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Quantum Field Theory

University of Cambridge Part III Mathematical Tripos

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Recommended Books and Resources

- M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*

This is a very clear and comprehensive book, covering everything in this course at the right level. It will also cover everything in the "Advanced Quantum Field Theory" course, much of the "Standard Model" course, and will serve you well if you go on to do research. To a large extent, our course will follow the first section of this book.

There is a vast array of further Quantum Field Theory texts, many of them with redeeming features. Here I mention a few very different ones.

- S. Weinberg, *The Quantum Theory of Fields, Vol 1*

This is the first in a three volume series by one of the masters of quantum field theory. It takes a unique route to through the subject, focussing initially on particles rather than fields. The second volume covers material lectured in "AQFT".

- L. Ryder, *Quantum Field Theory*

This elementary text has a nice discussion of much of the material in this course.

- A. Zee, *Quantum Field Theory in a Nutshell*

This is charming book, where emphasis is placed on physical understanding and the author isn't afraid to hide the ugly truth when necessary. It contains many gems.

- M Srednicki, *Quantum Field Theory*

A very clear and well written introduction to the subject. Both this book and Zee's focus on the path integral approach, rather than canonical quantization that we develop in this course.

There are also resources available on the web. Some particularly good ones are listed on the course webpage: http://www.damtp.cam.ac.uk/user/tong/qft.html

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Acknowledgements

0. Introduction

> "There are no real one-particle systems in nature, not even few-particle systems. The existence of virtual pairs and of pair fluctuations shows that the days of fixed particle numbers are over."
>
> *Viki Weisskopf*

The concept of wave-particle duality tells us that the properties of electrons and photons are fundamentally very similar. Despite obvious differences in their mass and charge, under the right circumstances both suffer wave-like diffraction and both can pack a particle-like punch.

Yet the appearance of these objects in classical physics is very different. Electrons and other matter particles are postulated to be elementary constituents of Nature. In contrast, light is a derived concept: it arises as a ripple of the electromagnetic field. If photons and particles are truely to be placed on equal footing, how should we reconcile this difference in the quantum world? Should we view the particle as fundamental, with the electromagnetic field arising only in some classical limit from a collection of quantum photons? Or should we instead view the field as fundamental, with the photon appearing only when we correctly treat the field in a manner consistent with quantum theory? And, if this latter view is correct, should we also introduce an "electron field", whose ripples give rise to particles with mass and charge? But why then didn't Faraday, Maxwell and other classical physicists find it useful to introduce the concept of matter fields, analogous to the electromagnetic field?

The purpose of this course is to answer these questions. We shall see that the second viewpoint above is the most useful: the field is primary and particles are derived concepts, appearing only after quantization. We will show how photons arise from the quantization of the electromagnetic field and how massive, charged particles such as electrons arise from the quantization of matter fields. We will learn that in order to describe the fundamental laws of Nature, we must not only introduce electron fields, but also quark fields, neutrino fields, gluon fields, W and Z-boson fields, Higgs fields and a whole slew of others. There is a field associated to each type of fundamental particle that appears in Nature.

Why Quantum Field Theory?

In classical physics, the primary reason for introducing the concept of the field is to construct laws of Nature that are *local*. The old laws of Coulomb and Newton involve "action at a distance". This means that the force felt by an electron (or planet) changes

immediately if a distant proton (or star) moves. This situation is philosophically unsatisfactory. More importantly, it is also experimentally wrong. The field theories of Maxwell and Einstein remedy the situation, with all interactions mediated in a local fashion by the field.

The requirement of locality remains a strong motivation for studying field theories in the quantum world. However, there are further reasons for treating the quantum field as fundamental[1]. Here I'll give two answers to the question: Why quantum field theory?

Answer 1: Because the combination of quantum mechanics and special relativity implies that particle number is not conserved.

Particles are not indestructible objects, made at the beginning of the universe and here for good. They can be created and destroyed. They are, in fact, mostly ephemeral and fleeting. This experimentally verified fact was first predicted by Dirac who understood how relativity implies the necessity of anti-particles. An extreme demonstration of particle creation is shown in the picture, which comes from the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, Long Island. This machine crashes gold nuclei together, each containing 197 nucleons. The resulting explosion contains up to 10,000 particles, captured here in all their beauty by the STAR detector.
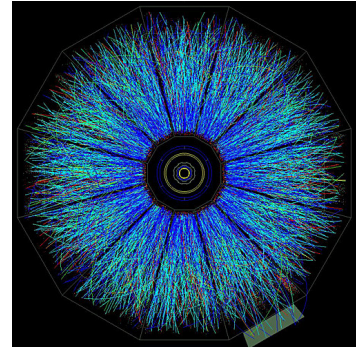
Figure 1:

We will review Dirac's argument for anti-particles later in this course, together with the better understanding that we get from viewing particles in the framework of quantum field theory. For now, we'll quickly sketch the circumstances in which we expect the number of particles to change. Consider a particle of mass m trapped in a box of size L. Heisenberg tells us that the uncertainty in the momentum is $\Delta p \geq \hbar/L$. In a relativistic setting, momentum and energy are on an equivalent footing, so we should also have an uncertainty in the energy of order $\Delta E \geq \hbar c/L$. However, when the uncertainty in the energy exceeds $\Delta E = 2mc^2$, then we cross the barrier to pop particle anti-particle pairs out of the vacuum. We learn that particle-anti-particle pairs are expected to be important when a particle of mass m is localized within a distance of order

$$\lambda = \frac{\hbar}{mc}$$

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[1] A concise review of the underlying principles and major successes of quantum field theory can be found in the article by Frank Wilczek, http://arxiv.org/abs/hep-th/9803075

At distances shorter than this, there is a high probability that we will detect particle-anti-particle pairs swarming around the original particle that we put in. The distance λ is called the *Compton wavelength*. It is always smaller than the de Broglie wavelength $\lambda_{dB} = h/|\vec{p}|$. If you like, the de Broglie wavelength is the distance at which the wavelike nature of particles becomes apparent; the Compton wavelength is the distance at which the concept of a single pointlike particle breaks down completely.

The presence of a multitude of particles and antiparticles at short distances tells us that any attempt to write down a relativistic version of the one-particle Schrödinger equation (or, indeed, an equation for any *fixed* number of particles) is doomed to failure. There is no mechanism in standard non-relativistic quantum mechanics to deal with changes in the particle number. Indeed, any attempt to naively construct a relativistic version of the one-particle Schrödinger equation meets with serious problems. (Negative probabilities, infinite towers of negative energy states, or a breakdown in causality are the common issues that arise). In each case, this failure is telling us that once we enter the relativistic regime we need a new formalism in order to treat states with an unspecified number of particles. This formalism is quantum field theory (QFT).

Answer 2: Because all particles of the same type are the same

This sound rather dumb. But it's not! What I mean by this is that two electrons are identical in every way, regardless of where they came from and what they've been through. The same is true of every other fundamental particle. Let me illustrate this through a rather prosaic story. Suppose we capture a proton from a cosmic ray which we identify as coming from a supernova lying 8 billion lightyears away. We compare this proton with one freshly minted in a particle accelerator here on Earth. And the two are exactly the same! How is this possible? Why aren't there errors in proton production? How can two objects, manufactured so far apart in space and time, be identical in all respects? One explanation that might be offered is that there's a sea of proton "stuff" filling the universe and when we make a proton we somehow dip our hand into this stuff and from it mould a proton. Then it's not surprising that protons produced in different parts of the universe are identical: they're made of the same stuff. It turns out that this is roughly what happens. The "stuff" is the proton field or, if you look closely enough, the quark field.

In fact, there's more to this tale. Being the "same" in the quantum world is not like being the "same" in the classical world: quantum particles that are the same are truely indistinguishable. Swapping two particles around leaves the state completely unchanged — apart from a possible minus sign. This minus sign determines the statistics of the particle. In quantum mechanics you have to put these statistics in by hand

and, to agree with experiment, should choose Bose statistics (no minus sign) for integer spin particles, and Fermi statistics (yes minus sign) for half-integer spin particles. In quantum field theory, this relationship between spin and statistics is not something that you have to put in by hand. Rather, it is a consequence of the framework.

What is Quantum Field Theory?

Having told you why QFT is necessary, I should really tell you what it is. The clue is in the name: it is the quantization of a classical field, the most familiar example of which is the electromagnetic field. In standard quantum mechanics, we're taught to take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory are *operator valued functions of space and time*. This means that we are dealing with an infinite number of degrees of freedom — at least one for every point in space. This infinity will come back to bite on several occasions.

It will turn out that the possible interactions in quantum field theory are governed by a few basic principles: locality, symmetry and renormalization group flow (the decoupling of short distance phenomena from physics at larger scales). These ideas make QFT a very robust framework: given a set of fields there is very often an almost unique way to couple them together.

What is Quantum Field Theory Good For?

The answer is: almost everything. As I have stressed above, for any relativistic system it is a necessity. But it is also a very useful tool in non-relativistic systems with many particles. Quantum field theory has had a major impact in condensed matter, high-energy physics, cosmology, quantum gravity and pure mathematics. It is literally the language in which the laws of Nature are written.

0.1 Units and Scales

Nature presents us with three fundamental dimensionful constants; the speed of light c, Planck's constant (divided by 2π) $\hbar$ and Newton's constant G. They have dimensions

$$[c] = LT^{-1}$$
$$[\hbar] = L^2 M T^{-1}$$
$$[G] = L^3 M^{-1} T^{-2}$$

Throughout this course we will work with "natural" units, defined by

$$c = \hbar = 1 \tag{0.1}$$

which allows us to express all dimensionful quantities in terms of a single scale which we choose to be mass or, equivalently, energy (since $E = mc^2$ has become $E = m$). The usual choice of energy unit is eV, the electron volt or, more often $GeV = 10^9 eV$ or $TeV = 10^{12} eV$. To convert the unit of energy back to a unit of length or time, we need to insert the relevant powers of c and $\hbar$. For example, the length scale λ associated to a mass m is the Compton wavelength

$$\lambda = \frac{\hbar}{mc}$$

With this conversion factor, the electron mass $m_e = 10^6 eV$ translates to a length scale $\lambda_e \sim 10^{-12} m$. (The Compton wavelength is also defined with an extra factor of 2π: $\lambda = 2\pi\hbar/mc$.)

Throughout this course we will refer to the *dimension* of a quantity, meaning the mass dimension. If X has dimensions of $(\text{mass})^d$ we will write $[X] = d$. In particular, the surviving natural quantity G has dimensions $[G] = -2$ and defines a mass scale,

$$G = \frac{\hbar c}{M_p^2} = \frac{1}{M_p^2} \tag{0.2}$$

where $M_p \approx 10^{19} GeV$ is the *Planck scale*. It corresponds to a length $l_p \approx 10^{-33} cm$. The Planck scale is thought to be the smallest length scale that makes sense: beyond this quantum gravity effects become important and it's no longer clear that the concept of spacetime makes sense. The largest length scale we can talk of is the size of the cosmological horizon, roughly $10^{60} l_p$.
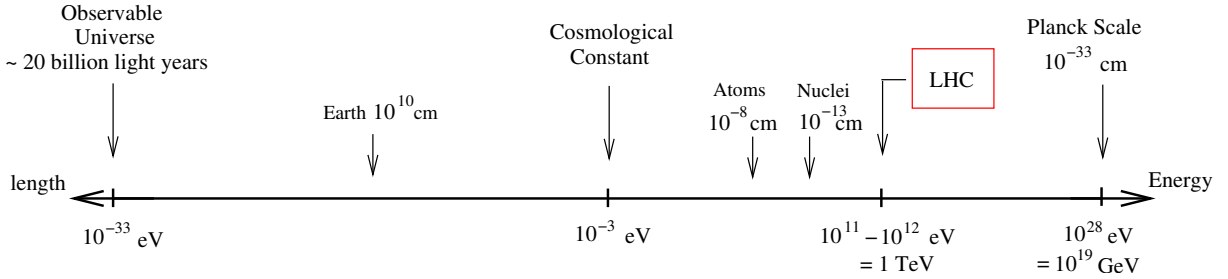
Figure 2: Energy and Distance Scales in the Universe

Some useful scales in the universe are shown in the figure. This is a logarithmic plot, with energy increasing to the right and, correspondingly, length increasing to the left. The smallest and largest scales known are shown on the figure, together with other

relevant energy scales. The standard model of particle physics is expected to hold up to about the TeV. This is precisely the regime that is currently being probed by the Large Hadron Collider (LHC) at CERN. There is a general belief that the framework of quantum field theory will continue to hold to energy scales only slightly below the Planck scale — for example, there are experimental hints that the coupling constants of electromagnetism, and the weak and strong forces unify at around 10^{18} GeV.

For comparison, the rough masses of some elementary (and not so elementary) particles are shown in the table,

Particle	Mass
neutrinos	$\sim 10^{-2}$ eV
electron	0.5 MeV
Muon	100 MeV
Pions	140 MeV
Proton, Neutron	1 GeV
Tau	2 GeV
W,Z Bosons	80-90 GeV
Higgs Boson	125 GeV

1. Classical Field Theory

In this first section we will discuss various aspects of classical fields. We will cover only the bare minimum ground necessary before turning to the quantum theory, and will return to classical field theory at several later stages in the course when we need to introduce new ideas.

1.1 The Dynamics of Fields

A *field* is a quantity defined at every point of space and time $(\vec{x}, t)$. While classical particle mechanics deals with a finite number of generalized coordinates $q_a(t)$, indexed by a label a, in field theory we are interested in the dynamics of fields

$$\phi_a(\vec{x}, t) \tag{1.1}$$

where both a and $\vec{x}$ are considered as labels. Thus we are dealing with a system with an infinite number of degrees of freedom — at least one for each point $\vec{x}$ in space. Notice that the concept of position has been relegated from a dynamical variable in particle mechanics to a mere label in field theory.

An Example: The Electromagnetic Field

The most familiar examples of fields from classical physics are the electric and magnetic fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. Both of these fields are spatial 3-vectors. In a more sophisticated treatement of electromagnetism, we derive these two 3-vectors from a single 4-component field $A^\mu(\vec{x}, t) = (\phi, \vec{A})$ where $\mu = 0, 1, 2, 3$ shows that this field is a vector in *spacetime*. The electric and magnetic fields are given by

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \tag{1.2}$$

which ensure that two of Maxwell's equations, $\nabla \cdot \vec{B} = 0$ and $d\vec{B}/dt = -\nabla \times \vec{E}$, hold immediately as identities.

The Lagrangian

The dynamics of the field is governed by a Lagrangian which is a function of $\phi(\vec{x}, t)$, $\dot{\phi}(\vec{x}, t)$ and $\nabla\phi(\vec{x}, t)$. In all the systems we study in this course, the Lagrangian is of the form,

$$L(t) = \int d^3x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{1.3}$$

where the official name for $\mathcal{L}$ is the *Lagrangian density*, although everyone simply calls it the Lagrangian. The action is,

$$S = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} \; = \int d^4x \, \mathcal{L} \qquad (1.4)$$

Recall that in particle mechanics L depends on q and $\dot{q}$, but not $\ddot{q}$. In field theory we similarly restrict to Lagrangians $\mathcal{L}$ depending on ϕ and $\dot{\phi}$, and not $\ddot{\phi}$. In principle, there's nothing to stop $\mathcal{L}$ depending on $\nabla\phi$, $\nabla^2\phi$, $\nabla^3\phi$, etc. However, with an eye to later Lorentz invariance, we will only consider Lagrangians depending on $\nabla\phi$ and not higher derivatives. Also we will not consider Lagrangians with explicit dependence on x^μ; all such dependence only comes through ϕ and its derivatives.

We can determine the equations of motion by the principle of least action. We vary the path, keeping the end points fixed and require $\delta S = 0$,

$$\delta S = \int d^4x \; \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right]$$
$$= \int d^4x \; \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \qquad (1.5)$$

The last term is a total derivative and vanishes for any $\delta\phi_a(\vec{x}, t)$ that decays at spatial infinity and obeys $\delta\phi_a(\vec{x}, t_1) = \delta\phi_a(\vec{x}, t_2) = 0$. Requiring $\delta S = 0$ for all such paths yields the Euler-Lagrange equations of motion for the fields ϕ_a,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \qquad (1.6)$$

1.1.1 An Example: The Klein-Gordon Equation

Consider the Lagrangian for a real scalar field $\phi(\vec{x}, t)$,

$$\mathcal{L} = \tfrac{1}{2} \eta^{\mu\nu} \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 \qquad (1.7)$$
$$= \tfrac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

where we are using the Minkowski space metric

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad (1.8)$$

Comparing (1.7) to the usual expression for the Lagrangian $L = T - V$, we identify the kinetic energy of the field as

$$T = \int d^3x \; \tfrac{1}{2}\dot{\phi}^2 \qquad (1.9)$$

and the potential energy of the field as

$$V = \int d^3x \; \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 \tag{1.10}$$

The first term in this expression is called the gradient energy, while the phrase "potential energy", or just "potential", is usually reserved for the last term.

To determine the equations of motion arising from (1.7), we compute

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \equiv (\dot{\phi}, -\nabla\phi) \tag{1.11}$$

The Euler-Lagrange equation is then

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0 \tag{1.12}$$

which we can write in relativistic form as

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{1.13}$$

This is the *Klein-Gordon Equation.* The Laplacian in Minkowski space is sometimes denoted by $\Box$. In this notation, the Klein-Gordon equation reads $\Box\phi + m^2\phi = 0$.

An obvious generalization of the Klein-Gordon equation comes from considering the Lagrangian with arbitrary potential $V(\phi)$,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \quad \Rightarrow \quad \partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0 \tag{1.14}$$

1.1.2 Another Example: First Order Lagrangians

We could also consider a Lagrangian that is linear in time derivatives, rather than quadratic. Take a complex scalar field ψ whose dynamics is defined by the real Lagrangian

$$\mathcal{L} = \frac{i}{2}(\psi^\star\dot{\psi} - \dot{\psi}^\star\psi) - \nabla\psi^\star \cdot \nabla\psi - m\psi^\star\psi \tag{1.15}$$

We can determine the equations of motion by treating ψ and $\psi^\star$ as independent objects, so that

$$\frac{\partial\mathcal{L}}{\partial\psi^\star} = \frac{i}{2}\dot{\psi} - m\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\star} = -\frac{i}{2}\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\nabla\psi^\star} = -\nabla\psi \tag{1.16}$$

This gives us the equation of motion

$$i\frac{\partial\psi}{\partial t} = -\nabla^2\psi + m\psi \tag{1.17}$$

This looks very much like the Schrödinger equation. Except it isn't! Or, at least, the interpretation of this equation is very different: the field ψ is a classical field with none of the probability interpretation of the wavefunction. We'll come back to this point in Section 2.8.

The initial data required on a Cauchy surface differs for the two examples above. When $\mathcal{L} \sim \dot{\phi}^2$, both ϕ and $\dot{\phi}$ must be specified to determine the future evolution; however when $\mathcal{L} \sim \psi^\star \dot{\psi}$, only ψ and $\psi^\star$ are needed.

1.1.3 A Final Example: Maxwell's Equations

We may derive Maxwell's equations in the vacuum from the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2} \left(\partial_\mu A_\nu \right) \left(\partial^\mu A^\nu \right) + \tfrac{1}{2} (\partial_\mu A^\mu)^2 \tag{1.18}$$

Notice the funny minus signs! This is to ensure that the kinetic terms for A_i are positive using the Minkowski space metric (1.8), so $\mathcal{L} \sim \tfrac{1}{2} \dot{A}_i^2$. The Lagrangian (1.18) has no kinetic term $\dot{A}_0^2$ for A_0. We will see the consequences of this in Section 6. To see that Maxwell's equations indeed follow from (1.18), we compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + (\partial_\rho A^\rho) \, \eta^{\mu\nu} \tag{1.19}$$

from which we may derive the equations of motion,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \equiv -\partial_\mu F^{\mu\nu} \tag{1.20}$$

where the *field strength* is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. You can check using (1.2) that this reproduces the remaining two Maxwell's equations in a vacuum: $\nabla \cdot \vec{E} = 0$ and $\partial \vec{E}/\partial t = \nabla \times \vec{B}$. Using the notation of the field strength, we may rewrite the Maxwell Lagrangian (up to an integration by parts) in the compact form

$$\mathcal{L} = -\tfrac{1}{4} \, F_{\mu\nu} F^{\mu\nu} \tag{1.21}$$

1.1.4 Locality, Locality, Locality

In each of the examples above, the Lagrangian is *local*. This means that there are no terms in the Lagrangian coupling $\phi(\vec{x}, t)$ directly to $\phi(\vec{y}, t)$ with $\vec{x} \neq \vec{y}$. For example, there are no terms that look like

$$L = \int d^3x \, d^3y \, \phi(\vec{x}) \phi(\vec{y}) \tag{1.22}$$

A priori, there's no reason for this. After all, $\vec{x}$ is merely a label, and we're quite happy to couple other labels together (for example, the term $\partial_3 A_0 \, \partial_0 A_3$ in the Maxwell Lagrangian couples the $\mu = 0$ field to the $\mu = 3$ field). But the closest we get for the $\vec{x}$ label is a coupling between $\phi(\vec{x})$ and $\phi(\vec{x} + \delta\vec{x})$ through the gradient term $(\nabla\phi)^2$. This property of locality is, as far as we know, a key feature of *all* theories of Nature. Indeed, one of the main reasons for introducing field theories in classical physics is to implement locality. In this course, we will only consider local Lagrangians.

1.2 Lorentz Invariance

The laws of Nature are relativistic, and one of the main motivations to develop quantum field theory is to reconcile quantum mechanics with special relativity. To this end, we want to construct field theories in which space and time are placed on an equal footing and the theory is invariant under Lorentz transformations,

$$x^\mu \longrightarrow (x')^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.23}$$

where $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda^\mu{}_\sigma \, \eta^{\sigma\tau} \, \Lambda^\nu{}_\tau = \eta^{\mu\nu} \tag{1.24}$$

For example, a rotation by θ about the x^3-axis, and a boost by $v < 1$ along the x^1-axis are respectively described by the Lorentz transformations

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.25}$$

with $\gamma = 1/\sqrt{1 - v^2}$. The Lorentz transformations form a Lie group under matrix multiplication. You'll learn more about this in the "Symmetries and Particle Physics" course.

The Lorentz transformations have a *representation* on the fields. The simplest example is the scalar field which, under the Lorentz transformation $x \to \Lambda x$, transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x) \tag{1.26}$$

The inverse Λ^{-1} appears in the argument because we are dealing with an *active* transformation in which the field is truly shifted. To see why this means that the inverse appears, it will suffice to consider a non-relativistic example such as a temperature field. Suppose we start with an initial field $\phi(\vec{x})$ which has a hotspot at, say, $\vec{x} = (1, 0, 0)$. After a rotation $\vec{x} \to R\vec{x}$ about the z-axis, the new field $\phi'(\vec{x})$ will have the hotspot at $\vec{x} = (0, 1, 0)$. If we want to express $\phi'(\vec{x})$ in terms of the old field ϕ, we need to place ourselves at $\vec{x} = (0, 1, 0)$ and ask what the old field looked like where we've come from at $R^{-1}(0, 1, 0) = (1, 0, 0)$. This R^{-1} is the origin of the inverse transformation. (If we were instead dealing with a passive transformation in which we relabel our choice of coordinates, we would have instead $\phi(x) \to \phi'(x) = \phi(\Lambda x)$).

The definition of a Lorentz invariant theory is that if $\phi(x)$ solves the equations of motion then $\phi(\Lambda^{-1}x)$ also solves the equations of motion. We can ensure that this property holds by requiring that the action is Lorentz invariant. Let's look at our examples:

Example 1: The Klein-Gordon Equation

For a real scalar field we have $\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$. The derivative of the scalar field transforms as a vector, meaning

$$(\partial_\mu \phi)(x) \rightarrow (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(y)$$

where $y = \Lambda^{-1}x$. This means that the derivative terms in the Lagrangian density transform as

$$
\begin{aligned}
\mathcal{L}_{deriv}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) \eta^{\mu\nu} \quad \longrightarrow \quad & (\Lambda^{-1})^\rho{}_\mu (\partial_\rho \phi)(y) \, (\Lambda^{-1})^\sigma{}_\nu (\partial_\sigma \phi)(y) \, \eta^{\mu\nu} \\
= \quad & (\partial_\rho \phi)(y) \, (\partial_\sigma \phi)(y) \, \eta^{\rho\sigma} \\
= \quad & \mathcal{L}_{deriv}(y) \qquad\qquad\qquad\qquad (1.27)
\end{aligned}
$$

The potential terms transform in the same way, with $\phi^2(x) \rightarrow \phi^2(y)$. Putting this all together, we find that the action is indeed invariant under Lorentz transformations,

$$S = \int d^4x \, \mathcal{L}(x) \longrightarrow \int d^4x \, \mathcal{L}(y) = \int d^4y \, \mathcal{L}(y) = S \qquad\qquad (1.28)$$

where, in the last step, we need the fact that we don't pick up a Jacobian factor when we change integration variables from $\int d^4x$ to $\int d^4y$. This follows because $\det \Lambda = 1$. (At least for Lorentz transformation connected to the identity which, for now, is all we deal with).

Example 2: First Order Dynamics

In the first-order Lagrangian (1.15), space and time are not on the same footing. ($\mathcal{L}$ is linear in time derivatives, but quadratic in spatial derivatives). The theory is not Lorentz invariant.

In practice, it's easy to see if the action is Lorentz invariant: just make sure all the Lorentz indices $\mu = 0, 1, 2, 3$ are contracted with Lorentz invariant objects, such as the metric $\eta_{\mu\nu}$. Other Lorentz invariant objects you can use include the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ and the matrices γ_μ that we will introduce when we come to discuss spinors in Section 4.

Example 3: Maxwell's Equations

Under a Lorentz transformation $A^\mu(x) \to \Lambda^\mu_{\ \nu} A^\nu(\Lambda^{-1}x)$. You can check that Maxwell's Lagrangian (1.21) is indeed invariant. Of course, historically electrodynamics was the first Lorentz invariant theory to be discovered: it was found even before the concept of Lorentz invariance.

1.3 Symmetries

The role of symmetries in field theory is possibly even more important than in particle mechanics. There are Lorentz symmetries, internal symmetries, gauge symmetries, supersymmetries.... We start here by recasting Noether's theorem in a field theoretic framework.

1.3.1 Noether's Theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved *current* $j^\mu(x)$ such that the equations of motion imply

$$\partial_\mu j^\mu = 0 \tag{1.29}$$

or, in other words, $\partial j^0 / \partial t + \nabla \cdot \vec{j} = 0$.

A Comment: A conserved current implies a conserved charge Q, defined as

$$Q = \int_{\mathbf{R}^3} d^3x \ j^0 \tag{1.30}$$

which one can immediately see by taking the time derivative,

$$\frac{dQ}{dt} = \int_{\mathbf{R}^3} d^3x \ \frac{\partial j^0}{\partial t} = - \int_{\mathbf{R}^3} d^3x \ \nabla \cdot \vec{j} = 0 \tag{1.31}$$

assuming that $\vec{j} \to 0$ sufficiently quickly as $|\vec{x}| \to \infty$. However, the existence of a current is a much stronger statement than the existence of a conserved charge because it implies that charge is conserved *locally*. To see this, we can define the charge in a finite volume V,

$$Q_V = \int_V d^3x \ j^0 \tag{1.32}$$

Repeating the analysis above, we find that

$$\frac{dQ_V}{dt} = - \int_V d^3x \ \nabla \cdot \vec{j} = - \int_A \vec{j} \cdot d\vec{S} \tag{1.33}$$

where A is the area bounding V and we have used Stokes' theorem. This equation means that any charge leaving V must be accounted for by a flow of the current 3-vector $\vec{j}$ out of the volume. This kind of local conservation of charge holds in any local field theory.

Proof of Noether's Theorem: We'll prove the theorem by working infinitesimally. We may always do this if we have a continuous symmetry. We say that the transformation

$$\delta\phi_a(x) = X_a(\phi) \tag{1.34}$$

is a symmetry if the Lagrangian changes by a total derivative,

$$\delta\mathcal{L} = \partial_\mu F^\mu \tag{1.35}$$

for some set of functions $F^\mu(\phi)$. To derive Noether's theorem, we first consider making an *arbitrary* transformation of the fields $\delta\phi_a$. Then

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a) \\
&= \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)
\end{aligned} \tag{1.36}
$$

When the equations of motion are satisfied, the term in square brackets vanishes. So we're left with

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \tag{1.37}$$

But for the symmetry transformation $\delta\phi_a = X_a(\phi)$, we have by definition $\delta\mathcal{L} = \partial_\mu F^\mu$. Equating this expression with (1.37) gives us the result

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) - F^\mu(\phi) \tag{1.38}$$

1.3.2 An Example: Translations and the Energy-Momentum Tensor

Recall that in classical particle mechanics, invariance under spatial translations gives rise to the conservation of momentum, while invariance under time translations is responsible for the conservation of energy. We will now see something similar in field theories. Consider the infinitesimal translation

$$x^\nu \to x^\nu - \epsilon^\nu \quad \Rightarrow \quad \phi_a(x) \to \phi_a(x) + \epsilon^\nu\partial_\nu\phi_a(x) \tag{1.39}$$

(where the sign in the field transformation is plus, instead of minus, because we're doing an active, as opposed to passive, transformation). Similarly, once we substitute a specific field configuration $\phi(x)$ into the Lagrangian, the Lagrangian itself also transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x) \tag{1.40}$$

Since the change in the Lagrangian is a total derivative, we may invoke Noether's theorem which gives us four conserved currents $(j^\mu)_\nu$, one for each of the translations ϵ^ν with $\nu = 0, 1, 2, 3$,

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \equiv T^\mu{}_\nu \tag{1.41}$$

$T^\mu{}_\nu$ is called the *energy-momentum* tensor. It satisfies

$$\partial_\mu T^\mu{}_\nu = 0 \tag{1.42}$$

The four conserved quantities are given by

$$E = \int d^3x \, T^{00} \quad \text{and} \quad P^i = \int d^3x \, T^{0i} \tag{1.43}$$

where E is the total energy of the field configuration, while P^i is the total momentum of the field configuration.

An Example of the Energy-Momentum Tensor

Consider the simplest scalar field theory with Lagrangian (1.7). From the above discussion, we can compute

$$T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{1.44}$$

One can verify using the equation of motion for ϕ that this expression indeed satisfies $\partial_\mu T^{\mu\nu} = 0$. For this example, the conserved energy and momentum are given by

$$E = \int d^3x \, \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 \tag{1.45}$$

$$P^i = \int d^3x \, \dot{\phi} \, \partial^i \phi \tag{1.46}$$

Notice that for this example, $T^{\mu\nu}$ came out symmetric, so that $T^{\mu\nu} = T^{\nu\mu}$. This won't always be the case. Nevertheless, there is typically a way to massage the energy momentum tensor of any theory into a symmetric form by adding an extra term

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu} \tag{1.47}$$

where $\Gamma^{\rho\mu\nu}$ is some function of the fields that is anti-symmetric in the first two indices so $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$. This guarantees that $\partial_\mu \partial_\rho \Gamma^{\rho\mu\nu} = 0$ so that the new energy-momentum tensor is also a conserved current.

A Cute Trick

One reason that you may want a symmetric energy-momentum tensor is to make contact with general relativity: such an object sits on the right-hand side of Einstein's field equations. In fact this observation provides a quick and easy way to determine a symmetric energy-momentum tensor. Firstly consider coupling the theory to a curved background spacetime, introducing an arbitrary metric $g_{\mu\nu}(x)$ in place of $\eta_{\mu\nu}$, and replacing the kinetic terms with suitable covariant derivatives using "minimal coupling". Then a symmetric energy momentum tensor in the flat space theory is given by

$$\Theta^{\mu\nu} = -\left.\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{1.48}$$

It should be noted however that this trick requires a little more care when working with spinors.

1.3.3 Another Example: Lorentz Transformations and Angular Momentum

In classical particle mechanics, rotational invariance gave rise to conservation of angular momentum. What is the analogy in field theory? Moreover, we now have further Lorentz transformations, namely boosts. What conserved quantity do they correspond to? To answer these questions, we first need the infinitesimal form of the Lorentz transformations

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{1.49}$$

where $\omega^{\mu}{}_{\nu}$ is infinitesimal. The condition (1.24) for Λ to be a Lorentz transformation becomes

$$(\delta^{\mu}{}_{\sigma} + \omega^{\mu}{}_{\sigma})(\delta^{\nu}{}_{\tau} + \omega^{\nu}{}_{\tau})\,\eta^{\sigma\tau} = \eta^{\mu\nu}$$
$$\Rightarrow \qquad \omega^{\mu\nu} + \omega^{\nu\mu} = 0 \tag{1.50}$$

So the infinitesimal form $\omega^{\mu\nu}$ of the Lorentz transformation must be an anti-symmetric matrix. As a check, the number of different 4×4 anti-symmetric matrices is $4\times3/2 = 6$, which agrees with the number of different Lorentz transformations (3 rotations + 3 boosts). Now the transformation on a scalar field is given by

$$\begin{aligned}
\phi(x) \rightarrow \phi'(x) &= \phi(\Lambda^{-1}x) \\
&= \phi(x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}) \\
&= \phi(x^{\mu}) - \omega^{\mu}{}_{\nu}\,x^{\nu}\,\partial_{\mu}\phi(x) \tag{1.51}
\end{aligned}$$

from which we see that

$$\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi \tag{1.52}$$

By the same argument, the Lagrangian density transforms as

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L}) \tag{1.53}$$

where the last equality follows because $\omega^\mu{}_\mu = 0$ due to anti-symmetry. Once again, the Lagrangian changes by a total derivative so we may apply Noether's theorem (now with $F^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$) to find the conserved current

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \omega^\rho{}_\nu x^\nu \partial_\rho \phi + \omega^\mu{}_\nu x^\nu \mathcal{L}$$

$$= -\omega^\rho{}_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\rho \phi - \delta^\mu{}_\rho x^\nu \mathcal{L} \right] = -\omega^\rho{}_\nu\, T^\mu{}_\rho x^\nu \tag{1.54}$$

Unlike in the previous example, I've left the infinitesimal choice of $\omega^\mu{}_\nu$ in the expression for this current. But really, we should strip it out to give six different currents, i.e. one for each choice of $\omega^\mu{}_\nu$. We can write them as

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \tag{1.55}$$

which satisfy $\partial_\mu(\mathcal{J}^\mu)^{\rho\sigma} = 0$ and give rise to 6 conserved charges. For $\rho, \sigma = 1, 2, 3$, the Lorentz transformation is a rotation and the three conserved charges give the total angular momentum of the field.

$$Q^{ij} = \int d^3x \, (x^i T^{0j} - x^j T^{0i}) \tag{1.56}$$

But what about the boosts? In this case, the conserved charges are

$$Q^{0i} = \int d^3x \, (x^0 T^{0i} - x^i T^{00}) \tag{1.57}$$

The fact that these are conserved tells us that

$$0 = \frac{dQ^{0i}}{dt} = \int d^3x \, T^{0i} + t \int d^3x \, \frac{\partial T^{0i}}{\partial t} - \frac{d}{dt} \int d^3x \, x^i T^{00}$$

$$= P^i + t\frac{dP^i}{dt} - \frac{d}{dt} \int d^3x \, x^i T^{00} \tag{1.58}$$

But we know that P^i is conserved, so $dP^i/dt = 0$, leaving us with the following consequence of invariance under boosts:

$$\frac{d}{dt} \int d^3x \, x^i T^{00} = \text{constant} \tag{1.59}$$

This is the statement that the center of energy of the field travels with a constant velocity. It's kind of like a field theoretic version of Newton's first law but, rather surprisingly, appearing here as a conservation law.

1.3.4 Internal Symmetries

The above two examples involved transformations of spacetime, as well as transformations of the field. An *internal symmetry* is one that only involves a transformation of the fields and acts the same at every point in spacetime. The simplest example occurs for a complex scalar field $\psi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2}$. We can build a real Lagrangian by

$$\mathcal{L} = \partial_\mu \psi^\star \, \partial^\mu \psi - V(|\psi|^2) \tag{1.60}$$

where the potential is a general polynomial in $|\psi|^2 = \psi^\star \psi$. To find the equations of motion, we could expand ψ in terms of ϕ_1 and ϕ_2 and work as before. However, it's easier (and equivalent) to treat ψ and $\psi^\star$ as independent variables and vary the action with respect to both of them. For example, varying with respect to $\psi^\star$ leads to the equation of motion

$$\partial_\mu \partial^\mu \psi + \frac{\partial V(\psi^\star \psi)}{\partial \psi^\star} = 0 \tag{1.61}$$

The Lagrangian has a continuous symmetry which rotates ϕ_1 and ϕ_2 or, equivalently, rotates the phase of ψ:

$$\psi \to e^{i\alpha}\psi \quad \text{or} \quad \delta\psi = i\alpha\psi \tag{1.62}$$

where the latter equation holds with α infinitesimal. The Lagrangian remains invariant under this change: $\delta\mathcal{L} = 0$. The associated conserved current is

$$j^\mu = i(\partial^\mu \psi^\star)\psi - i\psi^\star(\partial^\mu \psi) \tag{1.63}$$

We will later see that the conserved charges arising from currents of this type have the interpretation of electric charge or particle number (for example, baryon or lepton number).

Non-Abelian Internal Symmetries

Consider a theory involving N scalar fields ϕ_a, all with the same mass and the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{N} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}\sum_{a=1}^{N} m^2 \phi_a^2 - g\left(\sum_{a=1}^{N} \phi_a^2\right)^2 \tag{1.64}$$

In this case the Lagrangian is invariant under the non-Abelian symmetry group $G = SO(N)$. (Actually $O(N)$ in this case). One can construct theories from complex fields in a similar manner that are invariant under an $SU(N)$ symmetry group. Non-Abelian symmetries of this type are often referred to as *global* symmetries to distinguish them from the "local gauge" symmetries that you will meet later. Isospin is an example of such a symmetry, albeit realized only approximately in Nature.

Another Cute Trick

There is a quick method to determine the conserved current associated to an internal symmetry $\delta\phi = \alpha\phi$ for which the Lagrangian is invariant. Here, α is a constant real number. (We may generalize the discussion easily to a non-Abelian internal symmetry for which α becomes a matrix). Now consider performing the transformation but where α depends on spacetime: $\alpha = \alpha(x)$. The action is no longer invariant. However, the change must be of the form

$$\delta\mathcal{L} = (\partial_\mu \alpha)\, h^\mu(\phi) \tag{1.65}$$

since we know that $\delta\mathcal{L} = 0$ when α is constant. The change in the action is therefore

$$\delta S = \int d^4x \,\, \delta\mathcal{L} = -\int d^4x \,\, \alpha(x)\, \partial_\mu h^\mu \tag{1.66}$$

which means that when the equations of motion are satisfied (so $\delta S = 0$ *for all* variations, including $\delta\phi = \alpha(x)\phi$) we have

$$\partial_\mu h^\mu = 0 \tag{1.67}$$

We see that we can identify the function $h^\mu = j^\mu$ as the conserved current. This way of viewing things emphasizes that it is the derivative terms, not the potential terms, in the action that contribute to the current. (The potential terms are invariant even when $\alpha = \alpha(x)$).

1.4 The Hamiltonian Formalism

The link between the Lagrangian formalism and the quantum theory goes via the path integral. In this course we will not discuss path integral methods, and focus instead on canonical quantization. For this we need the Hamiltonian formalism of field theory. We start by defining the *momentum* $\pi^a(x)$ conjugate to $\phi_a(x)$,

$$\pi^a(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a} \tag{1.68}$$

The conjugate momentum $\pi^a(x)$ is a function of x, just like the field $\phi_a(x)$ itself. It is not to be confused with the total momentum P^i defined in (1.43) which is a single number characterizing the whole field configuration. The *Hamiltonian density* is given by

$$\mathcal{H} = \pi^a(x)\dot{\phi}_a(x) - \mathcal{L}(x) \tag{1.69}$$

where, as in classical mechanics, we eliminate $\dot{\phi}_a(x)$ in favour of $\pi^a(x)$ everywhere in $\mathcal{H}$. The Hamiltonian is then simply

$$H = \int d^3x \,\, \mathcal{H} \tag{1.70}$$

An Example: A Real Scalar Field

For the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi) \tag{1.71}$$

the momentum is given by $\pi = \dot{\phi}$, which gives us the Hamiltonian,

$$H = \int d^3x \; \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) \tag{1.72}$$

Notice that the Hamiltonian agrees with the definition of the total energy (1.45) that we get from applying Noether's theorem for time translation invariance.

In the Lagrangian formalism, Lorentz invariance is clear for all to see since the action is invariant under Lorentz transformations. In contrast, the Hamiltonian formalism is *not* manifestly Lorentz invariant: we have picked a preferred time. For example, the equations of motion for $\phi(x) = \phi(\vec{x}, t)$ arise from Hamilton's equations,

$$\dot{\phi}(\vec{x}, t) = \frac{\partial H}{\partial \pi(\vec{x}, t)} \quad \text{and} \quad \dot{\pi}(\vec{x}, t) = -\frac{\partial H}{\partial \phi(\vec{x}, t)} \tag{1.73}$$

which, unlike the Euler-Lagrange equations (1.6), do not look Lorentz invariant. Nevertheless, even though the Hamiltonian framework doesn't *look* Lorentz invariant, the physics must remain unchanged. If we start from a relativistic theory, all final answers must be Lorentz invariant even if it's not manifest at intermediate steps. We will pause at several points along the quantum route to check that this is indeed the case.

2. Free Fields

> "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."
>
> *Sidney Coleman*

2.1 Canonical Quantization

In quantum mechanics, canonical quantization is a recipe that takes us from the Hamiltonian formalism of classical dynamics to the quantum theory. The recipe tells us to take the generalized coordinates q_a and their conjugate momenta p^a and promote them to operators. The Poisson bracket structure of classical mechanics morphs into the structure of commutation relations between operators, so that, in units with $\hbar = 1$,

$$[q_a, q_b] = [p^a, p^b] = 0$$
$$[q_a, p^b] = i\, \delta_a^b \tag{2.1}$$

In field theory we do the same, now for the field $\phi_a(\vec{x})$ and its momentum conjugate $\pi^b(\vec{x})$. Thus a *quantum field* is an operator valued function of space obeying the commutation relations

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(\vec{x}), \pi^b(\vec{y})] = 0$$
$$[\phi_a(\vec{x}), \pi^b(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})\, \delta_a^b \tag{2.2}$$

Note that we've lost all track of Lorentz invariance since we have separated space $\vec{x}$ and time t. We are working in the Schrödinger picture so that the operators $\phi_a(\vec{x})$ and $\pi^a(\vec{x})$ do not depend on time at all — only on space. All time dependence sits in the states $|\psi\rangle$ which evolve by the usual Schrödinger equation

$$i\frac{d|\psi\rangle}{dt} = H\,|\psi\rangle \tag{2.3}$$

We aren't doing anything different from usual quantum mechanics; we're merely applying the old formalism to fields. Be warned however that the notation $|\psi\rangle$ for the state is deceptively simple: if you were to write the wavefunction in quantum field theory, it would be a *functional*, that is a function of every possible configuration of the field ϕ.

The typical information we want to know about a quantum theory is the spectrum of the Hamiltonian H. In quantum field theories, this is usually *very* hard. One reason for this is that we have an infinite number of degrees of freedom — at least one for every point $\vec{x}$ in space. However, for certain theories — known as *free theories* — we can find a way to write the dynamics such that each degree of freedom evolves independently

from all the others. Free field theories typically have Lagrangians which are quadratic in the fields, so that the equations of motion are linear. For example, the simplest relativistic free theory is the classical Klein-Gordon (KG) equation for a real scalar field $\phi(\vec{x}, t)$,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{2.4}$$

To exhibit the coordinates in which the degrees of freedom decouple from each other, we need only take the Fourier transform,

$$\phi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \phi(\vec{p}, t) \tag{2.5}$$

Then $\phi(\vec{p}, t)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2) \right) \phi(\vec{p}, t) = 0 \tag{2.6}$$

Thus, for each value of $\vec{p}$, $\phi(\vec{p}, t)$ solves the equation of a harmonic oscillator vibrating at frequency

$$\omega_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2} \tag{2.7}$$

We learn that the most general solution to the KG equation is a linear superposition of simple harmonic oscillators, each vibrating at a different frequency with a different amplitude. To quantize $\phi(\vec{x}, t)$ we must simply quantize this infinite number of harmonic oscillators. Let's recall how to do this.

2.1.1 The Simple Harmonic Oscillator

Consider the quantum mechanical Hamiltonian

$$H = \tfrac{1}{2} p^2 + \tfrac{1}{2} \omega^2 q^2 \tag{2.8}$$

with the canonical commutation relations $[q, p] = i$. To find the spectrum we define the creation and annihilation operators (also known as raising/lowering operators, or sometimes ladder operators)

$$a = \sqrt{\frac{\omega}{2}} \, q + \frac{i}{\sqrt{2\omega}} \, p \quad , \quad a^\dagger = \sqrt{\frac{\omega}{2}} \, q - \frac{i}{\sqrt{2\omega}} \, p \tag{2.9}$$

which can be easily inverted to give

$$q = \frac{1}{\sqrt{2\omega}} \, (a + a^\dagger) \quad , \quad p = -i\sqrt{\frac{\omega}{2}} \, (a - a^\dagger) \tag{2.10}$$

Substituting into the above expressions we find

$$[a, a^\dagger] = 1 \tag{2.11}$$

while the Hamiltonian is given by

$$\begin{aligned} H &= \tfrac{1}{2}\omega(aa^\dagger + a^\dagger a) \\ &= \omega(a^\dagger a + \tfrac{1}{2}) \end{aligned} \tag{2.12}$$

One can easily confirm that the commutators between the Hamiltonian and the creation and annihilation operators are given by

$$[H, a^\dagger] = \omega a^\dagger \quad \text{and} \quad [H, a] = -\omega a \tag{2.13}$$

These relations ensure that a and $a^\dagger$ take us between energy eigenstates. Let $|E\rangle$ be an eigenstate with energy E, so that $H|E\rangle = E|E\rangle$. Then we can construct more eigenstates by acting with a and $a^\dagger$,

$$Ha^\dagger |E\rangle = (E + \omega)a^\dagger |E\rangle \quad , \quad Ha |E\rangle = (E - \omega)a |E\rangle \tag{2.14}$$

So we find that the system has a ladder of states with energies

$$\ldots, E - \omega, E, E + \omega, E + 2\omega, \ldots \tag{2.15}$$

If the energy is bounded below, there must be a *ground state* $|0\rangle$ which satisfies $a|0\rangle = 0$. This has ground state energy (also known as zero point energy),

$$H|0\rangle = \tfrac{1}{2}\omega |0\rangle \tag{2.16}$$

Excited states then arise from repeated application of $a^\dagger$,

$$|n\rangle = (a^\dagger)^n |0\rangle \quad \text{with} \quad H|n\rangle = (n + \tfrac{1}{2})\omega |n\rangle \tag{2.17}$$

where I've ignored the normalization of these states so, $\langle n | n \rangle \neq 1$.

2.2 The Free Scalar Field

We now apply the quantization of the harmonic oscillator to the free scalar field. We write ϕ and π as a linear sum of an infinite number of creation and annihilation operators $a_{\vec{p}}^\dagger$ and $a_{\vec{p}}$, indexed by the 3-momentum $\vec{p}$,

$$\phi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right] \tag{2.18}$$

$$\pi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right] \tag{2.19}$$

Claim: The commutation relations for ϕ and π are equivalent to the following commutation relations for $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$

$$
\begin{aligned}
&[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \\
&[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
&[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \\
&[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})
\end{aligned}
\tag{2.20}
$$

Proof: We'll show this just one way. Assume that $[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$. Then

$$
\begin{aligned}
[\phi(\vec{x}), \pi(\vec{y})] &= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{(-i)}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \left(-[a_{\vec{p}}, a_{\vec{q}}^\dagger] e^{i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} + [a_{\vec{p}}^\dagger, a_{\vec{q}}] e^{-i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{(-i)}{2} \left(-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}\cdot(\vec{y}-\vec{x})} \right) \\
&= i\delta^{(3)}(\vec{x} - \vec{y}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
\tag{2.21}
$$

The Hamiltonian

Let's now compute the Hamiltonian in terms of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$. We have

$$
\begin{aligned}
H &= \frac{1}{2} \int d^3x\, \pi^2 + (\nabla\phi)^2 + m^2\phi^2 \\
&= \frac{1}{2} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6} \left[-\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}{2} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}})(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}) \right. \\
&\quad + \frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} (i\vec{p}\, a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - i\vec{p}\, a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}) \cdot (i\vec{q}\, a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - i\vec{q}\, a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}) \\
&\quad + \left. \frac{m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}})(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} + a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}) \right] \\
&= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \left[(-\omega_{\vec{p}}^2 + \vec{p}^2 + m^2)(a_{\vec{p}} a_{-\vec{p}} + a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger) + (\omega_{\vec{p}}^2 + \vec{p}^2 + m^2)(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}) \right]
\end{aligned}
$$

where in the second line we've used the expressions for ϕ and π given in (2.18) and (2.19); to get to the third line we've integrated over d^3x to get delta-functions $\delta^{(3)}(\vec{p}\pm\vec{q})$ which, in turn, allow us to perform the d^3q integral. Now using the expression for the frequency $\omega_{\vec{p}}^2 = \vec{p}^2 + m^2$, the first term vanishes and we're left with

$$
\begin{aligned}
H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}}^\dagger a_{\vec{p}} + \tfrac{1}{2}(2\pi)^3 \delta^{(3)}(0) \right]
\end{aligned}
\tag{2.22}
$$

Hmmmm. We've found a delta-function, evaluated at zero where it has its infinite spike. Moreover, the integral over $\omega_{\vec{p}}$ diverges at large p. What to do? Let's start by looking at the ground state where this infinity first becomes apparent.

2.3 The Vacuum

Following our procedure for the harmonic oscillator, let's define the vacuum $|0\rangle$ by insisting that it is annihilated by *all* $a_{\vec{p}}$,

$$a_{\vec{p}}|0\rangle = 0 \qquad \forall \; \vec{p} \tag{2.23}$$

With this definition, the energy E_0 of the ground state comes from the second term in (2.22),

$$H|0\rangle \equiv E_0|0\rangle = \left[\int d^3p \; \frac{1}{2}\omega_{\vec{p}}\delta^{(3)}(0)\right]|0\rangle = \infty |0\rangle \tag{2.24}$$

The subject of quantum field theory is rife with infinities. Each tells us something important, usually that we're doing something wrong, or asking the wrong question. Let's take some time to explore where this infinity comes from and how we should deal with it.

In fact there are two different ∞'s lurking in the expression (2.24). The first arises because space is infinitely large. (Infinities of this type are often referred to as *infra-red divergences* although in this case the ∞ is so simple that it barely deserves this name). To extract out this infinity, let's consider putting the theory in a box with sides of length L. We impose periodic boundary conditions on the field. Then, taking the limit where $L \to \infty$, we get

$$(2\pi)^3 \, \delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x \; e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0} = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x = V \tag{2.25}$$

where V is the volume of the box. So the $\delta(0)$ divergence arises because we're computing the total energy, rather than the energy density $\mathcal{E}_0$. To find $\mathcal{E}_0$ we can simply divide by the volume,

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2}\omega_{\vec{p}} \tag{2.26}$$

which is still infinite. We recognize it as the sum of ground state energies for each harmonic oscillator. But $\mathcal{E}_0 \to \infty$ due to the $|\vec{p}| \to \infty$ limit of the integral. This is a high frequency — or short distance — infinity known as an *ultra-violet divergence*. This divergence arises because of our hubris. We've assumed that our theory is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies. This is clearly absurd. The integral should be cut-off at high momentum in order to reflect the fact that our theory is likely to break down in some way.

We can deal with the infinity in (2.24) in a more practical way. In physics we're only interested in energy differences. There's no way to measure E_0 directly, so we can simply redefine the Hamiltonian by subtracting off this infinity,

$$H = \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \, a_{\vec{p}}^{\dagger} a_{\vec{p}} \tag{2.27}$$

so that, with this new definition, $H \left| 0 \right\rangle = 0$. In fact, the difference between this Hamiltonian and the previous one is merely an ordering ambiguity in moving from the classical theory to the quantum theory. For example, if we defined the Hamiltonian of the harmonic oscillator to be $H = (1/2)(\omega q - ip)(\omega q + ip)$, which is classically the same as our original choice, then upon quantization it would naturally give $H = \omega a^{\dagger} a$ as in (2.27). This type of ordering ambiguity arises a lot in field theories. We'll come across a number of ways of dealing with it. The method that we've used above is called *normal ordering*.

Definition: We write the *normal ordered* string of operators $\phi_1(\vec{x}_1) \ldots \phi_n(\vec{x}_n)$ as

$$: \phi_1(\vec{x}_1) \ldots \phi_n(\vec{x}_n) : \tag{2.28}$$

It is defined to be the usual product with all annihilation operators $a_{\vec{p}}$ placed to the right. So, for the Hamiltonian, we could write (2.27) as

$$: H := \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \, a_{\vec{p}}^{\dagger} a_{\vec{p}} \tag{2.29}$$

In the remainder of this section, we will normal order all operators in this manner.

2.3.1 The Cosmological Constant

Above I wrote "there's no way to measure E_0 directly". There is a BIG caveat here: gravity is supposed to see everything! The sum of all the zero point energies should contribute to the stress-energy tensor that appears on the right-hand side of Einstein's equations. We expect them to appear as a *cosmological constant* $\Lambda = E_0/V$,

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2.30}$$

Current observation suggests that 70% of the energy density in the universe has the properties of a cosmological constant with $\Lambda \sim (10^{-3} eV)^4$. This is much smaller than other scales in particle physics. In particular, the Standard Model is valid at least up to $10^{12} \, eV$. Why don't the zero point energies of these fields contribute to Λ? Or, if they do, what cancels them to such high accuracy? This is the cosmological constant problem. No one knows the answer!

2.3.2 The Casimir Effect

> "I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new... and he mumbled something about zero-point energy."
>
> *Hendrik Casimir*

Using the normal ordering prescription we can happily set $E_0 = 0$, while chanting the mantra that only energy differences can be measured. But we should be careful, for there is a situation where differences in the energy of vacuum fluctuations themselves can be measured.

To regulate the infra-red divergences, we'll make the x^1 direction periodic, with size L, and impose periodic boundary conditions such that

$$\phi(\vec{x}) = \phi(\vec{x} + L\vec{n}) \tag{2.31}$$

with $\vec{n} = (1, 0, 0)$. We'll leave y and z alone, but remember that we should compute all physical quantities per unit area A. We insert two reflecting plates, separated by a distance $d \ll L$ in the x^1 direction. The plates are such that they impose $\phi(x) = 0$ at the position of the plates. The presence of these plates affects the Fourier decomposition of the field and, in particular, means that the momentum of the field inside the plates is quantized as

$$\vec{p} = \left(\frac{n\pi}{d}, p_y, p_z\right) \qquad n \in \mathbf{Z}^+ \tag{2.32}$$
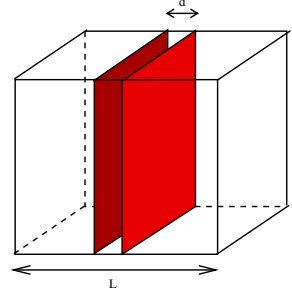
Figure 3:

For a *massless* scalar field, the ground state energy between the plates is

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2} \tag{2.33}$$

while the energy outside the plates is $E(L - d)$. The total energy is therefore

$$E = E(d) + E(L - d) \tag{2.34}$$

which – at least naively – depends on d. If this naive guess is true, it would mean that there is a force on the plates due to the fluctuations of the vacuum. This is the Casimir force, first predicted in 1948 and observed 10 years later. In the real world, the effect is due to the vacuum fluctuations of the electromagnetic field, with the boundary conditions imposed by conducting plates. Here we model this effect with a scalar.

But there's a problem. E is infinite! What to do? The problem comes from the arbitrarily high momentum modes. We could regulate this in a number of different ways. Physically one could argue that any real plate cannot reflect waves of arbitrarily high frequency: at some point, things begin to leak. Mathematically, we want to find a way to neglect modes of momentum $p \gg a^{-1}$ for some distance scale $a \ll d$, known as the ultra-violet (UV) cut-off. One way to do this is to change the integral (2.33) to,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \left(\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2} \right) e^{-a\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}} \tag{2.35}$$

which has the property that as $a \to 0$, we regain the full, infinite, expression (2.33). However (2.35) is finite, and gives us something we can easily work with. Of course, we made it finite in a rather ad-hoc manner and we better make sure that any physical quantity we calculate doesn't depend on the UV cut-off a, otherwise it's not something we can really trust.

The integral (2.35) is do-able, but a little complicated. It's a lot simpler if we look at the problem in $d = 1 + 1$ dimensions, rather than $d = 3 + 1$ dimensions. We'll find that all the same physics is at play. Now the energy is given by

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n \tag{2.36}$$

We now regulate this sum by introducing the UV cutoff a introduced above. This renders the expression finite, allowing us to start manipulating it thus,

$$
\begin{aligned}
E_{1+1}(d) &\to \frac{\pi}{2d} \sum_{n=1}^{\infty} n\, e^{-an\pi/d} \\
&= -\frac{1}{2} \frac{\partial}{\partial a} \sum_{n=1}^{\infty} e^{-an\pi/d} \\
&= -\frac{1}{2} \frac{\partial}{\partial a} \frac{1}{1 - e^{-a\pi/d}} \\
&= \frac{\pi}{2d} \frac{e^{a\pi/d}}{(e^{a\pi/d} - 1)^2} \\
&= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2)
\end{aligned}
\tag{2.37}
$$

where, in the last line, we've used the fact that $a \ll d$. We can now compute the full energy,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L - d) = \frac{L}{2\pi a^2} - \frac{\pi}{24} \left(\frac{1}{d} + \frac{1}{L - d} \right) + \mathcal{O}(a^2) \tag{2.38}$$

This is still infinite in the limit $a \to 0$, which is to be expected. However, the force is given by

$$\frac{\partial E_{1+1}}{\partial d} = \frac{\pi}{24d^2} + \ldots \tag{2.39}$$

where the ... include terms of size d/L and a/d. The key point is that as we remove both the regulators, and take $a \to 0$ and $L \to \infty$, the force between the plates remains finite. This is the Casimir force[2].

If we ploughed through the analogous calculation in $d = 3 + 1$ dimensions, and performed the integral (2.35), we would find the result

$$\frac{1}{A}\frac{\partial E}{\partial d} = \frac{\pi^2}{480d^4} \tag{2.40}$$

The true Casimir force is twice as large as this, due to the two polarization states of the photon.

2.4 Particles

Having dealt with the vacuum, we can now turn to the excitations of the field. It's easy to verify that

$$[H, a_{\vec{p}}^\dagger] = \omega_{\vec{p}} a_{\vec{p}}^\dagger \quad \text{and} \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \tag{2.41}$$

which means that, just as for the harmonic oscillator, we can construct energy eigenstates by acting on the vacuum $|0\rangle$ with $a_{\vec{p}}^\dagger$. Let

$$|\vec{p}\rangle = a_{\vec{p}}^\dagger |0\rangle \tag{2.42}$$

This state has energy

$$H |\vec{p}\rangle = \omega_{\vec{p}} |\vec{p}\rangle \quad \text{with} \quad \omega_{\vec{p}}^2 = \vec{p}^2 + m^2 \tag{2.43}$$

But we recognize this as the relativistic dispersion relation for a particle of mass m and 3-momentum $\vec{p}$,

$$E_{\vec{p}}^2 = \vec{p}^2 + m^2 \tag{2.44}$$

---

[2]The number 24 that appears in the denominator of the one-dimensional Casimir force plays a more famous role in string theory: the same calculation in that context is the reason the bosonic string lives in $26 = 24 + 2$ spacetime dimensions. (The $+2$ comes from the fact the string itself is extended in one space and one time dimension). You will need to attend next term's "String Theory" course to see what on earth this has to do with the Casimir force.

We interpret the state $|\vec{p}\rangle$ as the momentum eigenstate of a single particle of mass m. To stress this, from now on we'll write $E_{\vec{p}}$ everywhere instead of $\omega_{\vec{p}}$. Let's check this particle interpretation by studying the other quantum numbers of $|\vec{p}\rangle$. We may take the classical total momentum $\vec{P}$ given in (1.46) and turn it into an operator. After normal ordering, it becomes

$$\vec{P} = -\int d^3x \, \pi \vec{\nabla}\phi = \int \frac{d^3p}{(2\pi)^3} \, \vec{p} \, a_{\vec{p}}^\dagger a_{\vec{p}} \tag{2.45}$$

Acting on our state $|\vec{p}\rangle$ with $\vec{P}$, we learn that it is indeed an eigenstate,

$$\vec{P}|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle \tag{2.46}$$

telling us that the state $|\vec{p}\rangle$ has momentum $\vec{p}$. Another property of $|\vec{p}\rangle$ that we can study is its angular momentum. Once again, we may take the classical expression for the total angular momentum of the field (1.55) and turn it into an operator,

$$J^i = \epsilon^{ijk} \int d^3x \, (\mathcal{J}^0)^{jk} \tag{2.47}$$

It's not hard to show that acting on the one-particle state with zero momentum, $J^i \, |\vec{p} = 0\rangle = 0$, which we interpret as telling us that the particle carries no internal angular momentum. In other words, quantizing a scalar field gives rise to a spin 0 particle.

Multi-Particle States, Bosonic Statistics and Fock Space

We can create multi-particle states by acting multiple times with $a^\dagger$'s. We interpret the state in which n $a^\dagger$'s act on the vacuum as an n-particle state,

$$|\vec{p}_1, \ldots, \vec{p}_n\rangle = a_{\vec{p}_1}^\dagger \ldots a_{\vec{p}_n}^\dagger \, |0\rangle \tag{2.48}$$

Because all the $a^\dagger$'s commute among themselves, the state is symmetric under exchange of any two particles. For example,

$$|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle \tag{2.49}$$

This means that the particles are *bosons*.

The full Hilbert space of our theory is spanned by acting on the vacuum with all possible combinations of $a^\dagger$'s,

$$|0\rangle \;,\; a_{\vec{p}}^\dagger|0\rangle \;,\; a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger|0\rangle \;,\; a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger a_{\vec{r}}^\dagger|0\rangle \ldots \tag{2.50}$$

This space is known as a *Fock space*. The Fock space is simply the sum of the n-particle Hilbert spaces, for all $n \geq 0$. There is a useful operator which counts the number of particles in a given state in the Fock space. It is called the *number operator N*

$$N = \int \frac{d^3 p}{(2\pi)^3} \, a_{\vec{p}}^\dagger a_{\vec{p}} \tag{2.51}$$

and satisfies $N \left| \vec{p}_1, \ldots, \vec{p}_n \right\rangle = n \left| \vec{p}_1, \ldots, \vec{p}_n \right\rangle$. The number operator commutes with the Hamiltonian, $[N, H] = 0$, ensuring that particle number is conserved. This means that we can place ourselves in the n-particle sector, and stay there. This is a property of free theories, but will no longer be true when we consider interactions: interactions create and destroy particles, taking us between the different sectors in the Fock space.

Operator Valued Distributions

Although we're referring to the states $|\vec{p}\rangle$ as "particles", they're not localized in space in any way — they are momentum eigenstates. Recall that in quantum mechanics the position and momentum eigenstates are not good elements of the Hilbert space since they are not normalizable (they normalize to delta-functions). Similarly, in quantum field theory neither the operators $\phi(\vec{x})$, nor $a_{\vec{p}}$ are good operators acting on the Fock space. This is because they don't produce normalizable states. For example,

$$\langle 0 | \, a_{\vec{p}} a_{\vec{p}}^\dagger \, | 0 \rangle = \langle \vec{p} | \vec{p} \rangle = (2\pi)^3 \delta(0) \quad \text{and} \quad \langle 0 | \phi(\vec{x}) \, \phi(\vec{x}) \, | 0 \rangle = \langle \vec{x} | \vec{x} \rangle = \delta(0) \tag{2.52}$$

They are operator valued distributions, rather than functions. This means that although $\phi(\vec{x})$ has a well defined vacuum expectation value, $\langle 0 | \, \phi(\vec{x}) \, | 0 \rangle = 0$, the fluctuations of the operator at a fixed point are infinite, $\langle 0 | \, \phi(\vec{x}) \phi(\vec{x}) \, | 0 \rangle = \infty$. We can construct well defined operators by smearing these distributions over space. For example, we can create a wavepacket

$$|\varphi\rangle = \int \frac{d^3 p}{(2\pi)^3} \, e^{-i \vec{p} \cdot \vec{x}} \, \varphi(\vec{p}) \, |\vec{p}\rangle \tag{2.53}$$

which is partially localized in both position and momentum space. (A typical state might be described by the Gaussian $\varphi(\vec{p}) = \exp(-\vec{p}^{\,2}/2m^2)$).

2.4.1 Relativistic Normalization

We have defined the vacuum $|0\rangle$ which we normalize as $\langle 0 | 0 \rangle = 1$. The one-particle states $|\vec{p}\rangle = a_{\vec{p}}^\dagger |0\rangle$ then satisfy

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q}) \tag{2.54}$$

But is this Lorentz invariant? It's not obvious because we only have 3-vectors. What could go wrong? Suppose we have a Lorentz transformation

$$p^\mu \to (p')^\mu = \Lambda^\mu{}_\nu p^\nu \tag{2.55}$$

such that the 3-vector transforms as $\vec{p} \to \vec{p}'$. In the quantum theory, it would be preferable if the two states are related by a unitary transformation,

$$|\vec{p}\rangle \to |\vec{p}'\rangle = U(\Lambda) |\vec{p}\rangle \tag{2.56}$$

This would mean that the normalizations of $|\vec{p}\rangle$ and $|\vec{p}'\rangle$ are the same whenever $\vec{p}$ and $\vec{p}'$ are related by a Lorentz transformation. But we haven't been at all careful with normalizations. In general, we could get

$$|\vec{p}\rangle \to \lambda(\vec{p}, \vec{p}') \, |\vec{p}'\rangle \tag{2.57}$$

for some unknown function $\lambda(\vec{p}, \vec{p}')$. How do we figure this out? The trick is to look at an object which we know is Lorentz invariant. One such object is the identity operator on one-particle states (which is really the projection operator onto one-particle states). With the normalization (2.54) we know this is given by

$$1 = \int \frac{d^3p}{(2\pi)^3} \, |\vec{p}\rangle \langle \vec{p}| \tag{2.58}$$

This operator is Lorentz invariant, but it consists of two terms: the measure $\int d^3p$ and the projector $|\vec{p}\rangle\langle\vec{p}|$. Are these individually Lorentz invariant? In fact the answer is no.

Claim The Lorentz invariant measure is,

$$\int \frac{d^3p}{2E_{\vec{p}}} \tag{2.59}$$

Proof: $\int d^4p$ is obviously Lorentz invariant. And the relativistic dispersion relation for a massive particle,

$$p_\mu p^\mu = m^2 \quad \Rightarrow \quad p_0^2 = E_{\vec{p}}^2 = \vec{p}^2 + m^2 \tag{2.60}$$

is also Lorentz invariant. Solving for p_0, there are two branches of solutions: $p_0 = \pm E_{\vec{p}}$. But the choice of branch is another Lorentz invariant concept. So piecing everything together, the following combination must be Lorentz invariant,

$$\int d^4p \; \delta(p_0^2 - \vec{p}^2 - m^2)\Big|_{p_0>0} = \int \frac{d^3p}{2p_0}\Big|_{p_0=E_{\vec{p}}} \tag{2.61}$$

which completes the proof. $\qquad\qquad\square$

From this result we can figure out everything else. For example, the Lorentz invariant δ-function for 3-vectors is

$$2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{2.62}$$

which follows because

$$\int \frac{d^3p}{2E_{\vec{p}}}\, 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) = 1 \tag{2.63}$$

So finally we learn that the relativistically normalized momentum states are given by

$$|p\rangle = \sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\,a_{\vec{p}}^\dagger|0\rangle \tag{2.64}$$

Notice that our notation is rather subtle: the relativistically normalized momentum state $|p\rangle$ differs from $|\vec{p}\rangle$ by the factor $\sqrt{2E_{\vec{p}}}$. These states now satisfy

$$\langle p|\,q\rangle = (2\pi)^3\, 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{2.65}$$

Finally, we can rewrite the identity on one-particle states as

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,|p\rangle\,\langle p| \tag{2.66}$$

Some texts also define relativistically normalized creation operators by $a^\dagger(p) = \sqrt{2E_{\vec{p}}}\,a_{\vec{p}}^\dagger$. We won't make use of this notation here.

2.5 Complex Scalar Fields

Consider a complex scalar field $\psi(x)$ with Lagrangian

$$\mathcal{L} = \partial_\mu\psi^\star\,\partial^\mu\psi - M^2\psi^\star\psi \tag{2.67}$$

Notice that, in contrast to the Lagrangian (1.7) for a real scalar field, there is no factor of $1/2$ in front of the Lagrangian for a complex scalar field. If we write ψ in terms of real scalar fields by $\psi = (\phi_1 + i\phi_2)/\sqrt{2}$, we get the factor of $1/2$ coming from the $1/\sqrt{2}$'s. The equations of motion are

$$\partial_\mu\partial^\mu\,\psi + M^2\psi = 0$$
$$\partial_\mu\partial^\mu\psi^\star + M^2\psi^\star = 0 \tag{2.68}$$

where the second equation is the complex conjugate of the first. We expand the complex field operator as a sum of plane waves as

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(b_{\vec{p}}\,e^{+i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^\dagger\,e^{-i\vec{p}\cdot\vec{x}}\right)$$
$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(b_{\vec{p}}^\dagger\,e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\,e^{+i\vec{p}\cdot\vec{x}}\right) \tag{2.69}$$

Since the classical field ψ is not real, the corresponding quantum field ψ is not hermitian. This is the reason that we have different operators b and $c^\dagger$ appearing in the positive and negative frequency parts. The classical field momentum is $\pi = \partial\mathcal{L}/\partial\dot\psi = \dot\psi^\star$. We also turn this into a quantum operator field which we write as,

$$\pi = \int \frac{d^3p}{(2\pi)^3}\, i\, \sqrt{\frac{E_{\vec{p}}}{2}}\left(b_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} - c_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}}\right)$$

$$\pi^\dagger = \int \frac{d^3p}{(2\pi)^3}\, (-i)\, \sqrt{\frac{E_{\vec{p}}}{2}}\left(b_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} - c_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right) \tag{2.70}$$

The commutation relations between fields and momenta are given by

$$[\psi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \quad \text{and} \quad [\psi(\vec{x}), \pi^\dagger(\vec{y})] = 0 \tag{2.71}$$

together with others related by complex conjugation, as well as the usual $[\psi(\vec{x}), \psi(\vec{y})] = [\psi(\vec{x}), \psi^\dagger(\vec{y})] = 0$, etc. One can easily check that these field commutation relations are equivalent to the commutation relations for the operators $b_{\vec{p}}$ and $c_{\vec{p}}$,

$$[b_{\vec{p}}, b_{\vec{q}}^\dagger] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})$$

$$[c_{\vec{p}}, c_{\vec{q}}^\dagger] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}) \tag{2.72}$$

and

$$[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}^\dagger] = 0 \tag{2.73}$$

In summary, quantizing a complex scalar field gives rise to two creation operators, $b_{\vec{p}}^\dagger$ and $c_{\vec{p}}^\dagger$. These have the interpretation of creating two types of particle, both of mass M and both spin zero. They are interpreted as particles and anti-particles. In contrast, for a real scalar field there is only a single type of particle: for a real scalar field, the particle is its own antiparticle.

Recall that the theory (2.67) has a classical conserved charge

$$Q = i \int d^3x\, (\dot\psi^\star\psi - \psi^\star\dot\psi) = i \int d^3x\, (\pi\psi - \psi^\star\pi^\star) \tag{2.74}$$

After normal ordering, this becomes the quantum operator

$$Q = \int \frac{d^3p}{(2\pi)^3}\, (c_{\vec{p}}^\dagger c_{\vec{p}} - b_{\vec{p}}^\dagger b_{\vec{p}}) = N_c - N_b \tag{2.75}$$

so Q counts the number of anti-particles (created by $c^\dagger$) minus the number of particles (created by $b^\dagger$). We have $[H, Q] = 0$, ensuring that Q is conserved quantity in the quantum theory. Of course, in our free field theory this isn't such a big deal because both N_c and N_b are separately conserved. However, we'll soon see that in interacting theories Q survives as a conserved quantity, while N_c and N_b individually do not.

2.6 The Heisenberg Picture

Although we started with a Lorentz invariant Lagrangian, we slowly butchered it as we quantized, introducing a preferred time coordinate t. It's not at all obvious that the theory is still Lorentz invariant after quantization. For example, the operators $\phi(\vec{x})$ depend on space, but not on time. Meanwhile, the one-particle states evolve in time by Schrödinger's equation,

$$i\frac{d\,|\vec{p}(t)\rangle}{dt} = H\,|\vec{p}(t)\rangle \quad \Rightarrow \quad |\vec{p}(t)\rangle = e^{-iE_{\vec{p}}t}\,|\vec{p}\rangle \tag{2.76}$$

Things start to look better in the Heisenberg picture where time dependence is assigned to the operators $\mathcal{O}$,

$$\mathcal{O}_H = e^{iHt}\,\mathcal{O}_S\,e^{-iHt} \tag{2.77}$$

so that

$$\frac{d\mathcal{O}_H}{dt} = i[H, \mathcal{O}_H] \tag{2.78}$$

where the subscripts S and H tell us whether the operator is in the Schrödinger or Heisenberg picture. In field theory, we drop these subscripts and we will denote the picture by specifying whether the fields depend on space $\phi(\vec{x})$ (the Schrödinger picture) or spacetime $\phi(\vec{x}, t) = \phi(x)$ (the Heisenberg picture).

The operators in the two pictures agree at a fixed time, say, $t = 0$. The commutation relations (2.2) become equal time commutation relations in the Heisenberg picture,

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0$$
$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{2.79}$$

Now that the operator $\phi(x) = \phi(\vec{x}, t)$ depends on time, we can start to study how it evolves. For example, we have

$$\dot{\phi} = i[H, \phi] = \frac{i}{2}[\int d^3y\;\pi(y)^2 + \nabla\phi(y)^2 + m^2\phi(y)^2\;,\;\phi(x)]$$
$$= i\int d^3y\;\pi(y)\,(-i)\,\delta^{(3)}(\vec{y} - \vec{x}) = \pi(x) \tag{2.80}$$

Meanwhile, the equation of motion for π reads,

$$\dot{\pi} = i[H, \pi] = \frac{i}{2}\,[\int d^3y\;\pi(y)^2 + \nabla\phi(y)^2 + m^2\phi(y)^2\;,\;\pi(x)]$$

$$= \frac{i}{2} \int d^3y \ \left(\nabla_y[\phi(y), \pi(x)] \right) \nabla \phi(y) + \nabla \phi(y) \, \nabla_y[\phi(y), \pi(x)]$$

$$+ 2im^2 \phi(y) \, \delta^{(3)}(\vec{x} - \vec{y})$$

$$= - \left(\int d^3y \ \left(\nabla_y \, \delta^{(3)}(\vec{x} - \vec{y}) \right) \nabla_y \phi(y) \right) - m^2 \phi(x)$$

$$= \nabla^2 \phi - m^2 \phi \tag{2.81}$$

where we've included the subscript y on ∇_y when there may be some confusion about which argument the derivative is acting on. To reach the last line, we've simply integrated by parts. Putting (2.80) and (2.81) together we find that the field operator ϕ satisfies the Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{2.82}$$

Things are beginning to look more relativistic. We can write the Fourier expansion of $\phi(x)$ by using the definition (2.77) and noting,

$$e^{iHt} \, a_{\vec{p}} e^{-iHt} = e^{-iE_{\vec{p}} t} \, a_{\vec{p}} \quad \text{and} \quad e^{iHt} \, a_{\vec{p}}^\dagger e^{-iHt} = e^{+iE_{\vec{p}} t} \, a_{\vec{p}}^\dagger \tag{2.83}$$

which follows from the commutation relations $[H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}}$ and $[H, a_{\vec{p}}^\dagger] = +E_{\vec{p}} a_{\vec{p}}^\dagger$. This then gives,

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{+ip \cdot x} \right) \tag{2.84}$$

which looks very similar to the previous expansion (2.18) except that the exponent is now written in terms of 4-vectors, $p \cdot x = E_{\vec{p}} t - \vec{p} \cdot \vec{x}$. (Note also that a sign has flipped in the exponent due to our Minkowski metric contraction). It's simple to check that (2.84) indeed satisfies the Klein-Gordon equation (2.82).

Figure 4:

2.6.1 Causality

We're approaching something Lorentz invariant in the Heisenberg picture, where $\phi(x)$ now satisfies the Klein-Gordon equation. But there's still a hint of non-Lorentz invariance because ϕ and π satisfy *equal time* commutation relations,

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{2.85}$$
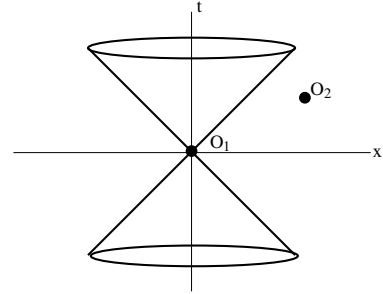
But what about arbitrary spacetime separations? In particular, for our theory to be *causal*, we must require that all spacelike separated operators commute,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \forall \ (x-y)^2 < 0 \tag{2.86}$$

This ensures that a measurement at x cannot affect a measurement at y when x and y are not causally connected. Does our theory satisfy this crucial property? Let's define

$$\Delta(x-y) = [\phi(x), \phi(y)] \tag{2.87}$$

The objects on the right-hand side of this expression are operators. However, it's easy to check by direct substitution that the left-hand side is simply a c-number function with the integral expression

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \tag{2.88}$$

What do we know about this function?

- It's Lorentz invariant, thanks to the appearance of the Lorentz invariant measure $\int d^3p/2E_{\vec{p}}$ that we introduced in (2.59).

- It doesn't vanish for timelike separation. For example, taking $x - y = (t, 0, 0, 0)$ gives $[\phi(\vec{x}, 0), \phi(\vec{x}, t)] \sim e^{-imt} - e^{+imt}$.

- It vanishes for space-like separations. This follows by noting that $\Delta(x-y) = 0$ at equal times for all $(x-y)^2 = -(\vec{x} - \vec{y})^2 < 0$, which we can see explicitly by writing

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = \tfrac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2 + m^2}} \left(e^{i\vec{p}\cdot(\vec{x} - \vec{y})} - e^{-i\vec{p}\cdot(\vec{x} - \vec{y})} \right) \tag{2.89}$$

  and noticing that we can flip the sign of $\vec{p}$ in the last exponent as it is an integration variable. But since $\Delta(x-y)$ Lorentz invariant, it can only depend on $(x-y)^2$ and must therefore vanish for all $(x-y)^2 < 0$.

We therefore learn that our theory is indeed causal with commutators vanishing outside the lightcone. This property will continue to hold in interacting theories; indeed, it is usually given as one of the axioms of local quantum field theories. I should mention however that the fact that $[\phi(x), \phi(y)]$ is a c-number function, rather than an operator, is a property of free fields only.

2.7 Propagators

We could ask a different question to probe the causal structure of the theory. Prepare a particle at spacetime point y. What is the amplitude to find it at point x? We can calculate this:

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{p}'}}} \langle 0| a_{\vec{p}} a_{\vec{p}'}^\dagger |0\rangle\, e^{-ip\cdot x + ip'\cdot y}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \equiv D(x-y) \tag{2.90}$$

The function $D(x-y)$ is called the *propagator*. For spacelike separations, $(x-y)^2 < 0$, one can show that $D(x-y)$ decays like

$$D(x-y) \sim e^{-m|\vec{x}-\vec{y}|} \tag{2.91}$$

So it decays exponentially quickly outside the lightcone but, nonetheless, is non-vanishing! The quantum field appears to leak out of the lightcone. Yet we've just seen that space-like measurements commute and the theory is causal. How do we reconcile these two facts? We can rewrite the calculation (2.89) as

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0 \ \ \text{if} \ \ (x-y)^2 < 0 \tag{2.92}$$

There are words you can drape around this calculation. When $(x-y)^2 < 0$, there is no Lorentz invariant way to order events. If a particle can travel in a spacelike direction from $x \to y$, it can just as easily travel from $y \to x$. In any measurement, the amplitudes for these two events cancel.

With a complex scalar field, it is more interesting. We can look at the equation $[\psi(x), \psi^\dagger(y)] = 0$ outside the lightcone. The interpretation now is that the amplitude for the particle to propagate from $x \to y$ cancels the amplitude for the *antiparticle* to travel from $y \to x$. In fact, this interpretation is also there for a real scalar field because the particle is its own antiparticle.

2.7.1 The Feynman Propagator

As we will see shortly, one of the most important quantities in interacting field theory is the *Feynman propagator*,

$$\Delta_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases} \tag{2.93}$$

where T stands for time ordering, placing all operators evaluated at later times to the left so,

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases} \tag{2.94}$$

Claim: There is a useful way of writing the Feynman propagator in terms of a 4-momentum integral.

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} \tag{2.95}$$

Notice that this is the first time in this course that we've integrated over 4-momentum. Until now, we integrated only over 3-momentum, with p^0 fixed by the mass-shell condition to be $p^0 = E_{\vec{p}}$. In the expression (2.95) for Δ_F, we have no such condition on p^0. However, as it stands this integral is ill-defined because, for each value of $\vec{p}$, the denominator $p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2$ produces a pole when $p^0 = \pm E_{\vec{p}} = \pm\sqrt{\vec{p}^2 + m^2}$. We need a prescription for avoiding these singularities in the p_0 integral. To get the Feynman propagator, we must choose the contour to be
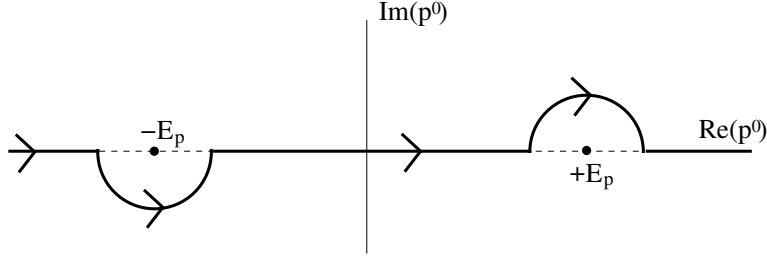
Figure 5: The contour for the Feynman propagator.

Proof:

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2} = \frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})} \tag{2.96}$$

so the residue of the pole at $p^0 = \pm E_{\vec{p}}$ is $\pm 1/2E_{\vec{p}}$. When $x^0 > y^0$, we close the contour in the lower half plane, where $p^0 \to -i\infty$, ensuring that the integrand vanishes since $e^{-ip^0(x^0-y^0)} \to 0$. The integral over p^0 then picks up the residue at $p^0 = +E_{\vec{p}}$ which is $-2\pi i/2E_{\vec{p}}$ where the minus sign arises because we took a clockwise contour. Hence when $x^0 > y^0$ we have

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4} \frac{-2\pi i}{2E_{\vec{p}}} \, i \, e^{-iE_{\vec{p}}(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} = D(x-y) \qquad (2.97)$$

which is indeed the Feynman propagator for $x^0 > y^0$. In contrast, when $y^0 > x^0$, we close the contour in an anti-clockwise direction in the upper half plane to get,

$$
\begin{aligned}
\Delta_F(x-y) &= \int \frac{d^3p}{(2\pi)^4} \frac{2\pi i}{(-2E_{\vec{p}})} \, i \, e^{+iE_{\vec{p}}(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(y^0-x^0)-i\vec{p}\cdot(\vec{y}-\vec{x})} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(y-x)} = D(y-x) \qquad (2.98)
\end{aligned}
$$

where to go to from the second line to the third, we have flipped the sign of $\vec{p}$ which is valid since we integrate over d^3p and all other quantities depend only on $\vec{p}^2$. Once again we reproduce the Feynman propagator. $\qquad\square$

Instead of specifying the contour, it is standard to write the Feynman propagator as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \qquad (2.99)$$

with $\epsilon > 0$, and infinitesimal. This has the effect of shifting the poles slightly off the real axis, so the integral along the real p^0 axis is equivalent to the contour shown in Figure 5. This way of writing the propagator is, for obvious reasons, called the "$i\epsilon$ prescription".
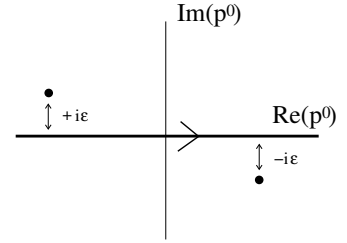
Figure 6:

2.7.2 Green's Functions

There is another avatar of the propagator: it is a Green's function for the Klein-Gordon operator. If we stay away from the singularities, we have

$$
\begin{aligned}
(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \left(-p^2 + m^2\right) e^{-ip\cdot(x-y)} \\
&= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \\
&= -i\,\delta^{(4)}(x-y) \qquad (2.100)
\end{aligned}
$$

Note that we didn't make use of the contour anywhere in this derivation. For some purposes it is also useful to pick other contours which also give rise to Green's functions.
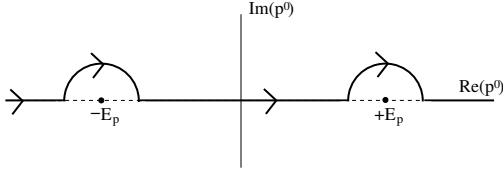
Figure 7: The retarded contour   **Figure 8:** The advanced contour
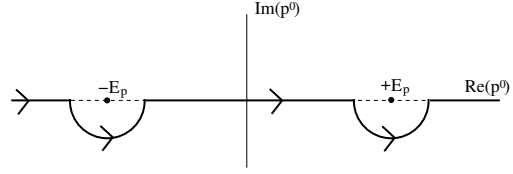
For example, the *retarded* Green's function $\Delta_R(x - y)$ is defined by the contour shown in Figure 7 which has the property

$$\Delta_R(x - y) = \begin{cases} D(x - y) - D(y - x) & x^0 > y^0 \\ 0 & y^0 > x^0 \end{cases} \tag{2.101}$$

The retarded Green's function is useful in classical field theory if we know the initial value of some field configuration and want to figure out what it evolves into in the presence of a source, meaning that we want to know the solution to the inhomogeneous Klein-Gordon equation,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = J(x) \tag{2.102}$$

for some fixed background function $J(x)$. Similarly, one can define the *advanced* Green's function $\Delta_A(x - y)$ which vanishes when $y^0 < x^0$, which is useful if we know the end point of a field configuration and want to figure out where it came from. Given that next term's course is called "Advanced Quantum Field Theory", there is an obvious name for the current course. But it got shot down in the staff meeting. In the quantum theory, we will see that the Feynman Green's function is most relevant.

2.8 Non-Relativistic Fields

Let's return to our classical complex scalar field obeying the Klein-Gordon equation. We'll decompose the field as

$$\psi(\vec{x}, t) = e^{-imt} \tilde{\psi}(\vec{x}, t) \tag{2.103}$$

Then the KG-equation reads

$$\partial_t^2 \psi - \nabla^2 \psi + m^2 \psi = e^{-imt} \left[\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2 \tilde{\psi} \right] = 0 \tag{2.104}$$

with the m^2 term cancelled by the time derivatives. The non-relativistic limit of a particle is $|\vec{p}| \ll m$. Let's look at what this does to our field. After a Fourier transform,

this is equivalent to saying that $|\ddot{\tilde{\psi}}| \ll m|\dot{\tilde{\psi}}|$. In this limit, we drop the term with two time derivatives and the KG equation becomes,

$$i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2m}\nabla^2\tilde{\psi} \tag{2.105}$$

This looks very similar to the Schrödinger equation for a non-relativistic free particle of mass m. Except it doesn't have any probability interpretation — it's simply a classical field evolving through an equation that's first order in time derivatives.

We wrote down a Lagrangian in section 1.1.2 which gives rise to field equations which are first order in time derivatives. In fact, we can derive this from the relativistic Lagrangian for a scalar field by again taking the limit $\partial_t\psi \ll m\psi$. After losing the tilde, so $\tilde{\psi} \to \psi$, the non-relativistic Lagrangian becomes

$$\mathcal{L} = +i\psi^\star\dot{\psi} - \frac{1}{2m}\nabla\psi^\star\nabla\psi \tag{2.106}$$

where we've divided by $1/2m$. This Lagrangian has a conserved current arising from the internal symmetry $\psi \to e^{i\alpha}\psi$. The current has time and space components

$$j^\mu = \left(-\psi^\star\psi, \frac{i}{2m}(\psi^\star\nabla\psi - \psi\nabla\psi^\star)\right) \tag{2.107}$$

To move to the Hamiltonian formalism we compute the momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\star \tag{2.108}$$

This means that the momentum conjugate to ψ is $i\psi^\star$. The momentum does not depend on time derivatives at all! This looks a little disconcerting but it's fully consistent for a theory which is first order in time derivatives. In order to determine the full trajectory of the field, we need only specify ψ and $\psi^\star$ at time $t = 0$: no time derivatives on the initial slice are required.

Since the Lagrangian already contains a "$p\dot{q}$" term (instead of the more familiar $\frac{1}{2}p\dot{q}$ term), the time derivatives drop out when we compute the Hamiltonian. We get,

$$\mathcal{H} = \frac{1}{2m}\nabla\psi^\star\nabla\psi \tag{2.109}$$

To quantize we impose (in the Schrödinger picture) the canonical commutation relations

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})] = 0$$
$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}) \tag{2.110}$$

We may expand $\psi(\vec{x})$ as a Fourier transform

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \tag{2.111}$$

where the commutation relations (2.110) require

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{2.112}$$

The vacuum satisfies $a_{\vec{p}}|0\rangle = 0$, and the excitations are $a_{\vec{p}_1}^\dagger \ldots a_{\vec{p}_n}^\dagger |0\rangle$. The one-particle states have energy

$$H|\vec{p}\rangle = \frac{\vec{p}^2}{2m}|\vec{p}\rangle \tag{2.113}$$

which is the non-relativistic dispersion relation. We conclude that quantizing the first order Lagrangian (2.106) gives rise to non-relativistic particles of mass m. Some comments:

- We have a complex field but only a single type of particle. The anti-particle is not in the spectrum. The existence of anti-particles is a consequence of relativity.

- A related fact is that the conserved charge $Q = \int d^3x : \psi^\dagger\psi :$ is the particle number. This remains conserved even if we include interactions in the Lagrangian of the form

$$\Delta\mathcal{L} = V(\psi^\star\psi) \tag{2.114}$$

  So in non-relativistic theories, particle number is conserved. It is only with relativity, and the appearance of anti-particles, that particle number can change.

- There is no non-relativistic limit of a real scalar field. In the relativistic theory, the particles are their own anti-particles, and there can be no way to construct a multi-particle theory that conserves particle number.

2.8.1 Recovering Quantum Mechanics

In quantum mechanics, we talk about the position and momentum operators $\vec{X}$ and $\vec{P}$. In quantum field theory, position is relegated to a label. How do we get back to quantum mechanics? We already have the operator for the total momentum of the field

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a_{\vec{p}}^\dagger a_{\vec{p}} \tag{2.115}$$

which, on one-particle states, gives $\vec{P} |\vec{p}\rangle = \vec{p} |\vec{p}\rangle$. It's also easy to construct the position operator. Let's work in the non-relativistic limit. Then the operator

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \, a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \tag{2.116}$$

creates a particle with δ-function localization at $\vec{x}$. We write $|\vec{x}\rangle = \psi^\dagger(\vec{x}) |0\rangle$. A natural position operator is then

$$\vec{X} = \int d^3x \, \vec{x} \, \psi^\dagger(\vec{x}) \, \psi(\vec{x}) \tag{2.117}$$

so that $\vec{X} |\vec{x}\rangle = \vec{x} |\vec{x}\rangle$.

Let's now construct a state $|\varphi\rangle$ by taking superpositions of one-particle states $|\vec{x}\rangle$,

$$|\varphi\rangle = \int d^3x \, \varphi(\vec{x}) |\vec{x}\rangle \tag{2.118}$$

The function $\varphi(\vec{x})$ is what we would usually call the Schrödinger wavefunction (in the position representation). Let's make sure that it indeed satisfies all the right properties. Firstly, it's clear that acting with the position operator $\vec{X}$ has the right action of $\varphi(\vec{x})$,

$$X^i |\varphi\rangle = \int d^3x \, x^i \, \varphi(\vec{x}) |\vec{x}\rangle \tag{2.119}$$

but what about the momentum operator $\vec{P}$? We will now show that

$$P^i |\varphi\rangle = \int d^3x \, \left(-i \frac{\partial \varphi}{\partial x^i} \right) |\vec{x}\rangle \tag{2.120}$$

which tells us that P^i acts as the familiar derivative on wavefunctions $|\varphi\rangle$. To see that this is the case, we write

$$
\begin{aligned}
P^i |\varphi\rangle &= \int \frac{d^3x \, d^3p}{(2\pi)^3} \, p^i a_{\vec{p}}^\dagger a_{\vec{p}} \, \varphi(\vec{x}) \, \psi^\dagger(\vec{x}) \, |0\rangle \\
&= \int \frac{d^3x \, d^3p}{(2\pi)^3} \, p^i a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \, \varphi(\vec{x}) \, |0\rangle
\end{aligned} \tag{2.121}
$$

where we've used the relationship $[a_{\vec{p}}, \psi^\dagger(\vec{x})] = e^{-i\vec{p}\cdot\vec{x}}$ which can be easily checked. Proceeding with our calculation, we have

$$
\begin{aligned}
P^i |\varphi\rangle &= \int \frac{d^3x \, d^3p}{(2\pi)^3} \, a_{\vec{p}}^\dagger \left(i \frac{\partial}{\partial x^i} e^{-i\vec{p}\cdot\vec{x}} \right) \varphi(\vec{x}) \, |0\rangle \\
&= \int \frac{d^3x \, d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \left(-i \frac{\partial \varphi}{\partial x^i} \right) a_{\vec{p}}^\dagger |0\rangle \\
&= \int d^3x \, \left(-i \frac{\partial \varphi}{\partial x^i} \right) |\vec{x}\rangle
\end{aligned} \tag{2.122}
$$

which confirms (2.120). So we learn that when acting on one-particle states, the operators $\vec{X}$ and $\vec{P}$ act as position and momentum operators in quantum mechanics, with $[X^i, P^j]\,|\varphi\rangle = i\delta^{ij}\,|\varphi\rangle$. But what about dynamics? How does the wavefunction $\varphi(\vec{x}, t)$ change in time? The Hamiltonian (2.109) can be rewritten as

$$H = \int d^3x \, \frac{1}{2m} \, \nabla\psi^\star \, \nabla\psi = \int \frac{d^3p}{(2\pi)^3} \, \frac{\vec{p}^2}{2m} \, a_{\vec{p}}^\dagger a_{\vec{p}} \qquad (2.123)$$

so we find that

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m} \, \nabla^2\varphi \qquad (2.124)$$

But this is the same equation obeyed by the original field (2.105)! Except this time, it really is the Schrödinger equation, complete with the usual probabilistic interpretation for the wavefunction φ. Note in particular that the conserved charge arising from the Noether current (2.107) is $Q = \int d^3x \, |\varphi(\vec{x})|^2$ which is the total probability.

Historically, the fact that the equation for the classical field (2.105) and the one-particle wavefunction (2.124) coincide caused some confusion. It was thought that perhaps we are quantizing the wavefunction itself and the resulting name "second quantization" is still sometimes used today to mean quantum field theory. It's important to stress that, despite the name, we're not quantizing anything twice! We simply quantize a classical field once. Nonetheless, in practice it's useful to know that if we treat the one-particle Schrödinger equation as the equation for a quantum field then it will give the correct generalization to multi-particle states.

Interactions

Often in quantum mechanics, we're interested in particles moving in some fixed background potential $V(\vec{x})$. This can be easily incorporated into field theory by working with a Lagrangian with explicit $\vec{x}$ dependence,

$$\mathcal{L} = i\psi^\star \dot{\psi} - \frac{1}{2m} \, \nabla\psi^\star \, \nabla\psi - V(\vec{x}) \, \psi^\star \psi \qquad (2.125)$$

Note that this Lagrangian doesn't respect translational symmetry and we won't have the associated energy-momentum tensor. While such Lagrangians are useful in condensed matter physics, we rarely (or never) come across them in high-energy physics, where all equations obey translational (and Lorentz) invariance.

One can also consider interactions *between* particles. Obviously these are only important for n particle states with $n \geq 2$. We therefore expect them to arise from additions to the Lagrangian of the form

$$\Delta\mathcal{L} = \psi^\star(\vec{x})\,\psi^\star(\vec{x})\,\psi(\vec{x})\,\psi(\vec{x}) \qquad (2.126)$$

which, in the quantum theory, is an operator which destroys two particles before creating two new ones. Such terms in the Lagrangian will indeed lead to inter-particle forces, both in the non-relativistic and relativistic setting. In the next section we explore these types of interaction in detail for relativistic theories.

3. Interacting Fields

The free field theories that we've discussed so far are very special: we can determine their spectrum, but nothing interesting then happens. They have particle excitations, but these particles don't interact with each other.

Here we'll start to examine more complicated theories that include interaction terms. These will take the form of higher order terms in the Lagrangian. We'll start by asking what kind of *small* perturbations we can add to the theory. For example, consider the Lagrangian for a real scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\,\phi^n \tag{3.1}$$

The coefficients λ_n are called *coupling constants*. What restrictions do we have on λ_n to ensure that the additional terms are small perturbations? You might think that we need simply make "$\lambda_n \ll 1$". But this isn't quite right. To see why this is the case, let's do some dimensional analysis. Firstly, note that the action has dimensions of angular momentum or, equivalently, the same dimensions as $\hbar$. Since we've set $\hbar = 1$, using the convention described in the introduction, we have $[S] = 0$. With $S = \int d^4x\,\mathcal{L}$, and $[d^4x] = -4$, the Lagrangian density must therefore have

$$[\mathcal{L}] = 4 \tag{3.2}$$

What does this mean for the Lagrangian (3.1)? Since $[\partial_\mu] = 1$, we can read off the mass dimensions of all the factors to find,

$$[\phi] = 1 \quad , \quad [m] = 1 \quad , \quad [\lambda_n] = 4 - n \tag{3.3}$$

So now we see why we can't simply say we need $\lambda_n \ll 1$, because this statement only makes sense for dimensionless quantities. The various terms, parameterized by λ_n, fall into three different categories

- $[\lambda_3] = 1$: For this term, the dimensionless parameter is λ_3/E, where E has dimensions of mass. Typically in quantum field theory, E is the energy scale of the process of interest. This means that $\lambda_3\,\phi^3/3!$ is a small perturbation at high energies $E \gg \lambda_3$, but a large perturbation at low energies $E \ll \lambda_3$. Terms that we add to the Lagrangian with this behavior are called *relevant* because they're most relevant at low energies (which, after all, is where most of the physics we see lies). In a relativistic theory, $E > m$, so we can always make this perturbation small by taking $\lambda_3 \ll m$.

- $[\lambda_4] = 0$: this term is small if $\lambda_4 \ll 1$. Such perturbations are called *marginal*.

- $[\lambda_n] < 0$ for $n \geq 5$: The dimensionless parameter is $(\lambda_n E^{n-4})$, which is small at low-energies and large at high energies. Such perturbations are called *irrelevant*.

As you'll see later, it is typically impossible to avoid high energy processes in quantum field theory. (We've already seen a glimpse of this in computing the vacuum energy). This means that we might expect problems with irrelevant operators. Indeed, these lead to "non-renormalizable" field theories in which one cannot make sense of the infinities at arbitrarily high energies. This doesn't necessarily mean that the theory is useless; just that it is incomplete at some energy scale.

Let me note however that the naive assignment of relevant, marginal and irrelevant is not always fixed in stone: quantum corrections can sometimes change the character of an operator.

An Important Aside: Why QFT is Simple

Typically in a quantum field theory, only the relevant and marginal couplings are important. This is basically because, as we've seen above, the irrelevant couplings become small at low-energies. This is a huge help: of the infinite number of interaction terms that we could write down, only a handful are actually needed (just two in the case of the real scalar field described above).

Let's look at this a little more. Suppose that we some day discover the true superduper "theory of everything unimportant" that describes the world at very high energy scales, say the GUT scale, or the Planck scale. Whatever this scale is, let's call it Λ. It is an energy scale, so $[\Lambda] = 1$. Now we want to understand the laws of physics down at our puny energy scale $E \ll \Lambda$. Let's further suppose that down at the energy scale E, the laws of physics are described by a real scalar field. (They're not of course: they're described by non-Abelian gauge fields and fermions, but the same argument applies in that case so bear with me). This scalar field will have some complicated interaction terms (3.1), where the precise form is dictated by all the stuff that's going on in the high energy superduper theory. What are these interactions? Well, we could write our dimensionful coupling constants λ_n in terms of dimensionless couplings g_n, multiplied by a suitable power of the relevant scale Λ,

$$\lambda_n = \frac{g_n}{\Lambda^{n-4}} \qquad (3.4)$$

The exact values of dimensionless couplings g_n depend on the details of the high-energy superduper theory, but typically one expects them to be of order 1: $g_n \sim \mathcal{O}(1)$. This

means that for experiments at small energies $E \ll \Lambda$, the interaction terms of the form ϕ^n with $n > 4$ will be suppressed by powers of $(E/\Lambda)^{n-4}$. This is usually a suppression by many orders of magnitude. (e.g for the energies E explored at the LHC, $E/M_{\rm pl} \sim 10^{-16}$). It is this simple argument, based on dimensional analysis, that ensures that we need only focus on the first few terms in the interaction: those which are relevant and marginal. It also means that if we only have access to low-energy experiments (which we do!), it's going to be very difficult to figure out the high energy theory (which it is!), because its effects are highly diluted except for the relevant and marginal interactions. The discussion given above is a poor man's version of the ideas of *effective field theory* and *Wilson's renormalization group*, about which you can learn more in the "Statistical Field Theory" course.

Examples of Weakly Coupled Theories

In this course we'll study only weakly coupled field theories i.e. ones that can truly be considered as small perturbations of the free field theory at all energies. In this section, we'll look at two types of interactions

1) ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.5}$$

with $\lambda \ll 1$. We can get a hint for what the effects of this extra term will be. Expanding out ϕ^4 in terms of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$, we see a sum of interactions that look like

$$a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger \quad \text{and} \quad a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger a_{\vec{p}} \quad \text{etc.} \tag{3.6}$$

These will create and destroy particles. This suggests that the ϕ^4 Lagrangian describes a theory in which particle number is not conserved. Indeed, we could check that the number operator N now satisfies $[H, N] \neq 0$.

2) Scalar Yukawa Theory

$$\mathcal{L} = \partial_\mu\psi^\star\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\psi^\star\psi - \frac{1}{2}m^2\phi^2 - g\psi^\star\psi\phi \tag{3.7}$$

with $g \ll M, m$. This theory couples a complex scalar ψ to a real scalar ϕ. While the individual particle numbers of ψ and ϕ are no longer conserved, we do still have a symmetry rotating the phase of ψ, ensuring the existence of the charge Q defined in (2.75) such that $[Q, H] = 0$. This means that the number of ψ particles minus the number of ψ anti-particles is conserved. It is common practice to denote the anti-particle as $\bar{\psi}$.

The scalar Yukawa theory has a slightly worrying aspect: the potential has a stable local minimum at $\phi = \psi = 0$, but is unbounded below for large enough $-g\phi$. This means we shouldn't try to push this theory too far.

A Comment on Strongly Coupled Field Theories

In this course we restrict attention to weakly coupled field theories where we can use perturbative techniques. The study of strongly coupled field theories is much more difficult, and one of the major research areas in theoretical physics. For example, some of the amazing things that can happen include

- **Charge Fractionalization:** Although electrons have electric charge 1, under the right conditions the elementary excitations in a solid have fractional charge $1/N$ (where $N \in 2\mathbf{Z} + 1$). For example, this occurs in the fractional quantum Hall effect.

- **Confinement:** The elementary excitations of quantum chromodynamics (QCD) are quarks. But they *never* appear on their own, only in groups of three (in a baryon) or with an anti-quark (in a meson). They are confined.

- **Emergent Space:** There are field theories in four dimensions which at strong coupling become quantum gravity theories in ten dimensions! The strong coupling effects cause the excitations to act as if they're gravitons moving in higher dimensions. This is quite extraordinary and still poorly understood. It's called the AdS/CFT correspondence.

3.1 The Interaction Picture

There's a useful viewpoint in quantum mechanics to describe situations where we have small perturbations to a well-understood Hamiltonian. Let's return to the familiar ground of quantum mechanics with a finite number of degrees of freedom for a moment. In the Schrödinger picture, the states evolve as

$$i\frac{d|\psi\rangle_S}{dt} = H |\psi\rangle_S \tag{3.8}$$

while the operators $\mathcal{O}_S$ are independent of time.

In contrast, in the Heisenberg picture the states are fixed and the operators change in time

$$\begin{aligned} \mathcal{O}_H(t) &= e^{iHt}\mathcal{O}_S\, e^{-iHt} \\ |\psi\rangle_H &= e^{iHt} |\psi\rangle_S \end{aligned} \tag{3.9}$$

The *interaction picture* is a hybrid of the two. We split the Hamiltonian up as

$$H = H_0 + H_{\text{int}} \tag{3.10}$$

The time dependence of operators is governed by H_0, while the time dependence of states is governed by H_{int}. Although the split into H_0 and H_{int} is arbitrary, it's useful when H_0 is soluble (for example, when H_0 is the Hamiltonian for a free field theory). The states and operators in the interaction picture will be denoted by a subscript I and are given by,

$$\begin{aligned} |\psi(t)\rangle_I &= e^{iH_0 t} |\psi(t)\rangle_S \\ \mathcal{O}_I(t) &= e^{iH_0 t} \mathcal{O}_S e^{-iH_0 t} \end{aligned} \tag{3.11}$$

This last equation also applies to H_{int}, which is time dependent. The interaction Hamiltonian in the interaction picture is,

$$H_I \equiv (H_{\text{int}})_I = e^{iH_0 t} (H_{\text{int}})_S \, e^{-iH_0 t} \tag{3.12}$$

The Schrödinger equation for states in the interaction picture can be derived starting from the Schrödinger picture

$$i\frac{d|\psi\rangle_S}{dt} = H_S |\psi\rangle_S \quad \Rightarrow \quad i\frac{d}{dt}\left(e^{-iH_0 t} |\psi\rangle_I\right) = (H_0 + H_{\text{int}})_S \, e^{-iH_0 t} |\psi\rangle_I$$

$$\Rightarrow \quad i\frac{d|\psi\rangle_I}{dt} = e^{iH_0 t}(H_{\text{int}})_S \, e^{-iH_0 t} \, |\psi\rangle_I \tag{3.13}$$

So we learn that

$$i\frac{d|\psi\rangle_I}{dt} = H_I(t) |\psi\rangle_I \tag{3.14}$$

3.1.1 Dyson's Formula

"Well, Birmingham has much the best theoretical physicist to work with, Peierls; Bristol has much the best experimental physicist, Powell; Cambridge has some excellent architecture. You can make your choice."

Oppenheimer's advice to Dyson on which university position to accept.

We want to solve (3.14). Let's write the solution as

$$|\psi(t)\rangle_I = U(t, t_0) |\psi(t_0)\rangle_I \tag{3.15}$$

where $U(t, t_0)$ is a unitary time evolution operator such that $U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$ and $U(t, t) = 1$. Then the interaction picture Schrödinger equation (3.14) requires that

$$i \frac{dU}{dt} = H_I(t) \, U \tag{3.16}$$

If H_I were a function, then we could simply solve this by

$$U(t, t_0) \stackrel{?}{=} \exp\left(-i \int_{t_0}^t H_I(t') \, dt' \right) \tag{3.17}$$

But there's a problem. Our Hamiltonian H_I is an operator, and we have ordering issues. Let's see why this causes trouble. The exponential of an operator is defined in terms of the expansion,

$$\exp\left(-i \int_{t_0}^t H_I(t') \, dt' \right) = 1 - i \int_{t_0}^t H_I(t') \, dt' + \frac{(-i)^2}{2} \left(\int_{t_0}^t H_I(t') \, dt' \right)^2 + \ldots \tag{3.18}$$

But when we try to differentiate this with respect to t, we find that the quadratic term gives us

$$-\frac{1}{2} \left(\int_{t_0}^t H_I(t') \, dt' \right) H_I(t) - \frac{1}{2} H_I(t) \left(\int_{t_0}^t H_I(t') \, dt' \right) \tag{3.19}$$

Now the second term here looks good, since it will give part of the $H_I(t) U$ that we need on the right-hand side of (3.16). But the first term is no good since the $H_I(t)$ sits the wrong side of the integral term, and we can't commute it through because $[H_I(t'), H_I(t)] \neq 0$ when $t' \neq t$. So what's the way around this?

Claim: The solution to (3.16) is given by *Dyson's Formula*. (Essentially first figured out by Dirac, although the compact notation is due to Dyson).

$$U(t, t_0) = T \exp\left(-i \int_{t_0}^t H_I(t') \, dt' \right) \tag{3.20}$$

where T stands for *time ordering* where operators evaluated at later times are placed to the left

$$T \left(\mathcal{O}_1(t_1) \, \mathcal{O}_2(t_2) \right) = \begin{cases} \mathcal{O}_1(t_1) \, \mathcal{O}_2(t_2) & t_1 > t_2 \\ \mathcal{O}_2(t_2) \, \mathcal{O}_1(t_1) & t_2 > t_1 \end{cases} \tag{3.21}$$

Expanding out the expression (3.20), we now have

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' \, H_I(t') + \frac{(-i)^2}{2} \left[\int_{t_0}^t dt' \int_{t'}^t dt'' \, H_I(t'') H_I(t') \right.$$
$$\left. + \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \, H_I(t') H_I(t'') \right] + \ldots$$

Actually these last two terms double up since

$$\int_{t_0}^{t} dt' \int_{t'}^{t} dt'' \, H_I(t'')H_I(t') = \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' \, H_I(t'')H_I(t')$$

$$= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t')H_I(t'') \qquad (3.22)$$

where the range of integration in the first expression is over $t'' \geq t'$, while in the second expression it is $t' \leq t''$ which is, of course, the same thing. The final expression is the same as the second expression by a simple relabelling. This means that we can write

$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt' \, H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t')H_I(t'') + \ldots \qquad (3.23)$$

Proof: The proof of Dyson's formula is simpler than explaining what all the notation means! Firstly observe that under the T sign, all operators commute (since their order is already fixed by the T sign). Thus

$$i\frac{\partial}{\partial t} \, T \exp\left(-i \int_{t_0}^{t} dt' \, H_I(t')\right) = T \left[H_I(t) \, \exp\left(-i \int_{t_0}^{t} dt' \, H_I(t')\right) \right]$$

$$= H_I(t) \, T \exp\left(-i \int_{t_0}^{t} dt' \, H_I(t')\right) \qquad (3.24)$$

since t, being the upper limit of the integral, is the latest time so $H_I(t)$ can be pulled out to the left. $\qquad \square$

Before moving on, I should confess that Dyson's formula is rather formal. It is typically very hard to compute time ordered exponentials in practice. The power of the formula comes from the expansion which is valid when H_I is small and is very easily computed.

3.2 A First Look at Scattering

Let us now apply the interaction picture to field theory, starting with the interaction Hamiltonian for our scalar Yukawa theory,

$$H_{\text{int}} = g \int d^3x \, \psi^\dagger \psi \phi \qquad (3.25)$$

Unlike the free theories discussed in Section 2, this interaction doesn't conserve particle number, allowing particles of one type to morph into others. To see why this is, we use

the interaction picture and follow the evolution of the state: $|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$, where $U(t, t_0)$ is given by Dyson's formula (3.20) which is an expansion in powers of H_{int}. But H_{int} contains creation and annihilation operators for each type of particle. In particular,

- $\phi \sim a + a^\dagger$: This operator can create or destroy ϕ particles. Let's call them *mesons*.

- $\psi \sim b + c^\dagger$: This operator can destroy ψ particles through b, and create anti-particles through $c^\dagger$. Let's call these particles *nucleons*. Of course, in reality nucleons are spin $1/2$ particles, and don't arise from the quantization of a scalar field. But we'll treat our scalar Yukawa theory as a toy model for nucleons interacting with mesons.

- $\psi^\dagger \sim b^\dagger + c$: This operator can create nucleons through $b^\dagger$, and destroy anti-nucleons through c.

Importantly, $Q = N_c - N_b$ remains conserved in the presence of H_{int}. At first order in perturbation theory, we find terms in H_{int} like $c^\dagger b^\dagger a$. This kills a meson, producing a nucleon-anti-nucleon pair. It will contribute to meson decay $\phi \to \psi\bar{\psi}$.

At second order in perturbation theory, we'll have more complicated terms in $(H_{\mathrm{int}})^2$, for example $(c^\dagger b^\dagger a)(cba^\dagger)$. This term will give contributions to scattering processes $\psi\bar{\psi} \to \phi \to \psi\bar{\psi}$. The rest of this section is devoted to computing the quantum amplitudes for these processes to occur.

To calculate amplitudes we make an important, and slightly dodgy, assumption:

<center>Initial and final states are eigenstates of the free theory</center>

This means that we take the initial state $|i\rangle$ at $t \to -\infty$, and the final state $|f\rangle$ at $t \to +\infty$, to be eigenstates of the free Hamiltonian H_0. At some level, this sounds plausible: at $t \to -\infty$, the particles in a scattering process are far separated and don't feel the effects of each other. Furthermore, we intuitively expect these states to be eigenstates of the individual number operators N, which commute with H_0, but not H_{int}. As the particles approach each other, they interact briefly, before departing again, each going on its own merry way. The amplitude to go from $|i\rangle$ to $|f\rangle$ is

$$\lim_{t_\pm \to \pm\infty} \langle f| U(t_+, t_-) |i\rangle \equiv \langle f| S |i\rangle \tag{3.26}$$

where the unitary operator S is known as the S-matrix. (S is for scattering). There are a number of reasons why the assumption of non-interacting initial and final states is shaky:

- Obviously we can't cope with bound states. For example, this formalism can't describe the scattering of an electron and proton which collide, bind, and leave as a Hydrogen atom. It's possible to circumvent this objection since it turns out that bound states show up as poles in the S-matrix.

- More importantly, a single particle, a long way from its neighbors, is never alone in field theory. This is true even in classical electrodynamics, where the electron sources the electromagnetic field from which it can never escape. In quantum electrodynamics (QED), a related fact is that there is a cloud of *virtual* photons surrounding the electron. This line of thought gets us into the issues of renormalization — more on this next term in the "AQFT" course. Nevertheless, motivated by this problem, after developing scattering theory using the assumption of non-interacting asymptotic states, we'll mention a better way.

3.2.1 An Example: Meson Decay

Consider the relativistically normalized initial and final states,

$$|i\rangle = \sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^\dagger |0\rangle$$
$$|f\rangle = \sqrt{4E_{\vec{q_1}}E_{\vec{q_2}}}\, b_{\vec{q_1}}^\dagger\, c_{\vec{q_2}}^\dagger |0\rangle \tag{3.27}$$

The initial state contains a single meson of momentum p; the final state contains a nucleon-anti-nucleon pair of momentum q_1 and q_2. We may compute the amplitude for the decay of a meson to a nucleon-anti-nucleon pair. To leading order in g, it is

$$\langle f| S |i\rangle = -ig \langle f| \int d^4x\, \psi^\dagger(x)\psi(x)\phi(x) |i\rangle \tag{3.28}$$

Let's go slowly. We first expand out $\phi \sim a + a^\dagger$ using (2.84). (Remember that the ϕ in this formula is in the interaction picture, which is the same as the Heisenberg picture of the free theory). The a piece will turn $|i\rangle$ into something proportional to $|0\rangle$, while the $a^\dagger$ piece will turn $|i\rangle$ into a two meson state. But the two meson state will have zero overlap with $\langle f|$, and there's nothing in the ψ and $\psi^\dagger$ operators that lie between them to change this fact. So we have

$$\langle f| S |i\rangle = -ig \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec{p}}}}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}}\, a_{\vec{p}}^\dagger e^{-ik\cdot x} |0\rangle$$
$$= -ig \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) e^{-ip\cdot x} |0\rangle \tag{3.29}$$

where, in the second line, we've commuted $a_{\vec{k}}$ past $a_{\vec{p}}^\dagger$, picking up a $\delta^{(3)}(\vec{p} - \vec{k})$ delta-function which kills the d^3k integral. We now similarly expand out $\psi \sim b + c^\dagger$ and

$\psi^\dagger \sim b^\dagger + c$. To get non-zero overlap with $\langle f|$, only the $b^\dagger$ and $c^\dagger$ contribute, for they create the nucleon and anti-nucleon from $|0\rangle$. We then have

$$\langle f| S |i\rangle = -ig \langle 0| \int \int \frac{d^4x\, d^3k_1 d^3k_2}{(2\pi)^6} \frac{\sqrt{E_{\vec{q}_1} E_{\vec{q}_2}}}{\sqrt{E_{\vec{k}_1} E_{\vec{k}_2}}} c_{\vec{q}_2} b_{\vec{q}_1} c^\dagger_{\vec{k}_1} b^\dagger_{\vec{k}_2} |0\rangle\, e^{i(k_1+k_2-p)\cdot x}$$

$$= -ig\, (2\pi)^4\, \delta^{(4)}(q_1 + q_2 - p) \tag{3.30}$$

and so we get our first quantum field theory amplitude.

Notice that the δ-function puts constraints on the possible decays. In particular, the decay only happens at all if $m \geq 2M$. To see this, we may always boost ourselves to a reference frame where the meson is stationary, so $p = (m, 0, 0, 0)$. Then the delta function imposes momentum conservation, telling us that $\vec{q}_1 = -\vec{q}_2$ and $m = 2\sqrt{M^2 + |\vec{q}|^2}$.

Later you will learn how to turn this quantum amplitude into something more physical, namely the lifetime of the meson. The reason this is a little tricky is that we must square the amplitude to get the probability for decay, which means we get the square of a δ-function. We'll explain how to deal with this in Section 3.6 below, and again in next term's "Standard Model" course.

3.3 Wick's Theorem

From Dyson's formula, we want to compute quantities like $\langle f| T \{H_I(x_1) \ldots H_I(x_n)\} |i\rangle$, where $|i\rangle$ and $|f\rangle$ are eigenstates of the free theory. The ordering of the operators is fixed by T, time ordering. However, since the H_I's contain certain creation and annihilation operators, our life will be much simpler if we can start to move all annihilation operators to the right where they can start killing things in $|i\rangle$. Recall that this is the definition of normal ordering. Wick's theorem tells us how to go from time ordered products to normal ordered products.

3.3.1 An Example: Recovering the Propagator

Let's start simple. Consider a real scalar field which we decompose in the Heisenberg picture as

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{3.31}$$

where

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}}\, e^{-ip\cdot x}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a^\dagger_{\vec{p}}\, e^{+ip\cdot x} \tag{3.32}$$

where the $\pm$ signs on $\phi^{\pm}$ make little sense, but apparently you have Pauli and Heisenberg to blame. (They come about because $\phi^+ \sim e^{-iEt}$, which is sometimes called the positive frequency piece, while $\phi^- \sim e^{+iEt}$ is the negative frequency piece). Then choosing $x^0 > y^0$, we have

$$
\begin{aligned}
T\,\phi(x)\phi(y) &= \phi(x)\phi(y) \\
&= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\
&= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + [\phi^+(x),\phi^-(y)] + \phi^-(x)\phi^-(y)
\end{aligned}
\tag{3.33}
$$

where the last line is normal ordered, and for our troubles we have picked up the extra term $D(x-y) = [\phi^+(x), \phi^-(y)]$ which is the propagator we met in (2.90). So for $x^0 > y^0$ we have

$$
T\,\phi(x)\phi(y) =: \phi(x)\phi(y): + D(x-y)
\tag{3.34}
$$

Meanwhile, for $y^0 > x^0$, we may repeat the calculation to find

$$
T\,\phi(x)\phi(y) =: \phi(x)\phi(y): + D(y-x)
\tag{3.35}
$$

So putting this together, we have the final expression

$$
T\,\phi(x)\phi(y) =: \phi(x)\phi(y): + \Delta_F(x-y)
\tag{3.36}
$$

where $\Delta_F(x-y)$ is the Feynman propagator defined in (2.93), for which we have the integral representation

$$
\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}
\tag{3.37}
$$

Let me reiterate a comment from Section 2: although $T\,\phi(x)\phi(y)$ and $: \phi(x)\phi(y):$ are both operators, the difference between them is a c-number function, $\Delta_F(x-y)$.

Definition: We define the *contraction* of a pair of fields in a string of operators $\dots \phi(x_1) \dots \phi(x_2) \dots$ to mean replacing those operators with the Feynman propagator, leaving all other operators untouched. We use the notation,

$$
\dots \overbrace{\phi(x_1) \dots \phi(x_2)} \dots
\tag{3.38}
$$

to denote contraction. So, for example,

$$
\overbrace{\phi(x)\phi(y)} = \Delta_F(x-y)
\tag{3.39}
$$

A similar discussion holds for complex scalar fields. We have

$$T\psi(x)\psi^\dagger(y) =: \psi(x)\psi^\dagger(y) : +\Delta_F(x-y) \tag{3.40}$$

prompting us to define the contraction

$$\overbrace{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overbrace{\psi(x)\psi(y)} = \overbrace{\psi^\dagger(x)\psi^\dagger(y)} = 0 \tag{3.41}$$

3.3.2 Wick's Theorem

For any collection of fields $\phi_1 = \phi(x_1)$, $\phi_2 = \phi(x_2)$, etc, we have

$$T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + : \text{all possible contractions} : \tag{3.42}$$

To see what the last part of this equation means, let's look at an example. For $n = 4$, the equation reads

$$T(\phi_1\phi_2\phi_3\phi_4) = : \phi_1\phi_2\phi_3\phi_4 : + \overbrace{\phi_1\phi_2} : \phi_3\phi_4 : + \overbrace{\phi_1\phi_3} : \phi_2\phi_4 : + \text{four similar terms}$$

$$+ \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3}\,\overbrace{\phi_2\phi_4} + \overbrace{\phi_1\phi_4}\,\overbrace{\phi_2\phi_3} \tag{3.43}$$

Proof: The proof of Wick's theorem proceeds by induction and a little thought. It's true for $n = 2$. Suppose it's true for $\phi_2 \ldots \phi_n$ and now add ϕ_1. We'll take $x_1^0 > x_k^0$ for all $k = 2, \ldots, n$. Then we can pull ϕ_1 out to the left of the time ordered product, writing

$$T(\phi_1\phi_2 \ldots \phi_n) = (\phi_1^+ + \phi_1^-) \, (: \phi_2 \ldots \phi_n : + : \text{contractions} :) \tag{3.44}$$

The ϕ_1^- term stays where it is since it is already normal ordered. But in order to write the right-hand side as a normal ordered product, the ϕ_1^+ term has to make its way past the crowd of ϕ_k^- operators. Each time it moves past ϕ_k^-, we pick up a factor of $\overbrace{\phi_1\phi_k} = \Delta_F(x_1 - x_k)$ from the commutator. (Try it!) $\qquad\qquad\square$

3.3.3 An Example: Nucleon Scattering

Let's look at $\psi\psi \to \psi\psi$ scattering. We have the initial and final states

$$|i\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\, b_{\vec{p}_1}^\dagger\, b_{\vec{p}_2}^\dagger\, |0\rangle \equiv |p_1, p_2\rangle$$

$$|f\rangle = \sqrt{2E_{\vec{p}_1'}}\sqrt{2E_{\vec{p}_2'}}\, b_{\vec{p}_1'}^\dagger b_{\vec{p}_2'}^\dagger\, |0\rangle \equiv |p_1', p_2'\rangle \tag{3.45}$$

We can then look at the expansion of $\langle f | S | i\rangle$. In fact, we really want to calculate $\langle f | S - 1 | i\rangle$ since we're not interested in situations where no scattering occurs. At order g^2 we have the term

$$\frac{(-ig)^2}{2} \int d^4x_1 d^4x_2\, T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right) \tag{3.46}$$

Now, using Wick's theorem we see there is a piece in the string of operators which looks like

$$: \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : \overbrace{\phi(x_1)\phi(x_2)} \tag{3.47}$$

which will contribute to the scattering because the two ψ fields annihilate the ψ particles, while the two $\psi^\dagger$ fields create ψ particles. Any other way of ordering the ψ and $\psi^\dagger$ fields will give zero contribution. This means that we have

$$\langle p_1', p_2' | : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : | p_1, p_2 \rangle$$
$$= \langle p_1', p_2' | \psi^\dagger(x_1)\psi^\dagger(x_2) | 0 \rangle \langle 0 | \psi(x_1)\psi(x_2) | p_1, p_2 \rangle$$
$$= \left(e^{ip_1' \cdot x_1 + ip_2' \cdot x_2} + e^{ip_1' \cdot x_2 + ip_2' \cdot x_1} \right) \left(e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2} + e^{-ip_1 \cdot x_2 - ip_2 \cdot x_1} \right)$$
$$= e^{ix_1 \cdot (p_1' - p_1) + ix_2 \cdot (p_2' - p_2)} + e^{ix_1 \cdot (p_2' - p_1) + ix_2 \cdot (p_1' - p_2)} + (x_1 \leftrightarrow x_2) \tag{3.48}$$

where, in going to the third line, we've used the fact that for relativistically normalized states,

$$\langle 0 | \psi(x) | p \rangle = e^{-ip \cdot x} \tag{3.49}$$

Now let's insert this into (3.46), to get the expression for $\langle f | S | i \rangle$ at order g^2,

$$\frac{(-ig)^2}{2} \int d^4 x_1 d^4 x_2 \left[e^{i \cdots} + e^{i \cdots} + (x_1 \leftrightarrow x_2) \right] \int \frac{d^4 k}{(2\pi)^4} \frac{ie^{ik \cdot (x_1 - x_2)}}{k^2 - m^2 + i\epsilon} \tag{3.50}$$

where the expression in square brackets is (3.48), while the final integral is the ϕ propagator which comes from the contraction in (3.47). Now the $(x_1 \leftrightarrow x_2)$ terms double up with the others to cancel the factor of $1/2$ out front. Meanwhile, the x_1 and x_2 integrals give delta-functions. We're left with the expression

$$(-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i(2\pi)^8}{k^2 - m^2 + i\epsilon} \left[\delta^{(4)}(p_1' - p_1 + k)\, \delta^{(4)}(p_2' - p_2 - k) \right.$$
$$\left. + \delta^{(4)}(p_2' - p_1 + k)\, \delta^{(4)}(p_1' - p_2 - k) \right] \tag{3.51}$$

Finally, we can trivially do the $d^4 k$ integral using the delta-functions to get

$$i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2 + i\epsilon} + \frac{1}{(p_1 - p_2')^2 - m^2 + i\epsilon} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

In fact, for this process we may drop the $+i\epsilon$ terms since the denominator is never zero. To see this, we can go to the center of mass frame, where $\vec{p}_1 = -\vec{p}_2$ and, by

momentum conservation, $|\vec{p}_1| = |\vec{p}_1{}'|$. This ensures that the 4-momentum of the meson is $k = (0, \vec{p} - \vec{p}\,')$, so $k^2 < 0$. We therefore have the end result,

$$
i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \qquad (3.52)
$$

We will see another, much simpler way to reproduce this result shortly using Feynman diagrams. This will also shed light on the physical interpretation.

This calculation is also relevant for other scattering processes, such as $\bar{\psi}\bar{\psi} \to \bar{\psi}\bar{\psi}$, $\psi\bar{\psi} \to \psi\bar{\psi}$. Each of these comes from the term (3.48) in Wick's theorem. However, we will never find a term that contributes to scattering $\psi\psi \to \bar{\psi}\bar{\psi}$, for this would violate the conservation of Q charge.

Another Example: Meson-Nucleon Scattering

If we want to compute $\psi\phi \to \psi\phi$ scattering at order g^2, we would need to pick out the term

$$
: \psi^\dagger(x_1)\phi(x_1)\psi(x_2)\phi(x_2) : \; \overbrace{\psi(x_1)\psi^\dagger(x_2)} \qquad (3.53)
$$

and a similar term with ψ and $\psi^\dagger$ exchanged. Once more, this term also contributes to similar scattering processes, including $\bar{\psi}\phi \to \bar{\psi}\phi$ and $\phi\phi \to \psi\bar{\psi}$.

3.4 Feynman Diagrams

> "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses."
>
> *Julian Schwinger*

As the above example demonstrates, to actually compute scattering amplitudes using Wick's theorem is rather tedious. There's a much better way. It requires drawing pretty pictures. These pictures represent the expansion of $\langle f | S | i \rangle$ and we will learn how to associate numbers (or at least integrals) to them. These pictures are called *Feynman diagrams*.

The object that we really want to compute is $\langle f | S - 1 | i \rangle$, since we're not interested in processes where no scattering occurs. The various terms in the perturbative expansion can be represented pictorially as follows

- Draw an external line for each particle in the initial state $|i\rangle$ and each particle in the final state $|f\rangle$. We'll choose dotted lines for mesons, and solid lines for nucleons. Assign a directed momentum p to each line. Further, add an arrow to

solid lines to denote its charge; we'll choose an incoming (outgoing) arrow in the initial state for ψ ($\bar{\psi}$). We choose the reverse convention for the final state, where an outgoing arrow denotes ψ.

- Join the external lines together with trivalent vertices
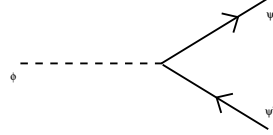
Each such diagram you can draw is in 1-1 correspondence with the terms in the expansion of $\langle f | S - 1 | i \rangle$.

3.4.1 Feynman Rules

To each diagram we associate a number, using the *Feynman rules*

- Add a momentum k to each internal line

- To each vertex, write down a factor of

$$(-ig)\,(2\pi)^4\,\delta^{(4)}\!\Big(\sum_i k_i\Big) \tag{3.54}$$

where $\sum k_i$ is the sum of all momenta flowing *into* the vertex.

- For each internal dotted line, corresponding to a ϕ particle with momentum k, we write down a factor of

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \tag{3.55}$$

We include the same factor for solid internal ψ lines, with m replaced by the nucleon mass M.

3.5 Examples of Scattering Amplitudes

Let's apply the Feynman rules to compute the amplitudes for various processes. We start with something familiar:

Nucleon Scattering Revisited

Let's look at how this works for the $\psi\psi \to \psi\psi$ scattering at order g^2. We can write down the two simplest diagrams contributing to this process. They are shown in Figure 9.
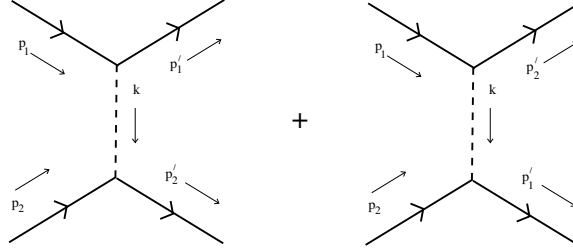

Figure 9: The two lowest order Feynman diagrams for nucleon scattering.

Applying the Feynman rules to these diagrams, we get

$$i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \qquad (3.56)$$

which agrees with the calculation (3.51) that we performed earlier. There is a nice physical interpretation of these diagrams. We talk, rather loosely, of the nucleons exchanging a meson which, in the first diagram, has momentum $k = (p_1 - p_1') = (p_2' - p_2)$. This meson doesn't satisfy the usual energy dispersion relation, because $k^2 \neq m^2$: the meson is called a *virtual particle* and is said to be *off-shell* (or, sometimes, off mass-shell). Heuristically, it can't live long enough for its energy to be measured to great accuracy. In contrast, the momentum on the external, nucleon legs all satisfy $p^2 = M^2$, the mass of the nucleon. They are *on-shell*. One final note: the addition of the two diagrams above ensures that the particles satisfy Bose statistics.

There are also more complicated diagrams which will contribute to the scattering process at higher orders. For example, we have the two diagrams shown in Figures 10 and 11, and similar diagrams with p_1' and p_2' exchanged. Using the Feynman rules, each of these diagrams translates into an integral that we will not attempt to calculate here. And so we go on, with increasingly complicated diagrams, all appearing at higher order in the coupling constant g.
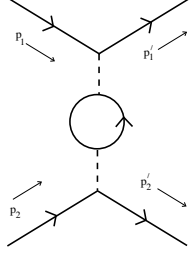
Figure 10: A contribution at $\mathcal{O}(g^4)$.
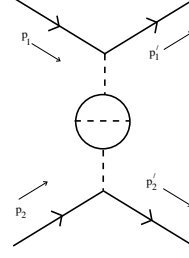
Figure 11: A contribution at $\mathcal{O}(g^6)$

Amplitudes

Our final result for the nucleon scattering amplitude $\langle f | S - 1 | i \rangle$ at order g^2 was

$$i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

The δ-function follows from the conservation of 4-momentum which, in turn, follows from spacetime translational invariance. It is common to all S-matrix elements. We will define the amplitude $\mathcal{A}_{fi}$ by stripping off this momentum-conserving delta-function,

$$\langle f | S - 1 | i \rangle = i \, \mathcal{A}_{fi} \, (2\pi)^4 \delta^{(4)}(p_F - p_I) \tag{3.57}$$

where p_I (p_F) is the sum of the initial (final) 4-momenta, and the factor of i out front is a convention which is there to match non-relativistic quantum mechanics. We can now refine our Feynman rules to compute the amplitude $i\mathcal{A}_{fi}$ itself:

- Draw all possible diagrams with appropriate external legs and impose 4-momentum conservation at each vertex.

- Write down a factor of $(-ig)$ at each vertex.

- For each internal line, write down the propagator

- Integrate over momentum k flowing through each loop $\int d^4 k / (2\pi)^4$.

This last step deserves a short explanation. The diagrams we've computed so far have no loops. They are *tree level* diagrams. It's not hard to convince yourself that in tree diagrams, momentum conservation at each vertex is sufficient to determine the momentum flowing through each internal line. For diagrams with loops, such as those shown in Figures 10 and 11, this is no longer the case.
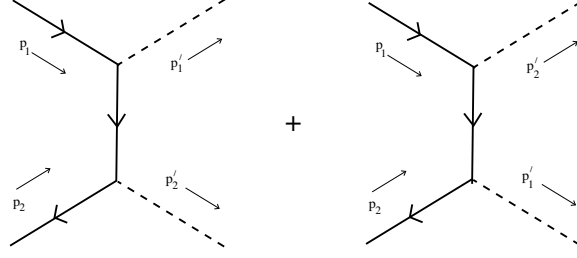
Figure 12: The two lowest order Feynman diagrams for nucleon to meson scattering.
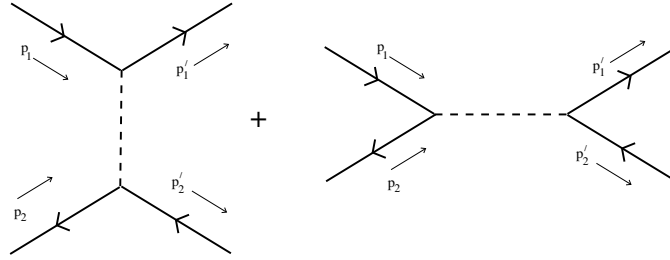
Nucleon to Meson Scattering

Let's now look at the amplitude for a nucleon-anti-nucleon pair to annihilate into a pair of mesons: $\psi\bar{\psi} \to \phi\phi$. The simplest Feynman diagrams for this process are shown in Figure 12 where the virtual particle in these diagrams is now the nucleon ψ rather than the meson ϕ. This fact is reflected in the denominator of the amplitudes which are given by

$$ i\mathcal{A} = (-ig)^2 \left[\frac{i}{(p_1 - p_1')^2 - M^2} + \frac{i}{(p_1 - p_2')^2 - M^2} \right] \tag{3.58}$$

As in (3.52), we've dropped the $i\epsilon$ from the propagators as the denominator never vanishes.

Nucleon-Anti-Nucleon Scattering

Figure 13: The two lowest order Feynman diagrams for nucleon-anti-nucleon scattering.

For the scattering of a nucleon and an anti-nucleon, $\psi\bar{\psi} \to \psi\bar{\psi}$, the Feynman diagrams are a little different. At lowest order, they are given by the diagrams of Figure 13. It is a simple matter to write down the amplitude using the Feynman rules,

$$ i\mathcal{A} = (-ig)^2 \left[\frac{i}{(p_1 - p_1')^2 - m^2} + \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon} \right] \tag{3.59}$$

Notice that the momentum dependence in the second term is different from that of nucleon-nucleon scattering (3.56), reflecting the different Feynman diagram that contributes to the process. In the center of mass frame, $\vec{p}_1 = -\vec{p}_2$, the denominator of the second term is $4(M^2 + \vec{p}_1^2) - m^2$. If $m < 2M$, then this term never vanishes and we may drop the $i\epsilon$. In contrast, if $m > 2M$, then the amplitude corresponding to the second diagram diverges at some value of $\vec{p}$. In this case it turns out that we may also neglect the $i\epsilon$ term, although for a different reason: the meson is unstable when $m > 2M$, a result we derived in (3.30). When correctly treated, this instability adds a finite imaginary piece to the denominator which

Figure 14:

overwhelms the $i\epsilon$. Nonetheless, the increase in the scattering amplitude which we see in the second diagram when $4(M^2 + \vec{p}^2) = m^2$ is what allows us to discover new particles: they appear as a resonance in the cross section. For example, the Figure 14 shows the cross-section (roughly the amplitude squared) plotted vertically for $e^+e^- \to \mu^+\mu^-$ scattering from the ALEPH experiment in CERN. The horizontal axis shows the center of mass energy. The curve rises sharply around 91 GeV, the mass of the Z-boson.
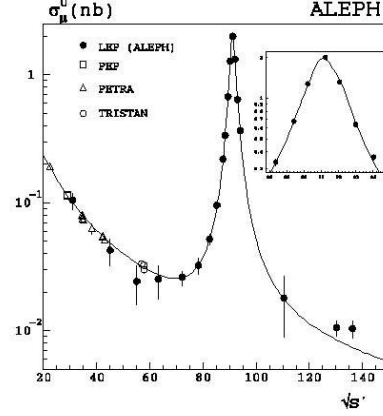
Meson Scattering

For $\phi\phi \to \phi\phi$, the simplest diagram we can write down has a single loop, and momentum conservation at each vertex is no longer sufficient to determine every momentum passing through the diagram. We choose to assign the single undetermined momentum k to the right-hand propagator. All other momenta are then determined. The amplitude corresponding to the diagram shown in the figure is
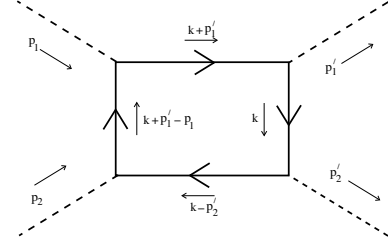
Figure 15:

$$(-ig)^4 \int \frac{d^4k}{(2\pi)^4} \; \frac{1}{(k^2 - M^2 + i\epsilon)((k + p_1')^2 - M^2 + i\epsilon)}$$
$$\times \frac{1}{((k + p_1' - p_1)^2 - M^2 + i\epsilon)((k - p_2')^2 - M^2 + i\epsilon)}$$

These integrals can be tricky. For large k, this integral goes as $\int d^4k/k^8$, which is at least convergent as $k \to \infty$. But this won't always be the case!

3.5.1 Mandelstam Variables

We see that in many of the amplitudes above — in particular those that include the exchange of just a single particle — the same combinations of momenta are appearing frequently in the denominators. There are standard names for various sums and differences of momenta: they are known as *Mandelstam variables*. They are

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p'_1 + p'_2)^2 \\
t &= (p_1 - p'_1)^2 = (p_2 - p'_2)^2 \\
u &= (p_1 - p'_2)^2 = (p_2 - p'_1)^2
\end{aligned}
\tag{3.60}
$$

where, as in the examples above, p_1 and p_2 are the momenta of the two initial particles, and p'_1 and p'_2 are the momenta of the final two particles. We can define these variables whether the particles involved in the scattering are the same or different. To get a feel for what these variables mean, let's assume all four particles are the same. We sit in the center of mass frame, so that the initial two particles have four-momenta

$$
p_1 = (E, 0, 0, p) \quad \text{and} \quad p_2 = (E, 0, 0, -p)
\tag{3.61}
$$

The particles then scatter at some angle θ and leave with momenta

$$
p'_1 = (E, 0, p\sin\theta, p\cos\theta) \quad \text{and} \quad p'_2 = (E, 0, -p\sin\theta, -p\cos\theta)
\tag{3.62}
$$

Then from the above definitions, we have that

$$
s = 4E^2 \quad \text{and} \quad t = -2p^2(1 - \cos\theta) \quad \text{and} \quad u = -2p^2(1 + \cos\theta)
\tag{3.63}
$$

The variable s measures the total center of mass energy of the collision, while the variables t and u are measures of the momentum exchanged between particles. (They are basically equivalent, just with the outgoing particles swapped around). Now the amplitudes that involve exchange of a single particle can be written simply in terms of the Mandelstam variables. For example, for nucleon-nucleon scattering, the amplitude (3.56) is schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. For the nucleon-anti-nucleon scattering, the amplitude (3.59) is $\mathcal{A} \sim (t - m^2)^{-1} + (s - m^2)^{-1}$. We say that the first case involves "t-channel" and "u-channel" diagrams. Meanwhile the nucleon-anti-nucleon scattering is said to involve "t-channel" and "s-channel" diagrams. (The first diagram indeed includes a vertex that looks like the letter "T").

Note that there is a relationship between the Mandelstam variables. When all the masses are the same we have $s + t + u = 4M^2$. When the masses of all 4 particles differ, this becomes $s + t + u = \sum_i M_i^2$.

3.5.2 The Yukawa Potential

So far we've computed the quantum amplitudes for various scattering processes. But these quantities are a little abstract. In Section 3.6 below (and again in next term's "Standard Model" course) we'll see how to turn amplitudes into measurable quantities such as cross-sections, or the lifetimes of unstable particles. Here we'll instead show how to translate the amplitude (3.52) for nucleon scattering into something familiar from Newtonian mechanics: a potential, or force, between the particles.

Let's start by asking a simple question in classical field theory that will turn out to be relevant. Suppose that we have a fixed δ-function source for a real scalar field ϕ, that persists for all time. What is the profile of $\phi(\vec{x})$? To answer this, we must solve the static Klein-Gordon equation,

$$-\nabla^2 \phi + m^2 \phi = \delta^{(3)}(\vec{x}) \tag{3.64}$$

We can solve this using the Fourier transform,

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \, \tilde{\phi}(\vec{k}) \tag{3.65}$$

Plugging this into (3.64) tells us that $(\vec{k}^2 + m^2)\tilde{\phi}(\vec{k}) = 1$, giving us the solution

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2} \tag{3.66}$$

Let's now do this integral. Changing to polar coordinates, and writing $\vec{k}\cdot\vec{x} = kr\cos\theta$, we have

$$
\begin{aligned}
\phi(\vec{x}) &= \frac{1}{(2\pi)^2} \int_0^\infty dk \, \frac{k^2}{k^2 + m^2} \frac{2\sin kr}{kr} \\
&= \frac{1}{(2\pi)^2 r} \int_{-\infty}^{+\infty} dk \, \frac{k \sin kr}{k^2 + m^2} \\
&= \frac{1}{2\pi r} \text{Re} \left[\int_{-\infty}^{+\infty} \frac{dk}{2\pi i} \frac{k e^{ikr}}{k^2 + m^2} \right]
\end{aligned}
\tag{3.67}
$$

We compute this last integral by closing the contour in the upper half plane $k \to +i\infty$, picking up the pole at $k = +im$. This gives

$$\phi(\vec{x}) = \frac{1}{4\pi r} e^{-mr} \tag{3.68}$$

The field dies off exponentially quickly at distances $1/m$, the Compton wavelength of the meson.

Now we understand the profile of the ϕ field, what does this have to do with the force between ψ particles? We do very similar calculations to that above in electrostatics where a charged particle acts as a δ-function source for the gauge potential: $-\nabla^2 A_0 = \delta^{(3)}(\vec{x})$, which is solved by $A_0 = 1/4\pi r$. The profile for A_0 then acts as the potential energy for another charged (test) particle moving in this background. Can we give the same interpretation to our scalar field? In other words, is there a classical limit of the scalar Yukawa theory where the ψ particles act as δ-function sources for ϕ, creating the profile (3.68)? And, if so, is this profile then felt as a static potential? The answer is essentially yes, at least in the limit $M \gg m$. But the correct way to describe the potential felt by the ψ particles is not to talk about classical fields at all, but instead work directly with the quantum amplitudes.

Our strategy is to compare the nucleon scattering amplitude (3.52) to the corresponding amplitude in non-relativistic quantum mechanics for two particles interacting through a potential. To make this comparison, we should first take the non-relativistic limit of (3.52). Let's work in the center of mass frame, with $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $\vec{p}\,' \equiv \vec{p}_1' = -\vec{p}_2'$. The non-relativistic limit means $|\vec{p}| \ll M$ which, by momentum conservation, ensures that $|\vec{p}\,'| \ll M$. In fact one can check that, for this particular example, this limit doesn't change the scattering amplitude (3.52): it's given by

$$i\mathcal{A} = +ig^2 \left[\frac{1}{(\vec{p} - \vec{p}\,')^2 + m^2} + \frac{1}{(\vec{p} + \vec{p}\,')^2 + m^2} \right] \tag{3.69}$$

How do we compare this to scattering in quantum mechanics? Consider two particles, separated by a distance $\vec{r}$, interacting through a potential $U(\vec{r})$. In non-relativistic quantum mechanics, the amplitude for the particles to scatter from momentum states $\pm\vec{p}$ into momentum states $\pm\vec{p}\,'$ can be computed in perturbation theory, using the techniques described in Section 3.1. To leading order, known in this context as the Born approximation, the amplitude is given by

$$\langle \vec{p}\,'| U(\vec{r}) |\vec{p}\rangle = -i \int d^3r \, U(\vec{r}) e^{-i(\vec{p} - \vec{p}\,')\cdot\vec{r}} \tag{3.70}$$

There's a relative factor of $(2M)^2$ that arises in comparing the quantum field theory amplitude $\mathcal{A}$ to $\langle \vec{p}\,'| U(\vec{r}) |\vec{p}\rangle$, that can be traced to the relativistic normalization of the states $|p_1, p_2\rangle$. (It is also necessary to get the dimensions of the potential to work out correctly). Including this factor, and equating the expressions for the two amplitudes, we get

$$\int d^3r \, U(\vec{r}) \, e^{-i(\vec{p} - \vec{p}\,')\cdot\vec{r}} = \frac{-\lambda^2}{(\vec{p} - \vec{p}\,')^2 + m^2} \tag{3.71}$$

where we've introduced the dimensionless parameter $\lambda = g/2M$. We can trivially invert this to find,

$$U(\vec{r}) = -\lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{\vec{p}^2 + m^2} \tag{3.72}$$

But this is exactly the integral (3.66) we just did in the classical theory. We have

$$U(\vec{r}) = \frac{-\lambda^2}{4\pi r} e^{-mr} \tag{3.73}$$

This is the *Yukawa potential*. The force has a range $1/m$, the Compton wavelength of the exchanged particle. The minus sign tells us that the potential is attractive.

Notice that quantum field theory has given us an entirely new perspective on the nature of forces between particles. Rather than being a fundamental concept, the force arises from the virtual exchange of other particles, in this case the meson. In Section 6 of these lectures, we will see how the Coulomb force arises from quantum field theory due to the exchange of virtual photons.

We could repeat the calculation for nucleon-anti-nucleon scattering. The amplitude from field theory is given in (3.59). The first term in this expression gives the same result as for nucleon-nucleon scattering *with the same sign*. The second term vanishes in the non-relativisitic limit (it is an example of an interaction that doesn't have a simple Newtonian interpretation). There is no longer a factor of $1/2$ in (3.70), because the incoming/outgoing particles are not identical, so we learn that the potential between a nucleon and anti-nucleon is again given by (3.73). This reveals a key feature of forces arising due to the exchange of scalars: they are universally attractive. Notice that this is different from forces due to the exchange of a spin 1 particle — such as electromagnetism — where the sign flips when we change the charge. However, for forces due to the exchange of a spin 2 particle — i.e. gravity — the force is again universally attractive.

3.5.3 ϕ^4 Theory

Let's briefly look at the Feynman rules and scattering amplitudes for the interaction Hamiltonian

$$H_{\text{int}} = \frac{\lambda}{4!} \phi^4 \tag{3.74}$$

The theory now has a single interaction vertex, which comes with a factor of $(-i\lambda)$, while the other Feynman rules remain the same. Note that we assign $(-i\lambda)$ to the

vertex rather than $(-i\lambda/4!)$. To see why this is, we can look at $\phi\phi \to \phi\phi$ scattering, which has its lowest contribution at order λ, with the term

$$\frac{-i\lambda}{4!} \langle p_1', p_2'| : \phi(x)\phi(x)\phi(x)\phi(x) : |p_1, p_2\rangle \tag{3.75}$$

Any one of the fields can do the job of annihilation or creation. This gives 4! different contractions, which cancels the 1/4! sitting out front.

Feynman diagrams in the ϕ^4 theory sometimes come with extra combinatoric factors (typically 2 or 4) which are known as symmetry factors that one must take into account. For more details, see the book by Peskin and Schroeder.
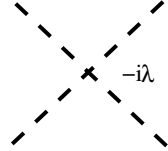
Figure 16:

Using the Feynman rules, the scattering amplitude for $\phi\phi \to \phi\phi$ is simply $i\mathcal{A} = -i\lambda$. Note that it doesn't depend on the angle at which the outgoing particles emerge: in ϕ^4 theory the leading order two-particle scattering occurs with equal probability in all directions. Translating this into a potential between two mesons, we have

$$U(\vec{r}) = \frac{\lambda}{(2m)^2} \int \frac{d^3p}{(2\pi)^3} \, e^{+i\vec{p}\cdot\vec{r}} = \frac{\lambda}{(2m)^2}\delta^{(3)}(\vec{r}) \tag{3.76}$$

So scattering in ϕ^4 theory is due to a δ-function potential. The particles don't know what hit them until it's over.

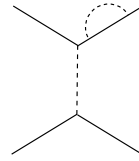### 3.5.4 Connected Diagrams and Amputated Diagrams

We've seen how one can compute scattering amplitudes by writing down all Feynman diagrams and assigning integrals to them using the Feynman rules. In fact, there are a couple of caveats about what Feynman diagrams you should write down. Both of these caveats are related to the assumption we made earlier that "initial and final states are eigenstates of the free theory" which, as we mentioned at the time, is not strictly accurate. The two caveats which go some way towards ameliorating the problem are the following

- We consider only connected Feynman diagrams, where every part of the diagram is connected to at least one external line. As we shall see shortly, this will be related to the fact that the vacuum $|0\rangle$ of the free theory is not the true vacuum $|\Omega\rangle$ of the interacting theory. An example of a diagram that is not connected is shown in Figure 17.

- We do not consider diagrams with loops on external lines, for example the diagram shown in the Figure 18. We will not explain how to take these into account in this course, but you will discuss them next term. They are related to the fact that the one-particle states of the free theory are not the same as the one-particle states of the interacting theory. In particular, correctly dealing with these diagrams will account for the fact that particles in interacting quantum field theories are never alone, but surrounded by a cloud of virtual particles. We will refer to diagrams in which all loops on external legs have been cut-off as "amputated".
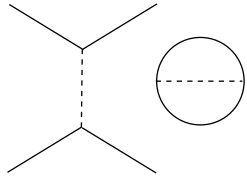
Figure 17: A disconnected diagram.   **Figure 18:** An un-amputated diagram

3.6 What We Measure: Cross Sections and Decay Rates

So far we've learnt to compute the quantum amplitudes for particles decaying or scattering. As usual in quantum theory, the probabilities for things to happen are the (modulus) square of the quantum amplitudes. In this section we will compute these probabilities, known as decay rates and cross sections. One small subtlety here is that the S-matrix elements $\langle f | S - 1 | i \rangle$ all come with a factor of $(2\pi)^4 \delta^{(4)}(p_F - p_I)$, so we end up with the square of a delta-function. As we will now see, this comes from the fact that we're working in an infinite space.

3.6.1 Fermi's Golden Rule

Let's start with something familiar and recall how to derive Fermi's golden rule from Dyson's formula. For two energy eigenstates $|m\rangle$ and $|n\rangle$, with $E_m \neq E_n$, we have to leading order in the interaction,

$$
\begin{aligned}
\langle m | U(t) | n \rangle &= -i \langle m | \int_0^t dt \, H_I(t) | n \rangle \\
&= -i \langle m | H_{\text{int}} | n \rangle \int_0^t dt' \, e^{i\omega t'} \\
&= - \langle m | H_{\text{int}} | n \rangle \, \frac{e^{i\omega t} - 1}{\omega}
\end{aligned} \tag{3.77}
$$

where $\omega = E_m - E_n$. This gives us the probability for the transition from $|n\rangle$ to $|m\rangle$ in time t, as

$$P_{n \to m}(t) = |\langle m| U(t) |n\rangle|^2 = 2|\langle m| H_{\text{int}} |n\rangle|^2 \left(\frac{1 - \cos \omega t}{\omega^2} \right) \tag{3.78}$$

The function in brackets is plotted in Figure 19 for fixed t. We see that in time t, most transitions happen in a region between energy eigenstates separated by $\Delta E = 2\pi/t$. As $t \to \infty$, the function in the figure starts to approach a delta-function. To find the normalization, we can calculate
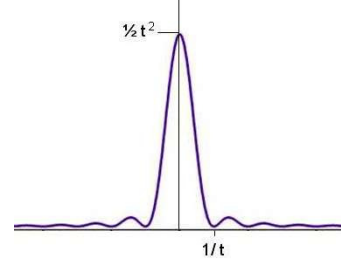
$$\int_{-\infty}^{+\infty} d\omega \left(\frac{1 - \cos \omega t}{\omega^2} \right) = \pi t$$

$$\Rightarrow \quad \left(\frac{1 - \cos \omega t}{\omega^2} \right) \to \pi t \delta(\omega) \quad \text{as} \quad t \to \infty$$

Figure 19:

Consider now a transition to a cluster of states with density $\rho(E)$. In the limit $t \to \infty$, we get the transition probability

$$
\begin{aligned}
P_{n \to m} &= \int dE_m \, \rho(E_m) \, 2|\langle m| H_{\text{int}} |n\rangle|^2 \left(\frac{1 - \cos \omega t}{\omega^2} \right) \\
&\to 2\pi \, |\langle m| H_{\text{int}} |n\rangle|^2 \, \rho(E_n) t
\end{aligned} \tag{3.79}
$$

which gives a constant probability for the transition per unit time for states around the same energy $E_n \sim E_m = E$.

$$\dot{P}_{n \to m} = 2\pi |\langle m| H_{\text{int}} |n\rangle|^2 \rho(E) \tag{3.80}$$

This is Fermi's Golden Rule.

In the above derivation, we were fairly careful with taking the limit as $t \to \infty$. Suppose we were a little sloppier, and first chose to compute the amplitude for the state $|n\rangle$ at $t \to -\infty$ to transition to the state $|m\rangle$ at $t \to +\infty$. Then we get

$$-i \langle m| \int_{t=-\infty}^{t=+\infty} H_I(t) |n\rangle = -i \langle m| H_{int} |n\rangle \, 2\pi \delta(\omega) \tag{3.81}$$

Now when squaring the amplitude to get the probability, we run into the problem of the square of the delta-function: $P_{n \to m} = |\langle m| H_{\text{int}} |n\rangle|^2 (2\pi)^2 \delta(\omega)^2$. Tracking through the previous computations, we realize that the extra infinity is coming because $P_{m \to n}$

is the probability for the transition to happen in infinite time $t \rightarrow \infty$. We can write the delta-functions as

$$(2\pi)^2 \delta(\omega)^2 = (2\pi)\delta(\omega)\, T \qquad (3.82)$$

where T is shorthand for $t \rightarrow \infty$ (we used a very similar trick when looking at the vacuum energy in (2.25)). We now divide out by this power of T to get the transition probability per unit time,

$$\dot{P}_{n\rightarrow m} = 2\pi |\langle m| H_{\text{int}} |n\rangle|^2 \, \delta(\omega) \qquad (3.83)$$

which, after integrating over the density of final states, gives us back Fermi's Golden rule. The reason that we've stressed this point is because, in our field theory calculations, we've computed the amplitudes in the same way as (3.81), and the square of the $\delta^{(4)}$-functions will just be re-interpreted as spacetime volume factors.

3.6.2 Decay Rates

Let's now look at the probability for a single particle $|i\rangle$ of momentum p_I (I=initial) to decay into some number of particles $|f\rangle$ with momentum p_i and total momentum $p_F = \sum_i p_i$. This is given by

$$P = \frac{|\langle f| S |i\rangle|^2}{\langle f| f\rangle \langle i| i\rangle} \qquad (3.84)$$

Our states obey the relativistic normalization formula (2.65),

$$\langle i| i\rangle = (2\pi)^3 \, 2E_{\vec{p}_I} \, \delta^{(3)}(0) = 2E_{\vec{p}_I} V \qquad (3.85)$$

where we have replaced $\delta^{(3)}(0)$ by the volume of 3-space. Similarly,

$$\langle f| f\rangle = \prod_{\text{final states}} 2E_{\vec{p}_i} V \qquad (3.86)$$

If we place our initial particle at rest, so $\vec{p}_I = 0$ and $E_{\vec{p}_I} = m$, we get the probability for decay

$$P = \frac{|\mathcal{A}_{fi}|^2}{2mV} (2\pi)^4 \delta^{(4)}(p_I - p_F) VT \prod_{\text{final states}} \frac{1}{2E_{\vec{p}_i} V} \qquad (3.87)$$

where, as in the second derivation of Fermi's Golden Rule, we've exchanged one of the delta-functions for the volume of spacetime: $(2\pi)^4 \delta^{(4)}(0) = VT$. The amplitudes $\mathcal{A}_{fi}$ are, of course, exactly what we've been computing. (For example, in (3.30), we saw

that $\mathcal{A} = -g$ for a single meson decaying into two nucleons). We can now divide out by T to get the transition function per unit time. But we still have to worry about summing over all final states. There are two steps: the first is to integrate over all possible momenta of the final particles: $V \int d^3 p_i / (2\pi)^3$. The factors of spatial volume V in this measure cancel those in (3.87), while the factors of $1/2E_{\vec{p}_i}$ in (3.87) conspire to produce the Lorentz invariant measure for 3-momentum integrals. The result is an expression for the density of final states given by the Lorentz invariant measure

$$d\Pi = (2\pi)^4 \delta^{(4)}(p_F - p_I) \prod_{\text{final states}} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \tag{3.88}$$

The second step is to sum over all final states with different numbers (and possibly types) of particles. This gives us our final expression for the decay probability per unit time, $\Gamma = \dot{P}$.

$$\Gamma = \frac{1}{2m} \sum_{\text{final states}} \int |\mathcal{A}_{fi}|^2 \, d\Pi \tag{3.89}$$

Γ is called the width of the particle. It is equal to the reciprocal of the half-life $\tau = 1/\Gamma$.

3.6.3 Cross Sections

Collide two beams of particles. Sometimes the particles will hit and bounce off each other; sometimes they will pass right through. The fraction of the time that they collide is called the *cross section* and is denoted by σ. If the incoming flux F is defined to be the number of incoming particles per area per unit time, then the total number of scattering events N per unit time is given by,

$$N = F\sigma \tag{3.90}$$

We would like to calculate σ from quantum field theory. In fact, we can calculate a more sensitive quantity $d\sigma$ known as the *differential cross section* which is the probability for a given scattering process to occur in the solid angle (θ, ϕ). More precisely

$$d\sigma = \frac{\text{Differential Probability}}{\text{Unit Time} \times \text{Unit Flux}} = \frac{1}{4E_1 E_2 V} \frac{1}{F} |\mathcal{A}_{fi}|^2 \, d\Pi \tag{3.91}$$

where we've used the expression for probability per unit time that we computed in the previous subsection. E_1 and E_2 are the energies of the incoming particles. We now need an expression for the unit flux. For simplicity, let's sit in the center of mass frame of the collision. We've been considering just a single particle per spatial volume V,

meaning that the flux is given in terms of the 3-velocities $\vec{v}_i$ as $F = |\vec{v}_1 - \vec{v}_2|/V$. This then gives,

$$d\sigma = \frac{1}{4E_1 E_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} |\mathcal{A}_{fi}|^2 \, d\Pi \tag{3.92}$$

If you want to write this in terms of momentum, then recall from your course on special relativity that the 3-velocities $\vec{v}_i$ are related to the momenta by $\vec{v} = \vec{p}/m\sqrt{1 - v^2} = \vec{p}/p^0$.

Equation (3.92) is our final expression relating the S-matrix to the differential cross section. You may now take your favorite scattering amplitude, and compute the probability for particles to fly out at your favorite angles. This will involve doing the integral over the phase space of final states, with measure $d\Pi$. Notice that different scattering amplitudes have different momentum dependence and will result in different angular dependence in scattering amplitudes. For example, in ϕ^4 theory the amplitude for tree level scattering was simply $\mathcal{A} = -\lambda$. This results in isotropic scattering. In contrast, for nucleon-nucleon scattering we have schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. This gives rise to angular dependence in the differential cross-section, which follows from the fact that, for example, $t = -2|\vec{p}|^2(1 - \cos\theta)$, where θ is the angle between the incoming and outgoing particles.

3.7 Green's Functions

So far we've learnt to compute scattering amplitudes. These are nice and physical (well – they're directly related to cross-sections and decay rates which are physical) but there are many questions we want to ask in quantum field theory that aren't directly related to scattering experiments. For example, we might want to compute the viscosity of the quark gluon plasma, or the optical conductivity in a tentative model of strange metals, or figure out the non-Gaussianity of density perturbations arising in the CMB from novel models of inflation. All of these questions are answered in the framework of quantum field theory by computing elementary objects known as *correlation functions*. In this section we will briefly define correlation functions, explain how to compute them using Feynman diagrams, and then relate them back to scattering amplitudes. We'll leave the relationship to other physical phenomena to other courses.

We'll denote the true vacuum of the interacting theory as $|\Omega\rangle$. We'll normalize H such that

$$H |\Omega\rangle = 0 \tag{3.93}$$

and $\langle \Omega | \Omega \rangle = 1$. Note that this is different from the state we've called $|0\rangle$ which is the vacuum of the free theory and satisfies $H_0 |0\rangle = 0$. Define

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Omega | T \phi_H(x_1) \ldots \phi_H(x_n) |\Omega\rangle \tag{3.94}$$

where ϕ_H is ϕ in the Heisenberg picture of the full theory, rather than the interaction picture that we've been dealing with so far. The $G^{(n)}$ are called correlation functions, or *Green's functions*. There are a number of different ways of looking at these objects which tie together nicely. Let's start by asking how to compute $G^{(n)}$ using Feynman diagrams. We prove the following result

Claim: We use the notation $\phi_1 = \phi(x_1)$, and write ϕ_{1H} to denote the field in the Heisenberg picture, and ϕ_{1I} to denote the field in the interaction picture. Then

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Omega | T \phi_{1H} \ldots \phi_{nH} |\Omega\rangle = \frac{\langle 0 | T \phi_{1I} \ldots \phi_{nI} \, S \, |0\rangle}{\langle 0 | S |0\rangle} \tag{3.95}$$

where the operators on the right-hand side are evaluated on $|0\rangle$, the vacuum of the free theory.

Proof: Take $t_1 > t_2 > \ldots > t_n$. Then we can drop the T and write the numerator of the right-hand side as

$$\langle 0 | U_I(+\infty, t_1)\phi_{1I} \, U(t_1, t_2) \, \phi_{2I} \ldots \phi_{nI} \, U_I(t_n, -\infty) |0\rangle$$

We'll use the factors of $U_I(t_k, t_{k+1}) = T \exp(-i \int_{t_k}^{t_{k+1}} H_I)$ to convert each of the ϕ_I into ϕ_H and we choose operators in the two pictures to be equal at some arbitrary time t_0. Then we can write

$$\langle 0 | U_I(+\infty, t_1)\phi_{1I} \, U(t_1, t_2) \, \phi_{2I} \, \ldots \, \phi_{nI} \, U_I(t_n, -\infty) |0\rangle$$
$$= \langle 0 | U_I(+\infty, t_0)\phi_{1H} \ldots \phi_{nH} \, U_I(t_0, -\infty) |0\rangle$$

Now let's deal with the two remaining $U(t_0, \pm\infty)$ at either end of the string of operators. Consider an arbitrary state $|\Psi\rangle$ and look at

$$\langle \Psi | U_I(t, -\infty) |0\rangle = \langle \Psi | U(t, -\infty) |0\rangle \tag{3.96}$$

where $U(t, -\infty)$ is the Schrödinger evolution operator, and the equality above follows because $H_0 |0\rangle = 0$. Now insert a complete set of states, which we take to be energy eigenstates of $H = H_0 + H_{\text{int}}$,

$$\langle \Psi | U(t, -\infty) |0\rangle = \langle \Psi | U(t, -\infty) \left[|\Omega\rangle \langle \Omega | + \sum_{n \neq 0} |n\rangle \langle n| \right] |0\rangle$$
$$= \langle \Psi | \Omega \rangle \langle \Omega | 0 \rangle + \lim_{t' \to -\infty} \sum_{n \neq 0} e^{iE_n(t'-t)} \langle \Psi | n \rangle \langle n | 0 \rangle \tag{3.97}$$

But the last term vanishes. This follows from the Riemann-Lebesgue lemma which says that for any well-behaved function

$$\lim_{\mu \to \infty} \int_a^b dx \, f(x) e^{i\mu x} = 0 \tag{3.98}$$

Why is this relevant? The point is that the $\sum_n$ in (3.97) is really an integral $\int dn$, because all states are part of a continuum due to the momentum. (There is a caveat here: we want the vacuum $|\Omega\rangle$ to be special, so that it sits on its own, away from the continuum of the integral. This means that we must be working in a theory with a mass gap – i.e. with no massless particles). So the Riemann-Lebesgue lemma gives us

$$\lim_{t' \to -\infty} \langle \Psi | \, U(t, t') \, |0\rangle = \langle \Psi | \, \Omega \rangle \, \langle \Omega | \, 0\rangle \tag{3.99}$$

(Notice that to derive this result, Peskin and Schroeder instead send $t \to -\infty$ in a slightly imaginary direction, which also does the job). We now apply the formula (3.99), to the top and bottom of the right-hand side of (3.95) to find

$$\frac{\langle 0 | \, \Omega \rangle \, \langle \Omega | \, T \phi_{1H} \ldots \phi_{nH} \, |\Omega\rangle \, \langle \Omega | \, 0\rangle}{\langle 0 | \, \Omega \rangle \, \langle \Omega | \, \Omega \rangle \, \langle \Omega | \, 0\rangle} \tag{3.100}$$

which, using the normalization $\langle \Omega | \, \Omega \rangle = 1$, gives us the left-hand side, completing the proof. $\qquad \square$.

3.7.1 Connected Diagrams and Vacuum Bubbles

We're getting closer to our goal of computing the Green's functions $G^{(n)}$ since we can compute both $\langle 0 | \, T \phi_I(x_1) \ldots \phi_I(x_n) \, S \, |0\rangle$ and $\langle 0 | \, S \, |0\rangle$ using the same methods we developed for S-matrix elements; namely Dyson's formula and Wick's theorem or, alternatively, Feynman diagrams. But what about dividing one by the other? What's that all about? In fact, it has a simple interpretation. For the following discussion, we will work in ϕ^4 theory. Since there is no ambiguity in the different types of line in Feynman diagrams, we will represent the ϕ particles as solid lines, rather than the dashed lines that we used previously. Then we have the diagramatic expansion for $\langle 0 | \, S \, |0\rangle$.

$$\langle 0 | \, S \, |0\rangle = 1 + 8 + \left(\, 8 + \oplus + 8 \, 8 \, \right) + \ldots \tag{3.101}$$

These diagrams are called vacuum bubbles. The combinatoric factors (as well as the symmetry factors) associated with each diagram are such that the whole series sums

to an exponential,

$$\langle 0| \, S \, |0\rangle = \exp\left(\, \text{⧖} \; + \; \text{⧗} \; + \; \text{⊕} \; + \; ... \right) \tag{3.102}$$

So the amplitude for the vacuum of the free theory to evolve into itself is $\langle 0| \, S \, |0\rangle =$ exp(all distinct vacuum bubbles). A similar combinatoric simplification occurs for generic correlation functions. Remarkably, the vacuum diagrams all add up to give the same exponential. With a little thought one can show that

$$\langle 0| \, T\phi_1 \ldots \phi_n S \, |0\rangle = \left(\sum \text{connected diagrams} \right) \langle 0| \, S \, |0\rangle \tag{3.103}$$

where "connected" means that every part of the diagram is connected to at least one of the external legs. The upshot of all this is that dividing by $\langle 0| \, S \, |0\rangle$ has a very nice interpretation in terms of Feynman diagrams: we need only consider the connected Feynman diagrams, and don't have to worry about the vacuum bubbles. Combining this with (3.95), we learn that the Green's functions $G^{(n)}(x_1 \ldots, x_n)$ can be calculated by summing over all connected Feynman diagrams,

$$\langle \Omega| \, T \, \phi_H(x_1) \ldots \phi_H(x_n) \, |\Omega\rangle = \sum \text{Connected Feynman Graphs} \tag{3.104}$$

An Example: The Four-Point Correlator: $\langle \Omega| \, T\phi_H(x_1) \ldots \phi_H(x_4) \, |\Omega\rangle$

As a simple example, let's look at the four-point correlation function in ϕ^4 theory. The sum of connected Feynman diagrams is given by,

All of these are connected diagrams, even though they don't look that connected! The point is that a connected diagram is defined by the requirement that every line is joined to an external leg. An example of a diagram that is not connected is shown in the figure. As we have seen, such diagrams are taken care of in shifting the vacuum from $|0\rangle$ to $|\Omega\rangle$.
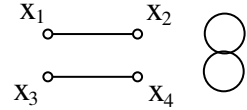

Figure 20:

Feynman Rules

The Feynman diagrams that we need to calculate for the Green's functions depend on $x_1, \ldots, x_n$. This is rather different than the Feynman diagrams that we calculated for

the S-matrix elements, where we were working primarily with momentum eigenstates, and ended up integrating over all of space. However, it's rather simple to adapt the Feynman rules that we had earlier in momentum space to compute $G^{(n)}(x_1 \ldots, x_n)$. For ϕ^4 theory, we have

- Draw n external points $x_1, \ldots, x_n$, connected by the usual propagators and vertices. Assign a spacetime position y to the end of each line.

- For each line $\text{x} \!\!\!-\!\!\!-\!\!\!-\!\!\! \text{y}$ from x to y write down a factor of the Feynman propagator $\Delta_F(x - y)$.

- For each vertex $\diagdown\!\!\!\!\diagup$ y at position y, write down a factor of $-i\lambda \int d^4y$.

3.7.2 From Green's Functions to S-Matrices

Having described how to compute correlation functions using Feynman diagrams, let's now relate them back to the S-matrix elements that we already calculated. The first step is to perform the Fourier transform,

$$\tilde{G}^{(n)}(p_1, \ldots, p_n) = \int \left[\prod_{i=1}^{n} d^4x_i e^{-ip_i \cdot x_i} \right] G^{(n)}(x_1, \ldots, x_n) \tag{3.105}$$

These are very closely related to the S-matrix elements that we've computed above. The difference is that the Feynman rules for $G^{(n)}(x_1, \ldots, x_n)$, effectively include propagators Δ_F for the external legs, as well as the internal legs. A related fact is that the 4-momenta assigned to the external legs is arbitrary: they are not on-shell. Both of these problems are easily remedied to allow us to return to the S-matrix elements: we need to simply cancel off the propagators on the external legs, and place their momentum back on shell. We have

$$\langle p'_1, \ldots, p'_{n'} | S - 1 | p_1 \ldots, p_n \rangle = (-i)^{n+n'} \prod_{i=1}^{n'} (p_i'^2 - m^2) \prod_{j=1}^{n} (p_j^2 - m^2) \tag{3.106}$$

$$\times \tilde{G}^{(n+n')}(-p'_1, \ldots, -p'_{n'}, p_1, \ldots, p_n)$$

Each of the factors $(p^2 - m^2)$ vanishes once the momenta are placed on-shell. This means that we only get a non-zero answer for diagrams contributing to $G^{(n)}(x_1, \ldots, x_n)$ which have propagators for each external leg.

So what's the point of all of this? We've understood that ignoring the unconnected diagrams is related to shifting to the true vacuum $|\Omega\rangle$. But other than that, introducing the Green's functions seems like a lot of bother for little reward. The important point

is that this provides a framework in which to deal with the true particle states in the interacting theory through renormalization. Indeed, the formula (3.106), suitably interpreted, remains true even in the interacting theory, taking into account the swarm of virtual particles surrounding asymptotic states. This is the correct way to consider scattering. In this context, (3.106) is known as the LSZ reduction formula. You will derive it properly next term.

4. The Dirac Equation

> "A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928. Dirac himself remarked in one of his talks that his equation was more intelligent than its author. It should be added, however, that it was Dirac who found most of the additional insights."
>
> *Weisskopf on Dirac*

So far we've only discussed scalar fields such that under a Lorentz transformation $x^\mu \to (x')^\mu = \Lambda^\mu_\nu x^\nu$, the field transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x) \tag{4.1}$$

We have seen that quantization of such fields gives rise to spin 0 particles. But most particles in Nature have an intrinsic angular momentum, or spin. These arise naturally in field theory by considering fields which themselves transform non-trivially under the Lorentz group. In this section we will describe the Dirac equation, whose quantization gives rise to fermionic spin $1/2$ particles. To motivate the Dirac equation, we will start by studying the appropriate representation of the Lorentz group.

A familiar example of a field which transforms non-trivially under the Lorentz group is the vector field $A_\mu(x)$ of electromagnetism,

$$A^\mu(x) \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1} x) \tag{4.2}$$

We'll deal with this in Section 6. (It comes with its own problems!). In general, a field can transform as

$$\phi^a(x) \to D[\Lambda]^a_b \, \phi^b(\Lambda^{-1} x) \tag{4.3}$$

where the matrices $D[\Lambda]$ form a *representation* of the Lorentz group, meaning that

$$D[\Lambda_1] D[\Lambda_2] = D[\Lambda_1 \Lambda_2] \tag{4.4}$$

and $D[\Lambda^{-1}] = D[\Lambda]^{-1}$ and $D[1] = 1$. How do we find the different representations? Typically, we look at infinitesimal transformations of the Lorentz group and study the resulting Lie algebra. If we write,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \tag{4.5}$$

for infinitesimal ω, then the condition for a Lorentz transformation $\Lambda^\mu_\sigma \Lambda^\nu_\rho \eta^{\sigma\rho} = \eta^{\mu\nu}$ becomes the requirement that ω is anti-symmetric:

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0 \tag{4.6}$$

Note that an antisymmetric 4×4 matrix has $4 \times 3/2 = 6$ independent components, which agrees with the 6 transformations of the Lorentz group: 3 rotations and 3 boosts. It's going to be useful to introduce a basis of these six 4×4 anti-symmetric matrices. We could call them $(\mathcal{M}^A)^{\mu\nu}$, with $A = 1, \ldots, 6$. But in fact it's better for us (although initially a little confusing) to replace the single index A with a pair of antisymmetric indices $[\rho\sigma]$, where $\rho, \sigma = 0, \ldots, 3$, so we call our matrices $(\mathcal{M}^{\rho\sigma})^{\mu}_{\ \nu}$. The antisymmetry on the ρ and σ indices means that, for example, $\mathcal{M}^{01} = -\mathcal{M}^{10}$, etc, so that ρ and σ again label six different matrices. Of course, the matrices are also antisymmetric on the $\mu\nu$ indices because they are, after all, antisymmetric matrices. With this notation in place, we can write a basis of six 4×4 antisymmetric matrices as

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu} \tag{4.7}$$

where the indices μ and ν are those of the 4×4 matrix, while ρ and σ denote which basis element we're dealing with. If we use these matrices for anything practical (for example, if we want to multiply them together, or act on some field) we will typically need to lower one index, so we have

$$(\mathcal{M}^{\rho\sigma})^{\mu}_{\ \nu} = \eta^{\rho\mu}\delta^{\sigma}_{\ \nu} - \eta^{\sigma\mu}\delta^{\rho}_{\ \nu} \tag{4.8}$$

Since we lowered the index with the Minkowski metric, we pick up various minus signs which means that when written in this form, the matrices are no longer necessarily antisymmetric. Two examples of these basis matrices are,

$$(\mathcal{M}^{01})^{\mu}_{\ \nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\mathcal{M}^{12})^{\mu}_{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

The first, $\mathcal{M}^{01}$, generates boosts in the x^1 direction. It is real and symmetric. The second, $\mathcal{M}^{12}$, generates rotations in the (x^1, x^2)-plane. It is real and antisymmetric. We can now write any $\omega^{\mu}_{\ \nu}$ as a linear combination of the $\mathcal{M}^{\rho\sigma}$,

$$\omega^{\mu}_{\ \nu} = \tfrac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}_{\ \nu} \tag{4.10}$$

where $\Omega_{\rho\sigma}$ are just six numbers (again antisymmetric in the indices) that tell us what Lorentz transformation we're doing. The six basis matrices $\mathcal{M}^{\rho\sigma}$ are called the *generators* of the Lorentz transformations. The generators obey the Lorentz Lie algebra relations,

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau} \tag{4.11}$$

where we have suppressed the matrix indices. A finite Lorentz transformation can then be expressed as the exponential

$$\Lambda = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \tag{4.12}$$

Let me stress again what each of these objects are: the $\mathcal{M}^{\rho\sigma}$ are six 4×4 basis elements of the Lorentz Lie algebra; the $\Omega_{\rho\sigma}$ are six numbers telling us what kind of Lorentz transformation we're doing (for example, they say things like rotate by $\theta = \pi/7$ about the x^3-direction and run at speed $v = 0.2$ in the x^1 direction).

4.1 The Spinor Representation

We're interested in finding other matrices which satisfy the Lorentz algebra commutation relations (4.11). We will construct the spinor representation. To do this, we start by defining something which, at first sight, has nothing to do with the Lorentz group. It is the *Clifford algebra*,

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1} \tag{4.13}$$

where γ^μ, with $\mu = 0, 1, 2, 3$, are a set of four matrices and the $\mathbf{1}$ on the right-hand side denotes the unit matrix. This means that we must find four matrices such that

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{when } \mu \neq \nu \tag{4.14}$$

and

$$(\gamma^0)^2 = 1 \quad , \quad (\gamma^i)^2 = -1 \quad i = 1, 2, 3 \tag{4.15}$$

It's not hard to convince yourself that there are no representations of the Clifford algebra using 2×2 or 3×3 matrices. The simplest representation of the Clifford algebra is in terms of 4×4 matrices. There are many such examples of 4×4 matrices which obey (4.13). For example, we may take

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.16}$$

where each element is itself a 2×2 matrix, with the σ^i the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.17}$$

which themselves satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

One can construct many other representations of the Clifford algebra by taking $V\gamma^\mu V^{-1}$ for any invertible matrix V. However, up to this equivalence, it turns out that there is a unique irreducible representation of the Clifford algebra. The matrices (4.16) provide one example, known as the *Weyl* or *chiral representation* (for reasons that will soon become clear). We will soon restrict ourselves further, and consider only representations of the Clifford algebra that are related to the chiral representation by a unitary transformation V.

So what does the Clifford algebra have to do with the Lorentz group? Consider the commutator of two γ^μ,

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \left\{ \begin{array}{cc} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{array} \right\} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma} \tag{4.18}$$

Let's see what properties these matrices have:

Claim 4.1: $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$

Proof: When $\mu \neq \nu$ we have

$$\begin{aligned}
[S^{\mu\nu}, \gamma^\rho] &= \tfrac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\
&= \tfrac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \tfrac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu \\
&= \tfrac{1}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu - \tfrac{1}{2}\{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu \\
&= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \hspace{5cm} \square
\end{aligned}$$

Claim 4.2: The matrices $S^{\mu\nu}$ form a representation of the Lorentz algebra (4.11), meaning

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho} \tag{4.19}$$

Proof: Taking $\rho \neq \sigma$, and using Claim 4.1 above, we have

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] \\
&= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \tfrac{1}{2}\gamma^\rho[S^{\mu\nu}, \gamma^\sigma] \\
&= \tfrac{1}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \tfrac{1}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \tfrac{1}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \tfrac{1}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu}
\end{aligned} \tag{4.20}$$

Now using the expression (4.18) to write $\gamma^\mu\gamma^\sigma = 2S^{\mu\sigma} + \eta^{\mu\sigma}$, we have

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu} \tag{4.21}$$

which is our desired expression. $\hspace{8cm}\square$

4.1.1 Spinors

The $S^{\mu\nu}$ are 4×4 matrices, because the γ^μ are 4×4 matrices. So far we haven't given an index name to the rows and columns of these matrices: we're going to call them $\alpha, \beta = 1, 2, 3, 4$.

We need a field for the matrices $(S^{\mu\nu})^\alpha{}_\beta$ to act upon. We introduce the Dirac *spinor* field $\psi^\alpha(x)$, an object with four complex components labelled by $\alpha = 1, 2, 3, 4$. Under Lorentz transformations, we have

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \, \psi^\beta(\Lambda^{-1}x) \tag{4.22}$$

where

$$\Lambda = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \tag{4.23}$$

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}S^{\rho\sigma}\right) \tag{4.24}$$

Although the basis of generators $\mathcal{M}^{\rho\sigma}$ and $S^{\rho\sigma}$ are different, we use the same six numbers $\Omega_{\rho\sigma}$ in both Λ and $S[\Lambda]$: this ensures that we're doing the same Lorentz transformation on x and ψ. Note that we denote both the generator $S^{\rho\sigma}$ and the full Lorentz transformation $S[\Lambda]$ as "S". To avoid confusion, the latter will always come with the square brackets $[\Lambda]$.

Both Λ and $S[\Lambda]$ are 4×4 matrices. So how can we be sure that the spinor representation is something new, and isn't equivalent to the familiar representation $\Lambda^\mu{}_\nu$? To see that the two representations are truly different, let's look at some specific transformations.

Rotations

$$S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\,\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad \text{(for } i \neq j\text{)} \tag{4.25}$$

If we write the rotation parameters as $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ (meaning $\Omega_{12} = -\varphi^3$, etc) then the rotation matrix becomes

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{+i\,\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{4.26}$$

where we need to remember that $\Omega_{12} = -\Omega_{21} = -\varphi^3$ when following factors of 2. Consider now a rotation by 2π about, say, the x^3-axis. This is achieved by $\vec{\varphi} = (0, 0, 2\pi)$,

and the spinor rotation matrix becomes,

$$S[\Lambda] = \begin{pmatrix} e^{+i\pi\sigma^3} & 0 \\ 0 & e^{+i\pi\sigma^3} \end{pmatrix} = -1 \tag{4.27}$$

Therefore under a 2π rotation

$$\psi^\alpha(x) \to -\psi^\alpha(x) \tag{4.28}$$

which is definitely not what happens to a vector! To check that we haven't been cheating with factors of 2, let's see how a vector would transform under a rotation by $\vec{\varphi} = (0, 0, \varphi^3)$. We have

$$\Lambda = \exp\left(\tfrac{1}{2}\, \Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi^3 & 0 \\ 0 & -\varphi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.29}$$

So when we rotate a vector by $\varphi^3 = 2\pi$, we learn that $\Lambda = 1$ as you would expect. So $S[\Lambda]$ is definitely a different representation from the familiar vector representation $\Lambda^\mu{}_\nu$.

Boosts

$$S^{0i} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{4.30}$$

Writing the boost parameter as $\Omega_{i0} = -\Omega_{0i} = \chi_i$, we have

$$S[\Lambda] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{4.31}$$

Representations of the Lorentz Group are not Unitary

Note that for rotations given in (4.26), $S[\Lambda]$ is unitary, satisfying $S[\Lambda]^\dagger S[\Lambda] = 1$. But for boosts given in (4.31), $S[\Lambda]$ is not unitary. In fact, there are *no* finite dimensional unitary representations of the Lorentz group. We have demonstrated this explicitly for the spinor representation using the chiral representation (4.16) of the Clifford algebra. We can get a feel for why it is true for a spinor representation constructed from any representation of the Clifford algebra. Recall that

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}S^{\rho\sigma}\right) \tag{4.32}$$

so the representation is unitary if $S^{\mu\nu}$ are anti-hermitian, i.e. $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$. But we have

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger] \tag{4.33}$$

which can be anti-hermitian if all γ^μ are hermitian or all are anti-hermitian. However, we can never arrange for this to happen since

$$
\begin{aligned}
(\gamma^0)^2 = 1 &\quad\Rightarrow\quad \text{Real Eigenvalues} \\
(\gamma^i)^2 = -1 &\quad\Rightarrow\quad \text{Imaginary Eigenvalues}
\end{aligned}
\tag{4.34}
$$

So we could pick γ^0 to be hermitian, but we can only pick γ^i to be anti-hermitian. Indeed, in the chiral representation (4.16), the matrices have this property: $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. In general there is no way to pick γ^μ such that $S^{\mu\nu}$ are anti-hermitian.

4.2 Constructing an Action

We now have a new field to work with, the Dirac spinor ψ. We would like to construct a Lorentz invariant equation of motion. We do this by constructing a Lorentz invariant action.

We will start in a naive way which won't work, but will give us a clue how to proceed. Define

$$
\psi^\dagger(x) = (\psi^\star)^T(x)
\tag{4.35}
$$

which is the usual adjoint of a multi-component object. We could then try to form a Lorentz scalar by taking the product $\psi^\dagger \psi$, with the spinor indices summed over. Let's see how this transforms under Lorentz transformations,

$$
\begin{aligned}
\psi(x) &\rightarrow S[\Lambda]\,\psi(\Lambda^{-1}x) \\
\psi^\dagger(x) &\rightarrow \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger
\end{aligned}
\tag{4.36}
$$

So $\psi^\dagger(x)\psi(x) \rightarrow \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger S[\Lambda]\psi(\Lambda^{-1}x)$. But, as we have seen, for some Lorentz transformation $S[\Lambda]^\dagger S[\Lambda] \neq 1$ since the representation is not unitary. This means that $\psi^\dagger\psi$ isn't going to do it for us: it doesn't have any nice transformation under the Lorentz group, and certainly isn't a scalar. But now we see why it fails, we can also see how to proceed. Let's pick a representation of the Clifford algebra which, like the chiral representation (4.16), satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. Then for all $\mu = 0, 1, 2, 3$ we have

$$
\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger
\tag{4.37}
$$

which, in turn, means that

$$
(S^{\mu\nu})^\dagger = \frac{1}{4}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = -\gamma^0 S^{\mu\nu}\gamma^0
\tag{4.38}
$$

so that

$$S[\Lambda]^\dagger = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \gamma^0\,S[\Lambda]^{-1}\gamma^0 \tag{4.39}$$

With this in mind, we now define the *Dirac adjoint*

$$\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0 \tag{4.40}$$

Let's now see what Lorentz covariant objects we can form out of a Dirac spinor ψ and its adjoint $\bar{\psi}$.

Claim 4.3: $\bar{\psi}\psi$ is a Lorentz scalar.

Proof: Under a Lorentz transformation,

$$\begin{aligned}
\bar{\psi}(x)\,\psi(x) &= \psi^\dagger(x)\,\gamma^0\,\psi(x) \\
&\rightarrow \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger\gamma^0\,S[\Lambda]\psi(\Lambda^{-1}x) \\
&= \psi^\dagger(\Lambda^{-1}x)\,\gamma^0\,\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)\,\psi(\Lambda^{-1}x)
\end{aligned} \tag{4.41}$$

which is indeed the transformation law for a Lorentz scalar. $\qquad\square$

Claim 4.4: $\bar{\psi}\,\gamma^\mu\,\psi$ is a Lorentz vector, which means that

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \rightarrow \Lambda^\mu{}_\nu\,\bar{\psi}(\Lambda^{-1}x)\,\gamma^\nu\,\psi(\Lambda^{-1}x) \tag{4.42}$$

This equation means that we can treat the $\mu = 0, 1, 2, 3$ index on the γ^μ matrices as a true vector index. In particular we can form Lorentz scalars by contracting it with other Lorentz indices.

Proof: Suppressing the x argument, under a Lorentz transformation we have,

$$\bar{\psi}\,\gamma^\mu\,\psi \rightarrow \bar{\psi}\,S[\Lambda]^{-1}\,\gamma^\mu\,S[\Lambda]\psi \tag{4.43}$$

If $\bar{\psi}\,\gamma^\mu\,\psi$ is to transform as a vector, we must have

$$S[\Lambda]^{-1}\,\gamma^\mu\,S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu \tag{4.44}$$

We'll now show this. We work infinitesimally, so that

$$\Lambda = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \approx 1 + \tfrac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma} + \dots \tag{4.45}$$

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}S^{\rho\sigma}\right) \approx 1 + \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots \tag{4.46}$$

so the requirement (4.44) becomes

$$-[S^{\rho\sigma}, \gamma^\mu] = (\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu \qquad (4.47)$$

where we've suppressed the α, β indices on γ^μ and $S^{\mu\nu}$, but otherwise left all other indices explicit. In fact equation (4.47) follows from Claim 4.1 where we showed that $[S^{\rho\sigma}, \gamma^\mu] = \gamma^\rho \eta^{\sigma\mu} - \gamma^\sigma \eta^{\mu\rho}$. To see this, we write the right-hand side of (4.47) by expanding out $\mathcal{M}$,

$$
\begin{aligned}
(\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu &= (\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)\gamma^\nu \\
&= \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho
\end{aligned}
\qquad (4.48)
$$

which means that the proof follows if we can show

$$-[S^{\rho\sigma}, \gamma^\mu] = \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho \qquad (4.49)$$

which is exactly what we proved in Claim 4.1. $\qquad\square$

Claim 4.5: $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor. More precisely, the symmetric part is a Lorentz scalar, proportional to $\eta^{\mu\nu}\,\bar{\psi}\psi$, while the antisymmetric part is a Lorentz tensor, proportional to $\bar{\psi}S^{\mu\nu}\psi$.

Proof: As above. $\qquad\square$

We are now armed with three bilinears of the Dirac field, $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^\nu\psi$, each of which transforms covariantly under the Lorentz group. We can try to build a Lorentz invariant action from these. In fact, we need only the first two. We choose

$$S = \int d^4x \; \bar{\psi}(x) \; (i\gamma^\mu\partial_\mu - m) \; \psi(x) \qquad (4.50)$$

This is the Dirac action. The factor of "i" is there to make the action real; upon complex conjugation, it cancels a minus sign that comes from integration by parts. (Said another way, it's there for the same reason that the Hermitian momentum operator $-i\nabla$ in quantum mechanics has a factor i). As we will see in the next section, after quantization this theory describes particles and anti-particles of mass $|m|$ and spin $1/2$. Notice that the Lagrangian is first order, rather than the second order Lagrangians we were working with for scalar fields. Also, the mass appears in the Lagrangian as m, which can be positive or negative.

4.3 The Dirac Equation

The equation of motion follows from the action (4.50) by varying with respect to ψ and $\bar{\psi}$ independently. Varying with respect to $\bar{\psi}$, we have

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \qquad (4.51)$$

This is the *Dirac equation*. It's completely gorgeous. Varying with respect to ψ gives the conjugate equation

$$i\partial_\mu \bar{\psi}\,\gamma^\mu + m\bar{\psi} = 0 \qquad (4.52)$$

The Dirac equation is first order in derivatives, yet miraculously Lorentz invariant. If we tried to write down a first order equation of motion for a scalar field, it would look like $v^\mu \partial_\mu \phi = \ldots$, which necessarily includes a privileged vector in spacetime v^μ and is not Lorentz invariant. However, for spinor fields, the magic of the γ^μ matrices means that the Dirac Lagrangian is Lorentz invariant.

The Dirac equation mixes up different components of ψ through the matrices γ^μ. However, each individual component itself solves the Klein-Gordon equation. To see this, write

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\,\psi = -\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2\right)\psi = 0 \qquad (4.53)$$

But $\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu$, so we get

$$-(\partial_\mu \partial^\mu + m^2)\psi = 0 \qquad (4.54)$$

where this last equation has no γ^μ matrices, and so applies to each component ψ^α, with $\alpha = 1, 2, 3, 4$.

The Slash

Let's introduce some useful notation. We will often come across 4-vectors contracted with γ^μ matrices. We write

$$A_\mu \gamma^\mu \equiv \slashed{A} \qquad (4.55)$$

so the Dirac equation reads

$$(i\slashed{\partial} - m)\psi = 0 \qquad (4.56)$$

4.4 Chiral Spinors

When we've needed an explicit form of the γ^μ matrices, we've used the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.57}$$

In this representation, the spinor rotation transformation $S[\Lambda_{\text{rot}}]$ and boost transformation $S[\Lambda_{\text{boost}}]$ were computed in (4.26) and (4.31). Both are block diagonal,

$$S[\Lambda_{\text{rot}}] = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \quad \text{and} \quad S[\Lambda_{\text{boost}}] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{4.58}$$

This means that the Dirac spinor representation of the Lorentz group is *reducible*. It decomposes into two irreducible representations, acting only on two-component spinors $u_\pm$ which, in the chiral representation, are defined by

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{4.59}$$

The two-component objects $u_\pm$ are called *Weyl spinors* or *chiral spinors*. They transform in the same way under rotations,

$$u_\pm \to e^{i\vec{\varphi}\cdot\vec{\sigma}/2}\, u_\pm \tag{4.60}$$

but oppositely under boosts,

$$u_\pm \to e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm \tag{4.61}$$

In group theory language, u_+ is in the $(\frac{1}{2},0)$ representation of the Lorentz group, while u_- is in the $(0,\frac{1}{2})$ representation. The Dirac spinor ψ lies in the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation. (Strictly speaking, the spinor is a representation of the double cover of the Lorentz group $SL(2, \mathbf{C})$).

4.4.1 The Weyl Equation

Let's see what becomes of the Dirac Lagrangian under the decomposition (4.59) into Weyl spinors. We have

$$\mathcal{L} = \bar{\psi}(i\,\slashed{\partial} - m)\psi = iu_-^\dagger \sigma^\mu \partial_\mu u_- + iu_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+) = 0 \tag{4.62}$$

where we have introduced some new notation for the Pauli matrices with a $\mu = 0, 1, 2, 3$ index,

$$\sigma^\mu = (1, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (1, -\sigma^i) \tag{4.63}$$

From (4.62), we see that a massive fermion requires both u_+ and u_-, since they couple through the mass term. However, a massless fermion can be described by u_+ (or u_-) alone, with the equation of motion

$$i\bar{\sigma}^\mu \partial_\mu u_+ = 0$$
$$\text{or} \quad i\sigma^\mu \partial_\mu u_- = 0 \tag{4.64}$$

These are the *Weyl equations*.

Degrees of Freedom

Let me comment here on the degrees of freedom in a spinor. The Dirac fermion has 4 complex components = 8 real components. How do we count degrees of freedom? In classical mechanics, the number of degrees of freedom of a system is equal to the dimension of the configuration space or, equivalently, half the dimension of the phase space. In field theory we have an infinite number of degrees of freedom, but it makes sense to count the number of degrees of freedom per spatial point: this should at least be finite. For example, in this sense a real scalar field ϕ has a single degree of freedom. At the quantum level, this translates to the fact that it gives rise to a single type of particle. A classical complex scalar field has two degrees of freedom, corresponding to the particle and the anti-particle in the quantum theory.

But what about a Dirac spinor? One might think that there are 8 degrees of freedom. But this isn't right. Crucially, and in contrast to the scalar field, the equation of motion is first order rather than second order. In particular, for the Dirac Lagrangian, the momentum conjugate to the spinor ψ is given by

$$\pi_\psi = \partial \mathcal{L}/\partial \dot{\psi} = i\psi^\dagger \tag{4.65}$$

It is not proportional to the time derivative of ψ. This means that the phase space for a spinor is therefore parameterized by ψ and $\psi^\dagger$, while for a scalar it is parameterized by ϕ and $\pi = \dot{\phi}$. So the *phase space* of the Dirac spinor ψ has 8 real dimensions and correspondingly the number of real degrees of freedom is 4. We will see in the next section that, in the quantum theory, this counting manifests itself as two degrees of freedom (spin up and down) for the particle, and a further two for the anti-particle.

A similar counting for the Weyl fermion tells us that it has two degrees of freedom.

4.4.2 γ^5

The Lorentz group matrices $S[\Lambda]$ came out to be block diagonal in (4.58) because we chose the specific representation (4.57). In fact, this is why the representation (4.57) is called the chiral representation: it's because the decomposition of the Dirac spinor ψ is simply given by (4.59). But what happens if we choose a different representation γ^μ of the Clifford algebra, so that

$$\gamma^\mu \to U\gamma^\mu U^{-1} \quad \text{and} \quad \psi \to U\psi \ ? \tag{4.66}$$

Now $S[\Lambda]$ will not be block diagonal. Is there an invariant way to define chiral spinors? We can do this by introducing the "fifth" gamma-matrix

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.67}$$

You can check that this matrix satisfies

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{and} \quad (\gamma^5)^2 = +1 \tag{4.68}$$

The reason that this is called γ^5 is because the set of matrices $\tilde{\gamma}^A = (\gamma^\mu, i\gamma^5)$, with $A = 0, 1, 2, 3, 4$ satisfy the five-dimensional Clifford algebra $\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\eta^{AB}$. (You might think that γ^4 would be a better name! But γ^5 is the one everyone chooses - it's a more sensible name in Euclidean space, where $A = 1, 2, 3, 4, 5$). You can also check that $[S_{\mu\nu}, \gamma^5] = 0$, which means that γ^5 is a scalar under rotations and boosts. Since $(\gamma^5)^2 = 1$, this means we may form the Lorentz invariant projection operators

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma^5\right) \tag{4.69}$$

such that $P_+^2 = P_+$ and $P_-^2 = P_-$ and $P_+P_- = 0$. One can check that for the chiral representation (4.57),

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.70}$$

from which we see that the operators $P_\pm$ project onto the Weyl spinors $u_\pm$. However, for an arbitrary representation of the Clifford algebra, we may use γ^5 to define the chiral spinors,

$$\psi_\pm = P_\pm\psi \tag{4.71}$$

which form the irreducible representations of the Lorentz group. ψ_+ is often called a "left-handed" spinor, while ψ_- is "right-handed". The name comes from the way the spin precesses as a massless fermion moves: we'll see this in Section 4.7.2.

4.4.3 Parity

The spinors $\psi_\pm$ are related to each other by *parity*. Let's pause to define this concept. The Lorentz group is defined by $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ such that

$$\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho} \tag{4.72}$$

So far we have only considered transformations Λ which are continuously connected to the identity; these are the ones which have an infinitesimal form. However there are also two discrete symmetries which are part of the Lorentz group. They are

$$\text{Time Reversal } T : x^0 \to -x^0 \ ; \ x^i \to x^i$$
$$\text{Parity } P : x^0 \to x^0 \ ; \ x^i \to -x^i \tag{4.73}$$

We won't discuss time reversal too much in this course. (It turns out to be represented by an anti-unitary transformation on states. See, for example the book by Peskin and Schroeder). But parity has an important role to play in the standard model and, in particular, the theory of the weak interaction.

Under parity, the left and right-handed spinors are exchanged. This follows from the transformation of the spinors under the Lorentz group. In the chiral representation, we saw that the rotation (4.60) and boost (4.61) transformations for the Weyl spinors $u_\pm$ are

$$u_\pm \xrightarrow{\text{rot}} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_\pm \quad \text{and} \quad u_\pm \xrightarrow{\text{boost}} e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm \tag{4.74}$$

Under parity, rotations don't change sign. But boosts do flip sign. This confirms that parity exchanges right-handed and left-handed spinors, $P : u_\pm \to u_\mp$, or in the notation $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$, we have

$$P : \psi_\pm(\vec{x}, t) \to \psi_\mp(-\vec{x}, t) \tag{4.75}$$

Using this knowledge of how chiral spinors transform, and the fact that $P^2 = 1$, we see that the action of parity on the Dirac spinor itself can be written as

$$P : \psi(\vec{x}, t) \to \gamma^0 \psi(-\vec{x}, t) \tag{4.76}$$

Notice that if $\psi(\vec{x}, t)$ satisfies the Dirac equation, then the parity transformed spinor $\gamma^0 \psi(-\vec{x}, t)$ also satisfies the Dirac equation, meaning

$$(i\gamma^0 \partial_t + i\gamma^i \partial_i - m)\gamma^0 \psi(-\vec{x}, t) = \gamma^0 (i\gamma^0 \partial_t - i\gamma^i \partial_i - m)\psi(-\vec{x}, t) = 0 \tag{4.77}$$

where the extra minus sign from passing γ^0 through γ^i is compensated by the derivative acting on $-\vec{x}$ instead of $+\vec{x}$.

4.4.4 Chiral Interactions

Let's now look at how our interaction terms change under parity. We can look at each of our spinor bilinears from which we built the action,

$$P: \quad \bar{\psi}\psi(\vec{x}, t) \to \bar{\psi}\psi(-\vec{x}, t) \tag{4.78}$$

which is the transformation of a scalar. For the vector $\bar{\psi}\gamma^\mu\psi$, we can look at the temporal and spatial components separately,

$$
\begin{aligned}
P: &\quad \bar{\psi}\gamma^0\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\psi(-\vec{x}, t) \\
P: &\quad \bar{\psi}\gamma^i\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^i\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^i\psi(-\vec{x}, t)
\end{aligned}
\tag{4.79}
$$

which tells us that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector, with the spatial part changing sign. You can also check that $\bar{\psi}S^{\mu\nu}\psi$ transforms as a suitable tensor.

However, now we've discovered the existence of γ^5, we can form another Lorentz scalar and another Lorentz vector,

$$\bar{\psi}\gamma^5\psi \quad \text{and} \quad \bar{\psi}\gamma^5\gamma^\mu\psi \tag{4.80}$$

How do these transform under parity? We can check:

$$P: \quad \bar{\psi}\gamma^5\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^5\gamma^0 \ \psi(-\vec{x}, t) = -\bar{\psi}\gamma^5\psi(-\vec{x}, t) \tag{4.81}$$

$$P: \quad \bar{\psi}\gamma^5\gamma^\mu\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\psi(-\vec{x}, t) = \begin{cases} -\bar{\psi}\gamma^5\gamma^0\psi(-\vec{x}, t) & \mu = 0 \\ +\bar{\psi}\gamma^5\gamma^i\psi(-\vec{x}, t) & \mu = i \end{cases}$$

which means that $\bar{\psi}\gamma^5\psi$ transforms as a *pseudoscalar*, while $\bar{\psi}\gamma^5\gamma^\mu\psi$ transforms as an *axial vector*. To summarize, we have the following spinor bilinears,

$$
\begin{aligned}
\bar{\psi}\psi : &\quad \text{scalar} \\
\bar{\psi}\gamma^\mu\psi : &\quad \text{vector} \\
\bar{\psi}S^{\mu\nu}\psi : &\quad \text{tensor} \\
\bar{\psi}\gamma^5\psi : &\quad \text{pseudoscalar} \\
\bar{\psi}\gamma^5\gamma^\mu\psi : &\quad \text{axial vector}
\end{aligned}
\tag{4.82}
$$

The total number of bilinears is $1 + 4 + (4 \times 3/2) + 4 + 1 = 16$ which is all we could hope for from a 4-component object.

We're now armed with new terms involving γ^5 that we can start to add to our Lagrangian to construct new theories. Typically such terms will break parity invariance of the theory, although this is not always true. (For example, the term $\phi\bar{\psi}\gamma^5\psi$ doesn't break parity if ϕ is itself a pseudoscalar). Nature makes use of these parity violating interactions by using γ^5 in the weak force. A theory which treats $\psi_\pm$ on an equal footing is called a *vector-like theory*. A theory in which ψ_+ and ψ_- appear differently is called a *chiral theory*.

4.5 Majorana Fermions

Our spinor ψ^α is a complex object. It has to be because the representation $S[\Lambda]$ is typically also complex. This means that if we were to try to make ψ real, for example by imposing $\psi = \psi^\star$, then it wouldn't stay that way once we make a Lorentz transformation. However, there is a way to impose a reality condition on the Dirac spinor ψ. To motivate this possibility, it's simplest to look at a novel basis for the Clifford algebra, known as the *Majorana basis*.

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \;, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \;, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \;, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

These matrices satisfy the Clifford algebra. What is special about them is that they are all pure imaginary $(\gamma^\mu)^\star = -\gamma^\mu$. This means that the generators of the Lorentz group $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, and hence the matrices $S[\Lambda]$ are real. So with this basis of the Clifford algebra, we can work with a real spinor simply by imposing the condition,

$$\psi = \psi^\star \tag{4.83}$$

which is preserved under Lorentz transformation. Such spinors are called *Majorana spinors*.

So what's the story if we use a general basis for the Clifford algebra? We'll ask only that the basis satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. We then define the *charge conjugate* of a Dirac spinor ψ as

$$\psi^{(c)} = C\psi^\star \tag{4.84}$$

Here C is a 4×4 matrix satisfying

$$C^\dagger C = 1 \quad \text{and} \quad C^\dagger \gamma^\mu C = -(\gamma^\mu)^\star \tag{4.85}$$

Let's firstly check that (4.84) is a good definition, meaning that $\psi^{(c)}$ transforms nicely under a Lorentz transformation. We have

$$\psi^{(c)} \to CS[\Lambda]^\star \psi^\star = S[\Lambda]C\psi^\star = S[\Lambda]\psi^{(c)} \tag{4.86}$$

where we've made use of the properties (4.85) in taking the matrix C through $S[\Lambda]^\star$. In fact, not only does $\psi^{(c)}$ transform nicely under the Lorentz group, but if ψ satisfies the Dirac equation, then $\psi^{(c)}$ does too. This follows from,

$$
\begin{aligned}
(i\,\partial\!\!\!/ - m)\psi = 0 \quad &\Rightarrow \quad (-i\,\partial\!\!\!/^\star - m)\psi^\star = 0 \\
&\Rightarrow \quad C(-i\,\partial\!\!\!/^\star - m)\psi^\star = (+i\,\partial\!\!\!/ - m)\psi^{(c)} = 0
\end{aligned}
$$

Finally, we can now impose the Lorentz invariant reality condition on the Dirac spinor, to yield a Majorana spinor,

$$
\psi^{(c)} = \psi \tag{4.87}
$$

After quantization, the Majorana spinor gives rise to a fermion that is its own anti-particle. This is exactly the same as in the case of scalar fields, where we've seen that a real scalar field gives rise to a spin 0 boson that is its own anti-particle. (Be aware: In many texts an extra factor of γ^0 is absorbed into the definition of C).

So what is this matrix C? Well, for a given representation of the Clifford algebra, it is something that we can find fairly easily. In the Majorana basis, where the gamma matrices are pure imaginary, we have simply $C_{\text{Maj}} = 1$ and the Majorana condition $\psi = \psi^{(c)}$ becomes $\psi = \psi^\star$. In the chiral basis (4.16), only γ^2 is imaginary, and we may take $C_{\text{chiral}} = i\gamma^2 = \left(\begin{smallmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{smallmatrix} \right)$. (The matrix $i\sigma^2$ that appears here is simply the anti-symmetric matrix $\epsilon^{\alpha\beta}$). It is interesting to see how the Majorana condition (4.87) looks in terms of the decomposition into left and right handed Weyl spinors (4.59). Plugging in the various definitions, we find that $u_+ = i\sigma^2 u_-^\star$ and $u_- = -i\sigma^2 u_+^\star$. In other words, a Majorana spinor can be written in terms of Weyl spinors as

$$
\psi = \begin{pmatrix} u_+ \\ -i\sigma^2 u_+^\star \end{pmatrix} \tag{4.88}
$$

Notice that it's not possible to impose the Majorana condition $\psi = \psi^{(c)}$ at the same time as the Weyl condition ($u_- = 0$ or $u_+ = 0$). Instead the Majorana condition relates u_- and u_+.

An Aside: Spinors in Different Dimensions: The ability to impose Majorana or Weyl conditions on Dirac spinors depends on both the dimension and the signature of spacetime. One can always impose the Weyl condition on a spinor in even dimensional Minkowski space, basically because you can always build a suitable "γ^5" projection matrix by multiplying together all the other γ-matrices. The pattern for when the Majorana condition can be imposed is a little more sporadic. Interestingly, although the Majorana condition and Weyl condition cannot be imposed simultaneously in four dimensions, you can do this in Minowski spacetimes of dimension $2, 10, 18, \ldots$.

4.6 Symmetries and Conserved Currents

The Dirac Lagrangian enjoys a number of symmetries. Here we list them and compute the associated conserved currents.

Spacetime Translations

Under spacetime translations the spinor transforms as

$$\delta\psi = \epsilon^\mu \partial_\mu \psi \tag{4.89}$$

The Lagrangian depends on $\partial_\mu \psi$, but not $\partial_\mu \bar\psi$, so the standard formula (1.41) gives us the energy-momentum tensor

$$T^{\mu\nu} = i\bar\psi \gamma^\mu \partial^\nu \psi - \eta^{\mu\nu}\mathcal{L} \tag{4.90}$$

Since a current is conserved only when the equations of motion are obeyed, we don't lose anything by imposing the equations of motion already on $T^{\mu\nu}$. In the case of a scalar field this didn't really buy us anything because the equations of motion are second order in derivatives, while the energy-momentum is typically first order. However, for a spinor field the equations of motion are first order: $(i\partial\!\!\!/ - m)\psi = 0$. This means we can set $\mathcal{L} = 0$ in $T^{\mu\nu}$, leaving

$$T^{\mu\nu} = i\bar\psi \gamma^\mu \partial^\nu \psi \tag{4.91}$$

In particular, we have the total energy

$$E = \int d^3x \, T^{00} = \int d^3x \, i\bar\psi\gamma^0\dot\psi = \int d^3x \, \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m)\psi \tag{4.92}$$

where, in the last equality, we have again used the equations of motion.

Lorentz Transformations

Under an infinitesimal Lorentz transformation, the Dirac spinor transforms as (4.22) which, in infinitesimal form, reads

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \psi^\beta \tag{4.93}$$

where, following (4.10), we have $\omega^\mu{}_\nu = \tfrac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu$, and $\mathcal{M}^{\rho\sigma}$ are the generators of the Lorentz algebra given by (4.8)

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu \tag{4.94}$$

which, after direct substitution, tells us that $\omega^{\mu\nu} = \Omega^{\mu\nu}$. So we get

$$\delta\psi^\alpha = -\omega^{\mu\nu}\left[x_\nu\partial_\mu\psi^\alpha - \tfrac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right] \tag{4.95}$$

The conserved current arising from Lorentz transformations now follows from the same calculation we saw for the scalar field (1.54) with two differences: firstly, as we saw above, the spinor equations of motion set $\mathcal{L} = 0$; secondly, we pick up an extra piece in the current from the second term in (4.95). We have

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - i\bar\psi\gamma^\mu S^{\rho\sigma}\psi \tag{4.96}$$

After quantization, when $(\mathcal{J}^\mu)^{\rho\sigma}$ is turned into an operator, this extra term will be responsible for providing the single particle states with internal angular momentum, telling us that the quantization of a Dirac spinor gives rise to a particle carrying spin $1/2$.

Internal Vector Symmetry

The Dirac Lagrangian is invariant under rotating the phase of the spinor, $\psi \to e^{-i\alpha}\psi$. This gives rise to the current

$$j_V^\mu = \bar\psi\gamma^\mu\psi \tag{4.97}$$

where "V" stands for *vector*, reflecting the fact that the left and right-handed components $\psi_\pm$ transform in the same way under this symmetry. We can easily check that j_V^μ is conserved under the equations of motion,

$$\partial_\mu j_V^\mu = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi) = im\bar\psi\psi - im\bar\psi\psi = 0 \tag{4.98}$$

where, in the last equality, we have used the equations of motion $i\not\partial\psi = m\psi$ and $i\partial_\mu\bar\psi\gamma^\mu = -m\bar\psi$. The conserved quantity arising from this symmetry is

$$Q = \int d^3x\, \bar\psi\gamma^0\psi = \int d^3x\, \psi^\dagger\psi \tag{4.99}$$

We will see shortly that this has the interpretation of electric charge, or particle number, for fermions.

Axial Symmetry

When $m = 0$, the Dirac Lagrangian admits an extra internal symmetry which rotates left and right-handed fermions in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi \quad \text{and} \quad \bar\psi \to \bar\psi e^{i\alpha\gamma^5} \tag{4.100}$$

Here the second transformation follows from the first after noting that $e^{-i\alpha\gamma^5}\gamma^0 = \gamma^0 e^{+i\alpha\gamma^5}$. This gives the conserved current,

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \qquad (4.101)$$

where A is for "axial" since j_A^μ is an axial vector. This is conserved only when $m = 0$. Indeed, with the full Dirac Lagrangian we may compute

$$\partial_\mu j_A^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi = 2im\bar{\psi}\gamma^5\psi \qquad (4.102)$$

which vanishes only for $m = 0$. However, in the quantum theory things become more interesting for the axial current. When the theory is coupled to gauge fields (in a manner we will discuss in Section 6), the axial transformation remains a symmetry of the classical Lagrangian. But it doesn't survive the quantization process. It is the archetypal example of an *anomaly*: a symmetry of the classical theory that is not preserved in the quantum theory.

4.7 Plane Wave Solutions

Let's now study the solutions to the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad (4.103)$$

We start by making a simple ansatz:

$$\psi = u(\vec{p})\, e^{-ip\cdot x} \qquad (4.104)$$

where $u(\vec{p})$ is a four-component spinor, independent of spacetime x which, as the notation suggests, can depend on the 3-momentum $\vec{p}$. The Dirac equation then becomes

$$(\gamma^\mu p_\mu - m)u(\vec{p}) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{pmatrix} u(\vec{p}) = 0 \qquad (4.105)$$

where we're again using the definition,

$$\sigma^\mu = (1, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (1, -\sigma^i) \qquad (4.106)$$

Claim: The solution to (4.105) is

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \qquad (4.107)$$

for any 2-component spinor ξ which we will normalize to $\xi^\dagger \xi = 1$.

Proof: Let's write $u(\vec{p})^T = (u_1, u_2)$. Then equation (4.105) reads

$$(p \cdot \sigma) \, u_2 = m u_1 \quad \text{and} \quad (p \cdot \bar{\sigma}) u_1 = m u_2 \tag{4.108}$$

Either one of these equations implies the other, a fact which follows from the identity $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p_j \delta^{ij} = p_\mu p^\mu = m^2$. To start with, let's try the ansatz $u_1 = (p \cdot \sigma) \xi'$ for some spinor ξ'. Then the second equation in (4.108) immediately tells us that $u_2 = m \xi'$. So we learn that any spinor of the form

$$u(\vec{p}) = A \begin{pmatrix} (p \cdot \sigma) \, \xi' \\ m \xi' \end{pmatrix} \tag{4.109}$$

with constant A is a solution to (4.105). To make this more symmetric, we choose $A = 1/m$ and $\xi' = \sqrt{p \cdot \bar{\sigma}} \, \xi$ with constant ξ. Then $u_1 = (p \cdot \sigma)\sqrt{p \cdot \bar{\sigma}} \, \xi = m\sqrt{p \cdot \sigma} \, \xi$. So we get the promised result (4.107) $\qquad \square$

Negative Frequency Solutions

We get further solutions to the Dirac equation from the ansatz

$$\psi = v(\vec{p}) \, e^{+ip \cdot x} \tag{4.110}$$

Solutions of the form (4.104), which oscillate in time as $\psi \sim e^{-iEt}$, are called positive frequency solutions. Those of the form (4.110), which oscillate as $\psi \sim e^{+iEt}$, are negative frequency solutions. It's important to note however that both are solutions to the classical field equations and both have positive energy (4.92). The Dirac equation requires that the 4-component spinor $v(\vec{p})$ satisfies

$$(\gamma^\mu p_\mu + m)v(\vec{p}) = \begin{pmatrix} m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & m \end{pmatrix} v(\vec{p}) = 0 \tag{4.111}$$

which is solved by

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \, \eta \\ -\sqrt{p \cdot \bar{\sigma}} \, \eta \end{pmatrix} \tag{4.112}$$

for some 2-component spinor η which we take to be constant and normalized to $\eta^\dagger \eta = 1$.

4.7.1 Some Examples

Consider the positive frequency solution with mass m and 3-momentum $\vec{p} = 0$,

$$u(\vec{p}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \tag{4.113}$$

where ξ is any 2-component spinor. Spatial rotations of the field act on ξ by (4.26),

$$\xi \to e^{+i\vec{\varphi}\cdot\vec{\sigma}/2}\xi \tag{4.114}$$

The 2-component spinor ξ defines the *spin* of the field. This should be familiar from quantum mechanics. A field with spin up (down) along a given direction is described by the eigenvector of the corresponding Pauli matrix with eigenvalue +1 (-1 respectively). For example, $\xi^T = (1,0)$ describes a field with spin up along the z-axis. After quantization, this will become the spin of the associated particle. In the rest of this section, we'll indulge in an abuse of terminology and refer to the classical solutions to the Dirac equations as "particles", even though they have no such interpretation before quantization.

Consider now boosting the particle with spin $\xi^T = (1,0)$ along the x^3 direction, with $p^\mu = (E,0,0,p)$. The solution to the Dirac equation becomes

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{E-p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \tag{4.115}$$

In fact, this expression also makes sense for a massless field, for which $E = p^3$. (We picked the normalization (4.107) for the solutions so that this would be the case). For a massless particle we have

$$u(\vec{p}) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.116}$$

Similarly, for a boosted solution of the spin down $\xi^T = (0,1)$ field, we have

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{E+p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow{m\to 0} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.117}$$

4.7.2 Helicity

The helicity operator is the projection of the angular momentum along the direction of momentum,

$$h = \tfrac{i}{2}\epsilon_{ijk}\hat{p}^i \, S^{jk} = \tfrac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{4.118}$$

where S^{ij} is the rotation generator given in (4.25). The massless field with spin $\xi^T = (1,0)$ in (4.116) has helicity $h = 1/2$: we say that it is *right-handed*. Meanwhile, the field (4.117) has helicity $h = -1/2$: it is *left-handed*.

4.7.3 Some Useful Formulae: Inner and Outer Products

There are a number of identities that will be very useful in the following section, regarding the inner (and outer) products of the spinors $u(\vec{p})$ and $v(\vec{p})$. It's firstly convenient to introduce a basis ξ^s and η^s, $s = 1, 2$ for the two-component spinors such that

$$\xi^{r\dagger}\xi^s = \delta^{rs} \quad \text{and} \quad \eta^{r\dagger}\eta^s = \delta^{rs} \tag{4.119}$$

for example,

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.120}$$

and similarly for η^s. Let's deal first with the positive frequency plane waves. The two independent solutions are now written as

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \tag{4.121}$$

We can take the inner product of four-component spinors in two different ways: either as $u^\dagger \cdot u$, or as $\bar{u} \cdot u$. Of course, only the latter will be Lorentz invariant, but it turns out that the former is needed when we come to quantize the theory. Here we state both:

$$u^{r\dagger}(\vec{p}) \cdot u^s(\vec{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}\,, & \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$

$$= \xi^{r\dagger}p\cdot\sigma\xi^s + \xi^{r\dagger}p\cdot\bar{\sigma}\xi^s = 2\xi^{r\dagger}p_0\xi^s = 2p_0\delta^{rs} \tag{4.122}$$

while the Lorentz invariant inner product is

$$\bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma}\,, & \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = 2m\delta^{rs} \tag{4.123}$$

We have analogous results for the negative frequency solutions, which we may write as

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta^s \\ -\sqrt{p \cdot \bar\sigma}\, \eta^s \end{pmatrix} \quad \begin{array}{l} \text{with} \quad v^{r\,\dagger}(\vec{p}) \cdot v^s(\vec{p}) = 2p_0 \delta^{rs} \\ \text{and} \quad \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) = -2m\delta^{rs} \end{array} \tag{4.124}$$

We can also compute the inner product between u and v. We have

$$\bar{u}^r(\vec{p}) \cdot v^s(\vec{p}) = \left(\xi^{r\,\dagger}\sqrt{p \cdot \sigma}\,,\ \xi^{r\,\dagger}\sqrt{p \cdot \bar\sigma} \right) \gamma^0 \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta^s \\ -\sqrt{p \cdot \bar\sigma}\, \eta^s \end{pmatrix}$$

$$= \xi^{r\dagger}\sqrt{(p \cdot \bar\sigma)(p \cdot \sigma)}\eta^s - \xi^{r\dagger}\sqrt{(p \cdot \bar\sigma)(p \cdot \sigma)}\eta^s = 0 \tag{4.125}$$

and similarly, $\bar{v}^r(\vec{p}) \cdot u^s(\vec{p}) = 0$. However, when we come to $u^\dagger \cdot v$, it is a slightly different combination that has nice properties (and this same combination appears when we quantize the theory). We look at $u^{r\,\dagger}(\vec{p}) \cdot v^s(-\vec{p})$, with the 3-momentum in the spinor v taking the opposite sign. Defining the 4-momentum $(p')^\mu = (p^0, -\vec{p})$, we have

$$u^{r\,\dagger}(\vec{p}) \cdot v^s(-\vec{p}) = \left(\xi^{r\,\dagger}\sqrt{p \cdot \sigma}\,,\ \xi^{r\,\dagger}\sqrt{p \cdot \bar\sigma} \right) \begin{pmatrix} \sqrt{p' \cdot \sigma}\, \eta^s \\ -\sqrt{p' \cdot \bar\sigma}\, \eta^s \end{pmatrix}$$

$$= \xi^{r\dagger}\sqrt{(p \cdot \sigma)(p' \cdot \sigma)}\eta^s - \xi^{r\dagger}\sqrt{(p \cdot \bar\sigma)(p' \cdot \bar\sigma)}\eta^s \tag{4.126}$$

Now the terms under the square-root are given by $(p \cdot \sigma)(p' \cdot \sigma) = (p_0 + p_i \sigma^i)(p_0 - p_i \sigma^i) = p_0^2 - \vec{p}^2 = m^2$. The same expression holds for $(p \cdot \bar\sigma)(p' \cdot \bar\sigma)$, and the two terms cancel. We learn

$$u^{r\,\dagger}(\vec{p}) \cdot v^s(-\vec{p}) = v^{r\,\dagger}(\vec{p}) \cdot u^s(-\vec{p}) = 0 \tag{4.127}$$

Outer Products

There's one last spinor identity that we need before we turn to the quantum theory. It is:

Claim:

$$\sum_{s=1}^{2} u^s(\vec{p})\, \bar{u}^s(\vec{p}) = \not{p} + m \tag{4.128}$$

where the two spinors are not now contracted, but instead placed back to back to give a 4×4 matrix. Also,

$$\sum_{s=1}^{2} v^s(\vec{p})\, \bar{v}^s(\vec{p}) = \not{p} - m \tag{4.129}$$

Proof:

$$\sum_{s=1}^{2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\,\dagger}\sqrt{p\cdot\bar{\sigma}}\,,\ \xi^{s\,\dagger}\sqrt{p\cdot\sigma}\right) \tag{4.130}$$

But $\sum_s \xi^s\,\xi^{s\,\dagger} = \mathbf{1}$, the 2×2 unit matrix, which then gives us

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} \tag{4.131}$$

which is the desired result. A similar proof works for $\sum_s v^s(\vec{p})\bar{v}^s(\vec{p})$. $\square$

5. Quantizing the Dirac Field

We would now like to quantize the Dirac Lagrangian,

$$\mathcal{L} = \bar{\psi}(x)\left(i\,\partial\!\!\!/ - m\right)\psi(x) \tag{5.1}$$

We will proceed naively and treat ψ as we did the scalar field. But we'll see that things go wrong and we will have to reconsider how to quantize this theory.

5.1 A Glimpse at the Spin-Statistics Theorem

We start in the usual way and define the momentum,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger \tag{5.2}$$

For the Dirac Lagrangian, the momentum conjugate to ψ is $i\psi^\dagger$. It does not involve the time derivative of ψ. This is as it should be for an equation of motion that is first order in time, rather than second order. This is because we need only specify ψ and $\psi^\dagger$ on an initial time slice to determine the full evolution.

To quantize the theory, we promote the field ψ and its momentum $\psi^\dagger$ to operators, satisfying the canonical commutation relations, which read

$$[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})] = 0$$
$$[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})] = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x} - \vec{y}) \tag{5.3}$$

It's this step that we'll soon have to reconsider.

Since we're dealing with a free theory, where any classical solution is a sum of plane waves, we may write the quantum operators as

$$\psi(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^s\, u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s\,\dagger}\, v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\psi^\dagger(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^{s\,\dagger}\, u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^s\, v^s(\vec{p})^\dagger e^{+i\vec{p}\cdot\vec{x}}\right] \tag{5.4}$$

where the operators $b_{\vec{p}}^{s\,\dagger}$ create particles associated to the spinors $u^s(\vec{p})$, while $c_{\vec{p}}^{s\,\dagger}$ create particles associated to $v^s(\vec{p})$. As with the scalars, the commutation relations of the fields imply commutation relations for the annihilation and creation operators

Claim: The field commutation relations (5.3) are equivalent to

$$[b_{\vec{p}}^r, b_{\vec{q}}^{s\,\dagger}] = (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q})$$
$$[c_{\vec{p}}^r, c_{\vec{q}}^{s\,\dagger}] = -(2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q}) \tag{5.5}$$

with all other commutators vanishing. Note the strange minus sign in the $[c, c^\dagger]$ term. It's not yet obvious that it's bad, but we should be aware of it. For now, let's just carry on.

Proof: Let's show that the $[b, b^\dagger]$ and $[c, c^\dagger]$ commutators reproduce the field commutators (5.3),

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \sum_{r,s} \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}} \left([b_{\vec{p}}^r, b_{\vec{q}}^{s\,\dagger}] u^r(\vec{p}) u^s(\vec{q})^\dagger e^{i(\vec{x}\cdot\vec{p} - \vec{y}\cdot\vec{q})}\right.$$

$$\left. + [c_{\vec{p}}^{r\,\dagger}, c_{\vec{q}}^s] v^r(\vec{p}) v^s(\vec{q})^\dagger e^{-i(\vec{x}\cdot\vec{p} - \vec{y}\cdot\vec{q})}\right)$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(u^s(\vec{p})\bar{u}^s(\vec{p})\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s(\vec{p})\bar{v}^s(\vec{p})\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \tag{5.6}$$

At this stage we use the outer product formulae (4.128) and (4.129) which tell us $\sum_s u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m$ and $\sum_s v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m$, so that

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left((\not{p} + m)\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + (\not{p} - m)\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left((p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 + (p_0\gamma^0 - p_i\gamma^i - m)\gamma^0\right) e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}$$

where, in the second term, we've changed $\vec{p} \to -\vec{p}$ under the integration sign. Now, using $p_0 = E_{\vec{p}}$ we have

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int \frac{d^3p}{(2\pi)^3} e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x} - \vec{y}) \tag{5.7}$$

as promised. Notice that it's a little tricky in the middle there, making sure that the $p_i\gamma^i$ terms cancel. This was the reason we needed the minus sign in the $[c, c^\dagger]$ commutator terms in (5.5). $\qquad\square$

5.1.1 The Hamiltonian

To proceed, let's construct the Hamiltonian for the theory. Using the momentum $\pi = i\psi^\dagger$, we have

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{5.8}$$

which means that $H = \int d^3x\, \mathcal{H}$ agrees with the conserved energy computed using Noether's theorem (4.92). We now wish to turn the Hamiltonian into an operator. Let's firstly look at

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^s(-\gamma^i p_i + m)u^s(\vec{p})\,e^{+i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s\,\dagger}(\gamma^i p_i + m)v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

where, for once we've left the sum over $s = 1,2$ implicit. There's a small subtlety with the minus signs in deriving this equation that arises from the use of the Minkowski metric in contracting indices, so that $\vec{p}\cdot\vec{x} \equiv \sum_i x^i p^i = -x^i p_i$. Now we use the defining equations for the spinors $u^s(\vec{p})$ and $v^s(\vec{p})$ given in (4.105) and (4.111), to replace

$$(-\gamma^i p_i + m)u^s(\vec{p}) = \gamma^0 p_0 u^s(\vec{p}) \quad \text{and} \quad (\gamma^i p_i + m)v^s(\vec{p}) = -\gamma^0 p_0 v^s(\vec{p}) \tag{5.9}$$

so we can write

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\,\gamma^0\left[b_{\vec{p}}^s u^s(\vec{p})\,e^{+i\vec{p}\cdot\vec{x}} - c_{\vec{p}}^{s\,\dagger}v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right] \tag{5.10}$$

We now use this to write the operator Hamiltonian

$$\begin{aligned}
H &= \int d^3x\,\psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \\
&= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6}\sqrt{\frac{E_{\vec{p}}}{4E_{\vec{q}}}}\left[b_{\vec{q}}^{r\,\dagger}u^r(\vec{q})^\dagger e^{-i\vec{q}\cdot\vec{x}} + c_{\vec{q}}^r v^r(\vec{q})^\dagger\,e^{+i\vec{q}\cdot\vec{x}}\right]\cdot \\
&\qquad\qquad\qquad\qquad\qquad \left[b_{\vec{p}}^s u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} - c_{\vec{p}}^{s\,\dagger}v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[b_{\vec{p}}^{r\,\dagger}b_{\vec{p}}^s[u^r(\vec{p})^\dagger\cdot u^s(\vec{p})] - c_{\vec{p}}^r c_{\vec{p}}^{s\,\dagger}[v^r(\vec{p})^\dagger\cdot v^s(\vec{p})]\right. \\
&\qquad\qquad\qquad\qquad \left. - b_{\vec{p}}^{r\,\dagger}c_{-\vec{p}}^{s\,\dagger}[u^r(\vec{p})^\dagger\cdot v^s(-\vec{p})] + c_{\vec{p}}^r b_{-\vec{p}}^s[v^r(\vec{p})^\dagger\cdot u^s(-\vec{p})]\right]
\end{aligned}$$

where, in the last two terms we have relabelled $\vec{p} \to -\vec{p}$. We now use our inner product formulae (4.122), (4.124) and (4.127) which read

$$u^r(\vec{p})^\dagger\cdot u^s(\vec{p}) = v^r(\vec{p})^\dagger\cdot v^s(\vec{p}) = 2p_0\delta^{rs} \quad \text{and} \quad u^r(\vec{p})^\dagger\cdot v^s(-\vec{p}) = v^r(\vec{p})^\dagger\cdot u^s(-\vec{p}) = 0$$

giving us

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b_{\vec{p}}^{s\,\dagger}b_{\vec{p}}^s - c_{\vec{p}}^s c_{\vec{p}}^{s\,\dagger}\right) \tag{5.11}$$

$$= \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b_{\vec{p}}^{s\,\dagger}b_{\vec{p}}^s - c_{\vec{p}}^{s\,\dagger}c_{\vec{p}}^s + (2\pi)^3\delta^{(3)}(0)\right) \tag{5.12}$$

The $\delta^{(3)}$ term is familiar and easily dealt with by normal ordering. However the $-c^\dagger c$ term is a disaster! The Hamiltonian is not bounded below, meaning that our quantum theory makes no sense. Taken seriously it would tell us that we could tumble to states of lower and lower energy by continually producing $c^\dagger$ particles. As the English would say, it's all gone a bit Pete Tong. (No relation).

Since the above calculation was a little tricky, you might think that it's possible to rescue the theory to get the minus signs to work out right. You can play around with different things, but you'll always find this minus sign cropping up somewhere. And, in fact, it's telling us something important that we missed.

5.2 Fermionic Quantization

The key piece of physics that we missed is that spin $1/2$ particles are fermions, meaning that they obey Fermi-Dirac statistics with the quantum state picking up a minus sign upon the interchange of any two particles. This fact is embedded into the structure of relativistic quantum field theory: the spin-statistics theorem says that integer spin fields must be quantized as bosons, while half-integer spin fields must be quantized as fermions. Any attempt to do otherwise will lead to an inconsistency, such as the unbounded Hamiltonian we saw in (5.12).

So how do we go about quantizing a field as a fermion? Recall that when we quantized the scalar field, the resulting particles obeyed bosonic statistics because the creation and annihilation operators satisfied the commutation relations,

$$[a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \quad \Rightarrow \quad a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger |0\rangle \equiv |\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle \tag{5.13}$$

To have states obeying fermionic statistics, we need anti-commutation relations, $\{A, B\} \equiv AB + BA$. Rather than (5.3), we will ask that the spinor fields satisfy

$$\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = \{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = 0$$
$$\{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x} - \vec{y}) \tag{5.14}$$

We still have the expansion (5.4) of ψ and $\psi^\dagger$ in terms of $b, b^\dagger, c$ and $c^\dagger$. But now the same proof that led us to (5.5) tells us that

$$\{b_{\vec{p}}^r, b_{\vec{q}}^{s\,\dagger}\} = (2\pi)^3 \delta^{rs}\,\delta^{(3)}(\vec{p} - \vec{q})$$
$$\{c_{\vec{p}}^r, c_{\vec{q}}^{s\,\dagger}\} = (2\pi)^3 \delta^{rs}\,\delta^{(3)}(\vec{p} - \vec{q}) \tag{5.15}$$

with all other *anti-commutators* vanishing,

$$\{b_{\vec{p}}^r, b_{\vec{q}}^s\} = \{c_{\vec{p}}^r, c_{\vec{q}}^s\} = \{b_{\vec{p}}^r, c_{\vec{q}}^{s\,\dagger}\} = \{b_{\vec{p}}^r, c_{\vec{q}}^s\} = \ldots = 0 \tag{5.16}$$

The calculation of the Hamiltonian proceeds as before, all the way through to the penultimate line (5.11). At that stage, we get

$$H = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \left[b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} - c_{\vec{p}}^{s} c_{\vec{p}}^{s\,\dagger} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \left[b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} + c_{\vec{p}}^{s\,\dagger} c_{\vec{p}}^{s} - (2\pi)^3 \delta^{(3)}(0) \right] \tag{5.17}$$

The anti-commutators have saved us from the indignity of an unbounded Hamiltonian. Note that when normal ordering the Hamiltonian we now throw away a negative contribution $-(2\pi)^3 \delta^{(3)}(0)$. In principle, this could partially cancel the positive contribution from bosonic fields. Cosmological constant problem anyone?!

5.2.1 Fermi-Dirac Statistics

Just as in the bosonic case, we define the vacuum $|0\rangle$ to satisfy,

$$b_{\vec{p}}^{s}|0\rangle = c_{\vec{p}}^{s}|0\rangle = 0 \tag{5.18}$$

Although b and c obey anti-commutation relations, the Hamiltonian (5.17) has nice commutation relations with them. You can check that

$$[H, b_{\vec{p}}^{r}] = -E_{\vec{p}} b_{\vec{p}}^{r} \quad \text{and} \quad [H, b_{\vec{p}}^{r\,\dagger}] = E_{\vec{p}} b_{\vec{p}}^{r\,\dagger}$$

$$[H, c_{\vec{p}}^{r}] = -E_{\vec{p}} c_{\vec{p}}^{r} \quad \text{and} \quad [H, c_{\vec{p}}^{r\,\dagger}] = E_{\vec{p}} c_{\vec{p}}^{r\,\dagger} \tag{5.19}$$

This means that we can again construct a tower of energy eigenstates by acting on the vacuum by $b_{\vec{p}}^{r\,\dagger}$ and $c_{\vec{p}}^{r,\dagger}$ to create particles and antiparticles, just as in the bosonic case. For example, we have the one-particle states

$$|\vec{p}, r\rangle = b_{\vec{p}}^{r\,\dagger} |0\rangle \tag{5.20}$$

The two particle states now satisfy

$$|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle \equiv b_{\vec{p}_1}^{r_1\,\dagger} b_{\vec{p}_2}^{r_2\,\dagger} |0\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle \tag{5.21}$$

confirming that the particles do indeed obey Fermi-Dirac statistics. In particular, we have the Pauli-Exclusion principle $|\vec{p}, r; \vec{p}, r\rangle = 0$. Finally, if we wanted to be sure about the spin of the particle, we could act with the angular momentum operator (4.96) to confirm that a stationary particle $|\vec{p} = 0, r\rangle$ does indeed carry intrinsic angular momentum $1/2$ as expected.

5.3 Dirac's Hole Interpretation

"In this attempt, the success seems to have been on the side of Dirac rather than logic"

Pauli on Dirac

Let's pause our discussion to make a small historical detour. Dirac originally viewed his equation as a relativistic version of the Schrödinger equation, with ψ interpreted as the wavefunction for a single particle with spin. To reinforce this interpretation, he wrote $(i\,\slashed{\partial} - m)\psi = 0$ as

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \equiv \hat{H}\psi \tag{5.22}$$

where $\vec{\alpha} = -\gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. Here the operator $\hat{H}$ is interpreted as the one-particle Hamiltonian. This is a very different viewpoint from the one we now have, where ψ is a classical field that should be quantized. In Dirac's view, the Hamiltonian of the system is $\hat{H}$ defined above, while for us the Hamiltonian is the field operator (5.17). Let's see where Dirac's viewpoint leads.

With the interpretation of ψ as a single-particle wavefunction, the plane-wave solutions (4.104) and (4.110) to the Dirac equation are thought of as energy eigenstates, with

$$
\begin{aligned}
\psi = u(\vec{p})\,e^{-ip\cdot x} \quad &\Rightarrow \quad i\frac{\partial\psi}{\partial t} = E_{\vec{p}}\psi \\
\psi = v(\vec{p})\,e^{+ip\cdot x} \quad &\Rightarrow \quad i\frac{\partial\psi}{\partial t} = -E_{\vec{p}}\psi
\end{aligned}
\tag{5.23}
$$

which look like positive and negative energy solutions. The spectrum is once again unbounded below; there are states $v(\vec{p})$ with arbitrarily low energy $-E_{\vec{p}}$. At first glance this is disastrous, just like the unbounded field theory Hamiltonian (5.12). Dirac postulated an ingenious solution to this problem: since the electrons are fermions (a fact which is put in by hand to Dirac's theory) they obey the Pauli-exclusion principle. So we could simply stipulate that in the true vacuum of the universe, all the negative energy states are filled. Only the positive energy states are accessible. These filled negative energy states are referred to as the *Dirac sea*. Although you might worry about the infinite negative charge of the vacuum, Dirac argued that only charge differences would be observable (a trick reminiscent of the normal ordering prescription we used for field operators).

Having avoided disaster by floating on an infinite sea comprised of occupied negative energy states, Dirac realized that his theory made a shocking prediction. Suppose that a negative energy state is excited to a positive energy state, leaving behind a hole. The hole would have all the properties of the electron, except it would carry positive charge. After flirting with the idea that it may be the proton, Dirac finally concluded that the hole is a new particle: the positron. Moreover, when a positron comes across

an electron, the two can annihilate. Dirac had predicted anti-matter, one of the greatest achievements of theoretical physics. It took only a couple of years before the positron was discovered experimentally in 1932.

Although Dirac's physical insight led him to the right answer, we now understand that the interpretation of the Dirac spinor as a single-particle wavefunction is not really correct. For example, Dirac's argument for anti-matter relies crucially on the particles being fermions while, as we have seen already in this course, anti-particles exist for both fermions and bosons. What we really learn from Dirac's analysis is that there is no consistent way to interpret the Dirac equation as describing a single particle. It is instead to be thought of as a classical field which has only positive energy solutions because the Hamiltonian (4.92) is positive definite. Quantization of this field then gives rise to both particle and anti-particle excitations.

This from Julian Schwinger:

> *"Until now, everyone thought that the Dirac equation referred directly to physical particles. Now, in field theory, we recognize that the equations refer to a sublevel. Experimentally we are concerned with particles, yet the old equations describe fields.... When you begin with field equations, you operate on a level where the particles are not there from the start. It is when you solve the field equations that you see the emergence of particles."*

5.4 Propagators

Let's now move to the Heisenberg picture. We define the spinors $\psi(\vec{x}, t)$ at every point in spacetime such that they satisfy the operator equation

$$\frac{\partial \psi}{\partial t} = i[H, \psi] \tag{5.24}$$

We solve this by the expansion

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^s u^s(\vec{p}) e^{-ip\cdot x} + c_{\vec{p}}^{s\,\dagger} v^s(\vec{p}) e^{+ip\cdot x} \right]$$

$$\psi^{\dagger}(x) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^{s\,\dagger} u^s(\vec{p})^{\dagger} e^{+ip\cdot x} + c_{\vec{p}}^s v^s(\vec{p})^{\dagger} e^{-ip\cdot x} \right] \tag{5.25}$$

Let's now look at the anti-commutators of these fields. We define the fermionic propagator to be

$$iS_{\alpha\beta} = \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} \tag{5.26}$$

In what follows we will often drop the indices and simply write $iS(x-y) = \{\psi(x), \bar{\psi}(y)\}$, but you should remember that $S(x-y)$ is a 4×4 matrix. Inserting the expansion (5.25), we have

$$iS(x-y) = \int \frac{d^3p\,d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}} \left[\{b^s_{\vec{p}}, b^{r\,\dagger}_{\vec{q}}\} u^s(\vec{p})\bar{u}^r(\vec{q}) e^{-i(p\cdot x - q\cdot y)} \right.$$

$$\left. +\{c^{s\,\dagger}_{\vec{p}}, c^r_{\vec{q}}\} v^s(\vec{p})\bar{v}^r(\vec{q}) e^{+i(p\cdot x - q\cdot y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[u^s(\vec{p})\bar{u}^s(\vec{p}) e^{-ip\cdot(x-y)} + v^s(\vec{p})\bar{v}^s(\vec{p}) e^{+ip\cdot(x-y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[(\not{p} + m) e^{-ip\cdot(x-y)} + (\not{p} - m) e^{+ip\cdot(x-y)} \right] \tag{5.27}$$

where to reach the final line we have used the outer product formulae (4.128) and (4.129). We can then write

$$iS(x-y) = (i\not{\partial}_x + m)(D(x-y) - D(y-x)) \tag{5.28}$$

in terms of the propagator for a real scalar field $D(x-y)$ which, recall, can be written as (2.90)

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \tag{5.29}$$

Some comments:

- For spacelike separated points $(x-y)^2 < 0$, we have already seen that $D(x-y) - D(y-x) = 0$. In the bosonic theory, we made a big deal of this since it ensured that

$$[\phi(x), \phi(y)] = 0 \qquad (x-y)^2 < 0 \tag{5.30}$$

outside the lightcone, which we trumpeted as proof that our theory was causal. However, for fermions we now have

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0 \qquad (x-y)^2 < 0 \tag{5.31}$$

outside the lightcone. What happened to our precious causality? The best that we can say is that all our observables are bilinear in fermions, for example the Hamiltonian (5.17). These still commute outside the lightcone. The theory remains causal as long as fermionic operators are not observable. If you think this is a little weak, remember that no one has ever seen a physical measuring apparatus come back to minus itself when you rotate by 360 degrees!

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- At least away from singularities, the propagator satisfies

$$(i\slashed{\partial}_x - m)S(x - y) = 0 \tag{5.32}$$

which follows from the fact that $(\slashed{\partial}_x^2 + m^2)D(x - y) = 0$ using the mass shell condition $p^2 = m^2$.

5.5 The Feynman Propagator

By a similar calculation to that above, we can determine the vacuum expectation value,

$$\langle 0 | \psi_\alpha(x)\bar{\psi}_\beta(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (\slashed{p} + m)_{\alpha\beta} \, e^{-ip\cdot(x-y)}$$

$$\langle 0 | \bar{\psi}_\beta(y)\psi_\alpha(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (\slashed{p} - m)_{\alpha\beta} \, e^{+ip\cdot(x-y)} \tag{5.33}$$

We now define the Feynman propagator $S_F(x - y)$, which is again a 4×4 matrix, as the time ordered product,

$$S_F(x - y) = \langle 0 | T\psi(x)\bar{\psi}(y) | 0 \rangle \equiv \begin{cases} \langle 0 | \psi(x)\bar{\psi}(y) | 0 \rangle & x^0 > y^0 \\ \langle 0 | -\bar{\psi}(y)\psi(x) | 0 \rangle & y^0 > x^0 \end{cases} \tag{5.34}$$

Notice the minus sign! It is necessary for Lorentz invariance. When $(x-y)^2 < 0$, there is no invariant way to determine whether $x^0 > y^0$ or $y^0 > x^0$. In this case the minus sign is necessary to make the two definitions agree since $\{\psi(x), \bar{\psi}(y)\} = 0$ outside the lightcone. We have the 4-momentum integral representation for the Feynman propagator,

$$S_F(x - y) = i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\gamma \cdot p + m}{p^2 - m^2 + i\epsilon} \tag{5.35}$$

which satisfies $(i\slashed{\partial}_x - m)S_F(x - y) = i\delta^{(4)}(x - y)$, so that S_F is a Green's function for the Dirac operator.

The minus sign that we see in (5.34) also occurs for any string of operators inside a time ordered product $T(\ldots)$. While bosonic operators commute inside T, fermionic operators anti-commute. We have this same behaviour for normal ordered products as well, with fermionic operators obeying $: \psi_1\psi_2 := - : \psi_2\psi_1 :$. With the understanding that all fermionic operators anti-commute inside T and $::$, Wick's theorem proceeds just as in the bosonic case. We define the contraction

$$\overbrace{\psi(x)\bar{\psi}(y)} = T(\psi(x)\bar{\psi}(y)) - : \psi(x)\bar{\psi}(y) : = S_F(x - y) \tag{5.36}$$

5.6 Yukawa Theory

The interaction between a Dirac fermion of mass m and a real scalar field of mass μ is governed by the Yukawa theory,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\phi\bar{\psi}\psi \tag{5.37}$$

which is the proper version of the baby scalar Yukawa theory we looked at in Section 3. Couplings of this type appear in the standard model, between fermions and the Higgs boson. In that context, the fermions can be leptons (such as the electron) or quarks.

Yukawa originally proposed an interaction of this type as an effective theory of nuclear forces. With an eye to this, we will again refer to the ϕ particles as mesons, and the ψ particles as nucleons. Except, this time, the nucleons have spin. (This is still not a particularly realistic theory of nucleon interactions, not least because we're omitting isospin. Moreover, in Nature the relevant mesons are pions which are pseudoscalars, so a coupling of the form $\phi\bar{\psi}\gamma^5\psi$ would be more appropriate. We'll turn to this briefly in Section 5.7.3).

Note the dimensions of the various fields. We still have $[\phi] = 1$, but the kinetic terms require that $[\psi] = 3/2$. Thus, unlike in the case with only scalars, the coupling is dimensionless: $[\lambda] = 0$.

We'll proceed as we did in Section 3, firstly computing the amplitude of a particular scattering process then, with that calculation as a guide, writing down the Feynman rules for the theory. We start with:

5.6.1 An Example: Putting Spin on Nucleon Scattering

Let's study $\psi\psi \to \psi\psi$ scattering. This is the same calculation we performed in Section (3.3.3) except now the fermions have spin. Our initial and final states are

$$|i\rangle = \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}}^{s\,\dagger} b_{\vec{q}}^{r\,\dagger}\, |0\rangle \equiv |\vec{p}, s; \vec{q}, r\rangle$$
$$|f\rangle = \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, b_{\vec{p}'}^{s'\,\dagger} b_{\vec{q}'}^{r'\,\dagger}\, |0\rangle \equiv |\vec{p}', s'; \vec{q}', r'\rangle \tag{5.38}$$

We need to be a little cautious about minus signs, because the $b^\dagger$'s now anti-commute. In particular, we should be careful when we take the adjoint. We have

$$\langle f| = \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, \langle 0|\, b_{\vec{q}'}^{r'} b_{\vec{p}'}^{s'} \tag{5.39}$$

We want to calculate the order λ^2 terms from the S-matrix element $\langle f|\, S - 1\, |i\rangle$.

$$\frac{(-i\lambda)^2}{2} \int d^4x_1 d^4x_2\, T\left(\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\, \bar{\psi}(x_2)\psi(x_2)\phi(x_2)\right) \tag{5.40}$$

where, as usual, all fields are in the interaction picture. Just as in the bosonic calculation, the contribution to nucleon scattering comes from the contraction

$$: \bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2) : \overbrace{\phi(x_1)\phi(x_2)} \tag{5.41}$$

We just have to be careful about how the spinor indices are contracted. Let's start by looking at how the fermionic operators act on $|i\rangle$. We expand out the ψ fields, leaving the $\bar{\psi}$ fields alone for now. We may ignore the $c^\dagger$ pieces in ψ since they give no contribution at order λ^2. We have

$$: \bar{\psi}(x_1)\psi(x_1)\, \bar{\psi}(x_2)\psi(x_2) : b_{\vec{p}}^{s\,\dagger} b_{\vec{q}}^{r\,\dagger} |0\rangle = -\int \frac{d^3k_1\, d^3k_2}{(2\pi)^6}\, [\bar{\psi}(x_1) \cdot u^m(\vec{k}_1)]\, [\bar{\psi}(x_2) \cdot u^n(\vec{k}_2)]$$
$$\frac{e^{-ik_1 \cdot x_1 - ik_2 \cdot x_2}}{\sqrt{4E_{\vec{k}_1} E_{\vec{k}_2}}}\, b_{\vec{k}_1}^m b_{\vec{k}_2}^n b_{\vec{p}}^{s\,\dagger} b_{\vec{q}}^{r\,\dagger} |0\rangle \tag{5.42}$$

where we've used square brackets $[\cdot]$ to show how the spinor indices are contracted. The minus sign that sits out front came from moving $\psi(x_1)$ past $\bar{\psi}(x_2)$. Now anti-commuting the b's past the $b^\dagger$'s, we get

$$= \frac{-1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\, \left([\bar{\psi}(x_1) \cdot u^r(\vec{q})]\,[\bar{\psi}(x_2) \cdot u^s(\vec{p})]e^{-ip \cdot x_2 - iq \cdot x_1}\right.$$
$$\left. - [\bar{\psi}(x_1) \cdot u^s(\vec{p})]\,[\bar{\psi}(x_2) \cdot u^r(\vec{q})]e^{-ip \cdot x_1 - iq \cdot x_2}\right) |0\rangle \tag{5.43}$$

Note, in particular, the relative minus sign that appears between these two terms. Now let's see what happens when we hit this with $\langle f|$. We look at

$$\langle 0|\, b_{\vec{q}'}^{r'} b_{\vec{p}'}^{s'}\, [\bar{\psi}(x_1) \cdot u^r(\vec{q})]\, [\bar{\psi}(x_2) \cdot u^s(\vec{p})]\, |0\rangle = \frac{e^{+ip' \cdot x_1 + iq' \cdot x_2}}{2\sqrt{E_{\vec{p}'}E_{\vec{q}'}}}[\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})]\, [\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})]$$
$$- \frac{e^{+ip' \cdot x_2 + iq' \cdot x_1}}{2\sqrt{E_{\vec{p}'}E_{\vec{q}'}}}[\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})]\, [\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})]$$

The $[\bar{\psi}(x_1) \cdot u^s(\vec{p})]\,[\bar{\psi}(x_2) \cdot u^r(\vec{q})]$ term in (5.43) doubles up with this, cancelling the factor of $1/2$ in front of (5.40). Meanwhile, the $1/\sqrt{E}$ terms cancel the relativistic state normalization. Putting everything together, we have the following expression for $\langle f|\, S - 1\, |i\rangle$

$$(-i\lambda)^2 \int \frac{d^4x_1 d^4x_2 d^4k}{(2\pi)^4}\, \frac{ie^{ik \cdot (x_1 - x_2)}}{k^2 - \mu^2 + i\epsilon}\, \left([\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})]e^{+ix_1 \cdot (q' - q) + ix_2 \cdot (p' - p)}\right.$$
$$\left. - [\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})]e^{ix_1 \cdot (p' - q) + ix_2 \cdot (q' - p)}\right)$$

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where we've put the ϕ propagator back in. Performing the integrals over x_1 and x_2, this becomes,

$$\int d^4k \frac{(2\pi)^4 i(-i\lambda)^2}{k^2 - \mu^2 + i\epsilon} \; \Big([\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})] \, [\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})] \delta^{(4)}(q' - q + k) \delta^{(4)}(p' - p - k)$$
$$- [\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})] \, [\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})] \delta^{(4)}(p' - q + k) \delta^{(4)}(q' - p - k) \Big)$$

And we're almost there! Finally, writing the S-matrix element in terms of the amplitude in the usual way, $\langle f | S - 1 | i \rangle = i\mathcal{A}(2\pi)^4 \delta^{(4)}(p + q - p' - q')$, we have

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})] \, [\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})]}{(p' - p)^2 - \mu^2 + i\epsilon} - \frac{[\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})] \, [\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})]}{(q' - p)^2 - \mu^2 + i\epsilon} \right)$$

which is our final answer for the amplitude.

5.7 Feynman Rules for Fermions

It's important to bear in mind that the calculation we just did kind of blows. Thankfully the Feynman rules will once again encapsulate the combinatoric complexities and make life easier for us. The rules to compute amplitudes are the following

- To each incoming fermion with momentum p and spin r, we associate a spinor $u^r(\vec{p})$. For outgoing fermions we associate $\bar{u}^r(\vec{p})$.
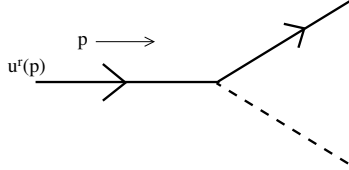
Figure 21: An incoming fermion
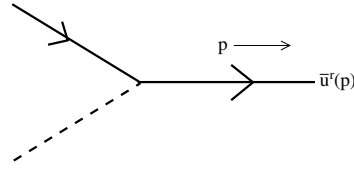
Figure 22: An outgoing fermion

- To each incoming anti-fermion with momentum p and spin r, we associate a spinor $\bar{v}^r(\vec{p})$. For outgoing anti-fermions we associate $v^r(\vec{p})$.
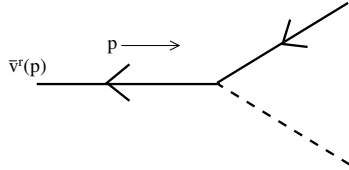
Figure 23: An incoming anti-fermion
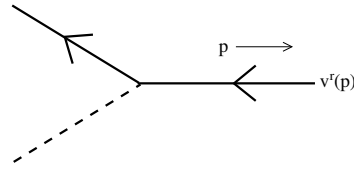
Figure 24: An outgoing anti-fermion

- Each vertex gets a factor of $-i\lambda$.

- Each internal line gets a factor of the relevant propagator.

$$\frac{i}{p^2 - \mu^2 + i\epsilon} \quad \text{for scalars}$$

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \quad \text{for fermions} \tag{5.44}$$

The arrows on the fermion lines must flow consistently through the diagram (this ensures fermion number conservation). Note that the fermionic propagator is a 4×4 matrix. The matrix indices are contracted at each vertex, either with further propagators, or with external spinors u, $\bar{u}$, v or $\bar{v}$.

- Impose momentum conservation at each vertex, and integrate over undetermined loop momenta.

- Add extra minus signs for statistics. Some examples will be given below.

5.7.1 Examples

Let's run through the same examples we did for the scalar Yukawa theory. Firstly, we have

Nucleon Scattering

For the example we worked out previously, the two lowest order Feynman diagrams are shown in Figure 25. We've drawn the second Feynman diagram with the legs crossed
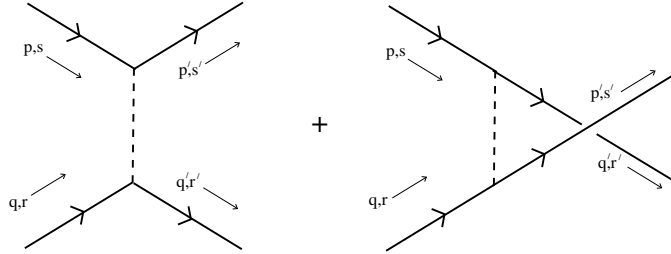
Figure 25: The two Feynman diagrams for nucleon scattering

to emphasize the fact that it picks up a minus sign due to statistics. (Note that the way the legs point in the Feynman diagram doesn't tell us the direction in which the particles leave the scattering event: the momentum label does that. The two diagrams above are different because the incoming legs are attached to different outgoing legs). Using the Feynman rules we can read off the amplitude.

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})][\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})]}{(p - p')^2 - \mu^2} - \frac{[\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})]}{(p - q')^2 - \mu^2} \right) \tag{5.45}$$

The denominators in each term are due to the meson propagator, with the momentum determined by conservation at each vertex. This agrees with the amplitude we computed earlier using Wick's theorem.

Nucleon to Meson Scattering

Let's now look at $\psi\bar{\psi} \to \phi\phi$. The two lowest order Feynman diagrams are shown in Figure 26.
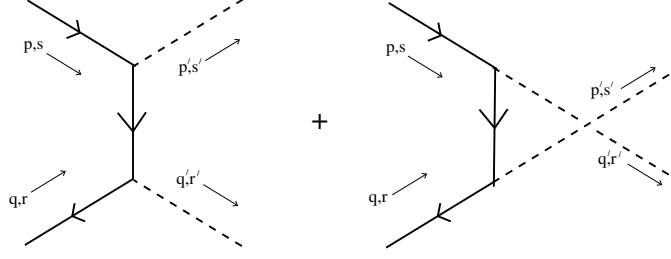
Figure 26: The two Feynman diagrams for nucleon to meson scattering

Applying the Feynman rules, we have

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - p'_\mu) + m]u^s(\vec{p})}{(p - p')^2 - m^2} + \frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - q'_\mu) + m]u^s(\vec{p})}{(p - q')^2 - m^2} \right)$$

Since the internal line is now a fermion, the propagator contains $\gamma_\mu(p_\mu - p'_\mu) + m$ factors. This is a 4×4 matrix which sits on the top, sandwiched between the two external spinors. Now the exchange statistics applies to the final meson states. These are bosons and, correspondingly, there is no relative minus sign between the two diagrams.

Nucleon-Anti-Nucleon Scattering

For $\psi\bar{\psi} \to \psi\bar{\psi}$, the two lowest order Feynman diagrams are of two distinct types, just like in the bosonic case. They are shown in Figure 27. The corresponding amplitude is given by,

$$\mathcal{A} = (-i\lambda)^2 \left(-\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})] [\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q}')]}{(p - p')^2 - \mu^2} + \frac{[\bar{v}^r(\vec{q}) \cdot u^s(\vec{p})] [\bar{u}^{s'}(\vec{p}') \cdot v^{r'}(\vec{q}')]}{(p + q)^2 - \mu^2 + i\epsilon} \right) \tag{5.46}$$

As in the bosonic diagrams, there is again the difference in the momentum dependence in the denominator. But now the difference in the diagrams is also reflected in the spinor contractions in the numerator.
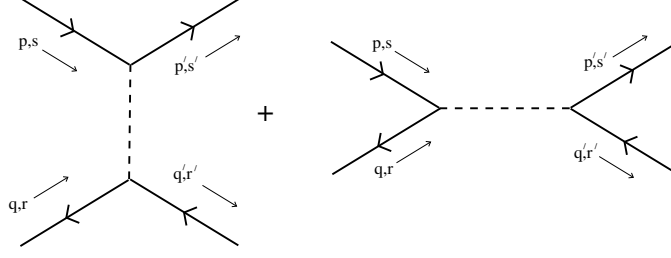
Figure 27: The two Feynman diagrams for nucleon-anti-nucleon scattering

More subtle are the minus signs. The fermionic statistics mean that the first diagram has an extra minus sign relative to the $\psi\psi$ scattering of Figure 25. Since this minus sign will be important when we come to figure out whether the Yukawa force is attractive or repulsive, let's go back to basics and see where it comes from. The initial and final states for this scattering process are

$$
\begin{aligned}
|i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}}^{s\,\dagger}\, c_{\vec{q}}^{r\,\dagger}\, |0\rangle \equiv |\vec{p}, s; \vec{q}, r\rangle \\
|f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, b_{\vec{p}'}^{s'\,\dagger}\, c_{\vec{q}'}^{r'\,\dagger}\, |0\rangle \equiv |\vec{p}', s'; \vec{q}', r'\rangle
\end{aligned}
\tag{5.47}
$$

The ordering of $b^\dagger$ and $c^\dagger$ in these states is crucial and reflects the scattering $\psi\bar{\psi} \to \psi\bar{\psi}$, as opposed to $\psi\bar{\psi} \to \bar{\psi}\psi$ which would differ by a minus sign. The first diagram in Figure 27 comes from the term in the perturbative expansion,

$$
\langle f| : \bar{\psi}(x_1)\psi(x_1)\, \bar{\psi}(x_2)\psi(x_2) : b_{\vec{p}}^{s\,\dagger}\, c_{\vec{q}}^{r\,\dagger}\, |0\rangle \sim \langle f| [\bar{v}^m(\vec{k}_1) \cdot \psi(x_1)]\, [\bar{\psi}(x_2) \cdot u^n(\vec{k}_2)] c_{\vec{k}_1}^m b_{\vec{k}_2}^n b_{\vec{p}}^{s\,\dagger} c_{\vec{q}}^{r\,\dagger}\, |0\rangle
$$

where we've neglected a bunch of objects in this equation like $\int d^4k_i$ and exponential factors because we only want to keep track of the minus signs. Moving the annihilation operators past the creation operators, we have

$$
+ \langle f| [\bar{v}^r(\vec{q}) \cdot \psi(x_1)]\, [\bar{\psi}(x_2) \cdot u^s(\vec{p})]\, |0\rangle
\tag{5.48}
$$

Repeating the process by expanding out the $\psi(x_1)$ and $\bar{\psi}(x_2)$ fields and moving them to the left to annihilate $\langle f|$, we have

$$
\langle 0| c_{\vec{q}'}^{r'} b_{\vec{p}'}^{s'} c_{\vec{l}_1}^{m\,\dagger} b_{\vec{l}_2}^{n\,\dagger} [\bar{v}^r(\vec{q}) \cdot v^m(\vec{l}_1)]\, [\bar{u}^n(\vec{l}_2) \cdot u^s(\vec{p})]\, |0\rangle \sim -[\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q}')]\, [\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})]
$$

where the minus sign has appeared from anti-commuting $c_{\vec{l}_1}^{m\,\dagger}$ past $b_{\vec{p}'}^{s'}$. This is the overall minus sign found in (5.46). One can also follow similar contractions to compute the second diagram in Figure 27.

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Meson Scattering

Finally, we can also compute the scattering of $\phi\phi \to \phi\phi$ which, as in the bosonic case, picks up its leading contribution at one-loop. The amplitude for the diagram shown in the figure is
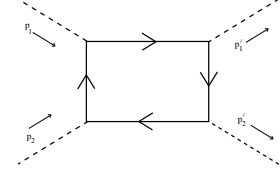

Figure 28:

$$i\mathcal{A} = -(-i\lambda)^4 \int \frac{d^4k}{(2\pi)^4} \ \mathrm{Tr} \frac{\slashed{k} + m}{(k^2 - m^2 + i\epsilon)} \frac{\slashed{k} + \slashed{p}_1' + m}{((k+p_1')^2 - m^2 + i\epsilon)}$$
$$\times \frac{\slashed{k} + \slashed{p}_1' - \slashed{p}_1 + m}{((k+p_1'-p_1)^2 - m^2 + i\epsilon)} \frac{\slashed{k} - \slashed{p}_2' + m}{((k-p_2')^2 - m^2 + i\epsilon)}$$

Notice that the high momentum limit of the integral is $\int d^4k/k^4$, which is no longer finite, but diverges logarithmically. You will have to wait until next term to make sense of this integral.

There's an overall minus sign sitting in front of this amplitude. This is a generic feature of diagrams with fermions running in loops: each fermionic loop in a diagram gives rise to an extra minus sign. We can see this rather simply in the diagram
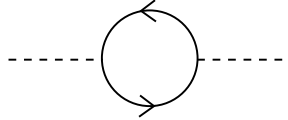
which involves the expression

$$\overbrace{\bar{\psi}_\alpha(x) \ \psi_\alpha(x)\bar{\psi}_\beta(y)}^{} \ \psi_\beta(y) = -\overbrace{\psi_\beta(y)\bar{\psi}_\alpha(x)}^{} \ \overbrace{\psi_\alpha(x)\bar{\psi}_\beta(y)}^{}$$
$$= -\mathrm{Tr} \ (S_F(y-x)S_F(x-y))$$

After passing the fermion fields through each other, a minus sign appears, sitting in front of the two propagators.
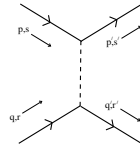
5.7.2 The Yukawa Potential Revisited

We saw in Section 3.5.2, that the exchange of a real scalar particle gives rise to a universally attractive Yukawa potential between two spin zero particles. Does the same hold for the spin 1/2 particles?

Recall that the strategy to compute the potential is to take the non-relativistic limit of the scattering amplitude, and compare with the analogous result from quantum mechanics. Our new amplitude now also includes the spinor degrees of freedom $u(\vec{p})$ and $v(\vec{p})$. In the non-relativistic limit, $p \to (m, \vec{p})$, and

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \to \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ -\sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \to \sqrt{m} \begin{pmatrix} \xi \\ -\xi \end{pmatrix} \tag{5.49}$$

In this limit, the spinor contractions in the amplitude for $\psi\psi \to \psi\psi$ scattering (5.45) become $\bar{u}^{s'} \cdot u^s = 2m\delta^{ss'}$ and the amplitude is

$= -i(-i\lambda)^2 (2m) \left(\dfrac{\delta^{s's}\delta^{r'r}}{(\vec{p} - \vec{p}') + \mu^2} - \dfrac{\delta^{s'r}\delta^{r's}}{(\vec{p} - \vec{q}') + \mu^2} \right) \tag{5.50}$

The δ symbols tell us that spin is conserved in the non-relativistic limit, while the momentum dependence is the same as in the bosonic case, telling us that once again the particles feel an attractive Yukawa potential,

$$U(\vec{r}) = -\frac{\lambda^2 e^{-\mu r}}{4\pi r} \tag{5.51}$$

Repeating the calculation for $\psi\bar{\psi} \to \psi\bar{\psi}$, there are two minus signs which cancel each other. The first is the extra overall minus sign in the scattering amplitude (5.46), due to the fermionic nature of the particles. The second minus sign comes from the non-relativistic limit of the spinor contraction for anti-particles in (5.46), which is $\bar{v}^{s'} \cdot v^s = -2m\delta^{ss'}$. These two signs cancel, giving us once again an attractive Yukawa potential (5.51).

5.7.3 Pseudo-Scalar Coupling

Rather than the standard Yukawa coupling, we could instead consider

$$\mathcal{L}_{\text{Yuk}} = -\lambda\phi\bar{\psi}\gamma^5\psi \tag{5.52}$$

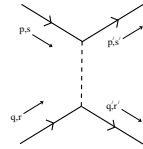This still preserves parity if ϕ is a pseudoscalar, i.e.

$$P : \phi(\vec{x}, t) \to -\phi(-\vec{x}, t) \tag{5.53}$$

We can compute in this theory very simply: the Feynman rule for the interaction vertex is now changed to a factor of $-i\lambda\gamma^5$. For example, the Feynman diagrams for $\psi\psi \to \psi\psi$ scattering are again given by Figure 25, with the amplitude now

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}')\gamma^5 u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}')\gamma^5 u^r(\vec{q})]}{(p-p')^2 - \mu^2} - \frac{[\bar{u}^{s'}(\vec{p}')\gamma^5 u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}')\gamma^5 u^s(\vec{p})]}{(p-q')^2 - \mu^2} \right)$$

We could again try to take the non-relativistic limit for this amplitude. But this time, things work a little differently. Using the expressions for the spinors (5.49), we have $\bar{u}^{s'}\gamma^5 u^s \to 0$ in the non-relativistic limit. To find the non-relativistic amplitude, we must go to next to leading order. One can easily check that $\bar{u}^{s'}(\vec{p}')\gamma^5 u^s(\vec{p}) \to m\,\xi^{s'\,T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^s$. So, in the non-relativistic limit, the leading order amplitude arising from pseudoscalar exchange is given by a spin-spin coupling,

$$\to +im(-i\lambda)^2 \frac{[\xi^{s'\,T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^s]\,[\xi^{r'\,T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^r]}{(\vec{p}-\vec{p}')^2 + \mu^2} \qquad (5.54)$$

6. Quantum Electrodynamics

In this section we finally get to quantum electrodynamics (QED), the theory of light interacting with charged matter. Our path to quantization will be as before: we start with the free theory of the electromagnetic field and see how the quantum theory gives rise to a photon with two polarization states. We then describe how to couple the photon to fermions and to bosons.

6.1 Maxwell's Equations

The Lagrangian for Maxwell's equations in the absence of any sources is simply

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{6.1}$$

where the field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.2}$$

The equations of motion which follow from this Lagrangian are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial_\mu F^{\mu\nu} = 0 \tag{6.3}$$

Meanwhile, from the definition of $F_{\mu\nu}$, the field strength also satisfies the Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{6.4}$$

To make contact with the form of Maxwell's equations you learn about in high school, we need some 3-vector notation. If we define $A^\mu = (\phi, \vec{A})$, then the electric field $\vec{E}$ and magnetic field $\vec{B}$ are defined by

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \tag{6.5}$$

which, in terms of $F_{\mu\nu}$, becomes

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.6}$$

The Bianchi identity (6.4) then gives two of Maxwell's equations,

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{6.7}$$

These remain true even in the presence of electric sources. Meanwhile, the equations of motion give the remaining two Maxwell equations,

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} \tag{6.8}$$

As we will see shortly, in the presence of charged matter these equations pick up extra terms on the right-hand side.

6.1.1 Gauge Symmetry

The massless vector field A_μ has 4 components, which would naively seem to tell us that the gauge field has 4 degrees of freedom. Yet we know that the photon has only two degrees of freedom which we call its polarization states. How are we going to resolve this discrepancy? There are two related comments which will ensure that quantizing the gauge field A_μ gives rise to 2 degrees of freedom, rather than 4.

- The field A_0 has no kinetic term $\dot{A}_0$ in the Lagrangian: it is not dynamical. This means that if we are given some initial data A_i and $\dot{A}_i$ at a time t_0, then the field A_0 is fully determined by the equation of motion $\nabla \cdot \vec{E} = 0$ which, expanding out, reads

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0 \tag{6.9}$$

This has the solution

$$A_0(\vec{x}) = \int d^3x' \, \frac{(\nabla \cdot \partial \vec{A}/\partial t)(\vec{x}')}{4\pi |\vec{x} - \vec{x}'|} \tag{6.10}$$

So A_0 is not independent: we don't get to specify A_0 on the initial time slice. It looks like we have only 3 degrees of freedom in A_μ rather than 4. But this is still one too many.

- The Lagrangian (6.3) has a *very* large symmetry group, acting on the vector potential as

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x) \tag{6.11}$$

for any function $\lambda(x)$. We'll ask only that $\lambda(x)$ dies off suitably quickly at spatial $\vec{x} \rightarrow \infty$. We call this a *gauge symmetry*. The field strength is invariant under the gauge symmetry:

$$F_{\mu\nu} \rightarrow \partial_\mu(A_\nu + \partial_\nu \lambda) - \partial_\nu(A_\mu + \partial_\mu \lambda) = F_{\mu\nu} \tag{6.12}$$

So what are we to make of this? We have a theory with an infinite number of symmetries, one for each function $\lambda(x)$. Previously we only encountered symmetries which act the same at all points in spacetime, for example $\psi \to e^{i\alpha}\psi$ for a complex scalar field. Noether's theorem told us that these symmetries give rise to conservation laws. Do we now have an infinite number of conservation laws?

The answer is no! Gauge symmetries have a very different interpretation than the global symmetries that we make use of in Noether's theorem. While the latter take a physical state to another physical state with the same properties, the gauge symmetry is to be viewed as a redundancy in our description. That is, two states related by a gauge symmetry are to be identified: they are the same physical state. (There is a small caveat to this statement which is explained in Section 6.3.1). One way to see that this interpretation is necessary is to notice that Maxwell's equations are not sufficient to specify the evolution of A_μ. The equations read,

$$\left[\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu\right] A^\nu = 0 \tag{6.13}$$

But the operator $\left[\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu\right]$ is not invertible: it annihilates any function of the form $\partial_\mu \lambda$. This means that given any initial data, we have no way to uniquely determine A_μ at a later time since we can't distinguish between A_μ and $A_\mu + \partial_\mu \lambda$. This would be problematic if we thought that A_μ is a physical object. However, if we're happy to identify A_μ and $A_\mu + \partial_\mu \lambda$ as corresponding to the same physical state, then our problems disappear.

Since gauge invariance is a redundancy of the system, we might try to formulate the theory purely in terms of the local, physical, gauge invariant objects $\vec{E}$ and $\vec{B}$. This is fine for the free classical theory: Maxwell's equations were, after all, first written in terms of $\vec{E}$ and $\vec{B}$. But it is not possible to describe certain quantum phenomena, such as the Aharonov-Bohm effect, without using the gauge potential A_μ. We will see shortly that we also require the gauge potential to describe classically charged fields. To describe Nature, it appears that we have to introduce quantities A_μ that we can never measure.
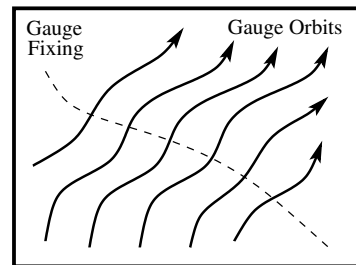
Figure 29:

The picture that emerges for the theory of electromagnetism is of an enlarged phase space, foliated by gauge orbits as shown in the figure. All states that lie along a given

line can be reached by a gauge transformation and are identified. To make progress, we pick a representative from each gauge orbit. It doesn't matter which representative we pick — after all, they're all physically equivalent. But we should make sure that we pick a "good" gauge, in which we cut the orbits.

Different representative configurations of a physical state are called different *gauges*. There are many possibilities, some of which will be more useful in different situations. Picking a gauge is rather like picking coordinates that are adapted to a particular problem. Moreover, different gauges often reveal slightly different aspects of a problem. Here we'll look at two different gauges:

- **Lorentz Gauge:** $\partial_\mu A^\mu = 0$

  To see that we can always pick a representative configuration satisfying $\partial_\mu A^\mu = 0$, suppose that we're handed a gauge field A'_μ satisfying $\partial_\mu (A')^\mu = f(x)$. Then we choose $A_\mu = A'_\mu + \partial_\mu \lambda$, where

  $$\partial_\mu \partial^\mu \lambda = -f \tag{6.14}$$

  This equation always has a solution. In fact this condition doesn't pick a unique representative from the gauge orbit. We're always free to make further gauge transformations with $\partial_\mu \partial^\mu \lambda = 0$, which also has non-trivial solutions. As the name suggests, the Lorentz gauge[3] has the advantage that it is Lorentz invariant.

- **Coulomb Gauge:** $\nabla \cdot \vec{A} = 0$

  We can make use of the residual gauge transformations in Lorentz gauge to pick $\nabla \cdot \vec{A} = 0$. (The argument is the same as before). Since A_0 is fixed by (6.10), we have as a consequence

  $$A_0 = 0 \tag{6.15}$$

  (This equation will no longer hold in Coulomb gauge in the presence of charged matter). Coulomb gauge breaks Lorentz invariance, so may not be ideal for some purposes. However, it is very useful to exhibit the physical degrees of freedom: the 3 components of $\vec{A}$ satisfy a single constraint: $\nabla \cdot \vec{A} = 0$, leaving behind just 2 degrees of freedom. These will be identified with the two polarization states of the photon. Coulomb gauge is sometimes called radiation gauge.

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[3]Named after Lorenz who had the misfortune to be one letter away from greatness.

6.2 The Quantization of the Electromagnetic Field

In the following we shall quantize free Maxwell theory twice: once in Coulomb gauge, and again in Lorentz gauge. We'll ultimately get the same answers and, along the way, see that each method comes with its own subtleties.

The first of these subtleties is common to both methods and comes when computing the momentum π^μ conjugate to A_μ,

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i \tag{6.16}$$

so the momentum π^0 conjugate to A_0 vanishes. This is the mathematical consequence of the statement we made above: A_0 is not a dynamical field. Meanwhile, the momentum conjugate to A_i is our old friend, the electric field. We can compute the Hamiltonian,

$$H = \int d^3x \; \pi^i \dot{A}_i - \mathcal{L}$$

$$= \int d^3x \; \tfrac{1}{2}\vec{E} \cdot \vec{E} + \tfrac{1}{2}\vec{B} \cdot \vec{B} - A_0(\nabla \cdot \vec{E}) \tag{6.17}$$

So A_0 acts as a Lagrange multiplier which imposes Gauss' law

$$\nabla \cdot \vec{E} = 0 \tag{6.18}$$

which is now a constraint on the system in which $\vec{A}$ are the physical degrees of freedom. Let's now see how to treat this system using different gauge fixing conditions.

6.2.1 Coulomb Gauge

In Coulomb gauge, the equation of motion for $\vec{A}$ is

$$\partial_\mu \partial^\mu \vec{A} = 0 \tag{6.19}$$

which we can solve in the usual way,

$$\vec{A} = \int \frac{d^3p}{(2\pi)^3} \; \vec{\xi}(\vec{p}) \, e^{ip \cdot x} \tag{6.20}$$

with $p_0^2 = |\vec{p}|^2$. The constraint $\nabla \cdot \vec{A} = 0$ tells us that $\vec{\xi}$ must satisfy

$$\vec{\xi} \cdot \vec{p} = 0 \tag{6.21}$$

which means that $\vec{\xi}$ is perpendicular to the direction of motion $\vec{p}$. We can pick $\vec{\xi}(\vec{p})$ to be a linear combination of two orthonormal vectors $\vec{\epsilon}_r$, $r = 1, 2$, each of which satisfies $\vec{\epsilon}_r(\vec{p}) \cdot \vec{p} = 0$ and

$$\vec{\epsilon}_r(\vec{p}) \cdot \vec{\epsilon}_s(\vec{p}) = \delta_{rs} \qquad r, s = 1, 2 \tag{6.22}$$

These two vectors correspond to the two polarization states of the photon. It's worth pointing out that you can't consistently pick a continuous basis of polarization vectors for every value of $\vec{p}$ because you can't comb the hair on a sphere. But this topological fact doesn't cause any complications in computing QED scattering processes.

To quantize we turn the Poisson brackets into commutators. Naively we would write

$$[A_i(\vec{x}), A_j(\vec{y})] = [E^i(\vec{x}), E^j(\vec{y})] = 0$$
$$[A_i(\vec{x}), E^j(\vec{y})] = i\delta_i^j \, \delta^{(3)}(\vec{x} - \vec{y}) \tag{6.23}$$

But this can't quite be right, because it's not consistent with the constraints. We still want to have $\nabla \cdot \vec{A} = \nabla \cdot \vec{E} = 0$, now imposed on the operators. But from the commutator relations above, we see

$$[\nabla \cdot \vec{A}(\vec{x}), \nabla \cdot \vec{E}(\vec{y})] = i\nabla^2 \, \delta^{(3)}(\vec{x} - \vec{y}) \neq 0 \tag{6.24}$$

What's going on? In imposing the commutator relations (6.23) we haven't correctly taken into account the constraints. In fact, this is a problem already in the classical theory, where the Poisson bracket structure is already altered[4]. The correct Poisson bracket structure leads to an alteration of the last commutation relation,

$$[A_i(\vec{x}), E_j(\vec{y})] = i\left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) \delta^{(3)}(\vec{x} - \vec{y}) \tag{6.25}$$

To see that this is now consistent with the constraints, we can rewrite the right-hand side of the commutator in momentum space,

$$[A_i(\vec{x}), E_j(\vec{y})] = i\int \frac{d^3p}{(2\pi)^3} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{6.26}$$

which is now consistent with the constraints, for example

$$[\partial_i A_i(\vec{x}), E_j(\vec{y})] = i\int \frac{d^3p}{(2\pi)^3} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right) ip_i \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = 0 \tag{6.27}$$

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[4]For a nice discussion of the classical and quantum dynamics of constrained systems, see the small book by Paul Dirac, "*Lectures on Quantum Mechanics*"

We now write $\vec{A}$ in the usual mode expansion,

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a_{\vec{p}}^r e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{r\dagger} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\vec{E}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{|\vec{p}|}{2}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a_{\vec{p}}^r e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{r\dagger} e^{-i\vec{p}\cdot\vec{x}} \right] \qquad (6.28)$$

where, as before, the polarization vectors satisfy

$$\vec{\epsilon}_r(\vec{p}) \cdot \vec{p} = 0 \quad \text{and} \quad \vec{\epsilon}_r(\vec{p}) \cdot \vec{\epsilon}_s(\vec{p}) = \delta_{rs} \qquad (6.29)$$

It is not hard to show that the commutation relations (6.25) are equivalent to the usual commutation relations for the creation and annihilation operators,

$$[a_{\vec{p}}^r, a_{\vec{q}}^s] = [a_{\vec{p}}^{r\dagger}, a_{\vec{q}}^{s\dagger}] = 0$$

$$[a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \qquad (6.30)$$

where, in deriving this, we need the completeness relation for the polarization vectors,

$$\sum_{r=1}^{2} \epsilon_r^i(\vec{p}) \epsilon_r^j(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \qquad (6.31)$$

You can easily check that this equation is true by acting on both sides with a basis of vectors $(\vec{\epsilon}_1(\vec{p}), \vec{\epsilon}_2(\vec{p}), \vec{p})$.

We derive the Hamiltonian by substituting (6.28) into (6.17). The last term vanishes in Coulomb gauge. After normal ordering, and playing around with $\vec{\epsilon}_r$ polarization vectors, we get the simple expression

$$H = \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \sum_{r=1}^{2} a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r \qquad (6.32)$$

The Coulomb gauge has the advantage that the physical degrees of freedom are manifest. However, we've lost all semblance of Lorentz invariance. One place where this manifests itself is in the propagator for the fields $A_i(x)$ (in the Heisenberg picture). In Coulomb gauge the propagator reads

$$D_{ij}^{\mathrm{tr}}(x - y) \equiv \langle 0| T A_i(x) A_j(y) |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \right) e^{-ip\cdot(x-y)} \qquad (6.33)$$

The *tr* superscript on the propagator refers to the "transverse" part of the photon. When we turn to the interacting theory, we will have to fight to massage this propagator into something a little nicer.

6.2.2 Lorentz Gauge

We could try to work in a Lorentz invariant fashion by imposing the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The equations of motion that follow from the action are then

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{6.34}$$

Our approach to implementing Lorentz gauge will be a little different from the method we used in Coulomb gauge. We choose to change the theory so that (6.34) arises directly through the equations of motion. We can achieve this by taking the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \tag{6.35}$$

The equations of motion coming from this action are

$$\partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\mu A^\mu) = \partial_\mu \partial^\mu A^\nu = 0 \tag{6.36}$$

(In fact, we could be a little more general than this, and consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \tag{6.37}$$

with arbitrary α and reach similar conclusions. The quantization of the theory is independent of α and, rather confusingly, different choices of α are sometimes also referred to as different "gauges". We will use $\alpha = 1$, which is called "Feynman gauge". The other common choice, $\alpha = 0$, is called "Landau gauge".)

Our plan will be to quantize the theory (6.36), and only later impose the constraint $\partial_\mu A^\mu = 0$ in a suitable manner on the Hilbert space of the theory. As we'll see, we will also have to deal with the residual gauge symmetry of this theory which will prove a little tricky. At first, we can proceed very easily, because both π^0 and π^i are dynamical:

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$$
$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial^i A^0 - \dot{A}^i \tag{6.38}$$

Turning these classical fields into operators, we can simply impose the usual commutation relations,

$$[A_\mu(\vec{x}), A_\nu(\vec{y})] = [\pi^\mu(\vec{x}), \pi^\nu(\vec{y})] = 0$$
$$[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = i\eta_{\mu\nu}\, \delta^{(3)}(\vec{x} - \vec{y}) \tag{6.39}$$

and we can make the usual expansion in terms of creation and annihilation operators and 4 polarization vectors $(\epsilon_\mu)^\lambda$, with $\lambda = 0, 1, 2, 3$.

$$A_\mu(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\lambda\dagger} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\pi^\mu(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} (+i) \sum_{\lambda=0}^{3} (\epsilon^\mu)^\lambda(\vec{p}) \left[a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\lambda\dagger} e^{-i\vec{p}\cdot\vec{x}} \right] \qquad (6.40)$$

Note that the momentum π^μ comes with a factor of $(+i)$, rather than the familiar $(-i)$ that we've seen so far. This can be traced to the fact that the momentum (6.38) for the classical fields takes the form $\pi^\mu = -\dot{A}^\mu + \ldots$. In the Heisenberg picture, it becomes clear that this descends to $(+i)$ in the definition of momentum.

There are now four polarization 4-vectors $\epsilon^\lambda(\vec{p})$, instead of the two polarization 3-vectors that we met in the Coulomb gauge. Of these four 4-vectors, we pick ϵ^0 to be timelike, while $\epsilon^{1,2,3}$ are spacelike. We pick the normalization

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'} \qquad (6.41)$$

which also means that

$$(\epsilon_\mu)^\lambda (\epsilon_\nu)^{\lambda'} \eta_{\lambda\lambda'} = \eta_{\mu\nu} \qquad (6.42)$$

The polarization vectors depend on the photon 4-momentum $p = (|\vec{p}|, \vec{p})$. Of the two spacelike polarizations, we will choose ϵ^1 and ϵ^2 to lie transverse to the momentum:

$$\epsilon^1 \cdot p = \epsilon^2 \cdot p = 0 \qquad (6.43)$$

The third vector ϵ^3 is the longitudinal polarization. For example, if the momentum lies along the x^3 direction, so $p \sim (1, 0, 0, 1)$, then

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (6.44)$$

For other 4-momenta, the polarization vectors are the appropriate Lorentz transformations of these vectors, since (6.43) are Lorentz invariant.

We do our usual trick, and translate the field commutation relations (6.39) into those for creation and annihilation operators. We find $[a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda'}] = [a_{\vec{p}}^{\lambda\dagger}, a_{\vec{q}}^{\lambda'\dagger}] = 0$ and

$$[a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda'\dagger}] = -\eta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \qquad (6.45)$$

The minus signs here are odd to say the least! For spacelike $\lambda = 1, 2, 3$, everything looks fine,

$$[a_{\vec{p}}^{\lambda}, a_{\vec{q}}^{\lambda' \dagger}] = \delta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \qquad \lambda, \lambda' = 1, 2, 3 \tag{6.46}$$

But for the timelike annihilation and creation operators, we have

$$[a_{\vec{p}}^0, a_{\vec{q}}^{0\dagger}] = -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6.47}$$

This is very odd! To see just how strange this is, we take the Lorentz invariant vacuum $|0\rangle$ defined by

$$a_{\vec{p}}^{\lambda} |0\rangle = 0 \tag{6.48}$$

Then we can create one-particle states in the usual way,

$$|\vec{p}, \lambda\rangle = a_{\vec{p}}^{\lambda \dagger} |0\rangle \tag{6.49}$$

For spacelike polarization states, $\lambda = 1, 2, 3$, all seems well. But for the timelike polarization $\lambda = 0$, the state $|\vec{p}, 0\rangle$ has negative norm,

$$\langle \vec{p}, 0 | \vec{q}, 0 \rangle = \langle 0 | a_{\vec{p}}^0 a_{\vec{q}}^{0\dagger} |0\rangle = -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6.50}$$

Wtf? That's very very strange. A Hilbert space with negative norm means negative probabilities which makes no sense at all. We can trace this negative norm back to the wrong sign of the kinetic term for A_0 in our original Lagrangian: $\mathcal{L} = +\frac{1}{2}\dot{\vec{A}}^2 - \frac{1}{2}\dot{A}_0^2 + \dots$.

At this point we should remember our constraint equation, $\partial_\mu A^\mu = 0$, which, until now, we've not imposed on our theory. This is going to come to our rescue. We will see that it will remove the timelike, negative norm states, and cut the physical polarizations down to two. We work in the Heisenberg picture, so that

$$\partial_\mu A^\mu = 0 \tag{6.51}$$

makes sense as an operator equation. Then we could try implementing the constraint in the quantum theory in a number of different ways. Let's look at a number of increasingly weak ways to do this

- We could ask that $\partial_\mu A^\mu = 0$ is imposed as an equation on operators. But this can't possibly work because the commutation relations (6.39) won't be obeyed for $\pi^0 = -\partial_\mu A^\mu$. We need some weaker condition.

- We could try to impose the condition on the Hilbert space instead of directly on the operators. After all, that's where the trouble lies! We could imagine that there's some way to split the Hilbert space up into good states $|\Psi\rangle$ and bad states that somehow decouple from the system. With luck, our bad states will include the weird negative norm states that we're so disgusted by. But how can we define the good states? One idea is to impose

$$\partial_\mu A^\mu \, |\Psi\rangle = 0 \tag{6.52}$$

on all good, physical states $|\Psi\rangle$. But this can't work either! Again, the condition is too strong. For example, suppose we decompose $A_\mu(x) = A_\mu^+(x) + A_\mu^-(x)$ with

$$A_\mu^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda a_{\vec{p}}^\lambda e^{-ip\cdot x}$$

$$A_\mu^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda a_{\vec{p}}^{\lambda\dagger} e^{+ip\cdot x} \tag{6.53}$$

  Then, on the vacuum $A_\mu^+ \, |0\rangle = 0$ automatically, but $\partial^\mu A_\mu^- \, |0\rangle \neq 0$. So not even the vacuum is a physical state if we use (6.52) as our constraint

- Our final attempt will be the correct one. In order to keep the vacuum as a good physical state, we can ask that physical states $|\Psi\rangle$ are defined by

$$\partial^\mu A_\mu^+ \, |\Psi\rangle = 0 \tag{6.54}$$

This ensures that

$$\langle \Psi' | \, \partial_\mu A^\mu \, |\Psi\rangle = 0 \tag{6.55}$$

so that the operator $\partial_\mu A^\mu$ has vanishing matrix elements between physical states. Equation (6.54) is known as the *Gupta-Bleuler* condition. The linearity of the constraint means that the physical states $|\Psi\rangle$ span a physical Hilbert space $\mathcal{H}_{\text{phys}}$.

So what does the physical Hilbert space $\mathcal{H}_{\text{phys}}$ look like? And, in particular, have we rid ourselves of those nasty negative norm states so that $\mathcal{H}_{\text{phys}}$ has a positive definite inner product defined on it? The answer is actually no, but almost!

Let's consider a basis of states for the Fock space. We can decompose any element of this basis as $|\Psi\rangle = |\psi_T\rangle \, |\phi\rangle$, where $|\psi_T\rangle$ contains only transverse photons, created by

$a_{\vec{p}}^{1,2\,\dagger}$, while $|\phi\rangle$ contains the timelike photons created by $a_{\vec{p}}^{0\,\dagger}$ and longitudinal photons created by $a_{\vec{p}}^{3\,\dagger}$. The Gupta-Bleuler condition (6.54) requires

$$(a_{\vec{p}}^{3} - a_{\vec{p}}^{0})\,|\phi\rangle = 0 \tag{6.56}$$

This means that the physical states must contain combinations of timelike and longitudinal photons. Whenever the state contains a timelike photon of momentum $\vec{p}$, it must also contain a longitudinal photon with the same momentum. In general $|\phi\rangle$ will be a linear combination of states $|\phi_n\rangle$ containing n pairs of timelike and longitudinal photons, which we can write as

$$|\phi\rangle = \sum_{n=0}^{\infty} C_n\,|\phi_n\rangle \tag{6.57}$$

where $|\phi_0\rangle = |0\rangle$ is simply the vacuum. It's not hard to show that although the condition (6.56) does indeed decouple the negative norm states, all the remaining states involving timelike and longitudinal photons have zero norm

$$\langle\phi_m|\,\phi_n\rangle = \delta_{n0}\delta_{m0} \tag{6.58}$$

This means that the inner product on $\mathcal{H}_{\text{phys}}$ is positive semi-definite. Which is an improvement. But we still need to deal with all these zero norm states.

The way we cope with the zero norm states is to treat them as gauge equivalent to the vacuum. Two states that differ only in their timelike and longitudinal photon content, $|\phi_n\rangle$ with $n \geq 1$ are said to be physically equivalent. We can think of the gauge symmetry of the classical theory as descending to the Hilbert space of the quantum theory. Of course, we can't just stipulate that two states are physically identical unless they give the same expectation value for all physical observables. We can check that this is true for the Hamiltonian, which can be easily computed to be

$$H = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}|\left(\sum_{i=1}^{3} a_{\vec{p}}^{i\,\dagger}a_{\vec{p}}^{i} - a_{\vec{p}}^{0\,\dagger}a_{\vec{p}}^{0}\right) \tag{6.59}$$

But the condition (6.56) ensures that $\langle\Psi|\,a_{\vec{p}}^{3\,\dagger}a_{\vec{p}}^{3}\,|\Psi\rangle = \langle\Psi|\,a_{\vec{p}}^{0\,\dagger}a_{\vec{p}}^{0}\,|\Psi\rangle$ so that the contributions from the timelike and longitudinal photons cancel amongst themselves in the Hamiltonian. This also renders the Hamiltonian positive definite, leaving us just with the contribution from the transverse photons as we would expect.

In general, one can show that the expectation values of all gauge invariant operators evaluated on physical states are independent of the coefficients C_n in (6.57).

Propagators

Finally, it's a simple matter to compute the propagator in Lorentz gauge. It is given by

$$\langle 0| \, T \, A_\mu(x) A_\nu(y) \, |0\rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} \, e^{-ip\cdot(x-y)} \tag{6.60}$$

This is a lot nicer than the propagator we found in Coulomb gauge: in particular, it's Lorentz invariant. We could also return to the Lagrangian (6.37). Had we pushed through the calculation with arbitrary coefficient α, we would find the propagator,

$$\langle 0| \, T \, A_\mu(x) A_\nu(y) \, |0\rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + i\epsilon} \left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2} \right) e^{-ip\cdot(x-y)} \tag{6.61}$$

6.3 Coupling to Matter

Let's now build an interacting theory of light and matter. We want to write down a Lagrangian which couples A_μ to some matter fields, either scalars or spinors. For example, we could write something like

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{6.62}$$

where j^μ is some function of the matter fields. The equations of motion read

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{6.63}$$

so, for consistency, we require

$$\partial_\mu j^\mu = 0 \tag{6.64}$$

In other words, j^μ must be a conserved current. But we've got lots of those! Let's look at how we can couple two of them to electromagnetism.

6.3.1 Coupling to Fermions

The Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i \, \slashed{\partial} - m)\psi \tag{6.65}$$

has an internal symmetry $\psi \to e^{-i\alpha}\psi$ and $\bar{\psi} \to e^{+i\alpha}\bar{\psi}$, with $\alpha \in \mathbf{R}$. This gives rise to the conserved current $j_V^\mu = \bar{\psi}\gamma^\mu\psi$. So we could look at the theory of electromagnetism coupled to fermions, with the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\slashed{\partial} - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi \tag{6.66}$$

where we've introduced a coupling constant e. For the free Maxwell theory, we have seen that the existence of a gauge symmetry was crucial in order to cut down the physical degrees of freedom to the requisite 2. Does our interacting theory above still have a gauge symmetry? The answer is yes. To see this, let's rewrite the Lagrangian as

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{6.67}$$

where $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$ is called the *covariant derivative*. This Lagrangian is invariant under gauge transformations which act as

$$A_\mu \to A_\mu + \partial_\mu\lambda \quad \text{and} \quad \psi \to e^{-ie\lambda}\psi \tag{6.68}$$

for an arbitrary function $\lambda(x)$. The tricky term is the derivative acting on ψ, since this will also hit the $e^{-ie\lambda}$ piece after the transformation. To see that all is well, let's look at how the covariant derivative transforms. We have

$$\begin{aligned} D_\mu\psi &= \partial_\mu\psi + ieA_\mu\psi \\ &\to \partial_\mu(e^{-ie\lambda}\psi) + ie(A_\mu + \partial_\mu\lambda)(e^{-ie\lambda}\psi) \\ &= e^{-ie\lambda}D_\mu\psi \end{aligned} \tag{6.69}$$

so the covariant derivative has the nice property that it merely picks up a phase under the gauge transformation, with the derivative of $e^{-ie\lambda}$ cancelling the transformation of the gauge field. This ensures that the whole Lagrangian is invariant, since $\bar{\psi} \to e^{+ie\lambda(x)}\bar{\psi}$.

Electric Charge

The coupling e has the interpretation of the electric charge of the ψ particle. This follows from the equations of motion of classical electromagnetism $\partial_\mu F^{\mu\nu} = j^\nu$: we know that the j^0 component is the charge density. We therefore have the total charge Q given by

$$Q = e \int d^3x\, \bar{\psi}(\vec{x})\gamma^0\psi(\vec{x}) \tag{6.70}$$

After treating this as a quantum equation, we have

$$Q = e \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} (b_{\vec{p}}^{s\dagger}b_{\vec{p}}^s - c_{\vec{p}}^{s\dagger}c_{\vec{p}}^s) \tag{6.71}$$

which is the number of particles, minus the number of antiparticles. Note that the particle and the anti-particle are required by the formalism to have opposite electric

charge. For QED, the theory of light interacting with electrons, the electric charge is usually written in terms of the dimensionless ratio α, known as the fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137} \tag{6.72}$$

Setting $\hbar = c = 1$, we have $e = \sqrt{4\pi\alpha} \approx 0.3$.

There's a small subtlety here that's worth elaborating on. I stressed that there's a radical difference between the interpretation of a global symmetry and a gauge symmetry. The former takes you from one physical state to another with the same properties and results in a conserved current through Noether's theorem. The latter is a redundancy in our description of the system. Yet in electromagnetism, the gauge symmetry $\psi \rightarrow e^{+ie\lambda(x)}\psi$ seems to lead to a conservation law, namely the conservation of electric charge. This is because among the infinite number of gauge symmetries parameterized by a function $\lambda(x)$, there is also a single global symmetry: that with $\lambda(x) = $ constant. This is a true symmetry of the system, meaning that it takes us to another physical state. More generally, the subset of global symmetries from among the gauge symmetries are those for which $\lambda(x) \rightarrow \alpha = $ constant as $x \rightarrow \infty$. These take us from one physical state to another.

Finally, let's check that the 4×4 matrix C that we introduced in Section 4.5 really deserves the name "charge conjugation matrix". If we take the complex conjugation of the Dirac equation, we have

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0 \quad \Rightarrow \quad (-i(\gamma^\mu)^\star\partial_\mu - e(\gamma^\mu)^\star A_\mu - m)\psi^\star = 0$$

Now using the defining equation $C^\dagger\gamma^\mu C = -(\gamma^\mu)^\star$, and the definition $\psi^{(c)} = C\psi^\star$, we see that the charge conjugate spinor $\psi^{(c)}$ satisfies

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi^{(c)} = 0 \tag{6.73}$$

So we see that the charge conjugate spinor $\psi^{(c)}$ satisfies the Dirac equation, but with charge $-e$ instead of $+e$.

6.3.2 Coupling to Scalars

For a real scalar field, we have no suitable conserved current. This means that we can't couple a real scalar field to a gauge field.

Let's now consider a complex scalar field φ. (For this section, I'll depart from our previous notation and call the scalar field φ to avoid confusing it with the spinor). We have a symmetry $\varphi \to e^{-i\alpha}\varphi$. We could try to couple the associated current to the gauge field,

$$\mathcal{L}_{\text{int}} = -i((\partial_\mu \varphi^\star)\varphi - \varphi^\star \partial_\mu \varphi)A^\mu \tag{6.74}$$

But this doesn't work because

- The theory is no longer gauge invariant

- The current j^μ that we coupled to A_μ depends on $\partial_\mu \varphi$. This means that if we try to compute the current associated to the symmetry, it will now pick up a contribution from the $j^\mu A_\mu$ term. So the whole procedure wasn't consistent.

We solve both of these problems simultaneously by remembering the covariant derivative. In this scalar theory, the combination

$$\mathcal{D}_\mu \varphi = \partial_\mu \varphi + ieA_\mu \varphi \tag{6.75}$$

again transforms as $\mathcal{D}_\mu \varphi \to e^{-ie\lambda}\mathcal{D}_\mu \varphi$ under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$ and $\varphi \to e^{-ie\lambda}\varphi$. This means that we can construct a gauge invariant action for a charged scalar field coupled to a photon simply by promoting all derivatives to covariant derivatives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu \varphi^\star \mathcal{D}^\mu \varphi - m^2 |\varphi|^2 \tag{6.76}$$

In general, this trick works for any theory. If we have a $U(1)$ symmetry that we wish to couple to a gauge field, we may do so by replacing all derivatives by suitable covariant derivatives. This procedure is known as *minimal coupling*.

6.4 QED

Let's now work out the Feynman rules for the full theory of quantum electrodynamics (QED) – the theory of electrons interacting with light. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\slashed{D} - m)\psi \tag{6.77}$$

where $D_\mu = \partial_\mu + ieA_\mu$.

The route we take now depends on the gauge choice. If we worked in Lorentz gauge previously, then we can jump straight to Section 6.5 where the Feynman rules for QED are written down. If, however, we worked in Coulomb gauge, then we still have a bit of work in front of us in order to massage the photon propagator into something Lorentz invariant. We will now do that.

In Coulomb gauge $\nabla \cdot \vec{A} = 0$, the equation of motion arising from varying A_0 is now

$$-\nabla^2 A_0 = e\psi^\dagger\psi \equiv ej^0 \qquad (6.78)$$

which has the solution

$$A_0(\vec{x}, t) = e \int d^3x' \, \frac{j^0(\vec{x}', t)}{4\pi|\vec{x} - \vec{x}'|} \qquad (6.79)$$

In Coulomb gauge we can rewrite the Maxwell part of the Lagrangian as

$$
\begin{aligned}
L_{\text{Maxwell}} &= \int d^3x \; \tfrac{1}{2}\vec{E}^2 - \tfrac{1}{2}\vec{B}^2 \\
&= \int d^3x \; \tfrac{1}{2}(\dot{\vec{A}} + \nabla A_0)^2 - \tfrac{1}{2}\vec{B}^2 \\
&= \int d^3x \; \tfrac{1}{2}\dot{\vec{A}}^2 + \tfrac{1}{2}(\nabla A_0)^2 - \tfrac{1}{2}\vec{B}^2
\end{aligned} \qquad (6.80)
$$

where the cross-term has vanished using $\nabla \cdot \vec{A} = 0$. After integrating the second term by parts and inserting the equation for A_0, we have

$$L_{\text{Maxwell}} = \int d^3x \; \left[\tfrac{1}{2}\dot{\vec{A}}^2 - \tfrac{1}{2}\vec{B}^2 + \frac{e^2}{2} \int d^3x' \, \frac{j_0(\vec{x})j_0(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \right] \qquad (6.81)$$

We find ourselves with a nonlocal term in the action. This is exactly the type of interaction that we boasted in Section 1.1.4 never arises in Nature! It appears here as an artifact of working in Coulomb gauge: it does not mean that the theory of QED is nonlocal. For example, it wouldn't appear if we worked in Lorentz gauge.

We now compute the Hamiltonian. Changing notation slightly from previous chapters, we have the conjugate momenta,

$$
\begin{aligned}
\vec{\Pi} &= \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \dot{\vec{A}} \\
\pi_\psi &= \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger
\end{aligned} \qquad (6.82)
$$
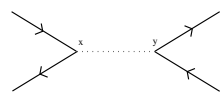
which gives us the Hamiltonian

$$H = \int d^3x \; \left[\tfrac{1}{2}\dot{\vec{A}}^2 + \tfrac{1}{2}\vec{B}^2 + \bar{\psi}(-i\gamma^i\partial_i + m)\psi - e\vec{j} \cdot \vec{A} + \frac{e^2}{2} \int d^3x' \, \frac{j^0(\vec{x})j^0(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \right]$$

where $\vec{j} = \bar{\psi}\vec{\gamma}\psi$ and $j^0 = \bar{\psi}\gamma^0\psi$.

6.4.1 Naive Feynman Rules

We want to determine the Feynman rules for this theory. For fermions, the rules are the same as those given in Section 5. The new pieces are:

- We denote the photon by a wavy line. Each end of the line comes with an $i, j = 1, 2, 3$ index telling us the component of $\vec{A}$. We calculated the transverse photon propagator in (6.33): it is ~~~~~ and contributes $D_{ij}^{\text{tr}} = \dfrac{i}{p^2 + i\epsilon} \left(\delta_{ij} - \dfrac{p_i p_j}{|\vec{p}|^2} \right)$

- The vertex contributes $-ie\gamma^i$. The index on γ^i contracts with the index on the photon line.

- The non-local interaction which, in position space, is given by contributes a factor of $\dfrac{i(e\gamma^0)^2 \delta(x^0 - y^0)}{4\pi |\vec{x} - \vec{y}|}$

These Feynman rules are rather messy. This is the price we've paid for working in Coulomb gauge. We'll now show that we can massage these expressions into something much more simple and Lorentz invariant. Let's start with the offending instantaneous interaction. Since it comes from the A_0 component of the gauge field, we could try to redefine the propagator to include a D_{00} piece which will capture this term. In fact, it fits quite nicely in this form: if we look in momentum space, we have

$$\frac{\delta(x^0 - y^0)}{4\pi |\vec{x} - \vec{y}|} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip \cdot (x-y)}}{|\vec{p}|^2} \tag{6.83}$$

so we can combine the non-local interaction with the transverse photon propagator by defining a new photon propagator

$$D_{\mu\nu}(p) = \begin{cases} +\dfrac{i}{|\vec{p}|^2} & \mu, \nu = 0 \\[3mm] \dfrac{i}{p^2 + i\epsilon} \left(\delta_{ij} - \dfrac{p_i p_j}{|\vec{p}|^2} \right) & \mu = i \neq 0, \nu = j \neq 0 \\[3mm] 0 & \text{otherwise} \end{cases} \tag{6.84}$$
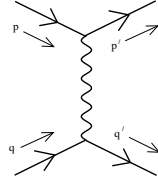
With this propagator, the wavy photon line now carries a $\mu, \nu = 0, 1, 2, 3$ index, with the extra $\mu = 0$ component taking care of the instantaneous interaction. We now need to change our vertex slightly: the $-ie\gamma^i$ above gets replaced by $-ie\gamma^\mu$ which correctly accounts for the $(e\gamma^0)^2$ piece in the instantaneous interaction.

The D_{00} piece of the propagator doesn't look a whole lot different from the transverse photon propagator. But wouldn't it be nice if they were both part of something more symmetric! In fact, they are. We have the following:

Claim: We can replace the propagator $D_{\mu\nu}(p)$ with the simpler, Lorentz invariant propagator

$$D_{\mu\nu}(p) = -i\frac{\eta_{\mu\nu}}{p^2} \tag{6.85}$$

Proof: There is a general proof using current conservation. Here we'll be more pedestrian and show that we can do this for certain Feynman diagrams. In particular, we focus on a particular tree-level diagram that contributes to $e^-e^- \to e^-e^-$ scattering,

 $\sim e^2[\bar{u}(p')\gamma^\mu u(p)]\, D_{\mu\nu}(k)\, [\bar{u}(q')\gamma^\nu u(q)]$ \qquad (6.86)

where $k = p - p' = q' - q$. Recall that $u(\vec{p})$ satisfies the equation

$$(\not{p} - m)u(\vec{p}) = 0 \tag{6.87}$$
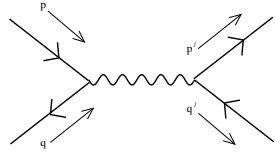
Let's define the spinor contractions $\alpha^\mu = \bar{u}(\vec{p}')\gamma^\mu u(\vec{p})$ and $\beta^\nu = \bar{u}(\vec{q}')\gamma^\nu u(\vec{q})$. Then since $k = p - p' = q' - q$, we have

$$k_\mu \alpha^\mu = \bar{u}(\vec{p}')(\not{p} - \not{p}')u(\vec{p}) = \bar{u}(\vec{p}')(m - m)u(\vec{p}) = 0 \tag{6.88}$$

and, similarly, $k_\nu \beta^\nu = 0$. Using this fact, the diagram can be written as

$$
\begin{aligned}
\alpha^\mu D_{\mu\nu}\beta^\nu &= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{(\vec{\alpha}\cdot\vec{k})(\vec{\beta}\cdot\vec{k})}{k^2|\vec{k}|^2} + \frac{\alpha^0\beta^0}{|\vec{k}|^2}\right) \\
&= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{k_0^2\alpha_0\beta_0}{k^2|\vec{k}|^2} + \frac{\alpha_0\beta_0}{|\vec{k}|^2}\right) \\
&= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{1}{k^2|\vec{k}|^2}(k_0^2 - k^2)\alpha_0\beta_0\right) \\
&= -\frac{i}{k^2}\alpha\cdot\beta = \alpha^\mu\left(-\frac{i\eta_{\mu\nu}}{k^2}\right)\beta^\nu
\end{aligned}
\tag{6.89}
$$

– 142 –

which is the claimed result. You can similarly check that the same substitution is legal in the diagram
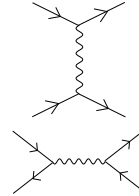
 $\qquad \sim \; e^2[\bar{v}(\vec{q})\gamma^\mu u(\vec{p})]D_{\mu\nu}(k)[\bar{u}(\vec{p}')\gamma^\nu v(\vec{q}')]$ $\qquad$ (6.90)

In fact, although we won't show it here, it's a general fact that in every Feynman diagram we may use the very nice, Lorentz invariant propagator $D_{\mu\nu} = -i\eta_{\mu\nu}/p^2$. $\qquad$ □

Note: This is the propagator we found when quantizing in Lorentz gauge (using the Feynman gauge parameter). In general, quantizing the Lagrangian (6.37) in Lorentz gauge, we have the propagator

$$D_{\mu\nu} = -\frac{i}{p^2}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right) \qquad (6.91)$$
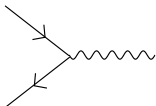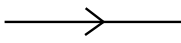
Using similar arguments to those given above, you can show that the $p_\mu p_\nu/p^2$ term cancels in all diagrams. For example, in the following diagrams the $p_\mu p_\nu$ piece of the propagator contributes as

 $\qquad \sim \bar{u}(p')\gamma^\mu u(p)\, k_\mu = \bar{u}(p')(\not{p} - \not{p}')u(p) = 0$

 $\qquad \sim \bar{v}(p)\gamma^\mu u(q)\, k_\mu = \bar{u}(p)(\not{p} + \not{q}')u(q) = 0$ $\qquad$ (6.92)

6.5 Feynman Rules

Finally, we have the Feynman rules for QED. For vertices and internal lines, we write

- Vertex: $-ie\gamma^\mu$

- Photon Propagator: $-\dfrac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$

- Fermion Propagator: $\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

For external lines in the diagram, we attach

- Photons: We add a polarization vector $\epsilon^\mu_{\text{in}}/\epsilon^\mu_{\text{out}}$ for incoming/outgoing photons. In Coulomb gauge, $\epsilon^0 = 0$ and $\vec{\epsilon}\cdot\vec{p} = 0$.

- Fermions: We add a spinor $u^r(\vec{p})/\bar{u}^r(\vec{p})$ for incoming/outgoing fermions. We add a spinor $\bar{v}^r(\vec{p})/v^r(\vec{p})$ for incoming/outgoing anti-fermions.

6.5.1 Charged Scalars

> "Pauli asked me to calculate the cross section for pair creation of scalar particles by photons. It was only a short time after Bethe and Heitler had solved the same problem for electrons and positrons. I met Bethe in Copenhagen at a conference and asked him to tell me how he did the calculations. I also inquired how long it would take to perform this task; he answered, "It would take me three days, but you will need about three weeks." He was right, as usual; furthermore, the published cross sections were wrong by a factor of four."
>
> *Viki Weisskopf*

The interaction terms in the Lagrangian for charged scalars come from the covariant derivative terms,

$$\mathcal{L} = \mathcal{D}_\mu \psi^\dagger \, \mathcal{D}^\mu \psi = \partial_\mu \psi^\dagger \partial^\mu \psi - ieA_\mu(\psi^\dagger \partial^\mu \psi - \psi \partial^\mu \psi^\dagger) + e^2 A_\mu A^\mu \psi^\dagger \psi \qquad (6.93)$$

This gives rise to two interaction vertices. But the cubic vertex is something we haven't seen before: it contains kinetic terms. How do these appear in the Feynman rules? After a Fourier transform, the derivative term means that the interaction is stronger for fermions with higher momentum, so we include a momentum factor in the Feynman rule. There is also a second, "seagull" graph. The two Feynman rules are
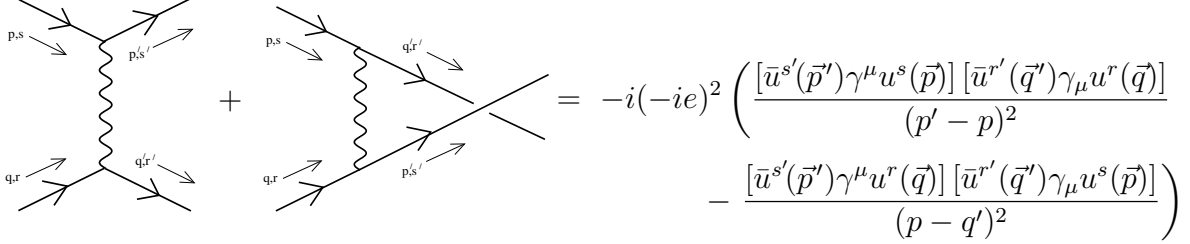
 $- ie(p + q)_\mu$ and $+ 2ie^2 \eta_{\mu\nu}$

The factor of two in the seagull diagram arises because of the two identical particles appearing in the vertex. (It's the same reason that the $1/4!$ didn't appear in the Feynman rules for ϕ^4 theory).

6.6 Scattering in QED

Let's now calculate some amplitudes for various processes in quantum electrodynamics, with a photon coupled to a single fermion. We will consider the analogous set of processes that we saw in Section 3 and Section 5. We have

Electron Scattering

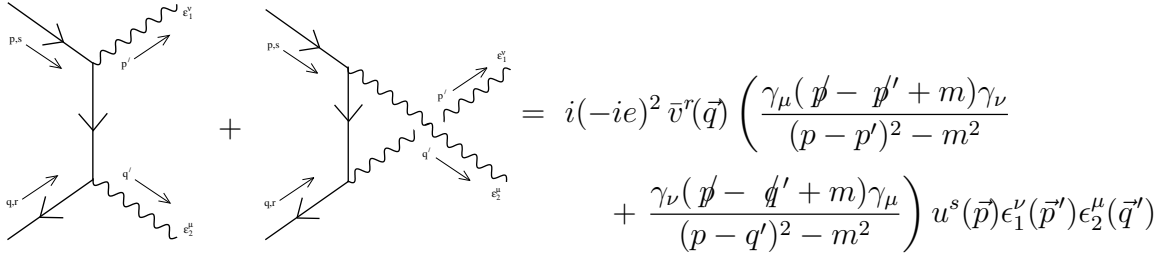Electron scattering $e^- e^- \rightarrow e^- e^-$ is described by the two leading order Feynman diagrams, given by

$$= -i(-ie)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}')\gamma^\mu u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}')\gamma_\mu u^r(\vec{q})]}{(p'-p)^2} \right.$$
$$\left. - \frac{[\bar{u}^{s'}(\vec{p}')\gamma^\mu u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}')\gamma_\mu u^s(\vec{p})]}{(p-q')^2} \right)$$

The overall $-i$ comes from the $-i\eta_{\mu\nu}$ in the propagator, which contract the indices on the γ-matrices (remember that it's really positive for $\mu,\nu = 1,2,3$).

Electron Positron Annihilation
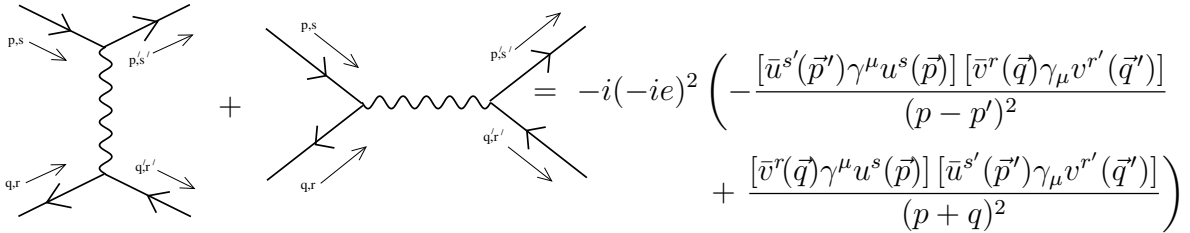
Let's now look at $e^- e^+ \rightarrow 2\gamma$, two gamma rays. The two lowest order Feynman diagrams are,

$$= i(-ie)^2\, \bar{v}^r(\vec{q}) \left(\frac{\gamma_\mu(\not{p}-\not{p}'+m)\gamma_\nu}{(p-p')^2 - m^2} \right.$$
$$\left. + \frac{\gamma_\nu(\not{p}-\not{q}'+m)\gamma_\mu}{(p-q')^2 - m^2} \right) u^s(\vec{p})\epsilon_1^\nu(\vec{p}')\epsilon_2^\mu(\vec{q}')$$

Electron Positron Scattering

For $e^- e^+ \rightarrow e^- e^+$ scattering (sometimes known as Bhabha scattering) the two lowest order Feynman diagrams are
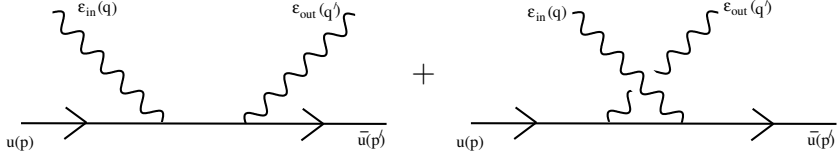

$$= -i(-ie)^2 \left(-\frac{[\bar{u}^{s'}(\vec{p}')\gamma^\mu u^s(\vec{p})]\,[\bar{v}^r(\vec{q})\gamma_\mu v^{r'}(\vec{q}')]}{(p-p')^2} \right.$$
$$\left. + \frac{[\bar{v}^r(\vec{q})\gamma^\mu u^s(\vec{p})]\,[\bar{u}^{s'}(\vec{p}')\gamma_\mu v^{r'}(\vec{q}')]}{(p+q)^2} \right)$$

Compton Scattering

The scattering of photons (in particular x-rays) off electrons $e^-\gamma \rightarrow e^-\gamma$ is known as Compton scattering. Historically, the change in wavelength of the photon in the

scattering process was one of the conclusive pieces of evidence that light could behave as a particle. The amplitude is given by,

$$= \ i(-ie)^2 \bar{u}^{r'}(\vec{p}') \left(\frac{\gamma_\mu (\not{p} + \not{q} + m)\gamma_\nu}{(p+q)^2 - m^2} + \frac{\gamma_\nu (\not{p} - \not{q}' + m)\gamma_\mu}{(p-q')^2 - m^2} \right) u^s(\vec{p}) \, \epsilon^\nu_{\text{in}} \, \epsilon^\mu_{\text{out}}$$

This amplitude vanishes for longitudinal photons. For example, suppose $\epsilon_{\text{in}} \sim q$. Then, using momentum conservation $p + q = p' + q'$, we may write the amplitude as

$$i\mathcal{A} = i(-ie)^2 \bar{u}^{r'}(\vec{p}') \left(\frac{\not{\epsilon}_{\text{out}}(\not{p} + \not{q} + m)\not{q}}{(p+q)^2 - m^2} + \frac{\not{q}(\not{p}' - \not{q} + m)\not{\epsilon}_{\text{out}}}{(p'-q)^2 - m^2} \right) u^s(\vec{p})$$

$$= i(-ie)^2 \bar{u}^{r'}(\vec{p}') \, \not{\epsilon}_{\text{out}} u^s(\vec{p}) \left(\frac{2p \cdot q}{(p+q)^2 - m^2} + \frac{2p' \cdot q}{(p'-q)^2 - m^2} \right) \qquad (6.94)$$

where, in going to the second line, we've performed some γ-matrix manipulations, and invoked the fact that q is null, so $\not{q}\not{q} = 0$, together with the spinor equations $(\not{p} - m)u(\vec{p})$ and $\bar{u}(\vec{p}')(\not{p}' - m) = 0$. We now recall the fact that q is a null vector, while $p^2 = (p')^2 = m^2$ since the external legs are on mass-shell. This means that the two denominators in the amplitude read $(p+q)^2 - m^2 = 2p \cdot q$ and $(p'-q) - m^2 = -2p' \cdot q$. This ensures that the combined amplitude vanishes for longitudinal photons as promised. A similar result holds when $\not{\epsilon}_{\text{out}} \sim q'$.

Photon Scattering

In QED, photons no longer pass through each other unimpeded. At one-loop, there is a diagram which leads to photon scattering. Although naively logarithmically divergent, the diagram is actually rendered finite by gauge invariance.
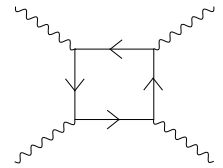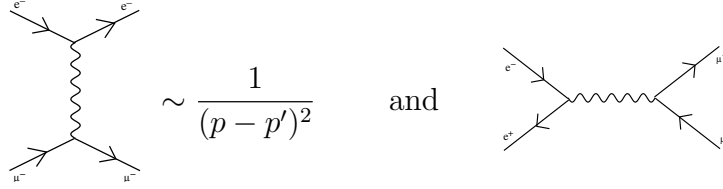

Figure 30:

Adding Muons

Adding a second fermion into the mix, which we could identify as a muon, new processes become possible. For example, we can now have processes such as $e^- \mu^- \to e^- \mu^-$ scattering, and $e^+ e^-$ annihilation into a muon anti-muon pair. Using our standard notation of p and q for incoming momenta, and p' and q' for outgoing

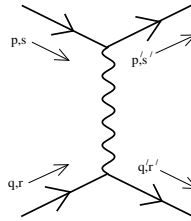momenta, we have the amplitudes given by

$$\sim \frac{1}{(p-p')^2} \qquad \text{and} \qquad \sim \frac{1}{(p+q)^2} \qquad (6.95)$$

6.6.1 The Coulomb Potential

We've come a long way. We've understood how to compute quantum amplitudes in a large array of field theories. To end this course, we use our newfound knowledge to rederive a result you learnt in kindergarten: Coulomb's law.

To do this, we repeat our calculation that led us to the Yukawa force in Sections 3.5.2 and 5.7.2. We start by looking at $e^-e^- \to e^-e^-$ scattering. We have

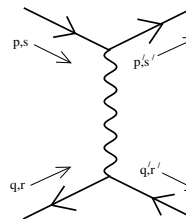$$= -i(-ie)^2 \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})] \, [\bar{u}(\vec{q}')\gamma_\mu u(\vec{q})]}{(p'-p)^2} \qquad (6.96)$$

Following (5.49), the non-relativistic limit of the spinor is $u(p) \to \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$. This means that the γ^0 piece of the interaction gives a term $\bar{u}^s(\vec{p})\gamma^0 u^r(\vec{q}) \to 2m\delta^{rs}$, while the spatial γ^i, $i=1,2,3$ pieces vanish in the non-relativistic limit: $\bar{u}^s(\vec{p})\gamma^i u^r(\vec{q}) \to 0$. Comparing the scattering amplitude in this limit to that of non-relativistic quantum mechanics, we have the effective potential between two electrons given by,

$$U(\vec{r}) = +e^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{|\vec{p}|^2} = +\frac{e^2}{4\pi r} \qquad (6.97)$$

We find the familiar repulsive Coulomb potential. We can trace the minus sign that gives a repulsive potential to the fact that only the A_0 component of the intermediate propagator $\sim -i\eta_{\mu\nu}$ contributes in the non-relativistic limit.

For $e^-e^+ \to e^-e^+$ scattering, the amplitude is

$$= +i(-ie)^2 \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})] \, [\bar{v}(\vec{q})\gamma_\mu v(\vec{q}')]}{(p'-p)^2} \qquad (6.98)$$
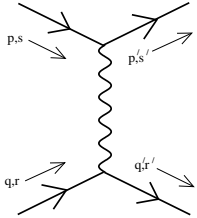
The overall + sign comes from treating the fermions correctly: we saw the same minus sign when studying scattering in Yukawa theory. The difference now comes from looking at the non-relativistic limit. We have $\bar{v}\gamma^0 v \to 2m$, giving us the potential between opposite charges,

$$U(\vec{r}) = -e^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{|\vec{p}|^2} = -\frac{e^2}{4\pi r} \tag{6.99}$$

Reassuringly, we find an attractive force between an electron and positron. The difference from the calculation of the Yukawa force comes again from the zeroth component of the gauge field, this time in the guise of the γ^0 sandwiched between $\bar{v}\gamma^0 v \to 2m$, rather than the $\bar{v}v \to -2m$ that we saw in the Yukawa case.

The Coulomb Potential for Scalars

There are many minus signs in the above calculation which somewhat obscure the crucial one which gives rise to the repulsive force. A careful study reveals the offending sign to be that which sits in front of the A_0 piece of the photon propagator $-i\eta_{\mu\nu}/p^2$. Note that with our signature $(+---)$, the propagating A_i have the correct sign, while A_0 comes with the wrong sign. This is simpler to see in the case of scalar QED, where we don't have to worry about the gamma matrices. From the Feynman rules of Section 6.5.1, we have the non-relativistic limit of scalar $e^- e^-$ scattering,

$$= -i\eta_{\mu\nu}(-ie)^2 \frac{(p+p')^\mu (q+q')_\nu}{(p'-p)^2} \rightarrow -i(-ie)^2 \frac{(2m)^2}{-(\vec{p}-\vec{p}')^2}$$

where the non-relativistic limit in the numerator involves $(p+p')\cdot(q+q') \approx (p+p')^0(q+q')_0 \approx (2m)^2$ and is responsible for selecting the A_0 part of the photon propagator rather than the A_i piece. This shows that the Coulomb potential for spin 0 particles of the same charge is again repulsive, just as it is for fermions. For $e^- e^+$ scattering, the amplitude picks up an extra minus sign because the arrows on the legs of the Feynman rules in Section 6.5.1 are correlated with the momentum arrows. Flipping the arrows on one pair of legs in the amplitude introduces the relevant minus sign to ensure that the non-relativistic potential between $e^- e^+$ is attractive as expected.

6.7 Afterword

In this course, we have laid the foundational framework for quantum field theory. Most of the developments that we've seen were already in place by the middle of the 1930s, pioneered by people such as Jordan, Dirac, Heisenberg, Pauli and Weisskopf[5].

Yet by the end of the 1930s, physicists were ready to give up on quantum field theory. The difficulty lies in the next terms in perturbation theory. These are the terms that correspond to Feynamn diagrams with loops in them, which we have scrupulously avoided computing in this course. The reason we've avoided them is because they typically give infinity! And, after ten years of trying, and failing, to make sense of this, the general feeling was that one should do something else. This from Dirac in 1937,

> *Because of its extreme complexity, most physicists will be glad to see the end of QED*

But the leading figures of the day gave up too soon. It took a new generation of postwar physicists — people like Schwinger, Feynman, Tomonaga and Dyson — to return to quantum field theory and tame the infinities. The story of how they did that will be told in next term's course.

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[5]For more details on the history of quantum field theory, see the excellent book "QED and the Men who Made it" by Sam Schweber.