2.5 Gaussian Elimination (cont.)

We can also find the exact solutions to systems which have infinitely many solutions.

Determine the solution set to

$$\begin{aligned} 5x_1 - 6x_2 + x_3 &= 4 \\ 2x_1 - 3x_2 + x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right] \underset{\sim}{\overset{A_{31}(-1)}{}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right]$$

$$\underset{\sim}{\overset{A_{12}(-2)}{\overline{A_{13}(-4)}}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$$\underset{\sim}{\overset{M_2(1/3)}{}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$$\underset{\sim}{M_{23}(-9)} \begin{array}{c} \\ \end{array} \begin{array}{cccc} & & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice the equation in the last row is $0x_1 + 0x_2 + 0x_3 = 0$, which is true for <u>any</u> x_1, x_2 and x_3.

So although we have three variables we only have two equations relating them and we are free to specify the third variable arbitrarily.

The variable we chose to specify is called a <u>free variable</u>

Rule

Chose free variables for every column which has no corresponding leading 1.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$

$x_2 - t = 1 \Rightarrow x_2 = 1 + t$

$x_1 - 3(1+t) + 2t = -1 \Rightarrow$

$x_1 = 3 + 3t - 2t - 1$

$= 2 + t$

The solution set S to the system is

$$S = \{(2+t, 1+t, t) \mid t \in \mathbb{R}\}$$

$$= \{(2,1,0) + t(1,1,1) \mid t \in \mathbb{R}\}$$

Ex/ Use Gaussian elimination to solve the system

$$
\begin{aligned}
x_1 - 2x_2 + 2x_3 - x_4 &= 3 \\
3x_1 + x_2 + 6x_3 + 11x_4 &= 16 \\
2x_1 - x_2 + 4x_3 + 4x_4 &= 9
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
1 & -2 & 2 & -1 & 3 \\
3 & 1 & 6 & 11 & 16 \\
2 & -1 & 4 & 4 & 9
\end{array}\right] \sim
$$

$$
\begin{array}{c}
\begin{array}{ccccc} & & x_3 & x_4 & \end{array} \\
\left[\begin{array}{cccc|c}
1 & -2 & 2 & -1 & 3 \\
0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

Let $x_4 = t$

$x_3 = s$

$x_2 + 2t = 1 \Rightarrow x_2 = 1 - 2t$

$x_1 - 2(1-2t) + 2s - t = 3$

$x_1 = 2 - 4t - 2s + t + 3$

$= 5 - 2s - 3t$

$S = \{(5-2s-3t, 1-2t, s, t) \mid s, t \in \mathbb{R}\}$

$= \{(5,1,0,0) + s(-2,0,1,0) + t(-3,-2,0,1) \mid s,t \in \mathbb{R}\}$

$\boxed{\text{HW 6}}$ Using Gauss-Jordan elimination determine the solution set to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 + x_3 &= 3 \\ 2x_1 + 6x_2 + 7x_3 &= 1 \end{aligned}$$

$\boxed{\text{HW7}}$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 14 \\ 3x_1 + x_2 - 2x_3 &= -1 \\ 7x_1 + 2x_2 - 3x_3 &= 3 \\ 5x_1 - x_2 - 2x_3 &= 5 \end{aligned}$$

$\boxed{\text{HW8}}$

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 - 2x_5 &= 3 \\ x_3 + 4x_4 - 3x_5 &= 2 \\ 2x_1 + 4x_2 - x_3 - 10x_4 + 5x_5 &= 0 \end{aligned}$$

Solve the following systems

$\boxed{\text{HW 9}}$

$$
\begin{aligned}
2x_1 - 4x_2 + 6x_3 &= 0 \\
3x_1 - 6x_2 + 9x_3 &= 0 \\
x_1 - 2x_2 + 3x_3 &= 0 \\
5x_1 - 10x_2 + 15x_3 &= 0
\end{aligned}
$$

$\boxed{\text{HW10}}$

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 - x_4 &= 3 \\
3x_1 + 2x_2 + x_3 - 5x_4 &= -6 \\
x_1 - 2x_2 + 3x_3 + x_4 &= 6
\end{aligned}
$$

2.6 The Inverse of a Square Matrix

In this section we consider a different viewpoint for solving a system of linear equations. Although computationally it is no more efficient than previous methods it gives us a much better conceptual understanding of linear algebra

In this section we assume all matrices are $n \times n$ square matrices

Say we want to solve a system

$Ax = b$

Assume we know a matrix B where

$$AB = BA = I_n$$

Then instead of using Gaussian elimination, we can easily solve the system using the fact that

$$BAx = Bb$$

$$Ix = Bb \Rightarrow x = Bb$$

Unfortunately finding this matrix B (if it exists) is no more computationally efficient than our earlier methods for solving systems of equations.

Let A be an $n \times n$ matrix if there exists a matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n$$

then we call A^{-1} the inverse of A

We say A is invertible if A^{-1} exists

Note if there is an inverse matrix

it is <u>unique</u>

To see this note if $AB = BA = I_n$ and $AC = CA = I_n$ then

$$C = CI = C(AB) = (CA)B = I \cdot B = B$$

The existence of an inverse (even if we don't know explicitly what it is) allows us to establish many important properties without working directly with explicit matrices

Thm/ If A^{-1} exists, there is a <u>unique</u> solution to the system of equations $Ax = b$

Pf/ (Existence)

$Ax = b \Rightarrow$

$A^{-1}Ax = A^{-1}b \Rightarrow$

$Ix = A^{-1}b \Rightarrow$

$x = A^{-1}b$

(Uniqueness)

Assume $Ax_1 = b$

$Ax_2 = b$

Then $A^{-1}Ax_1 = A^{-1}b$

$A^{-1}Ax_2 = A^{-1}b \Rightarrow$

$x_1 = A^{-1}b = x_2$

The following is a very important equivalence we use throughout the semester.

Thm A is invertible $\iff$ $\text{rank}(A) = n$

Note so far we haven't said how to explicitly find what A^{-1} is

One method is essentially the same thing as Gauss - Jordan elimination

We call it the Gauss - Jordan technique.

Write the matrices A and I next to each other

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

Then use Gauss-Jordan elimination on both A and I to rewrite A in reduced row echelon form. The resulting matrix on the right is A^{-1}.

$$[A \mid I] \sim \cdots \sim [I \mid A^{-1}]$$

Ex/ Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -10 & -3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -14 & -3 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3/14 & 2/14 & -1/14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 5/14 & -6/14 & 3/14 \\ 0 & 1 & 0 & -6/14 & 10/14 & 2/14 \\ 0 & 0 & 1 & 3/14 & 2/14 & -1/14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/14 & -16/14 & 1/14 \\ 0 & 1 & 0 & -6/14 & 10/14 & 2/14 \\ 0 & 0 & 1 & 3/14 & 2/14 & -1/14 \end{array}\right]$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 11 & -16 & 1 \\ -6 & 10 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$