# Improper Integral

WU-Guoning

March, 2020

# Content

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- Absolute Convergence of an Improper Integral
- Conditional Convergence of an Improper Integral

# Improper Integral and computation

## Definition 0.1

Suppose that $x \to f(x)$ is defined on the interval $[a, +\infty)$ and integrable on every closed interval $[a, b]$ contained in that interval. If the limit below exists,

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper Riemann integral or the improper integral of the function $f$ over the interval $[a, +\infty)$.

The expression $\int_a^{\infty} f(x)\,\mathrm{d}x$ itself is also called an improper integral, and in that case:

1. The integral converges if the limit exists;
2. The integral diverges if the limit does not exist.

# Improper Integral and computation

**Definition 0.2**

Suppose that $x \to f(x)$ is defined on the interval $[a, B)$ and integrable on any closed interval $[a, b] \subset [a, B)$. If the limit below exists:

$$\int_a^B f(x)\,\mathrm{d}x = \lim_{b \to B-0} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper integral of $f$ over the interval $[a, B)$.

### Example 0.3

Let us investigate the values of the parameters $\alpha$ for which the integral

$$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x$$

converges.

## Example 0.4

Let us investigate the values of the parameters $\alpha$ for which the integral

$$\int_0^{+\infty} e^{-\alpha x}\,\mathrm{d}x$$

converges.

**Example 0.5**

Compute the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

**Example 0.6**

Let us investigate the integral

$$\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^2} \, \mathrm{d}x.$$

## Example 0.7

Compute the integral

$$\int_0^1 \ln x \, \mathrm{d}x.$$

**Example 0.8**

Compute the integral

$$\int_0^{+\infty} e^{-x} x^n \, \mathrm{d}x. (n \in \mathbb{Z}^+)$$

**Example 0.9**

Compute the integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, \mathrm{d}x.$$

## Example 0.10

Compute the integral

$$\int_0^{+\infty} \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x.$$

# Convergence of an Improper Integral

Let $[a, \omega)$ be a finite or infinite interval and $x \to f(x)$ a function defined on that interval and integrable over every closed interval $[a, b] \subset [a, \omega)$. Then by definition

$$\int_a^{\omega} f(x)\,\mathrm{d}x = \lim_{b \to \omega} \int_a^{b} f(x)\,\mathrm{d}x, \tag{1}$$

if this limit exists as $b \to \omega$, $b \in [a, \omega)$.

# Convergence of an Improper Integral

The convergence of the improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ is equivalent to the existence of a limit for the function

$$\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x \tag{2}$$

as $b \to \omega$, $b \in [a, \omega)$.

# Convergence of an Improper Integral

**Theorem 0.11**

If the function $x \to f(x)$ is defined on the interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$, then the integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges if and only is for every $\epsilon > 0$ there exists $B \in [a, \omega)$, such that the relation

$$\left| \int_{b_1}^{b_2} f(x)\,\mathrm{d}x \right| \leq \epsilon$$

for any $b_1, b_2 \in [a, \omega)$ satisfying $B < b_1$ and $B < b_2$.

# Absolute Convergence of an Improper Integral

**Definition 0.12**

The improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges absolutely if the integral $\int_a^\omega |f|\,\mathrm{d}x$ converges.

# Absolute Convergence of an Improper Integral

**Theorem 0.13**

If a function $f \geq 0$ and integrable on every $[a, b] \subset [a, \omega)$, then the improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ exists if and only if the function $\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x$ is bounded on $[a, \omega)$.

# Absolute Convergence of an Improper Integral

**Theorem 0.14**

Suppose that the function $x \to f(x)$ and $x \to g(x)$ are defined on the interval $[a, \omega)$ and integrable on any closed interval $[a, b] \subset [a, \omega)$. If

$$0 \leq f(x) \leq g(x)$$

on $[a, \omega)$, then the convergence of $\int_a^\omega g(x)\,\mathrm{d}x$ implies convergence of $\int_a^\omega f(x)\,\mathrm{d}x$, and the inequality

$$\int_a^\omega f(x)\,\mathrm{d}x \leq \int_a^\omega g(x)\,\mathrm{d}x$$

holds. Divergence of the integral $\int_a^\omega f(x)\,\mathrm{d}x$ implies divergence of $\int_a^\omega g(x)\,\mathrm{d}x$.

# Absolute Convergence of an Improper Integral

**Example 0.15**

Let us discuss the integral

$$\int_0^{+\infty} \frac{\sqrt{x}}{\sqrt{1+x^4}}\,\mathrm{d}x$$

# Absolute Convergence of an Improper Integral

## Example 0.16

Let us discuss the integral

$$\int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$$

**Example 0.17**

Let us discuss the integral

$$\int_{1}^{+\infty} e^{-x^2}\,\mathrm{d}x$$

# Absolute Convergence of an Improper Integral

**Example 0.18**

Let us discuss the integral

$$\int_e^{+\infty} \frac{1}{\ln x}\,\mathrm{d}x$$

# Absolute Convergence of an Improper Integral

**Example 0.19**

Let us discuss the Euler integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x$$

# Absolute Convergence of an Improper Integral

## Example 0.20

Let us discuss the elliptic integral

$$\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,\mathrm{d}x (0 < k^2 < 1)$$

## Example 0.21

Let us discuss the integral

$$\int_0^1 \frac{1}{\cos\theta - \cos\varphi}\,\mathrm{d}x$$

# Conditional Convergence of an Improper Integral

**Definition 0.22**

If an improper integral converges but not absolutely, we say that it converges conditionally.

## Example 0.23

The integral

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = -\left.\frac{\cos x}{x}\right|_{\frac{\pi}{2}}^{+\infty} - \int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$$

# Conditional Convergence of an Improper Integral

## Theorem 0.24

Let $x \to f(x)$ and $x \to g(x)$ be functions defined on an interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$. Suppose that $g$ is monotonic. Then a sufficient condition for convergence of the improper integral

$$\int_a^\omega (fg)\,\mathrm{d}x$$

is that the one of the following pairs of conditions hold:

1. 
   1. the integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges;
   2. the function $g$ is bound on $[a, \omega)$.
2. 
   1. the function $\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x$ is bound on $[a, \omega)$;
   2. the integral $g(x)$ converges to zero as $x \to \omega, x \in [a, \omega)$.

# Conditional Convergence of an Improper Integral

**Example 0.25**

Let us discuss the **Euler-Possion** integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$$

# Conditional Convergence of an Improper Integral

## Example 0.26

Let us discuss the integral

$$\int_0^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x$$

# Conditional Convergence of an Improper Integral

## Example 0.27

Let us discuss the integral

$$\int_0^{+\infty} \frac{\sin x}{x^\alpha} \, \mathrm{d}x$$

The last slide!