2.6 The Inverse of a Square Matrix (cont.)

In this section we assume all matrices are $n \times n$ square matrices

Let A be an $n \times n$ matrix if there exists a matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n$$

then we call A^{-1} the inverse of A

We say A is invertible if A^{-1} exists

Thm/ If A^{-1} exists, there is a unique solution to the system of equations $Ax=b$

Thm/ A is invertible $\iff$ $\text{rank}(A)=n$

We found A^{-1} using Gauss-Jordan technique

$$[A \mid I] \sim \ldots \sim [I \mid A^{-1}]$$

HW1 Assume $A^3 = 0$. Show $I - 2A$ is invertible and $(I - 2A)^{-1} = I + 2A + 4A^2$.

HW2 Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 11 \\ 4 & -3 & 10 \end{bmatrix}$. Find A^{-1}.

HW3 Use A^{-1} to find the solution to the system of equations

$$\begin{aligned} x_1 + x_2 - 2x_3 &= -2 \\ x_2 + x_3 &= 3 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \end{aligned}$$

Properties of Inverse

If A and B are $n \times n$ invertible matrices

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Pf/

3. $A^T \cdot (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I$

$(A^{-1})^T \cdot A^T = (A \cdot A^{-1})^T = I^T = I$

2.7 Elementary Matrices

Although we introduced elementary row operations as operations to perform on matrices we could have expressed them in terms of matrix multiplication.

Any matrix obtained by performing a single elementary row operation on the identity matrix is called an elementary matrix

Fact: all elementary row operations can be represented by elementary matrices.

A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

1. The permutation P_{ij} is the permutation matrix that swaps rows i and j of the identity matrix

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{12} A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

2. $M_i(k)$ is represented by the diagonal matrix $diag(1, 1, \dots, k, \dots, 1)$ where k appears in the (i, i) th position.

Ex

$$M_3(2)A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{bmatrix}$$

3. $A_{ij}(k)$ is represented by the matrix with ones on the main diagonal k in the (j,i) position and 0s elsewhere

Ex/

$$A_{32}(-1)A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -3 & -3 & -3 \\ 7 & 8 & 9 \end{bmatrix}$$

Since elementary row operations can be performed on a matrix by premultiplication by an elementary matrix, we can reduce any matrix to row echelon form by multiplying by a sequence of elementary matrices on the left.

In other words, if A is a matrix and U is a row echelon form of A then there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = U$$

Ex/ Reduce $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to row echelon form using elementary matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A_{12}(-3) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$M_2(-1/2) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

In general, if A is any matrix and U is a row echelon form of A then there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = U$$

Since each elementary row operation is reversible it follows each elementary matrix is invertible

It is easy to see what the inverse matrices look like

$$M_i(k)^{-1} = M_i(1/k)$$

$$P_{ij}^{-1} = P_{ij}$$

$$A_{ij}(k)^{-1} = A_{ij}(-k)$$

Note if A is an invertible $n \times n$ matrix the unique reduced row echelon form of the matrix is the identity matrix I_n

It follows there are elementary matrices

$E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \ldots E_2 E_1 A = I_n$$

In particular

$$A^{-1} = \underline{E_k E_{k-1} \ldots E_2 E_1}$$

Thm Let A be an $n \times n$ matrix. Then

A is invertible $\Leftrightarrow$ A is a product of

elementary matrices.

Pf/ "$\Rightarrow$"

Assume A is invertible.

So $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$

$$E_k E_{k-1} \cdots E_2 E_1 A = I$$

$$(E_k E_{k-1} \cdots E_2 E_1)^{-1} =$$

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$$\cancel{(E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1})(E_k E_{k-1} \cdots E_2 E_1)} A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

these are elementary matrices

"$\Leftarrow$" Similar

$\boxed{\text{HW4}}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Write A as a product of elementary matrices.

$\boxed{\text{HW5}}$ Reduce $A = \begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ to row echelon form using elementary matrices.