Math 225: Linear Algebra — Fall 22

Exam 1 — 7 October 2022

First Name: SOLUTIONS (as in student record)

Last Name: (as in student record)

USC ID: Signature:

- This exam has 5 problems, and will last 50 minutes.
- You may use one page of notes, but no calculator.
- Show all of your work and justify every answer to receive full credit.
- Feel free to continue answers on other pages as long as you clearly indicate to the grader where they can find your solution.
- Work quickly, but carefully. Good luck!

Do not write in the box below:

Q01	Q02	Q03	Q04	Q05
/10	/10	/10	/10	/10

/50

Question 1 (10 points). Determine whether the following statements are **true** or **false**. You do not need to justify your answer.

(a) Let A and B be $n \times n$ matrices where $RREF(A) = RREF(B)$. Then A is invertible if and only if B is invertible.

True

(b) Assume AB is invertible. Then A and B are both invertible.

True

(c) Every vector space has a unique minimal spanning set.

False

(d) Assume A and B are $n \times n$ matrices where $det(AB) = det(BA)$. Then $AB = BA$

False

(e) Assume S and T are subspaces of a vector space V. Then

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

is a subspace of V.

False

Question 2 (10 points).

(a) Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} a-d & b-e & c-f \\ 2g & 2h & 2i \\ -d & -e & -f \end{bmatrix}$ and $det(A) = 4$.

Find $det(ABA^TB^{-2})$.

$$det(B) = (2)(-1)(-1)det(A) = 8$$

$$det(ABA^TB^{-2}) = det(A)det(B)\,det(A)\,\frac{1}{det(B)^2}$$

$$= \frac{det(A)^2}{det(B)} = 2$$

(b) Let $A = \begin{bmatrix} 2 & k & k \\ 2 & 1 & 4 \\ 1 & k & 0 \end{bmatrix}$. Find all values k for which the matrix fails to be invertible.

$$2\begin{vmatrix} 1 & 4 \\ k & 0 \end{vmatrix} - k\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix}$$

$$2(-4k) - k(-4) + k(2k-1)$$

$$-8k + 4k + 2k^2 - k$$

$$2k^2 - 5k = 0$$

$$k(2k-5) = 0$$

$k = 0 \qquad k = 5/2$

Question 3 (10 points). Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Can A be written as a product of elementary matrices? Justify your answer.

If your answer is yes, write A as a product of elementary matrices.

$$\det(A) = 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$$

so can be written as product

$$A_{32}(-2)\, A_{13}(-1)\, P_{23}\, A \qquad I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 4 (10 points)**.** Let S be the solution set to the system of linear equations

$$\begin{aligned} x - y + 3z - 2w &= 0 \\ 2x + y + 3z - 4w &= 0 \\ 3x + 4y + 2z - 6w &= 0 \end{aligned}$$

(a) Find S.

$$\begin{bmatrix} 1 & -1 & 3 & -2 \\ 2 & 1 & 3 & -4 \\ 3 & 4 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 = t$

$x_3 = s$

$x_2 = s$

$x_1 = -2s + 2t$

$\{(-2s + 2t, s, s, t) \mid s, t \in \mathbb{R}\}$

(b) Show that S is a subspace of $\mathbb{R}^4$.

Addition (i) $(-2s + 2t, s, s, t) + (-2u + 2v, u, u, v) =$

$-2(s+u) + 2(t+v), s+u, s+u, t+v) \in S$

Scalar Mult (ii) $k(-2s + 2t, s, s, t) = (-2ks + 2kt, ks, ks, kt) \in S$

Question 5 (10 points)**.** Let S be the subspace of $M_2(\mathbb{R})$ consisting of all 2×2 matrices whose four elements sum to zero.

Find three vectors v_1, v_2, v_3 in S where $span\{v_1, v_2, v_3\} = S$. Justify your answer.

An arbitrary vector in S looks like

$$\begin{bmatrix} a & b \\ c & -a + -b + -c \end{bmatrix}$$

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ span S