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VARIATION OF EXTREMAL LENGTH FUNCTION ON TEICHMÜLLER SPACE

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Abstract. Extremal length is an important conformal invariant on Riemann surface which is closely related to the geometry of Teichmüller metric. By identifying extremal length functions with energy of harmonic maps from Riemann surfaces to $\mathbb{R}$-trees, we study the second variation of extremal length functions along Weil-Petersson geodesics. We show that the extremal length of any simple closed curve is a strictly pluri-subharmonic function on Teichmüller space.

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1. Introduction

Let S be a smooth closed surface of negative Euler characteristic. The Teichmüller space $\mathcal{T}(S)$ is the space of (isotopy classes of) marked hyperbolic structures on S. The aim of this paper is to study the second variation of extremal length functions on $\mathcal{T}(S)$. Note that the first variational formula of extremal length functions along any differential path on Teichmüller space has known for a long time (see for example Gardiner [9]). A new proof using harmonic maps is given by Wentworth [26].

Let γ be a fixed simple closed curve on S. Denote by

$$\mathrm{Ext}_\gamma(\cdot) : \mathcal{T}(S) \to \mathbb{R}_+$$

the extremal length function of γ. The notion of extremal length is due to Ahlfors and Beurling [2]. We will give the definition of extremal length on Section 2. It is directly related to Teichmüller metric due to Kerckhoff's distance formula [15]. Estimates of extremal length along Teichmüller geodesics are important to study the geometry Teichmüller space with the Teichmüller metric [18, 19, 20, 7].

By results of Wolf [28, 29], for any $X \in \mathcal{T}(S)$, $\mathrm{Ext}_\gamma(X)$ can be identified with the energy of a harmonic map from X to a $\mathbb{R}$-tree. To be more precise, there is a $\pi_1(S)$-equivalent harmonic map from the universal cover of X to a $\mathbb{R}$-tree determined by the leaf structure of the measured foliation homotopic to γ. The Hopf differential of such a

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harmonic map realizes the Hubbard-Masur differential of γ, and it turns out that the energy of the harmonic map (restricted on a fundamental domain of X) is equal to $\mathrm{Ext}_\gamma(X)$ (see Section 2 for more details). Such an observation allows us to understand the second variation of $\mathrm{Ext}_\gamma(\cdot)$ along Weil-Petersson or Teichmüller geodesics.

Our main result is:

Theorem 1. Let γ be a simple closed curve on S. The the extremal length function $\mathrm{Ext}_\gamma(\cdot)$ is strictly pluri-subharmonic on $\mathcal{T}(S)$.

The above theorem is proved by showing that

Theorem 2. For any two Weil-Petersson geodesics $\Gamma_i(t), i = 1, 2$ on $\mathcal{T}(S)$ with $\Gamma_1(0) = \Gamma_2(0) = X$ and $\frac{d}{dt}\Gamma_1(0) = \mu$, $\frac{d}{dt}\Gamma_2(0) = i\mu$, where μ is a Beltrami differential represented a tangent vector to $\mathcal{T}(S)$ at X and i denotes the almost complex structure on $\mathcal{T}(S)$, we have

$$\frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_\gamma(\Gamma_1(t)) + \frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_\gamma(\Gamma_2(t)) > 0.$$

Note that extremal length is a conformal invariant, while Weil-Petersson metric is defined by using hyperbolic geometry. Hence Theorem 2 gives some analytic characterization of the extremal length in Teichmüller space in terms of hyperbolic (Weil-Petersson) geometry.

It turns out that harmonic map theory, which has been successfully used by Wolf [27] in the study of Thurston compactification and Weil-Petersson geometry, may be useful to study extremal length functions in Teichmüller space.

In contrast with extremal length, variational formulas for hyperbolic length were throughly studied by Kerckhoff [16], Wolpert [31, 32, 33] and Wolf [30]. The convexity of hyperbolic length was used to solve the Nielsen realization problem [16, 32]. The estimates of Weil-Petersson gradient and Hessian of hyperbolic length are important in the study of Weil-Petersson curvature expansions and geodesic flow [34, 5].

1.1. **Remark.** Note that Tromba [23] has proved that the Dirichlet energy function (with varied domains and fixed target) is strictly pluri-subharmonic on $\mathcal{T}(S)$. More explicitly, fix a hyperbolic structure g_0 on S, for any $X \in \mathcal{T}(S)$, there exists a unique harmonic map $h : X \to (S, g_0)$ homotopic to the identity map of S. Denote the energy of h by $E(X)$. It was shown by Tromba [23] that $E(\cdot)$ defines a strictly pluri-subharmonic function on $\mathcal{T}(S)$. The fact that the target surface (S, g_0) has constant curvature -1 plays an important role in Tromba's proof. While in our situation, the target is flat.

On the other hand, the Dirichlet energy function, with fixed domain and varied targets, is convex along Weil-Petersson geodesics (see [27, 23, 35]). In this case, the Hessian of Dirichlet energy function realizes the Weil-Petersson Riemannian metric. It is then possible to develop Teichmüller theory in terms of harmonic maps by systematic investigation of variations of the target hyperbolic structures, see Wolf [27] and Jost [14] for references.

1.2. **Organization of the paper.** We shall recall Wolf's treatment of extremal length of simple closed curves as energy of harmonic maps from Riemann surfaces to $\mathbb{R}$-trees in Section 2. In Section 3, we obtain a formal formula for the second variation of extremal length functions of simple closed curves on Teichmüller space. Such a formula is simplified in Section 4, where we study the variational vector fields of harmonic maps. Section 5 is devoted to the proof of Theorem 2. Finally we discuss some relation between the variation of extremal length of simple closed curve and Teichmüller metric on Teichmüller space.

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2. Preliminaries

Let S be a smooth closed surface of genus $g > 1$. Let $\mathcal{M}_{-1}$ be the space of hyperbolic metrics on S (complete Riemannian metrics of constant curvature -1 on S) and let Diff_0 be the group of diffeomorphisms of S isotopic to the identity. The Teichmüller space $\mathcal{T}(S)$ of S is defined to be the quotient space $\mathcal{M}_{-1}/\mathrm{Diff}_0$, where Diff_0 acts on $\mathcal{M}_{-1}$ by pulling back.

Note that every hyperbolic metric on S corresponds to a unique conformal structure on S. A surface with a conformal structure is a *Riemann surface*. The Teichmüller space $\mathcal{T}(S)$ is also the set of equivalence classes of marked Riemann surfaces homeomorphic to S. For further background on Teichmüller theory we refer to the book [10].

Throughout the paper, we shall identify a hyperbolic metric on S with its corresponding conformal structure. A hyperbolic metric on S is usually denoted by (S, g) where, in conformal coordinates, g is locally of the form $g = g(z)|dz|^2$. Furthermore, we shall denote an element of $\mathcal{T}(S)$ by (S, g), without explicit reference to the equivalent relation.

2.1. **Extremal length.** Let X be a Riemann surface. A *conformal metric* ρ on X is locally of the form $\rho(z)|dz|$ where z is the locally conformal coordinate of X and $\rho(z) \geq 0$ is Borel measurable. We define the *ρ-area* of X by

$$\mathrm{Area}_\rho(X) = \int_X \rho^2(z)|dz|^2.$$

Recall that an essential simple closed curve on S is *essential* if it is not homotopic to a point. In the following, we shall denote by γ an essential simple closed curve on S or its free isotopy class on S. The

ρ-length of γ is defined by

$$L_\rho(\gamma) = \inf_{\gamma'} \int_{\gamma'} \rho(z)|dz|,$$

where the infimum is taken over all simple closed curves γ' in the isotopy class of γ.

Definition 2.1. With the above notations, the *extremal length* of γ on X is defined by

$$\mathrm{Ext}_\gamma(X) = \sup_\rho \frac{L_\rho^2(\gamma)}{\mathrm{Area}_\rho(X)},$$

where $\rho(z)|dz|$ ranges over all conformal metrics on X with $0 < \mathrm{Area}_\rho(X) < \infty$.

The above definition is called the *analytic definition* of extremal length. There is a geometric definition of $\mathrm{Ext}_\gamma(X)$:

$$\mathrm{Ext}_\gamma(X) = \inf_C \frac{1}{\mathrm{mod}(C)},$$

where the infimum is taken over all embedded cylinders C in X with core curves isotopic to γ and $\mathrm{mod}(C)$ is the conformal modulus of C. As pointed out by Kerckhoff [15], in the estimation of extremal length of simple closed curve on Riemann surface, the analytic definition is useful for finding lower bounds, while the geometric definition is useful for finding upper bounds.

2.2. **Measured foliation and Quadratic differential.** A *measured foliation* on S is a foliation (with a finite number of singularities) endowed with a transversely invariant measure. The allowed singularities are topologically the same as those that occur at $z = 0$ in the line field $z^{p-2}dz^2, p \geq 3$. The *intersection number* $i(\gamma, \mathcal{F})$ of a simple closed curve γ with a measured foliation $\mathcal{F}$ endowed with transverse measure μ is defined by

$$i(\gamma, \mathcal{F}) = \inf_{\gamma'} \int_{\gamma'} d\mu,$$

where the infimum is taken over all simple closed curves γ' in the isotopy class of γ. Two measured laminations $\mathcal{F}$ and $\mathcal{F}'$ are said to be *measure equivalent* if, for all simple closed curves γ on S, $i(\gamma, \mathcal{F}) = i(\gamma, \mathcal{F}')$. Denote by $\mathcal{MF}$ the space of equivalence classes of measured foliations on S.

The leaves passing through a singularity are the *critical leaves* of the measured foliation. There is a special class of measured foliations that have the property that the complement of the critical leaves is homeomorphic to a cylinder. The leaves of the measured foliation on the cylinder are all freely homotopic to a simple closed curves γ. Such a measured foliation is completely determined as a point in $\mathcal{MF}$ by the height a of the cylinder and the isotopy class of γ. Denote such a foliation by (γ, a) or $a\gamma$.

Let $\mathcal{S}$ be the set of isotopy classes of essential simple closed curves on S. Thurston (see [8]) showed that $\mathcal{MF}$ is homeomorphic to a open ball of dimension $6g-g$ and the embedding $\mathcal{S} \times \mathbb{R}_+ \mapsto \mathcal{MF}$ is dense in $\mathcal{MF}$.

A *holomorphic quadratic differential* Φ on (S, g) is a $(2,0)$-tensor locally given by $\Phi = \Phi(z)dz^2$, where $\Phi(z)$ is holomorphic. Any holomorphic quadratic differential $\Phi = \Phi(z)dz^2$ determines a singular metric $|\Phi(z)||dz|^2$, with finitely many singular points corresponding to the zeros of Φ. The total area of S in this metric is given by

$$\int_S |\Phi| = \int_S |\Phi(z)|dz|^2.$$

Let $QD(g)$ be the space of holomorphic quadratic differentials on (S, g). There is a natural identification of $QD(g)$ with the holomorphic cotangent space of $\mathcal{T}(S)$ at (S, g).

Let us describe the relation between quadratic differentials and measured foliations. Firstly, an element $\Phi \in QD(g)$ gives rise to a pair of transverse measured foliations $\mathcal{F}_h(\Phi)$ and $\mathcal{F}_v(\Phi)$ on S, called the *horizontal foliation* and *vertical foliation* of Φ, respectively. The leaves of these foliations are given by setting the imaginary part (resp. real part) of Ψ equal to a constant. In a neighborhood of a nonsingular point, there are natural coordinates $z = x + iy$ so that the leaves of $\mathcal{F}_h$ are given by $y =$ constant, and the transverse measure of $\mathcal{F}_h$ is $|dy|$. The leaves of $\mathcal{F}_v$ are given by $x =$ constant, and the transverse measure is $|dx|$. The foliations $\mathcal{F}_h(\Phi)$ and $\mathcal{F}_v(\Phi)$ have zero set of q as their common singular set, and at each zero of order k they have a $k+2$-pronged singularity, locally modeled on the singularity at the origin of $z^k dz^2$.

On the other hand, according to a fundamental theorem of Hubbard and Masur [12], if $\mathcal{F}$ is a measured foliation on (S, g), then there is a unique holomorphic quadratic differential $\Phi(\mathcal{F}) \in QD(g)$ such that $\mathcal{F}$ is measured equivalent to $\mathcal{F}_v(\Phi(\mathcal{F}))$, the vertical measured foliation of $\Phi(\mathcal{F})$. $\Phi(\mathcal{F})$ is called the *Hubbard-Masur differential* of $\mathcal{F}$. A quadratic differential whose vertical foliation is measure equivalent to a (γ, a) is called a *one-cylinder Strebel differential.*

2.3. **Extending the definition of extremal length.** For any simple closed curve γ on S and $a > 0$, we know that $\mathrm{Ext}_{a\gamma}(X) = a^2 \mathrm{Ext}_\gamma(X)$. Based on the result that $\mathcal{S} \times \mathbb{R}_+$ is dense in $\mathcal{MF}$, Kerckhoff [15] generalized the definition of extremal length of simple closed curves to that of measured foliations.

The following fact is due to Kerckhoff [15].

Proposition 2.2. *The extremal length of any measured foliation $\mathcal{F}$ on (S,g) is equal to the area of $\Phi(\mathcal{F})$, that is,*

$$\mathrm{Ext}_{\mathcal{F}}(g) = \int_S |\Phi(\mathcal{F})|.$$

Proof. The proof given here can be found in Ivanov [13]. We include it for the sake of completeness.

By continuity and the density of weighted simple closed curves in $\mathcal{MF}$ [15], it suffices to prove the proposition for the case that $\mu = a\gamma \in \mathcal{MF}$, where γ is a simple closed curve on S and $a > 0$.

Let Φ be the one-cylinder Strebel differential on X determined by $a\gamma$. The complement of the vertical critical leaves of Φ is a cylinder foliated by circles isotopic to γ. We set $\rho = |\Phi|^{1/2}|dz|$. Then ρ is a flat metric on S, with a finite number of singular points, which are conical singularities of Φ. Measured in the flat metric ρ, the circumference and height of the cylinder are equal to $L_\rho(\gamma)$ and a, respectively. By a theorem of Jenkins-Strebel [22], the extremal length $\mathrm{Ext}_\gamma(g)$ of γ is equal to

$$\mathrm{Ext}_\gamma(g) = \frac{L_\rho(\gamma)}{a},$$

where $\rho = |\Phi|^{1/2}|dz|$.

Since the ρ-area $\mathrm{Area}(\rho) = \int_S |\Phi(z)||dz|^2$ of the cylinder is equal to $aL_\rho(\gamma)$, we have

$$\mathrm{Ext}_{a\gamma}(g) = a^2\mathrm{Ext}_\gamma(g) = aL_\rho(\gamma) = \mathrm{Area}(\rho).$$

□

2.4. **Realizing extremal length by energy of harmonic map.** Consider a measured foliation $(\mathcal{F},\mu)$ on (S,g). It lifts to a measured foliation $(\widetilde{\mathcal{F}},\widetilde{\mu})$ on the universal cover $(\widetilde{S},g)$. Let T be the leaf space of $\widetilde{\mathcal{F}}$. There is a natural projection $\pi : \widetilde{S} \to T$ given by projecting every leaf of $\widetilde{\mathcal{F}}$ to a point. We can define a metric d on T by pushing forward the measure $\widetilde{\mu}$ by the projection π, a.e., $d = \pi_*\widetilde{\mu}$. In this way, (T,d) becomes a real tree (and the real line whenever $\mathcal{F}$ is a simple closed curve). Note that the definition of (T,d) only depends on $(\mathcal{F},\mu)$ and the choice of lifting. The fundamental group $\pi_1(S)$ acts by isometries on (T,d) and the map π is equivalent with respect to this action. See Figure 1.

With the above terminology, now we may state Wolf's result in the form that we need in this paper.

Proposition 2.3 (Proposition 3.1, [29])**.** *There is a $\pi_1(S)$-equivariant map $\omega : (\widetilde{S},g) \to (T,d)$ which is equivariantly homotopic to $\pi : (\widetilde{S},g) \to (T,d)$. Off a discrete set, ω is locally a harmonic projection to a Euclidean line.*

Moreover, the vertical measured foliation of the Hopf differential $\Phi = <\omega_z, \omega_z>_d dz^2$ *of* ω *is measured equivalent to* $(\mathcal{F}, \mu)$*.*

In this paper, the map ω in Proposition 2.3 is called an *equivariant harmonic map* from $(\widetilde{S}, g)$ to (T, d) or, for simplicity, an equivariant harmonic map from (S, g) to the R-tree associated with $(\mathcal{F}, \mu)$.

Definition 2.4. The energy of ω is defined by

$$E(\omega, g) = \frac{1}{2}\int_S |\omega_z|_d^2 + |\omega_{\overline{z}}|_d^2 dz d\overline{z},$$

where the integral domain S is considered as a fundamental domain of $\pi_1(S)$ on the universal cover $\widetilde{S}$.

Since the harmonic map is $\pi_1(S)$-equivariant, the energy is well-defined.

Proposition 2.5. *The extremal length* $\mathrm{Ext}_\gamma(g)$ *is realized as the energy of the harmonic map* ω*.*

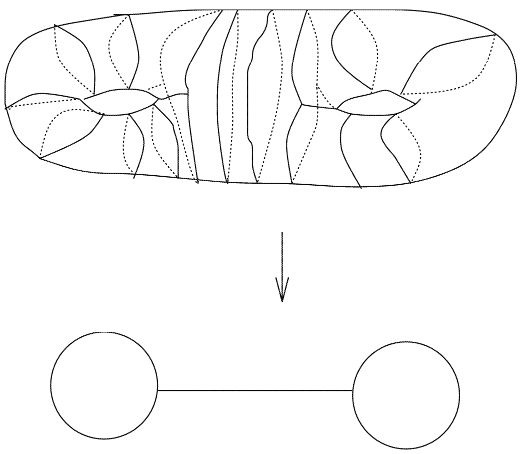

FIGURE 1. The picture is taken from [28]. The projection in the figure lifts to the projection π on the universal cover. The measured foliation in the figure corresponds to the union of three disjoint simple closed curves on the surface. The leaf space of the measured foliation is a $\mathbb{R}$-graph, whose universal cover is a $\mathbb{R}$-tree.

Proof. The proposition follows from the following equations:

$$\begin{aligned} E(\omega, g) &= \frac{1}{2}\int_S |\omega_z|_d^2 + |\omega_{\overline{z}}|_d^2 dz d\overline{z} \\ &= \int_S |\omega_z|_d^2 dz d\overline{z} \\ &= \int_S |\Phi| \\ &= \mathrm{Ext}_{\mathcal{F}}(g). \end{aligned}$$

The second equality holds since the Jacobian $|\omega_z|^2 - |\omega_{\overline{z}}|^2 = 0$ almost everywhere. The last two equalities hold since the Hopf differential Φ of ω is equal to the Hubbard-Masur differential $\Phi(\mathcal{F})$ and, by Proposition 2.2, the area of $\Phi(\mathcal{F})$ is equal to $\mathrm{Ext}_{\mathcal{F}}(g)$. □

3. Second variation of extremal length

3.1. **Weil-Petersson geometry.** The Teichmüller space $\mathcal{T}(S)$ is a complex manifold which can be endowed with a Kähler metric, the Weil-Petersson metric. Recall that tangent vectors to $\mathcal{T}(S)$ at a point (S, g) can be represented by harmonic Beltrami differentials of the form $\mu = \bar{\Phi}/g$, where Φ is a holomorphic quadratic differential on (S, g). The Weil-Petersson Riemannian inner product of two such tangent vectors is the L^2 inner product

$$< \frac{\bar{\Phi}}{g}, \frac{\bar{\Psi}}{g} >= \mathrm{Re}\int_S \frac{\Phi}{g}\frac{\bar{\Psi}}{g} d\mathrm{Area}_g = \mathrm{Re}\int_S \frac{\Phi(z)\overline{\Psi(z)}}{g(z)}|dz|^2.$$

The Weil-Petersson metric is not complete, but it is geodesically convex [32]. This means that any two points in $\mathcal{T}(S)$ can be joined by a unique Weil-Petersson geodesic.

Let (S, g_t) be a smooth family of hyperbolic metrics on S. Denote g_t by $g_t(z_t)|dz_t|^2$ in the conformal coordinates z_t of (S, g_t). For simplicity, we shall denote by $g = g_0$ and $z = z_0$. We always assume that the Beltrami differentials $\mu(t) = \frac{\partial z_t}{\partial \bar{z}} / \frac{\partial z_t}{\partial z}$ satisfy

$$\mu(t) = t\mu + o(t),$$

where $\|\mu\|_\infty < \infty$. We may consider (S, g_t) or $\mu(t)$ as a path in $\mathcal{T}(S)$, with tangent vector μ at the basis point (S, g).

If (S, g_t) agree through second order with a Weil-Petersson geodesic through (S, g), then we say that (S, g_t) is Weil-Petersson geodesic at (S, g). Although the equation for a family of hyperbolic metrics (S, g_t) to be a Weil-Petersson geodesic is unknown, by Ahlfors [1], $\mu(t) = t\frac{\overline{\Phi}}{g}$ is Weil-Petersson geodesic at $t = 0$.

Assumption: Since we mainly consider the second variation along Weil-Petersson geodesics, in the following discussion, we always assume that $\mu(t) = t\mu + o(t)$ which satisfies $\mu = \frac{\overline{\Phi}}{g}$ and $\ddot{\mu} = \frac{d^2}{dt^2}\mu(t)|_{t=0} \equiv 0$.

Remark 3.1. We may assume that (S, g_t) is given by

$$g_t = t\Phi dz^2 + g\left(\mathcal{H}(t) + \frac{t^2|\Phi|^2}{g^2\mathcal{H}(t)}\right) dzd\bar{z} + t\bar{\Phi}d\bar{z}^2$$

where $\mathcal{H}(t)$ is the solution of the Bochner equation

$$\Delta_g \log \mathcal{H}(t) = 2\mathcal{H}(t) - 2\frac{t^2|\Phi|^2}{g^2\mathcal{H}(t)} - 2.$$

Then the identity map id : $(S, g) \to (S, g_t)$ is harmonic; $\mu(t)$ is Weil-Petersson geodesic at $t = 0$. See Wolf [27].

Let $\mathbb{H}^2 = \widetilde{S}$ and Γ be the Fuchsian group of (S, g), that is, $(S, g) = \mathbb{H}^2/\Gamma$. Any Beltrami differential μ on (S, g) lifts to a automorphic form $\widetilde{\mu}(z)$ on $\mathbb{H}^2$, a.e.,

$$\widetilde{\mu}(h(z))\overline{h'(z)}/h'(z) = \widetilde{\mu}(z).$$

for all $h \in \Gamma$.

Assume that $\mu(t) = \frac{\partial z_t}{\partial \bar{z}}/\frac{\partial z_t}{\partial z} = t\mu + o(t)$. We can lift the quasiconformal map $z \to z_t$ to the universal covers such that it satisfies the Beltrami equation

$$f^t_{\bar{z}}(z) = \widetilde{\mu}(t) f^t_z(z)$$

with normalized conditions $f^t(0) = 0, f^t(1) = 1, f^t(\infty) = \infty$.

Denote by

$$V(z) = \lim_{t\to 0} \frac{f^t(z) - z}{t}.$$

The following lemma is due to Ahlfors [1]:

Lemma 3.2. $\frac{\partial}{\partial \bar{z}}V(z) = \widetilde{\mu}(z)$ *holds in the sense of distribution.*

Since $\mu = \mu(z)\frac{d\bar{z}}{dz}$ is a tensor of type $(-1, 1)$, $V(z)$ projects to a tensor of type -1 (a vector field) on (S, g). We shall denote such a vector field by $\dot{z}$.

3.2. **Variations of extremal length.** In this section, we will establish the second variation formula of $\mathrm{Ext}_\gamma(\cdot)$ along Weil-Petersson geodesics, where γ is a simple closed curve on S. One may wishes to extend the formula to the extremal length function of any measured foliation $\mathcal{F}$. However, we don't know about the regularity of the function $\mathrm{Ext}_{\mathcal{F}}(\cdot)$ in general. Another possible generalization is to study the variation of $\mathrm{Ext}_{\mathcal{F}_t}(g_t)$ where $\mathcal{F}_t$ is varied (such a situation appears when one study the Teichmüller distance function between a Teichmüller geodesic and a fixed point).

Let (T, d) be the $\mathbb{R}$-tree associated with γ. Up to scaling, we may assume that (T, d) is isometric to the the real line $\mathbb{R}$ endowed with

Euclidean metric. Let $\omega^t : (S, g_t) \to (T, d)$ be the corresponding harmonic maps. Denote by $E(\omega^t, g_t)$ the energy of ω^t, which is equal to the extremal length of γ on (S, g_t). That is,

$$E(\omega^t, g_t) = \int_S |\frac{\partial \omega^t}{\partial z_t}|^2 dz_t d\overline{z_t}.$$

It follows from the result of Eells-Lemaire [6] that $E(\omega^t, g_t)$ is a real analytic function of t (see also the proof of Lemma 5.6 in [25]). This is equivalent to say that the extremal length of γ is a real analytic function on Teichmüller space. Moreover, the uniqueness of ω^t up to parallel translation in T is due to Hartman [11]. In particular, we may choose a fixed point p in S and choose ω^t such that $\omega^t(p) = 0$ for all t.

Definition 3.3. The variational vector field of ω^t at $t = 0$ is the vector field $\dot{\omega} = \dot{\omega}(z)\frac{\partial}{\partial z}$ satisfying

$$\dot{\omega}(z) = \lim_{t \to 0} \frac{\omega^t(z) - \omega^0(z)}{t}$$

in locally conformal coordinates of (S, g).

Note that $\dot{\omega}_z = \frac{\partial \dot{\omega}(z)}{\partial z}$ is a function on (S, g), and $\dot{\omega}_{\bar{z}} = \frac{\partial \dot{\omega}(z)}{\partial \bar{z}} \frac{d\bar{z}}{dz}$ is a $(-1, 1)$-form on (S, g). As a result, $|\dot{\omega}_z|$ and $|\dot{\omega}_{\bar{z}}|$ are well-defined and $|\dot{\omega}_z|^2$ gives an area form on (S, g).

To study the second variation, first we separate the overall variation into a term that refers only to the second variation of the metrics g_t and another term that refers only to the second variation of the maps ω^t. Such a separation is quite standard for a variational functional (see Wolf [30]).

Formally, we set

$$g_t = g_0 + t\dot{g} + o(t)$$

and

$$\omega^t = \omega^0 + t\dot{\omega} + o(t).$$

One can consider $\dot{g}$ as a tangent vector to $\mathcal{M}_{-1}$ at g_0 and $\dot{\omega}$ as a tangent vector to $C^\infty(S)$ at ω^0. In the following, we denote by $D_1E[\dot{\omega}]$ the directional derivative of E in the direction $\dot{\omega}$ and D_2E the directional derivative of E in the direction $\dot{g}$. The directional derivative of $D_2E[\dot{g}]$ in the direction $\dot{g}$ is denoted by $D^2_{22}E[\dot{g}, \dot{g}]$ and vice versa.

Proposition 3.4 (Separation of second variation)**.**

$$\frac{d^2}{dt^2}|_{t=0} E(\omega^t, g_t) = -D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D^2_{22}E(\omega^t, g_t)[\dot{g}, \dot{g}].$$

Proof. Since the energy of a harmonic map is stationary with respect to variations of the map, we have $D_1E(\omega^t, g_t)[\dot{\omega}] = 0$ and $D_1E(\omega^t, g_t)[\ddot{\omega}] = 0$.

It follows that

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} D_1 E(\omega^t, g_t)[\dot{\omega}] \\ &= D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D^2_{12}E(\omega^t, g_t)[\dot{\omega}, \dot{g}] + D_1 E(\omega^t, g_t)[\ddot{\omega}] \\ &= D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D^2_{12}E(\omega^t, g_t)[\dot{\omega}, \dot{g}]. \end{aligned}$$

As a result,

$$D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] = -D^2_{12}E(\omega^t, g_t)[\dot{\omega}, \dot{g}].$$

The second variation of extremal length function is given by

$$\frac{d^2}{dt^2}|_{t=0} E(\omega^t, g_t) = D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + 2D^2_{12}E(\omega^t, g_t)[\dot{\omega}, \dot{g}]$$
$$+D^2_{22}E(\omega^t, g_t)[\dot{g}, \dot{g}].$$

Thus, we have

$$\frac{d^2}{dt^2}|_{t=0} E(\omega^t, g_t) = -D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D^2_{22}E(\omega^t, g_t)[\dot{g}, \dot{g}]. \tag{1}$$

□

For simplicity, we denote $\omega^0 = \omega$. The terms $D^2_{22}E(\omega^t, g_t)[\dot{g}, \dot{g}]$ and $D^2_{11}E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}]$ are formulated in the following lemmas.

Lemma 3.5. $D^2_{22}E(\omega^t, g_t)[\dot{g}, \dot{g}] = \frac{d^2}{dt^2}|_{t=0} E(\omega, g_t)$ *and*

$$\frac{d^2}{dt^2}|_{t=0} E(\omega, g_t) = 2\int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\overline{z}}|^2) dz d\overline{z}.$$

Proof. By definition, we have

$$E(\omega^t, g_t) = \int_S |\frac{\partial \omega^t}{\partial z_t}|^2 dz_t d\overline{z_t}.$$

Note that

$$dz_t = (z_t)_z dz + (z_t)_{\overline{z}} d\overline{z},$$
$$d\overline{z}_t = (\overline{z}_t)_z dz + (\overline{z}_t)_{\overline{z}} d\overline{z},$$
$$\mu(t) = \frac{(z_t)_{\overline{z}}}{(z_t)_z},$$

$$dz_t d\overline{z_t} = |(z_t)_z|^2 (1 - |\mu(t)|^2) dz d\overline{z}. \tag{2}$$

It follows from the chain rule of differential that

$$\begin{aligned} \omega_{z_t} &= \frac{\omega_z (\overline{z_t})_{\overline{z}} - \omega_{\overline{z}} (\overline{z_t})_z}{|(z_t)_z|^2 (1 - |\mu(t)|^2)} \\ &= \frac{\omega_z - \overline{\mu(t)} \omega_{\overline{z}}}{(z_t)_z (1 - |\mu(t)|^2)}. \end{aligned}$$

As a result,

$$
(3)\qquad |\omega_{z_t}|^2 = \frac{(\omega_z - \overline{\mu(t)}\omega_{\overline{z}})\overline{(\omega_z - \overline{\mu(t)}\omega_{\overline{z}})}}{|(z_t)_z|^2(1-|\mu(t)|^2)^2} = \frac{|\omega_z|^2 + |\mu(t)|^2|\omega_{\overline{z}}|^2 - 2\mathrm{Re}(\mu(t)\omega_z\overline{\omega}_z)}{|(z_t)_z|^2(1-|\mu(t)|^2)^2}.
$$

Combining equation (2) with (3), we have

$$
E(\omega, g_t) = \int_S \frac{|\omega_z|^2 + |\mu(t)|^2|\omega_{\overline{z}}|^2 - 2\mathrm{Re}(\mu(t)\omega_z\overline{\omega}_z)}{1-|\mu(t)|^2} dz d\bar{z}.
$$

Assume that $\mu(t) = t\mu + o(t^2)$, that is, $\dot{\mu} = \mu$ and $\ddot{\mu} \equiv 0$. Then we have

$$
E(\omega, g_t) = \int_S \frac{|\omega_z|^2 + t^2|\mu|^2|\omega_{\overline{z}}|^2 - t2\mathrm{Re}(\mu\omega_z\overline{\omega}_z)}{1-t^2|\mu|^2} dz d\bar{z} + o(t^2).
$$

Now we consider the variation of $E(\omega, g_t)$. Note that

$$
\begin{aligned}
\frac{d}{dt}E(\omega, g_t) &= \int_S \frac{2t|\mu|^2|\omega_{\overline{z}}|^2 - 2\mathrm{Re}(\mu\omega_z\overline{\omega}_z)}{1-t^2|\mu|^2} dz d\bar{z} \\
&\quad - \int_S (|\omega_z|^2 + t^2|\mu|^2|\omega_{\overline{z}}|^2 - t2\mathrm{Re}(\mu\omega_z\overline{\omega}_z)) \frac{-2t|\mu|^2}{|1-t^2|\mu|^2|^2} dz d\bar{z} \\
&\quad + o(t).
\end{aligned}
$$

It follows that

$$
(4)\qquad \frac{d^2}{dt^2}|_{t=0} E(\omega, g_t) = 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\overline{z}}|^2) dz d\bar{z}.
$$

□

Remark 3.6. It follows from the above proof that

$$
\frac{d}{dt}|_{t=0} E(\omega, g_t) = -2\mathrm{Re}\int_S \mu\omega_z\overline{\omega}_z dz d\overline{z}.
$$

Since

$$
\frac{d}{dt}|_{t=0} E(\omega^t, g_t) = D_1E(\omega_0, g_0)[\dot{\omega}] + D_2E(\omega_0, g_0)[\dot{g}]
$$

and

$$
D_1E(\omega_0, g_0)[\dot{\omega}] = 0,
$$

we get the following first variation formula:

$$
(5)\qquad \frac{d}{dt}|_{t=0} E(\omega^t, g_t) = -2\mathrm{Re}\int_S \Phi\mu
$$

where $\Phi = \omega_z\overline{\omega}_z dz^2 \in QD(g_0)$, which is the Hubbard-Masur differential for γ at (S, g_0). Actually, the above formula is valid for any measured foliation (see Gardiner [9] and Wentworth [26]).

The next lemma is to evaluate the term $D^2_{22}E(\omega_0, g_0)[\dot\omega, \dot\omega] = \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0)$ in (1).

Lemma 3.7. *We have*

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) = 2\int_S |\dot\omega_z|^2 dzd\overline{z}.$$

Proof. Set $\omega^t = \omega + t\dot\omega + \frac{t^2}{2}\ddot\omega + o(t^2)$. Then

$$E(\omega^t, g_0) = \int_S |\omega^t_z|^2 dzd\overline{z}$$

and

$$\begin{aligned} |\frac{\partial \omega^t}{\partial z}|^2 &= (\omega_z + t\dot\omega_z + \frac{t^2}{2}\ddot\omega_z + o(t^2))\overline{(\omega_z + t\dot\omega_z + \frac{t^2}{2}\ddot\omega_z + o(t^2))} \\ &= |\omega_z|^2 + t2\mathrm{Re}(\dot\omega_z\overline{\omega}_{\overline{z}}) + t^2(|\dot\omega_z|^2 + \mathrm{Re}(\omega\overline{\ddot\omega}_{z\overline{z}})) + o(t^2). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt}|_{t=0}E(\omega^t, g_0) &= 2\mathrm{Re}\int_S \dot\omega_z\overline{\omega}_{\overline{z}}dzd\overline{z} \\ \text{(integration by parts)} &= -2\mathrm{Re}\int_S \dot\omega\overline{\omega_{z\overline{z}}}dzd\overline{z} \\ \text{(by harmonicity)} &= 0. \end{aligned}$$

And then

$$\begin{aligned} \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) &= 2\int_S |\dot\omega_z|^2 dzd\overline{z} + 2\mathrm{Re}\int_S \omega_z\overline{\ddot\omega}_{\overline{z}}dzd\overline{z} \\ &= 2\int_S |\dot\omega_z|^2 dzd\overline{z} - 2\mathrm{Re}\int_S \omega_{z\overline{z}}\overline{\ddot\omega}dzd\overline{z} \\ \text{(since } \omega_{z\overline{z}} = 0) &= 2\int_S |\dot\omega_z|^2 dzd\overline{z}. \end{aligned} \tag{6}$$

□

Combining (1), (4) and (6), we have

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g^t) = 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\overline{z}}|^2)dzd\overline{z} - 2\int_S |\dot\omega_z|^2 dzd\overline{z}.$$

In conclusion, we have the following:

Theorem 3.8. *For any simple closed curve γ, if g_t is a Weil-Petersson geodesic in $\mathcal{T}(S)$ with Beltrami differential $\mu(t) = t\mu + o(t^2)$ at $t = 0$, and ω is the harmonic map from $(\widetilde{S}, g_0)$ to the $\mathbb{R}$-tree determined by γ, then*

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_\gamma(g_t) &= 2\int_S |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2)dzd\bar z - 2\int_S |\dot\omega_z|^2 dzd\bar z\\ &= 4\int_S |\mu|^2|\omega_z|^2 dzd\bar z - 2\int_S |\dot\omega_z|^2 dzd\bar z.\end{aligned}$$

4. The vector field $\dot\omega$

We have shown that

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t,g_0) = 2\int_S |\dot\omega_z|^2 dzd\bar z.$$

In order to compare this formula to formula (4), it is important to find an expression for $\dot\omega$ or $\dot\omega_z$ in terms of μ and ω. Note that we can take ω to be real and then (4) becomes

$$\frac{d^2}{dt^2}|_{t=0}E(\omega,g_t) = 4\int_S |\mu|^2|\Phi(z)|dzd\bar z,$$

where $\Phi(z)dz^2 = \omega_z\overline{\omega}_z dz^2$ is the Hopf differential of ω. It will be interesting to give an expression of $\dot\omega$ or $\dot\omega_z$ in term of μ and Φ.

Consider the harmonic map equation $\frac{\partial^2\omega^t}{\partial z_t\partial\overline{z_t}} = 0$, denoted by $H(\omega^t,z_t) = 0$. Differentiating in t, we have

$$-\frac{d}{dt}|_{t=0}H(\omega,z_t) = \frac{d}{dt}|_{t=0}H(\omega^t,z) = \dot\omega_{z\bar z}. \tag{7}$$

To compute $\frac{d}{dt}|_{t=0}H(\omega,z_t)$, we express the operator $\frac{\partial^2}{\partial z_t\partial\overline{z_t}}$ as

$$\begin{aligned}&\{\frac{1}{1-t^2|\mu|^2}\frac{1}{(z_t)_z}(\partial_z - t\overline{\mu}\partial_{\bar z})\}\circ\{\frac{1}{1-t^2|\mu|^2}\frac{1}{\overline{(z_t)_z}}(-t\mu\partial_z+\partial_{\bar z})\} + o(t)\\ &\quad = \{\frac{1}{(z_t)_z}(\partial_z - t\overline{\mu}\partial_{\bar z})\}\circ\{\frac{1}{\overline{(z_t)_z}}(-t\mu\partial_z+\partial_{\bar z})\} + o(t).\end{aligned}$$

It follows that

$$\begin{aligned}H(\omega,z_t) &= -t(\mu_z\omega_z+\mu\omega_{zz}) + (-t\mu\omega_z+\omega_{\bar z})(-\overline{(z_t)_{z\bar z}})\\ &\quad -t\overline{\mu}(\omega_{\bar z\bar z}+\overline{(z_t)_{z\bar z}}\omega_{\bar z}) + o(t).\end{aligned}$$

As a result,

$$-\frac{d}{dt}|_{t=0}H(\omega,z_t) = \mu_z\omega_z+\mu\omega_{zz}+\omega_{\bar z}\overline{\dot z_{z\bar z}}+\overline{\mu}\omega_{\bar z\bar z}.$$

By Lemma 3.2, $\mu = \dot z_{\bar z}$, hence $\overline{\dot z_{z\bar z}} = \overline{\mu_z}$, and then

$$\begin{aligned}-\frac{d}{dt}|_{t=0}H(\omega,z_t) &= \mu\omega_{zz}+\overline{\mu}\omega_{\bar z\bar z}+\overline{\mu_z}+\mu_z\omega_z\\ &= \frac{\partial}{\partial z}(\mu\omega_z)+\frac{\partial}{\partial\bar z}(\overline{\mu}\omega_{\bar z}).\end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$\dot{\omega}_{z\bar{z}} = \frac{\partial}{\partial z}(\mu\omega_z) + \frac{\partial}{\partial \bar{z}}(\overline{\mu}\omega_{\bar{z}}). \tag{9}$$

Using (9) and integration by part, we have

$$\begin{aligned}
\int_S |\dot{\omega}_z|^2 dz d\bar{z} &= -\int_S \bar{\dot{\omega}}\dot{\omega}_{z\bar{z}} dz d\bar{z} \\
&= -\int_S \frac{\partial}{\partial z}(\mu\omega_z)\bar{\dot{\omega}} + \frac{\partial}{\partial \bar{z}}(\overline{\mu}\omega_{\bar{z}})\bar{\dot{\omega}} dz d\bar{z} \\
&= \int_S \mu\omega_z\bar{\dot{\omega}}_z + \overline{\mu}\omega_{\bar{z}}\bar{\dot{\omega}}_{\bar{z}} dz d\bar{z}. \qquad (10)
\end{aligned}$$

The equality (10) will be used in next section to prove Theorem 2.

5. Pluri-subharmonicity

In this section we prove Theorem 1 by showing that

Theorem 5.1. *Let γ be a simple closed curve on S and let $E(\cdot) : \mathcal{T}(S) \to \mathbb{R}$ be the energy of harmonic maps from Riemann surfaces to the* R*-tree associated with γ. Then $E(\cdot)$ is strictly pluri-subharmonic on $\mathcal{T}(S)$.*

A real valued function F on a complex manifold M is *(strictly) pluri-subharmonic* if the *Levi-form $\partial\overline{\partial}F$* is (positive) definite at each point of M. Recall that $\partial\overline{\partial}F$ is a 2-form defined by

$$\partial\overline{\partial}F = \frac{\partial^2 F}{\partial z^\alpha \partial \overline{z}^\beta} dz^\alpha d\overline{z}^\beta$$

in holomorphic coordinates. If $\xi = \{\xi^\alpha\}$ and $\eta = \{\eta^\alpha\}$ are tangent vectors to M at a point z, then the value of this 2-form on tangent vectors is

$$\frac{\partial^2 F(z)}{\partial z^\alpha \partial \overline{z}^\beta}\xi^\alpha \overline{\eta}^\beta.$$

Since on a complex manifold the transition maps are holomorphic, the sign of $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \overline{z}^\beta}\xi^\alpha \overline{\eta}^\beta$ is independent of the choice of the holomorphic coordinates. As a consequence, if $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \overline{z}^\beta}\xi^\alpha \overline{\eta}^\beta \geq 0$ for any $\xi = \{\xi^\alpha\}$ and $\eta = \{\eta^\alpha\}$, we say that F is pluri-subharmonic at z, and if $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \overline{z}^\beta}\xi^\alpha \overline{\eta}^\beta > 0$ for any $\xi = \{\xi^\alpha\}$ and $\eta = \{\eta^\alpha\}$, we say that F is strictly pluri-subharmonic.

Pluri-subharmonic function is a natural generalization of subharmonic function of single complex variable, and it is related to several important notions in several complex variable theory, such as domain of holomorphy and so on.

Remark 5.2. Pulling back a convex functional by a harmonic map, one get a subharmonic function, while pulling back a pluri-subharmonic functional by a holomorphic map, one get a subharmonic function. Note that a holomorphic map into a Kähler manifold is harmonic.

In order to compute the Levi-form of the extremal length function $E(\cdot) = \mathrm{Ext}_\gamma(\cdot)$, we consider the Weil-Petersson metric on $\mathcal{T}(S)$. Since the Weil-Petersson metric is Kähler, due to a result of Deligne, Griffiths, Morgan and Sullivan [4] (see also [23]), the computation of the Levi-form is reduced to compute

$$\frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t))$$

where $\mu_k(t), k = 1, 2$ are two Weil-Petersson geodesics, such that $\mu_1(0) = \mu_2(0)$ and $\frac{d}{dt}\mu_1(0) = \mu$, $\frac{d}{dt}\mu_2(0) = i\mu$. Note that here we have identified the family of Beltrami differentials $\mu_k(t), k = 1, 2$ as two Weil-Petersson paths in $\mathcal{T}(S)$, which are corresponding to two families of hyperbolic metrics $(S, g_t|dz_t|^2)$ satisfying $\bar{\partial} z_t = \mu_k(t)\partial z_t, k = 1, 2$, respectively.

It follows that to prove Theorem 5.1, it suffices to show that

Theorem 5.3. *Let γ be a simple closed curve on S and let $E(\cdot) : \mathcal{T}(S) \to \mathbb{R}$ be the energy of harmonic maps from Riemann surfaces to the* R*-tree associated with γ. Then for any two Weil-Petersson geodesics $\mu_k(t), k = 1, 2$ on $\mathcal{T}(S)$ with $\mu_1(0) = \mu_2(0) = (S, g)$ and $\frac{d}{dt}\mu_1(0) = \mu$, $\frac{d}{dt}\mu_2(0) = i\mu$, we have*

$$\frac{d^2}{dt^2}|_{t=0}E_\gamma(\Gamma_1(t)) + \frac{d^2}{dt^2}|_{t=0}E_\gamma(\Gamma_2(t)) > 0.$$

Theorem 5.3 is equivalent to Theorem 2. The proof is given by the remaining part of this section.

5.1. **Proof of pluri-subharmonicity.** Let $\mu_k(t), k = 1, 2$ be two Weil-Petersson geodesics, such that $\mu_1(0) = \mu_2(0)$ and $\frac{d}{dt}\mu_1(0) = \mu$, $\frac{d}{dt}\mu_2(0) = i\mu$.

Corresponding to the two families of hyperbolic metrics, there are two families of harmonic maps whose energy realize the extremal length functions as we discussed before. We denote by $\dot{\omega}(\mu)$ and $\dot{\omega}(i\mu)$ the variational fields of harmonic maps corresponding to the direction μ and $i\mu$, respectively.

By our discussion in Section 3, we have

$$\begin{cases} \frac{d^2}{dt^2}E(\mu_1(t)) = -D^2_{11}E[\dot{\omega}(\mu), \dot{\omega}(\mu)] + D^2_{22}E[\mu, \mu]. \\ \frac{d^2}{dt^2}E(\mu_2(t)) = -D^2_{11}E[\dot{\omega}(i\mu), \dot{\omega}(i\mu)] + D^2_{22}E[i\mu, i\mu]. \end{cases}$$

As we have shown,

$$\begin{cases} D^2_{22}E[\mu, \mu] = 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z} \\ D^2_{22}E[\dot{\omega}(\mu), \dot{\omega}(\mu)] = 2\int_S |\frac{\partial \dot{\omega}(\mu)}{\partial z}|^2 dzd\bar{z} \end{cases}$$

In the second equation, $\frac{\partial \dot{\omega}(\mu)}{\partial z}$ denotes the partial derivative of $\dot{\omega}(\mu)$ with respect to z (while in previous sections, when $\dot{\omega} = \dot{\omega}(\mu)$, it was denoted by $\dot{\omega}_z$ for simplicity).

Apply Theorem 3.8, we have

$$\begin{aligned}
(11) \qquad & \frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t)) \\
&= 4\int_S |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2)dzd\bar z \\
&-2\int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z - 2\int_S |\frac{\partial\dot\omega(i\mu)}{\partial z}|^2 dzd\bar z.
\end{aligned}$$

By equation (10), we have

$$\int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z = \int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial z}\mu\omega_z + \int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}\overline{\mu}\omega_{\bar z}dzd\bar z,$$

and

$$\begin{aligned}
\int_S |\frac{\partial\dot\omega(i\mu)}{\partial z}|^2 dzd\bar z &= \int_S \frac{\partial\overline{\dot\omega(i\mu)}}{\partial z}(i\mu)\omega_z + \int_S \frac{\partial\overline{\dot\omega(i\mu)}}{\partial \bar z}\overline{(i\mu)}\omega_{\bar z}dzd\bar z \\
&= \int_S i\frac{\partial\overline{\dot\omega(i\mu)}}{\partial z}\mu\omega_z + \int_S -i\frac{\partial\overline{\dot\omega(i\mu)}}{\partial \bar z}\overline{\mu}\omega_{\bar z}dzd\bar z.
\end{aligned}$$

Since ω is real, we have

$$\begin{aligned}
& 2\int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z + 2\int |\frac{\partial\dot\omega(i\mu)}{\partial z}|^2 dzd\bar z \\
= \;& 2\int_S \mu\omega_z[\frac{\partial\overline{\dot\omega(\mu)}}{\partial z} + i\frac{\partial\overline{\dot\omega(i\mu)}}{\partial z}]dzd\bar z + 2\int_S \overline{\mu}\omega_{\bar z}[\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z} - i\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}]dzd\bar z \\
\leq \;& 2\int_S |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2)dzd\bar z \\
+ \;& \frac{1}{2}\int_S |\frac{\partial\overline{\dot\omega(\mu)}}{\partial z} + i\frac{\partial\overline{\dot\omega(\mu)}}{\partial z}|^2 dzd\bar z + \frac{1}{2}\int_S |\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z} - i\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}|^2 dzd\bar z \\
= \;& 2\int_S |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2)dzd\bar z + \int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z + \int_S |\frac{\partial\dot\omega(\mu)}{\partial \bar z}|^2 dzd\bar z \\
& -i\mathrm{Re}\Big(\int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial z}\frac{\partial\dot\omega(i\mu)}{\partial \bar z}dzd\bar z - \int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}\frac{\partial\dot\omega(i\mu)}{\partial z}dzd\bar z\Big).
\end{aligned}$$

In the above computation, we have applied the following inequality:

$$2\int_S |fg|dzd\bar z \leq 2\int |f|^2 dzd\bar z + \frac{1}{2}\int |g|^2 dzd\bar z.$$

It follows from integration by part that

$$\begin{aligned}
& \int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial z}\frac{\partial\dot\omega(i\mu)}{\partial \bar z}dzd\bar z - \int_S \frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}\frac{\partial\dot\omega(i\mu)}{\partial z}dzd\bar z \\
= \;& -\int_S \frac{\partial^2\overline{\dot\omega(\mu)}}{\partial z\partial\bar z}\dot\omega(i\mu)dzd\bar z + \int_S \frac{\partial^2\overline{\dot\omega(\mu)}}{\partial z\partial\bar z}\dot\omega(i\mu)dzd\bar z \\
= \;& 0.
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
& 2\int_S |\frac{\partial \dot\omega(\mu)}{\partial z}|^2 dz d\bar z + 2\int |\frac{\partial \dot\omega(i\mu)}{\partial z}|^2 dz d\bar z \\
\leq\ & 2\int_S |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2) dz d\bar z + \int_S |\frac{\partial \dot\omega(\mu)}{\partial z}|^2 dz d\bar z + \int_S |\frac{\partial \dot\omega(\mu)}{\partial \bar z}|^2 dz d\bar z.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \int_S |\frac{\partial \dot\omega(\mu)}{\partial z}|^2 dz d\bar z + 2\int |\frac{\partial \dot\omega(i\mu)}{\partial z}|^2 dz d\bar z \\
& \leq 2\int |\mu|^2(|\omega_z|^2+|\omega_{\bar z}|^2) dz d\bar z.
\end{aligned}
$$

Combined with (11), we conclude that

$$
\frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t)) \geq 0. \tag{12}
$$

As a result, we have shown that $E(\cdot)$ is pluri-subharmonic.

We have the following observation:

Proposition 5.4. *Then equation* (11) *can be wrote as*

$$
\frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t)) = 8\int_S |\mu|^2|\omega_z|^2 dz d\bar z - 8\int_S |\mu_{\bar z}|^2|\omega_z|^2 dz d\bar z. \tag{13}
$$

Proof. We know that ω is real. Write down the equation (9) as

$$
\frac{\partial^2 \dot\omega(\mu)}{\partial z \partial \bar z} = 2\mathrm{Re}\frac{\partial}{\partial z}(\mu\omega_z).
$$

Similarly, we have

$$
\frac{\partial^2 \dot\omega(i\mu)}{\partial z \partial \bar z} = 2\mathrm{Re}\frac{\partial}{\partial z}(i\mu\omega_z) = -2\mathrm{Im}\frac{\partial}{\partial z}(\mu\omega_z).
$$

Then

$$
|\dot\omega_z(\mu)|^2 + |\dot\omega_z(i\mu)|^2 = 4|\frac{\partial}{\partial z}(\mu\omega_z)|^2.
$$

And then

$$
\begin{aligned}
\int_S |\dot\omega_z(\mu)|^2 + |\dot\omega_z(i\mu)|^2 dz d\bar z &= 4\int_S |\frac{\partial}{\partial z}(\mu\omega_z)|^2 dz d\bar z \\
\text{(integration by part)} &= -4\int_S (\mu\omega_z)_{z\bar z}\overline{\mu\omega_z} dz d\bar z, \\
\text{(since } \omega_{z\bar z}=0) &= -4\int_S (\mu_{\bar z}\omega_z)_z \overline{\mu\omega_z} dz d\bar z, \\
\text{(integration by part)} &= 4\int_S \mu_{\bar z}\omega_z(\overline{\mu\omega_z})_z dz d\bar z \\
\text{(since } \omega_{z\bar z}=0 \text{ again)} &= 4\int_S \mu_{\bar z}\omega_z(\overline{\mu_{\bar z}\omega_z}) dz d\bar z, \\
&= 4\int_S |\mu_{\bar z}|^2|\omega_z|^2.
\end{aligned}
$$

Thus equation (11) is equal to

(14)

$$\frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t)) = 8\int_S |\mu|^2|\omega_z|^2 - 8\int_S |\mu_{\bar z}|^2|\omega_z|^2.$$

□

5.2. **Proof of strictly pluri-subharmonicity.** Next we show that the left hand side of (12) is actually positive.

Since the image of ω is a $\mathbb{R}$-tree, the Jacobian $|\omega_z|^2 - |\omega_{\bar z}|^2 = 0$. Moreover we can take ω to be real and then $\omega_{\bar z} = \overline{\omega_z}, \dot\omega_{\bar z} = \overline{\dot\omega_z}$.

Note that in the proof of pluri-subharmonicity, we have applied the Schwarz inequality to show that

$$\begin{aligned}
& 2\int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z + 2\int |\frac{\partial\dot\omega(i\mu)}{\partial z}|^2 dzd\bar z \\
= \;& 2\int_S \mu\omega_z[\frac{\partial\overline{\dot\omega(\mu)}}{\partial z} + i\frac{\partial\overline{\dot\omega(i\mu)}}{\partial z}]dzd\bar z + 2\int_S \bar\mu\omega_{\bar z}[\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z} - i\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}]dzd\bar z \\
\le \;& 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z \\
+ \;& \frac{1}{2}\int_S |\frac{\partial\overline{\dot\omega(\mu)}}{\partial z} + i\frac{\partial\overline{\dot\omega(\mu)}}{\partial z}|^2 dzd\bar z + \frac{1}{2}\int_S |\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z} - i\frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}|^2 dzd\bar z \\
= \;& 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z + \int_S |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z + \int_S |\frac{\partial\dot\omega(\mu)}{\partial \bar z}|^2 dzd\bar z
\end{aligned}$$

We denote the above inequality by $(*)$. Then we separate the discussion into two cases:

Case 1: the inequality $(*)$ is a strictly inequality.

In this case, we know from formula (11) that the extremal length is strictly pluri-subharmonic.

Case 2: the inequality $(*)$ is an equality.

If $\omega_z = 0$ at a point, then $\omega_{\bar z} = 0$ at this point too. Let $S_0 = \{z \in S : \omega_z = 0\}$. Then the integration

$$\int_{S_0} |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z = 0,$$

$$\int_{S_0} |\frac{\partial\dot\omega(\mu)}{\partial z}|^2 dzd\bar z = \int_{S_0} \frac{\partial\overline{\dot\omega(\mu)}}{\partial z}\mu\omega_z + \int_{S_0} \frac{\partial\overline{\dot\omega(\mu)}}{\partial \bar z}\bar\mu\omega_{\bar z} dzd\bar z = 0,$$

and

$$\int_{S_0} |\frac{\partial\dot\omega(i\mu)}{\partial z}|^2 dzd\bar z = 0.$$

As a result, the points in S_0 have no contribution to the inequality $(*)$ and the inequality

$$\begin{aligned}&\int_S |\frac{\partial \dot{\omega}(\mu)}{\partial z}|^2 dz d\bar{z} + 2\int |\frac{\partial \dot{\omega}(i\mu)}{\partial z}|^2 dz d\bar{z}\\ \leq\ & 2\int |\mu|^2(|\omega_z|^2+|\omega_{\bar{z}}|^2) dz d\bar{z}.\end{aligned}$$

Therefore, we only need to consider the inequality $(*)$ in any neighborhood $\mathcal{U}\subset S$ where $\omega_z \neq 0$.

Suppose that $\omega_z\neq 0$ in $\mathcal{U}$, Let

$$\tau(z)=\frac{\dot{\omega}(\mu)-i\dot{\omega}(i\mu)}{\omega_z}.$$

Since ω is real, we have

$$\overline{\tau}(z)=\frac{\dot{\omega}(\mu)+i\dot{\omega}(i\mu)}{\omega_{\bar{z}}}.$$

Since $\omega_{z\bar{z}}=0$, we have

$$\tau_{\bar{z}}=\left(\frac{\partial\dot{\omega}(\mu)}{\partial\bar{z}}-i\frac{\partial\dot{\omega}(i\mu)}{\partial\bar{z}}\right)/\omega_z$$

and

$$\bar{\tau}_z=\left(\frac{\partial\dot{\omega}(\mu)}{\partial z}+i\frac{\partial\dot{\omega}(i\mu)}{\partial z}\right)/\omega_{\bar{z}}.$$

Note that the inequality $(*)$ becomes equality only if

$$\tau_{\bar{z}}\omega_z=f(z)\bar{\mu}\omega_z, \bar{\tau}_z\omega_{\bar{z}}=g(z)\bar{\mu}\omega_{\bar{z}},$$

where f(z) and g(z) are real valued functions defined on $\mathcal{U}$.

Assume that the inequality $(*)$ becomes equality. Since $\omega_z\neq 0$ and $\omega_{\bar{z}}\neq 0$ in $\mathcal{U}$, we have $\tau_{\bar{z}}=f(z)\bar{\mu}, \bar{\tau}_z=g(z)\bar{\mu}$. In this case we claim that $\tau_{\bar{z}}=0$ in $\mathcal{U}$.

In fact,

$$\tau_{\bar{z}}=f(z)\bar{\mu}=g(z)\mu.$$

By assumption, $\mu=\frac{\bar{\Phi}}{g}$ where Φ is a holomorphic quadratic differential. If z is a point in $\mathcal{U}$ such that $f(z)\neq 0$ (and then $g(z)\neq 0$), then we have

$$\frac{\Phi^2(z)}{|\Phi(z)|^2}=\frac{\Phi(z)}{\bar{\Phi}(z)}=\frac{g(z)}{f(z)}.$$

It turns out that z should be a zero point of Φ. Otherwise, the value of $\Phi^2(z)$ is real in a neighborhood of z, which is impossible (note that $\Phi^2(z)$ is holomorphic). As a result, $\tau_{\bar{z}}=0$ in $\mathcal{U}$.

It follows that

$$\frac{\partial\dot{\omega}(\mu)}{\partial\bar{z}}-i\frac{\partial\dot{\omega}(i\mu)}{\partial\bar{z}}=0$$

and

$$\frac{\partial \dot{\omega}(\mu)}{\partial z} + i \frac{\partial \dot{\omega}(i\mu)}{\partial z} = 0$$

in $\mathcal{U}$. In this case we have (recall the first equality in $(*)$)

$$0 = \int_{\mathcal{U}} |\frac{\partial \dot{\omega}(\mu)}{\partial z}|^2 dz d\bar{z} + \int_{\mathcal{U}} |\frac{\partial \dot{\omega}(i\mu)}{\partial z}|^2 dz d\bar{z} < 2 \int_{\mathcal{U}} |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.$$

Since the Hopf differential of the harmonic map ω is non-trivial, $\omega_z \neq 0$ almost everywhere. We can cover the Riemann surface (S, g) by a family of open sets $\mathcal{U}$ satisfy $\omega_z \neq 0$ for $z \in \mathcal{U}$. It follows that

$$0 = \int_S |\dot{\omega}_z(\mu)|^2 dz\bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz\bar{z} < 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.$$

It is obviously from formula (11) that the strictly pluri-subharmonicity of $E(\cdot)$ holds.

Since in both two cases, we have shown that $E(\cdot)$ is a strictly pluri-subharmonic function, we finish the proof of Theorem 5.3.

5.3. **Application.** In general, an upper semi-continuous function F on a complex manifold M is said to be *plurisubharmonic* if and only if for any holomorphic map $\phi : \Delta \to M$, the function $F \circ \phi : \Delta \to M$ is subharmonic, where Δ denotes the unit disk. By density of wighted simple closed curves in the space of measured foliations, we have the following corollary.

Corollary 5.5. *Given any measured foliation $\mathcal{F}$ on S, the extremal length function $\mathrm{Ext}_{\mathcal{F}}(\cdot)$ is pluri-subharmonic on $\mathcal{T}(S)$.*

Using a different method, Vasil'ev [24] showed the extremal length of some maximally rational measured foliations are locally harmonic in $\mathcal{T}(S)$.

We also obtain a new proof of the following result of Bers-Erenpreis [3] (see also [23, 32]).

Corollary 5.6. *Teichmüller space is a Stein manifold.*

Proof. Let $\{\gamma_i\}$ be a finite set of simple closed curves filling the surface S and set

$$L(\cdot) = \sum_{\gamma_i} \mathrm{Ext}_{\gamma_i}(\cdot).$$

By definition of extremal length, $L(\cdot)$ has a universal lower bound

$$\sum_{\gamma_i} \ell^2_{\gamma_i}(\cdot)/2\pi|\chi(S)|, \tag{15}$$

where $\ell_{\gamma_i}(\cdot)$ denotes the hyperbolic length of γ_i. Since (15) is a proper function on $\mathcal{T}(S)$ (see Kerckhoff [16]), $L(\cdot)$ is also proper. It follows from Theorem 1 that $L(\cdot)$ defines a proper strictly pluri-subharmonic function on $\mathcal{T}(S)$. This implies that $\mathcal{T}(S)$ is a Stein manifold.

□

6. Relation with Teichmüller metric

In the following, we apply (10) to present an inequality about the second variation of $E(\omega^t, g_t) = Ext_\gamma(g_t)$ along a Teichmüller geodesic.

Proposition 6.1. *Let (S, g_t) be a Teichmüller geodesic in $\mathcal{T}(S)$ with Beltrami differential $\mu(t) = t\mu$, where $\|\mu\|_\infty = 1$. Then the following inequality holds:*

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_t) &\geq -2\int_S (|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z \\ &= -4\int_S |\omega_z|^2 dzd\bar z = -4E(\omega, g_0).\end{aligned}$$

Proof. It is not hard to see that the discussion in Section 3 and Section 4 applies second variations along Teichmüller geodesics. As before, we can assume that ω is real, and it follows that $\dot\omega_z = \overline{\dot\omega}_z$. By (10), we have

$$\begin{aligned}\int |\dot\omega_z|^2 dzd\bar z &= |\int_S \overline{\dot\omega}_z \mu\omega_z + \int_S \overline{\dot\omega_z}\overline{\mu}\omega_{\bar z} dzd\bar z| \\ &\leq \int_S |\overline{\dot\omega}_z\mu\omega_z| dzd\bar z + \int_S |\overline{\dot\omega_z}\overline{\mu}\omega_{\bar z}| dzd\bar z \\ &= \int_S |\dot\omega_z| [|\mu|(|\omega_z| + |\omega_{\bar z})] dzd\bar z \\ &\leq \frac{1}{2}\int_S |\dot\omega_z|^2 dzd\bar z + \frac{1}{2}\int_S |\mu|^2(|\omega_z| + |\omega_{\bar z}|)^2 dzd\bar z.\end{aligned}$$

Then

$$\frac{1}{2}\int |\dot\omega_z|^2 dzd\bar z \leq \frac{1}{2}\int |\mu|^2(|\omega_z| + |\omega_{\bar z}|)^2 dzd\bar z.$$

Equivalently,

$$\int_S |\dot\omega_z|^2 dzd\bar z \leq \int_S |\mu|^2(|\omega_z| + |\omega_{\bar z}|)^2 dzd\bar z.$$

As the image of ω is $\mathbb{R}$-tree, the Jacobian is zero. We have

$$0 = |\omega_z|^2 - |\omega_{\bar z}|^2.$$

Then

$$\begin{aligned}\int_S |\dot\omega_z|^2 dzd\bar z &\leq \int_S |\mu|^2(2|\omega_z|)^2 dzd\bar z \\ &\leq 4\int_S |\mu|^2|\omega_z|^2 dzd\bar z \\ &= 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2) dzd\bar z.\end{aligned}$$

Combining the above inequality with Theorem 3.8, we have

$$\begin{aligned}
\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_t) &= \frac{d^2}{dt^2}|_{t=0}E(\omega, g_t) - \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) \\
&= 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z - 2\int_S |\dot\omega_z|^2 dzd\bar z \\
&\geq 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z - 4\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z \\
&= -2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z.
\end{aligned}$$

By assumption $\|\mu\|_\infty = 1$, thus

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_t) \geq -2\int_S (|\omega_z|^2 + |\omega_{\bar z}|^2)dzd\bar z = -4E(\omega, g_0).$$

□

Kerckhoff [15] has discovered an elegant and useful way to compute the Teichmüller distance in terms of extremal length. It is an open problem that whether Teichmüller geodesic balls are convex. Recently, Lenzhen and Rafi [17] proved that extremal length functions are quasiconvex along any Teichmüller geodesic. As a corollary, they proved the quasiconvexity of Teichmüller geodesic balls. We hope that our study of extremal length variations may be applied in the study of the convexity of Teichmüller metric.

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