φ, ψ ... FO-formulas $\qquad \alpha, \beta, \gamma$: prop. formulas

$\hat{\varphi}, \hat{\psi}$ - propositional encodings $G(\varphi) = G(\hat{\varphi})$

COMPLETENESS

$$\begin{array}{c} \varphi \\ \Big\| \, cl, wd \\ \psi \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \hat{\varphi} \\ \Big\| \, cl, wd \\ \hat{\psi} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} G(\varphi) \\ \downarrow f \\ G(\psi) \end{array}$$

where f is skew fibration

need to show: f is also a fibration

Pf: each cl and wd is a fibration
- fibrations compose

$\Rightarrow$ f is a fibration

f is a skew bifibration

SOUNDNESS

$$\begin{array}{c} Q = G(\varphi) \\ \Big\downarrow f \\ P = G(\psi) \end{array} \quad \longrightarrow \quad \begin{array}{c} \hat{\varphi} \\ \| \, m \\ \alpha \\ \| \, acl \\ \beta \\ \| \, wd \\ \hat{\psi} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \hat{\varphi} \\ \| \, m \\ \alpha \\ \| \, acl \\ \hat{\psi}' = \beta \\ \| \, wd \\ \hat{\psi} \end{array}$$

f skew-bifibr.

let $P' = f(Q)$ then $P' = G(\beta)$

claim: $\beta = \hat{\psi}'$ for some ψ' (need that f is a fibration)

let x be a prop. formula

Define x^* as follows

$a^* = a$ if a is an ordinary atom

$(U_x \vee \dots \vee U_x)^* = U_x$

$(E_x \vee \dots \vee E_x)^* = E_x$

$(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$

$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$

this way get γ s.

$$\begin{array}{c} \alpha \\ \| \, acl \\ \gamma \\ \| \, acl \\ \beta \end{array} \quad \text{and} \quad \begin{array}{c} \beta^* \\ \| \, acl \\ \gamma^* = \beta^* \end{array}$$

only on prop. atoms, no U_x or E_x

and $\beta = \beta^*$

$$\rightsquigarrow \quad \begin{array}{c} \hat{\varphi}^* \\ \| \, m^* \\ \alpha^* \\ \| \, acl \\ \beta^* = \gamma^* = \beta = \hat{\psi}' \\ \| \, wd \\ \hat{\psi} \end{array}$$

$\longleftarrow$ is propositional enc. of FO formula

next: we show that each line in

$$\begin{array}{c} \hat{\varphi}^* \\ \Big\| \, m^* \\ \alpha^* \end{array}$$

is a prop encoding of a FO formula

- $\alpha^* = \hat{\theta}$ for some FO formula θ
- finally, each m^* is either an m $\dfrac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}$ or $\dfrac{(E_x \wedge C) \vee (E_x \wedge D)}{E_x \wedge (C \vee D)}$

which is the propos. encoding of $m_\exists$ $\dfrac{\exists x C \vee \exists x D}{\exists x (C \vee D)}$

Hence, we can go at each step from conclusion to premise, to conclude that $\hat{\varphi}^*$ is a prop. encoding

$\Rightarrow$ But not necessarily $\hat{\varphi} = \hat{\varphi}^*$ because the number of U_x might be messed up.

But this can be fixed with $m_\forall$ $\dfrac{\forall x C \vee \forall x D}{\forall x (C \vee D)}$

$$\Longrightarrow \quad \begin{array}{c} \varphi \\ \Big\| \, m, m_\forall, m_\exists \\ \varphi' \\ \Big\| \, acl \\ \psi' \\ \Big\| \, wd \\ \psi \end{array}$$

Attention: we need to take care of variables:

In an ordinary instance of contraction $\dfrac{(\forall x. D) \vee (\forall x. D)}{\forall x. D}$ or $\dfrac{(\exists x. D) \vee (\exists x. D)}{\exists x. D}$

the premise is not rectified. We should in fact write

$$\frac{\forall y. D \vee \forall z. D}{\forall x. D} \quad \text{and} \quad \frac{\exists y. D \vee \exists z. D}{\exists x. D}$$