不定积分、定积分

1 $\int x\arcsin x\,dx = \int \arcsin x\, d\frac{x^2}{2} = \frac{x^2}{2}\arcsin x - \int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}dx$

$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx + \frac{1}{2}\int\sqrt{1-x^2}\,dx$

$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\arcsin x + \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right) + C$.

2. $\int e^{\sin x}\sin 2x\,dx = \cancel{-\frac{1}{2}\int e^{\sin x}\,d\cos 2x = -\frac{1}{2}e^{\sin x}\cos 2x + \frac{1}{2}\int \cos 2x\, e^{\sin x}\cdot\cos x\,dx}$

$= 2\int e^{\sin x}\cos x\cdot\sin x\,dx = 2\int \sin x\, de^{\sin x} = 2\sin x\, e^{\sin x} - 2\int e^{\sin x}\cos x\,dx$

$= 2\sin x\, e^{\sin x} - 2e^{\sin x} + C$

3. $\int e^{\sqrt{x}}dx \xlongequal{\sqrt{x}=t} \int e^t\,dt^2 = \cancel{e^t t^2} = \int 2te^t\,dt = 2\int t\,de^t = 2te^t - 2\int e^t\,dt$

$= 2te^t - 2e^t + C$

4. $\int\frac{1-\tan x}{1+\tan x}dx = \int\frac{\cos x-\sin x}{\cos x+\sin x}dx = \int\frac{d(\cos x+\sin x)}{\cos x+\sin x} = \ln|\cos x+\sin x| + C$

5. $\int\frac{1}{\cos^4 x}dx = \int\sec^4 x\,dx = \int(1+\tan^2 x)\,d\tan x = \tan x + \frac{1}{3}\tan^3 x + C$

6. $\int\frac{\cos x}{\sin x+\cos x}dx \triangleq A$. $\int\frac{\sin x}{\sin x+\cos x} = B$ $\quad\begin{cases} A+B = x \\ A-B = \ln|\sin x+\cos x| \end{cases}$

$\Longrightarrow A = \frac{1}{2}(x + \ln|\sin x+\cos x|) + C$ 7题: $B = \frac{1}{2}(x - \ln|\sin x+\cos x|) + C$

8. $\int\frac{1}{1+x^4}dx = \frac{1}{2}\int\frac{1+x^2-(x^2-1)}{1+x^4}dx = \frac{1}{2}\int\frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$= \frac{1}{2}\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} - \frac{1}{2}\int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-(\sqrt{2})^2} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}$

$-\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$

9. $\int \frac{1+x^2}{1+x^4}dx = \int \frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} + C$

10. $\int \frac{1}{x(1+x)(1+x^2)}dx = \int \frac{A}{x} + \frac{B}{1+x} + \frac{ax+b}{1+x^2}\,dx = \int \frac{1}{x} + \frac{-\frac{1}{2}}{1+x} + \frac{-\frac{1}{2}(x+1)}{1+x^2}\,dx$

$= \ln|x| - \frac{1}{2}\ln|1+x| - \frac{1}{2}\int \frac{x+1}{1+x^2}dx$

$= \ln|x| - \frac{1}{2}\ln|1+x| - \frac{1}{2}\int \frac{\frac{1}{2}d(1+x^2)+1}{1+x^2} = \ln|x| - \frac{1}{2}\ln|1+x| - \frac{1}{2}\arctan x$

$- \frac{1}{2}\ln|1+x^2| + C.$

11. $\int \sin^n x\,dx = -\int \sin^{n-1}x\,d\cos x = -\sin^{n-1}x\cos x + \int \cos^2 x\cdot(n-1)\sin^{n-2}x\,dx$

$= -\sin^{n-1}x\cos x + (n-1)\cdot\int \sin^{n-2}x - \sin^n x\,dx$

$\Rightarrow \int \sin^n x\,dx = \frac{n-1}{n}\int \sin^{n-2}x\,dx - \frac{1}{n}\sin^{n-1}x\cos x$.

12. $\int \frac{\arcsin x}{x^2}dx = -\int \arcsin x\,d\frac{1}{x} = -\arcsin x\cdot\frac{1}{x} + \int \frac{1}{x}\frac{1}{\sqrt{1-x^2}}dx$

$= -\arcsin x\cdot\frac{1}{x} + \int \frac{1}{x^2}\frac{dx}{\sqrt{\frac{1}{x^2}-1}} = -\arcsin x\cdot\frac{1}{x} - \ln\left|1+\sqrt{\frac{1}{x^2}-1}\right|$

$+ C$

13. $\int \tan^n x\,dx = \int \tan^{n-2}x\,d(\sec^2 x-1) = \int \tan^{n-2}x\,d\tan x - \int \tan^{n-2}x\,dx$

$= \frac{1}{n-1}\tan^{n-1}x - I_{n-2}.$

二. 1. $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right) = \int_0^1 \frac{1}{1+x}dx = \ln 2$

2. $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n^2}} + \frac{1}{1+(\frac{2}{n})^2} + \cdots + \frac{1}{1+(\frac{n}{n})^2}\right) = \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}.$

三. 1 $\int_0^1 \frac{1-x^2}{1+x^2}dx = \int_0^1 \frac{-1-x^2+2}{1+x^2}dx = -x\Big|_0^1 + 2\arctan x\Big|_0^1 = -1+\frac{\pi}{2}$

2. $\int_e^{e^2} \frac{1}{x\ln x}dx = \int_e^{e^2} \frac{d\ln x}{\ln x} = \ln(\ln x)\Big|_e^{e^2} = \ln 2 - 0$

3. $\int_4^9 \sqrt{x} + \frac{1}{\sqrt{x}}dx = \left(\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right)\Big|_4^9 = \frac{2}{3}27 + 6 - \frac{2}{3}\cdot 8 - 4$

4. $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x\, dx = -\int_0^{\frac{\pi}{2}} 2\cos^5 x \cos x\, d\cos x = -2\cdot\frac{1}{7}\cos^7 x\Big|_0^{\frac{\pi}{2}} = \frac{2}{7}$.

5 $\int_0^1 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi\cdot 4 = \pi$

6. $\int_0^1 \frac{1}{e^x+e^{-x}}dx = \int_0^1 \frac{e^x}{1+e^{2x}}dx = \arctan e^x\Big|_0^1 = \arctan e - \frac{\pi}{4}$.

7. $\int_0^{\frac{\pi}{2}} \frac{d\sin x}{1+\sin^2 x} = \arctan \sin x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$.

8. $\int_{\frac{1}{e}}^{e} |\ln x|dx = \int_{\frac{1}{e}}^{e} -\ln x\, dx = -x\ln x\Big|_{\frac{1}{e}}^{e} + \int_{\frac{1}{e}}^{e} 1\,dx = -e - \frac{1}{e} + e - \frac{1}{e} = -\frac{2}{e}$.

四. 1 $\lim_{x\to 0} \frac{1}{x}\int_0^x \cos t^2 dt = \lim_{x\to 0} \frac{\cos x^2}{1} = 1$

2. $\lim_{x\to\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt} = \lim_{x\to\infty} \frac{2\int_0^x e^{t^2}dt\cdot e^{x^2}}{e^{2x^2}} = \lim_{x\to\infty} \frac{2\int_0^x e^{t^2}dt}{e^{x^2}} = 0$

五.

1 $\int_0^{\frac{\pi}{2}} f(\sin x)dx \overset{x=\frac{\pi}{2}-t}{=\!=\!=} \int_{\frac{\pi}{2}}^{0} f(\cos t)d(-t) = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

2. $\int_0^{\pi} x f(\sin x)dx \overset{x=\pi-t}{=\!=\!=} \int_{\pi}^{0} (\pi-t)f(\sin t)d(-t)$

$\Rightarrow \int_0^{\pi} x f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.