Ch 4: Vector Spaces

We spent the first ~ 1/3 of the course discussing some important computations in linear algebra / considering linear algebra from a computational standpoint.

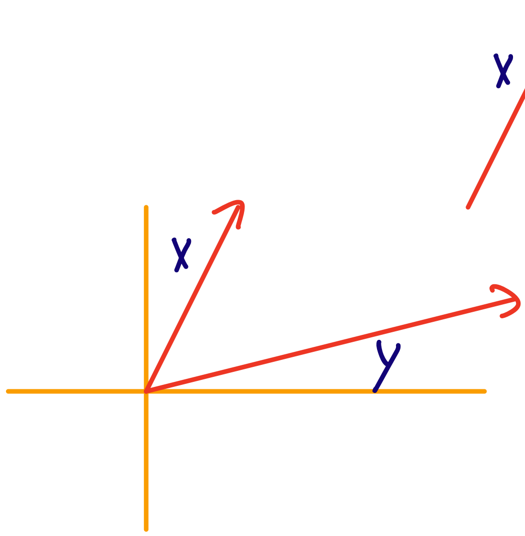
We spend the next part of the course trying to understand linear algebra from a geometric standpoint. This is a better way of understanding what is really going on.

4.1 Vectors in $\mathbb{R}^n$

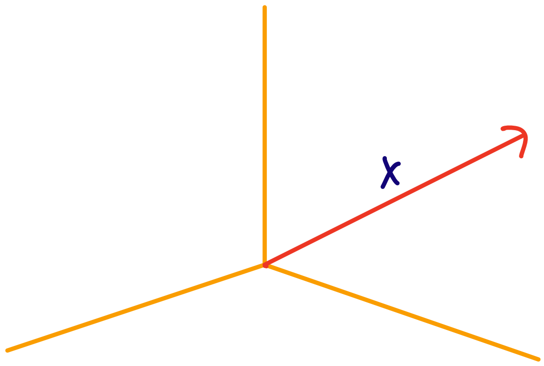
A vector in $\mathbb{R}^n$ has a magnitude and direction (but no fixed starting location)

We identify vectors with ordered pairs of real numbers in $\mathbb{R}^n$

Ex/ in $\mathbb{R}^2$



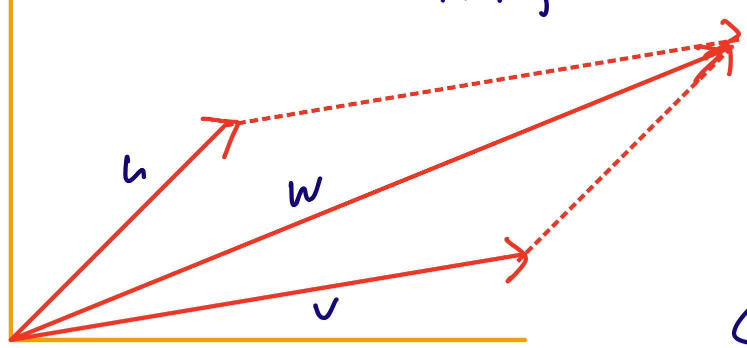
in $\mathbb{R}^3$



We can add and scalar multiply vectors

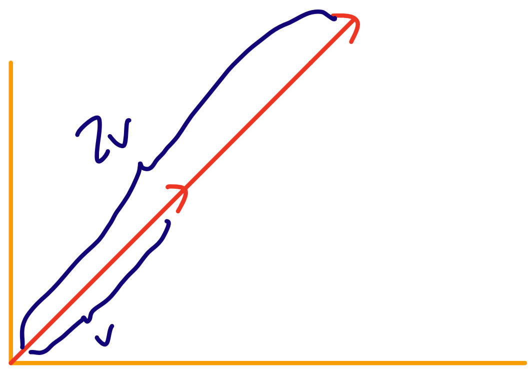
Addition

Parallelogram Rule



$$\underset{\langle 4,5\rangle}{u} + \underset{\langle 12,2\rangle}{v} = \underset{\langle 16,7\rangle}{w}$$

Scalar Multiplication



$v = \langle 4,4 \rangle$

$2v = 2\langle 4,4 \rangle = \langle 8,8 \rangle$

Addition and scalar multiplication of vectors clearly satisfy the following properties

For vectors x, y, z in $\mathbb{R}^n$ addition satisfies the properties

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$

3. there is a vector 0 such that $x + 0 = x$

4. there is a vector $-x$ such that $x + (-x) = 0$

For all vectors x, y and scalars r, s, t

5. $1x = x$

6. $(st)x = s(tx)$

7. $r(x+y) = rx + ry$

8. $(s+t)x = sx + tx$

We revisit these properties in a more general setting in the next section.

Note we have not defined vector multiplication, and there is a reason for this. There is no general notion of multiplication of vectors in $\mathbb{R}^n$.

(A cross product can be defined in $\mathbb{R}^3$, but this doesn't extend to $\mathbb{R}^n$)

We often express vectors in $\mathbb{R}^n$ in terms of standard "basis vectors"

~~For~~ $\mathbb{R}^2$

let $i = \langle 1, 0 \rangle$ $j = \langle 0, 1 \rangle$

Then any vector $\langle a, b \rangle$ in $\mathbb{R}^2$ can be written

$$\langle a, b \rangle = ai + bj = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle$$

$$= \langle a, 0 \rangle + \langle 0, b \rangle$$

$$= \langle a, b \rangle$$

$\mathbb{R}^3$ let $i := \langle 1, 0, 0 \rangle$ $j = \langle 0, 1, 0 \rangle$ $k = \langle 0, 0, 1 \rangle$

Then any vector $\langle a, b, c \rangle$ in $\mathbb{R}^3$ can be written

$$\langle a, b, c \rangle = ai + bj + ck$$

HW1 If $x = \langle 3, 4, 5, 6, 7 \rangle$ and $z = \langle -1, 0, -4, 1, 2 \rangle$ find y such that $2x + -3y = -z$.

HW2 Verify the associative law of addition for vectors in $\mathbb{R}^4$

4.2 Definition of a Vector Space

In the last section we looked at properties of vectors in $\mathbb{R}^n$ with addition and scalar multiplication.

In this section we generalize this example to the concept of a vector space.

This is one of the most important objects in mathematics.

From now on, let V be a nonempty set, where addition and scalar multiplication makes sense.

For instance,

V could be

- $\mathbb{R}$
- vectors in $\mathbb{R}^n$
- matrices
- functions
- $\vdots$
- etc.

By convention, we call objects in V vectors (although they may be matrices, etc.)

Vector Addition

For vectors u and v in V we write $u+v$ to denote the result of adding u and v

Scalar multiplication

For a (real or complex) scalar k we write kv to denote the result of multiplying v by k

We can now give the general definition of a vector space.

Let V be a nonempty set (whose elements are called vectors) on which addition and scalar multiplication is defined. Let F be the set of scalars.

V is a vector space if the following conditions hold:

(A1) (Closure under addition) For each pair of vectors u and v in V, then sum $u+v$ is in V.

(A2) (Closure under scalar multiplication) For each vector v in V and each scalar k in F, the scalar multiple kv is also in V

(A3) (Commutativity of addition) For all $u, v \in V$

$$u + v = v + u$$

(A4) (Associativity of addition)

For all $u, v, w \in V$ we have

$$(u+v)+w = u+(v+w)$$

(A5) (Existence of a zero vector in V)

There is a vector 0 in V satisfying

$$v+0=v \quad \text{for all } v \in V$$

(A6) (Existence of additive inverses in V)

For each $v \in V$ there is a vector $-v$ such that $v+(-v)=0$

(A7) (Unit property)

For all $v \in V$, $1v = v$

(A8) (Associativity of scalar multiplication)

For all $v \in V$ and all scalars $r, s \in F$

$(rs)v = r(sv)$

(A9) (Distributive property of scalar multiplication)

For all $u, v \in V$ and all scalars $r \in F$

$r(u+v) = ru + rv$

(A10) (Distributive property of scalar mult over scalar addition)

For all $v \in V$ $r, s \in F$

$(r+s)v = rv + sv$

This is a very general concept.
There are infinitely many vector spaces so we are not going to imagine all of them.

To show something is a vector space, we must verify all of conditions (A1) - (A10) hold.

Examples of Vector Spaces

1. Set of all real numbers $\mathbb{R}$ with addition and multiplication

2. Vectors in $\mathbb{R}^n$ with addition and scalar multiplication

We saw last section these satisfy properties (A1) - (A10).

Ex/ $M_2(\mathbb{R})$

Let V be the set of all 2×2 matrices with real entries, matrix addition and scalar multiplication forms of vector space.

We must check (A1) - (A10)

Pf/ Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$,

$C=\begin{bmatrix} i & j \\ k & l \end{bmatrix}$, $r,s\in\mathbb{R}$

(A1) Pick $A, B \in M_2(\mathbb{R})$

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in M_2(\mathbb{R})$$

(A2) Pick $A \in M_2(\mathbb{R})$, $r \in \mathbb{R}$ then

$$rA = r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \in M_2(\mathbb{R})$$

(A3) Pick $A, B \in M_2(\mathbb{R})$

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= B + A$$

(A4) Pick $A, B, C \in M_2(\mathbb{R})$

$(A + B) + C =$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) + \begin{bmatrix} i & j \\ k & l \end{bmatrix} =$$

$$\begin{bmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{bmatrix} =$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right)$$

(A5) Pick $A \in M_2(\mathbb{R})$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A + O = O + A = A$$

(A6) Pick $A \in M_2(\mathbb{R})$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A + (-A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(A7) For any $A \in M_2(\mathbb{R})$

$$1 \cdot A = A$$

(A8) Pick $r, s \in \mathbb{R}$ $A \in M_2(\mathbb{R})$

$$(rs)A = \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix}$$

$$= r(sA)$$

(A9) $r(A+B) =$

$$\begin{bmatrix} ra + re & rb + rf \\ rc + rg & rd + rh \end{bmatrix} =$$

$$rA + rB$$

(A10) $(r+s)A =$

$$\begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix} =$$

$rA + sA$

Ex/ Real valued functions

Let V be the set of all real valued functions defined on an interval I.

If $f, g \in V$ and k is any real number define:

$$f+g(x) = f(x) + g(x) \quad \text{for all } x \in I$$
$$(kf)(x) = k f(x) \quad \text{for all } x \in I$$

V is a vector space.

. As above, we can verify $(A1)-(A10)$ hold.

To show V is a vector space we need to check all of (A1) - (A10) hold.

To show V is not a vector space we only need to show one of (A1) - (A10) fails

HW3 Show $P_2(\mathbb{R})$ the real vector space of all real valued polynomials of degree ≤ 2 with real coefficients is a vector space. That is,

$$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

with addition and scalar multiplication being the usual operations for polynomials.

HW4 On $M_2(\mathbb{R})$ define the operation of addition $\oplus$ by

$$A \oplus B = AB$$

and use the usual scalar multiplication. Show $M_2(\mathbb{R})$ is not a vector space under $\oplus$ and scalar multiplication

$\boxed{\text{HW5}}$ On $\mathbb{R}^2$ define the operations of addition and scalar multiplication by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

$$k \odot (x_1, y_1) = (kx_1, y_1)$$

Which axioms for a vector space are satisfied by $\mathbb{R}^2$ with these operations?