《数学分析I》第一章测試解答

一. 求下列函數的極限

1. $\lim\limits_{n\to\infty}\sqrt[n]{n^3+3^n}=3$ (夹逼原理)

2. $\lim\limits_{n\to\infty}\frac{n^5}{e^n}=0$ $\left(\frac{n}{e^{\frac{1}{5}}}\right)^{5n}\longrightarrow 0$

3. $\lim\limits_{n\to\infty}(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})=\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n+2}+\sqrt{n+1}}-\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)=0$

4. $\lim\limits_{n\to\infty}\frac{\ln n}{n^\alpha}\ (\alpha\geq 1)=0$ (利用stoltz公式)

二. 证明题. 设 $\lim\limits_{n\to\infty}a_n=a$ 證明:

1. $\lim\limits_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=a$

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right|=\left|\frac{a_1-a+a_2-a+\cdots+a_n-a}{n}\right|\leq\left|\frac{a_1-a+\cdots+a_N-a}{n}\right|$$

$$+\frac{|a_{N+1}-a|+\cdots+|a_n-a|}{n}\leq\frac{M}{n}+\frac{n-N}{n}\frac{\varepsilon}{2}\quad\left(|a_n-a|<\frac{\varepsilon}{2}.\ n>N_1\right).$$

$\frac{M}{n}<\frac{\varepsilon}{2}$ $(n>N_2)$ $N=\max\{N_1,N_2\}$. $n>N$时有

$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

2. $\lim\limits_{n\to\infty}(a_1\cdot a_2\cdots a_n)^{\frac{1}{n}}=a$

$$e^{\frac{1}{n}(\ln a_1+\ln a_2+\cdots+\ln a_n)}\longrightarrow e^{\ln a}=a.$$

三. 证明题

设 $a>0$, $\sigma>0$, $a_1=\frac{1}{2}\left(a+\frac{\sigma}{a}\right)$, $a_{n+1}=\frac{1}{2}\left(a_n+\frac{\sigma}{a_n}\right)$ 证明数列 $\{a_n\}$ 收斂,並極限為 $\sqrt{\sigma}$.

证明: $a_{n+1}=\frac{1}{2}\left(a_n+\frac{\sigma}{a_n}\right)\geq\sqrt{\sigma}$

$a_{n+1} = \frac{a_n}{2}(1+\frac{\sigma}{a_n^2}) \le \frac{a_n}{2}\cdot 2 = a_n$ $\{a_n\}\downarrow$ 有界. ∴

$\{a_n\}$收敛。 $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty}\frac{1}{2}(a_n+\frac{\sigma}{a_n}) \Rightarrow A = \frac{1}{2}(A+\frac{\sigma}{A})$.

$\Rightarrow A=\sqrt{\sigma}$. 11

四.解答题

叙述数列$\{a_n\}$发散的柯西判定法则. 并利用该法则证明以下数列发散。

1. $a_n = \sin\frac{n\pi}{2}$. 2. $a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

a_n发散 $\Leftrightarrow \exists \varepsilon_0>0$, ~~当N~~ $\forall N>0$, $\exists n_1, n_2>N$ 有 $|a_{n_1}-a_{n_2}|\ge\varepsilon_0$。

(1) 取 $n_1=2m+1$, $n_2=2m$ $\left|\sin\frac{(2m+1)\pi}{2}-\sin\frac{2m\pi}{2}\right|=1>\frac{1}{2}$

(2) $|a_{2n}-a_n| = \left|\frac{1}{n+1}+\cdots+\frac{1}{2n}\right| > \frac{n}{2n} = \frac{1}{2}$.

五. 叙述$\{a_n\}$无界的定义。若$\{a_n\}$,$\{b_n\}$是無界数列。$\{a_nb_n\}$不定的

无界数列, 例如 ~~a_n~~ $1, 0, 2, 0, 3, 0, \cdots n, 0, \cdots$

$0, 1, 0, 2, 0, 3, \cdots 0, n, \cdots$

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《数学分析 I》第二章测试卷

一、

1. $\lim\limits_{x\to1}\frac{x^m-1}{x^n-1}$ $(m,n\in N^+)$

$\frac{x^m-1}{x^n-1}=\frac{(x-1)(x^{m-1}+x^{m-2}+x^{m-3}+\cdots+1)}{(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\cdots+1)}\longrightarrow\frac{m}{n}$

2. $\lim\limits_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}=\lim\limits_{x\to+\infty}\frac{\sqrt{1+\sqrt{\frac{x}{x^2}+\sqrt{\frac{x}{x^4}}}}}{\sqrt{1+\frac{1}{x}}}=1$

3. $\lim\limits_{x\to4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim\limits_{x\to4}\frac{\sqrt{x}+2}{x-4}\cdot\frac{2x-8}{\sqrt{1+2x}+3}=\frac{2\cdot4}{6}=\frac{4}{3}.$

4. $\lim\limits_{x\to0}\frac{(1+x)^{\frac{1}{n}}-1}{x}=\lim\limits_{x\to0}\frac{\frac{1}{n}x}{x}=\frac{1}{n}$

5. $\lim\limits_{x\to0}\frac{\sin5x-\sin3x}{\sin x}=5-3=2.$

6. $\lim\limits_{x\to1}(x-1)\tan\frac{\pi x}{2}\overset{x-1=t}{=\!=\!=}\lim\limits_{t\to0}t\tan\frac{\pi}{2}(t+1)=\lim\limits_{t\to0}t\cdot\frac{\sin(\frac{\pi}{2}+\frac{\pi}{2}t)}{\cos(\frac{\pi}{2}+\frac{\pi}{2}t)}$

$=\lim\limits_{t\to0}t\cdot\frac{\cos\frac{\pi}{2}t}{-\sin\frac{\pi}{2}t}=1\cdot(-\frac{2}{\pi})=-\frac{2}{\pi}.$

7. $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\frac{\sin(\frac{x+a}{2}+\frac{x-a}{2})-\sin(\frac{x+a}{2}-\frac{x-a}{2})}{x-a}=\lim\limits_{x\to a}\frac{\cancel{2}\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{\frac{x-a}{2}}$

$=\cos a.$

8. $\lim\limits_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}=\lim\limits_{x\to0}\frac{1}{x^3}\frac{\tan x-\sin x}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}$

$=\lim\limits_{x\to0}\frac{1}{x^3}\cdot\frac{\tan x(1-\cos x)}{2}=1/4.$

9. $\lim\limits_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}=\lim\limits_{x\to a}\left(1+\frac{\sin x}{\sin a}-1\right)^{\frac{1}{x-a}}$

$\because\ \frac{\sin x-\sin a}{\sin a}\cdot\frac{1}{x-a}\longrightarrow\tan a$

$\therefore\ \longrightarrow e^{\tan a}.$

10. $\lim\limits_{n\to\infty}\tan^n(\frac{\pi}{4}+\frac{1}{n})=\lim\limits_{n\to\infty}\left(\frac{1+\tan\frac{1}{n}}{1-\tan\frac{1}{n}}\right)^n=\cancel{\lim\limits_{n\to\infty}\left(1+\frac{-1}{1-\tan\frac{1}{n}}\right)}$

$$\lim_{n\to\infty}\left(1+\frac{2\tan\frac{1}{n}}{1-\tan\frac{1}{n}}\right)^n \longrightarrow e^2.$$

二. 设 $x\to 0^+$, 证明下列等式

1. $\frac{2x-x^2}{x}\longrightarrow 2.$ $\therefore\ 2x-x^2=O(x)$

2. $\frac{x\sin\sqrt{x}}{x^{\frac{3}{2}}}\longrightarrow 1$ $\therefore\ x\sin\sqrt{x}=O(x^{\frac{3}{2}})$

3. $\left|\frac{x\sin\frac{1}{x}}{|x|}\right|\le 1.$ $\therefore\ x\sin\frac{1}{x}=O(|x|)$

4. $\frac{\ln x}{\frac{1}{x^\varepsilon}}\longrightarrow 0$ $\therefore\ \ln x=o(\frac{1}{x^\varepsilon})$.

三. 用 $\varepsilon-\delta$ 证明下列函数的连续性.

1 $|\sqrt{x}-\sqrt{x_0}|=\frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}}\le\frac{|x-x_0|}{\sqrt{x_0}}$. 令 $\delta=\varepsilon\cdot\frac{1}{\sqrt{x_0}}$.

2. $|\sin x-\sin x_0|=\left|2\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}\right|\le 2\left|\frac{x-x_0}{2}\right|$. 令 $\delta=\varepsilon$.

四. 1. $x-\lfloor x\rfloor$ 的间断点为 $n\in\mathbb{Z}$. 为跳跃型.

2. $\lfloor x\rfloor\sin\pi x$ 的间断点为 $n\in\mathbb{Z}$ 且 $n\neq 0$. 为跳跃型.

3. $$\lim_{n\to\infty}\frac{1}{1+x^n}=\begin{cases}1, & |x|<1\\ \frac{1}{2} & x=1\\ 0, & |x|>1\end{cases}$$ 间断点为 $x=1$. 跳跃型间断点.

五. 证明黎曼函数 $f(x)=\begin{cases}\frac{1}{n}, & x=\frac{m}{n},\ (m,n)=1\\ 0. & x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$

下面证明 $f(x)$ 在 $[0,1]$ 上的情况, 因为 $f(x)$ 为周期函数.

$\forall\varepsilon>0$, $\left|\frac{1}{n}\right|\geq\varepsilon \Rightarrow n<\frac{1}{\varepsilon}$, 故 $n<\frac{1}{\varepsilon}$ 的点的个数为有限个,

记为 $n_1, n_2, \cdots n_K$. 故分母为有限个. 记为 $m_1, m_2, \cdots m_N$

若 $x_0 \in [0,1]$ 为无理数, 则取 $\delta = \min\{|x_0 - m_1|, \cdots |x_0 - m_N|\}$, 当 $|x - x_0| < \delta$ 时.

有 $|f(x) - 0| < \varepsilon$. $\therefore \lim\limits_{x \to x_0} f(x) = f(x_0)$.

若 $x_0 = \frac{n_1}{m_1}$. 可以找到无理数列 $\{x_n'\} \to x_0$, 故

$\cancel{|f(x_n')|}$ $f(x_n') = 0$. $\lim\limits_{n \to \infty} f(x_n') \neq f(\frac{n_1}{m_1})$ 不连续。 //.

数学分析 测试三.

一. 求下列函数导数.

1. $\frac{d}{dx}(e^{2x}\sin 3x) = 2e^{2x}\sin 3x + 3e^{2x}\cos 3x$

2. $\frac{d}{dx}\left(\sqrt{1+x} - \ln(x+\sqrt{x+1})\right) = \frac{1}{2}\frac{1}{\sqrt{1+x}} - \frac{1+\frac{1}{2}\frac{1}{\sqrt{x+1}}}{x+\sqrt{x+1}}$.

3. $\frac{d}{dx}(\arcsin e^{-x^2}) = \frac{1}{\sqrt{1-(e^{-x^2})^2}} \cdot e^{-x^2} \cdot (-2x)$

4. $\frac{d}{dx}\left(x\sqrt{a^2-x^2} + \frac{x}{a^2-x^2}\right) = \sqrt{a^2-x^2} + \frac{\frac{1}{2}x(-2x)}{\sqrt{a^2-x^2}} + \frac{a^2-x^2+2x^2}{(a^2-x^2)}$

5. $\frac{d}{dx}\ln\sin x = \frac{\cos x}{\sin x} = \operatorname{ctg} x$

6. $\frac{d}{dx}\ln(x+\sqrt{x^2+a^2}) = \frac{1}{x+\sqrt{x^2+a^2}} \cdot \left(1+\frac{x}{\sqrt{x^2+a^2}}\right) = \frac{1}{\sqrt{x^2+a^2}}$

7. $\frac{d}{dx}\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)$

$= \frac{1}{2}\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}} + a^2 \cdot \frac{\frac{1}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right)$

8. $\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x\ln x}) = x^x \cdot (\ln x+1)$

9. $\frac{d}{dx}\left(x^{x^a} + x^{a^x} + a^{x^x}\right) = \frac{d}{dx}\left(e^{x^a\ln x} + e^{a^x\ln x} + e^{x^x\ln a}\right)$

$= x^{x^a} \cdot (ax^{a-1}\ln x + x^{a-1}) + x^{a^x}(a^x\ln a\ln x + a^x \cdot \frac{1}{x})$

$+ a^{x^x}(x^x(\ln x+1)\ln a)$

10. $\frac{d}{dx}|x^3(x-1)(x+7)| = \frac{x^3(x-1)(x+7)}{|x^3(x-1)(x+7)|} \cdot (3x^2(x-1)(x+7)$

$+ x^3(x+7) + x^3(x-1))$.

二. 1. $\frac{d}{dx}\left(f\left(\frac{1}{\ln x}\right)\right) = f'\left(\frac{1}{\ln x}\right) * \frac{-\frac{1}{x}}{(\ln x)^2}$

2. $\frac{d}{dx}(\arctan f(x)) = \frac{f'(x)}{1+f^2(x)}$ 3. $\frac{d}{dx}(f(f(e^{x^2}))) = f'(f(e^{x^2}))\cdot f'(e^{x^2})\cdot e^{x^2}\cdot 2x$

4. $\frac{d}{dx}(f[\frac{1}{f(x)}]) = f'[\frac{1}{f(x)}] \cdot \frac{-f'(x)}{f^2(x)}$. 5. $\frac{d}{dx}[\frac{1}{f(f(x))}] = \frac{-f'(f(x))\cdot f'(x)}{f^2(f(x))}$.

三. 求下列隐函数的导数.

1. $y = x + \arctan y$ $\quad \frac{dy}{dx} = 1 + \frac{\frac{dy}{dx}}{1+y^2}$

$$\frac{dy}{dx} = \frac{1}{(1-\frac{1}{1+y^2})} = \frac{1+y^2}{y^2}.$$

2. $y + xe^y = 1$. $\quad y' + e^y + xe^y\cdot y' = 0 \Rightarrow y' = \frac{-e^y}{1+xe^y}$

3. $x^3+y^3-3axy=0$ $\quad 3x^2+3y^2\cdot y' - 3ay + 3axy' \Rightarrow y' = \frac{3ay-3x^2}{3y^2+3ax}$

4. $2y\sin x + x\ln y = 0$ $\quad 2y'\sin x + 2y\cos x + \ln y + \frac{y'}{y}x = 0$

$$y' = \frac{2y\cos x + \ln y}{-(2\sin x + \frac{x}{y})}$$

四. 1. $\frac{dy}{dx} = \frac{3bt^2}{2at} = \frac{3}{2}\frac{b}{a}t$ 2. $\frac{dy}{dx} = \frac{b\sin bt}{a\cos at} = \frac{b}{a}\tan ht$.

五. 1. $(\sin^2\omega x)^{(n)} = (\frac{1-\cos 2\omega x}{2})^{(n)} = \begin{cases} \frac{1-\cos 2\omega x}{2}, & n=0 \\ (-\frac{1}{2})(2\omega)^n\cos(2\omega x+\frac{n\pi}{2}). & n\neq 0 \end{cases}$

2. $y = xe^x$. $\quad (xe^x)^{(n)} = xe^x + ne^x$

3. $(\frac{1}{x^2-5x+6})^{(n)} = [(\frac{1}{x-2} - \frac{1}{x-3})]^{(n)} = (-1)\cdot(\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-3)^{n+1}})$.

4. $\left(e^{\alpha x}\cos\beta x\right)^{(n)} = \left((\alpha^2+\beta^2)^{\frac{1}{2}}\cdot e^{\alpha x}\cos(\beta x+\theta)\right)^{(n-1)}$

$$= (\alpha^2+\beta^2)^{\frac{n}{2}}\cdot e^{\alpha x}(\cos\beta x + n\theta).$$

$$\theta = \arctan\frac{\alpha}{\alpha^2+\beta^2}$$

$$\alpha e^{\alpha x}\cos\beta x - e^{\alpha x}\cdot\beta\cdot\sin\beta x$$

$$\cancel{= e^{\alpha x}\cdot\frac{1}{(\alpha^2+\beta^2)^{\frac{1}{2}}}}$$

$$= e^{\alpha x}(\alpha^2+\beta^2)^{\frac{1}{2}}\left[\cos\theta\cos\beta x - \sin\theta\sin\beta x\right]$$

$$= (\alpha^2+\beta^2)^{\frac{1}{2}}e^{\alpha x}\cos(\theta+\beta x)$$

六 1. $\begin{cases} x = at^2 \\ y = bt^3 \end{cases}$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{3bt^2}{2at^2}\right) = \dfrac{d}{dt}\left(\dfrac{3b}{2a}\right) \Big/ \dfrac{dx}{dt}$

$$= \frac{3b}{2a}\cdot\frac{1}{2at} = \frac{3b}{4a^2t}$$

七. $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$

$$\frac{d^3x}{dy^3} = \frac{d^2}{dy^2}\left(\frac{dx}{dy}\right) = \frac{d^2}{dy^2}\cdot\left(\frac{1}{y'}\right) = \frac{d}{dy}\left(\frac{d}{dx}\left(\frac{1}{y'}\right)\cdot\frac{dx}{dy}\right)$$

$$= \frac{d}{dy}\left(\frac{-y''}{(y')^2}\cdot\frac{1}{y'}\right) = \frac{d}{dy}\left(\frac{-y''}{(y')^3}\right) = \frac{d}{dx}\left(-\frac{y''}{(y')^3}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{y'''(y')^3 - y''\cdot 3(y')^2\cdot y''}{(y')^6}\cdot\frac{1}{y'} = \frac{3(y'')^2 - y'y'''}{(y')^5}.$$