Combinatorial Proofs and Decomposition Theorems for First-order Logic

***Abstract*—We uncover a close relationship between combinatorial and syntactic proofs for first-order logic. Whereas syntactic proofs are formalized in a deductive proof system based on inference rules, a combinatorial proof is a syntax-fre presentation of a proof that is independent from any set of inference rules. We show that the two proof representations are related via a deep inference decomposition theorem that establishes a new kind of normal form for syntactic proofs. This yields (a) a simple proof of soundness and completeness for first-order combinatorial proofs, and (b) a full completeness theorem: every combinatorial proof is the image of a syntactic proof.**

I. Introduction

First-order predicate logic is a cornerstone of modern logic. Since its formalisation by Frege [1] it has seen a growing usage in many fields of mathematics and computer science. Upon the development of proof theory by Hilbert [2], *proofs* became first-class citizens as mathematical objects that could be studied on their own. Since Gentzen's *sequent calculus* [3], [4], many other proof systems have been developed that allow the implementation of efficient proof search, for example *analytic tableaux* [5] or *resolution* [6]. Despite the immense progress made in proof theory in general and in the area of automated and interactive theorem provers in particular, we still have no satisfactory notion of proof identity for first-order logic. In this respect, proof theory is quite different from any other mathematical field. For example in group theory, two groups are *the same* iff they are isomorphic; in topology, two spaces are *the same* iff they are homeomorphic; etc. In proof theory, we have no such notion telling us when two proofs are *the same*, even though Hilbert was considering this problem as a possible 24th problem for his famous lecture [7] in 1900 [8], before proof theory existed as a mathematical field.

The main reason for this problem is that formal proofs, as they are usually studied in logic, are inextricably tied to the syntactic (inference rule based) proof system in which they are carried out. And it is difficult to compare two proofs that are produced within two different syntactic proof systems, based on different sets of inference rules. Just consider the derivations in Figure 1, showing two proofs of the formula $((\overline{p} \vee q) \wedge \overline{p}) \vee p$ and two proofs of the formula $\exists x.(\overline{p}x \vee (\forall y.py))$, one in the sequent calculus (top) and one in a deep inference system (bottom). It is, *a priori*, not clear how to compare them.

This is where *combinatorial proofs* come in. They were introduced by Hughes [9] for classical propositional logic as a syntax-free notion of proof, and as a potential solution to Hilbert's 24th problem [10] (see also [11]). The basic idea is to abstract away from the syntax of the inference rules used in inductively-generated proofs and consider the proof as a combinatorial object, more precisely as a special kind of graph homomorphism. For example, a propositional combinatorial

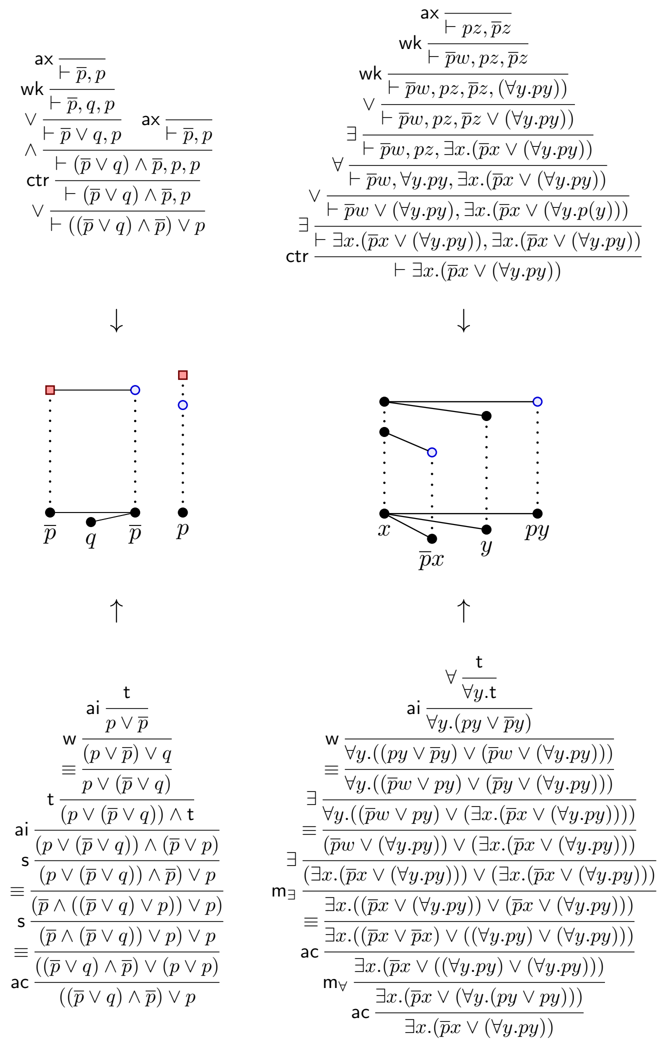


Fig. 1. Left: syntactic proofs in sequent calculus (above) and the calculus of structures (below) which translate to the same propositional combinatorial proof (centre). Right: syntactic proofs in sequent calculus (above) and the new calculus $\mathsf{KS1}$ introduced in this paper (below), which translate to the same first-order combinatorial proof (centre).

proof of Peirce's law $((p \Rightarrow q) \Rightarrow p) \Rightarrow p = ((\overline{p} \vee q) \wedge \overline{p}) \vee p$ is shown mid-left in Fig. 1, a homomorphism from a coloured graph to a graph labelled with propositional variables.

Several authors have illustrated how syntactic proofs in various proof systems can be translated to propositional combinatorial proofs: for sequent proofs in [10], for deep inference proofs in [12], for Frege systems in [13], and for tableaux systems and resolution in [14]. This enables a natural definition of proof identity for propositional logic: two proofs are *the same*, if they are mapped to the same combinatorial proof. For example, the left side of Fig. 1 translates syntactic proofs from sequent calculus and the calculus of structures into the same combinatorial proofs, witnessing that the two syntactic proofs, from different systems, are *the same*.

Recently, Acclavio and Straßburger extended this notion to relevant logics [15] and to modal logics [16], and Heijltjes, Hughes and Straßburger have provided combinatorial proofs for intuitionistic propositional logic [17].

In this paper we advance the idea that combinatorial proofs can provide a notion of proof identity for first-order logic. *First-order combinatorial proofs* were introduced by Hughes in [18]. For example, a first-order combinatorial proof of Smullyan's "drinker paradox" $\exists x(px \Rightarrow \forall y\, py) = \exists x.(\bar{p}x \vee (\forall y.py))$ is shown on the right of Fig. 1, a homomorphism from a partially coloured graph to a labelled graph. However, even though Hughes proves soundness and completeness, the proof is highly unsatisfactory: (1) the soundness argument is extremely long, intricate and cumbersome, and (2) the completeness proof does not allow a syntactic proof to be read back from a combinatorial proof, i.e., completeness is not *sequentializable* [24] nor *full* [19]. A fundamental problem is that not all combinatorial proofs can be obtained as translations of sequent calculus proofs.

We solve these issues by moving to a deep inference system. More precisely, we introduce a new proof system, called $\mathsf{KS1}$, for first-order logic, that (a) reflects every combinatorial proof, i.e., there is a surjection from $\mathsf{KS1}$ proofs to combinatorial proofs, (b) yields far simpler proofs of soundness and completeness for combinatorial proofs, and (c) admits new decomposition theorems establishing a precise correspondence between certain syntactic inference rules and certain combinatorial notions. The right side of Fig. 1 illustrates the surjection in (a), and since the syntactic proofs of the two systems both translate the same combinatorial proof, they can be considered *the same*.

In general, a *decomposition theorem* provides normal forms of proofs, separating subsets of inference rules of a proof system. A prominent example of a decomposition theorem is Herbrand's theorem [20], which allows a separation between the propositional part and the quantifier part in a first-order proof [4], [21]. Through the advent of deep inference, new kinds of proof decompositions became possible, most notably the separation between the linear part of a proof and the resource management of a proof. It has been shown by Straßburger [22] that a proof in classical propositional logic can be decomposed into a proof of multiplicative linear logic, followed by a proof consisting only of contractions and weakenings. In this paper we show that the same is possible for first-order logic.

Combinatorial proofs and deep inference can be seen as opposite ends of a spectrum: whereas deep inference allows for a very fine granularity of inference rules—one inference rule in a standard formalism, like sequent calculus or semantic tableaux, is usually simulated by a sequence of different deep inference rules—combinatorial proofs have completely abolished the concept of inference rule. And yet, there is a close relationship between the two, realized through a decomposition theorem, as we establish in this paper.

II. Preliminaries: First-order Logic

A. Terms and Formulas

Fix pairwise disjoint countable sets $\mathrm{VAR} = \{x, y, z, \ldots\}$ of variables, $\mathrm{FUN} = \{f, g, \ldots\}$ of function symbols, and $\mathrm{PRED} = \{p, q, \ldots\}$ of predicate symbols. Each function symbol and each predicate symbol has a finite arity. The grammars below generate the set TERM of ***terms***, denoted by $s, t, u, \ldots$, the set ATOM of ***atoms***, denoted by $a, b, c, \ldots$, and the set FORM of ***formulas***, denoted by $A, B, C, \ldots$:

$$\begin{aligned} t &::= \; x \mid f(t_1, \ldots, t_n) \\ a &::= \; \mathsf{t} \mid \mathsf{f} \mid p(t_1, \ldots, t_n) \mid \bar{p}(t_1, \ldots, t_n) \\ A &::= \; a \mid A \wedge A \mid A \vee A \mid \exists x.A \mid \forall x.A \end{aligned}$$

where the arity of f and p is n. For better readability of often omit parentheses and write simply $ft_1 \ldots t_n$ or $pt_1 \ldots t_n$. We consider the truth constants t (*true*) and f (*false*) as additional atoms, and we consider all formulas in negation normal form, where ***negation*** $(\bar{\cdot})$ is defined on atoms and formulas via De Morgan's laws:

$$\begin{array}{lll} \bar{\bar{a}} = a & \bar{\mathsf{t}} = \mathsf{f} & \overline{p(t_1, \ldots, t_n)} = \bar{p}(t_1, \ldots, t_n) \\ & \bar{\mathsf{f}} = \mathsf{t} & \overline{\bar{p}(t_1, \ldots, t_n)} = p(t_1, \ldots, t_n) \\ \multicolumn{2}{c}{\overline{\exists x.A} = \forall x.\bar{A}} & \overline{A \wedge B} = \bar{A} \;\vee\; \bar{B} \\ \multicolumn{2}{c}{\overline{\forall x.A} = \exists x.\bar{A}} & \overline{A \vee B} = \bar{A} \;\wedge\; \bar{B} \end{array}$$

Then we write $A \Rightarrow B$ as abbreviation for $\bar{A} \vee B$.

A formula is ***rectified*** if all bound variables are distinct from one another and from all free variables. Every formula can be transformed into a logically equivalent rectified form, by simply renaming the bound variables. If we consider formulas equivalent modulo α-conversion (renaming of bound variables), then the rectified form of a formula A is uniquely defined, and we denote it by $\widehat{A}$.

A ***substitution*** is a function $\sigma\colon \mathrm{VAR} \to \mathrm{TERM}$ that is the identity almost everywhere. We denote substitutions as $\sigma = [x_1/t_1, \ldots, x_n/t_n]$, where $\sigma(x_i) = t_i$ for $i = 1..n$ and $\sigma(x) = x$ for all $x \notin \{x_1, \ldots, x_n\}$. Write $A\sigma$ for the formula obtained from A by applying σ, i.e., by simultaneously replacing all occurrences of x_i by t_i. A ***variable renaming*** is a substitution ρ with $\rho(x) \in \mathrm{VAR}$ for all variables x.

B. Sequent Calculus $\mathsf{LK1}$

Sequents, denoted by $\Gamma, \Delta, \ldots$, are finite multisets of formulas, written as lists, separated by comma. The ***corresponding formula*** of a (non-empty) sequent $\Gamma = A_1, A_2, \ldots, A_n$ is the disjunction of its formulas: $\bigvee(\Gamma) = A_1 \vee A_2 \vee \cdots \vee A_n$. A sequent is ***rectified*** iff its corresponding formula is.

In this paper we use the sequent calculus $\mathsf{LK1}$, shown in Figure 2, which is a one-sided variant of Gentzen's original calculus [3] for first-order logic. To simplify some technicalities later in this paper, we include the mix-rule.

Theorem 1. $\mathsf{LK1}$ *is sound and complete for first-order logic.*

For a proof, see any standard textbook, e.g. [23].

$$\mathsf{ax}\ \frac{}{\vdash a, \overline{a}} \qquad \vee\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \wedge\ \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$$

$$\mathsf{t}\ \frac{}{\vdash \mathsf{t}} \qquad \mathsf{f}\ \frac{\vdash \Gamma}{\vdash \Gamma, \mathsf{f}} \qquad \mathsf{mix}\ \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

$$\exists\ \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A} \qquad \forall\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A}\ (x \text{ not free in } \Gamma)$$

$$\mathsf{ctr}\ \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{wk}\ \frac{\vdash \Gamma}{\vdash \Gamma, A}$$

Fig. 2. Sequent calculi $\mathsf{LK1}$ (all rules) and $\mathsf{MLL1^X}$ (rules in the dashed box)

The linear fragment of $\mathsf{LK1}$, i.e., the fragment without the rules ctr (*contraction*) and wk (*weakening*) defines *first-order multiplicative linear logic* [24], [25] *with mix* [26], [27] (MLL1+mix). We denote that system here with $\mathsf{MLL1^X}$ (shown in Figure 2 in the dashed box).

We will use the cut elimination theorem. The ***cut*** rule is

$$\mathsf{cut}\ \frac{\vdash \Gamma, A \quad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta} \tag{1}$$

Theorem 2. *If a sequent* $\vdash \Gamma$ *is provable in* $\mathsf{LK1}$+cut *then it is also provable in* $\mathsf{LK1}$*. Furthermore, if* $\vdash \Gamma$ *is provable in* $\mathsf{MLL1^X}$+cut *then it is also provable in* $\mathsf{MLL1^X}$.

As before, this is standard, see e.g. [23] for a proof.

III. Preliminaries: First-order Graphs

A. Graphs

A ***graph*** $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ is a pair where $V_\mathcal{G}$ is a finite set of ***vertices*** and $E_\mathcal{G}$ is a finite set of ***edges***, which are two-element subsets of $V_\mathcal{G}$. We write vw for an edge $\{v, w\}$.

The ***complement*** of a graph $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ is the graph $\mathcal{G}^\mathsf{C} = \langle V_\mathcal{G}, E_\mathcal{G}^\mathsf{C} \rangle$ where $vw \in E_\mathcal{G}^\mathsf{C}$ iff $vw \notin E_\mathcal{G}$.

Let $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ and $\mathcal{H} = \langle V_\mathcal{H}, E_\mathcal{H} \rangle$ be graphs such that $V_\mathcal{G} \cap V_\mathcal{H} = \emptyset$. A ***homomorphism*** $\varphi\colon \mathcal{G} \to \mathcal{H}$ is a function $\varphi\colon V_\mathcal{G} \to V_\mathcal{H}$ such that if $vw \in E_\mathcal{G}$ then $\varphi(v)\varphi(w) \in E_\mathcal{H}$. The ***union*** $\mathcal{G} + \mathcal{H}$ is the graph $\langle V_\mathcal{G} \cup V_\mathcal{H}, E_\mathcal{G} \cup E_\mathcal{H} \rangle$ and the ***join*** $\mathcal{G} \times \mathcal{H}$ is the graph $\langle V_\mathcal{G} \cup V_\mathcal{H}, E_\mathcal{G} \cup E_\mathcal{H} \cup \{vw \mid v \in V_\mathcal{G}, w \in V_\mathcal{H}\} \rangle$. A graph $\mathcal{G}$ is ***disconnected*** if $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ for two non-empty graphs $\mathcal{G}_1, \mathcal{G}_2$, otherwise it is ***connected***. It is ***coconnected*** if its complement is connected.

A graph $\mathcal{G}$ is ***labelled*** in a set L if each vertex $v \in V_\mathcal{G}$ has an element $\ell(v) \in L$ associated with it, its ***label***. A graph $\mathcal{G}$ is (partially) ***coloured*** if it carries a partial equivalence relation $\sim_\mathcal{G}$ on $V_\mathcal{G}$; each equivalence class is a ***colour***. A ***vertex renaming*** of $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ along a bijection $\hat{(\cdot)}\colon V_\mathcal{G} \to \hat{V}_\mathcal{G}$ is the graph $\hat{\mathcal{G}} = \langle \hat{V}_\mathcal{G}, \{\hat{v}\hat{w} \mid vw \in E_\mathcal{G}\} \rangle$, with colouring and/or labelling inherited (i.e., $\hat{v} \sim \hat{w}$ if $v \sim w$, and $\ell(\hat{v}) = \ell(v)$). Following standard graph theory, we identify graphs modulo vertex renaming.

A ***directed graph*** $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ is a set $V_\mathcal{G}$ of ***vertices*** and a set $E_\mathcal{G} \subseteq V_\mathcal{G} \times V_\mathcal{G}$ of ***direct edges***. A ***directed graph homomorphism*** $\varphi\colon \langle V_\mathcal{G}, E_\mathcal{G} \rangle \to \langle V_\mathcal{H}, E_\mathcal{H} \rangle$ is a function $\varphi\colon V_\mathcal{G} \to V_\mathcal{H}$ such that if $(v, w) \in E_\mathcal{G}$ then $(\varphi(v), \varphi(w)) \in E_\mathcal{H}$.

B. Cographs

A graph $\mathcal{H} = \langle V_\mathcal{H}, E_\mathcal{H} \rangle$ is a ***subgraph*** of a graph $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ if $V_\mathcal{H} \subseteq V_\mathcal{G}$ and $E_\mathcal{H} \subseteq E_\mathcal{G}$. It is ***induced*** if $v, w \in V_\mathcal{H}$ and $vw \in E_\mathcal{G}$ implies $vw \in E_\mathcal{H}$. An induced subgraph of $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ is uniquely determined by its set of vertices V and we denote it by $\mathcal{G}[V]$. A graph is ***$\mathcal{H}$-free*** if it does not contain $\mathcal{H}$ as an induced subgraph. The graph $\mathbf{P_4}$ is the (undirected) graph $\langle \{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4\} \rangle$. A ***cograph*** is a P_4-free undirected graph. The interest in cographs for our paper comes from the following well-known fact.

Theorem 3 ([28])**.** *A graph is a cograph iff it can be constructed from the singletons via the operations* $+$ *and* $\times$.

In a graph $\mathcal{G}$, the ***neighbourhood*** $N(v)$ of a vertex $v \in V_\mathcal{G}$ is defined as the set $\{w \mid vw \in E_\mathcal{G}\}$. A ***module*** is a set $M \subseteq V_\mathcal{G}$ such that $N(v) \setminus M = N(w) \setminus M$ for all $v, w \in M$. A module M is ***strong*** if for every module M', we have $M' \subseteq M$, $M \subseteq$ or $M \cap M' = \emptyset$. A module is ***proper*** if it has two or more vertices.

C. Fographs

A cograph is ***logical*** if every vertex is labelled by either an atom or variable, and it has at least one atom-labelled vertex. An atom-labelled vertex is called a ***literal*** and a variable-labelled vertex is called a ***binder***. A binder labelled with x is called an ***x-binder***. The ***scope*** of a binder b is the smallest proper strong module containing b. An ***x-literal*** is a literal whose atom contains the variable x. An x-binder ***binds*** every x-literal in its scope. In a logical cograph $\mathcal{G}$, a binder b is ***existential*** (resp. ***universal***) if, for every other vertex v in its scope, we have $bv \in E_\mathcal{G}$ (resp. $bv \notin E_\mathcal{G}$). An x-binder is ***legal*** if its scope contains no other x-binder and at least one literal.

Definition 4. A ***first-order graph*** or ***fograph*** is a logical cograph with legal binders. The ***binding graph*** of a fograph $\mathcal{G}$ is the directed graph $\vec{\mathcal{G}} = \langle V_\mathcal{G}, \{(b, l) \mid b \text{ binds } l\} \rangle$.

We define a mapping $[\![\cdot]\!]$ from formulas to (labelled) graphs, inductively as follows:

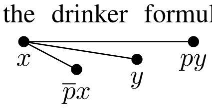
$$[\![a]\!] = \bullet a \quad \text{(for any atom } a\text{)}$$

$$[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![\exists x.A]\!] = \bullet x \times [\![A]\!]$$

$$[\![A \wedge B]\!] = [\![A]\!] \times [\![B]\!] \qquad [\![\forall x.A]\!] = \bullet x + [\![A]\!]$$

where we write $\bullet\alpha$ for a single-vertex labelled by α.

Example 5. Here is the fograph of the drinker formula $\exists x(px \Rightarrow \forall y\, py) = \exists x.(\overline{p}x \vee (\forall y.py))$:



Lemma 6. *If A is a rectified formula then $[\![A]\!]$ is a fograph.*

Proof. That $[\![A]\!]$ is a logical cograph follows immediately from the definition and Theorem 3. The fact that every binder of $[\![A]\!]$ is legal can be proved by structural induction on A. $\square$

Remark 7. Note that $[\![A]\!]$ is not necessarily a fograph if A is not rectified. If $A = (\forall x.px) \vee (\forall x.qx)$, then $[\![A]\!] = \bullet x\ \bullet px\ \bullet x\ \bullet qx$, the scope of each x-binder contains all the vertices, in particular, the other x-binder. On the other hand, there are non-rectified formulas which are translated to fographs by $[\![\cdot]\!]$. For example, in the graph of $(\exists x.px) \vee (\exists x.qx)$, both x-binders are legal, as they are not in each other's scope: $x\bullet\!\!-\!\!\bullet px\ \ x\bullet\!\!-\!\!\bullet qx$.

We define a congruence relation $\equiv$ on formulas by the following equations:

$$\begin{array}{cc} A \wedge B \equiv B \wedge A & (A \wedge B) \wedge C \equiv A \wedge (B \wedge C) \\ A \vee B \equiv B \vee A & (A \vee B) \vee C \equiv A \vee (B \vee C) \\ \forall x.\forall y.A \equiv \forall y.\forall x.A & \forall x.(A \vee B) \equiv (\forall x.A) \vee B \\ \exists x.\exists y.A \equiv \exists y.\exists x.A & \exists x.(A \wedge B) \equiv (\exists x.A) \wedge B \end{array} \tag{2}$$

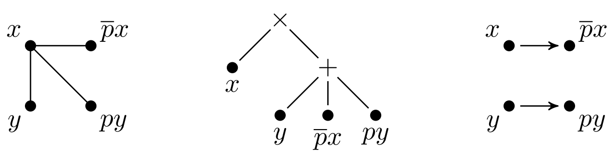
where x must not be free in B in the last two equations. Two formulas A and B are ***equivalent*** if $A \equiv B$. The following theorem shows that the set of fographs can been seen as the quotient FORM$/\equiv$.

Theorem 8. *Let A, B be rectified formulas. Then*

$$A \equiv B \iff [\![A]\!] = [\![B]\!]$$

Proof. By a straightforward induction on A. $\square$

Example 9. Both $\exists x.(\overline{p}x \vee (\forall y.py))$ and $\exists x \forall y\,(py \vee \overline{p}x)$, which are equivalent modulo $\equiv$, have the same (rectified) fograph $\mathcal{D}$, shown below-left.



Above-middle we show the *cotree* of the underlying cograph (illustrating the idea behind Theorem 3) and above-right is its binding graph $\vec{\mathcal{D}}$.

IV. First-order Combinatorial Proofs

A. Fonets

Two atoms are ***pre-dual*** if they are not $\mathtt{t}$ or $\mathtt{f}$, and their predicate symbols are dual (e.g. $p(x, y)$ and $\overline{p}(y, z)$) and two literals are ***pre-dual*** if their labels (atoms) are pre-dual. A ***linked fograph*** $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ is a coloured fograph $\mathcal{C}$ such that every colour (i.e., equivalence class of $\sim_{\mathcal{C}}$), called a ***link***, consists of two pre-dual literals, and every literal is either $\mathtt{t}$-labelled or in a link. Hence, in a linked fograph no vertex is labeled $\mathtt{f}$.

Let $\mathcal{C}$ be a linked fograph. The set of links can be seen as a unification problem by identifying dual predicate symbols. A ***dualizer*** of $\mathcal{C}$ is a substitution δ unifying all the links of $\mathcal{C}$. Since a first-order unification problem is either unsolvable or has a most general unifier, we can define the notion of ***most general dualizer***. A ***dependency*** is a pair $\{\bullet x, \bullet y\}$ of an existential binder $\bullet x$ and a universal binder $\bullet y$ such that the most general dualizer assigns to x a term containing y. A ***leap*** is either a link or a dependency. The ***leap graph*** $\mathcal{C}^{\mathsf{L}}$ of $\mathcal{C}$ is the undirected graph $\langle V_{\mathcal{C}}, L_{\mathcal{C}} \rangle$ where $L_{\mathcal{C}}$ is the set of leaps of $\mathcal{C}$. A vertex set $W \subseteq V_{\mathcal{C}}$ induces a ***matching*** in $\mathcal{C}$ if $W \neq \emptyset$ and for all $w \in W$, $N(w) \cap W$ is a singleton. We say that W induces a ***bimatching*** in $\mathcal{C}$ if it induces a matching in $\mathcal{C}$ and a matching in $\mathcal{C}^{\mathsf{L}}$.

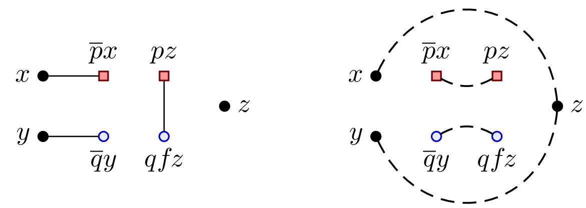


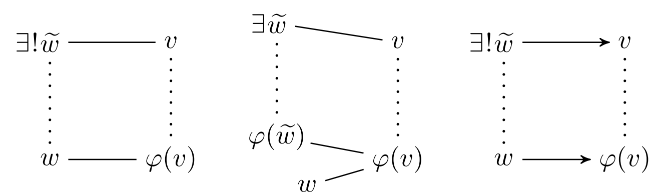
Fig. 3. A fonet (left) with dualizer $[x/z, y/fz]$ and its leap graph (right).

Definition 10. A ***first-order net*** or ***fonet*** is a linked fograph which has a dualizer but no induced bimatching.

Figure 3 shows a fonet with a unique dualizer, and its leap graph.

B. Skew Bifibrations

A graph homomorphism $\varphi\colon \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \to \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ is a ***fibration*** if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$, there exists a unique $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$ (indicated below-left), and is a ***skew fibration*** if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$ there exists $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w})w \notin E_{\mathcal{H}}$ (indicated below-centre). A directed graph homomorphism is a ***fibration*** if for all $v \in V_{\mathcal{G}}$ and $(w, \varphi(v)) \in E_{\mathcal{H}}$, there exists a unique $\tilde{w} \in V_{\mathcal{G}}$ such that $(\tilde{w}, v) \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$ (indicated below-right).



A ***fograph homomorphism*** $\varphi = \langle \varphi, \rho_{\varphi} \rangle$ is a pair where $\varphi\colon \mathcal{G} \to \mathcal{H}$ is a graph homomorphism between the underlying graphs, and ρ_{φ}, also called the ***substitution induced by*** φ is a variable renaming such that for all $v \in V_{\mathcal{G}}$ we have $\ell(\varphi(v)) = \rho_{\varphi}(\ell(v))$, and ρ_{φ} is the identity on variables not in $\mathcal{G}$. Note that φ necessarily maps binders to binders and literals to literals. Since ρ_{φ} is fully determined by φ alone, we often leave ρ_{φ} implicit. A fograph homomorphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ ***preserves existentials*** if for all existential binders b in $\mathcal{G}$, the binder $\varphi(b)$ is a existential in $\mathcal{H}$.

Definition 11. Let $\mathcal{G}$ and $\mathcal{H}$ be fographs. A ***skew bifibration*** $\varphi\colon \mathcal{G} \to \mathcal{H}$ is an existential-preserving fograph homomorphism that is a skew fibration on $\langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \to \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ and a fibration on the binding graphs $\vec{\mathcal{G}} \to \vec{\mathcal{H}}$.

Example 12. Below-left is a skew bifibration, whose binding fibration is below-centre. When the labels on the source fograph can be inferred (modulo renaming), we often omit the labeling in the upper graph, as below-right.

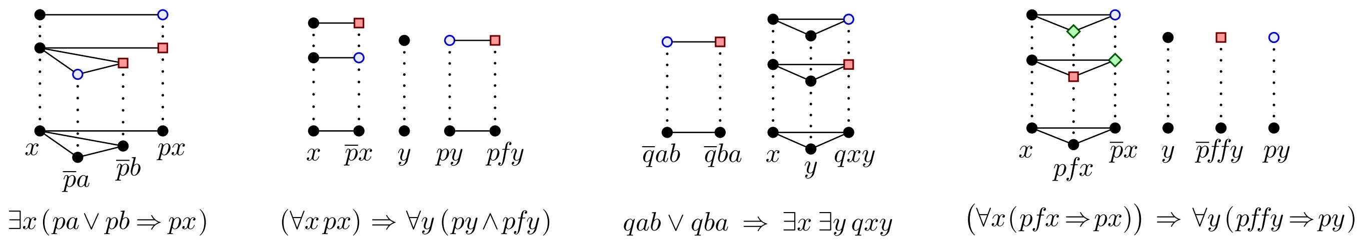


Fig. 4. Four combinatorial proofs, each shown above the formula proved. Here x and y are variables, f is a unary function symbol, a and b are constants (nullary function symbols), p is a unary predicate symbol, and q is a binary predicate symbol.

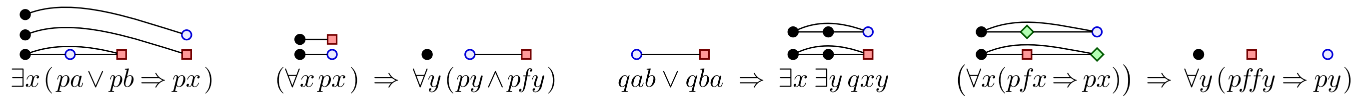
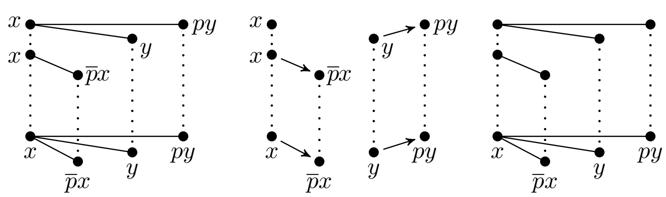


Fig. 5. Condensed forms of the four combinatorial proofs in Figure 4. We do not show the lower graph, and indicate the mapping by the position of the vertices of the upper graph.



Definition 13. A ***first-order combinatorial proof*** (***FOCP***) of a fograph $\mathcal{G}$ is a skew bifibration $\varphi\colon \mathcal{C} \to \mathcal{G}$ where $\mathcal{C}$ is a fonet. A ***first-order combinatorial proof*** of a formula A is a combinatorial proof of its graph $[\![A]\!]$.

Figure 4 shows examples of FOCPs (taken from [18]), each above the formula it proves. The same FOCPs are shown in Figure 5 in a "condensed form".

Theorem 14 ([18])**.** *FOCPs are sound and complete for first-order logic.*

Remark 15. Our definition of FOCP is slightly more lax than the original definition of [18], as we allow for a variable renaming ρ_φ which was forced to be the identity in [18].

V. First-order Deep Inference system KS1

In contrast to to standard proof formalisms, like sequent calculi or tableaux, where inference rules decompose the principal formula along its root connective, *deep inference rules* apply like rewriting rules inside any (positive) formula or sequent ***context***, which is denoted as $S\{\cdot\}$, and which is a formula (resp. sequent) with exactly one occurrence of the ***hole*** $\{\cdot\}$ in the position of an atom. Then $S\{A\}$ is the result of replacing the hole $\{\cdot\}$ in $S\{\cdot\}$ with A.

Figure 6 shows the inference rules for the deep inference system $\mathsf{KS1}$ that we introduce in this paper. It is a slight variation of the systems presented by Brünnler [29] and Ralph [30] in their PhD-theses. The main differences are (i) the explicit presence of the mix-rule, (ii) a different choice of how the formula equivalence $\equiv$ is defined, and (iii) an explicit rule for the equivalence.

$$\forall\,\frac{S\{\mathsf{t}\}}{S\{\forall x.\mathsf{t}\}} \qquad \mathsf{ai}\,\frac{S\{\mathsf{t}\}}{S\{a \vee \bar{a}\}} \qquad \mathsf{t}\,\frac{S\{A\}}{S\{A \wedge \mathsf{t}\}}$$

$$\mathsf{s}\,\frac{S\{A \wedge (B \vee C)\}}{S\{(A \wedge B) \vee C\}} \qquad \mathsf{mix}\,\frac{S\{A \wedge B\}}{S\{A \vee B\}}$$

$$\exists\,\frac{S\{A[x/t]\}}{S\{\exists x.A\}} \qquad \equiv\,\frac{S\{B\}}{S\{A\}}\ (\text{where } A \equiv B)$$

$$\mathsf{w}\,\frac{S\{A\}}{S\{A \vee B\}} \qquad \mathsf{m}\,\frac{S\{(A \wedge C) \vee (B \wedge D)\}}{S\{(A \vee B) \wedge (C \vee D)\}} \qquad \mathsf{ac}\,\frac{S\{a \vee a\}}{S\{a\}}$$

$$\mathsf{m}_\forall\,\frac{S\{(\forall x.A) \vee (\forall x.B)\}}{S\{\forall x.(A \vee B)\}} \qquad \mathsf{m}_\exists\,\frac{S\{(\exists x.A) \vee (\exists x.B)\}}{S\{\exists x.(A \vee B)\}}$$

$$\mathsf{w}_\forall\,\frac{S\{A\}}{S\{\forall x.A\}}\ (x \text{ not free in } A) \qquad \mathsf{c}_\forall\,\frac{S\{\forall x.\forall x.A\}}{S\{\forall x.A\}}$$

Fig. 6. Deep inference systems $\mathsf{KS1}$ (all rules) and $\mathsf{MLS1^X}$ (rules in the dashed box)

We consider here only the cut-free fragment, as cut-elimination for deep inference systems has already been discussed elsewhere (e.g. [21], [31]).[1] As with the sequent system $\mathsf{LK1}$, we also need for $\mathsf{KS1}$ the *linear fragment*, $\mathsf{MLS1^X}$, and that is shown in Figure 6 in the dashed box.

We write $\mathsf{S}\,\Big\Vert{\scriptstyle\Phi}\ \genfrac{}{}{0pt}{}{B}{A}$ to denote a derivation Φ from B to A using the rules from system S. A formula A is ***provable*** in a system S if there is a derivation in S from t to A.

In the course of this paper we will employ the general (non-

[1] In the deep inference literature, the cut-free fragment is also called the *down-fragment*. But as we do not discuss the *up-fragment* here, we omit the down-arrows $\downarrow$ in the rule names.

atomic) version of the contraction rule:

$$\mathsf{c}\,\frac{S\{A \vee A\}}{S\{A\}} \tag{3}$$

VI. MAIN RESULTS

We state the main results of this paper here, and prove them in later sections. The first is routine and expected, but needs to be proved nonetheless:

Theorem 16. $\mathsf{KS1}$ *is sound and complete for first-order logic.*

Our second result is more surprising, as it is a very strong decomposition result for first order logic.

Theorem 17. *For every derivation* $\mathsf{KS1}\,\begin{array}{c}\mathsf{t}\\ \Vert\,\Phi\\ A\end{array}$ *there are* f*-free formulas* $A_1, \ldots, A_5$*, such that there is a derivation:*

$$\begin{array}{rc}
 & \mathsf{t} \\
\{\forall,\mathsf{ai},\mathsf{t}\} & \Vert \\
 & A_5 \\
\{\mathsf{s},\mathsf{mix},\equiv\} & \Vert \\
 & A_4 \\
\{\exists\} & \Vert \\
 & A_3 \\
\{\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\} & \Vert \\
 & A_2 \\
\{\mathsf{ac},\mathsf{c}_\forall\} & \Vert \\
 & A_1 \\
\{\mathsf{w},\mathsf{w}_\forall,\equiv\} & \Vert \\
 & A
\end{array}$$

This theorem is stronger than the existing decompositions for first-order logic, which either separate only atomic contraction and atomic weakening [29] or only contraction [30] or only the quantifiers in form of a Herbrand theorem [32], [30].

Theorem 17 is also the reason why we have the rules $\mathsf{w}_\forall$ and $\mathsf{c}_\forall$ in system $\mathsf{KS1}$, as these rules are derivable with the other rules. However, they are needed to obtain this decomposition.

Figure **??** shows an example of a decomposed derivation in $\mathsf{KS1}$ of the formula $(\exists x.\bar{p}x) \vee (\forall y.(py \wedge pfy))$.

There is a weaker version of Theorem 17 that will also be useful:

Theorem 18. *For every derivation* $\mathsf{KS1}\,\begin{array}{c}\mathsf{t}\\ \Vert\,\Phi\\ A\end{array}$ *there is a formula* A_1 *not containing any occurrence of* f*, such that there is a derivation:*

$$\begin{array}{rc}
 & \mathsf{t} \\
\mathsf{MLS1}^{\mathsf{X}} & \Vert \\
 & A_1 \\
\{\mathsf{w},\mathsf{c},\equiv\} & \Vert \\
 & A
\end{array}$$

Let us now establish the connection between derivations in $\mathsf{KS1}$ and combinatorial proofs.

$$\begin{array}{rc}
\forall & \dfrac{\mathsf{t}}{\forall y.\mathsf{t}} \\
\mathsf{t} & \overline{\forall y.(\mathsf{t} \wedge \mathsf{t})} \\
\mathsf{ai} & \overline{\forall y.((\bar{p}y \vee py) \wedge \mathsf{t})} \\
\mathsf{ai} & \overline{\forall y.((\bar{p}y \vee py) \wedge (pfy \vee \bar{p}fy))} \\
\equiv & \overline{\forall y.(\bar{p}y \vee (py \wedge (pfy \vee \bar{p}fy)))} \\
\mathsf{s} & \overline{\forall y.(\bar{p}y \vee ((py \wedge pfy) \vee \bar{p}fy))} \\
\equiv & \overline{\forall y.((\bar{p}y \vee \bar{p}fy) \vee (py \wedge pfy))} \\
\exists & \overline{\forall y.((\bar{p}y \vee (\exists x.\bar{p}x)) \vee (py \wedge pfy))} \\
\exists & \overline{\forall y.(((\exists x.\bar{p}x) \vee (\exists x.\bar{p}x)) \vee (py \wedge pfy))} \\
\equiv & \overline{((\exists x.\bar{p}x) \vee (\exists x.\bar{p}x)) \vee (\forall y.(py \wedge pfy)} \\
\mathsf{m}_\exists & \overline{(\exists x.(\bar{p}x \vee \bar{p}x) \vee (\forall y.(py \wedge pfy))} \\
\mathsf{ac} & \overline{(\exists x.\bar{p}x) \vee (\forall y.(py \wedge pfy))}
\end{array}$$

Fig. 7. Example derivation in decomposed form of Theorem 17

Theorem 19. *Let* $\varphi\colon \mathcal{C} \to \mathcal{A}$ *be a combinatorial proof and let* A *be a formula with* $\mathcal{A} = [\![A]\!]$*. Then there is a derivation*

$$\begin{array}{rc}
 & \mathsf{t} \\
\mathsf{MLS1}^{\mathsf{X}} & \Vert\,\Phi_1 \\
 & A' \\
\{\mathsf{w},\mathsf{w}_\forall,\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\} & \Vert\,\Phi_2 \\
 & A
\end{array} \tag{4}$$

for some $A' \equiv C\rho_\varphi$ *where* C *is a formula with* $[\![C]\!] = \mathcal{C}$ *and* ρ_φ *is the variable renaming substitution induced by* φ*. Conversely, whenever we have a derivation as in* (9) *above, such that* f *does not occur in* A'*, then there is a combinatorial proof* $\varphi\colon \mathcal{C} \to [\![A]\!]$ *such that* $\mathcal{C} = [\![\widehat{A'}]\!]$.

Furthermore, in the proof of Theorem 20, we will see that (i) the links in the fonet $\mathcal{C}$ correspond precisely to the pairs of atoms that meet in the instances of the ai-rule in the derivation Φ_1, and (ii) the "flow-graph" of Φ_2 that traces the quantifier- and atom-occurrences in the derivation corresponds exactly to the vertex-mapping induced by φ. To give an example, consider the derivation in Figure **??** which corresponds to the right-most combinatorial proof in Figures 4 and 5.

Thus, combinatorial proofs are closely related to derivations of the form (9), and since by Theorem 17 every derivation can be transformed into that form, we can say that combinatorial proofs form a canonical proof representation for first order logic, similarly to what proof nets are for linear logic [33].

Finally, Theorems 16, 17 and 20 imply Theorem 14, which means that we have here an alternative proof of the soundness and completeness of first-order combinatorial proofs which is simpler than the one given in [18].

VII. TRANSLATING BETWEEN LK1 AND KS1

In this section we prove Theorems 16, 17, and 19, mainly by translating derivations to and from the sequent calculus, and by rule permutation arguments.

A. *The Linear Fragments* $\mathsf{MLL1^X}$ *and* $\mathsf{MLS1^X}$

In this section we show the equivalence of $\mathsf{MLL1^X}$ and $\mathsf{MLS1^X}$.

Lemma 20. *If* $\vdash \Gamma$ *is provable in* $\mathsf{MLL1^X}$ *then* $\bigvee(\Gamma)$ *is provable in* $\mathsf{MLS1^X}$.

Proof. This is a straightforward induction on the proof of $\vdash \Gamma$ in $\mathsf{MLL1^X}$, making a case analysis on the bottommost rule instance. We show here only the case of $\forall\,\dfrac{\vdash \Delta, A}{\vdash \Delta, \forall x.A}$ (all other cases are simpler or have been shown before, e.g. [29]): By induction hypothesis, there is a proof of $\bigvee(\Delta) \vee A$ in $\mathsf{MLS1^X}$. We can prefix every line in that proof by $\forall x$ and then compose the following derivation:

$$\equiv\,\dfrac{\begin{array}{c}\forall\,\dfrac{\mathsf{t}}{\forall x.\mathsf{t}}\\ \mathsf{MLS1^X}\big\Vert\\ \forall x.\bigvee(\Delta)\vee A\end{array}}{\bigvee(\Delta)\vee\forall x.A}$$

where we can apply the $\equiv$-rule because x is not free in Δ. □

Lemma 21. *Let* $\mathsf{r}\,\dfrac{S\{A\}}{S\{B\}}$ *be an inference rule in* $\mathsf{MLS1^X}$. *Then the sequent* $\vdash \overline{A}, B$ *is provable in* $\mathsf{MLL1^X}$.

Proof. This is a straightforward exercise. □

Lemma 22. *Let* A, B *be formulas, and let* $S\{\cdot\}$ *be a (positive) context. If* $\vdash \overline{A}, B$ *is provable in* $\mathsf{MLL1^X}$, *then so is* $\vdash \overline{S\{A\}}, S\{B\}$.

Proof. Straightforward induction on $S\{\cdot\}$. (see e.g. [34]) □

Lemma 23. *If a formula* C *is provable in* $\mathsf{MLS1^X}$ *then* $\vdash C$ *is provable in* $\mathsf{MLL1^X}$.

Proof. We proceed by induction on the number of inference steps in the proof of C in $\mathsf{MLS1^X}$. Consider the bottommost rule instance $\mathsf{r}\,\dfrac{S\{A\}}{S\{B\}}$. By induction hypothesis we have a $\mathsf{MLL1^X}$ proof Π of $\vdash S\{A\}$. By Lemmas 22 and 23, we have a $\mathsf{MLL1^X}$ proof of $\vdash \overline{S\{A\}}, S\{B\}$. We can compose them via cut:

$$\mathsf{cut}\,\dfrac{\vdash S\{A\} \quad \vdash \overline{S\{A\}}, S\{B\}}{\vdash S\{B\}}$$

and then apply Theorem 2. □

B. *Contraction and Weakening*

The first observation here is that Lemmas 21–24 from above also hold for $\mathsf{LK1}$ and $\mathsf{KS1}$. We therefore immediately have:

Theorem 24. *For every sequent* Γ, *we have that* $\vdash \Gamma$ *is provable in* $\mathsf{LK1}$ *if and only if* $\bigvee(\Gamma)$ *is provable in* $\mathsf{KS1}$.

Then Theorem 16 is an immediate consequence. Let us now proceed with providing further lemmas that will be needed for the other results.

Lemma 25. *The* c*-rule is derivable in* $\{\mathsf{ac}, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\}$.

Proof. This can be shown by a straightforward induction on A (for details, see e.g. [29]). □

Lemma 26. $\mathsf{w}_\forall, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists$ *are derivable in* $\{\mathsf{w}, \mathsf{c}, \equiv\}$.

Proof. We only show the cases for $\mathsf{w}_\forall$ and $\mathsf{c}_\forall$ (for the others see [29]):

$$\mathsf{c}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{A}{A \vee (\forall x.A)}}{\forall x.(A \vee A)}}{\forall x.A} \qquad \mathsf{c}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.\forall x.A}{\forall x.((\forall x.A) \vee A)}}{(\forall x.A) \vee (\forall x.A)}}{\forall x.A} \tag{5}$$

where in the first derivation, x is not free in A, and in the second one not free in $\forall x.A$. □

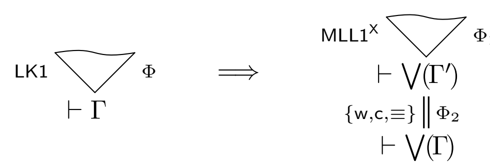
Lemma 27. *Let* A *and* B *be formulas. Then*

$$\begin{array}{c} A \\ {\scriptstyle\{\mathsf{w},\mathsf{c},\equiv\}}\big\Vert \\ B \end{array} \quad\Longleftrightarrow\quad \begin{array}{c} A \\ {\scriptstyle\{\mathsf{w},\mathsf{w}_\forall,\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}}\big\Vert \\ B \end{array}$$

Proof. Immediately from Lemmas 26 and 27. □

C. *Rule Permutations*

Theorem 28. *Let* Γ *be a sequent. If* $\vdash \Gamma$ *is provable in* $\mathsf{LK1}$ *(as depicted on the left below) then there is a sequent* Γ' *not containing any* f, *such that there is a derivation as shown on the right below:*



Proof. First, we can replace every instance of the f-rule in Φ by wk. Then the instances of wk and ctr are replaced by w and c, which can then be permuted down. (Details are in Appendix **??**. □

Lemma 29. *For every derivation* $\begin{array}{c}\mathsf{t}\\ {\scriptstyle\mathsf{MLS1^X}}\big\Vert\\ A\end{array}$ *there are formulas* A' *and* A'' *such that*

$$\begin{array}{c} \mathsf{t} \\ {\scriptstyle\{\forall,\mathsf{ai},\mathsf{t}\}}\big\Vert \\ A'' \\ {\scriptstyle\{\mathsf{s},\mathsf{mix},\equiv\}}\big\Vert \\ A' \\ {\scriptstyle\{\exists\}}\big\Vert \\ A \end{array}$$

Proof. First, observe that the $\exists$ rule can be permuted downwards over all the other rules since $A[x/t]$ has the same structure as A and none of the other rules has a premise of the form $S\{\exists x.A\}$. It suffices now to prove that all rules in $\{\forall, \mathsf{ai}, \mathsf{t}\}$ can be permuted over the rules in $\{\mathsf{s}, \mathsf{mix}, \equiv\}$, which is straightforward (see [37] for details). □

Lemma 30. *For every derivation* $\begin{array}{c} A \\ \{\mathsf{w},\mathsf{w}_\forall,\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\} \Big\| \\ B \end{array}$ *there are formulas A' and B' such that*

$$\begin{array}{c} A \\ \{\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\} \Big\| \\ A' \\ \{\mathsf{ac},\mathsf{c}_\forall\} \Big\| \\ B' \\ \{\mathsf{w},\mathsf{w}_\forall,\equiv\} \Big\| \\ B \end{array}$$

Proof. We first permute all instances of w and $\mathsf{w}_\forall$ to the bottom of the derivation and then permute in a second step the rules c and $\mathsf{c}_\forall$ below $\{\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists\}$. This involves a tedious but straightforward case analysis. However, unlike most other rule permutations in this paper this has not been done before in the deep inference literature. For this reason, we give the full case analysis in Appendix **??**. Note that this Lemma is the reason for the presence of the rules $\mathsf{w}_\forall$ and $\mathsf{c}_\forall$, as without them the permutation cases in (**??**) could not be resolved. □

We can now complete the proof of Theorems 17 and 19.

Proof of Theorem 19. Assume we have a proof of A in $\mathsf{KS1}$. By Theorem 25 we have a proof of $\vdash A$ in $\mathsf{LK1}$ to which we can apply Theorem 29. Finally, we apply Lemma 21 to get the desired shape. □

Proof of Theorem 17. Assume we have a proof of A in $\mathsf{KS1}$. We first apply Theorem 19, and then Lemma 30 to the upper half and Lemmas **??** and 31 to the lower half. □

VIII. Fonets and Linear Proofs

A. From $\mathsf{MLL1}^\mathsf{X}$ Proofs to Fonets

Let Π be a $\mathsf{MLL1}^\mathsf{X}$ proof of a rectified sequent $\vdash \Gamma$ not containing f. We now show how Π is translated into a linked fograph $[\![\Pi]\!] = \langle [\![\Gamma]\!], \sim_\Pi \rangle$. We proceed inductively, making a case analysis on the last rule in Π. At the same time we are constructing a dualizer δ_Π, so that in the end we can conclude that $[\![\Pi]\!]$ is in fact a fonet.

1) Π is $\mathsf{ax}\,\dfrac{}{\vdash a, \bar{a}}$: Then the only link is $\{a, \bar{a}\}$, and δ_Π is empty.
2) Π is $\mathsf{t}\,\dfrac{}{\vdash \mathsf{t}}$: Then $\sim_\Pi$ and δ_Π are both empty.
3) The last rule in Π is $\mathsf{mix}\,\dfrac{\vdash \Gamma' \quad \vdash \Gamma''}{\vdash \Gamma', \Gamma''}$: By induction hypothesis, we have proofs Π' and Π'' of Γ' and Γ'', respectively. We have $[\![\Gamma]\!] = [\![\Gamma']\!] + [\![\Gamma'']\!]$ and we can let $\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''}$ and $\delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$.
4) The last rule in Π is $\vee\,\dfrac{\vdash \Gamma_1, A, B}{\vdash \Gamma_1, A \vee B}$: By induction hypothesis, there is a proof Π' of $\Gamma' = \Gamma_1, A, B$. We have $[\![\Gamma]\!] = [\![\Gamma']\!]$ and let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.
5) The last rule in Π is $\wedge\,\dfrac{\vdash \Gamma_1, A \quad \vdash B, \Gamma_2}{\vdash \Gamma_1, A \wedge B, \Gamma_2}$: By induction hypothesis, we have proofs Π' and Π'' of $\Gamma' = \Gamma_1, A$ and $\Gamma'' = B, \Gamma_2$, respectively. We have $[\![\Gamma]\!] = [\![\Gamma_1]\!] + ([\![A]\!] \times [\![B]\!]) + [\![\Gamma_2]\!]$ and we let $\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''}$ and $\delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$.
6) The last rule in Π is $\exists\,\dfrac{\vdash \Gamma_1, A[x/t]}{\vdash \Gamma_1, \exists x.A}$: By induction hypothesis, there is a proof Π' of $\Gamma' = \Gamma_1, A[x/t]$. For each atom in $\Gamma' = \Gamma_1, A[x/t]$, there is a corresponding atom in $\Gamma = \Gamma_1, \exists x.A$. We can therefore define the linking $\sim_\Pi$ from the linking $\sim_{\Pi'}$ via this correspondence. Then, we let δ_Π be $\delta_{\Pi'} + [x/t]$. Since Γ is rectified x does not yet occur in $\delta_{\Pi'}$. Hence δ_Π is a dualizer of $[\![\Pi]\!]$.
7) The last rule in Π is $\forall\,\dfrac{\vdash \Gamma_1, A}{\vdash \Gamma_1, \forall x.A}\,(x \text{ not free in } \Gamma_1)$: By induction hypothesis, there is a proof Π' of $\Gamma' = \Gamma_1, A$, which has the same atoms as in $\Gamma = \Gamma_1, \forall x.A$. Hence, we can let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.

Theorem 31. *If Π is a $\mathsf{MLL1}^\mathsf{X}$ proof of a rectified f-free sequent $\vdash \Gamma$, then $[\![\Pi]\!]$ is a fonet and δ_Π a dualizer for it.*

Proof. We have to show that none of the operations above can introduce a bimatching. For cases 1–6, this is immediate. For case 7, observe that there is a potential dependency from each existential binder in $[\![\Gamma']\!]$ to the new x-binder $\bullet x$ in $[\![\Gamma]\!]$. However, observe that this $\bullet x$ vertex is not connected to any vertex in $[\![\Gamma']\!]$, and hence no such new dependency can be extended to a bimatching. That δ_Π is a dualizer for $[\![\Pi]\!]$ follows immediately from the construction. Hence, $[\![\Pi]\!]$ is a fonet. □

B. From $\mathsf{MLS1}^\mathsf{X}$ Proofs to Fonets

There is a more direct path from a $\mathsf{MLL1}^\mathsf{X}$ proof Π of a rectified sequent Γ to the linked fograph $[\![\Pi]\!]$: simply take the fograph $[\![\Gamma]\!]$, and let the equivalence classes of $\sim_\Pi$ be all the atom pairs that meet in an instance of ax, and δ_Π is simply the collection of all substitutions of all the instances of the $\exists$-rule in Π. We have chosen the more cumbersome path above because it gives us a direct proof of Theorem 32. However, for translating $\mathsf{MLS1}^\mathsf{X}$ derivation into fonets, we employ exactly that direct path.

First observe that in a derivation in $\mathsf{MLS1}^\mathsf{X}$ where the conclusion is rectified, every line is also rectified, as the only rules involving bound variables are $\forall$ and $\exists$ which both remove a binder. Therefore, we can call such a derivation ***rectified***, and for a non-rectified $\mathsf{MLS1}^\mathsf{X}$ derivation Φ we can define its ***rectification*** $\widehat{\Phi}$ inductively, by rectifying each line, proceeding step-wise from conclusion to premise.[2]

A rectified derivation $\begin{array}{c} \mathsf{t} \\ \mathsf{MLS1}^\mathsf{X} \Big\| \Phi \\ A \end{array}$ determines a substitution which maps the existential bound variables occurring in A to the terms substituted for them in the instances of the $\exists$-rule in

[2] As for formulas, the rectification of a derivation is unique up to renaming of bound variables.

Φ. We denote this substitution with δ_Φ and call it the ***dualizer*** of Φ. Furthermore, every atom occurring in the conclusion A must be consumed by a unique instance of the rule ai in Φ. This allows us to define a (partial) equivalence relation $\sim_\Phi$ on the atom occurrences in A by $a \sim_\Phi b$ if a and b are consumed by the same instance of ai in Φ. We call $\sim_\Phi$ the ***linking*** of Φ, and define $[\![\Phi]\!] = \langle [\![A]\!], \sim_\Phi \rangle$.

Theorem 32. *Let* $\mathsf{MLS1}^{\mathsf{X}} \overset{\mathsf{t}}{\underset{A}{\Big\Vert}} \Phi$ *be a rectified derivation where* A *is* f*-free. Then* $[\![\Phi]\!]$ *is a fonet and* δ_Φ *a dualizer for it.*

For proving this theorem, we have to show that no inference rule in $\mathsf{MLS1}^{\mathsf{X}}$ can introduce a bimatching. To simplify the argument, we introduce the ***frame*** [35] of the fograph $\mathcal{C}$, which is a linked (proposional) cograph in which the dependencies between the binders in $\mathcal{C}$ are encoded as links.

More formally, let C be a formula with $[\![C]\!] = \mathcal{C}$, to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent $C^\star$:

1) **Encode dependencies as fresh links.** For each dependency $\{\bullet x_i, \bullet y_j\}$ in $\mathcal{C}$, with corresponding subformulas $\exists x_i.A$ and $\forall y_j.B$ in C, we pick a fresh (nullary) predicate symbol $q_{i,j}$, and then replace $\exists x_i.A$ by $\overline{q}_{i,j} \wedge \exists x_i.A$, and replace $\forall y_j.B$ by $q_{i,j} \vee \forall y_j.B$.
2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace $\exists x_i.A$ by A and replace $\forall y_j.B$ by B everywhere.
3) **Simplify atoms.** After step 2, replace every predicate $p(t_1 \cdots t_n)$ (resp. $\overline{p}(t_1 \cdots t_n)$) with a nullary predicate symbol p (resp. $\overline{p}$)

Then $\sim_{C^\star}$ consists of the pairs induced by $\sim_\mathcal{C}$ and the new pairs $\{q_{i,j}, \overline{q}_{i,j}\}$ introduced in step 1 above. We call $C^\star$ the ***frame*** of C and we define the ***frame*** of $\mathcal{C}$, denoted $\mathcal{C}^\star$, as $\langle [\![C^\star]\!], \sim_{C^\star} \rangle$.

Lemma 33. *If a linked fograph* $\mathcal{C}$ *has an induced bimatching then so does its frame* $\mathcal{C}^\star$.

Proof. Immediately from the construction of the frame. □

Proof of Theorem 34. From Φ we construct a derivation $\Phi^\star$ of $A^\star$ in the propositional fragment of $\mathsf{MLS1}^{\mathsf{X}}$, such that $[\![\Phi^\star]\!] = [\![\Phi]\!]^\star$. The rules $\mathsf{ai}, \mathsf{t}, \mathsf{mix}$ and s are translated trivially, and for $\equiv$, it suffices to observe that the frame construction is invariant under $\equiv$. Finally, for the rules $\forall$ and $\exists$, proceed as follows. Every instance of $\forall$ is replaced by the derivation on the right below:[3]

$$\forall \frac{S\{\mathsf{t}\}}{S\{\forall y_j.\mathsf{t}\}} \quad \rightsquigarrow \quad \begin{array}{c} \mathsf{t} \\ \{\mathsf{ai},\mathsf{t}\} \Big\Vert \Psi_1 \\ S\{(q_{h_1,j} \vee \overline{q}_{h_1,j}) \wedge \cdots \wedge (q_{h_j,j} \vee \overline{q}_{h_j,j}) \wedge \mathsf{t}\} \\ \{\mathsf{s},\equiv\} \Big\Vert \Psi_2 \\ S\{q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\overline{q}_{h_1,j} \wedge \cdots \wedge \overline{q}_{h_j,j} \wedge \mathsf{t})\} \end{array}$$

where $h_1, \ldots, h_j$ range over the indices of the existential binders dependend on that y_j. It is easy to see how Ψ_1 is constructed, and for Ψ_2 see, e.g. [?], [34], [36], [37]. Then, every occurrence of $\forall y_j.F$ is replaced by $q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\overline{q}_{h_1,j} \wedge \cdots \wedge \overline{q}_{h_j,j} \wedge F)$ in the derivation below that $\forall$-instance. Now, observe that all instances of the $\exists$-rule introducing x_i dependend on y_j must occur below in the derivation (otherwise Φ would not be rectified). Now consider such an instance $\exists \dfrac{S\{B[x_i/t]\}}{S\{\exists x_i.B\}}$. Its context $S\{\cdot\}$ must contain all the $\forall y_j$ the $\exists x_i$ depends on, such that B is in their scope. Following the translation of the $\forall$ rules above, we can therefore translate the $\exists$-rule instance by the following derivation

$$\begin{array}{c} S_0\{\overline{q}_{i,k_1} \wedge S_1\{\overline{q}_{i,k_2} \wedge \cdots S_{k_i-1}\{\overline{q}_{i,k_i} \wedge S_{k_i}\{B'\}\} \cdots\}\} \\ \{\mathsf{s},\equiv\} \Big\Vert \Psi_3 \\ S_0\{S_1\{\cdots S_{k_i-1}\{S_{k_i}\{\overline{q}_{i,k_1} \wedge \overline{q}_{i,k_2} \wedge \cdots q_{i,k_i} \wedge B'\}\} \cdots\}\} \end{array}$$

where $k_1, \ldots, k_i$ are the indices of of the universal binders on which that x_i depends, and B' is B in which all predicates are replaced by a nullary one (step 3 in the frame construction). The derivation Ψ_3 can be constructed in the same way as Ψ_2. Doing this to all instances of the rules $\forall$ and $\exists$ in Φ yields indeed a propositional derivation $\Phi^\star$ with $[\![\Phi^\star]\!] = [\![\Phi]\!]^\star$. It has been shown by Retoré [?] and rediscovered by Straßburger [37] that $[\![\Phi^\star]\!] = \langle [\![C^\star]\!], \sim_{\Phi^\star} \rangle$ can not contain an induced bimatching. By Lemma 37, $[\![\Phi]\!]$ does not have an induced bimatching either. Furthemore, it followed from the definition of δ_Φ that it is a dualizer for $[\![\Phi]\!]$. □

Remark 34. There is an alternative path of proving Theorem 34 by translating Φ to an $\mathsf{MLL1}^{\mathsf{X}}$-proof Π, observing that this process preserves the linking and the dualizer. However, for this, we have to extent the construction above to the cut-rule, and then show that linking and dualizer of a sequent proof Π are invariant under cut elimination. This can be done similarly to unification nets in [35].

C. From Fonets to $\mathsf{MLL1}^{\mathsf{X}}$ Proofs

Now we are going to show how from a given fonet $\langle \mathcal{C}, \sim_\mathcal{C} \rangle$ we can construct a sequent proof Π in $\mathsf{MLL1}^{\mathsf{X}}$ such that $[\![\Pi]\!] = \langle \mathcal{C}, \sim_\mathcal{C} \rangle$. In the proof net literature, this operation is also called *sequentialization*. The basic idea behind our sequentialization is to use the frame of $\mathcal{C}$, to which we can apply the *splitting tensor theorem*, and then reconstruct the sequent proof Π.

Let Γ be a propositional sequent and $\sim_\Gamma$ be a linking for $[\![\Gamma]\!]$. A conjunction formula $A \wedge B$ is ***splitting*** or a ***splitting tensor*** if $\Gamma = \Gamma', A \wedge B, \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that $\sim_1$ is a linking for $[\![\Gamma', A]\!]$ and $\sim_2$ is a linking for $[\![B, \Gamma'']\!]$, i.e., removing the $\wedge$ from $A \wedge B$ splits the linked fograph $\langle [\![\Gamma]\!], \sim_\Gamma \rangle$ into two fographs. We say that $\langle [\![\Gamma]\!], \sim_\Gamma \rangle$ is ***mixed*** iff $\Gamma = \Gamma', \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that $\sim_1$ is a linking for $[\![\Gamma']\!]$ and $\sim_2$ is a linking for $[\![\Gamma'']\!]$. Finally, $\langle [\![\Gamma]\!], \sim_\Gamma \rangle$ is ***splittable*** if it is mixed or has a splitting tensor.

Theorem 35. *Let* Γ *be a* f*-free propositional sequent containing only atoms and* $\wedge$*-formulas, and* $\sim_\Gamma$ *be a linking for* $[\![\Gamma]\!]$*. If* $\langle [\![\Gamma]\!], \sim_\Gamma \rangle$ *does not induce a bimatching then it is splittable.*

This is the well-known splitting-tensor-theorem [24], [38], adapted for the presence of mix. In the setting of linked

[3]For better readability we omit superfluous parentheses, knowing that we always have $\equiv$ incorporating associativity and commutativity of $\wedge$ and $\vee$.

cographs, it has first been proved by Retoré [39], [40] and then rediscovered by Hughes [9]. We use it now for our sequentialization:

Theorem 36. *Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let Γ be a sequent with $[\![\Gamma]\!] = \mathcal{C}$. Then there is an $\mathsf{MLL1^X}$-proof Π of Γ, such that $[\![\Pi]\!] = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$.*

Proof. Let $\delta_{\mathcal{C}}$ be the dualizer of $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. We proceed by induction on the size of Γ (i.e., the number of symbols in it, without counting the commas). If Γ contains a formula with $\vee$-root, or a formula $\forall x.A$, we can immediately apply the $\vee$-rule or the $\forall$-rule of $\mathsf{MLL1^X}$ and proceed by induction hypothesis. If Γ contains a formula $\exists x.A$ such that the corresponding binder $\bullet x$ in $\mathcal{C}$ has no dependency, then we can apply the $\exists$-rule, choosing the term t as determined by $\delta_{\mathcal{C}}$, and proceed by induction hypothesis. Hence, we can now assume that Γ contains only atoms, $\wedge$-formulas, or formulas of shape $\exists x.A$, where the vertex $\bullet x$ has dependencies. Then the frame $\langle [\![\Gamma^\star]\!], \sim_{\Gamma^\star} \rangle$ does not induce a bimatching and contains only atoms and $\wedge$-formulas, and is therefore splittable. If it is mixed, then we can apply the mix-rule to Γ and apply the induction hypothesis to the two components. If it is not mixed then there must be a splitting tensor. If the splitting $\wedge$ is already in Γ, then we can apply the $\wedge$-rule and proceed by induction hypothesis on the two branches. However, if $\Gamma^\star$ is not mixed and all splitting tensors are $\wedge$-formulas introduced in step 1 of the frame construction, then we get a contradiction as in that case there must be a $\vee$- or $\forall$-formula in Γ. □

D. From Fonets to $\mathsf{MLS1^X}$ Proofs

We can now straightforwardly obtain the same result for $\mathsf{MLS1^X}$:

Theorem 37. *Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let C be a formula with $[\![C]\!] = \mathcal{C}$. Then there is a derivation $\begin{array}{c} \mathsf{t} \\ \mathsf{MLS1^X} \big\Vert \Phi \\ C \end{array}$ such that $[\![\Phi]\!] = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$.*

Proof. We apply Theorem 39 to obtain a sequent proof Π of $\vdash C$ with $[\![\Pi]\!] = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. Then we apply Lemma 21, observing that the translation from $\mathsf{MLL1^X}$ to $\mathsf{MLS1^X}$ preserves linking and dualizer. □

Remark 38. Note that it is also possible to do a direct "sequentialization" into the deep inference system $\mathsf{MLS1^X}$, using the techniques presented in [37] and [41].

IX. Skew Bifibrations and Resource Management

In this section we establish the relation between skew bifibrations and derivations in $\{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\}$. However, if a derivation Φ contains instances of the rules $\mathsf{c}_\forall$, $\mathsf{m}_\forall$, and $\mathsf{m}_\exists$ we can no longer naively define the rectification $\widehat{\Phi}$ as in the previous section for $\mathsf{MLS1^X}$, as these two rules cannot be applied if premise and conclusion are rectified. For this reason we define here rectified versions $\widehat{\mathsf{c}_\forall}$, $\widehat{\mathsf{m}_\forall}$ and $\widehat{\mathsf{m}_\exists}$, shown below:

$$\widehat{\mathsf{c}_\forall}\,\frac{S\{\forall y.\forall x.Ax\}}{S\{\forall x.Ax\}} \qquad \widehat{\mathsf{m}_\forall}\,\frac{S\{(\forall y.Ay) \vee (\forall z.Bz)\}}{S\{\forall x.(Ax \vee Bx)\}} \qquad \widehat{\mathsf{m}_\exists}\,\frac{S\{(\exists y.Ay) \vee (\exists z.Bz)\}}{S\{\exists x.(Ax \vee Bx)\}}$$

Here, we use the notation $A\cdot$ for a formula A with occurrences of a placeholder $\cdot$ for a variable. Then Ax stands for the results of replacing that placeholder with x, and also indicating that x must not occur in $A\cdot$. Then $\forall x.Ax$ and $\forall y.Ay$ are the same formula modulo renaming of the bound variable bound by the outermost $\forall$-quantifier. We also demand that the variables x, y, and z do not occur in the context $S\{\cdot\}$.

Note that in an instance of $\widehat{\mathsf{m}_\forall}$ or $\widehat{\mathsf{m}_\exists}$ (as shown above), we can have $x = y$ or $x = z$, but not both if the premise is rectified. If $x = y$ and $x = z$ we have $\mathsf{m}_\forall$ and $\mathsf{m}_\exists$ as special cases of $\widehat{\mathsf{m}_\forall}$ and $\widehat{\mathsf{m}_\exists}$, respectively. And similarly, if $x = y$ then $\mathsf{c}_\forall$ is a special case of $\widehat{\mathsf{c}_\forall}$.

For a derivation Φ in $\{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\}$, we can now construct the ***rectification*** $\widehat{\Phi}$ by rectifying each line of Φ, yielding a derivation in $\{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\}$.

For each instance $\mathsf{r}\,\dfrac{Q}{P}$ of an inference rule in $\{\mathsf{w}, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\}$ we can define the ***induced map*** $\lfloor \mathsf{r} \rfloor \colon V_{[\![Q]\!]} \to V_{[\![P]\!]}$ which acts as the identity for $\mathsf{r} \in \{\mathsf{m}, \equiv\}$ and as the canonical injection for $\mathsf{r} \in \{\mathsf{w}, \mathsf{w}_\forall\}$. For $\mathsf{r} = \mathsf{ac}$ it maps the vertices corresponding to the two atoms in the premise to vertex of the contracted atom in the conclusion, and for $\mathsf{r} \in \{\widehat{\mathsf{c}_\forall}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}\}$ it maps the two vertices corresponding to the quantifiers in the premise to the one in the conclusion (and acts as the identity on all other vertices). For a derivation Φ in $\{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\}$ we can then define the ***induced map*** $\lfloor \Phi \rfloor$ as the composition of the induced maps of the rule instances in Φ.

Lemma 39. *Let $\begin{array}{c} A \\ \{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\} \big\Vert \Phi \\ B \end{array}$ be given. Then there is a rectified derivation $\begin{array}{c} \widehat{A} \\ \{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\} \big\Vert \widehat{\Phi} \\ \widehat{B} \end{array}$, such that the induced maps $\lfloor \Phi \rfloor \colon [\![A]\!] \to [\![B]\!]$ and $\lfloor \widehat{\Phi} \rfloor \colon [\![\widehat{A}]\!] \to [\![\widehat{B}]\!]$ are equal up to a variable renaming of the vertex labels.*

Proof. Immediate from the definition. □

A. From Contraction and Weakening to Skew Bifibrations

Lemma 40. *Let $\begin{array}{c} A \\ \{\mathsf{w}, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\} \big\Vert \Phi \\ B \end{array}$ be a rectified derivation. Then the induced map $\lfloor \Phi \rfloor \colon [\![A]\!] \to [\![B]\!]$ is a skew bifibration.*

Before we show the proof of this lemma, we introduce another useful concept: the ***propositional encoding*** A° of a formula A, which is a propositional formula with the property that $[\![A^\circ]\!] = [\![A]\!]$. For this, we introduce new propositional variables that have the same names as the (first-order) variables $x \in \textsf{VAR}$. Then A° is defined inductively by:

$$
\begin{array}{rclcrcl}
a^\circ &=& a & & & & \\
(A \vee B)^\circ &=& A^\circ \vee B^\circ & \qquad & (\forall x A)^\circ &=& x \vee A^\circ \\
(A \wedge B)^\circ &=& A^\circ \wedge B^\circ & & (\exists x A)^\circ &=& x \wedge A^\circ
\end{array}
$$

Lemma 41. *For every formula A, we have $[\![A^\circ]\!] = [\![A]\!]$.*

Proof. Straightforward induction on A. □

We use $\equiv^\circ$ to denote the restriction of $\equiv$ to propositional formulas, i.e., the first two lines in (2).

Proof of Lemma 43. First, observe that for every inference rule $\mathsf{r} \in \{\mathsf{w}, \mathsf{w}_\forall, \mathsf{ac}, \widehat{\mathsf{c}_\forall}, \mathsf{m}, \widehat{\mathsf{m}_\forall}, \widehat{\mathsf{m}_\exists}, \equiv\}$ the induced map $\lfloor \mathsf{r} \rfloor \colon V_{[\![Q]\!]} \to V_{[\![P]\!]}$ defines a existential preserving graph homomorphism $[\![Q]\!] \to [\![P]\!]$ and a fibration on the corresponding binding graphs. Therefore, their composition $\lfloor \Phi \rfloor$ has the same properties fibration.

For showing that it is also a skew fibration, we construct for Φ its propositional encoding Φ° by translating every line into its propositional encoding. The instances of the rules $\widehat{\mathsf{m}_\forall}$ and $\widehat{\mathsf{m}_\exists}$ are replaced by:

$$
\widehat{\mathsf{ac}}\,\dfrac{\equiv\,\dfrac{S\{(y \vee (Ay)^\circ) \vee (z \vee (Bz)^\circ)\}}{S\{(y \vee z) \vee ((Ay)^\circ \vee (Bz)^\circ)\}}}{S\{x \vee ((Ax)^\circ \vee (Bx)^\circ)\}}
\qquad
\widehat{\mathsf{ac}}\,\dfrac{\mathsf{m}\,\dfrac{S\{(y \wedge (Ay)^\circ) \vee (z \wedge (Bz)^\circ)\}}{S\{(y \vee z) \wedge ((Ay)^\circ \vee (Bz)^\circ)\}}}{S\{x \wedge ((Ax)^\circ \vee (Bx)^\circ)\}}
$$

respectively, where $\widehat{\mathsf{ac}}$ is a ac that renames the variables—the propositional variable, as well as the first-order variable of the same name—as everything is rectified, there is no ambiguity here. Any instance of a rule w, ac, m, or $\equiv$ is translated to an instance of the same rule, and $\widehat{\mathsf{c}_\forall}$ is translated to $\widehat{\mathsf{ac}}$.

This gives us a derivation $\begin{array}{c} A^\circ \\ {\scriptstyle \{\mathsf{w},\mathsf{ac},\widehat{\mathsf{ac}},\mathsf{m},\equiv^\circ\}} \big\| {\scriptstyle \Phi^\circ} \\ B^\circ \end{array}$ such that $\lfloor \Phi^\circ \rfloor = \lfloor \Phi \rfloor$. It has been shown in [22] that $\lfloor \Phi^\circ \rfloor$ is a skew fibration. Hence, $\lfloor \Phi \rfloor$ is a skew fibration. □

B. From Skew Bifibrations to Contraction and Weakening

Lemma 42. *Let $\mathcal{A}$ and $\mathcal{B}$ be fographs, let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be a skew bifibration, and let A and B be formulas with $[\![A]\!] = \mathcal{A}$ and $[\![B]\!] = \mathcal{B}$. Then there are derivations*

$$
\begin{array}{c} A \\ {\scriptstyle \{\mathsf{w},\mathsf{ac},\widehat{\mathsf{c}_\forall},\mathsf{m},\widehat{\mathsf{m}_\forall},\widehat{\mathsf{m}_\exists},\equiv\}} \big\| {\scriptstyle \widehat{\Phi}} \\ B \end{array}
\quad \textit{and} \quad
\begin{array}{c} A\rho_\varphi \\ {\scriptstyle \{\mathsf{w},\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}} \big\| {\scriptstyle \Phi} \\ B \end{array}
$$

such that $\lfloor \widehat{\Phi} \rfloor = \varphi$ and $\widehat{\Phi}$ is a rectification of Φ, and ρ_φ is the substitution induced by φ.

In the proof of this lemma, we make use of the following concept: Let $\begin{array}{c} P \\ {\scriptstyle \mathsf{S}} \big\| {\scriptstyle \Psi} \\ Q \end{array}$ be a derivation where P and Q are propositional formulas (possibly using variable $x \in \textsf{VAR}$ at the places of atoms). We say that Ψ can be ***lifted*** to S' if there are (first-order) formulas C and D such that $P = C^\circ$ and $Q = D^\circ$ and there is a derivation $\begin{array}{c} C \\ {\scriptstyle \mathsf{S}'} \big\| {\scriptstyle \Psi'} \\ D \end{array}$.

Proof of Lemma 45. By Lemma 44 we have $\mathcal{A} = [\![A^\circ]\!]$ and $\mathcal{B} = [\![B^\circ]\!]$. Let $V'_{\mathcal{B}} \subseteq V_{\mathcal{B}}$ be the image of φ, and let $\mathcal{B}_1$ be the subgraph of $\mathcal{B}$ induced by $V'_{\mathcal{B}}$. Hence, we have two maps $\varphi'' \colon \mathcal{A} \to \mathcal{B}_1$ being a surjection and $\varphi' \colon \mathcal{B}_1 \to \mathcal{B}$ being an injection that reflects edges. Both, φ' and φ'' remain skew bifibrations. Let us first look at φ'. Let $\tilde{B}_1$ be the propositional formula obtained from B° by removing all atoms that are not represented by vertices in $V'_{\mathcal{B}}$. Then $[\![\tilde{B}_1]\!] = \mathcal{B}_1$. By [22, Proposition 7.6.1], we have a derivation $\begin{array}{c} \tilde{B}_1 \\ {\scriptstyle \{\mathsf{w},\equiv^\circ\}} \big\| {\scriptstyle \Phi_1^\circ} \\ B^\circ \end{array}$. A subformula of B° is called *weak* if it has been introduced by a weakening. All subformulas of a weak formula are also weak. Two weak subformulas B' and B'' of B° form a *weak pair* if $B^\circ \equiv S\{B' \vee B''\}$ for some context $S\{\cdot\}$. We can assume without loss of generality that whenever weak subformulas B' and B'' form a weak pair, they have been introduced by the same instance of w in Φ_1°.[4] Now we show that Φ_1° can be lifted. For this, observe that whenever a weakening in Φ_1° deletes an atom $x \in \textsf{VAR}$, it must also delete all atoms in the scope of the corresponding quantifier, because φ' is a fibration on the binding graph. Hence, each line in Φ_1° is the propositional encoding P° of a first-order formula P. We now have to show that each instance of w is indeed a correct application in first-order logic. For this we have to inspect the cases a weakening happens inside a subformula $x \vee C$ or $x \wedge C$ in Φ_1°. There are the following cases:

$$
\dfrac{S\{x \vee C\}}{S\{x \vee D \vee C\}}
\qquad
\dfrac{S\{x \wedge C\}}{S\{x \wedge (D \vee C)\}}
\qquad
\dfrac{S\{x \wedge C\}}{S\{(x \vee D) \wedge C\}}
$$

In the first case the weakening happens inside the scope of a $\forall$-quantifier, and in the second case inside the scope of a $\exists$-quantifier. Both cases are unproblematic in first-order logic. However, in the third case an $\exists$-quantifier would be transformed into an $\forall$-quantifier. But as φ has to preserve existentials, this third case cannot occur. Thus we have a first order derivation $\begin{array}{c} B_1 \\ {\scriptstyle \{\mathsf{w},\equiv\}} \big\| {\scriptstyle \Phi_1} \\ B \end{array}$ with $B_1^\circ = \tilde{B}_1$.

Let us now look at φ''. Let $\mathcal{A}_1 = \mathcal{A}\sigma_\varphi$ be the graph obtained from $\mathcal{A}$ by applying σ_φ to all the labels. Note that $\mathcal{A}_1$ is not necessarily a fograph, as binders might be illegal. But it still is a cograph, and we have a surjective skew fibration $\varphi'' \colon \mathcal{A}_1 \to \mathcal{B}_1$ that preserves the labels. Therefore, by [42, Proposition 7.5], there is a derivation $\begin{array}{c} A_1^\circ \\ {\scriptstyle \{\mathsf{ac},\mathsf{m},\equiv^\circ\}} \big\| {\scriptstyle \Phi_2^\circ} \\ B_1^\circ \end{array}$, where $A_1^\circ = A^\circ\sigma_\varphi$ is the result of applying σ_φ to A°. Note that

[4] If Φ_1° is not of that shape, it can brought into this form by simple rule permutations, and then collapsing several weakenings into a single one.

$A_1^\circ = (A\sigma_\varphi)^\circ$ and B_1° are both proposional encodings. We plan to show that Φ_2 can be lifted to $\{\mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\}$. However, observe that not every formula occurring in Φ_2 is a propositional encoding. There are two reasons for this: (i) we might have $P \equiv^\circ Q$ where P is a propositional encoding but Q is not, and (ii) the rule ac can duplicate an atom $x \in \mathrm{VAR}$. Let us write ac_x for such instances. The problem with (i) is that we could have the following situation

$$\equiv^\circ \frac{S\{(x \wedge (E \wedge C)) \vee (x \wedge (F \wedge D))\}}{\mathsf{m}\,\dfrac{S\{((x \wedge E) \wedge C) \vee ((x \wedge F) \wedge D)\}}{S\{((x \wedge E) \vee (x \wedge F)) \wedge (C \vee D)\}}} \tag{6}$$

where x occurs in $C \vee D$. Then premise and conclusion are both proposional encodings, but the whole derivation cannot be lifted. However, since we demand that the the mapping is a fibration (and therefore a momomorphism) on the binding graphs, there must be another instance of m further below in the derivation:

$$\mathsf{m}\,\frac{S'\{(x \wedge E) \vee (x \wedge F)\}}{S'\{(x \vee x) \wedge (E \vee F)\}} \tag{7}$$

We can permute both instances via the following more general scheme (see [22], [**?**] for a general discussion on permutations of the m-rule):

$$\mathsf{m}\,\frac{S\{(G \wedge E \wedge C) \vee (G \wedge F \wedge D)\}}{\mathsf{m}\,\dfrac{S\{((G \wedge E) \vee (H \wedge F)) \wedge (C \vee D)\}}{S\{(G \vee H) \wedge (E \vee F) \wedge (C \vee D)\}}} \leftrightarrow \mathsf{m}\,\frac{S\{(G \wedge E \wedge C) \vee (G \wedge F \wedge D)\}}{\mathsf{m}\,\dfrac{S\{(G \vee H) \wedge ((E \wedge C) \vee (F \wedge D))\}}{S\{(G \vee H) \wedge (E \vee F) \wedge (C \vee D)\}}} \tag{8}$$

We omitted some instances of $\equiv^\circ$ and some parentheses. We now call instances of m as in (4) ***illegal***, and we can transform Φ_2° through m-permutations (6) into a derivation that does not contain any illegal m-instances. To address (ii), we also apply a permutation argument, permuting all instances of ac_x up until they either reach the top of the derivation or an instance of m which separates the two atoms in the premise. More precisely, we consider the following inference rule

$$\mathsf{ac}_x^{\equiv}\,\frac{S_0\{S_1\{x\} \vee S_2\{x\}\}}{S\{x\}} \tag{9}$$

where $S_1\{\cdot\} \equiv \{\cdot\} \vee E$ and $S_2\{\cdot\} \equiv \{\cdot\} \vee F$ and $S\{\cdot\} \equiv S_0\{\{\cdot\} \vee E \vee F\}$ for some formulas E and F, where E or F or both might be empty. The rule $\mathsf{ac}_x^{\equiv}$ permutes over $\equiv$, ac, and other instances of $\mathsf{ac}_x^{\equiv}$, and over instances of m if they occur inside S_0 or S_1 or S_2. The only situation in which $\mathsf{ac}_x^{\equiv}$ cannot be permuted up is the following:

$$\mathsf{m}\,\frac{S\{(R_1\{x\} \wedge C) \vee (R_2\{x\} \wedge D)\}}{\mathsf{ac}_x^{\equiv}\,\dfrac{S\{(R_1\{x\} \vee R_2\{x\}) \wedge (C \vee D)\}}{S\{R\{x\} \wedge (C \vee D)\}}} \tag{10}$$

We can therefore assume that all instances of ac_x, that contract an atom $x \in \mathrm{VAR}$ are either at the top of Φ_2° or below a m-instance as in (8). We now lift Φ_2° to $\{\mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists, \equiv\}$, proceed by induction on the height of Φ_2°, beginning at the top, making a case analysis on the topmost rule that is not a $\equiv$.

- ac_x: We know that the premisse of (7) is a propositional encoding. Hence, $S_1\{\cdot\} = \{\cdot\} \vee E^\circ$ and $S_2\{\cdot\} = \{\cdot\} \vee F^\circ$ and both x are universals, and $E^\circ \vee F^\circ$ contains all occurrences of x bound by that universal. We have the following subcases:
  - E and F are both non-empty: We have

    $$\mathsf{ac}_x^{\equiv}\,\frac{S^\circ\{(x \vee E^\circ) \vee (x \vee F^\circ)\}}{S^\circ\{x \vee (E^\circ \vee F^\circ)\}}$$

    which can be lifted to

    $$\mathsf{m}_\forall\,\frac{S\{(\forall x.E) \vee (\forall x.F)\}}{S\{\forall x.(E \vee F)\}}$$

    where $S^\circ\{\cdot\}, E^\circ, F^\circ$ are the propositional encodings of $S\{\cdot\}, E, F$, respectively.
  - E° is empty and F° is non-empty: We have

    $$\mathsf{ac}_x^{\equiv}\,\frac{S^\circ\{x \vee (x \vee F^\circ)\}}{S^\circ\{x \vee F^\circ)\}}$$

    which can be lifted to

    $$\mathsf{c}_\forall\,\frac{S\{\forall x.\forall x.F\}}{S\{\forall x.F\}}$$

  - E° is non-empty and F° is empty: This is similar to the previous case.
  - E° and F° are both empty: This is impossible as the premise would not be a propositional encoding.
- ac (contracting an ordinary atom): This can trivially be lifted.
- m: There are several cases to consider.
  - If none of the four principal formulas in the premise is x or $x \vee F$ for some formula F and $x \in \mathrm{VAR}$, then this instance of m can trivially be lifted, and and we can proceed by induction hypothesis.
  - If exactly one of the four principal formulas in the premise is x for some $x \in \mathrm{VAR}$, then this x is the encoding of an existential in the premise and of an universal in the conclusion. This is impossible, as φ has to preserve existentials.
  - If two of the four principal formulas in the premise are x for some $x \in \mathrm{VAR}$, then we are in the following special case of (8):

    $$\mathsf{m}\,\frac{S\{(x \wedge C) \vee (x \wedge D)\}}{\mathsf{ac}_x^{\equiv}\,\dfrac{S\{(x \vee x) \wedge (C \vee D)\}}{S\{x \wedge (C \vee D)\}}}$$

    which can be lifted immediately to

    $$\mathsf{m}_\exists\,\frac{S\{(\exists x.C) \vee (\exists x.D)\}}{S\{\exists x.(C \vee D)\}}$$

  - We have a situation (8) where $R_1\{x\} \equiv x \vee E$ for some E and $R_2\{x\} \equiv x \vee F$ for some F with

$R\{x\} \equiv x \vee E \vee F$ (Otherwise, the application of $\mathsf{ac}^{\equiv}_x$ would not be correct.) That means, we have:

$$\mathsf{ac}^{\equiv}_x \frac{\mathsf{m}\,\dfrac{S\{((x \vee E) \wedge C) \vee ((x \vee F) \wedge D)\}}{S\{((x \vee E) \vee (x \vee F)) \wedge (C \vee D)\}}}{S\{(x \vee E \vee F) \wedge (C \vee D)\}}$$

which can be lifted to

$$\mathsf{m}_\forall \frac{\mathsf{m}\,\dfrac{S\{((\forall x.E) \wedge C) \vee ((\forall x.F) \wedge D)\}}{S\{((\forall x.E) \vee (\forall x.F)) \wedge (C \vee D)\}}}{S\{(\forall x.(E \vee F)) \wedge (C \vee D)\}}$$

– In all other cases (e.g. exactly one of the principal formulas is of shape $x \vee F$ (and none is x), we can trivially lift the m-instance, as the quantifier structure is not affected.

Thus Φ_2° can be lifted to $\begin{array}{c} A\sigma_\varphi \\ \{\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}\,\Big\Vert\,\Phi_2 \\ B_1 \end{array}$. We construct Φ by composing Φ_2 and Φ_1. Then $\widehat{\Phi}$ can be constructed by rectifying Φ, where the variables to be used in A are already given. That $\varphi = \lfloor\widehat{\Phi}\rfloor$ follows immediately from the construction. □

X. Summary and Proof of Main Result

The only theorem of Section VI that has not yet been proved is Theorem 20 establishing the full correspondence between decomposed proofs in KS1 and combinatorial proofs. We show the proof here, by summarizing the results of the previous two Sections VIII and IX.

Proof of Theorem 20. First, asssume we have a combinatorial proof $\varphi\colon \mathcal{C} \to \mathcal{A}$ be a combinatorial proof and a formula A with $\mathcal{A} = [\![A]\!]$. Let C be a formula with $[\![C]\!] = \mathcal{C}$, and let σ_φ be the substitution induced by φ. By Lemma 45 there is a derivation

$$\begin{array}{c} C\sigma_\varphi \\ \{\mathsf{w},\mathsf{ac},\mathsf{c}_\forall,\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}\,\Big\Vert\,\Phi_2 \\ A \end{array}$$

Since $\mathcal{C}$ is a fonet, we have by Theorem 40 a derivation

$$\begin{array}{c} \mathsf{t} \\ \mathsf{MLS1}^\mathsf{X}\,\Big\Vert\,\Phi_1' \\ C \end{array}$$

This derivation remains valid if we apply the substitution σ_φ to every line in Φ_1', yielding the derivation Φ_1 of $C\sigma_\varphi$ as desired.

Conversely, assume we have a decomposed derivation

$$\begin{array}{c} \mathsf{t} \\ \mathsf{MLS1}^\mathsf{X}\,\Big\Vert\,\Phi_1 \\ A' \\ \{\mathsf{w},\mathsf{ac},\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}\,\Big\Vert\,\Phi_2 \\ A \end{array} \tag{11}$$

Then we can transform Φ_1 into a rectified form $\widehat{\Phi}_1$, proving $\widehat{A}'$. By Theorem 34, the linked fograph $[\![\widehat{\Phi}_1]\!] = \langle [\![\widehat{A}']\!], \sim_{\widehat{\Phi}_1} \rangle$ is a fonet. Then, by Lemma 42, there is a rectified derivation $\begin{array}{c} \widehat{A'} \\ \{\mathsf{w},\widehat{\mathsf{ac}},\widehat{\mathsf{c}_\forall},\mathsf{m},\widehat{\mathsf{m}_\forall},\widehat{\mathsf{m}_\exists},\equiv\}\,\Big\Vert\,\widehat{\Phi_2} \\ \widehat{A} \end{array}$ whose induced map $\lfloor\widehat{\Phi_2}\rfloor\colon [\![\widehat{A'}]\!] \to [\![\widehat{A}]\!]$ is the same a the induced map $\lfloor\Phi_2\rfloor\colon [\![A']\!] \to [\![A]\!]$ of Φ_2. By Lemma 43, this map is a skew bifibration. Hence, we have a combinatorial proof $\varphi\colon \mathcal{C} \to [\![A]\!]$ with $\mathcal{C} = [\![\widehat{A'}]\!]$. **[[Lutz:** shit, something's wrong...**]]** □

Note that Theorem 20 shows at the same time soundness, completeness, and full completeness, as

1) every proof in KS1 can be translated into a combinatorial proof, and
2) every combinatorial proof is the image of a KS1-proof under that translation.

XI. Conclusion

In this paper we have uncovered a close correspondence between first-order combinatorial proofs and decomposed deep inference derivations of system KS1, and we have shown that every proof in KS1 has such a decomposed form.

The most surprising discovery for us was that all technical difficulties in our work could be reduced (in a non-trivial way) to the propositional setting.

The obvious next step in our research is to investigate proof composition and normalisation of first-order combinatorial proofs. Even in the propositional setting, the normalization of combinatorial proofs is underdeveloped. There exist two different procedures for cut elimination [10], [12], but both have their insufficiencies, and there is no general theory.

[[Lutz: do we want/can say more here?**]]**

[[TODO: mention CERES**]]**

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Appendix

A. Proof of Theorem 29

Proof of Theorem 29. Note that the instances of w, c in Φ_2 are deep, but inside sequent contexts.

First, if an instance of $\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, A}$ is followed by a rule in which A is not in the principal formula, it can be permuted downwards. Otherwise, the proof can be transformed using the following rewriting rules.

$$\wedge\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, A} \qquad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \rightsquigarrow \mathsf{wk}\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdots}}{\vdash \Gamma, A \wedge B, \Delta}$$

$$\vee\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \vee B} \rightsquigarrow \mathsf{w}\,\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}$$

$$\exists\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, A[x/t]}}{\vdash \Gamma, \exists x.A} \rightsquigarrow \mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, \exists x.A}$$

$$\forall\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, A}}{\vdash \Gamma, \forall x.A} \rightsquigarrow \mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, \forall x.A}$$

$$\mathsf{ctr}\,\dfrac{\mathsf{wk}\,\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, A}}{\vdash \Gamma, A} \rightsquigarrow \vdash \Gamma, A$$

Note that in the case of $\vee$, we use the deep rule w which can be permuted under all the rules. By using these rewriting rules, we can eventually get a derivation with all the instances of wk and w at the bottom. Now observe that the instances of ctr in Φ can be transformed using the following rule:

$$\mathsf{ctr}\,\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \rightsquigarrow \mathsf{c}\,\dfrac{\vee\,\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A}}{\vdash \Gamma, A}$$

Knowing that c can be permuted under all the rules of $\mathsf{MLL1^X}$, we eventually obtain a derivation:

$$\begin{array}{c} \mathsf{MLL1^X} \;\bigtriangledown\; \Phi'_1 \\ \vdash \Gamma' \\ \{\mathsf{wk},\mathsf{w},\mathsf{c},\equiv\} \,\Big\|\, \Phi'_2 \\ \vdash \Gamma \end{array}$$

Note that $\equiv$ is required here since the permutation of formulas is implicit in $\mathsf{MLL1^X}$.

By transforming each sequent of Φ'_2 into its corresponding formula, and by considering the following rewriting rule:

$$\mathsf{wk}\,\dfrac{\vdash \Gamma}{\vdash \Gamma, A} \rightsquigarrow \mathsf{w}\,\dfrac{\vdash \bigvee(\Gamma)}{\vdash \bigvee(\Gamma) \vee A}$$

, we obtain a derivation

$$\begin{array}{c} \mathsf{MLL1^X} \;\bigtriangledown\; \Phi_1 \\ \vdash \bigvee(\Gamma') \\ \{\mathsf{w},\mathsf{c},\equiv\} \,\Big\|\, \Phi_2 \\ \vdash \bigvee(\Gamma) \end{array}$$

where Φ_1 can be obtained from Φ'_1 by applying the $\vee$ rule. $\square$

B. Rule permutation for the proof of Lemma 31

We first study the interactions between two rules (only non-trivial cases are presented here):

- $\mathsf{r}_1/\mathsf{r}_2$, where $\mathsf{r}_1 \in \{\mathsf{w}, \mathsf{w}_\forall\}$ and $\mathsf{r}_2 \in \{\mathsf{ac}, \mathsf{c}_\forall, \mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists\}$:

$$\mathsf{ac}\,\dfrac{\mathsf{w}\,\dfrac{a}{a \vee a}}{a} \rightsquigarrow a$$

$$\mathsf{m}\,\dfrac{\mathsf{w}\,\dfrac{A \wedge C}{(A \wedge C) \vee (B \wedge D)}}{(A \vee B) \wedge (C \vee D)} \rightsquigarrow \mathsf{w}\,\dfrac{\mathsf{w}\,\dfrac{A \wedge C}{(A \vee B) \wedge C}}{(A \vee B) \wedge (C \vee D)}$$

$$\mathsf{m}_\forall\,\dfrac{\mathsf{w}\,\dfrac{\forall x.A}{(\forall x.A) \vee (\forall x.B)}}{\forall x.(A \vee B)} \rightsquigarrow \mathsf{w}\,\dfrac{\forall x.A}{\forall x.(A \vee B)}$$

$$\mathsf{c}_\forall\,\dfrac{\mathsf{w}_\forall\,\dfrac{\forall x.A}{\forall x.\forall x.A}\,(x \notin fv(\forall x.A))}{\forall x.A} \rightsquigarrow \forall x.A$$

$$\mathsf{m}_\forall\,\dfrac{\mathsf{w}_\forall\,\dfrac{A \vee (\forall x.B)}{(\forall x.A) \vee (\forall x.B)}\,(x \notin fv(A))}{\forall x.(A \vee B)} \rightsquigarrow \equiv\,\dfrac{A \vee (\forall x.B)}{\forall x.(A \vee B)}\,(x \notin fv(A))$$

- $\mathsf{r}_1/\mathsf{r}_2$, where $\mathsf{r}_1 \in \{\mathsf{ac}, \mathsf{c}_\forall\}$ and $\mathsf{r}_2 \in \{\mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists\}$:

$$\mathsf{m}_\forall\,\dfrac{\mathsf{c}_\forall\,\dfrac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{(\forall x.A) \vee (\forall x.B)\}}}{S\{\forall x.(A \vee B)\}} \rightsquigarrow \mathsf{m}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{m}_\forall\,\dfrac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{\forall x.(\forall x.A \vee B)\}}}{S\{(\forall x.A) \vee (\forall x.B)\}}}{S\{\forall x.(A \vee B)\}}$$

- $\mathsf{c}_\forall/\equiv$:

$$\equiv\,\dfrac{\mathsf{c}_\forall\,\dfrac{\forall x.\forall x.\forall y.A}{\forall x.\forall y.A}}{\forall y.\forall x.A} \rightsquigarrow \mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.\forall x.\forall y.A}{\forall y.\forall x.\forall x.A}}{\forall y.\forall x.A}$$

$$\equiv\,\dfrac{\mathsf{c}_\forall\,\dfrac{\forall x.\forall x.(A \vee B)}{\forall x.(A \vee B)}}{(\forall x.A) \vee B}\,(x \notin fv(B)) \rightsquigarrow \mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.\forall x.(A \vee B)}{(\forall x.\forall x.A) \vee B}\,(x \notin fv(B))}{(\forall x.A) \vee B}$$

$$\equiv\,\dfrac{\mathsf{c}_\forall\,\dfrac{(\forall x.\forall x.A) \vee B}{(\forall x.A) \vee B}}{\forall x.(A \vee B)}\,(x \notin fv(B)) \rightsquigarrow \mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{(\forall x.\forall x.A) \vee B}{\forall x.\forall x.(A \vee B)}\,(x \notin fv(B))}{\forall x.(A \vee B)}$$

- $\mathsf{w}/\equiv$:

$$\equiv\,\dfrac{\mathsf{w}\,\dfrac{A}{A \vee B}}{B \vee A}$$

$$\equiv\,\dfrac{\mathsf{w}\,\dfrac{A \vee C}{(A \vee B) \vee C)}}{A \vee (B \vee C)}$$

$$\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.A}{\forall x.(A \vee B)}}{(\forall x.A) \vee B}\,(x \notin fv(B)) \rightsquigarrow \mathsf{w}\,\dfrac{\forall x.A}{(\forall x.A) \vee B}$$

$$\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.A}{(\forall x.A) \vee B}}{\forall x.(A \vee B)}\,(x \notin fv(B)) \rightsquigarrow \mathsf{w}\,\dfrac{\forall x.A}{\forall x.(A \vee B)}$$

- $\mathsf{w}_\forall/\equiv$:
In the following two cases, we assume $x \neq y$ (otherwise they are trivial).

$$\equiv\,\dfrac{\mathsf{w}_\forall\,\dfrac{\forall y.A}{\forall x.\forall y.A}\,(x \notin fv(\forall y.A))}{\forall y.\forall x.A} \rightsquigarrow \mathsf{w}_\forall\,\dfrac{\forall y.A}{\forall y.\forall x.A}\,(x \notin fv(A))$$

$$\equiv\,\dfrac{\mathsf{w}_\forall\,\dfrac{\forall y.A}{\forall y.\forall x.A}\,(x \notin fv(A))}{\forall x.\forall y.A} \rightsquigarrow \mathsf{w}_\forall\,\dfrac{\forall y.A}{\forall x.\forall y.A}\,(x \notin fv(\forall y.A))$$

$$\equiv\,\dfrac{\mathsf{w}_\forall\,\dfrac{A \vee B}{\forall x.(A \vee B)}\,(x \notin fv(A \vee B))}{(\forall x.A) \vee B}\,(x \notin fv(B)) \rightsquigarrow \mathsf{w}_\forall\,\dfrac{A \vee B}{(\forall x.A) \vee B}\,(x \notin fv(A))$$

$$\equiv\,\dfrac{\mathsf{w}_\forall\,\dfrac{A \vee B}{(\forall x.A) \vee B}\,(x \notin fv(A))}{\forall x.(A \vee B)}\,(x \notin fv(B)) \rightsquigarrow \mathsf{w}_\forall\,\dfrac{A \vee B}{\forall x.(A \vee B)}\,(x \notin fv(A \vee B))$$

- $\equiv/\mathsf{c}_\forall$:

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.\forall y.\forall x.A}{\forall x.\forall x.\forall y.A}}{\forall x.\forall y.A}$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.\forall y.\forall x.A}{\forall y.\forall x.\forall x.A}}{\forall y.\forall x.A}$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.((\forall x.A) \vee B)}{(\forall x.\forall x.A) \vee B}}{(\forall x.A) \vee B}\ (x \notin fv(B))$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.((\forall x.A) \vee B)}{\forall x.\forall x.(A \vee B)}}{\forall x.(A \vee B)}\ (x \notin fv(B))$$

- $\equiv/\mathsf{m}$:

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{(C \wedge A) \vee (B \wedge D)}{(A \wedge C) \vee (B \wedge D)}}{(A \vee B) \wedge (C \vee D)}$$

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{(B \wedge D) \vee (A \wedge C)}{(A \wedge C) \vee (B \wedge D)}}{(A \vee B) \wedge (C \vee D)} \rightsquigarrow \equiv\,\dfrac{\mathsf{m}\,\dfrac{(B \wedge D) \vee (A \wedge C)}{(B \vee A) \wedge (D \vee C)}}{(A \vee B) \wedge (C \vee D)}$$

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{((A \wedge C) \wedge E) \vee (B \wedge D)}{(A \wedge (C \wedge E)) \vee (B \wedge D)}}{(A \vee B) \wedge ((C \wedge E) \vee D)}$$

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{(\forall x.(A \wedge C)) \vee (B \wedge D)}{\forall x.((A \wedge C) \vee (B \wedge D))}}{\forall x.((A \vee B) \wedge (C \vee D))}\ (x \notin fv(B \wedge D))$$

- $\equiv/\mathsf{m}_\forall$:

$$\mathsf{m}_\forall\,\dfrac{\equiv\,\dfrac{(\forall x.B) \vee (\forall x.A)}{(\forall x.A) \vee (\forall x.B)}}{\forall x.(A \vee B)} \rightsquigarrow \equiv\,\dfrac{\mathsf{m}_\forall\,\dfrac{(\forall x.B) \vee (\forall x.A)}{\forall x.(B \vee A)}}{\forall x.(A \vee B)}$$

$$\mathsf{m}_\forall\,\dfrac{\equiv\,\dfrac{(\forall y.\forall x.A) \vee (\forall x.B)}{(\forall x.\forall y.A) \vee (\forall x.B)}}{\forall x.((\forall y.A) \vee B)}$$

$$\mathsf{m}_\forall\,\dfrac{\equiv\,\dfrac{\forall x.(A \vee (\forall x.B))}{(\forall x.A) \vee (\forall x.B)}}{\forall x.(A \vee B)}$$

- $\equiv/\mathsf{m}_\exists$: similar to $\equiv/\mathsf{m}_\forall$

Interactions between two non-$\equiv$ rules with the presence of $\equiv$ in between:

- $\mathsf{c}_\forall/\equiv/\mathsf{r}$ where $\mathsf{r} \in \{\mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists\}$: First permute $\mathsf{c}_\forall$ under $\equiv$ and then permute $\mathsf{c}_\forall$ under r.
- $\mathsf{ac}/\equiv/\mathsf{r}$ where $\mathsf{r} \in \{\mathsf{m}, \mathsf{m}_\forall, \mathsf{m}_\exists\}$: First permute ac under $\equiv$ and then permute ac under r.
- $\mathsf{w}/\equiv/\mathsf{c}_\forall$: We only list non-trivial cases here:

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.\forall x.A}{\forall x.((\forall x.A) \vee B)}}{(\forall x.\forall x.A) \vee B}}{(\forall x.A) \vee B}\ (x \notin fv(B)) \rightsquigarrow \mathsf{w}\,\dfrac{\mathsf{c}_\forall\,\dfrac{\forall x.\forall x.A}{(\forall x.A) \vee B}}{(\forall x.A) \vee B}$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{\forall x.(B \vee (\forall x.A))}}{(\forall x.\forall x.A) \vee B}}{(\forall x.A) \vee B}\ (x \notin fv(B)) \rightsquigarrow \equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{\forall x.(B \vee A)}}{(\forall x.A) \vee B}\ (x \notin fv(B))$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.\forall x.A}{\forall x.((\forall x.A) \vee B)}}{\forall x.\forall x.(A \vee B)}}{\forall x.(A \vee B)}\ (x \notin fv(B)) \rightsquigarrow \mathsf{w}\,\dfrac{\mathsf{c}_\forall\,\dfrac{\forall x.\forall x.A}{\forall x.A}}{\forall x.(A \vee B)}$$

$$\mathsf{c}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{\forall x.(B \vee (\forall x.A))}}{\forall x.\forall x.(A \vee B)}}{\forall x.(A \vee B)}\ (x \notin fv(B)) \rightsquigarrow \equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{\forall x.(B \vee A)}}{\forall x.(A \vee B)}$$

- $\mathsf{w}/\equiv/\mathsf{ac}$:

$$\mathsf{ac}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{a \vee B}{(a \vee B) \vee a}}{(a \vee a) \vee B}}{a \vee B} \rightsquigarrow a \vee B$$

$$\mathsf{ac}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{a}{a \vee (a \vee B)}}{(a \vee a) \vee B}}{a \vee B} \rightsquigarrow \mathsf{w}\,\dfrac{a}{a \vee B}$$

$$\mathsf{ac}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.a}{(\forall x.a) \vee a}}{\forall x.(a \vee a)}}{\forall x.a}\ (x \notin fv(a)) \rightsquigarrow \forall x.a$$

$$\mathsf{ac}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{a}{a \vee (\forall x.a)}}{\forall x.(a \vee a)}}{\forall x.a}\ (x \notin fv(a)) \rightsquigarrow \mathsf{w}_\forall\,\dfrac{a}{\forall x.a}\ (x \notin fv(a))$$

- $\mathsf{w}/\equiv/\mathsf{m}$:

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{C \wedge A}{(C \wedge A) \vee (B \wedge D)}}{(A \wedge C) \vee (B \wedge D)}}{(A \vee B) \wedge (C \vee D)} \rightsquigarrow \mathsf{w}\,\dfrac{\mathsf{w}\,\dfrac{\equiv\,\dfrac{C \wedge A}{A \wedge C}}{(A \vee B) \wedge C}}{(A \vee B) \wedge (C \vee D)}$$

$$\mathsf{m}\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{B \wedge D}{(B \wedge D) \vee (\forall x.(A \wedge C))}}{\forall x.((A \wedge C) \vee (B \wedge D))}\,(x \notin fv(B \wedge D))}{\forall x.((A \vee B) \wedge (C \vee D))} \rightsquigarrow \equiv\,\dfrac{\mathsf{w}\,\dfrac{\mathsf{w}\,\dfrac{\forall\,\dfrac{B \wedge D}{\forall x.(B \wedge D)}\,(x \notin fv(B \wedge D))}{\forall x.((B \vee A) \wedge D)}}{\forall x.((B \vee A) \wedge (D \vee C))}}{\forall x.((A \vee B) \wedge (C \vee D))}$$

- $\mathsf{w}/\equiv/\mathsf{m}_\forall$:

$$\mathsf{m}_\forall\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{(\forall x.B) \vee (\forall x.A)}}{(\forall x.A) \vee (\forall x.B)}}{\forall x.(A \vee B)} \rightsquigarrow \equiv\,\dfrac{\mathsf{w}\,\dfrac{\forall x.B}{\forall x.(B \vee A)}}{\forall x.(A \vee B)}$$

- $\mathsf{w}/\equiv/\mathsf{m}_\exists$:

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$$\mathsf{m}_\exists\,\dfrac{\equiv\,\dfrac{\mathsf{w}\,\dfrac{\exists x.B}{(\exists x.B) \vee (\exists x.A)}}{(\exists x.A) \vee (\exists x.B)}}{\exists x.(A \vee B)} \rightsquigarrow \equiv\,\dfrac{\mathsf{w}\,\dfrac{\exists x.B}{\exists x.(B \vee A)}}{\exists x.(A \vee B)}$$