4.4 Spanning Sets (cont.)

Since the only operations defined in a vector space V are addition and scalar multiplication the most complicated elements in V have the form:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_k v_k$$

We give a special name for this

An expression of the form:

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

is called a <u>linear combination</u> of $v_1, v_2, \dots, v_k$

It is natural to ask what vectors can be written as linear combinations of $v_1, v_2, \ldots, v_k$.

If every vector in a vector space V can be written as a linear combination of $v_1, v_2, \ldots, v_k$ we say V is <u>spanned</u> or <u>generated</u> by $v_1, v_2, \ldots, v_k$ and call the set of vectors $\{v_1, v_2, \ldots, v_k\}$ a <u>spanning set</u> of V. In this case we say $\{v_1, v_2, \ldots, v_k\}$ <u>spans</u> V.

Ex/ Consider the vector $v = (5, 3, -6)$

Does v lie in $\text{span}\{(-1,1,2), (3,1,-4)\}$?

$$a(-1,1,2) + b(3,1,-4) = (5,3,-6)$$

$$\begin{aligned} -a + 3b &= 5 \\ a + b &= 3 \\ 2a - 4b &= -6 \end{aligned} \Rightarrow \quad 4b = 8 \quad b = 2 \Rightarrow a = 1$$

Note $2 \cdot 1 - 4(2) = -6$ so

$$1(-1,1,2) + 2(3,1,-4) = (5,3,-6)$$

Thm/ Let $v_1, v_2 \dots v_k$ be vectors in $\mathbb{R}^n$

Then $\{v_1, v_2, \dots v_k\}$ spans $\mathbb{R}^n$ $\Leftrightarrow$ for the matrix $A = [v_1, v_2, \dots, v_k]$ the linear system $Ac = v$ is consistent for every $v \in \mathbb{R}^n$

Pf/ "$\Rightarrow$" Assume $\{v_1, \dots, v_k\}$ spans $\mathbb{R}^n$

Pick $v \in \mathbb{R}^n$

Then $v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$

Let $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

Then $[v_1, v_2, \dots v_k] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = c_1 v_1 + c_2 v_2 + \dots + c_k v_k = v$ ✓

"$\subseteq$" Pick $w \in \mathbb{R}^n$. $Ac = w$

is consistent so

$$w = c_1 v_1 + \dots + c_k v_k$$

so $\{v_1, \dots, v_k\}$ spans $\mathbb{R}^n$.

Ex/ Verify that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

span $M_2(\mathbb{R})$.

Pick $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$

Want

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} a &= c_1 + c_2 + c_3 + c_4 \\ b &= c_2 + c_3 + c_4 \\ c &= c_3 + c_4 \\ d &= c_4 \end{aligned} \implies \begin{aligned} c_3 &= c - d \\ c_2 &= b - c \\ c_1 &= a - b \end{aligned}$$

So

$$\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = (a-b)A_1 + (b-c)A_2 + (c-d)A_3 + dA_4$$

$\boxed{\text{HW1}}$ Determine whether the vectors $v_1 = (1,-3,6)$

$v_2 = (1,-4,2)$ and $v_3 = (-2,10,4)$ span $\mathbb{R}^3$.

$\boxed{\text{HW2}}$ Let $p(x) = 2x^2 - x + 2$. Does $p(x)$

lie in $\operatorname{span}\{x-4,\ x^2 - x + 3\}$?

$\boxed{\text{HW3}}$ Show $v_1 = (-1,3,2)$ $v_2 = (1,-2,1)$

$v_3 = (2,1,1)$ span $\mathbb{R}^3$ and express

$v = (x,y,z)$ as a linear combination

of v_1, v_2, v_3.

Let $v_1, v_2, \dots, v_k$ be vectors in a vector space V. Forming all possible linear combinations of $v_1, v_2, \dots, v_k$ generates a subset of V called the linear span of $\{v_1, v_2, \dots, v_k\}$ denoted $\text{span}\{v_1, v_2, \dots, v_k\}$

In other words

$$\text{span}\{v_1, v_2, \dots, v_k\} =$$

$$\{v \in V : v = c_1 v_2 + c_2 v_2 + \dots + c_k v_k, \ c_1, c_2, \dots, c_k\}$$

Thm / Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. Then $\text{span}\{v_1, v_2, \ldots, v_k\}$ is a subspace of V.

Pf / (i) Pick $u, w \in \text{span}\{v_1, v_2, \ldots, v_k\}$

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$w = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k$$

$$u + w = (c_1 + b_1) v_1 + (c_2 + b_2) v_2 + \cdots + (c_k + b_k) v_k$$

$$\in \text{span}\{v_1, v_2, \ldots, v_k\}$$

(ii) Pick $u \in \text{span}\{v_1, \ldots, v_k\}$ Pick $r \in \mathbb{R}$.

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$ru = (rc_1) v_1 + (rc_2) v_2 + \cdots (rc_k) v_k \in$$

$$\text{span}\{v_1, v_2, \ldots, v_k\}$$

Ex/ Determine the subspace of $P_2(\mathbb{R})$ spanned by

$$p_1(x) = 1 + 3x \quad , \quad p_2(x) = x + x^2$$

and determine whether $\{p_1(x), p_2(x)\}$ is a spanning set for $P_2(\mathbb{R})$.

$$\text{Span}\{p_1, p_2\} = \{p(x) \in P_2(\mathbb{R}) \mid c_1(1+3x) + c_2(x+x^2)\}$$

$$= \{p(x) \in P_2(\mathbb{R}) \mid c_1 + (3c_1 + c_2)x + c_2 x^2\}$$

This is not a spanning set. Why?

Claim: $1 + x^2 \notin \text{span}\{p_1, p_2\}$

$c_1 = 1$, $c_2 = 1$ so the coefficient of x must be 4.

$\boxed{HW4}$ Consider the vectors

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

in $M_2(\mathbb{R})$. Determine $\text{span}\{A_1, A_2, A_3\}$

$\boxed{HW5}$ Find a spanning set for the nullspace of the matrix

$$A = \begin{bmatrix} -1 & 5 & 3 \\ 2 & -10 & -6 \end{bmatrix}$$

$\boxed{HW6}$ If $V = \mathbb{R}^3$ and $v_1 = (1, 0, 1)$ and $v_2 = (0, 1, 1)$ determine the subspace of $\mathbb{R}^3$ spanned by v_1 and v_2. Does $w = (1, 1, -1)$ lie in this subspace?