Set Theory

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目录

1 Intro

习题 40%

考试闭卷

期中 8 周

集合 $\{x, y, z, ...\}$ 外延 $\{x \mid x \text{ is } ...\}$ 内涵

Theorem 1.1 (Cantor-Bendixen)**.** 闭集 $X \subseteq \mathbb{R}$ *uncountable, then* $X = Y_1 \cup Y_2$*, where* Y_1 *is countable and* Y_2 *is perfect*

if Y_2 is perfect, then $|Y_2| = |\mathbb{R}|$.
Hence CH is true for close set
Suppose X is a set and $Y = \{x \mid x \in X \wedge x \notin x\}$ is a set
$Y \in Y$ => $Y \in X$ and $Y \notin Y$, a contradiction. Hence $Y \notin Y$.
$Y \notin Y$ => $Y \notin X$ or $Y \in Y$ => $Y \notin X$
Thus we have

Proposition 1.2. 1. *For any set* X*, there exists a set* Y *s.t.* $Y \notin X$*.*

2. *collection of all sets is not a set*

Notation: we call $\{x \mid \varphi(x)\}$ a class
$V = \{x \mid x = x\}$, $\{x \mid x \notin x\}$ is not a set

Proposition 1.3. $\{x \mid x \neq x\}$ *is a set*

证明. From Existential Axiom, there is a set X_0.
Claim: $\{x \mid x \neq x\} = \{x \mid x \in X_0 \wedge x \neq x\}$
for any z, $z \neq z$, we need to prove $z \in X_0$. But $z \neq z$ is always false. □

We write $\emptyset = \{x \mid x \neq x\}$
For any set $X \neq \emptyset$, its arbitrary intersection

$$\bigcap X = \{u \mid \forall Y (Y \in X \to u \in Y)\}$$

Definition 1.4. $(x, y) = \{\{x\}, \{x, y\}\}$

Proposition 1.5. $(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$

证明. 作业 □

partial ordering + well-founded => linear ordering
since for arbitrary $\{x, y\}$, it has a minimal element

Definition 1.6. For any set α, if $\in$ is a well-ordering on α, then α is an **ordinal**

Definition 1.7. **successor** of x is $S(x) = x \cup \{x\}$, or x^+

ordinal that is not 0 and successor is a limit ordinal
极限是否存在
let $\omega = \{n \mid n = 0 \vee n \text{ is successor and } \forall m < n (m \text{ is successor})\}$
Then we need to show that ω is a set. So we need new axiom

Proposition 1.8. X 是归纳集，则 $\omega \subseteq X$

证明. Otherwise we have least $n \in \omega$ and $n \notin X$. Let $n = S(m)$, then $m \in X$. Hence we get a contradiction □

Theorem 1.9. *For any $X \subset \omega$, if X is inductive, then $X = \omega$*

$[\varphi(0) \wedge \varphi(n) \to \varphi(n+1)] \Rightarrow \forall n \varphi(n)$

Theorem 1.10. *ω is a ordinal and is a limit ordinal*

证明. $\in$ is a well-ordering on ω, then we need show

1. $\in$ is a partial ordering
   (a) transitive. let $\varphi(x)$: if $m < n$, then for any x, $n < x \to m < x$
   $x = 0$
   $x = k + 1$, $n < k + 1$, $n \in k \cup \{k\}$. then $n = k$ or $n \in k$
2. $\in$ is well-founded
   $X \subseteq \omega$, $X \neq \emptyset$, then $\exists x_0 \in X$. consider $\omega - X$. Suppose X has no minimum element
   (a) $0 \notin X \Rightarrow 0 \in \omega$
   (b) if $n \in \omega - X$, then $S(n) \in \omega - X$. Suppose $n \notin X$ and $S(n) \in X$. Then $S(n)$ is minimum

   then $\omega - X$ is inductive

for any $\alpha < \omega$, $S(\alpha) \neq \omega$, hence ω is limit □

2 序数

Definition 2.1. 一个集合 X 成为 **传递**的，如果对任意 $x \in X$，都有 $x \subseteq X$

Proposition 2.2. *假设 X 是传递集。如果 X 的所有元素也是传递集，则 $\in$ 在 X 上是一个传递关系。反之亦然。*

证明. 注意在证明反方向时，$\in$ 是 X 上的传递关系，如果 $x \in b, b \in a$，我们先说明 $x, b, a \in X$，然后因为 $\in$ 是传递关系，于是 $x \in a$ □

Exercise 2.0.1. 如果 $(\mathcal{T}, \in)$ 是传递集，则外延公理在 $\mathcal{T}$ 中成立，即：对任意 $X, Y \in \mathcal{T}$，$X = Y$ 当且仅当对任意 $a \in \mathcal{T}$,$a \in X$ 当且仅当 $a \in Y$。即 $X \cap \mathcal{T} = Y \cap \mathcal{T}$

如果 $\mathcal{T}$ 不传递，存在 $X \in \mathcal{T}$，有 $X \not\subseteq \mathcal{T}$，有 $a \in X$ 且 $a \notin \mathcal{T}$，假设 X 与 Y 仅有 a 差别，但是 $\mathcal{T}$ 分辨不出

Definition 2.3. 对任意集合 α，如果 $\in$ 是 α 上的良序，就称 α 是 **序数**

$\{\omega, \omega+1, \omega+2, \dots\}$ 是集合吗？这需要替换公理

考虑 $V_{\omega+\omega}$，我们定义集合是它的元素，替换公理在这里不对，$f(\omega) \notin V_{\omega+\omega}$ 不是集合。因此我们需要替换公理来保证它是集合

3 基数与选择公理

Proposition 3.1. *TFAE*

1. *set X is finite*
2. *there is a linear order $\leq$ on X satisfying for any nonempty subset there is a maximum and a minimum*
3. *$\forall Y \subseteq \mathcal{P}(X)$ and $Y \neq \emptyset$, it has a maximal under $\subseteq$*

证明. $2 \to 1$. Let $x_0 = \inf(X)$. For any $k \in \mathbb{N}$, let $x_{k+1} = \inf(X - \{x_0, \dots, x_k\})$ if $x_k \neq \sup(X)$.

$1 \to 3$. Find a maximum cardinality

$3 \to 1$. If X is infinite. $Y = \{Z \subseteq X \mid Z \text{ is finite}\}$ □

Theorem 3.2. 一个序数是 α 是至多可数的，当且仅当存在 $\mathbb{R}$ 的子集 A，$ot(A)=\alpha$

证明. 首先假设 $A\subseteq\mathbb{R}$ 并且 $ot(A)=\alpha$，即 $A=\{a_\beta\mid\beta<\alpha\}$，并且 $a_\beta<a_\gamma$ 当且仅当 $\beta<\gamma$。对任意 $\beta<\alpha$，令 $I_\beta=(a_\beta,a_{\beta+1})$ 为实数的区间。如果 $\alpha=\eta+1$ 是后继序数，则令 $I_\eta=(a_\eta,a_\eta+1)$。这样的区间只有可数多 因为每个区间都有有理数，但是有理数只有可数多 □

Proposition 3.3 (2.2.15)**.** 对任意无穷基数 κ,λ

1. $\kappa^{<\lambda}=\sup\{\kappa^\eta\mid\eta\text{是基数并且}\eta<\lambda\}$

证明. 1. TTT

$$\begin{aligned}\kappa^{<\lambda}&=\left|\bigcup\{X^\beta\mid\beta<\lambda\}\right|\\&=\{\bigcup_{\beta<\lambda}\kappa^\beta\}=\bigoplus_{\beta<\lambda}|\kappa^\beta|\\&=\sup\{|\kappa^\beta|\mid\beta<\lambda\}\\&=\sup\{\kappa^{|\beta|}\mid\beta<\lambda\}\end{aligned}$$

2. 对于 $f\in\kappa^\lambda$，f 都是 $\lambda\otimes\kappa$ 的子集且 $f\in\{X\subseteq\lambda\otimes\kappa\mid|X|=\lambda\}$

□

Exercise 3.0.1. 若 κ 是不可达的，则 $V_\kappa\vDash ZF$。

Corollary 3.4. $ZF\nvDash\exists\kappa(\kappa\text{是 } inaccessible)$

证明. 由 Gödel 第二不完全性定理，如果存在了，就能证明有模型了，就证明一致了 □

大基数：κ 是基数且 $ZF\nvDash\exists\kappa$

Proposition 3.5. κ 不可达，$|V_\kappa|=\kappa$.

证明. $\kappa\leq|V_\kappa|$

证明对任意 $\alpha<\kappa$, $|V_\alpha|<\kappa$,

根据 $2^\alpha<\kappa$ □

证明 $f : |X| = \alpha \to V_\kappa$ 有界

若 $\beta < \alpha \cdot \omega$,$\beta = \alpha \cdot \xi + \eta$，其中 ξ 有穷

$\beta > \alpha$, $\beta = \alpha \cdot \xi + \eta$. $\alpha + \beta = \alpha + \alpha \cdot \xi + \eta = \alpha \cdot \xi + \eta$. $\alpha + \alpha \cdot \xi = \alpha \cdot \xi$. 只要证明 $\alpha \cdot (\xi + 1) = \alpha + \alpha \cdot \xi$

Lemma 3.6. $\left|\bigcup_{\gamma<\omega_\alpha} V_\gamma\right| = 2^{\aleph_\beta}$

证明. $2^{\aleph_\beta} = \sup\{2^{\aleph_\gamma} \mid \gamma < \beta\}$. Thus $2^{\aleph_\beta} \leq \left|\bigcup_{\gamma<\omega_\alpha} V_\gamma\right|$ □

$|V_{\omega+1}| = 2^{\aleph_0}, |V_{\omega+2}| = 2^{|2^{\aleph_0}|} > 2^{\aleph_0}$

$\left|V_{\omega_1}\right| \geq |V_{\omega+2}| > 2^{\aleph_0} = \beth(1)$

$\left|V_{\omega_1}\right| = \beth_{\omega_1}$

Proposition 3.7. 若 κ 不可达，$X \in V_\kappa$,$f : X \to V_\kappa$，则 $f[X] \in V_\kappa$

证明. $|X| < \kappa$，对不可达基数 $|V_\kappa| = \kappa$

因此 $|f(X)| < \kappa$

已知 $f[X] \subseteq V_\kappa$。令 $\lambda = \sup\{\text{rank}(y) \mid y \in f[X]\}$, 因为 $y \in V_\kappa$ 而 κ 是极限序数，因此存在 $\alpha < \kappa$ 使得 $y \in V_\alpha$，于是 $\text{rank}(y) < \alpha + 1 < \kappa$, 因此 λ 是 $< \kappa$ 个小于 κ 的上界，又因为 κ 是正则的，$\lambda < \kappa$，于是 $f[X] \subseteq V_\lambda$，于是 $f[X] \in V_{\lambda+1} \subseteq V_\kappa$ □

Proposition 3.8. 令 β 为任意序数, α 为任意极限序数, 证明: 如果 $\alpha+\beta = \beta$, 则 $\beta \geq \alpha \cdot \omega$

证明. $\alpha + \beta = \beta \Rightarrow \alpha \leq \beta \Rightarrow \exists \delta, \gamma(\beta = \alpha \cdot \delta + \gamma \wedge \gamma < \alpha)$

若 $\delta \geq \omega$ 就对了

若 $\delta < \omega$，$\alpha + \beta = \alpha + (\alpha \cdot \delta) + \gamma = \alpha(1 + \delta) + \delta > \alpha$ □

$\aleph_1 \leq 2^{\aleph_0}$，因此 $\aleph_1^{\aleph_0} \leq 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$

Proposition 3.9. 令 $X = \{f : \omega \to \omega_1 \mid f \text{ 1-1}\}$，证明 $|X| = 2^\omega$

证明. $Y = \{F \mid F : \aleph_0 \to \aleph_0 \times \aleph_1, \text{1-1}\}$

$G = \aleph_1^{\aleph_0} \to Y$ s.t. $G(f) = F \in Y$ s.t. $F(n) = (n, f(n))$, then G is 1-1

Hence $\aleph_1^{\aleph_0} \leq |Y| = (\aleph_0 \times \aleph_1)^{\aleph_0} = \aleph_1^{\aleph_0}$ □

证明. □

4 滤、理想与无界闭集

4.1 滤

Proposition 4.1. *1.* $-0 = 1$

2. $-1 = 0$

3. $a \cdot 1 = a$

4. $a + 0 = a$

5. $a + a = a$

6. $a \cdot a = a$

7. $1 + a = 1$

8. $0 \cdot a = 0$

9. $a + b = 1 \wedge a \cdot b = 0 \Rightarrow b = -a$

10. $-(a \cdot b) = (-a) + (-b)$

11. $-(a + b) = (-a) \cdot (-b)$

证明. 1. $1 = 0 + (-0) = (0 \cdot (-0)) + (-0) = -0$

2. $0 = 1 \cdot (-1) = (1 + (-1)) \cdot (-1) = -1$

3. $a \cdot 1 = a \cdot (a + (-a)) = a$

5. $a + a = a + (a \cdot 1) = a$

7. $1 + a = (a+1) \cdot 1 = (a+1) \cdot (a + -a) = a \cdot a + 0 + a + -a = a + -a = 1$

8. $0 \cdot a = (a \cdot (-a)) \cdot a = a \cdot a \cdot (-a) = a \cdot (-a) = 0$

9. $-a = (-a) \cdot 1 = (-a) \cdot (a + b) = (-a) \cdot a + (-a) \cdot b = (-a) \cdot b.$

$ab + (-a)b = -a.$ $b(a + (-a)) = b = -a$

10. $ab+(-a)+(-b) = ab+(-a)+(-b)\cdot 1 = ab+(-a)+(-b)a+(-b)(-a) = a(b+(-b))+(-a)+(-b)(-a) = 1+(-b)(-a) = 1$

$\square$

G 有有穷交性质 $F = \{b \in B \mid \exists g \in G(g \leq b)\}$

若 $a, b \in F$，则 $g_1 \leq a, g_2 \leq b$

Rasiowa-Sikorski [mathematics for metamathematics] 是为了证明一阶完全性，$\sum D$ 等价于 $\exists x$

令 $\phi(x, \bar{y})$, $M_\phi = \{[\varphi(t, \bar{y})] \mid t \text{ a term}\} \subset B$

Claim $\sum M_\phi = [\exists x \varphi(x, \bar{y})]$

那么如果 U 是完全的，那么 $\sum M_\phi \in U$，于是 $\exists t[\phi(t, \bar{y})] \in U$。类似于极大一致 Henkin 集

under CH $|A| < 2^{\aleph_0} = \aleph_1$ iff $|A| \leq \aleph_0$

cichon diagram

$f : M \to N$, $f \restriction A : A \to N$, $\forall a \in A, f \restriction A(a) = f(a)$

4.2 Clut set

Definition 4.2. α limit, $C \subseteq \alpha$ is a **club set** in α if

1. unbounded $\sup C = \alpha$, that is, for any $\beta < \alpha$, there is $\gamma \in C$ s.t. $\beta < \gamma$
2. closed: for any limit $\gamma < \alpha$, $\sup(C \cap \gamma) = \gamma \Rightarrow \gamma \in C$

If $A \subseteq C \subseteq \alpha$ and $\sup C = \gamma < \alpha$, then γ is the **limit point** of C. C is closed in α iff all limit point of C belong to C

Lemma 4.3. *α limit and* $\mathrm{cf}(\alpha) > \omega$, *then*

1. *α is a club set of itself*
2. *$\forall \beta < \alpha$, $\{\beta < \alpha \mid \delta > \beta\}$ is a club set in α*
3. *$X = \{\beta < \alpha \mid \beta \text{ limit}\}$ is a club set in α*

4. *If X is unbounded in α, then $X' = \{\gamma \in X \mid \gamma < \alpha \wedge \gamma$ is a limit point of $X\}$ is a club set in α*

证明. 3. X is closed. For any $\xi \in \alpha$, define

$$\xi = \xi_0, \xi_1, \cdots, \xi_n, \cdots \quad (n \in \omega)$$

s.t. $\xi_{n+1} = \min\{\alpha - \xi_n\}$. Let $\eta = \sup \xi_i$. Then $\xi < \eta \in X$. $\eta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$

4. Like 3, for any $\xi \in \alpha$, define $\xi_{n+1} = \min\{\xi' > \xi : X - \{\xi_1, \ldots, \xi_n\}\}$, this works since X is unbounded.

   For any limit point $\eta < \alpha$ of X', that is, $\sup(X' \cap \eta) = \eta$, then for any $\sigma < \eta$, there is limit point $\xi < \eta$ of X s.t. $\sigma + 1 < \xi$. By definition of limit point, $\exists \mu \in X \cap \xi$ s.t. $\sigma < \mu$, so $\sup(X \cap \eta) = \eta$ and η is a limit point of X, thus $\eta \in X'$

   Limit of limits of X is still a limit of X

□

Lemma 4.4. *if α is limit and $\mathrm{cf}(\alpha) > \omega$, and $f : \alpha \to \alpha$ is strictly increasing and continuous, that is, for any limit $\beta < \alpha$, $f(\beta) = \bigcup_{\gamma<\beta} f(\gamma)$, then*

1. *$\mathrm{im}(f)$ is a club set in α*

2. *if α is regular, then every club set C in α is the image of such a function*

证明. 2. suppose $ot(C) = \tau$. $f : (\tau, <) \cong (C, <)$, then f is strictly increasing and continuous. Since C is unbounded and α is regular, $\tau \geq \mathrm{cf}(\alpha) = \alpha$. $\forall \eta < \tau, \eta \leq f(\eta)$, so $\tau \leq \sup f(\eta) = \alpha$, thus $\tau = \alpha$

□

Proposition 4.5. *Suppose α is a limit ordinal and $\mathrm{cf}(\alpha) > \omega$, then for any $\gamma < \mathrm{cf}(\alpha)$, if $(C_\xi)_{\xi<\gamma}$ is a sequence of club sets in α, then $\bigcap_{\xi<\gamma} C_\xi$ is a club set in α*

证明. Suppose $\gamma = 2$. Intersection of closed sets is still closed. We prove that $C_1 \cap C_2$ is unbounded in α. $\forall \delta < \kappa$, $\exists \xi \in C_1, \eta \in C_2$ s.t. $\delta < \xi < \eta$, let

$$\xi_0 < \eta_0 < \xi_1 < \eta_1 < \dots$$

where $\xi_0 = \xi, \eta_0 = \eta$ and for any $n \in \omega$, $\xi_n \in C_1$, $\eta_n \in C_2$. Let μ be the limit of this sequence, then $\sup(C_1 \cap \mu) = \mu$ and $\sup(C_2 \cap \mu) = \mu$, hence $\mu \in C_1 \cap C_2$.

Suppose γ is a successor ordinal

Suppose γ is a limit ordinal, let $D = \bigcap_{\xi<\gamma} C_\xi$, we prove it is unbounded. For any $\eta < \gamma$, if $D_\eta = \bigcap\{C_\xi \mid \xi < \eta\}$, then η_n is a club set and $D = \bigcap_{\eta<\gamma} D_\eta$ and $\eta < \eta' < \gamma$ implies $D_\eta \supset D_{\eta'}$. For any $\mu < \alpha$, let

$$\xi_0 < \xi_1 < \cdots < \xi_\eta < \cdots$$

where $\xi_0 > \mu$, and for any $\eta < \gamma$, $\xi_\eta \in D_\eta$ is the minimum element larger than $\sup\{\xi_\alpha \mid \alpha < \eta\}$. Since $\mathrm{cf}(\alpha) > \gamma$, $\xi = \sup\{\xi_n \mid \eta < \gamma\} < \alpha$. for any $\eta < \lambda$, $\xi \in D_\eta$, thus $\xi \in D$ and $\mu < \xi$ $\square$

Definition 4.6. For any limit $\mathrm{cf}(\alpha) > \omega$

$$F_{CB}(\alpha) = \{X \subseteq \alpha \mid \exists C(C \text{ is a club set in } \alpha \wedge C \subseteq X)\}$$

is a filter, called **club filter** in α

Corollary 4.7. *If κ is uncountable regular cardinal, then club filter in κ is κ-complete*

Definition 4.8. for any ordinal α, $(X_\xi \mid \xi < \alpha)$ is a sequence of subsets of α

1. **diagonal intersection** of X_ξ

$$\triangle_{\xi<\alpha} X_\xi = \{\eta < \alpha \mid \eta \in \bigcap_{\xi<\eta} X_\xi\}$$

2. **diagonal union** of X_ξ

$$\triangledown_{\xi<\alpha} X_\xi = \{\eta < \alpha \mid \eta \in \bigcup_{\xi<\eta} X_\xi\}$$

Remark. Let $Y_\xi = \{\eta \in X_\xi \mid \eta > \xi\}$, then $\triangle_{\xi<\alpha} X_\xi = \triangle_{\xi<\alpha} Y_\xi$

Proposition 4.9. *for any uncountable regular κ, and a sequence of club sets $(X_\gamma \mid \gamma < \kappa)$ in κ, $\triangle_{\gamma<\kappa} X_\gamma$ is a club set in κ.*

证明. Let $C_\gamma = \bigcap_{\xi<\gamma} X_\xi$, then $\triangle X_\gamma = \triangle C_\gamma$

$$\eta \in \triangle C_\gamma \Leftrightarrow \forall \xi < \eta, \eta \in C_\xi = \bigcap_{\zeta<\xi} X_\zeta$$

$$\Leftrightarrow \forall \zeta < \xi < \eta, \eta \in X_\zeta$$

guess should be $C_\gamma = \bigcap_{\xi\leq\gamma} X_\xi$

let

$$C_0 \supset C_1 \supset \cdots \supset C_\gamma \supset \cdots \quad (\gamma < \kappa)$$

Define $C = \triangle C_\gamma$. To prove C is closed, let η be the limit point of C. We need to prove $\eta \in C$, that is, $\forall \xi < \eta, \eta \in C_\xi$. For any $\xi < \eta$, define $X = \{\nu \in C \mid \xi < \nu < \eta\}$, then $X \subset C_\xi$; by Theorem 4.7, C_ξ is a club in κ, therefore $\eta = \sup X \in C_\xi$, hence $\eta \in C$

Unboundedness: for any $\mu < \kappa$, define $(\beta_n \mid n \in \omega)$: let $\mu < \beta_0 \in C_0$, and $\beta_n < \beta_{n+1} \in C_{\beta_n}$. Since C_{β_n} is unbounded, such β_{n+1} can always be found. Also

$$C_{\beta_0} \supset C_{\beta_1} \supset C_{\beta_2} \supset \cdots$$

thus for any $m > n$, $\beta_m \in C_{\beta_{m+1}} \subset C_{\beta_n}$. Now we prove $\beta = \sup\{\beta_n \mid n \in \omega\} \in C$, which is suffice to show that for any $\xi < \beta$, $\beta \in C_\xi$. But if $\xi < \beta$, there is n s.t. $\xi < \beta_n$ and for any $m > n$, $\beta_m \in C_{\beta_n} \subset C_\xi$. Since C_ξ is closed, $\beta \in C_\xi$. Thus $\beta \in C$ □

Corollary 4.10. *For any uncountable regular cardinal κ, if $f : \kappa \to \kappa$ is a function ,then*

$$D = \{\alpha < \kappa \mid \forall \beta < \alpha (f(\beta) < \alpha)\}$$

is a club set

证明. For any $\alpha < \kappa$, let $C_\alpha = \{\beta < \kappa \mid f(\alpha) < \beta\}$, which is a club set. Then $D = \triangle C_\alpha$ $\square$

Definition 4.11. α limit and cf$(\alpha) > \omega$)

1. If $S \subseteq \alpha$ and for any club set C in α $S \cap C \neq \emptyset$, then S is called **stationary set** in α

2. $I_{NS}(\alpha) = \{X \subseteq \alpha \mid \exists C(C \text{ is a club set in } \alpha \wedge X \cap C = \emptyset)\}$ is called a **non-stationary ideal** in α

Proposition 4.12. *limit ordinal α with* cf$(\alpha) > \omega$

1. *club set in α is stationary. if S is stationary and $S \subseteq T \subseteq \alpha$, then T is stationary*

2. *stationary set in α is unbounded*

3. *there is unbounded $T \subseteq \alpha$ that is not stationary*

证明.

1. 4.5

2. If S is stationary, for any $\beta < \alpha$, $\{\gamma < \alpha \mid \beta < \gamma\}$ is a club set in α and the elements of the intersection of it with S is larger than β

3. $T = \{\alpha + 1 \mid \alpha < \kappa\}$ is unbounded but not stationary, since the club set of all limit ordinal doesn't intersect with it

$\square$

Proposition 4.13. *limit ordinal α with* cf$(\alpha) > \omega$ *and* $\lambda <$ cf(α) *is regular, then*

$$E^\alpha_\lambda = \{\beta < \alpha \mid \mathrm{cf}(\beta) = \lambda\}$$

is stationary in α

证明. For any club set C in α, define a strictly increasing sequence of C:

$$\alpha_0 < \alpha_1 < \cdots < \alpha_\xi < \cdots \quad (\xi < \lambda)$$

such sequence exists since λ is regular, $\lambda < \text{cf}(\alpha)$ and C is unbounded. suppose δ is the supremem of the sequence. Since C is closed, $\delta \in C$, Since $\text{cf}(\delta) = \lambda$, $\delta \in E^{\alpha}_{\lambda}$ $\square$

Proposition 4.14. *for any uncountable regular cardinal κ, if $(X_\xi \mid \xi < \kappa)$ is a sequence of non-stationary sets, then $\bigtriangledown_{\xi<\kappa} X_\xi$ is non-stationary. That is, $I_{NS}(\kappa)$ is closed under diagonal intersection*

证明. For any X_ξ, there is C_ξ s.t. $X_\xi \cap C_\xi = \emptyset$. Let $C = \triangle C_\xi$, then C is a club set. Let $X = \bigtriangledown X_\xi$, then $X \cap C = \emptyset$ $\square$

Definition 4.15. For a ordinal set S and $\text{dom}(f) = S$, if for any $0 \neq \alpha \in S$, $f(\alpha) < \alpha$, then f is **regressive**

Theorem 4.16 (Fodor)**.** *For any uncountable regular cardinal κ, stationary $S \subseteq \kappa$, if* $\text{dom}(f) = S$ *is regressive, then there is a stationary $T \subseteq S$ and ordinal $\gamma < \kappa$ s.t. for any $\alpha \in T$, $f(\alpha) = \gamma$*

证明. If for any $\gamma < \kappa$, $A_\gamma = \{\alpha \in S \mid f(\alpha) = \gamma\}$ is non-stationary, and there is a club set C_γ s.t. $A_\gamma \cap C_\gamma = \emptyset$, that is, for any $\alpha \in S \cap C_\gamma$, $f(\alpha) \neq \gamma$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$. Then $\alpha \in C$ iff $\forall \gamma < \alpha$, $\alpha \in C_\gamma$ iff $\forall \gamma < \alpha$, $f(\alpha) \neq \gamma$. Hence for any $\alpha \in C$, $f(\alpha) \geq \alpha$. Since C is a club set, $S \cap C \neq \emptyset$, but for any $\alpha \in S$, $f(\alpha) < \alpha$ $\square$

Lemma 4.17. *uncountable regular cardinal κ, $S \subseteq \kappa$ stationary, f is a regressive function on S. If for any $\eta < \kappa$,*

$$X_\eta = \{\alpha \in S \mid f(\alpha) \geq \eta\}$$

is stationary, then S can be partitioned into κ disjoint stationary sets

证明. For any $\eta < \kappa$, $f \upharpoonright X_\eta$ is a regressive function on X_η. By Fodor's, there is $\eta < \gamma_\eta < \kappa$ s.t. $S_{\gamma_n} = \{\alpha \in S \mid f(\alpha) = \gamma_n\}$ is stationary

Define $g : \kappa \to \kappa$: $g(0) = 0$, $g(\eta) = \sup\{\gamma_{g(\xi)} + 1 \mid \xi < \eta\}$. If $\xi < \eta < \kappa$, then $\gamma_{g(\xi)} < g(\eta) \leq \gamma_{g(\eta)}$, hence $\eta \mapsto \gamma_{g(\eta)}$ is a increasing cofinal function from κ to κ. Thus $\{S_{\gamma_{g(\eta)}} \mid \eta < \kappa\}$ has cardinality κ and is pairwise disjoint $\square$

Lemma 4.18. *uncountable regular κ, $\lambda < \kappa$ is regular, any stationary subset of*

$$E^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$$

can be partitioned into κ disjoint stationary subsets

证明. Stationary $S \subseteq E^\kappa_\lambda$, $\forall \alpha \in S$, choose a strictly increasing cofinal function $f_\alpha : \lambda \to \alpha$. $\forall \xi < \lambda$, define $g_\xi : \kappa \to \kappa$:

$$g_\xi(\alpha) = \begin{cases} 0 & \alpha \notin S \\ f_\alpha(\xi) & \alpha \in S \end{cases}$$

$g_\xi \upharpoonright S$ is regressive

$\forall \eta < \kappa \forall \xi < \lambda$, let

$$X^\eta_\xi = \{\alpha \in S \mid g_\xi(\alpha) \geq \eta\}$$

We prove: $\exists \xi < \lambda \forall \eta < \kappa$, X^η_ξ is stationary. Otherwise, $\forall \xi < \lambda$, there is a club C_ξ and a ordinal $\eta_\xi < \kappa$ s.t. $C_\xi \cap X^{\eta_\xi}_\xi = \emptyset$. Let $C = \bigcap_{\xi<\lambda} C_\xi$, $\eta = \sup\{\eta_\xi \mid \xi < \lambda\}$, then C is a club. But for any $\alpha \in C \cap S$, $\forall \xi < \lambda$, $g_\xi(\alpha) < \eta$ since $C \cap X^\eta_\xi = \emptyset$, therefore $C \cap S \subseteq \eta$, a contradiction since C is a club

Fix a $\xi < \lambda$ s.t. for any $\eta < \kappa$, X^η_ξ is stationary. By 4.17, S can be partitioned into κ disjoint stationary sets $\square$

Corollary 4.19. *uncountable regular κ, $X = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \alpha\}$. If $S \subseteq X$ is stationary, then S can be partitioned into κ disjoint stationary sets*

证明. Let $f : \kappa \to \kappa$ be $f(\alpha) = \mathrm{cf}(\alpha)$. Then $f \upharpoonright S$ is regressive. By Fodor's lemma, there is $\lambda < \kappa$, $S_\lambda = \{\alpha \in S \mid f(\alpha) = \lambda\}$ is stationary. Note that $S_\lambda \subseteq E^\kappa_\lambda$, hence S_λ can be partitioned into κ disjoint stationary sets $\square$

Lemma 4.20 (skip)**.** *uncountable regular κ, $S \subseteq \kappa$ stationary, $f : S \to \kappa$ regressive. for any $\beta < \kappa$, define*

$$S_\beta = \{\alpha \in S \mid f(\alpha) = \beta\}$$

Let $I = \{S_\beta \mid S_\beta$ stationary$\}$, then exactly one of below is true

1. $|I| = \kappa$
2. $|I| < \kappa$ *and there is a club C,* $\mathrm{im}(f \restriction C \cap S)$ *is bounded in* κ

Lemma 4.21. *uncountable regular κ, $R = \{\omega < \gamma < \kappa \mid \mathrm{cf}(\gamma) = \gamma\}$, define*

$$D = \{\gamma \in R \mid R \cap \gamma \in I_{NS}(\gamma)\}$$

If R is stationary in κ, then so is D

证明. If D is not stationary, there is club C s.t. $C \cap D = \emptyset$. Let C' be the set of limit points of C. Let $\gamma = \min(C' \cap R)$, $\gamma \in R - D$, thus $R \cap \gamma$ is stationary in γ

Now consider $C \cap \gamma$, since γ is a limit point of C, this set is unbounded in γ. By 4.3 (4), $C' \cap \gamma$ is a club set in γ, thus $R \cap C' \cap \gamma \neq \emptyset$, which contradicts the minimality of γ in $R \cap C'$ □

Theorem 4.22 (Soloway)**.** *Any stationary set in uncountable regular cardinal κ can be partitioned into κ disjoint stationary sets*

证明. stationary $S \subseteq \kappa$, let

$$S_0 = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \alpha\}$$
$$S_1 = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha\}$$

Then $S = S_0 \cup S_1$, hence either S_0 or S_1 is stationary ?

If S_0 is stationary, by 4.19, S_0 can be partitioned into κ disjoint stationary sets

Now suppose S_1 is stationary, let $D = \{\alpha \in S_1 \mid S_1 \cap \alpha \in I_{NS}(\alpha)\}$ □

4.3 Ultrafilter and large cardinal

4.3.1 Regular ultrafilter

Definition 4.23. limit ordinal α, cf$(\alpha) > \omega$, F is a filter on α. If F is closed under diagonal intersection, then F is **regular**

Example 4.1. For any limit ordinal α with cf$(\alpha) > \omega$, $F_{CB}(\alpha)$ is regular

Let elements in F have measure 1, otherwise has measure 0

Definition 4.24. limit ordinal α, cf$(\alpha) > \omega$, F is a filter on α, if $\forall X \in F$, $Y \cap X \neq \emptyset$, then $Y \subseteq \alpha$ has **positive measure**

Lemma 4.25. *uncountable regular cardinal κ, F is a filter on κ. F is regular $\Leftrightarrow$ for any $f : \kappa \to \kappa$, if there is a X with positive measure s.t. $f \upharpoonright X$ is regressive, then there exists $\gamma < \kappa$ s.t. $X_\gamma = \{\alpha \in X \mid f(\alpha) = \gamma\}$ has positive measure*

证明. $\Rightarrow$. $f : \kappa \to \kappa$, Y has positive measure and $f \upharpoonright Y$ is regressive. If for all $\gamma < \kappa$, $Y_\gamma = \{\alpha \in Y \mid f(\alpha) = \gamma\}$ doesn't have positive measure, then there is $X_\gamma \in F$ s.t. $Y_\gamma \cap X_\gamma = \emptyset$. Let $X = \triangle X_\gamma \in F$ since F is regular. Since Y has positive measure, $X \cap Y \neq \emptyset$. For any $\gamma \in X \cap Y$, since $\gamma \in X$, then for any $\beta < \gamma$, $f(\gamma) \neq \beta$, therefore $f(\gamma) \geq \gamma$, contradicting the fact that f is regressive on Y.

$\Leftarrow$. Suppose for any $\beta < \kappa$, $X_\beta \in F$ and $X = \triangle X_\beta \notin F$, then

$$Y = \kappa - X = \{\alpha < \kappa \mid \exists \beta < \alpha(\alpha \notin X_\beta)\}$$

has positive measure. Define $f : \kappa \to \kappa$:

$$f(\alpha) = \begin{cases} \min\{\beta \mid \beta < \alpha \wedge \alpha \notin X_\beta\} & \alpha \in Y \\ 0 & \end{cases}$$

that is, if $\alpha \notin X$, there is $\beta < \alpha$ s.t. $\alpha \notin X_\beta$

Then $f \upharpoonright Y$ is regressive, hence there is $0 < \gamma < \kappa$, $Y_\gamma = \{\alpha \in Y \mid f(\alpha) = \gamma\}$ has positive measure. But $\alpha \in Y_\gamma \Rightarrow \alpha \notin X_\gamma$, hence $X_\gamma \cap Y_\gamma = \emptyset$, a contradiction □

4.3.2 Measurable cardinal

Lemma 4.26. *there is no $\aleph_1$-complete non-principal ultrafilter on $2^\omega = \{f \mid f : \omega \to \{0, 1\}\}$*

证明. If U is a $\aleph_1$-complete non-principal ultrafilter on 2^ω. Let $L = \{f \in 2^\omega \mid f(0) = 0\}$, $R = \{f \in 2^\omega \mid f(0) = 1\}$, then $2^\omega = L \cup R$, and only one of them belongs to U. Define h and a sequence $(X_n)_{n\in\omega}$ of subsets of 2^ω as follows:

1. If $L \in U$, then $h(0) = 0$, $X_0 = R$. If $R \in U$, then let $h(0) = 1$, $X_0 = L$
2. Let $h(n)$ and X_n is defined, then

$$Y = \{f \in 2^\omega \mid \forall i \leq n(f(i) = h(i))\} \in U$$

   Let $Y^L = \{f \in Y \mid f(n+1) = 0\}$, $Y^R = \{f \in Y \mid f(n+1) = 1\}$, then only one of them belongs to U. If $Y^L \in U$, let $h(n+1) = 0$, $X_{n+1} = Y^R$; otherwise, $h(n+1) = 1$, $X_{n+1} = Y_L$

For any $f \in 2^\omega$, if $f \neq h$, then there is a smallest $i \in \omega$ s.t. $f(i) \neq h(i)$, which implies $f \in X_i$, thus

$$\{h\} \cup \bigcup_{n\in\omega} X_n = 2^\omega \in U$$

But $\forall n \in \omega$, $X_n \notin U$, U is $\aleph_1$-complete implying $\bigcup_{n\in\omega} X_n \notin U$. $\bigcup_{n\in\omega} X_n \notin U \Leftrightarrow \overline{\bigcup_{n\in\omega} X_n} \in U \Leftrightarrow \bigcap_{n\in\omega} \overline{X_n} \in U \Leftarrow \forall n \in \omega(\overline{X_n} \in U)$ And U is not principal, thus $\{h\} \notin U$. □

Lemma 4.27. *Let κ be the minimum cardinal with a $\aleph_1$-complete non-principal ultrafilter on it, then*

1. *any $\aleph_1$-complete non-principal ultrafilter on κ is κ-complete*
2. *κ is uncountable and regular*

证明. 1. U is an $\aleph_1$-complete non-principal ultrafilter on κ. If U is not κ-complete, then there is $\gamma < \kappa$, $(X_\beta)_{\beta<\gamma}$ a sequence of pairwise disjoint subsets of κ s.t. $\bigcup_{\beta<\gamma} X_\beta \in U$ and $\forall \beta < \gamma (X_\beta \notin U)$

Now we define a filter on γ. First, for any $Y \subseteq \gamma$, let

$$X_Y = \{\delta < \kappa \mid \exists \beta \in Y (\delta \in X_\beta)\}$$

that is, $X_Y = \bigcup_{\beta \in Y} X_\beta$. Let $F = \{Y \subseteq \gamma \mid X_Y \in U\}$, we prove that F is an $\aleph_1$-complete non-principal ultrafilter on γ, contradicting the minimality of κ

Since

2. Let $(X_\beta)_{\beta<\gamma}$ be a sequence of subsets of κ, $\gamma < \kappa$ and for any $\beta < \gamma$, $|X_\beta| < \kappa$, now we prove $\left|\bigcup_{\beta<\gamma} X_\beta\right| \neq \kappa$

By 1, let U be a κ-complete non-principal ultrafilter on κ. For any $\beta < \kappa$, since $|X_\beta| < \kappa$, $X_\beta \notin U$. But U is κ-complete, so $\bigcup_{\beta<\gamma} X_\beta \notin U$

□

5 复习

没有良序集同构于真前段

递归定理