4.3 Subspaces (cont.)

Let S be a nonempty subset of a vector space V. If S is itself a vector space under the same operations of addition and scalar multiplication as used in V, then we say S is a <u>subspace</u> of V.

Thm/ Let S be a nonempty subset of a vector space V. Then S is a subspace of V if and only if S is closed under the operations of addition and scalar multiplication.

Ex/ Let S be the set of all solutions to

$$x_1 - 4x_2 + 6x_3 = 0$$
$$-3x_1 + 10x_2 - 10x_3 = 0$$

Express S in set notation and verify S is a subspace of $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} 1 & -4 & 6 & 0 \\ -3 & 10 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

let $x_3 = t$

$x_2 - 4t = 0 \Rightarrow x_2 = 4t$

$x_1 - 10t = 0 \Rightarrow x_1 = 10t$

So $S = \{ t\langle 10, 4, 1\rangle \mid t \in \mathbb{R} \}$

Claim: S is a subspace of $\mathbb{R}^3$.

Pf: Check (A1) and (A2)

(A1) Pick $t_1\langle 10, 4, 1\rangle$ and $t_2\langle 10, 4, 1\rangle$ in S.

Then $t_1\langle 10,9,1\rangle + t_2\langle 10,9,1\rangle =$

$(t_1+t_2)\langle 10,9,1\rangle \in S$

(A2) Pick $t_1\langle 10,9,1\rangle \in S$

and pick $r \in \mathbb{R}$.

Then $(rt_1)\langle 10,9,1\rangle \in S$

So S is a subspace of $\mathbb{R}^3$.

Thm Let V be a vector space with zero vector 0. Then $S = \{0\}$ is a subspace of V.

Pf

(i) $0 + 0 = 0 \in S$

(ii) Pick $r \in \mathbb{R}$

$r \cdot 0 = 0 \in S$

Let A be an $m \times n$ matrix. The solution set to the corresponding homogeneous linear system $Ax = 0$ is called the null space of A and is denoted

$$\text{nullspace}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

Thm / The nullspace of A is a subspace of $\mathbb{R}^n$.

(i) Pick $x, y \in$ nullspace(A)

Then $Ax = 0$ and $Ay = 0$

But $Ax + Ay = 0$

$$A(x+y) = 0$$

$x+y \in$ nullspace(A)

(ii) Pick $r \in \mathbb{R}$, $x \in$ nullspace(A)

Then $Ax = 0$ and $r \cdot Ax = 0$

$A(rx) = 0 \Rightarrow rx \in$ nullspace(A)

HW10

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Calculate nullspace(A).

HW11

Let $V = M_2(\mathbb{R})$ and S be the subset of all 2×2 matrices with $\det(A) = 1$. Determine whether S is a subspace of V.

HW12

Let $V = P_2$ and S be the subset consisting of all polynomials of the form $p(x) = ax^2 + b$. Determine whether S is a subspace of V.

4.4 Spanning Sets

Since the only operations defined in a vector space V are addition and scalar multiplication the most complicated elements in V have the form:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_k v_k$$

We give a special name for this

An expression of the form:

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

is called a linear combination of $v_1, v_2, \dots, v_k$

It is natural to ask what vectors can be written as linear combinations of $v_1, v_2, \ldots, v_k$.

If every vector in a vector space V can be written as a linear combination of $v_1, v_2, \ldots, v_k$ we say V is <u>spanned</u> or <u>generated</u> by $v_1, v_2, \ldots, v_k$ and call the set of vectors $\{v_1, v_2, \ldots, v_k\}$ a <u>spanning set</u> of V. In this case we say $\{v_1, v_2, \ldots, v_k\}$ <u>spans</u> V.

Ex/ Show $\mathbb{R}^2$ is spanned by vectors

$v_1 = (1,1)$ $\quad$ $v_2 = (-3, 1)$

Pick $(c,d) \in \mathbb{R}^2$

Want to write (c,d) as

$x(1,1) + y(-3,1) = (c,d) \Rightarrow$

$$\begin{aligned} a - 3b &= x \\ a + b &= y \end{aligned} \Rightarrow$$

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

We can find a solution for $\begin{bmatrix} x \\ y \end{bmatrix}$ if

A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

$\det(A) = 4$

So $\{v_1, v_2\}$ is a spanning set.