25

Black-Body Radiation

The spectral density of black body radiation ... represents something absolute, and since the search for the absolutes has always appeared to me to be the highest form of research, I applied myself vigorously to its solution.

Max Planck, German physicist (1858–1947), Nobel Prize 1918

25.1 Black Bodies

In physics, the expression 'black body' refers to an object that absorbs all radiation incident on it and reflects nothing. It is, of course, an idealization, but one that can be approximated very well in the laboratory.

A black body is *not* really black. Although it does not reflect light, it can and does radiate light arising from its thermal energy. This is, of course, necessary if the black body is ever to be in thermal equilibrium with another object.

The purpose of the current chapter is to calculate the spectrum of radiation emanating from a black body. The calculation was originally carried out by Max Planck in 1900 and published the following year. This was before quantum mechanics had been invented—or perhaps it could be regarded the first step in its invention. In any case, Planck investigated the consequences of the assumption that light could only appear in discrete amounts given by the quantity

$$\Delta\epsilon_\omega = h\nu = \hbar\omega, \tag{25.1}$$

where ν is the frequency, $\omega = 2\pi\nu$ is the angular frequency, h is Planck's constant, and $\hbar = h/2\pi$. This assumption is well accepted today, but it was pretty daring in 1900 when Max Planck introduced it.

25.2 Universal Frequency Spectrum

If two black bodies at the same temperature are in equilibrium with each other, the frequency spectrum must be the same for each object. To see why, suppose that two black bodies, A and B, are in equilibrium with each other, but that A emits more power than B in a particular frequency range. Place a baffle between the objects that transmits radiation

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well in that frequency range, but is opaque to other frequencies. This would have the effect of heating B and cooling A, in defiance of the Second Law of Thermodynamics. Since that cannot happen, the frequency spectrum must be the same for all black bodies.

Since the radiation spectrum does not depend on the object, we might as well take advantage of the fact and carry out our calculations for the simplest object we can think of.

25.3 A Simple Model

We will consider a cubic cavity with dimensions $L \times L \times L$. The sides are made of metal and it contains electromagnetic radiation, but no matter. Radiation can only come in and out of the cavity through a very small hole in one side. Since the radiation must be reflected off the walls many times before returning to the hole, we can assume that it has been absorbed along the way, making this object—or at least the hole—a black body.

The only thing inside the cavity is electromagnetic radiation at temperature T. We wish to find the frequency spectrum of the energy stored in that radiation, which will also give us the frequency spectrum of light emitted from the hole.

25.4 Two Types of Quantization

In analyzing the simple model described in the previous section, we must be aware of the two kinds of quantization that enter the problem.

As a result of the boundary conditions due to the metal walls of the container, the frequencies of allowed standing waves are quantized. This is an entirely classical effect, similar to the quantization of frequency in the vibrations of a guitar string.

The second form of quantization is due to quantum mechanics, which specifies that the energy stored in an electromagnetic wave with angular frequency ω comes in multiples of $\hbar\omega$ (usually called photons).

The theory of electrodynamics gives us the wave equation for the electric field $\vec{E}(\vec{r})$ in a vacuum,

$$\nabla^2 \vec{E}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}. \tag{25.2}$$

In eq. (25.2),

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{25.3}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{25.4}$$

The solutions to eq. (25.2) must satisfy the boundary conditions that the component of the electric field parallel to a wall must vanish at that wall.

Using the symmetry of the model, we find that there are solutions of the following form:

$$E_x(\vec{r},t) = E_{x,o}\sin(\omega t)\cos(k_x x)\sin\left(k_y y\right)\sin(k_z z) \tag{25.5}$$

$$E_y(\vec{r},t) = E_{y,o}\sin(\omega t)\sin(k_x x)\cos\left(k_y y\right)\sin(k_z z) \tag{25.6}$$

$$E_z(\vec{r},t) = E_{z,o}\sin(\omega t)\sin(k_x x)\sin\left(k_y y\right)\cos(k_z z)\,. \tag{25.7}$$

The values of $E_{x,o}$, $E_{y,o}$, and $E_{z,o}$ are the amplitudes of the corresponding components of the electric field in the cavity.

Eq. (25.5) through (25.7) were written to impose the boundary condition that the parallel components of the electric field vanish at the walls of the cube where x, y, or z is equal to zero. To impose the same boundary condition at the remaining walls, where x, y, or z is equal to L, we have the conditions

$$k_x L = n_x \pi \tag{25.8}$$

$$k_y L = n_y \pi \tag{25.9}$$

$$k_z L = n_z \pi \tag{25.10}$$

where n_x, n_y, n_z are integers. Only positive integers are counted, because negative integers give exactly the same solutions.

Substituting eqs. (25.5), (25.6), and (25.7) into eq. (25.2), we find a relationship between the frequency and the wave numbers,

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{25.11}$$

This equation can also be written in terms of the integers n_x, n_y, n_z,

$$\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2 = \frac{\omega^2}{c^2}. \tag{25.12}$$

Clearly the value of ω must depend on the vector $\vec{n} = (n_x, n_y, n_z)$, We will indicate the $\vec{n}$-dependence by a subscript and solve eq. (25.12) for $\omega_{\vec{n}}$

$$\omega_{\vec{n}}^2 = \left(n_x^2 + n_y^2 + n_z^2\right)\left(\frac{\pi c}{L}\right)^2. \tag{25.13}$$

Taking the square root to find $\omega_{\vec{n}}$, we obtain

$$\omega_{\vec{n}} = \frac{\pi c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{n\pi c}{L} = \omega_n \tag{25.14}$$

where $n = |\vec{n}|$.

Because the wavelengths that contribute significantly to black-body radiation are very small in comparison with the size of the cavity, energy differences between neighboring points in $\vec{n}$-space are very small. This makes the frequency spectrum quasi-continuous and allows us to change sums over the discrete wavelengths into integrals. Furthermore, the dependence of the frequency on $\vec{n}$ shown in eq. (25.14) is rotationally symmetric, which simplifies the integrals further, as shown in the next section.

25.5 Black-Body Energy Spectrum

The first step in calculating the black-body energy spectrum is to find the density of states. From the solutions to the wave equation in Section 25.4, we expressed the individual modes in terms of the vectors $\vec{n}$. Note that the density of points in $\vec{n}$-space is one, since the components of $\vec{n}$ are all integers,

$$P_{\vec{n}}(\vec{n}) = 1. \tag{25.15}$$

To find the density of states as a function of frequency $P_\omega(\omega)$, integrate eq. (25.15) over $\vec{n}$-space,

$$P_\omega(\omega) = 2\frac{1}{8}\int_0^\infty 4\pi n^2 dn\, \delta(\omega - nc\pi/L) = \pi \left(\frac{L}{c\pi}\right)^3 \omega^2. \tag{25.16}$$

The factor of 2 is for the two polarizations of electromagnetic radiation, and the factor of 1/8 corrects for counting both positive and negative values of the components of $\vec{n}$.

Since each photon with frequency ω has energy $\hbar\omega$, the average energy can be found by summing over all numbers of photons weighted by the Boltzmann factor $\exp(-\beta\hbar\omega)$. Since this sum is formally identical to that for the simple harmonic oscillator, we can just write down the answer,

$$\langle \epsilon_\omega \rangle = \frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1}. \tag{25.17}$$

Note that eq. (25.17) does not include the ground-state energy that might be expected for a simple harmonic oscillator. The reason is a bit embarrassing. Since there is an infinite number of modes, the sum of the ground-state energies is infinite. The simplest way to deal with the problem is to ignore it on the grounds that a constant ground-state energy cannot affect the results for the radiation spectrum. That is what other textbooks do, and that is what I will do for the rest of the book. I suggest you do the same.

The energy density spectrum for black-body radiation as a function of the angular frequency ω is found by multiplying the density of states in eq. (25.16) by the average energy per state in eq. (25.17) and dividing by the volume $V = L^3$,

$$u_\omega = \left(\frac{1}{V}\right)\pi\left(\frac{L}{c\pi}\right)^3\omega^2\frac{\hbar\omega}{\exp(\beta\hbar\omega)-1} = \frac{\hbar}{\pi^2c^3}\omega^3\left(\exp(\beta\hbar\omega)-1\right)^{-1}. \tag{25.18}$$

Knowing the energy per unit volume contained in the black-body cavity from eq. (25.30), and the fact that light travels with the speed of light (if you will pardon the tautology), the energy per unit area radiated from the hole in the cavity, $\mathcal{J}_U$, is clearly proportional to cU/V. The actual equation includes a geometric factor of $1/4$, the calculation of which will be left to the reader.

Multiplying u_ω by the factor of $c/4$ to derive the radiated power gives us the Planck law for black-body radiation,

$$j_\omega = \frac{1}{4}cu_\omega = \left(\frac{\hbar}{4\pi^2c^2}\right)\frac{\omega^3}{\exp(\beta\hbar\omega)-1}. \tag{25.19}$$

Fig. 25.1 shows a plot of eq. (25.19) in dimensionless units; that is, $x^3/(\exp(x)-1)$, where $x = \beta\hbar\omega$. The function has a maximum at $x_{max} \approx 2.82144$.

It is very important to understand how the spectrum of black-body radiation scales as a function of T, which will be discussed in the following subsections.

25.5.1 Frequency of Maximum Intensity

Since $\omega = x(k_BT/\hbar)$, the location of the maximum is

$$\omega_{max} = x_{max}\,k_BT/\hbar \approx 2.82144\left(\frac{k_BT}{\hbar}\right), \tag{25.20}$$

which is proportional to the temperature, T.

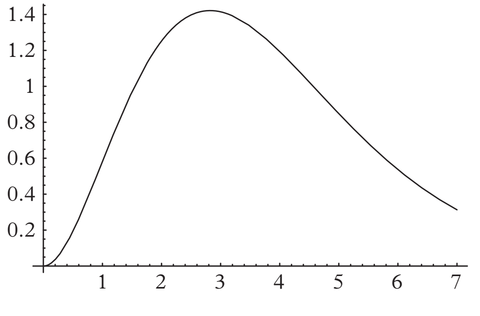

Fig. 25.1 *Plot of black-body radiation spectrum, $x^3/(\exp(x) - 1)$, in dimensionless units. For comparison with the black-body spectrum in eq. (25.19), note that $x = \beta\hbar\omega$. The integral under this dimensionless function was given as $\pi^4/15$ in eq. (25.29).*

The value of j_ω at its maximum is then

$$j_{\omega_{max}} = \left(\frac{\hbar}{4\pi^2 c^2}\right)\frac{(x_{max}k_BT/\hbar)^3}{\exp(x_{max})-1} = \left(\frac{x_{max}^3}{4\pi^2 c^2\hbar^2}\right)\frac{(k_BT)^3}{\exp(x_{max})-1}, \tag{25.21}$$

which is proportional to T^3. Clearly, the integral over the black-body spectrum is proportional to $T^3 \times T = T^4$, as expected from the Stefan–Boltzmann Law in eq. (25.31).

25.5.2 Low-Frequency Spectrum

The low-frequency energy spectrum is found for $x = \beta\hbar\omega << x_{max} \approx 2.82$. For small values of ω, we find

$$\begin{aligned} j_\omega &= \frac{1}{4}cu_\omega \qquad (25.22)\\ &= \left(\frac{\hbar}{4\pi^2 c^2}\right)\frac{\omega^3}{\exp(\beta\hbar\omega)-1}\\ &\approx \left(\frac{\hbar}{4\pi^2 c^2}\right)\frac{\omega^3}{1+\beta\hbar\omega-1}\\ &= \left(\frac{\hbar}{4\pi^2 c^2}\right)\frac{\omega^3}{\beta\hbar\omega}\\ &= \left(\frac{1}{4\pi^2 c^2}\right)\omega^2 k_BT. \end{aligned}$$

This expression can be understood because the small ω region corresponds to $\beta\hbar\omega \ll 1$, or $k_BT \gg \hbar\omega$, which is the condition that the classical theory is valid. In this limit,

$$\langle\epsilon_n\rangle \to k_BT, \tag{25.23}$$

independent of the value of ω. The factor of ω^2 in eq. (25.22) comes from the factor of ω^2 in eq. (25.16), which, in turn, came from the n^2 dependence of the surface of a sphere in $\vec{n}$-space.

25.5.3 High-Frequency Spectrum

At high frequencies, $\beta\hbar\omega \gg x_{max} > 1$, so that we can make the approximation that $\exp(\beta\hbar\omega) \gg 1$ in the expression for the spectrum of black-body radiation in eq. (25.22),

$$j_\omega = \left(\frac{\hbar}{4\pi^2 c^2}\right)\frac{\omega^3}{\exp(\beta\hbar\omega)-1} \approx \left(\frac{\hbar}{4\pi^2 c^2}\right)\omega^3\exp(-\beta\hbar\omega). \tag{25.24}$$

To understand the high-frequency behavior of the energy spectrum, first note that two factors of ω come from the n^2 dependence of the surface of a sphere in $\vec{n}$-space, as noted in the previous subsection. The third factor of ω comes from $\hbar\omega$, which is the energy of a single photon, while the factor $\exp(-\beta\hbar\omega)$ gives the relatively low probability of a high-frequency photon being excited.

Note that although the factor ω^3 diverges as ω increases, the factor of $\exp(-\beta\hbar\omega)$ goes to zero much more rapidly, giving the shape of the curve shown in Fig. 25.1.

25.6 Total Energy

The total quantum mechanical energy in the cavity radiation at temperature T is given by summing up the average energy in each mode. Frequency spectrum of a one-dimensional harmonic solid, as given in eq. (26.33).

$$U = 2\sum_{\vec{n}}\langle\epsilon_{\vec{n}}\rangle = 2\sum_{\vec{n}}\hbar\omega_{\vec{n}}\left(\exp(\beta\hbar\omega_{\vec{n}}) - 1\right)^{-1}. \tag{25.25}$$

The sum in eq. (25.25) is restricted to the positive octant in n-space in order to avoid double counting, and factor of two accounts for the two polarizations of light associated with every spatial mode. (Note that we have again omitted the ground-state energy for the electromagnetic modes for the same dubious reason given in Section 25.5.)

We again use the fact that the frequency spectrum is a quasi-continuum to write the sum in eq. (25.25) as an integral, which we can evaluate explicitly using the density of states, $P_\omega(\omega)$, found in eq. (25.16),

$$\begin{aligned} U &= \int_0^\infty \langle\epsilon_\omega\rangle P_\omega(\omega)d\omega \\ &= \pi\left(\frac{L}{c\pi}\right)^3\int_0^\infty \langle\epsilon_\omega\rangle\omega^2 d\omega \\ &= \pi\left(\frac{L}{c\pi}\right)^3\int_0^\infty \frac{\hbar\omega}{\exp(\beta\hbar\omega)-1}\omega^2 d\omega. \end{aligned} \tag{25.26}$$

At this point we can simplify the equation for U by introducing a dimensionless integration variable,

$$x = \beta\hbar\omega. \tag{25.27}$$

The expression for U becomes

$$U = \pi\beta^{-1}\left(\frac{L}{\beta\hbar\pi c}\right)^3\int_0^\infty dx\frac{x^3}{e^x-1}. \tag{25.28}$$

By a stroke of good fortune, the dimensionless integral in this equation is known exactly

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}, \tag{25.29}$$

Noting that the volume of the cavity is $V = L^3$, the average energy per volume in the cavity can be given exactly

$$\frac{U}{V} = u = \left(\frac{\pi^2}{15\hbar^3 c^3}\right)(k_B T)^4. \tag{25.30}$$

Since the energy of a black body is proportional to T^4, the specific heat per unit volume must be proportional to T^3. This might not seem terribly significant now, but keep it in mind for later in Chapter 26, when we discuss the specific heat of an insulator, which is also proportional to T^3 at low temperatures.

25.7 Total Black-Body Radiation

Multiplying the total energy by a factor of $c/4$, as explained in Section 25.5, we find the total black-body radiation density

$$\mathcal{J}_U = \frac{cU}{4V} = \frac{c}{4}\left(\frac{\pi^2}{15\hbar^3 c^3}\right)(k_B T)^4 = \sigma_B T^4 \tag{25.31}$$

where

$$\sigma_B = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}. \tag{25.32}$$

The constant σ_B is known as the Stefan–Boltzmann constant. It is named after the Austrian physicist Joseph Stefan (1835–1893), who first suggested that the energy radiated by a hot object was proportional to T^4, and his student Boltzmann, who found a theoretical argument for the fourth power of the temperature. The value of the Stefan–Boltzmann constant had been known experimentally long before Planck calculated it theoretically in 1900. Planck got the value right!

25.8 Significance of Black-Body Radiation

Perhaps the most famous occurrence of black-body radiation is in the background radiation of the universe, which was discovered in 1964 by Arno Allan Penzias (German physicist who became an American citizen, 1933–, Nobel Prize 1978) and Robert Woodrow Wilson (American astronomer, 1936–, Nobel Prize 1978).

Shortly after the Big Bang, the universe contained electromagnetic radiation at a very high temperature. With the expansion of the universe, the gas of photons cooled—much as a gas of particles would cool as the size of the container increased. Current measurements show that the background radiation of the universe is described extremely well by eq. (25.19) at a temperature of $2.725\,K$.

25.9 Problems

PROBLEM 25.1
Generalized energy spectra

For black-body radiation the frequency of the low-lying modes was proportional to the magnitude of the wave vector $\vec{k}$. Now consider a system in d dimensions for which the relationship between the frequencies of the modes and the wave vector is given by

$$\omega = Ak^s \text{ where } A \text{ and } s \text{ are constants.}$$

What is the temperature dependence of the specific heat at low temperatures?

PROBLEM 25.2
Radiation from the sun

1. The sun's radiation can be approximated by a black body. The surface temperature of the sun is about $5800\,K$ and its radius is $0.7 \times 10^9\,m$. The distance from the earth to the sun is $1.5 \times 10^{11}\,m$. The radius of the earth is $6.4 \times 10^6\,m$. Estimate the average temperature of the earth from this information. Be careful to state your assumptions and approximations explicitly.
2. From the nature of your assumptions in the previous question (rather than your knowledge of the actual temperature), would you expect a more accurate calculation with this data to give a higher or lower answer? Explain your reasons clearly to obtain one bonus point per reason.

PROBLEM 25.3
Black-body radiation

On the *same* graph, sketch the energy density of black-body radiation $u(\omega)$ vs. the frequency ω for the two temperatures T and $2T$. (Do *not* change the axes so that the two curves become identical.)

The graph should be *large*—filling the page, so that details can be seen. Small graphs are not acceptable.