

Combinatorial Proofs and Decomposition Theorems for First-order Logic

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Abstract—In this paper we uncover a close relation between combinatorial and syntactic proofs for first-order logic. Whereas syntactic proofs are formalized in some deductive proof system based on inference rules, a combinatorial proof is a “syntax-free” presentation of a proof that is independent from any set of inference rules. We show that the two proof representations are related via a deep inference decomposition theorem that establishes a new kind of normal form of syntactic proofs. As a consequence, we obtain (i) a simple proof of the soundness and completeness of first-order combinatorial proofs, and (ii) a full completeness theorem: every combinatorial proof is the image of a syntactic proof.

1 **[[[TODO: Examples,examples,examples]]]**

2 I. INTRODUCTION

3 First-order predicate logic is one of the cornerstones of
4 modern logic. Since its formalisation by Frege [1] it has
5 seen a growing usage in many fields of mathematics and
6 computer science. Since the development of proof theory by
7 Hilbert [2], *proofs* became first-class citizens as mathematical
8 objects that could be studied on its own. Since Gentzen’s
9 *sequent calculus* [3], [4], many other proof systems have
10 been developed that allow the implementation of efficient
11 proof search, for example *analytic tableaux* [5] or *resolu-*
12 *tion* [6]. Despite the immense progress that has been made
13 in proof theory in general and in the area of automated and
14 interactive theorem provers in particular[7], [8]**[[[TODO: find**
15 **references]]]****[[[Jui-Hsuan: done]]]**, we still have no satisfactory
16 notion of proof identity for first-order logic. In this respect,
17 proof theory is quite different from any other mathematical
18 field. For example in group theory, two groups are *the same*
19 iff they are isomorphic; in topology, two spaces are *the same*
20 iff they are homeomorphic; etc. In proof theory, we have no
21 such notion telling us when two proofs are *the same*, even
22 though Hilbert was considering this problem as possible 24th
23 problem for his famous lecture [9] in 1900 [10], before proof
24 theory existed as a mathematical field.

25 The main reason for this problem is that formal proofs, as
26 they are usually studied in logic, are inextricably tied to the
27 syntactic (inference rule based) proof system in which they are
28 carried out. And it is difficult to compare two proofs that are
29 produced within two different syntactic proof systems, based
30 on different sets of inference rules. **[[[Lutz: an example here?]]]**

31 This is where *combinatorial proofs* come in. They have been
32 introduced by Hughes [11] for classical propositional logic as

“syntax-free” notion of proof, and as a possible solution to
Hilbert’s 24th problem [12] (see also [13]). The basic idea is to
abstract away from the syntax of the inference rules used in the
proof and consider the proof as a combinatorial object, more
precisely as a special kind of homomorphism between two
graphs obeying certain properties. **[[[Lutz: an example here?]]]**

It has been shown by several authors how syntactic proofs
in various proof systems can be translated to propositional
combinatorial proofs: for sequent proofs in [12], for deep
inference proofs in [14], for Frege systems in [15], and for
tableaux systems and resolution in [16]. This allows to define
a natural notion of proof identity for propositional logic:
two proofs are *the same*, if they are mapped to the same
combinatorial proof.

Recently, Acclavio and Straßburger extended this notion to
relevant logics [17] and to modal logics [18]; and Heijlties,
Hughes and Straßburger have provided combinatorial proofs
for intuitionistic propositional logic [19].

In this paper we would like to push forward the idea that
combinatorial proofs can also for first order logic be used as a
notion of proof identity. *First-order combinatorial proofs* have
been introduced by Hughes in [20]. But even though Hughes
shows that the conclusion of every first-order combinatorial
proof is a valid formula, his proof is not really satisfactory,
as (i) it is long and cumbersome, and (ii) it does not allow to
read back a syntactic proof based inference rules. In fact, there
is the fundamental problem that not all combinatorial proofs
can be obtained as translations of sequent calculus proofs.

In this paper we solve this issue by moving to a deep
inference system. More precisely, we introduce a new proof
system, called KS1, for first-order logic, that (i) reflects every
combinatorial proof, i.e., there is a surjective mapping from
proofs in KS1 to combinatorial proofs, that (ii) allows to
provide simpler proofs of soundness and completeness of
combinatorial proofs, and (iii) admits new decomposition the-
orems establishing a precise correspondence between certain
syntactic inference rules and certain combinatorial notions.

In general, a *decomposition theorem* provides normal forms
of proofs, separating subsets of inference rules of a proof
system. A prominent example of a decomposition theorem is
Herbrand’s theorem [21], which allows a separation between
the propositional part and the quantifier part in a first-order
proof [4], [22]. Through the advent of deep inference, new

kinds of proof decompositions became possible, most notably the separation between the linear part of a proof and the resource management of a proof. It has been shown by Straßburger [23] that a proof in classical propositional logic can be decomposed into a proof of (multiplicative) linear logic, followed by a proof consisting only of contractions and weakenings. In this paper we show that the same is possible for first-order logic.

Combinatorial proofs and deep inference can be seen as opposite ends of a spectrum: whereas deep inference allows for a very fine granularity of inference rules—one inference rule in a standard formalism, like sequent calculus or semantic tableaux, is usually simulated by a sequence of different deep inference rules—have combinatorial proofs completely abolished the concept of inference rule. Nonetheless, there is a close relationship between the two, realized through a decomposition theorem, as we establish it in this paper.

II. PRELIMINARIES: FIRST-ORDER LOGIC

A. Terms and Formulas

We start from a countable set $\text{VAR} = \{x, y, z, \dots\}$ of variables, a countable set $\text{FUN} = \{f, g, \dots\}$ of function symbols, and a countable set $\text{PRED} = \{p, q, \dots\}$ of predicate symbols. Each function symbol and each predicate symbol has a finite arity. The grammars below generate the set **TERM** of *terms*, denoted by s, t, u, \dots , the set **ATOM** of *atoms*, denoted by a, b, c, \dots , and the set **FORM** of *formulas*, denoted by A, B, C, \dots :

$$\begin{aligned} t &::= x \mid f(t_1, \dots, t_n) \\ a &::= \mathbf{t} \mid \mathbf{f} \mid p(t_1, \dots, t_n) \mid \bar{p}(t_1, \dots, t_n) \\ A &::= a \mid A \wedge A \mid A \vee A \mid \exists x.A \mid \forall x.A \end{aligned}$$

where the arity of f and p is n . Note, that in this paper we consider the truth constants \mathbf{t} (*true*) and \mathbf{f} (*false*) as atoms, and we consider all formulas in negation normal form. The **negation** ($\bar{\cdot}$) is defined for all atoms and formulas via the De Morgan laws as follows:

$$\begin{aligned} \bar{\bar{a}} &= a & \bar{\mathbf{t}} &= \mathbf{f} & \overline{p(t_1, \dots, t_n)} &= \bar{p}(t_1, \dots, t_n) \\ \bar{\mathbf{f}} &= \mathbf{t} & \overline{\bar{p}(t_1, \dots, t_n)} &= p(t_1, \dots, t_n) \\ \overline{\exists x.A} &= \forall x.\bar{A} & \overline{A \wedge B} &= \bar{A} \vee \bar{B} \\ \overline{\forall x.A} &= \exists x.\bar{A} & \overline{A \vee B} &= \bar{A} \wedge \bar{B} \end{aligned}$$

A formula is **rectified** if all bound variables are distinct from one another and from all free variables. Every formula can be transformed into a logically equivalent rectified form, by simply renaming the bound variables. If we consider formulas equivalent modulo α -conversion (renaming of bound variables), then the rectified form of a formula A is uniquely defined, and we denote it by \bar{A} .

A **substitution** is a function $\sigma: \text{VAR} \rightarrow \text{TERM}$ that is the identity almost everywhere. We denote substitutions as $\sigma = [x_1/t_1, \dots, x_n/t_n]$, where $\sigma(x_i) = t_i$ for $i = 1..n$ and $\sigma(x) = x$ for all $x \notin \{x_1, \dots, x_n\}$. We write $A\sigma$ for the formula obtained from A by applying σ , i.e., by simultaneously replacing all occurrences of x_i by t_i . A **variable renaming** is a substitution ρ with $\rho(x) \in \text{VAR}$ for all variables x .

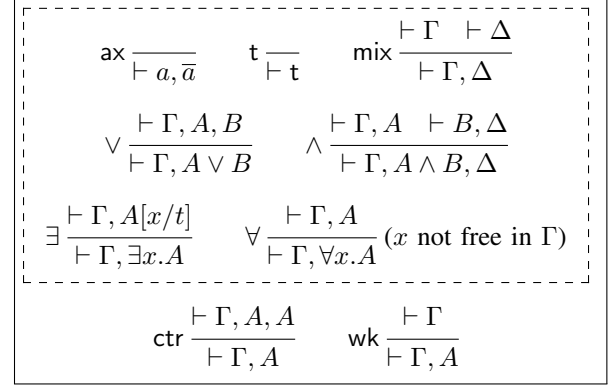


Fig. 1. Sequent calculi LK1 (all rules) and MLL1^X (rules in the dashed box)

B. Sequent Calculus LK1

Sequents, denoted by Γ, Δ, \dots , are finite multisets of formulas, written as lists, separated by comma. The **corresponding formula** of a sequent $\Gamma = A_1, A_2, \dots, A_n$ is the disjunction of its formulas: $\text{fm}(\Gamma) = A_1 \vee A_2 \vee \dots \vee A_n$. A sequent is **rectified** iff its corresponding formula is.

In this paper we use the sequent calculus LK1, shown in Figure 1, which is an one-sided variant of Gentzen's original calculus [3] for first-order logic. To simplify some technicalities later in this paper, we also include here the mix-rule.

Theorem 1. LK1 is sound and complete for first-order logic.

For a proof we refer to reader to any standard textbook, e.g. [24].

The linear fragment of LK1, i.e., the fragment without the rules ctr (*contraction*) and wk (*weakening*) defines *first-order multiplicative linear logic* [25], [26] with mix [27], [28] (MLL1+mix). We denote that system here with MLL1^X (shown in Figure 1 in the dashed box).

In this paper we make also use of the cut elimination theorem. The **cut** rule is

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \bar{A}, \Delta}{\vdash \Gamma, \Delta} \quad (1)$$

Theorem 2. If a sequent $\vdash \Gamma$ is provable in LK1+cut then it is also provable in LK1. Furthermore, if $\vdash \Gamma$ is provable in MLL1^X+cut then it is also provable in MLL1^X.

As before, this is standard, see e.g. [24] for a proof.

III. PRELIMINARIES: FIRST-ORDER GRAPHS

A. Graphs

A **graph** $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is a pair where $V_{\mathcal{G}}$ is a finite set of **vertices** and $E_{\mathcal{G}}$ is a finite set of **edges**, which are two-element subsets of $V_{\mathcal{G}}$. We write vw for an edge $\{v, w\}$.

The **complement** of a graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is the graph $\mathcal{G}^c = \langle V_{\mathcal{G}}, E_{\mathcal{G}}^c \rangle$ where $vw \in E_{\mathcal{G}}^c$ iff $vw \notin E_{\mathcal{G}}$.

Let $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ and $\mathcal{H} = \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ be graphs such that $V_{\mathcal{G}} \cap V_{\mathcal{H}} = \emptyset$. A **homomorphism** $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is a function

141 $\varphi: V_G \rightarrow V_H$ such that if $vw \in E_G$ then $\varphi(v)\varphi(w) \in E_H$.
 142 The **union** $\mathcal{G} + \mathcal{H}$ is the graph $\langle V_G \cup V_H, E_G \cup E_H \rangle$ and the
 143 **join** $\mathcal{G} \times \mathcal{H}$ is the graph $\langle V_G \cup V_H, E_G \cup E_H \cup \{vw \mid v \in$
 144 $V_G, w \in V_H\} \rangle$. A graph \mathcal{G} is **disconnected** if $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ for
 145 two non-empty graphs $\mathcal{G}_1, \mathcal{G}_2$, otherwise it is **connected**. It is
 146 **coconnected** if its complement is connected.

147 A graph \mathcal{G} is **labelled** in a set L if each vertex $v \in V_G$ has
 148 an element $\ell(v) \in L$ associated with it, its **label**. A graph
 149 \mathcal{G} is (partially) **coloured** if it carries a partial equivalence
 150 relation \sim_G on V_G ; each equivalence class is a **colour**. A **vertex**
 151 **renaming** of $\mathcal{G} = \langle V_G, E_G \rangle$ along a bijection $(\cdot): V_G \rightarrow \hat{V}_G$
 152 is the graph $\hat{\mathcal{G}} = \langle \hat{V}_G, \{\hat{v}\hat{w} \mid vw \in E_G\} \rangle$, with colouring and/or
 153 labelling inherited (i.e., $\hat{v} \sim \hat{w}$ if $v \sim w$, and $\ell(\hat{v}) = \ell(v)$).
 154 Following standard graph theory, we identify graphs modulo
 155 vertex renaming.

156 A **directed graph** $\mathcal{G} = \langle V_G, E_G \rangle$ is a set V_G of **vertices** and a
 157 set $E_G \subseteq V_G \times V_G$ of **direct edges**. A **directed graph homomor-**
 158 **phism** $\varphi: \langle V_G, E_G \rangle \rightarrow \langle V_H, E_H \rangle$ is a function $\varphi: V_G \rightarrow V_H$
 159 such that if $(v, w) \in E_G$ then $(\varphi(v), \varphi(w)) \in E_H$.

160 B. Cographs

161 A graph $\mathcal{H} = \langle V_H, E_H \rangle$ is a **subgraph** of a graph $\mathcal{G} =$
 162 $\langle V_G, E_G \rangle$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. It is **induced** if
 163 $v, w \in V_H$ and $vw \in E_G$ implies $vw \in E_H$. An induced
 164 subgraph of $\mathcal{G} = \langle V_G, E_G \rangle$ is uniquely determined by its set
 165 of vertices V and we denote it by $\mathcal{G}[V]$. A graph is **\mathcal{H} -free** if it
 166 does not contain \mathcal{H} as an induced subgraph. The graph \mathbf{P}_4 is
 167 the (undirected) graph $\langle \{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4\} \rangle$. A
 168 **cograph** is a \mathbf{P}_4 -free undirected graph. The interest in cographs
 169 for our paper comes from the following well-known fact.

170 **Theorem 3** ([29]). *A graph is a cograph iff it can be*
 171 *constructed from the singletons via the operations $+$ and \times .*

172 **[[TODO: ad the reference]] [[Jui-Hsuan: done]]**

173 In a graph \mathcal{G} , the **neighbourhood** $N(v)$ of a vertex $v \in V_G$ is
 174 defined as the set $\{w \mid vw \in E_G\}$. A **module** is a set $M \subseteq V_G$
 175 such that $N(v) \setminus M = N(w) \setminus M$ for all $v, w \in M$. A module
 176 M is **strong** if for every module M' , we have $M' \subseteq M$, $M \subseteq$
 177 $M' \cap M = \emptyset$. A module is **proper** if it has two or more
 178 vertices.

179 Modules in cographs correspond precisely to the subtrees
 180 of the cotrees (the term-trees constructing the graph via $+$
 181 and \times).

182 C. Fographs

183 A cograph is **logical** if every vertex is labelled either by an
 184 atom or variable, and it has at least one atom-labelled vertex.
 185 We write $\bullet\alpha$ for an α -labelled vertex. An atom-labelled vertex
 186 is called a **literal** and a variable-labelled vertex is called a
 187 **binder**. A binder labelled with x is called an **x -binder**. The
 188 **scope** of a binder b is the smallest proper strong module
 189 containing b . An **x -literal** is a literal whose atom contains the
 190 variable x . An x -binder **binds** every x -literal in its scope. In
 191 a logical cograph \mathcal{G} , a binder b is **existential** (resp. **universal**)
 192 if, for every other vertex v in its scope, we have $bv \in E_G$

(resp. $bv \notin E_G$). An x -binder is **legal** if its scope contains no
 other x -binder and at least one literal.

Definition 4. A **first-order graph** or **fograph** is a logical
 cograph with legal binders. The **binding graph** of a fograph
 \mathcal{G} is the directed graph $\vec{\mathcal{G}} = \langle V_G, \{(b, l) \mid b \text{ binds } l\} \rangle$.

We now define a mapping $\llbracket \cdot \rrbracket$ from formulas to (labelled)
 graphs, inductively as follows:

For a formula A , we can define its associated fograph $\llbracket A \rrbracket$
 inductively by:

$$\begin{aligned} \llbracket a \rrbracket &= \bullet a & (\text{for any atom } a) \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket + \llbracket B \rrbracket & \llbracket \exists x.A \rrbracket &= \bullet x \times \llbracket A \rrbracket \\ \llbracket A \wedge B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket & \llbracket \forall x.A \rrbracket &= \bullet x + \llbracket A \rrbracket \end{aligned}$$

where $\bullet a$ (resp. $\bullet x$) is a single-vertex graph whose vertex is
 labelled by a (resp. x).

Lemma 5. *If A is a rectified formula then $\llbracket A \rrbracket$ is a fograph.*

Proof. That $\llbracket A \rrbracket$ is a logical cograph follows immediately from
 the definition and Theorem 3. The fact that every binder of
 $\llbracket A \rrbracket$ is legal can be proved by structural induction on A . \square

Note that $\llbracket A \rrbracket$ is not necessarily a fograph if A is not
 rectified. If $A = (\forall x.p(x)) \vee (\forall x.q(x))$, then $\llbracket A \rrbracket = \bullet x \bullet p(x) \bullet$
 $x \bullet q(x)$, the scope of each x -binder contains all the vertices,
 in particular, the two x -binders. On the other hand, there are
 non-rectified formulas which are translated to fographs by $\llbracket \cdot \rrbracket$.
 For example, in the graph of $(\exists x.p(x)) \vee (\exists x.q(x))$, both x -
 binders are legal, as they are not in each others scope. **[[TODO:**
draw the picture]]. For this reason, we call a formula **clean** if
 it does not contain subformulas of the form $(\forall x.A) \vee (\forall x.B)$
 or $(\exists x.A) \wedge (\exists x.B)$, and no x -quantified formula occurs as
 subformula of another x -quantified formula. Then we have:

Lemma 6. *If A is clean iff $\llbracket A \rrbracket$ is a fograph.*

Proof. Induction on A , using Theorem 3. \square

Note that even though for every formula A we can obtain
 an equivalent clean formula by simply renaming some bound
 variables, this is not unique up to α -conversion, as it is the
 case for rectified formulas.

We define a congruence relation \equiv on formulas by the
 following equations:

$$\begin{aligned} A \wedge B &\equiv B \wedge A & (A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\ A \vee B &\equiv B \vee A & (A \vee B) \vee C &\equiv A \vee (B \vee C) \\ \forall x.\forall y.A &\equiv \forall y.\forall x.A & \forall x.(A \vee B) &\equiv (\forall x.A) \vee B \\ \exists x.\exists y.A &\equiv \exists y.\exists x.A & \exists x.(A \wedge B) &\equiv (\exists x.A) \wedge B \end{aligned} \quad (2)$$

where $x \notin \text{fv}(B)$ in the last two equations. Two formulas
 A and B are **equivalent** if $A \equiv B$. The following theorem
 shows that the set of fographs can be seen as the quotient
 FORM/\equiv .

Theorem 7. *Let A, B be rectified formulas. Then*

$$A \equiv B \iff \llbracket A \rrbracket = \llbracket B \rrbracket$$

Proof. By a straightforward induction on A . \square

A. Fonets

Two atoms are **pre-dual** if their predicate symbols are dual (e.g. $p(x, y)$ and $\bar{p}(y, z)$) and two literals are **pre-dual** if their labels (atoms) are pre-dual. A **linked fograph** $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ is a coloured fograph \mathcal{C} such that every colour (i.e., equivalence class of $\sim_{\mathcal{C}}$), called a **link**, consists of two pre-dual literals, and every literal is either t-labelled or in a link.

Let \mathcal{C} be a linked fograph. The set of links can be seen as a unification problem by identifying dual predicate symbols. A **dualizer** of \mathcal{C} is a substitution δ unifying all the links of \mathcal{C} . Since a first-order unification problem is either unsolvable or has a most general unifier, we can define the notion of **most general dualizer**. A **dependency** is a pair $\{\bullet x, \bullet y\}$ of an existential binder $\bullet x$ and a universal binder $\bullet y$ such that the most general dualizer assigns to x a term containing y . A **leap** is either a link or a dependency. The **leap graph** \mathcal{C}^L of \mathcal{C} is the undirected graph $\langle V_{\mathcal{C}}, L_{\mathcal{C}} \rangle$ where $L_{\mathcal{C}}$ is the set of leaps of \mathcal{C} . A vertex set $W \subseteq V_{\mathcal{C}}$ induces a **matching** in \mathcal{C} if for all $w \in W$, $N(w) \cap W$ is a singleton. We say that W induces a **bimatching** in \mathcal{C} if it induces a matching in \mathcal{C} and a matching in \mathcal{C}^L .

Definition 8. A **first-order net** or **fonet** is a linked fograph which has dualizer but no induced bimatching.

B. Skew Bifibrations

A graph homomorphism $\varphi: \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ is a **fibration** if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$, there exists a unique $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$, and is a **skew fibration** if for all $v \in V_{\mathcal{G}}$ and $w\varphi(v) \in E_{\mathcal{H}}$ there exists $\tilde{w} \in V_{\mathcal{G}}$ such that $\tilde{w}v \in E_{\mathcal{G}}$ and $\varphi(\tilde{w})w \notin E_{\mathcal{H}}$. A directed graph homomorphism is a **fibration** if for all $v \in V_{\mathcal{G}}$ and $(w, \varphi(v)) \in E_{\mathcal{H}}$, there exists a unique $\tilde{w} \in V_{\mathcal{G}}$ such that $(\tilde{w}, v) \in E_{\mathcal{G}}$ and $\varphi(\tilde{w}) = w$.

A **fograph homomorphism** $\varphi = \langle \varphi, \rho_{\varphi} \rangle$ is a pair where $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is a graph homomorphism between the underlying graphs, and ρ_{φ} , also called the **substitution induced by φ** is a variable renaming such that for all $v \in V_{\mathcal{G}}$ we have $\ell(\varphi(v)) = \rho_{\varphi}(\ell(v))$. Note that this entails that φ maps binders to binders and literals to literals. We say that a fograph homomorphism $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is **existential-preserving** if for all existential binders b in \mathcal{G} , the vertex $\varphi(b)$ is an existential binder in \mathcal{H} .

Definition 9. Let \mathcal{G} and \mathcal{H} be fographs. A **skew bifibration** $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ is an existential-preserving fograph homomorphism that is a skew fibration on $\langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle \rightarrow \langle V_{\mathcal{H}}, E_{\mathcal{H}} \rangle$ and a fibration on the binding graphs $\bar{\mathcal{G}} \rightarrow \bar{\mathcal{H}}$.

Definition 10. A **first-order combinatorial proof (FOCP)** of a fograph \mathcal{G} is a skew bifibration $\varphi: \mathcal{C} \rightarrow \mathcal{G}$ where \mathcal{C} is a fonet. A **first-order combinatorial proof** of a formula A is a combinatorial proof of its graph $\llbracket A \rrbracket$.

Theorem 11 ([20]). FOCPs are sound and complete for first-order logic.

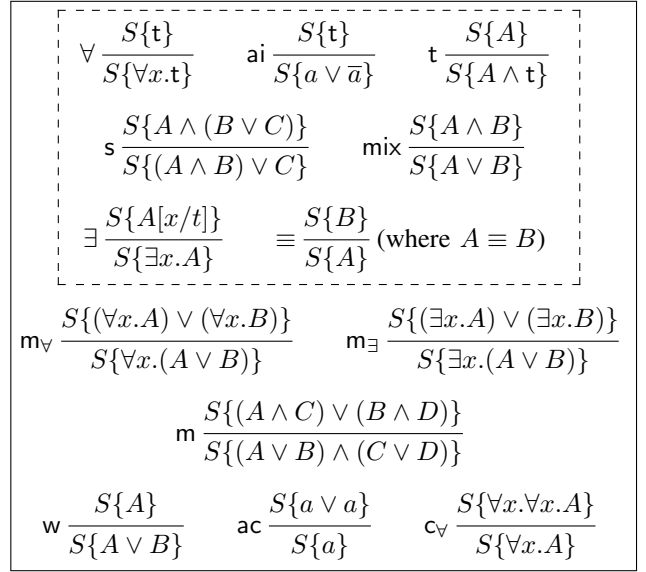


Fig. 2. Deep inference systems KS1 (all rules) and MLS1^X (rules in the dashed box)

Remark 12. In our definition of FOCP, we are slightly laxer than the original definition of [20], as we allow for a variable renaming σ_{φ} which was forced to be the identity in [20].

V. FIRST-ORDER DEEP INFERENCE SYSTEM KS1

Contrary to standard proof formalisms, like sequent calculi or tableaux, where inference rules decompose the principal formula along its root connective, can **deep inference rules** be applied like rewriting rules inside any (positive) formula or sequent **context**, which is denoted as $S\{\cdot\}$, and which is a formula (resp. sequent) with exactly one occurrence of the **hole** $\{\cdot\}$ in the position of an atom. Then $S\{A\}$ is the result of replacing the hole $\{\cdot\}$ in $S\{\cdot\}$ with A .

Figure 2 shows the inference rules for the deep inference system KS1 that we are using in this paper. It is a slight variation of the systems presented by Br  nnler [30] and Ralph [31] in their PhD-theses. The main differences being that we have (i) the explicit presence of the mix-rule, (ii) a different choice of how the formula equivalence \equiv is defined, and (iii) an explicit rule for the equivalence.

We consider here only the cut-free fragment, as cut-elimination for deep inference systems has already been discussed elsewhere (e.g. [22], [32]).¹

As with the sequent system LK1, we also need for KS1 the **linear fragment**, that we call here MLS1^X, and that is shown in Figure 2 in the dashed box.

B

We write $S \Vdash \Phi$ to denote a derivation Φ from B to A using

A

the rules from system S. A formula A is **provable** in a system S if there is a derivation in S from t to A .

¹In the deep inference literature, the cut-free fragment is also called the **down-fragment**. But as we do not discuss the **up-fragment** here, we omit the down-arrows \downarrow in the rule names.

In the course of this paper we are also going to make use of the general (non-atomic) version of the contraction rule:

$$\text{c} \frac{S\{A \vee A\}}{S\{A\}}$$

VI. MAIN RESULTS

We are now ready to see the main results of this paper. We only state them here and give the proofs in the later sections of the paper. The first one is routine and expected, but needs to be proved nonetheless:

Theorem 13. *KS1 is sound and complete for first-order logic.*

Our second result is more surprising, as it is a very strong decomposition result for first order logic.

Theorem 14. *For every derivation $\text{KS1} \parallel_{\Phi}^t$ there are formulas A_1, \dots, A_5 , such that there is a derivation:*

$$\begin{array}{c} t \\ \{\forall, \text{ai}, t\} \parallel \\ A_5 \\ \{s, \text{mix}, \equiv\} \parallel \\ A_4 \\ \{\exists\} \parallel \\ A_3 \\ \{m, m_{\forall}, m_{\exists}, \equiv\} \parallel \\ A_2 \\ \{ac, c_{\forall}\} \parallel \\ A_1 \\ \{w, \equiv\} \parallel \\ A \end{array}$$

This theorem is stronger than the existing decompositions for first-order logic, which either separated only atomic contraction and atomic weakening [30] or only contraction [31] or only the quantifiers in form of a Herbrand theorem [33], [31].

There is a weaker version of Theorem 14 that will also be useful:

Theorem 15. *For every derivation $\text{KS1} \parallel_{\Phi}^t$ there is a formula A_1 , such that there is a derivation:*

$$\begin{array}{c} t \\ \text{MLS1}^x \parallel \\ A_1 \\ \{w, c, \equiv\} \parallel \\ A \end{array}$$

Let us now establish the connection between derivations in KS1 and combinatorial proofs.

Theorem 16. *Let A be a rectified formula, and assume we have a combinatorial proof $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$. Then there is a derivation*

$$\begin{array}{c} t \\ \text{MLS1}^x \parallel_{\Phi_1} \\ A' \\ \{w, ac, m, m_{\forall}, m_{\exists}, \equiv\} \parallel_{\Phi_2} \\ A \end{array} \quad (3)$$

for some A' such that $\llbracket A' \rrbracket = \mathcal{C}$. Conversely, whenever we have a derivation as in (3) above, then there is a combinatorial proof $\varphi: \mathcal{C} \rightarrow \llbracket A \rrbracket$ such that $\mathcal{C} = \llbracket A' \rrbracket$.

Furthermore, in the proof of Theorem 16, we will see that (i) the links in the fonet \mathcal{C} correspond precisely to the pairs of atoms that meet in the instances of the ai-rule in the derivation Φ_1 , and (ii) the "flow-graph" of Φ_2 that traces the quantifier- and atom-occurrences in the derivation corresponds exactly to the vertex-mapping induced by φ .

Thus, combinatorial proofs are closely related to derivations of the form (3), and since by Theorem 14 every derivation can be transformed into that form, we can say that combinatorial proofs form a canonical proof representation for first order logic, similarly to what proof nets are for linear logic [34].

Finally, Theorems 13, 14 and 16 imply Theorem 11, which means that we have here an alternative proof of the soundness and completeness of first-order combinatorial proofs which is simpler than the one given in [20].

VII. TRANSLATING BETWEEN LK1 AND KS1

In this section we prove Theorems 13, 14, and 15, mainly by translating derivations to and from the sequent calculus, and by rule permutation arguments.

A. The Linear Fragments MLL1^x and MLS1^x

In this section we show the equivalence of MLL1^x and MLS1^x .

Lemma 17. *If $\vdash \Gamma$ is provable in MLL1^x then $\text{fm}(\Gamma)$ is provable in MLS1^x .*

Proof. This is a straightforward induction on the proof of $\vdash \Gamma$ in MLL1^x , making a case analysis on the bottommost rule instance. We show here only the case of $\forall \frac{\vdash \Delta, A}{\vdash \Delta, \forall x.A}$ (all other cases are simpler and have been shown before, e.g. [30]): By induction hypothesis, there is a proof of $\text{fm}(\Delta) \vee A$ in MLS1^x . We can prefix every line in that proof by $\forall x$ and then compose the following derivation:

$$\begin{array}{c} \forall \frac{t}{\forall x.t} \\ \text{MLS1}^x \parallel \\ \forall x.\text{fm}(\Delta) \vee A \\ \equiv \\ \text{fm}(\Delta) \vee \forall x.A \end{array}$$

where we can apply the \equiv -rule because x is not free in Δ . \square

353 **Lemma 18.** Let $r \frac{S\{A\}}{S\{B\}}$ be an inference rule in MLS1^X other
 354 than ai. Then the sequent $\vdash \overline{A}, B$ is provable in MLL1^X .

355 *Proof.* This is a straightforward exercise that we leave to the
 356 reader. (Note that the ax-rule can be applied to $\vdash f, t$ in the
 357 cases of $r = \forall$.) \square

358 **Lemma 19.** Let A, B be formulas, and let $S\{\cdot\}$ be a (pos-
 359 itive) context. If $\vdash \overline{A}, B$ is provable in MLL1^X , then so is
 360 $\vdash \overline{S\{A\}}, S\{B\}$.

361 *Proof.* Straightforward induction on $S\{\cdot\}$. (see e.g. [35]) \square

362 **Lemma 20.** If a formula C is provable in MLS1^X then $\vdash C$
 363 is provable in MLL1^X .

Proof. We proceed by induction on the number of inference
 steps in the proof of C in MLS1^X . Consider the bottommost
 rule instance $r \frac{S\{A\}}{S\{B\}}$. By induction hypothesis we have a
 MLL1^X proof Π of $\vdash S\{A\}$. If r is ai $\frac{S\{t\}}{S\{a \vee \overline{a}\}}$, we replace
 in Π all corresponding occurrences of t with $a \vee \overline{a}$ and the

rule instance $t \frac{}{\vdash t}$ with the derivation $\frac{\text{ax } \frac{}{\vdash a, \overline{a}}}{\vdash a \vee \overline{a}}$. This gives

us a proof of $\vdash S\{a \vee \overline{a}\}$. In all other cases, by Lemmas 18
 and 19, we have a MLL1^X proof of $\vdash \overline{S\{A\}}, S\{B\}$. We can
 compose them via cut:

$$\text{cut} \frac{\vdash S\{A\} \quad \vdash \overline{S\{A\}}, S\{B\}}{\vdash S\{B\}}$$

364 and then apply Theorem 2. \square

365 B. Contraction and Weakening

366 The first observation here is that Lemmas 17–20 from above
 367 also hold for LK1 and KS1. We therefore immediately have:

368 **Theorem 21.** For every sequent Γ , we have that $\vdash \Gamma$ is
 369 provable in LK1 if and only if $\text{fm}(\Gamma)$ is provable in KS1.

370 Then Theorem 13 is an immediate consequence. Let us now
 371 proceed with providing further lemmas that will be needed for
 372 the other results.

373 **Lemma 22.** The c -rule is derivable in $\{\text{ac}, m, m_\forall, m_\exists, \equiv\}$.

Proof. We show that there is always a derivation

$$\frac{A \vee A}{s \parallel} \quad A$$

374 , where $S = \{\text{ac}, m, m_\forall, m_\exists, \equiv\}$, by induction on A :

375 • If $A = a$, then we have $\text{ac} \frac{a \vee a}{a}$.

• If $A = B \wedge C$, then we have $m \frac{(B \wedge C) \vee (B \wedge C)}{(B \vee B) \wedge (C \vee C)}$.
 $s \parallel$
 $B \wedge (C \vee C)$ 376
 $s \parallel$
 $B \wedge C$ 377
 $\equiv \frac{(B \vee C) \vee (B \vee C)}{(B \vee B) \vee (C \vee C)}$
 • If $A = B \vee C$, then we have $s \parallel$
 $B \vee (C \vee C)$ 378
 $s \parallel$
 $B \vee C$ 379
 $m_\exists \frac{(\exists x.A') \vee (\exists x.A')}{\exists x.(A' \vee A')}$
 • If $A = \exists x.A'$, then we have $s \parallel$
 $\exists x.A'$ 380
 $m_\forall \frac{(\forall x.A') \vee (\forall x.A')}{\forall x.(A' \vee A')}$
 • If $A = \forall x.A'$, then we have $s \parallel$
 $\forall x.A'$ 381
 382 383 384 385

383 **TODO:** **Jui-Hsuan:** done. Maybe just keep one
 384 case. **Lutz:** yes, but we do that at the end. don't think about
 385 space right now. \square

386 **Lemma 23.** $c_\forall, m, m_\forall, m_\exists$ are derivable in $\{w, c, \equiv\}$.

387 *Proof.* **TODO:**

We have the following derivations:

$$\frac{w \frac{\forall x. \forall x.A}{\forall x. ((\forall x.A) \vee A)} \quad \equiv \frac{(\forall x.A) \vee (\forall x.A)}{c \frac{\forall x.A}{\forall x.A}} (x \notin fv(\forall x.A))$$

$$\frac{w \frac{(A \wedge C) \vee (B \wedge D)}{((A \vee B) \wedge C) \vee (B \wedge D)} \quad w \frac{((A \vee B) \wedge (C \vee D)) \vee (B \wedge D)}{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge (D \vee C))} \quad w \frac{((A \vee B) \wedge (C \vee D)) \vee ((B \vee A) \wedge (D \vee C))}{c \frac{(A \vee B) \wedge (C \vee D)}{(A \vee B) \wedge (C \vee D)}}$$

$$\frac{w \frac{(\exists x.A) \vee (\exists x.B)}{(\exists x.(A \vee B)) \vee (\exists x.B)} \quad w \frac{(\exists x.(A \vee B)) \vee (\exists x.(B \vee A))}{\equiv \frac{\exists x.(A \vee B) \vee (\exists x.(A \vee B))}{c \frac{\exists x.(A \vee B)}{\exists x.(A \vee B)}}}$$

$$\begin{array}{c}
(\forall x.A) \vee (\forall x.B) \\
\text{w} \frac{}{(\forall x.(A \vee B)) \vee (\forall x.B)} \\
\text{w} \frac{}{(\forall x.(A \vee B)) \vee (\forall x.(B \vee A))} \\
\equiv \\
\text{c} \frac{}{\forall x.(A \vee B)}
\end{array}$$

388 . **Jui-Hsuan:** done. If needed, we can introduce the notion
389 of open deduction to reduce the size of derivations...**Lutz:**
390 I was thinking about that, but (i) it is probably not worth the
391 effort, as we won't have many derivations, and (ii) it is hard to
392 define rectified derivations this way. \square

Lemma 24. Let A and B be formulas. Then

$$\frac{A}{\{w, c, \equiv\}} \parallel \quad \Longleftrightarrow \quad \frac{A}{\{w, ac, c\vee, m, m\vee, m\exists, \equiv\}} \parallel \quad B$$

393 *Proof.* This follows immediately from Lemmas 22 and 23.
394 \square

395 C. Rule Permutations

Theorem 25. Let Γ be a sequent. If $\vdash \Gamma$ is provable in LK1 (as depicted on the left below) then there is a sequent Γ' such that there is a derivation as shown on the right below:

$$\text{LK1} \frac{}{\vdash \Gamma} \Phi \quad \Longrightarrow \quad \text{MLL1}^x \frac{}{\vdash \Gamma'} \Phi_1$$

$\{w, c, \equiv\} \parallel \Phi_2$
 $\vdash \text{fm}(\Gamma)$

396 *Proof.* Note that the instances of w, c in Φ_2 are deep, but
397 inside sequent contexts.

398 First, if an instance of $\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A}$ is followed by a rule in
399 which A is not in the principal formula, it can be permuted
400 downwards. Otherwise, the proof can be transformed using the
401 following rewriting rules.

$$\begin{array}{l}
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \wedge \frac{}{\vdash \Gamma, A \wedge B, \Delta} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A \wedge B, \Delta} \\
\text{wk} \frac{\vdash \Gamma, A}{\vdash \Gamma, A, B} \vee \frac{}{\vdash \Gamma, A \vee B} \rightsquigarrow \text{w} \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \\
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A[x/t]} \exists \frac{}{\vdash \Gamma, \exists x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \exists x.A} \\
\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \forall \frac{}{\vdash \Gamma, \forall x.A} \rightsquigarrow \text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, \forall x.A}
\end{array}$$

$$\begin{array}{c}
\vdash \Gamma, A \\
\text{wk} \frac{}{\vdash \Gamma, A, A} \rightsquigarrow \vdash \Gamma, A \\
\text{ctr} \frac{}{\vdash \Gamma, A}
\end{array}$$

Note that in the case of \vee , we use the deep rule w which can
be permuted down over all the rules. By using these rewriting
rules, we can eventually get a derivation with all the instances
of wk and w at the bottom. Now observe that the instances of
 ctr in Φ can be transformed using the following rule:

$$\text{ctr} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\text{c} \frac{}{\vdash \Gamma, A \vee A}}$$

Knowing that c can be permuted down over all the rules of
 MLL1^x , we eventually obtain a derivation:

$$\begin{array}{c}
\text{MLL1}^x \frac{}{\vdash \Gamma_0} \Phi'_1 \\
\{wk, w, c, \equiv\} \parallel \Phi'_2 \\
\vdash \Gamma
\end{array}$$

Note that \equiv is required here since the permutation of formulas
is implicit in MLL1^x .

By transforming each sequent of Φ'_2 into its corresponding
formula, and by considering the following rewriting rule:

$$\text{wk} \frac{\vdash \Gamma}{\vdash \Gamma, A} \rightsquigarrow \text{w} \frac{\vdash \text{fm}(\Gamma)}{\vdash \text{fm}(\Gamma) \vee A}$$

, we obtain a derivation

$$\begin{array}{c}
\text{MLL1}^x \frac{}{\vdash \Gamma'} \Phi_1 \\
\{w, c, \equiv\} \parallel \Phi_2 \\
\vdash \text{fm}(\Gamma)
\end{array}$$

where $\Gamma' = \text{fm}(\Gamma_0)$ and Φ_1 can be obtained from Φ'_1 by
applying the \vee rule. **TO CHECK:** **Jui-Hsuan:** This
might be a bit long... \square

Lemma 26. For every derivation $\text{MLS1}^x \frac{t}{A}$ there are formulas
 A' and A'' such that

$$\begin{array}{c}
t \\
\{ \forall, ai, t \} \parallel \\
A'' \\
\{ s, mix, \equiv \} \parallel \\
A' \\
\{ \exists \} \parallel \\
A
\end{array}$$

Proof. First, observe that the \exists rule can be permuted down-
wards over all the other rules since $A[x/t]$ has the same
structure as A and none of the other rules has a premise

of the form $S\{\exists x.A\}$. It suffices now to prove that for all $r_1 \in \{\forall, \text{ai}, \text{t}\}$, for all $r_2 \in \{\text{s}, \text{mix}, \equiv\}$, we can permute r_1 upwards over r_2 . We show some cases here, and leave the others to the reader.

$$\begin{array}{c} \text{s} \frac{S\{A \wedge (\text{t} \vee C)\}}{S\{(A \wedge \text{t}) \vee C\}} \\ \text{ai} \frac{\quad}{S\{(A \wedge (a \vee \bar{a})) \vee C\}} \end{array} \rightsquigarrow \begin{array}{c} \text{ai} \frac{S\{A \wedge (\text{t} \vee C)\}}{S\{A \wedge ((a \vee \bar{a}) \vee C)\}} \\ \text{s} \frac{\quad}{S\{(A \wedge (a \vee \bar{a})) \vee C\}} \end{array}$$

$$\begin{array}{c} \text{mix} \frac{S\{A \wedge B\}}{S\{A \vee B\}} \\ \text{t} \frac{\quad}{S\{(A \vee (B \wedge \text{t}))\}} \end{array} \rightsquigarrow \begin{array}{c} \text{t} \frac{S\{A \wedge B\}}{S\{A \wedge (B \wedge \text{t})\}} \\ \text{mix} \frac{\quad}{S\{(A \vee (B \wedge \text{t}))\}} \end{array}$$

412 **[[TO CHECK:]]**

□

Lemma 27. For every derivation $\{w, \text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists, \equiv\} \parallel \begin{array}{c} A \\ B \end{array}$ there are formulas A' and B' such that

$$\begin{array}{c} A \\ \{m, m_\forall, m_\exists, \equiv\} \parallel \\ A' \\ \{\text{ac}, \text{c}_\forall\} \parallel \\ B' \\ \{w, \equiv\} \parallel \\ B \end{array}$$

Proof. First, a derivation consisting of an instance of w followed by $r \in \{\text{ac}, \text{c}_\forall, \text{m}, \text{m}_\forall, \text{m}_\exists\}$ can be either replaced by a derivation consisting of w only or the instance of w can be permuted downwards. We show some cases here, and leave the others to the reader.

$$\begin{array}{c} \text{w} \frac{S\{\forall x.A\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \\ \text{m}_\forall \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{w} \frac{S\{\forall x.A\}}{S\{\forall x.(A \vee B)\}} \end{array}$$

$$\begin{array}{c} \text{w} \frac{S\{A \wedge C\}}{S\{(A \wedge C) \vee (B \wedge D)\}} \\ \text{m} \frac{\quad}{S\{(A \vee B) \wedge (C \vee D)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{w} \frac{S\{A \wedge C\}}{S\{(A \vee B) \wedge C\}} \\ \text{w} \frac{\quad}{S\{(A \vee B) \wedge (C \vee D)\}} \end{array}$$

$$\begin{array}{c} \text{w} \frac{S\{a\}}{S\{a \vee a\}} \\ \text{ac} \frac{\quad}{S\{a\}} \end{array} \rightsquigarrow S\{a\}$$

For $r_1 \in \{\text{m}, \text{m}_\forall, \text{m}_\exists\}$, $r_2 \in \{\text{ac}, \text{c}_\forall\}$, r_1 can be permuted upwards over r_2 (with some \equiv inserted). The only non-trivial case is shown below:

$$\begin{array}{c} \text{c}_\forall \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{(\forall x.A) \vee (\forall x.B)\}} \\ \text{m}_\forall \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array} \rightsquigarrow \begin{array}{c} \text{m}_\forall \frac{S\{(\forall x.\forall x.A) \vee (\forall x.B)\}}{S\{\forall x.(A \vee B)\}} \\ \equiv \frac{\quad}{S\{\forall x.(A \vee B)\}} \end{array}$$

413 **[[TODO: permutation with \equiv]]**

□

414 We can now complete the proof of Theorems 14 and 15.

Proof of Theorem 15. Assume we have a proof of A in KS1. By Theorem 21 we have a proof of $\vdash A$ in LK1 to which we can apply Theorem 25. Finally, we apply Lemma 17 to get the desired shape. □

Proof of Theorem 14. Assume we have a proof of A in KS1. We first apply Theorem 15, and then Lemma 26 to the upper half and Lemma 27 to the lower half. □

VIII. FONETS AND LINEAR PROOFS

A. From MLL1^X Proofs to Fonets

Let Π be a MLL1^X proof of a rectified sequent $\vdash \Gamma$. We now show how Π is translated into a linked fograph $[\Pi] = \langle [\Gamma], \sim_\Pi \rangle$. We proceed inductively, making a case analysis on the last rule in Π . At the same time we are constructing a dualizer δ_Π , so that in the end we can conclude that $[\Pi]$ is in fact a fonet.

- 1) Π is $\text{ax} \frac{\quad}{\vdash a, \bar{a}}$: Then the only link is $\{a, \bar{a}\}$, and δ_Π is empty.
- 2) Π is $\text{t} \frac{\quad}{\vdash \text{t}}$: Then \sim_Π and δ_Π are both empty.
- 3) The last rule in Π is $\text{mix} \frac{\vdash \Gamma' \quad \vdash \Gamma''}{\vdash \Gamma', \Gamma''}$: By induction hypothesis, we have proofs Π' and Π'' of Γ' and Γ'' , respectively. We have $[\Gamma] = [\Gamma'] + [\Gamma'']$ and let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

- 4) The last rule in Π is $\vee \frac{\vdash \Gamma_1, A, B}{\vdash \Gamma_1, A \vee B}$: By induction hypothesis, there is proofs Π' of $\Gamma' = \Gamma_1, A, B$. We have $[\Gamma] = [\Gamma']$ and let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.
- 5) The last rule in Π is $\wedge \frac{\vdash \Gamma_1, A \quad \vdash B, \Gamma_2}{\vdash \Gamma_1, A \wedge B, \Gamma_2}$: By induction hypothesis, we have proofs Π' and Π'' of $\Gamma' = \Gamma_1, A$ and $\Gamma'' = B, \Gamma_2$, respectively. We have $[\Gamma] = [\Gamma_1] + ([A] \times [B]) + [\Gamma_2]$ and we let

$$\sim_\Pi = \sim_{\Pi'} \cup \sim_{\Pi''} \quad \text{and} \quad \delta_\Pi = \delta_{\Pi'} \cup \delta_{\Pi''}$$

- 6) The last rule in Π is $\exists \frac{\vdash \Gamma_1, A[x/t]}{\vdash \Gamma_1, \exists x.A}$: By induction hypothesis, there is a Π' of $\Gamma' = \Gamma_1, A[x/t]$. For each atom in $\Gamma' = \Gamma_1, A[x/t]$, there is a corresponding atom in $\Gamma = \Gamma_1, \exists x.A$. We can therefore define the linking \sim_Π from the linking $\sim_{\Pi'}$ via this correspondence. Then, we let δ_Π be $\delta_{\Pi'} + [x/t]$. Since Γ is rectified x does not yet occur in $\delta_{\Pi'}$. Hence δ_Π is a dualizer of $[\Pi]$.
- 7) The last rule in Π is $\forall \frac{\vdash \Gamma_1, A}{\vdash \Gamma_1, \forall x.A}$ (x not free in Γ_1): By induction hypothesis, there is a proof Π' of $\Gamma' = \Gamma_1, A$, which has the same atoms as in $\Gamma = \Gamma_1, \forall x.A$. Hence, we can let $\sim_\Pi = \sim_{\Pi'}$ and $\delta_\Pi = \delta_{\Pi'}$.

Theorem 28. If Π is a MLL1^X proof of a rectified sequent $\vdash \Gamma$, then $[\Pi]$ is a fonet and δ_Π is a dualizer for it.

Proof. We have to show that none of the operations above can introduce a bimatting. For cases 1–6, this is immediate. For case 7, observe that there is a potential dependency from each existential binder in $\llbracket \Gamma' \rrbracket$ to the new x -binder $\bullet x$ in $\llbracket \Gamma \rrbracket$. However, observe that this $\bullet x$ vertex is not connected to any vertex in $\llbracket \Gamma' \rrbracket$, and hence no such new dependency can be extended to a bimatting. That δ_Π is a dualizer for $\llbracket \Pi \rrbracket$ follows immediately from the construction. Hence, $\llbracket \Pi \rrbracket$ is a fonet. \square

B. From MLS1^\times Proofs to Fonets

There is a more direct path from a MLL1^\times proof Π of a rectified sequent Γ to the linked fograph $\llbracket \Pi \rrbracket$: simply take the fograph $\llbracket \Gamma \rrbracket$, and let the equivalence classes of \sim_Π be all the atom pairs that meet in an instance of ax , and δ_Π is simply the collection of all substitutions of all the instances of the \exists -rule in Π . We have chosen the more cumbersome path above because it gives us a direct proof of Theorem 28. However, for translating MLS1^\times derivation into fonets, we employ exactly that direct path.

A derivation Φ in MLS1^\times is **rectified** if every line in Φ is rectified.

Lemma 29. *Let Φ be a MLS1^\times proof of a formula A . Then Φ is rectified iff A is rectified.*

Proof. The only rules involving bound variables are \forall and \exists which both remove a binder (and all occurrences of the variable it binds). \square

Hence, for a non-rectified MLS1^\times derivation Φ in MLS1^\times we can define its **rectification** $\hat{\Phi}$ inductively, by rectifying each line, proceeding step-wise from conclusion to premise.²

A rectified derivation $\text{MLS1}^\times \parallel \Phi$ determines a substitution A

which maps the existential bound variables occurring in A to the terms substituted for them in the instances of the \exists -rule in Φ . We denote this substitution with δ_Φ and call it the **dualizer** of Φ . Furthermore, every atom occurring in the conclusion A must be consumed by a unique instance of the rule ai in Φ . This allows us to define a (partial) equivalence relation \sim_Φ on the atom occurrences in A by $a \sim_\Phi b$ if a and b are consumed by the same instance of ai in Φ . We call \sim_Φ the **linking** of Φ , and define $\llbracket \Phi \rrbracket = \langle \llbracket A \rrbracket, \sim_\Phi \rangle$.

TODO: example here

Theorem 30. *Let $\text{MLS1}^\times \parallel \Phi$ be a rectified derivation. Then $\llbracket \Phi \rrbracket$ is a fonet and δ_Φ a dualizer for it.*

For proving this theorem, we have to show that no inference rule in MLS1^\times can introduce a bimatting. To simplify the argument, we introduce the **frame** [37] of the fograph \mathcal{C} , which is a linked (propositional) cograph in which the dependencies between the binders in \mathcal{C} are encoded as links.

²As for formulas, the rectification of a derivation is unique up to renaming of bound variables.

More formally, let C be a formula with $\llbracket C \rrbracket = \mathcal{C}$, to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent C^* :

- 1) **Encode dependencies as fresh links.** For each dependency $\{\bullet x_i, \bullet y_j\}$ in \mathcal{C} , with corresponding subformulas $\exists x_i.A$ and $\forall y_j.B$ in C , we pick a fresh (nullary) predicate symbol $q_{i,j}$, and then replace $\exists x_i.A$ by $\bar{q}_{i,j} \wedge \exists x_i.A$, and replace $\forall y_j.B$ by $q_{i,j} \vee \forall y_j.B$.
- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace $\exists x_i.A$ by A and replace $\forall y_j.B$ by B everywhere.
- 3) **Simplify atoms.** After step 2, replace every predicate $p(t_1 \cdots t_n)$ (resp. $\bar{p}(t_1 \cdots t_n)$) with a nullary predicate symbol p (resp. \bar{p}).

The \sim_{C^*} consists of the pairs induced by $\sim_{\mathcal{C}}$ and the new pairs $\{q_{i,j}, \bar{q}_{i,j}\}$ introduced in step 1 above. We call C^* the **frame** of C and we define the **frame** of \mathcal{C} , denoted \mathcal{C}^* , as $\langle \llbracket C^* \rrbracket, \sim_{C^*} \rangle$.

Lemma 31. *A linked fograph \mathcal{C} has an induced bimatting iff its frame \mathcal{C}^* has an induced bimatting.*

Proof. This immediately follows from the construction of the frame. **Lutz: is it really an “iff”? It is easy to construct from a bimatting in \mathcal{C} a bimatting in the frame. (and I think we only need that direction). But what about the other direction?** \square

Proof of Theorem 30. From Φ we construct a derivation Φ^* of A^* in the propositional fragment of MLS1^\times , such that $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$. The rules ai , t , mix and s are translated trivially, and for \equiv , it suffices to observe that the frame construction is invariant under \equiv . Finally, for the rules \forall and \exists , proceed as follows. Every instance of \forall is replaced by the derivation on the right below:³

$$\forall \frac{S\{t\}}{S\{\forall y_j.t\}} \rightsquigarrow \frac{\frac{\text{t} \quad \{a_i, t\} \parallel \Psi_1}{S\{(q_{h_1,j} \vee \bar{q}_{h_1,j}) \wedge \cdots \wedge (q_{h_j,j} \vee \bar{q}_{h_j,j}) \wedge t\}}}{\{s, \equiv\} \parallel \Psi_2} S\{q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge t)\}$$

where h_1, \dots, h_j range over the indices of the existential binders dependent on that y_j . It is easy to see how Ψ_1 is constructed, and for Ψ_2 see, e.g. [?, [35], [36]] **Lutz: check if it is really there. otherwise [?]** Then, every occurrence of $\forall y_j.F$ is replaced by $q_{h_1,j} \vee \cdots \vee q_{h_j,j} \vee (\bar{q}_{h_1,j} \wedge \cdots \wedge \bar{q}_{h_j,j} \wedge F)$ in the derivation below that \forall -instance. Now, observe that all instances of the \exists -rule introducing x_i depend on y_j must occur below in the derivation (otherwise Φ would not be rectified). Now consider such an instance $\exists \frac{S\{B[x_i/t]\}}{S\{\exists x_i.B\}}$. Its context $S\{\cdot\}$ must contain all the $\forall y_j$ the $\exists x_i$ depends on, such that B is in their scope. Following the translation of the

³For better readability we omit superfluous parentheses, knowing that we always have \equiv incorporating associativity and commutativity of \wedge and \vee .

\forall rules above, we can therefore translate the \exists -rule instance by the following derivation

$$\begin{aligned} & S_0\{\bar{q}_{i,k_1} \wedge S_1\{\bar{q}_{i,k_2} \wedge \cdots S_{k_i-1}\{\bar{q}_{i,k_i} \wedge S_{k_i}\{B'\}\}\cdots\}\} \\ & \quad \quad \quad \{s, \equiv\} \parallel \Psi_3 \\ & S_0\{S_1\{\cdots S_{k_i-1}\{S_{k_i}\{\bar{q}_{i,k_1} \wedge \bar{q}_{i,k_2} \wedge \cdots q_{i,k_i} \wedge B'\}\}\cdots\}\} \end{aligned}$$

where k_1, \dots, k_i are the indices of the universal binders on which that x_i depends, and B' is B in which all predicates are replaced by nullary one (step 3 in the frame construction). The derivation Ψ_3 can be constructed in the same way as Ψ_2 above.

Doing this to all instances of the rules \forall and \exists in Φ yields indeed a propositional derivation Φ^* with $\llbracket \Phi^* \rrbracket = \llbracket \Phi \rrbracket^*$. It has been shown by Retoré [?] and rediscovered by Straßburger [?] that $\llbracket \Phi^* \rrbracket = \langle \llbracket C^* \rrbracket, \sim_{\Phi^*} \rangle$ can not contain an induced bimatching. By Lemma 33, $\llbracket \Phi \rrbracket$ does not have an induced bimatching either. Furthermore, it followed from the definition of δ_Φ that it is a dualizer for $\llbracket \Phi \rrbracket$. Hence $\llbracket \Phi \rrbracket$ is a fonet. \square

Remark 32. There is an alternative path of proving Theorem 30 by translating Φ to an MLL1^\times -proof Π , observing that this process preserves the linking and the dualizer. However, for this, we have to extend the construction above to the cut-rule, and then show that linking and dualizer of a sequent proof Π are invariant under cut elimination. This can be done similarly to unification nets in [37].

C. From Fonets to MLL1^\times Proofs

Now we are going to show how from a given fonet $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ we can construct a sequent proof Π in MLL1^\times such that $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. In the proof net literature, this operation is also called *sequentialization*. The basic idea behind our sequentialization is to construct a propositional linked cograph, called the **frame** [37] of \mathcal{C} , in which the dependencies between the binders in \mathcal{C} are encoded as links. Then we can apply the *splitting tensor theorem* to the frame, and then reconstruct the sequent proof Π .

More formally, let Γ be a sequent with $\llbracket \Gamma \rrbracket = \mathcal{C}$, to which we exhaustively apply the following subformula rewriting steps, to obtain a sequent Γ^* :

- 1) **Encode dependencies as fresh links.** For each dependency $(\bullet x, \bullet y)$ in \mathcal{C} , with corresponding subformulas $\exists x A$ and $\forall y B$ in Γ , we pick a fresh (nullary) predicate symbol q , and then replace $\exists x A$ by $q \wedge \exists x A$, and replace $\forall y B$ by $\bar{q} \vee \forall y B$.
- 2) **Erase quantifiers.** After step 1, remove all the quantifiers, i.e., replace $\exists x A$ by A and replace $\forall y B$ by B everywhere.
- 3) **Simplify atoms.** After step 2, replace every predicate $p(t_1 \cdots t_n)$ (resp. $\bar{p}(t_1 \cdots t_n)$) with a nullary predicate symbol p (resp. \bar{p})

The \sim_{Γ^*} consists of the pairs induced by $\sim_{\mathcal{C}}$ and the new pairs $\{q, \bar{q}\}$ introduced in step 1 above. We call Γ^* the **frame** of Γ and we define the **frame** of \mathcal{C} , denoted \mathcal{C}^* , as $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$, and we immediately have the following:

Lemma 33. *A linked fograph \mathcal{C} induces a bimatching iff its frame \mathcal{C}^* has an induced bimatching.*

Let Γ be a propositional sequent and \sim_Γ be a linking for $\llbracket \Gamma \rrbracket$. A conjunction formula $A \wedge B$ is **splitting** or a **splitting tensor** if $\Gamma = \Gamma', A \wedge B, \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that \sim_1 is a linking for $\llbracket \Gamma', A \rrbracket$ and \sim_2 is a linking for $\llbracket B, \Gamma'' \rrbracket$, i.e., removing the \wedge from $A \wedge B$ splits the linked fograph $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ into two fographs. We say that $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ is **mixed** iff $\Gamma = \Gamma', \Gamma''$ and $\sim_\Gamma = \sim_1 \cup \sim_2$, such that \sim_1 is a linking for $\llbracket \Gamma' \rrbracket$ and \sim_2 is a linking for $\llbracket \Gamma'' \rrbracket$. Finally, $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ is **splittable** if it is mixed or has a splitting tensor.

The purpose of introducing the frame is the following theorem.

Theorem 34. *Let Γ be a propositional sequent containing only atoms and \wedge -formulas, and \sim_Γ be a linking for $\llbracket \Gamma \rrbracket$. If $\langle \llbracket \Gamma \rrbracket, \sim_\Gamma \rangle$ does not induce a bimatching then it is splittable.*

This is the well-know splitting-tensor-theorem [25], [?], adapted for the presence of mix. In the setting of linked cographs, it has first been proved by Retoré [38], [?]. We use it now for our sequentialization:

Theorem 35. *Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let Γ be a sequent with $\llbracket \Gamma \rrbracket = \mathcal{C}$. Then there is an MLL1^\times -proof Π of Γ , such that $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$.*

Proof. Let $\delta_{\mathcal{C}}$ be the dualizer of $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. We proceed by induction on the size of Γ (i.e., the number of symbols in it, without counting the commas). If Γ contains a formula with \vee -root, or a formula $\forall x.A$, we can immediately apply the \vee -rule or the \forall -rule of MLL1^\times and proceed by induction hypothesis. If Γ contains a formula $\exists x.A$ such that the corresponding binder $\bullet x$ in \mathcal{C} has no dependency, then we can apply the \exists -rule, choosing the term t as determined by $\delta_{\mathcal{C}}$, and proceed by induction hypothesis. Hence, we can now assume that Γ contains only atoms, \wedge -formulas, or formulas of shape $\exists x.A$, where the vertex $\bullet x$ has dependencies. Then the frame $\langle \llbracket \Gamma^* \rrbracket, \sim_{\Gamma^*} \rangle$ does not induce a bimatching and contains only atoms and \wedge -formulas, and is therefore splittable. If it is mixed, then we can apply the mix-rule to Γ and apply the induction hypothesis to the two components. If it is not mixed then there must be a splitting tensor. If the splitting \wedge is already in Γ , then we can apply the \wedge -rule and proceed by induction hypothesis on the two branches. However, if Γ^* is not mixed and all splitting tensors are \wedge -formulas introduced in step 1 of the frame construction, then we get a contradiction as in that case there must be a \vee - or \forall -formula in Γ . **Lutz: can anyone give a good argument here?** \square

D. From Fonets to MLS1^\times Proofs

We can now straightforwardly obtain the same result for MLS1^\times :

Theorem 36. *Let $\langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$ be a fonet, and let C be a formula with $\llbracket C \rrbracket = \mathcal{C}$. Then there is a derivation $\text{MLS1}^\times \parallel_\Phi^t$ such that*

617 $\llbracket \Phi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$.

618 *Proof.* We apply Theorem 35 to obtain a sequent proof Π of \vdash
 619 C with $\llbracket \Pi \rrbracket = \langle \mathcal{C}, \sim_{\mathcal{C}} \rangle$. Then we apply Lemma 17, observing
 620 that the translation from MLL^{\times} to MLS^{\times} preserves linking
 621 and dualizer. \square

622 **Remark 37.** Note that it is also possible to do a direct
 623 “sequentialization” into the deep inference system MLS^{\times} ,
 624 using the techniques presented in [?] and [?].

625 IX. SKEW BIFIBRATIONS AND RESOURCE MANAGEMENT

In this section we establish the relation between skew bi-
 fibrations and derivations in $\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\}$. How-
 ever, if a derivation Φ contains instances of the rules c_{\forall} , m_{\forall} ,
 and m_{\exists} we can no longer naively define the rectification $\hat{\Phi}$ as
 in the previous section for MLS^{\times} , as these two rules cannot be
 applied if premise and conclusion are rectified. For this reason
 we define here rectified versions \hat{c}_{\forall} , \hat{m}_{\forall} and \hat{m}_{\exists} , shown below:

$$\begin{array}{c} \hat{c}_{\forall} \frac{S\{\forall y. \forall x. Ax\}}{S\{\forall x. Ax\}} \\ \hat{m}_{\forall} \frac{S\{(\forall y. Ay) \vee (\forall z. Bz)\}}{S\{\forall x. (Ax \vee Bx)\}} \\ \hat{m}_{\exists} \frac{S\{(\exists y. Ay) \vee (\exists z. Bz)\}}{S\{\exists x. (Ax \vee Bx)\}} \end{array}$$

626 Here, we use the notation $A \cdot$ for a formula A with occurrences
 627 of a placeholder \cdot for a variable. Then Ax stands for the results
 628 of replacing that placeholder with x , and also indicating that
 629 x must not occur in $A \cdot$. Then $\forall x. Ax$ and $\forall y. Ay$ are the same
 630 formula modulo renaming of the bound variable bound by the
 631 outermost \forall -quantifier. We also demand that the variables x ,
 632 y , and z do not occur in the context $S\{\cdot\}$.

633 Note that in an instance of \hat{m}_{\forall} or \hat{m}_{\exists} (as shown above),
 634 we can have $x = y$ or $x = z$, but not both if the premise is
 635 rectified. If $x = y$ and $x = z$ we have m_{\forall} and m_{\exists} as special
 636 cases of \hat{m}_{\forall} and \hat{m}_{\exists} , respectively. And similarly, if $x = y$ then
 637 c_{\forall} is a special case of \hat{c}_{\forall} .

638 For a derivation Φ in $\{w, ac, c_{\forall}, m, m_{\forall}, m_{\exists}, \equiv\}$, we can
 639 now construct the **rectification** $\hat{\Phi}$ by rectifying each line of
 640 Φ , yielding a derivation in $\{w, ac, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$.

641 For each instance $r \frac{Q}{P}$ of an inference rule in
 642 $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$ we can define the **induced map**
 643 $\llbracket r \rrbracket: V_{\llbracket Q \rrbracket} \rightarrow V_{\llbracket P \rrbracket}$ which acts as the identity for $r \in \{m, \equiv\}$
 644 and as the canonical injection for $r = w$. For $r = ac$ it maps
 645 the vertices corresponding to the two atoms in the premise
 646 to vertex of the contracted atom in the conclusion, and for
 647 $r \in \{\hat{c}_{\forall}, \hat{m}_{\forall}, \hat{m}_{\exists}\}$ it maps the two vertices corresponding to
 648 the quantifiers in the premise to the one in the conclusion (as
 649 acts as the identity on all other vertices). For a derivation Φ
 650 in $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$ we can then define the **induced**
 651 **map** $\llbracket \Phi \rrbracket$ as the composition of the induced maps of the
 652 rule instances in Φ . **[[Jui-Hsuan: maybe mention at least**
 653 **that the induced maps define graph homomorphisms. Do we**
 654 **need to talk about the contexts $S\{\cdot\}$ here (induced maps act**
 655 **clearly as the identity on contexts but we need them for the**

composition)?]] **[[Lutz: For the context, I already say it is the**
 identity. For the homom, it comes later]]

Lemma 38. Let $\{w, ac, m, m_{\forall}, m_{\exists}, \equiv\} \parallel \Phi$ be a derivation. Then
 there is a rectified derivation $\{w, \hat{ac}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\} \parallel \hat{\Phi}$, such that
 the induced maps $\llbracket \Phi \rrbracket: \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ and $\llbracket \hat{\Phi} \rrbracket: \llbracket \hat{A} \rrbracket \rightarrow \llbracket \hat{B} \rrbracket$
 are equal up to a variable renaming of the vertex labels.

Proof. Immediate from the definition. \square

[[TODO: example]]

664 A. From Contraction and Weakening to Skew Bifibrations

We say that a derivation Φ is **sane** if for every line Q in
 Φ we have that $\llbracket D \rrbracket$ is a fograph (i.e., all binders are legal).
 Clearly, every rectified derivation is sane, but not vice versa,
 as we might have multiple occurrences of bound variables in
 Q , such that $\llbracket Q \rrbracket$ is still a fograph.

Lemma 39. Let $\{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\} \parallel \Phi$ be a sane derivation.
 Then the induced map $\llbracket \Phi \rrbracket: \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ is a skew bifibration.

Before we show the proof of this lemma, we introduce
 another useful concept: the **propositional encoding** A° of a
 formula A , which is a propositional formula with the property
 that $\llbracket A^{\circ} \rrbracket = \llbracket A \rrbracket$. For this, we introduce new propositional
 variables that have the same names as the (first-order) variables
 $x \in \text{VAR}$. Then A° is defined inductively by:

$$\begin{array}{ll} a^{\circ} = a & (\forall x A)^{\circ} = x \vee A^{\circ} \\ (A \vee B)^{\circ} = A^{\circ} \vee B^{\circ} & (\exists x A)^{\circ} = x \wedge A^{\circ} \\ (A \wedge B)^{\circ} = A^{\circ} \wedge B^{\circ} & \end{array}$$

Lemma 40. For every formula A , we have $\llbracket A^{\circ} \rrbracket = \llbracket A \rrbracket$.

Proof. Straightforward induction on A . \square

We use \equiv° to denote the restriction of \equiv to propositional
 formulas, i.e., the first two lines in (2).

Proof of Lemma 39. First, observe that for every infer-
 ence rule $r \in \{w, ac, \hat{c}_{\forall}, m, \hat{m}_{\forall}, \hat{m}_{\exists}, \equiv\}$ the induced map
 $\llbracket r \rrbracket: V_{\llbracket Q \rrbracket} \rightarrow V_{\llbracket P \rrbracket}$ defines a existential preserving graph ho-
 momorphism $\llbracket Q \rrbracket \rightarrow \llbracket P \rrbracket$ and a fibration on the corresponding
 binding graphs. **[[Jui-Hsuan: we may need to have some**
explication here.]] **[[Lutz: no]]** Therefore, their composition
 $\llbracket \Phi \rrbracket$ has the same properties fibration.

For showing that it is also a skew fibration, we construct
 for Φ its propositional encoding Φ° by translating every line
 into its propositional encoding. **[[Jui-Hsuan: maybe mention**
 that an instance of one of the other rules can be translated
 into an instance of the same rule. It’s trivial but may be worth

mentioning. $\llbracket \llbracket \text{Lutz: done below} \rrbracket \rrbracket$ The instances of the rules \widehat{m}_\forall and \widehat{m}_\exists are replaced in two steps by:

$$\widehat{ac} \frac{\frac{S\{(y \vee (Ay)^\circ) \vee (z \vee (Bz)^\circ)\}}{S\{(y \vee z) \vee ((Ay)^\circ \vee (Bz)^\circ)\}}}{S\{x \vee ((Ax)^\circ \vee (Bx)^\circ)\}}$$

and

$$\widehat{m} \frac{\frac{S\{(y \wedge (Ay)^\circ) \vee (z \wedge (Bz)^\circ)\}}{S\{(y \vee z) \wedge ((Ay)^\circ \vee (Bz)^\circ)\}}}{S\{x \wedge ((Ax)^\circ \vee (Bx)^\circ)\}}$$

respectively, where \widehat{ac} is a ac that renames the variables—the propositional variable, as well as the first-order variable of the same name—as everything is recified, there is no ambiguity here. Any instance of a rule w , ac , m , or \equiv is translated to an instance of the same rule, and \widehat{c}_\forall is translated to \widehat{ac} .

This gives us a derivation $\{w, ac, \widehat{ac}, m, \equiv\} \parallel_{B^\circ} \Phi^\circ$ such that $[\Phi^\circ] = [\Phi]$. It has been shown in [23] that $[\Phi^\circ]$ is a skew fibration (see also [12], [?], [15]). Hence, $[\Phi]$ is a skew fibration. \square

B. From Skew Bifibrations to Contraction and Weakening

Lemma 41. *Let \mathcal{A} and \mathcal{B} be fographs, let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a skew bifibration, and let A and B be formulas with $\llbracket A \rrbracket = \mathcal{A}$ and $\llbracket B \rrbracket = \mathcal{B}$. Then there are derivations*

$$\frac{A}{\{w, ac, \widehat{c}_\forall, m, \widehat{m}_\forall, \widehat{m}_\exists, \equiv\} \parallel_{\widehat{B}} \widehat{\Phi}} \quad \text{and} \quad \frac{A\sigma_\varphi}{\{w, ac, \widehat{c}_\forall, m, m_\forall, m_\exists, \equiv\} \parallel_{\widehat{B}} \widehat{\Phi}}$$

such that $[\widehat{\Phi}] = \varphi$ and $\widehat{\Phi}$ is a rectification of Φ , and σ_φ is the substitution induced by φ .

In the proof of this lemma, we make use of the following concept: Let $s \parallel_\Psi P$ be a derivation where P and Q are propositional formulas (possibly using variable $x \in \text{VAR}$ at the places of atoms). We say that Ψ can be *lifted* to S' if there are (first-order) formulas C and D such that $P = C^\circ$ and $Q = D^\circ$ and there is a derivation $s' \parallel_{D'} \Psi'$.

Proof of Lemma 41. By Lemma 40 we have $\mathcal{A} = \llbracket A^\circ \rrbracket$ and $\mathcal{B} = \llbracket B^\circ \rrbracket$. Let $V'_B \subseteq V_B$ be the image of φ , and let B_1 be the subgraph of \mathcal{B} induced by V'_B . Hence, we have two maps $\varphi': \mathcal{A} \rightarrow B_1$ being a surjection and $\varphi'': B_1 \rightarrow \mathcal{B}$ being an injection that reflects edges. $\llbracket \text{Jui-Hsuan: what do you mean by "reflect edges"} \rrbracket \rrbracket \llbracket \text{Lutz: edge downstairs implies edge upstairs} \rrbracket \rrbracket$ Both, φ' and φ'' remain skew bifibrations. Let us first look at φ' . Let \tilde{B}_1 be the propositional formula obtained from B° by removing all atoms that are not represented by vertices in V'_B . Then $\llbracket \tilde{B}_1 \rrbracket = B_1$. By [23, Proposition 7.6.1], we have

a derivation $\{w, \equiv\} \parallel_{\Phi_1^\circ} \tilde{B}_1$. A subformula of B° is called *weak* if

it has been introduced by a weakening. All subformulas of a weak formula are also weak. Two weak subformulas B' and B'' of B° form a *weak pair* if $B^\circ \equiv S\{B' \vee B''\}$ for some context $S\{\cdot\}$. We can assume without loss of generality that whenever weak subformulas B' and B'' form a weak pair, they have been introduced by the same instance of w in Φ_1° .⁴ Now we show that Φ_1° can be lifted. For this, observe that whenever a weakening in Φ_1° deletes an atom $x \in \text{VAR}$, it must also delete all atoms in the scope of the corresponding quantifier, because φ' is a fibration on the binding graph. Hence, each line in Φ_1° is the propositional encoding P° of a first-order formula P . We now have to show that each instance of w is indeed a correct application in first-order logic. For this we have to inspect the cases a weakening happens inside a subformula $x \vee C$ or $x \wedge C$ in Φ_1° . There are the following cases:

$$\frac{S\{x \vee C\}}{S\{x \vee (D \vee C)\}} \quad \frac{S\{x \wedge C\}}{S\{x \wedge (D \vee C)\}} \quad \frac{S\{x \wedge C\}}{S\{(x \vee D) \wedge C\}}$$

In the first case the weakening happens inside the scope of a \forall -quantifier, and in the second case inside the scope of a \exists -quantifier. Both cases are unproblematic in first-order logic. However, in the third case an \exists -quantifier would be transformed into an \forall -quantifier. But as φ has to preserve existentials, this third case cannot occur. Thus we have a first

order derivation $\{w, \equiv\} \parallel_{\Phi_1} B_1$ with $B_1^\circ = \tilde{B}_1$.

Let us now look at φ'' . Let $\mathcal{A}_1 = \mathcal{A}\sigma_\varphi$ be the graph obtained from \mathcal{A} by applying σ_φ to all the labels. Note that \mathcal{A}_1 is not necessarily a fograph, as binders might be illegal. But it still is a cograph, and we have a surjective skew fibration $\varphi'': \mathcal{A}_1 \rightarrow \mathcal{B}_1$ that preserves the labels. Therefore, by [?,

Proposition 7.5], there is a derivation $\{ac, m, \equiv\} \parallel_{\Phi_2^\circ} A_1^\circ$, where

$A_1^\circ = A^\circ \sigma_\varphi$ is the result of applying σ_φ to A° . Note that $A_1^\circ = (A\sigma_\varphi)^\circ$ and B_1° are both propositional encodings. We plan to show that Φ_2 can be lifted to $\{ac, \widehat{c}_\forall, m, m_\forall, m_\exists, \equiv\}$. However, observe that not every formula occurring in Φ_2 is a propositional encoding. There are two reasons for this: (i) we might have $P \equiv^\circ Q$ where P is a propositional encoding but Q is not, and (ii) the rule ac can duplicate an atom $x \in \text{VAR}$. Let us write ac_x for such instances. To address (i), we consider here formulas equivalent modulo \equiv , always knowing that we can add instances of \equiv as needed.⁵ $\llbracket \text{Jui-Hsuan: this does not seem clear to me. What if from } A_1^\circ \text{ to } B_1^\circ \text{ there are just a bunch of } \equiv^\circ? \text{ What do we do in this case?} \rrbracket \rrbracket \llbracket \text{Lutz: see footnote} \rrbracket \rrbracket$ To address (ii), we apply a permutation argument, permuting all instances of ac_x up until they either reach the

⁴If Φ_1° is not of that shape, it can brought into this form by simple rule permutations, and then collapsing several weakenings into a single one.

⁵Note that whenever we have formulas P and Q with $P^\circ \equiv^\circ Q^\circ$ then $P \equiv Q$.

top of the derivation or an instance of m which separates the two atoms in the premise. More precisely, we consider the following inference rule

$$\text{ac}_x^{\equiv} \frac{S_0\{S_1\{x\} \vee S_2\{x\}\}}{S\{x\}} \quad (4)$$

where $S_1\{\cdot\} \equiv \{\cdot\} \vee E$ and $S_2\{\cdot\} \equiv \{\cdot\} \vee F$ and $S\{\cdot\} \equiv S_0\{\{\cdot\} \vee E \vee F\}$ for some formulas E and F , where E or F or both might be empty. The rule ac_x^{\equiv} permutes over \equiv , ac , and other instances of ac_x^{\equiv} , and over instances of m if they occur inside S_0 or S_1 or S_2 . The only situation in which ac_x^{\equiv} cannot be permuted up is the following:

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(R_1\{x\} \wedge C) \vee (R_2\{x\} \wedge D)\}}}{S\{(R_1\{x\} \vee R_2\{x\}) \wedge (C \vee D)\}} \quad (5)$$

We can therefore assume that all instances of ac_x , that contract an atom $x \in \text{VAR}$ are either at the top of Φ_2° or below a m -instance as in (5). We now lift Φ_2° to $\{\text{ac}, \text{c}_\forall, m, m_\forall, m_\exists, \equiv\}$, proceed by induction on the height of Φ_2° , beginning at the top, making a case analysis on the topmost rule that is not \equiv .

- ac_x : We know that the premisses of (4) is a propositional encoding. Hence, $S_1\{\cdot\} = \{\cdot\} \vee E^\circ$ and $S_2\{\cdot\} = \{\cdot\} \vee F^\circ$ and both x are universals, and $E^\circ \vee F^\circ$ contains all occurrences of x bound by that universal. We have the following subcases:

- E and F are both non-empty: We have

$$\text{ac}_x^{\equiv} \frac{S^\circ\{(x \vee E^\circ) \vee (x \vee F^\circ)\}}{S^\circ\{x \vee (E^\circ \vee F^\circ)\}}$$

which can be lifted to

$$m_\forall \frac{S\{(\forall x.E) \vee (\forall x.F)\}}{S\{\forall x.(E \vee F)\}}$$

where $S^\circ\{\cdot\}$, E° , F° are the propositional encodings of $S\{\cdot\}$, E , F , respectively.

- E° is empty and F° is non-empty: We have

$$\text{ac}_x^{\equiv} \frac{S^\circ\{x \vee (x \vee F^\circ)\}}{S^\circ\{x \vee F^\circ\}}$$

which can be lifted to

$$\text{c}_\forall \frac{S\{\forall x.\forall x.F\}}{S\{\forall x.F\}}$$

- E° is non-empty and F° is empty: This is similar to the previous case.
- E° and F° are both empty: This is impossible as the premise would not be a propositional encoding.

- ac (contracting an ordinary atom): This can trivially be lifted.
- m that is not in the situation of (5): Then now encoding of a quantifier is affected and the instance of m can be lifted. **TODO: medial permutation!!!**

- m/ac_x as in situation (5): We must have $R_1\{x\} \equiv x \vee E$ for some E and $R_2\{x\} \equiv x \vee F$ for some F with $R\{x\} \equiv x \vee E \vee F$. Otherwise, the application of ac_x^{\equiv} would not be correct. We have the following four cases:

- E and F are both non-empty: Then (5) is (modulo omitted applications of \equiv):

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{((x \vee E) \wedge C) \vee ((x \vee F) \wedge D)\}}}{S\{((x \vee E) \vee (x \vee F)) \wedge (C \vee D)\}}$$

which can be lifted to

$$m_\forall \frac{\frac{m}{S\{((\forall x.E) \wedge C) \vee ((\forall x.F) \wedge D)\}}}{S\{((\forall x.E) \vee (\forall x.F)) \wedge (C \vee D)\}}$$

Jui-Hsuan: maybe need some words to exclude the case in which C (or D) is a propositional variable. **Lutz:** shit. (you mean a “first order variable”) this actually can happen. then we have another m_\exists

- E is empty and F is not: Then (5) becomes

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(x \wedge C) \vee ((x \vee F) \wedge D)\}}}{S\{(x \vee (x \vee F)) \wedge (C \vee D)\}}$$

The conclusion is the propositional encoding of $S\{(\forall x.F) \wedge (C \vee D)\}$ and the premise is the propositional encoding of $S\{(\exists x.C) \vee ((\forall x.F) \vee D)\}$. Also note that no m -instance can break up the conjunction in $x \wedge C$ in the premise. Hence, φ maps an existential to a universal, which is ruled out by the definition. Hence, this case cannot occur.

- E is non-empty and F is empty: This case is similar to the previous subcase.
- E and F are both empty: Then (5) is

$$\text{ac}_x^{\equiv} \frac{\frac{m}{S\{(x \wedge C) \vee (x \wedge D)\}}}{S\{(x \vee x) \wedge (C \vee D)\}}$$

which can be lifted immediately to

$$m_\exists \frac{S\{(\exists x.C) \vee (\exists x.D)\}}{S\{\exists x.(C \vee D)\}}$$

Thus Φ_2° can be lifted to $\{\text{ac}, \text{c}_\forall, m, m_\forall, m_\exists, \equiv\} \parallel \Phi_2$. We construct B_1

Φ by composing Φ_2 and Φ_1 . Then $\hat{\Phi}$ can be constructed by rectifying Φ , where the variables to be used in A are already given. That $\varphi = [\hat{\Phi}]$ follows immediately from the construction. \square

X. SUMMARY AND PROOFS OF MAIN RESULTS

XI. CONCLUSION

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858 A. Unification Nets

859 **[[TODO:]]**

860 In this paragraph, we associate each formula A with its
861 **formula tree** $\mathcal{F}(A)$, a directed tree with leaves labelled by
862 atoms, internal nodes labelled by connectives and quantifiers,
863 and edges directed from leaves to the root. For a sequent
864 $\Gamma = A_1, \dots, A_n$, we denote with $\mathcal{F}(\Gamma)$, the forest formed by
865 $\mathcal{F}(A_1), \dots, \mathcal{F}(A_n)$, i.e., the disjoint union of $\mathcal{F}(A_i)$'s. The
866 **roots** of $\mathcal{F}(\Gamma)$ are the roots of A_i 's

867 Let Γ be a sequent in MLL1^\times . Consider the forest $\mathcal{F}(\Gamma)$.
868 A **link** on Γ is a pair of leaves whose atoms are pre-dual. A
869 **linking** λ on Γ is a set of disjoint links such that each leaf
870 of $\mathcal{F}(\Gamma)$ is either labelled by t or in exactly one link. Similar
871 to the set of links in linked fographs, a linking can be seen
872 as a unification problem, and a **dualizer** δ of the linking λ is
873 an assignment unifying all the links in λ . There exists a **most**
874 **general dualizer** of λ if λ has a dualizer. **[[Jui-Hsuan: Now**
875 **I use the same terminology as for linked fographs]]** **[[Lutz:**
876 **use δ for the dualizer (or even better, make it a macro)]]** A
877 **dependency** is a pair $(\bullet\exists x, \bullet\forall y)$ of nodes such that the most
878 general dualizer assigns to x a term containing y .

879 Let λ is a linking on Γ that has a dualizer. The **unification**
880 **structure** $\mathcal{U}(\lambda)$ associated with λ is the forest $\mathcal{F}(\Gamma)$ together
881 with an undirected edge between leaves l and l' for every link
882 $\{l, l'\}$ in λ and a directed edge from $\bullet\exists x$ to $\bullet\forall y$ for every
883 dependency $(\bullet\exists x, \bullet\forall y)$.

884 A **switching graph** of a unification structure $\mathcal{U}(\lambda)$ is any
885 derivative of $\mathcal{U}(\lambda)$ obtained by keeping only one edge into
886 each \vee and \forall and undirecting remaining edges. A linking is
887 **correct** if it is unifiable and all of the switching graphs of its
888 associated unification structure are acyclic.

889 **Definition 42.** A **unification net** on a sequent Γ is a correct
890 linking on Γ .

891 B. Translation between Unification Nets and MLL1^\times

892 **[[TODO:]]**

893 **Theorem 43.** If a sequent is provable in MLL1^\times , then there
894 exists a unification net on it.

895 *Proof.* We proceed by induction on the proof of $\vdash \Gamma$ in
896 MLL1^\times , making a case analysis on the bottommost rule
897 instance:

- 898 • $\text{ax} \frac{}{\vdash a, \bar{a}}$: the linking $\{a, \bar{a}\}$ is correct.
- 899 • $\text{t} \frac{}{\vdash t}$: the empty linking is correct.
- 900 • $\text{mix} \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$: By induction hypothesis, there is a
901 correct linking on Γ and another one on Δ , their union
902 giving a correct linking on Γ, Δ .

- $\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$: By induction hypothesis, there is a correct
linking on Γ, A, B , and it is correct on $\Gamma, A \vee B$ as well.
- $\wedge \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$: By induction hypothesis, there is a
correct linking on Γ, A and another one on B, Δ , their
union giving a correct linking on $\Gamma, A \wedge B, \Delta$.
- $\exists \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A}$: By induction hypothesis, there is a correct
linking λ on $\Gamma, A[x/t]$. For each atom in $\Gamma, A[x/t]$, there
is a corresponding atom in $\Gamma, \exists x.A$. There is therefore a
linking λ' on $\Gamma, \exists x.A$ obtained from λ via this correspon-
dence, and it is not difficult to check that λ' is correct as
well.
- $\forall \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A}$ (x not free in Γ) : By induction hypothesis,
there is a correct linking on Γ, A , and it is easy to see
that it is a correct linking on $\Gamma, \forall x.A$ as well.

This allows to define a translation $[\cdot]$ from proofs in MLL1^\times
to unification nets. \square

Theorem 44. Any unification net can be obtained via the
translation $[\cdot]$ given in Theorem 43.

To prove this theorem, we need some basic lemmas about
connected components in switching graphs of unification nets.

Lemma 45. The number of connected components of an acyclic
graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is equal to $|E_{\mathcal{G}}| - |V_{\mathcal{G}}|$.

Proof. By a straightforward induction on $|V_{\mathcal{G}}|$. \square

Lemma 46. The number of connected components is the same
for any switching graph of a unification net.

Proof. An immediate consequence of Lemma 45. \square

In the proof, we also use the notion of **frame** introduced by
Hughes in [37].

Definition 47. Let λ be a unification net on an MLL1^\times sequent
 Γ . We define the **frame** of λ by exhaustively applying the
following subformula rewriting steps, to obtain a linking λ_m
on an $\text{MLL} + \text{mix}$ sequent Γ_m :

- 1) **Encode dependencies as fresh links.** For each depen-
dency $\exists x \rightarrow \forall y$, with corresponding subformulas $\exists x.A$
and $\forall y.B$, we add a fresh link as follows. Let P be a fresh
(nullary) predicate symbol. Replace $\exists x.A$ with $P \wedge \exists x.A$
and $\forall y.B$ with $\bar{P} \vee \forall y.B$, and add an axiom link between
 P and \bar{P} .
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers.
(We no longer need their leaps since they are encoded
as links in step 1.)
- 3) **Simplify atoms.** After step 2, replace every predicate
 $Pt_1 \dots t_n$ with a nullary predicate symbol P .

Note that the linking λ_m is a valid $\text{MLL} + \text{mix}$ proof net.

Lemma 48. Suppose that λ is a MLL + mix proof net which is connected and such that any of its switching graphs is not connected. Then there exists a \vee node in $\mathcal{U}(\lambda)$ such that λ is correct on the sequent Γ' obtained from Γ by replacing this \vee by a \wedge .

Proof. Suppose that such a \vee node does not exist. Then it is clear that for any two nodes, there exists a switching graph containing a path between them and this path corresponds to an AE -path in [38]. By [38, Proposition 3], λ corresponds to a sequent proof that does not use mix, which implies the connectedness of the switching graphs of λ . Contradiction. **TO CHECK:** \square

Lemma 49. Suppose that λ is a MLL1^X proof net which is connected and such that any of its switching graphs is not connected. Then there exists a \vee node in $\mathcal{U}(\lambda)$ such that λ is correct on the sequent Γ' obtained from Γ by replacing this \vee by a \wedge .

Proof. Consider the frame λ_m of λ . The number of any switching graph of $\mathcal{U}(\lambda)$ is equal to that of $\mathcal{U}(\lambda_m)$. Apply Lemma 48 and it is clear that such \vee cannot be one of the fresh \vee 's added during the frame construction. \square

We can now give the proof of Theorem 44.

Proof of Theorem 44. Let λ be a unification net on Γ . We proceed by induction on the number of connected components of the unification structure $\mathcal{U}(\lambda)$:

- If there is only one connected component, we proceed by induction on the number k of connected components of any switching graph of $\mathcal{U}(\lambda)$. If $k = 1$, we obtain a proof Φ in MLL1^X such that $[\Phi] = \lambda$ by applying [37, Theorem 3]. If $k > 1$, using the Lemma 49, we obtain a sequent Γ' on which λ is correct by transforming a \vee node into a \wedge . By induction hypothesis, there is a proof Φ' in MLL1^X whose translation is λ . By considering the \wedge rule instance corresponding to the \wedge node in Φ' , we

$$\text{have: } \Phi' = \wedge \frac{\frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A \wedge B, \Delta_2}}{\vdash \Gamma'}. \text{ We can thus obtain}$$

$$\text{a proof } \Phi \text{ of } \Gamma: \Phi = \frac{\text{mix} \frac{\frac{\frac{\Phi_1}{\vdash \Delta_1, A} \quad \frac{\Phi_2}{\vdash B, \Delta_2}}{\vdash \Delta_1, A, B, \Delta_2}}{\vdash \Delta_1, A \vee B, \Delta_2}}{\vdash \Gamma}$$

$$[\Phi] = \lambda.$$

- If there are $n > 1$ connected components, add a fresh \vee node connecting two formulas belonging to different

connected components of Γ to get a new sequent Γ' . Define a unification net λ' on Γ' using the same linking as λ . By induction hypothesis, since $\mathcal{U}(\lambda')$ has $n - 1$ connected components, there is a MLL1^X proof Φ' such that $[\Phi'] = \lambda'$. Consider the \vee rule instance corresponding to the \vee node in question. Since \vee is invertible, we can permute downwards this rule instance until it becomes the last rule of the proof (note that this transformation does not change the image of the proof by the translation $[\cdot]$) to get a new proof Φ'' of Γ' . By deleting the last rule instance from Φ'' , we obtain a proof Φ of Γ such that $[\Phi] = \lambda$. **TO CHECK:** \square

We proceed by induction on the number of connectives in Γ . In the base case, Γ is of the form

$$p_1(t_{11}, \dots, t_{1n_1}), \overline{p_1}(t_{11}, \dots, t_{1n_1}), \dots, p_k(t_{k1}, \dots, t_{kn_k}), \overline{p_k}(t_{k1}, \dots, t_{kn_k}), \underbrace{t, \dots, t}_{m \text{ times}}$$

and λ is the linking $\{(a_1, \overline{a_1}), \dots, (a_k, \overline{a_k})\}$, where $a_i = p_i(t_{i1}, \dots, t_{in_i})$, which equals to $[\Pi]$, where Π is the proof consisting of m instances of the t rule, n instances $\text{ax} \frac{}{\vdash a_i, \overline{a_i}}$ of the ax rule, and followed by $m + k - 1$ instances of the mix rule.

Now we consider the inductive cases:

- $\Gamma = \Delta, A \vee B$: Let $\Gamma' = \Delta, A, B$. Define λ' on Γ' using the same links as λ by identifying the leaves of $\mathcal{F}(\Gamma')$ with those of $\mathcal{F}(\Gamma)$. We now check that λ' is a unification net:
 - The most general dualizer of λ is also the most general dualizer of λ' as they correspond to the same unification problem. Hence, the unification structure $\mathcal{U}(\lambda')$ is equal to the restriction of $\mathcal{U}(\lambda)$ to the nodes of $\mathcal{F}(\Gamma')$.
 - Every switching graph of λ' is acyclic: if there were some switching graph of $\mathcal{U}(\lambda')$ containing a cycle, it would induce a switching graph of $\mathcal{U}(\lambda)$ containing also a cycle, by adding an edge from the root of $\mathcal{F}(A)$ to the \vee node in question.
- $\Gamma = \Delta, \forall x.A$: Let $\Gamma' = \Delta, A$. Define λ' on Γ' using the same links as λ . We now check that λ' is a unification net:
 - The most general dualizer of λ is also the most general dualizer of λ' as they correspond to the same unification problem.
 - Every switching graph of $\mathcal{U}(\lambda')$ is acyclic: if there were some switching graph of $\mathcal{U}(\lambda')$ containing a cycle, it would induce a switching graph of $\mathcal{U}(\lambda)$ containing also a cycle, by adding an edge from the root of $\mathcal{F}(A)$ to the \forall node in question.
- $\mathcal{F}(\Gamma)$ has a root $\exists x$ with no outgoing dependency edge:

C. Translation between Unification Nets and Fonets

1032 **TODO:**

1033 XII. FIRST-ORDER COMBINATORIAL PROOFS

1034 A. First-order Logic

1035 In this paper, we also use some *deep inference* [36] rules
1036 that are in the following form:

$$\frac{\vdash S\{A\}}{\vdash S\{B\}}$$

1038 where $S\{ \}$ stands for a **context**, which corresponds to a
1039 sequent with a hole taking the place of an atom, and $S\{A\}$
1040 represents the sequent or formula obtained by replacing the
1041 hole in $S\{ \}$ with the formula A . Formally,

$$1042 C ::= \Box \mid A \vee C \mid C \wedge A \mid \exists x C \mid \forall x C.$$

$$1043 S ::= C \mid A, S \mid S, A$$

1044 where A is a formula. The above rule can be thus seen as the
1045 rewriting rule $A \rightarrow B$.

1046 We use the notation $\parallel_{\mathcal{P}}^A$ for denoting that there is a
1047 derivation from premise $\vdash S\{A\}$ to conclusion $\vdash S\{B\}$ in
1048 system \mathcal{P} for any context S .

1049 B. Graphs

1050 C. First-order combinatorial proofs

1051 D. MLL1^X and Unification Nets

1052 In MLL1^X, terms, atoms, formulas are defined as in first-
1053 order logic. For simplicity, we choose to use \vee and \wedge instead of
1054 \wp and \otimes which are generally used in the presentation of linear
1055 logic. A formula A is identified with its **formula tree** $\mathcal{F}(A)$,
1056 a directed tree with leaves labelled by atoms, internal nodes
1057 labelled by connectives and quantifiers, and edges directed
1058 from leaves to the root. A **sequent** Γ is simply a disjoint union
1059 of formulas. We write comma for disjoint union.

Sequents are proved using the inference rules of MLL1^X:

$$\begin{array}{c} \frac{}{\vdash A, \neg A} \text{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta} \text{cut} \\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \wedge \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A} \forall (x \notin fv(\Gamma)) \quad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A} \exists \end{array}$$

Fig. 3. Sequent calculus for MLL1^X

1060 We also consider the mix rule:

$$1061 \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \text{mix}$$

1063 Let Γ be a sequent in MLL1 + mix. A **link** on Γ is a pair
1064 of leaves whose atoms are pre-dual. A **linking** on Γ is a
1065 set of disjoint links such that each leaf of Γ is in exactly

one link. Similar to the set of links in the linked fograph, a
linking can be seen as a unification problem, and a link is said
unifiable if the corresponding unification problem is solvable.
Dependencies are defined as previously.

1070 XIII. FROM FIRST-ORDER LOGIC TO COMBINATORIAL PROOFS

1072 A. Decomposition Theorem

Consider the following deep inference rules [36]:

$$\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}} \text{c} \quad \frac{\vdash S\{f\}}{\vdash S\{A\}} \text{w}$$

Note that the ctr (resp. wk) rule in LK is derivable in $\{c, \vee\}$
(resp. $\{w, f\}$) and that c and w rules permute downwards with
the non-structural rules of LK.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \frac{}{\vdash \Gamma, A} \text{c}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \frac{}{\vdash \Gamma, A} \text{w}$$

We also give an example to show how rule permutation
works:

$$\frac{\frac{\Gamma, A \vee A}{\Gamma, A} \text{c} \quad \Delta, B}{\Gamma, \Delta, A \wedge B} \wedge \rightsquigarrow \frac{\Gamma, A \vee A \quad \Delta, B}{\Gamma, \Delta, (A \vee A) \wedge B} \wedge \frac{}{\Gamma, \Delta, A \wedge B} \text{c}$$

We want to establish the following theorem:

Theorem 50. *Let Γ be a sequent. Then there is a proof of Γ in LK + mix iff there is a proof of some sequent Δ in MLL1 + mix and a derivation from Δ to Γ consisting of the c and w rules only.*

Proof. (\Rightarrow) This direction comes from the above observation:
it suffices to permute downwards all the instances of the c and
w rules.

(\Leftarrow) We regard the proof in MLL1 + mix as a proof in
LK + mix. Then we put the derivation consisting of only c
and w under the proof in LK + mix. Now we try to permute
all the instances c and w upwards with the rules of LK and
mix. For the c part, the only non-trivial case is the permutation
with the \vee rule where the formula generated is $A \vee A$.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A} \vee \frac{}{\vdash \Gamma, A} \text{ctr} \rightsquigarrow \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr}$$

In this case, the permutation of this instance of c stops and
we continue with the remaining instances.

For the w part, the only non-trivial case is the permutation
with the f rule (or the instance of wk where f is introduced):

$$\frac{\vdash \Gamma}{\vdash \Gamma, f} \text{f} \frac{}{\vdash \Gamma, A} \text{wk} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk}$$

In this case, the permutation of this instance of w stops and we continue with the remaining instances. \square

D. Hughes proves in [37] the soundness and completeness of unification nets with respect to MLL1 + mix. In the following, we establish the equivalence between unification nets and fonets.

B. Equivalence between unification nets and fonets

In the following, we usually confound a vertex with its label.

Definition 51. A *switching path* of a unification structure $U(\lambda)$ is a path in a switching graph of $U(\lambda)$.

Definition 52. A *switching path* of a formula tree $\mathcal{F}(A)$ is a path in $\mathcal{F}(A)$ that does not go through both incoming edges of a \vee .

Proposition 53. In a formula tree, the root is connected to every vertex by a switching path.

Now we give the key proposition relating a fograph to its corresponding formula tree.

Proposition 54. Let u and v be two distinct vertices of a fograph $\llbracket \llbracket A \rrbracket \rrbracket$, then we have the equivalence between:

- u and v are adjacent in $\llbracket \llbracket A \rrbracket \rrbracket$
- u and v are connected by a switching path of $\mathcal{F}(A)$, and if one of them is a universal quantifier, then the other is not a descendant of the former.

Proof. By induction on A .

- If A is an atom, trivial.
- If $A = A_1 \wedge A_2$, then we distinguish two cases:
 - u and v are both in A_1 (resp. A_2): trivial by the induction hypothesis.
 - one of them is in A_1 and the other is in A_2 : they are adjacent in $\llbracket \llbracket A \rrbracket \rrbracket$ by definition. By Proposition 53, the one in A_1 (resp. A_2) is connected to the vertex representing A_1 (resp. A_2) by a switching path. Together with the two edges incident to $A_1 \wedge A_2$, we obtain a switching path connecting u and v .
- If $A = A_1 \vee A_2$, then we distinguish two cases:
 - u and v are both in A_1 (resp. A_2): trivial by the induction hypothesis.
 - one of them is in A_1 and the other is in A_2 : they are not adjacent in $\llbracket \llbracket A \rrbracket \rrbracket$ by definition. It is clear that they are not connected by a switching path.
- If $A = \exists x A'$, then we distinguish two cases:
 - u and v are both in A' : trivial by the induction hypothesis.
 - one of them is $\exists x$ and the other is in A' : trivial by Proposition 53
- If $A = \forall x A'$, then we distinguish two cases:
 - u and v are both in A' : trivial by the induction hypothesis.

- one of them is $\forall x$ and the other is in A' : they are not adjacent in $\llbracket \llbracket A \rrbracket \rrbracket$ by definition and it is clear that the former is a descendant of $\forall x$. \square

Proposition 55. If there exists an induced bimatching of the linked fograph $G = \llbracket \llbracket A \rrbracket \rrbracket$, then there exists a switching graph of the corresponding unification net which contains a cycle.

Proof. Suppose that there exists a set W inducing a bimatching in G . Then (W, E_G) and (W, L_G) are matchings.

Let E_W (resp. L_W) be the restriction of E_G (resp. L_G) to W . If $E_W \cap L_W \neq \emptyset$, then there exist u and v such that $uv \in E_G$ and $uv \in L_G$. By Proposition 54, there exists a switching path of the formula tree of A . Together with the leap uv , this path induces a cycle in a switching graph of the corresponding unification structure.

We can now suppose that E_W and L_W are disjoint. It is not difficult to see the existence of an alternating and elementary cycle in the bicoloured graph $(W, E_W \uplus L_W)$, i.e. a cycle of which the edges are alternately in E_W and L_W and containing no two equal vertices. By Proposition 54, this cycle induces a cycle in the unification structure. Now we want to construct a switching graph that contains this cycle.

Consider a universal quantifier $\forall x$. If $\forall x \notin W$, then we keep the incoming edge from its direct subformula and remove all the dependencies. Otherwise, since (W, L_G) is a matching, there exists a unique existential quantifier adjacent to $\forall x$ and we keep thus the corresponding edge in the unification structure.

Now consider a \vee . We distinguish three cases:

- the cycle goes through none of the two branches (incoming edges) of the \vee : we can choose an arbitrary switching for this \vee
- the cycle goes through exactly one branch: we choose the corresponding switching
- the cycle goes through both branches: this means that there exist $v_L \in W$ (resp. v_R) in the left (resp. right) branch, $u_L, u_R \in W$, such that $u_L v_L, u_R v_R \in E_W$ and that the corresponding switching path from u_L to v_L (resp. from u_R to v_R) goes through the left (resp. right) edge of \vee .

The red (resp. blue) path is the switching path corresponding to the edge $u_L v_L$ (resp. $u_R v_R$) in E_W .

It is clear that u_L (resp. u_R) is not in the branches of the \vee . Otherwise, there will be no switching path from u_L to v_L

By Proposition 54, we know that u_L and u_R are not universal quantifiers which are ancestors the \vee and that there exist one switching path from u_L to v_L and one from u_R to v_R . In particular, there exist one switching path from u_L to the \vee and one from the \vee to v_R , and by concatenating the two, we obtain a switching path from u_L to v_R . By Proposition 54, u_L and v_R are thus adjacent in (W, E_G) , which is impossible since (W, E_W) is a matching.

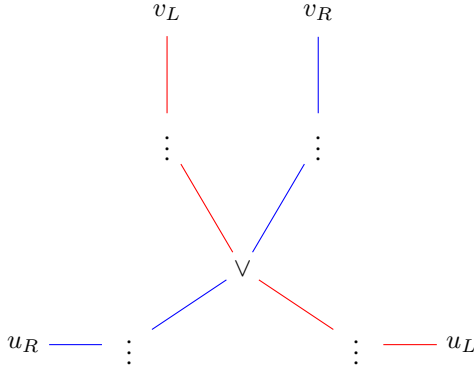


Fig. 4. A schema showing that the two branches of the same \vee cannot be used in the cycle at the same time.

Notice that the switching paths here are in the underlying formula tree. We have to verify that they are compatible with the choices of switching made for universal quantifiers. That is, if $uv \in E_W$, then for all the universal quantifiers $\forall x$ on the switching path (in the formula tree), we have chosen in the first part of the proof to keep the only edge from the child of $\forall x$ to itself. In fact, if there exists a universal quantifier $w \in W$ on the switching path $u \rightarrow v$, then one of u and v is not a descendant of w . Moreover, if u (resp. v) is a universal quantifier, then w is not in its scope. By Proposition 54, $\{wu, wv\} \cap E_W \neq \emptyset$, which is impossible since (W, E_W) is a matching. We have thus constructed a switching graph containing this cycle. \square

Proposition 56. *If one of the switching graphs of the unification structure of A contains a cycle or is not connected, then there exists an induced bimatching of the corresponding linked fograph.*

Proof. We use frames introduced by D. Hughes in Section 4 of [37].

Definition 57. Let θ be a unification structure on an MLL¹ \times sequent Γ . We define the **frame** of θ by exhaustively applying the following subformula rewriting steps, to obtain a proof structure θ_m on an MLL sequent Γ_m :

- 1) **Encode dependencies as fresh links.** For each dependency $\exists x \rightarrow \forall y$, with corresponding subformulas $\exists xA$ and $\forall yB$, we add a fresh link as follows. Let P be a fresh (nullary) predicate symbol. Replace $\exists xA$ with $P \wedge \exists xA$ and $\forall yB$ with $\overline{P} \vee \forall yB$, and add an axiom link between P and \overline{P} .
- 2) **Erase quantifiers.** After step 1, erase all the quantifiers. (We no longer need their leaps since they are encoded as links in step 1.)
- 3) **Simplify atoms.** After step 2, replace every predicate $Pt_1 \cdots t_n$ with a nullary predicate symbol P .

We have the following results:

Let u and v be atoms or quantifiers in a unification structure θ . Then they are connected by a switching path in the

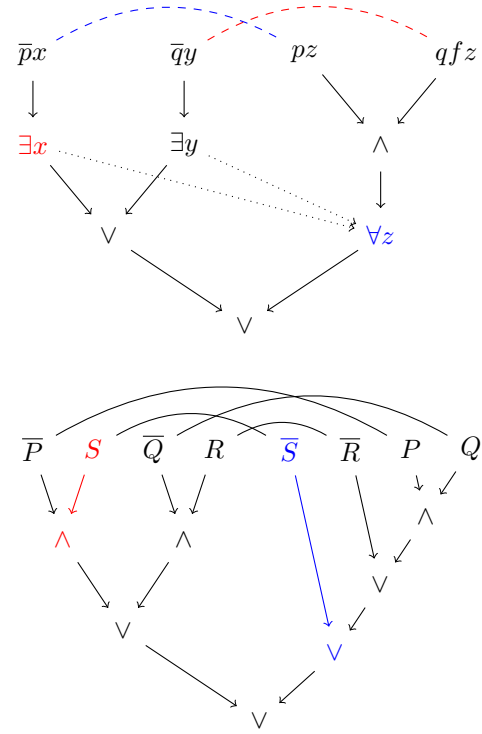


Fig. 5. A unification net and its frame. The colored part shows how the dependency $\exists x \rightarrow \forall z$ is transformed.

unification structure if, and only if, their corresponding nodes are connected by a switching path in θ_m .

Consider now a switching graph H of a unification structure θ of A .

If H contains a cycle, then the corresponding switching graph of θ_m also contains a cycle. Hence, by applying the propositional results (Theorem 7) from [38], we conclude that there exists a chordless, alternating, and elementary cycle in the bicoloured graph $(W, E_W \uplus L_W)$, which corresponds to an induced bimatching in the linked fograph. (Note that the linked fograph (cograph) corresponding to θ_m is equivalent to the one corresponding to θ .) \square

C. From contraction/weakening to skew bifibrations

We first introduce the atomic contraction rule, the medial rule, and two rules on quantifiers.

$$\begin{array}{c}
 \frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac} \quad \frac{\vdash S\{(A \wedge B) \vee (C \wedge D)\}}{\vdash S\{(A \vee C) \wedge (B \vee D)\}} \text{ m} \\
 \frac{\vdash S\{\exists x A \vee \exists x B\}}{\vdash S\{\exists x(A \vee B)\}} \text{ m}_1 \downarrow \quad \frac{\vdash S\{\forall x A \vee \forall x B\}}{\vdash S\{\forall x(A \vee B)\}} \text{ m}_2 \downarrow
 \end{array}$$

Here, we also consider the equivalence generated by the associativity, commutativity of \vee and the equations $t \vee A \equiv t$ and $f \vee A \equiv A$.

Now we have the following lemma:

Lemma 58. The contraction rule c is derivable for $\{ac, m, m_1\downarrow, m_2\downarrow\}$.

Proof. We prove that there is always $\frac{A \vee A}{A} \parallel_{\{ac, m, m_1\downarrow, m_2\downarrow\}}$ by structural induction on A .

- If $A = t$ or $A = f$, we have $\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}} \equiv$. (the premiss and the conclusion are equivalent)
- If $A = a$, then we have $\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} ac$
- If $A = A_1 \vee A_2$, then by the induction hypothesis, we have $\frac{\vdash S\{(A_1 \vee A_2) \vee (A_1 \vee A_2)\}}{\vdash S\{(A_1 \vee A_1) \vee (A_2 \vee A_2)\}} \equiv$

Hence, we have $\vdash S\{A_1 \vee (A_2 \vee A_2)\}$

- If $A = A_1 \wedge A_2$, then by the induction hypothesis, we have $\frac{\vdash S\{(A_1 \wedge A_2) \vee (A_1 \wedge A_2)\}}{\vdash S\{(A_1 \vee A_1) \wedge (A_2 \vee A_2)\}} m$

Hence, we have $\vdash S\{A_1 \wedge (A_2 \vee A_2)\}$

- If $A = \exists x A'$, then by the induction hypothesis, we have $\frac{\vdash S\{\exists x A' \vee \exists x A'\}}{\vdash S\{\exists x (A' \vee A')\}} m_1\downarrow$

Hence, we have $\vdash S\{\exists x A'\}$

- If $A = \forall x A'$, then by the induction hypothesis, we have $\frac{\vdash S\{\forall x A' \vee \forall x A'\}}{\vdash S\{\forall x (A' \vee A')\}} m_2\downarrow$

Hence, we have $\vdash S\{\forall x A'\}$

Lemma 59. The rules $m_1\downarrow$ and $m_2\downarrow$ are derivable for $\{w, c\}$.

Proof. We have:

$$\frac{\vdash S\{\exists x A\}}{\vdash S\{\exists x (A \vee f)\}} \equiv \quad \text{and} \quad \frac{\vdash S\{\exists x B\}}{\vdash S\{\exists x (f \vee B)\}} \equiv$$

Thus, we have:

$$\frac{\vdash S\{\exists x A \vee \exists x B\}}{\vdash S\{\exists x (A \vee B) \vee \exists x (A \vee B)\}} c$$

Similar for $m_2\downarrow$.

Now we define a propositional encoding for first-order formulas.

Definition 60. The propositional encoding A° of a formula A is defined inductively by:

$$\begin{aligned} a^\circ &= a \text{ for every atom } a \\ (A \vee B)^\circ &= A^\circ \vee B^\circ & (A \wedge B)^\circ &= A^\circ \wedge B^\circ \\ (\forall x A)^\circ &= U_x \vee A^\circ & (\exists x A)^\circ &= E_x \wedge A^\circ \end{aligned}$$

where U_x and E_x are fresh nullary atoms.

Similarly, we can define the propositional encoding S° of a context S inductively by setting $\square^\circ = \square$. Note that S° is also a context.

We have the following facts:

Proposition 61. For any context S and any formula A :

- A° is a formula containing no quantifier for any formula A .
- $\llbracket \llbracket A^\circ \rrbracket \rrbracket = \llbracket \llbracket A \rrbracket \rrbracket$ by confounding the atoms U_x, E_x with the variable x . Thus, a map $f : \llbracket \llbracket A^\circ \rrbracket \rrbracket \rightarrow \llbracket \llbracket B^\circ \rrbracket \rrbracket$ can be seen as a map $f : \llbracket \llbracket A \rrbracket \rrbracket \rightarrow \llbracket \llbracket B \rrbracket \rrbracket$.
- $(S\{A\})^\circ = S^\circ\{A^\circ\}$.

Proposition 62. Let A and B be two formulas such that $A \Delta \parallel_{\{w, c\}} B$. Then $A^\circ \Delta \parallel_{\{w, c\}} B^\circ$.

Proof. Trivial by induction.

Lemma 63. Given two formulas A and B and a derivation $\Delta \parallel_{\{w, c\}} A \rightarrow B$, then there exists a skew bifibration $G(A) \rightarrow G(B)$.

Proof. By Lemma 58, there exists a derivation $\Delta \parallel_{\{w, ac, m, m_1\downarrow, m_2\downarrow\}} A \rightarrow B$.

For each rule from $\{w, ac, m, m_1\downarrow, m_2\downarrow\}$, we define a map and show that it is a skew fibration.

- $\frac{\vdash S\{f\}}{\vdash S\{A\}} w$: the map wk maps f to anything and is identity elsewhere.

- 1321 • $\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}} \text{ ac:}$
 1322 the map ac maps the two a -labelled literals in the premise
 1323 to the a -labelled literal in the conclusion.
 $\vdash S\{(A \wedge B) \vee (C \wedge D)\}$
- 1324 • $\frac{\vdash S\{(A \vee C) \wedge (B \vee D)\}}{\vdash S\{\exists x A \vee \exists x B\}} m:$
 1325 the map m is the canonical identity that maps A to A ,
 1326 \dots, D to D .
 $\vdash S\{\exists x A \vee \exists x B\}$
- 1327 • $\frac{\vdash S\{\exists x(A \vee B)\}}{\vdash S\{\forall x A \vee \forall x B\}} m_1 \downarrow:$
 1328 the map m_1 maps the two x -labelled binders in the
 1329 premise to the x -labelled binder in the conclusion, A to
 1330 A and B to B .
 $\vdash S\{\forall x A \vee \forall x B\}$
- 1331 • $\frac{\vdash S\{\forall x(A \vee B)\}}{\vdash S\{\forall x(A \vee B)\}} m_2 \downarrow:$
 1332 the map m_2 maps the two x -labelled binders in the
 1333 premise to the x -labelled binder in the conclusion, A to
 1334 A and B to B .

1335 By considering propositional encodings, the maps defined
 1336 are label-preserving skew fibrations on the underlying fographs
 1337 according to [23].

1338 Now we prove that each map $g \in \{wk, ac, m, m_1, m_2\}$ is
 1339 a skew bifibration. To do that, it suffices to prove that g is a
 1340 fibration between the corresponding binding graphs since it is
 1341 already a skew fibration on the corresponding fographs and it
 1342 is label-preserving and existential-preserving.

1343 for each x -binder b in $\llbracket \langle \rangle B^\circ \rrbracket$, for each vertex
 1344 $v \in V(\llbracket \langle \rangle A^\circ \rrbracket)$ such that $g(v)$ is bound by b , there exists a
 1345 unique binder b' such that b' binds v .

- 1346 • wk and m are clearly fibrations: the binding relations of
 1347 the premise and the conclusion are exactly the same.
- 1348 • ac is a fibration: suppose that a that in the conclusion a
 1349 is bound by some quantifier b in S , then for each of its
 1350 preimages by ac , there exists exactly one binder (in fact,
 1351 b) in S that binds it.
- 1352 • m_1 and m_2 are fibrations: in the conclusion, for every
 1353 atom a in $A \vee B$ bound by the x -labelled quantifier, a has
 1354 exactly one preimage and it is bound by the x -labelled
 1355 quantifier in the premise.

1356 Therefore, all of these maps are skew bifibrations and since
 1357 skew bifibrations on fographs compose (Lemma 10.32, [20]),
 1358 there exists a skew bifibration from $\llbracket \langle \rangle A \rrbracket$ to $\llbracket \langle \rangle B \rrbracket$.
 1359 \square

1360 **Theorem 64.** *If a formula A is provable in LK, then it has a*
 1361 *combinatorial proof.*

1362 *Proof.* By Theorem 50, there exists a formula A' such that
 1363 there is a proof Π of A' in MLL1^X and a derivation D from
 1364 A' to A consisting of the w and c rules only. The proof Π
 1365 corresponds to a unique unification net which is equivalent to
 1366 the fonet corresponding to Π , i.e., the fograph $\llbracket \langle \rangle A' \rrbracket$ together
 1367 with the links of Π . By Lemma 63, there exists a skew
 1368 bifibration $\llbracket \langle \rangle A' \rrbracket \rightarrow \llbracket \langle \rangle A \rrbracket$. We have thus a combinatorial
 1369 proof of A .
 1370 \square

D. From skew bifibrations to contraction/weakening 1371

Theorem 65. *Let A and B be two formulas and $f : G(A) \rightarrow$* 1372
 $G(B)$ a skew bifibration. Then there exists a derivation 1373
 A
 $\Delta \parallel_{\{w, c\}}.$ 1374
 B

f can be seen as a skew fibration from $G(A^\circ)$ to $G(B^\circ)$,
 which gives the existence of the propositions A' and B' , and
 of the following derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B' \\ \Delta'' \parallel_w \\ B^\circ \end{array}$$

Lemma 66. *there exists B'' such that $B''^\circ = B'$.* 1375

Proof. Consider the derivation Δ'' . If some U_x (or E_x) is 1376
 introduced via weakening, then all the atoms it binds in B° 1377
 should also be introduced via weakening. In fact, an atom of 1378
 B° is introduced via weakening is equivalent to the fact that 1379
 its corresponding vertex is not in the image of f . Since there 1380
 is an edge from U_x (resp. E_x) to all the literals it binds in the 1381
 binding graph $\llbracket \langle \rangle B \rrbracket$, if one of the atoms is in the image, U_x 1382
 (resp. E_x) should also be in the image since f is a fibration 1383
 on binding graphs. 1384

This means that a such B'' can be obtained from B by 1385
 erasing all the U_x and E_x introduced via weakening and all 1386
 the atoms they bind. \square 1387

We introduce new (atomic) symbols E_x^* and U_x^* which are 1388
 used to represent disjunctions of E_x and U_x respectively. 1389

We define a translation $(\cdot)^*$ inductively by: 1390

- 1391 • $(E_x \vee \dots \vee E_x)^* = E_x$
- 1392 • $(U_x \vee \dots \vee U_x)^* = U_x$
- 1393 • structural recursion in all the other cases.

Then the derivation:

$$\begin{array}{c} A^\circ \\ \Delta \parallel_m \\ A' \\ \Delta' \parallel_{ac} \\ B''^\circ \end{array}$$

can be translated to the derivation:

$$\begin{array}{c} A^{\circ*} \\ \Delta^* \parallel \\ B''^{\circ*} \end{array}$$

where Δ^* is the derivation obtained by replacing all the 1394
 formulas F with F^* and by applying the following rule 1395
 transformation: 1396

$$\frac{S\{Q_x\}}{S\{Q_x\}} \text{ ac} \rightsquigarrow \frac{S\{Q_x\}}{S\{Q_x\}} = 1397$$

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m } \rightsquigarrow \frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m'}$$

where Q_x stands for E_x or U_x .

Δ^* can now be transformed into a valid derivation Δ_1 by using the two transformation rules above and by applying them in a bottom-up style:

$$\frac{A^{\circ*}}{\Delta_1 \parallel_{\text{ac, m, m'}} B''^{\circ*}}$$

Lemma 67. *Every line of Δ_1 is a propositional encoding.*

Proof. We proceed by bottom-up induction in the derivation. Clearly, $(B''^{\circ})^*$ is a propositional encoding as there is no disjunction of Q_x in it.

First consider the ac rule: $\frac{C \vee C}{C} \text{ ac}$

It is clear that if C is a propositional encoding, then so is $C \vee C$.

Now consider the m rule:

$$\frac{S\{(C \wedge D) \vee (E \wedge F)\}}{S\{(C \vee E) \wedge (D \vee F)\}} \text{ m}$$

Suppose that $(C \vee E) \wedge (D \vee F) = G^{\circ}$ for some G . Since $C \vee E$ cannot be Q_x (otherwise, the rule applied would be m'), G can be written as $G_1 \wedge G_2$ with $C \vee E = G_1^{\circ}$ and $D \vee F = G_2^{\circ}$.

We have thus $G_i = \forall x_i H_i$ or $J_i \vee K_i$ ($i = 1, 2$).

If $G_i = \forall x_i H_i$ for some i , then there will be a conjunction of U_x and some formula which can never be eliminated by the rules m , m' and ac . However, there exists no such conjunction in $A^{\circ*}$, which leads to a contradiction.

Hence, G_i can be written as $J_i \vee K_i$ for $i = 1, 2$. We now have $(C \wedge D) \vee (E \wedge F) = ((J_1 \wedge J_2) \vee (K_1 \wedge K_2))^{\circ}$.

Finally, consider the m' rule:

$$\frac{S\{(E_x \wedge C) \vee (E_x \wedge D)\}}{S\{E_x \wedge (C \vee D)\}} \text{ m'}$$

Suppose that $E_x \wedge (C \vee D) = F^{\circ}$ for some F . It is clear that $F = \exists x G$ with $G^{\circ} = C \vee D$ for some G . We distinguish two cases:

- $G = \forall y H$: in this case, $(E_x \wedge C) \vee (E_x \wedge D)$ has a subformula $(E_x \wedge U_y)$, which cannot be eliminated by the rules m , m' , ac . It is clear that $A^{\circ*}$ does not have a subformula of this form, which leads to a contradiction.
- $G = G_1 \vee G_2$: in this case, $(E_x \wedge C) \vee (E_x \wedge D) = ((\exists x G_1) \vee (\exists x G_2))^{\circ}$.

□