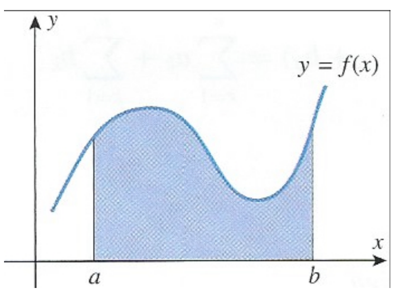
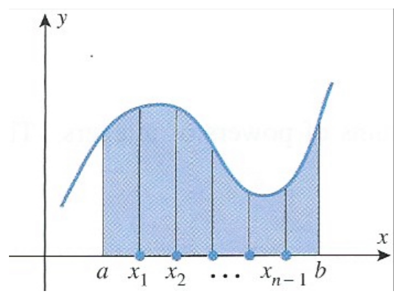
Multiple Integrals

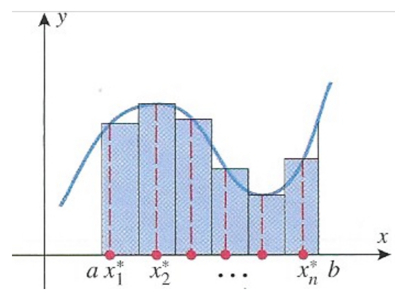
1 Double Integrals

Definite integrals appear when one solves

Area problem. Find the area A of the region R bounded above by the curve $y = f(x)$, below by the x-axis, and on the sides by $x = a$ and $x = b$.







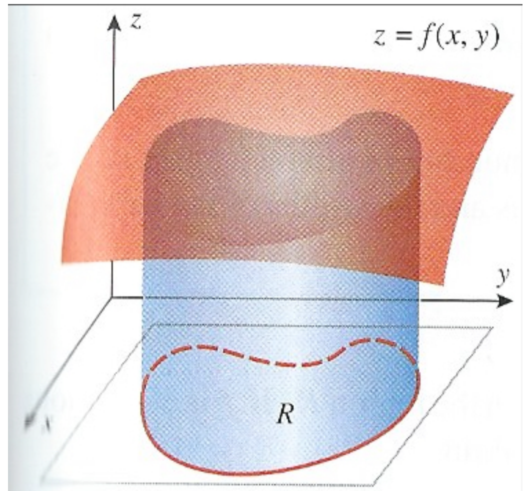
$$A = \int_a^b f(x)\,dx = \lim_{\max\ \Delta x_i \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Mass problem. Find the mass M of a rod of length L whose linear density (the mass per unit length) is a function $\delta(x)$, where x is the distance from a point of the rod to one of the rod's ends.

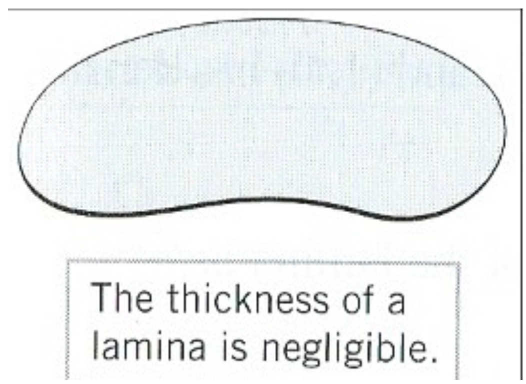
$$M = \int_0^L \delta(x)\,dx$$

Double integrals appear when one solves

Volume problem. Find the volume V of the solid G enclosed between the surface $z = f(x, y)$ and a region R in the xy-plane where $f(x, y)$ is continuous and nonnegative on R.

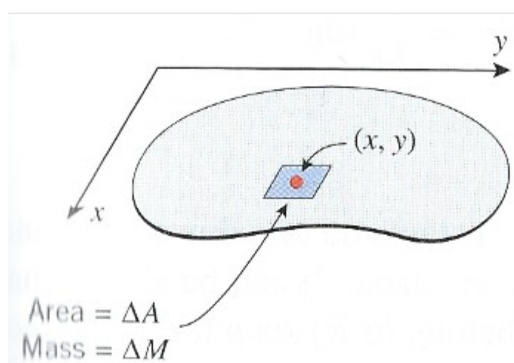


Mass problem. Find the mass M of a lamina (a region R in the xy-plane) whose density (the mass per unit area) is a continuous nonnegative function $\delta(x, y)$ defined as

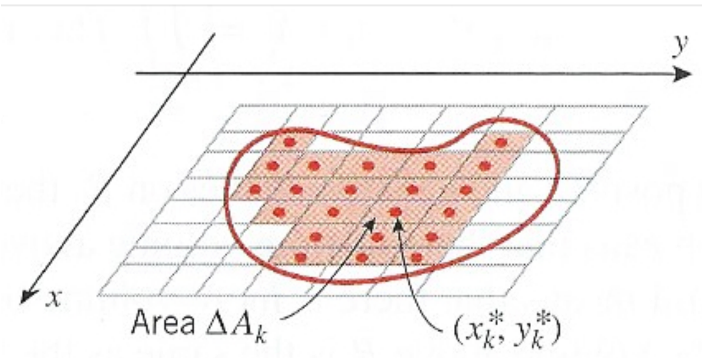


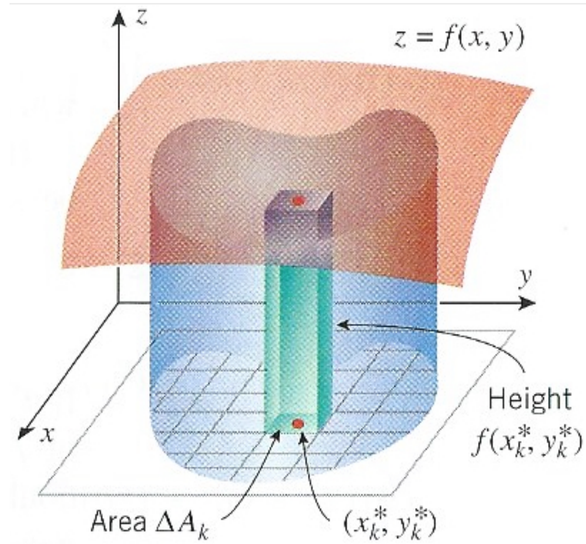
$$\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A}$$

where ΔM is the mass of the small rectangle of area ΔA which contains (x, y).



Let us consider the volume problem.





1. Divide the rectangle enclosing R into subrectangles, and exclude all those rectangles that contain points outside of R. Let n be the number of all the rectangles inside R, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the k-th subrectangle.

2. Choose any point (x_k^*, y_k^*) in the k-th subrectangle. The volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$ is $\Delta V_k = f(x_k^*, y_k^*)\Delta A_k$. Thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

   This sum is called the **Riemann sum**.

3. Take the sides of all the subrectangles to 0, and therefore the number of them to infinity, and get

$$V = \lim_{\max\ \Delta x_i, \Delta y_i \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \iint_R f(x, y)\, dA$$

   The last term is the notation for the limit of the Riemann sum, and it is called the **double integral** of $f(x, y)$ over R.

In what follows we identify

$$\lim_{\max\ \Delta x_i, \Delta y_i \to 0} \sum_{k=1}^{n} \cdots \equiv \lim_{n\to\infty} \sum_{k=1}^{n} \cdots$$

If f is continuous but not nonnegative on R then the limit represents a difference of volumes – above and below the xy-plane. It is called the net signed volume between R and the surface $z = f(x, y)$, and it is given by the limit of the corresponding Riemann sum that is the double integral of $f(x, y)$ over R

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

Similarly, the mass M of a lamina with density $\delta(x, y)$ is

$$M = \iint_R \delta(x,y)\,dA$$

Properties of double integrals

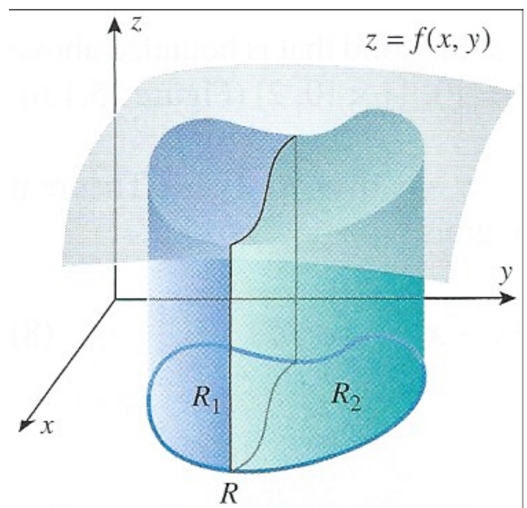
1. If f, g are continuous on R, and c, d are constants, then

$$\iint_R \big(c\, f(x,y) + d\, g(x,y)\big)\, dA = c \iint_R f(x,y)\, dA + d \iint_R g(x,y)\, dA$$

2. If R is divided into two regions R_1 and R_2, then

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.



Double integrals over rectangular regions

The symbols

$$\int_a^b f(x,y)\,dx \quad \text{and} \quad \int_c^d f(x,y)\,dy$$

where in the first integral y is fixed while in the second integral x is fixed, denote **partial definite integrals**.

Examples.

$$(i)\ \int_1^2 \sin(2x-3y)\,dx\,, \quad (ii)\ \int_0^1 \sin(2x-3y)\,dy\,.$$

We can then integrate the resulting functions of y and x with respect to y and x, respectively. This two-stage integration process is called **iterated or repeated integration**.
Notation

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

These integrals are called **iterated integrals**.

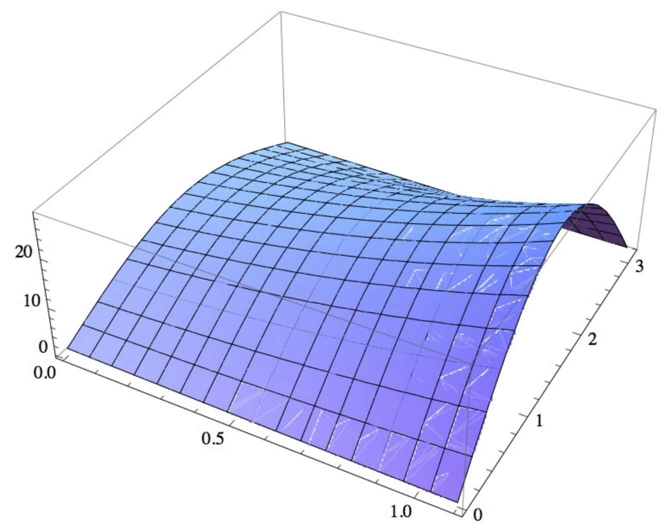
Example.

$$(i)\ \int_0^1 \int_1^2 \sin(2x-3y)\,dx\,dy\,, \quad (ii)\ \int_1^2 \int_0^1 \sin(2x-3y)\,dy\,dx$$

Theorem. Let R be the rectangle $a \le x \le b$, $c \le y \le d$. If $f(x, y)$ is continuous on R then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example. Use a double integral to find V under the surface



$$z = 3\pi e^x \sin y + e^{-x}$$

and over the rectangle

$$R = \{(x, y) : 0 \le x \le \ln 3\,, 0 \le y \le \pi\}$$

$$V = \frac{38}{3}\pi \approx 39.7935 > 0$$

2 Double integrals over nonrectangular regions

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

Replace $c \to g_1(x)$, $d \to g_2(x)$. Then

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

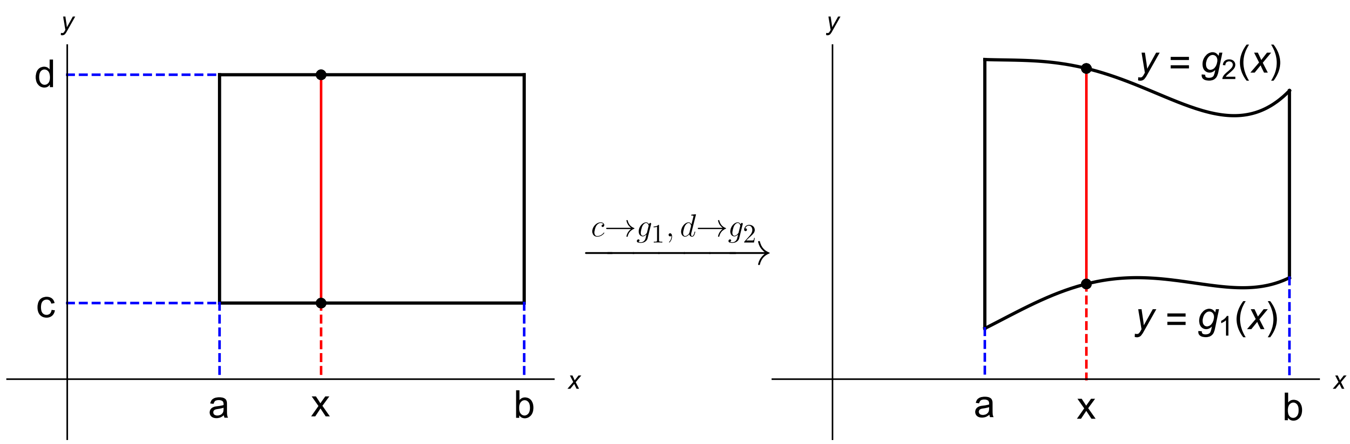


Figure 1: The rectangle becomes a type I region: $g_2(x) \geq g_1(x)$.

Theorem. If R is a type I region on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Example. Find the volume V of the solid enclosed by the surfaces $z = 0$, $y^2 = x$, and $x + z = 1$.

$$V = \frac{8}{15}$$

Similarly, in

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

we replace $a \to h_1(y)$, $b \to h_2(y)$. Then

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

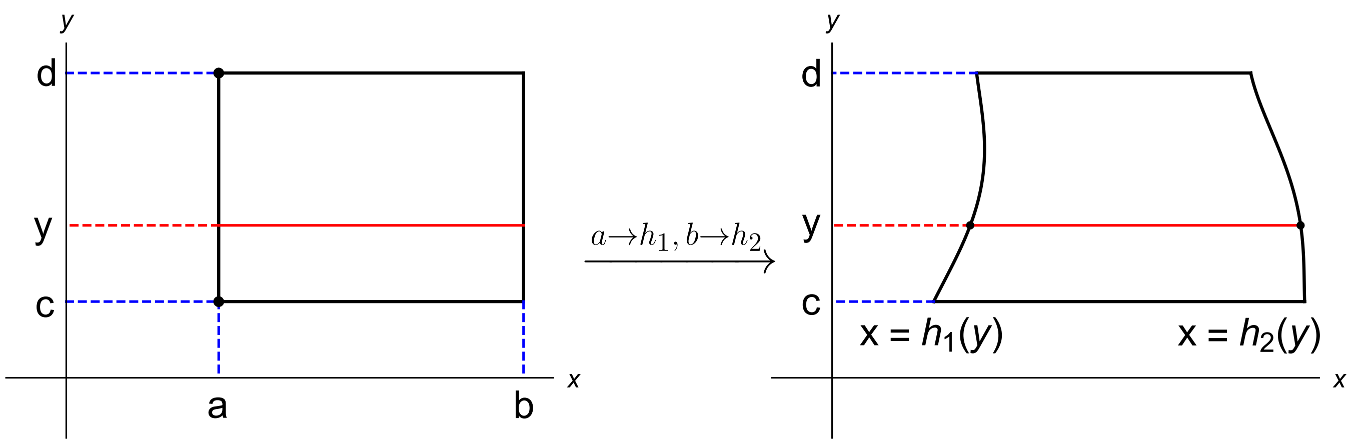


Figure 2: The rectangle becomes a type II region: $h_2(y) \geq h_1(y)$.

Theorem. If R is a type II region on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example. Find the volume V of the solid enclosed by the surfaces $z = 0$, $y^2 = x$, and $x + z = 1$.

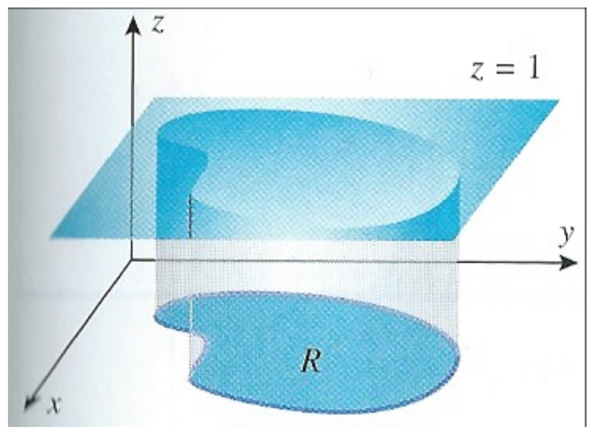
Some regions belong to both type I and II, e.g. the region from the previous example, a disc and a triangle.

Reversing the order of integration. Example.

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy = \frac{1}{3}\left(e^8 - 1\right)$$

Area calculated as a double integral

The solid enclosed between the plane $z = 1$, and a region R is a right cylinder with base R, height $h = 1$, and with cross-sectional area A equal to the area of R.
Its volume is

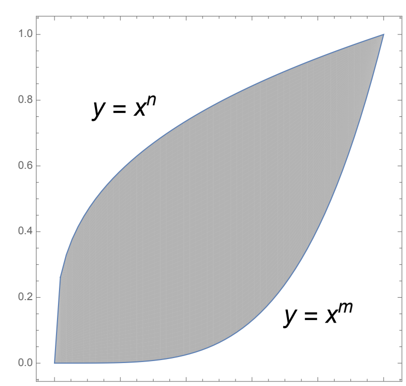


$$V = A\,h = A = \iint_R 1\, dA$$

Thus

$$\text{Area of } R = \iint_R 1\, dA = \iint_R dA$$

Example. Find A of R enclosed between $y = x^m$ and $y = x^n$.



3 Double integrals in polar coordinates

Polar coordinates

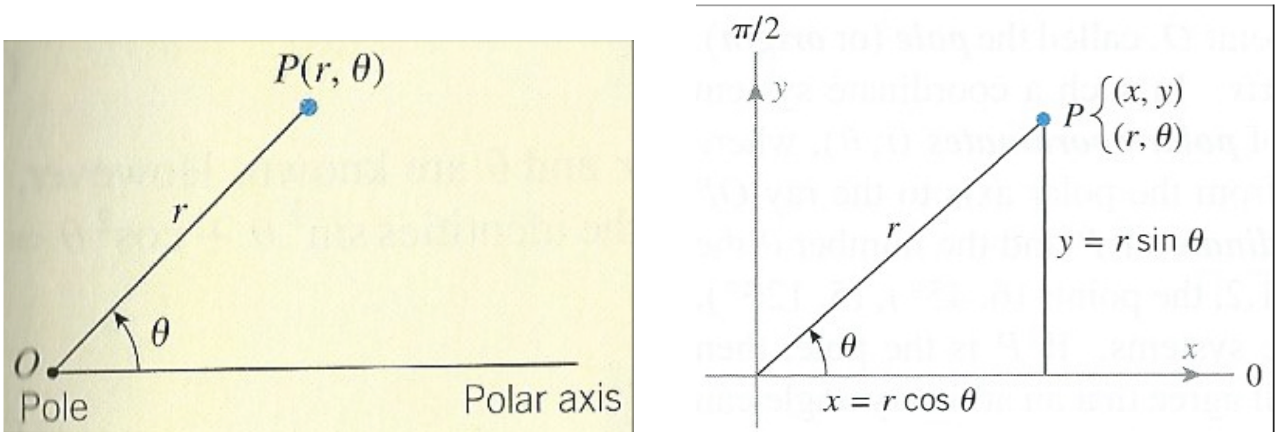


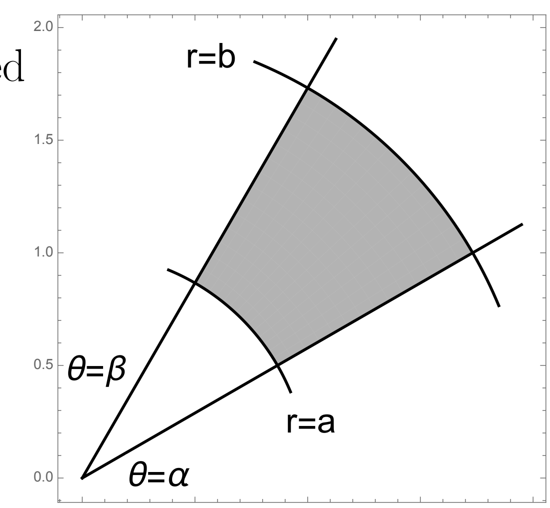
Figure 3: $x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad r^2 = x^2 + y^2\,, \quad \tan\theta = \frac{y}{x}\,.$

We identify $$(r, \theta + 2\pi n) \sim (r, \theta)\,, \quad n = \pm 1\,, \pm 2\,, \ldots$$

and $$(-r, \theta) \sim (r, \theta + \pi)$$

because they give the same x, y coordinates.

The graph of $r = a$ is the circle of radius a centred at O.
The graph of $\theta = \alpha$, $r \geq 0$ is the ray making an angle of α with the polar axis.



A polar rectangle is a region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two circles $r = a$, $r = b$.

Its area is

$A = \frac{1}{2}b^2(\beta - \alpha) - \frac{1}{2}a^2(\beta - \alpha)$

$\quad = \frac{1}{2}(b + a)(b - a)(\beta - \alpha)$

$\quad = \bar{r}\,\Delta r\,\Delta\theta$, where $\bar{r} = \frac{1}{2}(b + a)$,

$\Delta r = b - a$ is the radial thickness, $\Delta\theta = \beta - \alpha$ is the central angle.

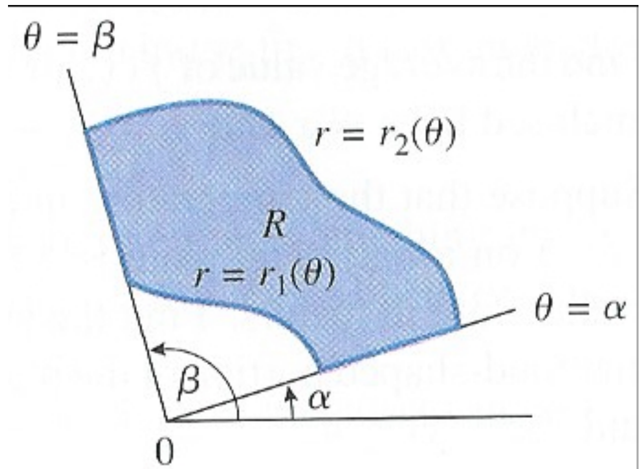
A simple polar region

is a region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two continuous polar curves $r = r_1(\theta)$, $r = r_2(\theta)$ which satisfy

(i) $\alpha \leq \beta$,

(ii) $\beta - \alpha \leq 2\pi$,

(iii) $0 \leq r_1(\theta) \leq r_2(\theta)$.



Examples.

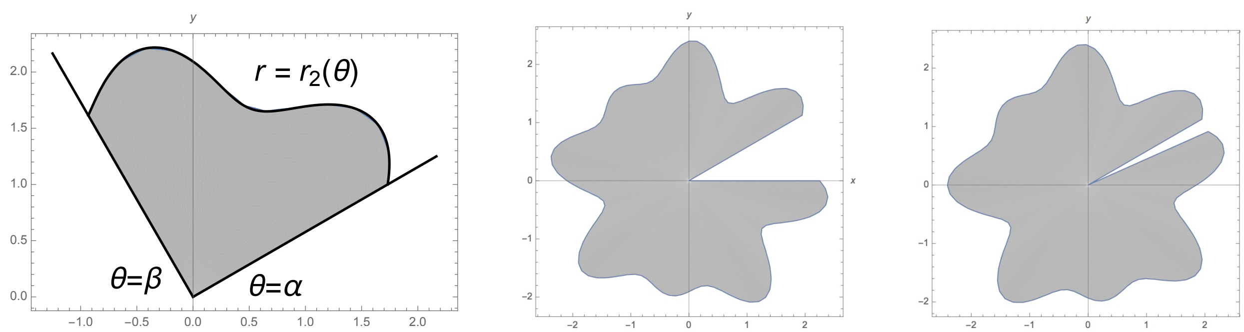
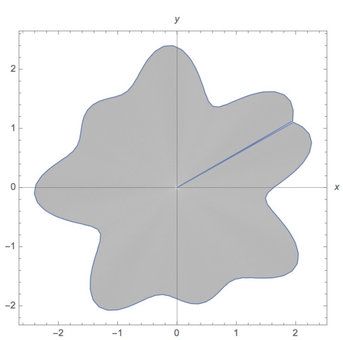
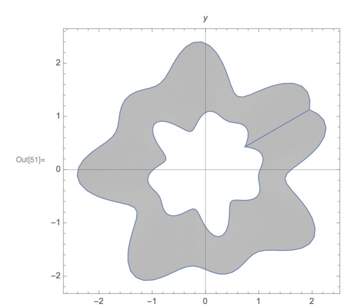


Figure 4: $r_1(\theta) = 0$ and $\beta - \alpha < 2\pi$

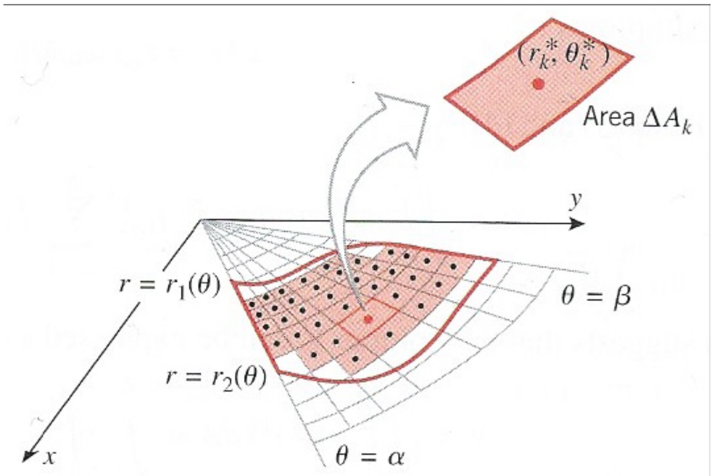


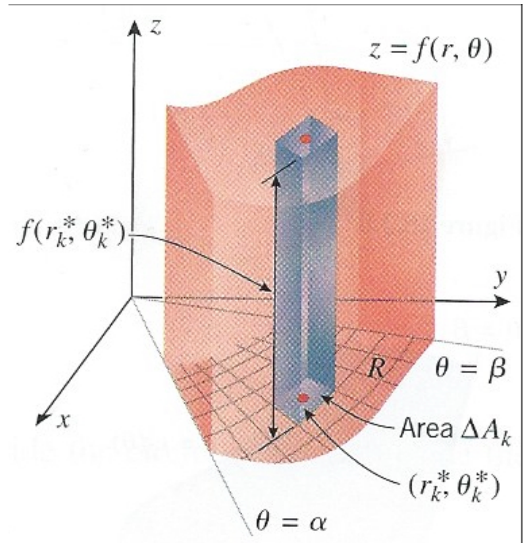
$r_1(\theta) = 0$ and $\beta - \alpha = 2\pi$



$r_1(\theta) \neq 0$ and $\beta - \alpha = 2\pi$

Let us consider the volume problem in polar coordinates.





1. Divide the rectangle enclosing R into **polar** subrectangles, and exclude all those rectangles that contain points outside of R. Let n be the number of all the rectangles inside R, and let

   $\Delta A_k = \bar{r}_k \, \Delta r_k \, \Delta\theta_k$ be the area of the k-th polar subrectangle.

2. Choose any point (r_k^*, θ_k^*) in the k-th subrectangle. The volume of a right cylinder with base area ΔA_k and height $f(r_k^*, \theta_k^*)$ is $\Delta V_k = f(r_k^*, \theta_k^*)\Delta A_k$. Thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\, \bar{r}_k \, \Delta r_k \, \Delta\theta_k$$

   This sum is called the **polar Riemann sum**.

3. Take the sides of all the subrectangles to 0, and therefore the number of them to infinity, and get

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k = \iint_R f(r, \theta)\, dA$$

   The last term is the notation for the limit of the Riemann sum, and it is called the **polar double integral** of $f(r, \theta)$ over R.

The limit of the polar Riemann sum is the same for any choice of points (r_k^*, θ_k^*). Let us choose (r_k^*, θ_k^*) to be the centre of the k-th polar rectangle, that is $r_k^* = \bar{r}_k$, $\theta_k^* = \bar{\theta}_k = \frac{1}{2}(\theta_{k-1} + \theta_k)$. Then, the polar double integral is given by

$$\iint_R f(r, \theta)\, dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{r}_k, \bar{\theta}_k)\, \bar{r}_k\, \Delta r_k\, \Delta\theta_k$$

This formula is similar to the one for the double integral in rectangular coordinates, and it is valid for any region R.

Theorem. If R is a simple polar region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two continuous polar curves $r = r_1(\theta)$, $r = r_2(\theta)$, and if $f(r, \theta$ is continuous on R, then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

Example. Find the volume of the solid below $z = 1 - x^2 - y^2$, inside of $x^2 + y^2 - x = 0$, and above $z = 0$. $Answer:\ V = 5\pi/32$

$$\text{Area of } R = \iint_R dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} r\, dr\, d\theta = \frac{1}{2} \int_\alpha^\beta \left(r_2(\theta)^2 - r_1(\theta)^2\right) d\theta$$

Example. Find A of R in the first quadrant that is outside $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$. $Answer:\ A = 4 + \pi/2$

Example. Find A enclosed by the three-petaled rose $r = \cos 3\theta$. $Answer:\ A = \pi/4$

Converting double integrals from rectangular to polar coordinates

$$\iint_R f(x,y)\,dA = \iint_{\substack{\text{appropriate}\\\text{limits}}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

It is especially useful if $f(x,y) = g(x^2+y^2) = g(r)$

or $f(x,y) = g(y/x) = g(\tan\theta)$

Example.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \frac{\pi}{5}$$

Example.

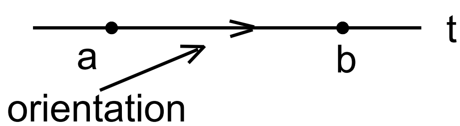
$$\int_{0}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \frac{3}{2}$$

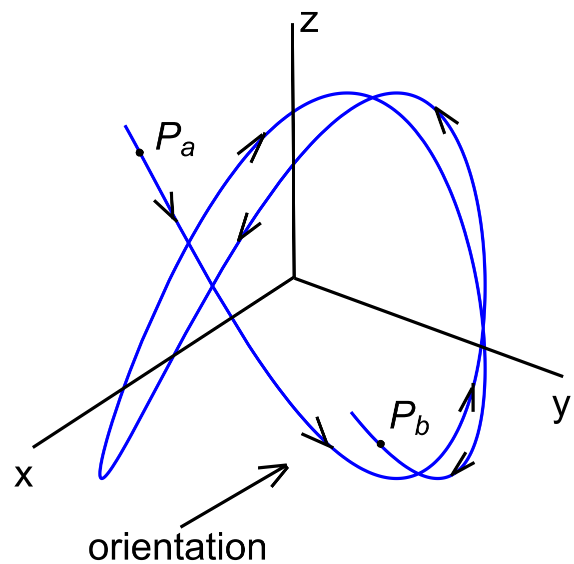
Example.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

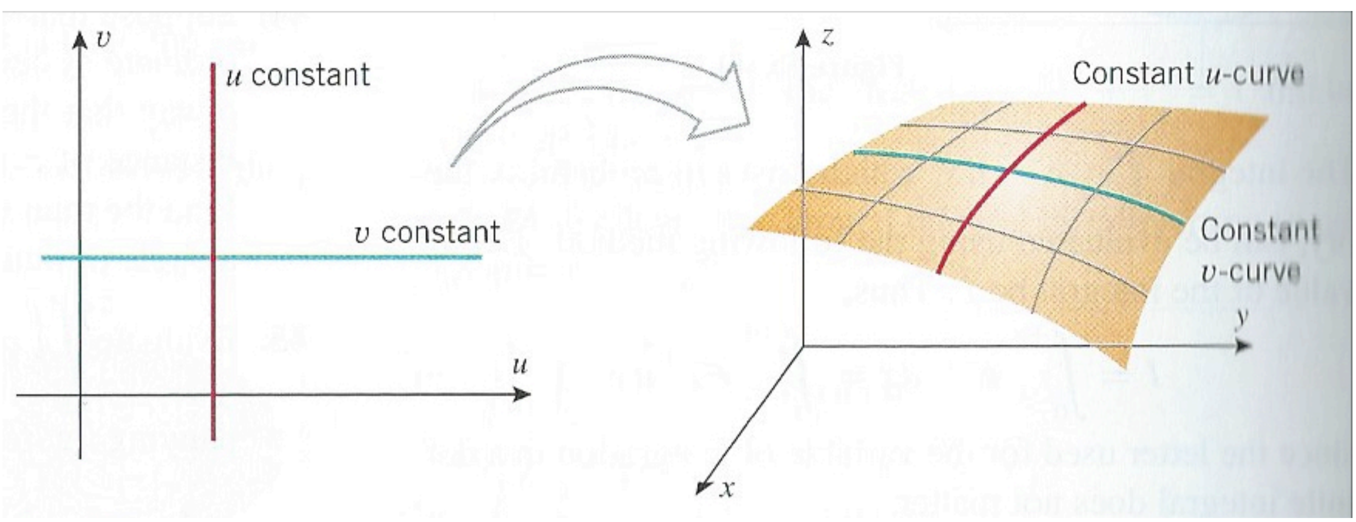
4 Parametric surfaces

A curve: $x = x(t)\,,\ y = y(t)\,,\ z = z(t)\,,\ t_0 \le t \le t_1$





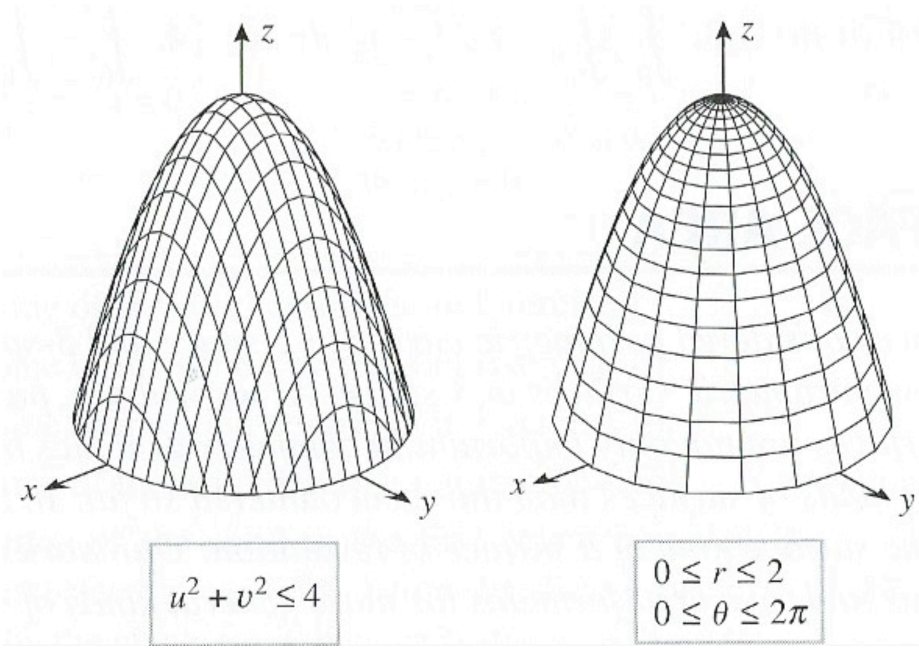
A surface: $x = x(u,v)\,,\ y = y(u,v)\,,\ z = z(u,v)\,,\ (u,v) \in R$

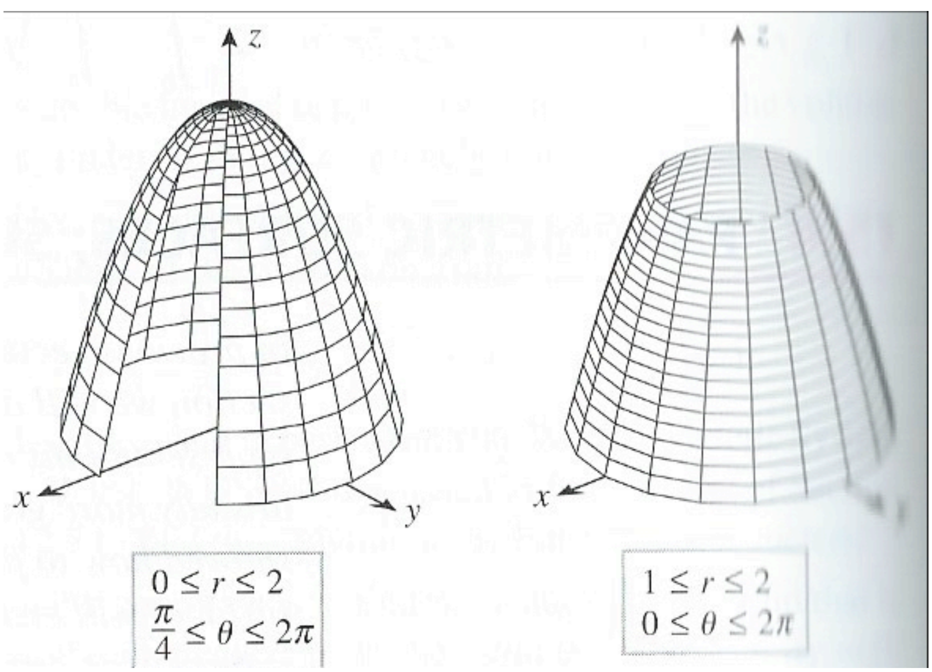


Example. $z = 4 - x^2 - y^2$

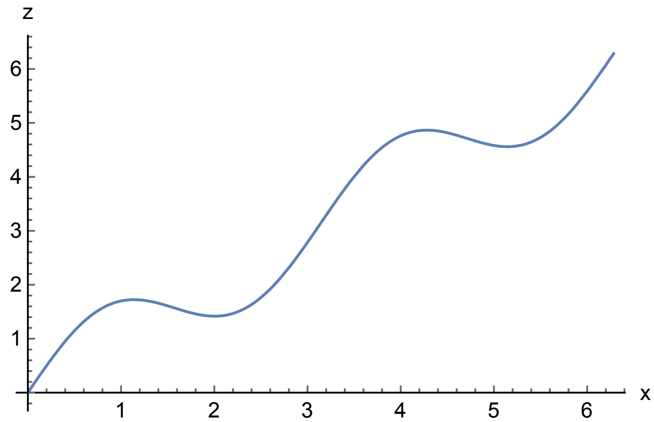
Rectangular coordinates: $x = u$, $y = v$, $z = 4 - u^2 - v^2$

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = 4 - r^2$



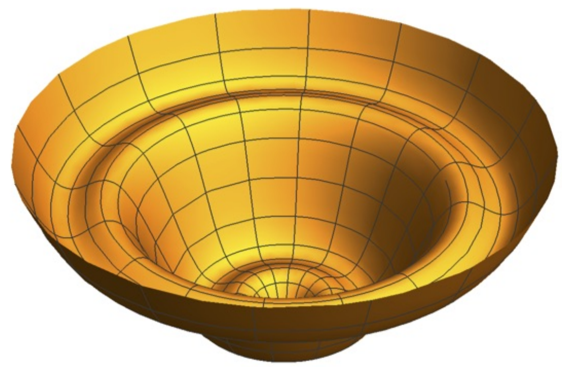


Polar coordinates are useful for **surfaces of revolution** generated by revolving a curve $z = f(x)$, $x \geq 0$ in the xz-plane, or, equivalently, a curve $z = f(y)$, $y \geq 0$ in the yz-plane about the z-axis.



These surfaces are graphs of functions $z = f(\sqrt{x^2 + y^2})$, and can be parametrised as

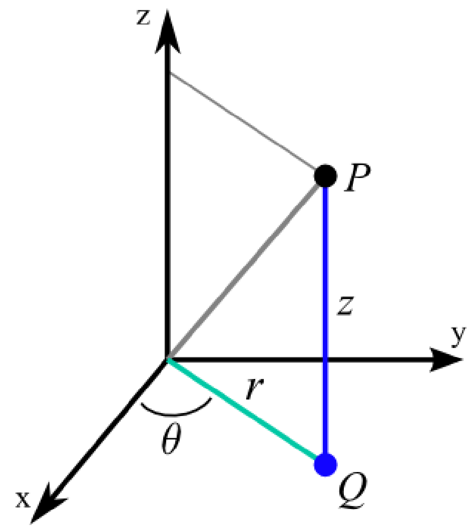
$$x = r\cos\theta,\ y = r\sin\theta,\ z = f(r)$$



Example. A right cone of height h and base radius a oriented along the z-axis, with vertex pointing up, and with the base located at $z = 0$.

$$x = r\cos\theta\,,\ y = r\sin\theta\,,\ z = h(1 - \frac{r}{a})\,, \quad 0 \leq r \leq a\,,\ 0 \leq \theta \leq 2\pi$$

Cylindrical coordinates



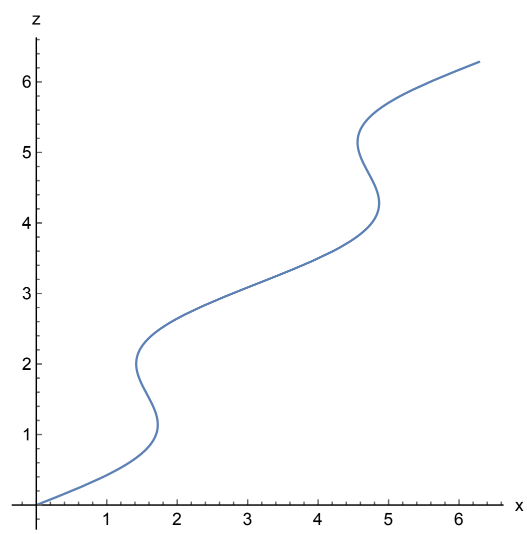
$$x = r\cos\theta\,,\ \ y = r\sin\theta\,,\ \ z$$
$$r \geq 0\,,\ \ 0 \leq \theta \leq 2\pi$$

Plane: $z = \text{const}$

Circular cylinder: $r = \text{const}$

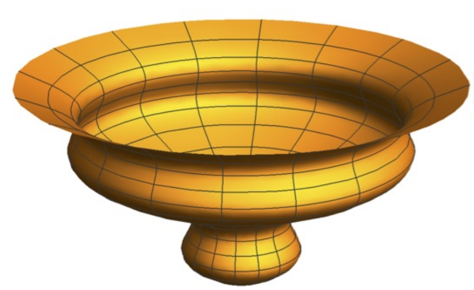
Half-plane: $\theta = \text{const}$

Cylindrical coordinates are useful for **surfaces of revolution** generated by revolving a curve $x = f(z)$ in the xz-plane or, equivalently, a curve $y = f(z)$ in the yz-plane about the z-axis.



These surfaces are parametrised as

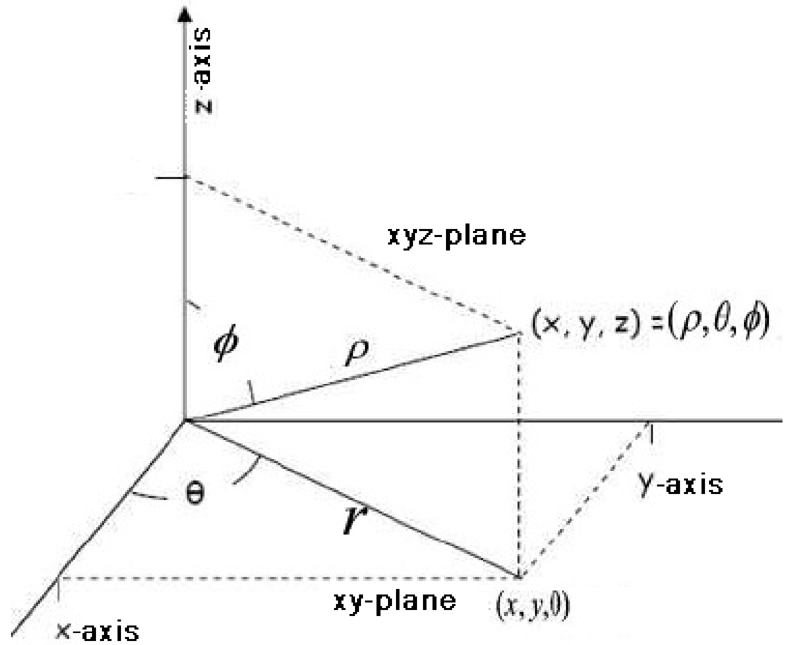
$$x = f(\zeta)\cos\theta,\ y = f(\zeta)\sin\theta,\ z = \zeta$$



Example. A right cone of height h and base radius a oriented along the z-axis, with vertex pointing up, and with the base located at $z = 0$.

$$x = (1-\frac{\zeta}{h})a\cos\theta\,,\ y = (1-\frac{\zeta}{h})a\sin\theta\,,\ z = \zeta\,,\quad 0 \leq \zeta \leq h\,,\ 0 \leq \theta \leq 2\pi$$

Spherical coordinates



$x = \rho \sin\phi \cos\theta\,,$

$y = \rho \sin\phi \sin\theta\,,$

$z = \rho \cos\phi\,,$

$\rho \geq 0\,,\ 0 \leq \theta \leq 2\pi\,,$

$0 \leq \phi \leq \pi\,.$

Sphere: $r = \text{const}$

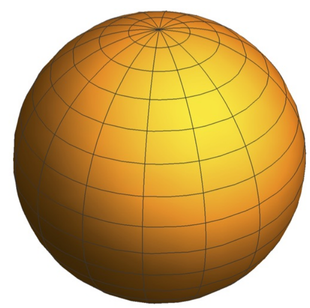
Half-plane: $\theta = \text{const}$

Cone: $\phi = \text{const}$

Example. $x^2 + y^2 + z^2 = 9 \;\Rightarrow\; \rho = 3$

$x = 3\sin\phi\cos\theta\,,\ y = 3\sin\phi\sin\theta\,,\ z = 3\cos\phi\,,$

$0 \leq \theta \leq 2\pi\,,\ 0 \leq \phi \leq \pi\,.$



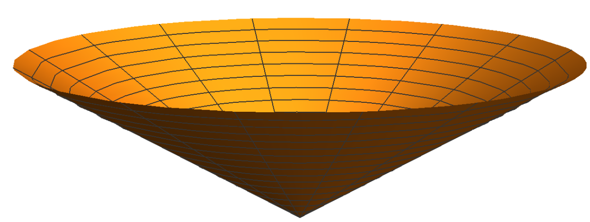
The constant ϕ-curves are the lines of **latitude**.

The constant θ-curves are the lines of **longitude**.

Example. $z = \sqrt{\frac{x^2+y^2}{3}} \;\Rightarrow\; \phi = \frac{\pi}{3}$

$x = \frac{\sqrt{3}}{2}\rho\cos\theta\,,\ y = \frac{\sqrt{3}}{2}\rho\sin\theta\,,\ z = \frac{1}{2}\rho\,,$

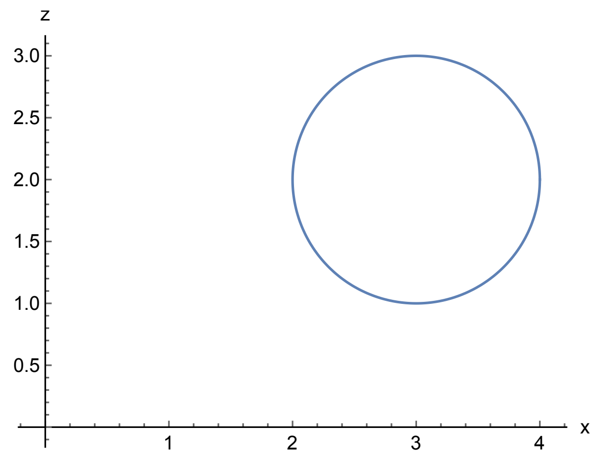
$\rho \geq 0\,,\ 0 \leq \theta \leq 2\pi\,.$



The constant ρ-curves are circles.

The constant θ-curves are half-lines.

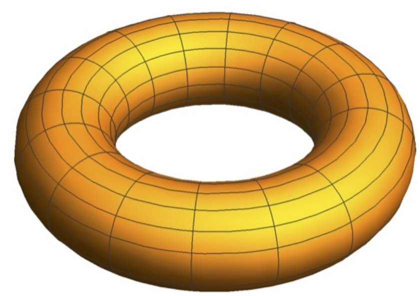
Sphere is an example of a **surface of revolution** generated by revolving a parametric curve $x = f(t)$, $z = g(t)$ or, equivalently, a parametric curve $y = f(t)$, $z = g(t)$ about the z-axis.



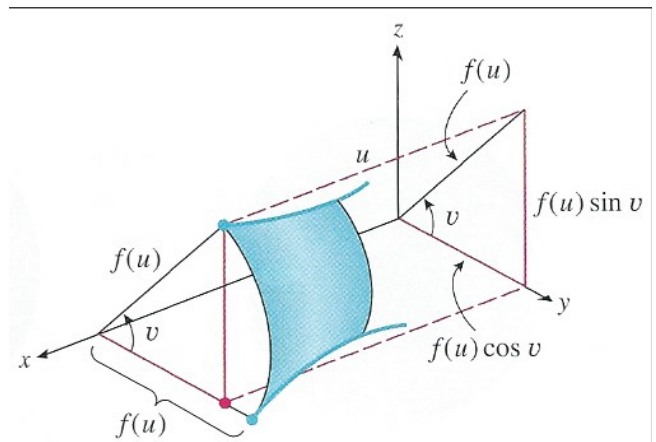
Such a surface is parametrised as

$$x = f(t)\cos\theta\,,\ \ y = f(t)\sin\theta\,,\ \ z = g(t)\,.$$

Example. Torus.



Similarly, we can get surfaces of revolution by revolving a parametric curve $y = f(u)$, $x = g(u)$ or, equivalently, a parametric curve $z = f(u)$, $x = g(u)$ about the x-axis.



They are parametrised as

$$\begin{aligned} x &= g(u)\,, \\ y &= f(u)\cos v\,, \\ z &= f(u)\sin v \end{aligned}$$

Finally, we can get surfaces of revolution by revolving a parametric curve $z = f(u)$, $y = g(u)$ or, equivalently, a parametric curve $x = f(u)$, $y = g(u)$ about the y-axis.

They are parametrised as

$$x = f(u)\sin v\,,\ \ y = g(u)\,,\ \ z = f(u)\cos v$$

Vector-valued functions of two variables

The vector form of parametric eqs for a surface

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$\vec{r}(u,v)$ is a vector-valued functions of two variables.

Partial derivatives

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\,\vec{i} + \frac{\partial y}{\partial u}\,\vec{j} + \frac{\partial z}{\partial u}\,\vec{k}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}\,\vec{i} + \frac{\partial y}{\partial v}\,\vec{j} + \frac{\partial z}{\partial v}\,\vec{k}$$

Tangent planes to parametric surfaces

Let σ be a parametric surface in 3-space.

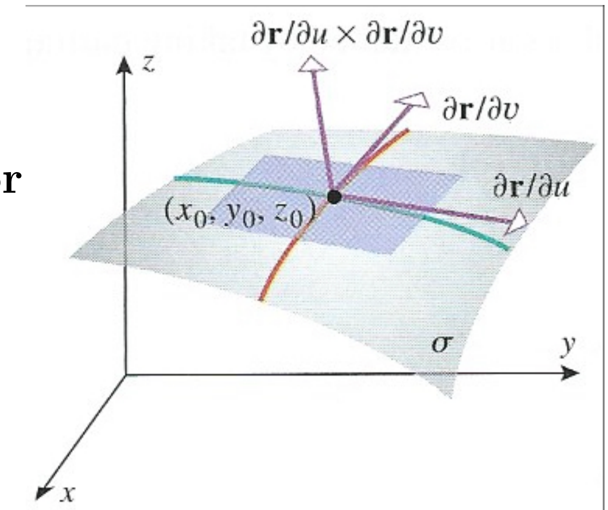
Definition. A plane is said to be tangent to σ at P_0 provided a line through P_0 lies in the plane if and only if it is a tangent line at P_0 to a curve on σ.

$\sigma:\ \vec{r}(u,v)\,,\ P_0(a,b,c) \in \sigma$

$a = x(u_0,v_0)\,,\ b = y(u_0,v_0)\,,\ c = z(u_0,v_0)$

If $\frac{\partial \vec{r}}{\partial u} \neq 0$ then it is tangent to the constant v-curve.
If $\frac{\partial \vec{r}}{\partial v} \neq 0$ then it is tangent to the constant u-curve.
Thus, if $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0$ at (u_0,v_0) then it is orthogonal to both tangent vectors and is normal to the tangent plane and the surface at P_0.

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}$$



is called the **principle normal vector** to the surface $\vec{r} = \vec{r}(u, v)$ at (u_0, v_0). Thus, the tangent plane equation is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

Example. Tangent plane to

$$x = uv\,,\ y = u\,, z = v^2 \quad \text{at} \quad (2, -1)$$

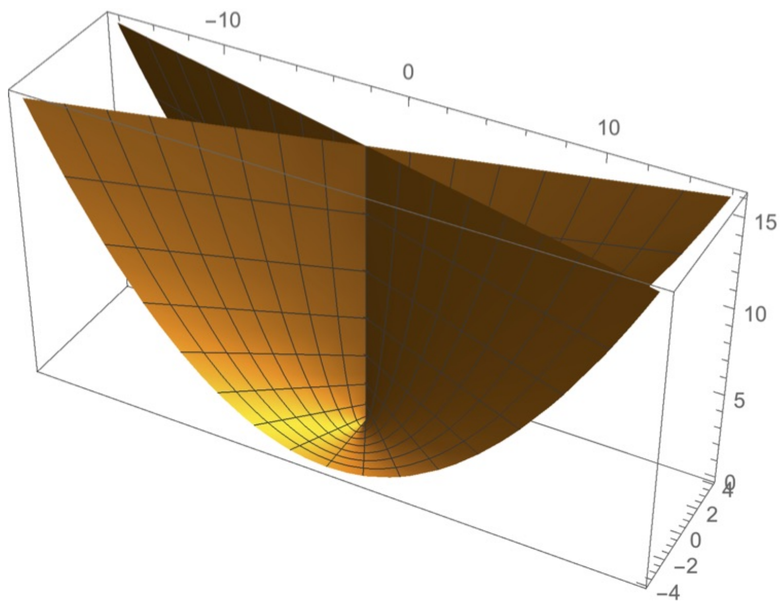
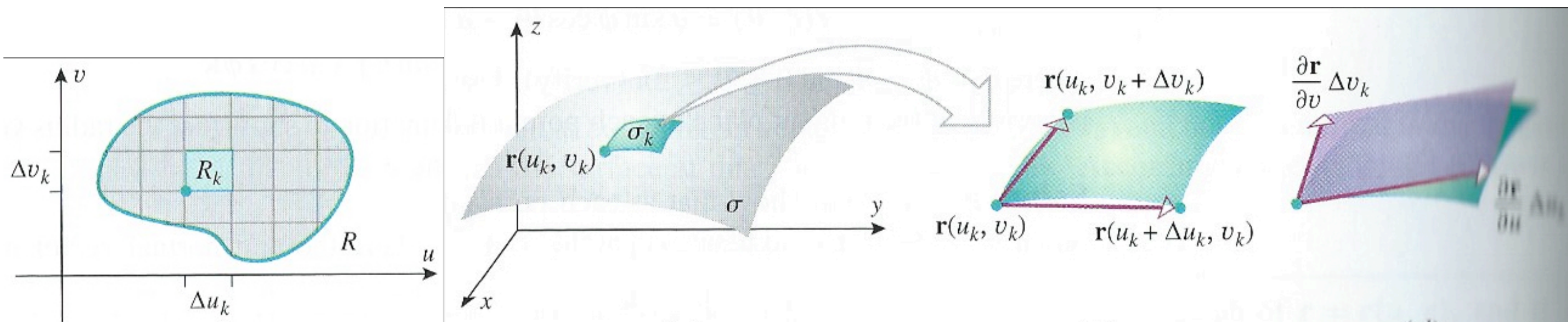


Figure 5: Whitney's umbrella.

$Answer:\ x + y + z = 1$

Surface area

Let $\vec{r} = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$ be a smooth parametric surface on a region R of the uv-plane, that is $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0$ on R, and therefore there is a tangent plane for every $(u,v) \in R$.



$R_k: \;\; \Delta A_k = \Delta u_k \Delta v_k$

$$\Delta S_k \approx \text{Area of parallelogram} = \left| \frac{\partial \vec{r}}{\partial u}\Delta u_k \times \frac{\partial \vec{r}}{\partial v}\Delta v_k \right|$$

$$= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u_k \Delta v_k = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta A_k$$

Thus,

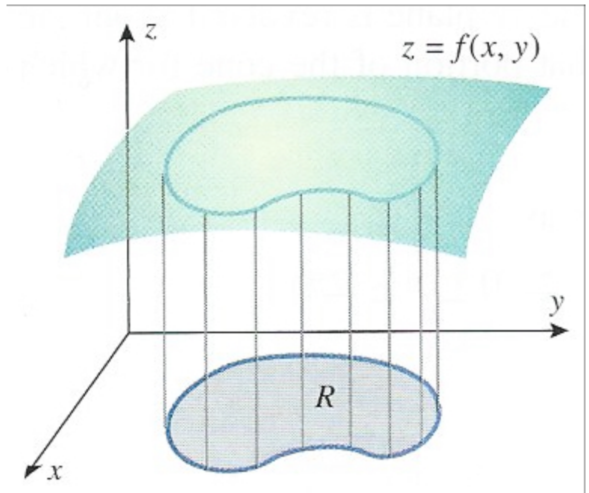
$$S \approx \sum_{k=1}^{n} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta A_k$$

In the limit $n \to \infty$

$$S = \iint_R \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

Examples. Surfaces of revolution: Sphere, cone, torus

Surface area of surfaces of the form $z = f(x,y)$



$$x = u\,,\ y = v\,,\ z = f(u,v)\,, \quad \vec{r} = u\,\vec{i} + v\,\vec{j} + z\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial u} = \vec{i} + \frac{\partial z}{\partial u}\,\vec{k}\,, \quad \frac{\partial \vec{r}}{\partial v} = \vec{j} + \frac{\partial z}{\partial v}\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \vec{k} - \frac{\partial z}{\partial v}\,\vec{j} - \frac{\partial z}{\partial u}\,\vec{i}$$

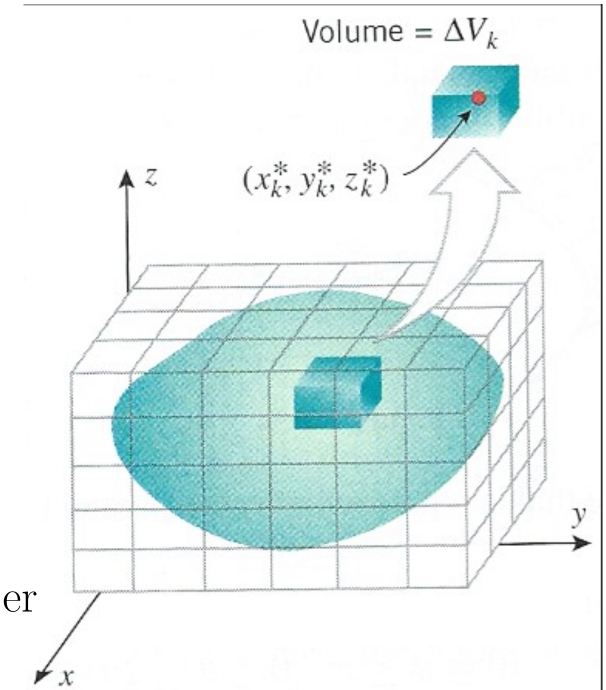
$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{1 + \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}$$

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example. S of $z = x^2 + y^2$ below $z = 1$.
Answer : $\frac{\pi}{6}(5\sqrt{5} - 1) \approx 5.330$

5 Triple Integrals

Mass problem. Find the mass M of a solid G whose density (the mass per unit volume) is a continuous nonnegative function $\delta(x, y, z)$.



1. Divide the box enclosing G into subboxes, and exclude all those subboxes that contain points outside of G. Let n be the number of all the subboxes inside G, and let $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$ be the volume of the k-th subbox.

2. Choose any point (x_k^*, y_k^*, z_k^*) in the k-th subbox. The mass of the k-th subbox is $\Delta M_k \approx \delta(x_k^*, y_k^*, z_k^*)\Delta V_k$. Thus,

$$M \approx \sum_{k=1}^{n} \Delta M_k = \sum_{k=1}^{n} \delta(x_k^*, y_k^*, z_k^*)\Delta V_k = \sum_{k=1}^{n} \delta(x_k^*, y_k^*, z_k^*)\Delta x_k \Delta y_k \Delta z_k$$

   This sum is called the **Riemann sum**.

3. Take the sides of all the subboxes to 0, and therefore the number of them to infinity, and get

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k^*, y_k^*, z_k^*)\Delta V_k = \iiint_G \delta(x, y, z)\, dV$$

   The last term is the notation for the limit of the Riemann sum, and it is called the **triple integral** of $\delta(x, y, z)$ over G.

Properties of triple integrals

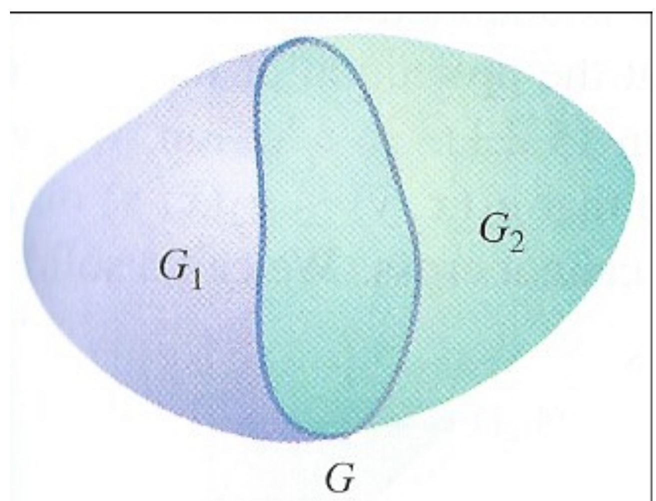
1. If f, g are continuous on G, and c, d are constants, then

$$\iiint_G \big(c\, f(x,y,z) + d\, g(x,y,z)\big)\, dV = c \iiint_G f(x,y,z)\, dV + d \iiint_G g(x,y,z)\, dV$$

2. If G is divided into two solids G_1 and G_2, then

$$\iiint_G f(x,y,z)\, dV = \iiint_{G_1} f(x,y,z)\, dV + \iiint_{G_2} f(x,y,z)\, dV$$

The mass of the entire solid is the sum of the masses of the solids G_1 and G_2.



Evaluating triple integrals over rectangular boxes

$$a \le x \le b\,, \quad c \le y \le d\,, \quad k \le z \le l$$

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx$$

or any permutation, e.g.

$$\iiint_G f(x,y,z)\,dV = \int_c^d \int_k^l \int_a^b f(x,y,z)\,dx\,dz\,dy$$

Example. Find the mass of the box

$$\frac{1}{3} \le x \le \frac{1}{2}\,, \quad 0 \le y \le \pi\,, \quad 0 \le z \le 1$$

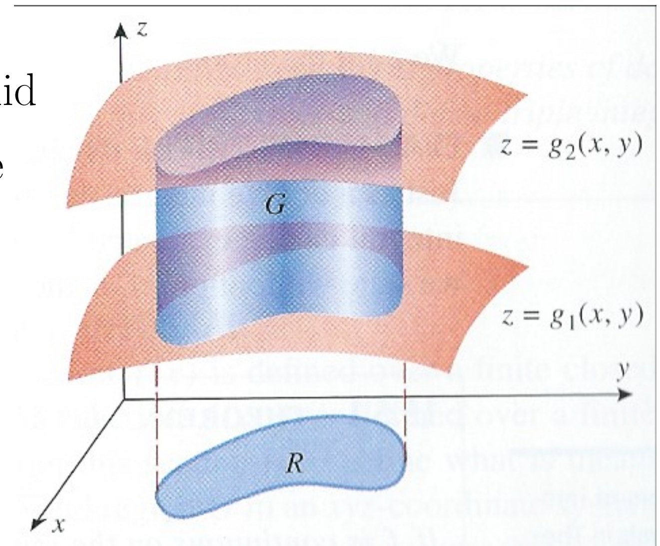
if its density is

$$\delta(x,y,z) = xz\sin(xy)$$

$Answer: \quad M = \frac{1}{12} + \frac{\sqrt{3}}{4\pi} - \frac{1}{2\pi} \approx 0.0620106$

Evaluating triple integrals over simple xy-, xz-, yz-solids

A solid G is called a simple xy-solid if it is bounded above by a surface $z = g_2(x,y)$, below by a surface $z = g_1(x,y)$, and its projection on the xy plane is a region R.



$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA$$

Example. Find the mass of the solid G defined by the inequalities

$$\frac{\pi}{6} \le y \le \frac{\pi}{2}, \quad y \le x \le \frac{\pi}{2}, \quad 0 \le z \le xy$$

if its density is

$$\delta(x,y,z) = \cos(z/y)$$

$Answer: \quad M = \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \approx 0.442972$

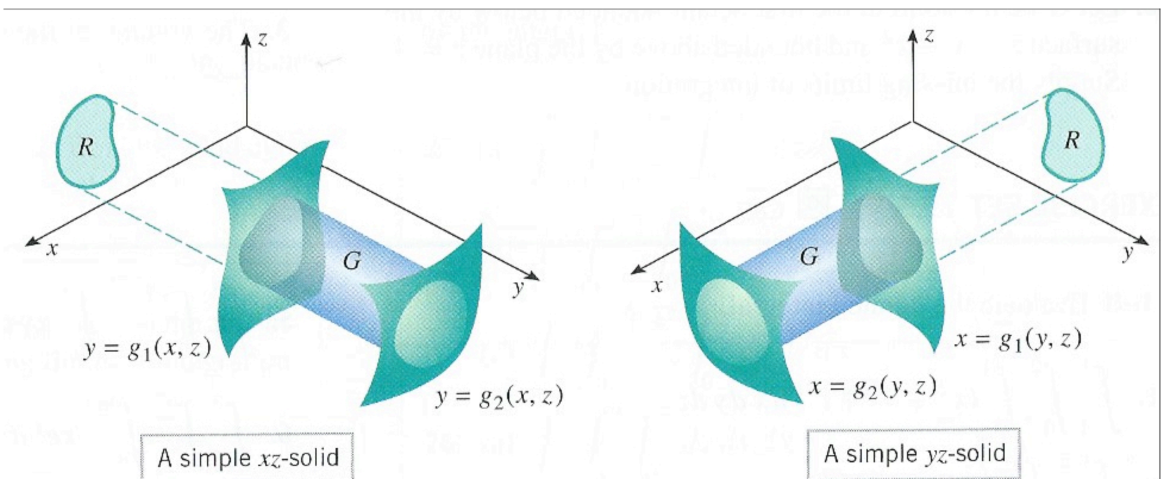
$$\text{Volume of } G = \iiint_G dV$$

Example. Find V of G bounded by the surfaces

$$y = x^2\,, \quad y + z = 4\,, \quad z = 0$$

Answer : $V = \frac{256}{15}$

Integration in other orders



A simple xz-solid

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\right] dA$$

A simple yz-solid

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right] dA$$

6 Centre of gravity and centroid

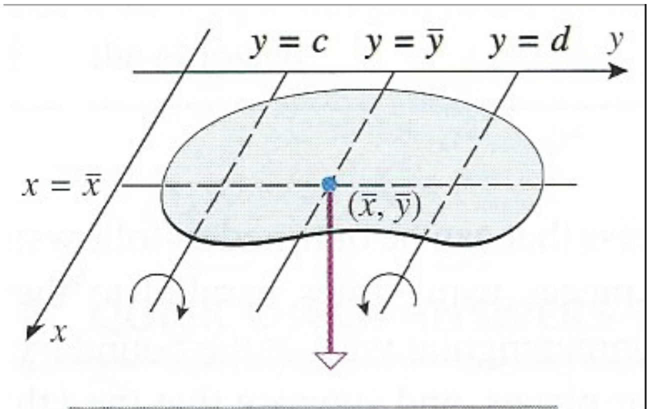
Mass, centre of gravity and centroid of a lamina

Recall that a lamina is an idealised flat object that is thin enough to be viewed as a 2-d plane region R.

The mass M of a lamina with density $\delta(x, y)$ is

$$M = \iint_R \delta(x, y)\, dA$$

The **centre of gravity** of a lamina is a unique point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force acting at the point $(\bar{x}, \bar{y})$



Force of gravity acting on the center of gravity of the lamina

$$\bar{x} = \frac{1}{M} \iint_R x\, \delta(x, y)\, dA\,, \quad \bar{y} = \frac{1}{M} \iint_R y\, \delta(x, y)\, dA$$

Example. Find the centre of gravity of a lamina with density $\delta(x, y) = x + 1$ bounded by $x^2 + (y + 1)^2 = 1$

$\quad Answer: \quad M = \pi\,,\ \bar{x} = 1/4\,,\ \bar{y} = -1$

For a **homogeneous** lamina with $\delta(x, y) = \text{const}$,
the centre of gravity is called the **centroid** of the lamina
or the centroid of the region R
because it does not depend on $\delta(x, y) = \text{const}$.

$$\bar{x} = \frac{1}{A} \iint_R x \, dA \,, \quad \bar{y} = \frac{1}{A} \iint_R y \, dA \,, \quad A = \iint_R dA$$

Example. Find the centroid of a region bounded by

$$(x-1)^2 + y^2 = 1 \,, \text{ and } \; (x-2)^2 + y^2 = 4$$

$Answer: \quad A = 3\pi \,, \; \bar{x} = 7/3 \,, \; \bar{y} = 0$

Mass, centre of gravity and centroid of a solid

The **centre of gravity** of a solid G with density $\delta(x,y,z)$ is a unique point $(\bar{x},\bar{y},\bar{z})$ such that the effect of gravity on the solid is equivalent to that of a single force acting at the point $(\bar{x},\bar{y},\bar{z})$

$$\bar{x} = \frac{1}{M}\iiint_G x\,\delta(x,y,z)\,dV\,, \quad \bar{y} = \frac{1}{M}\iiint_G y\,\delta(x,y,z)\,dV$$

$$\bar{z} = \frac{1}{M}\iiint_G z\,\delta(x,y,z)\,dV\,, \quad M = \iiint_G \delta(x,y,z)\,dV$$

Example. Find the centre of gravity of a solid G bounded by the surfaces $x^2+y^2=1$ and $x^2+y^2=4$, above by the surface $z=5-x^2-y^2$, and below by the surface $z=0$ with the density

$$\delta(x,y,z) = e^{5-x^2-y^2-z}$$

Answer : $M = \pi(e^4-e-3) \approx 153.561\,,\ \bar{z} = \frac{2e^4-2e-21}{2\left(e^4-e-3\right)} \approx 0.85$

For a **homogeneous** solid with $\delta(x,y) = $const, the centre of gravity is called the **centroid** of the solid.

$$\bar{x} = \frac{1}{V}\iiint_G x\,dV\,, \quad \bar{y} = \frac{1}{V}\iiint_G y\,dV$$

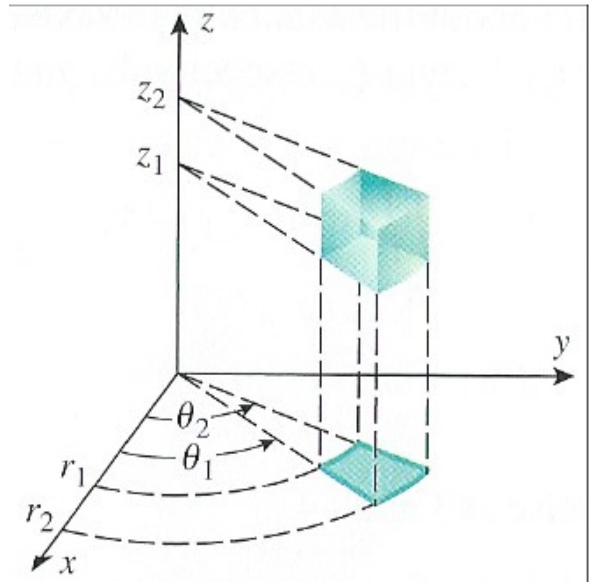
$$\bar{z} = \frac{1}{V}\iiint_G z\,dV\,, \quad V = \iiint_G dV$$

Example. Find the centroid of the solid below $z=10-x^2-y^2$, inside of $x^2+y^2=1$, and above $z=0$

Answer : $V = 19\pi/2\,,\ \bar{x}=0\,,\ \bar{y}=0\,,\ \bar{z} = 271/57 \approx 4.75439$

7 Triple integrals in cylindrical and spherical coordinates

Cylindrical coordinates



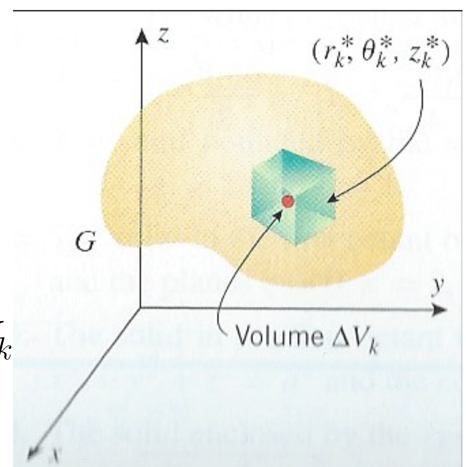
Cylindrical wedge or cylindrical element of volume is interior of intersection of

two cylinders: $r = r_1\,,\ r = r_2$

two half-planes: $\theta = \theta_1\,,\ \theta = \theta_2$

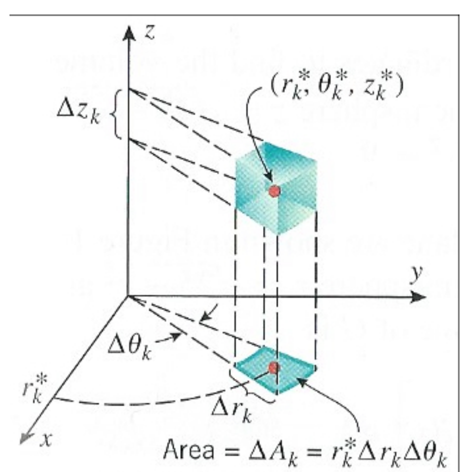
two planes: $z = z_1\,,\ z = z_2$

The dimensions: $\theta_2 - \theta_1\,,\ r_2 - r_1\,,\ z_2 - z_1$ are called the central angle, thickness and height of the wedge.



Divide G by cylindrical wedges

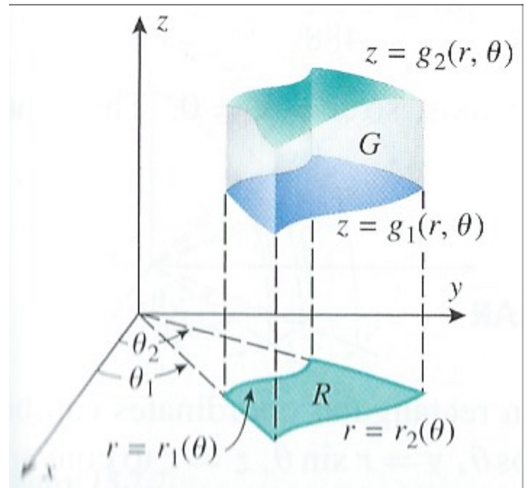
$$\iiint_G f(r,\theta,z)\,dV = \lim_{n\to\infty}\sum_{n=1}^{\infty} f(r_k^*, \theta_k^*, z_k^*)\Delta V_k$$



$$\begin{aligned}\Delta V_k &= [\,\text{area of base}\,]\cdot[\,\text{height}\,]\\ &= r_k^*\Delta r_k\Delta\theta_k\Delta z_k\end{aligned}$$

Theorem.

Let G be a solid whose upper suface is $z = g_2(r, \theta)$ and whose lower suface is $z = g_1(r, \theta)$ in cylindrical coordinates. If projection of G on the xy-plane is a simple polar region R, and if $f(r, \theta, z)$ is continuous on G, then



$$\iiint_G f(r,\theta,z)\,dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\right] dA$$
$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$

Example. V and centroid of G bounded above by $z = \sqrt{25 - x^2 - y^2}$, below by $z = 0$, and laterally by $x^2 + y^2 = 9$.

Answer : $V = 122\pi/3\,,\ \bar{z} = 1107/488$

Converting triple integrals from rectangular to cylindrical coordinates

$$\iiint_G f(x,y,z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Example.

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx = \frac{243}{4}\pi$$

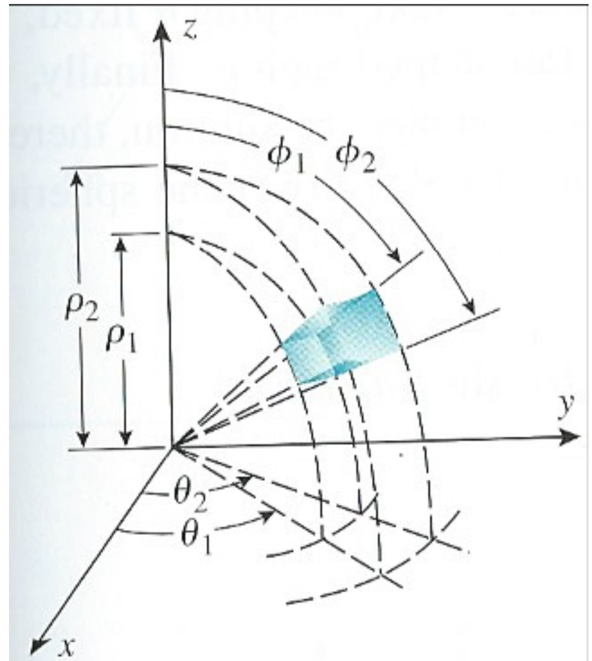
Spherical coordinates

Spherical wedge or spherical element of volume is interior of intersection of

two spheres: $\rho = \rho_1\,,\ \rho = \rho_2$

two half-planes: $\theta = \theta_1\,,\ \theta = \theta_2$

nappes of two right circular cones: $\phi = \phi_1\,,\ \phi = \phi_2$

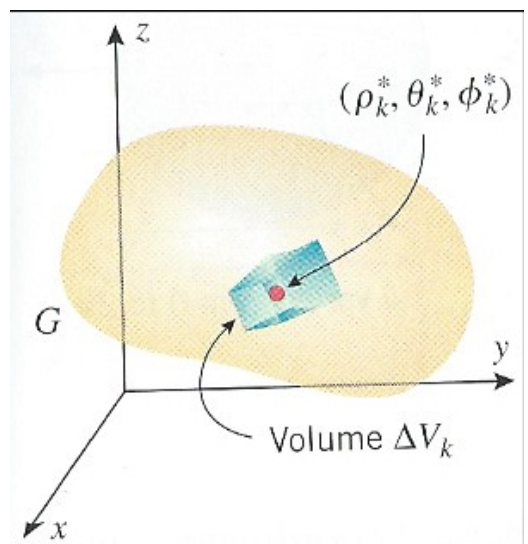


The numbers:

$\theta_2 - \theta_1\,,\ \rho_2 - \rho_1\,,\ \phi_2 - \phi_1$

are the dimensions of the wedge.

Divide G by spherical wedges



$$\iiint_G f(r,\theta,\phi)\,dV = \lim_{n\to\infty}\sum_{n=1}^{\infty} f(r_k^*,\theta_k^*,\phi_k^*)\Delta V_k$$

$$\Delta V_k = (\rho_k^*)^2 \sin\phi_k^* \Delta\rho_k \Delta\phi_k \Delta\theta_k$$

$$\iiint_G f(r,\theta,\phi)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r,\theta,\phi)\,\rho^2\,\sin\phi\,d\rho\,d\phi\,d\theta$$

Example. V and centroid of G bounded above by

$x^2 + y^2 + z^2 = 16$ and below by $z = \sqrt{x^2 + y^2}$.

Answer : $V = 64(2 - \sqrt{2})\pi/3 > 0\,,\ \bar{z} = 3/2(2 - \sqrt{2}) \approx 2.56$

Converting triple integrals from rectangular to spherical coordinates

$$\iiint_G f(x, y, z)\, dV = \underset{\substack{\text{appropriate}\\ \text{limits}}}{\iiint} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2\, \sin\phi\, d\rho\, d\phi\, d\theta$$

Example.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx = \frac{64}{9}\pi$$