4.2 Definition of a Vector Space (cont.)

Ex/ On $\mathbb{R}^+$, the set of all positive real numbers, define the operations of addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$x \oplus y = xy$$
$$c \odot x = x^c$$

Determine whether $\mathbb{R}^+$ together with these operations is a vector space.

(A1) $x, y \in \mathbb{R}^+ \Rightarrow \underset{x \oplus y}{\overset{\shortparallel}{xy}} \in \mathbb{R}^+$

(A2) $x \in \mathbb{R}^+,\ r \in \mathbb{R} \Rightarrow x^r = r \odot x \in \mathbb{R}$

(A3) $x \oplus y = xy = yx = y \oplus x$

(A4) $x \oplus (y \oplus z) = xyz = (x \oplus y) \oplus z$

(A5) Let the zero vector 0_v be 1

$x \oplus 1 = 1 \oplus x = x \quad \forall x \in \mathbb{R}^+$

(A6) $\neg x = 1/x$

$x \oplus 1/x = 1/x \oplus x = 1/x \cdot x = 1 = 0_v$

(A7) $1 \cdot x = x^1 = x$

(A8) $(rs) \odot x = x^{rs} = (x^s)^r$

$= r \odot (sx)$

(A9) $r \odot (x \oplus y) = (xy)^r = x^r y^r = r \odot x \oplus s \odot y$

(A10) $(r+s) \odot x = x^{r+s} = x^r \cdot x^s = r \odot x \oplus s \odot x$

This is a vector space

More Properties of Vector Spaces

Let V be a vector space over F

1. The zero vector is unique.
2. $0v = 0$ for all $v \in V$
3. $k0 = 0$ for all scalars $k \in F$
4. The additive inverse of each element in V is unique
5. For all $v \in V$, $-v = (-1)v$
6. If k is a scalar and $v \in V$ such that $kv = 0$, then $k = 0$ or $v = 0$.

List of Important Vector Spaces

$\mathbb{R}^n$, the real vector space of ordered n-tuples of real numbers

$\mathbb{C}^n$, the complex vector space of ordered n-tuples of complex numbers

$M_{m\times n}(\mathbb{R})$ the real vector space of all $m\times n$ matrices with real elements

$M_n(\mathbb{R})$ the real vector space of all $n\times n$ matrices with real elements

$C^k(I)$, the vector space of all real valued functions that are continuous and have (at least) k continuous derivatives on an interval I in $\mathbb{R}$.

$P_n(\mathbb{R})$ the real vector space of all real valued polynomials of degree $\leq n$ with real coefficients. That is

$$P_n(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_0, a_1, \dots, a_n \in \mathbb{R}\}$$

4.3 Subspaces

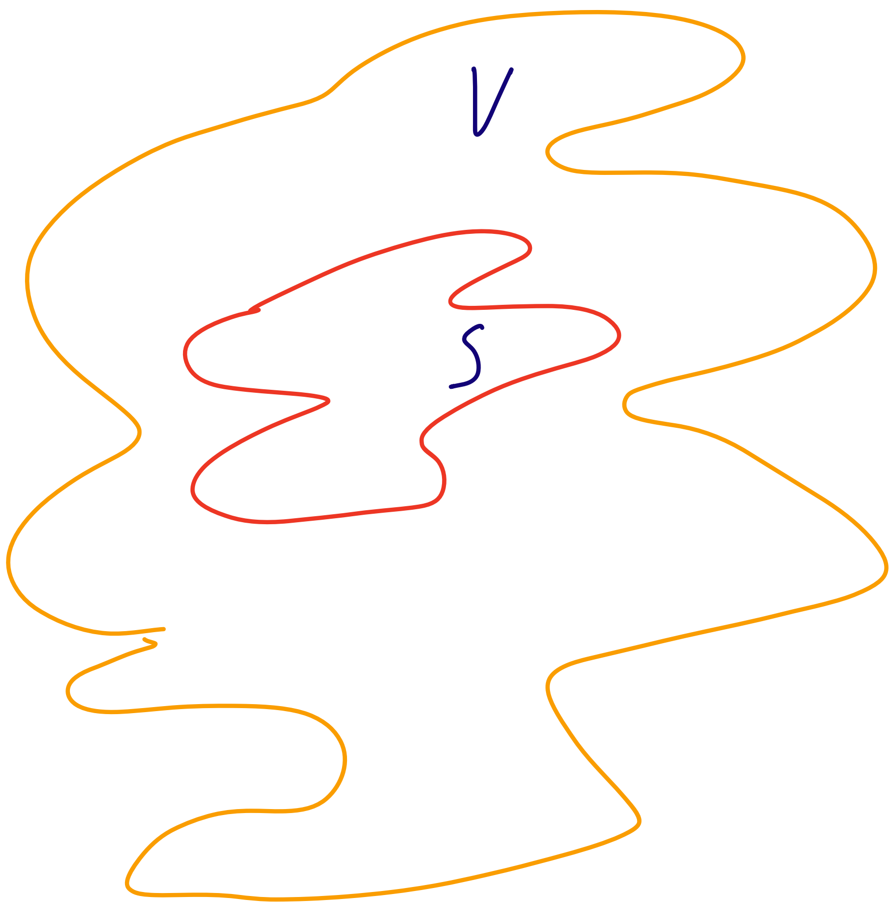
We often encounter subsets of vector spaces, for instance subsets of $\mathbb{R}^n$

It is important to know whether these subsets inherit the structure of the ambient space.

(This is generally important in math)

For instance, let $Ax = b$ represent an $n \times n$ system of linear equations

Then the solution set S to this system is a <u>subset</u> of $\mathbb{R}^n$. But does the solution set also form a vector space?



Is S a vector space?

Let S be a nonempty subset of a vector space V. If S is itself a vector space under the same operations of addition and scalar multiplication as used in V, then we say S is a subspace of V.

To show V was a vector space we needed to check all of conditions (A1) - (A10).

Fortunately we don't need to check all of these axioms to show S is a subspace of V.

Thm/ Let S be a nonempty subset of a vector space V. Then S is a subspace of V if and only if S is closed under the operations of addition and scalar multiplication.

Pf/

" $\Rightarrow$ " Assume S is a subspace of V.

So S is a vector space in particular it satisfies (A1) and (A2)

"$\Leftarrow$" Assume S is closed under addition and scalar multiplication.

In theory, we need to check (A1) – (A10) hold

but (A3), (A4), (A7) – (A10) follow immediately from S being a subset of the vector space V.

So we only need to check (A5) and (A6)

(A5) Since S is closed under scalar multiplication we have $0 \cdot v = 0 \in S$

(A6) Since S is closed under addition plus scalar mult for any $u \in S$ $-u \in S$ and $u + (-u) = 0 \in S$

Note in particular if a subset S of a vector space fails to contain 0, then it can not be a subspace.

Ex/ Let S be the set of all solutions to

$$x_1 - 4x_2 + 6x_3 = 0$$
$$-3x_1 + 10x_2 - 10x_3 = 0$$

Express S in set notation and verify S is a subspace of $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} 1 & -4 & 6 & 0 \\ -3 & 10 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

let $x_3 = t$

$x_2 - 4t = 0 \Rightarrow x_2 = 4t$

$x_1 - 10t = 0 \Rightarrow x_1 = 10t$

So $S = \{t\langle 10,4,1\rangle \mid t \in \mathbb{R}\}$

Claim: S is a subspace of $\mathbb{R}^3$.

Pf: Check (A1) and (A2)

(A1) Pick $t_1\langle 10,4,1\rangle$ and $t_2\langle 10,4,1\rangle$ in S.

Then $t_1\langle 10,9,1\rangle + t_2\langle 10,9,1\rangle =$

$(t_1+t_2)\langle 10,9,1\rangle \in S$

(A2) Pick $t_1\langle 10,9,1\rangle \in S$

and pick $r \in \mathbb{R}$.

Then $(rt_1)\langle 10,9,1\rangle \in S$

So S is a subspace of $\mathbb{R}^3$.

HW 6 Show that the set S where

$$S = \{A \in M_n(\mathbb{R}) \mid A^T = -A\}$$

is a subspace of $M_n(\mathbb{R})$.

HW7 Determine whether the set

$$S = \{(x, x+1, 0) \mid x \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$.

HW8 Let $V = M_2(\mathbb{R})$ and S be the subset of all 2×2 matrices with $\det(A) = 1$. Determine whether S is a subspace of V.

$\boxed{\text{HW 9}}$ Let $V = P_2$ and S be the subset consisting of all polynomials of the form $p(x) = ax^2 + b$. Determine whether S is a subspace of V.