Teaching unit in university of cologne

UNIVERSITÄT
ZU KÖLN

Oscillating Electronic Circuits

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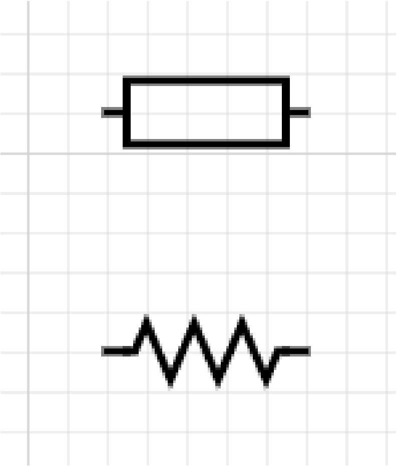
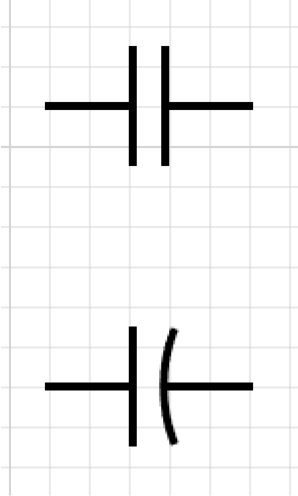
10th July 2024

Recap --- three components

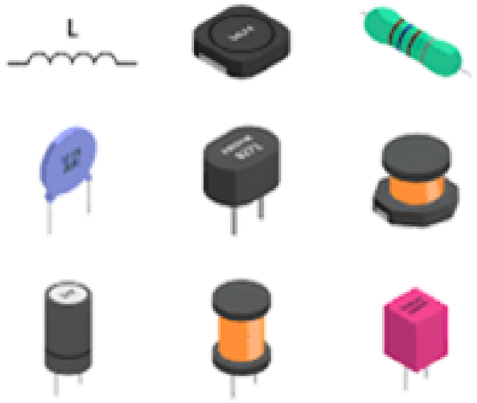
Resistor

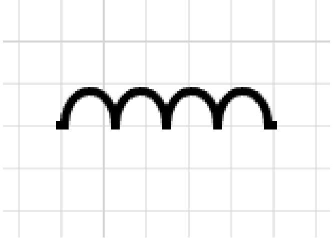


Capacitor



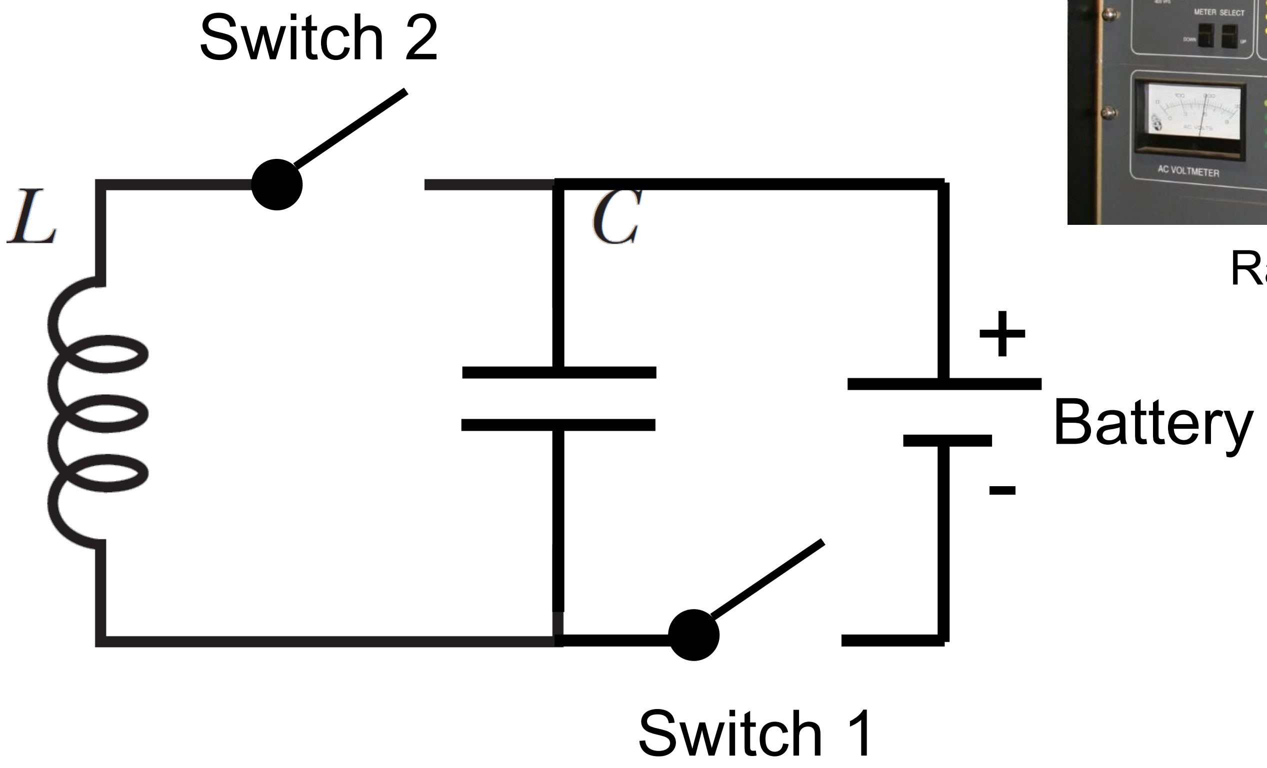
Inductor

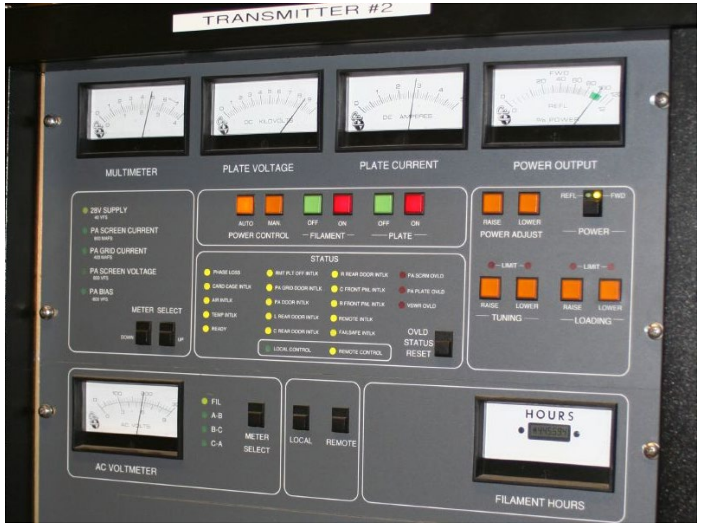




	Unit	Energy	Energy	
Resistor (*R*)	Ohm (Ω)	Dissipation	i^2R	$i = \frac{V}{R}$
Capacitor (*C*)	Farad (F)	Storage in electric field	$\frac{q^2}{2C}$	$i = C\frac{dV}{dt}$
Inductor (*L*)	Henry (H)	Storage in magnetic field	$\frac{Li^2}{2}$	$V = L\frac{di}{dt}$

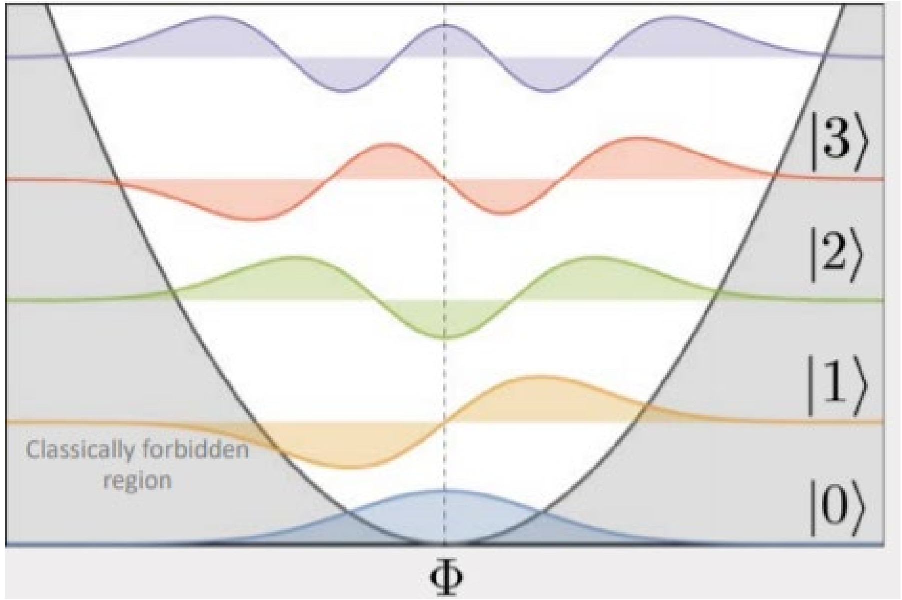
LC circuit





Radio transmitter

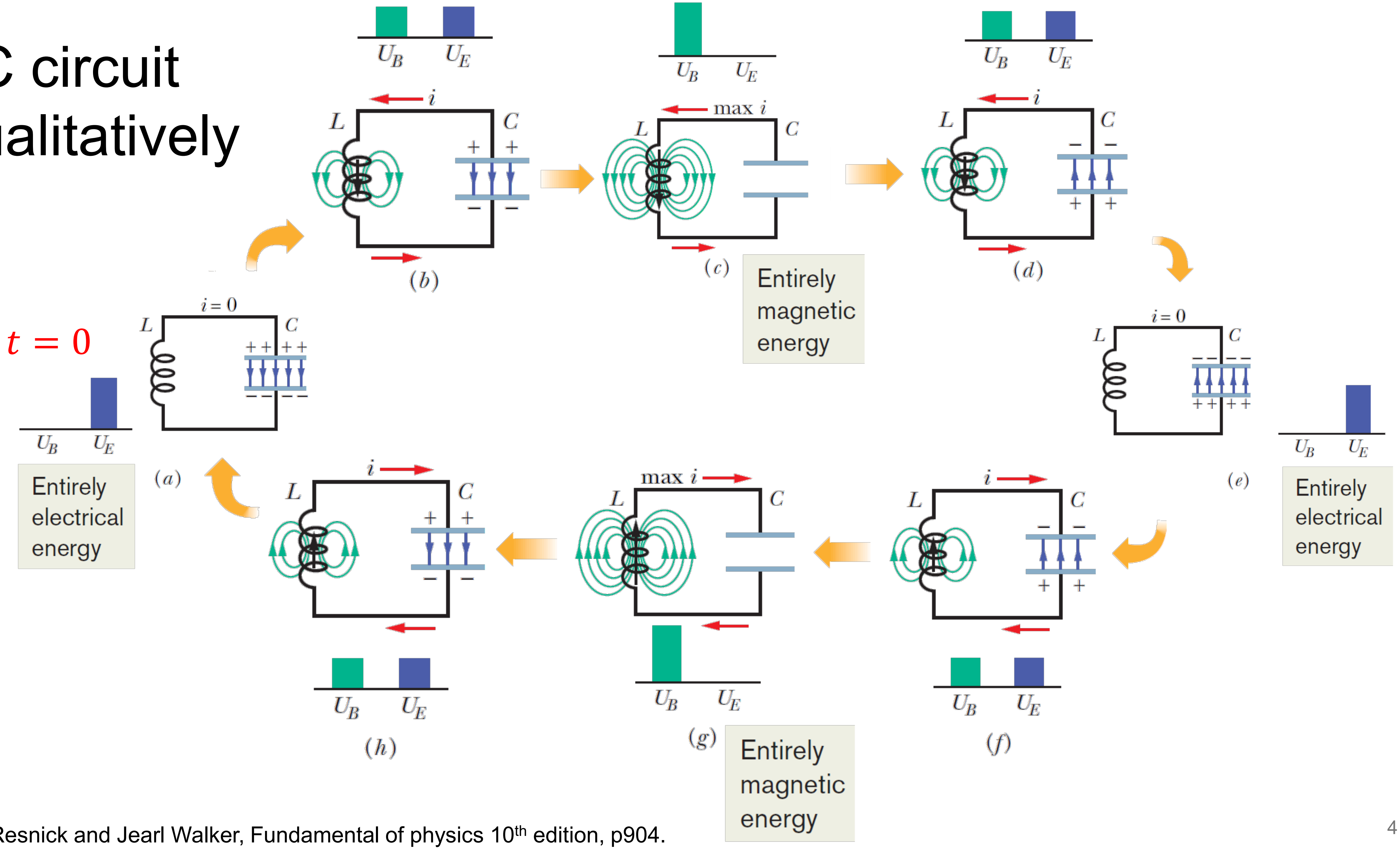
Radio receiver



Quantum region --- harmonic oscillator

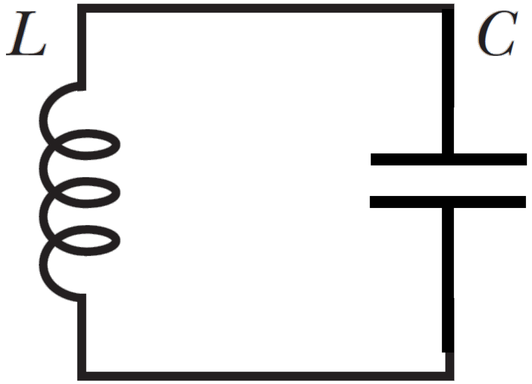
https://en.wikipedia.org/wiki/Transmitter
https://en.wikipedia.org/wiki/Shortwave_radio_receiver
https://grishmaprs.medium.com/introduction-to-transmon-qubits-and-qiskit-pulses-f62621d768c0

LC circuit qualitatively



Halliday, Resnick and Jearl Walker, Fundamental of physics 10th edition, p904.

LC circuit quantitatively



$$q = q(t) \qquad i = i(t)$$

$$U_E = \frac{q^2}{2C} \qquad U_B = \frac{Li^2}{2} \qquad U = U_E + U_B$$

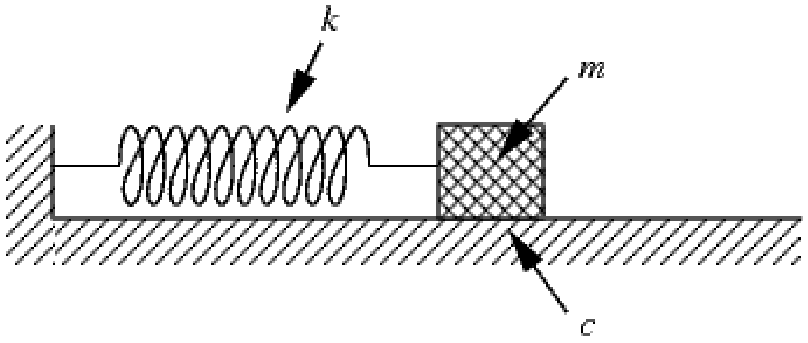
an ideal case   Law of conservation of energy $\frac{dU}{dt} = 0$

$$\frac{1}{2C}\frac{d(q^2)}{dt} + \frac{L}{2}\frac{d(i^2)}{dt} = 0 \qquad \frac{1}{C}q\frac{dq}{dt} + Li\frac{di}{dt} = 0 \qquad i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

LC circuit --- in analogy to the block-spring oscillator

$$U_k = \frac{kx^2}{2} \qquad U_b = \frac{mv^2}{2} \qquad U = U_k + U_b$$



Law of conservation of energy $\quad \frac{dU}{dt} = 0$

$$\frac{1}{2}k\frac{d(x^2)}{dt} + \frac{m}{2}\frac{d(v^2)}{dt} = 0 \qquad kx\frac{dx}{dt} + mv\frac{dv}{dt} = 0 \qquad v = \frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

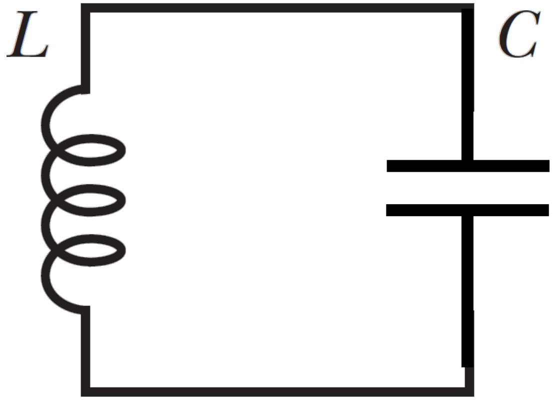
phase constant

$$x = x_0 \cos(\omega t + \phi)$$

amplitude

angular frequency $\quad \omega = \sqrt{\frac{k}{m}}$

LC circuit



$$m\frac{d^2x}{dt^2} + kx = 0$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

$$m \to L \qquad k \to \frac{1}{C}$$

$$q = q_0 \cos(\omega t + \phi)$$

$$i = \frac{dq}{dt} = -q_0\omega \sin(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m}} \to \omega = \frac{1}{\sqrt{LC}}$$

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

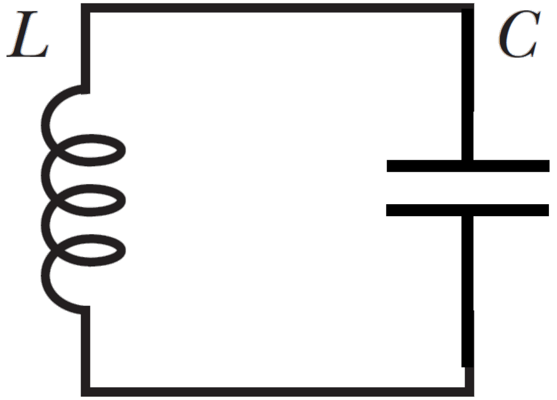
LC circuit

$$q = q_0 \cos(\omega t + \phi)$$

$$i = \frac{dq}{dt} = -q_0 \omega \sin(\omega t + \phi)$$

$$U_E = \frac{q^2}{2C} = \frac{q_0^2}{2C}\cos^2(\omega t + \phi)$$

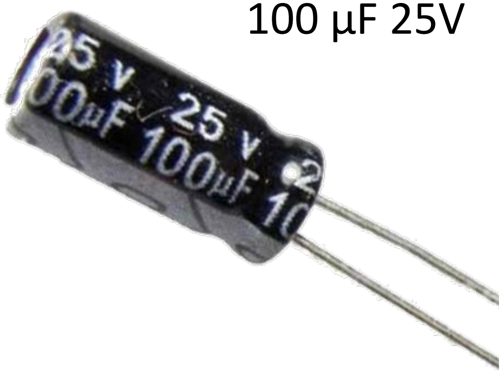
$$U_B = \frac{Li^2}{2} = \frac{Lq_0^2\omega^2}{2}\sin^2(\omega t + \phi)$$

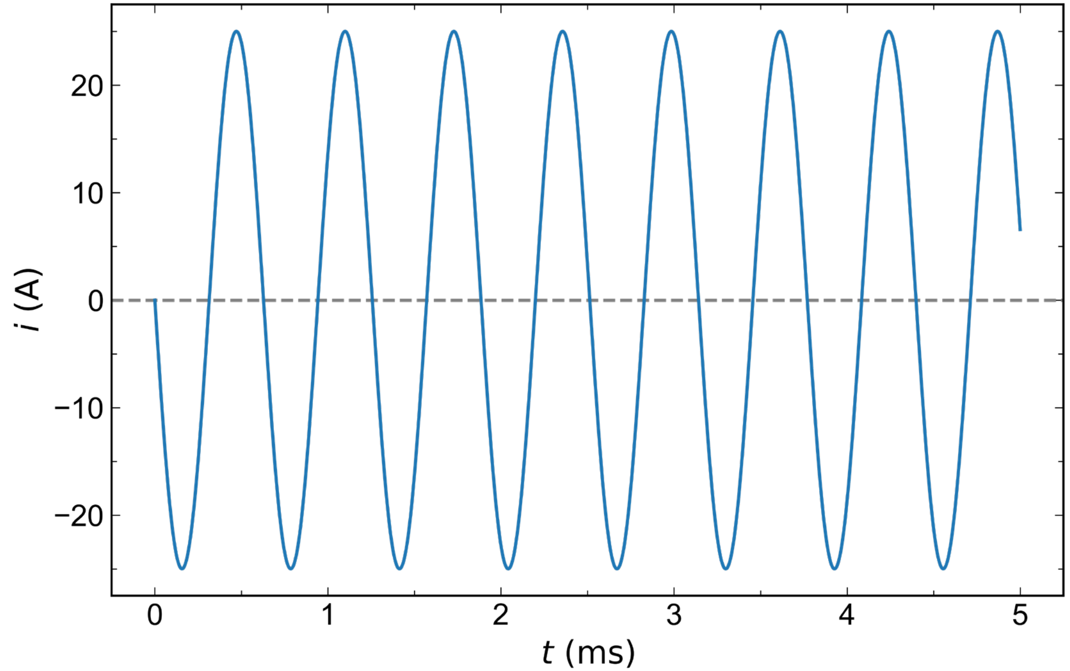


$C = 100\ \mu F$	$L = 100\ \mu H$
$q_0 = 2500\ \mu C$	$\phi = 0$

$$i = -25 \sin(10^4 t) \quad (A)$$

100 µF 25V



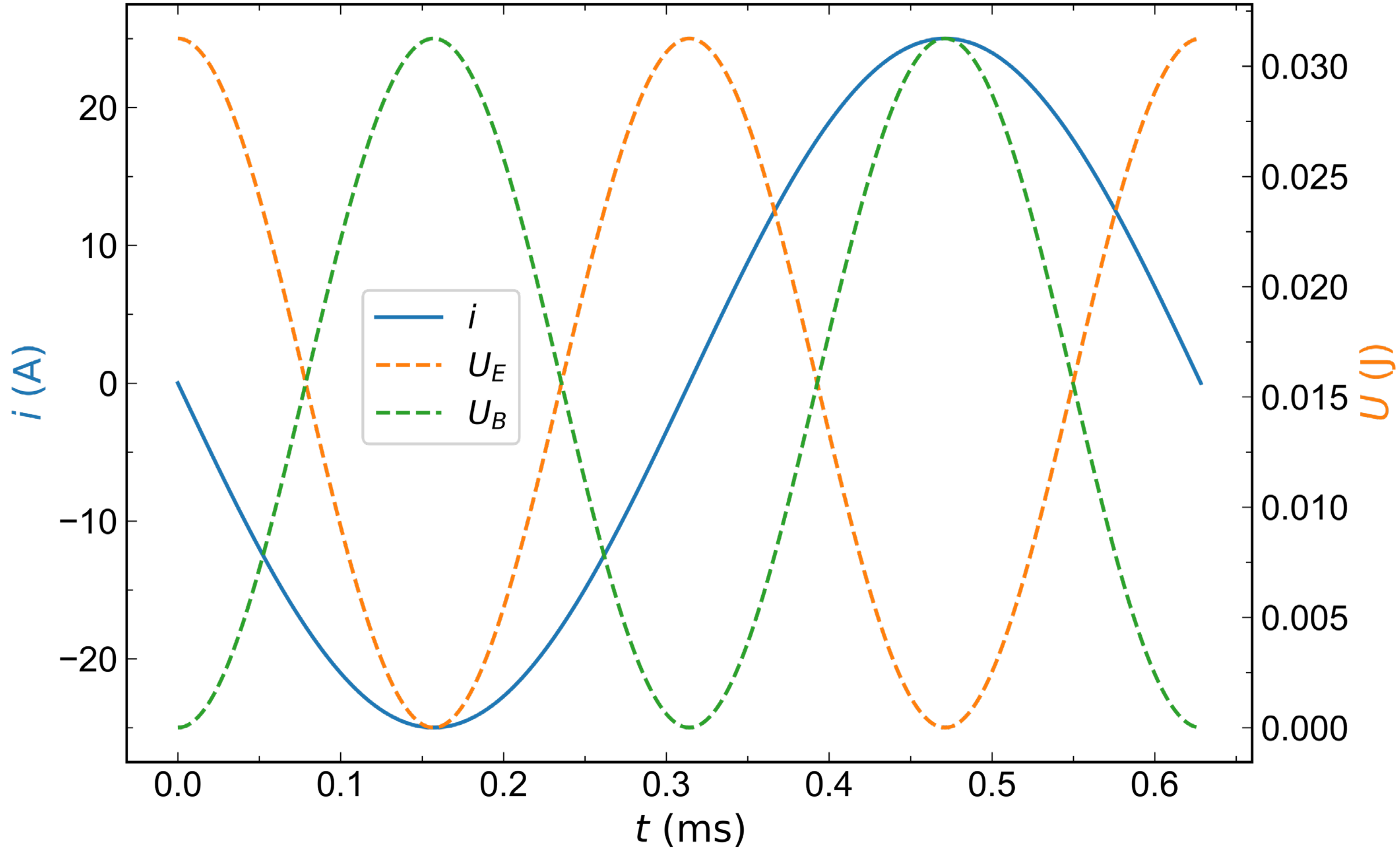


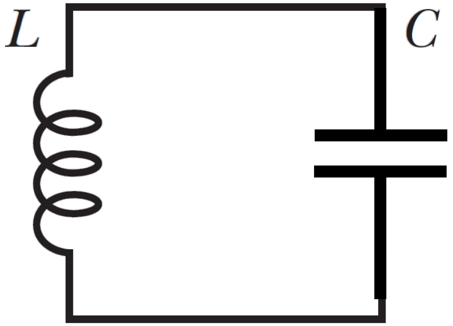
alternating current

LC circuit

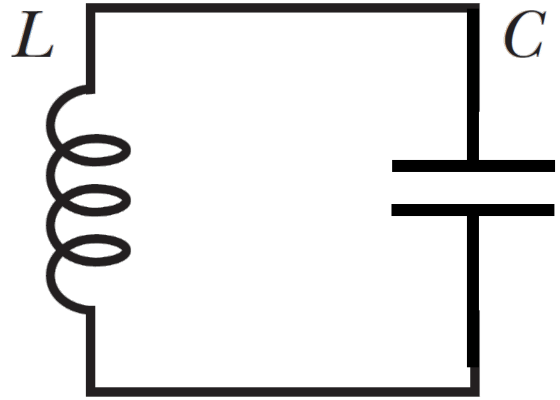
$$U_E = \frac{q^2}{2C} = \frac{q_0^2}{2C}\cos^2(\omega t + \phi)$$
$$= 3.125 \times 10^{-2}\cos^2(10^4 t) \quad (J)$$

$$U_B = \frac{Li^2}{2} = \frac{Lq_0^2\omega^2}{2}\sin^2(\omega t + \phi)$$
$$= 3.125 \times 10^{-2}\sin^2(10^4 t) \quad (J)$$





Question



How to use Kirchhoff's laws to obtain the equation of the LC circuit?

capacitor

inductor

V_c

$$V_L = L\frac{dI_L}{dt}$$

$$L\frac{dI_L}{dt} + V_c = 0$$

$$I_C = C\frac{dV_C}{dt}$$

I_L

$$C\frac{dV_C}{dt} = I_L$$

$$LC\frac{d^2 i}{dt^2} + i = 0$$

A bit more practical (RLC circuit)

$$U_E = \frac{q^2}{2C} \qquad U_B = \frac{Li^2}{2} \qquad U = U_E + U_B$$

$$\frac{dU}{dt} = -i^2 R \quad \Rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

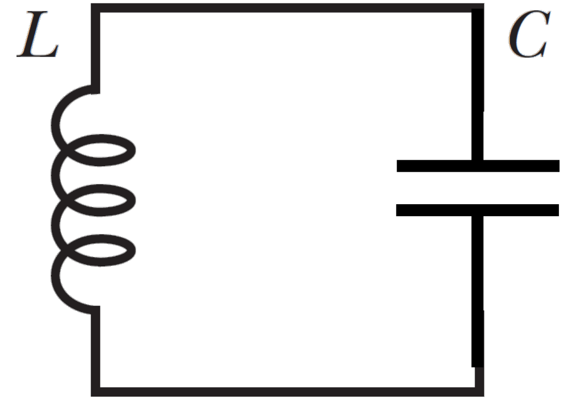
energy dissipation

$$Lx^2 + Rx + \frac{1}{C} = 0$$

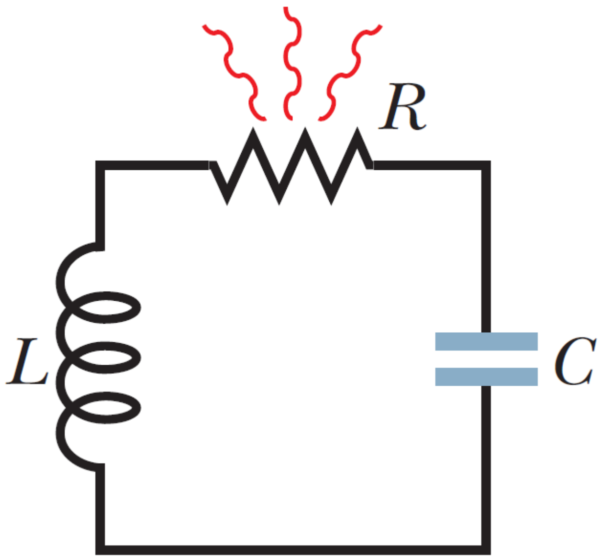
$$x_{1,2} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$
$$= -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$

can be complex numbers

$$\alpha = R/2L$$



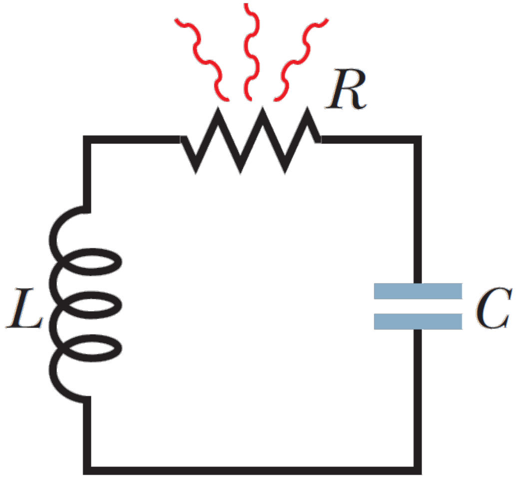
Ideal: resistance=0



RLC circuit

RLC circuit

$$x_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$



A scenario: $\boxed{\alpha = \omega}$

$$x_{1,2} = -\alpha \qquad q = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t}$$

$$q\Big|_{t=0} = K_1 = q_0$$

$$i\Big|_{t=0} = K_2 - \alpha K_1 = 0$$

$$q = q_0 e^{-\alpha t} + \alpha q_0 t e^{-\alpha t}$$

$$i = -\alpha^2 q_0 t e^{-\alpha t}$$

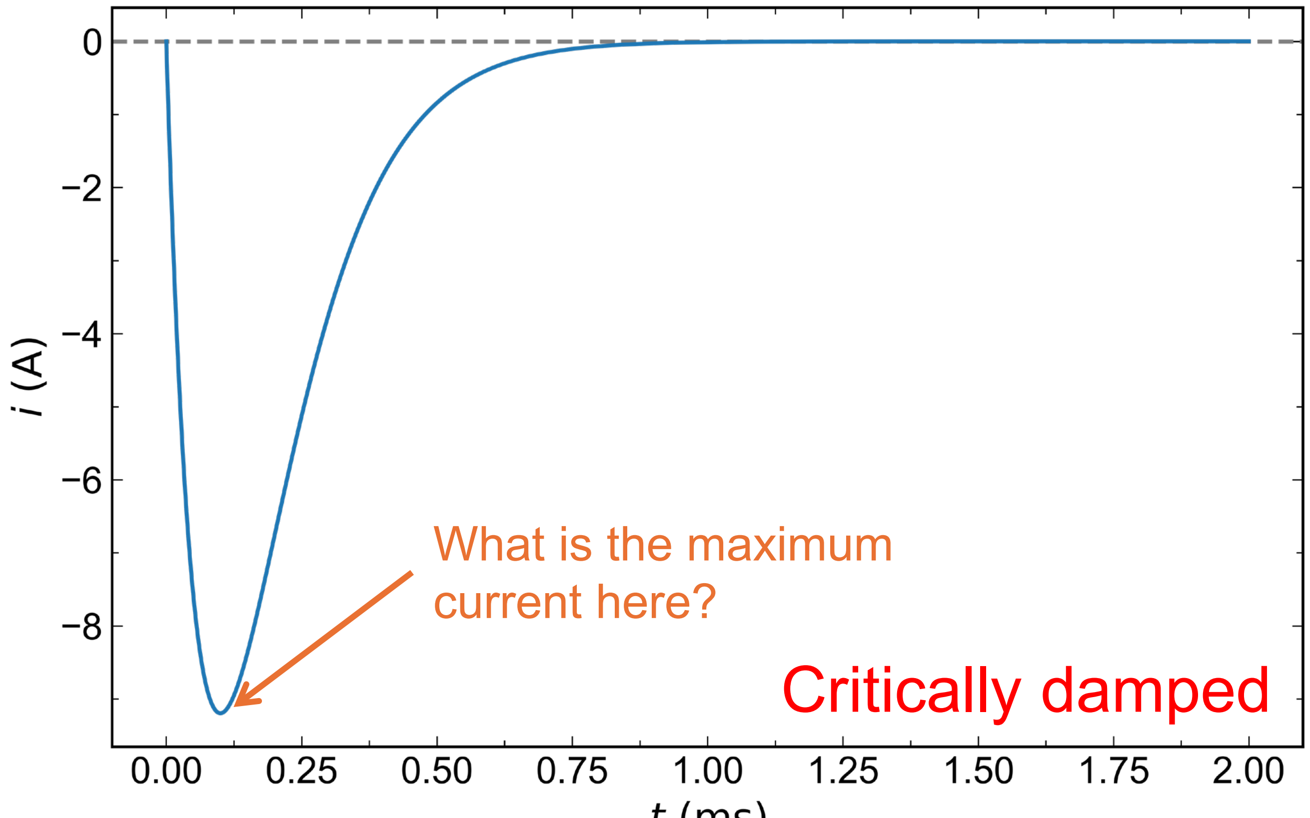
$C = 100\ \mu F \qquad L = 100\ \mu H$

$q_0 = 2500\ \mu C \qquad \phi = 0$

$R = 2\sqrt{L/C} = 2\ \Omega$

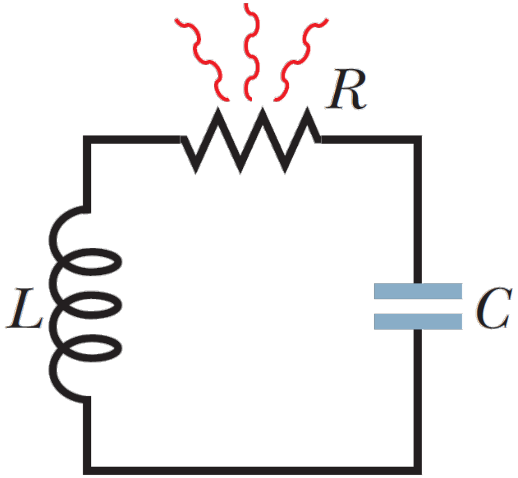
$$q = 2.5 \times 10^{-3} e^{-10^4 t} \quad (C)$$

$$i = -2.5 \times 10^5 t e^{-10^4 t} \quad (A)$$



RLC circuit

$$x_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$



Another scenario: $\boxed{\alpha > \omega}$

$$q = K_1 e^{x_1 t} + K_2 e^{x_2 t}$$

$$q\Big|_{t=0} = K_1 + K_2 = q_0$$

$$i\Big|_{t=0} = K_1 x_1 + K_2 x_2 = 0$$

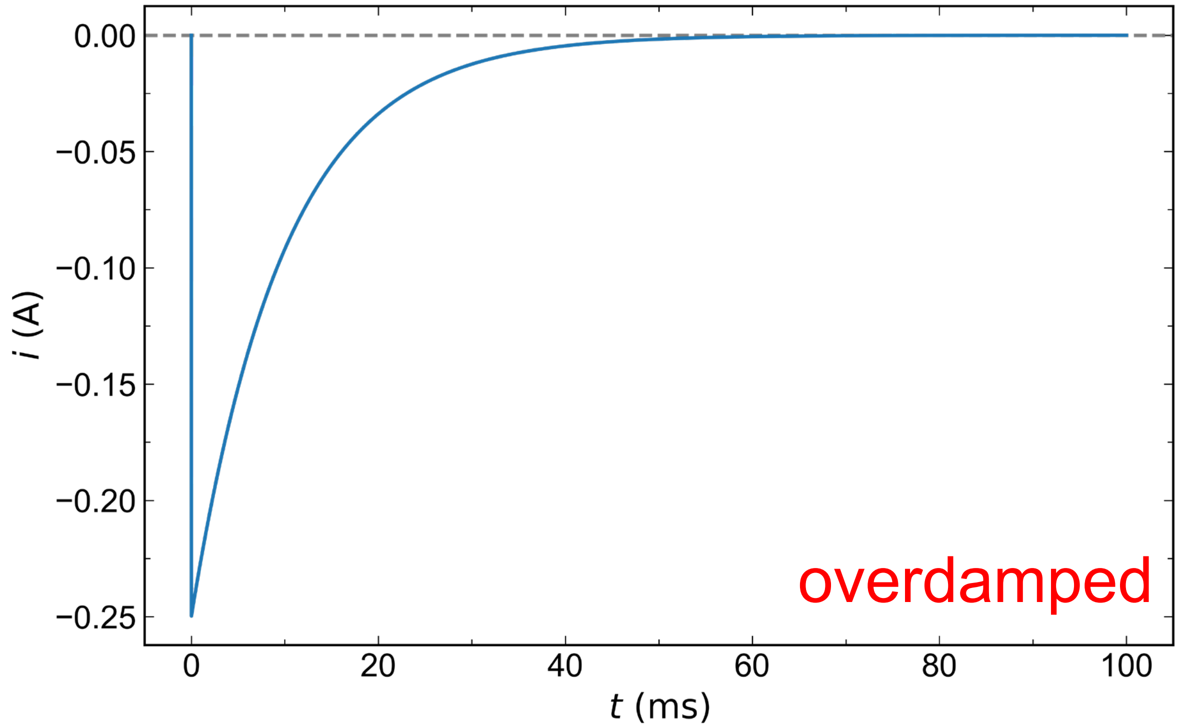
$$\boxed{\begin{aligned} q &= \frac{q_0}{x_1 - x_2}\left(x_1 e^{x_2 t} - x_2 e^{x_1 t}\right) \\ i &= -\frac{q_0 x_1 x_2}{x_1 - x_2}\left(e^{x_1 t} - e^{x_2 t}\right) \end{aligned}}$$

$C = 100\ \mu F \qquad L = 100\ \mu H$

$q_0 = 2500\ \mu C \qquad \phi = 0$

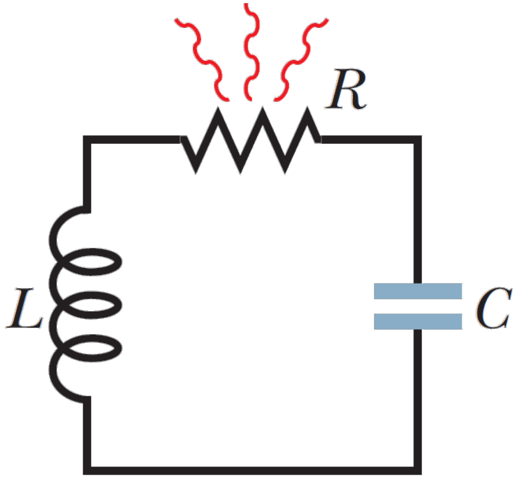
$R = 100\ \Omega > 2\sqrt{L/C}$

$$i \approx 0.25 e^{-999900t} - 0.25 e^{-100t}$$



RLC circuit

$$x_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$



Another scenario: $\boxed{\alpha < \omega}$ $\qquad \beta = \sqrt{\omega^2 - \alpha^2}$

$$q = K_1 e^{(-\alpha + j\beta)t} + K_2 e^{(-\alpha - j\beta)t}$$

Euler's formula $\quad e^{jx} = \cos x + j \sin x$

$$q = (K_1 + K_2)e^{-\alpha t}\cos(\beta t) + j(K_1 - K_2)e^{-\alpha t}\sin(\beta t)$$

Note: K_1 and K_2 can be complex number here

$$A_1 = K_1 + K_2;\ A_2 = K_1 - K_2$$

$$i = -A_1 e^{-\alpha t}[\alpha\cos(\beta t) + \beta\sin(\beta t)] + jA_2 e^{-\alpha t}[\beta\cos(\beta t) - \alpha\sin(\beta t)]$$

$$q\Big|_{t=0} = A_1 = q_0$$

$$i\Big|_{t=0} = -\alpha A_1 + j\beta A_2 = 0$$

$$A_2 = -j\frac{\alpha}{\beta}A_1$$

$$\Rightarrow$$

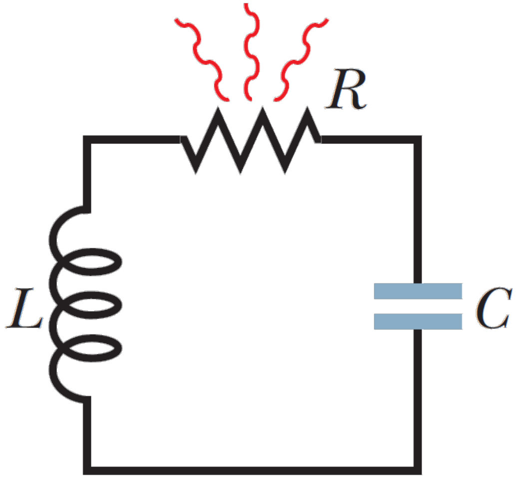
$$\boxed{\begin{aligned} q &= \frac{q_0}{\beta}e^{-\alpha t}[\beta\cos(\beta t) + \alpha\sin(\beta t)] \\ i &= -\frac{q_0(\alpha^2 + \beta^2)}{\beta}e^{-\alpha t}\sin(\beta t) \end{aligned}}$$

RLC circuit

$C = 100\ \mu F \qquad L = 100\ \mu H$

$q_0 = 2500\ \mu C \qquad \phi = 0$

$R = 1\ \Omega < 2\sqrt{L/C}$

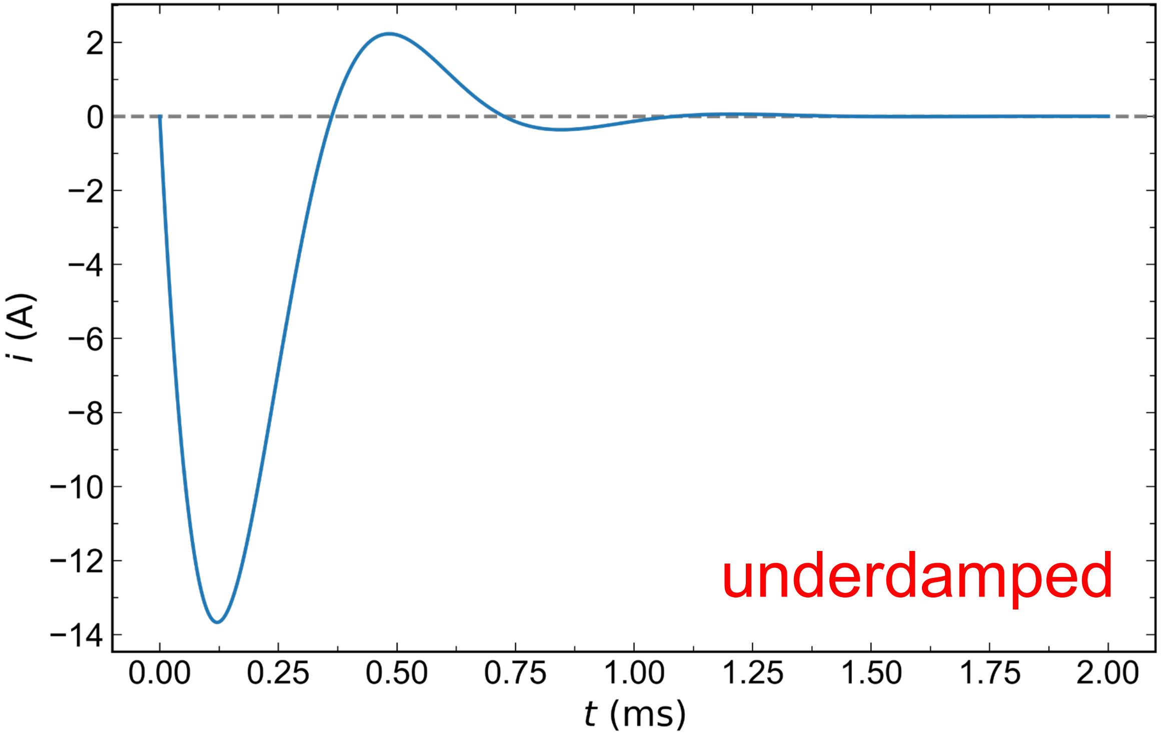


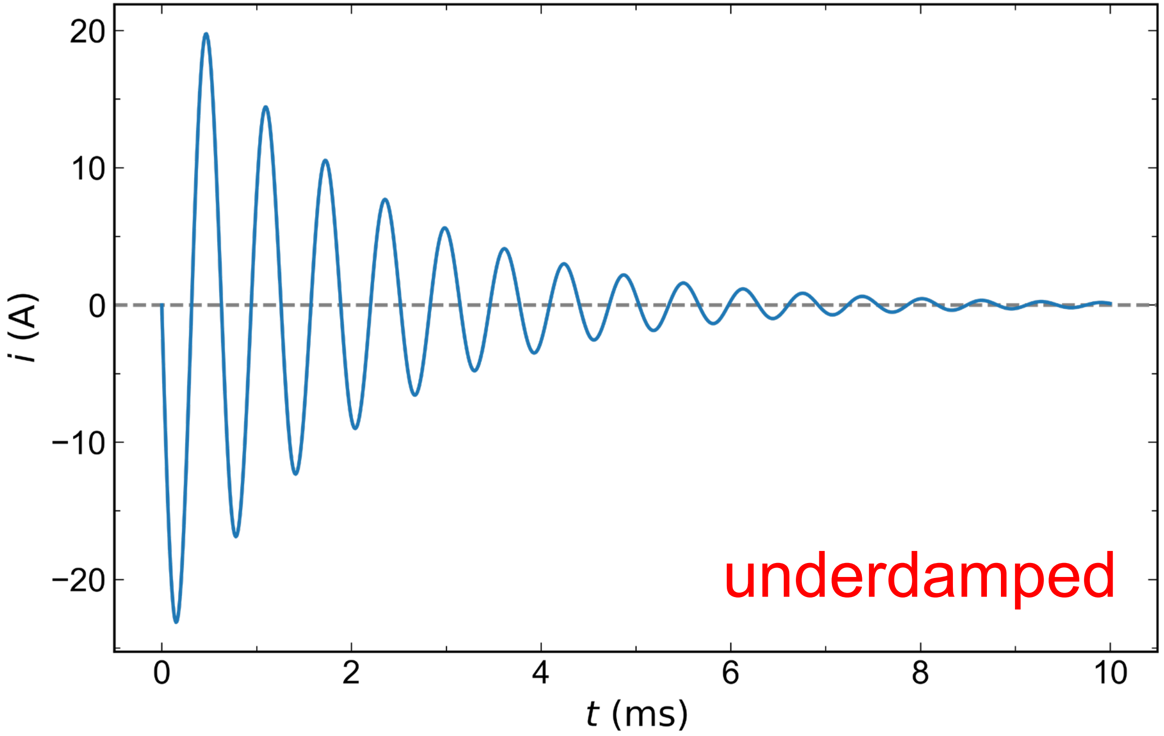
$$q = 0.00289e^{-5000t}\cos(8660t - 0.524)$$

$$i \approx -28.9e^{-5000t}\sin(8660t)$$

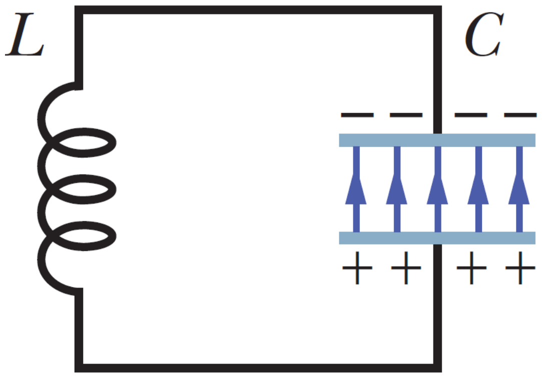
$R = 0.1\ \Omega \ll 2\sqrt{L/C}$

$$i \approx -25e^{-500t}\sin(9987.5t)$$

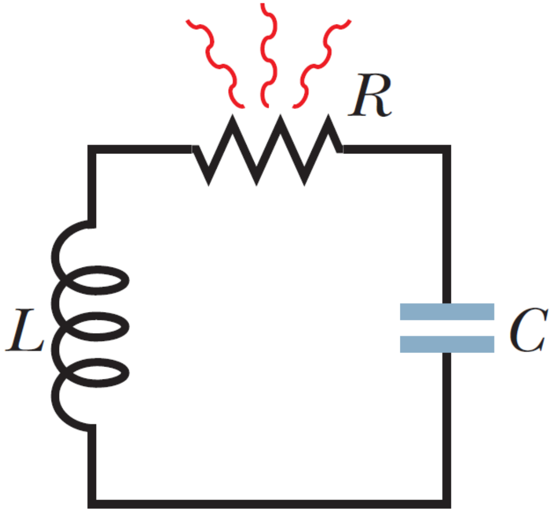




RLC circuit



Ideal: resistance=0



RLC circuit

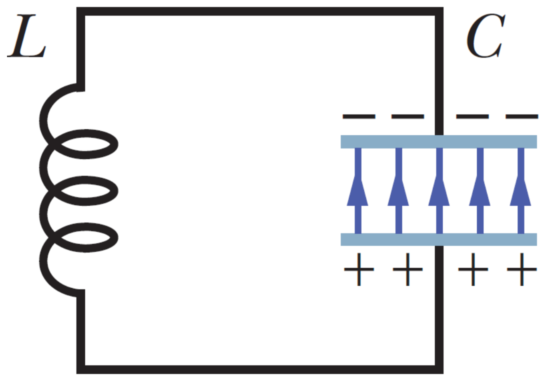
Question: even in an ideal LC circuit, is the energy conserved?

summary

LC circuit:

✓ shows oscillations with a frequency of $\omega = 1/\sqrt{LC}$

✓ Alternatively stores energy in electric and magnetic fields



LRC circuit:

✓ According to the magnitude of resistance, shows overdamped, underdamped or critically damped behavior