For additional time spent working problems attend the Open Discussions on Thursdays 10-12pm in KAP 265

## 2.3 Terminology for Systems of Linear Equations

(cont.)

Some of the major questions for this course are:

1. Does a system of linear equations have a solution?

2. If yes, how many solutions are there?

3. How do we determine all solutions?

Note:

If the collection of equations are not linear, there is no general method for finding solutions to the system

So restricting to linear equations severely limits what is possible.

It is easiest to see how through some geometry.

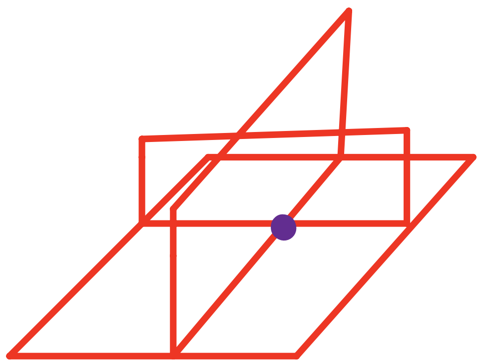
$\boxed{\text{2D}}$ (Last class)

$\boxed{\text{3D}}$ A linear equation in three variables has the form: $a_1x_1 + a_2x_2 + a_3x_3 = b$

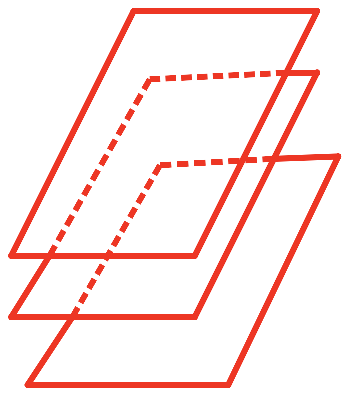
The graph of a linear equation in $\mathbb{R}^3$ is a plane

So consider a collection of linear equations involving three variables.

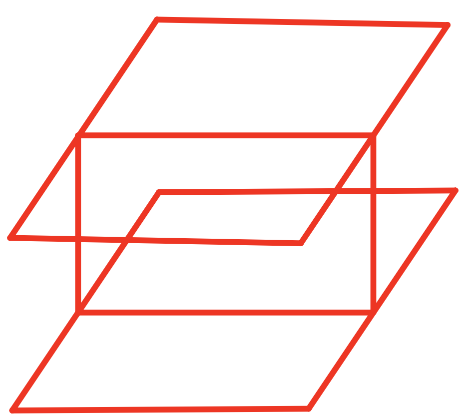
What can happen?



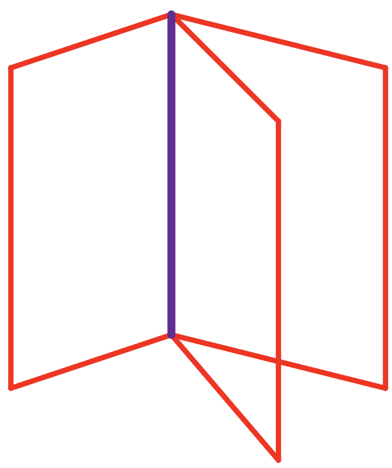
planes intersect at a point:
(unique solution)



parallel planes
(no solution)



no common intersection:
(no solution)



planes intersect in a line:
(infinite number of solutions)

Notice these are exactly the same three possibilities as in the 2D case!

In fact, although we can no longer draw pictures for higher dimensions, this is true for any system of linear equations in $n$ variables.

(We will show this later)

A system of equations that has at least one solution is said to be consistent whereas a system that has no solution is inconsistent

There is a natural way to associate a linear system of equations with a matrix - we associate the entries of the matrix with the coefficients of the equations

For the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n
\end{aligned}
$$

the matrix of coefficients

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

## the augmented matrix

$$A^{\#} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Write the system of equations with the augmented matrix

$$\left[\begin{array}{cccc|c} -2 & 0 & 5 & -1 & 6 \\ 4 & -1 & 2 & 2 & -2 \\ -7 & -6 & 0 & 4 & -8 \end{array}\right]$$

$$-2x_1 + 0x_2 + 5x_3 - x_4 = 6$$

$$4x_1 - 1x_2 + 2x_3 + 2x_4 = -2$$

$$-7x_1 - 6x_2 + 0x_3 + 4x_4 = -8$$

$\boxed{HW5}$ Determine the matrix $A^{\#}$ of the system

$$
\begin{aligned}
x_1 + 2x_2 - 3x_4 &= 1 \\
2x_1 + 4x_2 - 5x_3 &= 2 \\
7x_1 + 2x_3 - x_4 &= 3
\end{aligned}
$$

Conversely, if we have an $m \times n$ matrix system:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

this can be written as the <u>vector equation</u>

$$Ax = b$$

with $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Note that the (real) solutions to the vector equation $Ax = b$ are elements in $\mathbb{R}^n$ .

We write the set of all real solutions of $Ax=b$:

$$S=\{x\in\mathbb{R}^n : Ax=b\}$$

where $S$ is a subset of $\mathbb{R}^n$.

We will spend the next sections developing methods to explicitly determine $S$.

$\boxed{\text{HW 6}}$ Consider the $m \times n$ system $Ax = 0$.

If $x = [x_1\ x_2, \dots, x_n]^T$ and

$y = [y_1\ y_2, \dots, y_n]^T$

are solutions to $Ax = 0$

show that $z = x + y$ and

$w = cx$ are also solutions.

Show this is not true when

$Ax = b$ and $b \neq 0$.

# 2.4 Row Echelon Matrices and Elementary Row Operations

In this section we learn an algorithm for solving systems of linear equations

It is based on techniques you used to solve systems of two linear equations in two unknowns in middle school.

Recall: if you have a system

$$\begin{cases} 2x + y = 6 \\ x + 2y = 3 \end{cases}$$

you may solve it either using substitution or elimination.

We may also use these techniques to solve larger systems

$\boxed{\text{HW 7}}$

Solve $\begin{cases} 2x + y = 6 \\ x + 2y = 3 \end{cases}$

Note if we have a system

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 3 \quad \text{Solve for } x_1 \\
x_2 - 4x_3 &= 1 \quad \text{Solve for } x_2 \\
x_3 &= 4
\end{aligned}
$$

this system is easy solved using

back substitution

The augmented matrix for this system is

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 3 \\
0 & 1 & -4 & 1 \\
0 & 0 & 1 & 4
\end{array}\right]
$$

Note it has all zeros beneath the main diagonal.

This motivates the following definition

A $m \times n$ matrix is called a row-echelon matrix if it satisfies the following conditions:

1. If there are any rows of all $0$s they occur at the bottom of the matrix

2. The first nonzero element in any nonzero row is a 1 (called a leading 1)

3. The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

Eg

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row echelon

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

n.t row echelon

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

n.t rowechelon