Solution to Math 225: Quiz 3

1.

Determine the solution set to the system $A\mathbf{x} = 0$ for the given matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{bmatrix}$$

Solution 0.0.1 *The augmented matrix*

$$A^{\#} = \begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{pmatrix}$$

Let us do row reduction: First $\sim$ we add row 1 to row 2 and subtract row 1 to row 3, Second $\sim$ we subtract 2 times row 2 to row 3,

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{pmatrix}$$

Now we convert the reduced $A^{\#}$ back to equation form:

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 0 \\ + x_2 + x_3 + x_4 &= 0 \\ - 2x_3 + x_4 &= 0 \end{aligned}$$

We see that we have one free variable in the sense that once we fix it then we can determine the rest of the variables. Set $x_4 = 2t$, we have that $x_3 = \frac{x_4}{2} = t, x_2 = -x_4 - x_3 = -2t - t = -3t, x_1 = x_4 - 2x_3 - x_2 = 2t - 2t - (-3t) = 3t$. Thus

$$\boldsymbol{x} = \begin{pmatrix} 3t \\ -3t \\ t \\ 2t \end{pmatrix} = t \begin{pmatrix} 3 \\ -3 \\ 1 \\ 2 \end{pmatrix}$$

where $t \in \mathbb{R}$. This is our final solution.

2.

For the following questions, decide whether the given statements are true or false. If true, give a brief justification. If false, provide an example showing that this is the case.

1. If A and B are invertible 2×2 matrices, then AB is invertible.

   Solution 0.0.2 *True, since A, B are invertible, they have inverses A^{-1}, B^{-1}. Use these we claim matrix AB has a inverse, namely $B^{-1}A^{-1}$, we can check as follows:*

   $$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

   By definition, AB is invertible

2. If A and B are invertible 2×2 matrices, then $A + B$ is invertible.

   Solution 0.0.3 *False, take $A = I, B = -I$, their inverses are $A^{-1} = I, B^{-1} = -I$, and $A + B = 0_2$ which is clearly non-invertible.*

3. If A is an invertible matrix such that $A^2 = A$, then A is the identity matrix.

   Solution 0.0.4 *True, Apply A^{-1} on equation $A^2 = A$ we have that*

   $$A^{-1}A^2 = A^{-1}A \Rightarrow A^{-1}AA = I \Rightarrow (A^{-1}A)A = I \Rightarrow A = I$$

   *When we remove the invertibility condition, A is not necessarily the identity matrix any more but the **projection** matrix.*