Elements of Vector Analysis and Field Theory

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1 Maxwell's Equations

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism, quantum field theory, classical optics, and electric circuits. They underpin all electric, optical and radio technologies, including power generation, electric motors, wireless communication, cameras, televisions, computers etc. Maxwell's equations describe how electric and magnetic fields are generated by charges, currents, and changes of each other. One important consequence of the equations is that they demonstrate how fluctuating electric and magnetic fields propagate at the speed of light. Known as electromagnetic radiation, these waves may occur at various wavelengths to produce a spectrum from radio waves to γ-rays. The equations are named after the physicist and mathematician James Clerk Maxwell, who between 1861 and 1862 published an early form of the equations, and first proposed that light is an electromagnetic phenomenon.

The equations have two major variants. The microscopic Maxwell equations have universal applicability, but are unwieldy for common calculations. They relate the electric and magnetic fields to total charge and total current, including the complicated charges and currents in materials at the atomic scale. The

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"macroscopic" Maxwell equations define two new auxiliary fields that describe the large-scale behaviour of matter without having to consider atomic scale details. However, their use requires experimentally determining parameters for a phenomenological description of the electromagnetic response of materials.

The term "Maxwell's equations" is often used for equivalent alternative formulations. Versions of Maxwell's equations based on the electric and magnetic potentials are preferred for explicitly solving the equations as a boundary value problem, analytical mechanics, or for use in quantum mechanics. The spacetime formulations (i.e., on spacetime rather than space and time separately), are commonly used in high energy and gravitational physics because they make the compatibility of the equations with special and general relativity manifest.[note 1] In fact, Einstein developed special and general relativity to accommodate the absolute speed of light that drops out of the Maxwell equations with the principle that only relative movement has physical consequences.

2 Vector Operations in Curvilinear Coordinates

a. Just as, for example, the sphere $x^2 + y^2 + z^2 = a^2$ has a particularly simple equation $R = a$ in spherical coordinates, vector fields $x \to \boldsymbol{A}(x)$ in $\mathbb{R}^3$ (or $\mathbb{R}^n$) often assume a simpler expression in a coordinate system that is not Cartesian. For that reason we now wish to find explicit formulas from which one can find grad, curl and div in a rather extensive class of curvilinear coordinates. But first it is necessary to be precise as to what is meant by the coordinate expression for a field $\boldsymbol{A}$ in a curvilinear system.

We begin with two introductory examples of a descriptive character.

Example 1 *Suppose we have a fixed Cartesian coordinate system x^1, x^2 in the Euclidean plane $\mathbb{R}^2$. When we say that a vector field $(A^1, A^2)(x)$ is defined in $\mathbb{R}^2$, we mean that some vector $A(x) \in \boldsymbol{T}\mathbb{R}^2$ is connected with each point $x = (x^1, x^2) \in R^2$, and in the basis of $\boldsymbol{T}\mathbb{R}^2$ consisting of the unit vectors $\boldsymbol{e}_1(x), \boldsymbol{e}_2(x)$ in the coordinate directions we have the expansion $\boldsymbol{A}(x) = A^1(x)\boldsymbol{e}_1(x) + A^2(x)\boldsymbol{e}_2(x)$ (see Figure 1).In this case the basis $\{\boldsymbol{e}_1(x), \boldsymbol{e}_2(x)\}$ of $\boldsymbol{T}\mathbb{R}^2$ is essentially independent of x.*

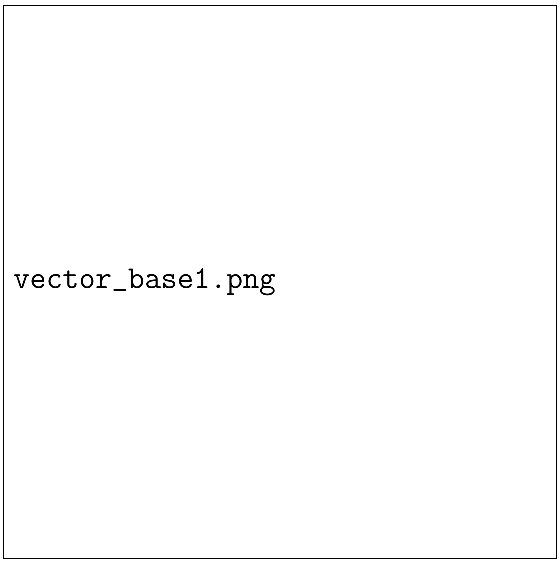

Figure 1: Cartesian Coordinate.

Example 2 *In the case when polar coordinate* (r, φ) *are defined in the same plane* $\mathbb{R}^2$*, at each point* $x \in \mathbb{R}^2 \backslash 0$ *one can also attach unit vectors* $\boldsymbol{e}_1(x) = \boldsymbol{e}_r(x), \boldsymbol{e}_2(x) = \boldsymbol{e}_\varphi(x)$ *(see Figure 2) in the coordinate directions. They also form a basis in* $\boldsymbol{T}\mathbb{R}^2$ *with respect to which one can expand the vector* $\boldsymbol{A}(x)$ *of the field* $\boldsymbol{A}$ *attached to* $x : \boldsymbol{A}(x) = A^1(x)\boldsymbol{e}_1(x) + A^2(x)\boldsymbol{e}_2(x)$*. It is then natural to regard the ordered pair of functions* $(A^1, A^2)(x)$ *as the expression for the field* $\boldsymbol{A}$ *in polar coordinate.*

Thus, if $(A^1, A^2)(x) \equiv (1, 0)$*, this is a field of unit vectors in* $\mathbb{R}^2$ *pointing radially away from the center 0. The field* $(A^1, A^2)(x) \equiv (0, 1)$ *can be obtained from the preceding field by rotating each vector in it counterclockwise by the angle* $\pi/2$*.*

These are not constant fields in $\mathbb{R}^2$, although the components of their coordinate representation are constant. The point is that the basis in which the expression is taken varies synchronously with the vector of the field in a transition from one point to another. It is clear that the components of the coordinate representation of these field in Cartesian coordinates would not be constant at all.

b. After these introductory consideration, let us consider more formally the problem of defining vector fields in curvilinear coordinate systems.

We recall first of all that a curvilinear coordinate system t^1, t^2, t^3 in a domain

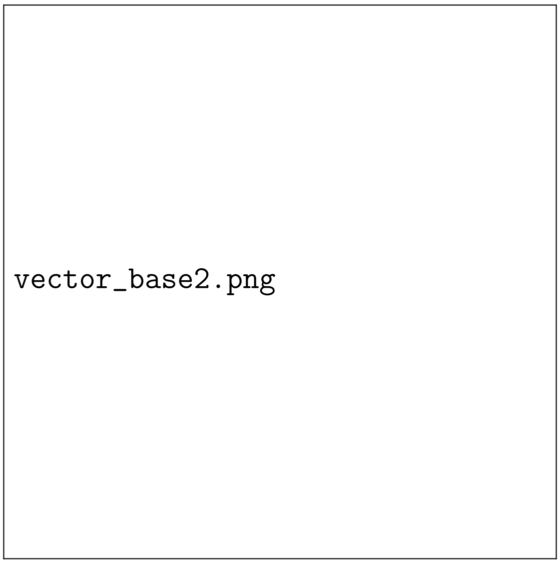

Figure 2: Polar Coordinate.

$D \subset \mathbb{R}^2$ is a diffeomorphism $\varphi : D_t \to D$ of a domain D_t in the Euclidean parameter space $\mathbb{R}^3$ onto the domain D, as a result of which each point $x = \varphi(t) \in D_t$ requires the Cartesian coordinates t^1, t^2, t^3 of the corresponding point $t \in D_t$.

Since φ is a diffeomorphism, the tangent mapping $\varphi'(t) : \boldsymbol{T}\mathbb{R}^3_t \to \boldsymbol{T}\mathbb{R}^3_x$ is a vector-space isomorphism. To the canonical basis $\xi_1(t) = (1,0,0)\,, \xi_2(t) = (0,1,0)\,, \xi_3\,(0,0,1)$ of $\boldsymbol{T}\mathbb{R}^3_t$ corresponds the basis of $\boldsymbol{T}\mathbb{R}^3_{x=\varphi(t)}$ consisting of the vectors $\xi_i(x) = \varphi'(t)\xi_i(t) = \frac{\partial\varphi(t)}{\partial t^i}, i = 1,2,3$, giving the coordinate directions. To the expansion $\boldsymbol{A}(x) = \alpha_1\xi_1(x) + \alpha_2\xi_2(x) + \alpha_3\xi_3(x)$ of any vector $\boldsymbol{A}(x) \in \boldsymbol{T}\mathbb{R}^3_x$ in this basis there corresponds the same expression $\boldsymbol{A}(t) = \alpha_1\xi_1(t) + \alpha_2\xi_2(t) + \alpha_3\xi_3(t)$ (with the same components $\alpha_1, \alpha_2, \alpha_3$) of the vector $\boldsymbol{A}(t) = \varphi^{-1}\boldsymbol{A}(x)$ in the canonical basis $\xi_1(t), \xi_2(t), \xi_3(t)$ in $\boldsymbol{T}\mathbb{R}^3_t$. In the absence of a Euclidean structure in $\mathbb{R}^3$, the number $\alpha_1, \alpha_2, \alpha_3$ would be the most natural coordinate expression for the vector $\boldsymbol{A}(x)$ connected with this curvilinear coordinate system.

c. However, adopting such a coordinate representation would not be quite consistent with what we agreed to in Example above. The point is that the basis $\xi_1(x), \xi_2(x), \xi_3(x)$ in $\boldsymbol{T}\mathbb{R}^3_x$ corresponding to the canonical basis $\xi_1(t), \xi_2(t), \xi_3(t)$ in $\boldsymbol{T}\mathbb{R}^3_t$, although it is consists of vectors in the coordinate directions, is not at all required to consist of *unit vector* in those directions, that is, in general $\langle\xi_i, \xi_i\rangle(t) \neq 1$.

We shall now take account of this circumstance which results from the presence of a Euclidean structure in $\mathbb{R}^3$ and consequently in each vector space $\boldsymbol{T}\mathbb{R}^3_x$ also.

Because of the isomorphism $\varphi'(t) : \boldsymbol{T}\mathbb{R}^3_t \to \boldsymbol{T}\mathbb{R}^3_{x=\varphi(t)}$ we can transfer the Euclidean structure of $\boldsymbol{T}\mathbb{R}^3_x$ into $\boldsymbol{T}\mathbb{R}^3_t$ by setting $\langle\tau_1,\tau_2\rangle := \langle\varphi'\tau_1,\varphi'\tau_2\rangle$ for every pair vector $\tau_1\,\tau_2 \in \boldsymbol{T}\mathbb{R}^3_t$. In particular, we obtain from this the following expression for the square of the length of a vector:

$$\begin{aligned}\langle\tau,\tau\rangle &= \langle\varphi'(t)\tau,\varphi'(t)\tau\rangle\langle\frac{\partial\varphi}{\partial t^i}\tau^i,\frac{\partial\varphi}{\partial t^j}\tau\rangle\\ &= \langle\frac{\partial\varphi}{\partial t^i},\frac{\partial\varphi}{\partial t^j}\rangle(t)\tau^i\tau^j = \langle\xi_i,\xi_j\rangle(t)\tau^i\tau^j\\ &= g_{ij}(t)\tau^i\tau^j = ds^2.\end{aligned} \tag{1}$$

The quadratic form

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}t^i\mathrm{d}t^j \tag{2}$$

whose coefficients are the pairwise inner products of the vectors in the canonical basis determines the inner product on $\boldsymbol{T}\mathbb{R}^3_t$ completely. If such a form is defined at each point of a domain $D_t \subset \mathbb{R}^3_t$, then, as is known from geometry, one say that a *Riemannian metric* is defined in this domain. A Riemannian metric makes it possible to introduce a Euclidean structure in each tangent space $\boldsymbol{T}\mathbb{R}^3_t(t \in D_t)$ within the context of rectilinear coordinate t^1, t^2, t^3 in $\mathbb{R}^3_t$, corresponding to the curved embedding $\varphi : D_t \to D$ of the domain D_t in the Euclidean space $\mathbb{R}^3$.