Solution to Math 225: Quiz 4

1.

Determine all values of the constant k for which the given system has a unique solution:

$$\begin{cases} x_1 + kx_2 = 2, \\ kx_1 + x_2 + x_3 = 1, \\ x_1 + x_2 + x_3 = 1. \end{cases}$$

Solution 0.0.1 *We know that the given system has a unique solution if and only if (iff) the coefficient matrix* $A = \begin{pmatrix} 1 & k & 0 \\ k & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *is invertible, equivalently,* $det(A) \neq 0$*. So we compute* $det(A)$ *as follows: subtract Row 3 from Row 2 we get* $A' = \begin{pmatrix} 1 & k & 0 \\ k-1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$*. Notice that adding a row to another row does not change the determinant so* $det(A') = det(A)$*, we do cofactor expansion for* A' *starting from Row 2: Notice that the only non-zero element in Row 2 is the first entry, thus we only need to compute the cofactor of the minor* $M_{2,1} = \begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}$*. Thus* $det(A) = det(A') = (-1)(k-1)(k \cdot 1 - 1 \cdot 0) = -k(k-1)$*. It does not equal to zero iff* $k \neq \begin{cases} 0 \\ 1 \end{cases}$*. Thus the given system has a unique solution if and only if* $k \neq \begin{cases} 0 \\ 1 \end{cases}$*.*

2.

Let $V = \mathbb{R}^3$ and let S be the subset of all vectors $(x, y, z) \in V$ such that

$$z = 3x \quad \text{and} \quad y = 2x.$$

Determine whether S is a subspace of V. You must justify your answer.

Solution 0.0.2 *Yes, S is a subspace of V. Notice by description*

$$S = \{\begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} | x \in \mathbb{R}\} = \{x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} | x \in \mathbb{R}\}$$

It is clear now that $\mathbf{0} \in S$ since we can set $x = 0$ and thus $S \neq \varnothing$.

Now we see that S is closed under vector space addition: $\forall v_1, v_2 \in S$ we have that

$$v_1 = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = x_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

for some $x_1, x_2 \in \mathbb{R}$, thus

$$v_1 + v_2 = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (x_1 + x_2) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in S$$

Notice here in the last equation we used distributivity on scalar of scalar multiplication(Axiom 9).

Now we see that S is closed under scalar multiplication: $\forall v \in S$ we have that $v = x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ for some $x \in \mathbb{R}$, thus

$$cv = c(x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}) = (cx) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in S$$

Notice here in the last equation we used associativity on scalar of scalar multiplication (Axiom 8).
Geometrically speaking, S is the vector subspace of a line passing the origin in vector space $V = \mathbb{R}^3$

3.

For the following questions, decide whether the given statements are true or false. If true, give a brief justification. If false, provide an example showing that this is the case.

1. If A is an $n \times n$ matrix, then $\det(A^2)$ cannot be negative.

   Solution 0.0.3 *True, because* $det(A^2) = det(A) \cdot det(A) = (det(A))^2 \geq 0$

2. Assume A and B are square matrices and B is row equivalent to A. Then $\det(A) = \det(B)$.

   Solution 0.0.4 *False, take* $A = I, B = 2I$ *clearly* $RREF(A) = RREF(B)$ *but* $det(A) = 1 \neq 4 = det(B)$ *(row equivalence forgets scaling of rows).*

3. If $R = \mathbb{R}^3$ and S consists of all *(vectors with end)* points on the xy-plane and the xz-plane, then S is a subspace of $\cancel{V}$ R.

   Solution 0.0.5 *False, take* $v_1 = (0, 1, 0)^T$ *from* xy*-plane and* $v_2 = (0, 0, 1)^T$ *from* xz*-plane, we see that* $v_1 + v_2 = (0, 1, 1)^T$ *lying on* yz*-plane, thus* $v_1 + v_2 \notin S$*. The closedness of vector addition fails.*

   Nevertheless, the closedness for scalar multiplication does not fail here. It is a good example to illustrate that the closedness for addition and multiplication does not imply each other. (More generally, union of proper vector subspaces over field consisting infinite elements can never be a vector space).