Simulation

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公式推导

目标是

$$\min_{\boldsymbol{\beta}} \frac{1}{n}\sum_{i=1}^{n}\rho_\tau\left(Y_i - X_i^T\boldsymbol{\beta}\right) - N\left(\widetilde{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)^T\boldsymbol{\Sigma^{-1}}\left(\widetilde{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)/2$$

用一光滑函数$H(\frac{x}{h})$代替示性函数$\mathbf{1}\left\{x\geq 0\right\}$,得到checkfunction的近似

$$K_h\left(x\right) = x\Big[H\left(\frac{x}{h}\right)+\tau-1\Big]$$

$$\frac{\partial K_h\left(x\right)}{\partial x} = H\left(\frac{x}{h}\right)+\tau-1+\frac{x}{h}H^{'}\left(\frac{x}{h}\right)$$

目标变为

$$\min_{\boldsymbol{\beta}}\boldsymbol{L}\left(\boldsymbol{\beta}\right) = \min_{\boldsymbol{\beta}}\frac{1}{n}\sum_{i=1}^{n}K_h\left(Y_i - X_i^T\boldsymbol{\beta}\right) - N\left(\widetilde{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)^T\boldsymbol{\Sigma^{-1}}\left(\widetilde{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)/2$$

注意到

$$\frac{\partial K_h\left(Y_i - X_i^T\boldsymbol{\beta}\right)}{\partial\boldsymbol{\beta}} = \frac{\partial K_h\left(Y_i - X_i^T\boldsymbol{\beta}\right)}{\partial(Y_i - X_i^T\boldsymbol{\beta})}\times\frac{\partial(Y_i - X_i^T\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}$$

$$=(-X_i)\left[H\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)+\tau-1+\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}H^{'}\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)\right]$$

且

$$\frac{\partial\Big[N\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\Big]}{\partial\boldsymbol{\beta}} = \frac{\partial\Big[N\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\Big]}{\partial\Big(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\Big)} = \boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)$$

所以

$$\frac{\partial\boldsymbol{L}\left(\boldsymbol{\beta}\right)}{\partial\boldsymbol{\beta}} = \frac{1}{n}\sum_{i=1}^{n}(-X_i)\left[H\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)+\tau-1+\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}H^{'}\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)\right] - N\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)$$

令$\frac{\partial\boldsymbol{L}(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}=0$,有

$$\sum_{i=1}^{n}X_i\Big[H\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)+\tau-1+\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}H^{'}\left(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\right)\Big] + nN\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)=0$$

即

$$\sum_{i=1}^{n}X_i\Big\{H\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)+\tau-1+\frac{Y_i}{h}H'\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)\Big\} - \sum_{i=1}^{n}X_iX_i^T\frac{1}{h}H'\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)\boldsymbol{\beta} + nN\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)=0$$

$$\sum_{i=1}^{n} X_i \Big\{ H\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big) + \tau - 1 + \frac{Y_i}{h} H'\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)\Big\} - \sum_{i=1}^{n} X_i X_i^T \frac{1}{h} H'\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)\widetilde{\boldsymbol{\beta}}$$
$$+ \sum_{i=1}^{n} X_i X_i^T \frac{1}{h} H'\Big(\frac{Y_i - X_i^T\boldsymbol{\beta}}{h}\Big)\left(\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}\right) + nN\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}\right) = 0$$

记

$$\boldsymbol{U} = \sum_{i=1}^{n} X_i \left\{ H\left(\frac{Y_i - \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}_0}{h}\right) + \tau - 1 + \frac{Y_i}{h} H'\left(\frac{Y_i - lX_i'\widehat{\boldsymbol{\beta}}_0}{h}\right)\right\} - \sum_{i=1}^{n} X_i X_i^T \frac{1}{h} H'\Big(\frac{Y_i - X_i^T\widehat{\boldsymbol{\beta}}_0}{h}\Big)\widetilde{\boldsymbol{\beta}}$$
$$\boldsymbol{V} = \sum_{i=1}^{n} X_i X_i' \frac{1}{h} H'\left(\frac{Y_i - \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}_h}{h}\right) - nN\boldsymbol{\Sigma^{-1}}$$

则有

$$\boldsymbol{V}\left(\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}\right) = \boldsymbol{U}$$

从而

$$\widehat{\boldsymbol{\beta}}_h = \widetilde{\boldsymbol{\beta}} + \boldsymbol{V}^{-1}\boldsymbol{U}$$

即

$$\widehat{\boldsymbol{\beta}}_h = \widetilde{\boldsymbol{\beta}} + \left\{\sum_{i=1}^{n} X_i X_i' \frac{1}{h} H'\left(\frac{Y_i - \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}_h}{h}\right) - nN\boldsymbol{\Sigma^{-1}}\right\}^{-1} \times$$
$$\left\{\sum_{i=1}^{n} X_i \Big\{ H\left(\frac{Y_i - \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}_0}{h}\right) + \tau - 1 + \frac{Y_i}{h} H'\left(\frac{Y_i - lX_i'\widehat{\boldsymbol{\beta}}_0}{h}\right)\Big\} - \sum_{i=1}^{n} X_i X_i^T \frac{1}{h} H'\Big(\frac{Y_i - X_i^T\widehat{\boldsymbol{\beta}}_0}{h}\Big)\widetilde{\boldsymbol{\beta}}\right\}$$

做了100次simulation结果如下，0.46是用最小二乘估计作为初值的误差，一共迭代了10次，后面10个数是每次迭代的误差

算法收敛很快，迭代3到4次之后误差就不变了，好像有一个误差下界，跟窗宽h和Σ的选取都有关系

```
load('结果.RData')
res$mean
```

```
##  [1] 0.460551519 0.009979337 0.009974852 0.009974849 0.009974849 0.009974849
##  [7] 0.009974849 0.009974849 0.009974849 0.009974849 0.009974849
```

```
cat('最终收敛的误差为:', res$mean[11])
```

```
## 最终收敛的误差为: 0.009974849
```

```
cat('不带辅助信息的误差为:',mean(single.error))
```

```
## 不带辅助信息的误差为: 0.01013438
```