Derivative

1. Differentiable Functions

Definition 1.1 A function $f: E \to \mathbb{R}$ defined on a set E is differentiable at a point $a \in E$ that is a limit point of E, if there exists a linear function $A(x-a)$ of the increment $x-a$ of the argument such that $f(x) - f(a)$ can be represented as

$$f(x) - f(a) = A(x-a) + \circ(x-a)x \to a, a \in E \tag{1}$$

In other words, a function is differentiable at a point a if the change in its values in a neighborhood of the point in question is linear up to a correction that is infinitesimal compared with the magnitude of the displacement $x - a$ for the point a.

Definition 1.2 The linear function $A(x-a)$ in \eqref{eq:eq1} is called the differential of the function f at a. The number A is unambiguously determined due to the uniqueness of the limit.

Definition 1.3 The number \begin{equation} f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} {\tag 2} \label{eq:eq2} \end{equation} is called the derivative of the function f at a. Relation \eqref{eq:eq2} can be rewritten in the equivalent form $$\frac{f(x) - f(a)}{x - a} = f'(a) + \alpha(x)$$ where $\alpha(x) \to 0$ as $x \to a, x \in E$, which is equivalent to $$f(x) - f(a) = f'(a)(x-a) + o(x-a)$$ as $x \to a, x \in E$.

Graphically, this definition says that the derivative of f at a is the slope of the tangent line to $y = f(x)$ at a, which is the limit as $x \to a$ of the slopes of the lines through $(x,f(x))$ and $(a,f(a))$.

We can also write $$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Definition 1.4 A function $f: E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is differentiable at a point $x \in E$ that is a limit point of E, if \begin{equation} f(x+h) - f(x) = A(x)h + \alpha(x;h){\tag 3} \label{eq:eq3} \end{equation} where $h \to A(x)h$ is a linear function in h and $\alpha(x;h) = \circ(h)$as $h \to 0, x+h \in E$.

Definition 1.5 The function $h \to A(x)h$ of Definition \eqref{eq:eq3} , which is linear in h, is called the differential of the function $f: E \to \mathbb{R}$ at the point $x \in E$ and is denoted as $df(x)$ or $Df(x)$. Thus, $df(x)(h) = A(x)h.$

From definitions \eqref{eq:eq2} and \eqref{eq:eq3} we have $$\Delta f(x;h) - df(x)(h) = \alpha(x;h)$$

The Tangent line, Geometric Meaning of the Derivative and Differential

If we were seeking a polynomial $P_n(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n + o((x - x_0)^n)$ such that $$f(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n + o((x - x_0)^n)$$as $x \to x_0, x \in E$ We could find successively, $$ \begin{array}{c} \displaystyle c_0 = \lim_{x \to x_0} f(x) \newline \displaystyle c_1 = \lim_{x \to x_0} \frac{f(x) - c_0}{x - x_0} \newline \vdots \newline \displaystyle c_n = \lim_{x \to x_0} \frac{f(x) - [c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^{n-1}] }{(x - x_0)^n} \end{array} $$

Proposition 1 A function $f: E \to \mathbb{R}$ that is continuous at a point $x_0 \in E$ that is a limit of $E \subset \mathbb{R}$ admits a linear approximation $$f(x) = c_0 + c_1(x - x_0) + o(x - x_0)$$ if and only if it is differentiable at the point.

Definition If a function $f:E \to \mathbb{R}$ is defined on a set $E \subset \mathbb{R}$ and differentiable at a point $x \in E$, the line defined by $$y - f'(x_0) = f'(x_0)(x - x_0)$$ is called the tangent to the graph of this function at the point $(x_0, f(x_0))$.

Some Examples

Example 1 Let $f(x) = \sin x$. We shall show that $f'(x) = \cos x$.

Example 2 We shall show that $\cos'(x) = - \sin x$.

Example 3 If $f(t) = r\sin \omega t$, then $f'(t) = r\omega \cos \omega t.$ If $f(t) = r\cos \omega t$, then $f'(t) = -r\omega \sin \omega t.$

Example 4 The instantaneous velocity and instantaneous acceleration of a point mass. Suppose a point mass is moving in a plane and that in some given coordinate system its motion is described by differentiable function of time $$x = x(t), y=y(t)$$ In particular, this motion is written as in the form $$r(t) = \left(r\cos (\omega t + \alpha), r\sin (\omega t+ \alpha)\right)$$ **Example 5** The optic property of a parabolic mirror. Let us consider the parabola $y = \frac{1}{2p}x^2(p>0)$, and construct the tangent to it at the point $(x_0, y_0) = (x_0, \frac{1}{2p}x_0^2)$.

Example 6 $$f(x) = \left\{\begin{array}{cc} x^2\sin \frac{1}{x}, &\text{ if } x \ne 0 \newline 0, &\text{ if } x = 0. \end{array} \right.$$

Example 7 We shall show that $$e^{x+h} - e^x = e^xh + \circ(h) \quad as \quad h \to 0$$. **Example 8** If $a > 0$, then $a^{x+h} - a^x = a^h(\ln a)h + \circ(h)$ as $h \to 0.$

Homework

1. 設$\displaystyle f(x) = \left\{\begin{array}{cc}x^2, & x \ge 3 \newline ax+b, & x < 3 \end{array}\right.$試確定$a,b$的值，使$f$在$x=3$處可導。
2. 求下列曲線在指定點處的切線，法線方程。（1） $\displaystyle y = \frac{x^2}{4}, P(2,1)$ (2) $\displaystyle y = \cos x, P(0,1)$
3. 求下列函數的導數：(1)$\displaystyle f(x) = |x|^3$, (2) $\displaystyle f(x) = \left\{\begin{array}{cc} x+1, & x \ge 0 \newline 1, & x < 0 \end{array}\right.$
4. 設函數$$f(x) = \left\{\begin{array}{cc} x^{\alpha}\sin \frac{1}{x}, & x \ne 0 \newline 0, & x = 0 \end{array}\right.$$試問：（1）α為何值時，函數在$x=0$點連續；（2）α為何值時，函數在$x=0$點可導.

2. The Basic Rules of Differentiation

Differentiation and the Arithmetic Operations

Theorem 1 If function $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are differentiable at a point $x \in X$, then (1) their sum is differentiable at x, and $$(f+g)'(x) = \left(f' + g'\right)(x)$$ (2) their product is differentiable at x, and $$\left(f\cdot g\right)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$ (3) their quotient is differentiable at x if $g(x) \ne 0$, and $$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Corollary 1.1 The derivative of a linear combination of differentiable functions equals the same linear combination of the derivatives of these functions.

Corollary 1.2 If the functions $f_1, \cdots, f_n$ are differentiable at x, then $$\left(f_1 f_2 \cdots f_n\right)' (x) = f_1'f_2 \cdots f_n + f_1f_2' \cdots f_n + \cdots + f_1f_2 \cdots f_n' $$

Corollary 1.3 It follows from the relation between the derivative and the differential that we have: $$ d(f+g)(x) = df(x) + dg(x)$$ $$ d(f\cdot g)(x) = g(x)df(x) + f(x)dg(x)$$ $$d\left(\frac{f}{g}\right)(x) = \frac{g(x)df(x) -f(x)dg(x)}{g^2(x)}$$

Example 1 Find the derivative of $\displaystyle \tan x$ and $\cot x$.

Differentiation of a Composite Function (chain rule)

Theorem 2 If the function: $f: X \to Y \subset \mathbb{R}$ is differentiable at a point $x \in X$ and the function $g: Y \to \mathbb{R}$ is differentiable at the point $y = f(x) \in Y$, then the composite function $g \circ f: X \to \mathbb{R}$ is differentiable at x, and the differential $\rm{d} (g \circ f)(x): T\mathbb{R} \to T\mathbb{R}g(f(x))$ of their composition equals the composition $\rm{d}g(y) \circ \rm{d}f(x)$ of their differentials.

Corollary 2.1 The derivative $(g \circ f)'(x)$ of the composition of differentiable real-valued functions equals the product $g'(f(x)) \cdot f'(x)$ of the derivatives of these functions computed at the corresponding points. \begin{equation} \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}{\tag 4} \end{equation}

Example 2 Let us show that for $\alpha \in \mathbb{R}$ we have $\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}x} = \alpha x^{\alpha -1}$ in the domain $x > 0$, that is, $\mathrm{d} x^{\alpha} = \alpha x^{\alpha - 1}\mathrm{d}x$

Example 3 The derivative of the logarithm of the absolute value of a differentiable function is often called its logarithmic derivative. \begin{equation} \mathrm{d}\left(\ln |f|\right)(x) = \frac{f'(x)}{f(x)} \mathrm{d}x = \frac{\mathrm{d}f(x)}{f(x)}.{\tag 5} \end{equation}

Example 4 The absolute and relative errors in the value of a differentiable function caused by errors in the data for the argument. \begin{equation} f(x+h) - f(x) = f'(x)h + \alpha(x;h){\tag 6} \end{equation} \begin{equation} \frac{|f'(x)h|}{|f(x)|} = \frac{|\mathrm{d}f(x)h|}{|f(x)|}{\tag 7} \end{equation}

Example 5 Find the derivative of the following function:

1. $\displaystyle y = \sin x^2$
2. $\displaystyle y = \sqrt{x^2 + 1}$
3. $\displaystyle y = \ln (x + \sqrt{x^2 + 1})$
4. $\displaystyle y = \tan^2 \frac{1}{x}$
5. $\displaystyle y = u(x)^{v(x)}$

Differentiation of an Inverse Function

Theorem 3 Let the function $f: X \to Y$ and $f^{-1}: Y \to X$ be mutually inverse and continuous at points x_0 and $f(x_0) = y_0 \in Y$ respectively. If f is differentiable at x_0 and $f'(x_0) \ne 0$, then f^{-1} is also differentiable at the point y_0, and $$\left(f^{-1}\right)'(y_0)=\left(f'(x_0)\right)^{-1} $$

Remark

> (1)If we knew in advance that the function f^{-1} was differentiable at y_0, we would find immediately by the identity $\left(f^{-1} \circ f\right)(x) = x$ and the theorem on differentiation of a composite function that $\left(f^{-1}\right)'(y_0)\cdot f'(x_0) = 1.$

> (2)The condition $f'(x_0) \ne 0$ is obviously equivalent to the statement that the mapping $h \to f'(x_0)h$ realized by the differential $\mathrm{d}f(x_0):T\mathbb{R}(x_0) \to T\mathbb{R}(y_0)$ is invertible mapping $\left[\mathrm{d}f(x_0)\right]^{-1}: T\mathbb{R}(y_0)\to T\mathbb{R}(x_0)$ given by the formula $ \tau \to \left(f'(x_0)\right)^{-1}\tau$.

Example 5 We shall show that $\arcsin'y = \frac{1}{1-y^2}$ for $|y| < 1$. **Example 6** $\displaystyle \mathrm{arccot}'y = -\frac{1}{1+y^2}, \arctan'y = \frac{1}{1+y^2}$ **Example 7** The hyperbolic and inverse hyperbolic functions and their derivatives. The function

$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ \cosh x &= \frac{1}{2}(e^x - e^{-x})\end{aligned}$ *$$arerespectivelythehyperbolicsineandhyperboliccosineof$x$.Thesefunctions,whichforthetimebeinghavebeenintroducedpurelyformally,arisejustasnaturallyinmanyproblemsasthecircularf*

```
\begin{split}
    \sinh (-x) = -\sinh x \\
    \cosh (-x) =  \cosh x
\end{split}
```

$$ Moreover, the following basic identity is obvious

$$\cosh^2 x - \sinh^2 x = 1$$

The graphs of the functions $y = \sinh x$ and $y = \cosh x$ are shown in `![](./Figs/hyps.png)` The inverse of the hyperbolic sine is $

```
x = \ln (y + \sqrt{1+y^2})
```

Thus,

```
\sinh^{-1} y = \ln(y + \sqrt{1+y^2})
```

Similarly, usingthemonotonicityofthefunction$y = \cosh x$onitsdefinition, wehave

```
\begin{split}
    \cosh^{-1}_-(y) = \ln \left(y - \sqrt{y^2 - 1}\right)\newline
    \cosh^{-1}_+(y) = \ln \left(y + \sqrt{y^2 - 1}\right)
\end{split}
```

Fromthedefinitionsgivenabove, wefind

```
\begin{split}
    \sinh'x = \cosh x,\\
    \cosh'x = \sinh x,
\end{split}
```

$$ and by the theorem on the derivative of an inverse function, we find

$$\begin{aligned}\left(\sinh^{-1} y\right)' &= \frac{1}{\sinh' x} = \frac{1}{\cosh' x} = \frac{1}{\sqrt{1+y^2}} \\ \left(\cosh_-^{-1} y\right)' = \frac{1}{\cosh' x} &= \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = -\frac{1}{\sqrt{y^2-1}}, y > 1 \\ \left(\cosh_+^{-1} y\right)' &= \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = -\frac{1}{\sqrt{y^2-1}}, y > 1\end{aligned}$$

Like $\tan x$ and $\cot x$ one can consider the functions

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x}, \\ \coth x &= \frac{\cosh x}{\sinh x}\end{aligned}$$

called the hyperbolic tangent and hyperbolic cotangent respectively, and also the functions inverse to them, the area tangent $

```
\begin{split}
\tanh^{-1} y = \frac{1}{2}\ln\frac{1+y}{1-y}, |y| < 1,\newline
\coth^{-1} y = \frac{1}{2}\ln\frac{y+1}{y-1}, |y| > 1,
\end{split}
```

Bytherulesfordifferentiationwehave

```
\begin{split}
    \tanh'x = \frac{1}{\cosh^2x}\newline
    \coth'x = -\frac{1}{\sinh x}
\end{split}
```

You can't use 'macro parameter character #' in math mode

f''(x), \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}

andifwewishtoindicateexplicitlythevariableofdifferentiationinthefirstcase, wealsowrite, forexample$ $f''_{xx}(x)$. $*Thefollowingnotati*

f^{(n)}(x), \frac{\mathrm{d}^nf(x)}{\mathrm{d}x^n}.

Alsobyconvention, $ $f^{(0)}(x) = f(x)$. $ ∗ ∗*Example*(*Leibniz*′*sformular*) ∗ ∗*Let*$ $u(x)$ $*and*$ $v(x)$ $*befunctionshavingderivativesuptoorder*

P_n(x) = P_n^{(0)}(0) + \frac{1}{1!}P_n^{(1)}(0)x + \frac{1}{2!}P_n^{(2)}(0)x^2 + \cdots + \frac{1}{n!}P_n^{(n)}(0)x^n

You can't use 'macro parameter character #' in math mode

f(x) = \left\{

$$\begin{matrix} x^2 & \text{for} & -1 \le x < 2 \\ 4 & \text{for} & 2 \le x \end{matrix}$$

\right.

You can't use 'macro parameter character #' in math mode

\dot{{a}}(t) = (\cos t, \sin t)

and

\vert {v} \vert = 1

Theparticleisatthesamepint$ $r(0) = r(2\pi)$ $*, theequalitymeansthat*

{r}(2\pi) - {r}(0) = {v}(\xi)(2\pi - 0)

wouldmean$ $t = 2\pi$ $*. Butthisisimpossible. Evenso, weshalllearnthatthereisstillarelationbetweenthedisplacementoveratimeintervalandvelocity. Itconsistsofthe*

x'(\tau)(y(b) - y(a)) = y'(\tau)(x(b) - x(a))

Ifinaddition$ $x'(t) \neq 0$ $*foreach*$ $t \in]a, b[$ $*, then*$ $x(a) \neq x(b)$ $*andwehavetheequality*

\frac{y(b) - y(a)}{x(b) - x(a)} = \frac{y'(\tau)}{x'(\tau)}

You can't use 'macro parameter character #' in math mode

```
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) +
\cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x_0;x)
```

−[] ∗ ∗*Theorem* ∗ ∗*Ifthefunction*$ f $*iscontinuousontheclosedintervalwithendpoints*$ x_0 $*and*$ x $*alongwithitsfirst*$ n $*derivatives, andithasaderivativeoforder*$ $n+1$ $*attheinteriorpointsofthisinterval, thenforany*

```
r_n(x_0;x) = \frac{1}{n!}f^{(n+1)}(\xi)(x-\xi)^{n+1}(x - x_0)
```

Aparticularelegantformularesultsifweset$ $\phi(t) = (x-t)^{n+1}$ $. > ∗ ∗*Corollary*(*Lagrange*′*sformfortheremainderterm*) ∗ ∗

```
r_n(x_0; x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x - x_0)^{n+1}.
```

Letusconsidersomeexamples. ∗ ∗*Example* ∗ ∗*Forthefunction*$ $f(x) = e^x$ $*with*$ $x_0 = 0$ $*Taylor*′*sformulahastheform*

```
e^x = 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + r_n(0;x)
```

andtheremainderis

```
r_n(0;x) = \frac{1}{(n+1)!}e^{\xi}x^{n+1}
```

where$ $|\xi| < |x|$ $*. Thus*

```
\vert r_n(0;x)\vert  = \frac{1}{(n+1)!}e^{\xi}\vert x \vert^{n+1} < \frac{\vert x \vert^{n+1}}
{(n+1)!}e^{\vert x \vert}.
```

But for each fixed $x \in \mathbb{R}$, if $n \to \infty$, the quantity $\frac{|x|^{n+1}}{(n+1)!}$, as we know, tends to zero. Hence

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots, \forall x \in \mathbb{R}$$

Example The function $a^x, 0 < a, a \neq 1$, similarly:

$$a^x = 1 + \frac{\ln a}{1!}x + \frac{\ln^2a}{2!}x^2 + \cdots + \frac{\ln^na}{n!}x^n + \cdots$$

Example

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots$$

Example

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots, \forall x \in \mathbb{R}$$

Example

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1}+ \cdots, \forall x \in \mathbb{R}$$

Example

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{1}{(2n)!}x^{2n}+ \cdots, \forall x \in \mathbb{R}$$

Example For the function $f(x) = \ln(1+x)$, we have

$$\ln (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{(n-1)}}{n}x^n + r_n(0;x).$$

where

$$r_n(0;x) = \frac{1}{n!}\frac{(-1)^nn!}{(1+\xi)^{n+1}}(x - \xi)^nx,$$

or

$$r_n(0;x) = (-1)^nx\frac{(x - \xi)^n}{(1+\xi)^n}\frac{1}{(1+\xi)}$$

where ξ lies between 0 and x. If $|x| < 1$, it follows from the condition that ξ lies between 0 and x that

$$\begin{aligned}\frac{\vert x - \xi \vert}{\vert 1 + \xi \vert} &= \frac{\vert x \vert - \vert \xi \vert}{1 - \vert \xi \vert} \\ &= 1 - \frac{1 - \vert x \vert}{1 - \vert \xi \vert} \le 1 - \frac{1 - \vert x \vert}{1 - \vert 0 \vert} \\ &= \vert x \vert.\end{aligned}$$

Thus for $|x| < 1$

$$\vert r_n(0;x) \vert \le \vert x \vert^{n+1},$$

and consequently the following expansion is valid for $|x| < 1$:

$$\ln (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n}x^n + \cdots$$

$$**Example** For the function \$(1+x)^{\alpha}\$, where \$\alpha \in \mathbb{R}\$, we have \$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}\$, so that the Talylor's formula at \$x_0 = 0\$ for this function has the form (1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha}{2 ...$$

$$+ \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + r_n(0;x) \text{ Using Cauchy's remainder, we find}$$

```
r_n(0;x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}(1+\xi)^{\alpha - n -1}
(x - \xi)^nx,
```

$$ where ξ lies between x.

If $\vert x \vert < 1$,then, using the estimates, we have \begin{equation} \vert r_n(0;x) \vert \le \left\vert \alpha \left(1-\frac{\alpha}{1}\right) \cdots \left(1-\frac{\alpha}{n}\right)\right\vert \left(1+\xi\right)^{\alpha-1} \vert x \vert^{n+1}. \end{equation} When n is increased by 1, the right side of Eq.~\ref{eq:eq3} is multiplied by $\displaystyle \left\vert \left(1 - \frac{\alpha}{n+1}\right) x\right\vert$. But since $\vert x \vert < 1$, we shall have $\left\vert\left(1 - \frac{\alpha}{n+1}\right)x\right\vert < q < 1$, independently of the value α, provided $\vert x \vert < q < 1$ and n is sufficiently large.

It follows from this that $r_m(0;x) \to 0$ as $n \to \infty$ for $\alpha \in \mathbb{R}$ and any x in the open interval $\vert x \vert < 1$: \begin{equation} (1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots \end{equation} In this case $f(x) = (1+x)^n$, we write the follwoing equality:

$$(1+x)^n = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

> **Definition** If the function $f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at a point x_0, the series

$$f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \cdots \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + \cdots$$
is called the Taylor Series of f at the point x_0. It should not be thought that the Taylor series of an infinitely differentiable function converges in some neighborhood of x_0, for given any sequence $c_0, c_1, \cdots, c_n, \cdots$ of numbers, one can construct (although this is not simple to do) a function $f(x)$ such that $f^{(n)}(x_0) = c_n, n \in \mathbb{N}$. It should also not be thought that if the Taylor series converges, it necessarily converges to the function that generated it. A Taylor series converges to the function that generated it only when the generating function belongs to the class of so-called **analytic function.** Here is Cauchy's example of a non-analytic function:
$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases}$$
In conclusion, we discuss a local version of Taylor's formula. We wish to choose a polynomial $P_n(x) = x_0 + c_1(x-x_0) + \cdots + c_n(x-x_0)^n$ so as to have $

```
f(x) = P_n(x) + \circ((x - x_0)^n)
```

$$> **Proposition** If there exists a polynomial \$P_n(x) = c_0 + c_1(x-x_0) + \cdots + c_n(x-x_0)^n\$ satisfying f(x) = P_n(x) + \circ((x-x_0)^n), that polynomial is unique. ** Proof: ** Indeed, form Eq. ??? we see that$$

c_0 = \lim*{x \to x_0}f(x), \ c_1 = \lim*{x \to x_0}\frac{f(x) - c_0}{x - x_0},\ \vdots \ c_n = \lim*{x \to x_0}\frac{f(x) - c_0 - c_1(x - x_0) - \cdots - c*{n-1}(x - x_0)^{n-1}}{(x-x_0)^n}

$$> **Proposition** Let \$E\$ be a closed interval having \$x_0 \in \mathbb{R}\$ as an endpoint. If the function \$f : E \to \mathbb{R}\$ has derivatives \$f'(x_0), f''(x_0), \cdots, f^{(n-1)}(x_0)\$ up to order \$n\$ inclusive at the point \$x_0\$, then the following represe...$$

```
\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(n-1)}(x_0) = 0
```

$$then$$

```
\phi(x) = \circ\left((x - x_0)^n\right)
```

$$ as $x \to x_0$

Let us summaries our results. We have defined the Taylor polynomial

$$P_n(x_0;x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

written the Taylor formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x_0;x)$$

and obtained the following important specific form of it:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

where ξ is a point between x_0 and x.

If f has derivatives of orders up to $n \ge 1$ inclusive at the point x_0, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \circ\left((x-x_0)^n\right)$$

In particular, we can now write the following table of asympototic formulas as $x \to 0$

$$\begin{aligned} e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + O(x^{n+1}) \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + O(x^{2n+2}) \\ \sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + O(x^{2n+3}) \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n}x^n + O(x^{n+1}) \end{aligned}$$

Example Show that $\tan x = x + \frac{1}{3}x^3 + \circ(x^3)$ as $x \to 0$

Example Show that $\displaystyle \ln\cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \bigcirc(x^8)$ as $x \to 0$

Example Let us find the values of the first six derivatives of the function $\ln \cos x$ at $x = 0$.

Example Let $f(x)$ be an infinitely differentiable function at the point x_0, and suppose we know the expansion

$$f'(x) = c'_0 + c'_1 x + \cdots + c'_n x^n + \bigcirc(x^{n+1})$$

of its derivatives in a neighborhood of zero. Then, from the uniqueness of the Taylor expansion we have $

```
    \left(f'(x)\right)^{(k)} (0) = k!c'_k,
```

$$ and so $\displaystyle f^{(k+1)}(0) = k!c'_k$. Thus for the function $f(x)$ itself we have the expansion

$$f(x) = f(0) + \frac{c'_0}{1!}x + \frac{1!c'_1}{2!}x^2 + \cdots + \frac{n!c'_n}{(n+1)!}x^{n+1} + \bigcirc(x^{n+2}).$$

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or, after simplication,
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$$f(x) = f(0) + \frac{c'_0}{1}x + \frac{c'_1}{2}x^2 + \cdots + \frac{c'_n}{(n+1)}x^{n+1} + \bigcirc(x^{n+2}).$$

Example Let us find the Taylor expansion of the function $f(x) = \tan^{-1} x$ at 0.

Example Let us find the Taylor expansion of the function $f(x) = \sin^{-1}x$ at 0.

Example Find the limit $\displaystyle \lim_{x \to 0} \frac{\tan^{-1}x - \sin x}{\tan x - \sin^{-1}x}$. **Example** Let f be a function that is differentiable n times on an interval I. Prove the following statements. (1). If f vanishes at $n+1$ points of I, there exists a point $\xi \in I$ such that $f^{(n)}(\xi) = 0$. (2). If $x_1, x_2, \cdots x_n$ are points of the interval I, there exist a unique polynomial $L(x)$(\textbf{the Lagrange interpolation polynomial}) of degree at most $n-1$ such that $f(x_i) = L(x_i), i=1,2,\cdots,n$. In addition, for $x \in I$ there exist a point $\xi \in I$ such that

$$f(x) - L(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!}f^{(n)}(\xi).$$

(3). If $x_1 < x_2 < \cdots < x_p$ are points of I and $n_i, 1\le i \le p$, are natural numbers such that $n_1 + n_2 + \cdots + n_p = n$ and $f^{(k)}(x_i) = 0, 0 \le k \le n_i-1$, then there exists a point ξ in the closed interval $[x_1, x_p]$ at which $f^{(n-1)}(\xi) = 0$. (4). There exists a unique polynomial $H(x)$ (the Hermite interpolating polynomial) of degree $n-1$ such that $f^{(k)}(x_i) = H^{(k)}(x_i), 0 \le k \le n_i-1$. Moreover, inside the smallest interval containing the points x and $x_i, i=1,2,\cdots,p$, there is a point ξ such that

$$f(x) = H(x) + \frac{(x-x_1)^{n1}\cdots(x-x_p)^{n_p}}{n!}f^{(n)}(\xi).$$

This formula is called the \textbf{Hermite interpolation formula}. The points $x_1, x_2, \cdots x_p$, are called the interpolation nodes of multiplicity $n_i$respectively. Special cases of the Hermite interpolation formula are the Lagrange interpolation formula.

4. The Study of Functions Using the Methods of Differential Calculus

Conditions for a Function to be Monotonic

> **Proposition** The following relations hold between the monotonicity properties of a function $f: E \to \mathbb{R}$ that is differentiable on an open interval $]a,b[= E$ and the sign (positivity) of its derivative f' on that interval: [\begin{array}{lcl} f'(x) > 0 \Rightarrow & f \text{ is increasing } & \Rightarrow f'(x) \ge 0 \ f'(x) \ge 0 \Rightarrow & f \text{ is non-decreasing } & \Rightarrow f'(x) \ge 0 \ f'(x) \equiv 0 \Rightarrow & f \equiv \text{ const } & \Rightarrow f'(x) \equiv 0 \ f'(x) \le 0 \Rightarrow & f \text{ is non-increasing } & \Rightarrow f'(x) \le 0 \ f'(x) < 0 \Rightarrow & f \text{ is decreasing } & \Rightarrow f'(x) \le 0 \end{array}] \end{proposition} \begin{example} \normalfont Let $f(x) = x^3 - 3x + 2$ on $\mathbb{R}$. Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, and we can say that the function is increasing on the open interval $]-\infty, -1[$, decreasing on $]-1,1[$, and increasing again on $]1, +\infty[$. \end{example}

\subsection{Conditions for an Interior Extremum of a Function} \begin{proposition} \normalfont In order for a point x_0 to be an extremum of a funtion $f: U(x_0) \to \mathbb{R}$ defined on a neighborhood $U(x_0)$ of that point, a necessary condition is that one of the following two conditions hold: either the function is not differentiable at x_0 or $f'(x_0) = 0$. \end{proposition}

Simple examples show that these necessary conditions are not sufficient. \begin{example} \normalfont Let $f(x) = x^3$ on $\mathbb{R}$. Then $f'(0) = 0$, but there is no extremum at x_0. \end{example}

\begin{example} \normalfont Let [f(x) = \left{\begin{array}{cl} x & \text{ for } x > 0\ 2x & \text{ for } x < 0 \end{array}\right.] \end{example}

\begin{proposition}{\rm (Sufficient conditions for an extremum in terms of the first derivative).} \normalfont Let $f: U(x_0) \to \mathbb{R}$ be a function defined on a neighborhood $U(x_0)$ of the point x_0, which is continuous at the point itself and differentiable in a deleted neighborhood $U(x_0)\setminus x_0$. Let $\mathring{U}^-(x_0) = \left\{x \in U(x_0)\vert x < x_0\right\}$ and $\mathring{U}^+(x_0) = \left\{x \in U(x_0) \vert x > x_0\right\}$.

```
Then the following conclusions are valid:
\begin{enumerate}[label=(\alph*)]
    \item $\forall x \in \mathring{U}^-(x_0),f'(x) < 0 \wedge \forall x \in \mathring{U}^+(x_0),
        f'(x) < 0 \Rightarrow f \text{ has no extremum at } x_0 $
    \item $\forall x \in \mathring{U}^-(x_0),f'(x) < 0 \wedge \forall x \in \mathring{U}^+(x_0),
        f'(x) > 0 \Rightarrow x_0 \text{ is a strict local minimum }  $
    \item $\forall x \in \mathring{U}^-(x_0),f'(x) > 0 \wedge \forall x \in \mathring{U}^+(x_0),
        f'(x) < 0 \Rightarrow x_0 \text{ is a strict local maximum }$
    \item $\forall x \in \mathring{U}^-(x_0),f'(x) > 0 \wedge \forall x \in \mathring{U}^+(x_0),
        f'(x) > 0 \Rightarrow f \text{ has no extremum at } x_0 $
\end{enumerate}
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\end{proposition}

Briefly, but less precisely, one can say that if the derivative changes sign in passing through the point, then the point is an extremum, while if the derivative does not change sign, the point is not an extremum.

We remark immediately, however, that these sufficient conditions are not necessary for an extremum, as one can verify using the following example:

\begin{example} \normalfont [\displaystyle f(x) = \left{\begin{array}{cl} 2x^2 + x^2\sin\frac{1}{x} & \text{ for } x \ne 0 \ 0 & \text{ for } x =0 \end{array} \right.] \label{ex:ex1} \end{example}

\graphicspath{ {./fun1/} } \begin{figure}[htbp] \centering \includegraphics[width=0.6\textwidth]{fun1-crop.pdf} \label{fig:fig1} \caption{figure of Example~\ref{ex:ex1}} \end{figure}

Since $x^2 \le f(x) \le 2x^2$, it is clear that the function has a strict local minimum at $x_0 = 0$.

\begin{proposition}{\rm (Sufficient conditions for an extremum in terms of higher-order derivatives)} \normalfont Suppose a function $f: U(x_0) \to \mathbb{R}$ defined on a neighborhood $U(x_0)$ of x_0 has derivatives of order up to n inclusive at $x_0, (n \ge 1)$.

```
If $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \ne 0$, then there is
no extrmum at $x_0$ if $n$ is odd. If $n$ is even, the point $x_0$ is a local extremum,
in fact a strict local minimum if $f^{(n)}>0$ and a strict local maximum if
$f^{(n)}<0$.
```

\end{proposition}

\begin{example} \normalfont The law of refraction in geometric optics(Snell's law). According to Fermat's principle, the actual trajectory of a light ray between two points is such that the ray requires minimum time to pass from one point to the other compared with all paths joining the two points. \end{example}

\begin{example} \normalfont We shall show that for $x > 0$ \begin{equation} x^{\alpha} - \alpha x + \alpha -1 \le 0, \text{ when } 0 < \alpha < 1, \end{equation} \begin{equation} x^{\alpha} - \alpha x + \alpha -1 \ge 0, \text{ when } 0 > \alpha \text{ or } \alpha > 1. \end{equation} \end{example}

\begin{example}{\rm \textbf{Young's inequality}} \normalfont If $a > 0$ and $b > 0$, and the number p and q such that $p \ne 0,1$ and $q \ne 0,1$, and $\displaystyle \frac{1}{p} + \frac{1}{q} = 1$, then \begin{equation} a^{\frac{1}{p}}b^{\frac{1}{q}} \le \frac{1}{p}a + \frac{1}{q}b, \text{ if } p > 1 \label{eq:eq6} \end{equation} \begin{equation} a^{\frac{1}{p}}b^{\frac{1}{q}} \ge \frac{1}{p}a + \frac{1}{q}b, \text{ if } p < 1 \label{eq:eq7} \end{equation} and equality holds in formula~\ref{eq:eq6} and ~\ref{eq:eq7} only when $a = b$. \end{example} \begin{proof} It suffices to set $x = \frac{a}{b}$ and $\alpha = \frac{1}{p}$. \end{proof}

\begin{example}{\rm \textbf{Holder's inequality}} \normalfont Let $x_i,i=1,2,\cdots n, y_i \ge 0, i=1,2,\cdots,n$ and $\frac{1}{p} + \frac{1}{q} = 1$.

```
Then
\begin{equation}
    \sum_{i=1}^{n}x_iy_i \le \left(\sum_{i=1}^{n}x_i^p\right)^{1/p}
    \left(\sum_{i=1}^ny_i^q\right)^{1/q} \text{ for } p > 1
\end{equation}
and
\begin{equation}
    \sum_{i=1}^{n}x_iy_i \ge \left(\sum_{i=1}^{n}x_i^p\right)^{1/p}
    \left(\sum_{i=1}^ny_i^q\right)^{1/q} \text{ for } p < 1, p \ne 0
\end{equation}
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\end{example}

\begin{example}{\rm \textbf{Minkowski's inequality}} \normalfont Let $x_i\ge 0, y_i \ge 0, i=1,2,\cdots,n$. Then \begin{equation} \left(\sum{i=1}^n\left(x_i+y_i\right)^p\right)^{1/p} \le \left(\sum{i=1}^nx_i^p\right)^{1/p}+ \left(\sum{i=1}^ny_i^p\right)^{1/p} , \text{ when } p > 1, \end{equation} \begin{equation} \left(\sum{i=1}^n\left(x_i+y_i\right)^p\right)^{1/p} \ge \left(\sum{i=1}^nx_i^p\right)^{1/p}+ \left(\sum{i=1}^ny_i^p\right)^{1/p} , \text{ when } p < 1, \end{equation} \end{example}

\section{Conditions for a function to be Convex} \begin{definition} \normalfont A function $f: [a,b] \to \mathbb{R}$ defined on an open interval $]a,b[\subset \mathbb{R}$ is convex if the inequality holds \begin{equation} f(\alpha_1x_1 + \alpha_2x_2) \le \alpha_1f(x_1) + \alpha_2f(x_2) \label{eq:eq8} \end{equation} holds for any points $x_1, x_2 \in]a,b[$ and any numbers $\alpha_1\ge 0, \alpha_2 \ge 0$ such that $\alpha_1 + \alpha_2 = 1.$ If this inequality is strict whenever $x_1 \ne x_2$ and $\alpha_1\alpha_2 \ne 0$ on $]a,b[$. \end{definition}

\graphicspath{ {./Figs/} } \begin{figure}[htbp] \centering \includegraphics[width=0.5\textwidth]{convex.png} \caption{Convex function} \label{fig:fig2} \end{figure}

\begin{definition} \normalfont If the opposite inequality holds for a function $f:]a,b[\to \mathbb{R}$, that function is said to be concave on the interval $]a,b[$, or, more often, convex upward in the interval, as opposed to a convex function, which is then said to be convex downward on $]a,b[.$ \end{definition}

In the relations $x = \alpha_1 x_1 + \alpha_2 x_2, \alpha_1 + \alpha_2 = 1,$ we have [\alpha_1 = \frac{x_2 - x}{x_2 - x_1}, \alpha_2 = \frac{x - x_1}{x_2 - x_1}] so that formula~\ref{eq:eq8} can be rewritten as [f(x) \le \frac{x_2 - x}{x_2 - x_1}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2)] Taking account the inequalities $x_1 < x < x_2$ and $x_1 < x_2$, we then obtain [\left(x_2 - x\right)f(x_1) + \left(x_1 - x_2\right)f(x) + \left(x - x_1\right)f(x_2) \ge 0] Remarking that $x_2 - x_1 = (x_2 - x) + (x - x_1)$ we obtain from the last inequality, after elementary transformations \begin{equation} \frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \label{eq:eq9} \end{equation} for $x_1 < x < x_2$.

Inequality~\ref{eq:eq9} is another way of writing the definition of convexity of the function $f(x)$ on an open interval $]a,b[$. Geometrically, \ref{eq:eq9} means (see Figure~\ref{fig:fig2})that the slope of the chord I joining $x_1, f(x_1)$ to $x, f(x)$ is not larger than (and in the case of strict convexity is less than) the slope of the chord II joining $x, f(x)$ to $x_2, f(x_2)$.

Now let us assume that the function $f:]a,b[\to \mathbb{R}$ is differentiable on $]a,b[$. Then, letting x in Eq.~\ref{eq:eq9} tend first to x_1, the tend to x_2, we obtain [f'(x_1) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le f(x_2)] which establishes that the derivative of f is monotonic.

Taking this fact into account, for a strictly convex function we find, using Lagrange's theorem, that [f'(x_1) \le f'(\xi_1) = \frac{f(x) - f(x_1)}{x - x_1} < \frac{x_2 - x}{x_2 - x} = f'(\xi_2) \le f'(x_2)] for $x_1 < \xi_1 < x < \xi_2 < x_2$, that is, strict convexity implies that the derivative is strictly monotonic.

Thus, if a differentiable function f is convex on an open interval $]a,b[$, then f' is nondecreasing on $]a,b[$, and in the case when f is strictly convex, its derivative f' is increasing on $]a,b[$.

These conditions turns out to be not only necessary, but also sufficient for convexity of a differentiable function.

\begin{proposition} \normalfont A necessary and sufficient condition for a function $f:]a,b[\to \mathbb{R}$ that is differentiable on the open interval $]a,b[$ to be convex (downward) on that interval is that its derivative f' be non-decreasing on $]a,b[$. A strictly increasing f' corresponds to a strictly convex function. \end{proposition}

\begin{corollary} \normalfont A necessary and sufficient condition for a function $f:]a,b[\to \mathbb{R}$ that having a second derivative on the open interval $]a,b[$ to be convex on $]a,b[$ is that $f''(x) \ge 0$ on that interval. The condition $f''(x) > 0$ on $]a, b[$ is sufficient to guarantee that f is strictly convex. \end{corollary}

\begin{example} \normalfont Let us examine the convex of the following functions: [x^{\alpha}, a^x, \log_ax, \sin x] \end{example}

\begin{proposition} \normalfont A function $f:]a,b[\to \mathbb{R} $ that is differentiable on the open interval $]a,b[$ is convex(downward) on $]a,b[$ if and only if its graph contains no points below any tangent drawn to it. In that case, a necessary and sufficient condition for strict convexity is that all points of the graph except the point of tangency lie strictly above the tangent line. \end{proposition}

\begin{example} \normalfont Using the proposition to prove that [e^x \ge 1 + x] and this inequality is strict for $x \ne 0.$ Similarly, using the convexity of $\ln x$, one can verify that [\ln x \le x - 1] holds for $x > 0$, the inequality being strict for $x \ne 1$. \end{example}

\begin{definition} \normalfont Let $f: U(x_0) \to \mathbb{R}$ be a function defined and differentiable on a neighborhood $U(x_0)$ of $x_0 \in \mathbb{R}$. If the function is convex downward (resp. upward) on the set $\mathring{U}^-(x_0) = \left\{x \in U(x_0) \vert x < x_0\right\}$ and convex upward (resp. downward) on $\mathring{U}^+(x_0) = \left\{x\in U(x_0) \vert x > x_0\right\} $, then $\left(x_0, f(x_0)\right)$ is called an \textbf{infection point}. \end{definition}

An analytic criterion for the abscissa x_0 of a point of infection is easy to surmise. If $f(x)$ is twice differentiable at x_0, then since $f'(x)$ has either a maximum or minimum at x_0, we must have $f''(x_0) = 0$.

If the second derivative $f''(x)$ is defined on $U(x_0)$ and one has one sign everywhere on $\mathring{U}^-(x_0)$ and the opposite sign on $\mathring{U}^+(x_0)$,so that the point $\left(x_0,f(x_0)\right)$ is a point of inflection.

\begin{example} When consider the function $f(x) = \sin x$, we shall show that the abscissas $x = \pi k, k \in \mathbb{Z}$ are points of inflection. \end{example}

\begin{example} \normalfont It should not be thought that the passing of a curve from one side of its tangent line to the other at a point is a sufficient condition for the point to be a inflection point. It may, after all, happen that the curve does not have any constant convexity on either a left or a right neighborhood of the point. A example (see Fig.~\ref{fig:fig3}) is [f(x) = \left{\begin{array}{cl} 2x^3 + x^2\sin \frac{1}{x^2} & \text{ for } x \ne 0, \ 0 & \text{ for } x = 0 \end{array} \right.] \label{ex:ex2} \end{example}

\begin{proposition}{\rm \textbf{Jensen's Inequality}} \normalfont If $f:]a,b[\to \mathbb{R}$ is a convex function, $x_1, x_2, \cdots, x_n$ are points of $]a,b[$, and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then [f(\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n) \le \alpha_1f(x_1) + \alpha_2f(x_2) + \cdots + \alpha_nf(x_n)] \end{proposition} \graphicspath{ {./fun2/} } \begin{figure}[htbp] \centering \includegraphics[height=0.4\textwidth]{fun2.pdf} \caption{Figure of example~\ref{ex:ex2}} \label{fig:fig3} \end{figure}

\begin{example} \normalfont The function $f(x) = \ln x$ is strictly convex upward on the set of positive numbers. ans so we have [\alpha_1\ln x_1 + \alpha_2\ln x_2 + \cdots + \alpha_n\ln x_n \le \ln\left(\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n\right)] or [x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \le \alpha_1x_1 + \alpha_2x_2 + \alpha_nx_n] for $x_i \ge 0, i=1,2,\cdots,n, \sum_{i=1}^n\alpha_i = 1$. In partiular, if $\alpha_i = \frac{1}{n}, i=1,2,\cdots,n$, we obtain the classical inequality [\sqrt[n]{x_1x_2\cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}] \end{example}

\begin{example} \normalfont Let $f(x) = x^p, x \ge 0, p>1$. Since such a funtion is convex, we have [\left(\sum{i=1}^n\alpha_i x_i\right)^p \le \sum{i=1^n}\alpha_ix_i^p.] Setting $\displaystyle q = \frac{p}{p-1}, \alpha_i = b_i^q\left(\sum_{i=1}^nb_i^q\right)^{-1}$, and $\displaystyle x_i = a_ib_i^{-1/(p_i-1)}\sum_{i=1}^nb_i^q$ here, we obtain the Holder's inequality: [\sum{i=1}^na_ib_i \le \left(\sum{i=1}^na_i^p\right)^{1/p}\left(\sum_{i=1}^na_i^q\right)^{1/q}] \end{example}

\section{L'H$\ddot{\rm \textbf{o}}$spital Role} We now pause to discuss a special, but very useful device for finding the limit of a ratio of functions, known as L'Hospital \footnote{G.F.de l'Hospital(1661-1704), French mathematician, a capable student of Johann Bernoulli.}rule.

\begin{proposition} \normalfont Suppose the functions $f:]a,b[\to \mathbb{R}$ and $g:]a,b[\to \mathbb{R}$ are differentiable on the open interval $]a,b[$ with $g'(x) \ne 0$ on $]a,b[$ and [\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a+0 (-\infty \le A \le +\infty),] then [\frac{f(x)}{g(x)} \to A \text{ as } x \to a+0] in each of the following two cases: \begin{enumerate} \item $\displaystyle f(x) \to 0 \wedge g(x) \to 0, \text{ as } x \to a+0,i$ \item $\displaystyle g(x) \to \infty, \text{ as } x \to a+0.$ \end{enumerate} A similar assertion holds as $x \to b-0$. \end{proposition} \begin{proof} \normalfont [\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}] [\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(\xi)}{g'(\xi)} \left(1 - \frac{g(y)}{g(x)}\right)] \end{proof}

\begin{example} \normalfont Find the following limits using L'Hospital rule. [\lim{x \to 0}\frac{1 - \cos 2x}{x^2}] [\lim{x \to \infty}\frac{\frac{\pi}{2} - \tan^{-1}x}{\sin \frac{1}{x}}] [\lim{x \to +\infty}\frac{x^a}{e^{bx}},(a>0,b>0)] [\lim{x \to 0}x \ln x] [\lim{x \to 0^+}\ln^x\frac{1}{x}] [\lim{x \to 0}x^x] [\lim{x\to {\frac{\pi}{2}}}\frac{1 + \sin x}{1 - \cos x}] [\lim{x \to \infty} \frac{x + \cos x}{x}] \end{example} \section{Asymptotic Line} \begin{definition} \normalfont The line $c_0 + c_1x$ is called the \textbf{asympotote} of the graph of the function $y = f(x)$ as $x \to -\infty$(or $x \to +\infty$) if [f(x) - (c_0 + c_1x) = \circ(1) \text{ as } x \to -\infty (or x \to +\infty)] \label{def:def1} \end{definition} It obviously from the Def.\ref{def:def1} that [c_1 = \lim{x \to -\infty}\frac{f(x)}{x}] and [c_0 = \lim{x \to -\infty} \left(f(x) - c_1x\right)] In general, if $\displaystyle f(x) - (c_0 + c_1x + \cdots + c_nx^n) = 0$ as $x \to -\infty$, then [c_n = \lim{x \to -\infty}\frac{f(x)}{x^n}] [c{n-1} = \lim{x \to -\infty} \frac{f(x) - c_nx^n}{x^{n-1}}] [\vdots] [c_0 = \lim{x \to -\infty} f(x) - (c_1x + c_2x^2 + \cdots + c_nx^n)] These relations, written out here for $x \to -\infty$, are of course also valid in the case $x \to +\infty$ and can be used to describe the asymptotic behavior of the graph of a function $f(x)$ using the graph of the corresponding algebraic polynomial \$c_0 + c_1x + \cdots + c_nx^n\$.

\begin{example} \normalfont The graph of the function [y = x + \tan^{-1}(x^3 - 1)] is well approximated by the line \$y = x - \frac{\pi}{2}\$, as \$x \to -\infty\$, and by the line \$y = x + \frac{\pi}{2}\$, as \$x \to +\infty\$. \end{example}

\subsection{Examples of Sketches of Graphs of Functions(Without Application of the Differential Calculus)} \begin{example} \normalfont Let us construct a sketch of the graphs of the functions [h = \log_{x^2-3x -2}2] [y = \sin x^2] \end{example}

\graphicspath{ {./Figs/} } \begin{figure} \centering \begin{minipage}[htbp]{0.4\textwidth} \includegraphics[width = \textwidth]{funplot1.png} \caption{Graph of \$\log_{x^2 - 3x -2}2\$} \end{minipage} \begin{minipage}[htbp]{0.4\textwidth} \includegraphics[width = \textwidth]{funplot2.png} \caption{Graph of \$\sin x^2\$} \end{minipage} \label{fig:fig4} \end{figure}

\subsection{The Use of Differential Calculus in Constructing the Graph of a Function} **Example** Construct the graph of the function (see Figure~\ref{fig:fig5})

$$f(x) = |x + 2|e^{-1/x}$$