

Fourier Series and the Fourier Transform

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1 Basic General Concept Connected with Fourier Series

1.1 Orthogonal Systems of Functions

a. Expansion of a vector in a vector space. During this course of analysis we have mentioned several times that certain classes of functions form vector spaces in relation to the standard arithmetic operations. Such, for example, are the basic classes of analysis, which consist of smooth, continuous, or integrable real, complex, or vector-valued functions on a domain $X \subset \mathbb{R}^n$.

In analysis, as a rule, it is necessary to consider "infinite linear combinations" - series of functions of the form

$$f = \sum_{k=1}^{\infty} \alpha_k f_k$$

The definition of the sum of the series requires that some topology (in particular, a metric) be defined in the vector space in question, making it possible to judge whether the difference $f - S_n$ tends to zero or not, where $S_n = \sum_{k=1}^n \alpha_k f_k$.

The main device used in classical analysis to introduce a metric on a vector space is to define some norm of a vector or inner product of vectors in that space. We are now going to consider only spaces endowed with an inner product (which, as before, we shall denote \langle, \rangle)

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Definition 1.1. The vectors \mathbf{x} and \mathbf{y} in a vector space endowed with an inner product \langle, \rangle are orthogonal (with respect to that inner product) if $\langle x, y \rangle = 0$.

Definition 1.2. The system of vectors $\{x_k : k \in K\}$ is orthogonal if the vectors in it corresponding to different values of the index k are pairwise orthogonal.

Definition 1.3. The system of vectors $\{e_k : k \in K\}$ is orthonormalized (or orthonormal) if $\langle e_i, e_j \rangle = \delta_{ij}$ for every pair of indices $i, j \in K$, where $\delta_{i,j}$ is the Kronecker symbol, that is

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Definition 1.4. A finite system of vectors x_1, x_2, \dots, x_n is linearly independent if the equality $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0$ is possible only when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

The main question that will interest us now is the question of expanding a vector in a given system of linearly independent vectors.

As is known from analytic geometry, expansions in orthogonal and orthonormal systems have many technical advantages over expansions in arbitrary linearly independent systems. In the orthogonal expansion, the coefficients of the expansion are easy to compute, it is easy to compute the inner product of two vectors from their coefficients in an orthogonal basis, and so on.

1.2 Examples of Orthogonal Systems of Functions

The inner product is defined as

$$\langle f, g \rangle = \int_X f \cdot \bar{g}(x) \, dx \quad (1)$$

on the vector space $\mathcal{R}_2(X, \mathbb{C})$ consisting of functions on the set $X \subset \mathbb{R}^n$ that are locally square-integrable.

Since $|f \cdot \bar{g}| \leq \frac{1}{2} (|f|^2 + |g|^2)$, the integral 1 converges and hence defines $\langle f, g \rangle$ unambiguously.

If we discussing real-valued functions, relations 1 is the real space $\mathcal{R}_2(x, \mathbb{R})$ reduces to the equality

$$\langle f, g \rangle = \int_X (f \cdot g)(x) \, dx. \quad (2)$$

Examples 1. We recall for integers m and n

$$\int_{-\pi}^{\pi} e^{imx} \cdot e^{-inx} dx = \begin{cases} 0, & \text{if } m \neq n, \\ 2\pi, & \text{if } m = n \end{cases} \quad (3)$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \neq 0 \\ 2\pi, & \text{if } m = n = 0 \end{cases} \quad (4)$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0 \quad (5)$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \neq 0 \\ 0, & \text{if } m = n = 0 \end{cases} \quad (6)$$

These relations show that $\{e^{inx}; n \in \mathbb{Z}\}$ is an orthogonal system of vectors in the space $\mathcal{R}_2([-\pi, \pi]; \mathbb{C})$ relative to the inner product 1. The **trigonometric system** $\{1, \cos nx, \sin nx; n \in \mathbb{N}\}$ is orthogonal in $\mathcal{R}_2([-\pi, \pi]; \mathbb{R})$. If we allow linear combinations, then by Euler's formulas $e^{inx} = \cos nx + i \sin nx$, $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, $\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$. We see that these two systems can be expressed linearly in terms of each other, that is, they are algebraically equivalent. For that reason the exponential system $\{e^{inx}; n \in \mathbb{Z}\}$ is called the trigonometric system or more precisely the trigonometric system in complex notation.

If the closed interval $[-\pi, \pi]$ is replaced by an arbitrary closed interval $[-l, l] \subset \mathbb{R}$, then by a change of variable one can obtain the analogous systems $\{e^{i\frac{\pi}{l}nx}; n \in \mathbb{Z}\}$ and $\{1, \cos \frac{\pi}{l}nx, \sin \frac{\pi}{l}nx; n \in \mathbb{N}\}$, which are orthogonal in the space $(\mathcal{R}_2([-l, l]), \mathbb{C})$ and $(\mathcal{R}_2([-l, l]), \mathbb{R})$ and also the corresponding orthonormal systems

$$\left\{ \frac{1}{\sqrt{2l}} e^{i\frac{\pi}{l}nx}; n \in \mathbb{Z} \right\}$$

$$\left\{ \frac{1}{\sqrt{2l}}, \frac{1}{\sqrt{l}} \cos \frac{\pi}{l}nx, \frac{1}{\sqrt{l}} \sin \frac{\pi}{l}nx \right\}$$

Examples 2. Let I_x be an interval in \mathbb{R}^m and I_y an interval in \mathbb{R}^n , and let $f_i(x)$ be an orthogonal system of functions in $\mathcal{R}_2(I_x, \mathbb{R})$ and $g_j(y)$ an orthogonal system of functions in $\mathcal{R}_2(I_y, \mathbb{R})$. Then, the system of functions $\{u_{ij}(x, y) = f_i(x)g_j(y)\}$ is orthogonal in $\mathcal{R}_2(I_x \times I_y, \mathbb{R})$.

1.3 Orthogonalization

It is well-known that in a finite-dimensional Euclidean space, starting with a linearly independent system of vectors, there is a canonical way of constructing an orthogonal and even orthonormal system of vectors equivalent to the given system, using the Gram-Schmit orthogonalization process. By the same method one can obviously orthonormalize any linearly independent system of vectors $\psi_1, \psi_2, \dots, \psi_n$ in any vector space having an inner product. The Gram-Schmidt orthogonalization

$$\begin{aligned} (1) \quad \varphi_1 &= \frac{\psi_1}{\|\psi_1\|} \\ (2) \quad \varphi_2 &= \frac{\psi_2 - \langle \psi_2, \varphi_1 \rangle \varphi_1}{\|\psi_2 - \langle \psi_2, \varphi_1 \rangle \varphi_1\|} \quad \dots \\ (3) \quad \varphi_n &= \frac{\psi_n - \sum_{k=1}^{n-1} \langle \psi_n, \varphi_k \rangle \varphi_k}{\|\psi_n - \langle \psi_n, \varphi_k \rangle \varphi_k\|} \end{aligned}$$

Examples 3. *The process of orthogonalizing the linear independent system*

$$1, x, x^2, \dots$$

*in $\mathcal{R}_2([-1, 1], \mathbb{R})$ leads to the system of orthogonal polynomials known as the **Legendre polynomials**,*

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n (x^2 - 1)^n}{dx^n}$$

The orthonormalized Legendre polynomials have the form

$$\hat{P}_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x)$$

Theorem 1.1. *(Pythagoras)*

1. If $\{x_i\}$ is a system of mutually orthogonal vectors and $x = \sum_i x_i$, then

$$\|x\|^2 = \sum_i \|x_i\|^2$$

2. If $\{e_i\}$ is a system of mutually orthonormal vectors and $x = \sum_i x^i e_i$,

$$\text{then } \|x\|^2 = \sum_i |x^i|^2$$

2 Fourier Series and Fourier Coefficients

2.1 Definition of the Fourier Coefficients and the Fourier Series

Let $\{e_i\}$ be an orthonormal system and $\{l_i\}$ be an orthogonal system of vectors in a space X with inner product \langle, \rangle .

Suppose that $x = \sum x^i l_i$, the coefficient x^i in this expansion of the vector x can be found:

$$x^i = \frac{\langle x, l_i \rangle}{\langle l_i, l_i \rangle}$$

If $l_i = e_i$, the expansion become even simple:

$$x^i = \langle x, e_i \rangle$$

Definition 2.1. The number $\left\{ \frac{\langle x, l_i \rangle}{\langle l_i, l_i \rangle} \right\}$ are the **Fourier coefficients** of the vector $x \in X$ in the orthogonal system $\{l_i\}$.

Definition 2.2. If X is a vector space with inner product \langle, \rangle and $l_1, l_2, \dots, l_n, \dots$ is an orthogonal system of nonzero vectors in X , then for each vector $x \in X$ one can form the series

$$x \sim \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle} l_k \quad (7)$$

This series is the **Fourier series** of x in the orthogonal system $\{l_k\}$. In the case of orthonormal system $\{e_k\}$ the Fourier series of a vector $x \in X$ has a particular simple expression:

$$x \sim \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \quad (8)$$

Examples 4. Let $f(x) \in X = \mathcal{R}_2([- \pi, \pi], \mathbb{R})$ there corresponds a Fourier series

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} a_k(f) \cos kx + b_k(f) \sin kx$$

in this system. The Fourier coefficients are defined as:

$$a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, k = 0, 1, 2, \dots \quad (9)$$

$$b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, k = 0, 1, 2, \dots \quad (10)$$

Examples 5. Let us consider the orthogonal system $\{e^{ikx}, k \in \mathbb{Z}\}$ in the space $\mathcal{R}_2([-\pi, \pi], \mathbb{C})$. Let $f(x) \in \mathcal{R}_2([-\pi, \pi], \mathbb{C})$, then the coefficients

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

It can be proved that

$$c_k = \begin{cases} \frac{1}{2}(a_k - ib_k), & \text{if } k \geq 0 \\ \frac{1}{2}(a_{-k} + ib_{-k}), & \text{if } k < 0 \end{cases} \quad (11)$$

2.2 Basic General Properties of Fourier Coefficients and Series

Lemma 2.1. 1. Let $\{l_k\}$ be a finite or countable system of non-zero pairwise orthogonal vectors in X , and suppose the Fourier series of $x \in X$ in the system $\{l_k\}$ converges to $x_l \in X$.

2. Then in the representation $x = x_l + h$ the vector h is orthogonal to x_l , moreover, h is orthogonal to the entire linear subspace generated by the system of vectors $\{l_k\}$, and also to its closure in X .

2.3 Bessel's Inequality

Taking account of the orthogonality of the vectors x_l and h in the decomposition $x = x_l + h$, we find by the Pythagorean theorem that $\|x\|^2 = \|x_l\|^2 + \|h\|^2 \geq \|x_l\|^2$. This relation, written in terms of Fourier coefficients, is called **Bessel's inequality**.

$$\|x_l\|^2 = \sum_k \left| \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle} \right|^2 \langle l_k, l_k \rangle \leq \|x\|^2 \quad (12)$$

Hence

$$\sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle} \leq \|x\|^2 \quad (13)$$

This is Bessel's inequality. It has a particularly simply appearance for an orthonormal system of vectors $\{e_k\}$:

$$\sum_k |\langle x, e_k \rangle|^2 \leq \|x\|^2 \quad (14)$$

Examples 6. For the trigonometric system Bessel's inequality has the form

$$\frac{|a_0(f)|^2}{2} + \sum_{k=1}^{\infty} |a_k(f)|^2 + |b_k(f)|^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2(x) dx \quad (15)$$

For the system $\{e^{ikx}; k \in \mathbb{Z}\}$ Bessel's inequality can be written

$$\sum_{-\infty}^{\infty} |c_k(f)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2(x) dx. \quad (16)$$

2.4 The Extremal Property of the Fourier Coefficients

Examples 7.

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 0, & -\pi < x < 0 \end{cases}$$

Examples 8.

$$f(x) = \begin{cases} x^2, & 0 < x < \pi \\ 0, & x = \pi \\ -x^2, & \pi < x < 2\pi \end{cases}$$

A more compact way of writing the Fourier series of a function $f(x)$, with period 2π , uses the variable subscript $n = 1, 2, \dots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos nx + b_n \sin nx]$$

We need to work out the Fourier coefficients (a_0, a_n and b_n) for given functions $f(x)$.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{2\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{2\pi} f(x) \cos nx dx \\ b_n &= \frac{1}{\pi} \int_{2\pi} f(x) \sin nx dx \end{aligned}$$