2.4 Row Echelon Matrices and Elementary Row Operations (cont.)

In this section we learn an algorithm for solving any system of linear equations.

It is based on techniques you used to solve systems of two linear equations in two unknowns in middle school.

Last class:

A $m \times n$ matrix is called a row-echelon matrix if it satisfies the following conditions:

1. If there are any rows of all 0s they occur at the bottom of the matrix

2. The first nonzero element in any nonzero row is a $\underline{1}$ (called a leading 1)

3. The leading $\underline{1}$ of any row below the first row is to the right of the leading $\underline{1}$ of the row above it.

In order to find the solution set of a vector equation $Ax=b$ it is helpful to reduce A to a particular row echelon form.

A $m\times n$ matrix is called a reduced row echelon matrix if it satisfies the following conditions:

1. It is a row-echelon matrix.
2. Any column that contains a leading 1 has zeros everywhere else.

Ex/ Determine whether the following matrices are row echelon, reduced row echelon or neither

$$\begin{bmatrix} 1 & 0 & 12 \\ 0 & 0 & 11 \\ 0 & 0 & 01 \\ 0 & 0 & 01 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & -1 & 70 \\ 0 & 0 & 11 \end{bmatrix}$$

row echelon

$$\begin{bmatrix} 1 & 0 & 25 \\ 1 & 0 & 02 \\ 0 & 1 & 10 \end{bmatrix}$$

row echelon

It's easy to solve systems of linear equations if the corresponding matrix is in row echelon/reduced row echelon form.

Two systems of linear equations are <u>equivalent</u> if they have the same solution set.

The next step is to show for any linear system there is an equivalent system whose corresponding matrix has row echelon form.

We do this using a basic computational tool called elementary row operations.

Idea: Say you have a linear system

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 1 \\
2x_1 + 2x_2 + x_3 &= 2 \\
x_1 + 3x_2 + 3x_3 &= 3
\end{aligned}
$$

Clearly the following operations do not change the solutions to this system

1. Swapping the position of two equations
2. Multiplying an equation by a nonzero number
3. Adding a multiple of one equation to another equation.

When looking at the corresponding matrix this yields the following operations on rows of the augmented matrix

1. P_{ij}: Permuting the ith and jth row of A

2. $M_i(k)$: Multiplying every element of the ith row of A by a nonzero scalar k.

3. $A_{ij}(k)$: Adding k times the ith row to the jth row of A.

$A \sim B$ means the matrix B has been obtained from A by a series of elementary row operations

A matrix B obtained from A by a finite sequence of elementary row operations is <u>row equivalent</u> to A

<u>Thm</u> Every matrix is row equivalent to a row echelon matrix

We prove this by giving an algorithm.

Let's introduce it by example

Ex/ Reduce $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$ to

row echelon form.

Step 1: Put a leading 1 in the (1,1) position

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \overset{P_{12}}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

Step 2: Use the leading 1 to put zeros beneath it in column 1.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \underset{\substack{A_{12}(-2) \\ A_{13}(4) \\ A_{14}(-2)}}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

Step 3: Put a leading 1 in (2,2) position

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \overset{A_{32}(-1)}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

Step 4: Use the leading 1 in the (2,2) position to put zeros beneath it in column 2.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \overset{\substack{A_{23}(-2) \\ A_{24}(-2)}}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

Step 5: Put a leading 1 in the (3,3) position

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix} \overset{M_3(1/13)}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

Step 6: Use the leading 1 in the (3,3) position to put zeros beneath it in column 3.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 9 & 9 \end{bmatrix} \overset{A_{34}(-9)}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final matrix is in row echelon form.

$\boxed{\text{HW1}}$ Put the matrix in row echelon form.

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Although a $m \times n$ matrix is row equivalent to many row echelon matrices it is equivalent to a unique reduced row echelon matrix.

Note once we reduce A to row echelon form it is easy to continue reducing A to reduced row echelon form by eliminating entries above each leading 1.

$\boxed{HW2}$ Determine the reduced row echelon form

of $$\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$\boxed{HW3}$ Determine reduced row echelon form

of

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$