

Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen.

(Von Herrn Cantor in Halle a. S.)

Unter einer reellen algebraischen Zahl wird allgemein eine reelle Zahlgrosse ω verstanden, welche einer nicht identischen Gleichung von der Form genügt:

(1.) $a_0 \omega^n + a_1 \omega^{n-1} + \dots + a_n = 0,$

wo n, a_0, a_1, \dots, a_n ganze Zahlen sind; wir können uns hierbei die Zahlen n und a_0 positiv, die Coefficienten a_0, a_1, \dots, a_n ohne gemeinschaftlichen Theiler und die Gleichung (1.) irreductibel denken; mit diesen Festsetzungen wird erreicht, dass nach den bekannten Grundsätzen der Arithmetik und Algebra die Gleichung (1.), welcher eine reelle algebraische Zahl genügt, eine völlig bestimmte ist; umgekehrt gehören bekanntlich zu einer Gleichung von der Form (1.) höchstens soviel reelle algebraische Zahlen ω , welche ihr genügen, als ihr Grad n angiebt. Die reellen algebraischen Zahlen bilden in ihrer Gesamtheit einen Inbegriff von Zahlgrossen, welcher mit (ω) bezeichnet werde; es hat derselbe, wie aus einfachen Betrachtungen hervorgeht, eine solche Beschaffenheit, dass in jeder Nähe irgend einer gedachten Zahl α unendlich viele Zahlen aus (ω) liegen; um so auffallender dürfte daher für den ersten Anblick die Bemerkung sein, dass man den Inbegriff (ω) dem Inbegriffe aller ganzen positiven Zahlen ν , welcher durch das Zeichen (ν) angedeutet werde, eindeutig zuordnen kann, so dass zu jeder algebraischen Zahl ω eine bestimmte ganze positive Zahl ν und umgekehrt zu jeder positiven ganzen Zahl ν eine völlig bestimmte reelle algebraische Zahl ω gehört, dass also, um mit anderen Worten dasselbe zu bezeichnen, der Inbegriff (ω) in der Form einer unendlichen gesetzmässigen Reihe:

(2.) $\omega_0 \omega_1 \dots \omega_r \dots$

Georg Cantor (left) and his legendary 1874 publication "Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen" in Journal für die Reine und Angewandte Mathematik (1874).

BIOGRAPHY

The Nature of Infinity — and Beyond

An introduction to Georg Cantor and his transfinite paradise



Jørgen Veisdal Following
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G eorg Cantor showed a markedly artistic streak as a child and was reportedly an outstanding violinist. His surname ‘Cantor’ in latin means ‘singer’ or ‘musician’. When he in 1867 — at the age of 22 — completed his doctoral thesis at the University of Berlin, he named it *In re mathematica ars propendi pluris facienda est quam solvendi* which in English reads “In mathematics, the art of asking questions is more valuable than solving problems”. Later renowned as a legendarily deep thinker, Cantor would grow up to be the man who dared both ask and answer one of the deepest and most fundamental questions of all:

How big is infinity?

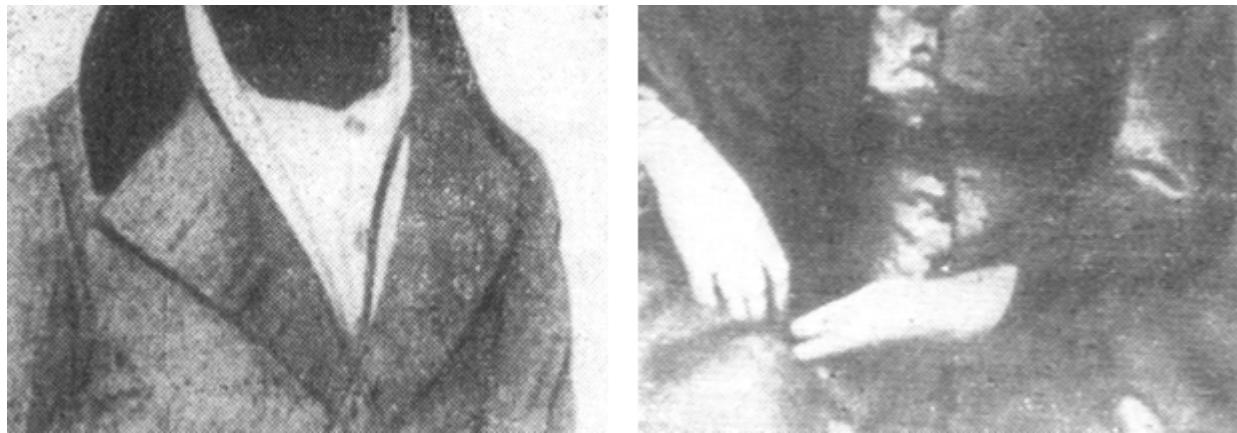
Commonly thought unanswerable, Cantor in the 1870s, ‘80s and ‘90s introduced radical new ideas about the answer to this question which established set theory as a new branch of pure mathematics. This article hopes to introduce you to his most notable work, and its implications.

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Early life (1845–69)

Georg Cantor was in a sense lucky to have been born when he was, in Saint Petersburg on the 3rd of March 1845. His parents were Danish. His mother Marie (family name Meyer) came from a family of musicians with Russian origins and his father Georg Woldemar was a very successful business man, first as a wholesaling agent in St. Petersburg, and later as a broker on the city’s stock market.





Cantor's father and mother, G.W. and Marie Cantor

Cantor was greatly influenced by his father, a man of great cultural and philosophical interests who would throughout his son's school and university years provide him with much well-meaning advice about his life and career. Yet, by some accounts, despite recognizing his son's mathematical abilities would still obstinately try to force his son into engineering as a more promising profession than mathematics.

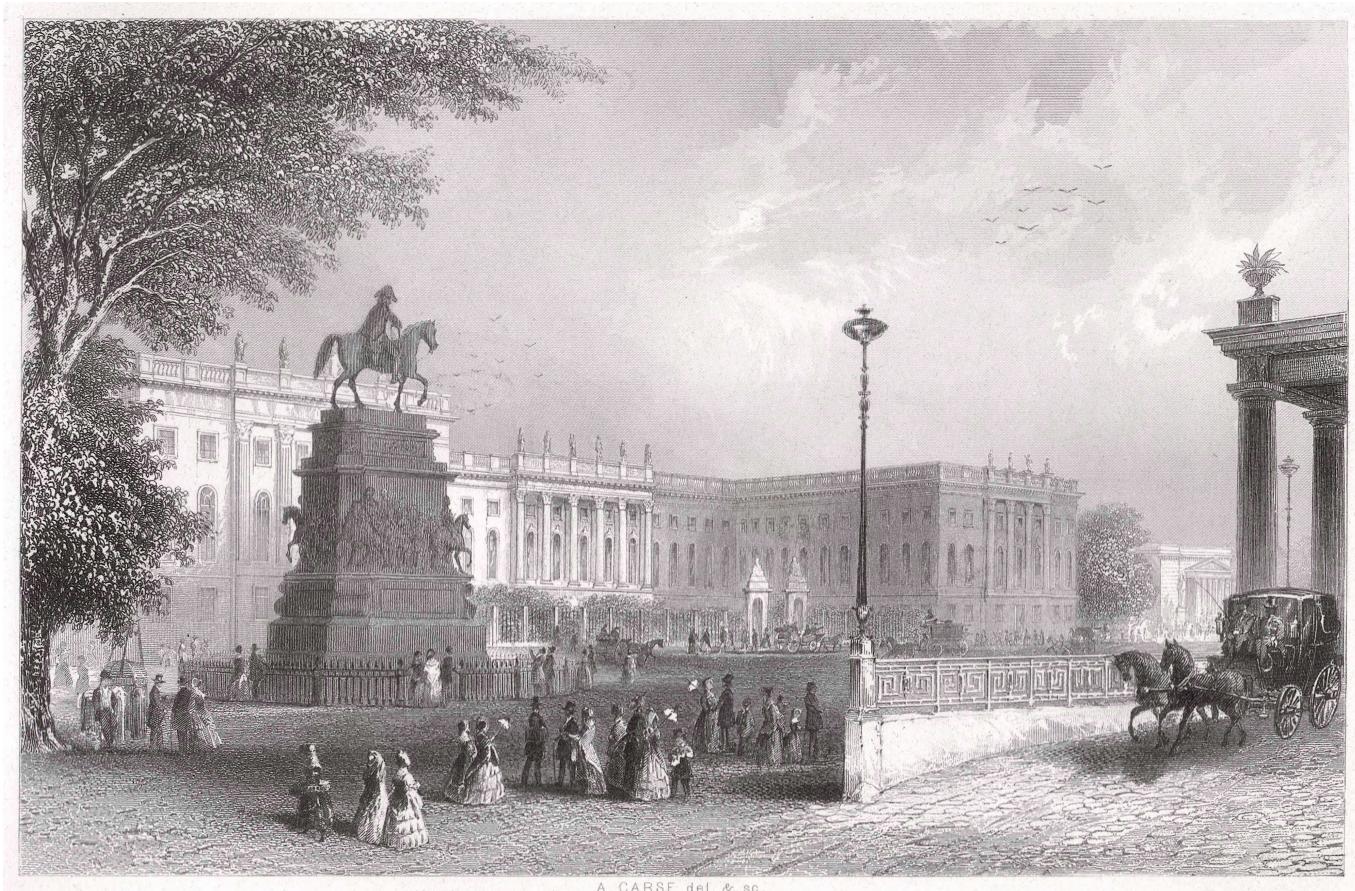
Education (1860-69)

	Hausmann	Jackson	Wiesbaden	Johnson	Salamanca	Koenigsberg	St. Petersburg	St. Petersburg	Göttingen	Bremen	Harrison	Pape
Gr. Festia	Religious	R.P.	Wiesbaden	Johnson	Salamanca	Koenigsberg	St. Petersburg	St. Petersburg	Göttingen	Bremen	Harrison	Pape
Reyes		5	4	5	4	5	4	5	5	5		
Reyes		3½	3	4	2½	3½	3	5	3½			
Schoffer		4	2	4	2½	3½	2	5	4			
Götz		3½	3	2½	1	—	3	—	4			
Aug		8	0	8	7	7½	0	8	8			
Cantor	5	4½	5	4½	—	5	5	5	5			5
M. P.	4	3	4	4	—	4½	4½	5	3½			4
Schoffer	4	3	3½	4	—	4½	4	5	4			4½
Götz	—	3½	3½	3	—	—	4½	—	4			5
Aug	E.P.	L	7½	8	—	E.P.	E.P.	8	8			E.P.

Cantor's grades at age 8, when he attended the St.Petri-Schule for German speaking people in St.Petersburg.

Cantor's school career was like that of many highly gifted mathematicians — an early recognition of his talent (before the age of fifteen) and an absorbing interest in his studies. As early as in Saint Petersburg, Cantor received private tutoring lessons. In Germany, he first attended private school in Frankfurt at the Darmstadt nonclassical school, before entering the Wiesbaden Gymnasium in 1860. He graduated with

distinction from the Realschule in Darmstadt, and in 1862 began his university studies at Höheren Gewerbschule where he studied engineering for two years before transferring to the Swiss Federal Politechnic (ETH Zurich) to pursue mathematics. Following the death of his father from tuberculosis the next year, he received a substantial inheritance (half a million marks) and shifted his studies to the University of Berlin.



Humboldt University of Berlin in 1850 (then Friedrich Wilhelm University)

In Berlin Cantor attended lectures by Ernst Kummer, Leopold Kronecker and Karl Weierstrass, whose interest in arithmetics exerted a strong influence on his earliest work. In 1866, he spent the summer semester at the University of Göttingen, at that time the world's capital of mathematical thought. Both his dissertation "*De aequationibus secundi gradus indeterminatis*" in 1867 and his habilitation "*De transformatione formarum ternariarum quadraticarum*" in 1869 regarded number theory, in particular an outstanding problem left over from Gauss' *Disquisitiones Arithmeticae* concerning the solutions to the indeterminate diophantine equation $ax^2+by^2+cz^2 = 0$, also known as Legendre's equation.



Georg Cantor around 1870

Although lauded as *docta et ingeniosa* (“learned and clever”) his “severely classical dissertation”, his age notwithstanding gave no particular hint of the genius that was to come. He passed his oral examination *magna cum laude*. Upon receiving his Ph.D. he left Berlin to assume a position as a *Privatdozent* (a lecturer who lives off the fees he can collect from his students) at Halle University, replacing his friend K.H.A. Schwartz (who went to Zurich) and assuming work under Eduard Heine, the professor of mathematics there.

Early career (1870–73)

Some have argued that the antecedents of Cantor’s later groundbreaking work can be traced as far back as to his earliest post-graduate publications. In fact, in Cantor’s research devoted the theory of trigonometric series, one can indeed find traces of his early interest in the “continuum”. Following the influences of both Weierstrass in Berlin and Heine in Halle, Cantor’s first paper *Über einen die trigonometrischen Reihen betreffenden Lehrsatz* (“On a theorem concerning the trigonometric series”) was completed for publication in March of 1870 and positioned to “advance the

understanding of the convergence properties of the representation of an arbitrarily given function by means of infinite trigonometric series". Starting from the trigonometric series and the work on functions of a complex variable done by Riemann, Cantor in the paper showed the following theorem:

Cantor's Uniqueness Theorem (1870): Every function $f: \mathbb{R} \rightarrow \mathbb{R}$ can have at most one representation by a trigonometric series.

If a function $f(x)$ is represented by a trigonometric series that is convergent for all x , then that representation is unique. In 1871, he strengthened the result, proving that the uniqueness holds even if the series diverges at a finite number of points in any given interval. This result was one previously attempted by many of the greatest minds at the time, including Heine, Peter Dirichlet and Bernhard Riemann, who thus far had only been able to show that it held under certain limited circumstances.

His next paper, published in 1872, extended the result even further. The paper, *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen* ("On the generalization of a theorem from the theory of trigonometric series") provides a definition of a boundary point of a point-set P to be any point such that every neighborhood of the point contains infinitely many points of P . The first derivative of P (designated P') is the set of all boundary points of P , the second derivative P'' is the set of all boundary points of P' , and so on. This definition laid the foundation for point-set topology, later extended in particular by Felix Hausdorff, Émile Borel and Maurice René Fréchet. Cantor used the definition to improve his uniqueness theorem, by showing that the theorem holds even if the trigonometric series diverges at an *infinite* number of points, provided that the set of points is of finite order (a point-set P is of finite order if, for some integer n , the n th derivative $P^{(n)}$ of P is a finite set).

Viewed in retrospect, the paper connects Cantor's early work in analysis with what is now considered his most important work on the study of transfinite sets, for instance in its focus on infinite point sets and the definition of real numbers he provides:

Cantor's definition of real numbers \mathbb{R} (1872): A real number is an infinite series of rational numbers:

$a_1, a_2, \dots, a_u, \dots$

such that for any given ε there exists an u_1 such that for $u \geq u_1$ and for any positive integer v , $|a_{u+v} - a_u| < \varepsilon$.

In the paper, Cantor discusses this definition and compares it to those previously given by his former and current mentors, Weierstrass in Berlin and Heine at Halle. The results added to his previous work and was sufficient to promote Cantor to *außerordentlicher Professor* (associate professor) at Halle University in 1872.

Correspondence with Richard Dedekind (1872–73)

Later in the same year, Cantor met Richard Dedekind for the first time, who at that point was professor of mathematics at the Technische Hochschule at Brunswick. Dedekind had previously published a paper that provided an axiomatic analysis of the structure of the set of real numbers \mathbb{R} . His definition was of the real numbers as a complete, ordered field. Cantor and Dedekind exchanged letters over a period of many years. The mathematical portions of their letters were later published by Noether and Cavailleès (1937) and are now kept at the University of Evansville in Indiana.

On November 29th, 1873 Cantor sent a letter to Dedekind asking whether or not the collection of natural numbers and the collection of positive real numbers “*can be corresponded so that each individual of one collection corresponds to one and only one individual of the other?*” to which Dedekind replied by writing that he did not know the answer, adding however that the question was not of much practical interest. At this point, it seems Cantor agreed with this contention, stating that his interest in the matter were related to Joseph Liouville’s 1844 theorem proving the existence of transcendental numbers:

Halle, December 2nd 1873

I was exceptionally pleased to receive your answer to my last letter. I put my question to you because I had wondered about it already several years ago, and was never certain whether the difficulty I found was subjective or whether it was inherent in the subject. Since you write that you too are unable to answer it, I may assume the latter. In addition, I should like to add that I have never seriously occupied myself with it, because it has no special practical interest for me. And I entirely agree with you when you say that for this reason it does not deserve much effort. But it

would be good if it could be answered; e.g. if it could be answered with no, then one would have a new proof of Liouville's theorem that there are transcendental numbers.

– G. Cantor

From Cantor's next letter a few days later however it seems clear that his interest in the topic was not quite as fleeting as he had expressed to Dedekind, although he at this time does not outline any particularly important implications:

Halle, December 7th 1873

"..In the last days I have had the time to pursue more thoroughly the conjecture I spoke to you about; only today do I believe myself to have finished with the thing; but if I should be deceiving myself, I should certainly find no more indulgent judge than you."

In the letter, Cantor next proceeds with the first draft of a proof of why the real numbers **cannot** be put in one-to-one correspondence with the natural numbers. Not two days later, he sends Dedekind a revised and simpler proof, along with his apologies for occupying his time with the matter:

Halle, December 9th 1873

I have already found a simplified proof of the theorem just proved, so that the decomposition of the sequence into (1),(2),(3), ... is no longer necessary. I show directly that if I start with a sequence

(i) $\omega_1, \omega_2, \dots, \omega_u,$

then in every given interval $(\alpha \dots \beta)$ I can determine a number η that is not contained in (i). From this it follows at once that the totality (x) cannot be correlated one-to-one with the totality (u); and I infer that there exist essential differences among totalities and value-sets that I was until recently unable to fathom.

Now I must ask your forgiveness for having taken so much of your time with this question. Confirming the receipt of your friendly lines of the 8th of December, allow me to assure you that nothing can give me more pleasure than to have been lucky enough to arouse in you an interest for certain questions of analysis.

– G. Cantor

Dedekind's notes from the period make clear the chronology of events:

Brunswick, December 7th 1873

Cantor communicates to me a rigorous proof, found on the same day, of the theorem that the totality of all positive numbers $\omega < 1$ cannot be one-to-one correlated with the totality (n).

I answered this letter, received on the 8th of December, on the same day with congratulations for the fine success. At the same time, I rephrase much more simply the core of the proof (which was still quite complicated).

– Richard Dedekind

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Set Theory

Described by the Stanford Encyclopaedia of Philosophy as “one of the greatest achievements of modern mathematics”, set theory is widely acknowledged to have been founded by the paper that resulted from the work Cantor did in the period 1873–1884. In particular, the origins of set theory is traced back to a single paper published in 1874 by Cantor, entitled *Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, (“On a Property of the Collection of All Real Algebraic Numbers”). The fundamental and most consequential result it presents is the uncountability of the real numbers, and as consequence, the invention of a distinction between numbers that belong to “*the continuum*” and those that belong to “*a collection like the totality of real algebraic numbers*”. The paper appeared in *Journal für die reine und angewandte Mathematik* (“Crelle’s Journal”) just before Cantor turned 30 years old. As he wrote to Dedekind about two weeks after arriving at his proof:

Berlin, December 25th 1873

"..Although I did not yet wish to publish the subject I recently for the first time discussed with you, I have nevertheless unexpectedly been caused to do so. I communicated my results to Herr Weierstrass on the 22nd; however, there was no time to go into details; already on the 23d I had the pleasure of a visit from him, at which I could communicate the proofs to him. He was of the opinion that I must publish the thing at least in so far as it concerns the algebraic numbers. So I wrote a short paper with the title: On a property of the set of all real algebraic numbers and sent it to Professor Borchardt to be considered for the Journal fur Math.

As you will see, your comments (which I value highly) and your manner of putting some of the points were of great assistance to me."

– G. Cantor

In five short pages, Cantor's paper presents three important results:

1. The set of real algebraic numbers is countable; and
2. In every interval $[a,b]$ there are infinitely many numbers not included in any sequence; and as a consequence that
3. The set of real numbers are uncountably infinite;

The rest of this article is devoted to explaining the implications of the third result, on the uncountability of real numbers. For this, we begin with a few fundamental concepts.

What is a set?

"A set is a Many which allows itself to be thought of as a One" — Georg Cantor

A set is a collection of elements. The set consisting of the numbers 3,4 and 5 is denoted by $\{3, 4, 5\}$. For larger sets and the sake of simplicity, an ellipsis is often used if the reader can easily guess what the missing elements are. Cantor's original definition of an "aggregate" (set), translated, went as follows:

Cantor's Definition of a Set

By a set we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the "elements" of M .

Countability

A countable set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers.

The property of countability is an important one in set theory. An intuitive interpretation of countability is “listability”, that the elements of a set may be written down in a list.

The most inherently countable set is the natural numbers \mathbb{N} , in that the elements of \mathbb{N} are the counting numbers themselves (1,2,3, ...). As we know, they are infinite in number, and so termed “countably infinite”, or “denumerable”. For other sets, formally, by stating that a set is countable one means that the elements of the set can be put in one-to-one correspondence with elements of the set of natural numbers \mathbb{N} , i.e. that:

Countable sets

A set S is countable if there exists an injective function f from S to the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. If such an f can be found that is also surjective (and therefore bijective), then S is called a countably infinite set, or denumerable.

For instance, for the set of even numbers ($2n | n \in \mathbb{N}$):

$$\begin{array}{ccccccc} 2 & 4 & 6 & 8 & 10 & \dots & 2n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ 1 & 2 & 3 & 4 & 5 & \dots & n \end{array}$$

We see that the elements of the two sets may be put in one-to-one correspondence with one other, and so we can determine that the set of even numbers is also countable.

The countability property makes it possible to compare sets in terms of the number of elements they contain without actually counting anything, and in this way make inferences about the relative sizes of both finite and infinite sets. For practical reasons let us illustrate the finite case by imagining a classroom with 100 seats. Filled with students, one can make an inference about the size of the set of students in relation to size of the set of seats. If seats are vacant, the set of seats is larger than the set of students. If no seats are vacant and some students are standing, the size of the set of students is larger than that of seats, and so on.

The Countability of Rational Numbers (1873)

Cantor's first published investigation into the countability of sets occurred in 1873 when he proved that the rational numbers \mathbb{Q} (fractions/ratios) are countable. His rather elegant and intuitive proof went as follows:

Proof of the Countability of the Rational numbers \mathbb{Q}

Let us first propose that the set of rational numbers \mathbb{Q} is countable. To prove this assertion, let us arrange all the rational numbers (ratios of natural numbers) in an infinite table as such:

1/1	1/2	1/3	1/4	1/5	...
2/1	2/2	2/3	2/4	2/5	...
3/1	3/2	3/3	3/4	3/5	...
4/1	4/2	4/3	4/4	4/5	...
5/1	5/2	5/3	5/4	5/5	...
...

Next, starting in the upper lefthand corner, move through the diagonals from left to right at 45 degrees, starting with 1/1, then 1/2 and 2/1, then 3/1, 2/2 and 1/3 and so on. Write down every new number you come across. You will obtain the following ordering:

1/1, 1/2, 2/1, 3/1, 2/2, ...
 1 2 3 4 5 ...

Which is not just a well-ordering, but also in one-to-one correspondence with the natural numbers in their natural order. This proves the countability of the rational numbers \mathbb{Q} .

The Countability of Real Algebraic Numbers (1874)

A year later, in his 1884 paper, Cantor showed that the real algebraic numbers are countable. Real algebraic numbers are real numbers ω which satisfy equations of the form: $a_0 \omega^u + a^1 \omega^{u-1} + \dots + a_u = 0$. That is to say, real algebraic numbers are roots of non-zero real polynomials. They are countable, i.e:

The Countability of Real Algebraic Numbers

The collection of all algebraic reals can be written as an infinite sequence.

Cantor showed it in his 1874 paper by the following proof:

Proof of the Countability of Real Algebraic Numbers (1874)

For each polynomial equation of the form

$$a_0\omega^u + a_1\omega^{u-1} + \dots + a_u = 0$$

with integer coefficients a_i , define its index to be the sum of the absolute values of the coefficients plus the degree of the equation:

$$|a_0| + |a_1| + \dots + |a_u|$$

The only equation of index 2 is $\omega = 0$, so its solution, 0, is the first algebraic number. The four equations of index 3 are $2x = 0$, $x + 1 = 0$, $x - 1 = 0$, and $x^2 = 0$. They have roots 0, -1, 1, so he included the new values -1 and 1 as the second and third entries on his list of algebraic numbers.

Observe that for each index there are only finitely many equations and that each equation only has finitely many roots. Listing the new roots by order of index and by increasing magnitude within each index, one establishes a systematic method for listing all the algebraic numbers. As with rationals, the one-to-one correspondence with the natural numbers proved that the set of algebraic numbers have to countably infinite.

The Uncountability of Real Numbers (1874)

Cantor's most fruitful use of countability as a concept occurred in the third result of his 1874 paper when he demonstrated the *uncountability* of the real numbers — the first set shown to lack this property. A real number \mathbb{R} is a value of a continuous quantity that can represent a distance along a line. Any real number can be determined by a possibly infinite decimal representation, such as that of e.g. 8.632, 0.00001, 10.1 and so on, where each consecutive digit is measured in units one tenth the size of the previous one. The statement that the real numbers are uncountable is equivalent to the statement:

The Uncountability of Real Numbers

Given any sequence of real numbers and any interval $[\alpha \dots \beta]$, one can determine a number η in $[\alpha \dots \beta]$ that does not belong to the sequence. Hence, one can determine infinitely many such numbers η in $[\alpha \dots \beta]$.

As we've seen from his letter exchanges with Dedekind in 1873, we know how Cantor worked towards the momentous result. His original proof (Cantor's First Uncountability Proof) went as follows, and is based on the Bolzano-Weierstrass theorem:

Proof of the Uncountability of the Real numbers \mathbb{R} (1874)

Suppose we have an infinite sequence of real numbers,

(i) $\omega_1, \omega_2, \dots, \omega_v, \dots$

where the sequence is generated according to any law and the numbers are distinct from each other. Then in any given interval $(\alpha \dots \beta)$ a number η (and consequently infinitely many such numbers) can be determined such that it does not occur in the series (i).

To prove this, we go to the end of the interval $[\alpha \dots \beta]$, which has been given to us arbitrarily and in which $\alpha < \beta$. The first two numbers of our sequence (i) which lie in the interior of this interval (with the exception of the boundaries), can be designated by α' , β' , letting $\alpha' < \beta'$. Similarly, let us designate the first two numbers of our sequence which lie in the interior of $(\alpha' \dots \beta')$ by α'' , β'' and let $\alpha'' < \beta''$. In the same way, construct the next interval, and so on.

Here, therefore, α' , $\alpha'' \dots$ are by definition determinate numbers of our sequence (i), whose indices are continually increasing. The same goes for the sequence β' , $\beta'' \dots$; Furthermore, the numbers α' , $\alpha'' \dots$ are always increasing in size, while the numbers β' , $\beta'' \dots$ are always decreasing in size. Of the intervals $[\alpha \dots \beta]$, $[\alpha' \dots \beta']$, $[\alpha'' \dots \beta''] \dots$ each encloses all of those that follow. Here, only two cases are conceivable.

In the first case, the number of intervals so formed is finite. In this case, let the last of them be $(\alpha^v \dots \beta^v)$. Since its interior can be at most one number of the sequence (i), a number η can be chosen from this interval which is not contained in (i), thereby proving the theorem.

In the second case, the number of constructed intervals is infinite. Then, because they are always increasing in size without growing into the infinite, the numbers $\alpha, \alpha', \alpha'', \dots$ have a determinate boundary value α^w . The same holds for the numbers $\beta, \beta', \beta'', \dots$ because they are always decreasing in size. Let their boundary value be β^w . If $\alpha^w = \beta^w$, then one easily persuades oneself, if one only looks back to the definition of the intervals that the number $\eta = \alpha^w = \beta^w$ cannot be contained in our sequence (i). However, if $\alpha^w < \beta^w$, then every number η in the interior of the interval $[\alpha^w \dots \beta^w]$ as well as its boundaries satisfies the requirement that it is not contained in the sequence (i).

Cantor's Diagonal argument (1891)

Cantor seventeen years later provided a simpler proof using what has become known as Cantor's diagonal argument, first published in an 1891 paper entitled *Über eine*

elementere Frage der Mannigfaltigkeitslehre (“On an elementary question of Manifold Theory”). I include it here for its elegance and simplicity. Generalized, the now famous argument goes as follows:

Proof: Cantor's diagonal argument (1891)

In his paper, Cantor considers the set M of all infinite sequences of the binary numbers m and w . Sequences such as:

$$\begin{aligned}E_1 &= (m, m, m, m, m, \dots), \\E_2 &= (w, w, w, w, w, \dots), \\E_3 &= (m, w, m, w, m, \dots), \\E_4 &= (w, m, w, m, w, \dots), \\E_5 &= (m, m, w, w, m, \dots)\end{aligned}$$

Cantor asserts that there exists a set M that does not have the “breath” of the series $E_1, E_2, E_3 \dots$, meaning M is of a different size than the sum of each sequence E_n , i.e. that even though M is constructed of all the infinite sequences of the binary numbers m and w , he can always construct a new sequence E_0 which “is both an element of M and is not an element of M .”

The new sequence E_0 is constructed using the complements of one digit from each sequence $E_1, E_2, \dots E_n$. A complement of a binary number is defined as the value obtained by inverting the bits in the representation of the number (swapping m for w and visa versa). So, the new sequence is made up of the complement the first digit from the sequence E_1 (m), the complement of the second digit from the sequence E_2 (w), the complement of the third digit from the sequence E_3 (m) and so on to finally the complement of the n th digit from the sequence E_n . From the example sequences above, the new sequence E_0 would then be:

$$E_0 = (w, m, w, w, w, \dots)$$

By its construction, E_0 differs from each sequence E_n since their n th digits differ. Hence, E_0 cannot be one of the infinite sequences in the set M .

Applied to prove the uncountability of the real numbers \mathbb{R} :

Proof of the Uncountability of the Real numbers \mathbb{R}

This proof is by contradiction, i.e. we will assume that the real numbers \mathbb{R} are countable and derive a contradiction. If the reals are countable, then they could be listed:

1. 657.**8**53260...
 2. 2.**3**13333...
 3. 3.**1**41592...
 4. .**0**00307...
 5. 49.**4**94949...
 6. .**8**73257...
- ...

To obtain a contradiction, it suffices to show that there exists some real α that is missing from the list. The construction of such an α works by making its first decimal place different from the first decimal place of the first number of the list, by making the second decimal place different from the second decimal place of the second number, and in general by making the n th decimal place different from the n th decimal place of the n th number on the list.

Even simpler, for our α we'll make the n th decimal place **1** unless it is already **1**, in which case we'll make it **2**. By this process, for our example list of numbers, we obtain:

$$\alpha = .\mathbf{1}22\mathbf{1}\mathbf{1}\dots$$

Which, by construction cannot be a member of the list we created. And so, by contradiction, our list of all reals cannot contain every number, and so must be uncountable.

The conclusions of both proofs (1874 and 1891) are the same — although both the natural numbers and the real numbers are infinite in number and so go on forever, there “aren’t enough” natural numbers to create a one-to-one correspondence between them and the real numbers. Cantor’s brilliant discovery, in other words, showed rigorously that *infinity comes in different sizes*, some of which are larger than others.

There are more real numbers than there are natural numbers.

...

From Cantor’s correspondence with Dedekind around the time of the submission of the original proof in 1874, it seems clear that he was already in the process of pondering this particular implication of the result, although from known records he does not seem to explicitly state so to Dedekind. We do however see traces of his brilliantly creative and

questioning mind in his letters from the same time, such as in this excerpt from January of 1874 about the sizes of sets of different dimensions:

Halle, January 5th 1874

"..Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? It still seems to me at the moment that the answer to this question is very difficult – although here too one is so impelled to say no that one would like to hold the proof to be almost superfluous."

– G. Cantor

When Dedekind does not reply to the proposition directly, Cantor repeats the inquiry a few weeks later, indicating his awareness of the meaningful implications it has:

Halle, January 28th 1874

"..When you get around to answering me, I should be grateful to hear whether you had the same difficulty as I in answering the question I sent to you in January about the correlation of a line and a surface, or whether I am deceiving myself. In Berlin a friend to whom I presented the same problem told me the subject was somewhat absurd, because it is self-evident that two independent variables cannot be reduced to one."

– G. Cantor

From what we can deduce from known records, it would be three years until Dedekind and Cantor again spoke on the subject. From his letters, it is clear that Cantor's sophistication on the topic of one-to-one correspondence between infinite sets at this point has grown, and that his understanding of its implications in 1877 run much deeper than before:

Halle, June 20th 1877

“...I should like to know whether you consider an inference-procedure that I use to be arithmetically rigorous.

The problem is to show that surfaces, bodies, indeed even continuous structures of p dimensions can be correlated one-to-one with continuous lines, i.e. with structures of only one dimension—so that surfaces, bodies, indeed even continuous structures of p dimensions have the same power as curves. This idea seems to conflict with the one that is especially prevalent among the representatives of modern geometry, who speak of simply infinite, doubly, triply, . . . p -fold infinite structures. (Sometimes you even find the idea that the infinity of points of a surface or a body is obtained by as it were squaring or cubing the infinity of points of a line.)”

– G. Cantor

• • •

Infinite sets

“I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking”.

— C. F. Gauss, 1831

The elements of all the sets we've encountered so far have been infinite in number, meaning they go on forever. However, we've also shown that one of them is not of the same “size”, or at least, that it cannot be put into one-to-one correspondence with the natural numbers. Perhaps even more paradoxically, we've seen that an infinite subset (e.g. the even numbers) of an infinite set (the natural numbers) *can* be put into one-to-one correspondence, giving rise to a peculiar property of infinite sets, namely that:

Definition of an infinite set

A set A is infinite if, and only if, there is a one-to-one correspondence between A and a set X which is a proper subset of A .

This property, coined by Dedekind, seems paradoxical given the intuitive notion that there must always be more elements in a whole than in some parts of it (Euclid's so-

called Common Notion 5). It means that if two infinite sets contain the same number of elements when there is:

1. One-to-one correspondence between them; and
2. The size of any whole must be greater than that of any of its parts;

Then the number of elements in an infinite set cannot be thought to be a measure of its size. It suggests that the elements of an infinite set are in a sense “without number”, given that you can never count all of them, but also because the notion of a number as a measure of size in this realm makes little sense — All infinite sets seem to come out as being of the same size if one-to-one correspondence is taken as indicating sameness in terms of the size of a set.

Cardinal numbers

So just how does one go about studying the properties and differences of infinite sets? After his 1874 discovery of the existence on non-denumerable infinite sets, in 1878 Cantor turned to a more general study of what he called *powers*, or cardinal numbers — the study of the sizes of sets. The cardinality of the set A is usually denoted by $|A|$, sometimes $\text{card}(A)$.

Cantor's Definition of Cardinal Numbers

We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises from the set M when we make abstraction from the nature of its various elements m and of the order in which they are given.

Or stated more simply, cardinal numbers are a generalization of the natural numbers used to measure the cardinality (size) of sets. Using the cardinality property, Cantor was able to formally answer the question he repeatedly asked Dedekind, namely whether or not a square could be mapped onto a line with a one-to-one correspondence of the points on each, i.e.:

Theorem: The set \mathbb{R}^2 of all ordered pairs of real numbers (that is, the real plane) has the same size as \mathbb{R} .

The theorem emerged from Cantor's 1878 paper *Ein Beitrag zur Mannigfaltigkeitslehre* ("A contribution to manifold theory") and may elegantly be proven in the following/modern way (attributed by to Julius König):

Proof that $|\mathbb{R}^2| = |\mathbb{R}|$

It suffices to prove that the set of all pairs (x,y) , $0 < x,y < 1$ can be mapped bijectively onto $(0,1]$. Consider the pair (x,y) and write x,y in their unique non-terminating decimal expansion as in the following example:

$$\begin{array}{ccccccc} x = & 0.3 & 01 & 2 & 007 & 08 & \dots \\ y = & 0.009 & 2 & 05 & 1 & 0008 & \dots \end{array}$$

Note that the digits of x and y have been separated into groups by always going to the next nonzero digit, inclusive. Now we associate to (x,y) the number $z \in (0,1]$ by writing down the first x -group, after that the first y -group, then the second x -group, and so on. Thus, in our example, we obtain:

$$z = 0.3\ 009\ 01\ 2\ 2\ 05\ 007\ 1\ 08\ 0008\ \dots$$

Since neither x nor y exhibits only zeroes from a certain point on, we find that the expression for z is again a non-terminating decimal expansion. Conversely, from the expansion of z we can immediately read off the preimage (x,y) and the map is bijective.

So, again paradoxically, the two-dimensional plane \mathbb{R}^2 can indeed be mapped bijectively (with one-to-one correspondence) onto the one-dimensional line \mathbb{R} . Inductively, we can extend the result to higher dimensions. Its counterintuitive nature lead Cantor to famously announce:

Halle, June 29th 1877

"..Please excuse my zeal for the subject if I make so many demands upon your kindness and patience; the communications which I lately sent you are even for me so unexpected, so new, that I can have no peace of mind until I obtain from you, honored friend, a decision about their correctness. So long as you have not agreed with me, I can only say: je le vois, mais je ne le crois pas."

“I see it, but I do not believe it”.

• • •

Infinite cardinal numbers

When Cantor in 1878 then turned to study *infinite* cardinal numbers, he was already aware of the existence of two such “powers” (Mächtigkeiten): point-sets (e.g. the natural numbers) and the continuum (e.g. real numbers). In his 1883 paper entitled *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (“Foundations of a General Theory of Manifolds”) he introduced a distinction between two infinities, the transfinite and the absolute:

Transfinite numbers are numbers that are "infinite" in the sense that they are larger than all finite numbers, yet not necessarily absolutely infinite.

The absolute infinite Ω , also introduced by Cantor, can be thought as a number which is bigger than any conceivable or inconceivable quantity, either finite or transfinite. Transfinite numbers are increasable in magnitude, while the absolute is unincreasable. The particular transfinite numbers he had in mind were those that he became aware of through his studies of the countability of some infinite sets (e.g. the natural numbers) and the uncountability of other sets (e.g. the real numbers). He labeled their cardinalities \aleph_0 (aleph naught) and \aleph_1 (aleph one), respectively, the first two “orders of infinity”, both smaller than absolute infinity Ω .

The Continuum Hypothesis (1878)

There are no infinite cardinal numbers strictly in between the cardinality of the natural numbers \aleph_0 and the cardinality of the real numbers \aleph_1 .

No introduction to Cantor would be complete without discussing the infamous hypothesis that has become forever linked with his life work, Cantor’s Continuum Hypothesis (CH). Much of his work on the conjecture was published in the six-part

treatise *Über unendliche, lineare Punktmannichfaltigkeiten* (“On infinite, linear manifolds of points”) in the journal *Mathematische Annalen* between 1879 and 1884.



Georg Cantor (left) and his six-part treatise *Über unendliche, lineare Punktmannichfaltigkeiten* in the journal *Mathematische Annalen*.

Its first appearance however, came in the 1878 paper *Ein Beitrag zur Mannigfaltigkeitslehre* (“A contribution to manifold theory”), where he states:

The question arises how the different parts of a continuous straight line, i.e. the different infinite manifolds of points that can be conceived in it, are related with respect to their powers.

Let us divest this problem of its geometric guise, and understand by a linear manifold of real numbers, every conceivable totality of infinitely many, distinct real numbers. Then the questions arises,

into *how many* and *which* classes do the linear manifolds fall, if manifolds of the same power are placed into the same class, and manifolds of different power into different classes?

By an inductive procedure, whose more exact presentation will not be given here, the theorem is suggested that the number of classes of linear manifolds that this principle of sorting gives rise to is finite, and indeed, equals two.

We know the cardinal numbers $0, 1, 2, \dots$ and the infinite cardinal number \aleph_0 , and further that the cardinality of the real numbers is bigger than \aleph_0 . Cantor's contention in his statement of the continuum hypothesis is that the cardinality of the real numbers is the next transfinite number after \aleph_0 , i.e. that

$$c = |\mathbb{R}| = \aleph_1$$

Meaning that no set can have a cardinality that is larger than that of the natural numbers \aleph_0 and smaller than c , and that c is the cardinality of the real numbers. \aleph_1 in this sense lies beyond any countable set of cardinal numbers other than itself, and can only be “reached” by adding together other cardinal numbers with the power of \aleph_1 .

Attempted proofs

Cantor spent many of the remaining years of his life wrestling with providing a proof that the continuum hypothesis is true. His direct strategy was to use the derived sets $P^{(n)}$ of a point set P to measure its cardinality. As Bertrand Russell put it:

Popularly speaking, the first derivative consists of all points in whose neighborhood an infinite number of terms of the collection are heaped up; and subsequent derivatives give, as it were, different degrees of concentration in any neighborhood. Thus, it is easy to see why derivatives are relevant to continuity; to be continuous, a collection must be as concentrated as possible in every neighborhood containing any terms of the collection.

Because the process of taking derivatives does not necessarily terminate after a countably infinite number of iterations, Cantor continued the process into the transfinite. When the strategy failed, Cantor turned to his so-called “indirect strategy”

which is the main subject of the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (“Foundations of a General Theory of Aggregates”) published in 1883. The strategy was based on his theory of powers of cardinal numbers, i.e. on the introduction of a class of transfinite numbers that can be used to count the size of any infinite set. The continuum hypothesis would in this system be shown by determining where the power of the continuum lies on the “scale” of transfinite numbers — that it was the first non-denumerable transfinite number.

Cantor would spend many years attempting to solve the continuum hypothesis. When one day he thought he had found a proof of its truth, the next day he had found a proof of its falsehood: then next again he found a proof of its truth only to later realize that all of his proofs had been invalid.

. . .

Mental health

Cantor suffered his first serious mental breakdown in May of 1884, ten years after the publication of his first proof of the uncountability of the real numbers. Most historians believe the breakdown occurred as a result of an ongoing dispute Cantor had with Leopold Kronecker at the University of Berlin, coupled with the apparent intractability of the continuum hypothesis. As we can read from letters Cantor sent to Swedish mathematician Mittag-Leffler, Cantor’s first breakdown occurred just as he had returned from a joyous trip to Paris where he met, among other mathematicians, Henri Poincaré. Cantor writes that he liked Poincaré very much and was happy to learn that the great man understood his transfinite set theory and its applications. In addition, he writes that he spent time visiting galleries and museums, indulging in his love of the opera and theater. Cantor’s breakdown reportedly occurred shortly after he returned to Germany to attend to family affairs.





We do not know what caused Cantor's breakdown. Arthur Schoenflies argues that Cantor's bitterness with the tremendous opposition to his work, championed by his former professor Leopold Kronecker in Berlin, was the main driver of his distress. Kronecker, more than any other professional mathematician at the time, had been the most vocal opponent of Cantor's ideas going all the way back to Cantor's 1874 paper, which Cantor was afraid Kronecker would delay the publication of, as he had done one of Heine's articles. Due to these concerns, on Weierstrass' advice, Cantor left his uncountability theorem out of the initial draft of the article and only later added it during proofreading as a remark at the end of its introduction. In addition, Kronecker's influence is said to have led Cantor to use Dedekind's version of the proof of the countability of the real numbers, but deliberately leave out Dedekind's "principle of continuity", which Kronecker did not accept. Reportedly, every single one of the 52 letters Cantor sent Mittag-Leffler in 1884 mentioned Kronecker by name.

Kronecker disagreed fundamentally with the thrust of Cantor's work on set theory because, among other reasons, it asserted the existence of sets satisfying certain properties without giving examples of specific sets whose members satisfied these properties. Kronecker also only admitted mathematical concepts if they could be constructed in a finite number of steps from the natural numbers, which he took for a given. Kronecker had been Cantor's professor in Berlin, and headed the department of mathematics there until his death in 1891. Every time Cantor applied for a post in Berlin, he was declined, despite having become a renowned name in mathematical circles.

Cantor's ideas, in direct opposition to Kroneckers, eventually famously lead the latter to call Cantor a "corrupter of youth" who "must be stopped".

Final years

After his 1884 hospitalization, there is no record that Cantor was admitted to any sanatorium again until 1899. That year, his youngest son died and Cantor reportedly lost his passion for mathematics. When in 1903 Julius König presented a paper which attempted to disprove the basic tenants of transfinite set theory, Cantor perceived it as a grave public humiliation. Despite Ernst Zermelo demonstrating the paper's invalidity less than a day later, Cantor remained shaken and even momentarily began questioning the existence of God (Cantor was a devout Christian). The events preceded a series of additional hospitalizations at intervals of two to three years.



Cantor in 1917

Despite continuing to inquire for positions at the University of Berlin, Cantor would remain at Halle University until his death. He spent the last 20 years of his life in a state

of chronic depression, defending his controversial ideas about set theory and the validity of his proofs, mainly against criticism from other mathematicians in Germany. Cantor retired in 1913, living in poverty and suffering from malnourishment during World War I. In June 1917, he again entered a sanatorium, where he eventually died of a heart attack on January 6th, 1918. From his final admission until his death, he continually wrote his wife asking to be allowed to come home.

. . .

Paradise lost?

In 1900 German mathematician David Hilbert identified the continuum hypothesis as one of the 23 most significant problems to shape the future of mathematics in the 20th century. His prediction turned out accurate, as attempts by other mathematicians to prove or disprove Cantor's conjecture led to some of the deepest work in set theory thus far.

It would not be until 1940 that the Austro-Hungarian logician Kurt Gödel confirmed the consistency of the continuum hypothesis by showing that it could not be disproved from the other axioms of set theory. Twenty-three years later, American mathematician Paul Cohen established its independence by showing that the continuum hypothesis could not be proved from the other axioms of set theory. They in other words showed that the statement $c = \aleph_1$ is *independent* of the Zermelo–Fraenkel axiom system generally accepted as the most common foundation of mathematics. The consistency and independence of Cantor's conjecture meant that it was possible to build valid models of set theory that satisfied the continuum hypothesis and other models that did not. The realization of the existence of this and other unprovable statements changed the nature of mathematics as a rigorous, logical discipline, prompting Hilbert in 1926 to proclaim in defence of Cantorian set theory:

“From the paradise, that Cantor created for us, no-one can expel us” — David Hilbert

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