Limits

Guoning Wu

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> ”Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be” –Bertrand Russell

In discussing the various aspects of the concept of a real number we remarked in particular that in measuring real physical quantities we obtain sequences of approximate values with which one must then work. Such a state of affairs immediately raises at least the following three questions:

1. What relation does the sequence of approximations so obtained have to the quantity being measured? We have in mind the mathematical aspect of the question, that is, we wish to obtain an exact expression of what is meant in general by the expression ”sequence of approximate values” and the extent to which such a sequence describes the value of the quantity. Is the description unambiguous, or can the same sequence correspond to different values of the measured quantity?

2. How are operations on the approximate values connected with the same operations on the exact values?

3. How can one determine from a sequence of numbers whether it can be a sequence of arbitrarily precise approximations of the values of some quantity?

1 The Limit of a Sequence

1.1 Definitions and Examples

We recall the following definition.

Definition 1.1. A function $f : \mathbb{N} \to X$ whose domain of definition is the set of natural numbers is called a **sequence**.

The values $f(n)$ of the function f are called the terms of the sequence. We denote $x_n := f(n)$. In this connection the sequence itself is denoted $\{x_n\}$, and also written as $x_1, x_2, \cdots, x_n \cdots$. It is called a sequence in X.

The element x_n is called the *nth term of the sequence*.

Examples 1.

1. $f(n) = \dfrac{1}{n}$ generates the sequence $\dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \cdots$
2. $f(n) = \dfrac{(-1)^n}{n}$ generates the sequence $\dfrac{-1}{1}, \dfrac{1}{2}, \dfrac{-1}{3}, \cdots$
3. $f(n) = (-1)^{n+1}$ generates the sequence $1, -1, 1, \cdots$
4. $f(n) = \dfrac{n^3}{n+1}$ generates the sequence $\dfrac{1^3}{1+1}, \dfrac{2^3}{2+1}, \dfrac{3^3}{3+1}, \cdots$

Definition 1.2. A number $A \in \mathbb{R}$ is called the **limit of the numerical sequence** $\{x_n\}$ if for every neighborhood $V(A)$ of A there exists an index N (depending on $V(A)$) such that all terms of the sequence having index larger than N belong to the neighborhood $V(A)$.

We now write these formulations of the definitions of a limit in the language of symbolic logic.

$$\lim_{n\to\infty} x_n = A := \forall V(A), \exists N \in \mathbb{N}, \forall n > N, x_n \in V(A)$$

, and respectively

$$\lim_{n\to\infty} = A := \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |x_n - A| < \epsilon$$

.

Definition 1.3. If $\lim\limits_{n\to\infty} x_n = A$, we say the sequence $\{x_n\}$ converges to A and writes $x_n \to A$ as $n \to \infty$.

A sequence having a limit is said to be convergent. A sequence that does not have a limit is said to be divergent.

Let us consider some examples.

Examples 2. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, since $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \epsilon$ when $n > N = \left\lfloor \dfrac{1}{\epsilon} \right\rfloor$.

Examples 3. $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$.

Examples 4. $\lim\limits_{n\to\infty} \left(1 + \dfrac{(-1)^n}{n}\right) = 1$.

Examples 5. $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0.$

Examples 6. $\lim\limits_{n\to\infty} \dfrac{1}{q^n} = 0, \text{if} |q| > 1.$

Examples 7. *The sequence* $1, 2, \frac{1}{3}, 4, \frac{1}{5}, \cdots$ *whose* n *term is* $x_n = n^{(-1)^n}$ *is divergent.*

Examples 8. *The sequence* $1, -1, 1, -1, \cdots$ *for which* $x_n = (-1)^n$*, has no limit.*

Examples 9. $\lim\limits_{n\to\infty} \dfrac{1}{n^\alpha} = 0.(\alpha > 0)$

Examples 10. $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1.(\alpha > 0)$

Examples 11. Let $g(n) = n - \left\lfloor \dfrac{n}{2} \right\rfloor + \left\lfloor \dfrac{n}{3} \right\rfloor - \left\lfloor \dfrac{n}{4} \right\rfloor + \cdots$. Evaluate $\lim\limits_{n\to\infty} \dfrac{g(n)}{n}$.

1.2 Properties of the Limit of a Sequence

1.2.1 General Properties

Definition 1.4. If there exists a number A and an index N such that $x_n = A$ for all $n > N$, the sequence $\{x_n\}$ will be called **ultimately constant.**

Definition 1.5. A sequence $\{x_n\}$ is bounded if there exists M such that $|x_n| < M$ for all $n \in M$.

Theorem 1.1. 1) An ultimately constant sequence converges. 2) Any neighborhood of the limit of a sequence contains all but a finite number of terms of the sequence. 3) A convergent sequence cannot have two different limits. 4) A convergent sequence is bounded.

1.2.2 Passage to the Limit and the Arithmetic Operation

Definition 1.6. If $\{x_n\}$ and $\{y_n\}$ are two numerical sequences,their sum, product, and quotient are the sequences

$$\{x_n + y_n\}, \{x_n \cdot y_n\}, \left\{\frac{x_n}{y_n}\right\}$$

The quotient, of course, is defined only when $y_n \neq 0$ for all $n \in N$.

Theorem 1.2. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be numerical sequences. If* $\lim\limits_{n\to\infty} x_n = A$ *and* $\lim\limits_{n\to\infty} y_n = B$*, then*

a) $\lim\limits_{n\to\infty} (x_n + y_n) = A + B.$

b) $\lim\limits_{n\to\infty} (x_n \cdot y_n) = A \cdot B.$

c) $\lim\limits_{n\to\infty} (x_n \setminus y_n) = A \setminus B$*, provided that* $y_n \neq 0(n = 1, 2, \cdots,), B \neq 0.$

1.2.3 Passage to the Limit and Inequalities

Theorem 1.3. a) *Let $\{x_n\}$ and $\{y_n\}$ be numerical sequences. If $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$. If $A < B$, then there exists an index $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n > N$.*

Corollary 1.4. Suppose $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$. If there exists N such that for all $n > N$ we have

1. $x_n > y_n$, then $A \geq B$.
2. $x_n \geq y_n$, then $A \geq B$.
3. $x_n > B$, then $A \geq B$.
4. $x_n \geq B$, then $A \geq B$.

1.2.4 Questions Involving the Existence of the Limit of a Sequence

a. The Cauchy Criterion

Definition 1.7. A sequence x_n is called a **fundamental or Cauchy** sequence if for any $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ whenever $n > N$ and $m > N$.

Theorem 1.5. *(Cauchy's convergence criterion)* A numerical sequence converges if and only if it is a Cauchy sequence.

Proof.

$$a_n := \inf_{k\geq n} x_k$$

$$b_n := \sup_{k\geq n} x_k$$

Since $a_n \leq b_n$,

$$a_n = \inf_{k\geq n} x_k \leq x_k \leq \sup_{k\geq n} x_k = b_k$$

$$a_n \leq A \leq b_n$$

$$|x_k - A| \leq b_n - a_n$$

□

Examples 12. The sequence $(-1)^n, n = 1, 2, \cdots$ has no limit.

Examples 13. Let

$$x_1 = 0, x_2 = 0.\alpha_1, x_3 = 0.\alpha_1\alpha_2, \cdots, x_n = 0.\alpha_1\alpha_2\cdots\alpha_n, \cdots$$

be a sequence of finite binary fractions in which each successive fraction is obtained by adjoining a 0 or a 1 to its predecessor. Such a sequence always converges.

Examples 14. Suppose $\{x_n\}$ satisfies $|x_{n+1} - x_n| \leq k|x_n - x_{n-1}|, 0 < k < 1, n = 1, 2, \cdots$, then $\{x_n\}$ converges.

b. Some Criterions for the Existence of the Limit of Sequences

Definition 1.8. *A sequence x_n is increasing if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$, nondecreasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, nonincreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, and decreasing if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. Sequences of these four types are called monotonic sequences.*

Definition 1.9. *A sequence x_n is bounded above if there exists a number M such that $x_n < M$ for all $n \in \mathbb{N}$.*

Theorem 1.6. *(Weierstrass) In order for a non-decreasing sequence to have a limit it is necessary and sufficient that it is bounded above.*

Proof. Let $s = \sup_{n \in \mathbb{N}} x_n$, prove that

$$\lim_{n\to\infty} x_n = s$$

□

Examples 15.

$$\lim_{n\to\infty} \frac{n}{q^n} = 0, q > 1$$

Corollary 1.7.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Examples 16.

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \text{for any } a > 0$$

Examples 17.

$$\lim_{n\to\infty} \frac{q^n}{n!} = 0.$$

Examples 18.

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

Examples 19. *Suppose* $x_1 > 0, x_{n+1} = 1 + \frac{x_n}{1+x_n}, n = 1, 2, \cdots$. *Find the limit of* x_n.

Examples 20. *Suppose* $x_1 = \sqrt{a}, x_2 = \sqrt{a + x_1}, \cdots, x_n = \sqrt{a + x_{n-1}}, \cdots$, for $a \in \mathbb{R}^+$. *Find the limit* $\lim_{n\to\infty} x_n$.

Examples 21. *The Fibonacci sequence is:*

$$a_1 = 1, a_2 = 1, a_3 = a_1 + a_2, \cdots, a_{n+1} = a_n + a_{n-1}, \cdots$$

if we take $b_n = \dfrac{a_{n+1}}{a_n}, n = 1, 2, \cdots$. *Then find the* $\lim\limits_{n\to\infty} b_n$.

Proof.

$$b_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{b_{n-1}}$$

We find that, if $b_n > \dfrac{\sqrt{5}+1}{2}$, then $b_{n+1} < \dfrac{\sqrt{5}+1}{2}$ and vice versa. Furthermore, we find $b_{2k-1} \in \left(0, \dfrac{\sqrt{5}+1}{2}\right)$, $b_{2k} \in \left(\dfrac{\sqrt{5}+1}{2}, \infty\right)$

$$\begin{aligned} b_{2k+2} - b_{2k} &= 1 + \cfrac{1}{1 + \cfrac{1}{b_{2k}}} - b_{2k} \\ &= \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k}\right)\left(\frac{\sqrt{5}-1}{2} + b_{2k}\right)}{1 + b_{2k}} < 0 \end{aligned}$$

$$\begin{aligned} b_{2k+1} - b_{2k-1} &= 1 + \cfrac{1}{1 + \cfrac{1}{b_{2k-1}}} - b_{2k-1} \\ &= \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k-1}\right)\left(\frac{\sqrt{5}-1}{2} + b_{2k-1}\right)}{1 + b_{2k-1}} > 0 \end{aligned}$$

From $b_{2k+2} = \dfrac{1 + 2b_{2k}}{1 + b_{2k}}$, we derive,

$$a = \frac{1 + 2a}{1 + a}$$
$$a \approx 0.618$$

□

Examples 22. Suppose $a_n = 1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots + \dfrac{1}{n^p} (p > 0)$, prove that if $p > 1$, then the series converge, and if $0 < p \leq 1$, then the sequence does not converge.

c. The Number e

Examples 23. Let us prove that the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists.

In this case the limit is a number denoted by the letter e, after Euler [1]. This number is just as central to analysis as the number 1 to arithmetic or π to geometry. We begin by verifying the following inequality, sometimes called Jakob Bernoulli's inequality. [2]

$$(1+\alpha)^n \geq 1+n\alpha \text{ for } n\in\mathbb{N} \text{ and } \alpha>-1$$

Proof. The assertion is true for $n=1$. If it holds for $n\in\mathbb{N}$, then it must hold for $n+1$, since we have

$$(1+\alpha)^{n+1}=(1+\alpha)(1+\alpha)^n \geq (1+\alpha)(1+n\alpha)\geq 1+(n+1)\alpha$$

□

We now show that the sequence $y_n=\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing.

Proof.

$$\begin{aligned}\frac{y_{n-1}}{y_n}&=\frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}}=\frac{n^{2n}}{(n^2-1)^n}\cdot\frac{n}{n+1}=\left(1+\frac{1}{n^2-1}\right)^n\cdot\frac{n}{n+1}\\&\geq\left(1+\frac{n}{n^2-1}\right)\frac{n}{n+1}>\left(1+\frac{1}{n}\right)\frac{n}{n+1}=1\end{aligned}$$

Since the terms of the sequence are positive, the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$ exists. But we then have

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n&=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\left(1+\frac{1}{n}\right)^{-1}\\&=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\end{aligned}$$

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[1] **Leonhard Euler:**(1707-1783) was a Swiss mathematician, physicist, astronomer, logician and engineer who was made important and influential discoveries in many branches of mathematics like infinitesimal calculus and graph theory while also making pioneering contributions to several branches such as topology and analytic number theory. He is also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy, and music theory.

[2] Jakob Bernoulli (1654-1705) - Swiss mathematician, a member of the famous Bernoulli family of scholars. He was one of the founders of the calculus of variations and probability theory.

Figure 1: Leonhard Euler.

□

Definition 1.10.

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The Mathematical Constant e. The number e is a mathematical constant that is the base of the natural logarithm: the unique number whose natural logarithm is equal to one. It is approximately equal to 2.71828, and is the limit of (1 + 1/n)n as n approaches infinity, an expression that arises in the study of compound interest. It can also be calculated as the sum of the infinite series.

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots.$$

Sometimes called Euler's number after the Swiss mathematician Leonhard Euler, e is not to be confused with γ, the Euler–Mascheroni constant, sometimes called simply Euler's constant. The number e is also known as Napier's constant, but Euler's choice of the symbol e is said to have been retained in his honor. The constant was discovered by the Swiss mathematician Jacob Bernoulli while studying compound interest.

The number e is of eminent importance in mathematics, alongside 0, 1, π and i. All five of these numbers play important and recurring roles across mathematics, and are the five constants appearing in one formulation of Euler's identity.

$$e^{i\pi} + 1 = 0.$$

where e is Euler's number, the base of natural logarithms, i is the imaginary unit, which satisfies $i^2 = -1$, and π is pi, the ratio of the circumference of a circle to its diameter.

Euler's identity is named after the Swiss mathematician Leonhard Euler. It is considered to be an example of mathematical beauty. Like the constant π, e is irrational: it is not a ratio of integers. Also like π, e is transcendental: it is not a root of any non-zero polynomial with rational coefficients. The numerical value of e truncated to 50 decimal places is

2.71828182845904523536028747135266249775724709369 9.

The first references to the constant were published in 1618 in the table of an appendix of a work on logarithms by John Napier. However, this did not contain the constant itself, but simply a list of logarithms calculated from the constant. It is assumed that the table was written by William Oughtred. The discovery of the constant itself is credited to Jacob Bernoulli in 1683, who attempted to find the value of the following expression (which is in fact e):

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

The first known use of the constant, represented by the letter b, was in correspondence from Gottfried Leibniz to Christiaan Huygens in 1690 and 1691. Leonhard Euler introduced the letter e as the base for natural logarithms, writing in a letter to Christian Goldbach of 25 November 1731. Euler started to use the letter e for the constant in 1727 or 1728, in an unpublished paper on explosive forces in cannons, and the first appearance of e in a publication was in Euler's Mechanica (1736). While in the subsequent years some researchers used the letter c, e was more common and eventually became the standard.

Examples 24. *Find the limit of*

$$\lim_{n\to\infty} \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right\}$$

The limit of the sequence of Example 24 is called the Euler–Mascheroni constant.The numerical value of the Euler–Mascheroni constant, to 50 decimal places, is:

0.57721566490153286060651209008240243104215933593992. . . .

The Euer-Mascheroni Constant. The Euler–Mascheroni constant (also called Euler's constant) is a mathematical constant recurring in analysis and number theory, usually denoted by the lowercase Greek letter gamma (γ).

It is defined as the limiting difference between the harmonic series and the natural logarithm:

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n}\frac{1}{k}\right)\\ &= \int_1^{\infty}\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)dx.\end{aligned}$$

Here, $\lfloor\rfloor$ represents the floor function.

The constant first appeared in a 1734 paper by the Swiss mathematician Leonhard Euler, titled De Progressionibus harmonicis observationes (Eneström Index 43). Euler used the notations C and O for the constant. In 1790, Italian mathematician Lorenzo Mascheroni used the notations A and a for the constant. The notation γ appears nowhere in the writings of either Euler or Mascheroni, and was chosen at a later time perhaps because of the constant's connection to the gamma function. For example, the German mathematician Carl Anton Bretschneider used the notation γ in 1835 and Augustus De Morgan used it in a textbook published in parts from 1836 to 1842.

Examples 25.

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)=\ln 2.$$

Examples 26.

$$\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}+\cdots+(-1)^{n+1}\frac{1}{n}\right)=\ln 2.$$

d. Subsequences and Partial Limits of a Sequence

Definition 1.11. *If $x_1, x_2, \cdots, x_n, \cdots$ is a sequence and $n_1 < n_2 < \cdots < n_k < \cdots$ an increasing sequence of natural numbers, then the sequence $x_{n_1}, x_{n_2}, \cdots, x_{n_k}\cdots$ is a subsequence of the sequence x_n.*

Lemma 1.8. *(Bolzano-Weierstrass) Every bounded sequence of real numbers contains a convergent subsequence.*

Definition 1.12. *We shall write $x_n \to +\infty$ and say that the sequence x_n tends to positive infinity if for each number c there exist $N \in \mathbb{N}$ such that $x_n > c$ for all $n > N$.*

Lemma 1.9. *From each sequence of real numbers one can extract either a convergent subsequence or a subsequence that tends to infinity.*

Let x_k be an arbitrary sequence of real numbers. If it is bounded below, one can consider the sequence $i_n = \inf_{k \geq n} x_k$.

Definition 1.13. *The number* $l = \lim_{n\to\infty} \inf_{k\geq n} x_k$ *is called the inferior limit of the sequence* x_n *and denoted* $\varliminf_{k\to\infty} x_k$ *or* $\lim\inf_{k\to\infty} x_k$*. Ifi*$_n \to +\infty$*, it is said that the inferior limit of the sequence equals positive infinity, and we write* $\varliminf_{k\to\infty} x_k = +\infty$*. If the original sequence* x_n *is not bounded below, the we shall have* $i_n = \inf_{k\geq n} x_k = -\infty$ *for all* n*. In that case we say that the inferior limit of the sequence equals negative infinity and write* $\varliminf_{k\to\infty} x_k = -\infty$*.*

Definition 1.14.

$$\varlimsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup_{k\geq n} x_k$$

Examples 27. $x_k = (-1)^k, k \in \mathbb{N}$.

$$\varliminf_{k\to\infty} x_k = -1$$

$$\varlimsup_{k\to\infty} x_k = 1$$

Examples 28. $x_k = k^{(-1)^k}$.

$$\varliminf_{k\to\infty} k^{(-1)^k} = \lim_{n\to\infty} \inf_{k\geq n} k^{(-1)^k} = \lim_{n\to infty} 0 = 0$$

$$\varlimsup_{k\to\infty} k^{(-1)^k} = \lim_{n\to\infty} \sup_{k\geq n} k^{(-1)^k} = \lim_{n\to\infty} +\infty = +\infty$$

Examples 29. $x_k = k, k \in \mathbb{N}$.

$$\varliminf_{k\to\infty} k = \lim_{n\to\infty} \inf_{k\geq n} k = \lim_{n\to\infty} n = +\infty$$

$$\varlimsup_{k\to\infty} k = \lim_{n\to\infty} \sup_{k\geq n} k = \lim_{n\to\infty} +\infty = +\infty$$

Examples 30. $x_k = \dfrac{(-1)^k}{k}, k \in \mathbb{N}$.

$$\varliminf_{k\to\infty} k = \lim_{n\to\infty} \inf_{k\geq n} \frac{(-1)^k}{k} = 0$$

$$\varlimsup_{k\to\infty} k = \lim_{n\to\infty} \sup_{k\geq n} \frac{(-1)^k}{k} = 0$$

Examples 31. $x_k = -k^2, k \in \mathbb{N}$

$$\varliminf_{k\to\infty}(-k^2) = \lim_{n\to\infty}\inf_{k\geq n}(-k^2) = -\infty$$

$$\varlimsup_{k\to\infty}(-k^2) = \lim_{n\to\infty}\sup_{k\geq n}(-k^2) = -\infty$$

Examples 32. $x_k = (-1)^k k, k \in \mathbb{N}$

$$\varliminf_{k\to\infty}((-1)^k k) = \lim_{n\to\infty}\inf_{k\geq n}((-1)^k k) = -\infty$$

$$\varlimsup_{k\to\infty}((-1)^k k) = \lim_{n\to\infty}\sup_{k\geq n}((-1)^k k) = \infty$$

Definition 1.15. *A number (or the symbol ∞ or $-\infty$) is called a partial limit of a sequence, if the sequence contains a subsequence converging to that number.*

Proposition 1.1. *The inferior and superior limits of a bounded sequence are respectively the smallest and the largest partial limits of the sequence.*

Proposition 1.2. *For any sequence, the inferior limit is the smallest of its partial limits and the superior limit is the largest of its partial limits.*

Corollary 1.10. *A sequence has a limit or tends to negative or positive infinity if and only if its inferior and superior limits are same.*

Corollary 1.11. *A sequence converge if and only if every subsequence of it converge.*

e. Stolz Theorem

Theorem 1.12. Suppose $\{y_n\}$ is strictly increasing and tends to $+\infty$, and

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a(+\infty, -\infty),$$

then

$$\lim_{n\to\infty}\frac{x_n}{y_n} = a.$$

Proof. First, let we see the case of $a = 0$. Because $\lim\limits_{n\to\infty}\dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} = 0$, then we have $\forall\epsilon > 0, \exists N_1$ such that $\forall n > N_1$,

$$|x_n - x_{n-1}| < \epsilon\,(y_n - y_{n-1})$$

Since $\lim_{n\to\infty} y_n = +\infty$, we then suppose $y_{N_1} > 0$, and we have

$$\begin{aligned}|x_n - x_{N_1}| &\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{N_1+1} - x_{N_1}| \\ &< \epsilon \left[(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_{N_1+1} - y_{N_1})\right] \\ &= \epsilon(y_n - y_{N_1})\end{aligned}$$

Both sides divided by y_n, then

$$\left|\frac{x_n}{y_n} - \frac{x_{N_1}}{y_n}\right| \le \epsilon\left(1 - \frac{y_{N_1}}{y_n}\right) < \epsilon$$

For fixed N_1, we can find $N > N_1$ such that $\forall n > N$, $\left|\frac{x_{N_1}}{y_n}\right| < \epsilon$, then,

$$\left|\frac{x_n}{y_n}\right| \le \epsilon + \left|\frac{y_{N_1}}{y_n}\right| < 2\epsilon$$

If $\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a \neq 0$, $x'_n = x_n - ay_n$, then,

$$\lim_{n\to\infty} \frac{x'_n - x'_{n-1}}{y_n - y_{n-1}} \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} - a = 0$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x'_n}{y_n} + a = a$$

□

Examples 33.

$$\lim_{n\to\infty} \frac{1^k + 2^2 + \cdots + n^k}{n^{k+1}}$$

Examples 34. Suppose $\lim_{n\to\infty} a_n = a$, find

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n^2}$$

1.2.5 Elementary Facts about Series

a. The Sum of a Series and the Cauchy Criterion for Convergence of a Series We wish to give a precise meaning to the expression $a_1 + a_2 + \cdots + \cdots$, which expresses the sums of all the terms of the sequence $\{a_n\}$.

2 The Limit of a Function

2.1 Definitions and Examples

Let E be a subset of $\mathbb{R}$ and a a limit point of E. Let $f : E \to \mathbb{R}$ be a real-valued function defined on E.

Definition 2.1. We shall say (following Cauchy) that the function $f : E \to \mathbb{R}$ tends to A as x tends to a, or that A is the limit of f as x tends to a, if for every $\epsilon > 0$ there exist $\delta > 0$ such that $|f(x) - A| < \epsilon$ for every $x \in E$ such that $0 < |x - a| < \delta$.

In logical symbolism these condition are written as

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E \text{ and } 0 < |x - a| < \delta, \Rightarrow |f(x) - A| < \epsilon$$

Examples 35. Let $E = \mathbb{R} \setminus 0$, and $f(x) = x \sin \frac{1}{x}$. We shall verify that

$$\lim_{E \ni x \to 0} x \sin \frac{1}{x} = 0.$$

Definition 2.2. A **deleted neighborhood** of a point is a neighborhood of the point from which the point itself has been removed.

Definition 2.3.

$$\lim_{E \ni x \to a} f(x) = A := \forall V_{\mathbb{R}}(A), \exists \mathring{U}_E(a) \Rightarrow f\left(\mathring{U}_E(a)\right) \subset V_R(A).$$

Examples 36.

$$\lim_{x \to 0} e^x = 1$$

Examples 37.

$$\lim_{x \to 2} x^2 = 4$$

Examples 38. The function

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

is defined on the whole line. We shall show that it has no limit as x tends to 0.

The nonexistence of this limit is expressed by

$$\forall A \in \mathbb{R}, \exists V(A), \forall \overset{\circ}{U}(0) \Rightarrow \exists x \in \overset{\circ}{U}(0), f(x) \notin V(A).$$

Examples 39. The function

$$f(x) = \sin\frac{1}{x}$$

has no limit as $x \to 0$

Proposition 2.1. The relation $\lim_{E \ni x \to a} f(x) = A$ holds if and only if for every sequence $\{x_n\}$ of points $x_n \in E \setminus a$ converging to a, the sequence $\{f(x_n)\}$ converges to A.

Proof. The fact that $\lim_{E \ni x \to a} f(x) = A \Rightarrow \lim_{n\to\infty} f(x_n) = A$ follows immediately from the definitions. Indeed, if $\lim_{E \ni x \to a} f(x) = A$, then for any neighborhood $V(A)$ of A there exists a deleted neighborhood $\overset{\circ}{U}_E(a)$ of a point a in E such that for $x \in \overset{\circ}{U}_E(a)$ we have $f(x) \in V(A)$. If the sequence $\{x_n\}$ of points in E a converges to a, there exists an index N such that $x_n \in \overset{\circ}{U}_E(a)$ for $n > N$, and then $f(x_n) \in V(A)$.

We now prove the converse. If A is not the limit of $f(x)$ as $x \to a$, there exists a neighborhood $V(A)$ such that for any $n \in N$, there exist points x_n in the deleted $\frac{1}{n}$ neighborhood of a in E such that $f(x_n) \notin V(A)$. But this means that the sequence $\{f(x_n)\}$ does not converges to A, even though $\{x_n\}$ converges to a. □

2.2 Properties of the Limit of a Function

We now establish a number of properties of the limit of a function that are constantly being used. Many of them are analogous to the properties of the limit of a sequence that have already established.

We call the reader's attention to the fact that, in order to establish the properties of the limit of a function, we need only two properties of deleted neighborhoods of a limit point of a set:

1. $\overset{\circ}{U}_E(a) \neq \emptyset$, the deleted neighborhood of the point in E is non-empty.
2. $\forall \overset{\circ}{U}_{E_1}(a), \forall \overset{\circ}{U}_{E_2}(a), \exists \overset{\circ}{U}_E(a) \Rightarrow \overset{\circ}{U}_E(a) \subset \overset{\circ}{U}_{E_1}(a) \cap \overset{\circ}{U}_{E_2}(a).$

a. Properties of the Limit of a Function

Definition 2.4. A function $f : E \to \mathbb{R}$ assuming only one value is called constant. A function $f : E \to \mathbb{R}$ is called ultimately constant as $E \ni x \to a$ if it is constant in some deleted neighborhood $\overset{\circ}{U}_E(a)$, where a is a limit point of E.

Definition 2.5. A function $f : E \to \mathbb{R}$ is bounded, bounded above, or bounded below respectively if there is a number $C \in \mathbb{R}$ such that $|f(x)| < C$, $f(x) < C$ or $C < f(x)$, for all $x \in E$.

Examples 40. The function $f(x) = \left(\sin \dfrac{1}{x} + x \cos \dfrac{1}{x}\right)$ defined by this formula for $x \neq 0$ is not bounded on its domain of definition, but it is ultimately bounded as $x \to 0$.

Theorem 2.1.

1. $f : E \to \mathbb{R}$ is ultimately the constant A as $E \ni x \to a \Rightarrow \lim\limits_{E \ni x \to a} = A$
2. $\lim\limits_{E \ni x \to a} f(x) = A \Rightarrow f : E \to \mathbb{R}$ is ultimately bounded as $E \ni x \to a$
3. $\lim\limits_{E \ni x \to a} f(x) = A_1 \wedge \lim\limits_{E \ni x \to a} f(x) = A_2 \Rightarrow A_1 = A_2$

b. Passage to Limit and Arithmetic Operations

Definition 2.6. If two numerical-valued functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ have a common domain of definition E, their sum, product and quotient are respectively the functions defined on the same set by the following formulas:

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x),\\ (f \cdot g)(x) &:= f(x) \cdot g(x),\\ \left(\frac{f}{g}\right)(x) &:= \frac{f(x)}{g(x)}.\end{aligned}$$

Proposition 2.2.

1. If $\alpha : E \to \mathbb{R}$ and $\beta : E \to \mathbb{R}$ are infinitesimal functions as $E \ni x \to a$, then their sum $\alpha + \beta : E \to \mathbb{R}$ also infinitesimal as $E \ni x \to a$.
2. If $\alpha : E \to \mathbb{R}$ and $\beta : E \to \mathbb{R}$ are infinitesimal functions as $E \ni x \to a$, then their product $\alpha \cdot \beta : E \to \mathbb{R}$ also infinitesimal as $E \ni x \to a$.

3. If $\alpha : E \to \mathbb{R}$ is infinitesimal functions as $E \ni x \to a$, and $\beta : E \to \mathbb{R}$ is ultimately bounded as $E \ni x \to a$ then their product $\alpha \cdot \beta : E \to \mathbb{R}$ also infinitesimal as $E \ni x \to a$.

Remark.

$$\lim_{E \ni x \to a} f(x) = A \Leftrightarrow f(x) = A + \alpha(x) \wedge \lim_{E \ni x \to a} \alpha(x) = 0$$

Theorem 2.2. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two functions with a common domain of definition. If $\lim_{E \ni x \to a} f(x) = A$ and $\lim_{E \ni x \to a} g(x) = B$, then

$$\lim_{E \ni x \to a} (f + g)(x) = A + B,$$

$$\lim_{E \ni x \to a} (f \cdot g)(x) = A \cdot B,$$

$$\lim_{E \ni x \to a} \left(\frac{f}{g}\right)(x) = \frac{A}{B}, \text{ if } B \neq 0 \text{ and } g(x) \neq 0 \text{ for } x \in E$$

c. Passage to the Limit and Inequalities

Theorem 2.3.

1. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two functions with a common domain of definition. If $\lim_{E \ni x \to a} f(x) = A$ and $\lim_{E \ni x \to a} g(x) = B$ and $A < B$, then there exists a deleted neighborhood $\mathring{U}_E(a)$ of a in E at each point of which $f(x) < g(x)$.

2. If the relations $f(x) \leq g(x) \leq h(x)$ hold for functions $f : E \to \mathbb{R}, g : E \to \mathbb{R}$ and $h : E \to \mathbb{R}$,and if $\lim_{E \ni x \to a} f(x) = \lim_{E \ni x \to a} h(x) = C$, then the limit of $g(x)$ exists as $E \ni x \to a$,and $\lim_{E \ni x \to a} g(x) = C$.

Corollary 2.4. Suppose $\lim_{E \ni x \to a} f(x) = A$ and $\lim_{E \ni x \to a} g(x) = B$. Let $\mathring{U}_E(a)$ be a deleted neighborhood of a in E.

1. If $f(x) > g(x)$ for all $x \in \mathring{U}_E(a)$, then $A \geq B$,

2. If $f(x) \geq g(x)$ for all $x \in \mathring{U}_E(a)$, then $A \geq B$,

3. If $f(x) > B$ for all $x \in \mathring{U}_E(a)$, then $A \geq B$,

4. If $f(x) \geq B$ for all $x \in \mathring{U}_E(a)$, then $A \geq B$.

d. Two Important Examples

Examples 41.

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Proof.

$$\begin{aligned}
S_{\triangleleft OCD}&=\frac{1}{2}|OC||\widehat{CD}|=\frac{1}{2}\cos x(x\cos x)=\frac{1}{2}x\cos^2 x\\
&<S_{\Delta OAB}=\frac{1}{2}|OA||BC|=\frac{1}{2}\sin x\\
&<S_{\triangleleft OAB}=\frac{1}{2}|OA||\widehat{AB}|=\frac{1}{2}x
\end{aligned}$$

$$\cos^2 x<\frac{\sin x}{x}<1, 0<|x|<\frac{\pi}{2}$$

□

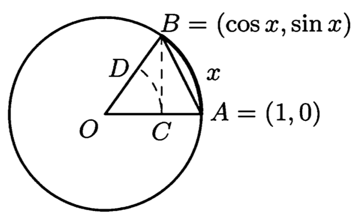

Figure 2: Sine Inequality.

Examples 42.

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=e$$

Examples 43.

$$\lim_{x\to 0}(1+\tan x)^{\cot x}=e$$

Examples 44.

$$\lim_{x\to 0}\left(1+\frac{\alpha}{x}\right)^{\beta x}=e$$

Examples 45.

$$\lim_{x\to +\infty}\frac{x}{q^x}=0, q>1$$

Examples 46.

$$\lim_{x\to +\infty}\frac{\log_a x}{x}=0, a>0$$

2.3 The General Definition of the Limit of a Function

When proving the Properties of limit of a function, we verified that the only requirements imposed on the deleted neighborhoods in which our functions were defined and which arose in the course of the proofs were the properties below:

Definition 2.7. A set $\mathcal{B}$ of subsets $B \subset X$ of a set X is called a base in X if the following conditions hold:

1. $\forall B \in \mathcal{B}, B \neq \emptyset$,
2. $\forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}, \exists B \in \mathcal{B} \subset B_1 \cap B_2$.

In other words, the elements of the collection $\mathcal{B}$ are non-empty subsets of X and the intersection of any two of them always contains an element of the same collection.

For example, the notation $E \ni x \to a + 0(resp. E \ni x \to a - 0)$ will be used instead of $x \to a, x \in E \cap E_a^+(resp. x \to a, x \in E \cap E_a^-)$. It means that x tends to a in E while remaining larger (resp. smaller) than a.

b. The Limit of a Function Over a Base

Definition 2.8. Let $f : X \to \mathbb{R}$ be a function defined on a set X and $\mathcal{B}$ a base in X. A number $A \in \mathbb{R}$ is called the limit of the function f over the base $\mathcal{B}$ if for every neighborhood $V(A)$ of A there is an element $B \in \mathcal{B}$ whose image $f(B)$ in contained in $V(A)$.

We now repeat the definition of the limit over a base in logical symbols:

$$\lim_{\mathcal{B}} f(x) = A := \forall V(A), \exists B \in \mathcal{B}, f(B) \subset V(A).$$

Thus,

$$\lim_{x \to a-0} f(x) = A := \forall \epsilon > 0, \exists \delta > 0, \forall x \in]a - \delta, a[, \Rightarrow |f(x) - A| < \epsilon.$$

Definition 2.9. A function $f : X \to \mathbb{R}$ is ultimately constant over the base $\mathcal{B}$ if there exists a number $A \in \mathbb{R}$ and an element $B \in \mathcal{B}$ such that $f(x) = A$ for all $x \in B$

Definition 2.10. A function $f : X \to \mathbb{R}$ is ultimately bounded over the base $\mathcal{B}$ if exists a number $c > 0$ and an element $B \in \mathcal{B}$ such that $|f(x)| < c$ for all $x \in B$.

Definition 2.11. A function $f : X \to \mathbb{R}$ is infinitesimal over the base $\mathcal{B}$ if $\lim_{\mathcal{B}} = 0$.

2.4 Existence of the Limit of a Function

a. The Cauchy Criterion

Definition 2.12. The oscillation of a function $f : X \to \mathbb{R}$ on a set $E \subset X$ is

$$\omega(f, E) := \sup_{x_1, x_2 \in E} |f(x_1) - f(x_2)| \, .$$

that is, the least upper bound of the absolute value of the difference of the values of the function at two arbitrary points $x_1, x_2 \in E$.

Examples 47.

$$\omega\left(x^2, [-1, 2]\right) = 4.$$

Examples 48.

$$\omega\left(x, [-1, 2]\right) = 3.$$

Theorem 2.5. *(The Cauchy Criterion for the existence of a limit of a function)* Let X be a set and $\mathcal{B}$ a base in X. A function $f : X \to \mathbb{R}$ has a limit over the base $\mathcal{B}$ if and only if for every $\epsilon > 0$ there exists $B \in \mathcal{B}$ such that the oscillation of f on B is less than ϵ.

b. The Relationship Between The Limit Of Function And Sequence

Theorem 2.6. $\lim\limits_{x \to x_0} f(x) = A$ if and only if for every sequence $\{x_n\}$ satisfies $x_n \neq x_0$ and $\lim\limits_{n \to \infty} x_n = x_0$, we have $\lim\limits_{n \to \infty} f(x_n) = A$

Examples 49. Prove that the limit $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ not exists.

c. The Limit of a Composite Function

Theorem 2.7. *(The limit of a composite function)* Let Y be a set, $\mathcal{B}_Y$ a base in Y, and $g : Y \to \mathbb{R}$ a mapping having a limit over a base $\mathcal{B}_Y$. Let X be a set, $\mathcal{B}_X$ a base in X and $f : X \to Y$ a mapping of X into Y such that for every $B_Y \in \mathcal{B}_Y$ there exists $B_X \in \mathcal{B}_X$ whose image $f(B_X)$ is contained in B_Y.

Under these hypotheses, the composition $g \circ f : X \to \mathbb{R}$ of the mappings f and g is defined and has a limit over the base $\mathcal{B}_X$ and

$$\lim_{\mathcal{B}_X} (g \circ f)(x) = \lim_{\mathcal{B}_Y} g(y)$$

Examples 50.

$$\lim_{x \to 0} \frac{\sin 7x}{x} = 7$$

Theorem 2.8. *(Criterion for the existence of a limit of a monotonic function)* A necessary and sufficient condition for a function $f : E \to \mathbb{R}$ that is non-decreasing on the set E to have a limit as $x \to \sup E$, is that it be bounded above. For this function to have a limit as $x \to \inf E$, it is necessary and sufficient that it is bounded below.

Comparison of the Asymptotic Behavior of Functions We begin with discussion with some examples to clarify the subject.

Let $\pi(x)$ be the number of primes not larger than a given number $x \in \mathbb{R}$. Although for any fixed x we can find (if only by explicit enumeration) the value of $\pi(x)$, we are nevertheless not in a position to say, for example, how the function behaves as $x \to +\infty$, or, what is the same, what the asymptotic law of distribution of prime numbers is. We have known sine the time of Euclid that $\pi(x) \to +\infty$ as $x \to +\infty$, but the proof that $\pi(x)$ grows approximately like $\frac{x}{\ln x}$ was achieved only in the nineteenth century by P. L. Chebyshev [3]

Definition 2.13. The function f is said to be infinitesimal compared with the the function g over the base $\mathcal{B}$, and write $f = \circ(g)$ over $\mathcal{B}$ if the relation $f(x) = \alpha(x)g(x)$ holds ultimately over the $\mathcal{B}$, where $\alpha(x)$ is a function that is infinitesimal over $\mathcal{B}$.

Examples 51.

$$x^2 = \circ(x) \text{ as } x \to 0$$

Examples 52.

$$x = \circ(x^2) \text{ as } x \to \infty.$$

Definition 2.14. If $f = \circ(g)$ and g is itself infinitesimal over $\mathcal{B}$, we say that f is an infinitesimal of higher order that g over $\mathbb{B}$.

Examples 53.

$$x^{-2} \text{ is an infinitesimal of higher order that } x^{-1} \text{ as } x \to \infty.$$

Definition 2.15. A function that tends to infinity over a given base is said to be an infinite function or simply an infinity over the given base.

Definition 2.16. If f and g are infinite functions over $\mathcal{B}$ and $f = \circ(g)$ over the base $\mathcal{B}$, we say that g is a higher order infinity than f over $\mathcal{B}$.

[3] P. L. Chebyshev (1821-1894) - outstanding Russian mathematician and specialist in theoretical mechanics, the founder of a large mathematical school in Russia.

Examples 54. $\dfrac{1}{x} = \circ\dfrac{1}{x^2}$ as $x \to 0$.

Examples 55. We shall show that for $a > 1$ and any $n \in \mathbb{Z}$

$$\lim_{x\to+\infty} \frac{x^n}{a^x} = 0$$

that is, $x^n = \circ(a^x)$ as $x \to +\infty$.

Examples 56. Let us show that

$$\lim_{x\to\infty} \frac{x^\alpha}{a^x} = 0,$$

for $a > 1$ and any $\alpha \in \mathbb{R}$.

Examples 57. Let us show that

$$\lim_{x\to 0} \frac{a^{-1/x}}{x^\alpha} = 0$$

for $a > 1$ and any $\alpha \in \mathbb{R}$.

Examples 58. Let us show that

$$\lim_{x\to+\infty} \frac{\log_a x}{x^\alpha} = 0$$

for $\alpha > 0$.

Examples 59. Let us show that

$$x^\alpha \log_a x = \circ(1)$$

as $x \to 0, x \in \mathbb{R}_+$.

Definition 2.17. Let us agree that the notation $f = \bigcirc(g)$ over the base $\mathcal{B}$ (read "f is big-oh g") means that the relation $f(x) = \beta(x)g(x)$ holds ultimately where $\beta(x)$ is ultimately bounded over the base $\mathcal{B}$.

Examples 60. Let us show

$$\left(\frac{1}{x} + \sin x\right) x = \bigcirc(x)$$

as $x \to \infty$

Definition 2.18. The function f and g are of the same order over $\mathcal{B}$, and we write $f \asymp g$ over $\mathcal{B}$, if $f = \bigcirc(g)$ and $g = \bigcirc(f)$ simutantaneously.

Examples 61. Let us show

$$(2+\sin x)x \asymp x$$

as $x \to \infty$.

Definition 2.19. If the relation $f(x) = \gamma(x)g(x)$ holds ultimately over $\mathcal{B}$ where $\lim_{\mathcal{B}} = 1$, we say that the function f behaves asymptotically like g over $\mathcal{B}$, or, more briefly, that f is equivalent to g over $\mathcal{B}$.

In this case we shall write $f \sim g$ over $\mathcal{B}$. The use of the word equivalent is justified by the relations:

1. $f \sim f$ over $\mathcal{B}$,
2. $f \sim g \Rightarrow g \sim f$ over $\mathcal{B}$,
3. $f \sim g \wedge g \sim h \Rightarrow f \sim h$ over $\mathcal{B}$.

It is useful to note that since the relation $\lim_{\mathcal{B}} \gamma(x) = 1$ is equivalent to $\gamma(x) = 1 + \alpha(x)$, where $\alpha(x) \to 0$ over $\mathcal{B}$, the relation $f \sim g$ over $\mathcal{B}$ is equivalent to $f(x) = g(x) + \alpha(x)g(x) = g(x) + \circ(g(x))$ over $\mathcal{B}$.

Examples 62.

$$x^2 + x = \left(1 + \frac{1}{x}\right) \sim x$$

as $x \to \infty$.

Examples 63. Since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, we have $\sin x \sim x$ as $x \to 0$, which can be written as $\sin x = x + \circ(x)$ as $x \to 0$.

Examples 64. Let us show that

$$\ln(1+x) \sim x$$

as $x \to 0$. Thus, $\ln(1+x) = x + \circ(x)$.

Examples 65. Let us show that

$$e^x = 1 + x + \circ(x)$$

as $x \to 0$. Thus, $e^x - 1 \sim x$ as $x \to 0$.

Examples 66. Let us show that

$$(1+x)^\alpha = 1 + \alpha x + \circ(x)$$

as $x \to 0$.

Proposition 2.3. If $f \sim \tilde{f}$, then $\lim_{\mathcal{B}} f(x)g(x) = \lim_{\mathcal{B}} \tilde{f}(x)g(x)$, provided one of these limit exists.

Examples 67. Let us show that

$$\lim_{x \to 0} \frac{\ln \cos x}{\sin (x^2)} = \frac{-1}{2}.$$

Examples 68. Let us show that

$$\sqrt{x^2 + x} \sim x$$

as $x \to +\infty$.

Examples 69. Let us show that

$$\lim_{x \to 0} \frac{\ln (1 + x^2)}{(e^{2x} - 1) \tan x} = \frac{1}{2}$$

Examples 70. Let us show that

$$\lim_{x \to \infty} x \left(\sqrt[3]{x^3 + x} - \sqrt[3]{x^3 - x} \right) = \frac{2}{3}$$

Examples 71. Let us show that

$$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}} \sqrt{e}$$