Functions of several variables

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1 The Space $\mathbb{R}^m$ and the Most Important Classed of its Subsets

1.1 The Set $\mathbb{R}^m$ and the distant in it

We make the convention that $\mathbb{R}^m$ denotes the set of ordered m-tuples $(x^1, x^1, \cdots, x^m)$ of real numbers $x^i \in \mathbb{R}$.

The function:

$$d(x_1, x_2) = \sqrt{\sum_{i=1}^{m}(x_1^i - x_2^i)^2}$$

obviously has the following properties:

1. $d(x_1, x_2) \geq 0$;
2. $d(x_1, x_2) = 0 \iff x_1 = x_2$
3. $d(x_1, x_2) = d(x_2, x_1)$;
4. $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$;

A function defined on pairs of points (x_1, x_2) of a set X and possessing the properties 1,2,3,4 is called a **metric or distance on** X.

1.2 Open and Closed Sets in $\mathbb{R}^m$

Definition 1.1. For each $\delta > 0$, the set

$$B(a, \delta) = \{x \in \mathbb{R}^m \,| d(a, x) < \delta\}$$

is called the ball with center $a \in \mathbb{R}^m$ of radius δ or the δ-neighborhood of the point $a \in \mathbb{R}^m$.

Definition 1.2. A set $G \subset \mathbb{R}^m$ is open in $\mathbb{R}^m$ if for every point $x \in G$ there is a ball $B(a, \delta)$ such that $B(a, \delta) \subset G$.

Definition 1.3. An open set in $\mathbb{R}^m$ containing a given point is called a neighborhood of that point in $\mathbb{R}^m$.

Examples 1. $\mathbb{R}^m$ is an open set in $\mathbb{R}^m$.

Examples 2. The empty set $\emptyset$ contains no points at all is an open set in $\mathbb{R}^m$.

Examples 3. A ball $B(a; r)$ is an open set in $\mathbb{R}^m$.

Definition 1.4. The set $F \subset \mathbb{R}^m$ is closed in $\mathbb{R}^m$ if its complement $F^C = \mathbb{R}^m \setminus F$ is open in $\mathbb{R}^m$.

Examples 4. The set $\bar{B}(a; r) = \{x \in \mathbb{R}^m | d(a, x) \leq r\}, r \geq 0$ is a close set.

Proposition 1.1.

1. The union $\bigcup\limits_{\alpha \in A} G_\alpha$ of the sets of any system $\{G_\alpha, \alpha \in A\}$ of open sets in $\mathbb{R}^m$ is an open set in $\mathbb{R}^m$.

2. The intersection $\bigcap\limits_{i=1}^{n} G_i$ of a finite number of open sets in $\mathbb{R}^m$ is an open set in $\mathbb{R}^m$.

3. The intersection $\bigcap\limits_{\alpha \in A} F_\alpha$ of the sets of any system $\{F_\alpha, \alpha \in A\}$ of closed sets in $\mathbb{R}^m$ is an closed set in $\mathbb{R}^m$.

4. The union $\bigcup\limits_{i=1}^{n} F_i$ of a finite number of closed sets in $\mathbb{R}^m$ is an closed set in $\mathbb{R}^m$.

Definition 1.5. In relation to a set $E \subset \mathbb{R}^m$ a point is

1. **an interior point** if some neighborhood of it is contained in E;

2. **an exterior point** if it is a interior point of the complement of E in $\mathbb{R}^m$;

3. **a boundary point** if it is neither an interior nor an exterior point of E.

Definition 1.6. A set $K \subset \mathbb{R}^m$ is compact if form every covering of K by sets that are open in $\mathbb{R}^m$ one can extract a finite covering.

Examples 5. *A closed interval $[a, b] \subset \mathbb{R}$ is compact by the finite covering lemma.*

2 Limits and Continuity of Functions of Several Variables

2.1 The Limit of a Function

In the next few sections we shall be consider functions $f : X \to \mathbb{R}^n$ defined on subsets of $\mathbb{R}^m$.

Definition 2.1. *A point $A \in \mathbb{R}^n$ is the **limit of the mapping** $f : X \to \mathbb{R}^n$ over a base $\mathcal{B}$ in X if for every neighborhood $V(A)$ of the point there exists an element $B \in \mathcal{B}$ of the base whose image $f(B)$ is contained in $V(A)$.*

In brief,

$$\lim_{\mathcal{B}} f(x) = A := \forall V(A), \exists B \in \mathcal{B}, f(B) \subset V(A)$$

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Examples 6. $\lim\limits_{(x,y)\to(2,1)} x^2 + xy + y^2 = 7$

Examples 7. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$, *where*

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

Theorem 2.1. $\lim\limits_{\substack{P\to P_0 \\ P\in D}} f(P) = A$ 的充要條件是：對於D 的任意子集E,只要P_0是E的聚點，就有

$$\lim_{\substack{P\to P_0 \\ P\in E}} f(P) = A$$

Corollary 2.2. 設$E_1 \subset D, P_0$是E_1的聚點，若$\lim\limits_{\substack{P\to P_0 \\ P\in E_1}} f(P)$ 不存在，則$\lim\limits_{\substack{P\to P_0 \\ P\in D}} f(P)$ 不存在。

Corollary 2.3. 設$E_1, E_2 \subset D, P_0$是它們的聚點，若$\lim\limits_{\substack{P\to P_0 \\ P\in E_1}} f(P) = A_1$，$\lim\limits_{\substack{P\to P_0 \\ P\in E_1}} f(P) = A_2$，但$A_1 \neq A_2$,則$\lim\limits_{\substack{P\to P_0 \\ P\in D}} f(P)$ 不存在。

Examples 8. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, *where*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

2.2 Continuity of a Function of Several Variables and Properties of Continuous Function

Let E be a subset of $\mathbb{R}^m$ and $f : E \to \mathbb{R}^n$ be a function defined on E with values in $\mathbb{R}^n$.

Definition 2.2. *The function $f : E \to \mathbb{R}^n$ is a continuous at $a \in E$ if for every neighborhood $V(f(a))$ of the value $f(a)$ that the function assumes at a, there exists a neighborhood $U_E(a)$ of a in E whose image $f(U_E(a))$ is contained in $V(f(a))$.*

Thus $f : E \to \mathbb{R}^n$ is continuous at $a \in E \iff \forall V(f(a)), \exists U_E(a), s.t. f(U_E(a)) \subset V(f(a))$.

It follows from the definition above that the mapping $f : E \to \mathbb{R}^n$ defined by the relation

$$(x^1, x^2, \cdots, x^m) = x \mapsto y = (y^1, y^2, \cdots, y^n) = (f^1(x^1, x^2, \cdots, x^n), \cdots, f^n(x^1, x^2, \cdots, x^n))$$

is continuous at a point if and only if each of the function $y^i = f^i(x^1, x^2, \cdots, x^m)$ is continuous at that point.

Local properties of continuous functions

a) A mapping $f : E \mapsto \mathbb{R}^n$ is continuous at a point $a \in E$ if and only is $\omega(f; a) = 0$.

b) A mapping $f : E \mapsto \mathbb{R}^n$ is continuous at a point $a \in E$ is bounded in some neighborhood $U_E(a)$ of that point.

c) If the mapping $g : Y \mapsto \mathbb{R}^k$ of the set $Y \subset \mathbb{R}^n$ is continuous at a point $y_0 \in Y$ and the mapping $f : X \mapsto Y$ of the set $X \subset \mathbb{R}^m$ is continuous at a point $x_0 \in X$ and $f(x_0) = y_0$, then the mapping $g \circ f : X \mapsto \mathbb{R}^k$ is defined, and is continuous at $x_0 \in X$.

Global Properties of Continuous function.

a) If a mapping $f : K \mapsto \mathbb{R}^n$ is continuous on a compact set $K \subset \mathbb{R}^m$, then it is uniformly continuous on K.

b) If a mapping $f : K \mapsto \mathbb{R}^n$ is continuous on a compact set $K \subset \mathbb{R}^m$, then it is bounded on K.

c) If a mapping $f : K \mapsto \mathbb{R}^n$ is continuous on a compact set $K \subset \mathbb{R}^m$, then it assumes its maximal and minimal values at some point of K.

d) If a mapping $f : K \mapsto \mathbb{R}^n$ is continuous on a connected set $E \subset \mathbb{R}^m$ and assumes the values $f(a) = A$ and $f(b) = B$ at points $a, b \in E$, then for any C between A and B, there is a point $c \in E$ at which $f(c) = C$.

3 作業

1. 求下列函數的極限

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1+x^2+y^2}{x^2+y^2}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}-1}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} (x+y)\sin\frac{1}{x^2+y^2}$

   (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

2. 討論下列函數在點$(0,0)$的重極限與累次極限

   (a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$

   (b) $f(x,y) = (x+y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}$

   (c) $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$

   (d) $f(x,y) = y\sin\dfrac{1}{x} + x\sin\dfrac{1}{y}$

3. 討論下列函數的連續性

   (a) $f(x,y) = \tan\left(x^2+y^2\right)$

   (b) $f(x,y) = \lfloor x+y \rfloor$

   (c) $f(x,y) = \begin{cases} \frac{\sin xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

   (d) $f(x,y) = \begin{cases} 0, x \text{ is an irrational number} \\ y, x \text{ is a rational number} \end{cases}$