

# Definite Integration

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February 23, 2020

## 1 Definition of the Integral and Description of the Set of Integrable Functions

### 1.1 Introduction

Suppose a point is moving along the real line, with  $s(t)$  being its coordinate at time  $t$  and  $s'(t) = v(t)$  its velocity at the same instant  $t$ . Assume that we know the position  $S(t_0)$  of the point at time  $t_0$  and that we receive information on its velocity. Having this function, we wish to compute  $s(t)$  for any given value of time  $t > t_0$ .

If we assume that the velocity  $v(t)$  varies continuously, the displacement of the point over small time interval can be computed approximately as the product  $v(\tau)\Delta t$  of the velocity at an arbitrary instant  $\tau$  belonging to that time interval and the magnitude  $\Delta t$  of the time interval itself. Taking this observation into account, we partition the interval  $[t_0, t]$  by marking some times  $t_i, i = 0, 1, \dots, n$  so that  $t_0 < t_1 < \dots < t_n = t$  and so the interval  $[t_{i-1}, t_i]$  are small. Let  $\Delta t_i = t_i - t_{i-1}$  and  $\tau_i \in [t_{i-1}, t_i]$ . Then we have the approximation equality

$$s(t) - s(t_0) \approx \sum_{i=1}^n v(\tau_i) \Delta t_i$$

The approximation will become more precise if we partition the close interval into smaller and smaller intervals. Thus we must conclude that in the limit as the length  $\lambda$  of the largest of these intervals tends to zero we shall obtain an exact equality

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n v(\tau_i) \Delta t_i = s(t) - s(t_0) \quad (1)$$

Such sums, called **Riemann sums**, are encountered in a wide variety of situations.

Let us attempt, for example, following Archimedes, to find the area under the parabola  $y = x^2$  above the closed interval  $[0, 1]$ .

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{3}$$

## 1.2 Definition of the Riemann Integral

### a. Partition

**Definition 1.1.** A **partition**  $P$  of a closed interval  $[a, b]$ ,  $a < b$ , is a finite system of points  $x_0, x_1, \dots, x_n$  of the interval such that  $a = t_0 < t_1 < \dots < t_n = b$ .

The intervals  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, n$  are called the intervals of the partitions  $P$ . The largest of the lengths of the intervals of the partition  $P$ , denoted  $\lambda(P)$ , is called the **mesh** of the partition.

**Definition 1.2.** We speak of a partition with distinguished points  $(P, \xi)$  on the closed interval  $[a, b]$  if we have a partition  $P$  of  $[a, b]$  and a point  $\xi \in [t_{i-1}, t_i]$  has been chosen in each of the intervals of the partition  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$ .

We denoted the set of point  $(\xi_1, \dots, \xi_n)$  by the single letter  $\xi$ .

**b. A Base in the Set of Partitions** In the set  $\mathcal{P}$  of partitions with distinguished points on a given interval  $[a, b]$ , we consider the following base  $\mathcal{B} = \{B_d\}$ . The element  $B_d$ ,  $d > 0$ , of the base  $\mathcal{B}$  consists of all partitions with distinguished points  $(P, \xi)$  on  $[a, b]$  for which  $\lambda(P) < d$ .

### c. Riemann Sums

**Definition 1.3.** If a function  $f$  is defined on the closed interval  $[a, b]$  and  $(P, \xi)$  is a partition with distinguished points on this closed interval, the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^n f(\xi_i) \Delta x_i, \quad (2)$$

where  $\Delta x_i = x_i - x_{i-1}$ , is the **Riemann sum** of the function  $f$  corresponding to the partition  $(P, \xi)$  with distinguished point on  $[a, b]$ .

Thus, when the function  $f$  is fixed, the Riemann sum  $\sigma(f; P, \xi)$  is a function  $\Phi(p) = \sigma(f; p)$  on the set  $\mathcal{P}$  of all partitions  $p = (P, \xi)$  with distinguished point on the closed interval  $[a, b]$ . Since there is a base  $\mathcal{B}$  in  $\mathcal{P}$ , one can ask about the limit of the function  $\Phi(p)$  over the base.

**d. The Riemann Integral** Let  $f$  be a function defined on a closed interval  $[a, b]$ .

**Definition 1.4.** The number  $I$  is the **Riemann integral** of the function  $f$  on the closed interval  $[a, b]$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\left| I - \sum_{i=1}^n f(\xi_i) \Delta x_i \right| < \epsilon$$

for any partition  $(P, \xi)$  with distinguished points on  $[a, b]$  whose mesh  $\lambda(P)$  is less than  $\delta$ .

Since the partition  $p = (P, \xi)$  for which  $\lambda(P) < \delta$  form the element  $B_\delta$  of the base  $\mathcal{B}$  introduced above in the set  $\mathcal{P}$  of partitions with distinguished points, the above definition is equivalent to

$$I = \lim_B \Phi(p)$$

The integral of  $f(x)$  over  $[a, b]$  is denoted

$$\int_a^b f(x) \, dx,$$

in which the number  $a$  and  $b$  are called respectively the lower and upper limits of integration. The function  $f$  is called the integrand,  $f(x)dx$  is called the differential form, and  $x$  is the variable of integration. Thus

$$\int_a^b f(x) \, dx = \lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (3)$$

**Definition 1.5.** A function  $f$  is Riemann integrable on the closed interval  $[a, b]$  if the limit of the Riemann sums in Eq. 3 exists as  $\lambda(P) \rightarrow 0$  (that is, the Riemann integral of  $f$  is defined).

The set of Riemann-integrable functions on a closed interval  $[a, b]$  will be denoted  $\mathcal{R}[a, b]$ .

### 1.3 The Set of Integrable Functions

The integrability or non-integrability of a function  $f$  on  $[a, b]$  depends on the existence of the limit below

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

By the Cauchy criterion, this limit exists if and only if for every  $\epsilon > 0$  there exists an element  $B_\delta \in \mathcal{B}$  in the base such that

$$|\Phi(p') - \Phi(p'')| < \epsilon$$

for any two points  $p', p'' \in B_\delta$ .

In more detailed notation, what has just been said means that for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|\sigma(f; P', \xi') - \sigma(f; P'', \xi'')| < \epsilon$$

or, what is the same,

$$\left| \sum_{i=1}^{n'} f(\xi'_i) \Delta x'_i - \sum_{i=1}^{n''} f(\xi''_i) \Delta x''_i \right| < \epsilon$$

for any partition  $(P', \xi')$  and  $(P'', \xi'')$  with distinguished points on the interval  $[a, b]$  with  $\lambda(P') < \delta$  and  $\lambda(P'') < \delta$ .

#### a. A Necessary Condition for Integrability.

**Proposition 1.1.** A necessary condition for a function  $f$  defined on a closed interval  $[a, b]$  to be Riemann integrable on  $[a, b]$  is that  $f$  be bounded on  $[a, b]$ .

**b. A Sufficient Condition for Integrability and the Most Important Classes of Integrable Functions** We begin with some notation and remarks that will be used in the explanation to follow.

We agree that when a partition  $P$

$$a = x_0 < x_1 < \cdots < x_n = b$$

is given on the interval  $[a, b]$ , we shall use the symbol  $\Delta_i$  to denote the interval  $[x_{i-1}, x_i]$  along with  $\Delta x_i$  as a notation for the difference  $x_i - x_{i-1}$ . If a partition  $\tilde{P}$  of the closed interval  $[a, b]$  is obtained from a partition  $P$  by the jointing new points to  $P$ , we call  $\tilde{P}$  a refinement of  $P$ . When a refinement  $\tilde{P}$  of a partition  $P$  is constructed, some of the closed intervals  $\Delta_i = [x_{i-1}, x_i]$  of the partition  $P$  themselves undergo partitioning:

$$x_{i-1} = x_{i0} < x_{i1} < \cdots < x_{in_i} = x_i.$$

**Proposition 1.2.** A sufficient condition for a bounded function  $f$  to be integrable on a closed interval  $[a, b]$  is that for every  $\epsilon > 0$  there exist a number  $\delta > 0$  such that

$$\sum_{i=0}^n \omega(f; \Delta_i) \Delta x_i < \epsilon$$

for any partition  $P$  of  $[a, b]$  with mesh  $\lambda(P) < \delta$ .

*Proof.* Let  $P$  be a partition of  $[a, b]$  and  $\tilde{P}$  a refinement of  $P$ . Let us estimate the difference between the Riemann sums  $\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi)$ . Using the notation introduced above, we can write

$$\begin{aligned}
\left| \sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi) \right| &= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_{ij}) \Delta x_{ij} - \sum_{i=1}^n f(\xi_i) \Delta x_i \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_{ij}) \Delta x_{ij} - \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_i) \Delta x_{ij} \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} (f(\xi_{ij}) - f(\xi_i)) \Delta x_{ij} \right| \leq \sum_{i=1}^n \sum_{j=1}^{n_j} |f(\xi_{ij}) - f(\xi_i)| \Delta x_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^{n_j} \omega(f; \Delta_i) \Delta x_{ij} = \sum_{i=1}^n \omega(f; \Delta x_i) \Delta x_i.
\end{aligned}$$

It follows from the estimation for the difference of the Riemann sums that if the function satisfies the sufficient condition given in the statement of the proposition, then for each  $\epsilon > 0$ , we can find  $\delta > 0$  such that

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi)| < \frac{\epsilon}{2}$$

Now if  $(P', \xi')$  and  $(P'', \xi'')$  are arbitrary partitions with distinguished points on  $[a, b]$  whose meshes satisfy  $\lambda(P') < \delta$  and  $\lambda(P'') < \delta$ , then, by what has been proved, the partition  $\tilde{P} = P' \cup P''$ , we have

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P', \xi')| < \frac{\epsilon}{2}$$

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P'', \xi'')| < \frac{\epsilon}{2}$$

It follows that

$$|\sigma(f; P', \xi') - \sigma(f; P'', \xi'')| < \epsilon$$

provided that  $\lambda(P') < \delta$ ,  $\lambda(P'') < \delta$ . Therefore, by the Cauchy criterion, the limit of the Riemann sums exists:

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i,$$

that is  $f \in \mathcal{R}[a, b]$ . □

**Corollary 1.1.**  $(f \in C[a, b]) \Rightarrow (f \in \mathcal{R}[a, b])$ , that is, every continuous function on a closed interval is integrable on that close interval.

**Corollary 1.2.** If a bounded function  $f$  on a closed interval  $[a, b]$  is continuous everywhere except at a finite set of points, then  $f \in \mathcal{R}[a, b]$ .

**Corollary 1.3.** A monotonic function on a closed interval is integrable on that interval.

**Definition 1.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a real valued function that is defined and bounded on the closed interval  $[a, b]$ , let  $P$  be a partition of  $[a, b]$ , and let  $\Delta_i (i = 1, 2, \dots, n)$  be the intervals of the partition  $P$ . Let  $m_i = \inf_{x \in \Delta_i} f(x)$  and  $M_i = \sup_{x \in \Delta_i} f(x), i = 1, 2, \dots, n$ .

The sums

$$s(f; P) = \sum_{i=1}^n m_i \Delta x_i$$

and

$$S(f; P) = \sum_{i=1}^n M_i \Delta x_i$$

are called respectively the lower and upper Riemann sums of the function  $f$  on the interval  $[a, b]$  corresponding to the partition  $P$  of the interval. The sums  $s(f; P)$  and  $S(f; P)$  are also called the lower and upper **Darboux** sums corresponding to the partition  $P$  of  $[a, b]$ .

If  $(P, \xi)$  is an arbitrary partition with distinguished points on  $[a, b]$ , then obviously

$$s(f; P) \leq \sigma(f; P, \xi) \leq S(f; P) \quad (4)$$

**Lemma 1.4.**

$$s(f; P) = \inf_{\xi} \sigma(f; P, \xi)$$

$$S(f; P) = \sup_{\xi} \sigma(f; P, \xi)$$

**Proposition 1.3.** A bounded real-valued function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  if and only if the following limit exist and are equal to each other:

$$\underline{I} = \lim_{\lambda(P) \rightarrow 0} s(f; P); \bar{I} = \lim_{\lambda(P) \rightarrow 0} S(f; P). \quad (5)$$

When this happens, the common value  $I = \underline{I} = \bar{I}$  is the integral

$$\int_a^b f(x) \, dx$$

**Proposition 1.4.** A necessary and sufficient condition for a function  $f : [a, b] \rightarrow \mathbb{R}$  defined on a closed interval  $[a, b]$  to be **Riemann integrable** on  $[a, b]$  is the following relation:

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n \omega(f; \Delta_i) \Delta x_i = 0 \quad (6)$$

### c. The Vector Space $\mathcal{R}[a, b]$

**Proposition 1.5.** If  $f, g \in \mathcal{R}[a, b]$ , then

1.  $(f + g) \in \mathcal{R}[a, b]$ ;
2.  $\alpha f \in \mathcal{R}[a, b]$ , where  $\alpha$  is a numerical coefficient;
3.  $|f| \in \mathcal{R}[a, b]$ ;
4.  $f|_{[c, d]} \in \mathcal{R}[a, b]$  if  $[c, d] \subset [a, b]$ ;
5.  $(f \cdot g) \in \mathcal{R}[a, b]$ .

## 2 Linearity, Additivity and Monotonicity of the Integral

### 2.1 The Integral as a Linear Function on the Space $\mathcal{R}[a, b]$

**Theorem 2.1.** If  $f, g \in \mathcal{R}[a, b]$ , then  $\alpha f + \beta g \in \mathcal{R}[a, b]$ , and

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx$$

**Remark.** To avoid any possible confusion, functions defined on functions are usually called functionals. Thus we have proved that the integral is a linear functional on the vector space  $\mathcal{R}[a, b]$  of integrable functions.

### 2.2 The Integral as a Additive Function of the Interval of Integration

The value of the integral  $\int_a^b f(x) \, dx = I(f; [a, b])$  depends on both the integrand and the closed interval  $[a, b]$  over which the integral is taken. For example, if  $f \in \mathcal{R}[a, b]$ , then, as we know,  $f|_{[\alpha, \beta]} \in \mathcal{R}[\alpha, \beta]$  if  $[\alpha, \beta] \subset [a, b]$ , that is  $\int_\alpha^\beta f(x) \, dx$  is defined.

**Lemma 2.2.** if  $a < b < c$  and  $f \in \mathcal{R}[a, c]$ , then  $f|_{[a, b]} \in \mathcal{R}[a, b]$ ,  $f|_{[b, c]} \in \mathcal{R}[b, c]$ , and the following equality holds:

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

From the definition of integral, we have: if  $a > b$  then,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

In this connection, it is also natural to set

$$\int_a^a f(x) dx = 0$$

**Theorem 2.3.** Let  $a, b, c \in \mathbb{R}$  and let  $f$  be a function integrable over the largest closed interval having two of these points as endpoints. Then the restriction of  $f$  to each of the other closed interval is also integrable over those intervals and the following equality holds:

$$\int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^a f(x) dx = 0$$

**Definition 2.1.** Suppose that to each  $(\alpha, \beta)$  of points  $\alpha, \beta \in [a, b]$  a number  $I(\alpha, \beta)$  is assigned so that

$$I(\alpha, \gamma) = I(\alpha, \beta) + I(\beta, \gamma)$$

for any triple point  $\alpha, \beta, \gamma$ . Then the function  $I(\alpha, \beta)$  is called an additive(oriented) interval function defined on intervals contained in  $[a, b]$ .

If  $f \in [A, B]$ , and  $a, b, c \in [A, B]$ , then, setting  $I(a, b) = \int_a^b f(x) dx$ , we conclude that

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

that is, the integral is an additive interval function on the interval of integration.

## 2.3 Estimation of the Integral, Monotonicity of the Integral, and the Mean-Value Theorem

### 2.3.1 A General Estimation of the Integral.

**Theorem 2.4.** If  $a \leq b$  and  $f \in \mathcal{R}[a, b]$ , then  $|f| \in \mathcal{R}[a, b]$  and the following inequality holds

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f|(x) dx$$



If  $|f|(x) \leq C$  on  $[a, b]$  then

$$\int_a^b |f| \, dx \leq C(b - a)$$

### 2.3.2 Monotonicity of the Integral and the First Mean-Value Theorem

**Theorem 2.5.** If  $a \leq b$ ,  $f_1, f_2 \in \mathcal{R}[a, b]$ ,  $f_1(x) \leq f_2(x)$ ,  $\forall x \in [a, b]$ , then

$$\int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx$$

**Corollary 2.6.** If  $a \leq b$ ,  $f \in \mathcal{R}[a, b]$ ,  $m \leq f(x) \leq M$ ,  $\forall x \in [a, b]$ , then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$$

**Corollary 2.7.** If  $a \leq b$ ,  $f \in \mathcal{R}[a, b]$ ,  $m = \inf_{x \in [a, b]} f(x)$ ,  $M = \sup_{x \in [a, b]} f(x)$ , then there exists a number  $\mu \in [m, M]$  such that

$$\int_a^b f(x) \, dx = \mu(b - a)$$

**Corollary 2.8.** If  $f \in C[a, b]$ , there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b f(x) \, dx = f(\xi)(b - a) \tag{7}$$

**Remark.** The equality Equation(7) is often called **the first mean-value theorem**. We, however, reserve that name for the following somewhat more general proposition.

**Theorem 2.9 ((First Mean-Value Theorem)).** Let  $f, g \in \mathcal{R}[a, b]$ ,  $m = \inf_{x \in [a, b]} f(x)$ ,  $M = \sup_{x \in [a, b]} f(x)$ . If  $g$  is nonnegative (or nonpositive) on  $[a, b]$ , then

$$\int_a^b (fg) \, dx = \mu \int_a^b g(x) \, dx$$

where  $\mu \in [m, M]$ . If, in addition, it is known that  $f \in C[a, b]$ , then there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b (fg) \, dx = f(\xi) \int_a^b g(x) \, dx$$

**Abel's Transformation** Let  $A_k = \sum_{i=1}^k a_i, A_0 = 0$ , then

$$\begin{aligned}
\sum_{i=1}^n a_i b_i &= \sum_{i=1}^n (A_i - A_{i-1}) b_i = \sum_{i=1}^n A_i b_i - \sum_{i=1}^n A_{i-1} b_i \\
&= \sum_{i=1}^n A_i b_i - \sum_{i=0}^{n-1} A_i b_{i+1} = A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}) \\
&= A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1}) \\
&= A_n b_n + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1})
\end{aligned}$$

**Lemma 2.10.** If the numbers  $A_k = \sum_{i=1}^k a_i (k = 1, 2, \dots, n)$  satisfy the inequality  $m \leq A_k \leq M$  and the numbers  $b_i, i = 1, 2, \dots, n$  are nonnegative and  $b_i \geq b_{i+1}$  for  $i = 1, 2, \dots, n-1$ , then,

$$mb_1 \leq \sum_{i=1}^n a_i b_i \leq Mb_1$$

**Lemma 2.11.** If  $f \in \mathcal{R}[a, b]$ , then for any  $x \in [a, b]$  the function

$$F(x) = \int_a^x f(t) dt$$

is defined and  $F(x) \in C[a, b]$ .

**Lemma 2.12.** If  $f, g \in \mathcal{R}[a, b]$  and  $g$  is a non-negative non-increasing function on  $[a, b]$  then there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b (fg) dx = g(a) \int_a^\xi f(x) dx.$$

**Theorem 2.13** (Second mean-value theorem for the integral). *If  $f, g \in \mathcal{R}[a, b]$  and  $g$  is a monotonic function on  $[a, b]$ , then there exists a point  $\xi \in [a, b]$  such that*

$$\int_a^b (fg) dx = g(a) \int_a^\xi f(x) dx + g(b) \int_\xi^b f(x) dx$$

## 3 The Integral and the Derivative

### 3.1 The Integral and the Primitive

Let  $f$  be a Riemann-integrable function on a closed interval  $[a, b]$ . On this interval let us consider the function

$$F(x) = \int_a^x f(t) dt \tag{8}$$

often called an integral with variable upper bound limit.

Since  $f \in \mathcal{R}[a, b]$ , it follows that  $f|_{[a, x]} \in \mathcal{R}[a, x]$  if  $[a, x] \subset [a, b]$ , therefore the function

$$x \rightarrow F(x)$$

is unambiguously defined for  $x \in [a, b]$ .

**Lemma 3.1.** If  $f \in \mathcal{R}[a, b]$  and the function  $f$  is continuous at a point  $x \in [a, b]$ , then the function  $F$  defined on  $[a, b]$  by Equation(8) is differentiable at the point  $x$ , and the following equality holds:

$$F'(x) = f(x)$$

**Theorem 3.2.** Every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  on the closed interval  $[a, b]$  has a primitive, and every primitive of  $f$  on  $[a, b]$  has the form

$$\mathcal{F}(x) = \int_a^x f(t) dt + C$$

**Definition 3.1.** A continuous function  $x \rightarrow F(x)$  on an interval of the real line is called a primitive (or generalized primitive) of the function  $x \rightarrow f(x)$  defined on the same interval if the relation  $\mathcal{F}'(x) = f(x)$  holds at all points of the interval, with only a finite number of exceptions.

### 3.2 The Newton-Leibniz Formula

**Theorem 3.3.** If  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function with a finite number of points of discontinuity, then  $f \in \mathcal{R}[a, b]$  and

$$\int_a^b f(x) dx = \mathcal{F}(b) - \mathcal{F}(a) \quad (9)$$

where  $\mathcal{F} : [a, b] \rightarrow \mathbb{R}$  is any primitive of  $f$  on  $[a, b]$ .

**Remark.** Relation(9), which is fundamental for all of analysis, is called the Newton-Leibniz formula( or fundamental theorem of calculus.)

### 3.3 Integration by Parts in the Definite Integral and Taylor's Formula

**Proposition 3.1.** If the function  $u(x)$  and  $v(x)$  are continuously differentiable on a closed interval with endpoints  $a$  and  $b$ , then

$$\int_a^b (u(x)v'(x)) dx = (uv)|_a^b - \int_a^b (v(x)u'(x)) dx$$

or

$$\int_a^b u(x) dv(x) = (uv)|_a^b - \int_a^b v(x) du(x)$$

As a corollary we now obtain the Taylor formula with integral form of the remainder. Suppose on the closed interval with endpoints  $a$  and  $x$  the function  $t \rightarrow f(t)$  has  $n$  continuous derivatives, we have

$$\begin{aligned}
f(x) - f(a) &= \int_a^x f'(t) \, dt = - \int_a^x \int_a^x f'(t)(x-t)' \, dt \\
&= -f'(t)(x-t)|_a^x + \int_a^x f''(t)(x-t) \, dt \\
&= f'(a)(x-a) - \frac{1}{2} \int_a^x f''(t)[(x-t)^2]' \, dt \\
&= f'(a)(x-a) - \frac{1}{2} f''(t)(x-t)^2|_a^x + \frac{1}{2} \int_a^x f'''(t)(x-t)^2 \, dt \\
&= f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 - \frac{1}{2 \cdot 3} \int_a^x f'''(t)[(x-t)^3]' \, dt \\
&= \dots \\
&= f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a; x)
\end{aligned}$$

where

$$r_{n-1}(a; x) = \frac{1}{(n-1)!} \int_a^x f^{(n)}(t)(x-t)^{n-1} \, dt. \quad (10)$$

**Proposition 3.2.** If the function  $t \rightarrow f(t)$  has continuous derivatives up to order  $n$  inclusive on the closed interval with endpoints  $a$  and  $x$ , then Taylor's formula holds:

$$f(x) = f(a) + \frac{1}{1!} f'(a)(x-a) + \dots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a; x),$$

with remainder term  $r_{n-1}(a; x)$  represented in the integral form (10).

We note that from the First Mean-Value Theorem, we can derive at the Lagrange remainder.

### 3.4 Change of Variable in an Integral

**Proposition 3.3.** If  $\varphi : [\alpha, \beta] \rightarrow [a, b]$  is a continuously differentiable mapping of the closed interval  $[\alpha, \beta]$  into the closed interval  $[a, b]$  such that  $\varphi(\alpha) = a$  and  $\varphi(\beta) = b$ , then for any continuous function  $f(x)$  on  $[a, b]$  the function  $f(\varphi(t))\varphi'(t)$  is continuous on the closed interval  $[\alpha, \beta]$ , and

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t) \, dt. \quad (11)$$

### 3.5 Some Examples

**Examples 1.**

$$\int_{-1}^1 \sqrt{1-x^2} \, dx$$

**Examples 2.**

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0,$$

$$\int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi$$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi,$$

$m, n \in \mathbb{N}$ .

**Examples 3.** Let  $f \in \mathcal{R}[-a, a]$ , we shall show that

$$\int_{-a}^a f(x) \, dx = \begin{cases} 2 \int_0^a f(x) \, dx, & \text{if } f \text{ is an even function.} \\ 0, & \text{if } f \text{ is an odd function.} \end{cases}$$

**Examples 4.** Let  $f$  be a function defined on the entire real line  $\mathbb{R}$  and having period  $T$ , that is  $f(x+T) = f(x)$  for all  $x \in \mathbb{R}$ . If  $f$  is integrable on each finite closed interval, then for any  $a \in \mathbb{R}$  we have the equality

$$\int_a^{a+T} f(x) \, dx = \int_0^T f(x) \, dx,$$

that is, the integral of a periodic function over an interval whose length equals the period  $T$  of the function is independent of the interval of integration on the real line.

**Examples 5.**

$$\lim_{x \rightarrow \infty} \frac{\left( \int_0^x e^{t^2} \, dt \right)^2}{\int_0^x e^{2t^2} \, dt}$$

**Examples 6.**

$$\int_0^{\frac{\pi}{2}} f(\sin x) \, dx = \int_0^{\frac{\pi}{2}} f(\cos x) \, dx$$

$$\int_0^{\pi} x f(\sin x) \, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) \, dx$$

**Examples 7.**

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

**Wallis formula**

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \left[ \frac{(2m)!!}{(2m-1)!!} \right]^2 \cdot \frac{1}{2m+1}$$

**Examples 8.** The quality  $\mu = \int_a^b f(x) \, dx$  is called the integral average value of the function on the closed interval  $[a, b]$ .

Let  $f(x)$  be a function that is defined on  $\mathbb{R}$  and integrable on any closed interval. We use  $f$  to construct the new function

$$F_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t) \, dt$$

whose value at the point  $x$  is the integral average value  $f$  in the  $\delta$ -neighborhood of  $x$ .

We shall show that  $F_\delta(x)$  is, compared to  $f$ , more regular. More precisely, if  $f$  is integrable on any close interval  $[a, b]$ , the  $F_\delta(x)$  is continuous on  $\mathbb{R}$ , and if  $f \in C(\mathbb{R})$ , then  $F_\delta(x) \in C^{(1)}(\mathbb{R})$ .

## 4 Some Applications of Integration

### 4.1 Additive Interval Functions and the Integral

An additive interval function is a function  $[\alpha, \beta] \rightarrow I(\alpha, \beta)$  that assigns a number  $I(\alpha, \beta)$  to each ordered pair of points  $(\alpha, \beta)$  in such a way that the following equality holds for any triple of point  $\alpha, \beta, \gamma \in [a, b]$ :

$$I(\alpha, \gamma) = I(\alpha, \beta) + I(\beta, \gamma). \quad (12)$$

Setting

$$\mathcal{F}(x) = I(a, x)$$

**Examples 9.** If  $f \in \mathcal{R}[a, b]$ , the function  $\mathcal{F}(x) = \int_a^x f(t) \, dt$  generates via formula 12 the additive function

$$I(\alpha, \beta) = \int_\alpha^\beta f(t) \, dt$$

**Proposition 4.1.** Suppose the additive function  $I[\alpha, \beta]$  defined for point  $\alpha, \beta$  of a closed interval  $[a, b]$  is such that there exists a function  $f \in \mathcal{R}[a, b]$  connected with  $I$  as follows: the relation

$$\inf_{x \in [\alpha, \beta]} f(x)(\beta - \alpha) \leq I(\alpha, \beta) \leq \sup_{x \in [\alpha, \beta]} f(x)(\beta - \alpha)$$

holds for any closed interval  $[\alpha, \beta]$  such that  $a \leq \alpha \leq \beta \leq b$ . Then

$$I[a, b] = \int_a^b f(x) dx$$

## 4.2 Arc Length

Let  $\Gamma : [a, b] \rightarrow \mathbb{R}^3$  be a smooth path that is defined by  $\mathbf{r}(t) = (x(t), y(t), z(t)), t \in [a, b]$ , and suppose that the velocity  $\mathbf{v}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$  are continuous, then

$$\inf_{t \in [\alpha, \gamma]} |\mathbf{v}(t)|(\gamma - \alpha) \leq l[\alpha, \gamma] \leq \sup_{t \in [\alpha, \gamma]} |\mathbf{v}(t)|(\gamma - \alpha)$$

and therefore

$$l[a, b] = \int_a^b |v(t)| dt = \int_a^b \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} dt$$

**Examples 10.** Suppose the point moves according to the law

$$\begin{cases} x = R \cos 2\pi t \\ y = R \sin 2\pi t \end{cases}$$

over the time interval  $[0, 1]$ , find the length of the path.

Let us consider the problem of computing the length of the graph of a function  $y = f(x)$  defined on a closed interval  $[a, b] \subset \mathbb{R}$ .

$$l[a, b] = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

**Examples 11.** Let us consider the semicircle

$$y = \sqrt{1 - x^2}, -1 \leq x \leq 1$$

of the circle  $x^2 + y^2 = 1$ .

**Proposition 4.2.** If a smooth path  $\tilde{\Gamma} : [\alpha, \beta] \rightarrow \mathbb{R}^3$  is obtained from a smooth path  $\Gamma : [a, b] \rightarrow \mathbb{R}$  by an admissible change of parameter, then the lengths of the two paths are equal.

$$\int_a^b \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} dt = \int_\alpha^\beta \sqrt{\dot{\tilde{x}}^2(\tau) + \dot{\tilde{y}}^2(\tau) + \dot{\tilde{z}}^2(\tau)} d\tau$$

**Examples 12.** Let us find the length of the ellipse defined by the canonical equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b > 0) \quad (13)$$

### 4.3 The Area of a Curvilinear Trapezoid

$$S[a, b] = \int_a^b f(x) \, dx$$

**Examples 13.** Let us compute the area of the ellipse given the canonical equation of Eq 13

### 4.4 Volume of a Solid of Revolution

$$V[a, b] = \pi \int_a^b f^2(x) \, dx$$

### 4.5 Work and Energy

## 5 Improper Integral

In the preceding section we encountered the need for somewhat broader concept of the Riemann integral. There, in studying a particular problem, we form an idea of the direction in which this should be done.

### 5.1 Definition, Examples, and Basic Properties of Improper Integrals

**Definition 5.1.** Suppose the function  $x \rightarrow f(x)$  is defined on the interval  $[a, +\infty)$  and integrable on every closed interval  $[a, b]$  contained in that interval. The quantity

$$\int_a^{+\infty} f(x) \, dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) \, dx,$$

if this limit exists, is called the **improper Riemann integral** or the improper integral of the function  $f$  over the interval  $[a, +\infty)$ .

**Examples 14.** Let us investigate the values of the parameter  $\alpha$  for which the improper integral

$$\int_1^{+\infty} \frac{dx}{x^\alpha}$$

converges, or what is the same, is defined.



**Definition 5.2.** Suppose the function  $x \rightarrow f(x)$  is defined on the interval  $[a, B)$  and integrable on any closed interval  $[a, b] \subset [a, B)$ . The quality

$$\int_a^B f(x) dx = \lim_{b \rightarrow B-0} \int_a^b f(x) dx,$$

if this limit exists, is called the improper integral of  $f$  over the interval  $[a, B)$ .

**Examples 15.** Let us investigate the values of the parameter  $\alpha$  for which the improper integral

$$\int_0^1 \frac{dx}{x^\alpha}$$

converges, or what is the same, is defined.

**Examples 16.**

$$\int_{-\infty}^0 e^x dx$$

**Proposition 5.1.** Suppose  $x \rightarrow f(x)$  and  $x \rightarrow g(x)$  are functions defined on an interval  $[a, \omega)$  and integrable on every closed interval  $[a, b] \subset [a, \omega)$ . Suppose the improper integrals

$$\int_a^\omega f(x) dx, \int_a^\omega g(x) dx,$$

are defined.

Then a) if  $\omega \in \mathbb{R}$  and  $f \in \mathcal{R}[a, \omega]$ , the values of the integral  $\int_a^\omega f(x) dx$  are the same, whether it is as a proper or an improper integral;

b) for any  $\lambda_1, \lambda_2 \in \mathbb{R}$  the function  $\lambda_1 f + \lambda_2 g$  is integrable in the improper sense on  $[a, \omega)$  and the following equality holds:

$$\int_a^\omega (\lambda_1 f + \lambda_2 g)(x) dx = \lambda_1 \int_a^\omega f(x) dx + \lambda_2 \int_a^\omega g(x) dx$$

c) if  $c \in [a, \omega)$ , then

$$\int_a^\omega f(x) dx = \int_a^c f(x) dx + \int_c^\omega f(x) dx$$

d) if  $\varphi : [\alpha, \gamma) \rightarrow [a, \omega)$  is a smooth monotonic mapping with  $\varphi(\alpha) = a$  and  $\varphi(\beta) \rightarrow \omega$  as  $\beta \rightarrow \gamma, \beta \in [\alpha, \gamma)$ , then the improper integral of the function  $t \rightarrow (f \circ \varphi)(t)\varphi'(t)$  over  $[\alpha, \gamma)$  exists and the following equality holds

$$\int_a^\omega f(x) dx = \int_\alpha^\gamma (f \circ \varphi)(t)\varphi'(t) dt$$

If  $f, g \in C^{(1)}[a, \omega)$  and the limit  $\lim_{x \rightarrow \omega} (f \cdot g)(x)$  exists, then the functions  $f g'$  and  $f' g$  are either both integrable or both nonintegrable in the improper sense on  $[a, \omega)$ , and when they are integrable the following equality holds:

$$\int_a^\omega (f \cdot g')(x) dx = (f \cdot g)(x)|_a^\omega - \int_a^\omega (f' \cdot g)(x) dx$$

## 5.2 Convergence of an Improper Integral

**a. The Cauchy Criterion** The convergence of the improper integral  $\int_a^\omega f(x) dx$  is equivalent to the existence of a limit for the function

$$\mathcal{F}(b) = \int_a^b f(x) dx$$

as  $b \rightarrow \omega, b \in [a, \omega)$

**Proposition 5.2.** (Cauchy Criterion for Convergence of an improper integral) If the function  $x \rightarrow f(x)$  is defined on the interval  $[a, \omega)$  and integrable on every closed interval  $[a, b] \subset [a, \omega)$ , then the integral  $\int_a^\omega f(x) dx$  converges if and only if for every  $\epsilon > 0$  there exists  $B \in [a, \omega)$  such that the relation

$$\left| \int_{b_1}^{b_2} f(x) dx \right| < \epsilon$$

holds for any  $b_1, b_2 \in [a, \omega)$  satisfying  $B < b_1, B < b_2$ .

### b. Absolute Convergence of an Improper Integral

**Definition 5.3.** The improper integral  $\int_a^\omega f(x) dx$  converges absolutely if the integral  $\int_a^\omega |f| dx$  converges.

**Proposition 5.3.** Let  $[a, \omega)$  be a finite or infinite interval and  $x \rightarrow f(x)$  a function defined on that interval and integrable over every closed interval  $[a, b] \subset [a, \omega]$ , and  $f(x) \geq 0$ , then the improper integral  $\int_a^\omega f(x) dx$  exists if and only if the function

$$\mathcal{F}(b) = \int_a^b f(x) dx$$

is bounded on  $[a, \omega)$

**Corollary 5.1. (Integral test for convergence of a series)** If the function  $x \rightarrow f(x)$  is defined on the interval  $[1, +\infty]$ , non-negative, non-increasing, and integrable on each closed interval  $[1, b] \subset [1, +\infty)$ , then the series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + \cdots$$

and the integral

$$\int_1^{+\infty} f(x) dx$$

either both converge or both diverge.

In particular, one can say that the result of

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

converges only for  $\alpha > 1$ .

**Theorem 5.2. Comparison theorem** Suppose the function  $x \rightarrow f(x)$  and  $x \rightarrow g(x)$  are defined on the interval  $[a, \omega)$  and integrable on any closed interval  $[a, b] \subset [a, \omega)$ .

If  $0 \leq f(x) \leq g(x)$  on  $[a, \omega)$ , then convergence of  $g(x)$  implies convergence of  $f(x)$ , and

$$\int_a^\omega f(x) dx \leq \int_a^\omega g(x) dx$$

holds. Divergence of  $f(x)$  implies divergence of  $g(x)$ .

**Examples 17.**

$$\begin{aligned} & \int_1^{+\infty} \frac{\sqrt{x}}{\sqrt{1+x^4}} dx, \int_1^{+\infty} \frac{\cos x}{x^2} dx, \int_0^{\frac{\pi}{2}} \ln \sin x dx, \\ & \int_1^{+\infty} e^{-x^2} dx, \int_e^{+\infty} \frac{1}{\ln x} dx, \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} dx, \end{aligned}$$

### C. Conditional Convergence of an Improper Integral

**Definition 5.4.** If an improper integral converges but not absolutely, we say that it converges conditionally.

**Examples 18.**

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x} dx$$

**Proposition 5.4. (Abel-Dirichlet test for convergence of an integral).** Let  $x \rightarrow f(x)$  and  $x \rightarrow g(x)$  be functions defined on an interval  $[a, \omega)$  and integrable on every closed interval  $[a, b] \subset [a, \omega)$ . Suppose that  $g$  is monotonic.

Then a sufficient condition for convergence of the improper integral

$$\int_a^\omega (f \cdot g)(x) dx$$

is that the one of the following pairs of conditions hold:

a1) the integral  $\int_a^\omega f(x) \, dx$  converges,

b1) the function  $g$  is bounded on  $[a, \omega)$ .

or

a2) the function  $\mathcal{F}(b) = \int_a^b f(x) \, dx$  is bounded on  $[a, \omega)$ ,

b2) the function  $g(x)$  tends to zero as  $x \rightarrow \omega, x \in [a, \omega)$ .

**Examples 19.**

$$\int_0^{\frac{1}{e}} \frac{1}{x^p \ln x} \, dx, p \in \mathbb{R}^+$$

**Examples 20.**

$$\int_0^1 \frac{1}{x^p} \sin \frac{1}{x} \, dx, (p < 2)$$

## 6 作業

### 6.1 證明題

1. 證明：若分割 $\tilde{P}$  是分割 $P$ 增加若干分點得到的分割，則有：

$$\sum_{\tilde{P}} \omega'_i \Delta x'_i \leq \sum_P \omega_i \Delta x_i$$

2. 證明：若 $f$ 在 $[a, b]$ 上可積， $[\alpha, \beta] \subset [a, b]$ ，則 $f$ 在 $[\alpha, \beta]$ 上也可積。
3. 設 $f, g$ 均為定義在 $[a, b]$ 上的有界函數，僅在有限個點處 $f(x) \neq g(x)$ ，證明：若 $f$ 在 $[a, b]$ 上可積，則 $g$ 在 $[a, b]$ 上也可積，且有：

$$\int_a^b f(x) dx = \int_a^b g(x) dx$$

4. 設 $f$ 在 $[a, b]$ 上有界， $\{a_n\} \subset [a, b]$ ， $\lim_{n \rightarrow \infty} a_n = c$ ，證明：若 $f$ 在 $[a, b]$ 上只有 $a_n, n = 1, 2, \dots$ 為其間斷點，則 $f$ 在 $[a, b]$ 上可積。
5. 證明：若 $f \in C[a, b]$  且 $f(x) \geq 0, \forall x \in [a, b]$  則以下結果成立：
- (a) 如果函數 $f(x)$ 存在一點 $f(x_0) > 0, x_0 \in [a, b]$ ，則有：

$$\int_a^b f(x) dx > 0$$

- (b) 若 $\int_a^b f(x) = 0$ ，則有 $f(x) \equiv 0$

6. 證明若 $f \in C[a, b], f(x) \geq 0, \forall x \in [a, b]$ ，且 $M = \max_{[a, b]} f(x)$ ，則

$$\lim_{n \rightarrow \infty} \left( \int_a^b f^n(x) dx \right)^{\frac{1}{n}} = M$$

7. 證明黎曼函數

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, p, q \text{互質}, q > p, \\ 0, & x = 0, 1 \text{ 其它}(0,1)\text{內無理數} \end{cases}$$

在區間 $[0, 1]$ 上可積。

8. 計算下列定積分

- (a)  $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x \, dx$
- (b)  $\int_0^1 \sqrt{4-x^2} \, dx$
- (c)  $\int_0^a x^2 \sqrt{a^2-x^2} \, dx (a > 0)$
- (d)  $\int_0^1 \frac{1}{(x^2-x+1)^{\frac{3}{2}}} \, dx (a > 0)$
- (e)  $\int_0^1 \frac{1}{e^x + e^{-x}} \, dx$
- (f)  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} \, dx$
- (g)  $\int_0^1 \arcsin x \, dx$
- (h)  $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$
- (i)  $\int_{\frac{1}{e}}^e |\ln x| \, dx$
- (j)  $\int_0^1 e^{\sqrt{x}} \, dx$
- (k)  $\int_0^a x^2 \sqrt{\frac{a-x}{a+x}} \, dx (a > 0)$
- (l)  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} \, dx$

9. 求下列極限

- (a)  $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x \cos t^2 \, dt$
- (b)  $\lim_{x \rightarrow \infty} \frac{\left( \int_0^x e^{t^2} \, dt \right)^2}{\int_0^x e^{2t^2} \, dt}$

10. 求下列曲線的弧長

- (a)  $y = x^{\frac{3}{2}}, 0 \leq x \leq 4$
- (b)  $x = a \cos^3 t, y = a \sin^3 t (a > 0), 0 \leq t \leq 2\pi$
- (c)  $r = a \sin^3 \frac{\theta}{3} (a > 0), 0 \leq \theta \leq 3\pi$

11. 求下列平面曲線繞旋轉軸所圍成立體的體積

- (a)  $y = \sin x, 0 \leq x \leq \pi$ , 繞  $x$  軸。
- (b)  $x = a(t - \sin t), y = a(1 - \cos t) (a > 0, 0 \leq t \leq 2\pi)$ , 繞  $x$  軸。
- (c)  $r = a(1 + \cos \theta) (a > 0)$ , 繞極軸。

12. 求下列平面曲線繞指定軸旋轉得到的面積

- (a)  $y = \sin x, 0 \leq x \leq \pi$ , 繞  $x$  軸。
- (b)  $x = a(t - \sin t), y = a(1 - \cos t) (a > 0, 0 \leq t \leq 2\pi)$ , 繞  $x$  軸。
- (c)  $r = a(1 + \cos \theta) (a > 0)$ , 繞極軸。

13. 討論下列積分是否收斂？若收斂，則求其極限。

- (a)  $\int_0^{+\infty} x e^{-x^2} dx$
- (b)  $\int_{-\infty}^{+\infty} x e^{-x^2} dx$
- (c)  $\int_1^{+\infty} \frac{1}{x^2(1+x)} dx$
- (d)  $\int_0^{+\infty} e^{-x} \sin x dx$
- (e)  $\int_0^{+\infty} \frac{1}{\sqrt{1+x^2}} dx$
- (f)  $\int_a^b \frac{1}{(x-a)^p} dx$
- (g)  $\int_0^1 \frac{1}{1-x^2} dx$
- (h)  $\int_0^1 \sqrt{\frac{x}{1-x}} dx$
- (i)  $\int_0^1 \frac{1}{x(\ln x)^p} dx$

14. 討論下列積分的收斂性

- (a)  $\int_0^{+\infty} \frac{1}{\sqrt[3]{x^4+1}} dx$
- (b)  $\int_1^{+\infty} \frac{x}{1-e^x} dx$
- (c)  $\int_1^{+\infty} \frac{x \arctan x}{x^3+1} dx$

$$(d) \int_0^{+\infty} \frac{x^m}{x^n + 1} dx (m, n \geq 0)$$

$$(e) \int_0^2 \frac{1}{(x-1)^2} dx$$

$$(f) \int_0^\pi \frac{\sin x}{x^{\frac{3}{2}}} dx$$

$$(g) \int_0^1 \frac{1}{x^\alpha} \sin \frac{1}{x} dx$$

$$(h) \int_0^{+\infty} e^{-x} \ln x dx$$

15. 討論下列去窮積分為絕對收斂還是條件收斂

$$(a) \int_1^{+\infty} \frac{\sin \sqrt{x}}{x} dx$$

$$(b) \int_e^{+\infty} \frac{\ln(\ln x)}{\ln x} \sin x dx$$