Math 540 Homework 8

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- We now express $\delta_i \circ F_j$. We know previously that $\sigma_i : \Delta_{n+1} \to Y$ as

  $$\sigma_i(\Delta_{n+1}) = H(\sigma(\delta_i(\Delta_{n+1}))) = H(\sigma(\delta_i(F(\Delta_n))))$$

  with triangulated square diagram we observe that $\delta_i \circ F_j$ can be represented as different projections from (Δ_{n+1}), when composing with face map F_i, now plug it in:

  $$\delta_i \circ F_j = (F_{j-1} \times Id_{[0,1]}) \circ \delta_i, i \leq j-2$$

  or

  $$\delta_i \circ F_j = (F_{j-1} \times Id_{[0,1]}) \circ \delta_{i-1}, i \geq j+1$$

  or

  $$\delta_i \circ F_j \text{ itself }, otherwise$$

  when we deal with $\sigma : \Delta_n \to \Delta_n \times \{pt\}$ the expression corresponds to border term i_0, i_1 which is given by

  $$i_0 = \sigma_1 \circ F_1, i_1 = \sigma_n \circ F_n$$

  .

- By Hatcher and by above computation,

  $$\partial K(\sigma) = \sum_{j \leq i} (-1)^i (-1)^j H \circ (\sigma \times Id) \circ \delta_j + + \sum_{j \geq i} (-1)^i (-1)^{j+1} H \circ (\sigma \times Id) \circ \delta_j$$

  $$K(\sigma)\partial = \sum_{j < i} (-1)^i (-1)^j H \circ (\sigma \times Id) \circ \delta_j + \sum_{j > i} (-1)^{i-1} (-1)^j H \circ (\sigma \times Id) \circ \delta_j$$

  The term with $i = j$ in two sums vanishes except $H \circ (\sigma \times Id) \circ \delta_0 = f \circ \sigma$ and $H \circ (\sigma \times Id) \circ \delta_n = -g \circ \sigma$, The term with $i \neq j$ is $-K(\sigma)\partial$ by above calculation again. Thus the sum gives that

  $$\partial K(\sigma) + K(\sigma)\partial = C_n(f) - C_n(g)$$

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- We split into two cases: $n \geq 1$, from definition it is clear that $\tilde{C}_n(X) = C_n(X)$, and morphism $\tilde{\partial}_n(X) = C_n(\partial)$. then accordingly

  $$\tilde{H}_n(X) = Ker(\tilde{\partial}_n)/Im(\tilde{\partial}_{n+1}) = Ker(\partial_n)/Im(\partial_{n+1} = H_n(X)$$

  .

  $n \leq -1$, by convention we know that $H_{-1}(X) = 0$, we explicitly compute the module $\tilde{H}_n(X)$, which $\tilde{H}_{-1}(X) = Ker(\tilde{\partial}_{-1})/Im(\tilde{\partial}_0)$ and we know that $\tilde{\partial}_{-1}$ is zero map thus eat entire $\tilde{C}_{-1}(X) = R$. Map $\tilde{\partial}_0$ as defined in problem is epimorphism since we can always find a suitable basis from $C_0(X)$ to map to some $\sum_{i=1}^k a_i$. Thus $Im(\tilde{\partial}_0) = R$, we have

$$Ker(\tilde{\partial}_{-1})/Im(\tilde{\partial}_0) = R/R = 0$$

This proves the claim.

We do not have to worry about smaller index homology groups since then $\tilde{C}_n(X)$ is 0 .

- To show that $\tilde{H}_0(X) = 0$ is equivalent to say that $Ker(\tilde{\partial}_0) = Im(\tilde{\partial}_1)$. We thus start with two sides inclusion. Notice we have path connected X, thus for elements in $Ker(\tilde{\partial}_0)$, we know they should have form

$$\sum_{i=1}^k a_i x_i, a_j = -\sum_{i=1, i\neq j}^k a_i$$

which indeed assembly point rays. Rewrite $\sum_{i=1}^k a_i x_i, a_j = -\sum_{i=1,i\neq j}^k a_i$ as $\sum_{i=1,i\neq j}^k (a_i x_i - a_i x_j)$ we see that this element is in $Im(\tilde{\partial}_1)$, since $\tilde{\partial}_1(\sigma(x_i, x_j)) = x_i - x_j$. We thus proved

$$Ker(\tilde{\partial}_0) \subset Im(\tilde{\partial}_1)$$

Then

$$Ker(\tilde{\partial}_0) \supset Im(\tilde{\partial}_1)$$

follows from that

$$\tilde{\partial}_0(x_i - x_j) = 1 - 1 = 0$$

.

Therefore we proved two side inclusion, the claim follows.

- Suppose that X has n path connected *components* $X_1, \ldots, X_n$. Show that $H_0(X) = R^{n-1}$.k We proceed this by induction, the base case is proved in part ii), now recall $\tilde{H}_0(X) = Ker(\tilde{\partial}_0)/Im(\tilde{\partial}_1)$ and assume that $\tilde{H}_0(X) = R^{n-1}$ when $X = \bigsqcup_{i=1}^n x_i$. When we let $X' = X \sqcup X_{n+1}$, the $Im(\tilde{\partial}_1) = R$ did not change since it is still $\{x_i - x_j\}$, at the mean while $Ker(\tilde{\partial}_0)$ split as

$$Ker(\tilde{\partial}_0) = \bigoplus_{i=1}^n Ker(\tilde{\partial}_0 | X_i)$$

then

$$\tilde{H}_0(X') = \frac{\bigoplus_{i=1}^n Ker(\tilde{\partial}_0|X_i) \oplus Ker(\tilde{\partial}_0|X_{n+1})}{Im(\tilde{\partial}_1)} = \frac{R^n \oplus R}{R} = R^n$$

By induction hypothesis thus the relation follows.