3.7. *If* $\bar{a}' \in \mathrm{dcl}(\bar{a}B)$ *or more generally if* $\bar{a}' \in \mathrm{acl}(\bar{a}B)$*, then* $\mathrm{U}(\bar{a}'/B) \leq \mathrm{U}(\bar{a}/B)$.

Proof. By the Lascar inequalities and Proposition 3.1,

$$\begin{aligned}\mathrm{U}(\bar{a}'/B) &= 0 + \mathrm{U}(\bar{a}'/B) \leq \mathrm{U}(\bar{a}/\bar{a}'B) + \mathrm{U}(\bar{a}'/B) \leq \mathrm{U}(\bar{a}\bar{a}'/B) \\ &= \mathrm{U}(\bar{a}'\bar{a}/B) \leq \mathrm{U}(\bar{a}'/\bar{a}B) \oplus \mathrm{U}(\bar{a}/B) = 0 \oplus \mathrm{U}(\bar{a}/B) = \mathrm{U}(\bar{a}/B).\end{aligned}$$ □

Proposition 3.7 is analogous to (May 5–7, Lemma 6.10), which says the same thing for Morley rank.

4 Lascar rank of sets

Continue to assume the theory is superstable.

Definition 4.1. Let X be an A-definable set. (As usual, A should be small.) Then $\mathrm{U}(X) = \sup_{\bar{b} \in X} \mathrm{U}(\bar{b}/A)$.

(When $X = \varnothing$, we define $\mathrm{U}(X)$ to be the special value $-\infty$.)

Lemma 4.2. *In Definition 4.1,* $\mathrm{U}(X)$ *depends only on* X*, not on* A*.*

Proof. Let $\mathrm{U}_A(X) = \sup_{\bar{b} \in X} \mathrm{U}(\bar{b}/A)$. Suppose X is A-definable and A'-definable for two different sets A, A'. We must show $\mathrm{U}_A(X) = \mathrm{U}_{A'}(X)$.

First consider the "comparable case" where $A \subseteq A'$. Proposition 2.4 shows

$$\mathrm{U}(\bar{b}/A) \geq \mathrm{U}(\bar{b}/A')$$

for any $\bar{b} \in X$, so certainly $\mathrm{U}_A(X) \geq \mathrm{U}_{A'}(X)$.

Claim. For any $\bar{b} \in X$, there is $\bar{b}' \in X$ with $\mathrm{U}(\bar{b}'/A') = \mathrm{U}(\bar{b}/A)$.

Proof. By the existence/extension property of forking, there is $\bar{b}' \equiv_A \bar{b}$ such that $\bar{b}' \mathop{\smile\!\!\!\!\!\!\mid}_A A'$. Then $\bar{b}' \in X$ because $\bar{b} \in X$ and X is A-definable. And

$$\mathrm{U}(\bar{b}'/A') = \mathrm{U}(\bar{b}'/A) = \mathrm{U}(\bar{b}/A).$$

The first equality is by Proposition 2.4 and the second holds because $\mathrm{tp}(\bar{b}'/A) = \mathrm{tp}(\bar{b}/A)$. $\square_{\mathrm{Claim}}$

The claim then shows $\mathrm{U}_A(X) \leq \mathrm{U}_{A'}(X)$. This completes the proof in the "comparable case" $A \subseteq A'$.

The general case then follows by two applications of the comparable case:

$$\mathrm{U}_A(X) = \mathrm{U}_{AA'}(X) = \mathrm{U}_{A'}(X). \qquad \square$$

Remark 4.3. Definition 4.1 is similar to (May 5–7, Proposition 6.9(2)), which showed that if X is A-definable, then

$$\mathrm{RM}(X) = \max_{\bar{b} \in X} \mathrm{RM}(\bar{b}/A).$$

One "bad" feature of Lascar rank, compared to Morley rank, is that in Definition 4.1 the maximum is not always attained. Another defect is that $\mathrm{U}(\bar{a}/B)$ is not the minimum of $\mathrm{U}(X)$ as X ranges over B-definable sets containing $\bar{a}$ (compare with May 5–7, Proposition 6.9(1)).

Proposition 4.4. *If T is totally transcendental, and X is a definable set, then* $\mathrm{RM}(X) \geq \mathrm{U}(X)$.

Proof. Take a small set A defining X. Then $\mathrm{RM}(X) = \sup_{\bar{b} \in X} \mathrm{RM}(\bar{b}/A) \geq \sup_{\bar{b} \in X} \mathrm{U}(\bar{b}/A) = \mathrm{U}(X)$ by Proposition 2.6. $\square$

$\mathrm{U}(X)$ can be thought of as the "dimension" of X. Here are some of its nice properties.

Theorem 4.5. *If X is definable, then $\mathrm{U}(X) > 0$ iff X is infinite.*

Proof. Take a small set A defining X. By Proposition 3.1, $\mathrm{U}(X) > 0$ iff there is $\bar{b} \in X$ with $\bar{b} \notin \mathrm{acl}(A)$. If X is finite, then $X \subseteq \mathrm{acl}(A)$ by definition of $\mathrm{acl}(A)$, so $\mathrm{U}(X) \leq 0$. If X is infinite, then X is large (by saturation) so $X \not\subseteq \mathrm{acl}(A)$ and there is $\bar{b} \in X \setminus \mathrm{acl}(A)$, showing $\mathrm{U}(X) > 0$. $\square$

Theorem 4.6. *Let X, Y be definable.*

1. *If $X \subseteq Y$, then $\mathrm{U}(X) \leq \mathrm{U}(Y)$.*

2. *$\mathrm{U}(X \cup Y) = \max(\mathrm{U}(X), \mathrm{U}(Y))$.*

Proof. Take A defining X and Y, and use it to calculate $\mathrm{U}(X), \mathrm{U}(Y), \mathrm{U}(X \cup Y)$. Then everything is obvious from Definition 4.1. $\square$

Theorem 4.7. *Let $f : X \to Y$ be a definable function.*

1. *If f is surjective, then $\mathrm{U}(X) \geq \mathrm{U}(Y)$.*

2. *If f is a bijection, then $\mathrm{U}(X) = \mathrm{U}(Y)$.*

3. *If f is an injection, then $\mathrm{U}(X) \leq \mathrm{U}(Y)$.*

Proof. 1. Take a small set C defining f, X, Y.

Claim. If $\bar{b} \in Y$, there is $\bar{a} \in X$ with $\mathrm{U}(\bar{a}/C) \geq \mathrm{U}(\bar{b}/C)$.

Proof. Take $\bar{a} \in X$ with $f(\bar{a}) = \bar{b}$. Then $\bar{b} \in \mathrm{dcl}(C\bar{a})$, so $\mathrm{U}(\bar{b}/C) \leq \mathrm{U}(\bar{a}/C)$ by Proposition 3.7. $\square_{\text{Claim}}$

The claim then implies $\mathrm{U}(Y) = \sup_{\bar{b} \in Y} \mathrm{U}(\bar{b}/C) \leq \sup_{\bar{a} \in X} \mathrm{U}(\bar{a}/C) = \mathrm{U}(X)$.

Then the proofs of (2) and (3) are like the proofs of (May 5–7, Proposition 6.4, parts (2) and (3)). $\square$

Theorem 4.8. *If $\mathrm{U}(X), \mathrm{U}(Y) < \omega$, then $\mathrm{U}(X \times Y) = \mathrm{U}(X) + \mathrm{U}(Y)$.*

Proof. Take C defining X and Y. If $(\bar{a}, \bar{b}) \in X \times Y$, then

$$\mathrm{U}(\bar{a}, \bar{b}/C) \leq \mathrm{U}(\bar{a}/C) + \mathrm{U}(\bar{b}/C) \leq \mathrm{U}(X) + \mathrm{U}(Y)$$

by Proposition 3.5. This implies $\mathrm{U}(X \times Y) \leq \mathrm{U}(X) + \mathrm{U}(Y)$.

Because $\mathrm{U}(X) < \omega$, $\max_{\bar{a} \in X} \mathrm{U}(\bar{a}/C)$ exists. Take $\bar{a} \in X$ with $\mathrm{U}(\bar{a}/C) = \mathrm{U}(X)$. Similarly, take $\bar{b} \in Y$ with $\mathrm{U}(\bar{b}/C) = \mathrm{U}(Y)$. Replacing $\bar{a}$ with $\bar{a}' \equiv_C \bar{a}$, we may assume $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$ by the existence/extension property. Then

$$\mathrm{U}(X \times Y) \geq \mathrm{U}(\bar{a}\bar{b}/C) = \mathrm{U}(\bar{a}/C) + \mathrm{U}(\bar{b}/C) = \mathrm{U}(X) + \mathrm{U}(Y)$$

by Proposition 3.5. $\square$

Theorem 4.9. *Let $f : X \to Y$ be a definable surjection. Suppose for every $\bar{b} \in Y$, the definable set $f^{-1}(\bar{b})$ has Lascar rank k. Suppose $k, \mathrm{U}(Y) < \omega$. Then $\mathrm{U}(X) = k + \mathrm{U}(Y)$.*

Proof. Take small C defining f, X, Y. Take $\bar{b} \in Y$ with $\mathrm{U}(\bar{b}/C) = \mathrm{U}(Y)$. The set $f^{-1}(\bar{b})$ is $C\bar{b}$-definable and has rank k, so there is $\bar{a} \in f^{-1}(\bar{b})$ with $\mathrm{U}(\bar{a}/C\bar{b}) = k$. Then $\mathrm{U}(\bar{a}, \bar{b}/C) = k + \mathrm{U}(Y)$ by the Lascar inequalities. Also, $\bar{b} = f(\bar{a})$, so $(\bar{a}, \bar{b}) \in \mathrm{dcl}(C\bar{a})$, and therefore $\mathrm{U}(X) \geq \mathrm{U}(\bar{a}/C) \geq \mathrm{U}(\bar{a}, \bar{b}/C) = k + \mathrm{U}(Y)$ by Proposition 3.7.

Conversely, we claim $\mathrm{U}(X) \leq k + \mathrm{U}(Y)$. Take $\bar{a} \in X$. It suffices to show $\mathrm{U}(\bar{a}/C) \leq k + \mathrm{U}(Y)$. Let $\bar{b} = f(\bar{a})$. Then $\bar{a}$ is in the $C\bar{b}$-definable set $f^{-1}(\bar{b})$ of rank k, so $\mathrm{U}(\bar{a}/C\bar{b}) \leq k$. Then

$$\mathrm{U}(\bar{a}/C) \leq \mathrm{U}(\bar{a}\bar{b}/C) = \mathrm{U}(\bar{a}/C\bar{b}) + \mathrm{U}(\bar{b}/C) \leq k + \mathrm{U}(Y). \qquad \square$$

Example 4.10.

1. If X and Y are definable of finite rank, applying Theorem 4.9 to the projection $X \times Y \to Y$ recovers Theorem 4.8. Each fiber $X \times \{\bar{a}\}$ has rank $k := \mathrm{U}(X)$, so $\mathrm{U}(X \times Y) = k + \mathrm{U}(Y)$.

2. Let X be a definable set of finite rank and let E be a definable equivalence relation. Suppose every equivalence class has rank k. The set $Y = X/E$ is definable in M^{eq} (which is also superstable). Applying Theorem 4.9 to $X \to X/E$, we see that $\mathrm{U}(Y) = \mathrm{U}(X) - k$.

3. If you know group theory, here is an instance of the previous point. Let G be a definable group of finite Lascar rank. Let H be a definable normal sugroup. Then the quotient group G/H has rank $\mathrm{U}(G/H) = \mathrm{U}(G) - \mathrm{U}(H)$. (The equivalence classes are the cosets of H, which are in definable bijection with H and all have the same rank as H.)

5 Lascar and Morley rank in strongly minimal theories

Suppose T is strongly minimal. Then T is totally transcendental (May 5–7, Example 7.5) and therefore superstable. The next two propositions give a concrete way of calculating $\mathrm{U}(\bar{a}/B)$:

Proposition 5.1. *If $a \in \mathbb{M}^1$ and $B \subseteq \mathbb{M}$, then*

$$\mathrm{U}(a/B) = \begin{cases} 0 & \text{if } a \in \mathrm{acl}(B) \\ 1 & \text{if } a \notin \mathrm{acl}(B). \end{cases}$$

Proof. By Proposition 2.6, $\mathrm{U}(a/B) \leq \mathrm{RM}(a/B)$. But $\mathrm{RM}(a/B) \leq \max_{a \in \mathbb{M}} \mathrm{RM}(a/B) = \mathrm{RM}(\mathbb{M}^1) = 1$ by (May 5–7, Example 6.3(4) and Proposition 6.9(2)). Thus $\mathrm{U}(a/B)$ is 0 or 1. Now use Proposition 3.1. $\square$

Proposition 5.2. *If $\bar{a} \in \mathbb{M}^n$ and $B \subseteq \mathbb{M}$, then*

$$\mathrm{U}(\bar{a}/B) = \sum_{i=1}^{n} \mathrm{U}(a_i/Ba_1 \cdots a_{i-1}).$$

Proof. By induction and the Lascar inequalities. $\square$

Let p be the global transcendental type. Recall that

$$a \models p \restriction B \iff a \notin \operatorname{acl}(B).$$

Let $p^{\otimes n} = \underbrace{p \otimes \cdots \otimes p}_{n \text{ times}}$. We say that $(a_1, \ldots, a_n)$ is *independent over* B if the following equivalent conditions hold:

1. $\bar{a} \models p^{\otimes n} \restriction B$
2. $a_i \notin \operatorname{acl}(Ba_1 \cdots a_{i-1})$ for each i.
3. $(a_1, \ldots, a_n)$ is a sequence of realizations of $p \restriction B$ that is independent over B.

The equivalence is basically by (April 21–28, Remark 14.3). Note that whether $\bar{a}$ is independent over B doesn't change if we permute the coordinates of $\bar{a}$.

Lemma 5.3. *If $\bar{a} \in \mathbb{M}^n$ is independent over B, then*

$$\mathrm{RM}(\bar{a}/B) = \mathrm{U}(\bar{a}/B) = n.$$

Proof. $\mathrm{RM}(\bar{a}/B) = n$ by (May 5–7, Theorem 6.11(3)), as $\operatorname{tp}(\bar{a}/B) = p^{\otimes n} \restriction B$. Meanwhile, $a_i \notin \operatorname{acl}(Ba_1 \cdots a_{i-1})$ implies $\mathrm{U}(a_i/Ba_1 \cdots a_{i-1}) = 1$ by Proposition 5.1. Then Proposition 5.2 gives

$$\mathrm{U}(\bar{a}/B) = \sum_{i=1}^{n} \mathrm{U}(a_i/Ba_1 \cdots a_{i-1}) = \sum_{i=1}^{n} 1 = n. \qquad \square$$

A *subtuple* of $(a_1, \ldots, a_n)$ is a tuple of the form $(a_{i_1}, \ldots, a_{i_k})$ forsome $0 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$. Here is another way to think about $\mathrm{U}(\bar{a}/B)$:

Proposition 5.4. *Suppose $\bar{a} \in \mathbb{M}^n$ and $B \subseteq \mathbb{M}$. Let $\bar{c}$ be a maximal subtuple of $\bar{a}$ that is independent over B. If k is the length of $\bar{c}$, then*

$$\mathrm{U}(\bar{a}/B) = \mathrm{RM}(\bar{a}/B) = k.$$

In particular, Morley rank and Lascar rank agree for complete types, and both are finite.

Proof. Rearranging the coordinates, we may assume $\bar{c} = (a_1, \ldots, a_k)$. If $a_{k+1} \notin \operatorname{acl}(Ba_1 \cdots a_k)$, then $(a_1, \ldots, a_{k+1})$ is a longer independent subtuple, a contradiction. Therefore, $a_{k+1} \in \operatorname{acl}(Ba_1 \cdots a_k)$. Similarly, $a_\ell \in \operatorname{acl}(Ba_1 \cdots a_k)$ for all $k < \ell \le n$. Therefore $\bar{a} \in \operatorname{acl}(B\bar{c})$. Also $\bar{c} \in \operatorname{dcl}(\bar{a}) \subseteq \operatorname{acl}(B\bar{a})$. By Proposition 3.7,

$$\mathrm{U}(\bar{a}/B) = \mathrm{U}(\bar{c}/B).$$

Similarly, $\mathrm{RM}(\bar{a}/B) = \mathrm{RM}(\bar{c}/B)$ by (May 5–7, Lemma 6.10). But

$$\mathrm{U}(\bar{c}/B) = \mathrm{RM}(\bar{c}/B) = k$$

by Lemma 5.3. $\square$

Theorem 5.5. *If $X \subseteq \mathbb{M}^n$ is definable, then* $\mathrm{U}(X) = \mathrm{RM}(X) \leq n$.

In particular, Lascar rank and Morley rank agree for definable sets, and both are finite.

Proof. If X is B-definable, then

$$\mathrm{U}(X) = \max_{\bar{a} \in X} \mathrm{U}(\bar{a}/B)$$
$$\mathrm{RM}(X) = \max_{\bar{a} \in X} \mathrm{RM}(\bar{a}/B)$$

by Definition 4.1 and (May 5–7, Proposition 6.9(2)). By Proposition 5.4 the right hand sides are equal, and are at most n. □

By the finiteness and equality of the two ranks, we see that Morley rank has the nice properties of Lascar rank. For example,

$$\mathrm{RM}(\bar{a}\bar{b}/C) = \mathrm{RM}(\bar{a}/\bar{b}C) + \mathrm{RM}(\bar{b}/C)$$
$$\mathrm{RM}(X \times Y) = \mathrm{RM}(X) + \mathrm{RM}(Y).$$

Lemma 5.6. *Let $\varphi(x_1, \ldots, x_n)$ be an $L(\mathbb{M})$-formula and let $X = \varphi(\mathbb{M}^n)$. Then* $\mathrm{U}(X) = n$ *iff $\varphi(\bar{x}) \in p^{\otimes n}$.*

Proof. $\mathrm{U}(X) = \mathrm{RM}(X) = \mathrm{RM}(\varphi(\bar{x}))$ which is defined to be

$$\max\{\mathrm{RM}(q) : q \in S_n(\mathbb{M}),\ \varphi(\bar{x}) \in q(\bar{x})\}.$$

By (May 5–7, Theorem 6.11), $\mathrm{RM}(q) = n \iff q = p^{\otimes n}$, and $\mathrm{RM}(q) < n$ otherwise. So $\mathrm{RM}(\varphi(\bar{x})) \geq n$ iff $\varphi(\bar{x}) \in p^{\otimes n}$. □

Lemma 5.7. *Suppose $X \subseteq \mathbb{M}^n$ is definable and $0 \leq k \leq n$. Then the following are equivalent:*

1. *There is a coordinate projection*[2], *$\pi : \mathbb{M}^n \to \mathbb{M}^k$ such that* $\mathrm{RM}(\pi(X)) = k$.
2. $\mathrm{RM}(X) \geq k$.

Proof. (1) $\implies$ (2): the map $X \to \pi(X)$ is a surjection; use Theorem 4.7(1).

(2) $\implies$ (1): Take B defining X. Let $d = \mathrm{U}(X) \geq k$. Take $\bar{a} \in X$ with $\mathrm{U}(\bar{a}/B) = d$. By Proposition 5.4 there is a subtuple $\bar{c}_0$ of $\bar{a}$ such that $\bar{c}_0$ is independent over B and $\bar{c}_0 \in \mathbb{M}^d$. Let $\bar{c}$ be a subtuple of $\bar{c}_0$ of length $k \leq d$. Then $\bar{c}$ is also independent over B. We can write $\bar{c}$ as $\pi(\bar{a})$ for some coordinate projection $\pi : \mathbb{M}^n \to \mathbb{M}^k$. Then $\bar{c} = \pi(\bar{a})$ is in the B-definable set $\pi(X)$, so

$$\mathrm{U}(\pi(X)) \geq \mathrm{U}(\bar{c}/B) = k.$$

But $\pi(X) \subseteq \mathbb{M}^k$, so $\mathrm{U}(\pi(X)) \leq k$, and therefore $\mathrm{U}(\pi(X)) = k$ as desired. □

[2] When $n = 3$ and $k = 2$ for example, this means that π is one of the following maps: $\pi(x, y, z) = (x, y)$, or $\pi(x, y, z) = (x, z)$, or $\pi(x, y, z) = (y, z)$. In general there are $\binom{n}{k}$ different coordinate projections $\mathbb{M}^n \to \mathbb{M}^k$.

Theorem 5.8 (Definability of Morley/Lascar rank)**.** *Let $\varphi(x_1, \ldots, x_n; \bar{y})$ be a formula. Each of the sets*

$$D_k = \{\bar{b} \in \mathbb{M} : \varphi(\mathbb{M}^n; \bar{b}) \text{ has rank } k\}$$

is definable.

Proof. Let D'_k be the set of $\bar{b} \in \mathbb{M}$ such that $\mathrm{U}(\varphi(\mathbb{M}^n; \bar{b})) \geq k$. Then $D_k = D'_k \setminus D'_{k-1}$, so it suffices to show each D'_k is definable. Fix k. Let $N = \binom{n}{k}$, and let $\pi_1, \ldots, \pi_N$ be the distinct coordinate projections $\mathbb{M}^n \to \mathbb{M}^k$. Let $D'_{k,i}$ be the set of $\bar{b}$ such that if $X = \varphi(\mathbb{M}^n; \bar{b})$, then $\pi_i(X)$ has rank k. By Lemma 5.7, $D'_k = \bigcup_{i=1}^N D'_{k,i}$, so it suffices to show that $D'_{k,i}$ is definable. Fix i. If $X = \varphi(\mathbb{M}^n; \bar{b})$, then $\pi_i(X) = \psi(\mathbb{M}^k; \bar{b})$, where

$$\psi(z_1, \ldots, z_k; \bar{y}) = (\exists x_1, \ldots, x_n \ (\varphi(\bar{x}, \bar{y}) \ \wedge \ \bar{z} = \pi_i(\bar{x}))).$$

By Lemma 5.6,

$$\mathrm{U}(\pi_i(X)) = k \iff \psi(\bar{z}; \bar{b}) \in p^{\otimes k}(\bar{z}).$$

That is,

$$D'_{k,i} = \{\bar{b} \in \mathbb{M} : \psi(\bar{z}; \bar{b}) \in p^{\otimes k}(\bar{z}).$$

This set is definable because $p^{\otimes k}$ is a definable type, by stability. □

The significance of Theorem 5.8 is that if $X_{\bar{b}}$ is a definable set depending definably on some parameter $\bar{b}$, then we can express things like "$\mathrm{RM}(X_{\bar{b}}) = k$" by some formula $\psi(\bar{b})$.

Next, we want to extend all the results of this section to uncountably categorical theories. We first need a couple tools.

6 Type-definable and $\vee$-definable sets

From now on, we omit tuple bars. Things like a, b, c, x, y, z can denote tuples of elements or variables. Usually the tuples are finite.

Let A be a small set.

Definition 6.1. A set $X \subseteq \mathbb{M}^n$ is *type-definable* over A if X is an (infinite) intersection of A-definable sets. X is $\vee$*-definable* over A if X is an (infinite) union of A-definable sets.

Note that X is type-definable over A iff X is defined by a partial type over A. X is $\vee$-definable over A iff $\mathbb{M}^n \setminus X$ is type-definable over A.

Lemma 6.2 ("Open criterion")**.** *$X \subseteq \mathbb{M}^n$ is $\vee$-definable over A iff the following property holds: for any $b \in X$, there is an A-definable set N with $b \in N \subseteq X$.*

Proof. Clear, if you think about it for a bit. □

I'm calling Lemma 6.2 the "open criterion" since it looks like the definition of open sets in metric spaces. In fact, $\vee$-definable sets over A correspond to open sets in $S_n(A)$.

Proposition 6.3. *Let D be definable and X be a subset. If X and $D \setminus X$ are $\vee$-definable, then X is definable.*

Proof. Note that X and $D \setminus X$ are type-definable. If $\Sigma(x)$ and $\Gamma(x)$ define them, then $\Sigma(x) \cup \Gamma(x)$ is inconsistent. Therefore there is a finite subtype $\Sigma_0(x)$ of X such that $\Sigma_0(x) \cup \Gamma(x)$ is inconsistent. Then for any $a \in D$,

$$a \in X \implies a \models \Gamma \implies a \models \Gamma_0 \implies a \not\models \Sigma \implies a \notin D \setminus X \implies a \in X$$

so all those things are equivalent, and in particular Γ_0 defines X. Sets defined by finite types are definable. $\square$

7 "α-isolation"

Suppose T is totally transcendental. Say that $\varphi(x) \in L(B)$ "α-isolates" $p \in S_n(B)$ if

$$\{p\} = \{q \in S_n(B) : \mathrm{RM}(q) \geq \alpha\}.$$

For example, φ 0-isolates p if φ isolates p.

Remark 7.1. If M is an $\aleph_0$-saturated model and $\varphi(x)$ α-isolates p, then $\mathrm{RM}(p) = \alpha$, because Morley rank agrees with Cantor-Bendixson rank over M (May 5–7, Lemma 6.2), and p is isolated in $E_\alpha = \{p \in S_n(M) : \mathrm{RM}(p) \geq \alpha\}$, so that $p \notin E_{\alpha+1} = E'_\alpha$.

Lemma 7.2. *If $\alpha = \mathrm{RM}(a/B)$, then $\mathrm{tp}(a/B)$ is α-isolated by some $\varphi(x) \in \mathrm{tp}(a/B)$.*

Proof. By (May 5–7, Proposition 6.9(1)) there is $\varphi_0 \in \mathrm{tp}(a/B)$ such that $\mathrm{RM}(\varphi_0(x)) = \alpha$. Let $(p_i : i < \kappa)$ be all the types of Morley rank α in $S_n(B)$ extending $\varphi_0(x)$. Let q_i be a global non-forking extension of p_i. By (May 5–7, Proposition 8.2), $\mathrm{RM}(q_i) = \mathrm{RM}(p_i) = \alpha$. If $\kappa \geq \aleph_0$, then $\mathrm{RM}(\varphi_0) \geq \alpha + 1$, a contradiction. (If there are infinitely many points in $[\varphi_0] \subseteq S_n(\mathbb{M})$ with Cantor-Bendixson rank at least α, then there is at least one of rank $\alpha+1$, by (May 5–7, Proposition 3.13).) Therefore κ is finite. Without loss of generality, $p = p_0$. For $i = 1, 2, \ldots, \kappa - 1$ let $\varphi_i(x)$ be a formula in p but not p_i. Take $\varphi(x) = \bigwedge_{i=1}^{\kappa} \varphi_i(x)$. $\square$

8 Preliminaries

Assume T is uncountably categorical in a countable language. By (May 12, Theorem 5.3), T is totally transcendental, and has no Vaught pairs. By (May 12, Lemma 2.5), $\exists^\infty$ is eliminated. By (May 12, Lemma 2.7), there is a strongly minimal set D defined over the prime model. *For simplicity, we assume D is $\varnothing$-definable.* Otherwise, we need to carry around the parameters defining D everywhere. Note that

$$1 \leq \mathrm{U}(D) \leq \mathrm{RM}(D) = 1$$

by Proposition 4.4, Theorem 4.5 and the fact that D is strongly minimal. If $b \in D$ and $C \subseteq \mathbb{M}$, then

$$\mathrm{U}(b/C) = \begin{cases} 0 & b \in \operatorname{acl}(C) \\ 1 & b \notin \operatorname{acl}(C). \end{cases}$$

as in Proposition 5.1.

Lemma 8.1. *Suppose M is a small model and $a \notin M$. Then there is $b \in D$ such that $b \in \operatorname{acl}(Ma) \setminus \operatorname{acl}(M)$.*

Proof. Let $M[a]$ be a prime model over $M \cup \{a\}$. By no Vaught pairs, $D(M[a]) \supsetneq D(M)$. Take $b \in D(M[a]) \setminus D(M)$. Then $b \notin M = \operatorname{acl}(M)$. By (May 12, Proposition 3.5), $M[a]$ is atomic over $M \cup \{a\}$, and so $\operatorname{tp}(b/Ma)$ is isolated by some formula $\varphi(y)$. If $b' \models \operatorname{tp}(b/Ma)$, then $b' \in D$ and $b' \notin M$, so $\varphi(\mathbb{M}) \subseteq D$ and $\varphi(\mathbb{M}) \cap D(M) = \varnothing$. As D is strongly minimal, $\varphi(\mathbb{M})$ is finite or cofinite in D. Since $\varphi(\mathbb{M})$ doesn't intersect the infinite set $D(M)$, $\varphi(\mathbb{M})$ must be finite, and then $b \in \operatorname{acl}(Ma)$. $\square$

Corollary 8.2. $\mathrm{U}(a/C) < \omega$ *for any C.*

Proof. Suppose $\mathrm{U}(a/C) \geq \omega$. By Remark 2.7 we may assume $\mathrm{U}(a/C) = \omega$. By Lemma 8.1 there is $b \in D$ with $b \in \operatorname{acl}(Ma) \setminus \operatorname{acl}(M)$. Then $\mathrm{U}(b/Ma) = 0 < \mathrm{U}(b/M)$, so $b \not\perp\!\!\!\perp_M a$. Then $a \not\perp\!\!\!\perp_M b$, so $\mathrm{U}(a/Mb) < \mathrm{U}(a/M) = \omega$. Then $\mathrm{U}(a/Mb)$ is a finite number $n < \omega$, and the Lascar inequalities give

$$\omega = \mathrm{U}(a/M) \leq \mathrm{U}(ab/M) \leq \mathrm{U}(a/bM) \oplus \mathrm{U}(b/M) = n + 1 < \omega. \qquad \square$$

Warning 8.3. We don't yet know that $\mathrm{U}(X) < \omega$ for definable sets X. (Perhaps $\{\mathrm{U}(a/C) : a \in X\} = \{0, 1, 2, 3, \ldots\}$.)

Lemma 8.4. *In Lemma 8.1,* $\mathrm{U}(a/Mb) = \mathrm{U}(a/M) - 1$. *(This makes sense because* $\mathrm{U}(a/M)$ *and* $\mathrm{U}(a/Mb)$ *are finite.)*

Proof. As $b \in \operatorname{acl}(Ma)$ and $b \notin \operatorname{acl}(M)$,

$$\begin{aligned} \mathrm{U}(ab/M) &= \mathrm{U}(b/Ma) + \mathrm{U}(a/M) = 0 + \mathrm{U}(a/M) = \mathrm{U}(a/M) \\ \mathrm{U}(ab/M) &= \mathrm{U}(a/Mb) + \mathrm{U}(b/M) = \mathrm{U}(a/Mb) + 1. \end{aligned} \qquad \square$$

9 The inductive step

Continue the assumptions of the previous section.

We prove the following three lemmas together, by induction on $k < \omega$.

Inductive Lemma 1. $\mathrm{RM}(a/B) = k \iff \mathrm{U}(a/B) = k$.

Inductive Lemma 2. *If X is definable,* $\mathrm{RM}(X) = k \iff \mathrm{U}(X) = k$.

Inductive Lemma 3. *If $\varphi(x;y)$ is a formula, then*

$$\{b \in \mathbb{M} : \mathrm{U}(\varphi(\mathbb{M};b)) = k\}$$

is $\vee$-definable over $\varnothing$.

First suppose $k = 0$. Inductive Lemma 1 holds because

$$\mathrm{U}(a/B) = 0 \iff a \in \mathrm{acl}(B) \iff \mathrm{RM}(a/B) = 0.$$

Inductive Lemma 2 holds because

$$\mathrm{U}(X) = 0 \iff |X| < \infty \iff \mathrm{RM}(X) = 0.$$

And Inductive Lemma 3 holds because

$$\{b \in \mathbb{M} : |\varphi(\mathbb{M};b)| < \infty\} = \bigcup_{j=0}^{\infty} \{b \in \mathbb{M} : |\varphi(\mathbb{M};b)| = j\}$$

and the sets in the union are $\varnothing$-definable.

Next suppose $k > 0$. k will be fixed for the rest of the section. By induction, we assume that Inductive Lemmas 1–3 hold for all smaller values of k. In particular, for $j < k$, $\mathrm{U} = j \iff \mathrm{RM} = j$, and we just say "rank $= j$".

Definition 9.1. An L-formula $\varphi(x;y)$ is "j-good" if for every $b \in \mathbb{M}$,

$$\varphi(\mathbb{M};b) = \varnothing \text{ or } \mathrm{U}(\varphi(\mathbb{M};b)) = j.$$

Lemma 9.2. *Suppose $j < k$, and $\varphi(\mathbb{M};b)$ has rank j (where φ is an L-formula and $b \in \mathbb{M}$). Then there is a j-good L-formula $\varphi'(x;y)$ such that*

$$\varphi'(x;y) \vdash \varphi(x;y)$$
$$\varphi'(\mathbb{M};b) = \varphi(\mathbb{M};b).$$

Proof. Let $D_j = \{c : \mathrm{U}(\varphi(\mathbb{M};c)) = j\}$. Then $b \in D_j$. By Inductive Lemma 3, D_j is $\vee$-definable over 0. By the "open criterion", there is a 0-definable set $\psi(\mathbb{M})$ with $b \in \psi(\mathbb{M}) \subseteq D_j$. Take $\varphi'(x;y) = \varphi(x,y) \wedge \psi(y)$. For any c, one of the following happens:

- $\mathbb{M} \models \psi(c)$, and then $\varphi'(\mathbb{M};c) = \varphi(\mathbb{M};c)$ has rank j as $c \in \psi(\mathbb{M}) \subseteq D_j$.
- $\mathbb{M} \models \neg\psi(c)$, and then $\varphi'(\mathbb{M};c) = \varnothing$.

Thus φ' is j-good. As $\mathbb{M} \models \psi(b)$, we also get $\varphi'(\mathbb{M};b) = \varphi(\mathbb{M};b)$. $\square$

Lemma 9.3. *Suppose M is a model, X is M-definable, $a \in X$, and $\mathrm{U}(a/M) = k$. Then there is $\varphi(x;c) \in \mathrm{tp}(a/M)$ such that*

- *$\varphi(x;y)$ is k-good*

- $\varphi(x; c)$ *k-isolates* $\mathrm{tp}(a/M)$
- $\varphi(\mathbb{M}; c) \subseteq X$.

The proof is complex, but I don't know a good way to simplify it.

Proof. By Lemma 8.1 and 8.4, there is $b \in D$ with $b \in \mathrm{acl}(Ma) \setminus \mathrm{acl}(M)$ and $\mathrm{U}(a/Mb) = k-1$. By Inductive Lemma 1, $\mathrm{RM}(a/Mb) = k - 1$. If $\psi(x, y, c) \in \mathrm{tp}(a, b/M)$ is strong enough, then

1. $\mathrm{tp}(a/Mb)$ is $(k-1)$-isolated by $\psi(x, b, c)$, by Lemma 7.2.
2. $\psi(a, \mathbb{M}, c)$ is finite, because $b \in \mathrm{acl}(Ma)$.

Strengthening $\psi(x, y, z)$ further, we can ensure

3. $\psi(a', \mathbb{M}, c')$ is finite for any a', c'.
4. $\psi(x, y, z)$ implies $x \in X$ and $y \in D$.

As $\mathrm{tp}(a/Mb)$ is $(k-1)$-isolated by $\psi(x, b, c)$, we have

$$\mathrm{RM}(\psi(x, b, c)) = \mathrm{RM}(a/Mb) = k - 1,$$

so $\mathrm{U}(\psi(x, b, c)) = k - 1$ by Inductive Lemma 2. By Lemma 9.2, we can further strengthen $\psi(x, y, z)$ and ensure

5. $\psi(x; y, z)$ is $(k-1)$-good: $\psi(\mathbb{M}; b', c')$ has Lascar/Morley rank $k - 1$ or is empty, for any b', c'.

Let $\varphi(x, z) = (\exists y)\psi(x, y, z)$ and $\delta(y, z) = (\exists x)\psi(x, y, z)$. Geometrically, $\varphi(\mathbb{M}, c')$ and $\delta(\mathbb{M}, c')$ are the projections of $\psi(\mathbb{M}, c') \subseteq X \times D$ onto X and D. Note that $\varphi(a, c)$ and $\delta(b, c)$ hold. As $b \notin \mathrm{acl}(M)$, the M-definable set $\delta(\mathbb{M}, c)$ must be infinite. Replacing $\psi(x, y, z)$ with

$$\psi(x, y, z) \wedge (\exists^\infty w)\delta(w, z),$$

we may assume

6. For any c', $\delta(\mathbb{M}, c')$ is infinite or empty.

Now we check that φ satisfies the three properties:

k-goodness: Fix c' such that $\varphi(\mathbb{M}, c') \subseteq X \times D$ is non-empty. By (6), $\delta(\mathbb{M}, c') \subseteq D$ is infinite, so $\mathrm{U}(\delta(\mathbb{M}, c')) = 1$. By (5), the fibers of the projection $\psi(\mathbb{M}, c') \to \delta(\mathbb{M}, c')$ have Lascar rank $k - 1$. By Theorem 4.9, $\psi(\mathbb{M}, c')$ has Lascar rank k. By (3), the fibers of the projection $\psi(\mathbb{M}, c') \to \varphi(\mathbb{M}, c')$ are finite, so $\varphi(\mathbb{M}, c')$ has Lascar rank k by Theorem 4.9 again.

k-isolation: First of all $\mathrm{tp}(a/M)$ has Morley rank at least k by Proposition 2.6. Suppose a' satisfies $\varphi(x;c)$. We must show that either $\mathrm{RM}(a'/M) < k$ or $\mathrm{tp}(a'/M) = \mathrm{tp}(a/M)$. Take b' such that $\psi(a', b'; c)$ holds.

- If $b' \in M$, then $\mathrm{U}(a'/M) = \mathrm{U}(a'/Mb') \leq \mathrm{U}(\psi(x; b', c)) \leq k-1$ by (5). By Inductive Lemma 1, $\mathrm{RM}(a'/M) \leq k-1$ as desired.
- If $b' \notin M$, then b and b' are both in D and both transcendental over M, so $b \equiv_M b'$ as D is strongly minimal. Moving a', b' by $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$, we may assume $b' = b$. Then a' satisfies the formula $\varphi(x, b, c)$ that $(k-1)$-isolates $\mathrm{tp}(a/Mb)$. One of two things happens:
  - $\mathrm{tp}(a'/Mb) = \mathrm{tp}(a/Mb)$. Then $a' \equiv_M a$ as desired.
  - $\mathrm{RM}(a'/Mb) < k-1$. By Inductive Lemma 1, $\mathrm{U}(a'/Mb) < k-1$. By the Lascar inequalities, $\mathrm{U}(a'/M) \leq \mathrm{U}(a'/Mb) + \mathrm{U}(b/M) < (k-1)+1 = k$. Then Inductive Lemma 1 gives $\mathrm{RM}(a'/M) < k$ as desired.

$\varphi(\mathbb{M}; c)$ contained in X: true by (4). □

Lemma 9.4. *Suppose* $\mathrm{U}(a/B) = k$. *Then* $\mathrm{RM}(a/B) = k$.

Proof. Take an $\aleph_0$-saturated model $M \supseteq B$. Moving M by an automorphism, we may assume $M \mathop{\smile\!\!\!\!\!\!|}_B a$, or equivalently, $a \mathop{\smile\!\!\!\!\!\!|}_B M$. By (May 5–7, Proposition 8.2) and Proposition 2.4, $\mathrm{U}(a/M) = \mathrm{U}(a/B)$ and $\mathrm{RM}(a/M) = \mathrm{U}(a/B)$. So we need to show

$$\mathrm{U}(a/M) = k \overset{?}{\Longrightarrow} \mathrm{RM}(a/M) = k.$$

By Lemma 9.3, there is $\varphi(x;c)$ which k-isolates $\mathrm{tp}(a/M)$. By Remark 7.1, $\mathrm{tp}(a/M)$ has Morley rank k. □

Then Inductive Lemma 1 follows. We need to prove:

$$\mathrm{U}(a/B) = k \overset{?}{\Longleftrightarrow} \mathrm{RM}(a/B) = k.$$

The $\Rightarrow$ direction is Lemma 9.4. Conversely, suppose $\mathrm{RM}(a/B) = k$ but $\mathrm{U}(a/B) \neq k = \mathrm{RM}(a/B)$. By Proposition 2.6, $\mathrm{U}(a/B) < \mathrm{RM}(a/B) = k$. But then Inductive Lemma 1 implies $\mathrm{U}(a/B) = \mathrm{RM}(a/B)$, a contradiction.

Next, Inductive Lemma 2 follows. We need to prove

$$\mathrm{U}(X) = k \overset{?}{\Longleftrightarrow} \mathrm{RM}(X) = k.$$

Take a small set B defining X. Then

$$\mathrm{U}(X) = k \iff (\exists a \in X : \mathrm{U}(a/B) = k \text{ and } \forall a \in X : \mathrm{U}(a/B) \leq k)$$
$$\mathrm{RM}(X) = k \iff (\exists a \in X : \mathrm{RM}(a/B) = k \text{ and } \forall a \in X : \mathrm{RM}(a/B) \leq k).$$

The right-hand sides are equal by Inductive Lemma 1 (including the level k just proved).

Lemma 9.5. *If* $\mathrm{U}(X) = k$*, then* X *can be written as a finite union* $\bigcup_{i=1}^{n} \varphi_i(\mathbb{M}; b_i)$ *where* φ_i *is* k_i*-good and* $\varphi_i(\mathbb{M}; b_i)$ *is non-empty.*

Proof. Fix a small model M defining X. If $a \in X$, then $\mathrm{U}(a/M) = j \le k$, so by Lemma 9.3 (including at lower levels), there is a j-good formula $\phi(x; y)$ and parameter $c \in M$ with

$$a \in \phi(\mathbb{M}; c) \subseteq X.$$

By compactness, finitely many of these sets cover X. $\square$

Finally we prove Inductive Lemma 3. Fix $\varphi(x; y)$. Let

$$D_k = \{b : \mathrm{U}(\varphi(\mathbb{M}; b)) = k\}.$$

We want to show that D_k is $\vee$-definable over $\varnothing$. We use the "open criterion" Lemma 6.2. Suppose $b_0 \in D_k$. We want a $\varnothing$-definable set D with $b_0 \in D \subseteq D_k$. By Lemma 9.5,

$$\varphi(\mathbb{M}; b_0) = \bigcup_{i=1}^{n} \psi_i(\mathbb{M}; c_i),$$

where ψ_i is k_i-good and $\psi_i(\mathbb{M}; c_i)$ is non-empty. Note

$$k = \mathrm{U}(\varphi(\mathbb{M}; b_0)) = \max\{\mathrm{U}(\psi_i(\mathbb{M}; c_i)) : 1 \le i \le n\} = \max(k_1, \ldots, k_n).$$

Let D be the set of b such that there are $c'_1, \ldots, c'_n$ such that

$$\varphi(\mathbb{M}; b) = \bigcup_{i=1}^{n} \psi_i(\mathbb{M}; c'_i)$$

and the sets $\psi_i(\mathbb{M}; c'_i)$ are all non-empty. Then D is definable, and $b_0 \in D$. If $b \in D$, then

$$\mathrm{U}(\varphi(\mathbb{M}; b)) = \max\{\mathrm{U}(\psi_i(\mathbb{M}; c'_i)) : 1 \le i \le n\} = \max(k_1, \ldots, k_n) = k$$

because ψ_i is k_i-good. Thus $D \subseteq D_k$. By Lemma 6.2, D_k is $\vee$-definable over $\varnothing$.

10 The conclusion

Continue to assume T is uncountably categorical.

Proposition 10.1. $\mathrm{RM}(a/B) < \omega$ *for any* a, B*.*

Proof. Suppose $\mathrm{RM}(a/B) \ge \omega$. By Inductive Lemma 1, $\mathrm{U}(a/B) \ne k$ for all $k < \omega$. By process of elimination, $\mathrm{U}(a/B) \ge \omega$, contradicting Corollary 8.2. $\square$

Then Inductive Lemma 1 shows that $\mathrm{RM}(a/B) = \mathrm{U}(a/B)$ for any a, B.

Theorem 10.2. *If* X *is a definable set, then* $\mathrm{U}(X) = \mathrm{RM}(X) < \omega$*.*

Proof. Suppose X is B-definable. By (May 5–7, Proposition 6.9(2)) there is $a \in X$ with $\mathrm{RM}(a/B) = \mathrm{RM}(X)$. By Proposition 10.1, $\mathrm{RM}(a/B) < \omega$, and then $\mathrm{RM}(X) = \mathrm{U}(X)$ by Inductive Lemma 2. $\square$

Theorem 10.3. *For any k and $\varphi(x, y)$, the set*

$$D_k = \{b : \mathrm{U}(\varphi(\mathbb{M}, b)) = k\}$$

is definable.

Proof. It is $\vee$-definable by Inductive Lemma 3, and its complement is the $\vee$-definable union $\bigcup_{i \neq k} D_i$. Use Proposition 6.3. $\square$

In conclusion, in uncountably categorical theories, Morley rank is finite, is definable, and has all the nice additivity properties of Lascar rank, like

$$\mathrm{RM}(X \times Y) = \mathrm{RM}(X) + \mathrm{RM}(Y).$$