

Combinatorial Proofs and Decomposition Theorems for First-order Logic

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Abstract—We uncover a close relationship between combinatorial and syntactic proofs for first-order logic. Whereas syntactic proofs are formalized in some deductive proof system based on inference rules, a combinatorial proof is a “syntax-free” presentation of a proof that is independent from any set of inference rules. We show that the two proof representations are related via a deep inference decomposition theorem that establishes a new kind of normal form of syntactic proofs. This yields (a) a simple proof of the soundness and completeness of first-order combinatorial proofs, and (b) a full completeness theorem: every combinatorial proof is the image of a syntactic proof.

[[[TODO: Examples, examples, examples]]]

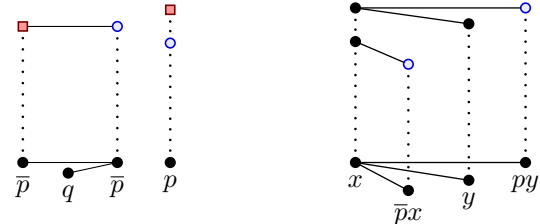
I. INTRODUCTION

First-order predicate logic is a cornerstone of modern logic. Since its formalisation by Frege [1] it has seen a growing usage in many fields of mathematics and computer science. Upon the development of proof theory by Hilbert [2], *proofs* became first-class citizens as mathematical objects that could be studied on their own. Since Gentzen’s *sequent calculus* [3], [4], many other proof systems have been developed that allow the implementation of efficient proof search, for example *analytic tableaux* [5] or *resolution* [6]. Despite the immense progress made in proof theory in general and in the area of automated and interactive theorem provers in particular, we still have no satisfactory notion of proof identity for first-order logic. In this respect, proof theory is quite different from any other mathematical field. For example in group theory, two groups are *the same* iff they are isomorphic; in topology, two spaces are *the same* iff they are homeomorphic; etc. In proof theory, we have no such notion telling us when two proofs are *the same*, even though Hilbert was considering this problem as a possible 24th problem for his famous lecture [9] in 1900 [10], before proof theory existed as a mathematical field.

The main reason for this problem is that formal proofs, as they are usually studied in logic, are inextricably tied to the syntactic (inference rule based) proof system in which they are carried out. And it is difficult to compare two proofs that are produced within two different syntactic proof systems, based on different sets of inference rules. [[Lutz: an example here?]]

This is where *combinatorial proofs* come in. They have been introduced by Hughes [11] for classical propositional logic as a “syntax-free” notion of proof, and as a potential solution to

Hilbert’s 24th problem [12] (see also [13]). The basic idea is to abstract away from the syntax of the inference rules used in the proof and consider the proof as a combinatorial object, more precisely as a special kind of graph homomorphism. For example, a propositional combinatorial proof of Peirce’s law $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ is depicted below-left, a homomorphism from a coloured graph to a graph labelled with propositional variables.



Several authors have illustrated how syntactic proofs in various proof systems can be translated to propositional combinatorial proofs: for sequent proofs in [12], for deep inference proofs in [14], for Frege systems in [15], and for tableaux systems and resolution in [16]. This enables a natural definition of proof identity for propositional logic: two proofs are *the same*, if they are mapped to the same combinatorial proof.

Recently, Acclavio and Straßburger extended this notion to relevant logics [17] and to modal logics [18], and Heijltjes, Hughes and Straßburger have provided combinatorial proofs for intuitionistic propositional logic [19].

In this paper we advance the idea that combinatorial proofs can provide a notion of proof identity for first-order logic. *First-order combinatorial proofs* were introduced by Hughes in [20]. For example, a first-order combinatorial proof of Smullyan’s “drinker paradox” $\exists x(px \Rightarrow \forall y py)$ is shown above-right, a homomorphism from a partially coloured graph to a labelled graph. However, even though Hughes proves soundness and completeness, the proof is highly unsatisfactory: (1) the soundness argument is extremely long, intricate and cumbersome, and (2) the completeness proof does not allow a syntactic proof to be read back from a combinatorial proof, i.e., completeness is not *full*. [[TODO: cite something for full comp?]] A fundamental problem is that not all combinatorial proofs can be obtained as translations of sequent calculus proofs.

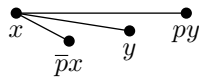
In this paper we solve this issue by moving to a deep inference system. More precisely, we introduce a new proof system, called KS1, for first-order logic, that (a) reflects every combinatorial proof, i.e., there is a surjective mapping from proofs in KS1 to combinatorial proofs, and (b) yields far simpler proofs of soundness and completeness of combinatorial proofs, and (c) admits new decomposition theorems establishing a precise correspondence between certain syntactic inference rules and certain combinatorial notions.

In general, a *decomposition theorem* provides normal forms of proofs, separating subsets of inference rules of a proof system. A prominent example of a decomposition theorem is Herbrand's theorem [21], which allows a separation between the propositional part and the quantifier part in a first-order proof [4], [22]. Through the advent of deep inference, new kinds of proof decompositions became possible, most notably the separation between the linear part of a proof and the resource management of a proof. It has been shown by Straßburger [23] that a proof in classical propositional logic can be decomposed into a proof of (multiplicative) linear logic, followed by a proof consisting only of contractions and weakenings. In this paper we show that the same is possible for first-order logic.

Combinatorial proofs and deep inference can be seen as opposite ends of a spectrum: whereas deep inference allows for a very fine granularity of inference rules—one inference rule in a standard formalism, like sequent calculus or semantic tableaux, is usually simulated by a sequence of different deep inference rules—have combinatorial proofs completely abolished the concept of inference rule. Nonetheless, there is a close relationship between the two, realized through a decomposition theorem, as we establish it in this paper.

A. Pictures that could be used elsewhere in the paper

The fograph of drinker formula $\exists x(px \Rightarrow \forall y py) = \exists x(\bar{p}x \vee \forall y py)$:



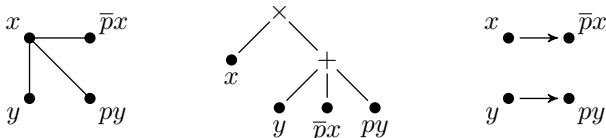
Condensed combinatorial proof of drinker formula(s):



Fig. 1 is a floating figure with four combinatorial proofs.

Fig. 2 is a floating figure with the condensed forms of the four combinatorial proofs in Fig. 1.

Both $\exists x(\bar{p}x \vee \forall y py)$ and $\exists x \forall y (py \vee \bar{p}x)$ have the same rectified fograph D , shown below-left.



Lifting diagrams:

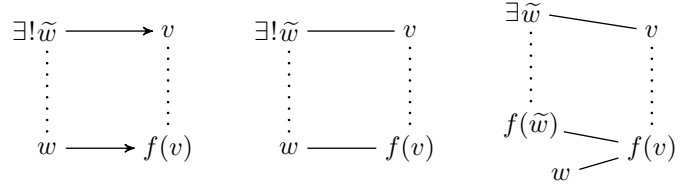
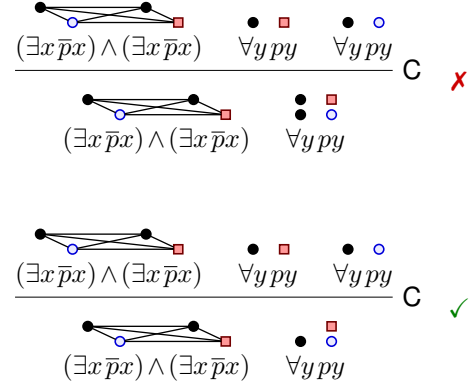


Fig. 3 shows the drinker bifibration, binding fibration, and skeleton.

Fig. 4 shows a fonet with a unique dualizer, and its leap graph.

Illustrating why we need to collapse twins during contraction, to preserve the target as a fograph:



Aligning both $\bullet \blacksquare$ and $\bullet \circ$ over a single copy of $\forall y py$ results in two uncoloured vertices \bullet over $\forall y$. The cover therefore fails to be a fograph: both uncoloured vertices are implicitly labelled with y and so are outer y -binders in the scope of each other. The correct operation is shown above-right, in which the troublesome pair is collapsed to a single uncoloured vertex \bullet over $\forall y$.

II. PRELIMINARIES: FIRST-ORDER LOGIC

A. Terms and Formulas

Fix pairwise disjoint countable sets $\text{VAR} = \{x, y, z, \dots\}$ of variables, $\text{FUN} = \{f, g, \dots\}$ of function symbols, and $\text{PRED} = \{p, q, \dots\}$ of predicate symbols. Each function symbol and each predicate symbol has a finite arity. The grammars below generate the set **TERM** of *terms*, denoted by s, t, u, \dots , the set **ATOM** of *atoms*, denoted by a, b, c, \dots , and the set **FORM** of *formulas*, denoted by A, B, C, \dots :

$$\begin{aligned} t &::= x \mid f(t_1, \dots, t_n) \\ a &::= \mathbf{t} \mid \mathbf{f} \mid p(t_1, \dots, t_n) \mid \bar{p}(t_1, \dots, t_n) \\ A &::= a \mid A \wedge A \mid A \vee A \mid \exists x.A \mid \forall x.A \end{aligned}$$

where the arity of f and p is n . We consider the truth constants \mathbf{t} (*true*) and \mathbf{f} (*false*) as additional atoms, and we consider all formulas in negation normal form, where *negation* ($\bar{\cdot}$) is defined on atoms and formulas via De Morgan's laws:

$$\begin{aligned} \overline{\bar{a}} &= a & \bar{\mathbf{t}} &= \mathbf{f} & \overline{p(t_1, \dots, t_n)} &= \bar{p}(t_1, \dots, t_n) \\ \bar{\mathbf{f}} &= \mathbf{t} & \overline{\bar{p}(t_1, \dots, t_n)} &= p(t_1, \dots, t_n) \\ \overline{\exists x.A} &= \forall x.\bar{A} & \overline{A \wedge B} &= \bar{A} \vee \bar{B} \\ \overline{\forall x.A} &= \exists x.\bar{A} & \overline{A \vee B} &= \bar{A} \wedge \bar{B} \end{aligned}$$

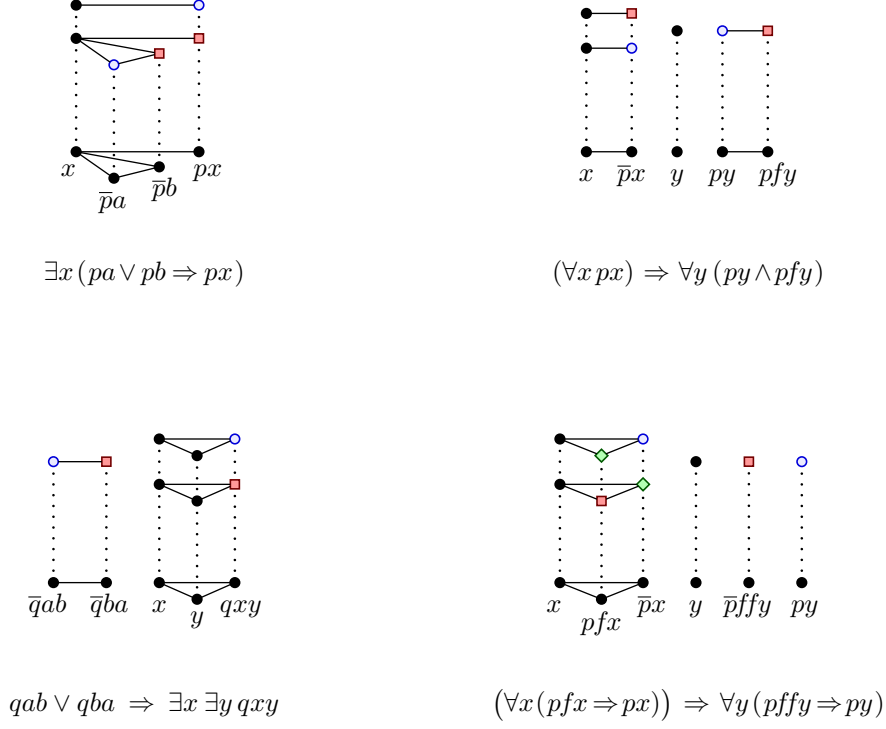


Fig. 1. Four combinatorial proofs, each shown above the formula proved. Here x and y are variables, f is a unary function symbol, a and b are constants (nullary function symbols), p is a unary predicate symbol, and q is a binary predicate symbol.

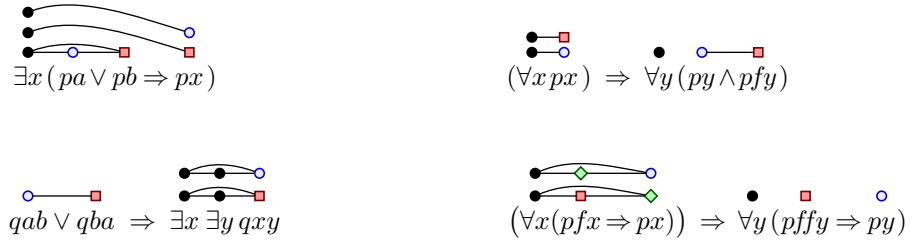


Fig. 2. Condensed forms of the four combinatorial proofs in Fig. 1.

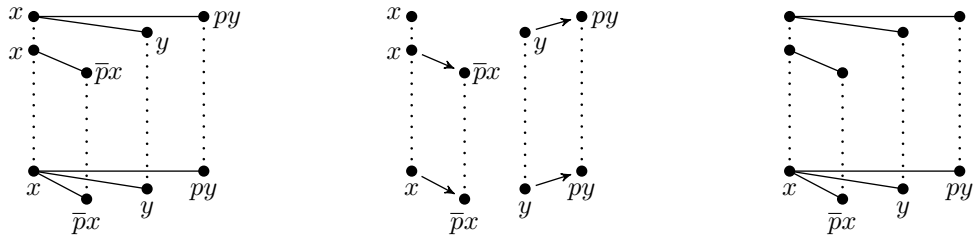


Fig. 3. A skew bifibration (left), its binding fibration (centre), and its skeleton (right).

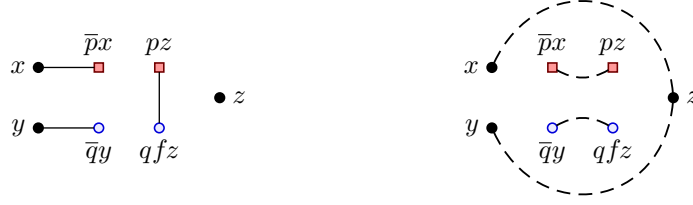


Fig. 4. A fonet (left) with unique dualizer $\{x \mapsto z, y \mapsto fz\}$ and its leap graph (right).

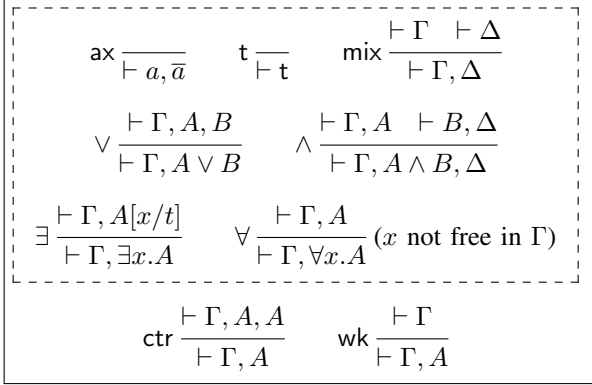


Fig. 5. Sequent calculi LK1 (all rules) and MLL1^X (rules in the dashed box)

A formula is **rectified** if all bound variables are distinct from one another and from all free variables. Every formula can be transformed into a logically equivalent rectified form, by simply renaming the bound variables. If we consider formulas equivalent modulo α -conversion (renaming of bound variables), then the rectified form of a formula A is uniquely defined, and we denote it by \bar{A} .

A **substitution** is a function $\sigma: \text{VAR} \rightarrow \text{TERM}$ that is the identity almost everywhere. We denote substitutions as $\sigma = [x_1/t_1, \dots, x_n/t_n]$, where $\sigma(x_i) = t_i$ for $i = 1..n$ and $\sigma(x) = x$ for all $x \notin \{x_1, \dots, x_n\}$. We write $A\sigma$ for the formula obtained from A by applying σ , i.e., by simultaneously replacing all occurrences of x_i by t_i . A **variable renaming** is a substitution ρ with $\rho(x) \in \text{VAR}$ for all variables x .

B. Sequent Calculus LK1

Sequents, denoted by Γ, Δ, \dots , are finite multisets of formulas, written as lists, separated by comma. The **corresponding formula** $\text{fm}(\Gamma)$ of a sequent $\Gamma = A_1, A_2, \dots, A_n$ is the disjunction of its formulas: $\text{fm}(\Gamma) = A_1 \vee A_2 \vee \dots \vee A_n$. A sequent is **rectified** iff its corresponding formula is.

In this paper we use the sequent calculus LK1, shown in Figure 5, which is a one-sided variant of Gentzen's original calculus [3] for first-order logic. To simplify some technicalities later in this paper, we include the mix-rule.

Theorem 1. LK1 is sound and complete for first-order logic.

For a proof we refer to reader to any standard textbook, e.g. [24].

The linear fragment of LK1, i.e., the fragment without the rules *ctr* (contraction) and *wk* (weakening) defines *first-order multiplicative linear logic* [25], [26] with *mix* [27], [28] (MLL1+mix). We denote that system here with MLL1^X (shown in Figure 5 in the dashed box).

We will employ the cut elimination theorem. The **cut** rule is

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \bar{A}, \Delta}{\vdash \Gamma, \Delta} \quad (1)$$

Theorem 2. If a sequent $\vdash \Gamma$ is provable in LK1+cut then it is also provable in LK1. Furthermore, if $\vdash \Gamma$ is provable in MLL1^X+cut then it is also provable in MLL1^X.

As before, this is standard, see e.g. [24] for a proof.

III. PRELIMINARIES: FIRST-ORDER GRAPHS

A. Graphs

A **graph** $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is a pair where $V_{\mathcal{G}}$ is a finite set of **vertices** and $E_{\mathcal{G}}$ is a finite set of **edges**, which are two-element subsets of $V_{\mathcal{G}}$. We write vw for an edge $\{v, w\}$.

The **complement** of a graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is the graph $\mathcal{G}^c = \langle V_{\mathcal{G}}, E_{\mathcal{G}}^c \rangle$ where $vw \in E_{\mathcal{G}}^c$ iff $vw \notin E_{\mathcal{G}}$.

Let $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ and $\mathcal{H} = \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ be graphs such that $V_{\mathcal{G}} \cap V_{\mathcal{H}} = \emptyset$. A **homomorphism** $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is a function $\varphi: V_{\mathcal{G}} \rightarrow V_{\mathcal{H}}$ such that if $vw \in E_{\mathcal{G}}$ then $\varphi(v)\varphi(w) \in E_{\mathcal{H}}$. The **union** $\mathcal{G} + \mathcal{H}$ is the graph $\langle V_{\mathcal{G}} \cup V_{\mathcal{H}}, E_{\mathcal{G}} \cup E_{\mathcal{H}} \rangle$ and the **join** $\mathcal{G} \times \mathcal{H}$ is the graph $\langle V_{\mathcal{G}} \cup V_{\mathcal{H}}, E_{\mathcal{G}} \cup E_{\mathcal{H}} \cup \{vw \mid v \in V_{\mathcal{G}}, w \in V_{\mathcal{H}}\} \rangle$. A graph \mathcal{G} is **disconnected** if $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ for two non-empty graphs $\mathcal{G}_1, \mathcal{G}_2$, otherwise it is **connected**. It is **coconnected** if its complement is connected.

A graph \mathcal{G} is **labelled** in a set L if each vertex $v \in V_{\mathcal{G}}$ has an element $\ell(v) \in L$ associated with it, its **label**. A graph \mathcal{G} is (partially) **coloured** if it carries a partial equivalence relation $\sim_{\mathcal{G}}$ on $V_{\mathcal{G}}$; each equivalence class is a **colour**. A **vertex renaming** of $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ along a bijection $(\cdot): V_{\mathcal{G}} \rightarrow \hat{V}_{\mathcal{G}}$ is the graph $\hat{\mathcal{G}} = \langle \hat{V}_{\mathcal{G}}, \{\hat{v}\hat{w} \mid vw \in E_{\mathcal{G}}\} \rangle$, with colouring and/or labelling inherited (i.e., $\hat{v} \sim \hat{w}$ if $v \sim w$, and $\ell(\hat{v}) = \ell(v)$). Following standard graph theory, we identify graphs modulo vertex renaming.

A **directed graph** $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is a set $V_{\mathcal{G}}$ of **vertices** and a set $E_{\mathcal{G}} \subseteq V_{\mathcal{G}} \times V_{\mathcal{G}}$ of **direct edges**. A **directed graph homomorphism** $\varphi: \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ is a function $\varphi: V_{\mathcal{G}} \rightarrow V_{\mathcal{H}}$ such that if $(v, w) \in E_{\mathcal{G}}$ then $(\varphi(v), \varphi(w)) \in E_{\mathcal{H}}$.

B. Cographs

A graph $\mathcal{H} = \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ is a **subgraph** of a graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ if $V_{\mathcal{H}} \subseteq V_{\mathcal{G}}$ and $E_{\mathcal{H}} \subseteq E_{\mathcal{G}}$. It is **induced** if $v, w \in V_{\mathcal{H}}$ and $vw \in E_{\mathcal{G}}$ implies $vw \in E_{\mathcal{H}}$. An induced subgraph of $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is uniquely determined by its set of vertices V and we denote it by $\mathcal{G}[V]$. A graph is **\mathcal{H} -free** if it does not contain \mathcal{H} as an induced subgraph. The graph \mathbf{P}_4 is the (undirected) graph $\langle \{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4\} \rangle$. A **cograph** is a \mathbf{P}_4 -free undirected graph. The interest in cographs for our paper comes from the following well-known fact.

Theorem 3 ([29]). *A graph is a cograph iff it can be constructed from the singletons via the operations $+$ and \times .*

In a graph \mathcal{G} , the **neighbourhood** $N(v)$ of a vertex $v \in V_{\mathcal{G}}$ is defined as the set $\{w \mid vw \in E_{\mathcal{G}}\}$. A **module** is a set $M \subseteq V_{\mathcal{G}}$ such that $N(v) \setminus M = N(w) \setminus M$ for all $v, w \in M$. A module M is **strong** if for every module M' , we have $M' \subseteq M$, $M \subseteq M' \cap M' = \emptyset$. A module is **proper** if it has two or more vertices.

Modules in cographs correspond precisely to the subtrees of the cotrees (the term-trees constructing the graph via $+$ and \times).
TODO: Cotree must alternate $+$ and \times . Skip this comment completely?!!!

C. Fographs

A cograph is **logical** if every vertex is labelled by either an atom or variable, and it has at least one atom-labelled vertex. We write $\bullet\alpha$ for an α -labelled vertex. An atom-labelled vertex is called a **literal** and a variable-labelled vertex is called a **binder**. A binder labelled with x is called an **x -binder**. The **scope** of a binder b is the smallest proper strong module containing b . An **x -literal** is a literal whose atom contains the variable x . An x -binder **binds** every x -literal in its scope. In a logical cograph \mathcal{G} , a binder b is **existential** (resp. **universal**) if, for every other vertex v in its scope, we have $bv \in E_{\mathcal{G}}$ (resp. $bv \notin E_{\mathcal{G}}$). An x -binder is **legal** if its scope contains no other x -binder and at least one literal.

Definition 4. A **first-order graph** or **fograph** is a logical cograph with legal binders. The **binding graph** of a fograph \mathcal{G} is the directed graph $\vec{\mathcal{G}} = \langle V_{\mathcal{G}}, \{(b, l) \mid b \text{ binds } l\} \rangle$.

We define a mapping $\llbracket \cdot \rrbracket$ from formulas to (labelled) graphs, inductively as follows:

For a formula A , we can define its associated fograph $\llbracket A \rrbracket$ inductively by:
TODO: duplicate two sentences above!!!

$$\llbracket a \rrbracket = \bullet a \quad (\text{for any atom } a)$$

$$\llbracket A \vee B \rrbracket = \llbracket A \rrbracket + \llbracket B \rrbracket \quad \llbracket \exists x.A \rrbracket = \bullet x \times \llbracket A \rrbracket$$

$$\llbracket A \wedge B \rrbracket = \llbracket A \rrbracket \times \llbracket B \rrbracket \quad \llbracket \forall x.A \rrbracket = \bullet x + \llbracket A \rrbracket$$

where $\bullet a$ (resp. $\bullet x$) is a single-vertex graph whose vertex is labelled by a (resp. x).

Lemma 5. *If A is a rectified formula then $\llbracket A \rrbracket$ is a fograph.*

Proof. That $\llbracket A \rrbracket$ is a logical cograph follows immediately from the definition and Theorem 3. The fact that every binder of $\llbracket A \rrbracket$ is legal can be proved by structural induction on A . \square

Note that $\llbracket A \rrbracket$ is not necessarily a fograph if A is not rectified. If $A = (\forall x.p(x)) \vee (\forall x.q(x))$, then $\llbracket A \rrbracket = \bullet x \bullet p(x) \bullet x \bullet q(x)$, the scope of each x -binder contains all the vertices, in particular, the two x -binders. On the other hand, there are non-rectified formulas which are translated to fographs by $\llbracket \cdot \rrbracket$. For example, in the graph of $(\exists x.p(x)) \vee (\exists x.q(x))$, both x -binders are legal, as they are not in each other's scope: $x \bullet \bullet p x \quad x \bullet \bullet q x$. For this reason, we call a formula **clean** if it does not contain subformulas of the form $(\forall x.A) \vee (\forall x.B)$ or $(\exists x.A) \wedge (\exists x.B)$, **TODO: need the four cases as in clear definition in fopws? Consider $(\forall x.A) \vee (B \vee (\forall x.C))$. But then Lemma 6 doesn't hold. (fopws proves only that there's a surjection from clear formulas onto fographs, whose kernel is the six formula equalities)!!!** and no x -quantified formula occurs as subformula of another x -quantified formula. Then we have:

Lemma 6. *If A is clean iff $\llbracket A \rrbracket$ is a fograph.*

Proof. Induction on A , using Theorem 3. \square

Even though for every formula A we can obtain an equivalent clean formula by simply renaming some bound variables, this is not unique up to α -conversion, as is the case for rectified formulas.

We define a congruence relation \equiv on formulas by the following equations:

$$\begin{aligned} A \wedge B &\equiv B \wedge A & (A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\ A \vee B &\equiv B \vee A & (A \vee B) \vee C &\equiv A \vee (B \vee C) \\ \forall x.\forall y.A &\equiv \forall y.\forall x.A & \forall x.(A \vee B) &\equiv (\forall x.A) \vee B \\ \exists x.\exists y.A &\equiv \exists y.\exists x.A & \exists x.(A \wedge B) &\equiv (\exists x.A) \wedge B \end{aligned} \quad (2)$$

where $x \notin fv(B)$ in the last two equations. Two formulas A and B are **equivalent** if $A \equiv B$. The following theorem shows that the set of fographs can be seen as the quotient FORM/\equiv .

Theorem 7. *Let A, B be rectified formulas. Then*

$$A \equiv B \iff \llbracket A \rrbracket = \llbracket B \rrbracket$$

Proof. By a straightforward induction on A . \square

IV. FIRST-ORDER COMBINATORIAL PROOFS

A. Fonets

Two atoms are **pre-dual** if their predicate symbols are dual (e.g. $p(x, y)$ and $\bar{p}(y, z)$) and two literals are **pre-dual** if their labels (atoms) are pre-dual. A **linked fograph** $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ is a coloured fograph \mathcal{C} such that every colour (i.e., equivalence class of $\sim_{\mathcal{C}}$), called a **link**, consists of two pre-dual literals, and every literal is either t-labelled or in a link.

Let \mathcal{C} be a linked fograph. The set of links can be seen as a unification problem by identifying dual predicate symbols. A **dualizer** of \mathcal{C} is a substitution δ unifying all the links of \mathcal{C} . Since a first-order unification problem is either unsolvable or has a most general unifier, we can define the notion of **most general dualizer**. A **dependency** is a pair $\{\bullet x, \bullet y\}$ of an existential binder $\bullet x$ and a universal binder $\bullet y$ such that the most general dualizer assigns to x a term containing y . A

284 *leap* is either a link or a dependency. The *leap graph* \mathcal{C}^L of \mathcal{C}
 285 is the undirected graph $\langle V_{\mathcal{C}}, L_{\mathcal{C}} \rangle$ where $L_{\mathcal{C}}$ is the set of leaps
 286 of \mathcal{C} . A vertex set $W \subseteq V_{\mathcal{C}}$ induces a *matching* in \mathcal{C} if for all
 287 $w \in W$, $N(w) \cap W$ is a singleton. We say that W induces a
 288 *bimatching* in \mathcal{C} if it induces a matching in \mathcal{C} and a matching
 289 in \mathcal{C}^L .

290 **Definition 8.** A *first-order net* or *fonet* is a linked fograph
 291 which has dualizer but no induced bimatching.

292 B. Skew Bifibrations

293 A graph homomorphism $\varphi: \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ is a
 294 *fibration* if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$, there exists
 295 a unique $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$, and
 296 is a *skew fibration* if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$ there
 297 exists $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w})w \notin E_{\mathcal{H}}$. A
 298 directed graph homomorphism is a *fibration* if for all $v \in V_{\mathcal{G}}$
 299 and $(w, \varphi(v)) \in E_{\mathcal{H}}$, there exists a unique $\tilde{w} \in V_{\mathcal{G}}$ such that
 300 $(\tilde{w}, v) \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$.

301 A *fograph homomorphism* $\varphi = \langle \varphi, \rho_{\varphi} \rangle$ is a pair where
 302 $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is a graph homomorphism between the underlying
 303 graphs, and ρ_{φ} , also called the *substitution induced by* φ
 304 is a variable renaming such that for all $v \in V_{\mathcal{G}}$ we have
 305 $\ell(\varphi(v)) = \rho_{\varphi}(\ell(v))$. Note that this entails that φ maps binders
 306 to binders and literals to literals. We say that a fograph
 307 homomorphism $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is *existential-preserving* if for
 308 all existential binders b in \mathcal{G} , the vertex $\varphi(b)$ is an existential
 309 binder in \mathcal{H} .

310 **Definition 9.** Let \mathcal{G} and \mathcal{H} be fographs. A *skew bifibration*
 311 $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is an existential-preserving fograph homomor-
 312 phism that is a skew fibration on $\langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ and
 313 a fibration on the binding graphs $\tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{H}}$.

314 **Definition 10.** A *first-order combinatorial proof (FOCP)* of
 315 a fograph \mathcal{G} is a skew bifibration $\varphi: \mathcal{C} \rightarrow \mathcal{G}$ where \mathcal{C} is a
 316 fonet. A *first-order combinatorial proof* of a formula A is a
 317 combinatorial proof of its graph $\llbracket A \rrbracket$.

318 **Theorem 11 ([20]).** *FOCPs are sound and complete for first-*
 319 *order logic.*

320 **Remark 12.** In or definition of FOCP, we are slightly laxer
 321 than the original definition of [20], as we allow for a variable
 322 renaming σ_{φ} which was forced to be the identity in [20].

323 V. FIRST-ORDER DEEP INFERENCE SYSTEM KS1

324 Contrary to standard proof formalisms, like sequent calculi
 325 or tableaux, where inference rules decompose the principal
 326 formula along its root connective, can *deep inference rules*
 327 be applied like rewriting rules inside any (positive) formula
 328 or sequent *context*, which is denoted as $S\{\cdot\}$, and which is
 329 a formula (resp. sequent) with exactly one occurrence of the
 330 *hole* $\{\cdot\}$ in the position of an atom. Then $S\{A\}$ is the result
 331 of replacing the hole $\{\cdot\}$ in $S\{\cdot\}$ with A .

332 Figure 6 shows the inference rules for the deep inference
 333 system KS1 that we are using in this paper. It is a slight
 334 variation of the systems presented by Br  nnler [30] and
 335 Ralph [31] in their PhD-theses. The main differences being

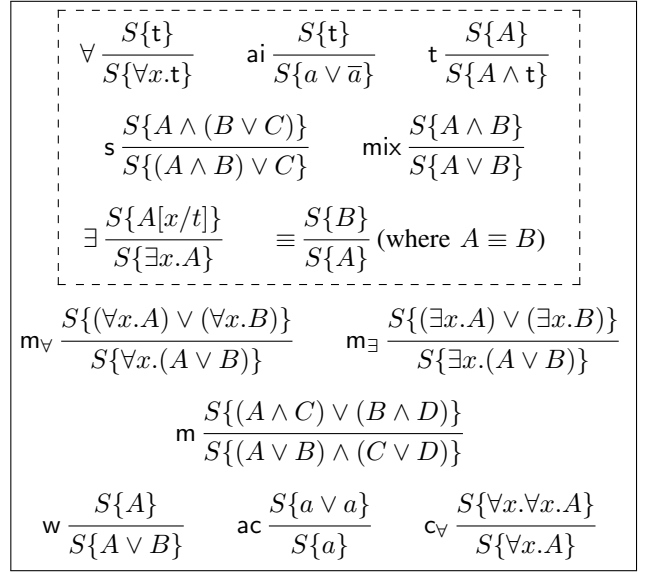


Fig. 6. Deep inference systems KS1 (all rules) and MLS1^X (rules in the dashed box)

that we have (i) the explicit presence of the mix-rule, (ii) a
 different choice of how the formula equivalence \equiv is defined,
 and (iii) an explicit rule for the equivalence.

We consider here only the cut-free fragment, as cut-
 elimination for deep inference systems has already been dis-
 cussed elsewhere (e.g. [22], [32]).¹

As with the sequent system LK1, we also need for KS1 the
linear fragment, that we call here MLS1^X , and that is shown
 in Figure 6 in the dashed box.

B

We write $s \Vdash_{\Phi}$ to denote a derivation Φ from B to A using
 A
 the rules from system S . A formula A is *provable* in a system
 S if there is a derivation in S from t to A .

In the course of this paper we are also going to make use
 of the general (non-atomic) version of the contraction rule:

$$c \frac{S\{A \vee A\}}{S\{A\}}$$

348 VI. MAIN RESULTS

349 We are now ready to see the main results of this paper. We
 350 only state them here and give the proofs in the later sections
 351 of the paper. The first one is routine and expected, but needs
 352 to be proved nonetheless:

353 **Theorem 13.** *KS1 is sound and complete for first-order logic.*

354 Our second result is more surprising, as it is a very strong
 355 decomposition result for first order logic.

¹In the deep inference literature, the cut-free fragment is also called the *down-fragment*. But as we do not discuss the *up-fragment* here, we omit the down-arrows \downarrow in the rule names.

Theorem 14. For every derivation $\text{KS1} \parallel_{\Phi}^t$ there are formulas A_1, \dots, A_5 , such that there is a derivation:

$$\begin{array}{c} t \\ \{\forall, \text{ai}, t\} \parallel \\ A_5 \\ \{s, \text{mix}, \equiv\} \parallel \\ A_4 \\ \{\exists\} \parallel \\ A_3 \\ \{m, m_{\forall}, m_{\exists}, \equiv\} \parallel \\ A_2 \\ \{ac, c_{\forall}\} \parallel \\ A_1 \\ \{w, \equiv\} \parallel \\ A \end{array}$$

This theorem is stronger than the existing decompositions for first-order logic, which either separated only atomic contraction and atomic weakening [30] or only contraction [31] or only the quantifiers in form of a Herbrand theorem [33], [31].

There is a weaker version of Theorem 14 that will also be useful:

Theorem 15. For every derivation $\text{KS1} \parallel_{\Phi}^t$ there is a formula A_1 , such that there is a derivation:

$$\begin{array}{c} t \\ \text{MLS1}^X \parallel \\ A_1 \\ \{w, c, \equiv\} \parallel \\ A \end{array}$$

Let us now establish the connection between derivations in KS1 and combinatorial proofs.

Theorem 16. Let $\varphi: \mathcal{C} \rightarrow \mathcal{A}$ be a combinatorial proof and let A be a formula with $\mathcal{A} = \llbracket A \rrbracket$. Then there is a derivation

$$\begin{array}{c} t \\ \text{MLS1}^X \parallel_{\Phi_1} \\ A' \\ \{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\} \parallel_{\Phi_2} \\ A \end{array} \quad (3)$$

for some $A' \equiv C\sigma_{\varphi}$ where C is a formula with $\llbracket C \rrbracket = \mathcal{C}$ and σ_{φ} is the variable renaming substitution induced by φ . Conversely, whenever we have a derivation as in (6) above, then there is a combinatorial proof $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$ such that $\mathcal{C} = \llbracket A' \rrbracket$.

Furthermore, in the proof of Theorem 16, we will see that (i) the links in the fonet \mathcal{C} correspond precisely to the pairs of atoms that meet in the instances of the ai-rule in the derivation

Φ_1 , and (ii) the "flow-graph" of Φ_2 that traces the quantifier- and atom-occurrences in the derivation corresponds exactly to the vertex-mapping induced by φ .

Thus, combinatorial proofs are closely related to derivations of the form (6), and since by Theorem 14 every derivation can be transformed into that form, we can say that combinatorial proofs form a canonical proof representation for first order logic, similarly to what proof nets are for linear logic [34].

Finally, Theorems 13, 14 and 16 imply Theorem 11, which means that we have here an alternative proof of the soundness and completeness of first-order combinatorial proofs which is simpler than the one given in [20].

VII. TRANSLATING BETWEEN LK1 AND KS1

In this section we prove Theorems 13, 14, and 15, mainly by translating derivations to and from the sequent calculus, and by rule permutation arguments.

A. The Linear Fragments MLL1^X and MLS1^X

In this section we show the equivalence of MLL1^X and MLS1^X .

Lemma 17. If $\vdash \Gamma$ is provable in MLL1^X then $\text{fm}(\Gamma)$ is provable in MLS1^X .

Proof. This is a straightforward induction on the proof of $\vdash \Gamma$ in MLL1^X , making a case analysis on the bottommost rule instance. We show here only the case of $\forall \frac{\vdash \Delta, A}{\vdash \Delta, \forall x.A}$ (all other cases are simpler and have been shown before, e.g. [30]): By induction hypothesis, there is a proof of $\text{fm}(\Delta) \vee A$ in MLS1^X . We can prefix every line in that proof by $\forall x$ and then compose the following derivation:

$$\begin{array}{c} t \\ \forall \frac{}{\forall x.t} \\ \text{MLS1}^X \parallel \\ \forall x.\text{fm}(\Delta) \vee A \\ \equiv \\ \text{fm}(\Delta) \vee \forall x.A \end{array}$$

where we can apply the \equiv -rule because x is not free in Δ . \square

Lemma 18. Let $r \frac{S\{A\}}{S\{B\}}$ be an inference rule in MLS1^X other than ai. Then the sequent $\vdash \overline{A}, B$ is provable in MLL1^X .

Proof. This is a straightforward exercise that we leave to the reader. (Note that the ax-rule can be applied to $\vdash f, t$ in the cases of $r = \forall$.) \square

Lemma 19. Let A, B be formulas, and let $S\{\cdot\}$ be a (positive) context. If $\vdash \overline{A}, B$ is provable in MLL1^X , then so is $\vdash S\{A\}, S\{B\}$.

Proof. Straightforward induction on $S\{\cdot\}$. (see e.g. [35]) \square

Lemma 20. If a formula C is provable in MLS1^X then $\vdash C$ is provable in MLL1^X .

Proof. We proceed by induction on the number of inference steps in the proof of C in MLS1^X . Consider the bottommost

rule instance $r \frac{S\{A\}}{S\{B\}}$. By induction hypothesis we have a MLL1^x proof Π of $\vdash S\{A\}$. If r is $ai \frac{S\{t\}}{S\{a \vee \bar{a}\}}$, we replace in Π all corresponding occurrences of t with $a \vee \bar{a}$ and the

rule instance $t \frac{}{\vdash t}$ with the derivation $ax \frac{}{\vdash a, \bar{a}} \vee \frac{}{\vdash a \vee \bar{a}}$. This gives

us a proof of $\vdash S\{a \vee \bar{a}\}$. In all other cases, by Lemmas 18 and 19, we have a MLL1^x proof of $\vdash \bar{S}\{A\}, S\{B\}$. We can compose them via cut:

$$cut \frac{\vdash S\{A\} \quad \vdash \bar{S}\{A\}, S\{B\}}{\vdash S\{B\}}$$

and then apply Theorem 2. \square

B. Contraction and Weakening

The first observation here is that Lemmas 17–20 from above also hold for LK1 and KS1. We therefore immediately have:

Theorem 21. *For every sequent Γ , we have that $\vdash \Gamma$ is provable in LK1 if and only if $\text{fm}(\Gamma)$ is provable in KS1.*

Then Theorem 13 is an immediate consequence. Let us now proceed with providing further lemmas that will be needed for the other results.

Lemma 22. *The c-rule is derivable in $\{ac, m, m_{\forall}, m_{\exists}, \equiv\}$.*

Proof. We show that there is always a derivation

$$\frac{A \vee A}{s \parallel} A$$

, where $S = \{ac, m, m_{\forall}, m_{\exists}, \equiv\}$, by induction on A :

• If $A = a$, then we have $ac \frac{a \vee a}{a}$.

• If $A = B \wedge C$, then we have $m \frac{(B \wedge C) \vee (B \wedge C)}{(B \vee B) \wedge (C \vee C)}$

• If $A = B \vee C$, then we have $s \parallel \frac{B \wedge (C \vee C)}{B \wedge C} \equiv \frac{(B \vee C) \vee (B \vee C)}{(B \vee B) \vee (C \vee C)}$

• If $A = \exists x.A'$, then we have $m_{\exists} \frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')}$

• If $A = \forall x.A'$, then we have $s \parallel \frac{B \vee C}{B \vee C} \equiv \frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')}$

$$m_{\exists} \frac{(\forall x.A') \vee (\forall x.A')}{\forall x.(A' \vee A')} \quad s \parallel \quad \forall x.A'$$

• If $A = \forall x.A'$, then we have

TODO: **Jui-Hsuan:** done. Maybe just keep one case. **Lutz:** yes, but we do that at the end. don't think about space right now. \square

Lemma 23. $c_{\forall}, m, m_{\forall}, m_{\exists}$ are derivable in $\{w, c, \equiv\}$.

Proof. **TODO:**

We have the following derivations:

$$\frac{w \frac{\forall x.\forall x.A}{\forall x.((\forall x.A) \vee A)} \quad \equiv \frac{(\forall x.A) \vee (\forall x.A)}{c \frac{\forall x.A}{\forall x.A}} (x \notin \text{fv}(\forall x.A))$$

$$\frac{w \frac{(A \wedge C) \vee (B \wedge D)}{w \frac{((A \vee B) \wedge C) \vee (B \wedge D)}{w \frac{((A \vee B) \wedge (C \vee D)) \vee (B \wedge D)}{w \frac{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge D)}{w \frac{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge (D \vee C))}{\equiv \frac{((A \vee B) \wedge (C \vee D)) \vee ((A \vee B) \wedge (C \vee D))}{c \frac{(A \vee B) \wedge (C \vee D)}{(A \vee B) \wedge (C \vee D)}}}}$$

$$\frac{w \frac{(\exists x.A) \vee (\exists x.B)}{w \frac{(\exists x.(A \vee B)) \vee (\exists x.B)}{w \frac{(\exists x.(A \vee B)) \vee (\exists x.(B \vee A))}{\equiv \frac{\exists x.(A \vee B) \vee (\exists x.(A \vee B))}{c \frac{\exists x.(A \vee B)}{\exists x.(A \vee B)}}}}$$

$$\frac{w \frac{(\forall x.A) \vee (\forall x.B)}{w \frac{(\forall x.(A \vee B)) \vee (\forall x.B)}{w \frac{(\forall x.(A \vee B)) \vee (\forall x.(B \vee A))}{\equiv \frac{(\forall x.(A \vee B)) \vee (\forall x.(A \vee B))}{c \frac{\forall x.(A \vee B)}{\forall x.(A \vee B)}}}}$$

Jui-Hsuan: done. If needed, we can introduce the notion of open deduction to reduce the size of derivations... **Lutz:** I was thinking about that, but (i) it is probably not worth the effort, as we won't have many derivations, and (ii) it is hard to define rectified derivations this way. \square

Lemma 24. *Let A and B be formulas. Then*

$$\frac{A}{\{w, c, \equiv\} \parallel B} \iff \frac{A}{\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\} \parallel B}$$

Proof. This follows immediately from Lemmas 22 and 23. \square

Theorem 25. Let Γ be a sequent. If $\vdash \Gamma$ is provable in LK1 (as depicted on the left below) then there is a sequent Γ' such that there is a derivation as shown on the right below:

$$\text{LK1} \frac{\vdash \Gamma}{\vdash \Gamma} \Phi \quad \Rightarrow \quad \text{MLL1}^X \frac{\vdash \Gamma'}{\vdash \Gamma'} \Phi_1$$

$$\{w, c, \equiv\} \parallel \Phi_2$$

$$\vdash \text{fm}(\Gamma)$$

438 *Proof.* Note that the instances of w, c in Φ_2 are deep, but
439 inside sequent contexts.

440 First, if an instance of $\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A}$ is followed by a rule in
441 which A is not in the principal formula, it can be permuted
442 downwards. Otherwise, the proof can be transformed using the
443 following rewriting rules.

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \wedge \frac{\vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A \wedge B, \Delta}$$

$$\text{wk} \frac{\vdash \Gamma, A}{\vdash \Gamma, A, B} \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \rightsquigarrow w \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}$$

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A[x/t]} \exists \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \exists x.A}$$

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \forall \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \forall x.A}$$

$$\text{wk} \frac{\vdash \Gamma, A}{\vdash \Gamma, A, A} \text{ctr} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \rightsquigarrow \vdash \Gamma, A$$

444 Note that in the case of \vee , we use the deep rule w which can
445 be permuted down over all the rules. By using these rewriting
446 rules, we can eventually get a derivation with all the instances
447 of wk and w at the bottom. Now observe that the instances of
448 ctr in Φ can be transformed using the following rule:

$$\text{ctr} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \rightsquigarrow \vee \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \text{c} \frac{\vdash \Gamma, A \vee A}{\vdash \Gamma, A}$$

Knowing that c can be permuted down over all the rules of MLL1^X , we eventually obtain a derivation:

$$\text{MLL1}^X \frac{\vdash \Gamma_0}{\vdash \Gamma_0} \Phi'_1$$

$$\{wk, w, c, \equiv\} \parallel \Phi'_2$$

$$\vdash \Gamma$$

Note that \equiv is required here since the permutation of formulas is implicit in MLL1^X .

By transforming each sequent of Φ'_2 into its corresponding formula, and by considering the following rewriting rule:

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \rightsquigarrow w \frac{\vdash \text{fm}(\Gamma)}{\vdash \text{fm}(\Gamma) \vee A}$$

, we obtain a derivation

$$\text{MLL1}^X \frac{\vdash \Gamma'}{\vdash \Gamma'} \Phi_1$$

$$\{w, c, \equiv\} \parallel \Phi_2$$

$$\vdash \text{fm}(\Gamma)$$

where $\Gamma' = \text{fm}(\Gamma_0)$ and Φ_1 can be obtained from Φ'_1 by applying the \vee rule. **TO CHECK:** **Jui-Hsuan:** This might be a bit long... \square

Lemma 26. For every derivation $\text{MLS1}^X \frac{t}{A}$ there are formulas A' and A'' such that

$$\frac{t}{A''} \parallel \{v, ai, t\}$$

$$\frac{t}{A'} \parallel \{s, mix, \equiv\}$$

$$\frac{t}{A} \parallel \{\exists\}$$

Proof. First, observe that the \exists rule can be permuted downwards over all the other rules since $A[x/t]$ has the same structure as A and none of the other rules has a premise of the form $S\{\exists x.A\}$. It suffices now to prove that for all $r_1 \in \{v, ai, t\}$, for all $r_2 \in \{s, mix, \equiv\}$, we can permute r_1 upwards over r_2 . We show some cases here, and leave the others to the reader.

$$s \frac{S\{A \wedge (t \vee C)\}}{S\{(A \wedge t) \vee C\}} \rightsquigarrow ai \frac{S\{A \wedge (t \vee C)\}}{S\{A \wedge ((a \vee \bar{a}) \vee C)\}} \text{ai} \frac{S\{A \wedge ((a \vee \bar{a}) \vee C)\}}{S\{(A \wedge (a \vee \bar{a})) \vee C\}} s \frac{S\{(A \wedge (a \vee \bar{a})) \vee C\}}{S\{(A \wedge (a \vee \bar{a})) \vee C\}}$$

$$mix \frac{S\{A \wedge B\}}{S\{A \vee B\}} \rightsquigarrow t \frac{S\{A \wedge B\}}{S\{A \wedge (B \wedge t)\}} t \frac{S\{A \wedge (B \wedge t)\}}{S\{(A \vee (B \wedge t))\}} mix \frac{S\{(A \vee (B \wedge t))\}}{S\{(A \vee (B \wedge t))\}}$$

TO CHECK:

\square 454

Lemma 27. For every derivation $\frac{A}{\{w, ac, c_v, m, m_v, m_\exists, \equiv\}} \parallel \frac{B}{B}$ there are formulas A' and B' such that

$$\frac{\frac{\frac{A}{\{m, m_v, m_\exists, \equiv\}} \parallel \frac{A'}{\{ac, c_v\}} \parallel \frac{B'}{\{w, \equiv\}} \parallel B}{B}}$$

Proof. First, a derivation consisting of an instance of w followed by $r \in \{ac, c_v, m, m_v, m_\exists\}$ can be either replaced by a derivation consisting of w only or the instance of w can be permuted downwards. We show some cases here, and leave the others to the reader.

$$\begin{aligned} \frac{w}{m_v} \frac{S\{\forall x.A\}}{S\{(\forall x.A) \vee (\forall x.B)\}} &\rightsquigarrow w \frac{S\{\forall x.A\}}{S\{\forall x.(A \vee B)\}} \\ \frac{w}{m} \frac{S\{A \wedge C\}}{S\{(A \wedge C) \vee (B \wedge D)\}} &\rightsquigarrow w \frac{S\{A \wedge C\}}{S\{(A \vee B) \wedge C\}} \\ \frac{w}{ac} \frac{S\{a\}}{S\{a \vee a\}} &\rightsquigarrow S\{a\} \end{aligned}$$

For $r_1 \in \{m, m_v, m_\exists\}$, $r_2 \in \{ac, c_v\}$, r_1 can be permuted upwards over r_2 (with some \equiv inserted). The only non-trivial case is shown below:

$$\frac{c_v}{m_v} \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \rightsquigarrow \frac{m_v}{m_v} \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \rightsquigarrow \frac{m_v}{m_v} \frac{S\{\forall x.(A \vee B)\}}{S\{\forall x.(A \vee B)\}}$$

455 **TODO: permutation with \equiv** \square

456 We can now complete the proof of Theorems 14 and 15.

457 *Proof of Theorem 15.* Assume we have a proof of A in KS1.
458 By Theorem 21 we have a proof of $\vdash A$ in LK1 to which we
459 can apply Theorem 25. Finally, we apply Lemma 17 to get
460 the desired shape. \square

461 *Proof of Theorem 14.* Assume we have a proof of A in KS1.
462 We first apply Theorem 15, and then Lemma 26 to the upper
463 half and Lemma 27 to the lower half. \square

VIII. FONETS AND LINEAR PROOFS

A. From MLL1^X Proofs to Fonets

Let Π be a MLL1^X proof of a rectified sequent $\vdash \Gamma$. We now show how Π is translated into a linked fograph $\llbracket \Pi \rrbracket = \langle \llbracket \Gamma \rrbracket, \sim_\Pi \rangle$. We proceed inductively, making a case analysis on the last rule in Π . At the same time we are constructing a dualizer δ_Π , so that in the end we can conclude that $\llbracket \Pi \rrbracket$ is in fact a fonet.

1) Π is $\text{ax} \frac{}{\vdash a, \bar{a}}$: Then the only link is $\{a, \bar{a}\}$, and δ_Π is empty.

2) Π is $\text{t} \frac{}{\vdash t}$: Then \sim_Π and δ_Π are both empty.

3) The last rule in Π is $\text{mix} \frac{\vdash \Gamma' \quad \vdash \Gamma''}{\vdash \Gamma', \Gamma''}$: By induction hypothesis, we have proofs Π' and Π'' of Γ' and Γ'' , respectively. We have $\llbracket \Gamma \rrbracket = \llbracket \Gamma' \rrbracket + \llbracket \Gamma'' \rrbracket$ and let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

4) The last rule in Π is $\vee \frac{\vdash \Gamma_1, A, B}{\vdash \Gamma_1, A \vee B}$: By induction hypothesis, there is proofs Π' of $\Gamma' = \Gamma_1, A, B$. We have $\llbracket \Gamma \rrbracket = \llbracket \Gamma' \rrbracket$ and let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.

5) The last rule in Π is $\wedge \frac{\vdash \Gamma_1, A \quad \vdash B, \Gamma_2}{\vdash \Gamma_1, A \wedge B, \Gamma_2}$: By induction hypothesis, we have proofs Π' and Π'' of $\Gamma' = \Gamma_1, A$ and $\Gamma'' = B, \Gamma_2$, respectively. We have $\llbracket \Gamma \rrbracket = \llbracket \Gamma_1 \rrbracket + (\llbracket A \rrbracket \times \llbracket B \rrbracket) + \llbracket \Gamma_2 \rrbracket$ and we let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

6) The last rule in Π is $\exists \frac{\vdash \Gamma_1, A[x/t]}{\vdash \Gamma_1, \exists x.A}$: By induction hypothesis, there is a Π' of $\Gamma' = \Gamma_1, A[x/t]$. For each atom in $\Gamma' = \Gamma_1, A[x/t]$, there is a corresponding atom in $\Gamma = \Gamma_1, \exists x.A$. We can therefore define the linking \sim_Π from the linking $\sim_{\Pi'}$ via this correspondence. Then, we let δ_Π be $\delta_{\Pi'} + [x/t]$. Since Γ is rectified x does not yet occur in $\delta_{\Pi'}$. Hence δ_Π is a dualizer of $\llbracket \Pi \rrbracket$.

7) The last rule in Π is $\forall \frac{\vdash \Gamma_1, A}{\vdash \Gamma_1, \forall x.A}$ (x not free in Γ_1): By induction hypothesis, there is a proof Π' of $\Gamma' = \Gamma_1, A$, which has the same atoms as in $\Gamma = \Gamma_1, \forall x.A$. Hence, we can let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.

Theorem 28. If Π is a MLL1^X proof of a rectified sequent $\vdash \Gamma$, then $\llbracket \Pi \rrbracket$ is a fonet and δ_Π is a dualizer for it.

Proof. We have to show that none of the operations above can introduce a bimatching. For cases 1–6, this is immediate. For case 7, observe that there is a potential dependency from each existential binder in $\llbracket \Gamma' \rrbracket$ to the new x -binder $\bullet x$ in $\llbracket \Gamma \rrbracket$. However, observe that this $\bullet x$ vertex is not connected to any vertex in $\llbracket \Gamma' \rrbracket$, and hence no such new dependency can be extended to a bimatching. That δ_Π is a dualizer for $\llbracket \Pi \rrbracket$ follows immediately from the construction. Hence, $\llbracket \Pi \rrbracket$ is a fonet. \square

B. From MLS1^X Proofs to Fonets

There is a more direct path from a MLL1^X proof Π of a rectified sequent Γ to the linked fograph $\llbracket \Pi \rrbracket$: simply take the fograph $\llbracket \Gamma \rrbracket$, and let the equivalence classes of \sim_Π be all the atom pairs that meet in an instance of ax , and δ_Π is simply the collection of all substitutions of all the instances of the \exists -rule in Π . We have chosen the more cumbersome path above because it gives us a direct proof of Theorem 28. However, for translating MLS1^X derivation into fonets, we employ exactly that direct path.

A derivation Φ in MLS1^X is **rectified** if every line in Φ is rectified.

Lemma 29. *Let Φ be a MLS1^X proof of a formula A . Then Φ is rectified iff A is rectified.*

Proof. The only rules involving bound variables are \forall and \exists which both remove a binder (and all occurrences of the variable it binds). \square

Hence, for a non-rectified MLS1^X derivation Φ in MLS1^X we can define its **rectification** $\hat{\Phi}$ inductively, by rectifying each line, proceeding step-wise from conclusion to premise.²

A rectified derivation $\text{MLS1}^X \llbracket \Phi \rrbracket$ determines a substitution A

which maps the existential bound variables occurring in A to the terms substituted for them in the instances of the \exists -rule in Φ . We denote this substitution with δ_Φ and call it the **dualizer** of Φ . Furthermore, every atom occurring in the conclusion A must be consumed by a unique instance of the rule ai in Φ . This allows us to define a (partial) equivalence relation \sim_Φ on the atom occurrences in A by $a \sim_\Phi b$ if a and b are consumed by the same instance of ai in Φ . We call \sim_Φ the **linking** of Φ , and define $\llbracket \Phi \rrbracket = \langle \llbracket A \rrbracket, \sim_\Phi \rangle$.

!!!TODO: example here!!!

Theorem 30. *Let $\text{MLS1}^X \llbracket \Phi \rrbracket$ be a rectified derivation. Then $\llbracket \Phi \rrbracket$ is a fonet and δ_Φ a dualizer for it.*

For proving this theorem, we have to show that no inference rule in MLS1^X can introduce a bimatching. To simplify the argument, we introduce the **frame** [37] of the fograph \mathcal{C} , which is a linked (propositional) cograph in which the dependencies between the binders in \mathcal{C} are encoded as links.

More formally, let C be a formula with $\llbracket C \rrbracket = \mathcal{C}$, to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent C^* :

- 1) **Encode dependencies as fresh links.** For each dependency $\{\bullet x_i, \bullet y_j\}$ in \mathcal{C} , with corresponding subformulas $\exists x_i.A$ and $\forall y_j.B$ in C , we pick a fresh (nullary) predicate symbol $q_{i,j}$, and then replace $\exists x_i.A$ by $\bar{q}_{i,j} \wedge \exists x_i.A$, and replace $\forall y_j.B$ by $q_{i,j} \vee \forall y_j.B$.

²As for formulas, the rectification of a derivation is unique up to renaming of bound variables.

- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace $\exists x_i.A$ by A and replace $\forall y_j.B$ by B everywhere.

- 3) **Simplify atoms.** After step 2, replace every predicate $p(t_1 \cdots t_n)$ (resp. $\bar{p}(t_1 \cdots t_n)$) with a nullary predicate symbol p (resp. \bar{p})

The \sim_{C^*} consists of the pairs induced by \sim_C and the new pairs $\{q_{i,j}, \bar{q}_{i,j}\}$ introduced in step 1 above. We call C^* the **frame** of C and we define the **frame** of \mathcal{C} , denoted \mathcal{C}^* , as $\langle \llbracket C^* \rrbracket, \sim_{C^*} \rangle$.

Lemma 31. *A linked fograph \mathcal{C} has an induced bimatching iff its frame \mathcal{C}^* has an induced bimatching.*

Proof. This immediately follows from the construction of the frame. !!!Lutz: is it really an “iff”? It is easy to construct from a bimatching in \mathcal{C} a bimatching in the frame. (and I think we only need that direction). But what about the other direction?!!! \square

Proof of Theorem 30. From Φ we construct a derivation Φ^* of A^* in the propositional fragment of MLS1^X , such that $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$. The rules ai , t , mix and s are translated trivially, and for \equiv , it suffices to observe that the frame construction is invariant under \equiv . Finally, for the rules \forall and \exists , proceed as follows. Every instance of \forall is replaced by the derivation on the right below:³

$$\forall \frac{S\{t\}}{S\{\forall y_j.t\}} \rightsquigarrow \frac{\frac{\frac{\text{t}}{\{ \text{ai}, t \} \parallel \Psi_1}}{S\{(q_{h_1,j} \vee \bar{q}_{h_1,j}) \wedge \cdots \wedge (q_{h_j,j} \vee \bar{q}_{h_j,j}) \wedge t\}}}{\{s, \equiv\} \parallel \Psi_2} S\{q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge t)\}$$

where h_1, \dots, h_j range over the indices of the existential binders dependent on that y_j . It is easy to see how Ψ_1 is constructed, and for Ψ_2 see, e.g. [?, [35], [36]!!!Lutz: check if it is really there. otherwise [?]] Then, every occurrence of $\forall y_j.F$ is replaced by $q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge F)$ in the derivation below that \forall -instance. Now, observe that all instances of the \exists -rule introducing x_i depend on y_j must occur below in the derivation (otherwise Φ would not be rectified). Now consider such an instance $\exists \frac{S\{B[x_i/t]\}}{S\{\exists x_i.B\}}$. Its

context $S\{\cdot\}$ must contain all the $\forall y_j$ the $\exists x_i$ depends on, such that B is in their scope. Following the translation of the \forall rules above, we can therefore translate the \exists -rule instance by the following derivation

$$\frac{S_0\{\bar{q}_{i,k_1} \wedge S_1\{\bar{q}_{i,k_2} \wedge \cdots \wedge S_{k_i-1}\{\bar{q}_{i,k_i} \wedge S_{k_i}\{B'\}\}\} \cdots\}}{\{s, \equiv\} \parallel \Psi_3} S_0\{S_1\{\cdots S_{k_i-1}\{S_{k_i}\{\bar{q}_{i,k_1} \wedge \bar{q}_{i,k_2} \wedge \cdots \wedge q_{i,k_i} \wedge B'\}\} \cdots\}\}$$

where k_1, \dots, k_i are the indices of the universal binders on which that x_i depends, and B' is B in which all predicates are replaced by nullary one (step 3 in the frame construction).

³For better readability we omit superfluous parentheses, knowing that we always have \equiv incorporating associativity and commutativity of \wedge and \vee .

The derivation Ψ_3 can be constructed in the same way as Ψ_2 above.

Doing this to all instances of the rules \forall and \exists in Φ yields indeed a propositional derivation Φ^* with $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$. It has been shown by Retoré [?] and rediscovered by Straßburger [?] that $\llbracket \Phi^* \rrbracket = \langle \llbracket C^* \rrbracket, \sim_{\Phi^*} \rangle$ can not contain an induced bimatching. By Lemma 33, $\llbracket \Phi \rrbracket$ does not have an induced bimatching either. Furthermore, it followed from the definition of δ_Φ that it is a dualizer for $\llbracket \Phi \rrbracket$. Hence $\llbracket \Phi \rrbracket$ is a fonet. \square

Remark 32. There is an alternative path of proving Theorem 30 by translating Φ to an MLL1^X -proof Π , observing that this process preserves the linking and the dualizer. However, for this, we have to extend the construction above to the cut-rule, and then show that linking and dualizer of a sequent proof Π are invariant under cut elimination. This can be done similarly to unification nets in [37].

C. From Fonets to MLL1^X Proofs

Now we are going to show how from a given fonet $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ we can construct a sequent proof Π in MLL1^X such that $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. In the proof net literature, this operation is also called *sequentialization*. The basic idea behind our sequentialization is to construct a propositional linked cograph, called the **frame** [37] of \mathcal{C} , in which the dependencies between the binders in \mathcal{C} are encoded as links. Then we can apply the *splitting tensor theorem* to the frame, and then reconstruct the sequent proof Π . [\[Lutz: if the proof of thm 30 is verified, we can delete the frame-def here\]](#)

More formally, let Γ be a sequent with $\llbracket \Gamma \rrbracket = \mathcal{C}$, to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent Γ^* :

- 1) **Encode dependencies as fresh links.** For each dependency $(\bullet x, \bullet y)$ in \mathcal{C} , with corresponding subformulas $\exists x.A$ and $\forall y.B$ in Γ , we pick a fresh (nullary) predicate symbol q , and then replace $\exists x.A$ by $q \wedge \exists x.A$, and replace $\forall y.B$ by $\bar{q} \vee \forall y.B$.
- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace $\exists x.A$ by A and replace $\forall y.B$ by B everywhere.
- 3) **Simplify atoms.** After step 2, replace every predicate $p(t_1 \cdots t_n)$ (resp. $\bar{p}(t_1 \cdots t_n)$) with a nullary predicate symbol p (resp. \bar{p})

The \sim_{Γ^*} consists of the pairs induced by $\sim_{\mathcal{C}}$ and the new pairs $\{q, \bar{q}\}$ introduced in step 1 above. We call Γ^* the **frame** of Γ and we define the **frame** of \mathcal{C} , denoted C^* , as $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$, and we immediately have the following:

Lemma 33. A linked fograph \mathcal{C} induces a bimatching iff its frame C^* has an induced bimatching.

Let Γ be a propositional sequent and \sim_Γ be a linking for $\llbracket \Gamma \rrbracket$. A conjunction formula $A \wedge B$ is **splitting** or a **splitting tensor** if $\Gamma = \Gamma', A \wedge B, \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that \sim_1 is a linking for $\llbracket \Gamma', A \rrbracket$ and \sim_2 is a linking for $\llbracket B, \Gamma'' \rrbracket$, i.e., removing the \wedge from $A \wedge B$ splits the linked fograph $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ into two fographs. We say that $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ is **mixed**

iff $\Gamma = \Gamma', \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that \sim_1 is a linking for $\llbracket \Gamma' \rrbracket$ and \sim_2 is a linking for $\llbracket \Gamma'' \rrbracket$. Finally, $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ is **splittable** if it is mixed or has a splitting tensor.

The purpose of introducing the frame is the following theorem.

Theorem 34. Let Γ be a propositional sequent containing only atoms and \wedge -formulas, and \sim_Γ be a linking for $\llbracket \Gamma \rrbracket$. If $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ does not induce a bimatching then it is splittable.

This is the well-know splitting-tensor-theorem [25], [?], adapted for the presence of mix. In the setting of linked cographs, it has first been proved by Retoré [38], [?]. We use it now for our sequentialization:

Theorem 35. Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let Γ be a sequent with $\llbracket \Gamma \rrbracket = \mathcal{C}$. Then there is an MLL1^X -proof Π of Γ , such that $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$.

Proof. Let $\delta_{\mathcal{C}}$ be the dualizer of $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. We proceed by induction on the size of Γ (i.e., the number of symbols in it, without counting the commas). If Γ contains a formula with \vee -root, or a formula $\forall x.A$, we can immediately apply the \vee -rule or the \forall -rule of MLL1^X and proceed by induction hypothesis. If Γ contains a formula $\exists x.A$ such that the corresponding binder $\bullet x$ in \mathcal{C} has no dependency, then we can apply the \exists -rule, choosing the term t as determined by $\delta_{\mathcal{C}}$, and proceed by induction hypothesis. Hence, we can now assume that Γ contains only atoms, \wedge -formulas, or formulas of shape $\exists x.A$, where the vertex $\bullet x$ has dependencies. Then the frame $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$ does not induce a bimatching and contains only atoms and \wedge -formulas, and is therefore splittable. If it is mixed, then we can apply the mix-rule to Γ and apply the induction hypothesis to the two components. If it is not mixed then there must be a splitting tensor. If the splitting \wedge is already in Γ , then we can apply the \wedge -rule and proceed by induction hypothesis on the two branches. However, if Γ^* is not mixed and all splitting tensors are \wedge -formulas introduced in step 1 of the frame construction, then we get a contradiction as in that case there must be a \vee - or \forall -formula in Γ . [\[Lutz: can anyone give a good argument here?\]](#) \square

D. From Fonets to MLS1^X Proofs

We can now straightforwardly obtain the same result for MLS1^X :

Theorem 36. Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let C be a formula with $\llbracket C \rrbracket = \mathcal{C}$. Then there is a derivation $\text{MLS1}^X \vdash C$ such that

$$\llbracket \Phi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle.$$

Proof. We apply Theorem 35 to obtain a sequent proof Π of $\vdash C$ with $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. Then we apply Lemma 17, observing that the translation from MLL1^X to MLS1^X preserves linking and dualizer. \square

Remark 37. Note that it is also possible to do a direct “sequentialization” into the deep inference system MLS1^X , using the techniques presented in [?] and [?].

In this section we establish the relation between skew bifibrations and derivations in $\{w, ac, c_\forall, m, m_\forall, m_\exists, \equiv\}$. However, if a derivation Φ contains instances of the rules c_\forall , m_\forall , and m_\exists we can no longer naively define the rectification $\widehat{\Phi}$ as in the previous section for $MLS1^X$, as these two rules cannot be applied if premise and conclusion are rectified. For this reason we define here rectified versions \widehat{c}_\forall , \widehat{m}_\forall and \widehat{m}_\exists , shown below:

$$\widehat{c}_\forall \frac{S\{\forall y. \forall x. Ax\}}{S\{\forall x. Ax\}} \quad \widehat{m}_\forall \frac{S\{\forall y. Ay\} \vee (\forall z. Bz)}{S\{\forall x. (Ax \vee Bx)\}} \quad \widehat{m}_\exists \frac{S\{\exists y. Ay\} \vee (\exists z. Bz)}{S\{\exists x. (Ax \vee Bx)\}}$$

Here, we use the notation $A \cdot$ for a formula A with occurrences of a placeholder \cdot for a variable. Then Ax stands for the results of replacing that placeholder with x , and also indicating that x must not occur in $A \cdot$. Then $\forall x. Ax$ and $\forall y. Ay$ are the same formula modulo renaming of the bound variable bound by the outermost \forall -quantifier. We also demand that the variables x , y , and z do not occur in the context $S\{\cdot\}$.

Note that in an instance of \widehat{m}_\forall or \widehat{m}_\exists (as shown above), we can have $x = y$ or $x = z$, but not both if the premise is rectified. If $x = y$ and $x = z$ we have m_\forall and m_\exists as special cases of \widehat{m}_\forall and \widehat{m}_\exists , respectively. And similarly, if $x = y$ then c_\forall is a special case of \widehat{c}_\forall .

For a derivation Φ in $\{w, ac, c_\forall, m, m_\forall, m_\exists, \equiv\}$, we can now construct the **rectification** $\widehat{\Phi}$ by rectifying each line of Φ , yielding a derivation in $\{w, ac, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\}$.

For each instance $r \frac{Q}{P}$ of an inference rule in $\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\}$ we can define the **induced map** $[r]: V_{[Q]} \rightarrow V_{[P]}$ which acts as the identity for $r \in \{m, \equiv\}$ and as the canonical injection for $r = w$. For $r = ac$ it maps the vertices corresponding to the two atoms in the premise to vertex of the contracted atom in the conclusion, and for $r \in \{\widehat{c}_\forall, \widehat{m}_\forall, \widehat{m}_\exists\}$ it maps the two vertices corresponding to the quantifiers in the premise to the one in the conclusion (as acts as the identity on all other vertices). For a derivation Φ in $\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\}$ we can then define the **induced map** $[\Phi]$ as the composition of the induced maps of the rule instances in Φ . **Jui-Hsuan:** maybe mention at least that the induced maps define graph homomorphisms. Do we need to talk about the contexts $S\{\cdot\}$ here (induced maps act clearly as the identity on contexts but we need them for the composition)? **Lutz:** For the context, I already say it is the identity. For the homom, it comes later

Lemma 38. Let $\{w, ac, c_\forall, m, m_\forall, m_\exists, \equiv\} \parallel \Phi$ be a derivation. Then there is a rectified derivation $\{w, \widehat{ac}, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\} \parallel \widehat{\Phi}$, such that the induced maps $[\Phi]: [A] \rightarrow [B]$ and $[\widehat{\Phi}]: [\widehat{A}] \rightarrow [\widehat{B}]$ are equal up to a variable renaming of the vertex labels.

Proof. Immediate from the definition. \square

TODO: example

A. From Contraction and Weakening to Skew Bifibrations

We say that a derivation Φ is **sane** if for every line Q in Φ we have that $[D]$ is a fograph (i.e., all binders are legal). Clearly, every rectified derivation is sane, but not vice versa, as we might have multiple occurrences of bound variables in Q , such that $[Q]$ is still a fograph.

Lemma 39. Let $\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\} \parallel \Phi$ be a sane derivation.

Then the induced map $[\Phi]: [A] \rightarrow [B]$ is a skew bifibration.

Before we show the proof of this lemma, we introduce another useful concept: the **propositional encoding** A° of a formula A , which is a propositional formula with the property that $[A^\circ] = [A]$. For this, we introduce new propositional variables that have the same names as the (first-order) variables $x \in \text{VAR}$. Then A° is defined inductively by:

$$\begin{aligned} a^\circ &= a & (\forall x A)^\circ &= x \vee A^\circ \\ (A \vee B)^\circ &= A^\circ \vee B^\circ & (\exists x A)^\circ &= x \wedge A^\circ \\ (A \wedge B)^\circ &= A^\circ \wedge B^\circ \end{aligned}$$

Lemma 40. For every formula A , we have $[A^\circ] = [A]$.

Proof. Straightforward induction on A . \square

We use \equiv° to denote the restriction of \equiv to propositional formulas, i.e., the first two lines in (2).

Proof of Lemma 39. First, observe that for every inference rule $r \in \{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\}$ the induced map $[r]: V_{[Q]} \rightarrow V_{[P]}$ defines a existential preserving graph homomorphism $[Q] \rightarrow [P]$ and a fibration on the corresponding binding graphs. **Jui-Hsuan:** we may need to have some explication here. **Lutz:** no Therefore, their composition $[\Phi]$ has the same properties fibration.

For showing that it is also a skew fibration, we construct for Φ its propositional encoding Φ° by translating every line into its propositional encoding. **Jui-Hsuan:** maybe mention that an instance of one of the other rules can be translated into an instance of the same rule. It's trivial but may be worth mentioning. **Lutz:** done below The instances of the rules \widehat{m}_\forall and \widehat{m}_\exists are replaced in two steps by:

$$\begin{aligned} & \frac{S\{(y \vee (Ay)^\circ) \vee (z \vee (Bz)^\circ)\}}{\equiv} \\ & \widehat{ac} \frac{S\{(y \vee z) \vee ((Ay)^\circ \vee (Bz)^\circ)\}}{S\{x \vee ((Ax)^\circ \vee (Bx)^\circ)\}} \end{aligned}$$

and

$$\begin{aligned} & \frac{S\{(y \wedge (Ay)^\circ) \vee (z \wedge (Bz)^\circ)\}}{m} \\ & \widehat{ac} \frac{S\{(y \vee z) \wedge ((Ay)^\circ \vee (Bz)^\circ)\}}{S\{x \wedge ((Ax)^\circ \vee (Bx)^\circ)\}} \end{aligned}$$

respectively, where \widehat{ac} is a ac that renames the variables—the propositional variable, as well as the first-order variable of the same name—as everything is rectified, there is no ambiguity

here. Any instance of a rule w , ac , m , or \equiv is translated to an instance of the same rule, and \widehat{c}_\forall is translated to \widehat{ac} .

This gives us a derivation $\{w, ac, \widehat{ac}, m, \equiv\} \parallel_{B^\circ}^{\Phi^\circ}$ such that $[\Phi^\circ] = [\Phi]$. It has been shown in [23] that $[\Phi^\circ]$ is a skew fibration (see also [12], [?], [15]). Hence, $[\Phi]$ is a skew fibration. \square

B. From Skew Bifibrations to Contraction and Weakening

Lemma 41. *Let \mathcal{A} and \mathcal{B} be fographs, let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a skew bifibration, and let A and B be formulas with $\llbracket A \rrbracket = \mathcal{A}$ and $\llbracket B \rrbracket = \mathcal{B}$. Then there are derivations*

$$\frac{A}{\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\} \parallel_{\widehat{B}}^{\widehat{\Phi}}} \quad \text{and} \quad \frac{A\sigma_\varphi}{\{w, ac, \widehat{c}_\forall, m, m_\forall, m_\exists, \equiv\} \parallel_{\widehat{B}}^{\widehat{\Phi}}}$$

such that $[\widehat{\Phi}] = \varphi$ and $\widehat{\Phi}$ is a rectification of Φ , and σ_φ is the substitution induced by φ .

In the proof of this lemma, we make use of the following

concept: Let $s \parallel_{\Psi}^P$ be a derivation where P and Q are propositional formulas (possibly using variable $x \in \text{VAR}$ at the places of atoms). We say that Ψ can be *lifted* to S' if there are (first-order) formulas C and D such that $P = C^\circ$ and $Q = D^\circ$ and

there is a derivation $s' \parallel_{\Psi'}^D$.

Proof of Lemma 41. By Lemma 40 we have $\mathcal{A} = \llbracket A^\circ \rrbracket$ and $\mathcal{B} = \llbracket B^\circ \rrbracket$. Let $V'_B \subseteq V_B$ be the image of φ , and let \mathcal{B}_1 be the subgraph of \mathcal{B} induced by V'_B . Hence, we have two maps $\varphi'': \mathcal{A} \rightarrow \mathcal{B}_1$ being a surjection and $\varphi': \mathcal{B}_1 \rightarrow \mathcal{B}$ being an injection that reflects edges. **Jui-Hsuan:** what do you mean by "reflect edges"? **Lutz:** edge downstairs implies edge upstairs Both, φ' and φ'' remain skew bifibrations. Let us first look at φ' . Let \tilde{B}_1 be the propositional formula obtained from B° by removing all atoms that are not represented by vertices in V'_B . Then $\llbracket \tilde{B}_1 \rrbracket = \mathcal{B}_1$. By [23, Proposition 7.6.1], we have

a derivation $\{w, \equiv\} \parallel_{\tilde{B}_1}^{\Phi_1^\circ}$. A subformula of B° is called *weak* if

it has been introduced by a weakening. All subformulas of a weak formula are also weak. Two weak subformulas B' and B'' of B° form a *weak pair* if $B^\circ \equiv S\{B' \vee B''\}$ for some context $S\{\cdot\}$. We can assume without loss of generality that whenever weak subformulas B' and B'' form a weak pair, they have been introduced by the same instance of w in Φ_1° .⁴ Now we show that Φ_1° can be lifted. For this, observe that whenever a weakening in Φ_1° deletes an atom $x \in \text{VAR}$, it must also delete all atoms in the scope of the corresponding quantifier, because φ' is a fibration on the binding graph. Hence, each line

⁴If Φ_1° is not of that shape, it can brought into this form by simple rule permutations, and then collapsing several weakenings into a single one.

in Φ_1° is the propositional encoding P° of a first-order formula P . We now have to show that each instance of w is indeed a correct application in first-order logic. For this we have to inspect the cases a weakening happens inside a subformula $x \vee C$ or $x \wedge C$ in Φ_1° . There are the following cases:

$$\frac{S\{x \vee C\}}{S\{x \vee D \vee C\}} \quad \frac{S\{x \wedge C\}}{S\{x \wedge (D \vee C)\}} \quad \frac{S\{x \wedge C\}}{S\{(x \vee D) \wedge C\}}$$

In the first case the weakening happens inside the scope of a \forall -quantifier, and in the second case inside the scope of a \exists -quantifier. Both cases are unproblematic in first-order logic. However, in the third case an \exists -quantifier would be transformed into an \forall -quantifier. But as φ has to preserve existentials, this third case cannot occur. Thus we have a first

order derivation $\{w, \equiv\} \parallel_{\Phi_1}^{\tilde{B}_1}$ with $B_1^\circ = \tilde{B}_1$.

Let us now look at φ'' . Let $\mathcal{A}_1 = \mathcal{A}\sigma_\varphi$ be the graph obtained from \mathcal{A} by applying σ_φ to all the labels. Note that \mathcal{A}_1 is not necessarily a fograph, as binders might be illegal. But it still is a cograph, and we have a surjective skew fibration $\varphi'': \mathcal{A}_1 \rightarrow \mathcal{B}_1$ that preserves the labels. Therefore, by [?,

Proposition 7.5], there is a derivation $\{ac, m, \equiv\} \parallel_{\Phi_2^\circ}^{\Phi_1^\circ}$, where

$A_1^\circ = A^\circ\sigma_\varphi$ is the result of applying σ_φ to A° . Note that $A_1^\circ = (A\sigma_\varphi)^\circ$ and B_1° are both propositional encodings. We plan to show that Φ_2 can be lifted to $\{ac, c_\forall, m, m_\forall, m_\exists, \equiv\}$. However, observe that not every formula occurring in Φ_2 is a propositional encoding. There are two reasons for this: (i) we might have $P \equiv^\circ Q$ where P is a propositional encoding but Q is not, and (ii) the rule ac can duplicate an atom $x \in \text{VAR}$. Let us write ac_x for such instances. To address (i), we consider here formulas equivalent modulo \equiv , always knowing that we can add instances of \equiv as needed.⁵ **Jui-Hsuan:** this does not seem clear to me. What if from A_1° to B_1° there are just a bunch of \equiv° ? What do we do in this case? **Lutz:** see footnote To address (ii), we apply a permutation argument, permuting all instances of ac_x up until they either reach the top of the derivation or an instance of m which separates the two atoms in the premise. More precisely, we consider the following inference rule

$$ac_x^\equiv \frac{S_0\{S_1\{x\} \vee S_2\{x\}\}}{S\{x\}} \quad (4)$$

where $S_1\{\cdot\} \equiv \{\cdot\} \vee E$ and $S_2\{\cdot\} \equiv \{\cdot\} \vee F$ and $S\{\cdot\} \equiv S_0\{\{\cdot\} \vee E \vee F\}$ for some formulas E and F , where E or F or both might be empty. The rule ac_x^\equiv permutes over \equiv , ac , and other instances of ac_x^\equiv , and over instances of m if they

⁵Note that whenever we have formulas P and Q with $P^\circ \equiv^\circ Q^\circ$ then $P \equiv Q$.

occur inside S_0 or S_1 or S_2 . The only situation in which ac_x^\equiv cannot be permuted up is the following:

$$\text{ac}_x^\equiv \frac{\text{m} \frac{S\{(R_1\{x\} \wedge C) \vee (R_2\{x\} \wedge D)\}}{S\{(R_1\{x\} \vee R_2\{x\}) \wedge (C \vee D)\}}}{S\{R\{x\} \wedge (C \vee D)\}} \quad (5)$$

We can therefore assume that all instances of ac_x , that contract an atom $x \in \text{VAR}$ are either at the top of Φ_2° or below a m-instance as in (5). We now lift Φ_2° to $\{\text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\}$, proceed by induction on the height of Φ_2° , beginning at the top, making a case analysis on the topmost rule that is not a \equiv .

- ac_x : We know that the premiss of (4) is a propositional encoding. Hence, $S_1\{\cdot\} = \{\cdot\} \vee E^\circ$ and $S_2\{\cdot\} = \{\cdot\} \vee F^\circ$ and both x are universals, and $E^\circ \vee F^\circ$ contains all occurrences of x bound by that universal. We have the following subcases:

- E and F are both non-empty: We have

$$\text{ac}_x^\equiv \frac{S^\circ\{(x \vee E^\circ) \vee (x \vee F^\circ)\}}{S^\circ\{x \vee (E^\circ \vee F^\circ)\}}$$

which can be lifted to

$$\text{m}_\forall \frac{S\{(\forall x.E) \vee (\forall x.F)\}}{S\{\forall x.(E \vee F)\}}$$

where $S^\circ\{\cdot\}$, E° , F° are the propositional encodings of $S\{\cdot\}$, E , F , respectively.

- E° is empty and F° is non-empty: We have

$$\text{ac}_x^\equiv \frac{S^\circ\{x \vee (x \vee F^\circ)\}}{S^\circ\{x \vee F^\circ\}}$$

which can be lifted to

$$\text{c}_\forall \frac{S\{\forall x.\forall x.F\}}{S\{\forall x.F\}}$$

- E° is non-empty and F° is empty: This is similar to the previous case.
- E° and F° are both empty: This is impossible as the premise would not be a propositional encoding.
- ac (contracting an ordinary atom): This can trivially be lifted.
- m that is not in the situation of (5): Then now encoding of a quantifier is affected and the instance of m can be lifted. **TODO: medial permutation!!!!**
- m/ac_x as in situation (5): We must have $R_1\{x\} \equiv x \vee E$ for some E and $R_2\{x\} \equiv x \vee F$ for some F with $R\{x\} \equiv x \vee E \vee F$. Otherwise, the application of ac_x^\equiv would not be correct. We have the following four cases:
 - E and F are both non-empty: Then (5) is (modulo omitted applications of \equiv):

$$\text{ac}_x^\equiv \frac{\text{m} \frac{S\{((x \vee E) \wedge C) \vee ((x \vee F) \wedge D)\}}{S\{((x \vee E) \vee (x \vee F)) \wedge (C \vee D)\}}}{S\{(x \vee E \vee F) \wedge (C \vee D)\}}$$

which can be lifted to

$$\text{m}_\forall \frac{\text{m} \frac{S\{((\forall x.E) \wedge C) \vee ((\forall x.F) \wedge D)\}}{S\{((\forall x.E) \vee (\forall x.F)) \wedge (C \vee D)\}}}{S\{(\forall x.(E \vee F)) \wedge (C \vee D)\}}$$

Jui-Hsuan: maybe need some words to exclude the case in which C (or D) is a propositional variable. **Lutz:** shit. (you mean a “first order variable”) this actually can happen. then we have another m_\exists

- E is empty and F is not: Then (5) becomes

$$\text{ac}_x^\equiv \frac{\text{m} \frac{S\{(x \wedge C) \vee ((x \vee F) \wedge D)\}}{S\{(x \vee (x \vee F)) \wedge (C \vee D)\}}}{S\{(x \vee F) \wedge (C \vee D)\}}$$

The conclusion is the propositional encoding of $S\{(\forall x.F) \wedge (C \vee D)\}$ and the premise is the propositional encoding of $S\{(\exists x.C) \vee ((\forall x.F) \vee D)\}$. Also note that no m -instance can break up the conjunction in $x \wedge C$ in the premise. Hence, φ maps an existential to a universal, which is ruled out by the definition. Hence, this case cannot occur.

- E is non-empty and F is empty: This case is similar to the previous subcase.
- E and F are both empty: Then (5) is

$$\text{ac}_x^\equiv \frac{\text{m} \frac{S\{(x \wedge C) \vee (x \wedge D)\}}{S\{(x \vee x) \wedge (C \vee D)\}}}{S\{x \wedge (C \vee D)\}}$$

which can be lifted immediately to

$$\text{m}_\exists \frac{S\{(\exists x.C) \vee (\exists x.D)\}}{S\{\exists x.(C \vee D)\}}$$

Thus Φ_2° can be lifted to $\{\text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\} \parallel \Phi_2$. We construct B_1

Φ by composing Φ_2 and Φ_1 . Then $\widehat{\Phi}$ can be constructed by rectifying Φ , where the variables to be used in A are already given. That $\varphi = \llbracket \widehat{\Phi} \rrbracket$ follows immediately from the construction. \square

X. SUMMARY AND PROOF OF MAIN RESULT

The only theorem of Section VI that has not yet been proved is Theorem 16 establishing the full correspondence between decomposed proofs in KS1 and combinatorial proofs. We show the proof here, by summarizing the results of the previous two Sections VIII and IX.

Proof of Theorem 16. First, assume we have a combinatorial proof $\varphi: \mathcal{C} \rightarrow \mathcal{A}$ be a combinatorial proof and a formula A with $\mathcal{A} = \llbracket A \rrbracket$. Let C be a formula with $\llbracket C \rrbracket = \mathcal{C}$, and let σ_φ be the substitution induced by φ . By Lemma 41 there is a derivation

$$\frac{C\sigma_\varphi}{\{\text{w}, \text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\} \parallel \Phi_2} A$$

Since \mathcal{C} is a fonet, we have by Theorem 36 a derivation

$$\begin{array}{c} \text{t} \\ \text{MLS1}^x \parallel \Phi'_1 \\ \mathcal{C} \end{array}$$

This derivation remains valid if we apply the substitution σ_φ to every line in Φ'_1 , yielding the derivation Φ_1 of $\mathcal{C}\sigma_\varphi$ as desired.

Conversely, assume we have a decomposed derivation

$$\begin{array}{c} \text{t} \\ \text{MLS1}^x \parallel \Phi_1 \\ A' \\ \{w, ac, m, m_\forall, m_\exists, \equiv\} \parallel \Phi_2 \\ A \end{array} \quad (6)$$

Then we can transform Φ_1 into a rectified form $\widehat{\Phi}_1$, proving \widehat{A}' . By Theorem 30, the linked fograph $\llbracket \widehat{\Phi}_1 \rrbracket = \langle \llbracket \widehat{A}' \rrbracket, \sim_{\widehat{\Phi}_1} \rangle$ is a fonet. Then, by Lemma 38, there is a rectified derivation

$$\begin{array}{c} \widehat{A}' \\ \{w, ac, c_\forall, m, m_\forall, m_\exists, \equiv\} \parallel \widehat{\Phi}_2 \end{array} \text{ whose induced map } [\widehat{\Phi}_2]: \llbracket \widehat{A}' \rrbracket \rightarrow \widehat{A}$$

$\llbracket \widehat{A} \rrbracket$ is the same as the induced map $[\Phi_2]: \llbracket A' \rrbracket \rightarrow \llbracket A \rrbracket$ of Φ_2 . By Lemma 39, this map is a skew bifibration. Hence, we have a combinatorial proof $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$ with $\mathcal{C} = \llbracket A' \rrbracket$.

[[Lutz: shit, something's wrong...]] \square

Note that Theorem 16 shows at the same time soundness, completeness, and full completeness, as

- 1) every proof in KS1 can be translated into a combinatorial proof, and
- 2) every combinatorial proof is the image of a KS1-proof under that translation.

XI. CONCLUSION

In this paper we have uncovered a close correspondence between first-order combinatorial proofs and decomposed deep inference derivations of system KS1, and we have shown that every proof in KS1 has such a decomposed form.

The most surprising discovery for us was that all technical difficulties in our work could be reduced (in a non-trivial way) to the propositional setting.

The obvious next step in our research is to investigate proof composition and normalisation of first-order combinatorial proofs. Even in the propositional setting, the normalization of combinatorial proofs is underdeveloped. There exist two different procedures for cut elimination [12], [?], but both have their insufficiencies, and there is no general theory.

[[Lutz: do we want/can say more here?]]

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936 A. Unification Nets

937 **[[TODO:]]**

938 In this paragraph, we associate each formula A with its
 939 **formula tree** $\mathcal{F}(A)$, a directed tree with leaves labelled by
 940 atoms, internal nodes labelled by connectives and quantifiers,
 941 and edges directed from leaves to the root. For a sequent
 942 $\Gamma = A_1, \dots, A_n$, we denote with $\mathcal{F}(\Gamma)$, the forest formed by
 943 $\mathcal{F}(A_1), \dots, \mathcal{F}(A_n)$, i.e., the disjoint union of $\mathcal{F}(A_i)$'s. The
 944 **roots** of $\mathcal{F}(\Gamma)$ are the roots of A_i 's

945 Let Γ be a sequent in MLL1^\times . Consider the forest $\mathcal{F}(\Gamma)$.
 946 A **link** on Γ is a pair of leaves whose atoms are pre-dual. A
 947 **linking** λ on Γ is a set of disjoint links such that each leaf
 948 of $\mathcal{F}(\Gamma)$ is either labelled by t or in exactly one link. Similar
 949 to the set of links in linked fographs, a linking can be seen
 950 as a unification problem, and a **dualizer** δ of the linking λ is
 951 an assignment unifying all the links in λ . There exists a **most**
 952 **general dualizer** of λ if λ has a dualizer. **[[Jui-Hsuan: Now**
 953 **I use the same terminology as for linked fographs]]** **[[Lutz:**
 954 **use δ for the dualizer (or even better, make it a macro)]]** A
 955 **dependency** is a pair $(\bullet\exists x, \bullet\forall y)$ of nodes such that the most
 956 general dualizer assigns to x a term containing y .

957 Let λ is a linking on Γ that has a dualizer. The **unification**
 958 **structure** $\mathcal{U}(\lambda)$ associated with λ is the forest $\mathcal{F}(\Gamma)$ together
 959 with an undirected edge between leaves l and l' for every link
 960 $\{l, l'\}$ in λ and a directed edge from $\bullet\exists x$ to $\bullet\forall y$ for every
 961 dependency $(\bullet\exists x, \bullet\forall y)$.

962 A **switching graph** of a unification structure $\mathcal{U}(\lambda)$ is any
 963 derivative of $\mathcal{U}(\lambda)$ obtained by keeping only one edge into
 964 each \vee and \forall and undirecting remaining edges. A linking is
 965 **correct** if it is unifiable and all of the switching graphs of its
 966 associated unification structure are acyclic.

967 **Definition 42.** A **unification net** on a sequent Γ is a correct
 968 linking on Γ .

969 B. Translation between Unification Nets and MLL1^\times

970 **[[TODO:]]**

971 **Theorem 43.** If a sequent is provable in MLL1^\times , then there
 972 exists a unification net on it.

973 *Proof.* We proceed by induction on the proof of $\vdash \Gamma$ in
 974 MLL1^\times , making a case analysis on the bottommost rule
 975 instance:

- 976 • $\text{ax} \frac{}{\vdash a, \bar{a}}$: the linking $\{a, \bar{a}\}$ is correct.
- 977 • $\text{t} \frac{}{\vdash t}$: the empty linking is correct.
- 978 • $\text{mix} \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$: By induction hypothesis, there is a
 979 correct linking on Γ and another one on Δ , their union
 980 giving a correct linking on Γ, Δ .

- $\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$: By induction hypothesis, there is a correct
 981 linking on Γ, A, B , and it is correct on $\Gamma, A \vee B$ as well. 982
- $\wedge \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$: By induction hypothesis, there is a
 983 correct linking on Γ, A and another one on B, Δ , their
 984 union giving a correct linking on $\Gamma, A \wedge B, \Delta$. 985
- $\exists \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A}$: By induction hypothesis, there is a correct
 986 linking λ on $\Gamma, A[x/t]$. For each atom in $\Gamma, A[x/t]$, there
 987 is a corresponding atom in $\Gamma, \exists x.A$. There is therefore a
 988 linking λ' on $\Gamma, \exists x.A$ obtained from λ via this correspon-
 989 dence, and it is not difficult to check that λ' is correct as
 990 well. 991
- $\forall \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A}$ (x not free in Γ) : By induction hypothesis,
 992 there is a correct linking on Γ, A , and it is easy to see
 993 that it is a correct linking on $\Gamma, \forall x.A$ as well. 994

This allows to define a translation $[\cdot]$ from proofs in MLL1^\times
 995 to unification nets. \square 996

Theorem 44. Any unification net can be obtained via the
 997 translation $[\cdot]$ given in Theorem 43. 998

To prove this theorem, we need some basic lemmas about
 999 connected components in switching graphs of unification nets. 1000

Lemma 45. The number of connected components of an acyclic
 1001 graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is equal to $|E_{\mathcal{G}}| - |V_{\mathcal{G}}|$. 1002

Proof. By a straightforward induction on $|V_{\mathcal{G}}|$. \square 1003

Lemma 46. The number of connected components is the same
 1004 for any switching graph of a unification net. 1005

Proof. An immediate consequence of Lemma 45. \square 1006

In the proof, we also use the notion of **frame** introduced by
 1007 Hughes in [37]. 1008

Definition 47. Let λ be a unification net on an MLL1^\times sequent
 1009 Γ . We define the **frame** of λ by exhaustively applying the
 1010 following subformula rewriting steps, to obtain a linking λ_m 1011
 1012 on an $\text{MLL} + \text{mix}$ sequent Γ_m :

- 1) **Encode dependencies as fresh links.** For each depen-
 1013 dency $\exists x \rightarrow \forall y$, with corresponding subformulas $\exists x.A$ 1014
 and $\forall y.B$, we add a fresh link as follows. Let P be a fresh
 1015 (nullary) predicate symbol. Replace $\exists x.A$ with $P \wedge \exists x.A$ 1016
 and $\forall y.B$ with $\bar{P} \vee \forall y.B$, and add an axiom link between
 1017 P and \bar{P} . 1018
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers.
 1019 (We no longer need their leaps since they are encoded
 1020 as links in step 1. 1021
- 3) **Simplify atoms.** After step 2, replace every predicate
 1022 $Pt_1 \dots t_n$ with a nullary predicate symbol P . 1023

Note that the linking λ_m is a valid $\text{MLL} + \text{mix}$ proof net. 1024

Lemma 48. Suppose that λ is a MLL + mix proof net which is connected and such that any of its switching graphs is not connected. Then there exists a \vee node in $\mathcal{U}(\lambda)$ such that λ is correct on the sequent Γ' obtained from Γ by replacing this \vee by a \wedge .

Proof. Suppose that such a \vee node does not exist. Then it is clear that for any two nodes, there exists a switching graph containing a path between them and this path corresponds to an AE -path in [38]. By [38, Propostion 3], λ corresponds to a sequent proof that does not use mix, which implies the connectedness of the switching graphs of λ . Contradiction. ■ **TO CHECK:** ■

Lemma 49. Suppose that λ is a MLL1^X proof net which is connected and such that any of its switching graphs is not connected. Then there exists a \vee node in $\mathcal{U}(\lambda)$ such that λ is correct on the sequent Γ' obtained from Γ by replacing this \vee by a \wedge .

Proof. Consider the frame λ_m of λ . The number of any switching graph of $\mathcal{U}(\lambda)$ is equal to that of $\mathcal{U}(\lambda_m)$. Apply Lemma 48 and it is clear that such \vee cannot be one of the fresh \vee 's added during the frame construction. ■

We can now give the proof of Theorem 44.

Proof of Theorem 44. Let λ be a unification net on Γ . We proceed by induction on the number of connected components of the unification structure $\mathcal{U}(\lambda)$:

- If there is only one connected component, we proceed by induction on the number k of connected components of any switching graph of $\mathcal{U}(\lambda)$. If $k = 1$, we obtain a proof Φ in MLL1^X such that $[\Phi] = \lambda$ by applying [37, Theorem 3]. If $k > 1$, using the Lemma 49, we obtain a sequent Γ' on which λ is correct by transforming a \vee node into a \wedge . By induction hypothesis, there is a proof Φ' in MLL1^X whose translation is λ . By considering the \wedge rule instance corresponding to the \wedge node in Φ' , we

$$\text{have: } \Phi' = \wedge \frac{\frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A \wedge B, \Delta_2}}{\vdash \Gamma'}. \text{ We can thus obtain}$$

$$\text{a proof } \Phi \text{ of } \Gamma: \Phi = \frac{\text{mix} \frac{\frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A, B, \Delta_2}}{\vdash \Delta_1, A \vee B, \Delta_2}}{\vdash \Gamma}$$

$$[\Phi] = \lambda.$$

- If there are $n > 1$ connected components, add a fresh \vee node connecting two formulas belonging to different

connected components of Γ to get a new sequent Γ' . Define a unification net λ' on Γ' using the same linking as λ . By induction hypothesis, since $\mathcal{U}(\lambda')$ has $n - 1$ connected components, there is a MLL1^X proof Φ' such that $[\Phi'] = \lambda'$. Consider the \vee rule instance corresponding to the \vee node in question. Since \vee is invertible, we can permute downwards this rule instance until it becomes the last rule of the proof (note that this transformation does not change the image of the proof by the translation $[\cdot]$) to get a new proof Φ'' of Γ' . By deleting the last rule instance from Φ'' , we obtain a proof Φ of Γ such that $[\Phi] = \lambda$. ■ **TO CHECK:** ■

We proceed by induction on the number of connectives in Γ . In the base case, Γ is of the form

$$p_1(t_{11}, \dots, t_{1n_1}), \overline{p_1}(t_{11}, \dots, t_{1n_1}), \dots, p_k(t_{k1}, \dots, t_{kn_k}), \overline{p_k}(t_{k1}, \dots, t_{kn_k}), \underbrace{t, \dots, t}_{m \text{ times}}$$

and λ is the linking $\{(a_1, \overline{a_1}), \dots, (a_k, \overline{a_k})\}$, where $a_i = p_i(t_{i1}, \dots, t_{in_i})$, which equals to $[\Pi]$, where Π is the proof consisting of m instances of the t rule, n instances $\text{ax} \frac{}{\vdash a_i, \overline{a_i}}$ of the ax rule, and followed by $m + k - 1$ instances of the mix rule.

Now we consider the inductive cases:

- $\Gamma = \Delta, A \vee B$: Let $\Gamma' = \Delta, A, B$. Define λ' on Γ' using the same links as λ by identifying the leaves of $\mathcal{F}(\Gamma')$ with those of $\mathcal{F}(\Gamma)$. We now check that λ' is a unification net:
 - The most general dualizer of λ is also the most general dualizer of λ' as they correspond to the same unification problem. Hence, the unification structure $\mathcal{U}(\lambda')$ is equal to the restriction of $\mathcal{U}(\lambda)$ to the nodes of $\mathcal{F}(\Gamma')$.
 - Every switching graph of λ' is acyclic: if there were some switching graph of $\mathcal{U}(\lambda')$ containing a cycle, it would induce a switching graph of $\mathcal{U}(\lambda)$ containing also a cycle, by adding an edge from the root of $\mathcal{F}(A)$ to the \vee node in question.
- $\Gamma = \Delta, \forall x.A$: Let $\Gamma' = \Delta, A$. Define λ' on Γ' using the same links as λ . We now check that λ' is a unification net:
 - The most general dualizer of λ is also the most general dualizer of λ' as they correspond to the same unification problem.
 - Every switching graph of $\mathcal{U}(\lambda')$ is acyclic: if there were some switching graph of $\mathcal{U}(\lambda')$ containing a cycle, it would induce a switching graph of $\mathcal{U}(\lambda)$ containing also a cycle, by adding an edge from the root of $\mathcal{F}(A)$ to the \forall node in question.
- $\mathcal{F}(\Gamma)$ has a root $\exists x$ with no outgoing dependency edge:

■

C. Translation between Unification Nets and Fonets

1110 **TODO:**

1111 XII. FIRST-ORDER COMBINATORIAL PROOFS

1112 A. First-order Logic

1113 In this paper, we also use some *deep inference* [36] rules
1114 that are in the following form:

$$\frac{\vdash S\{A\}}{\vdash S\{B\}}$$

1116 where $S\{ \}$ stands for a **context**, which corresponds to a
1117 sequent with a hole taking the place of an atom, and $S\{A\}$
1118 represents the sequent or formula obtained by replacing the
1119 hole in $S\{ \}$ with the formula A . Formally,

$$1120 C ::= \Box \mid A \vee C \mid C \wedge A \mid \exists x C \mid \forall x C.$$

$$1121 S ::= C \mid A, S \mid S, A$$

1122 where A is a formula. The above rule can be thus seen as the
1123 rewriting rule $A \rightarrow B$.

1124 We use the notation $\parallel_{\mathcal{P}}^A$ for denoting that there is a
1125 derivation from premise $\vdash S\{A\}$ to conclusion $\vdash S\{B\}$ in
1126 system \mathcal{P} for any context S .

1127 B. Graphs

1128 C. First-order combinatorial proofs

1129 D. MLL1^X and Unification Nets

1130 In MLL1^X, terms, atoms, formulas are defined as in first-
1131 order logic. For simplicity, we choose to use \vee and \wedge instead of
1132 \wp and \otimes which are generally used in the presentation of linear
1133 logic. A formula A is identified with its **formula tree** $\mathcal{F}(A)$,
1134 a directed tree with leaves labelled by atoms, internal nodes
1135 labelled by connectives and quantifiers, and edges directed
1136 from leaves to the root. A **sequent** Γ is simply a disjoint union
1137 of formulas. We write comma for disjoint union.

Sequents are proved using the inference rules of MLL1^X:

$$\begin{array}{c} \frac{}{\vdash A, \neg A} \text{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta} \text{cut} \\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \wedge \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A} \forall (x \notin fv(\Gamma)) \quad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A} \exists \end{array}$$

Fig. 7. Sequent calculus for MLL1^X

1138 We also consider the mix rule:

$$1139 \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \text{mix}$$

1141 Let Γ be a sequent in MLL1 + mix. A **link** on Γ is a pair
1142 of leaves whose atoms are pre-dual. A **linking** on Γ is a
1143 set of disjoint links such that each leaf of Γ is in exactly

one link. Similar to the set of links in the linked fograph, a
linking can be seen as a unification problem, and a link is said
unifiable if the corresponding unification problem is solvable.
Dependencies are defined as previously.

1144 XIII. FROM FIRST-ORDER LOGIC TO COMBINATORIAL 1145 PROOFS

1146 A. Decomposition Theorem

Consider the following deep inference rules [36]:

$$\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}} \text{c} \quad \frac{\vdash S\{f\}}{\vdash S\{A\}} \text{w}$$

Note that the ctr (resp. wk) rule in LK is derivable in $\{c, \vee\}$
(resp. $\{w, f\}$) and that c and w rules permute downwards with
the non-structural rules of LK.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \frac{}{\vdash \Gamma, A} \text{c}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \frac{}{\vdash \Gamma, A} \text{w}$$

We also give an example to show how rule permutation
works:

$$\frac{\frac{\Gamma, A \vee A}{\Gamma, A} \text{c} \quad \Delta, B}{\Gamma, \Delta, A \wedge B} \wedge \rightsquigarrow \frac{\Gamma, A \vee A \quad \Delta, B}{\Gamma, \Delta, (A \vee A) \wedge B} \wedge \frac{}{\Gamma, \Delta, A \wedge B} \text{c}$$

We want to establish the following theorem:

Theorem 50. *Let Γ be a sequent. Then there is a proof of Γ in LK + mix iff there is a proof of some sequent Δ in MLL1 + mix and a derivation from Δ to Γ consisting of the c and w rules only.*

Proof. (\Rightarrow) This direction comes from the above observation:
it suffices to permute downwards all the instances of the c and
w rules.

(\Leftarrow) We regard the proof in MLL1 + mix as a proof in
LK + mix. Then we put the derivation consisting of only c
and w under the proof in LK + mix. Now we try to permute
all the instances c and w upwards with the rules of LK and
mix. For the c part, the only non-trivial case is the permutation
with the \vee rule where the formula generated is $A \vee A$.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \frac{}{\vdash \Gamma, A} \text{ctr} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr}$$

In this case, the permutation of this instance of c stops and
we continue with the remaining instances.

For the w part, the only non-trivial case is the permutation
with the f rule (or the instance of wk where f is introduced):

$$\frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \frac{}{\vdash \Gamma, A} \text{w} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk}$$

1181 In this case, the permutation of this instance of w stops and
1182 we continue with the remaining instances. \square

1184 D. Hughes proves in [37] the soundness and completeness of
1185 unification nets with respect to MLL1 + mix. In the following,
1186 we establish the equivalence between unification nets and
1187 fonets.

1188 B. Equivalence between unification nets and fonets

1189 In the following, we usually confound a vertex with its label.

1190 **Definition 51.** A *switching path* of a unification structure
1191 $U(\lambda)$ is a path in a switching graph of $U(\lambda)$.

1192 **Definition 52.** A *switching path* of a formula tree $\mathcal{F}(A)$ is a
1193 path in $\mathcal{F}(A)$ that does not go through both incoming edges
1194 of a \vee .

1195 **Proposition 53.** *In a formula tree, the root is connected to*
1196 *every vertex by a switching path.*

1197 Now we give the key proposition relating a fograph to its
1198 corresponding formula tree.

1199 **Proposition 54.** *Let u and v be two distinct vertices of a*
1200 *fograph $\llbracket(\sqcup)A$, then we have the equivalence between:*

- 1201 • u and v are adjacent in $\llbracket(\sqcup)A$
- 1202 • u and v are connected by a switching path of $\mathcal{F}(A)$, and
1203 if one of them is a universal quantifier, then the other is
1204 not a descendant of the former.

1205 *Proof.* By induction on A .

- 1206 • If A is an atom, trivial.
- 1207 • If $A = A_1 \wedge A_2$, then we distinguish two cases:
 - 1208 – u and v are both in A_1 (resp. A_2): trivial by the
1209 induction hypothesis.
 - 1210 – one of them is in A_1 and the other is in A_2 : they are
1211 adjacent in $\llbracket(\sqcup)A$ by definition. By Proposition 53,
1212 the one in A_1 (resp. A_2) is connected to the vertex
1213 representing A_1 (resp. A_2) by a switching path.
1214 Together with the two edges incident to $A_1 \wedge A_2$,
1215 we obtain a switching path connecting u and v .
- 1216 • If $A = A_1 \vee A_2$, then we distinguish two cases:
 - 1217 – u and v are both in A_1 (resp. A_2): trivial by the
1218 induction hypothesis.
 - 1219 – one of them is in A_1 and the other is in A_2 : they are
1220 not adjacent in $\llbracket(\sqcup)A$ by definition. It is clear that
1221 they are not connected by a switching path.
- 1222 • If $A = \exists x A'$, then we distinguish two cases:
 - 1223 – u and v are both in A' : trivial by the induction
1224 hypothesis.
 - 1225 – one of them is $\exists x$ and the other is in A' : trivial by
1226 Proposition 53
- 1227 • If $A = \forall x A'$, then we distinguish two cases:
 - 1228 – u and v are both in A' : trivial by the induction
1229 hypothesis.

- one of them is $\forall x$ and the other is in A' : they are
not adjacent in $\llbracket(\sqcup)A$ by definition and it is clear that
the former is a descendant of $\forall x$. \square

Proposition 55. *If there exists an induced bimatching of the
linked fograph $G = \llbracket(\sqcup)A$, then there exists a switching graph
of the corresponding unification net which contains a cycle.*

Proof. Suppose that there exists a set W inducing a bimat-
ching in G . Then (W, E_G) and (W, L_G) are matchings.

Let E_W (resp. L_W) be the restriction of E_G (resp. L_G) to W .
If $E_W \cap L_W \neq \emptyset$, then there exist u and v such that $uv \in E_G$
and $uv \in L_G$. By Proposition 54, there exists a switching
path of the formula tree of A . Together with the leap uv , this
path induces a cycle in a switching graph of the corresponding
unification structure.

We can now suppose that E_W and L_W are disjoint. It is not
difficult to see the existence of an alternating and elementary
cycle in the bicoloured graph $(W, E_W \uplus L_W)$, i.e. a cycle of
which the edges are alternately in E_W and L_W and containing
no two equal vertices. By Proposition 54, this cycle induces a
cycle in the unification structure. Now we want to construct a
switching graph that contains this cycle.

Consider a universal quantifier $\forall x$. If $\forall x \notin W$, then we keep
the incoming edge from its direct subformula and remove all
the dependencies. Otherwise, since (W, L_G) is a matching,
there exists a unique existential quantifier adjacent to $\forall x$
and we keep thus the corresponding edge in the unification
structure.

Now consider a \vee . We distinguish three cases:

- the cycle goes through none of the two branches (incom-
ing edges) of the \vee : we can choose an arbitrary switching
for this \vee
- the cycle goes through exactly one branch: we choose the
corresponding switching
- the cycle goes through both branches: this means that
there exist $v_L \in W$ (resp. v_R) in the left (resp. right)
branch, $u_L, u_R \in W$, such that $u_L v_L, u_R v_R \in E_W$
and that the corresponding switching path from u_L to
 v_L (resp. from u_R to v_R) goes through the left (resp.
right) edge of \vee .

The red (resp. blue) path is the switching path corre-
sponding to the edge $u_L v_L$ (resp. $u_R v_R$) in E_W .

It is clear that u_L (resp. u_R) is not in the branches of the
 \vee . Otherwise, there will be no switching path from u_L
to v_L

By Proposition 54, we know that u_L and u_R are not
universal quantifiers which are ancestors the \vee and that
there exist one switching path from u_L to v_L and one
from u_R to v_R . In particular, there exist one switching
path from u_L to the \vee and one from the \vee to v_R , and
by concatenating the two, we obtain a switching path
from u_L to v_R . By Proposition 54, u_L and v_R are thus
adjacent in (W, E_G) , which is impossible since (W, E_W)
is a matching.

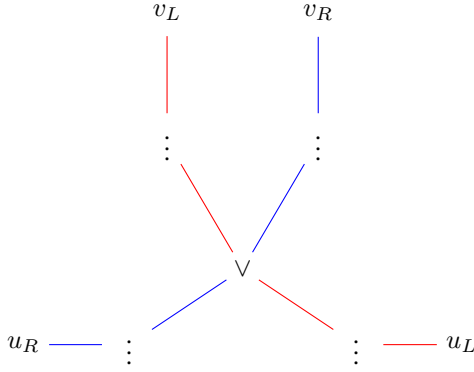


Fig. 8. A schema showing that the two branches of the same \vee cannot be used in the cycle at the same time.

Notice that the switching paths here are in the underlying formula tree. We have to verify that they are compatible with the choices of switching made for universal quantifiers. That is, if $uv \in E_W$, then for all the universal quantifiers $\forall x$ on the switching path (in the formula tree), we have chosen in the first part of the proof to keep the only edge from the child of $\forall x$ to itself. In fact, if there exists a universal quantifier $w \in W$ on the switching path $u \rightarrow v$, then one of u and v is not a descendant of w . Moreover, if u (resp. v) is a universal quantifier, then w is not in its scope. By Proposition 54, $\{wu, wv\} \cap E_W \neq \emptyset$, which is impossible since (W, E_W) is a matching. We have thus constructed a switching graph containing this cycle. \square

Proposition 56. *If one of the switching graphs of the unification structure of A contains a cycle or is not connected, then there exists an induced bimatching of the corresponding linked fograph.*

Proof. We use frames introduced by D. Hughes in Section 4 of [37].

Definition 57. Let θ be a unification structure on an MLL¹ sequent Γ . We define the **frame** of θ by exhaustively applying the following subformula rewriting steps, to obtain a proof structure θ_m on an MLL sequent Γ_m :

- 1) **Encode dependencies as fresh links.** For each dependency $\exists x \rightarrow \forall y$, with corresponding subformulas $\exists xA$ and $\forall yB$, we add a fresh link as follows. Let P be a fresh (nullary) predicate symbol. Replace $\exists xA$ with $P \wedge \exists xA$ and $\forall yB$ with $\overline{P} \vee \forall yB$, and add an axiom link between P and \overline{P} .
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers. (We no longer need their leaps since they are encoded as links in step 1.)
- 3) **Simplify atoms.** After step 2, replace every predicate $Pt_1 \cdots t_n$ with a nullary predicate symbol P .

We have the following results:

Let u and v be atoms or quantifiers in a unification structure θ . Then they are connected by a switching path in the

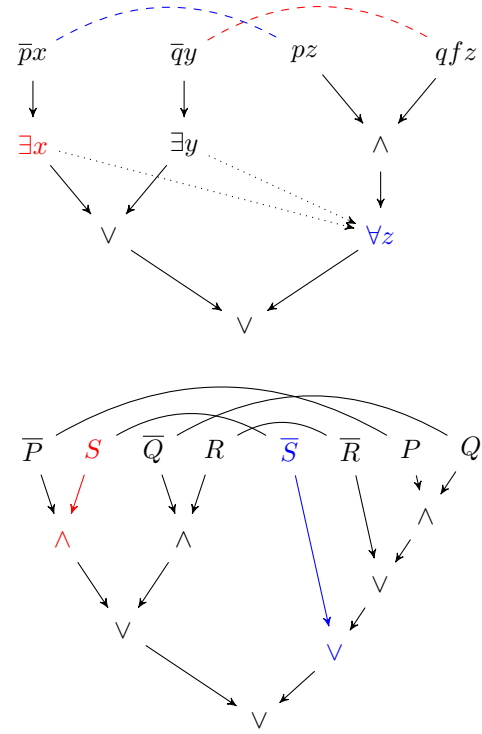


Fig. 9. A unification net and its frame. The colored part shows how the dependency $\exists x \rightarrow \forall z$ is transformed.

unification structure if, and only if, their corresponding nodes are connected by a switching path in θ_m .

Consider now a switching graph H of a unification structure θ of A .

If H contains a cycle, then the corresponding switching graph of θ_m also contains a cycle. Hence, by applying the propositional results (Theorem 7) from [38], we conclude that there exists a chordless, alternating, and elementary cycle in the bicoloured graph $(W, E_W \uplus L_W)$, which corresponds to an induced bimatching in the linked fograph. (Note that the linked fograph (cograph) corresponding to θ_m is equivalent to the one corresponding to θ .) \square

C. From contraction/weakening to skew bifibrations

We first introduce the atomic contraction rule, the medial rule, and two rules on quantifiers.

$$\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac} \quad \frac{\vdash S\{(A \wedge B) \vee (C \wedge D)\}}{\vdash S\{(A \vee C) \wedge (B \vee D)\}} \text{ m}$$

$$\frac{\vdash S\{\exists xA \vee \exists xB\}}{\vdash S\{\exists x(A \vee B)\}} \text{ m}_1\downarrow \quad \frac{\vdash S\{\forall xA \vee \forall xB\}}{\vdash S\{\forall x(A \vee B)\}} \text{ m}_2\downarrow$$

Here, we also consider the equivalence generated by the associativity, commutativity of \vee and the equations $t \vee A \equiv t$ and $f \vee A \equiv A$.

Now we have the following lemma:

Lemma 58. The contraction rule c is derivable for $\{ac, m, m_1\downarrow, m_2\downarrow\}$.

Proof. We prove that there is always $\frac{A \vee A}{A} \parallel_{\{ac, m, m_1\downarrow, m_2\downarrow\}}$ by structural induction on A .

- If $A = t$ or $A = f$, we have $\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}} \equiv$. (the premiss and the conclusion are equivalent)
- If $A = a$, then we have $\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} ac$
- If $A = A_1 \vee A_2$, then by the induction hypothesis, we have $\frac{\vdash S\{(A_1 \vee A_2) \vee (A_1 \vee A_2)\}}{\vdash S\{(A_1 \vee A_1) \vee (A_2 \vee A_2)\}} \equiv$

$$\frac{\vdash S\{(A_1 \vee A_2) \vee (A_1 \vee A_2)\}}{\vdash S\{(A_1 \vee A_1) \vee (A_2 \vee A_2)\}} \equiv$$

$$\vdash S\{A_1 \vee (A_2 \vee A_2)\}$$

Hence, we have

$$\vdash S\{A_1 \vee A_2\}$$

- If $A = A_1 \wedge A_2$, then by the induction hypothesis, we have $\frac{\vdash S\{(A_1 \wedge A_2) \vee (A_1 \wedge A_2)\}}{\vdash S\{(A_1 \vee A_1) \wedge (A_2 \vee A_2)\}} m$

$$\vdash S\{A_1 \wedge A_2\}$$

Hence, we have

$$\vdash S\{A_1 \wedge A_2\}$$

- If $A = \exists x A'$, then by the induction hypothesis, we have $\frac{\vdash S\{\exists x A' \vee \exists x A'\}}{\vdash S\{\exists x(A' \vee A')\}} m_1\downarrow$

$$\vdash S\{\exists x(A' \vee A')\}$$

Hence, we have

$$\vdash S\{\exists x A'\}$$

- If $A = \forall x A'$, then by the induction hypothesis, we have $\frac{\vdash S\{\forall x A' \vee \forall x A'\}}{\vdash S\{\forall x(A' \vee A')\}} m_2\downarrow$

$$\vdash S\{\forall x(A' \vee A')\}$$

Hence, we have

$$\vdash S\{\forall x A'\}$$

Lemma 59. The rules $m_1\downarrow$ and $m_2\downarrow$ are derivable for $\{w, c\}$.

Proof. We have:

$$\frac{\vdash S\{\exists x A\}}{\vdash S\{\exists x(A \vee f)\}} \equiv \quad \text{and} \quad \frac{\vdash S\{\exists x B\}}{\vdash S\{\exists x(f \vee B)\}} \equiv$$

Thus, we have:

$$\frac{\vdash S\{\exists x A \vee \exists x B\}}{\vdash S\{\exists x(A \vee B) \vee \exists x(A \vee B)\}} c$$

Similar for $m_2\downarrow$.

Now we define a propositional encoding for first-order formulas.

Definition 60. The propositional encoding A° of a formula A is defined inductively by:

$$\begin{aligned} a^\circ &= a \text{ for every atom } a \\ (A \vee B)^\circ &= A^\circ \vee B^\circ & (A \wedge B)^\circ &= A^\circ \wedge B^\circ \\ (\forall x A)^\circ &= U_x \vee A^\circ & (\exists x A)^\circ &= E_x \wedge A^\circ \end{aligned}$$

where U_x and E_x are fresh nullary atoms.

Similarly, we can define the propositional encoding S° of a context S inductively by setting $\square^\circ = \square$. Note that S° is also a context.

We have the following facts:

Proposition 61. For any context S and any formula A :

- A° is a formula containing no quantifier for any formula A .
- $\llbracket \llbracket A^\circ \rrbracket \rrbracket = \llbracket \llbracket A \rrbracket \rrbracket$ by confounding the atoms U_x, E_x with the variable x . Thus, a map $f : \llbracket \llbracket A^\circ \rrbracket \rrbracket \rightarrow \llbracket \llbracket B^\circ \rrbracket \rrbracket$ can be seen as a map $f : \llbracket \llbracket A \rrbracket \rrbracket \rightarrow \llbracket \llbracket B \rrbracket \rrbracket$.
- $(S\{A\})^\circ = S^\circ\{A^\circ\}$.

Proposition 62. Let A and B be two formulas such that $A \Delta \parallel_{\{w, c\}} B$. Then $A^\circ \Delta \parallel_{\{w, c\}} B^\circ$.

Proof. Trivial by induction.

Lemma 63. Given two formulas A and B and a derivation $\Delta \parallel_{\{w, c\}} A \rightarrow B$, then there exists a skew bifibration $G(A) \rightarrow G(B)$.

Proof. By Lemma 58, there exists a derivation $\Delta \parallel_{\{w, ac, m, m_1\downarrow, m_2\downarrow\}} A \rightarrow B$.

For each rule from $\{w, ac, m, m_1\downarrow, m_2\downarrow\}$, we define a map and show that it is a skew fibration.

- $\frac{\vdash S\{f\}}{\vdash S\{A\}} w$: the map wk maps f to anything and is identity elsewhere.

- 1399 • $\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac:}$
 1400 the map ac maps the two a -labelled literals in the premise
 1401 to the a -labelled literal in the conclusion.
 $\vdash S\{(A \wedge B) \vee (C \wedge D)\}$
- 1402 • $\frac{\vdash S\{(A \vee C) \wedge (B \vee D)\}}{\vdash S\{\exists x A \vee \exists x B\}} m:$
 1403 the map m is the canonical identity that maps A to A ,
 1404 \dots, D to D .
 $\vdash S\{\exists x A \vee \exists x B\}$
- 1405 • $\frac{\vdash S\{\exists x(A \vee B)\}}{\vdash S\{\forall x(A \vee B)\}} m_1 \downarrow:$
 1406 the map m_1 maps the two x -labelled binders in the
 1407 premise to the x -labelled binder in the conclusion, A to
 1408 A and B to B .
 $\vdash S\{\forall x(A \vee B)\}$
- 1409 • $\frac{\vdash S\{\forall x(A \vee B)\}}{\vdash S\{\forall x(A \vee B)\}} m_2 \downarrow:$
 1410 the map m_2 maps the two x -labelled binders in the
 1411 premise to the x -labelled binder in the conclusion, A to
 1412 A and B to B .

1413 By considering propositional encodings, the maps defined
 1414 are label-preserving skew fibrations on the underlying fographs
 1415 according to [23].

1416 Now we prove that each map $g \in \{wk, ac, m, m_1, m_2\}$ is
 1417 a skew bifibration. To do that, it suffices to prove that g is a
 1418 fibration between the corresponding binding graphs since it is
 1419 already a skew fibration on the corresponding fographs and it
 1420 is label-preserving and existential-preserving.

1421 for each x -binder b in $\llbracket \langle \rangle B^\circ \rrbracket$, for each vertex
 1422 $v \in V(\llbracket \langle \rangle A^\circ \rrbracket)$ such that $g(v)$ is bound by b , there exists a
 1423 unique binder b' such that b' binds v .

- 1424 • wk and m are clearly fibrations: the binding relations of
 1425 the premise and the conclusion are exactly the same.
- 1426 • ac is a fibration: suppose that a that in the conclusion a
 1427 is bound by some quantifier b in S , then for each of its
 1428 preimages by ac , there exists exactly one binder (in fact,
 1429 b) in S that binds it.
- 1430 • m_1 and m_2 are fibrations: in the conclusion, for every
 1431 atom a in $A \vee B$ bound by the x -labelled quantifier, a has
 1432 exactly one preimage and it is bound by the x -labelled
 1433 quantifier in the premise.

1434 Therefore, all of these maps are skew bifibrations and since
 1435 skew bifibrations on fographs compose (Lemma 10.32, [20]),
 1436 there exists a skew bifibration from $\llbracket \langle \rangle A \rrbracket$ to $\llbracket \langle \rangle B \rrbracket$.
 1437 \square

1438 **Theorem 64.** *If a formula A is provable in LK, then it has a*
 1439 *combinatorial proof.*

1440 *Proof.* By Theorem 50, there exists a formula A' such that
 1441 there is a proof Π of A' in MLL1^X and a derivation D from
 1442 A' to A consisting of the w and c rules only. The proof Π
 1443 corresponds to a unique unification net which is equivalent to
 1444 the fonet corresponding to Π , i.e., the fograph $\llbracket \langle \rangle A' \rrbracket$ together
 1445 with the links of Π . By Lemma 63, there exists a skew
 1446 bifibration $\llbracket \langle \rangle A' \rrbracket \rightarrow \llbracket \langle \rangle A \rrbracket$. We have thus a combinatorial
 1447 proof of A .
 1448 \square

D. From skew bifibrations to contraction/weakening 1449

Theorem 65. *Let A and B be two formulas and $f : G(A) \rightarrow$*
 $G(B)$ a skew bifibration. Then there exists a derivation
 A 1450
 $\Delta \parallel_{\{w, c\}}.$ 1451
 B 1452

f can be seen as a skew fibration from $G(A^\circ)$ to $G(B^\circ)$,
 which gives the existence of the propositions A' and B' , and
 of the following derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B' \\ \Delta'' \parallel_w \\ B^\circ \end{array}$$

Lemma 66. *there exists B'' such that $B''^\circ = B'$.* 1453

Proof. Consider the derivation Δ'' . If some U_x (or E_x) is
 1454 introduced via weakening, then all the atoms it binds in B°
 1455 should also be introduced via weakening. In fact, an atom of
 1456 B° is introduced via weakening is equivalent to the fact that
 1457 its corresponding vertex is not in the image of f . Since there
 1458 is an edge from U_x (resp. E_x) to all the literals it binds in the
 1459 binding graph $\llbracket \langle \rangle B \rrbracket$, if one of the atoms is in the image, U_x
 1460 (resp. E_x) should also be in the image since f is a fibration
 1461 on binding graphs.
 1462

This means that a such B'' can be obtained from B by
 1463 erasing all the U_x and E_x introduced via weakening and all
 1464 the atoms they bind. \square 1465

We introduce new (atomic) symbols E_x^* and U_x^* which are
 1466 used to represent disjunctions of E_x and U_x respectively. 1467

We define a translation $(\cdot)^*$ inductively by: 1468

- 1469 • $(E_x \vee \dots \vee E_x)^* = E_x$ 1469
- 1470 • $(U_x \vee \dots \vee U_x)^* = U_x$ 1470
- 1471 • structural recursion in all the other cases. 1471

Then the derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B''^\circ \end{array}$$

can be translated to the derivation:

$$\begin{array}{c} A^{\circ*} \\ \Delta^* \parallel \\ B''^{\circ*} \end{array}$$

where Δ^* is the derivation obtained by replacing all the
 1472 formulas F with F^* and by applying the following rule 1473
 transformation: 1474

$$\frac{S\{Q_x\}}{S\{Q_x\}} \text{ ac} \rightsquigarrow \frac{S\{Q_x\}}{S\{Q_x\}} =$$

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m } \rightsquigarrow \frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m'}$$

where Q_x stands for E_x or U_x .

Δ^* can now be transformed into a valid derivation Δ_1 by using the two transformation rules above and by applying them in a bottom-up style:

$$\frac{A^{\circ*}}{\Delta_1 \parallel_{\text{ac}, \text{m}, \text{m'}}} B''^{\circ*}$$

Lemma 67. Every line of Δ_1 is a propositional encoding.

Proof. We proceed by bottom-up induction in the derivation. Clearly, $(B''^{\circ})^*$ is a propositional encoding as there is no disjunction of Q_x in it.

First consider the ac rule: $\frac{C \vee C}{C} \text{ ac}$

It is clear that if C is a propositional encoding, then so is $C \vee C$.

Now consider the m rule:

$$\frac{S\{(C \wedge D) \vee (E \wedge F)\}}{S\{(C \vee E) \wedge (D \vee F)\}} \text{ m}$$

Suppose that $(C \vee E) \wedge (D \vee F) = G^{\circ}$ for some G . Since $C \vee E$ cannot be Q_x (otherwise, the rule applied would be m'), G can be written as $G_1 \wedge G_2$ with $C \vee E = G_1^{\circ}$ and $D \vee F = G_2^{\circ}$.

We have thus $G_i = \forall x_i H_i$ or $J_i \vee K_i$ ($i = 1, 2$).

If $G_i = \forall x_i H_i$ for some i , then there will be a conjunction of U_x and some formula which can never be eliminated by the rules m , m' and ac . However, there exists no such conjunction in $A^{\circ*}$, which leads to a contradiction.

Hence, G_i can be written as $J_i \vee K_i$ for $i = 1, 2$. We now have $(C \wedge D) \vee (E \wedge F) = ((J_1 \wedge J_2) \vee (K_1 \wedge K_2))^{\circ}$.

Finally, consider the m' rule:

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m'}$$

Suppose that $E_x \wedge (C \vee D) = F^{\circ}$ for some F . It is clear that $F = \exists x G$ with $G^{\circ} = C \vee D$ for some G . We distinguish two cases:

- $G = \forall y H$: in this case, $(E_x \wedge C) \vee (E_x \wedge D)$ has a subformula $(E_x \wedge U_y)$, which cannot be eliminated by the rules m , m' , ac . It is clear that $A^{\circ*}$ does not have a subformula of this form, which leads to a contradiction.
- $G = G_1 \vee G_2$: in this case, $(E_x \wedge C) \vee (E_x \wedge D) = ((\exists x G_1) \vee (\exists x G_2))^{\circ}$.

□