HW1

Ch 2 Matrices and Systems of Linear Equations

In this chapter we learn how to do arithmetic with matrices.

2.1 Matrices: Definitions and Notation

rows first columns second

A $m \times n$ matrix is a rectangular array of numbers arranged in m horizontal rows and n vertical columns.

$m \times n$ is the size of the matrix.

Matrices are usually denoted by upper case letters such as A and B.

The entries of a matrix are called the elements of the matrix.

Ex

$$A = \begin{bmatrix} 2 & 2 & 7 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 1 & 3 \\ 5 & 7 & 4 \end{bmatrix}$$ is a 5×3 matrix

We use indices to refer to the elements of a matrix

In the matrix B, b_{ij} refers to the entry in the ith row and jth column of B.

Ex

$$B = \begin{bmatrix} 1 & 2 & 4 & 6 & 1 \\ 3 & 7 & 5 & 6 & 2 \\ 1 & 2 & 3 & 1 & 4 \end{bmatrix}$$

$b_{23} =$ 5

$b_{32} =$ 2

$b_{15} =$ 1

$\boxed{HW1}$

$$B = \begin{bmatrix} 1 & 2 & 4 & 6 & 1 \\ 3 & 7 & 5 & 6 & 2 \\ 1 & 2 & 3 & 1 & 4 \end{bmatrix}$$ is a ______________ matrix.

$\boxed{HW2}$

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 3 \\ 4 & 5 & 2 \\ 2 & 4 & 4 \end{bmatrix}$$

Find c_{23}, c_{32}, and c_{41}

Using index notation, a general $m \times n$ matrix A is written

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or abbreviated by $A = [a_{ij}]$

Two matrices A and B are equal if

1. They both have same size $m \times n$

2. All corresponding elements in the matrices are equal $a_{ij} = b_{ij}$ $1 \le i \le m$, $1 \le j \le n$

Ex/
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

A $1 \times n$ matrix is called a

row vector (or row n-vector)

A $n \times 1$ matrix is called a

column vector (or column n-vector)

$$\begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

row vector

column vector

Note that a matrix is comprised of either row vectors or column vectors, depending on how you want to look at it.

Ex/ Pagerank matrix from last class

$$M = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

Let $a_1 = [\,0 \quad 0 \quad 1 \quad 1/2\,]$

$a_2 = [\,1/3 \quad 0 \quad 0 \quad 0\,]$

$a_3 = [\,1/3 \quad 1/2 \quad 0 \quad 1/2\,]$

$a_4 = [\,1/3 \quad 1/2 \quad 0 \quad 0\,]$

$$b_1 = \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix} \quad b_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad b_4 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

Then $M = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$

row vectors

column vectors

A row or column vector is typically written in bold print (although we will likely not do this at times)

If we interchange the row vectors and column vectors in an $m \times n$ matrix A we obtain an $n \times m$ matrix called the transpose of A.

We denote this matrix by A^T.

In index notation, the (i,j)th element of A^T denoted a_{ij}^T is given by

$a_{ij}^T = a_{ji}$.

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Note that the transpose of a row vector is a column vector (and vice versa)

An $n \times n$ matrix is called a square matrix

If A is an $n \times n$ square matrix the entries a_{ii} $1 \le i \le n$ make up the main diagonal of the matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

main diagonal of 4x4 matrix

The sum of the main diagonal elements of an $n \times n$ matrix A is called the trace of A and is written $tr(A)$.

Although it looks like a silly calculation it is one of the most important invariants in mathematics.

Ex/ $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 4 & 5 \\ 6 & 6 & 2 \end{bmatrix}$

$tr(A) = 7$

$\boxed{\text{HW3}}$

Let $B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 3 & 5 \\ 5 & 6 & 1 & 2 \end{bmatrix}$

Find B^T.

$\boxed{\text{HW4}}$

$B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 1 & 2 \end{bmatrix}$

Find $tr(B)$

An $n \times n$ matrix $A = [a_{ij}]$ is lower triangular if $a_{ij} = 0$ whenever $i < j$ and is upper triangular if $a_{ij} = 0$ whenever $i > j$

An $n \times n$ matrix D is **diagonal** if $a_{ij} = 0$ whenever $i \neq j$.

Ex

$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -7 \end{bmatrix}$ is lower Δ

$B = \begin{bmatrix} -3 & 3 & 4 \\ 0 & -5 & 1 \\ 0 & 0 & 9 \end{bmatrix}$ is upper Δ

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix

A square matrix satisfying $A^T = A$ is called a symmetric matrix

If $A = [a_{ij}]$ then $-A$ is the matrix with elements $-a_{ij}$.

A matrix satisfying $A^T = -A$ is called a skew-symmetric matrix.

The general form of a 3x3 matrix is:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

However the general form of a 3x3 symmetric matrix is:

$$\begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix}$$

$\boxed{\text{HW5}}$ Determine whether a matrix of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is skew-symmetric.

We will see that this matrix describes a geometric transformation of $\mathbb{R}^2$.

It yieds a counterclockwise rotation by angle θ in the xy-plane

In a picture, for $\theta = \pi/2$.

$$M = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix}$$