Math 540 Homework 9

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1

Since A is contractible, $H_n(A) = 0$ for all $n \neq 0$, we have diagram below:

$$\cdots \longrightarrow 0 \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, A) \longrightarrow$$

$$\longrightarrow 0 \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow$$

$$\longrightarrow 0 \longrightarrow H_{n-1}(X)M \longrightarrow H_{n-1}(X, A) \longrightarrow ...$$

The sequence about is exact, so we indeed have

$$0 \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, A) \longrightarrow 0$$

as an exact short 4 sequence the middle morphism has to be isomorphism, then claim $H_{n+1}(X) \cong H_{n+1}(X, A)$ follows.

(Dealt with 1 carefully: when $n = 1$ and X path connected we have

$$0 \longrightarrow H_1(X) \longrightarrow H_1(X, A) \longrightarrow R \longrightarrow R$$

but we notice that relative module $H_0(X, A)$ is 0 since $Z_0(X, A) = B_0(X, A)$ since ∂c is nothing, then we extend this to

$$0 \longrightarrow H_1(X) \longrightarrow H_1(X, A) \longrightarrow R \longrightarrow R \longrightarrow 0$$

Then $Im(H_1(A)) \to (H_1(X, A))$=$Ker((H_1(X, A)) \to R)$, but $Im(H_1(X, A)) \to R = Ker(R \to R)$, and if $Ker(R \to R) = 0$ (which follows from the forced relation $Im(R \to R) = Ker(R \to 0) = R$), then by Isomorphism theorems we have $Im(H_1(A)) \to (H_1(X, A))$=$Ker((H_1(X, A)) \to R) = (H_1(X, A))$, that is to say map

$$(H_1(A)) \to (H_1(X, A))$$

is surjective, but it is injective from initial exactness, thus this still exists as isomorphism. The case with connected components has the exactly same argument, just replacing the fourth R $\bigoplus_n R$)

2

Let us restrict $n \geq 2$

Since $A \subset X$ are both contractible, $H_n(A) = H_n(X) = 0$ for all $n \neq 0$. We again use the long exact sequence from 1) with replaced $H_n(X) = 0$, so we obtained short exact "4" sequence:

$$0 \longrightarrow 0 \longrightarrow H_{n+1}(X, A) \longrightarrow 0$$

This isomorphism of forces $H_{n+1}(X) = H_{n+1}(X, A) = 0$. Therefore we acquired desired $H_{n+1}(X, A) = 0 = H_n(A)$.

Now let $n = 1$, in general, from short 4 isomorphism we have $H_1(X, A) \cong H_1(X) = 0$, yet from the theorem in Chapter 7, $H_n(A) = R$ where R is the coefficient ring. Therefore the claim fails when $n = 1$, however, if the space= $\emptyset$ it holds.

(:Less *a priori* but conclusion holds.)

3

We know the deformation retraction of A and X are corresponding to isomorphic homology modules of $H_n(A) = H_n(X)$, therefore we adopt the (long) exact sequence given as:

$$\cdots \longrightarrow H_{n+1}(A) \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X)M \longrightarrow H_{n-1}(X, A) \longrightarrow \ldots$$

Now we find $H_n(X, A)$: Name $H_{n+1}(A) = H_{n+1}(X) = M$ if dimension is one down then $M^{(1)}$, we have some information:

$$ker(H_{n+1}(A) \to H_{n+1}(X)) = Im(0 \to H_{n+1}(A)) = 0$$

and

$$M/\{0\} = Im(H_{n+1}(A) \to H_{n+1}(X)) = ker(H_{n+1}(X) \to H_{n+1}(X, A)) = M$$

Apply Isomorphism theorem,

$$Im(H_{n+1}(A) \to H_{n+1}(X, A)) = ker(H_{n+1}(X, A) \to H_n(A)) = \{0\}$$

Thus

$$H_{n+1}(X, A)/\{0\} = Im(H_{n+1}(X, A) \to H_n(A)) = ker(H_n(A) \to H_n(X)) = \{0\}$$

Eventually we have $H_{n+1}(X, A) = \{0\}$. As our n is arbitrarily chosen, the claim follows.

4 Fun Problem

Some computation gives as below.

a) Take $[c] \in H_n(X, X_1)$, $\partial c \in C_{n-1}(X_1)$, say K_n is a chain map between $H_n(X_2, X_1 \cap X_2) \to H_n(X, X_1)$

i)

$$c = c_{X_1} + c_{X-X_1} + \partial K_n(c) + K_{n-1}(\partial c) \Leftrightarrow c' = \partial c = \partial c_1 + \partial c_2$$

ii)

$$c_{X-X_2} = c - c_{X_1} + \partial K_n(c) - K_{n-1}(\partial c) \in C_n(X \cap X_1)$$

Also as $\partial c' = 0$, we then choose appropriate c'_1, c'_2 we obtain the expression

$$c = c_1 + c_2 + c'_1 + c'_2 + \partial c'$$

iii)

$$\partial c_{X-X_2} = \partial c + \partial c_{X_1} + \partial\partial K_n(c) - \partial c'_2 \in C_{n-1}(X_1 \cap X_2)$$

Since every term is in $C_{n-1}(X_1 \cap X_2)$, we claim this c_{X-X_2} define a class in excised H_n, that is to say $[c_2 + c'_2] \in H_n(X_2, X_1 \cap X_2)$

iv) Now

$$H_n(i)c_{X-X_2} = [c - c_{X_1} + \partial K_n(c) - K_{n-1}(\partial c)] = [c] \in H_n(X, X_1)$$

Thus the map is surjective.

b) i,ii) map $i_{X_1 \to X}$ defines a natural inclusion of chain module $i^*_{X_1 \to X} : C_n(X_1) \to C_n(X)$, then map descends to

$$H_n(i)^*_{X_1 \to X} : C_n(X_1)/C_n(X_2) \to C_n(X)/C_n(X_2)$$

similarl we have

$$H_n(j)^*_{X_2 \to X} : C_n(X)/C_n(X_2) \to C_n(X)/C_n(X_1)$$

Now we need to show that $Im(H_n(i)) = Ker(H_n(j))$ (though topologicall trivial: $X_1 - X_2 = (X - X_2) - (X - X_1)$). We notice that $Im(\tilde{i}) = C_n(X_1)/C_n(X_2)$, and simplices are in linear space (then we just need to show below holds for one simplex),thus

$$c_2 - c_1 = \partial c'_1 + \partial c'_2$$

. Pick an element in $[c] \in Im(\tilde{i})$, we have a representative $c = c_{X_1} + c_{X_2}$, then

$$c_1 + \partial c_1 \in C_n(X_{12})$$

iii) Since $X \supset X_1$ we also have $C_n(X)/C_n(X_2) \supset C_n(X_1)/C_n(X_2)$, now

$$H_n(j)[c] = H_n(j)(c_{X_1} + c_{X_2}) = 0$$

Thus we showed the morphism is injective.

c) i) $\partial c_2 = -\partial c_1$ is equivalently $\partial(c_2 + c_1) = 0$. We see the element ∂c_2 is an element in $C_{n-1}(X_1)$ and $C_{n-1}(X_1)$ which precisely is in $C_{n-1}(X_1 \cap X_2)$

ii) We know $c_2 - c_{12} = \partial c'_2$ since left element is a boundary term, then the injectivity of given $H_n(j)$ tells the non-trivial $c_2 \in H_n(X_2 \cap X_1) \Leftrightarrow c_2$ is also in $C_n(X, X_1)$, thus it has to be some $\tilde{c} + c_{12}$ for $\tilde{c}$ non-trivial and vanishing under differential, then we pick $\tilde{c} = \partial c'_2, c'_2 \in C_{n+1}X_2$ as defined, which is compatible with given above..

iii) use ii), also since $c - c_2' \in C_{n+1}X$, $\partial c - \partial c_2' \in C_n(X_1)$ and $c - c_2'$ is not in the relative boundaries, we should have

$$(c - c_2' - \partial c'') \cap C_n(X - X_1) \neq \varnothing$$

, that says $c - c_2' - \partial c'' = c_2'' + c_1'$ where $c_2'' \in C_{n+1}(X_2), c_1' \in C_{n+1}(X_1)$

iv) Show $[c_1 + c_2] = 0$: that is to show $[c_1 + c_2]$ is exact but we have

$$c_1 + c_2 = c_1 + C_2 = \partial c = \partial c_2'' + \partial c_1' + \partial\partial c'' + \partial c_2' = \partial(c_2' + c_2'') + \partial c_1'$$

Since the element in the kernel is precisely 0 the map is injective,

d) It is a long and tedious job to repeat part a) so we hand the job to Allen Hatcher who performed an excellent job on this, Hooray!

e) This is an application of symmetry: Combine a),b),c),d) we have the two isomorphism implication; Apply this we know that $(X_2, X_1 \cap X_2) \to (X, X_1)$ is an excision map, therefore there has an isomorphism $H_n^{X_1 X_2} \to H_n^X$, this isomorphism implies $(X_1, X_1 \cap X_2) \to (X, X_2)$ since it is "iff" proposition.

5 Fun

from old homework

a) We know the deformation retraction of $\mathbb{R}^m - \{x_0\}$ and $\mathbb{R}^n - \{x_0\}$ are corresponding to S^{m-1} and S^{n-1}, but the homology modules of S^{m-1} and S^{n-1} are known to be different. If such homeomorphism exists then homology modules as homotopy invariant will be congruent. This reveals a contradiction.

b) Continue the fashion of contradiction, suppose such homeomorphism exists then homology modules $H_n(R^m) = H_n(R^m)$, therefore also holds for their relative homology $H_n(R^m, \{x_0\}) = H_n(R^n, \{y_0\})$, since the point itself is a deformation retraction, we thus have

$$H_n(R^m, \{x_0\}) = H_n(R^m - \{x_0\}, \{x_0\} - \{x_0\}) = H_n(R^n - \{y_0\}, \{y_0\} - \{y_0\}) = H_n(R^n, \{y_0\})$$

We notice, therefore $H_n(R^m - \{x_0\}) = H_n(R^n - \{y_0\})$, but this contradicts with the conclusion in the previous statement.