4.5 Linear Dependence and Independence (cont.)

We will also need to consider whether a given set of functions is linearly independent (or not).

We say functions $\{f_1, f_2, \ldots, f_n\}$ are linearly independent on an interval I if and only if the only values of the scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_k f_k(x) = 0 \quad \text{for all } x \in I$$

are $c_1 = c_2 = \ldots = c_k = 0$

Note: the condition must hold for all $x \in I$

Similar to above, the determinant can be used as a tool to check whether a collection of functions is linearly independent

Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. The Wronskian of these functions is the determinant defined by

$$W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

Ex/ $f_1(x) = \sin x$ $\quad f_2(x) = \cos(x)$ on $(-\infty, \infty)$

Find $W[f_1, f_2](x)$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$f_1(x) = x$ $\quad f_2(x) = x^2$ $\quad f_3(x) = x^3$ on $(-\infty, \infty)$

Find $W[f_1, f_2, f_3](x)$

$$\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$\boxed{\text{HW 9}}$ Let $f_1(x) = 1$ $f_2(x) = x$ $f_3(x) = 2x - 1$

Find $W[f_1, f_2, f_3](x)$ on $(-\infty, \infty)$

$= \dfrac{e^x + e^{-x}}{2}$

$\boxed{\text{HW 10}}$ Let $f_1(x) = e^x$ $f_2(x) = e^{-x}$ $f_3(x) = \cosh x$

Find $W[f_1, f_2, f_3](x)$ on $(-\infty, \infty)$

Thm/ Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point x_0 in I, then $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on I.

We will revisit this in Ch 8 when considering differential equations.

Note: It is only necessary for $W[f_1, f_2, \ldots, f_k](x)$ to be nonzero at a single point in I for $\{f_1, f_2, \ldots, f_k\}$ to be linearly independent on I

Theorem does not say if $W[f_1, f_2, \ldots, f_k] = 0$ for every $x \in I$ then $\{f_1, f_2, \ldots, f_k\}$ is linearly dependent on I.

If $W[f_1, f_2, \ldots, f_k](x) = 0$ for all $x \in I$, the theorem gives no information as to the linear dependence or independence of $\{f_1, f_2, \ldots, f_k\}$ on I.

Ex/ Determine whether the following functions are linearly dependent on $(-\infty, \infty)$

a) $f_1(x) = e^x$, $f_2(x) = x^2 e^x$

$$W[f_1, f_2](x) = \begin{vmatrix} e^x & x^2 e^x \\ e^x & 2xe^x + x^2 e^x \end{vmatrix}$$

$$= e^x(2xe^x + x^2 e^x) - e^x x^2 e^x$$

$$= 2x e^{2x} \neq 0$$

when $x = 1$

So $\{f_1, f_2\}$ are independent.

b) $f_1(x) = x$ $\qquad f_2(x) = x + x^2$ $\qquad f_3(x) = 2x - x^2$

$$\begin{vmatrix} x & x+x^2 & 2x-x^2 \\ 1 & 1+2x & 2-2x \\ 0 & 2 & -2 \end{vmatrix}$$

$$x\begin{vmatrix} 1+2x & 2-2x \\ 2 & -2 \end{vmatrix} - \begin{vmatrix} x+x^2 & 2x-x^2 \\ 2 & -2 \end{vmatrix}$$

$$x\left(-2-4x-(4-4x)\right) - \left(-2x-2x^2-(4x-2x^2)\right)$$

$$-\cancel{2x} - \cancel{4x^2} - \cancel{4x} + \cancel{4x^2} + \cancel{2x} + \cancel{2x^2} + \cancel{4x} - \cancel{2x^2}$$

$= 0$ for all x

so dependent

$\boxed{\text{HW11}}$ Show that $\{1, x, x^2, x^3\}$ is linearly independent on every interval.

$\boxed{\text{HW12}}$ Determine whether $f_1(x) = e^x$ $f_2(x) = e^{-x}$ $f_3(x) = \cosh x$ are linearly dependent on $(-\infty, \infty)$.