2.2 Matrix Algebra (cont.)

Last week we learned:

- Matrix Terminology
- Matrix Arithmetic

Let's confirm we remembered the basic points.

For an $n \times n$ matrix we use the usual power notation to denote the operation of multiplying A by itself

$$A^2 = A \cdot A$$

$$A^3 = A \cdot A \cdot A$$

The identity matrix I_n is the $n \times n$ matrix with 1s on the main diagonal and zeros elsewhere

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of I_n

$$A_{m\times n} I_n = A_{m\times n}$$

$$I_m A_{m\times n} = A_{m\times n}$$

$\boxed{HW1}$ If $A = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}$ calculate

A^2 and verify A satisfies

$$A^2 + 4A + 18 I_2 = O_2 .$$

$\boxed{HW2}$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 6 \\ 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Show that $(AB)^T = B^T A^T$.

In fact, this property holds in general.

Properties of Transpose

In 2.1 we introduced the transpose A^T of a matrix A. We may now establish some properties of transpose.

1. $(A^T)^T = A$ clear
2. $(A+C)^T = A^T + C^T$ showed last week
3. $(AB)^T = B^T A^T$

Let's prove Property 3.

$$(AB)^T_{ij} = AB_{ji}$$

$= $ dot product of jth row of A with ith column of B

$$= \sum_{k=1}^{n} a_{jk} \cdot b_{ki}$$

$$= \sum_{k=1}^{n} b_{ki} \cdot a_{jk}$$

$$= \sum_{k=1}^{n} (b_{ik})^T \cdot (a_{kj})^T$$

$$B^T A^T_{ij}$$

So $(AB)^T = B^T A^T$

2.3 Terminology for Systems of Linear Equations

As in the Google pagerank example, we will use matrix equations to solve systems of linear equations

First we introduce some terminology

A linear equation is an equation that may be put in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

A general $m \times n$ system of linear equations has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n
\end{aligned}
\qquad (*)
$$

where the system coefficients a_{ij} and the system constants b_j are given scalars and $x_1, \ldots, x_n$ denote the unknowns in the system. If $b_i = 0$ for all i, the system is called homogeneous, otherwise it is called nonhomogeneous.

A tuple of numbers $(c_1, c_2, \ldots, c_n)$ is a <u>solution</u> to the system (a) if substituting $(c_1, c_2, \ldots, c_n)$ for $(x_1, x_2, \ldots, x_n)$ yields a true numerical statement.

The set of all solutions to a system of equations is called the <u>solution set</u> to the system.

$\boxed{\text{HW3}}$ Verify that for all values of s and t

$$(s,\ s-2t,\ 2s+3t,\ t)$$

is a solution to the system

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0 \\ 2x_2 - x_3 + 7x_4 = 0 \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0 \end{cases}$$

Some of the major questions for this course are:

1. Does a system of linear equations have a solution?

2. If yes, how many solutions are there?

3. How do we determine all solutions?

Note:

If the collection of equations are <u>not linear</u>, there is <u>no general method</u> for finding solutions to the system.

So restricting to linear equations severely limits what is possible.

It is easiest to see how through some geometry.

$\boxed{2D}$

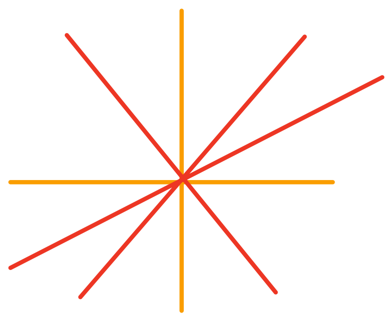
A linear equation of two variables has the form $a_1 x_1 + a_2 x_2 = b$

The graph of a linear equation in the xy-plane is a line

So consider a collection of linear equations involving two variables.

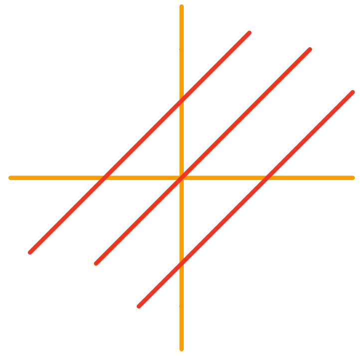
What can happen?

Case 1: All lines intersect in a point



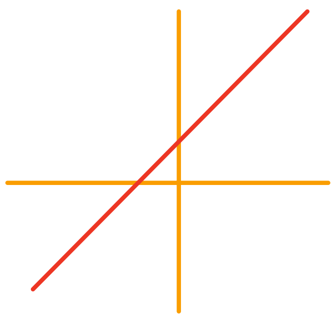
In this case there unique solution

Case 2: Its not the case all lines intersect in a point



In this case there no solution

Case 3: All equations describe the same line.



In this case infinitely many solutions

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So for a system of linear equations involving two unknowns there are exactly three possibilities

1) one solution

2) no solution

3) infinitely many solutions

$\boxed{\text{HW4}}$

Determine graphically how many solutions there are to the system of equations

$$\begin{cases} x_1 - x_2 = 1 \\ x_1 + 2x_2 = 0 \\ x_2 = -2 \end{cases}$$