2.4 Row Echelon Matrices and Elementary Row Operations (cont.)

Algorithm for reducing an $m \times n$ matrix A to row echelon form.

1. Start with an $m \times n$ matrix. If $A = 0$ go to (7).
2. Determine the leftmost nonzero column (this is called the pivot column) and the topmost position in the pivot column (this is called the pivot position)

3 Use elementary row operations to put a 1 in the pivot position.

4. Use elementary row operations to put zeros below the pivot position

5. If there are no more nonzero rows below the pivot position go to (7), otherwise go to (6).

6. Apply (2)-(5) to the submatrix consisting of the rows that lie below the pivot position.

7. The matrix is a row echelon matrix.

The number of nonzero rows in any row echelon form of a matrix A is called the rank of A and is denoted $\mathrm{rank}(A)$.

Ex/

Determine the rank of $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 4 & 3 & 6 \end{bmatrix}$

$$\overset{P_{12}}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 4 & 3 & 6 \end{bmatrix}$$

$$\overset{\substack{A_{12}(-2) \\ A_{13}(-4)}}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\overset{A_{23}(1)}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$

$\boxed{HW4}$ Determine the rank of $\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$

$\boxed{HW5}$ Determine the rank of

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

2.5 Gaussian Elimination

It is easy to find the solution set of a system of linear equations once its corresponding augmented matrix has been reduced to row echelon or reduced row echelon form.

Simplest case: A is an $n \times n$ matrix and $\operatorname{rank}(A) = n$.

Ex/ Determine the solution set to:

$$\begin{aligned} 4x_1 - 3x_2 + 6x_3 &= 2 \\ x_1 - 3x_2 + 6x_3 &= 5 \\ -2x_1 + 3x_2 - 8x_3 &= -6 \end{aligned} \qquad (*)$$

First, $(*)$ is represented as the augmented matrix A

$$A = \left[\begin{array}{ccc|c} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{array}\right]$$

Now

$$\left[\begin{array}{ccc|c} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{array}\right] \underset{\sim}{P_{12}}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 4 & -3 & 6 & 2 \\ -2 & 3 & -8 & -6 \end{array}\right] \sim$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Hence $(*)$ is equivalent to the system

$$x_1 - 3x_2 + 6x_3 = 5$$

$$x_2 - 2x_3 = -2$$

$$x_3 = 1$$

which can easily be solved using back substitution.

$$x_3 = 1 \Rightarrow x_2 = -2 + 2 = 0$$

$$\Rightarrow x_1 = 5 + 0 - 6 = -1$$

Solution set is $\{(-1, 0, 1)\}$

This process of reduction to row echelon form and back substitution is called <u>Gaussian elimination</u>

We can also encode the back substitution into matrix calculations by reducing A to a matrix with <u>reduced row echelon form</u>.

This process is called <u>Gauss-Jordan elimination</u>.

In all examples where we've found a solution set to a system of equations there has been a unique solution

How can we tell if there are infinitely many solutions (or no solution), and determine what these solutions are?

It is easy to determine the number of solutions if we know the rank.

Thm/ Consider the $m \times n$ linear system $Ax = b$. Let $A^\#$ be the augmented matrix for the system.

(i) If $\text{rank}(A) = n$, the system has a <u>unique</u> solution

(ii) If $\text{rank}(A) = \text{rank}(A^\#) = r < n$, then the system has an <u>infinite number</u> of solutions indexed by $n-r$ free variables

(iii) If $\text{rank}(A) < \text{rank}(A^\#)$ the system is <u>inconsistent</u> and has no solutions.

Cor/ A homogeneous system $Ax=0$ is consistent (it has at least one solution)

Pf/ Let $x=0$.