

# Combinatorial Proofs and Decomposition Theorems for First-order Logic

Dominic J. D. Hughes

Lutz Straßburger  
Inria, Equipe Partout  
Ecole Polytechnique, LIX  
France

Jui-Hsuan Wu  
Ecole Normale Supérieure  
France

**Abstract**—In this paper we uncover a close relation between combinatorial and syntactic proofs for first-order logic. Whereas syntactic proofs are formalized in some deductive proof system based on inference rules, a combinatorial proof is a “syntax-free” presentation of a proof that is independent from any set of inference rules. We show that the two proof representations are related via a deep inference decomposition theorem that establishes a new kind of normal form of syntactic proofs. As a consequence, we obtain (i) a simple proof of the soundness and completeness of first-order combinatorial proofs, and (ii) a full completeness theorem: every combinatorial proof is the image of a syntactic proof.

1 **[[[TODO: Examples,examples,examples]]]**

## 2 I. INTRODUCTION

3 First-order predicate logic is one of the cornerstones of  
4 modern logic. Since its formalisation by Frege [1] it has  
5 seen a growing usage in many fields of mathematics and  
6 computer science. Since the development of proof theory by  
7 Hilbert [2], *proofs* became first-class citizens as mathematical  
8 objects that could be studied on their own. Since Gentzen’s  
9 *sequent calculus* [3], [4], many other proof systems have  
10 been developed that allow the implementation of efficient  
11 proof search, for example *analytic tableaux* [5] or *resolu-*  
12 *tion* [6]. Despite the immense progress that has been made  
13 in proof theory in general and in the area of automated and  
14 interactive theorem provers in particular[7], [8]**[[[TODO: find**  
15 **references]]]****[[[Jui-Hsuan: done]]]**, we still have no satisfactory  
16 notion of proof identity for first-order logic. In this respect,  
17 proof theory is quite different from any other mathematical  
18 field. For example in group theory, two groups are *the same*  
19 iff they are isomorphic; in topology, two spaces are *the same*  
20 iff they are homeomorphic; etc. In proof theory, we have no  
21 such notion telling us when two proofs are *the same*, even  
22 though Hilbert was considering this problem as possible 24th  
23 problem for his famous lecture [9] in 1900 [10], before proof  
24 theory existed as a mathematical field.

25 The main reason for this problem is that formal proofs, as  
26 they are usually studied in logic, are inextricably tied to the  
27 syntactic (inference rule based) proof system in which they are  
28 carried out. And it is difficult to compare two proofs that are  
29 produced within two different syntactic proof systems, based  
30 on different sets of inference rules. **[[[Lutz: an example here?]]]**

31 This is where *combinatorial proofs* come in. They have been  
32 introduced by Hughes [11] for classical propositional logic as

“syntax-free” notion of proof, and as a possible solution to  
Hilbert’s 24th problem [12] (see also [13]). The basic idea is to  
abstract away from the syntax of the inference rules used in the  
proof and consider the proof as a combinatorial object, more  
precisely as a special kind of homomorphism between two  
graphs obeying certain properties. **[[[Lutz: an example here?]]]**

It has been shown by several authors how syntactic proofs  
in various proof systems can be translated to propositional  
combinatorial proofs: for sequent proofs in [12], for deep  
inference proofs in [14], for Frege systems in [15], and for  
tableaux systems and resolution in [16]. This allows to define  
a natural notion of proof identity for propositional logic:  
two proofs are *the same*, if they are mapped to the same  
combinatorial proof.

Recently, Acclavio and Straßburger extended this notion to  
relevant logics [17] and to modal logics [18]; and Heijlties,  
Hughes and Straßburger have provided combinatorial proofs  
for intuitionistic propositional logic [19].

In this paper we would like to push forward the idea that  
combinatorial proofs can also for first order logic be used as a  
notion of proof identity. *First-order combinatorial proofs* have  
been introduced by Hughes in [20]. But even though Hughes  
shows that the conclusion of every first-order combinatorial  
proof is a valid formula, his proof is not really satisfactory,  
as (i) it is long and cumbersome, and (ii) it does not allow to  
read back a syntactic proof based inference rules. In fact, there  
is the fundamental problem that not all combinatorial proofs  
can be obtained as translations of sequent calculus proofs.

In this paper we solve this issue by moving to a deep  
inference system. More precisely, we introduce a new proof  
system, called KS1, for first-order logic, that (i) reflects every  
combinatorial proof, i.e., there is a surjective mapping from  
proofs in KS1 to combinatorial proofs, that (ii) allows to  
provide simpler proofs of soundness and completeness of  
combinatorial proofs, and (iii) admits new decomposition the-  
orems establishing a precise correspondence between certain  
syntactic inference rules and certain combinatorial notions.

In general, a *decomposition theorem* provides normal forms  
of proofs, separating subsets of inference rules of a proof  
system. A prominent example of a decomposition theorem is  
Herbrand’s theorem [21], which allows a separation between  
the propositional part and the quantifier part in a first-order  
proof [4], [22]. Through the advent of deep inference, new

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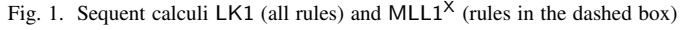
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B. *Sequent Calculus* LK1 109

*Sequents*, denoted by  $\Gamma, \Delta, \dots$ , are finite multisets of formulas, written as lists, separated by comma. The **corresponding formula** of a sequent  $\Gamma = A_1, A_2, \dots, A_n$  is the disjunction of its formulas:  $\text{fm}(\Gamma) = A_1 \vee A_2 \vee \dots \vee A_n$ . A sequent is **rectified** iff its corresponding formula is.

In this paper we use the sequent calculus LK1, shown in Figure 1, which is an one-sided variant of Gentzen’s original calculus [3] for first-order logic. To simplify some technicalities later in this paper, we also include here the mix-rule.

**Theorem 1.** *LK1 is sound and complete for first-order logic.* 120

For a proof we refer to reader to any standard textbook, e.g. [24].

The linear fragment of LK1, i.e., the fragment without the rules *ctr* (*contraction*) and *wk* (*weakening*) defines *first-order multiplicative linear logic* [25], [26] with *mix* [27], [28] (MLL1+mix). We denote that system here with MLL1<sup>x</sup> (shown in Figure 1 in the dashed box).

In this paper we make also use of the cut elimination theorem. The *cut* rule is

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta} \quad (1)$$

**Theorem 2.** *If a sequent  $\vdash \Gamma$  is provable in  $\text{LK1}+\text{cut}$  then it is also provable in  $\text{LK1}$ . Furthermore, if  $\vdash \Gamma$  is provable in  $\text{MLL1}^{\text{X}}+\text{cut}$  then it is also provable in  $\text{MLL1}^{\text{X}}$ .*

As before, this is standard, see e.g. [24] for a proof. 131

## III. PRELIMINARIES: FIRST-ORDER GRAPHS 132

## A. Graphs 133

A **graph**  $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$  is a pair where  $V_{\mathcal{G}}$  is a finite set of **vertices** and  $E_{\mathcal{G}}$  is a finite set of **edges**, which are two-element subsets of  $V_{\mathcal{G}}$ . We write  $vw$  for an edge  $\{v, w\}$ .

The **complement** of a graph  $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$  is the graph  $\mathcal{G}^c = \langle V_{\mathcal{G}}, E_{\mathcal{G}}^c \rangle$  where  $vw \in E_{\mathcal{G}}^c$  iff  $vw \notin E_{\mathcal{G}}$ .

Let  $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$  and  $\mathcal{H} = \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$  be graphs such that  $V_{\mathcal{G}} \cap V_{\mathcal{H}} = \emptyset$ . A **homomorphism**  $\varphi: \mathcal{G} \rightarrow \mathcal{H}$  is a function

141  $\varphi: V_G \rightarrow V_H$  such that if  $vw \in E_G$  then  $\varphi(v)\varphi(w) \in E_H$ .  
 142 The **union**  $\mathcal{G} + \mathcal{H}$  is the graph  $\langle V_G \cup V_H, E_G \cup E_H \rangle$  and the  
 143 **join**  $\mathcal{G} \times \mathcal{H}$  is the graph  $\langle V_G \cup V_H, E_G \cup E_H \cup \{vw \mid v \in$   
 144  $V_G, w \in V_H\} \rangle$ . A graph  $\mathcal{G}$  is **disconnected** if  $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$  for  
 145 two non-empty graphs  $\mathcal{G}_1, \mathcal{G}_2$ , otherwise it is **connected**. It is  
 146 **coconnected** if its complement is connected.

147 A graph  $\mathcal{G}$  is **labelled** in a set  $L$  if each vertex  $v \in V_G$  has  
 148 an element  $\ell(v) \in L$  associated with it, its **label**. A graph  
 149  $\mathcal{G}$  is (partially) **coloured** if it carries a partial equivalence  
 150 relation  $\sim_G$  on  $V_G$ ; each equivalence class is a **colour**. A **vertex**  
 151 **renaming** of  $\mathcal{G} = \langle V_G, E_G \rangle$  along a bijection  $(\cdot): V_G \rightarrow \hat{V}_G$   
 152 is the graph  $\hat{\mathcal{G}} = \langle \hat{V}_G, \{\hat{v}\hat{w} \mid vw \in E_G\} \rangle$ , with colouring and/or  
 153 labelling inherited (i.e.,  $\hat{v} \sim \hat{w}$  if  $v \sim w$ , and  $\ell(\hat{v}) = \ell(v)$ ).  
 154 Following standard graph theory, we identify graphs modulo  
 155 vertex renaming.

156 A **directed graph**  $\mathcal{G} = \langle V_G, E_G \rangle$  is a set  $V_G$  of **vertices** and a  
 157 set  $E_G \subseteq V_G \times V_G$  of **direct edges**. A **directed graph homomor-**  
 158 **phism**  $\varphi: \langle V_G, E_G \rangle \rightarrow \langle V_H, E_H \rangle$  is a function  $\varphi: V_G \rightarrow V_H$   
 159 such that if  $(v, w) \in E_G$  then  $(\varphi(v), \varphi(w)) \in E_H$ .

## 160 B. Cographs

161 A graph  $\mathcal{H} = \langle V_H, E_H \rangle$  is a **subgraph** of a graph  $\mathcal{G} =$   
 162  $\langle V_G, E_G \rangle$  if  $V_H \subseteq V_G$  and  $E_H \subseteq E_G$ . It is **induced** if  
 163  $v, w \in V_H$  and  $vw \in E_G$  implies  $vw \in E_H$ . An induced  
 164 subgraph of  $\mathcal{G} = \langle V_G, E_G \rangle$  is uniquely determined by its set  
 165 of vertices  $V$  and we denote it by  $\mathcal{G}[V]$ . A graph is  **$\mathcal{H}$ -free** if it  
 166 does not contain  $\mathcal{H}$  as an induced subgraph. The graph  $\mathbf{P}_4$  is  
 167 the (undirected) graph  $\langle \{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4\} \rangle$ . A  
 168 **cograph** is a  $\mathbf{P}_4$ -free undirected graph. The interest in cographs  
 169 for our paper comes from the following well-known fact.

170 **Theorem 3** ([29]). *A graph is a cograph iff it can be*  
 171 *constructed from the singletons via the operations  $+$  and  $\times$ .*

172 **[[TODO: ad the reference]] [[Jui-Hsuan: done]]**

173 In a graph  $\mathcal{G}$ , the **neighbourhood**  $N(v)$  of a vertex  $v \in V_G$  is  
 174 defined as the set  $\{w \mid vw \in E_G\}$ . A **module** is a set  $M \subseteq V_G$   
 175 such that  $N(v) \setminus M = N(w) \setminus M$  for all  $v, w \in M$ . A module  
 176  $M$  is **strong** if for every module  $M'$ , we have  $M' \subseteq M$ ,  $M \subseteq$   
 177  $M' \cap M = \emptyset$ . A module is **proper** if it has two or more  
 178 vertices.

179 Modules in cographs correspond precisely to the subtrees  
 180 of the cotrees (the term-trees constructing the graph via  $+$   
 181 and  $\times$ ).

## 182 C. Fographs

183 A cograph is **logical** if every vertex is labelled either by an  
 184 atom or variable, and it has at least one atom-labelled vertex.  
 185 We write  $\bullet\alpha$  for an  $\alpha$ -labelled vertex. An atom-labelled vertex  
 186 is called a **literal** and a variable-labelled vertex is called a  
 187 **binder**. A binder labelled with  $x$  is called an  **$x$ -binder**. The  
 188 **scope** of a binder  $b$  is the smallest proper strong module  
 189 containing  $b$ . An  **$x$ -literal** is a literal whose atom contains the  
 190 variable  $x$ . An  $x$ -binder **binds** every  $x$ -literal in its scope. In  
 191 a logical cograph  $\mathcal{G}$ , a binder  $b$  is **existential** (resp. **universal**)  
 192 if, for every other vertex  $v$  in its scope, we have  $bv \in E_G$

(resp.  $bv \notin E_G$ ). An  $x$ -binder is **legal** if its scope contains no  
 other  $x$ -binder and at least one literal.

**Definition 4.** A **first-order graph** or **fograph** is a logical  
 cograph with legal binders. The **binding graph** of a fograph  
 $\mathcal{G}$  is the directed graph  $\vec{\mathcal{G}} = \langle V_G, \{(b, l) \mid b \text{ binds } l\} \rangle$ .

We now define a mapping  $\llbracket \cdot \rrbracket$  from formulas to (labelled)  
 graphs, inductively as follows:

For a formula  $A$ , we can define its associated fograph  $\llbracket A \rrbracket$   
 inductively by:

$$\begin{aligned} \llbracket a \rrbracket &= \bullet a & (\text{for any atom } a) \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket + \llbracket B \rrbracket & \llbracket \exists x.A \rrbracket &= \bullet x \times \llbracket A \rrbracket \\ \llbracket A \wedge B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket & \llbracket \forall x.A \rrbracket &= \bullet x + \llbracket A \rrbracket \end{aligned}$$

where  $\bullet a$  (resp.  $\bullet x$ ) is a single-vertex graph whose vertex is  
 labelled by  $a$  (resp.  $x$ ).

**Lemma 5.** *If  $A$  is a rectified formula then  $\llbracket A \rrbracket$  is a fograph.*

*Proof.* That  $\llbracket A \rrbracket$  is a logical cograph follows immediately from  
 the definition and Theorem 3. The fact that every binder of  
 $\llbracket A \rrbracket$  is legal can be proved by structural induction on  $A$ .  $\square$

Note that  $\llbracket A \rrbracket$  is not necessarily a fograph if  $A$  is not  
 rectified. If  $A = (\forall x.p(x)) \vee (\forall x.q(x))$ , then  $\llbracket A \rrbracket = \bullet x \bullet p(x) \bullet$   
 $x \bullet q(x)$ , the scope of each  $x$ -binder contains all the vertices,  
 in particular, the two  $x$ -binders. On the other hand, there are  
 non-rectified formulas which are translated to fographs by  $\llbracket \cdot \rrbracket$ .  
 For example, in the graph of  $(\exists x.p(x)) \vee (\exists x.q(x))$ , both  $x$ -  
 binders are legal, as they are not in each others scope. **[[TODO:**  
**draw the picture]]**. For this reason, we call a formula **clean** if  
 it does not contain subformulas of the form  $(\forall x.A) \vee (\forall x.B)$   
 or  $(\exists x.A) \wedge (\exists x.B)$ , and no  $x$ -quantified formula occurs as  
 subformula of another  $x$ -quantified formula. Then we have:

**Lemma 6.** *If  $A$  is clean iff  $\llbracket A \rrbracket$  is a fograph.*

*Proof.* Induction on  $A$ , using Theorem 3.  $\square$

Note that even though for every formula  $A$  we can obtain  
 an equivalent clean formula by simply renaming some bound  
 variables, this is not unique up to  $\alpha$ -conversion, as it is the  
 case for rectified formulas.

We define a congruence relation  $\equiv$  on formulas by the  
 following equations:

$$\begin{aligned} A \wedge B &\equiv B \wedge A & (A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\ A \vee B &\equiv B \vee A & (A \vee B) \vee C &\equiv A \vee (B \vee C) \\ \forall x.\forall y.A &\equiv \forall y.\forall x.A & \forall x.(A \vee B) &\equiv (\forall x.A) \vee B \\ \exists x.\exists y.A &\equiv \exists y.\exists x.A & \exists x.(A \wedge B) &\equiv (\exists x.A) \wedge B \end{aligned} \quad (2)$$

where  $x \notin \text{fv}(B)$  in the last two equations. Two formulas  
 $A$  and  $B$  are **equivalent** if  $A \equiv B$ . The following theorem  
 shows that the set of fographs can be seen as the quotient  
 $\text{FORM}/\equiv$ .

**Theorem 7.** *Let  $A, B$  be rectified formulas. Then*

$$A \equiv B \iff \llbracket A \rrbracket = \llbracket B \rrbracket$$

*Proof.* By a straightforward induction on  $A$ .  $\square$

## A. Fonets

Two atoms are **pre-dual** if their predicate symbols are dual (e.g.  $p(x, y)$  and  $\bar{p}(y, z)$ ) and two literals are **pre-dual** if their labels (atoms) are pre-dual. A **linked fograph**  $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$  is a coloured fograph  $\mathcal{C}$  such that every colour (i.e., equivalence class of  $\sim_{\mathcal{C}}$ ), called a **link**, consists of two pre-dual literals, and every literal is either t-labelled or in a link.

Let  $\mathcal{C}$  be a linked fograph. The set of links can be seen as a unification problem by identifying dual predicate symbols. A **dualizer** of  $\mathcal{C}$  is a substitution  $\delta$  unifying all the links of  $\mathcal{C}$ . Since a first-order unification problem is either unsolvable or has a most general unifier, we can define the notion of **most general dualizer**. A **dependency** is a pair  $\{\bullet x, \bullet y\}$  of an existential binder  $\bullet x$  and a universal binder  $\bullet y$  such that the most general dualizer assigns to  $x$  a term containing  $y$ . A **leap** is either a link or a dependency. The **leap graph**  $\mathcal{C}^L$  of  $\mathcal{C}$  is the undirected graph  $\langle V_{\mathcal{C}}, L_{\mathcal{C}} \rangle$  where  $L_{\mathcal{C}}$  is the set of leaps of  $\mathcal{C}$ . A vertex set  $W \subseteq V_{\mathcal{C}}$  induces a **matching** in  $\mathcal{C}$  if for all  $w \in W$ ,  $N(w) \cap W$  is a singleton. We say that  $W$  induces a **bimatching** in  $\mathcal{C}$  if it induces a matching in  $\mathcal{C}$  and a matching in  $\mathcal{C}^L$ .

**Definition 8.** A **first-order net** or **fonet** is a linked fograph which has dualizer but no induced bimatching.

## B. Skew Bifibrations

A graph homomorphism  $\varphi: \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$  is a **fibration** if for all  $v \in V_{\mathcal{G}}$  and  $w\varphi(v) \in E_{\mathcal{H}}$ , there exists a unique  $\tilde{w} \in V_{\mathcal{G}}$  such that  $\tilde{w}v \in E_{\mathcal{G}}$  and  $\varphi(\tilde{w}) = w$ , and is a **skew fibration** if for all  $v \in V_{\mathcal{G}}$  and  $w\varphi(v) \in E_{\mathcal{H}}$  there exists  $\tilde{w} \in V_{\mathcal{G}}$  such that  $\tilde{w}v \in E_{\mathcal{G}}$  and  $\varphi(\tilde{w})w \notin E_{\mathcal{H}}$ . A directed graph homomorphism is a **fibration** if for all  $v \in V_{\mathcal{G}}$  and  $(w, \varphi(v)) \in E_{\mathcal{H}}$ , there exists a unique  $\tilde{w} \in V_{\mathcal{G}}$  such that  $(\tilde{w}, v) \in E_{\mathcal{G}}$  and  $\varphi(\tilde{w}) = w$ .

A **fograph homomorphism**  $\varphi = \langle \varphi, \rho_{\varphi} \rangle$  is a pair where  $\varphi: \mathcal{G} \rightarrow \mathcal{H}$  is a graph homomorphism between the underlying graphs, and  $\rho_{\varphi}$ , also called the **substitution induced by  $\varphi$**  is a variable renaming such that for all  $v \in V_{\mathcal{G}}$  we have  $\ell(\varphi(v)) = \rho_{\varphi}(\ell(v))$ . Note that this entails that  $\varphi$  maps binders to binders and literals to literals. We say that a fograph homomorphism  $\varphi: \mathcal{G} \rightarrow \mathcal{H}$  is **existential-preserving** if for all existential binders  $b$  in  $\mathcal{G}$ , the vertex  $\varphi(b)$  is an existential binder in  $\mathcal{H}$ .

**Definition 9.** Let  $\mathcal{G}$  and  $\mathcal{H}$  be fographs. A **skew bifibration**  $\varphi: \mathcal{G} \rightarrow \mathcal{H}$  is an existential-preserving fograph homomorphism that is a skew fibration on  $\langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$  and a fibration on the binding graphs  $\bar{\mathcal{G}} \rightarrow \bar{\mathcal{H}}$ .

**Definition 10.** A **first-order combinatorial proof (FOCP)** of a fograph  $\mathcal{G}$  is a skew bifibration  $\varphi: \mathcal{C} \rightarrow \mathcal{G}$  where  $\mathcal{C}$  is a fonet. A **first-order combinatorial proof** of a formula  $A$  is a combinatorial proof of its graph  $\llbracket A \rrbracket$ .

**Theorem 11 ([20]).** FOCPs are sound and complete for first-order logic.

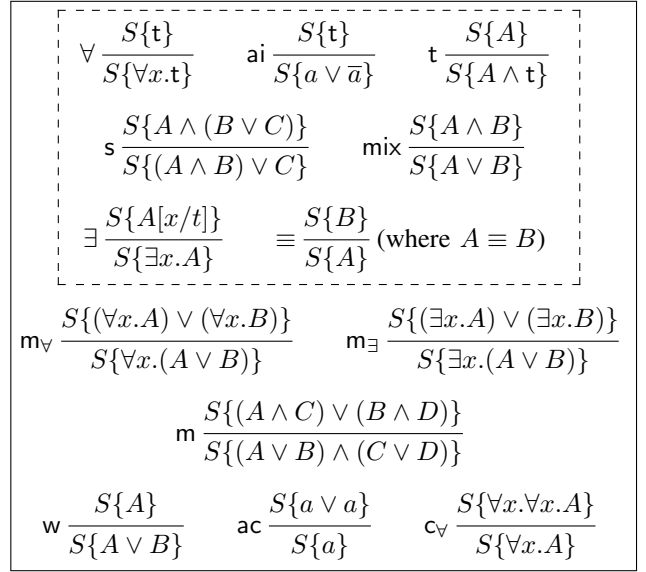


Fig. 2. Deep inference systems KS1 (all rules) and MLS1<sup>X</sup> (rules in the dashed box)

**Remark 12.** In our definition of FOCP, we are slightly laxer than the original definition of [20], as we allow for a variable renaming  $\sigma_{\varphi}$  which was forced to be the identity in [20].

## V. FIRST-ORDER DEEP INFERENCE SYSTEM KS1

Contrary to standard proof formalisms, like sequent calculi or tableaux, where inference rules decompose the principal formula along its root connective, can **deep inference rules** be applied like rewriting rules inside any (positive) formula or sequent **context**, which is denoted as  $S\{\cdot\}$ , and which is a formula (resp. sequent) with exactly one occurrence of the **hole**  $\{\cdot\}$  in the position of an atom. Then  $S\{A\}$  is the result of replacing the hole  $\{\cdot\}$  in  $S\{\cdot\}$  with  $A$ .

Figure 2 shows the inference rules for the deep inference system KS1 that we are using in this paper. It is a slight variation of the systems presented by Brünnler [30] and Ralph [31] in their PhD-theses. The main differences being that we have (i) the explicit presence of the mix-rule, (ii) a different choice of how the formula equivalence  $\equiv$  is defined, and (iii) an explicit rule for the equivalence.

We consider here only the cut-free fragment, as cut-elimination for deep inference systems has already been discussed elsewhere (e.g. [22], [32]).<sup>1</sup>

As with the sequent system LK1, we also need for KS1 the **linear fragment**, that we call here MLS1<sup>X</sup>, and that is shown in Figure 2 in the dashed box.

$B$

We write  $S \Vdash \Phi$  to denote a derivation  $\Phi$  from  $B$  to  $A$  using

$A$

the rules from system S. A formula  $A$  is **provable** in a system S if there is a derivation in S from  $t$  to  $A$ .

<sup>1</sup>In the deep inference literature, the cut-free fragment is also called the **down-fragment**. But as we do not discuss the **up-fragment** here, we omit the down-arrows  $\downarrow$  in the rule names.

In the course of this paper we are also going to make use of the general (non-atomic) version of the contraction rule:

$$\text{c} \frac{S\{A \vee A\}}{S\{A\}}$$

## VI. MAIN RESULTS

We are now ready to see the main results of this paper. We only state them here and give the proofs in the later sections of the paper. The first one is routine and expected, but needs to be proved nonetheless:

**Theorem 13.** *KS1 is sound and complete for first-order logic.*

Our second result is more surprising, as it is a very strong decomposition result for first order logic.

**Theorem 14.** *For every derivation  $\text{KS1} \parallel_{\Phi}^t$  there are formulas  $A_1, \dots, A_5$ , such that there is a derivation:*

$$\begin{array}{c} t \\ \{\forall, \text{ai}, t\} \parallel \\ A_5 \\ \{s, \text{mix}, \equiv\} \parallel \\ A_4 \\ \{\exists\} \parallel \\ A_3 \\ \{m, m_{\forall}, m_{\exists}, \equiv\} \parallel \\ A_2 \\ \{ac, c_{\forall}\} \parallel \\ A_1 \\ \{w, \equiv\} \parallel \\ A \end{array}$$

This theorem is stronger than the existing decompositions for first-order logic, which either separated only atomic contraction and atomic weakening [30] or only contraction [31] or only the quantifiers in form of a Herbrand theorem [33], [31].

There is a weaker version of Theorem 14 that will also be useful:

**Theorem 15.** *For every derivation  $\text{KS1} \parallel_{\Phi}^t$  there is a formula  $A_1$ , such that there is a derivation:*

$$\begin{array}{c} t \\ \text{MLS1}^x \parallel \\ A_1 \\ \{w, c, \equiv\} \parallel \\ A \end{array}$$

Let us now establish the connection between derivations in KS1 and combinatorial proofs.

**Theorem 16.** *Let  $\varphi: \mathcal{C} \rightarrow \mathcal{A}$  be a combinatorial proof and let  $A$  be a formula with  $\mathcal{A} = \llbracket A \rrbracket$ . Then there is a derivation*

$$\begin{array}{c} t \\ \text{MLS1}^x \parallel_{\Phi_1} \\ A' \\ \{\text{w}, \text{ac}, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\} \parallel_{\Phi_2} \\ A \end{array} \quad (3)$$

for some  $A' \equiv C\sigma_{\varphi}$  where  $C$  is a formula with  $\llbracket C \rrbracket = \mathcal{C}$  and  $\sigma_{\varphi}$  is the variable renaming substitution induced by  $\varphi$ . Conversely, whenever we have a derivation as in (6) above, then there is a combinatorial proof  $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$  such that  $\mathcal{C} = \llbracket A' \rrbracket$ .

Furthermore, in the proof of Theorem 16, we will see that (i) the links in the fonet  $\mathcal{C}$  correspond precisely to the pairs of atoms that meet in the instances of the ai-rule in the derivation  $\Phi_1$ , and (ii) the "flow-graph" of  $\Phi_2$  that traces the quantifier- and atom-occurrences in the derivation corresponds exactly to the vertex-mapping induced by  $\varphi$ .

Thus, combinatorial proofs are closely related to derivations of the form (6), and since by Theorem 14 every derivation can be transformed into that form, we can say that combinatorial proofs form a canonical proof representation for first order logic, similarly to what proof nets are for linear logic [34].

Finally, Theorems 13, 14 and 16 imply Theorem 11, which means that we have here an alternative proof of the soundness and completeness of first-order combinatorial proofs which is simpler than the one given in [20].

## VII. TRANSLATING BETWEEN LK1 AND KS1

In this section we prove Theorems 13, 14, and 15, mainly by translating derivations to and from the sequent calculus, and by rule permutation arguments.

### A. The Linear Fragments $\text{MLL1}^x$ and $\text{MLS1}^x$

In this section we show the equivalence of  $\text{MLL1}^x$  and  $\text{MLS1}^x$ .

**Lemma 17.** *If  $\vdash \Gamma$  is provable in  $\text{MLL1}^x$  then  $\text{fm}(\Gamma)$  is provable in  $\text{MLS1}^x$ .*

*Proof.* This is a straightforward induction on the proof of  $\vdash \Gamma$  in  $\text{MLL1}^x$ , making a case analysis on the bottommost rule instance. We show here only the case of  $\forall \frac{\vdash \Delta, A}{\vdash \Delta, \forall x.A}$  (all other cases are simpler and have been shown before, e.g. [30]): By induction hypothesis, there is a proof of  $\text{fm}(\Delta) \vee A$  in  $\text{MLS1}^x$ . We can prefix every line in that proof by  $\forall x$  and then compose the following derivation:

$$\begin{array}{c} t \\ \forall \frac{}{\forall x.t} \\ \text{MLS1}^x \parallel \\ \forall x.\text{fm}(\Delta) \vee A \\ \equiv \\ \text{fm}(\Delta) \vee \forall x.A \end{array}$$

where we can apply the  $\equiv$ -rule because  $x$  is not free in  $\Delta$ .  $\square$

355 **Lemma 18.** Let  $r \frac{S\{A\}}{S\{B\}}$  be an inference rule in  $\text{MLS1}^X$  other  
 356 than ai. Then the sequent  $\vdash \overline{A}, B$  is provable in  $\text{MLL1}^X$ .

357 *Proof.* This is a straightforward exercise that we leave to the  
 358 reader. (Note that the ax-rule can be applied to  $\vdash f, t$  in the  
 359 cases of  $r = \forall$ .)  $\square$

360 **Lemma 19.** Let  $A, B$  be formulas, and let  $S\{\cdot\}$  be a (pos-  
 361 itive) context. If  $\vdash \overline{A}, B$  is provable in  $\text{MLL1}^X$ , then so is  
 362  $\vdash \overline{S\{A\}}, S\{B\}$ .

363 *Proof.* Straightforward induction on  $S\{\cdot\}$ . (see e.g. [35])  $\square$

364 **Lemma 20.** If a formula  $C$  is provable in  $\text{MLS1}^X$  then  $\vdash C$   
 365 is provable in  $\text{MLL1}^X$ .

*Proof.* We proceed by induction on the number of inference  
 steps in the proof of  $C$  in  $\text{MLS1}^X$ . Consider the bottommost  
 rule instance  $r \frac{S\{A\}}{S\{B\}}$ . By induction hypothesis we have a  
 $\text{MLL1}^X$  proof  $\Pi$  of  $\vdash S\{A\}$ . If  $r$  is ai  $\frac{S\{t\}}{S\{a \vee \overline{a}\}}$ , we replace  
 in  $\Pi$  all corresponding occurrences of  $t$  with  $a \vee \overline{a}$  and the

rule instance  $t \frac{}{\vdash t}$  with the derivation  $\frac{\text{ax} \frac{}{\vdash a, \overline{a}}}{\vee \frac{}{\vdash a \vee \overline{a}}}$ . This gives

us a proof of  $\vdash S\{a \vee \overline{a}\}$ . In all other cases, by Lemmas 18  
 and 19, we have a  $\text{MLL1}^X$  proof of  $\vdash \overline{S\{A\}}, S\{B\}$ . We can  
 compose them via cut:

$$\text{cut} \frac{\vdash S\{A\} \quad \vdash \overline{S\{A\}}, S\{B\}}{\vdash S\{B\}}$$

366 and then apply Theorem 2.  $\square$

### 367 B. Contraction and Weakening

368 The first observation here is that Lemmas 17–20 from above  
 369 also hold for LK1 and KS1. We therefore immediately have:

370 **Theorem 21.** For every sequent  $\Gamma$ , we have that  $\vdash \Gamma$  is  
 371 provable in LK1 if and only if  $\text{fm}(\Gamma)$  is provable in KS1.

372 Then Theorem 13 is an immediate consequence. Let us now  
 373 proceed with providing further lemmas that will be needed for  
 374 the other results.

375 **Lemma 22.** The c-rule is derivable in  $\{\text{ac}, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\}$ .

*Proof.* We show that there is always a derivation

$$\frac{A \vee A}{s \parallel} \quad A$$

376 , where  $S = \{\text{ac}, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\}$ , by induction on  $A$ :

377 • If  $A = a$ , then we have ac  $\frac{a \vee a}{a}$ .

• If  $A = B \wedge C$ , then we have 
$$\frac{\frac{\frac{(B \wedge C) \vee (B \wedge C)}{(B \vee B) \wedge (C \vee C)} \quad s \parallel}{B \wedge (C \vee C)} \quad s \parallel}{B \wedge C} \quad \text{m} \quad \frac{(B \vee C) \vee (B \vee C)}{(B \vee B) \vee (C \vee C)} \quad \equiv$$
 378 379

• If  $A = B \vee C$ , then we have 
$$\frac{\frac{\frac{(B \vee C) \vee (B \vee C)}{(B \vee B) \vee (C \vee C)} \quad s \parallel}{B \vee (C \vee C)} \quad s \parallel}{B \vee C} \quad \text{m} \quad \frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')} \quad \equiv$$
 380 381

• If  $A = \exists x.A'$ , then we have 
$$\frac{\frac{\frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')} \quad s \parallel}{\exists x.A'} \quad \text{m}_\exists \quad \frac{(\forall x.A') \vee (\forall x.A')}{\forall x.(A' \vee A')} \quad \equiv$$
 382 383

• If  $A = \forall x.A'$ , then we have 
$$\frac{\frac{\frac{(\forall x.A') \vee (\forall x.A')}{\forall x.(A' \vee A')} \quad s \parallel}{\forall x.A'} \quad \text{m}_\forall \quad \frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')} \quad \equiv$$
 384 385

385 **TODO:** **Jui-Hsuan:** done. Maybe just keep one  
 386 case. **Lutz:** yes, but we do that at the end. don't think about  
 387 space right now.  $\square$

388 **Lemma 23.**  $c_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists$  are derivable in  $\{\text{w}, \text{c}, \equiv\}$ .

389 *Proof.* **TODO:**

We have the following derivations:

$$\frac{\frac{\frac{\forall x.\forall x.A}{\forall x.((\forall x.A) \vee A)} \quad \text{w} \quad \frac{(\forall x.A) \vee (\forall x.A)}{\forall x.A} \quad \text{c}}{(\forall x.A) \vee (\forall x.A)} \quad \equiv \quad \frac{\forall x.\forall x.A}{\forall x.A} \quad \text{w}$$

$$\frac{\frac{\frac{\frac{(A \wedge C) \vee (B \wedge D)}{((A \vee B) \wedge C) \vee (B \wedge D)} \quad \text{w} \quad \frac{((A \vee B) \wedge (C \vee D)) \vee (B \wedge D)}{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge (D \vee C))} \quad \text{w} \quad \frac{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge (D \vee C))}{((A \vee B) \wedge (C \vee D)) \vee ((A \vee B) \wedge (C \vee D))} \quad \equiv \quad \frac{((A \vee B) \wedge (C \vee D)) \vee ((A \vee B) \wedge (C \vee D))}{(A \vee B) \wedge (C \vee D)} \quad \text{c}$$

$$\frac{\frac{\frac{(\exists x.A) \vee (\exists x.B)}{(\exists x.(A \vee B)) \vee (\exists x.B)} \quad \text{w} \quad \frac{(\exists x.(A \vee B)) \vee (\exists x.(B \vee A))}{\exists x.(A \vee B) \vee (\exists x.(A \vee B))} \quad \equiv \quad \frac{\exists x.(A \vee B) \vee (\exists x.(A \vee B))}{\exists x.(A \vee B)} \quad \text{c}$$



$$\begin{array}{c}
(\forall x.A) \vee (\forall x.B) \\
\text{w} \frac{}{(\forall x.(A \vee B)) \vee (\forall x.B)} \\
\text{w} \frac{}{(\forall x.(A \vee B)) \vee (\forall x.(B \vee A))} \\
\equiv \\
\text{c} \frac{}{\forall x.(A \vee B)}
\end{array}$$

390 . **Jui-Hsuan:** done. If needed, we can introduce the notion  
391 of open deduction to reduce the size of derivations...**Lutz:**  
392 I was thinking about that, but (i) it is probably not worth the  
393 effort, as we won't have many derivations, and (ii) it is hard to  
394 define rectified derivations this way.  $\square$

**Lemma 24.** Let  $A$  and  $B$  be formulas. Then

$$\frac{A}{\{w, c, \equiv\}} \parallel B \iff \frac{A}{\{w, ac, cv, m, m\vee, m\exists, \equiv\}} \parallel B$$

395 *Proof.* This follows immediately from Lemmas 22 and 23.  
396  $\square$

### 397 C. Rule Permutations

**Theorem 25.** Let  $\Gamma$  be a sequent. If  $\vdash \Gamma$  is provable in LK1 (as depicted on the left below) then there is a sequent  $\Gamma'$  such that there is a derivation as shown on the right below:

$$\text{LK1} \frac{}{\vdash \Gamma} \Phi \implies \text{MLL1}^x \frac{}{\vdash \Gamma'} \Phi_1 \quad \frac{}{\{w, c, \equiv\} \parallel \Phi_2} \vdash \text{fm}(\Gamma)$$

398 *Proof.* Note that the instances of  $w, c$  in  $\Phi_2$  are deep, but  
399 inside sequent contexts.

400 First, if an instance of  $\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A}$  is followed by a rule in  
401 which  $A$  is not in the principal formula, it can be permuted  
402 downwards. Otherwise, the proof can be transformed using the  
403 following rewriting rules.

$$\begin{array}{l}
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \wedge \frac{}{\vdash \Gamma, A \wedge B, \Delta} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A \wedge B, \Delta} \\
\text{wk} \frac{\vdash \Gamma, A}{\vdash \Gamma, A, B} \vee \frac{}{\vdash \Gamma, A \vee B} \rightsquigarrow \text{w} \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \\
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A[x/t]} \exists \frac{}{\vdash \Gamma, \exists x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \exists x.A} \\
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \forall \frac{}{\vdash \Gamma, \forall x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \forall x.A}
\end{array}$$

$$\begin{array}{c}
\vdash \Gamma, A \\
\text{wk} \frac{}{\vdash \Gamma, A, A} \rightsquigarrow \vdash \Gamma, A \\
\text{ctr} \frac{}{\vdash \Gamma, A}
\end{array}$$

Note that in the case of  $\vee$ , we use the deep rule  $w$  which can  
be permuted down over all the rules. By using these rewriting  
rules, we can eventually get a derivation with all the instances  
of  $\text{wk}$  and  $w$  at the bottom. Now observe that the instances of  
 $\text{ctr}$  in  $\Phi$  can be transformed using the following rule:

$$\text{ctr} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\text{c} \frac{}{\vdash \Gamma, A \vee A}}$$

Knowing that  $c$  can be permuted down over all the rules of  
 $\text{MLL1}^x$ , we eventually obtain a derivation:

$$\begin{array}{c}
\text{MLL1}^x \frac{}{\vdash \Gamma_0} \Phi'_1 \\
\{wk, w, c, \equiv\} \parallel \Phi'_2 \\
\vdash \Gamma
\end{array}$$

Note that  $\equiv$  is required here since the permutation of formulas  
is implicit in  $\text{MLL1}^x$ .

By transforming each sequent of  $\Phi'_2$  into its corresponding  
formula, and by considering the following rewriting rule:

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \rightsquigarrow \text{w} \frac{\vdash \text{fm}(\Gamma)}{\vdash \text{fm}(\Gamma) \vee A}$$

, we obtain a derivation

$$\begin{array}{c}
\text{MLL1}^x \frac{}{\vdash \Gamma'} \Phi_1 \\
\{w, c, \equiv\} \parallel \Phi_2 \\
\vdash \text{fm}(\Gamma)
\end{array}$$

where  $\Gamma' = \text{fm}(\Gamma_0)$  and  $\Phi_1$  can be obtained from  $\Phi'_1$  by  
applying the  $\vee$  rule. **TO CHECK:** **Jui-Hsuan:** This  
might be a bit long...  $\square$

**Lemma 26.** For every derivation  $\text{MLS1}^x \frac{t}{A}$  there are formulas  
 $A'$  and  $A''$  such that

$$\begin{array}{c}
t \\
\{ \forall, ai, t \} \parallel \\
A'' \\
\{ s, mix, \equiv \} \parallel \\
A' \\
\{ \exists \} \parallel \\
A
\end{array}$$

*Proof.* First, observe that the  $\exists$  rule can be permuted down-  
wards over all the other rules since  $A[x/t]$  has the same  
structure as  $A$  and none of the other rules has a premise

of the form  $S\{\exists x.A\}$ . It suffices now to prove that for all  $r_1 \in \{\forall, \text{ai}, \text{t}\}$ , for all  $r_2 \in \{\text{s}, \text{mix}, \equiv\}$ , we can permute  $r_1$  upwards over  $r_2$ . We show some cases here, and leave the others to the reader.

$$\begin{array}{c} \text{s} \frac{S\{A \wedge (\text{t} \vee C)\}}{S\{(A \wedge \text{t}) \vee C\}} \\ \text{ai} \frac{\quad}{S\{(A \wedge (a \vee \bar{a})) \vee C\}} \end{array} \rightsquigarrow \begin{array}{c} \text{ai} \frac{S\{A \wedge (\text{t} \vee C)\}}{S\{A \wedge ((a \vee \bar{a}) \vee C)\}} \\ \text{s} \frac{\quad}{S\{(A \wedge (a \vee \bar{a})) \vee C\}} \end{array}$$

$$\begin{array}{c} \text{mix} \frac{S\{A \wedge B\}}{S\{A \vee B\}} \\ \text{t} \frac{\quad}{S\{(A \vee (B \wedge \text{t}))\}} \end{array} \rightsquigarrow \begin{array}{c} \text{t} \frac{S\{A \wedge B\}}{S\{A \wedge (B \wedge \text{t})\}} \\ \text{mix} \frac{\quad}{S\{(A \vee (B \wedge \text{t}))\}} \end{array}$$

414 **[[TO CHECK: ]]**

□

**Lemma 27.** For every derivation  $\{w, \text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\} \parallel \begin{array}{c} A \\ B \end{array}$  there are formulas  $A'$  and  $B'$  such that

$$\begin{array}{c} A \\ \{m, m_\forall, m_\exists, \equiv\} \parallel \\ A' \\ \{\text{ac}, \text{c}_\forall\} \parallel \\ B' \\ \{w, \equiv\} \parallel \\ B \end{array}$$

*Proof.* First, a derivation consisting of an instance of  $w$  followed by  $r \in \{\text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists\}$  can be either replaced by a derivation consisting of  $w$  only or the instance of  $w$  can be permuted downwards. We show some cases here, and leave the others to the reader.

$$\begin{array}{c} \text{w} \frac{S\{\forall x.A\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \\ \text{m}_\forall \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{w} \frac{S\{\forall x.A\}}{S\{\forall x.(A \vee B)\}} \end{array}$$

$$\begin{array}{c} \text{w} \frac{S\{A \wedge C\}}{S\{(A \wedge C) \vee (B \wedge D)\}} \\ \text{m} \frac{\quad}{S\{(A \vee B) \wedge (C \vee D)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{w} \frac{S\{A \wedge C\}}{S\{(A \vee B) \wedge C\}} \\ \text{w} \frac{\quad}{S\{(A \vee B) \wedge (C \vee D)\}} \end{array}$$

$$\begin{array}{c} \text{w} \frac{S\{a\}}{S\{a \vee a\}} \\ \text{ac} \frac{\quad}{S\{a\}} \end{array} \rightsquigarrow S\{a\}$$

For  $r_1 \in \{\text{m}, \text{m}_\forall, \text{m}_\exists\}$ ,  $r_2 \in \{\text{ac}, \text{c}_\forall\}$ ,  $r_1$  can be permuted upwards over  $r_2$  (with some  $\equiv$  inserted). The only non-trivial case is shown below:

$$\begin{array}{c} \text{c}_\forall \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \\ \text{m}_\forall \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{m}_\forall \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{\forall x.(A \vee B)\}} \\ \equiv \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array}$$

415 **[[TODO: permutation with  $\equiv$ ]]**

□

416 We can now complete the proof of Theorems 14 and 15.

*Proof of Theorem 15.* Assume we have a proof of  $A$  in KS1. By Theorem 21 we have a proof of  $\vdash A$  in LK1 to which we can apply Theorem 25. Finally, we apply Lemma 17 to get the desired shape. □

*Proof of Theorem 14.* Assume we have a proof of  $A$  in KS1. We first apply Theorem 15, and then Lemma 26 to the upper half and Lemma 27 to the lower half. □

## VIII. FONETS AND LINEAR PROOFS

### A. From MLL1<sup>X</sup> Proofs to Fonets

Let  $\Pi$  be a MLL1<sup>X</sup> proof of a rectified sequent  $\vdash \Gamma$ . We now show how  $\Pi$  is translated into a linked fograph  $[\Pi] = \langle [\Gamma], \sim_\Pi \rangle$ . We proceed inductively, making a case analysis on the last rule in  $\Pi$ . At the same time we are constructing a dualizer  $\delta_\Pi$ , so that in the end we can conclude that  $[\Pi]$  is in fact a fonet.

- 1)  $\Pi$  is  $\text{ax} \frac{\quad}{\vdash a, \bar{a}}$ : Then the only link is  $\{a, \bar{a}\}$ , and  $\delta_\Pi$  is empty.
- 2)  $\Pi$  is  $\text{t} \frac{\quad}{\vdash \text{t}}$ : Then  $\sim_\Pi$  and  $\delta_\Pi$  are both empty.
- 3) The last rule in  $\Pi$  is  $\text{mix} \frac{\vdash \Gamma' \quad \vdash \Gamma''}{\vdash \Gamma', \Gamma''}$ : By induction hypothesis, we have proofs  $\Pi'$  and  $\Pi''$  of  $\Gamma'$  and  $\Gamma''$ , respectively. We have  $[\Gamma] = [\Gamma'] + [\Gamma'']$  and let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

- 4) The last rule in  $\Pi$  is  $\vee \frac{\vdash \Gamma_1, A, B}{\vdash \Gamma_1, A \vee B}$ : By induction hypothesis, there is proofs  $\Pi'$  of  $\Gamma' = \Gamma_1, A, B$ . We have  $[\Gamma] = [\Gamma']$  and let  $\sim_\Pi = \sim_{\Pi'}$  and  $\delta_\Pi = \delta_{\Pi'}$ .
- 5) The last rule in  $\Pi$  is  $\wedge \frac{\vdash \Gamma_1, A \quad \vdash B, \Gamma_2}{\vdash \Gamma_1, A \wedge B, \Gamma_2}$ : By induction hypothesis, we have proofs  $\Pi'$  and  $\Pi''$  of  $\Gamma' = \Gamma_1, A$  and  $\Gamma'' = B, \Gamma_2$ , respectively. We have  $[\Gamma] = [\Gamma_1] + ([A] \times [B]) + [\Gamma_2]$  and we let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

- 6) The last rule in  $\Pi$  is  $\exists \frac{\vdash \Gamma_1, A[x/t]}{\vdash \Gamma_1, \exists x.A}$ : By induction hypothesis, there is a  $\Pi'$  of  $\Gamma' = \Gamma_1, A[x/t]$ . For each atom in  $\Gamma' = \Gamma_1, A[x/t]$ , there is a corresponding atom in  $\Gamma = \Gamma_1, \exists x.A$ . We can therefore define the linking  $\sim_\Pi$  from the linking  $\sim_{\Pi'}$  via this correspondence. Then, we let  $\delta_\Pi$  be  $\delta_{\Pi'} + [x/t]$ . Since  $\Gamma$  is rectified  $x$  does not yet occur in  $\delta_{\Pi'}$ . Hence  $\delta_\Pi$  is a dualizer of  $[\Pi]$ .
- 7) The last rule in  $\Pi$  is  $\forall \frac{\vdash \Gamma_1, A}{\vdash \Gamma_1, \forall x.A}$  ( $x$  not free in  $\Gamma_1$ ): By induction hypothesis, there is a proof  $\Pi'$  of  $\Gamma' = \Gamma_1, A$ , which has the same atoms as in  $\Gamma = \Gamma_1, \forall x.A$ . Hence, we can let  $\sim_\Pi = \sim_{\Pi'}$  and  $\delta_\Pi = \delta_{\Pi'}$ .

**Theorem 28.** If  $\Pi$  is a MLL1<sup>X</sup> proof of a rectified sequent  $\vdash \Gamma$ , then  $[\Pi]$  is a fonet and  $\delta_\Pi$  is a dualizer for it.



*Proof.* We have to show that none of the operations above can introduce a bmatching. For cases 1–6, this is immediate. For case 7, observe that there is a potential dependency from each existential binder in  $\llbracket \Gamma' \rrbracket$  to the new  $x$ -binder  $\bullet x$  in  $\llbracket \Gamma \rrbracket$ . However, observe that this  $\bullet x$  vertex is not connected to any vertex in  $\llbracket \Gamma' \rrbracket$ , and hence no such new dependency can be extended to a bmatching. That  $\delta_\Pi$  is a dualizer for  $\llbracket \Pi \rrbracket$  follows immediately from the construction. Hence,  $\llbracket \Pi \rrbracket$  is a fonet.  $\square$

### B. From $\text{MLS1}^\times$ Proofs to Fonets

There is a more direct path from a  $\text{MLL1}^\times$  proof  $\Pi$  of a rectified sequent  $\Gamma$  to the linked fograph  $\llbracket \Pi \rrbracket$ : simply take the fograph  $\llbracket \Gamma \rrbracket$ , and let the equivalence classes of  $\sim_\Pi$  be all the atom pairs that meet in an instance of  $\text{ax}$ , and  $\delta_\Pi$  is simply the collection of all substitutions of all the instances of the  $\exists$ -rule in  $\Pi$ . We have chosen the more cumbersome path above because it gives us a direct proof of Theorem 28. However, for translating  $\text{MLS1}^\times$  derivation into fonets, we employ exactly that direct path.

A derivation  $\Phi$  in  $\text{MLS1}^\times$  is **rectified** if every line in  $\Phi$  is rectified.

**Lemma 29.** *Let  $\Phi$  be a  $\text{MLS1}^\times$  proof of a formula  $A$ . Then  $\Phi$  is rectified iff  $A$  is rectified.*

*Proof.* The only rules involving bound variables are  $\forall$  and  $\exists$  which both remove a binder (and all occurrences of the variable it binds).  $\square$

Hence, for a non-rectified  $\text{MLS1}^\times$  derivation  $\Phi$  in  $\text{MLS1}^\times$  we can define its **rectification**  $\hat{\Phi}$  inductively, by rectifying each line, proceeding step-wise from conclusion to premise.<sup>2</sup>

A rectified derivation  $\text{MLS1}^\times \llbracket \Phi \rrbracket$  determines a substitution  $A$

which maps the existential bound variables occurring in  $A$  to the terms substituted for them in the instances of the  $\exists$ -rule in  $\Phi$ . We denote this substitution with  $\delta_\Phi$  and call it the **dualizer** of  $\Phi$ . Furthermore, every atom occurring in the conclusion  $A$  must be consumed by a unique instance of the rule  $\text{ai}$  in  $\Phi$ . This allows us to define a (partial) equivalence relation  $\sim_\Phi$  on the atom occurrences in  $A$  by  $a \sim_\Phi b$  if  $a$  and  $b$  are consumed by the same instance of  $\text{ai}$  in  $\Phi$ . We call  $\sim_\Phi$  the **linking** of  $\Phi$ , and define  $\llbracket \Phi \rrbracket = \langle \llbracket A \rrbracket, \sim_\Phi \rangle$ .

■**TODO: example here**■

**Theorem 30.** *Let  $\text{MLS1}^\times \llbracket \Phi \rrbracket$  be a rectified derivation. Then  $\llbracket \Phi \rrbracket$  is a fonet and  $\delta_\Phi$  a dualizer for it.*

For proving this theorem, we have to show that no inference rule in  $\text{MLS1}^\times$  can introduce a bmatching. To simplify the argument, we introduce the **frame** [38] of the fograph  $\mathcal{C}$ , which is a linked (propositional) cograph in which the dependencies between the binders in  $\mathcal{C}$  are encoded as links.

<sup>2</sup>As for formulas, the rectification of a derivation is unique up to renaming of bound variables.

More formally, let  $C$  be a formula with  $\llbracket C \rrbracket = \mathcal{C}$ , to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent  $C^*$ :

- 1) **Encode dependencies as fresh links.** For each dependency  $\{\bullet x_i, \bullet y_j\}$  in  $\mathcal{C}$ , with corresponding subformulas  $\exists x_i.A$  and  $\forall y_j.B$  in  $C$ , we pick a fresh (nullary) predicate symbol  $q_{i,j}$ , and then replace  $\exists x_i.A$  by  $\bar{q}_{i,j} \wedge \exists x_i.A$ , and replace  $\forall y_j.B$  by  $q_{i,j} \vee \forall y_j.B$ .
- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace  $\exists x_i.A$  by  $A$  and replace  $\forall y_j.B$  by  $B$  everywhere.
- 3) **Simplify atoms.** After step 2, replace every predicate  $p(t_1 \cdots t_n)$  (resp.  $\bar{p}(t_1 \cdots t_n)$ ) with a nullary predicate symbol  $p$  (resp.  $\bar{p}$ ).

The  $\sim_{C^*}$  consists of the pairs induced by  $\sim_{\mathcal{C}}$  and the new pairs  $\{q_{i,j}, \bar{q}_{i,j}\}$  introduced in step 1 above. We call  $C^*$  the **frame** of  $C$  and we define the **frame** of  $\mathcal{C}$ , denoted  $\mathcal{C}^*$ , as  $\langle \llbracket C^* \rrbracket, \sim_{C^*} \rangle$ .

**Lemma 31.** *A linked fograph  $\mathcal{C}$  has an induced bmatching iff its frame  $\mathcal{C}^*$  has an induced bmatching.*

*Proof.* This immediately follows from the construction of the frame. ■**Lutz: is it really an “iff”? It is easy to construct from a bmatching in  $\mathcal{C}$  a bmatching in the frame. (and I think we only need that direction). But what about the other direction?**■

*Proof of Theorem 30.* From  $\Phi$  we construct a derivation  $\Phi^*$  of  $A^*$  in the propositional fragment of  $\text{MLS1}^\times$ , such that  $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$ . The rules  $\text{ai}$ ,  $\text{t}$ ,  $\text{mix}$  and  $\text{s}$  are translated trivially, and for  $\equiv$ , it suffices to observe that the frame construction is invariant under  $\equiv$ . Finally, for the rules  $\forall$  and  $\exists$ , proceed as follows. Every instance of  $\forall$  is replaced by the derivation on the right below:<sup>3</sup>

$$\forall \frac{S\{t\}}{S\{\forall y_j.t\}} \rightsquigarrow \frac{\frac{\text{t} \quad \{a_i, t\} \parallel \Psi_1}{S\{(q_{h_1,j} \vee \bar{q}_{h_1,j}) \wedge \cdots \wedge (q_{h_j,j} \vee \bar{q}_{h_j,j}) \wedge t\}}}{S\{q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge t)\}} \parallel \Psi_2$$

where  $h_1, \dots, h_j$  range over the indices of the existential binders dependent on that  $y_j$ . It is easy to see how  $\Psi_1$  is constructed, and for  $\Psi_2$  see, e.g. [?, [35], [44]]**Lutz: check if it is really there. otherwise [37]**■ Then, every occurrence of  $\forall y_j.F$  is replaced by  $q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge F)$  in the derivation below that  $\forall$ -instance. Now, observe that all instances of the  $\exists$ -rule introducing  $x_i$  depend on  $y_j$  must occur below in the derivation (otherwise  $\Phi$  would not be rectified). Now consider such an instance  $\exists \frac{S\{B[x_i/t]\}}{S\{\exists x_i.B\}}$ . Its context  $S\{\cdot\}$  must contain all the  $\forall y_j$  the  $\exists x_i$  depends on, such that  $B$  is in their scope. Following the translation of the

<sup>3</sup>For better readability we omit superfluous parentheses, knowing that we always have  $\equiv$  incorporating associativity and commutativity of  $\wedge$  and  $\vee$ .

$\forall$  rules above, we can therefore translate the  $\exists$ -rule instance by the following derivation

$$\begin{aligned} & S_0\{\bar{q}_{i,k_1} \wedge S_1\{\bar{q}_{i,k_2} \wedge \cdots S_{k_i-1}\{\bar{q}_{i,k_i} \wedge S_{k_i}\{B'\}\}\cdots\}\} \\ & \quad \{s, \equiv\} \parallel \Psi_3 \\ & S_0\{S_1\{\cdots S_{k_i-1}\{S_{k_i}\{\bar{q}_{i,k_1} \wedge \bar{q}_{i,k_2} \wedge \cdots q_{i,k_i} \wedge B'\}\}\cdots\}\} \end{aligned}$$

where  $k_1, \dots, k_i$  are the indices of the universal binders on which that  $x_i$  depends, and  $B'$  is  $B$  in which all predicates are replaced by nullary one (step 3 in the frame construction). The derivation  $\Psi_3$  can be constructed in the same way as  $\Psi_2$  above.

Doing this to all instances of the rules  $\forall$  and  $\exists$  in  $\Phi$  yields indeed a propositional derivation  $\Phi^*$  with  $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$ . It has been shown by Retoré [?] and rediscovered by Straßburger [37] that  $\llbracket \Phi^* \rrbracket = \langle \llbracket C^* \rrbracket, \sim_{\Phi^*} \rangle$  can not contain an induced bimatching. By Lemma 33,  $\llbracket \Phi \rrbracket$  does not have an induced bimatching either. Furthermore, it followed from the definition of  $\delta_\Phi$  that it is a dualizer for  $\llbracket \Phi \rrbracket$ . Hence  $\llbracket \Phi \rrbracket$  is a fonet.  $\square$

**Remark 32.** There is an alternative path of proving Theorem 30 by translating  $\Phi$  to an  $\text{MLL1}^\times$ -proof  $\Pi$ , observing that this process preserves the linking and the dualizer. However, for this, we have to extend the construction above to the cut-rule, and then show that linking and dualizer of a sequent proof  $\Pi$  are invariant under cut elimination. This can be done similarly to unification nets in [38].

### C. From Fonets to $\text{MLL1}^\times$ Proofs

Now we are going to show how from a given fonet  $\langle C, \sim_C \rangle$  we can construct a sequent proof  $\Pi$  in  $\text{MLL1}^\times$  such that  $\llbracket \Pi \rrbracket = \langle C, \sim_C \rangle$ . In the proof net literature, this operation is also called *sequentialization*. The basic idea behind our sequentialization is to construct a propositional linked cograph, called the **frame** [38] of  $C$ , in which the dependencies between the binders in  $C$  are encoded as links. Then we can apply the *splitting tensor theorem* to the frame, and then reconstruct the sequent proof  $\Pi$ . **[[Lutz: if the proof of thm 30 is verified, we can delete the frame-def here]]**

More formally, let  $\Gamma$  be a sequent with  $\llbracket \Gamma \rrbracket = C$ , to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent  $\Gamma^*$ :

- 1) **Encode dependencies as fresh links.** For each dependency  $(\bullet x, \bullet y)$  in  $C$ , with corresponding subformulas  $\exists x A$  and  $\forall y B$  in  $\Gamma$ , we pick a fresh (nullary) predicate symbol  $q$ , and then replace  $\exists x A$  by  $q \wedge \exists x A$ , and replace  $\forall y B$  by  $\bar{q} \vee \forall y B$ .
- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace  $\exists x A$  by  $A$  and replace  $\forall y B$  by  $B$  everywhere.
- 3) **Simplify atoms.** After step 2, replace every predicate  $p(t_1 \cdots t_n)$  (resp.  $\bar{p}(t_1 \cdots t_n)$ ) with a nullary predicate symbol  $p$  (resp.  $\bar{p}$ )

The  $\sim_{\Gamma^*}$  consists of the pairs induced by  $\sim_C$  and the new pairs  $\{q, \bar{q}\}$  introduced in step 1 above. We call  $\Gamma^*$  the **frame** of  $\Gamma$

and we define the **frame** of  $C$ , denoted  $C^*$ , as  $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$ , and we immediately have the following:

**Lemma 33.** *A linked fograph  $C$  induces a bimatching iff its frame  $C^*$  has an induced bimatching.*

Let  $\Gamma$  be a propositional sequent and  $\sim_\Gamma$  be a linking for  $\llbracket \Gamma \rrbracket$ . A conjunction formula  $A \wedge B$  is **splitting** or a **splitting tensor** if  $\Gamma = \Gamma', A \wedge B, \Gamma''$  and  $\sim_\Gamma = \sim_1 \cup \sim_2$ , such that  $\sim_1$  is a linking for  $\llbracket \Gamma', A \rrbracket$  and  $\sim_2$  is a linking for  $\llbracket B, \Gamma'' \rrbracket$ , i.e., removing the  $\wedge$  from  $A \wedge B$  splits the linked fograph  $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$  into two fographs. We say that  $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$  is **mixed** iff  $\Gamma = \Gamma', \Gamma''$  and  $\sim_\Gamma = \sim_1 \cup \sim_2$ , such that  $\sim_1$  is a linking for  $\llbracket \Gamma' \rrbracket$  and  $\sim_2$  is a linking for  $\llbracket \Gamma'' \rrbracket$ . Finally,  $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$  is **splittable** if it is mixed or has a splitting tensor.

The purpose of introducing the frame is the following theorem.

**Theorem 34.** *Let  $\Gamma$  be a propositional sequent containing only atoms and  $\wedge$ -formulas, and  $\sim_\Gamma$  be a linking for  $\llbracket \Gamma \rrbracket$ . If  $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$  does not induce a bimatching then it is splittable.*

This is the well-know splitting-tensor-theorem [25], [39], adapted for the presence of mix. In the setting of linked cographs, it has first been proved by Retoré [40], [41]. We use it now for our sequentialization:

**Theorem 35.** *Let  $\langle C, \sim_C \rangle$  be a fonet, and let  $\Gamma$  be a sequent with  $\llbracket \Gamma \rrbracket = C$ . Then there is an  $\text{MLL1}^\times$ -proof  $\Pi$  of  $\Gamma$ , such that  $\llbracket \Pi \rrbracket = \langle C, \sim_C \rangle$ .*

*Proof.* Let  $\delta_C$  be the dualizer of  $\langle C, \sim_C \rangle$ . We proceed by induction on the size of  $\Gamma$  (i.e., the number of symbols in it, without counting the commas). If  $\Gamma$  contains a formula with  $\vee$ -root, or a formula  $\forall x.A$ , we can immediately apply the  $\vee$ -rule or the  $\forall$ -rule of  $\text{MLL1}^\times$  and proceed by induction hypothesis. If  $\Gamma$  contains a formula  $\exists x.A$  such that the corresponding binder  $\bullet x$  in  $C$  has no dependency, then we can apply the  $\exists$ -rule, choosing the term  $t$  as determined by  $\delta_C$ , and proceed by induction hypothesis. Hence, we can now assume that  $\Gamma$  contains only atoms,  $\wedge$ -formulas, or formulas of shape  $\exists x.A$ , where the vertex  $\bullet x$  has dependencies. Then the frame  $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$  does not induce a bimatching and contains only atoms and  $\wedge$ -formulas, and is therefore splittable. If it is mixed, then we can apply the mix-rule to  $\Gamma$  and apply the induction hypothesis to the two components. If it is not mixed then there must be a splitting tensor. If the splitting  $\wedge$  is already in  $\Gamma$ , then we can apply the  $\wedge$ -rule and proceed by induction hypothesis on the two branches. However, if  $\Gamma^*$  is not mixed and all splitting tensors are  $\wedge$ -formulas introduced in step 1 of the frame construction, then we get a contradiction as in that case there must be a  $\vee$ - or  $\forall$ -formula in  $\Gamma$ . **[[Lutz: can anyone give a good argument here?]]**  $\square$

### D. From Fonets to $\text{MLS1}^\times$ Proofs

We can now straightforwardly obtain the same result for  $\text{MLS1}^\times$ :

**Theorem 36.** Let  $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$  be a fonet, and let  $C$  be a formula with  $\llbracket C \rrbracket = \mathcal{C}$ . Then there is a derivation  $\text{MLS1}^{\times} \parallel_{\Phi}^t C$  such that  $\llbracket \Phi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ .

*Proof.* We apply Theorem 35 to obtain a sequent proof  $\Pi$  of  $\vdash C$  with  $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ . Then we apply Lemma 17, observing that the translation from  $\text{MLL1}^{\times}$  to  $\text{MLS1}^{\times}$  preserves linking and dualizer.  $\square$

**Remark 37.** Note that it is also possible to do a direct “sequentialization” into the deep inference system  $\text{MLS1}^{\times}$ , using the techniques presented in [37] and [42].

## IX. SKEW BIFIBRATIONS AND RESOURCE MANAGEMENT

In this section we establish the relation between skew bifibrations and derivations in  $\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\}$ . However, if a derivation  $\Phi$  contains instances of the rules  $c_{\forall}$ ,  $m_{\forall}$ , and  $m_{\exists}$  we can no longer naively define the rectification  $\hat{\Phi}$  as in the previous section for  $\text{MLS1}^{\times}$ , as these two rules cannot be applied if premise and conclusion are rectified. For this reason we define here rectified versions  $\hat{c}_{\forall}$ ,  $\hat{m}_{\forall}$  and  $\hat{m}_{\exists}$ , shown below:

$$\hat{c}_{\forall} \frac{S\{\forall y. \forall x. Ax\}}{S\{\forall x. Ax\}} \quad \hat{m}_{\forall} \frac{S\{(\forall y. Ay) \vee (\forall z. Bz)\}}{S\{\forall x. (Ax \vee Bx)\}} \quad \hat{m}_{\exists} \frac{S\{(\exists y. Ay) \vee (\exists z. Bz)\}}{S\{\exists x. (Ax \vee Bx)\}}$$

Here, we use the notation  $A \cdot$  for a formula  $A$  with occurrences of a placeholder  $\cdot$  for a variable. Then  $Ax$  stands for the results of replacing that placeholder with  $x$ , and also indicating that  $x$  must not occur in  $A \cdot$ . Then  $\forall x. Ax$  and  $\forall y. Ay$  are the same formula modulo renaming of the bound variable bound by the outermost  $\forall$ -quantifier. We also demand that the variables  $x$ ,  $y$ , and  $z$  do not occur in the context  $S\{\cdot\}$ .

Note that in an instance of  $\hat{m}_{\forall}$  or  $\hat{m}_{\exists}$  (as shown above), we can have  $x = y$  or  $x = z$ , but not both if the premise is rectified. If  $x = y$  and  $x = z$  we have  $m_{\forall}$  and  $m_{\exists}$  as special cases of  $\hat{m}_{\forall}$  and  $\hat{m}_{\exists}$ , respectively. And similarly, if  $x = y$  then  $c_{\forall}$  is a special case of  $\hat{c}_{\forall}$ .

For a derivation  $\Phi$  in  $\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\}$ , we can now construct the **rectification**  $\hat{\Phi}$  by rectifying each line of  $\Phi$ , yielding a derivation in  $\{w, ac, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$ .

For each instance  $r \frac{Q}{P}$  of an inference rule in  $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$  we can define the **induced map**  $\llbracket r \rrbracket: V_{\llbracket Q \rrbracket} \rightarrow V_{\llbracket P \rrbracket}$  which acts as the identity for  $r \in \{m, \equiv\}$  and as the canonical injection for  $r = w$ . For  $r = ac$  it maps the vertices corresponding to the two atoms in the premise to vertex of the contracted atom in the conclusion, and for  $r \in \{\hat{c}_{\forall}, \hat{m}_{\forall}, \hat{m}_{\exists}\}$  it maps the two vertices corresponding to the quantifiers in the premise to the one in the conclusion (as acts as the identity on all other vertices). For a derivation  $\Phi$  in  $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$  we can then define the **induced map**  $\llbracket \Phi \rrbracket$  as the composition of the induced maps of the rule instances in  $\Phi$ . **Jui-Hsuan:** maybe mention at least

that the induced maps define graph homomorphisms. Do we need to talk about the contexts  $S\{\cdot\}$  here (induced maps act clearly as the identity on contexts but we need them for the composition)? **Lutz:** For the context, I already say it is the identity. For the homom, it comes later

**Lemma 38.** Let  $\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\} \parallel_{\Phi}^A B$  be a derivation. Then there is a rectified derivation  $\{w, \hat{ac}, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\} \parallel_{\hat{\Phi}}^{\hat{A}} \hat{B}$ , such that the induced maps  $\llbracket \Phi \rrbracket: \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$  and  $\llbracket \hat{\Phi} \rrbracket: \llbracket \hat{A} \rrbracket \rightarrow \llbracket \hat{B} \rrbracket$  are equal up to a variable renaming of the vertex labels.

*Proof.* Immediate from the definition.  $\square$

**TODO: example**

### A. From Contraction and Weakening to Skew Bifibrations

We say that a derivation  $\Phi$  is **sane** if for every line  $Q$  in  $\Phi$  we have that  $\llbracket D \rrbracket$  is a fograph (i.e., all binders are legal). Clearly, every rectified derivation is sane, but not vice versa, as we might have multiple occurrences of bound variables in  $Q$ , such that  $\llbracket Q \rrbracket$  is still a fograph.

**Lemma 39.** Let  $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\} \parallel_{\Phi}^A B$  be a sane derivation. Then the induced map  $\llbracket \Phi \rrbracket: \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$  is a skew bifibration.

Before we show the proof of this lemma, we introduce another useful concept: the **propositional encoding**  $A^{\circ}$  of a formula  $A$ , which is a propositional formula with the property that  $\llbracket A^{\circ} \rrbracket = \llbracket A \rrbracket$ . For this, we introduce new propositional variables that have the same names as the (first-order) variables  $x \in \text{VAR}$ . Then  $A^{\circ}$  is defined inductively by:

$$\begin{aligned} a^{\circ} &= a & (\forall x A)^{\circ} &= x \vee A^{\circ} \\ (A \vee B)^{\circ} &= A^{\circ} \vee B^{\circ} & (\exists x A)^{\circ} &= x \wedge A^{\circ} \\ (A \wedge B)^{\circ} &= A^{\circ} \wedge B^{\circ} \end{aligned}$$

**Lemma 40.** For every formula  $A$ , we have  $\llbracket A^{\circ} \rrbracket = \llbracket A \rrbracket$ .

*Proof.* Straightforward induction on  $A$ .  $\square$

We use  $\equiv^{\circ}$  to denote the restriction of  $\equiv$  to propositional formulas, i.e., the first two lines in (2).

*Proof of Lemma 39.* First, observe that for every inference rule  $r \in \{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$  the induced map  $\llbracket r \rrbracket: V_{\llbracket Q \rrbracket} \rightarrow V_{\llbracket P \rrbracket}$  defines a existential preserving graph homomorphism  $\llbracket Q \rrbracket \rightarrow \llbracket P \rrbracket$  and a fibration on the corresponding binding graphs. **Jui-Hsuan:** we may need to have some explication here. **Lutz:** no Therefore, their composition  $\llbracket \Phi \rrbracket$  has the same properties fibration.

For showing that it is also a skew fibration, we construct for  $\Phi$  its propositional encoding  $\Phi^{\circ}$  by translating every line into its propositional encoding. **Jui-Hsuan:** maybe mention that an instance of one of the other rules can be translated into an instance of the same rule. It’s trivial but may be worth

mentioning.  $\llbracket \llbracket \text{Lutz: done below} \rrbracket$  The instances of the rules  $\widehat{m}_\forall$  and  $\widehat{m}_\exists$  are replaced in two steps by:

$$\widehat{ac} \frac{S\{(y \vee (Ay)^\circ) \vee (z \vee (Bz)^\circ)\}}{S\{(y \vee z) \vee ((Ay)^\circ \vee (Bz)^\circ)\}} \equiv \frac{S\{(y \vee (Ay)^\circ) \vee (z \vee (Bz)^\circ)\}}{S\{x \vee ((Ax)^\circ \vee (Bx)^\circ)\}}$$

and

$$\widehat{m} \frac{S\{(y \wedge (Ay)^\circ) \vee (z \wedge (Bz)^\circ)\}}{S\{(y \vee z) \wedge ((Ay)^\circ \vee (Bz)^\circ)\}} \equiv \frac{S\{(y \wedge (Ay)^\circ) \vee (z \wedge (Bz)^\circ)\}}{S\{x \wedge ((Ax)^\circ \vee (Bx)^\circ)\}}$$

687 respectively, where  $\widehat{ac}$  is a  $ac$  that renames the variables—the  
688 propositional variable, as well as the first-order variable of the  
689 same name—as everything is recified, there is no ambiguity  
690 here. Any instance of a rule  $w$ ,  $ac$ ,  $m$ , or  $\equiv$  is translated to an  
691 instance of the same rule, and  $\widehat{c}_\forall$  is translated to  $\widehat{ac}$ .

692 This gives us a derivation  $\{w, ac, \widehat{ac}, m, \equiv\} \parallel_{B^\circ} \Phi^\circ$  such that  
693  $[\Phi^\circ] = [\Phi]$ . It has been shown in [23] that  $[\Phi^\circ]$  is a skew  
694 fibration (see also [12], [43], [15]). Hence,  $[\Phi]$  is a skew  
695 fibration.  $\square$

#### 696 B. From Skew Bifibrations to Contraction and Weakening

**Lemma 41.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be fographs, let  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  be a skew bifibration, and let  $A$  and  $B$  be formulas with  $\llbracket A \rrbracket = \mathcal{A}$  and  $\llbracket B \rrbracket = \mathcal{B}$ . Then there are derivations*

$$\frac{A}{\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\} \parallel_{\widehat{B}} \widehat{\Phi}} \quad \text{and} \quad \frac{A\sigma_\varphi}{\{w, ac, \widehat{c}_\forall, m, m_\forall, m_\exists, \equiv\} \parallel_{\widehat{B}} \widehat{\Phi}}$$

697 such that  $[\widehat{\Phi}] = \varphi$  and  $\widehat{\Phi}$  is a rectification of  $\Phi$ , and  $\sigma_\varphi$  is  
698 the substitution induced by  $\varphi$ .

699 In the proof of this lemma, we make use of the following  
700 concept: Let  $s \parallel_P \Psi$  be a derivation where  $P$  and  $Q$  are proposi-

701 tional formulas (possibly using variable  $x \in \text{VAR}$  at the places  
702 of atoms). We say that  $\Psi$  can be **lifted** to  $S'$  if there are (first-  
703 order) formulas  $C$  and  $D$  such that  $P = C^\circ$  and  $Q = D^\circ$  and  
704 there is a derivation  $s' \parallel_{D'} \Psi'$ .

*Proof of Lemma 41.* By Lemma 40 we have  $\mathcal{A} = \llbracket A^\circ \rrbracket$  and  $\mathcal{B} = \llbracket B^\circ \rrbracket$ . Let  $V'_B \subseteq V_B$  be the image of  $\varphi$ , and let  $B_1$  be the subgraph of  $\mathcal{B}$  induced by  $V'_B$ . Hence, we have two maps  $\varphi': \mathcal{A} \rightarrow B_1$  being a surjection and  $\varphi'': B_1 \rightarrow \mathcal{B}$  being an injection that reflects edges.  $\llbracket \llbracket \text{Jui-Hsuan: what do you mean by "reflect edges"?} \rrbracket \rrbracket \llbracket \llbracket \text{Lutz: edge downstairs implies edge upstairs} \rrbracket \rrbracket$  Both,  $\varphi'$  and  $\varphi''$  remain skew bifibrations. Let us first look at  $\varphi'$ . Let  $\tilde{B}_1$  be the propositional formula obtained from  $B^\circ$  by removing all atoms that are not represented by vertices in  $V'_B$ . Then  $\llbracket \tilde{B}_1 \rrbracket = B_1$ . By [23, Proposition 7.6.1], we have

a derivation  $\{w, \equiv\} \parallel_{\Phi_1^\circ} \tilde{B}_1$ . A subformula of  $B^\circ$  is called *weak* if

it has been introduced by a weakening. All subformulas of a weak formula are also weak. Two weak subformulas  $B'$  and  $B''$  of  $B^\circ$  form a *weak pair* if  $B^\circ \equiv S\{B' \vee B''\}$  for some context  $S\{\cdot\}$ . We can assume without loss of generality that whenever weak subformulas  $B'$  and  $B''$  form a weak pair, they have been introduced by the same instance of  $w$  in  $\Phi_1^\circ$ .<sup>4</sup> Now we show that  $\Phi_1^\circ$  can be lifted. For this, observe that whenever a weakening in  $\Phi_1^\circ$  deletes an atom  $x \in \text{VAR}$ , it must also delete all atoms in the scope of the corresponding quantifier, because  $\varphi'$  is a fibration on the binding graph. Hence, each line in  $\Phi_1^\circ$  is the propositional encoding  $P^\circ$  of a first-order formula  $P$ . We now have to show that each instance of  $w$  is indeed a correct application in first-order logic. For this we have to inspect the cases a weakening happens inside a subformula  $x \vee C$  or  $x \wedge C$  in  $\Phi_1^\circ$ . There are the following cases:

$$\frac{S\{x \vee C\}}{S\{x \vee D \vee C\}} \quad \frac{S\{x \wedge C\}}{S\{x \wedge (D \vee C)\}} \quad \frac{S\{x \wedge C\}}{S\{(x \vee D) \wedge C\}}$$

In the first case the weakening happens inside the scope of a  $\forall$ -quantifier, and in the second case inside the scope of a  $\exists$ -quantifier. Both cases are unproblematic in first-order logic. However, in the third case an  $\exists$ -quantifier would be transformed into an  $\forall$ -quantifier. But as  $\varphi$  has to preserve existentials, this third case cannot occur. Thus we have a first

order derivation  $\{w, \equiv\} \parallel_{\Phi_1} B_1$  with  $B_1^\circ = \tilde{B}_1$ .

Let us now look at  $\varphi''$ . Let  $\mathcal{A}_1 = \mathcal{A}\sigma_\varphi$  be the graph obtained from  $\mathcal{A}$  by applying  $\sigma_\varphi$  to all the labels. Note that  $\mathcal{A}_1$  is not necessarily a fograph, as binders might be illegal. But it still is a cograph, and we have a surjective skew fibration  $\varphi'': \mathcal{A}_1 \rightarrow B_1$  that preserves the labels. Therefore, by [43,

Proposition 7.5], there is a derivation  $\{ac, m, \equiv\} \parallel_{\Phi_2^\circ} A_1^\circ$ , where

$A_1^\circ = A^\circ \sigma_\varphi$  is the result of applying  $\sigma_\varphi$  to  $A^\circ$ . Note that  $A_1^\circ = (A\sigma_\varphi)^\circ$  and  $B_1^\circ$  are both propositional encodings. We plan to show that  $\Phi_2$  can be lifted to  $\{ac, \widehat{c}_\forall, m, m_\forall, m_\exists, \equiv\}$ . However, observe that not every formula occurring in  $\Phi_2$  is a propositional encoding. There are two reasons for this: (i) we might have  $P \equiv^\circ Q$  where  $P$  is a propositional encoding but  $Q$  is not, and (ii) the rule  $ac$  can duplicate an atom  $x \in \text{VAR}$ . Let us write  $ac_x$  for such instances. To address (i), we consider here formulas equivalent modulo  $\equiv$ , always knowing that we can add instances of  $\equiv$  as needed.<sup>5</sup>  $\llbracket \llbracket \text{Jui-Hsuan: this does not seem clear to me. What if from } A_1^\circ \text{ to } B_1^\circ \text{ there are just a bunch of } \equiv^\circ? \text{ What do we do in this case?} \rrbracket \rrbracket \llbracket \llbracket \text{Lutz: see footnote} \rrbracket \rrbracket$  To address (ii), we apply a permutation argument, permuting all instances of  $ac_x$  up until they either reach the

<sup>4</sup>If  $\Phi_1^\circ$  is not of that shape, it can brought into this form by simple rule permutations, and then collapsing several weakenings into a single one.

<sup>5</sup>Note that whenever we have formulas  $P$  and  $Q$  with  $P^\circ \equiv^\circ Q^\circ$  then  $P \equiv Q$ .



top of the derivation or an instance of  $m$  which separates the two atoms in the premise. More precisely, we consider the following inference rule

$$\text{ac}_x^{\equiv} \frac{S_0\{S_1\{x\} \vee S_2\{x\}\}}{S\{x\}} \quad (4)$$

where  $S_1\{\cdot\} \equiv \{\cdot\} \vee E$  and  $S_2\{\cdot\} \equiv \{\cdot\} \vee F$  and  $S\{\cdot\} \equiv S_0\{\{\cdot\} \vee E \vee F\}$  for some formulas  $E$  and  $F$ , where  $E$  or  $F$  or both might be empty. The rule  $\text{ac}_x^{\equiv}$  permutes over  $\equiv$ ,  $\text{ac}$ , and other instances of  $\text{ac}_x^{\equiv}$ , and over instances of  $m$  if they occur inside  $S_0$  or  $S_1$  or  $S_2$ . The only situation in which  $\text{ac}_x^{\equiv}$  cannot be permuted up is the following:

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(R_1\{x\} \wedge C) \vee (R_2\{x\} \wedge D)\}}}{S\{(R_1\{x\} \vee R_2\{x\}) \wedge (C \vee D)\}} \quad (5)$$

We can therefore assume that all instances of  $\text{ac}_x$ , that contract an atom  $x \in \text{VAR}$  are either at the top of  $\Phi_2^\circ$  or below a  $m$ -instance as in (5). We now lift  $\Phi_2^\circ$  to  $\{\text{ac}, \text{c}_\forall, m, m_\forall, m_\exists, \equiv\}$ , proceed by induction on the height of  $\Phi_2^\circ$ , beginning at the top, making a case analysis on the topmost rule that is not  $\equiv$ .

•  $\text{ac}_x$ : We know that the premiss of (4) is a propositional encoding. Hence,  $S_1\{\cdot\} = \{\cdot\} \vee E^\circ$  and  $S_2\{\cdot\} = \{\cdot\} \vee F^\circ$  and both  $x$  are universals, and  $E^\circ \vee F^\circ$  contains all occurrences of  $x$  bound by that universal. We have the following subcases:

–  $E$  and  $F$  are both non-empty: We have

$$\text{ac}_x^{\equiv} \frac{S^\circ\{(x \vee E^\circ) \vee (x \vee F^\circ)\}}{S^\circ\{x \vee (E^\circ \vee F^\circ)\}}$$

which can be lifted to

$$m_\forall \frac{S\{(\forall x.E) \vee (\forall x.F)\}}{S\{\forall x.(E \vee F)\}}$$

where  $S^\circ\{\cdot\}$ ,  $E^\circ$ ,  $F^\circ$  are the propositional encodings of  $S\{\cdot\}$ ,  $E$ ,  $F$ , respectively.

–  $E^\circ$  is empty and  $F^\circ$  is non-empty: We have

$$\text{ac}_x^{\equiv} \frac{S^\circ\{x \vee (x \vee F^\circ)\}}{S^\circ\{x \vee F^\circ\}}$$

which can be lifted to

$$\text{c}_\forall \frac{S\{\forall x.\forall x.F\}}{S\{\forall x.F\}}$$

–  $E^\circ$  is non-empty and  $F^\circ$  is empty: This is similar to the previous case.

–  $E^\circ$  and  $F^\circ$  are both empty: This is impossible as the premise would not be a propositional encoding.

•  $\text{ac}$  (contracting an ordinary atom): This can trivially be lifted.

•  $m$  that is not in the situation of (5): Then now encoding of a quantifier is affected and the instance of  $m$  can be lifted. **TODO: medial permutation!!!**

•  $m/\text{ac}_x$  as in situation (5): We must have  $R_1\{x\} \equiv x \vee E$  for some  $E$  and  $R_2\{x\} \equiv x \vee F$  for some  $F$  with  $R\{x\} \equiv x \vee E \vee F$ . Otherwise, the application of  $\text{ac}_x^{\equiv}$  would not be correct. We have the following four cases:

–  $E$  and  $F$  are both non-empty: Then (5) is (modulo omitted applications of  $\equiv$ ):

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{((x \vee E) \wedge C) \vee ((x \vee F) \wedge D)\}}}{S\{((x \vee E) \vee (x \vee F)) \wedge (C \vee D)\}} \frac{S\{((x \vee E) \wedge C) \vee ((x \vee F) \wedge D)\}}{S\{(x \vee E \vee F) \wedge (C \vee D)\}}$$

which can be lifted to

$$m_\forall \frac{\frac{m}{S\{((\forall x.E) \wedge C) \vee ((\forall x.F) \wedge D)\}}}{S\{((\forall x.E) \vee (\forall x.F)) \wedge (C \vee D)\}} \frac{S\{((\forall x.E) \wedge C) \vee ((\forall x.F) \wedge D)\}}{S\{(\forall x.(E \vee F)) \wedge (C \vee D)\}}$$

**Jui-Hsuan:** maybe need some words to exclude the case in which  $C$  (or  $D$ ) is a propositional variable. **Lutz:** shit. (you mean a “first order variable”) this actually can happen. then we have another  $m_\exists$

–  $E$  is empty and  $F$  is not: Then (5) becomes

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(x \wedge C) \vee ((x \vee F) \wedge D)\}}}{S\{(x \vee (x \vee F)) \wedge (C \vee D)\}} \frac{S\{(x \wedge C) \vee ((x \vee F) \wedge D)\}}{S\{(x \vee F) \wedge (C \vee D)\}}$$

The conclusion is the propositional encoding of  $S\{(\forall x.F) \wedge (C \vee D)\}$  and the premise is the propositional encoding of  $S\{(\exists x.C) \vee ((\forall x.F) \vee D)\}$ . Also note that no  $m$ -instance can break up the conjunction in  $x \wedge C$  in the premise. Hence,  $\varphi$  maps an existential to a universal, which is ruled out by the definition. Hence, this case cannot occur.

–  $E$  is non-empty and  $F$  is empty: This case is similar to the previous subcase.

–  $E$  and  $F$  are both empty: Then (5) is

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(x \wedge C) \vee (x \wedge D)\}}}{S\{(x \vee x) \wedge (C \vee D)\}} \frac{S\{(x \wedge C) \vee (x \wedge D)\}}{S\{x \wedge (C \vee D)\}}$$

which can be lifted immediately to

$$m_\exists \frac{S\{(\exists x.C) \vee (\exists x.D)\}}{S\{\exists x.(C \vee D)\}}$$

Thus  $\Phi_2^\circ$  can be lifted to  $\{\text{ac}, \text{c}_\forall, m, m_\forall, m_\exists, \equiv\} \parallel \Phi_2$ . We construct  $B_1$

$\Phi$  by composing  $\Phi_2$  and  $\Phi_1$ . Then  $\hat{\Phi}$  can be constructed by rectifying  $\Phi$ , where the variables to be used in  $A$  are already given. That  $\varphi = [\hat{\Phi}]$  follows immediately from the construction.  $\square$

## X. SUMMARY AND PROOF OF MAIN RESULT

The only theorem of Section VI that has not yet been proved is Theorem 16 establishing the full correspondence between decomposed proofs in KS1 and combinatorial proofs. We show the proof here, by summarizing the results of the previous two Sections VIII and IX.

*Proof of Theorem 16.* First, assume we have a combinatorial proof  $\varphi: \mathcal{C} \rightarrow \mathcal{A}$  be a combinatorial proof and a formula  $A$  with  $\mathcal{A} = \llbracket A \rrbracket$ . Let  $C$  be a formula with  $\llbracket C \rrbracket = \mathcal{C}$ , and let  $\sigma_\varphi$  be the substitution induced by  $\varphi$ . By Lemma 41 there is a derivation

$$\frac{C\sigma_\varphi}{\{w, ac, c_\vee, m, m_\vee, m_\exists, \equiv\} \parallel \Phi_2} A$$

Since  $\mathcal{C}$  is a fonet, we have by Theorem 36 a derivation

$$\frac{t}{MLS1^x \parallel \Phi_1} C$$

This derivation remains valid if we apply the substitution  $\sigma_\varphi$  to every line in  $\Phi_1$ , yielding the derivation  $\Phi_1$  of  $C\sigma_\varphi$  as desired.

Conversely, assume we have a decomposed derivation

$$\frac{\frac{t}{MLS1^x \parallel \Phi_1} A'}{\{w, ac, m, m_\vee, m_\exists, \equiv\} \parallel \Phi_2} A \quad (6)$$

Then we can transform  $\Phi_1$  into a rectified form  $\widehat{\Phi}_1$ , proving  $\widehat{A}'$ . By Theorem 30, the linked fograph  $\llbracket \widehat{\Phi}_1 \rrbracket = \langle \llbracket \widehat{A}' \rrbracket, \sim_{\widehat{\Phi}_1} \rangle$  is a fonet. Then, by Lemma 38, there is a rectified derivation

$$\frac{\{w, \widehat{ac}, \widehat{c}_\vee, m, \widehat{m}_\vee, \widehat{m}_\exists, \equiv\} \parallel \widehat{\Phi}_2}{\widehat{A}} \text{ whose induced map } [\widehat{\Phi}_2]: \llbracket \widehat{A}' \rrbracket \rightarrow \widehat{A}$$

$\llbracket \widehat{A} \rrbracket$  is the same as the induced map  $[\Phi_2]: \llbracket A' \rrbracket \rightarrow \llbracket A \rrbracket$  of  $\Phi_2$ . By Lemma 39, this map is a skew bifibration. Hence, we have a combinatorial proof  $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$  with  $\mathcal{C} = \llbracket A' \rrbracket$ .  
**[[Lutz: shit, something's wrong...]]**  $\square$

Note that Theorem 16 shows at the same time soundness, completeness, and full completeness, as

- 1) every proof in KS1 can be translated into a combinatorial proof, and
- 2) every combinatorial proof is the image of a KS1-proof under that translation.

## XI. CONCLUSION

In this paper we have uncovered a close correspondence between first-order combinatorial proofs and decomposed deep inference derivations of system KS1, and we have shown that every proof in KS1 has such a decomposed form.

The most surprising discovery for us was that all technical difficulties in our work could be reduced (in a non-trivial way) to the propositional setting.

The obvious next step in our research is to investigate proof composition and normalisation of first-order combinatorial proofs. Even in the propositional setting, the normalization of combinatorial proofs is underdeveloped. There exist two different procedures for cut elimination [12], [?], but both have their insufficiencies, and there is no general theory.

**[[Lutz: do we want/can say more here?]]**

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912 A. Unification Nets

913 **[[TODO: ]]**

914 In this paragraph, we associate each formula  $A$  with its  
 915 **formula tree**  $\mathcal{F}(A)$ , a directed tree with leaves labelled by  
 916 atoms, internal nodes labelled by connectives and quantifiers,  
 917 and edges directed from leaves to the root. For a sequent  
 918  $\Gamma = A_1, \dots, A_n$ , we denote with  $\mathcal{F}(\Gamma)$ , the forest formed by  
 919  $\mathcal{F}(A_1), \dots, \mathcal{F}(A_n)$ , i.e., the disjoint union of  $\mathcal{F}(A_i)$ 's. The  
 920 **roots** of  $\mathcal{F}(\Gamma)$  are the roots of  $A_i$ 's

921 Let  $\Gamma$  be a sequent in  $\text{MLL1}^\times$ . Consider the forest  $\mathcal{F}(\Gamma)$ .  
 922 A **link** on  $\Gamma$  is a pair of leaves whose atoms are pre-dual. A  
 923 **linking**  $\lambda$  on  $\Gamma$  is a set of disjoint links such that each leaf  
 924 of  $\mathcal{F}(\Gamma)$  is either labelled by  $t$  or in exactly one link. Similar  
 925 to the set of links in linked fographs, a linking can be seen  
 926 as a unification problem, and a **dualizer**  $\delta$  of the linking  $\lambda$  is  
 927 an assignment unifying all the links in  $\lambda$ . There exists a **most**  
 928 **general dualizer** of  $\lambda$  if  $\lambda$  has a dualizer. **[[Jui-Hsuan: Now**  
 929 **I use the same terminology as for linked fographs]]** **[[Lutz:**  
 930 **use  $\delta$  for the dualizer (or even better, make it a macro)]]** A  
 931 **dependency** is a pair  $(\bullet\exists x, \bullet\forall y)$  of nodes such that the most  
 932 general dualizer assigns to  $x$  a term containing  $y$ .

933 Let  $\lambda$  is a linking on  $\Gamma$  that has a dualizer. The **unification**  
 934 **structure**  $\mathcal{U}(\lambda)$  associated with  $\lambda$  is the forest  $\mathcal{F}(\Gamma)$  together  
 935 with an undirected edge between leaves  $l$  and  $l'$  for every link  
 936  $\{l, l'\}$  in  $\lambda$  and a directed edge from  $\bullet\exists x$  to  $\bullet\forall y$  for every  
 937 dependency  $(\bullet\exists x, \bullet\forall y)$ .

938 A **switching graph** of a unification structure  $\mathcal{U}(\lambda)$  is any  
 939 derivative of  $\mathcal{U}(\lambda)$  obtained by keeping only one edge into  
 940 each  $\vee$  and  $\forall$  and undirecting remaining edges. A linking is  
 941 **correct** if it is unifiable and all of the switching graphs of its  
 942 associated unification structure are acyclic.

943 **Definition 42.** A **unification net** on a sequent  $\Gamma$  is a correct  
 944 linking on  $\Gamma$ .

945 B. Translation between Unification Nets and  $\text{MLL1}^\times$

946 **[[TODO: ]]**

947 **Theorem 43.** If a sequent is provable in  $\text{MLL1}^\times$ , then there  
 948 exists a unification net on it.

949 *Proof.* We proceed by induction on the proof of  $\vdash \Gamma$  in  
 950  $\text{MLL1}^\times$ , making a case analysis on the bottommost rule  
 951 instance:

- 952 •  $\text{ax} \frac{}{\vdash a, \bar{a}}$  : the linking  $\{a, \bar{a}\}$  is correct.
- 953 •  $\text{t} \frac{}{\vdash t}$  : the empty linking is correct.
- 954 •  $\text{mix} \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$  : By induction hypothesis, there is a  
 955 correct linking on  $\Gamma$  and another one on  $\Delta$ , their union  
 956 giving a correct linking on  $\Gamma, \Delta$ .

- $\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$  : By induction hypothesis, there is a correct  
 957 linking on  $\Gamma, A, B$ , and it is correct on  $\Gamma, A \vee B$  as well. 958
- $\wedge \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$  : By induction hypothesis, there is a  
 959 correct linking on  $\Gamma, A$  and another one on  $B, \Delta$ , their  
 960 union giving a correct linking on  $\Gamma, A \wedge B, \Delta$ . 961
- $\exists \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A}$  : By induction hypothesis, there is a correct  
 962 linking  $\lambda$  on  $\Gamma, A[x/t]$ . For each atom in  $\Gamma, A[x/t]$ , there  
 963 is a corresponding atom in  $\Gamma, \exists x.A$ . There is therefore a  
 964 linking  $\lambda'$  on  $\Gamma, \exists x.A$  obtained from  $\lambda$  via this correspon-  
 965 dence, and it is not difficult to check that  $\lambda'$  is correct as  
 966 well. 967
- $\forall \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A}$  ( $x$  not free in  $\Gamma$ ) : By induction hypothesis,  
 968 there is a correct linking on  $\Gamma, A$ , and it is easy to see  
 969 that it is a correct linking on  $\Gamma, \forall x.A$  as well. 970

This allows to define a translation  $[\cdot]$  from proofs in  $\text{MLL1}^\times$   
 971 to unification nets.  $\square$  972

**Theorem 44.** Any unification net can be obtained via the  
 973 translation  $[\cdot]$  given in Theorem 43. 974

To prove this theorem, we need some basic lemmas about  
 975 connected components in switching graphs of unification nets. 976

**Lemma 45.** The number of connected components of an acyclic  
 977 graph  $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$  is equal to  $|E_{\mathcal{G}}| - |V_{\mathcal{G}}|$ . 978

*Proof.* By a straightforward induction on  $|V_{\mathcal{G}}|$ .  $\square$  979

**Lemma 46.** The number of connected components is the same  
 980 for any switching graph of a unification net. 981

*Proof.* An immediate consequence of Lemma 45.  $\square$  982

In the proof, we also use the notion of **frame** introduced by  
 983 Hughes in [38]. 984

**Definition 47.** Let  $\lambda$  be a unification net on an  $\text{MLL1}^\times$  sequent  
 985  $\Gamma$ . We define the **frame** of  $\lambda$  by exhaustively applying the  
 986 following subformula rewriting steps, to obtain a linking  $\lambda_m$   
 987 on an  $\text{MLL} + \text{mix}$  sequent  $\Gamma_m$ : 988

- 1) **Encode dependencies as fresh links.** For each depen-  
 989 dency  $\exists x \rightarrow \forall y$ , with corresponding subformulas  $\exists x.A$   
 990 and  $\forall y.B$ , we add a fresh link as follows. Let  $P$  be a fresh  
 991 (nullary) predicate symbol. Replace  $\exists x.A$  with  $P \wedge \exists x.A$   
 992 and  $\forall y.B$  with  $\bar{P} \vee \forall y.B$ , and add an axiom link between  
 993  $P$  and  $\bar{P}$ . 994
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers.  
 995 (We no longer need their leaps since they are encoded  
 996 as links in step 1. 997
- 3) **Simplify atoms.** After step 2, replace every predicate  
 998  $Pt_1 \dots t_n$  with a nullary predicate symbol  $P$ . 999

Note that the linking  $\lambda_m$  is a valid  $\text{MLL} + \text{mix}$  proof net. 1000

**Lemma 48.** Suppose that  $\lambda$  is a MLL + mix proof net which is connected and such that any of its switching graphs is not connected. Then there exists a  $\vee$  node in  $\mathcal{U}(\lambda)$  such that  $\lambda$  is correct on the sequent  $\Gamma'$  obtained from  $\Gamma$  by replacing this  $\vee$  by a  $\wedge$ .

*Proof.* Suppose that such a  $\vee$  node does not exist. Then it is clear that for any two nodes, there exists a switching graph containing a path between them and this path corresponds to an  $AE$ -path in [40]. By [40, Propostion 3],  $\lambda$  corresponds to a sequent proof that does not use mix, which implies the connectedness of the switching graphs of  $\lambda$ . Contradiction. **TO CHECK:**  $\square$

**Lemma 49.** Suppose that  $\lambda$  is a MLL1<sup>X</sup> proof net which is connected and such that any of its switching graphs is not connected. Then there exists a  $\vee$  node in  $\mathcal{U}(\lambda)$  such that  $\lambda$  is correct on the sequent  $\Gamma'$  obtained from  $\Gamma$  by replacing this  $\vee$  by a  $\wedge$ .

*Proof.* Consider the frame  $\lambda_m$  of  $\lambda$ . The number of any switching graph of  $\mathcal{U}(\lambda)$  is equal to that of  $\mathcal{U}(\lambda_m)$ . Apply Lemma 48 and it is clear that such  $\vee$  cannot be one of the fresh  $\vee$ 's added during the frame construction.  $\square$

We can now give the proof of Theorem 44.

*Proof of Theorem 44.* Let  $\lambda$  be a unification net on  $\Gamma$ . We proceed by induction on the number of connected components of the unification structure  $\mathcal{U}(\lambda)$ :

- If there is only one connected component, we proceed by induction on the number  $k$  of connected components of any switching graph of  $\mathcal{U}(\lambda)$ . If  $k = 1$ , we obtain a proof  $\Phi$  in MLL1<sup>X</sup> such that  $[\Phi] = \lambda$  by applying [38, Theorem 3]. If  $k > 1$ , using the Lemma 49, we obtain a sequent  $\Gamma'$  on which  $\lambda$  is correct by transforming a  $\vee$  node into a  $\wedge$ . By induction hypothesis, there is a proof  $\Phi'$  in MLL1<sup>X</sup> whose translation is  $\lambda$ . By considering the  $\wedge$  rule instance corresponding to the  $\wedge$  node in  $\Phi'$ , we

$$\text{have: } \Phi' = \wedge \frac{\frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A \wedge B, \Delta_2}}{\vdash \Gamma'}. \text{ We can thus obtain}$$

$$\text{a proof } \Phi \text{ of } \Gamma: \Phi = \frac{\text{mix} \frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A, B, \Delta_2}}{\vdash \Delta_1, A \vee B, \Delta_2} \text{ such that}$$

$$\frac{\vdots}{\vdash \Gamma}$$

$$[\Phi] = \lambda.$$

- If there are  $n > 1$  connected components, add a fresh  $\vee$  node connecting two formulas belonging to different

connected components of  $\Gamma$  to get a new sequent  $\Gamma'$ . Define a unification net  $\lambda'$  on  $\Gamma'$  using the same linking as  $\lambda$ . By induction hypothesis, since  $\mathcal{U}(\lambda')$  has  $n - 1$  connected components, there is a MLL1<sup>X</sup> proof  $\Phi'$  such that  $[\Phi'] = \lambda'$ . Consider the  $\vee$  rule instance corresponding to the  $\vee$  node in question. Since  $\vee$  is invertible, we can permute downwards this rule instance until it becomes the last rule of the proof (note that this transformation does not change the image of the proof by the translation  $[\cdot]$ ) to get a new proof  $\Phi''$  of  $\Gamma'$ . By deleting the last rule instance from  $\Phi''$ , we obtain a proof  $\Phi$  of  $\Gamma$  such that  $[\Phi] = \lambda$ . **TO CHECK:**  $\square$

We proceed by induction on the number of connectives in  $\Gamma$ . In the base case,  $\Gamma$  is of the form

$$p_1(t_{11}, \dots, t_{1n_1}), \overline{p_1}(t_{11}, \dots, t_{1n_1}), \dots, p_k(t_{k1}, \dots, t_{kn_k}), \overline{p_k}(t_{k1}, \dots, t_{kn_k}), \underbrace{t, \dots, t}_{m \text{ times}}$$

and  $\lambda$  is the linking  $\{(a_1, \overline{a_1}), \dots, (a_k, \overline{a_k})\}$ , where  $a_i = p_i(t_{i1}, \dots, t_{in_i})$ , which equals to  $[\Pi]$ , where  $\Pi$  is the proof consisting of  $m$  instances of the  $t$  rule,  $n$  instances  $\text{ax} \frac{}{\vdash a_i, \overline{a_i}}$  of the  $\text{ax}$  rule, and followed by  $m + k - 1$  instances of the mix rule.

Now we consider the inductive cases:

- $\Gamma = \Delta, A \vee B$ : Let  $\Gamma' = \Delta, A, B$ . Define  $\lambda'$  on  $\Gamma'$  using the same links as  $\lambda$  by identifying the leaves of  $\mathcal{F}(\Gamma')$  with those of  $\mathcal{F}(\Gamma)$ . We now check that  $\lambda'$  is a unification net:
  - The most general dualizer of  $\lambda$  is also the most general dualizer of  $\lambda'$  as they correspond to the same unification problem. Hence, the unification structure  $\mathcal{U}(\lambda')$  is equal to the restriction of  $\mathcal{U}(\lambda)$  to the nodes of  $\mathcal{F}(\Gamma')$ .
  - Every switching graph of  $\lambda'$  is acyclic: if there were some switching graph of  $\mathcal{U}(\lambda')$  containing a cycle, it would induce a switching graph of  $\mathcal{U}(\lambda)$  containing also a cycle, by adding an edge from the root of  $\mathcal{F}(A)$  to the  $\vee$  node in question.
- $\Gamma = \Delta, \forall x.A$ : Let  $\Gamma' = \Delta, A$ . Define  $\lambda'$  on  $\Gamma'$  using the same links as  $\lambda$ . We now check that  $\lambda'$  is a unification net:
  - The most general dualizer of  $\lambda$  is also the most general dualizer of  $\lambda'$  as they correspond to the same unification problem.
  - Every switching graph of  $\mathcal{U}(\lambda')$  is acyclic: if there were some switching graph of  $\mathcal{U}(\lambda')$  containing a cycle, it would induce a switching graph of  $\mathcal{U}(\lambda)$  containing also a cycle, by adding an edge from the root of  $\mathcal{F}(A)$  to the  $\forall$  node in question.
- $\mathcal{F}(\Gamma)$  has a root  $\exists x$  with no outgoing dependency edge:

$\square$

### C. Translation between Unification Nets and Fonets

## 1087 XII. FIRST-ORDER COMBINATORIAL PROOFS

### 1088 A. First-order Logic

1089 In this paper, we also use some *deep inference* [44] rules  
1090 that are in the following form:

$$\frac{\vdash S\{A\}}{\vdash S\{B\}}$$

1092 where  $S\{ \}$  stands for a **context**, which corresponds to a  
1093 sequent with a hole taking the place of an atom, and  $S\{A\}$   
1094 represents the sequent or formula obtained by replacing the  
1095 hole in  $S\{ \}$  with the formula  $A$ . Formally,

$$1096 C ::= \Box \mid A \vee C \mid C \wedge A \mid \exists x C \mid \forall x C.$$

$$1097 S ::= C \mid A, S \mid S, A$$

1098 where  $A$  is a formula. The above rule can be thus seen as the  
1099 rewriting rule  $A \rightarrow B$ .

1100 We use the notation  $\parallel_{\mathcal{P}}^A$  for denoting that there is a  
1101 derivation from premise  $\vdash S\{A\}$  to conclusion  $\vdash S\{B\}$  in  
1102 system  $\mathcal{P}$  for any context  $S$ .

### 1103 B. Graphs

### 1104 C. First-order combinatorial proofs

### 1105 D. MLL1<sup>X</sup> and Unification Nets

1106 In MLL1<sup>X</sup>, terms, atoms, formulas are defined as in first-  
1107 order logic. For simplicity, we choose to use  $\vee$  and  $\wedge$  instead of  
1108  $\wp$  and  $\otimes$  which are generally used in the presentation of linear  
1109 logic. A formula  $A$  is identified with its **formula tree**  $\mathcal{F}(A)$ ,  
1110 a directed tree with leaves labelled by atoms, internal nodes  
1111 labelled by connectives and quantifiers, and edges directed  
1112 from leaves to the root. A **sequent**  $\Gamma$  is simply a disjoint union  
1113 of formulas. We write comma for disjoint union.

Sequents are proved using the inference rules of MLL1<sup>X</sup>:

$$\begin{array}{c} \frac{}{\vdash A, \neg A} \text{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta} \text{cut} \\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \wedge \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A} \forall (x \notin fv(\Gamma)) \quad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A} \exists \end{array}$$

Fig. 3. Sequent calculus for MLL1<sup>X</sup>

1114 We also consider the mix rule:

$$1115 \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \text{mix}$$

1117 Let  $\Gamma$  be a sequent in MLL1 + mix. A **link** on  $\Gamma$  is a pair  
1118 of leaves whose atoms are pre-dual. A **linking** on  $\Gamma$  is a  
1119 set of disjoint links such that each leaf of  $\Gamma$  is in exactly

one link. Similar to the set of links in the linked fograph, a  
linking can be seen as a unification problem, and a link is said  
**unifiable** if the corresponding unification problem is solvable.  
**Dependencies** are defined as previously.

## 1120 XIII. FROM FIRST-ORDER LOGIC TO COMBINATORIAL 1121 PROOFS

### 1122 A. Decomposition Theorem

Consider the following deep inference rules [44]:

$$\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}} \text{c} \quad \frac{\vdash S\{f\}}{\vdash S\{A\}} \text{w}$$

Note that the ctr (resp. wk) rule in LK is derivable in  $\{c, \vee\}$   
(resp.  $\{w, f\}$ ) and that c and w rules permute downwards with  
the non-structural rules of LK.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \frac{}{\vdash \Gamma, A} \text{c}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \frac{}{\vdash \Gamma, A} \text{w}$$

We also give an example to show how rule permutation  
works:

$$\frac{\frac{\Gamma, A \vee A}{\Gamma, A} \text{c} \quad \Delta, B}{\Gamma, \Delta, A \wedge B} \wedge \rightsquigarrow \frac{\Gamma, A \vee A \quad \Delta, B}{\Gamma, \Delta, (A \vee A) \wedge B} \wedge \frac{}{\Gamma, \Delta, A \wedge B} \text{c}$$

We want to establish the following theorem:

**Theorem 50.** *Let  $\Gamma$  be a sequent. Then there is a proof of  $\Gamma$  in LK + mix iff there is a proof of some sequent  $\Delta$  in MLL1 + mix and a derivation from  $\Delta$  to  $\Gamma$  consisting of the c and w rules only.*

*Proof.* ( $\Rightarrow$ ) This direction comes from the above observation:  
it suffices to permute downwards all the instances of the c and  
w rules.

( $\Leftarrow$ ) We regard the proof in MLL1 + mix as a proof in  
LK + mix. Then we put the derivation consisting of only c  
and w under the proof in LK + mix. Now we try to permute  
all the instances c and w upwards with the rules of LK and  
mix. For the c part, the only non-trivial case is the permutation  
with the  $\vee$  rule where the formula generated is  $A \vee A$ .

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr}$$

In this case, the permutation of this instance of c stops and  
we continue with the remaining instances.

For the w part, the only non-trivial case is the permutation  
with the f rule (or the instance of wk where f is introduced):

$$\frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk}$$

1157 In this case, the permutation of this instance of  $w$  stops and  
1158 we continue with the remaining instances.

1159  $\square$

1160 D. Hughes proves in [38] the soundness and completeness of  
1161 unification nets with respect to MLL1 + mix. In the following,  
1162 we establish the equivalence between unification nets and  
1163 fonets.

#### 1164 B. Equivalence between unification nets and fonets

1165 In the following, we usually confound a vertex with its label.

1166 **Definition 51.** A *switching path* of a unification structure  
1167  $U(\lambda)$  is a path in a switching graph of  $U(\lambda)$ .

1168 **Definition 52.** A *switching path* of a formula tree  $\mathcal{F}(A)$  is a  
1169 path in  $\mathcal{F}(A)$  that does not go through both incoming edges  
1170 of a  $\vee$ .

1171 **Proposition 53.** In a formula tree, the root is connected to  
1172 every vertex by a switching path.

1173 Now we give the key proposition relating a fograph to its  
1174 corresponding formula tree.

1175 **Proposition 54.** Let  $u$  and  $v$  be two distinct vertices of a  
1176 fograph  $\llbracket \llbracket A \rrbracket \rrbracket$ , then we have the equivalence between:

- 1177 •  $u$  and  $v$  are adjacent in  $\llbracket \llbracket A \rrbracket \rrbracket$
- 1178 •  $u$  and  $v$  are connected by a switching path of  $\mathcal{F}(A)$ , and  
1179 if one of them is a universal quantifier, then the other is  
1180 not a descendant of the former.

1181 *Proof.* By induction on  $A$ .

- 1182 • If  $A$  is an atom, trivial.
- 1183 • If  $A = A_1 \wedge A_2$ , then we distinguish two cases:
  - 1184 –  $u$  and  $v$  are both in  $A_1$  (resp.  $A_2$ ): trivial by the  
1185 induction hypothesis.
  - 1186 – one of them is in  $A_1$  and the other is in  $A_2$ : they are  
1187 adjacent in  $\llbracket \llbracket A \rrbracket \rrbracket$  by definition. By Proposition 53,  
1188 the one in  $A_1$  (resp.  $A_2$ ) is connected to the vertex  
1189 representing  $A_1$  (resp.  $A_2$ ) by a switching path.  
1190 Together with the two edges incident to  $A_1 \wedge A_2$ ,  
1191 we obtain a switching path connecting  $u$  and  $v$ .
- 1192 • If  $A = A_1 \vee A_2$ , then we distinguish two cases:
  - 1193 –  $u$  and  $v$  are both in  $A_1$  (resp.  $A_2$ ): trivial by the  
1194 induction hypothesis.
  - 1195 – one of them is in  $A_1$  and the other is in  $A_2$ : they are  
1196 not adjacent in  $\llbracket \llbracket A \rrbracket \rrbracket$  by definition. It is clear that  
1197 they are not connected by a switching path.
- 1198 • If  $A = \exists x A'$ , then we distinguish two cases:
  - 1199 –  $u$  and  $v$  are both in  $A'$ : trivial by the induction  
1200 hypothesis.
  - 1201 – one of them is  $\exists x$  and the other is in  $A'$ : trivial by  
1202 Proposition 53
- 1203 • If  $A = \forall x A'$ , then we distinguish two cases:
  - 1204 –  $u$  and  $v$  are both in  $A'$ : trivial by the induction  
1205 hypothesis.

- one of them is  $\forall x$  and the other is in  $A'$ : they are  
not adjacent in  $\llbracket \llbracket A \rrbracket \rrbracket$  by definition and it is clear that  
the former is a descendant of  $\forall x$ .

1209  $\square$

**Proposition 55.** If there exists an induced bimatching of the  
linked fograph  $G = \llbracket \llbracket A \rrbracket \rrbracket$ , then there exists a switching graph  
of the corresponding unification net which contains a cycle.

*Proof.* Suppose that there exists a set  $W$  inducing a bimat-  
ching in  $G$ . Then  $(W, E_G)$  and  $(W, L_G)$  are matchings.

Let  $E_W$  (resp.  $L_W$ ) be the restriction of  $E_G$  (resp.  $L_G$ ) to  $W$ .  
If  $E_W \cap L_W \neq \emptyset$ , then there exist  $u$  and  $v$  such that  $uv \in E_G$   
and  $uv \in L_G$ . By Proposition 54, there exists a switching  
path of the formula tree of  $A$ . Together with the leap  $uv$ , this  
path induces a cycle in a switching graph of the corresponding  
unification structure.

We can now suppose that  $E_W$  and  $L_W$  are disjoint. It is not  
difficult to see the existence of an alternating and elementary  
cycle in the bicoloured graph  $(W, E_W \uplus L_W)$ , i.e. a cycle of  
which the edges are alternately in  $E_W$  and  $L_W$  and containing  
no two equal vertices. By Proposition 54, this cycle induces a  
cycle in the unification structure. Now we want to construct a  
switching graph that contains this cycle.

Consider a universal quantifier  $\forall x$ . If  $\forall x \notin W$ , then we keep  
the incoming edge from its direct subformula and remove all  
the dependencies. Otherwise, since  $(W, L_G)$  is a matching,  
there exists a unique existential quantifier adjacent to  $\forall x$   
and we keep thus the corresponding edge in the unification  
structure.

Now consider a  $\vee$ . We distinguish three cases:

- the cycle goes through none of the two branches (incom-  
ing edges) of the  $\vee$ : we can choose an arbitrary switching  
for this  $\vee$
- the cycle goes through exactly one branch: we choose the  
corresponding switching
- the cycle goes through both branches: this means that  
there exist  $v_L \in W$  (resp.  $v_R$ ) in the left (resp. right)  
branch,  $u_L, u_R \in W$ , such that  $u_L v_L, u_R v_R \in E_W$   
and that the corresponding switching path from  $u_L$  to  
 $v_L$  (resp. from  $u_R$  to  $v_R$ ) goes through the left (resp.  
right) edge of  $\vee$ .

The red (resp. blue) path is the switching path corre-  
sponding to the edge  $u_L v_L$  (resp.  $u_R v_R$ ) in  $E_W$ .

It is clear that  $u_L$  (resp.  $u_R$ ) is not in the branches of the  
 $\vee$ . Otherwise, there will be no switching path from  $u_L$   
to  $v_L$

By Proposition 54, we know that  $u_L$  and  $u_R$  are not  
universal quantifiers which are ancestors the  $\vee$  and that  
there exist one switching path from  $u_L$  to  $v_L$  and one  
from  $u_R$  to  $v_R$ . In particular, there exist one switching  
path from  $u_L$  to the  $\vee$  and one from the  $\vee$  to  $v_R$ , and  
by concatenating the two, we obtain a switching path  
from  $u_L$  to  $v_R$ . By Proposition 54,  $u_L$  and  $v_R$  are thus  
adjacent in  $(W, E_G)$ , which is impossible since  $(W, E_W)$   
is a matching.

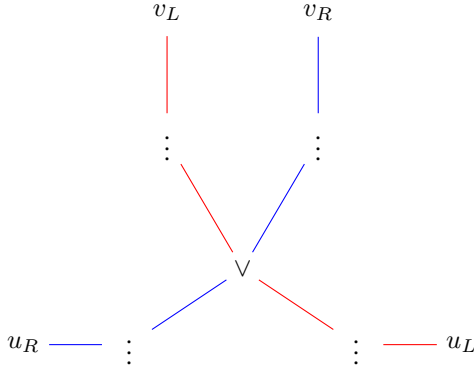


Fig. 4. A schema showing that the two branches of the same  $\vee$  cannot be used in the cycle at the same time.

Notice that the switching paths here are in the underlying formula tree. We have to verify that they are compatible with the choices of switching made for universal quantifiers. That is, if  $uv \in E_W$ , then for all the universal quantifiers  $\forall x$  on the switching path (in the formula tree), we have chosen in the first part of the proof to keep the only edge from the child of  $\forall x$  to itself. In fact, if there exists a universal quantifier  $w \in W$  on the switching path  $u \rightarrow v$ , then one of  $u$  and  $v$  is not a descendant of  $w$ . Moreover, if  $u$  (resp.  $v$ ) is a universal quantifier, then  $w$  is not in its scope. By Proposition 54,  $\{wu, wv\} \cap E_W \neq \emptyset$ , which is impossible since  $(W, E_W)$  is a matching. We have thus constructed a switching graph containing this cycle.  $\square$

**Proposition 56.** *If one of the switching graphs of the unification structure of  $A$  contains a cycle or is not connected, then there exists an induced bimatching of the corresponding linked fograph.*

*Proof.* We use frames introduced by D. Hughes in Section 4 of [38].

**Definition 57.** Let  $\theta$  be a unification structure on an MLL<sup>1</sup> $\times$  sequent  $\Gamma$ . We define the **frame** of  $\theta$  by exhaustively applying the following subformula rewriting steps, to obtain a proof structure  $\theta_m$  on an MLL sequent  $\Gamma_m$ :

- 1) **Encode dependencies as fresh links.** For each dependency  $\exists x \rightarrow \forall y$ , with corresponding subformulas  $\exists xA$  and  $\forall yB$ , we add a fresh link as follows. Let  $P$  be a fresh (nullary) predicate symbol. Replace  $\exists xA$  with  $P \wedge \exists xA$  and  $\forall yB$  with  $\overline{P} \vee \forall yB$ , and add an axiom link between  $P$  and  $\overline{P}$ .
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers. (We no longer need their leaps since they are encoded as links in step 1.)
- 3) **Simplify atoms.** After step 2, replace every predicate  $Pt_1 \cdots t_n$  with a nullary predicate symbol  $P$ .

We have the following results:

Let  $u$  and  $v$  be atoms or quantifiers in a unification structure  $\theta$ . Then they are connected by a switching path in the

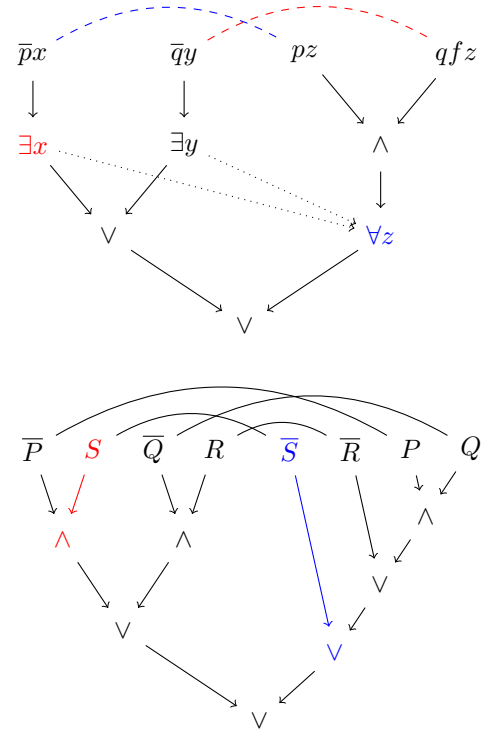


Fig. 5. A unification net and its frame. The colored part shows how the dependency  $\exists x \rightarrow \forall z$  is transformed.

unification structure if, and only if, their corresponding nodes are connected by a switching path in  $\theta_m$ .

Consider now a switching graph  $H$  of a unification structure  $\theta$  of  $A$ .

If  $H$  contains a cycle, then the corresponding switching graph of  $\theta_m$  also contains a cycle. Hence, by applying the propositional results (Theorem 7) from [40], we conclude that there exists a chordless, alternating, and elementary cycle in the bicoloured graph  $(W, E_W \uplus L_W)$ , which corresponds to an induced bimatching in the linked fograph. (Note that the linked fograph (cograph) corresponding to  $\theta_m$  is equivalent to the one corresponding to  $\theta$ .)  $\square$

### C. From contraction/weakening to skew bifibrations

We first introduce the atomic contraction rule, the medial rule, and two rules on quantifiers.

$$\begin{array}{c}
 \frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac} \quad \frac{\vdash S\{(A \wedge B) \vee (C \wedge D)\}}{\vdash S\{(A \vee C) \wedge (B \vee D)\}} \text{ m} \\
 \frac{\vdash S\{\exists xA \vee \exists xB\}}{\vdash S\{\exists x(A \vee B)\}} \text{ m}_1\downarrow \quad \frac{\vdash S\{\forall xA \vee \forall xB\}}{\vdash S\{\forall x(A \vee B)\}} \text{ m}_2\downarrow
 \end{array}$$

Here, we also consider the equivalence generated by the associativity, commutativity of  $\vee$  and the equations  $t \vee A \equiv t$  and  $f \vee A \equiv A$ .

Now we have the following lemma:





- 1375 •  $\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac:}$   
 1376 the map  $ac$  maps the two  $a$ -labelled literals in the premise  
 1377 to the  $a$ -labelled literal in the conclusion.  
 $\vdash S\{(A \wedge B) \vee (C \wedge D)\}$
- 1378 •  $\frac{\vdash S\{(A \vee C) \wedge (B \vee D)\}}{\vdash S\{\exists x A \vee \exists x B\}} m:$   
 1379 the map  $m$  is the canonical identity that maps  $A$  to  $A$ ,  
 1380  $\dots, D$  to  $D$ .  
 $\vdash S\{\exists x A \vee \exists x B\}$
- 1381 •  $\frac{\vdash S\{\exists x(A \vee B)\}}{\vdash S\{\forall x A \vee \forall x B\}} m_1 \downarrow:$   
 1382 the map  $m_1$  maps the two  $x$ -labelled binders in the  
 1383 premise to the  $x$ -labelled binder in the conclusion,  $A$  to  
 1384  $A$  and  $B$  to  $B$ .  
 $\vdash S\{\forall x A \vee \forall x B\}$
- 1385 •  $\frac{\vdash S\{\forall x(A \vee B)\}}{\vdash S\{\forall x(A \vee B)\}} m_2 \downarrow:$   
 1386 the map  $m_2$  maps the two  $x$ -labelled binders in the  
 1387 premise to the  $x$ -labelled binder in the conclusion,  $A$  to  
 1388  $A$  and  $B$  to  $B$ .

1389 By considering propositional encodings, the maps defined  
 1390 are label-preserving skew fibrations on the underlying fographs  
 1391 according to [23].

1392 Now we prove that each map  $g \in \{wk, ac, m, m_1, m_2\}$  is  
 1393 a skew bifibration. To do that, it suffices to prove that  $g$  is a  
 1394 fibration between the corresponding binding graphs since it is  
 1395 already a skew fibration on the corresponding fographs and it  
 1396 is label-preserving and existential-preserving.

1397 for each  $x$ -binder  $b$  in  $\llbracket \langle \rangle B^\circ \rrbracket$ , for each vertex  
 1398  $v \in V(\llbracket \langle \rangle A^\circ \rrbracket)$  such that  $g(v)$  is bound by  $b$ , there exists a  
 1399 unique binder  $b'$  such that  $b'$  binds  $v$ .

- 1400 •  $wk$  and  $m$  are clearly fibrations: the binding relations of  
 1401 the premise and the conclusion are exactly the same.
- 1402 •  $ac$  is a fibration: suppose that  $a$  that in the conclusion  $a$   
 1403 is bound by some quantifier  $b$  in  $S$ , then for each of its  
 1404 preimages by  $ac$ , there exists exactly one binder (in fact,  
 1405  $b$ ) in  $S$  that binds it.
- 1406 •  $m_1$  and  $m_2$  are fibrations: in the conclusion, for every  
 1407 atom  $a$  in  $A \vee B$  bound by the  $x$ -labelled quantifier,  $a$  has  
 1408 exactly one preimage and it is bound by the  $x$ -labelled  
 1409 quantifier in the premise.

1410 Therefore, all of these maps are skew bifibrations and since  
 1411 skew bifibrations on fographs compose (Lemma 10.32, [20]),  
 1412 there exists a skew bifibration from  $\llbracket \langle \rangle A \rrbracket$  to  $\llbracket \langle \rangle B \rrbracket$ .  
 1413  $\square$

1414 **Theorem 64.** *If a formula  $A$  is provable in LK, then it has a*  
 1415 *combinatorial proof.*

1416 *Proof.* By Theorem 50, there exists a formula  $A'$  such that  
 1417 there is a proof  $\Pi$  of  $A'$  in  $\text{MLL1}^X$  and a derivation  $D$  from  
 1418  $A'$  to  $A$  consisting of the  $w$  and  $c$  rules only. The proof  $\Pi$   
 1419 corresponds to a unique unification net which is equivalent to  
 1420 the fonet corresponding to  $\Pi$ , i.e., the fograph  $\llbracket \langle \rangle A' \rrbracket$  together  
 1421 with the links of  $\Pi$ . By Lemma 63, there exists a skew  
 1422 bifibration  $\llbracket \langle \rangle A' \rrbracket \rightarrow \llbracket \langle \rangle A \rrbracket$ . We have thus a combinatorial  
 1423 proof of  $A$ .  
 1424  $\square$

#### 1425 D. From skew bifibrations to contraction/weakening

**Theorem 65.** *Let  $A$  and  $B$  be two formulas and  $f : G(A) \rightarrow$*   
 1426  *$G(B)$  a skew bifibration. Then there exists a derivation*  
 1427  $A$   
 $\Delta \parallel_{\{w, c\}}.$   
 $B$  1428

$f$  can be seen as a skew fibration from  $G(A^\circ)$  to  $G(B^\circ)$ ,  
 which gives the existence of the propositions  $A'$  and  $B'$ , and  
 of the following derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B' \\ \Delta'' \parallel_w \\ B^\circ \end{array}$$

**Lemma 66.** *there exists  $B''$  such that  $B''^\circ = B'$ .* 1429

*Proof.* Consider the derivation  $\Delta''$ . If some  $U_x$  (or  $E_x$ ) is  
 1430 introduced via weakening, then all the atoms it binds in  $B^\circ$   
 1431 should also be introduced via weakening. In fact, an atom of  
 1432  $B^\circ$  is introduced via weakening is equivalent to the fact that  
 1433 its corresponding vertex is not in the image of  $f$ . Since there  
 1434 is an edge from  $U_x$  (resp.  $E_x$ ) to all the literals it binds in the  
 1435 binding graph  $\llbracket \langle \rangle B \rrbracket$ , if one of the atoms is in the image,  $U_x$   
 1436 (resp.  $E_x$ ) should also be in the image since  $f$  is a fibration  
 1437 on binding graphs.  
 1438

This means that a such  $B''$  can be obtained from  $B$  by  
 1439 erasing all the  $U_x$  and  $E_x$  introduced via weakening and all  
 1440 the atoms they bind.  $\square$  1441

We introduce new (atomic) symbols  $E_x^*$  and  $U_x^*$  which are  
 1442 used to represent disjunctions of  $E_x$  and  $U_x$  respectively. 1443

We define a translation  $(\cdot)^*$  inductively by: 1444

- 1445 •  $(E_x \vee \dots \vee E_x)^* = E_x$  1445
- 1446 •  $(U_x \vee \dots \vee U_x)^* = U_x$  1446
- 1447 • structural recursion in all the other cases. 1447

Then the derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B''^\circ \end{array}$$

can be translated to the derivation:

$$\begin{array}{c} A^{\circ*} \\ \Delta^* \parallel \\ B''^{\circ*} \end{array}$$

where  $\Delta^*$  is the derivation obtained by replacing all the  
 1448 formulas  $F$  with  $F^*$  and by applying the following rule  
 1449 transformation: 1450

$$\frac{S\{Q_x\}}{S\{Q_x\}} \text{ ac} \rightsquigarrow \frac{S\{Q_x\}}{S\{Q_x\}} =$$

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m} \rightsquigarrow \frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m}'$$

where  $Q_x$  stands for  $E_x$  or  $U_x$ .

$\Delta^*$  can now be transformed into a valid derivation  $\Delta_1$  by using the two transformation rules above and by applying them in a bottom-up style:

$$\frac{A^{\circ*}}{\Delta_1 \parallel_{\text{ac}, \text{m}, \text{m}'}} B''^{\circ*}$$

**Lemma 67.** Every line of  $\Delta_1$  is a propositional encoding.

*Proof.* We proceed by bottom-up induction in the derivation. Clearly,  $(B''^{\circ})^*$  is a propositional encoding as there is no disjunction of  $Q_x$  in it.

First consider the ac rule:  $\frac{C \vee C}{C} \text{ ac}$

It is clear that if  $C$  is a propositional encoding, then so is  $C \vee C$ .

Now consider the m rule:

$$\frac{S\{(C \wedge D) \vee (E \wedge F)\}}{S\{(C \vee E) \wedge (D \vee F)\}} \text{ m}$$

Suppose that  $(C \vee E) \wedge (D \vee F) = G^{\circ}$  for some  $G$ . Since  $C \vee E$  cannot be  $Q_x$  (otherwise, the rule applied would be  $\text{m}'$ ),  $G$  can be written as  $G_1 \wedge G_2$  with  $C \vee E = G_1^{\circ}$  and  $D \vee F = G_2^{\circ}$ .

We have thus  $G_i = \forall x_i H_i$  or  $J_i \vee K_i$  ( $i = 1, 2$ ).

If  $G_i = \forall x_i H_i$  for some  $i$ , then there will be a conjunction of  $U_x$  and some formula which can never be eliminated by the rules  $\text{m}$ ,  $\text{m}'$  and  $\text{ac}$ . However, there exists no such conjunction in  $A^{\circ*}$ , which leads to a contradiction.

Hence,  $G_i$  can be written as  $J_i \vee K_i$  for  $i = 1, 2$ . We now have  $(C \wedge D) \vee (E \wedge F) = ((J_1 \wedge J_2) \vee (K_1 \wedge K_2))^{\circ}$ .

Finally, consider the  $\text{m}'$  rule:

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m}'$$

Suppose that  $E_x \wedge (C \vee D) = F^{\circ}$  for some  $F$ . It is clear that  $F = \exists x G$  with  $G^{\circ} = C \vee D$  for some  $G$ . We distinguish two cases:

- $G = \forall y H$ : in this case,  $(E_x \wedge C) \vee (E_x \wedge D)$  has a subformula  $(E_x \wedge U_y)$ , which cannot be eliminated by the rules  $\text{m}$ ,  $\text{m}'$ ,  $\text{ac}$ . It is clear that  $A^{\circ*}$  does not have a subformula of this form, which leads to a contradiction.
- $G = G_1 \vee G_2$ : in this case,  $(E_x \wedge C) \vee (E_x \wedge D) = ((\exists x G_1) \vee (\exists x G_2))^{\circ}$ .

□