4.5 Linear Dependence and Independence

It is clear that a vector space can have many spanning sets

For example,

$$\{\langle 0,1\rangle, \langle 1,0\rangle\}, \quad \{\langle 1,0\rangle \langle 0,1\rangle, \langle 2,1\rangle\},$$
$$\{\langle 1,2\rangle, \langle 2,1\rangle\},$$
$$\{\langle 1,2\rangle, \langle 0,0\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$$ are

all spanning sets for $\mathbb{R}^2$.

Natural question: is there a "best" spanning set to work with?

Well, that depends on the context but one natural thing to consider is the size of a spanning set

For instance $\{\langle 1,0\rangle \langle 0,1\rangle, \langle 2,1\rangle\}$, contains an extra vector since

$\{\langle 0,1\rangle, \langle 1,0\rangle\}$ is already a spanning set of $\mathbb{R}^2$.

We say $\{\langle 0,1\rangle, \langle 1,0\rangle\}$ is a minimal spanning set since it contains the smallest number of vectors needed to span $\mathbb{R}^2$.

To determine the minimal spanning set we are faced with the question of determining whether a set contains "extra" vectors or not.

To make this precise we define the following

A finite nonempty set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space V is said to be linearly dependent if there exist scalars $c_1, c_2, \dots, c_k$ not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

.

A finite set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is said to be linearly independent if the only values of the scalars $c_1, c_2, \ldots, c_k$ for which

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

are $c_1 = c_2 = \cdots = c_k = 0$.

Thm Let $\{v_1, v_2, \ldots, v_k\}$ be a set of at least two vectors in a vector space V.

Then $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if at least one of the vectors in the set can be expressed as a linear combination of the others.

Ex/ Determine whether the set of polynomials $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ is linearly dependent or independent in $P_3(\mathbb{R})$ where. If dependent, find a minimal set whose span is equal to span $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$.

$p_1(x) = 1 - 4x^3 \quad p_2(x) = 2 + 2x \quad p_3(x) = 1 + 2x^3 \quad p_4(x) = 2x - x^3$

Dependent if

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

has nontrivial solution i.e. if

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has non trivial solution

We can use row reduction to determine this

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 8 & 6 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 8 & 6 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\tfrac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no leading 1

V_4 is a linear combination of $V_1 - V_3$ so set is dependent.

Since columns corresponding to $V_1 - V_3$ have leading 1s $\{V_1, V_2, V_3\}$ is a minimal set whose span is equal to span$\{V_1, V_2, V_3, V_4\}$

Facts

1. A linearly independent spanning set is a minimal spanning set

2. Any set of two vectors in V is linearly dependent iff one vector is a scalar multiple of the other.

3. Any set of vectors in V containing the zero vector is linearly dependent.

4. Any nonempty finite set of linearly dependent vectors in a vector space V contains a linearly independent subset that has the same span as the given set of vectors.

Thm/ Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and $A = [v_1, v_2, \ldots, v_k]$. Then $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if the linear system $Ac = 0$ has a nontrivial solution.

Let $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

$$Ac = [v_1, v_2, \ldots, v_k] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

$$= c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0$$

has a nontrivial solution

$\Leftrightarrow$

$\{v_1, v_2, \ldots, v_k\}$ are linearly dependent.

Corollary Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and $A = \{v_1, v_2, \ldots, v_k\}$

1. If $k > n$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

2. If $k = n$ then $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if $\det(A) = 0$

Ex/ Determine whether the following vectors are linearly dependent or linearly independent in $\mathbb{R}^4$.

$v_1 = (1,4,1,2)$ $v_2 = (3,-5,1,0)$ $v_3 = (2,0,0,0)$

$v_4 = (-2,3,0,0)$

Let $A = [v_3, v_4, v_2, v_1]$

$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\det(A) = 12 \neq 0$ so $\{v_1, v_2, v_3, v_4\}$ are independent.

$\boxed{\text{HW 7}}$ Determine whether the set of polynomials $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ is linearly dependent or independent in $P_3(\mathbb{R})$ where

$p_1(x) = 1 - 4x^3 \quad p_2(x) = 2 + 2x \quad p_3(x) = 1 - x^2 + 2x^3 \quad p_4(x) = 2x - x^3$

$\boxed{\text{HW8}}$ Determine all values of the constant k for which the given set of vectors is linearly independent in $\mathbb{R}^4$

$$\{(1,0,1,k), (-1,0,k,1), (2,0,1,3)\}$$