HW1

2.2 Matrix Algebra

In this section we learn to do arithmetic with matrices.

To start, it works like you would expect

If A and B are both $m \times n$ matrices then the sum $A+B$ is the $m \times n$ matrix whose elements are obtained by adding the corresponding elements of A and B.

In other words, if $A = [a_{ij}]$, $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}]$$

Ex/ $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 & 2 & 6 \\ 1 & 4 & 2 & 1 \end{bmatrix}$

$$A + B = \begin{bmatrix} 5 & 7 & 6 & 11 \\ 2 & 6 & 8 & 8 \end{bmatrix}$$

Properties of Matrix Addition

$A + B = B + A$ commutativity

$A + (B + C) = (A + B) + C$ associativity

If A is an $m \times n$ matrix and s is a scalar then we let sA denote the matrix obtained by multiplying every element of A by s. This is called <u>scalar multiplication</u>.

In other words if $A = [a_{ij}]$, then

$$sA = [s\, a_{ij}]$$

Scalar multiplication

Properties of Scalar Multiplication

1. $1A = A$
2. $s(A+B) = sA + sB$
3. $(s+t)A = sA + tA$
4. $s(tA) = (st)A = (ts)A = t(sA)$

If A and B are both $m \times n$ matrices then we define subtraction of these two matrices to be $A - B = A + (-1)B$

If $A = [a_{ij}]$ $B = [b_{ij}]$ then $A - B = [a_{ij} - b_{ij}]$

$\boxed{\text{HW6}}$ Show that $(A+C)^T = A^T + C^T$

$\boxed{\text{HW7}}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Find $2A - 3B$

The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zero. We use the notation $O_{m \times n}$ or O to refer to the zero matrix.

$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix Multiplication

So far matrix arithmetic has worked roughly as expected.

The first case where matrix arithmetic has noticably different properties than usual arithmetic is matrix multiplication.

In particular if $A=[a_{ij}]$ $B=[b_{ij}]$ are $n\times n$ matrices, in most cases:

$$\boxed{AB \neq [a_{ij} b_{ij}]}$$

We'll discuss the meaning of matrix multiplication at length later on, but first lets get used to how to calculate it.

First we define the dot product of a $1 \times n$ row vector and an $n \times 1$ column vector.

If $\mathbf{a} = [a_1 \; a_2 \; \cdots \; a_n]$ is a $1 \times n$ row vector

and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a $n \times 1$ column vector

Then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}$$

If you took Calc III, you saw for vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ $\mathbf{y} = \langle y_1, y_2, \ldots, y_n \rangle$ the dot product

$$\boxed{\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}$$

Note that the dot product is a number, not a vector

We use the dot product of row and column vectors to define matrix multiplication

Let's start from the simplest case.

$\boxed{\text{Case 1:}}$ Finding the product of a $\underline{1 \times n}$ row vector and an $\underline{n \times 1}$ column vector.

If $\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_n]$ is a $1 \times n$ row vector

and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a $n \times 1$ column vector

Then the matrix product $\mathbf{ab}$ is defined

$$\mathbf{ab} = [\mathbf{a} \cdot \mathbf{b}]$$
$$= [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$$

So the result of multiplying a $1 \times n$ row vector and a $n \times 1$ column vector is a 1×1 matrix, or a scalar.

Ex/ $a = \begin{bmatrix} 5 & 4 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$$a \cdot b = \begin{bmatrix} 5 \cdot 1 + 4 \cdot 2 + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 27 \end{bmatrix}$$

$\boxed{HW8}$

$x = [1\ 2\ 3\ 4]$

$y = [4\ 3\ 2\ 1]$

$xy^T =$

$\boxed{\text{Case II:}}$ Product of an $m \times n$ matrix and a $n \times 1$ column vector

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \\ \mathbf{a}_m \end{bmatrix}$

(where $\mathbf{a}_i$ is the row vector $[a_{i1} \ a_{i2} \ \ldots \ a_{in}]$)

Let $\mathbf{x} = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a $n \times 1$ column vector

Then the matrix product $A\mathbf{x}$ is defined

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_m \cdot x \end{bmatrix}$$

Ax is a $m \times 1$ matrix

Each entry of Ax is determined by taking the dot product of a row of A with the column vector x.

Ex

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Find Ax

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 2\cdot 1 + 3\cdot 4 \\ 4\cdot 1 + 2\cdot 2 + 1\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 12 \end{bmatrix}$$

$\boxed{\text{Case 3}}$: Product of an $m \times n$ matrix A and a $n \times p$ matrix B.

This is the general case for multiplying two matrices.

Note by the condition given above for matrix multiplication to be defined: we need:

$$\boxed{\begin{array}{c}\#\text{ of columns of } A = \\ \#\text{ of rows of } B\end{array}}$$

However unlike matrix addition the two matrices need not have the same shape to be multiplied.

As before, to find the entries of the matrix AB we take the dot product of rows of A with columns of B

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$$

(where $\mathbf{a}_i$ is a row vector of A)

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

(where $\mathbf{b}_j$ is a column vector of B)

Then the matrix product AB is the $m \times p$ matrix:

$$AB = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_p \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \cdots & a_2 \cdot b_p \\ \vdots & & & \vdots \\ a_m \cdot b_1 & a_m \cdot b_2 & \cdots & a_m \cdot b_p \end{bmatrix}$$

Note if $AB = C$ the c_{ij} entry of C is $c_{ij} = a_i \cdot b_j$

(the dot product of the ith row of A with the jth column of B.)

Furthermore if $A_{m\times n}$ and $B_{n\times p}$ then $A_{m\times n} B_{n\times p}$ is a $m \times p$ matrix.

Ex

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Find $AB = \begin{bmatrix} (1\cdot 3 + 2\cdot 1) & (1\cdot 2 + 2\cdot 4) \\ (3\cdot 3 + 4\cdot 1) & (3\cdot 2 + 4\cdot 4) \end{bmatrix}$

$= \begin{bmatrix} 5 & 10 \\ 13 & 22 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Find ABC

$\begin{bmatrix} 5 & 10 \\ 13 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 15 \\ 35 & 35 \end{bmatrix}$

We may also express matrix multiplication using index notation. If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = AB$ then

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad 1 \le i \le m, \quad 1 \le j \le p}$$

Sum notation

Read: "the sum from $k=1$ to $k=n$ of..."

Ex/ $A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 4 & 1 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \\ 4 & 1 & 1 \end{bmatrix}$

Find AB and BA (if possible)

$AB = \begin{bmatrix} 21 & 9 & 19 \\ 15 & 10 & 25 \end{bmatrix}$

BA not defined

$\boxed{\text{HW9}}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Find Ax

$\boxed{\text{HW10}}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Show that $AB \neq BA$.

(Hence the properties of matrix multiplication differ from usual multiplication of numbers)