ABSTRACT

A Beefy Frangi Filter for Noisy Vascular Segmentation and Network Connection in PCSVN

By

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May 2018

Recent statistical analysis of placental features has suggested the usefulness of studying key features of the placental chorionic surface vascular network (PCSVN) as a measure of overall neonatal health. A recent study has suggested that reliable reporting of these features may be useful in identifying risks of certain neurodevelopmental disorders at birth. The necessary features can be extracted from an accurate tracing of the surface vascular network, but such tracings must still be done manually, with significant user intervention. Automating this procedure would not only allow more data acquisition to study the potential effects of placental health on later conditions, but may ideally serve as a real-time diagnostic for neonatal risk factors as well.

Much work has been to develop reliable vascular extraction methods for well-known image domains (such as retinal MRA images) using Hessian-based filters, namely the (multiscale) Frangi filter. It is desirable to extend these arguments to placental images, but this approach is greatly hindered by the inherent irregularity of the placental surface as a whole, which introduces significant noise into the image domain. A recent attempt was made to apply an additional local curvilinear filter to the Frangi result in an effort to remove some noise from the final extraction.

Here we propose an alternate extraction method. First, we use arguments from Frangis original paper to provide a proper selection of parameters for our particular image domain. Using the same arguments from differential geometry that gave rise to the Frangi filter, we calculate the leading principal direction (eigenvector of the Hessian) to indicate the directionality of curvilinear

features at a particular scale. We are then able to apply an appropriately-oriented morphological filter to our Frangi targets at select scales to remove noise. This approach differs significantly from previous efforts in that morphological filtering will take place at each scale space, rather than being performed one time following multiscale synthesis. Noise removal performed in this way is expected to aide in coherent interpretation of targets that should appear in a connected network.

Finally, we discuss an important advancement in implementation–scale space conversion for differentiation (i.e. gaussian blur) via Fast Fourier Transform (FFT) rather than a more traditional convolution with a gaussian kernel, which offers a significant speedup. This thesis will also contain a general, in depth summary of both multiscale Hessian filters and scale-space theory.

We demonstrate the effectiveness of our improved vascular extraction technique on several of the following image domains: a private database of barium-injected samples provided by University of Rochester, uninjected/raw placental samples from Placental Analytics LLC, a collection of simulated images, the DRIVE and STARE databases of retinal MRAs, and a new collection of computer-generated images with significant curvilinear content.

Time permitting, this research will be extended to include a method of network connection, so that a logically connected vascular network is realized (i.e. network completion).

A Beefy Frangi Filter for Noisy Vascular Segmentation and Network Connection in PCSVN

A THESIS

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Master of Science in Applied Mathematics

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May 2018

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INTRODUCTION

The Applied Problem

Reference Nen's paper and latest autism risk paper.

Context in Image Processing

- brief background of math image processing methods
- what's been tried in this applied problem
  - nen [1]
  - catalina's paper
  - kara's paper
  - other domains

Research Goals

Segue from previous paragraph, talk about strengths and weaknesses of other methods and what this research aims to accomplish. Include 'research questions' that could allow a reader to answer the question "will this research work for my problem?". "Elevator pitch" maybe goes here.

Roadmap

Outline of the thesis ("firstly" bullshit)

CHAPTER 2

MATHEMATICAL METHODS

Overview of Differential Geometry in Image Processing

Basics, Definitions

Definition 2.1.1. *For theoretical purposes, we may view any 2D grayscale image as a continuous function* $L : \mathbb{R}^2 \to \mathbb{R}$ *with* $L \in \mathcal{C}^2\left(\mathbb{R}^2\right)$.

Definition 2.1.2. *In the context of differential geometry, we wish to refer to its graph* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *by* $(u,v) \mapsto (u,v,L(u,v))$.

Definition 2.1.3. *In situations where we wish to discuss a discrete image, we may refer to* $L_0 \in \mathbb{R}^{m\times n}$. *That is,* L_0 *is a matrix corresponding to the m-by-n digital grayscale image.*

Viewing the surface in R^3, we define the Hessian Hess of the surface L at a point (x,y) on the surface as the matrix of its second partial derivatives:

$$\text{Hess}(x,y) = \begin{bmatrix} L_{xx}(x,y) & L_{xy}(x,y) \\ L_{yx}(x,y) & L_{yy}(x,y) \end{bmatrix} \tag{2.1}$$

At any point (x,y) we denote the two eigenpairs of $\text{Hess}(x,y)$ as

$$\text{Hess} u_i = \kappa_i u_i\ , \quad i = 1,2 \tag{2.2}$$

where κ_i and u_i are known as the *principal curvatures* and *principal directions* of $L(x,y)$, respectively, and we label such that $|\kappa_2| \geq |\kappa_1|$. Notably, $\text{Hess}(x,y)$ is a real, symmetric matrix (since $L_{xy} = L_{yx}$ and L is a real function)and thus its eigenvalues are real and its eigenvectors are orthonormal to each other, as given by following lemma:

Lemma 2.1.1 (Principal Axis Theorem?)**.** *Let A be a real, symmetric matrix. The eigenvalues of A are real and its eigenvectors are orthonormal to each other.*

Proof. Let $x \neq 0$ so that $Ax = \lambda x$. Then

$$\begin{aligned} \|Ax\|_2^2 = \langle Ax, Ax\rangle &= (Ax)^* Ax \\ &= x^* A^* Ax = x^* A^T Ax = x * AAx \\ &= x^* A\lambda x = \lambda x^* Ax \\ &= \lambda x^* \lambda x = \lambda^2 x^* x = \lambda^2 \|x\|_2^2 \end{aligned}$$

Upon rearrangement, we have $\lambda^2 = \frac{\|Ax\|_2^2}{\|x\|_2^2} \geq 0 \implies \lambda$ is real.

To prove that a set of orthonormalizable eigenvectors exists, let A be real, symmetric as above and consider the eigenpairs $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2$ with $v_1, v_2 \neq 0$. [1]

In the case that $\lambda_1 \neq \lambda_2$, we have

$$\begin{aligned} (\lambda_1 - \lambda_2) v_1^T v_2 &= \lambda_1 v_1^T v_2 - \lambda_2 v_1^T v_2 \\ &= (\lambda_1 v_1)^T v_2 - v_1^T (\lambda_2 v_2) \\ &= (Av_1)^T v_2 - v_1^T (Av_2) \\ &= v_1^T A^T v_2 - v_1^T Av_2 \\ &= v_1^T Av_2 - v_1^T Av_2 = 0 \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $v_1^T v_2 = 0$.

In the case that $\lambda_1 = \lambda_2 =: \lambda$, we can define (as in Gram-Schmidt orthogonalization)

[1] To simplify notation, we simplify our argument to consider two explicit eigenvectors only, since we're only concerned with the 2×2 matrix Hess anyway.

$u = v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1$. This is an eigenvector for $\lambda = \lambda_2$, as

$$\begin{aligned}
Au &= A\left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) \\
&= Av_2 - \frac{v_1^T v_2}{v_1^T v_1} Av_1 \\
&= \lambda v_2 - \frac{v_1^T v_2}{v_1^T v_1} \lambda v_1 \\
&= \lambda\left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) = \lambda u
\end{aligned}$$

and is perpendicular to v_1, since

$$\begin{aligned}
v_1^T u &= v_1^T\left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) \\
&= v_1^T v_2 - \left(\frac{v_1^T v_2}{v_1^T v_1}\right) v_1^T v_1 \\
&= v_1^T v_2 - v_1^T v_2 (1) = 0.
\end{aligned}$$

□

Thus we see that the two principal directions form an orthonormal frame at each point (x,y) within the continuous image $L(x,y)$.

✠The following is an **unverified claim** (which might be useful for later): The frame varies continuously along paths in $\mathbb{R}^2$ except at points where $\mathrm{Hess}(x,y)$ is singular. To make this explicit:

Theorem 2.1.2 (Continuity of the leading principal direction)**.** *Let $\theta : I := [0,1] \to \mathbb{R}^2$ be a parametrized regular curve in $\mathbb{R}^2$ and $H_\theta := \mathrm{Hess}_f \circ \theta(t)$ be the matrix-valued function (where Hess_f is the 2×2 Hessian of the smooth surface f) Let $U : I \to \mathbb{R}^2$ be the implicitly-defined vector valued function s.t. $U(t)$ is the leading eigenvector of H_θ (and therefore the leading principal direction of f). That is,*

$$H_\theta\, U(t) = \lambda U(t) \quad \textit{with} \quad \lambda = \rho(H_\theta) \tag{2.3}$$

In other words, $|\lambda| \geq |\tilde{\lambda}|$ for any $\tilde{\lambda} : H_\theta\, u = \tilde{\lambda} u$ for some $u \neq 0$.

Then, $U(t)$ is continuous in t whenever $H_f(t)$ is non-singular. Note: Maybe fix this so that the path avoids any nonsingular points? $U(t)$ isn't even well-defined at such points anyway.

Proof. First, we show that $U(t)$ is a well-defined function at all points t where $H_f(t)$ is non-singular. □

✠TODO

Differential Geometry

[motivate the 'notational dump' and help explain why it's such a fucking mess]

We wish to describe the structure of an image as a surface. To do this, we develop the notion of curvature of a surface in R^3 in a standard way.

Preliminaries of Differential Geometry / Notational Dump

[Mention when you're talking about a general surface in $\mathbb{R}^3$ and when it's specifically a graph and make sure it's clear which case you're dealing with and motivate why you'd want to talk about non-graphs at all (to define shape operator in general)]

Given an open subset $U \subset \mathbb{R}^2$ and a twice differentiable function $h : U \to \mathbb{R}$. then we define its *graph* f as

$$f : U \to \mathbb{R}^3 \quad \text{by} \quad f(u_1, u_2) \;=\; \big(u_1, u_2, h(u_1, u_2)\big) \quad , u = (u_1, u_2) \in U$$

For any point $u \in U$, we associate a $p \in f[U]$, i.e. $p = f(u)$.

We define the tangent plane of f at point p as the map

$$T_u f := Df|_u\,(T_u U) \subset T_{f(u)}\mathbb{R}^3$$

where $Df|_x$ is the differential map of f at point x given by

$$Df|_x : U \to \mathbb{R}^3 \quad \text{by} \quad v \mapsto J_f(x) \cdot v$$

where $J_f(x)$ is the Jacobian of f evaluated at point x, i.e. the matrix

$$J_f(x) = \left[\frac{\partial f_i}{\partial u_j}\bigg|_x \right]_{i,j}$$

We shall denote a tangent vector $X \in T_u f$ at point p. We may expand any such vector X in terms of the basis $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$; that is, span $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\} = T_u f$.

For any immersion, $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ is a linearly independent set. We can directly see that f_{u_1} and f_{u_2} are linearly independent in the case of a graph, as $f_{u_1} = (1,0,h_{u_2})$ and $f_{u_2} = (0,1,h_{u_2})$, which are clearly linearly independent. They are not, in general, orthogonal. So the differential map of f is exactly the expansion of a point $v \in U$ along the basis $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$.

Given a closed interval $I \subset \mathbb{R}$, we can define a regular curve on the surface $c : I \to \mathbb{R}^3$ such that image$(c) \subset$ image(f). In other words, we can (for this particular curve) define an intermediary parametrization θ_c for this curve so that $c = f \circ \theta_c$ i.e.

$$\theta_c : I \to U \text{ by } \theta(t) = \big(\theta_1(t), \theta_2(t)\big)$$

and $c(t) = f(\theta(t))$.

[make this a definition] We also shall define at a point $p = f(u)$ the Gauss map normal to the tangent plane

$$\nu : U \to \mathbb{R}^3 \quad \text{by} \quad \nu(u_1,u_2) := \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\|\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}\|}$$

with each partial above understood to be evaluated at the input $u \in U$,that is, we calculate $\frac{\partial f}{\partial u_i}\Big|_u$. It is clear that $\nu \perp \frac{\partial f}{\partial u_i}$ each $i = 1,2$, and also that span $\left\{\frac{\partial \nu}{\partial u_1}, \frac{\partial \nu}{\partial u_2}\right\} \subset T_u f$ as well. [at this point if you can prove $\left\{\frac{\partial \nu}{\partial u_1}, \frac{\partial \nu}{\partial u_2}\right\}$ are linearly independent then do so, or just save for later, in which case don't use derivates at all]

To show that $\left\{\frac{\partial \nu}{\partial u_1}, \frac{\partial \nu}{\partial u_2}\right\} \in T_u f$, first note that at any particular $u \in U$, $\langle \nu, \nu \rangle = 1 \implies \frac{\partial}{\partial u_i}\langle \nu, \nu \rangle = 0$, and so by chain rule $2\langle \frac{\partial \nu}{\partial u_i}, \nu \rangle = 0 \implies \frac{\partial \nu}{\partial u_i} \perp \nu$. Since $\nu \perp$ span $\left\{\frac{\partial f}{\partial u_i}\right\}$ as well (since ν its outer product), in $\mathbb{R}^3$, this implies span $\left\{\frac{\partial \nu}{\partial u_i}\right\} \parallel$ span $\left\{\frac{\partial f}{\partial u_i}\right\}$.

Curvature of a surface

In the context of a regular arc-length parametrized curve $c : I \to \mathbb{R}^3$ parametrized along some closed interval $I \in \mathbb{R}$ (that is, a differentiable, one-to-one curve where $c'(s) = 1 \ \forall s \in I$), curvature at a point $s \in I$ is defined simply as the magnitude of the curve's acceleration: $\kappa(s) := \|c''(s)\|$.

To extend the notion of curvature of a surface f, we can consider the curvature of such a curve embedded within the surface: that is, $c[I] = f[\theta_c[I]]$. Considering a point $p \in I$ and its associated point $u = \theta_c(p)$, we wish to compare the curvatures of all curves at the point with a shared velocity. We now present a main result that provides a notion of curvature of a surface.

Theorem 2.2.1 (Theorem of Meusnier)**.** *Given a point $u \in U$ and a tangent direction $X \in T_u f$, any curve on the surface $c : I \to \text{image}(f)$ with $p \in I : \theta_c(p) = u$ where $c'(p) = X$ will have the same curvature.*

[PROVIDE A VISUALIZATION OF THIS]

In other words, any two curves on the surface with a common velocity at a given point on the surface will have the same curvature.

Proof. Considering any such curve where $\frac{\partial c}{\partial t}(p) = X$ where $X \in T_u f$ is a normalized vector tangent to the surface at the point u, we wish to decompose the curve's acceleration along the orthogonal vectors X and the Gauss map $\nu := \nu(u_1, u_2) = \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\|\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}\|}$. (Note that X and ν are indeed orthogonal, as $X \in \text{span}\left\{\frac{\partial f}{\partial u_i}\right\} = T_u f$, and $\nu \perp T_u f$). We then have (at this fixed point $u = \theta_c(p)$)

$$c'' = \langle c'', X \rangle X + \langle c'', \nu \rangle \nu \tag{2.4}$$

.

The first term is always zero for a regular curve:

$$\langle c'', X \rangle = \langle c'', c' \rangle = 0$$

since either $c'' \perp c'$ (which is generally true for a regular curve in $\mathbb{R}^3$ with a nontrivial curvature) or $c'' = 0$ itself. [see Kuhnel pg. 13, def 2.4. What points of differential geometry are too basic? what's off limits? Should I develop this too?]

We can rewrite the second coefficient of eq. (2.4) using the chain rule:

$$\langle c'', \nu \rangle = \frac{\partial}{\partial t}\left[\langle c', \nu \rangle\right] - \langle c', \frac{\partial \nu}{\partial t} \rangle \tag{2.5}$$

$$= \frac{\partial}{\partial t}\left[\langle X, \nu \rangle\right] - \langle c', \frac{\partial \nu}{\partial t} \rangle \tag{2.6}$$

$$= 0 - \langle X, \frac{\partial \nu}{\partial t} \rangle \tag{2.7}$$

Thus, we can express the curvature at this point on our selected curve as

$$\|c''\| = \|\langle c'', X \rangle X + \langle c'', \nu \rangle \nu\| = \|0 + \langle c'', \nu \rangle \nu\| \tag{2.8}$$

$$= -\langle X, \frac{\partial \nu}{\partial t} \rangle \|\nu\| \tag{2.9}$$

$$= -\langle X, \frac{\partial \nu}{\partial t} \rangle \tag{2.10}$$

$$= \langle X, -\frac{\partial \nu}{\partial t} \rangle \tag{2.11}$$

We may compute $-\frac{\partial \nu}{\partial t}$ via chain rule:

$$-\frac{d\nu}{dt} = -\frac{d}{dt}\left[\nu(u_1, u_2)\right] \tag{2.12}$$

$$= -\frac{d}{dt}\left[\nu(\theta_1(t), \theta_2(t))\right] \tag{2.13}$$

$$= \theta_1'(t)\left(-\frac{\partial \nu}{\partial u_1}\right) + \theta_2'(t)\left(-\frac{\partial \nu}{\partial u_2}\right) \tag{2.14}$$

Identifying $\left\{\frac{\partial \nu}{\partial u_i}\right\}_{i=1,2}$ as a subset of $T_u f$, we can identify a linear transformation $\mathsf{L} : T_u f \to T_u f$ which maps the basis $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$ to this subset, i.e. $\mathsf{L}(\frac{\partial f}{\partial u_i}) = -\frac{\partial \nu}{\partial u_i}$. This allows us to rewrite the time derivative of the Gauss map eq. (2.12) as

$$
\begin{aligned}
-\frac{d\nu}{dt} &= \theta_1'(t)\left(-\frac{\partial \nu}{\partial u_1}\right) + \theta_2'(t)\left(-\frac{\partial \nu}{\partial u_2}\right) && (2.15)\\
&= \theta_1'(t)\left(\mathsf{L}\left(-\frac{\partial f}{\partial u_1}\right)\right) + \theta_2'(t)\left(\mathsf{L}\left(-\frac{\partial f}{\partial u_2}\right)\right) && (2.16)\\
&= \mathsf{L}\left(\theta_1'(t)\left(-\frac{\partial f}{\partial u_1}\right) + \theta_2'(t)\left(-\frac{\partial f}{\partial u_2}\right)\right) && (2.17)\\
&= \mathsf{L}\left(\frac{d}{dt}\left[f\left(\theta(t)\right)\right]\right) = \mathsf{L}\left(\frac{d}{dt}\left[c(t)\right]\right) = \mathsf{L}\left(X\right) && (2.18)
\end{aligned}
$$

With this, we can re-express the curvature of our curve as

$$
\|c''\| = \langle X, -\frac{\partial \nu}{\partial t}\rangle = \langle X, \mathsf{L}(X)\rangle \tag{2.19}
$$

which only depends on the point u and the selected direction X, not on the particular curve at all. □

In fact, we refer to this quantity as the normal curvature of the surface.

Definition 2.2.1. *The normal curvature of a surface at point u in the direction X is given by* $\kappa_\nu := \langle X, \mathsf{L}(X)\rangle$.

The above proof shows that this is an intrinsic property of the surface. In other contexts (not necessary here), this quantity is referred to as the *second fundamental form* at the point $u \in U$; that is, $\mathbf{II}(X,X) := -\langle X, \mathsf{L}(X)\rangle$.

The map L introduced in the proof above is known as the Weingarten map and is implicitly defined at each $u \in U$. We wish to make its existence rigorous as well as find a matrix representation for it, using the standard motivation that $\mathsf{L}(\frac{\partial f}{\partial u_i}) = -\frac{\partial \nu}{\partial u_i}$.

Definition 2.2.2 (Weingarten map)**.** *The Weingarten map (or shape operator) is the map* $\mathsf{L} : T_u f \to T_u f$ *given by* $\mathsf{L} = D\nu \circ (Df)^{-1}$.

[Fix this notational nightmare]

That is, we may trace any $X \in T_u f$ which has been expanded in terms of the basis $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ and map it to the subset $\left\{-\frac{\partial \nu}{\partial u_1}, -\frac{\partial \nu}{\partial u_2}\right\}$.

The Weingarten map can be shown to be well-defined, invariant under coordinate transfomration [see Kuhnel]. We will content ourselves with simply constructing it in the context of a graph.

To find a matrix representation for L, (which we will denote $\widehat{\mathsf{L}} \in R^{2\times 2}$) we simply wish to find a linear transformation such that $\widehat{\mathsf{L}}\, \frac{\partial f}{\partial u_i}\Big|_{T_u f} = -\frac{\partial \nu}{\partial u_i}\Big|_{T_u f}$ for $i = 1,2$ where $-\,X|_{T_u f}$ denotes that $X \in T_u f$ is being represented in so-called 'local coordinates' for $T_u f$ (Strictly speaking, of course $T_u f \subset \mathbb{R}^3$ and thus $\frac{\partial f}{\partial u_i} \in \mathbb{R}^3$. Thus when we say $\frac{\partial f}{\partial u_i}\Big|_{T_u f}$ we are refering to this 3-vector expanded w.r.t. a basis for $T_u f$). In matrix form, we describe this situation as

$$\begin{bmatrix} \widehat{\mathsf{L}} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial f}{\partial u_1}\Big|_{T_u f} & \frac{\partial f}{\partial u_2}\Big|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ \widehat{\mathsf{L}}\,\frac{\partial f}{\partial u_1}\Big|_{T_u f} & \widehat{\mathsf{L}}\,\frac{\partial f}{\partial u_2}\Big|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.20}$$

$$= \begin{bmatrix} \uparrow & \uparrow \\ -\frac{\partial \nu}{\partial u_1}\Big|_{T_u f} & -\frac{\partial \nu}{\partial u_2}\Big|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.21}$$

Now, representing each vector in $T_u f$ with respect to the basis $\left\{\frac{\partial f}{\partial u_i}\right\}$, we have

$$\Longrightarrow \begin{bmatrix} \widehat{\mathsf{L}} \end{bmatrix} \begin{bmatrix} \leftarrow \frac{\partial f}{\partial u_1} \rightarrow \\ \leftarrow \frac{\partial f}{\partial u_2} \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow \frac{\partial f}{\partial u_1} \rightarrow \\ \leftarrow \frac{\partial f}{\partial u_2} \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ -\frac{\partial \nu}{\partial u_1} & -\frac{\partial \nu}{\partial u_2} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.22}$$

We can simplify this greatly by defining

$$g_{ij} := \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle \quad \text{and} \quad h_{ij} := \langle \frac{\partial f}{\partial u_i}, -\frac{\partial \nu}{\partial u_j} \rangle \tag{2.23}$$

so that

$$\begin{bmatrix} \widehat{\mathsf{L}} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \tag{2.24}$$

Then we rearrange to solve for $\widehat{\mathsf{L}}$ as

$$\widehat{\mathsf{L}} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \tag{2.25}$$

where $\left[g_{ij}\right]$ is clearly invertible, as the set $\left\{\frac{\partial f}{\partial u_j}\right\}$ is linearly independent.

It should be noted that this matrix representation is accurate not only for the surface of a graph, but for any *generalized* surface $f : U \to \mathbb{R}^3$ with $u \mapsto (x(u), y(u), z(u))$ as well. We shall later show that this calculation simplifies (somewhat) in the case that our surface is a graph.

Our final goal to is to characterize such normal curvatures. Namely, we wish to establish a method of determining in which directions an extremal normal curvature occurs. To do so, we shall consider the relationship between the direction X and the normal curvature κ_{v} in that direction at some specified u.

First, we need the following lemma:

Lemma 2.2.2. *If $A \in R^{n\times n}$ is a symmetric real matrix, $v \in R^n$ and given the dot product $\langle\cdot,\cdot\rangle$, we have $\nabla_v\langle v, Av\rangle = 2Av$. In particular, when $A = I$ the identity matrix, we have $\nabla_v\langle v, v\rangle = 2v$.*

Proof. The result is uninterestingly obtained by tracking each (the 'ith') component of $\nabla_v\langle v, Av\rangle$:

$$\Big(\nabla_v\langle v,Av\rangle\Big)_i = \frac{\partial}{\partial v_i}\Big[\langle v,Av\rangle\Big] = \frac{\partial}{\partial v_i}\left[\sum_{j=1}^n v_j(Av)_j\right] \tag{2.26}$$

$$= \frac{\partial}{\partial v_i}\left[\sum_{j=1}^n v_j \sum_{k=1}^n a_{jk}v_k\right] \tag{2.27}$$

$$= \frac{\partial}{\partial v_i}\left[a_{ii}v_i^2 + v_i\sum_{k\neq i} a_{ik}v_k + v_i\sum_{j\neq i} a_{ji}v_j + \sum_{j\neq i}\sum_{k\neq i} v_j a_{jk} v_k\right] \tag{2.28}$$

$$= 2a_{ii}v_i + \sum_{k\neq i} a_{ik}v_k + \sum_{j\neq i} a_{ji}v_j + 0 \tag{2.29}$$

$$= 2a_{ii}v_i + 2\sum_{k\neq i} a_{ik}v_k = 2\sum_{k=1}^n a_{ik}v_k = 2\big(Av\big)_i \tag{2.30}$$

$$\implies \nabla_v\langle v,Av\rangle = 2Av. \tag{2.31}$$

$\square$

We are now ready for the major result of this section.

Theorem 2.2.3 (Theorem of Olinde Rodrigues)**.** *Fixing a point $u \in U$, a direction $X \in T_u f$ minimizes the normal curvature $\kappa_\nu = \langle \mathsf{L}X, X\rangle$ subject to $\langle X, X\rangle = 1$ iff X is a (normalized) eigenvector of the Weingarten map L.*

Proof. In the following, we will assume that $X \in T_u f$ is expanded, in local coordinates, i.e. along a two dimensional basis (such as $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$) and thus can refer to L freely as the 2×2 matrix $\widehat{\mathsf{L}}$. Using the method of Lagrange multipliers, we define the Lagrangian:

$$\mathscr{L}(X;\lambda) := \langle \widehat{\mathsf{L}}X, X\rangle - \lambda\Big(\langle X, X\rangle - 1\Big) \tag{2.32}$$

Extremal values occur when $\nabla_{X,\lambda}\mathscr{L}(X;\lambda) = 0$, which becomes the two equations

$$\begin{cases} \nabla_X \langle \widehat{\mathsf{L}}X, X\rangle - \lambda \nabla_X \left(\langle X, X\rangle - 1\right) = 0 \\ \langle X, X\rangle - 1 = 0 \end{cases} \tag{2.33}$$

The second requirement is simply the constraint that X is normalized. Using the previous lemma, we can simplify the first result as follows:

$$\begin{aligned} \nabla_X \langle \widehat{\mathsf{L}}X, X\rangle - \lambda \nabla_X \left(\langle X, X\rangle - 1\right) &= 0 \\ 2\widehat{\mathsf{L}}X - \lambda(2X) &= 0 \\ \implies \quad \widehat{\mathsf{L}}X - \lambda X &= 0 \\ \implies \quad \widehat{\mathsf{L}}X &= \lambda X \end{aligned} \tag{2.34}$$

which implies that X is an eigenvector of $\widehat{\mathsf{L}}$ with corresponding eigenvalue λ ($X \neq 0$ from the second equation of eq. (2.33)). Thus the two hypotheses are exactly equivalent when X is normalized. It is also worth remarking that the corresponding eigenvalue λ is the Lagrangian multiplier itself. $\square$

Thus, to find the directions of greatest and least curvature of a surface at a point $u \in U$, we simply must calculate the Weingarten map and its eigenvectors. We refer to these directions as follows.

Definition 2.2.3 (Principal Curvatures and Principal Directions)**.** *The extremal values of normal curvature of a surface at a point $u \in U$ are referred to as **principal curvatures**. The corresponding directions at which normal curvature attains an extremal value are referred to as **principal directions**.*

Our final goal is to explicitly determine a (hopefully simplified) version of the Weingarten map in the case of a graph $f(u_1,u_2) = (u_1,u_2,h(u_1,u_2))$ and calculate the principal directions and curvatures in a simple example.

Theorem 2.2.4. *When $f : U \to \mathbb{R}^3$ is given by $(x,y) \mapsto (x,y,h(x,y))$, the matrix representation of the Weingarten map ~~is exactly the Hessian matrix given in (2.1)~~ is given by*

$$\widehat{\mathsf{L}} = \mathrm{Hess}(h)\tilde{G}\,, \quad \text{where} \quad \tilde{G} := \frac{1}{\sqrt{1+h_x^2+h_y^2}} \begin{bmatrix} 1+h_y^2 & -h_xh_y \\ -h_xh_y & 1+h_x^2 \end{bmatrix} \tag{2.35}$$

In particular, given a point $u = (x,y) \in U \subset \mathbb{R}^2$ where $h_x \approx h_y \approx 0$, we have $\tilde{G} \approx \mathrm{Id}$, and thus $\widehat{\mathsf{L}} \approx \mathrm{Hess}$.

Proof. First, we can (using chain rule) rewrite each component as in eq. (2.23):

$$h_{ij} = \langle \frac{\partial f}{\partial u_i}, -\frac{\partial \nu}{\partial u_j} \rangle = \langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \nu \rangle$$

Now, given our particular surface f, we can calculate each of these components directly. We have:

$$\begin{gathered} f_x = (1,0,h_x), \quad f_y = (0,1,h_y) \\ f_{xx} = (0,0,h_{xx}), \quad f_{xy} = (0,0,h_{xy}) = f_{yx}, \quad f_{yy} = (0,0,h_{yy}) \end{gathered} \tag{2.36}$$

and we have the unit normal vector (Gauss map)

$$\nu(u_1,u_2) = \frac{\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}}{\|\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\|} \tag{2.37}$$

$$= \frac{(1,0,h_x)\times(0,1,h_y)}{\|\cdots\|} \tag{2.38}$$

$$= \frac{(-h_x,-h_y,1)}{\sqrt{h_x^2+h_y^2+1}} \tag{2.39}$$

We then calculate each h_{ij} as

$$\begin{gathered} h_{11} = \langle \frac{\partial^2 f}{\partial x^2}, \nu \rangle = \frac{h_{xx}}{\sqrt{1+h_x^2+h_y^2}} \\ h_{12} = \langle \frac{\partial^2 f}{\partial x \partial y}, \nu \rangle = \frac{h_{xy}}{\sqrt{1+h_x^2+h_y^2}} = h_{21} \\ h_{22} = \langle \frac{\partial^2 f}{\partial y^2}, \nu \rangle = \frac{h_{yy}}{\sqrt{1+h_x^2+h_y^2}} \end{gathered} \tag{2.40}$$

and thus the first matrix in eq. (2.25) is given by

$$[h_{ij}] = \frac{1}{\sqrt{1+h_x^2+h_y^2}} \mathrm{Hess}(h) \tag{2.41}$$

To calcuate the second, we use

$$\begin{gathered} g_{ij} = \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle \\ g_{11} = \langle f_x, f_x \rangle = 1+h_x^2 \\ g_{12} = \langle f_x, f_y \rangle = h_x h_y = g_{21} \\ g_{22} = \langle f_y, f_y \rangle = 1+h_y^2 \end{gathered} \tag{2.42}$$

and thus

$$[g_{ij}]^{-1} = \begin{bmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1+h_y^2 & -h_x h_y \\ -h_x h_y & 1+h_x^2 \end{bmatrix} \tag{2.43}$$

Combining $[h_{ij}]$ and $[g_{ij}]^{-1}$ from eq. (2.43) and eq. (2.41) we arrive at eq. (2.35). $\square$

Thus the matrix of the Weingarten map $\widehat{\mathsf{L}}$ is the Hessian matrix exactly at a critical point $u \in U$, where $\nabla h(u) = (h_x(u), h_y(u)) = 0$. Of course this implies that $\widehat{\mathsf{L}}$ and $\mathrm{Hess}(h)$ have the same eigenvalues and eigenvectors at these points.

To make this a little more explicit, we will calculate the Weingarten map for a relatively simple graph.

The Weingarten map of a cylindrical ridge

Let f be the graph given by

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ by } f(x,y) = (x, y, h(x,y)), \text{ with } h(x,y) = \begin{cases} \sqrt{r^2 - x^2} & -r \leq x \leq r \\ 0 & \text{else} \end{cases} \tag{2.44}$$

We calculate the necessary partial derivatives of f as follows:

$$\frac{\partial f}{\partial x} = \left(1, 0, \frac{-x}{\sqrt{r^2 - x^2}}\right) \quad , \quad \frac{\partial^2 f}{\partial x^2} = \left(0, 0, \frac{-r^2}{\left(\sqrt{r^2 - x^2}\right)^3}\right) \tag{2.45}$$

$$\frac{\partial f}{\partial y} = (0, 1, 0) \quad , \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = 0 \tag{2.46}$$

The gauss map is given by

$$\nu(x,y) = \frac{\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}}{\|\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\|} = \left(\frac{x}{r},\ 0,\ \frac{\sqrt{r^2 - x^2}}{r}\right) \tag{2.47}$$

$$\implies \frac{\partial \nu}{\partial x} = \left(\frac{1}{r}\ ,\ 0\ ,\ \frac{-x}{r\sqrt{r^2 - x^2}}\right) \quad , \quad \frac{\partial \nu}{\partial y} = (0, 0, 0)\,. \tag{2.48}$$

We then calculate matrix elements of the Weingarten map's construction as given in

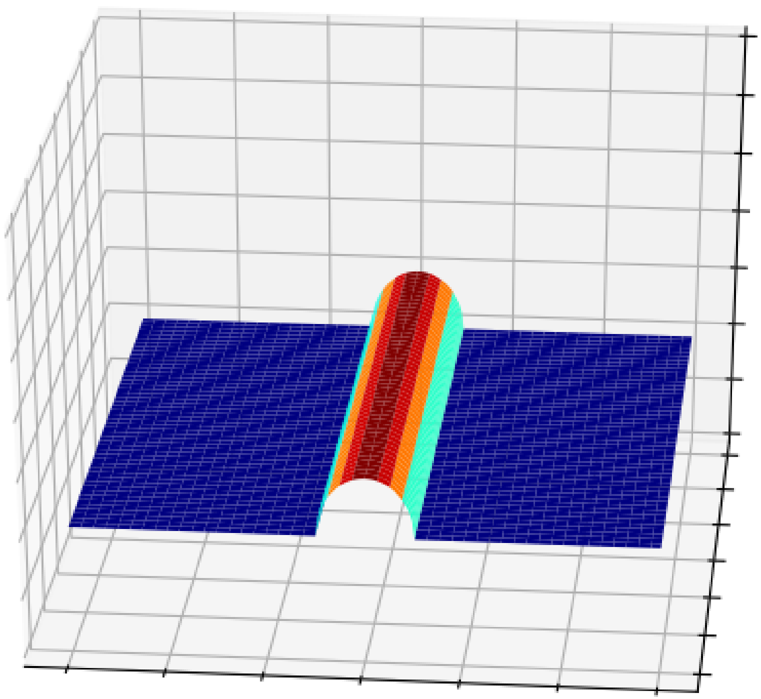

FIGURE 1. The graph of a cylindrical ridge of radius r, as given by $f(x,y)$ as given above.

eq. (2.41) and eq. (2.43) :

$$[h_{ij}] = \frac{1}{\sqrt{1+h_x^2+h_y^2}}\text{Hess}(h) = \frac{1}{\sqrt{1+\left(\frac{x^2}{r^2-x^2}\right)}}\begin{bmatrix} \frac{-r^2}{\sqrt{r^2-x^2}^3} & 0 \\ 0 & 0 \end{bmatrix} \qquad = \begin{bmatrix} \frac{-r}{r^2-x^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.49}$$

$$[g_{ij}]^{-1} = \begin{bmatrix} \frac{r^2-x^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.50}$$

$$\implies \widehat{\mathsf{L}} = [h_{ij}][g_{ij}]^{-1} = \begin{bmatrix} \frac{-r}{r^2-x^2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{r^2-x^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.51}$$

$$= \begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.52}$$

We see that $u_2 = (0,1)$ and $u_1 = (1,0)$ are eigenvectors for $\widehat{\mathrm{L}}$ with respective eigenvalues $\kappa_2 = -\frac{1}{r}, \kappa_1 = 0$. Given the theorem of Olinde Rodriguez suggests that u_2 points in the direction of maximum curvature of the surface, $-\frac{1}{r}$, which is predictably in the direction directly perpendicular to the trough, whereas the direction of least curvature is along the trough and is 0. The theorem of Meusnier suggests that the normal curvature $\kappa_2 = -\frac{1}{r}$ is reasonable– any curve on the trough perpendicular to the ridge should have the curvature of a circle (the negative simply indicates that we are on the "outside" of the surface). Finally, we note that at the ridge of the trough is exactly where $\nabla f = 0$, and the Weingarten map is exactly the Hessian matrix there.

Calculating Derivatives of Discrete Images

Discrete derivatives. ✠**TODO:** Develop the following:

- Take derivative with gradient / divided difference.
- Derivatives should be taken on Gaussian blur (equivalent to scale space development) [Lindeberg]. That is, you can take the derivative of either the convolved image, or you can take derivatives of the Gaussian itself, *then* convolve.
- Pseudocode for `np.gradient` which is used in calculating Hessian (code below)

```
gaussian_filtered = fftgauss(image, sigma=sigma)
Lx, Ly = np.gradient(gaussian_filtered)
Lxx, Lxy = np.gradient(Lx)
Lxy, Lyy = np.gradient(Ly)
```

The Frangi Filter

Intro to Hessian-based filters

Hessian-based filters are a family of curvilinear filters that employ the Hessian and its eigenspaces to determine regions of significant curvature within an image. Several such filters exist –see Sato [13] and Lorenz[10]. These filters use information about the principal curvatures (eigenvalues of the Hessian) at each point to

Overview of Frangi vesselness measure

The Frangi filter is a widely used [citation needed] Hessian-based filter that relies on the principal curvatures–that is, the eigenvalues of $\mathrm{Hess}_\sigma(x,y)$ at some particular scale σ at each point (x,y) in the image.

Implementation Details: Convolution Speedup via FFT

As described above, the actual computation of derivatives is achieved via convolution with a gaussian. In practice, this is very slow for large scales. In **TODO** .

- find image processing papers that find hessian from FFT / who uses this?
- with above: downsides?
- SIDE BY SIDE comparison?

2D Discrete Fourier Transform Convolution Theorem . [2]

Theorem 2.4.1 (2D DFT Convolution Theorem)**.** *Given two discrete functions are sequences with the same length*[3]*, that is: $f(x,y)$ and $h(x,y)$ for integers $0 < x < M$ and $0 < y < N$, we can take the discrete fourier transform (DFT) of each:*

$$F(u,v) := \mathcal{D}\{f(x,y)\} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{2.53}$$

$$H(u,v) := \mathcal{D}\{h(x,y)\} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} h(x,y)e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{2.54}$$

and given the convolution of the two functions

$$(f \star h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n) \tag{2.55}$$

[2] the following was adapted in a large part from DFT: an owner's manual. cite?

[3] If they're not actually the same length, DIP-GW suggests to make the final length at least $P = A + C - 1$ and $Q = B + D - 1$ in the case that the sizes are $A \times B$ and $C \times D$ for $f(x,y)$ and $h(x,y)$ respectively. Not sure if that matters.

then $(f \star h)(x,y)$ *and* $MN \cdot F(u,v)H(u,v)$ *are transform pairs, i.e.*

$$(f \star h)(x,y) = \mathcal{D}^{-1}\left\{MN \cdot F(u,v)H(u,v)\right\} \tag{2.56}$$

The proof follows from the definition of convolution, substituting in the inverse-DFT of f and h, and then rearrangement of finite sums.

Proof.

$$(f \star h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n) \tag{2.57}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)e^{2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\right)\left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{u(x-m)}{M}+\frac{v(y-n)}{N}\right)}\right) \tag{2.58}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)\left(\sum_{m=0}^{M-1} e^{2\pi i\left(\frac{m(p-u)}{M}\right)}\right)\left(\sum_{n=0}^{N-1} e^{2\pi i\left(\frac{n(q-v)}{N}\right)}\right)\right) \tag{2.59}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)\left(M\cdot\hat{\delta}_M(p-u)\right)\left(N\cdot\hat{\delta}_M(q-v)\right)\right) \tag{2.60}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\cdot MNF(u,v) \tag{2.61}$$

$$= MN\cdot\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{2.62}$$

$$= MN\cdot\mathcal{D}^{-1}\left\{FH\right\} \tag{2.63}$$

where

$$\hat{\delta}_N(k) = \begin{cases} 1 & \text{when } k = 0 \mod N \\ 0 & \text{else} \end{cases} \tag{2.64}$$

□

Above, we make use of the following lemma [add this before DFT convolution theorem and embed the definition of $\hat{\delta}_N$ inside]

Lemma 2.4.2. *Let j and k be integers and let N be a positive integer. Then*

$$\sum_{n=0}^{N-1} e^{2\pi i\left(\frac{n(j-k)}{N}\right)} = N\cdot\hat{\delta}_N(j-k) \tag{2.65}$$

Proof. Consider the complex number $e^{2\pi i(j-k)/N}$. Note first that this is an N-th root of unity, since

$$\left(e^{2\pi i(j-k)/N}\right)^N = e^{2\pi i(j-k)} = \left(e^{2\pi i}\right)^{(j-k)} = 1^{(j-k)} = 1$$

In other words, $e^{2\pi in(j-k)/N}$ is a root of $z^N-1=0$, which we can factor as

$$z^N-1 \;=\; (z-1)\left(z^{n-1}+\cdots+z+1\right) \;=\; (z-1)\sum_{n=0}^{N-1} z^n. \tag{2.66}$$

thus giving us

$$0 = \left(e^{2\pi i(j-k)/N}-1\right)\sum_{n=0}^{N-1} e^{2\pi in(j-k)/N} \tag{2.67}$$

To prove the claim in eq. (2.65), we consider two cases: First, if $j-k$ is a multiple of N, we of course have $e^{2\pi in(j-k)/N} = \left(e^{2\pi i}\right)^{n(j-k)/N} = 1$ and thus the left side of eq. (2.65) reduces to

$$\sum_{n=0}^{N-1}\left(e^{2\pi i}\right)^{n(j-k)/N} = \sum_{n=0}^{N-1}(1) = N$$

.

In the case that $j-k$ is *not* a multiple of N, we refer to eq. (2.67). The first factor is not zero since, $\left(e^{2\pi i(j-k)/N}\right)\neq 1$ (simply since $(j-k)/N$ is not an integer), and thus it must be that the second factor is 0:

$$\sum_{n=0}^{N-1}\left(e^{2\pi i(j-k)/N}\right)^n = 0$$

We can combine these two cases by invoking the definition of eq. (2.64), giving us the result. □

FFT .

As noted, the above result applies to the Discrete Fourier Transform. As noted, we actually achieve a convolution speedup using a Fast Fourier Transform (FFT) instead.

- Basic theory.

- Show speedup and equivalency.

Odds & Ends of Fourier Analysis .

- Sampling theory?

- Wraparound error?

- Any other kinks introduced in going from continuous to discrete.

Linear Scale Space Theory

[this is all as cited in GSST book (see starting in sec 6.3.1). as mentioned there, this is analogous to a discrete development (lindeberg 1990 lindeberg 1991 lindeberg 1994c lindeberg 1994e)]

Koenderink showed/asserted that ”any image can be embedded in a one-parameter family of derived images (with resolution as the parameter) in essentially only one unique way” given a few of the so-called *scale space axioms*. They showed in particular that any such family must satisfy the heat equation

$$\Delta K(x,y,\sigma) = K_\sigma(x,y,\sigma) \text{ for } \sigma \geq 0 \text{ such that } K(x,y,0) = u_0(x,y). \tag{2.68}$$

where $K : \mathbb{R}^3 \to \mathbb{R}$ and $u_0 : \mathbb{R}^2 \to \mathbb{R}$: is the original image (viewed as a continuous surface) and σ is a resolution parameter. Much work has been done to formalize this approach. There is a long list of desired properties quoted from gsst. goal is to try to find a minimal subset of axioms and show that other desired properties follow.

Axioms

Require existence of a continuous scale parameter.

Require linear shift-invariant smoothing. Shift invariant means that no position in the original signal is favored (makes sense, this should apply to any image.)

"Linearity implies that all-scale space properties valid for the original signal will transfer to its derivatives. Hence, there is no commitment to certain aspects of image structure, such as the zero-order representation, or its first- or second-order derivatives." [this is dubious, as mentioned by JvB]

Uniqueness of the Gaussian Kernel

This is the result of most of these axioms.

$G_\sigma \star u_0$ solves the heat equation

given u_0 as a continuous image (unscaled), we construct PDE with this as a boundary condition.

$$u : \mathbb{R}^2 \supset \Omega \to \mathbb{R} \text{ with } u(\boldsymbol{x},t) : \begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) \quad , t \geq 0 \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \end{cases} \tag{2.69}$$

We show that

$$u(\boldsymbol{x},t) = \left(G_{\sqrt{2t}} \star u_0\right)(\boldsymbol{x}) \tag{2.70}$$

solves (the above tagged equation), where

$$G_\sigma := \frac{1}{2\pi\sigma^2} e^{\left(-|x|^2/(2\sigma^2)\right)}$$

First, we need a quick lemma regarding differentiation a continuous convolution.

Lemma 2.5.1. *Derivative of a convolution is the way that it is (obviously rewrite this).*

Proof. For a single variable,

$$\frac{\partial}{\partial\alpha}[f(\alpha)\star g(\alpha)] = \frac{\partial}{\partial\alpha}\left[\int f(t)g(\alpha-t)dt\right] \tag{2.71}$$

$$= \int f(t)\frac{\partial}{\partial\alpha}[g(\alpha-t)]\,dt \tag{2.72}$$

$$= \int f(t)\left(\frac{\partial g}{\partial\alpha}\right)g(\alpha-t)dt \tag{2.73}$$

$$= f(\alpha)\star g'(\alpha) \tag{2.74}$$

By symmetry of convolution we can also conclude

$$\frac{\partial}{\partial\alpha}[f(\alpha)\star g(\alpha)] = f'(\alpha)\star g(\alpha)$$

If f and g are twice differentiable, we can compound this result to show a similar statement holds for second derivatives, and then, given the additivity of convolution, we may conclude

$$\Delta(f\star g) = \Delta(f)\star g = f\star\Delta(g) \tag{2.75}$$

□

Theorem 2.5.2. *$u(\boldsymbol{x},t) = \left(G_{\sqrt{2t}}\star u_0\right)(\boldsymbol{x})$ solves the heat equation.*

Proof. We focus on the particular kernel

$$G_{\sqrt{2t}} = \frac{1}{4\pi t}e^{\left(-|x|^2/(4t)\right)}$$

Then

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) = \frac{\partial}{\partial t}\left(G_{\sqrt{2t}}(\boldsymbol{x},t)\star u_0(\boldsymbol{x})\right) \tag{2.76}$$

$$= \frac{\partial}{\partial t}\left(G_{\sqrt{2t}}(\boldsymbol{x},t)\right)\star u_0(\boldsymbol{x}) \tag{2.77}$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}e^{\left(-|x|^2/(4t)\right)}\right)\star u_0(\boldsymbol{x}) \tag{2.78}$$

$$= \left[-\frac{1}{4\pi t^2}e^{\left(-|x|^2/(4t)\right)} + \frac{1}{4\pi t}\left(\frac{-|x|^2}{4t^2}\right)e^{-|x|^2/(4t)}\right]\star u_0(\boldsymbol{x}) \tag{2.79}$$

$$= -\frac{1}{4t^2}\left(e^{\left(-|x|^2/(4t)\right)} + |\boldsymbol{x}|^2 G_{\sqrt{2t}}(\boldsymbol{x},t)\right)\star u_0(\boldsymbol{x}) \tag{2.80}$$

and from the previous lemma,

$$\Delta u(\boldsymbol{x},t) = \Delta\left(G_{\sqrt{2t}} \star u_0(\boldsymbol{x})\right) = \Delta\left(G_{\sqrt{2t}}\right) \star u_0(\boldsymbol{x})$$

We explicitly calculate the Laplacian of $G_\sigma(x,y) = A\exp(-\frac{x^2+y^2}{2\sigma^2})$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} G_\sigma(x,y) &= A\left(\frac{-2x}{2\sigma^2}\right)\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \\
\implies \frac{\partial^2}{\partial^2 x} G_\sigma(x,y) &= A \cdot \frac{\partial}{\partial x}\left[-\frac{x}{\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right] \\
&= A\left[-\frac{1}{\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) + \frac{x}{\sigma^2}\cdot\frac{2x}{2\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right] \\
&= A\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\left[-\frac{1}{\sigma^2}+\frac{x^2}{\sigma^4}\right] \\
&= \frac{1}{\sigma^2}G_\sigma(x,y)\left[\frac{x^2}{\sigma^2}-1\right]
\end{aligned}$$

By symmetry of argument we also may conclude

$$\frac{\partial^2}{\partial y^2} G_\sigma(x,y) = \frac{1}{\sigma^2}G_\sigma(x,y)\left[\frac{y^2}{\sigma^2}-1\right]$$

and so

$$\Delta G_\sigma(x,y) = \frac{\partial^2}{\partial x^2}(G_\sigma) + \frac{\partial^2}{\partial y^2}(G_\sigma) = \frac{1}{\sigma^2}G_\sigma(x,y)\left[\frac{x^2+y^2}{\sigma^2}-2\right] \tag{2.81}$$

Then, given lemma 2.5.1, we conclude

$$\Delta\left[G_\sigma(x,y) \star u_0(x,y)\right] = \left(\frac{1}{\sigma^2}G_\sigma(x,y)\left[\frac{x^2+y^2}{\sigma^2}-2\right]\right) \star u_0(x,y) \tag{2.82}$$

For particular choices of $\sigma(t) = \sqrt{2t}$ and $A = \frac{1}{4\pi t}$, we see

$$\begin{aligned}
\Delta\left[G_{\sqrt{2t}}(x,y) \star u_0(x,y)\right] &= \left(\frac{1}{2t}G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{2t}-2\right]\right) \star u_0(x,y) && (2.83) \\
&= \left(G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{4t^2}-\frac{1}{t}\right]\right) \star u_0(x,y) && (2.84)
\end{aligned}$$

We then calculate the time derivative, using our particular choice of $\sigma(t) = \sqrt{2t}$ and $A = \frac{1}{4\pi t}$ as:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[G_{\sigma(t)}(x,y) \star u_0(x,y)\right] &= \frac{\partial}{\partial t}\left[G_{\sigma(t)}(x,y)\right] \star u_0(x,y) && (2.85)\\
&= \frac{\partial}{\partial t}\left[G_{\sqrt{2t}}(x,y)\right] \star u_0(x,y) && (2.86)\\
&= \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\exp\left(-\frac{x^2+y^2}{4t}\right)\right] \star u_0(x,y) && (2.87)\\
&= \left[-\frac{1}{4\pi t^2}\exp\left(-\frac{x^2+y^2}{4t}\right) + \frac{1}{4\pi t}\left(\frac{x^2+y^2}{4t^2}\exp\left(-\frac{x^2+y^2}{4t}\right)\right)\right] \star u_0(x,y) && (2.88)\\
&= \left(G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{4t^2} - \frac{1}{t}\right]\right) \star u_0(x,y) && (2.89)
\end{aligned}
$$

Combining these results, we find that

$$\frac{\partial}{\partial t}\left[G_{\sqrt{2t}} \star u_0\right] = \Delta\left[G_{\sqrt{2t}} \star u_0\right] \tag{2.90}$$

as desired. $\square$

Morphology

Methods of merging multiscale methods.

Other Odds and Ends

Finding the placental plate. Preprocessing. Could go in next section as well.

CHAPTER 3

RESEARCH PROTOCOL

List. All. Decisions. You Make. Be very explicit.

Pseudocode?

CHAPTER 4

RESULTS AND ANALYSIS

Data Set

Specifics of your data set. Barium placentas, other datasets.

NOTE: FIND YOUR TRACING PROTOCOL.

NOTE: Be specific about each so it's easy to gauge for others whether or not these methods would apply to their own research

Preprocessing

Results

Show stuff who the fuck knows

Answer Research Questions

CHAPTER 5

CONCLUSION

What it did well. What it didn't. People who want to expand–where should they start?

APPENDICES

APPENDIX A

APPENDIX TITLE

Put code here (and on github)

BIBLIOGRAPHY

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[1] Nen Huynh. *A filter bank approach to automate vessel extraction with applications*. PhD thesis, California State University, Long Beach, 2013.