3/4 points

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Exercise 1. $(\mathbb{C}, +, \cdot)$ is an algebraically closed field. Show that the algebraic set $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$ is reducible, i.e., not a variety

+1

Proof. Since $x^2+y^2 = (x+yi)(x-yi)$, $\{(x, y) \in \mathbb{C}^2 : x^2+y^2 = 0\} = \{(x, y) \in \mathbb{C}^2 : x + yi = 0\} \cup \{(x, y) \in \mathbb{C}^2 : x - yi = 0\}$ □

Exercise 2. Consider the theory of dense linear orders. Let $\varphi(x, y)$ be the formula $x < y$. One can show that $\varphi(x, y)$ has dichotomy property. Show by giving an example that D_3 is consistent

+1

Proof. Consider

- $y : 0$
  - $y_0 : -4$
    - $y_{00} : -6$
      - $x_{000} : -7$
      - $x_{001} : -5$
    - $y_{01} : -2$
      - $x_{010} : -3$
      - $x_{011} : -1$
  - $y_1 : 4$
    - $y_{10} : 2$
      - $x_{100} : 1$
      - $x_{101} : 3$
    - $y_{11} : 6$
      - $x_{110} : 5$
      - $x_{111} : 7$

□

Exercise 3. In the structure $M = (\mathbb{R}, +, \cdot, 0, 1, \leq)$, let $\varphi(\bar{x}, \bar{y})$ be the formula $x_1y_1 + x_2y_2 = 1$. Thus $\varphi(\mathbb{R}^2, \bar{b})$ is a line for most $\bar{b} \in \mathbb{R}^2$. It turns out that the formula φ does not have the dichotomy property. Find the largest n s.t. D_n is consistent

$\varphi(x,y;a,b)$ is $ax+by=1$, not $ab+xy=1$

No it's $n=2$.

+0.25 for a correct analysis of the wrong formula. (sorry that the x's and y's were confusing)

Proof. Largest n is 1. For a fixed $\bar{y}=(a,b)$ with $ab \neq 0$, we could take $\bar{x}_0$ on the line of $xy = 1-ab$ and $\bar{x}_0$ outside the line.

$ax+by=1$ not $ab+xy=1$ that's a hyperbola not a line

Now for $n=2$, suppose we have $\bar{y}=(a,b)$, $\bar{y}_0=(a_0,b_0)$, $\bar{y}_1=(a_1,b_1)$, $\bar{x}_{ij}=(a_{ij},b_{ij})$ for $i,j=0,1$ and D_n is consistent. Then since $\varphi(\bar{x}_{00},\bar{y})$ and $\varphi(\bar{x}_{01},\bar{y})$.

Suppose $ab=1$, then $a_{00}b_{00}=a_{01}b_{01}=0$. Since $\varphi(\bar{x}_{00},\bar{y}_0)$, $a_0b_0=1$ and hence $\varphi(\bar{x}_{01},\bar{y}_1)$, a contradiction.

Now since $ab \neq 1$, $\bar{x}_{00}$ and $\bar{x}_{01}$ are on the same line $xy=1-ab$, and there is no such $\bar{y}_0$ to get a line $xy=1-a_0b_0$ to isolate $\bar{x}_{00}$ and $\bar{x}_{01}$.

Thus D_2 is inconsistent $\square$

Exercise 4. Let T be a complete theory of the structure $(\mathbb{Z},+,-,0)$. Show that T is not $\aleph_0$-stable

+0.75 (lose 1/4 for several errors in the definitions)

Proof. Suppose we are working in base-2 system.

Given $\sigma \in 2^{<\omega}$, let $\phi_{\sigma 0}(x) = \exists y(x = y\cdot(10)^{\mathrm{lh}(\sigma)+2}+\sigma)$ and $\phi_{\sigma 1}(x) = \exists(x = y\cdot(10)^{\mathrm{lh}(\sigma)+2}+\sigma+1\cdot(10)^{\mathrm{lh}(\sigma)+1})$ where $\mathrm{lh}(\sigma)$ denotes the length of σ. Then $\phi_{\sigma i}(x) \Leftrightarrow x$ extends σi for $i=0,1$. Thus we have a tree

ϕ_{00} is $\exists y\,(x = y\cdot(10)^{3}+0)$ [$\mathrm{lh}(\sigma)+2 = 3$]

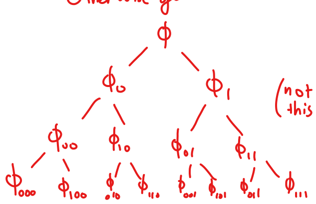
$(x = y \times 1000 + 0)$

so x extends 000

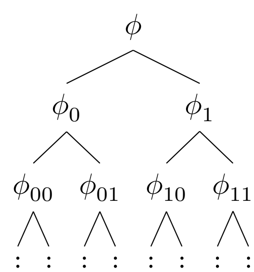
I think maybe the +2 and +1 are off by one.

Might want to reverse σ

Otherwise get



(not this →)



$\phi_{0110101}$ is

$\exists y\,(x = y\cdot 10^8 + 011010 + 10^7)$

$x = y\cdot 10^8 + 10000000 + 011010$

$= y\cdot 10^8 + 10011010$

x extends 10011010

↖ seems like I ended up in wrong end

0110101

10011010 ??

where ϕ is $x=x$.

Now note that for any $\sigma \in 2^{<\omega}$ $\phi_\sigma \leftrightarrow \phi_{\sigma 0} \vee \phi_{\sigma 1}$ and $\phi_{\sigma i} \vDash \neg\phi_{\sigma(1-i)}$ for $i=0,1$. For each $f:\omega\to 2$, $[\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \cdots$ and since $S_1(\mathbb{Z})$ is compact, there is $p_f \in \bigcap_{i\in\omega}[\phi_{f|i}]$. If $f,g\in 2^\omega$ and $f\neq g$, then there is n s.t. $f(n)\neq g(n)$ and $f \mid n = g \mid n$. Then since $\phi_{f|(n+1)} \vDash \neg\phi_{g|(n+1)}$, $[\phi_{f|n+1}] \cap [\phi_{g|n+1}] = \emptyset$ and hence $p_f \neq p_g$. Thus $|S_1(\mathbb{Z})| \geq 2^{\aleph_0}$ $\square$

You might want to clarify that "$y\cdot 100$" means $\underbrace{y+\cdots+y}_{100 \text{ times}}$, since multiplication isn't in the structure.