Homework1

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Exercise 1 (0). Show that $[a]_\sim = [b]_\sim$ if and only if $a \sim b$, and that $[a]_\sim \cap [b]_\sim = \emptyset$ if $a \nsim b$

Proof. If $[a]_\sim = [b]_\sim$, then for all $x \in [a]_\sim$, $x \in [b]_\sim$. As $a \in [a]_\sim$, $a \in [b]_\sim$, that is, $a \sim b$.

If $a \sim b$, then for all $x \in [a]_\sim$, as $x \sim a$ and $a \sim b$, then $x \sim b$ and hence $x \in [b]_\sim$. Thus $[a]_\sim \subset [b]_\sim$. Similarly, $[b]_\sim \subset [a]_\sim$. Hence $[a]_\sim = [b]_\sim$

If $[a]_\sim \cap [b]_\sim = \emptyset$ and suppose $a \sim b$. Then $[a]_\sim = [b]_\sim$. As $a \in [a]_\sim$, $[a]_\sim$ is not empty and hence $[a]_\sim \cap [b]_\sim \neq \emptyset$, a contradiction.

If $a \nsim b$ and suppose $[a]_\sim \cap [b]_\sim \neq \emptyset$, let $x \in [a]_\sim \cap [b]_\sim$. Then $x \sim a$ and $x \sim b$, and so $a \sim b$, a contradiction □

Exercise 2. Suppose a binary relation $(E', \approx)$ is elementary equivalent to an equivalence relation $(E, \sim)$. Show that $\approx$ is an equivalence relation

Proof. $S_\omega(E', E) \neq \emptyset$.

1. $\approx$ is reflexive. For any $x' \in E'$, as $\emptyset \in S_\omega(E', E)$, there is a $x \in E$ such that $\{(x', x)\}$ is a local isomorphism from E' to E, which means that $x' \approx x'$ if and only if $x \sim x$. Since $\sim$ is an equivalence relation, $x \sim x$, hence $x' \approx x'$

2. $\approx$ is symmetric. For any $x', y' \in E'$ and $x' \approx y'$, as $\emptyset \in S_\omega(E', E) \subset S_2(E', E)$, we have a local isomorphism $\{(x', x), (y', y)\}$ from E' onto E. Then $x \sim y$ and hence $y \sim x$ as $\sim$ is symmetric. Consequently $y' \approx x'$

3. $\approx$ is transitive. For any $x', y', z' \in E'$, $x' \approx y'$ and $y' \approx z'$. As $\emptyset \in S_\omega(E', E) \subset S_3(E', E)$, we have a local isomorphism $\{(x', x), (y', y), (z', z)\}$ from E' onto E. Then $x \sim y$ and $y \sim z$ and hence $x \sim z$. Thus we have $x' \approx z'$

□

Exercise 3. Show that $\mathcal{K}$ is non-empty

Proof. Take a injective function $f : \omega \to E$, and we define S_n for all nonzero $n \in \omega$ as

$$S_n = \left\{ f(i) \mid i \in \omega \wedge \frac{n(n-1)}{2} \leq i < \frac{n(n+1)}{2} \right\}$$

and $S_\omega = E \setminus f(\omega)$. Then

$$S = \{S_\omega\} \cup \bigcup_{n \in \omega} \{S_n\}$$

forms a partition of E. And for each nonzero $n \in \omega$, $|S_n| = n$.

Let $\sim_S = \{(x, y) \in E^2 \mid \exists \alpha \in \omega + 1 (x \in S_\alpha \wedge y \in S_\alpha)\}$ defined by S. Then $\sim_S \in \mathcal{K}$ and hence $\mathcal{K}$ is non-empty. □

Exercise 4. Suppose that $(E, \sim) \in \mathcal{K}$ and $(E', \approx)$ is elementarily equivalent to $(E, \sim)$. Show that $(E', \approx) \in \mathcal{K}$.

Proof. From Exercise 2 we know that $\approx$ is an equivalence relation on E'. For each nonzero $n \in \omega$, $(E, \sim)$ has exactly one equivalence class of size n, denoted by $S_n = \{e_1, \ldots, e_n\}$. As $\emptyset \in S_\omega(E, E') \subset S_{n+1}(E, E')$, we have a local isomorphism $s \in S_1(E, E')$ such that $s(e_i) = e'_i$ for all $i = 1, 2, \ldots, n$ and $e'_i \in E'$. Let $S'_n = \{e'_1, \ldots, e'_n\}$, then each pair of elements in S'_n is equivalent in the sense of $\approx$. If there is $e' \in E' \setminus S'_n$ such that $e' \approx e'_1$, then as s is an 1-isomorphism, there should be some other element e in E such that $e \sim e_1$, which is impossible. Hence S'_n is an equivalence class in E' of size n.

If there is another equivalence class S''_n of size n in E'. As $\emptyset \in S_\omega(E', E) \subset S_{2n+1}(E', E)$, we can construct a 1-isomorphism r with $\mathrm{dom}(r) = S'_n \cup S''_n$. As S'_n and S''_n are two distinct equivalence class, $\mathrm{im}(r)$ is also a union of two distinct equivalence class of size n. But $(E, \sim)$ only has exactly one equivalence class of size n, we get a contradiction. Thus for $n = 1, 2, 3, \ldots$, $(E', \approx)$ has exactly one equivalence class of size n. Thus $(E', \approx) \in \mathcal{K}$ □

Exercise 5. Suppose that $(E, \sim)$and $(E', \approx)$ are both in $\mathcal{K}$. Let s be a local isomorphism from $(E, \sim)$ to $(E', \approx)$, and let $p \geq 0$. Suppose that for every $a \in \mathrm{dom}(s)$, the $\sim$-equivalence class of a has the same size as the $\approx$-equivalence class of $s(a)$, or both equivalence classes have size greater than p. Then s is a p-isomorphism from $(E, \sim)$ to $(E', \cong)$

Proof. We prove this by induction on p.

If $p = 0$, then s is a 0-isomorphism by definition

If $p = n + 1$. We now prove that s is a $n + 1$-isomorphism.

For every $e \in E \setminus \mathrm{dom}(s)$, we choose $e' \in E'$ for different cases:

1. if there is a $x \in E$ such that $e \sim x$, then as $|[x]_\sim| = |[s(x)]_\approx|$ or both $|[x]_\sim|$ and $|[s(x)]_\approx|$ are greater than $n + 1$, we are able to choose a $e' \in E' \setminus \mathrm{im}(s)$ with $e' \sim s(x)$.

2. If there is no $x \in E$ with $e \sim x$ and $|[e]_\sim| = m \leq n$. If there is $y \in \mathrm{im}(s)$ with $|[y]_\approx| = |[e]_\sim|$, then there will be a $n - m + 1$-isomorphism $u \supset s$ maps a subset of $[s^{-1}(y)]_\sim$ onto $[y]_\approx$, otherwise there will be a contradiction if u is not onto. Also as $n - m + 1 \geq 1$, $[s^{-1}(y)]_\sim = [y]_\approx$, which leads to another contradiction. Hence there is no $y \in \mathrm{im}(s)$ with $|[y]_\approx|$. Then we pick an e' from the equivalence class of size m from E'.

3. If there is no $x \in E$ with $e \sim x$ and $|[e]_\sim| > n$, then choose a element e' from a equivalence class $S'_{>n}$ of size greater than n from E' with $S'_{>n} \cap \mathrm{im}(s) = \emptyset$.

Let $t = s \cup \{(e, e')\}$. Then t is a local isomorphism by our construction and the conditions in exercise are satisfied. Hence t is an n-isomorphism by induction. Thus forth condition is satisfied.

The back condition is similar.

Hence s is an $n + 1$-isomorphism $\square$

Exercise 6. Suppose that $(E, \sim)$ and $(E', \approx)$ are both in $\mathscr{K}$. Show that $(E, \sim)$ is elementarily equivalent to $(E', \approx)$

Proof. We need to prove that for every $p \in \omega$, $S_p(E, E')$ is not empty. But for local isomorphism $\emptyset$, the condition in Exercise 5 are always satisfied and $\emptyset \in S_p(E, E')$. Hence $S_p(E, E')$ are not empty for all $p \in \omega$.

Thus $(E, \sim)$ is elementarily equivalent to $(E', \approx)$. $\square$

Exercise 7. Construct two equivalence relations $(E, \sim)$ and $(E', \approx)$ s.t. $(E, \sim) \sim_\omega (E', \approx)$, but $(E, \sim) \nsim_\infty (E', \approx)$.

Proof. Let $E = \mathbb{Q}$, $E' = \mathbb{R}$, and $\sim = \{(x, y) \mid x - y \in \mathbb{Z}\} = \approx$. $\square$