2.6 The Inverse of a Square Matrix (cont.)

Recall: if A is an invertible $n \times n$ matrix the unique reduced row echelon form of the matrix is the identity matrix I_n

It follows there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = I_n$$

In particular

$$A^{-1} = \underline{E_k E_{k-1} \cdots E_2 E_1}$$

Thm Let A be an $n \times n$ matrix. Then

A is invertible $\iff$ A is a product of elementary matrices.

2.8 The Invertible Matrix Theorem 1

We can combine many of the results so far in this course into a single theorem. We will add to this list as the course progresses.

Thm (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following conditions on A are equivalent:

(a) A is invertible
(b) The equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$

(c) The equation $Ax=0$ has only the trivial solution $x=0$

(d) $\operatorname{rank}(A)=n$

(e) A can be expressed as a product of elementary matrices

(f) A is row-equivalent to I_n

Ch 3 Determinants

3.1-3.3 Definition and Properties of Determinant

In this chapter we attach a number to an $n \times n$ matrix which determines many of its important properties

Although we could give a more theoretical view of determinants, we choose to present it from a computational standpoint and explore its properties.

First we describe determinants for $n \times n$ matrices with $n = 1, 2, 3$ then we consider the general case.

$n = 1$

$$A = \begin{bmatrix} a \end{bmatrix}$$

Then the determinant $\det(A) = a$

$n = 2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $\det(A) = ad - bc$

Recall if $ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

This is not a coincidence

$n=3$

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det A = a(ei-fh)-b(di-fg)+c(dh-eg)$$

Notation: we also write $|A|$ for $\det(A)$

For example, if $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det(A) =$ determinant of A.

$\boxed{HW6}$

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$$

Find $\det(A)$

$\boxed{HW7}$

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix}$$

Find $\det(A)$

$\boxed{HW8}$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $|A|$

Now we give an algorithm for calculating determinants for any $n \times n$ matrix.

(Note: much of this is taken from Section 3.3)

Cofactor Expansion

Let A be an $n \times n$ matrix. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A

Ex/ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{23} \cdot a_{32}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11} a_{33} - a_{31} a_{13}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11} a_{23} - a_{21} a_{13}$$

Let A be an $n \times n$ matrix. The cofactor C_{ij} of the element a_{ij} is defined

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Ex

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$C_{12} = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= -1(a_{21}a_{33} - a_{31}a_{23})$$

Thm (Cofactor Expansion)

Let A be an $n \times n$ matrix. If we multiply the elements in any row or column of A by their cofactors, then the sum of the products is $\det(A)$

If we expand along row i, then

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$= \sum_{k=1}^{n} a_{ik}C_{ik}$$

Expanding along column j,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$= \sum_{k=1}^{n} a_{kj}\, C_{kj}$$

This process is called the cofactor expansion of A.

Which row or column we choose to expand depends on the matrix we are given to work with.

Ex

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

Find $\det A$

For ease of calculation we should

expand expand column three

$$\det(A) = 0 \cdot C_{13} - 0 \cdot C_{23} + 2 \cdot C_{33}$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \Rightarrow$$

$$\det(A) = -2$$

$\boxed{\text{HW9}}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $\det A$

$\boxed{\text{HW10}}$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 3 & 0 & 2 \end{bmatrix}$$

Find $\det B$.