The dichotomy property and definability of types

Advanced Model Theory

March 10, 2022

In class we proved

> If some type over a model is non-definable, then some formula has the dichotomy property.

Or equivalently,

> If no formula has the dichotomy property, then all types over models are definable.

This was Proposition 8 in the March 3 notes. Our proof followed Theorem 11.11 in the textbook, using strong heirs p^U coming from ultrapower. This document discussed a completely different proof, which might be easier to understand, and which tells us something stronger:

> If no formula has the dichotomy property, then all types over *arbitrary sets* are definable.

In Section 1 we review notation. In Section 2 we review the dichotomy property. In Section 3 we discuss the notion of "φ-types" and state the main theorem (Theorem 10). In Section 4 we prove the main theorem, and in Section 5 we give some applications.

1 Notation

A "binary string" is a sequence of 0's and 1's. If α is an ordinal, then 2^α is the set of binary strings of length α, and $2^{<\alpha} = \bigcup_\beta 2^\beta$. In these notes, we only care about $\alpha \leq \omega$. Then 2^ω is the set of binary strings of length ω, and $2^{<\omega}$ is the set of finite binary strings.

For example, some typical elements of $2^{<\omega}$ are

$$\text{(the empty string)},\ 010,\ 1,\ 11001101,$$

and a typical element of 2^ω is

$$0110100110010110100101100 1\cdots$$

If $\sigma \in 2^\alpha$ and $\beta \leq \alpha$, let $\sigma \restriction \beta$ be the initial segment of σ of length β. For example, $01001 \restriction 3 = 010$. If σ, τ are binary strings, then $\sigma \sqsubseteq \tau$ means that τ extends σ. Equivalently, $\sigma \sqsubseteq \tau$ if $\sigma = \tau \restriction \beta$, where β is the length of σ. For example, $0100 \sqsubseteq 01001$, but $1001 \not\sqsubseteq 01001$.

2 The dichotomy property

Fix a complete theory T and monster model $\mathbb{M}$. Fix a formula $\varphi(\bar{x};\bar{y})$.

Definition 1. "D_α is consistent" if there are $(\bar{a}_\sigma : \sigma \in 2^\alpha)$ and $(\bar{b}_\tau : \tau \in 2^{<\alpha})$ such that

$$\begin{aligned}\sigma \sqsupseteq \tau 0 &\implies \mathbb{M} \models \varphi(\bar{a}_\sigma, \bar{b}_\tau)\\ \sigma \sqsupseteq \tau 1 &\implies \mathbb{M} \models \neg\varphi(\bar{a}_\sigma, \bar{b}_\tau).\end{aligned}$$

For example, D_2 is consistent if there are $\bar{a}_{00}, \bar{a}_{01}, \bar{a}_{10}, \bar{a}_{11}, \bar{b}, \bar{b}_0, \bar{b}_1$ such that

$$\begin{aligned}\mathbb{M} \models &\varphi(\bar{a}_{00}, \bar{b}) \ \wedge\ \varphi(\bar{a}_{00}, \bar{b}_0)\\ &\wedge\ \varphi(\bar{a}_{01}, \bar{b}) \ \wedge\ \neg\varphi(\bar{a}_{01}, \bar{b}_0)\\ &\wedge\ \neg\varphi(\bar{a}_{10}, \bar{b}) \ \wedge\ \varphi(\bar{a}_{10}, \bar{b}_1)\\ &\wedge\ \neg\varphi(\bar{a}_{11}, \bar{b}) \ \wedge\ \neg\varphi(\bar{a}_{11}, \bar{b}_1).\end{aligned}$$

Remark 2. "D_α" is the name for the set of formulas $\{\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 0\} \cup \{\neg\varphi(\bar{x}_\sigma, \bar{y}_\tau) : \sigma \sqsupseteq \tau 1\}$, a partial type in the variables $(\bar{x}_\sigma : \sigma \in 2^\alpha)$ and $(\bar{y}_\tau : \tau \in 2^{<\alpha})$.

Definition 3. $\varphi(\bar{x};\bar{y})$ has the *dichotomy property* if D_ω is consistent.

Lemma 4. *If D_n is consistent for all $n < \omega$, then D_ω is consistent.*

Poizat thinks this is obvious.[1]

Proof. Let F be a finite fragment of D_ω. By compactness, it suffices to show that F is consistent. Take n bigger than the length of τ for any $\bar{y}_\tau$ appearing in F. Because D_n is consistent, there are $(\bar{a}^0_\sigma : \sigma \in 2^n)$ and $(\bar{b}^0_\tau : \tau \in 2^{<n})$ as in Definition 1. Define $(\bar{a}_\sigma : \sigma \in 2^\omega)$ and $(\bar{b}_\tau : \tau \in 2^{<\omega})$ as follows:

- If $\sigma \in 2^\omega$, then $\bar{a}_\sigma = \bar{a}^0_{\sigma \restriction n}$.
- If τ is such that $\bar{y}_\tau$ appears in F, then τ has length less than n, so we can take $\bar{b}_\tau = \bar{b}^0_\tau$.
- If τ is such that $\bar{y}_\tau$ doesn't appear in F, choose $\bar{b}_\tau$ to be anything.

Then the $(\bar{a}_\sigma : \sigma \in 2^\omega)$ and $(\bar{b}_\tau : \tau \in 2^{<\omega})$ satisfy F:

- If $\sigma \sqsupseteq \tau 0$ and $\varphi(\bar{x}_\sigma, \bar{y}_\tau) \in F$, then τ has length less than n, by choice of n. Then $(\sigma \restriction n) \supseteq \tau 0$ and so $\varphi(\bar{x}_{\sigma\restriction n}, \bar{y}_\tau) \in D_n$. Then $\varphi(\bar{a}^0_{\sigma\restriction n}, \bar{b}^0_\tau)$ holds. We chose $\bar{a}_\sigma = \bar{a}^0_{\sigma\restriction n}$ and $\bar{b}_\tau = \bar{b}^0_\tau$, so $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds.
- Similarly, if $\sigma \supseteq \tau 1$ and $\neg\varphi(\bar{x}_\sigma, \bar{y}_\tau) \in F$, then $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds. $\square$

[1] "Note that every finite fragment of D_α can be interpreted in D_n, for large enough n" (Poizat, page 235).

3 φ-types

Continue to fix $T, \mathbb{M}, \varphi(\bar{x}; \bar{y})$. Let n be the length of the variable tuple $\bar{x}$.

Definition 5. If $B \subseteq \mathbb{M}$ is a set and $\bar{a} \in \mathbb{M}^n$, then $\mathrm{tp}^\varphi(\bar{a}/B)$ is the partial type

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B,\ \mathbb{M} \models \varphi(\bar{a}, \bar{b})\} \ \cup\ \{\neg\varphi(\bar{x}; \bar{b}) : \bar{b} \in B,\ \mathbb{M} \models \neg\varphi(\bar{a}, \bar{b})\}.$$

In other words, $\mathrm{tp}^\varphi(\bar{a}/B)$ is the set of formulas in $\mathrm{tp}(\bar{a}/B)$ of the form φ or $\neg\varphi$.

Remark 6. Suppose $\bar{a}, \bar{b} \in \mathbb{M}^n$.

1. $\mathrm{tp}^\varphi(\bar{a}/C) = \mathrm{tp}^\varphi(\bar{b}/C)$ if and only if

   $$\forall \bar{c} \in C : \mathbb{M} \models (\varphi(\bar{a}, \bar{c}) \leftrightarrow \varphi(\bar{b}, \bar{c}))$$

   So $\bar{a}$ and $\bar{b}$ have the same φ-type over C iff they satisfy exactly the same formulas of the form $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$.

2. $\bar{b} \in \mathbb{M}^n$ realizes $\mathrm{tp}^\varphi(\bar{a}/C)$ if and only if $\mathrm{tp}^\varphi(\bar{a}/C) = \mathrm{tp}^\varphi(\bar{b}/C)$.

Warning 7. Some authors like Pillay in his book *Geometric Stability Theory* use "φ-type" to mean something slightly different. But the definition here is more common in contemporary research.

Definition 8. $S_\varphi(B)$ is the set $\{\mathrm{tp}^\varphi(\bar{a}/B) : \bar{a} \in \mathbb{M}^n\}$.

Definition 9. A φ-type $p \in S_\varphi(B)$ is *definable* if there is an $L(B)$-formula $\psi(\bar{y})$ such that

$$\text{For any } \bar{b} \in B,\ \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \mathbb{M} \models \psi(\bar{b}).$$

Theorem 10. *Suppose φ does not have the dichotomy property. Then every φ-type over any set is definable.*

4 Proof of Theorem 10

Continue to fix $T, \mathbb{M}, \varphi(\bar{x}; \bar{y})$.

Definition 11. Let $\Sigma(\bar{x})$ be a small set of $L(\mathbb{M})$-formulas. Define "$R_{\varphi,2}(\Sigma(\bar{x})) \geq n$" by recursion on n:

- $R_{\varphi,2}(\Sigma(\bar{x})) \geq 0$ iff $\Sigma(\bar{x})$ is consistent.
- $R_{\varphi,2}(\Sigma(\bar{x})) \geq n + 1$ iff there is $\bar{b} \in \mathbb{M}$ such that

  $$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n$$
  $$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) \geq n$$

Lemma 12. *$R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ if and only if there are $(\bar{a}_\sigma : \sigma \in 2^n)$ and $(\bar{b}_\tau : \tau \in 2^{<n})$ such that*

- *If σ extends $\tau 0$, then $\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds.*
- *If σ extends $\tau 1$, then $\neg\varphi(\bar{a}_\sigma, \bar{b}_\tau)$ holds.*
- *Each $\bar{a}_\sigma$ satisfies $\Sigma(\bar{x})$.*

Proof. By induction on n. (Try a few small values of n and you'll see how this works.) $\square$

Definition 13. $R_{\varphi,2}(\Sigma(\bar{x}))$ is the largest n such that $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$, or $-\infty$ if there is no such n (because $\Sigma(\bar{x})$ is inconsistent), or $+\infty$ if $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$ for all n.

$R_{\varphi,2}(\Sigma(\bar{x}))$ is called the "φ-2-rank" or "$R_{\varphi,2}$-rank" of Σ.

Remark 14 (Monotonicity)**.** If $\Sigma(\bar{x}) \vdash \Sigma'(\bar{x})$, then $R_{\varphi,2}(\Sigma(\bar{x})) \leq R_{\varphi,2}(\Sigma'(\bar{x}))$. This can either be proved by induction using Definition 11, or directly from the explicit description in Lemma 12.

We write $\{\bar{x} = \bar{x}\}$ for the partial type that is satisfies by all $\bar{a} \in \mathbb{M}$. (We could also write $\{\}$, but it is traditional to write $\{\bar{x} = \bar{x}\}$ instead.) Note $\Sigma(\bar{x}) \vdash \{\bar{x} = \bar{x}\}$ for any Σ.

Remark 15. From Lemma 12, we see that $R_{\varphi,2}(\{\bar{x} = \bar{x}\}) \geq n$ iff "D_n is consistent" in the sense of Definition 1. By Lemma 4,

$$R_{\varphi,2}(\{\bar{x} = \bar{x}\}) = +\infty \text{ iff } \varphi \text{ has the dichotomy property.}$$

In particular, if φ does *not* have the dichotomy property, then $R_{\varphi,2}(\{\bar{x} = \bar{x}\})$ is finite. By Monotonicity, $R_{\varphi,2}(\Sigma(\bar{x}))$ is finite for any $\Sigma(\bar{x})$.

Remark 16 (Definability)**.** Suppose $\Sigma(\bar{x})$ is a <u>finite</u> partial type over $A \subseteq \mathbb{M}$, and suppose $n < \omega$. Then the set

$$\{\bar{b} \in \mathbb{M} : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) \geq n\}$$

is A-definable. This can either be proved by induction on n using Definition 11, or directly from the explicit description in Lemma 12.

Now we can prove Theorem 10. Suppose $p \in S_\varphi(B)$. Take a finite subtype $\Sigma(\bar{x}) \subseteq_f p(\bar{x})$ minimizing $R_{\varphi,2}(\Sigma(\bar{x}))$. Then $\Sigma(\bar{x})$ is a partial type over B. Let $k = R_{\varphi,2}(\Sigma(\bar{x}))$.

Claim. If $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b})\}) = k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{b})\}) < k$$

Proof. Monotonicity gives

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) \leq R_{\varphi,2}(\Sigma(\bar{x})) = k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x},\bar{b})\}) \leq R_{\varphi,2}(\Sigma(\bar{x})) = k$$

If the first inequality is sharp, then $R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) < k$. But $\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}$ is a finite subtype of p (because $\varphi(\bar{x},\bar{b}) \in p(\bar{x})$), so this contradicts the choice of $\Sigma(\bar{x})$. Therefore the first inequality is an equality:

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) = k$$

If the second inequality is *not* sharp, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x},\bar{b})\}) = k$$

Then $R_{\varphi,2}(\Sigma(\bar{x})) \geq k+1$ by Definition 11, contradicting the choice of k. Therefore the second inequality is sharp:

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x},\bar{b})\}) < k.$$ □

A similar argument shows

Claim. If $\neg\varphi(\bar{x},\bar{b}) \in p(\bar{x})$, then

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) < k$$
$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\neg\varphi(\bar{x},\bar{b})\}) = k$$

Combining the two claims, we see that the set

$$\{\bar{b} \in B : \varphi(\bar{x},\bar{b}) \in p(\bar{x})\}$$

is exactly

$$\{\bar{b} \in B : R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) \geq k\}.$$

By Definability, there is an $L(B)$-formula $\psi(\bar{x})$ such that

$$R_{\varphi,2}(\Sigma(\bar{x}) \cup \{\varphi(\bar{x},\bar{b})\}) \geq k \iff \mathbb{M} \models \psi(\bar{b}).$$

Therefore

$$\{\bar{b} \in B : \varphi(\bar{x},\bar{b}) \in p(\bar{x})\} = \{\bar{b} \in B : \mathbb{M} \models \psi(\bar{b})\}.$$

It follows that p is a definable φ-type. This completes the proof of Theorem 10.

4.1 Remarks on the proof

1. In the proof of Theorem 10, the finite subtype $\Sigma(\bar{x}) \subseteq p(\bar{x})$ chosen to minimize $R_{\varphi,2}(\Sigma(\bar{x}))$ actually has
$$R_{\varphi,2}(\Sigma(\bar{x})) = R_{\varphi,2}(p(\bar{x})).$$
This comes from the following:

   Fact 17. *Let $\Sigma(\bar{x})$ be a partial type over $\mathbb{M}$.*

   (a) If $n < \omega$ and if $R_{\varphi,2}(\Sigma_0(\bar{x})) \geq n$ for every finite subtype $\Sigma_0(\bar{x})$, then $R_{\varphi,2}(\Sigma(\bar{x})) \geq n$.

   (b) $R_{\varphi,2}(\Sigma(\bar{x}))$ is the minimum of $R_{\varphi,2}(\Sigma_0(\bar{x}))$ as Σ_0 ranges over finite subtypes of Σ.

   Part (1) is an easy consequence of Lemma 12 and compactness.[2] Part (2) follows formally from part (1).[3]

2. We have discussed $R_{\varphi,2}$ for partial types, but we can also define it for definable sets. If D is a definable set, defined by a finite type $\Sigma(\bar{x})$, then $R_{\varphi,2}(D) := R_{\varphi,2}(\Sigma(\bar{x}))$. The choice of Σ doesn't matter by Remark 14. An equivalent definition in the style of Definition 11 is

   - $R_{\varphi,2}(D) \geq 0$ iff D is non-empty.
   - $R_{\varphi,2}(D) \geq n+1$ iff there is $\bar{b}$ such that
$$R_{\varphi,2}(D \cap \varphi(\mathbb{M}^n, \bar{b})) \geq n$$
$$R_{\varphi,2}(D \setminus \varphi(\mathbb{M}^n, \bar{b})) \geq n.$$

   More generally, we can allow D to be a type-definable set (a set defined by a partial type, or equivalent, a small intersection of definable sets.)

5 Consequences of Theorem 10

Theorem 18. *Suppose that no formula $\varphi(x_1, \ldots, x_n; \bar{y})$ has the dichotomy property. For any model $M \models T$ and any $p \in S_n(M)$, p is definable.*

Proof. Take $\bar{a} \in \mathbb{M}$ realizing p. Let $\varphi(\bar{x}; \bar{y})$ be a formula. By Theorem 10, $\mathrm{tp}^{\varphi}(\bar{a}/M)$ is definable, which means that there is an $L(M)$-formula $\psi(\bar{y})$ such that for any $\bar{b} \in M$,
$$\varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \iff \varphi(\bar{x}, \bar{b}) \in \mathrm{tp}^{\varphi}(\bar{a}/M) \iff \mathbb{M} \models \psi(\bar{b}).$$
This exactly means that p is definable. $\square$

[2] If you can get the leaves a_σ to satisfy Σ_0 for any Σ_0, you can get them to satisfy Σ.

[3] If $\Sigma_0(\bar{x}) > \Sigma(\bar{x})$ for all finite subtypes Σ_0, then take $n = \Sigma(\bar{x}) + 1$, and contradict part (1).

We also proved this in class by a completely different method (Proposition 8 in the March 3 notes).

Remark 19. When $A \subseteq \mathbb{M}$ is arbitrary (not necessarily a model), one says that $p \in S_n(A)$ is *definable* if for any $\varphi(\bar{x}; \bar{y})$ there is an $L(A)$-formula $\psi(\bar{y})$ such that for $\bar{b} \in A$,

$$\varphi(\bar{x}; \bar{b}) \in p(\bar{x}) \iff \mathbb{M} \models \psi(\bar{b}).$$

When A is a model, this recovers the definition of "definable type" over a model.

The proof of Theorem 18 shows more generally that

> If no formula has the dichotomy property, then any $p \in S_n(A)$ is definable for any A.

"No formula has the dichotomy property" is one of the definitions of "stable," and so we can conclude:

> In a stable theory, any type over *any* set is definable.

Warning 20. Definable types over arbitrary sets are not as well-behaved as definable types over models. For example:

1. A definable type over $A \subseteq \mathbb{M}$ can have more than one A-definable extension to $\mathbb{M}$.[4]

2. A definable type over $A \subseteq \mathbb{M}$ can have no A-definable extensions to $\mathbb{M}$.[5]

3. There is no good notion of "heirs" in this context.

Here is another important application of Theorem 10

Theorem 21. *Assume T is stable and $M \models T$. Let $D \subseteq M^n$ be $\varnothing$-definable. If $X \subseteq D$ is definable (with parameters from M), then X is definable over parameters from D.*

Proof. Let $\psi(\bar{y})$ be a formula defining D and let $\varphi(\bar{a}; \bar{y})$ be a formula defining X. Then $\mathrm{tp}^{\varphi}(\bar{a}/D)$ is definable by Theorem 10. Therefore there is a formula $\theta(\bar{y})$ with parameters from D such that if $\bar{b} \in D$, then

$$(M \models \varphi(\bar{a}; \bar{b})) \iff \varphi(\bar{x}; \bar{b}) \in \mathrm{tp}^{\varphi}(\bar{a}/D) \iff (M \models \theta(\bar{b}))$$

In other words, if $\psi(\bar{y})$ holds, then $\varphi(\bar{a}; \bar{y})$ is equivalent to $\theta(\bar{y})$. Therefore X is defined by $\psi(\bar{y}) \wedge \theta(\bar{y})$, a formula with parameters from D. ☐

[4]Example: in DLO, there is only one 1-type over $\varnothing$. It has two different $\varnothing$-definable extensions to $\mathbb{M}$: the types at $+\infty$ and $-\infty$.

[5]Example: in ACF, $\mathrm{tp}(\sqrt{2}/\mathbb{Q})$ is definable (because ACF is stable), but there is no $\mathbb{Q}$-definable extension to the monster model $\mathbb{M}$. Indeed, there are exactly two extensions to $\mathbb{M}$, namely $\mathrm{tp}(\sqrt{2}/\mathbb{M})$ and $\mathrm{tp}(-\sqrt{2}/\mathbb{M})$. These are exchanged by some automorphisms in $\mathrm{Aut}(\mathbb{M}/\mathbb{Q})$, so neither one can be $\mathbb{Q}$-definable.

Poizat calls Theorem 21 the *Parameter Separation Theorem* (Corollary 12.31). In modern literature, Theorem 21 is usually called "stable embeddedness."[6] Using Theorem 21, one can prove the following:

Fact 22. *Let M be a stable structure and $D \subseteq M^n$ be A-definable. Let N be the structure whose domain is D, with an m-ary relation symbol for each A-definable $X \subseteq D^m$. Then the definable subsets of D^m in M agree with the definable subsets of D^m in N.*

Note that we may as well take A to be finite, and then the language of N is no bigger than the language of M. Fact 22 is especially useful when M is a monster model. One can show that N is also a monster model (assuming A is small).

Here is the point of Fact 22: when studying a definable set D in a stable theory, we can essentially assume that D is the entire structure. For example, if G is a definable group in a stable theory, and we want to understand the internal structure of G, we can assume that G is the entire structure. If D is a strongly minimal set[7], and we want to understand definable relations on D, we can assume D is the entire structure (and then the theory is strongly minimal!).

[6] A set $A \subseteq M$ is "stably embedded" if for any definable set $D \subseteq M^n$, there is an A-definable set $D' \subseteq M^n$ such that $D \cap A^n = D' \cap A^n$. Theorem 21 says that any definable set in a stable theory is stably embedded. More generally, Theorem 10 says that *any* set in a stable theory is stably embedded.

[7] A strongly minimal *set* is an infinite definable set $D \subseteq \mathbb{M}$ that can't be written as a disjoint union $X \sqcup Y$ of two infinite definable sets. *This is the original motivation for the terminology "minimal."*