ABSTRACT

A Beefy Frangi Filter for Noisy Vascular Segmentation and Network Connection in PCSVN

By

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Recent statistical analysis of placental features has suggested the usefulness of studying key features of the placental chorionic surface vascular network (PCSVN) as a measure of overall neonatal health. A recent study has suggested that reliable reporting of these features may be useful in identifying risks of certain neurodevelopmental disorders at birth. The necessary features can be extracted from an accurate tracing of the surface vascular network, but such tracings must still be done manually, with significant user intervention. Automating this procedure would not only allow more data acquisition to study the potential effects of placental health on later conditions, but may ideally serve as a real-time diagnostic for neonatal risk factors as well.

Much work has been to develop reliable vascular extraction methods for well-known image domains (such as retinal MRA images) using Hessian-based filters, namely the (multiscale) Frangi filter. It is desirable to extend these arguments to placental images, but this approach is greatly hindered by the inherent irregularity of the placental surface as a whole, which introduces significant noise into the image domain. A recent attempt was made to apply an additional local curvilinear filter to the Frangi result in an effort to remove some noise from the final extraction.

Here we propose an alternate extraction method. First, we use arguments from Frangi's original paper to provide a proper selection of parameters for our particular image domain. Using the same arguments from differential geometry that gave rise to the Frangi filter, we calculate the leading principal direction (eigenvector of the Hessian) to indicate the directionality of curvilinear features at a particular scale. We are then able to apply an

appropriately-oriented morphological filter to our Frangi targets at select scales to remove noise. This approach differs significantly from previous efforts in that morphological filtering will take place at each scale space, rather than being performed one time following multiscale synthesis. Noise removal performed in this way is expected to aide in coherent interpretation of targets that should appear in a connected network.

Finally, we discuss an important advancement in implementation–scale space conversion for differentiation (i.e. gaussian blur) via Fast Fourier Transform (FFT) rather than a more traditional convolution with a gaussian kernel, which offers a significant speedup. This thesis will also contain a general, in depth summary of both multiscale Hessian filters and scale-space theory.

We demonstrate the effectiveness of our improved vascular extraction technique on several of the following image domains: a private database of barium-injected samples provided by University of Rochester, uninjected/raw placental samples from Placental Analytics LLC, a collection of simulated images, the DRIVE and STARE databases of retinal MRAs, and a new collection of computer-generated images with significant curvilinear content.

Time permitting, this research will be extended to include a method of network connection, so that a logically connected vascular network is realized (i.e. network completion).

A Beefy Frangi Filter for Noisy Vascular Segmentation and Network Connection in PCSVN

A THESIS

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  - catalina's paper
  - kara's paper
  - other domains

Research Goals

Segue from previous paragraph, talk about strengths and weaknesses of other methods and what this research aims to accomplish. Include 'research questions' that could allow a reader to answer the question "will this research work for my problem?". "Elevator pitch" maybe goes here.

Roadmap

Outline of the thesis ("firstly" bullshit)

CHAPTER 2

MATHEMATICAL METHODS

Our goal is establish a resource efficient method of finding curvilinear content in 2D grayscale digital images using concepts of differential geometry. We proceed by (i) establishing a standard method of viewing these images as 2D surfaces, (ii) developing a minimal yet rigorous distillation of differential geometry to obtain suitiable quantifiers for the study of curvilinear structure in 3D surfaces, (iii) establishing a filter based on these quantifiers, and finally (iv) developing methods necessary for efficient computation of the filter.

Problem Setup in Image Processing

A digital 2D grayscale image is given by a MxN array of pixels, whose intensity is given by an integer value between 0 and 255.

Definition 2.1 (Image as a pixel matrix)**.**

$$\mathtt{I} \in \mathbb{N}^{M \times N} \quad \textit{with} \quad 0 \leq \mathtt{I}_{ij} \leq 2^8 - 1$$

For theoretical purposes, we wish to consider any such picture to ultimately be a sampling of a 2D continuous surface. We also require that this surface is sufficiently continuous as to admit the existence of second partial derivatives.

Definition 2.2 (Image as an interpolated surface)**.**

$$h : \mathbb{R}^2 \to \mathbb{R} \quad \textit{with} \quad h \in \mathcal{C}^2\left(\mathbb{R}^2\right), \quad \textit{where} \quad h(i,j) = \mathtt{I}_{ij}\ \forall (i,j) \in \{0,...,M\} \times \{0,...,N\} \subset \mathbb{N}^2$$

That is, the function h is identical to the pixel matrix $\mathtt{I}$ at all integer inputs, and simply a "smooth enough" interpolation of those points for all other values.

It is of course necessary to admit that $\mathtt{I}$ is not really a perfect representation of the underlying "content" within the picture. Not only is information lost when $\mathtt{I}$ is stored as an integer, there are also elements of noise and anomalies of lighting that would constitute noise

to the original signal. There are multiple treatments of image processing that do address this discrepancy in a pragmatic way [5], especially when the goal is noise reduction. However, we will be content to simply represent the pixels of `I` as the ultimate "cause" of the surface h in definition 2.2, and worry not about how faithfully that sampling corresponds to the real world. Moreover, though our samples in the image domain have been carefully prepared (as outlined in), there are numerous shortcomings therein, and improvements to the veracity of our original signal could be made from many angles. Though we shall draw upon the notion of the pixel matrix `I` as a sampling again to motivate our development of scale space theory in section 2.4, we ultimately use these techniques because we find them successful to our problem.

Differential Geometry

We wish to describe the structure of an image as a surface. To do this, we develop the notion of curvature of a surface in $\mathbb{R}^3$ in a standard way, following [10] (although any undergraduate text in Differential Geometry should prove satisfactory).

Preliminaries of Differential Geometry

Given an open subset $U \subset \mathbb{R}^2$ and a twice differentiable function $h : U \to \mathbb{R}$ (as in definition 2.2) we define the graph, f, of h in the following definition.

Definition 2.3. *The surface f is a graph (of the function h) when*

$$f : U \to \mathbb{R}^3 \quad by \quad f(u_1, u_2) \;=\; \Big(u_1, u_2, h(u_1, u_2)\Big)\,, \quad u = (u_1, u_2) \in U \subset \mathbb{R}^2$$

Since the graph f is clearly one-to-one by definition, we may readily associate any input $u \in U$ with its corresponding output $p \in f[U]$, i.e. $p = f(u) = f(u_1, u_2) \;=\; \Big(u_1, u_2, h(u_1, u_2)\Big)$, depending on whether we wish to focus on a point of a graph in terms of its input or in terms of the structure of the graph itself.

Our development of curvature ultimately will hinge upon a careful consideration of the tangent plane of f at a point p, for we will require a concrete definition of both the tangent

space within the domain and image of f, as well as the so called "differential" of f, the lattermost of which we will only define for the immediate case required Seeing that f is one-to-one should make a lot of this futzing about complete overkill, but I've yet to find a way to distill it. That is, this development works for any parametrized surface element, not necessarily a graph. Whatever for now.

Definition 2.4 (Tangent space of U at u)**.**

$$T_u U = \{u\} \times \mathbb{R}^2$$

Definition 2.5 (Tangent space of $\mathbb{R}^3$ at p)**.**

$$T_p \mathbb{R}^3 = \{p\} \times \mathbb{R}^3$$

It is immediately clear that $T_u U$ and $T_p \mathbb{R}^3$ are isomorphic to $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and we can easily visualize elements of $T_u U$ are tangent vectors in $\mathbb{R}^2$ "originating" at the point u, and elements of $T_p \mathbb{R}^3$ are tangent vectors "originating" at the point p.

Definition 2.6 (The differential of f at a point u)**.** *$Df|_u$ is the map from $T_u U$ into $\mathbb{R}^3$ given by*

$$Df|_u : T_u U \to T_{f(u)} \mathbb{R}^3 \quad by \quad w \mapsto J_f(u) \cdot v$$

where $J_f(u)$ is the Jacobian of f evaluated at some fixed point $u \in U$, i.e. the matrix

$$J_f(u) = \left[\left. \frac{\partial f_i}{\partial u_j} \right|_u \right]_{i,j}$$

Although not necessary presently, we could just as easily consider the differential of an arbitrary function as a map between tangent vectors in the function's domain and tangent vectors in its range. We could also just identify this as mapping $U \to \mathbb{R}^3$ by the obvious isomorphism described above. and then differential of f at x is simply a linear transformation of between the tangent spaces $T_u U$ and $T_p \mathbb{R}^3$ where the transformation in question is given by the Jacobian. We can define such a differential at any point u in the domain.

With these three definitions, we are equipped to give a formal definition of $T_u f$, the tangent plane of f at an input u.

Definition 2.7 (Tangent plane of a graph)**.**

$$T_u f := Df|_u (T_u U) \subset T_{f(u)}\mathbb{R}^3 = T_p\mathbb{R}^3$$

This vectors of this plane can thus be identified as tangent vectors from $T_u U$ that have been passed through the differential mapping $Df|_u$. We shall denote a generic tangent vector $X \in T_u f$ at point p. We may expand any such vector X in terms of the basis $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$; that is, $\text{span}\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\} = T_u f$.

Given the level of abstraction above, it may be refreshing to explicitly show the linear independence of this set in the case of an arbitrary graph f.

Lemma 2.1. *When f is a graph, for all points $u \in U$, $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ is in fact a basis for the tangent plane $T_u f$.*

Quite obviously, we're assuming $(1,0), (0,1) \in U$. If this is not the case, we pick some α small enough so that $(\alpha, 0)$ and $(0, \alpha)$ are contained and this scaled version would serve as a basis instead.

Proof. Given the definition of a graph f as in definition 2.3, we can directly calculate the partial derivatives of f at a point u.

$$f_{u_1} = (1, 0, h_{u_1}(u)) \quad \text{and} \quad f_{u_2} = (0, 1, h_{u_2}(u))$$

which are obviously linearly independent. Then $Df|_u(1,0) = f_{u_1}$, and $Df|_u(0,1) = f_{u_2}$, which shows $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\} \in T_u f$. Thus $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ is a linearly independent subset of $T_u f$, and can serve as its basis. □

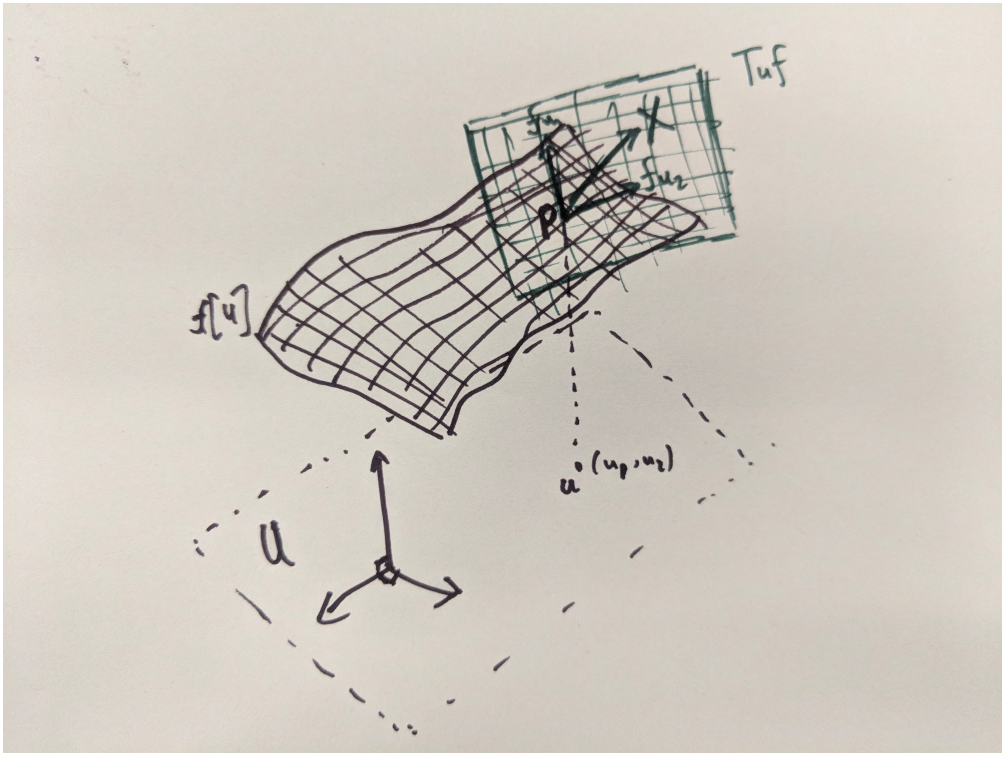


FIGURE 1: Tangent plane of a graph

The partials derivatives of f are not, in general, orthogonal at any point u, unless it happens that h_{u_1} or h_{u_2} is zero. A visualization of some of the above is given in fig. 1, although note that f_{u_1} and f_{u_2} accidentally appear orthogonal.

We now concern ourselves with developing the notion of curvature on a surface. First, we need to consider an arbitrary regular curve (i.e. differentiable, one-to-one, non-zero derivative) contained within the image of f.

Curvature of a surface and its calculation

In the context of a regular arc-length parametrized curve $c : I \to \mathbb{R}^3$ parametrized along some closed interval $I \in \mathbb{R}$ (that is, a differentiable, one-to-one curve where $c'(s) = 1 \ \forall s \in I$), curvature at a point $s \in I$ is defined simply as the magnitude of the curve's acceleration: $\kappa(s) := \|c''(s)\|$.

To extend the notion of curvature of a surface f, we can consider the curvature of such an arbitrary curve embedded within the surface.

Definition 2.8 (Surface curve)**.** *Given a closed interval $I \subset \mathbb{R}$, we call the regular curve $c : I \to \mathbb{R}^3$ a surface curve in the event that* $\text{image}(c) \subset \text{image}(f)$ *entirely. The one-to-one-ness of the graph f ensures that we can define (for the given curve) an intermediary parametrization θ_c so that $c = f \circ \theta_c$. That is,*

$$\theta_c : I \to U \text{ by } \theta(t) = \Big(\theta_1(t), \theta_2(t)\Big)$$

so that $c(t) = f(\theta_c(t)) \ \forall t \in I$, and $c[I] = f\left[\theta_c[I]\right]$..

Note as well that the velocity of this particular curve lies within $T_u f$. This can be seen by an elementary application of chain rule:

$$-\frac{dc}{dt} = -\frac{d}{dt}\Big[f(\theta_c(t))\Big] \tag{2.1}$$

$$= -\frac{d}{dt}\Big[f(\theta_1(t), \theta_2(t))\Big] \tag{2.2}$$

$$= \theta_1'(t)\left(\frac{\partial f}{\partial u_1}\right) + \theta_2'(t)\left(\frac{\partial f}{\partial u_2}\right) \in T_u f. \tag{2.3}$$

Considering a point $p \in I$ and its associated point $u = \theta_c(p)$, we wish to compare the curvatures of all (regular) surface curves passing through the point p at some particular velocity.

We now present a main result that provides a notion of curvature of a surface.

Theorem 2.2 (Theorem of Meusnier)**.** *Given a point $u \in U$ and a tangent direction $X \in T_u f$, any regular curve on the surface $c : I \to \text{image}(f)$ with $p \in I : \theta_c(p) = u$ where $c'(p) = X$ will have the same curvature.*

[TODO: provide a visualization of this]

In other words, any two curves on the surface with a common velocity at a given point on the surface will have the same curvature. To prove this, we'll require one final definition.

Definition 2.9 (The Gauss Map)**.** *The Gauss map at a point $p = f(u)$ is the unit normal to the tangent plane*

$$\nu : U \to \mathbb{R}^3 \quad by \quad \nu(u) := \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \|}$$

Each partial above understood to be evaluated at the input $u \in U$; that is, we calculate $\frac{\partial f}{\partial u_i}\Big|_u$. The existence of the cross product in its definition makes it clear that $\nu \perp \frac{\partial f}{\partial u_i}$ each $i = 1, 2$. A simple dimensionality argument of $\mathbb{R}^3$ implies that these must exist in $T_u f$. However, we can also show it directly:

To show that $\left\{ \frac{\partial \nu}{\partial u_1}, \frac{\partial \nu}{\partial u_2} \right\} \in T_u f$, first note that at any particular $u \in U$, $\langle \nu, \nu \rangle = 1 \implies \frac{\partial}{\partial u_i} \langle \nu, \nu \rangle = 0$, and so by chain rule $2 \langle \frac{\partial \nu}{\partial u_i}, \nu \rangle = 0 \implies \frac{\partial \nu}{\partial u_i} \perp \nu$. Since $\nu \perp \text{span} \left\{ \frac{\partial f}{\partial u_i} \right\}$ as well (since ν its outer product), in $\mathbb{R}^3$, this implies $\text{span} \left\{ \frac{\partial \nu}{\partial u_i} \right\} \parallel \text{span} \left\{ \frac{\partial f}{\partial u_i} \right\}$.

Thus, we have $\text{span} \left\{ \frac{\partial \nu}{\partial u_1}, \frac{\partial \nu}{\partial u_2} \right\} \subset T_u f$ as well and we can also use it as a basis.

We are finally ready to prove theorem 2.2, the Theorem of Meusnier.

Proof. Let $X \in T_u f$ be given and consider some curve where $\frac{\partial c}{\partial t}(u) = X$ where $X \in T_u f$. We wish to decompose the curve's acceleration along the orthogonal vectors X and the Gauss

map $\nu = \nu(u_1, u_2) = \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \|}$ as in definition 2.9. Note that X and ν are indeed orthogonal, as $X \in \text{span}\left\{ \frac{\partial f}{\partial u_i} \right\} = T_u f$, and $\nu \perp T_u f$). We then have (at this fixed point $u = \theta_c(p)$)

$$c'' = \langle c'', X \rangle X + \langle c'', \nu \rangle \nu \tag{2.4}$$

.

Because c is a regular curve, we either have $c'' = 0$, or $c' \perp c''$, since $\|c'\| = 1$ implies $0 = \frac{d}{dt}\langle c', c' \rangle = 2\langle c'', c' \rangle$. Thus

$$\langle c'', X \rangle = \langle c'', c' \rangle = 0$$

and we can rewrite the second coefficient of eq. (2.4) using the chain rule:

$$\begin{aligned}
\langle c'', \nu \rangle &= \frac{\partial}{\partial t}\left[\langle c', \nu \rangle \right] - \langle c', \frac{\partial \nu}{\partial t} \rangle && (2.5)\\
&= \frac{\partial}{\partial t}\left[\langle X, \nu \rangle \right] - \langle c', \frac{\partial \nu}{\partial t} \rangle && (2.6)\\
&= 0 - \langle X, \frac{\partial \nu}{\partial t} \rangle && (2.7)
\end{aligned}$$

Thus, we can express the curvature at this point on our selected curve as

$$\begin{aligned}
\|c''\| = \|\langle c'', X \rangle X + \langle c'', \nu \rangle \nu\| &= \|0 + \langle c'', \nu \rangle \nu\| && (2.8)\\
&= -\langle X, \frac{\partial \nu}{\partial t} \rangle \|\nu\| && (2.9)\\
&= -\langle X, \frac{\partial \nu}{\partial t} \rangle && (2.10)\\
&= \langle X, -\frac{\partial \nu}{\partial t} \rangle && (2.11)
\end{aligned}$$

We may compute the quantity $-\frac{\partial \nu}{\partial t}$ that appears in eq. (2.11) via chain rule:

$$\begin{aligned}
-\frac{d\nu}{dt} &= -\frac{d}{dt}\Big[\nu(u_1, u_2) \Big] && (2.12)\\
&= -\frac{d}{dt}\Big[\nu(\theta_1(t), \theta_2(t)) \Big] && (2.13)\\
&= \theta_1'(t)\left(-\frac{\partial \nu}{\partial u_1} \right) + \theta_2'(t)\left(-\frac{\partial \nu}{\partial u_2} \right) && (2.14)
\end{aligned}$$

Identifying span$\left\{-\frac{\partial \nu}{\partial u_i}\right\}_{i=1,2}$ as a subset of $T_u f$, we can define a linear transformation L which maps the basis $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$ to this subset:

Definition 2.10 (The Weingarten Map)**.**

$$\mathsf{L} : T_u f \to T_u f \quad \textit{given by the composition} \quad \mathsf{L} = D\nu \circ (Df)^{-1}.$$

That is, $\mathsf{L}(\frac{\partial f}{\partial u_i}) = -\frac{\partial \nu}{\partial u_i}$ for $i = 1, 2$, where the negative sign comes about from blind adherence to eq. (2.14) and eq. (2.11). We will This allows us to rewrite the time derivative of the Gauss map eq. (2.12) as

$$-\frac{d\nu}{dt} = \theta_1'(t)\left(-\frac{\partial \nu}{\partial u_1}\right) + \theta_2'(t)\left(-\frac{\partial \nu}{\partial u_2}\right) \tag{2.15}$$

$$= \theta_1'(t)\left(\mathsf{L}\left(\frac{\partial f}{\partial u_1}\right)\right) + \theta_2'(t)\left(\mathsf{L}\left(\frac{\partial f}{\partial u_2}\right)\right) \tag{2.16}$$

$$= \mathsf{L}\left[\theta_1'(t)\left(\frac{\partial f}{\partial u_1}\right) + \theta_2'(t)\left(\frac{\partial f}{\partial u_2}\right)\right] \tag{2.17}$$

$$= \mathsf{L}\left(\frac{d}{dt}\left[f\left(\theta(t)\right)\right]\right) = \mathsf{L}\left(\frac{d}{dt}\left[c(t)\right]\right) = \mathsf{L}\left(X\right) \tag{2.18}$$

With this, we can re-express the curvature of our curve from eq. (2.11) as the much simpler

$$\|c''\| = \langle X, -\frac{\partial \nu}{\partial t}\rangle = \langle X, \mathsf{L}(X)\rangle \tag{2.19}$$

The linear transformation L from definition 2.10, and thereby the computation of curvature given in eq. (2.19), depends only on the point u and the selected direction X, not on the particular curve c at all. □

To recap, given a point u on the surface and an arbitrary vector X in the tangent plane, we can calculate the curvature of any surface curve with velocity X there. In fact, we refer to this intrinsic quantity as the normal curvature of the surface.

Definition 2.11. *The normal curvature of a surface, denoted κ_ν at point u in the direction X is given by*

$$\kappa_\nu := \langle X, \mathsf{L}(X)\rangle$$

In fact, theorem 2.2 shows that the normal curvature is an intrinsic property of the surface–it depends only on the surface at a point, and no reference to any particular curve on the surface is necessary or implied.

The map L introduced in the proof above is known as the Weingarten map and is implicitly defined at each $u \in U$. We wish to make its existence rigorous as well as find a matrix representation for it, using the standard motivation that $\mathsf{L}(\frac{\partial f}{\partial u_i}) = -\frac{\partial \nu}{\partial u_i}$.

That is, we may trace any $X \in T_u f$ which has been expanded in terms of the basis $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ and map it to the span of $\left\{-\frac{\partial \nu}{\partial u_1}, -\frac{\partial \nu}{\partial u_2}\right\}$.

The Weingarten map can be formally shown to be well-defined, invariant under coordinate transformation [10] in the general case, which is certainly useful for surfaces f that are not graphs. The situation is much less delicate if f is a graph–the linear transformation may be simply constructed, and we proceed by simply calculating its matrix representation.

Lemma 2.3. *The Weingarten map as in definition 2.10 is well-defined for graphs.*

To find a matrix representation for L, (which we will denote $\widehat{\mathsf{L}} \in R^{2\times 2}$) we simply wish to find a linear transformation such that $\widehat{\mathsf{L}}\ \frac{\partial f}{\partial u_i}\Big|_{T_u f} = -\ \frac{\partial \nu}{\partial u_i}\Big|_{T_u f}$ for $i = 1, 2$ where $-\ X|_{T_u f}$ denotes that $X \in T_u f$ is being represented in so-called 'local coordinates' for $T_u f$ (Strictly speaking, of course $T_u f \subset \mathbb{R}^3$ and thus $\frac{\partial f}{\partial u_i} \in \mathbb{R}^3$. Thus when we say $\frac{\partial f}{\partial u_i}\Big|_{T_u f}$ we are referring to this 3-vector expanded with respect to a basis for $T_u f$). In matrix form, we describe this situation as

$$\begin{bmatrix} \\ \widehat{\mathsf{L}} \\ \\ \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \left.\frac{\partial f}{\partial u_1}\right|_{T_u f} & \left.\frac{\partial f}{\partial u_2}\right|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ \widehat{\mathsf{L}}\left.\frac{\partial f}{\partial u_1}\right|_{T_u f} & \widehat{\mathsf{L}}\left.\frac{\partial f}{\partial u_2}\right|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.20}$$

$$= \begin{bmatrix} \uparrow & \uparrow \\ -\left.\frac{\partial \nu}{\partial u_1}\right|_{T_u f} & -\left.\frac{\partial \nu}{\partial u_2}\right|_{T_u f} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.21}$$

Now, representing each vector in $T_u f$ with respect to the basis $\left\{\frac{\partial f}{\partial u_i}\right\}$, we have

$$\implies \begin{bmatrix} \\ \widehat{\mathsf{L}} \\ \\ \end{bmatrix} \begin{bmatrix} \leftarrow \frac{\partial f}{\partial u_1} \rightarrow \\ \leftarrow \frac{\partial f}{\partial u_2} \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow \frac{\partial f}{\partial u_1} \rightarrow \\ \leftarrow \frac{\partial f}{\partial u_2} \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ -\frac{\partial \nu}{\partial u_1} & -\frac{\partial \nu}{\partial u_2} \\ \downarrow & \downarrow \end{bmatrix} \tag{2.22}$$

We can simplify this greatly by defining

$$g_{ij} := \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle \quad \text{and} \quad h_{ij} := \langle \frac{\partial f}{\partial u_i}, -\frac{\partial \nu}{\partial u_j} \rangle \tag{2.23}$$

so that

$$\begin{bmatrix} \\ \widehat{\mathsf{L}} \\ \\ \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \tag{2.24}$$

Then we rearrange to solve for $\widehat{\mathsf{L}}$ as

$$\widehat{\mathsf{L}} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \tag{2.25}$$

where $[g_{ij}]$ is clearly invertible, as the set $\left\{\frac{\partial f}{\partial u_j}\right\}$ is linearly independent.

It should be noted that this matrix representation is accurate not only for the surface of a graph, but for any *generalized* surface $f : U \to \mathbb{R}^3$ with $u \mapsto (x(u), y(u), z(u))$ as well. We shall later show that this calculation simplifies (somewhat) in the case that our surface is a graph.

Our final goal to is to characterize such normal curvatures. Namely, we wish to establish a method of determining in which directions an extremal normal curvature occurs.

Principal Curvatures and Principal Directions

To do so, we shall consider the relationship between the direction X and the normal curvature κ_ν in that direction at some specified u.

First, we need the following lemma:

Lemma 2.4. *If $A \in R^{n\times n}$ is a symmetric real matrix, $v \in R^n$ and given the dot product $\langle\cdot,\cdot\rangle$, we have $\nabla_v\langle v, Av\rangle = 2Av$. In particular, when $A = I$ the identity matrix, we have $\nabla_v\langle v, v\rangle = 2v$.*

Proof. The result is uninterestingly obtained by tracking each (the 'ith') component of $\nabla_v\langle v, Av\rangle$:

$$\Big(\nabla_v\langle v, Av\rangle\Big)_i = \frac{\partial}{\partial v_i}\Big[\langle v, Av\rangle\Big] = \frac{\partial}{\partial v_i}\left[\sum_{j=1}^{n} v_j(Av)_j\right] \tag{2.26}$$

$$= \frac{\partial}{\partial v_i}\left[\sum_{j=1}^{n} v_j \sum_{k=1}^{n} a_{jk}v_k\right] \tag{2.27}$$

$$= \frac{\partial}{\partial v_i}\left[a_{ii}v_i^2 + v_i\sum_{k\neq i} a_{ik}v_k + v_i\sum_{j\neq i} a_{ji}v_j + \sum_{j\neq i}\sum_{k\neq i} v_j a_{jk}v_k\right] \tag{2.28}$$

$$= 2a_{ii}v_i + \sum_{k\neq i} a_{ik}v_k + \sum_{j\neq i} a_{ji}v_j + 0 \tag{2.29}$$

$$= 2a_{ii}v_i + 2\sum_{k\neq i} a_{ik}v_k = 2\sum_{k=1}^{n} a_{ik}v_k = 2\Big(Av\Big)_i \tag{2.30}$$

$$\implies \nabla_v\langle v, Av\rangle = 2Av. \tag{2.31}$$

□

We are now ready for the major result of this section, which ties the Weingarten map to

the notion of normal curvatures.

Theorem 2.5 (Theorem of Olinde Rodrigues)**.** *Fixing a point $u \in U$, a direction $X \in T_u f$ minimizes the normal curvature $\kappa_\nu = \langle \mathsf{L}X, X\rangle$ subject to $\langle X, X\rangle = 1$ iff X is a (normalized) eigenvector of the Weingarten map L.*

Proof. In the following, we will assume that $X \in T_u f$ is expanded, in local coordinates, i.e. along a two dimensional basis (such as $\left\{\frac{\partial f}{\partial u_i}\right\}_{i=1,2}$) and thus can refer to L freely as the 2×2 matrix $\widehat{\mathsf{L}}$. Using the method of Lagrange multipliers, we define the Lagrangian:

$$\mathscr{L}(X;\lambda) := \langle \widehat{\mathsf{L}}X, X\rangle - \lambda\Big(\langle X, X\rangle - 1\Big) \tag{2.32}$$

Extremal values occur when $\nabla_{X,\lambda}\mathscr{L}(X;\lambda) = 0$, which results in the two equations

$$\begin{cases} \nabla_X \langle \widehat{\mathsf{L}}X, X\rangle - \lambda\nabla_X\left(\langle X, X\rangle - 1\right) = 0 \\ \qquad\qquad\qquad\qquad \langle X, X\rangle - 1 = 0 \end{cases} \tag{2.33}$$

The second requirement is simply the constraint that X is normalized. Using the previous lemma, we can simplify the first result as follows:

$$\begin{aligned} \nabla_X \langle \widehat{\mathsf{L}}X, X\rangle - \lambda\nabla_X\left(\langle X, X\rangle - 1\right) &= 0 \\ 2\widehat{\mathsf{L}}X - \lambda\left(2X\right) &= 0 \\ \implies \quad \widehat{\mathsf{L}}X - \lambda X &= 0 \\ \implies \quad \widehat{\mathsf{L}}X &= \lambda X \end{aligned} \tag{2.34}$$

which implies that X is an eigenvector of $\widehat{\mathsf{L}}$ with corresponding eigenvalue λ ($X \neq 0$ from the second equation of eq. (2.33)). Thus the two hypotheses are exactly equivalent when X is normalized. It is also worth remarking that the corresponding eigenvalue λ is the Lagrangian multiplier itself. □

Thus, to find the directions of greatest and least curvature of a surface at a point $u \in U$, we simply must calculate the Weingarten map and its eigenvectors. We refer to these directions as follows.

Definition 2.12 (Principal Curvatures and Principal Directions)**.** *The extremal values of normal curvature of a surface at a point $u \in U$ are referred to as* ***principal curvatures****. The corresponding directions at which normal curvature attains an extremal value are referred to as* ***principal directions****.*

Our final goal is to explicitly determine a (hopefully simplified) version of the Weingarten map in the case of a graph $f(u_1, u_2) = (u_1, u_2, h(u_1, u_2))$ and calculate the principal directions and curvatures in a simple example.

Theorem 2.6. *When $f : U \to \mathbb{R}^3$ is given by $(x, y) \mapsto (x, y, h(x, y))$, the matrix representation of the Weingarten map ~~is exactly the Hessian matrix given in (2.1)~~ is given by*

$$\widehat{\mathsf{L}} = \mathrm{Hess}(h)\tilde{G}\,, \quad \text{where} \quad \tilde{G} := \frac{1}{\sqrt{1 + h_x^2 + h_y^2}} \begin{bmatrix} 1 + h_y^2 & -h_x h_y \\ -h_x h_y & 1 + h_x^2 \end{bmatrix} \tag{2.35}$$

In particular, given a point $u = (x, y) \in U \subset \mathbb{R}^2$ where $h_x \approx h_y \approx 0$, we have $\tilde{G} \approx \mathrm{Id}$, and thus $\widehat{\mathsf{L}} \approx \mathrm{Hess}$.

Proof. First, we can (using chain rule) rewrite each component as in eq. (2.23):

$$h_{ij} = \langle \frac{\partial f}{\partial u_i}, -\frac{\partial \nu}{\partial u_j} \rangle = \langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \nu \rangle$$

Now, given our particular surface f, we can calculate each of these components directly. We have:

$$\begin{gathered} f_x = (1, 0, h_x), \quad f_y = (0, 1, h_y) \\ f_{xx} = (0, 0, h_{xx}), \quad f_{xy} = (0, 0, h_{xy}) = f_{yx}, \quad f_{yy} = (0, 0, h_{yy}) \end{gathered} \tag{2.36}$$

and we have the unit normal vector (Gauss map)

$$\nu(u_1, u_2) = \frac{\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}}{\|\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\|} \tag{2.37}$$

$$= \frac{(1, 0, h_x) \times (0, 1, h_y)}{\|\cdots\|} \tag{2.38}$$

$$= \frac{(-h_x, -h_y, 1)}{\sqrt{h_x^2 + h_y^2 + 1}} \tag{2.39}$$

We then calculate each h_{ij} as

$$\begin{aligned} h_{11} &= \langle \frac{\partial^2 f}{\partial x^2}, \nu \rangle = \frac{h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}} \\ h_{12} &= \langle \frac{\partial^2 f}{\partial x \partial y}, \nu \rangle = \frac{h_{xy}}{\sqrt{1 + h_x^2 + h_y^2}} = h_{21} \\ h_{22} &= \langle \frac{\partial^2 f}{\partial y^2}, \nu \rangle = \frac{h_{yy}}{\sqrt{1 + h_x^2 + h_y^2}} \end{aligned} \tag{2.40}$$

and thus the first matrix in eq. (2.25) is given by

$$[h_{ij}] = \frac{1}{\sqrt{1 + h_x^2 + h_y^2}} \operatorname{Hess}(h) \tag{2.41}$$

To calcuate the second, we use

$$\begin{aligned} g_{ij} &= \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle \\ g_{11} &= \langle f_x, f_x \rangle = 1 + h_x^2 \\ g_{12} &= \langle f_x, f_y \rangle = h_x h_y = g_{21} \\ g_{22} &= \langle f_y, f_y \rangle = 1 + h_y^2 \end{aligned} \tag{2.42}$$

and thus

$$[g_{ij}]^{-1} = \begin{bmatrix} 1 + h_x^2 & h_x h_y \\ h_x h_y & 1 + h_y^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 + h_y^2 & -h_x h_y \\ -h_x h_y & 1 + h_x^2 \end{bmatrix} \tag{2.43}$$

Combining $[h_{ij}]$ and $[g_{ij}]^{-1}$ from eq. (2.43) and eq. (2.41) we arrive at eq. (2.35). □

Thus the matrix of the Weingarten map $\widehat{\mathsf{L}}$ is the Hessian matrix exactly at a critical point $u \in U$, where $\nabla h(u) = (h_x(u), h_y(u)) = 0$. Of course this implies that $\widehat{\mathsf{L}}$ and Hess(h) have the same eigenvalues and eigenvectors at these points.

To make this a little more explicit, we will calculate the Weingarten map for a relatively simple graph.

The Weingarten map and Principal Curvatures of a Cylindrical Ridge

Let f be the graph given by

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ by } f(x,y) = (x, y, h(x,y)), \text{ with } h(x,y) = \begin{cases} \sqrt{r^2 - x^2} & -r \le x \le r \\ 0 & \text{else} \end{cases} \tag{2.44}$$

We calculate the necessary partial derivatives of f as follows:

$$\frac{\partial f}{\partial x} = \left(1, 0, \frac{-x}{\sqrt{r^2 - x^2}}\right) \quad , \quad \frac{\partial^2 f}{\partial x^2} = \left(0, 0, \frac{-r^2}{\left(\sqrt{r^2 - x^2}\right)^3}\right) \tag{2.45}$$

$$\frac{\partial f}{\partial y} = (0, 1, 0) \quad , \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = 0 \tag{2.46}$$

The gauss map is given by

$$\nu(x,y) = \frac{\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}}{\| \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \|} = \left(\frac{x}{r},\, 0,\, \frac{\sqrt{r^2 - x^2}}{r}\right) \tag{2.47}$$

$$\implies \frac{\partial \nu}{\partial x} = \left(\frac{1}{r}\,,\, 0\,,\, \frac{-x}{r\sqrt{r^2 - x^2}}\right) \quad , \quad \frac{\partial \nu}{\partial y} = (0,0,0)\,. \tag{2.48}$$

We then calculate matrix elements of the Weingarten map's construction as given in

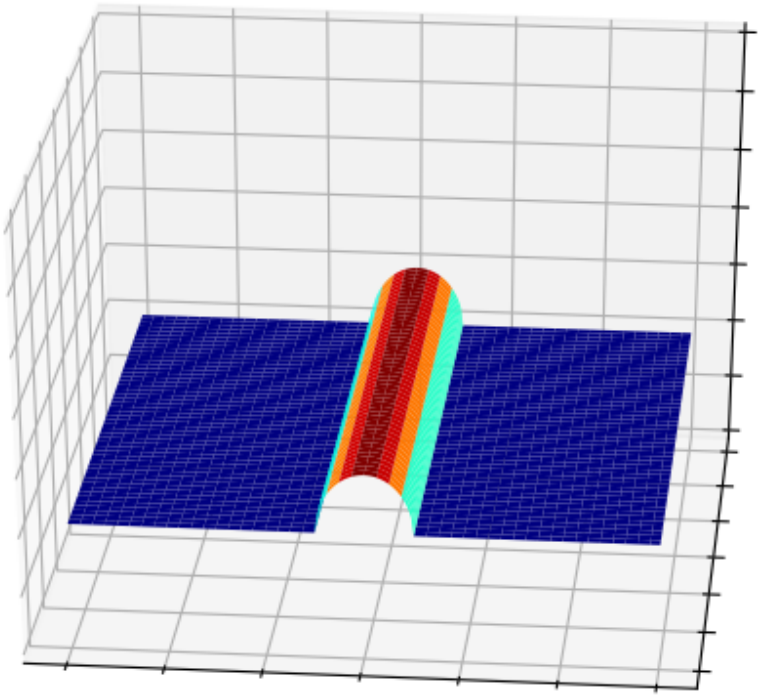


FIGURE 2: The graph of a cylindrical ridge of radius r, as given by $f(x,y)$ as given above.

eq. (2.41) and eq. (2.43) :

$$[h_{ij}] = \frac{1}{\sqrt{1+h_x^2+h_y^2}}\,\mathrm{Hess}(h) = \frac{1}{\sqrt{1+\left(\frac{x^2}{r^2-x^2}\right)}}\begin{bmatrix} \frac{-r^2}{\sqrt{r^2-x^2}^3} & 0 \\ 0 & 0 \end{bmatrix} \quad = \begin{bmatrix} \frac{-r}{r^2-x^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.49}$$

$$[g_{ij}]^{-1} = \begin{bmatrix} \frac{r^2-x^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.50}$$

$$\implies \widehat{\mathsf{L}} = [h_{ij}][g_{ij}]^{-1} = \begin{bmatrix} \frac{-r}{r^2-x^2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{r^2-x^2}{r^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.51}$$

$$= \begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.52}$$

We see that $u_2 = (0,1)$ and $u_1 = (1,0)$ are eigenvectors for $\widehat{\mathsf{L}}$ with respective eigenvalues $\kappa_2 = -\frac{1}{r}, \kappa_1 = 0$. Given the theorem of Olinde Rodriguez suggests that u_2 points in the direction of maximum curvature of the surface, $-\frac{1}{r}$, which is predictably in the direction directly perpendicular to the trough, whereas the direction of least curvature is along the trough and is 0. The theorem of Meusnier suggests that the normal curvature $\kappa_2 = -\frac{1}{r}$ is reasonable– any curve on the trough perpendicular to the ridge should have the curvature of a circle (the negative simply indicates that we are on the "outside" of the surface). Finally, we note that at the ridge of the trough is exactly where $\nabla f = 0$, and the Weingarten map is exactly the Hessian matrix there.

Viewing the surface in $\mathbb{R}^3$, we define the Hessian Hess of the surface L at a point (x,y) on the surface as the matrix of its second partial derivatives:

$$\text{Hess}(x,y) = \begin{bmatrix} L_{xx}(x,y) & L_{xy}(x,y) \\ L_{yx}(x,y) & L_{yy}(x,y) \end{bmatrix} \tag{2.53}$$

At any point (x,y) we denote the two eigenpairs of $\text{Hess}(x,y)$ as

$$\text{Hess} u_i = \kappa_i u_i \,, \quad i = 1,2 \tag{2.54}$$

where κ_i and u_i are known as the *principal curvatures* and *principal directions* of $L(x,y)$, respectively, and we label such that $|\kappa_2| \geq |\kappa_1|$. Notably, $\text{Hess}(x,y)$ is a real, symmetric matrix (since $L_{xy} = L_{yx}$ and L is a real function) and thus its eigenvalues are real and its eigenvectors are orthonormal to each other, as given by following lemma:

Lemma 2.7 (Principal Axis Theorem?)**.** *Let A be a real, symmetric matrix. The eigenvalues of A are real and its eigenvectors are orthonormal to each other.*

Proof. Let $x \neq 0$ so that $Ax = \lambda x$. Then

$$\begin{aligned}
\|Ax\|_2^2 = \langle Ax, Ax \rangle = (Ax)^* Ax \\
&= x^* A^* Ax = x^* A^T Ax = x * AAx \\
&= x^* A \lambda x = \lambda x^* Ax \\
&= \lambda x^* \lambda x = \lambda^2 x^* x = \lambda^2 \|x\|_2^2
\end{aligned}$$

Upon rearrangement, we have $\lambda^2 = \frac{\|Ax\|_2^2}{\|x\|_2^2} \geq 0 \implies \lambda$ is real.

To prove that a set of orthonormalizable eigenvectors exists, let A be real, symmetric as above and consider the eigenpairs $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$ with $v_1, v_2 \neq 0$. [1]

In the case that $\lambda_1 \neq \lambda_2$, we have

$$\begin{aligned}
(\lambda_1 - \lambda_2) v_1^T v_2 = \lambda_1 v_1^T v_2 - \lambda_2 v_1^T v_2 \\
&= (\lambda_1 v_1)^T v_2 - v_1^T (\lambda_2 v_2) \\
&= (Av_1)^T v_2 - v_1^T (Av_2) \\
&= v_1^T A^T v_2 - v_1^T Av_2 \\
&= v_1^T Av_2 - v_1^T Av_2 = 0
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $v_1^T v_2 = 0$.

In the case that $\lambda_1 = \lambda_2 =: \lambda$, we can define (as in Gram-Schmidt orthogonalization)

[1] To simplify notation, we simplify our argument to consider two explicit eigenvectors only, since we're only concerned with the 2×2 matrix Hess anyway.

$u = v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1$. This is an eigenvector for $\lambda = \lambda_2$, as

$$\begin{aligned} Au &= A\left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) \\ &= Av_2 - \frac{v_1^T v_2}{v_1^T v_1} Av_1 \\ &= \lambda v_2 - \frac{v_1^T v_2}{v_1^T v_1} \lambda v_1 \\ &= \lambda\left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) = \lambda u \end{aligned}$$

and is perpendicular to v_1, since

$$\begin{aligned} v_1^T u &= v_1^T \left(v_2 - \frac{v_1^T v_2}{v_1^T v_1} v_1\right) \\ &= v_1^T v_2 - \left(\frac{v_1^T v_2}{v_1^T v_1}\right) v_1^T v_1 \\ &= v_1^T v_2 - v_1^T v_2 (1) = 0. \end{aligned}$$

□

Thus we see that the two principal directions form an orthonormal frame at each point (x,y) within the continuous image $L(x, y)$.

We now seek to harness the ideas of this section to the task at hand: identifying curvilinear content within images.

The Frangi Filter: Uniscale

The Frangi filter, first described by Alejandro Frangi et al. in [4] is a widely used (cite) Hessian-based filter within image processing. Hessian-based filters make use of the logical "proximity" of the Hessian to notions of curvature of surfaces, as developed in section 2.2. Several such Hessian-based filters exist–see [19] and [16], as well as a comparison given in [18]. These filters use information about the principal curvatures, approximated as eigenvalues of the Hessian) at each point in the image to identify regions of significant curvature within an image.

Frangi's filter was orignally developed for vascular segmentation in images such as MRIs and it excels in that context.

The procedure for a single scale in a 2D image is as follows: Let λ_1, λ_2 be the two eigenvalues of the Hessian of the image at point (x, y), ordered such that $|\lambda_1| \leq |\lambda_2|$, and define the Frangi vesselness measure as:

$$V_\sigma(x_0, y_0) = \begin{cases} 0 & \text{if} \quad \lambda_2 > 0 \\ \exp\left(-\frac{A^2}{2\beta^2}\right)\left(1 - \exp(-\frac{S^2}{2\gamma^2})\right) & \text{otherwise} \end{cases} \tag{2.55}$$

where

$$A := |\lambda_1/\lambda_2| \quad \text{and} \quad S := \sqrt{\lambda_1^2 + \lambda_2^2} \tag{2.56}$$

and β and γ are tuning parameters. Before we discuss appropriate values for β and γ, we first seek to highlight the significance of eq. (2.55), and in particular, the ratios defined in eq. (2.56). A and S are known as the anisotropy measure and structureness measure, respectively.

Anisotropy Measure

The anisotropy (or directionality) measure A is simply the ratio of magnitudes of λ_1 and λ_2. Since at a ridge point of a tubular structure, we should have $\lambda_1 \approx 0$ and $|\lambda_2| \gg |\lambda_1|$, a very small value of A would be present at a ridge of a tubular structure.

In fig. 3, this situation is demonstrated. Here, u_1, u_2 form the orthogonal set of Hessian eigenvectors with corresponding eigenvalues λ_1 and λ_2. At such a ridgelike structure, we could predict the largest change in curvature to be straight down the ridge (in the direction of u_2), and the direction of least curvature to be directly along the ridge (in the direction of u_1). $\lambda_1 \approx 0$ and λ_2 is large and negative Note that the length of these vectors in this picture is not meant tor represent their magnitudes, as u_2 should have a much larger relative magnitude by design!

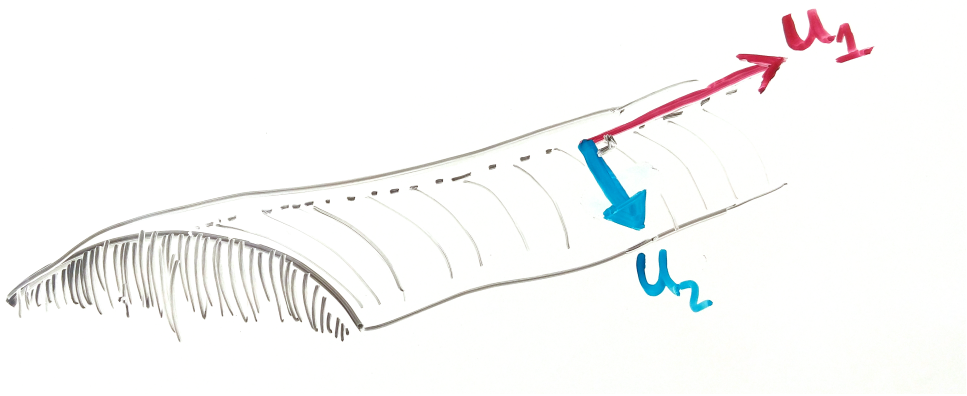


FIGURE 3: The principal eigenvectors at a ridge like structure

Of course, if the the ridge is perfectly circular along its cross section (as was in section 2.2, it is of course apparent that λ_2 would be the same value at any place along the ridge (not just at its crest), and λ_1 would likewise be 0 at any such point. One could also imagine a similar situation in which the dropoff from crest to bottom gets increasing steep. In such a case, λ_2 as a function of x would in fact be largest nearest to the bottom. This thought experiment should dispel a naive misunderstanding of the power of a Frangi filter: a high anisotropy measure (and a large structureness measure) will not in general identify the crests of a ridge-like structure–it only will highlight that such a pixel is on a ridge-like structure at all. Thus, the anisotropy measure will not necessarily be at a maximum at the crest of the ridge.

Similiarly, the vessel we we wish to identify can not be reasonably expected to behave as perfectly as our toy example. There will likely be small aberrations in a ridgelike structure, such as small divots or depressions in an overall ridge-like structure. Of importance in our data set later, there will be points where we seem to "lose" our ridgelike structure due to issues with the sample section 4.1.

Importantly, this formulation does not require λ_1 to be approximately zero, just that the

curvature in the downward direction is much more significant.

Also the crest could be really flat ("hangar shaped"), in which case both are around zero. At the crest of the ridge, we would actually expect both u_1 and u_2 to be around 0, whereas a point somewhere between the crest and the "foot" of the ridge to contain the maximum u_2.

We will fix some of these issues by casting this as a multiscale problem in section 2.5.

Two other ideas that could fix some other discrepancies mentioned above is to identify these ridges on their own, or also where the 'feet are'. We will discuss these ideas in **??**.

Structureness measure

There is another concern with using the pure ratio $S := |\lambda_1/\lambda_2|$ as an identifying feature of ridgelike structures apart from the ones listed above. We could still have $|\lambda_2| \gg |\lambda_1|$ in a relative sense, but still have $\lambda_2 \approx 0$. As a rather extreme example, we should certainly wish to differentiate a point on the surface where $\lambda_2 \approx 10^{-5}$ and $\lambda_1 \approx 10^{-10}$ from another point where $\lambda_2 \approx 10000$ and $\lambda_2 = 0.1$.

A natural fix to differentiate these points is to introduce a "structureness" measure to insure that there is in fact significant curvilinear activity at the point in question. Frangi used $S := \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$, which is in fact the 2-norm of the Hessian matrix. Thus the Frangi filter should also prefer areas of great curvilinear content in the image first of all.

The Frangi vesselness measure

Our goal then is to attach a numerical measure to each pixel in the image (at a particular scale σ) that is large when the anisotropy measure A and the structureness measure S is sufficiently large.

The form Frangi arrived at in eq. (2.55) in which a factor of $\exp\{...\}$ and $(1 - \exp\{\})$ are multiplied together are simply to ensure that the final vesselness measure V is largest when

A is small and S is large enough, with rapidly decay in other situations.

Frangi further strengthened the filter by adding an additional case to in eq. (2.55), ensuring that λ_2 is not positive. If we are indeed at a curvilinear ridge, we need the second derivative of the surface in the maximal direction to be negative, which hasn't been accounted for as yet in our formulation of A and S – we wish (for our purposes) to only indentify when we are finding crests. A will still be small and S will still be large however if we identify a "trough".

The only perceivable difference is that the maximum normal curvature will be positive–we are at a local minimum in the direction of u_2. In situations where we wish to only identify ridges (as is the case here) we simply exclude any points where there is not a negative curvature in the maximal direction.

The Frangi vesselness filter: Choosing parameters β and γ

The parameters β and γ are meant to scale so that the peaks of $\exp\{\cdots\}$ and $1-\exp\{\cdots\}$ coincide enough to be statistically significant but rapidly decay in areas not associated with curvilinear structure. What values of these parameters are appropriate is ultimately dependent on the context of the problem.

Frangi suggested for γ that half of the Frobenius norm of the Hessian matrix is appropriate, simply because the minimum value of S is zero, and its maximum value is approx the 2 norm of the Hessian. For β Frangi chose an innocuous intermediate point, $\beta = 1/2$ (and thus $2\beta^2 = 1/2$). As we will show later, choosing the structureness parameter γ is rather important for the context especially if the background (non-ridgelike structure) is significant and noisy. β should be strengthened/relaxed depending on how "flat" the ridgelike structure is. If there is a lot of gain then β should be smaller. If this is not the case, a stronger filter can be created by requiring A to be much smaller.

We now take a quick tangent from our description of the Frangi filter to develop and

justify our "multiscale" approach.

Linear Scale Space Theory

There is obviously a major disconnect in the ideas presented above. Although the ideas presented above require differentiation of continuous surfaces, our image is in fact a discrete pixel. That is, our previous discussions have been in terms of an image as the continuous surface in definition 2.2, rather than the more realistic discrete pixel matrix as in definition 2.1. The present section seeks to address this disconnect. In particular, we seek to mitigate the bias of our limited sampling of the "true" 3D surface. Our main goal is to counter against some of the bias of our particular sampling. In particular, we wish to not over-represent structures that are clear at our resolution without giving appropriate weight to larger structures as well. Koenderink [9] argued that "any image can be embedded in a one-parameter family of derived images (with resolution as the parameter) in essentially only one unique way" given a few of the so-called scale space axioms. He (and others) showed that a small set of intuitive axioms imply require that any such family of images must satisfy the heat equation

$$\Delta K(x,y,\sigma) = K_\sigma(x,y,\sigma) \text{ for } \sigma \geq 0 \text{ such that } K(x,y,0) = u_0(x,y). \tag{2.57}$$

where $K : \mathbb{R}^3 \to \mathbb{R}$ and $u_0 : \mathbb{R}^2 \to \mathbb{R}$: is the original image (viewed as a continuous surface) and σ is a resolution parameter. Much work has been done to formalize this approach [20]. There is a long list of desired properties–we will to try to identify a minimal subset of axioms and show that other desired properties follow.

Axioms

To make matters manageable, we require the one-parameter family of scaled images to be generated by an operation on the original image:

$$\{\ K(x,y;\sigma) = T_\sigma u_0 \mid \sigma \geq 0\ ,\ K(x,y,;0) = u_0\ \}$$

The following axioms are then requirements on what sort of operation T_σ should be.

Axiom 2.1 (Linear-shift and Rotational Invariance)**.** *Linear-shift (or translation) invariance means that no position in the original signal is favored. This is intuitive, as our operation should apply to any image fairly, regardless of where content is found in the image. Similarly, there should be not be favoritism toward any particular orientation of content within the image.*

Axiom 2.2 (Continuity of Scale Parameter)**.** *There is no reason for the scale parameter to be discrete; we may alter the resolution with whatever precision we desire. That is, we take the resolution parameter σ to be a nonzero real number (as opposed to an integer). Moreover, we require that the operator behaves continuously with respect to the scale parameter.*

What happens as $\sigma \downarrow 0$ is not immediately clear though. An argument from functional analysis (see [6]) implies that there is a so-called "infinitesimal generator" A which is a limit case of our desired operator T; that is

$$Au_0 = \lim_{\sigma \downarrow 0} \frac{T_\sigma u_0 - u_0}{\sigma} \tag{2.58}$$

and moreover that there is a resultant differential equation concerning the derivative of the family and A:

$$\partial_\sigma K(x,y;\sigma) = \lim_{\sigma \downarrow 0} \frac{K(\cdot;\sigma+h) - K(\cdot;\sigma)}{h} = A(T_\sigma u) = A(K(\cdot,\sigma)) \tag{2.59}$$

We shall return to this idea later and more concretely describe A once we actually characterize the generating operator T_σ.

Axiom 2.3 (Semigroup property)**.** *The semigroup property is simply that transforming the original image by some resolution σ should have the same overall effect of two successive transformations σ_1 and σ_2, i.e.*

$$T_\sigma u = T_{\sigma_1+\sigma_2} u \tag{2.60}$$

Axiom 2.4 (Causality Conditon)**.** *The following requirement has great implication, and is also very successful in encoding our intuitive sense of "resolution". The causality condition is the one that, as resolution decreases, no finer detail is introduced into the image. That is, as the scale increases, there will be no creation of local extrema that did not exist at a smaller scale.*

In other words, if $K(x_0, y_0; \sigma_0)$ is a local maximum (at the point (x_0, y_0), at this fixed σ_0) i.e. then an increase in scale can only weaken this peak, i.e.

$$\begin{cases} \nabla K(x_0, y_0; \sigma_0) = 0 \\ \\ \Delta K(x_0, y_0; \sigma_0) < 0 \end{cases} \quad \Longrightarrow \quad K(x_0, y_0; \sigma_1) \leq K(x_0, y_0; \sigma_0) \,\forall\, \sigma_1 \geq \sigma_0 \tag{2.61}$$

Similarly, if $K(x_0, y_0; \sigma_0)$ is a local minimum (with respect to space), then an increase in scale cannot make such a valley more profound, i.e.

$$\begin{cases} \nabla K(x_0, y_0; \sigma_0) = 0 \\ \\ \Delta K(x_0, y_0; \sigma_0) > 0 \end{cases} \quad \Longrightarrow \quad K(x_0, y_0; \sigma_1) \geq K(x_0, y_0; \sigma_0) \,\forall\, \sigma_1 \geq \sigma_0 \tag{2.62}$$

This implies that no image feature is sharpened by an decrease and resolution–the only result is a monotonic blurring of the image as scale parameter σ tends to infinity.

Uniqueness of the Gaussian Kernel

The above requirements are actually sufficient in proving not only that the operator T_σ is a convolution, but that the heat equation described in eq. (2.57) must hold. This has been

shown in various ways, both by Koenderink [9], Babaud [2], as well as Lindeberg in [20]. In fact, it is shown that the Gaussian is the unique convolution kernel that works.

To this, show that:

- a kernel satisfying the above axioms must satisfy the heat equation
- the gaussian kernel satisfies that.
- gaussian kernel is the only kernel that works.

That is,

$$K(x,y;\sigma) = T_\sigma u_0 = G_\sigma \star u_0 \quad \text{where} \quad G_\sigma := \frac{1}{2\pi\sigma^2} e^{\left(-|x|^2/(2\sigma^2)\right)} \tag{2.63}$$

We can show that this solution solves the heat equation. Given u_0 as a continuous image (unscaled), we construct PDE with this as a boundary condition.

$$u : \mathbb{R}^2 \supset \Omega \to \mathbb{R} \text{ with } u(\boldsymbol{x},t) : \begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) \quad , t \geq 0 \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \end{cases} \tag{2.64}$$

We show that

$$u(\boldsymbol{x},t) = \left(G_{\sqrt{2t}} \star u_0\right)(\boldsymbol{x}) \tag{2.65}$$

solves (the above tagged equation), where

s

First, we need a quick lemma regarding differentiation a continuous convolution.

Lemma 2.8. *Derivative of a convolution is the way that it is (obviously rewrite this).*

Proof. For a single variable,

$$\frac{\partial}{\partial\alpha}\left[f(\alpha)\star g(\alpha)\right] = \frac{\partial}{\partial\alpha}\left[\int f(t)g(\alpha-t)dt\right] \tag{2.66}$$

$$= \int f(t)\frac{\partial}{\partial\alpha}\left[g(\alpha-t)\right]dt \tag{2.67}$$

$$= \int f(t)\left(\frac{\partial g}{\partial\alpha}\right)g(\alpha-t)dt \tag{2.68}$$

$$= f(\alpha)\star g'(\alpha) \tag{2.69}$$

By symmetry of convolution we can also conclude

$$\frac{\partial}{\partial\alpha}\left[f(\alpha)\star g(\alpha)\right] = f'(\alpha)\star g(\alpha)$$

If f and g are twice differentiable, we can compound this result to show a similar statement holds for second derivatives, and then, given the additivity of convolution, we may conclude

$$\Delta\left(f\star g\right) = \Delta(f)\star g = f\star\Delta(g) \tag{2.70}$$

□

Theorem 2.9. *$u(\boldsymbol{x},t) = \left(G_{\sqrt{2t}}\star u_0\right)(\boldsymbol{x})$ solves the heat equation.*

Proof. We focus on the particular kernel

$$G_{\sqrt{2t}} = \frac{1}{4\pi t}e^{\left(-|x|^2/(4t)\right)}$$

Then

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) = \frac{\partial}{\partial t}\left(G_{\sqrt{2t}}(\boldsymbol{x},t)\star u_0(\boldsymbol{x})\right) \tag{2.71}$$

$$= \frac{\partial}{\partial t}\left(G_{\sqrt{2t}}(\boldsymbol{x},t)\right)\star u_0(\boldsymbol{x}) \tag{2.72}$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}e^{\left(-|x|^2/(4t)\right)}\right)\star u_0(\boldsymbol{x}) \tag{2.73}$$

$$= \left[-\frac{1}{4\pi t^2}e^{\left(-|x|^2/(4t)\right)} + \frac{1}{4\pi t}\left(\frac{-|x|^2}{4t^2}\right)e^{-|x|^2/(4t)}\right]\star u_0(\boldsymbol{x}) \tag{2.74}$$

$$= -\frac{1}{4t^2}\left(e^{\left(-|x|^2/(4t)\right)} + |\boldsymbol{x}|^2 G_{\sqrt{2t}}(\boldsymbol{x},t)\right)\star u_0(\boldsymbol{x}) \tag{2.75}$$

and from the previous lemma,

$$\Delta u(\boldsymbol{x},t) = \Delta\left(G_{\sqrt{2t}} \star u_0(\boldsymbol{x})\right) = \Delta\left(G_{\sqrt{2t}}\right) \star u_0(\boldsymbol{x})$$

We explicitly calculate the Laplacian of $G_\sigma(x,y) = A\exp(-\frac{x^2+y^2}{2\sigma^2})$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} G_\sigma(x,y) &= A\left(\frac{-2x}{2\sigma^2}\right)\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \\
\implies \frac{\partial^2}{\partial^2 x} G_\sigma(x,y) &= A\cdot\frac{\partial}{\partial x}\left[-\frac{x}{\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right] \\
&= A\left[-\frac{1}{\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) + \frac{x}{\sigma^2}\cdot\frac{2x}{2\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right] \\
&= A\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\left[-\frac{1}{\sigma^2}+\frac{x^2}{\sigma^4}\right] \\
&= \frac{1}{\sigma^2} G_\sigma(x,y)\left[\frac{x^2}{\sigma^2}-1\right]
\end{aligned}$$

By symmetry of argument we also may conclude

$$\frac{\partial^2}{\partial y^2} G_\sigma(x,y) = \frac{1}{\sigma^2} G_\sigma(x,y)\left[\frac{y^2}{\sigma^2}-1\right]$$

and so

$$\Delta G_\sigma(x,y) = \frac{\partial^2}{\partial x^2}\left(G_\sigma\right) + \frac{\partial^2}{\partial y^2}\left(G_\sigma\right) = \frac{1}{\sigma^2} G_\sigma(x,y)\left[\frac{x^2+y^2}{\sigma^2}-2\right] \tag{2.76}$$

Then, given lemma 2.8, we conclude

$$\Delta\left[G_\sigma(x,y)\star u_0(x,y)\right] = \left(\frac{1}{\sigma^2} G_\sigma(x,y)\left[\frac{x^2+y^2}{\sigma^2}-2\right]\right)\star u_0(x,y) \tag{2.77}$$

For particular choices of $\sigma(t) = \sqrt{2t}$ and $A = \frac{1}{4\pi t}$, we see

$$\begin{aligned}
\Delta\left[G_{\sqrt{2t}}(x,y)\star u_0(x,y)\right] &= \left(\frac{1}{2t} G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{2t}-2\right]\right)\star u_0(x,y) &(2.78)\\
&= \left(G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{4t^2}-\frac{1}{t}\right]\right)\star u_0(x,y) &(2.79)
\end{aligned}$$

We then calculate the time derivative, using our particular choice of $\sigma(t) = \sqrt{2t}$ and $A = \frac{1}{4\pi t}$ as:

$$\frac{\partial}{\partial t}\left[G_{\sigma(t)}(x,y) \star u_0(x,y)\right] = \frac{\partial}{\partial t}\left[G_{\sigma(t)}(x,y)\right] \star u_0(x,y) \tag{2.80}$$

$$= \frac{\partial}{\partial t}\left[G_{\sqrt{2t}}(x,y)\right] \star u_0(x,y) \tag{2.81}$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\exp\left(-\frac{x^2+y^2}{4t}\right)\right] \star u_0(x,y) \tag{2.82}$$

$$= \left[-\frac{1}{4\pi t^2}\exp\left(-\frac{x^2+y^2}{4t}\right) + \frac{1}{4\pi t}\left(\frac{x^2+y^2}{4t^2}\exp\left(-\frac{x^2+y^2}{4t}\right)\right)\right] \star u_0(x,y) \tag{2.83}$$

$$= \left(G_{\sqrt{2t}}(x,y)\left[\frac{x^2+y^2}{4t^2} - \frac{1}{t}\right]\right) \star u_0(x,y) \tag{2.84}$$

Combining these results, we find that

$$\frac{\partial}{\partial t}\left[G_{\sqrt{2t}} \star u_0\right] = \Delta\left[G_{\sqrt{2t}} \star u_0\right] \tag{2.85}$$

as desired. □

Scale Spaces over Discrete Structures

The above developments from scale space axioms have (since their first appearance) been recast in terms of discrete structures (rather than continuous surfaces) as in [13]. However, we've chosen to present the above in their original continuous surface for clarity of argument. The discrete case is not much different– we still have the same axioms, and it can be shown that the family of scaled images must simply satisfy a discrete version of the However, viewing our actual image definition 2.1 as a sample of a continuous surface definition 2.2, we might naïvely expect our convolution by the Gaussian to "commute" with our supposed sampling of the continuous signal, or even that we could simply convolve our discrete signal with a

discretely sampled Gaussian kernel. The latter in fact, seems to be an often implemented interpretation of scale space theory.

To be clear, the "sampled" 1D Gaussian Kernel we have in mind might be given by:

Definition 2.13 (Sampled Gaussian Kernel and Generated Family)**.**

$$g(n;\sigma) = \frac{1}{2\pi\sigma} e^{-n^2/2\sigma}\ , \quad -\infty < n < \infty$$

and the resulting (1D) convolution would be given by

$$K(x,\sigma) = \sum_{n=-\infty}^{\infty} g(n;\sigma) f(x-n) \quad \text{for} \quad x \in \mathbb{Z},\ \sigma > 0$$

The reality of the matter is that a discretely sampled Gaussian is not an appropriate kernel for creating discrete scale space. In [13] and in particular [12], Lindeberg demonstrated that the sampled Gaussian kernel violates not only semigroup property (axiom 2.3), but–much less forgivably–the causality property (axiom 2.3). There is absolutely no guarantee that convolution with a sampled Gaussian kernel will not create "spurious" structures as resolution increases.

Fortunately, Lindeberg was immediately able to remedy this by providing a discrete analogue of the Gaussian kernel, which does satisfy axiom 2.4 and axiom 2.3:

Definition 2.14 (Discrete Gaussian Kernel)**.** *The discrete Gaussian kernel, which can be shown to be a suitable generator for scale space, is given by*

$$T(n;\sigma) = e^{-\alpha\sigma} I_n(\alpha\sigma), \quad I_n(\sigma) = I_{-n}(\sigma) = (-1)^n J_n(i\sigma) \quad n \geq 0, \sigma, \alpha > 0 \tag{2.86}$$

where I_n are the modified Bessel functions of integer order based on the ordinary Bessel functions J_n, i.e.

$$I_n(x) = \sum_{m=0}^{\infty} \frac{1}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}\ , \quad n \geq 0$$

where we have taken the liberty of simplifying the typical definition [1] (which involves the gamma function), since we only desire Bessel functions of integer order. The parameter α above is simply an optional scaling parameter which is simply set to 1 hereforth.

The derived family of 1D signals is then given by

$$K(x,\sigma) = \sum_{n=-\infty}^{\infty} T(n;t) f(x-n) \quad \text{for} \quad x \in \mathbb{Z},\, t > 0 \tag{2.87}$$

The compatibility of scale space theory and derivatives on discrete structures and extension to two dimensions was also demonstrated by Lindeberg in [14] and [15]. In particular, we may take derivatives of the convolutions of our discrete images using, say, a central difference. Lastly, the 2D version of the family given in eq. (2.87) can be obtained by independent convolution of its dimensions (i.e. it is separable). We will make these ideas explicit in chapter 3 and the Appendix.

With the ideas of scale established, we may return to our discussion of the Frangi filter.

The Frangi Filter: A multiscale approach

Our ideas of scale developed in the previous section imply that, if the ridgelike structures we wish to detect are more prominent at different scales, then a multiscale approach is the natural one. Considering our developments in section 2.3, we wish to probe at multiple scales regions that would receive a high vesselness score at any range, and consider them all together. Frangi [4] approached this problem by simply aggregating vesselness measure over all scales:

$$V(x_0,y_0) = \max_{\sigma \in \Sigma} V_\sigma(x_0,y_0) \tag{2.88}$$

where $\Sigma := \{\sigma_0, \sigma_1, \cdots, \sigma_N\}$ is a range of parameters at which to probe. These should be chosen to be representative enough of all scales where meaningful content is expected to be found.

Thresholding

After this procedure, we are left with a matrix with as many samples/pixels as the original image, all with a vesselness measure between 0 and 1 for each pixel in the image:

$$\mathsf{V}_\Sigma := [V(x,y)]_{\substack{0\leq x<M \\ 0\leq y<N}} \tag{2.89}$$

Whereas Frangi [4] refrained from making any explicit decisions on whether something was definitely a vessel or not, we may wish to be final about the whole matter and ultimately say whether or not a pixel does in fact corresponds to a curvilinear structure. There are multiple methods of doing so. A straightforward enough approach is to simply threshold past some certain point. The resulting matrix can be given in terms of either eq. (2.88) or eq. (2.89)

$$V_{\Sigma,\alpha}(x,y) = \begin{cases} 1 & \text{if} \quad V(x,y) \geq \alpha \\ 0 & \text{else} \end{cases}, \quad \alpha > 0 \text{ for } \alpha \text{ fixed.} \tag{2.90}$$

We will discuss alternatives methods of aggregating results from our multiscale method, as well as optimal values for parameters and scales in chapter 3.

All that remains to describe mathematically is how to actually calculate the derivatives of our images and deal with the ultimately discrete nature of our samples.

Calculating 2D Hessian

According to section 2.4, we may calculate derivatives of our structure by calculating a gradient on our convolved image. Our method of calculating the gradient of a matrix uses a second-order accurate central difference, as in [3]. Specific implementation will be discussed in chapter 3.

We note in passing that we may take the derivative of the Gaussian kernel and then convolve it, and the effect will be the same as if we had taken the derivative subsequently

[5]. This could offer some computational speedup if we wish to run this procedure on many samples and fixed scale sizes, although we have implemented our scale spaces in the conventional way, as discussed in chapter 3.

Convolution Speedup via FFT

In practice, the convolutions described above are very slow for large scales (σ), as the size of the kernel is very large. Instead, we will perform a fast Fourier transform, which offers a speedup of N^2 operations vs $\mathcal{O}(n \cdot \log_2 n)$ operations .

Fourier Transforms

Fourier Transform of a continuous 1D signal

A periodic signal (real valued function) $f(t)$ of period T can be expanded in an infinite basis as follows:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T}t}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\frac{2\pi n}{T}t} dt \tag{2.91}$$

The Fourier transform of a 1D continuous function is defined by

$$F(\mu) := \mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{i2\pi\mu} dt \tag{2.92}$$

It can be shown that an inverse transform will then recover our original signal:

$$f(t) = \mathscr{F}^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu) e^{i2\pi\mu t} dt \tag{2.93}$$

Together, eq. (2.92) and eq. (2.93) are referred to as the *Fourier transform pair* of the signal $f(t)$.

Fourier Transform of a Discrete 1D signal

We wish to develop the Fourier transform pair for a discrete signal., following [5]. We frame the situation as follows: A continuous function $f(t)$ is represented as the sampled function

$\tilde{f}(t)$ by multiplying it by a sampling (or impulse) function, an infinite series of discrete impulses with equal spacing ΔT:

$$s_{\Delta T}(t) := \sum_{n=-\infty}^{\infty} \delta[t - n\Delta T]\,, \quad \delta[t] = \begin{cases} 1\,, & t = 0 \\ 0\,, & t \neq 0 \end{cases} \tag{2.94}$$

where $\delta[t]$ is the discrete unit impulse.

The discrete sample $f(t)$ is then constructed from $f(t)$ by

$$\tilde{f}(t) = f(t) s_{\Delta T}(t) \tag{2.95}$$

From this (and actually the convolution theorem) we can calculate $\tilde{F}(t)$. Given the discrete signal $\tilde{f}$, we construct the transform $\tilde{F}(\mu) = \mathscr{F}\{\tilde{f}(t)\}$. by simply transforming the definition eq. (2.95).

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-i2\pi\mu n\Delta T}\,, \quad f_n = \tilde{f}(n) = f(n\Delta T) \tag{2.96}$$

The transform is a continuous function with period $1/\Delta T$.

2D DFT Convolution Theorem

Theorem 2.10 (2D DFT Convolution Theorem)**.** *, that is: Given two discrete functions are sequences with the same length. $f(x,y)$ and $h(x,y)$ for integers $0 < x < M$ and $0 < y < N$, we can take the discrete fourier transform (DFT) of each:*

$$F(u,v) := \mathcal{D}\{f(x,y)\} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2\pi i\left(\frac{ux}{M} + \frac{vy}{N}\right)} \tag{2.97}$$

$$H(u,v) := \mathcal{D}\{h(x,y)\} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x,y) e^{-2\pi i\left(\frac{ux}{M} + \frac{vy}{N}\right)} \tag{2.98}$$

and given the convolution of the two functions

$$(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n) \tag{2.99}$$

then $(f \star h)(x,y)$ *and* $MN \cdot F(u,v)H(u,v)$ *are transform pairs, i.e.*

$$(f \star h)(x,y) = \mathcal{D}^{-1}\{MN \cdot F(u,v)H(u,v)\} \tag{2.100}$$

The proof follows from the definition of convolution, substituting in the inverse-DFT of f and h, and then rearrangement of finite sums.

Proof.

$$(f \star h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n) \tag{2.101}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)e^{2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\right)\left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{u(x-m)}{M}+\frac{v(y-n)}{N}\right)}\right) \tag{2.102}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)\left(\sum_{m=0}^{M-1} e^{2\pi i\left(\frac{m(p-u)}{M}\right)}\right)\left(\sum_{n=0}^{N-1} e^{2\pi i\left(\frac{n(q-v)}{N}\right)}\right)\right) \tag{2.103}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q)\left(M\cdot\hat{\delta}_M(p-u)\right)\left(N\cdot\hat{\delta}_M(q-v)\right)\right) \tag{2.104}$$

$$= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right)\cdot MNF(u,v) \tag{2.105}$$

$$= MN\cdot\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)H(u,v)e^{2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{2.106}$$

$$= MN\cdot\mathcal{D}^{-1}\{FH\} \tag{2.107}$$

where

$$\hat{\delta}_N(k) = \begin{cases} 1 & \text{when } k = 0 \mod N \\ 0 & \text{else} \end{cases} \tag{2.108}$$

□

Above, we make use of the following lemma

Lemma 2.11. *Let j and k be integers and let N be a positive integer. Then*

$$\sum_{n=0}^{N-1} e^{2\pi i\left(\frac{n(j-k)}{N}\right)} = N \cdot \hat{\delta}_N(j-k) \tag{2.109}$$

Proof. Consider the complex number $e^{2\pi i(j-k)/N}$. Note first that this is an N-th root of unity, since

$$\left(e^{2\pi i(j-k)/N}\right)^N = e^{2\pi i(j-k)} = \left(e^{2\pi i}\right)^{(j-k)} = 1^{(j-k)} = 1$$

In other words, $e^{2\pi in(j-k)/N}$ is a root of $z^N - 1 = 0$, which we can factor as

$$z^N - 1 \;=\; (z-1)\left(z^{n-1} + \cdots + z + 1\right) \;=\; (z-1)\sum_{n=0}^{N-1} z^n. \tag{2.110}$$

thus giving us

$$0 = \left(e^{2\pi i(j-k)/N} - 1\right)\sum_{n=0}^{N-1} e^{2\pi in(j-k)/N} \tag{2.111}$$

To prove the claim in eq. (2.109), we consider two cases: First, if $j-k$ is a multiple of N, we of course have $e^{2\pi in(j-k)/N} = \left(e^{2\pi i}\right)^{n(j-k)/N} = 1$ and thus the left side of eq. (2.109) reduces to

$$\sum_{n=0}^{N-1}\left(e^{2\pi i}\right)^{n(j-k)/N} = \sum_{n=0}^{N-1}(1) = N$$

.

In the case that $j-k$ is *not* a multiple of N, we refer to eq. (2.111). The first factor is not zero since, $\left(e^{2\pi i(j-k)/N}\right) \neq 1$ (simply since $(j-k)/N$ is not an integer), and thus it must be that the second factor is 0:

$$\sum_{n=0}^{N-1}\left(e^{2\pi i(j-k)/N}\right)^n = 0$$

We can combine these two cases by invoking the definition of eq. (2.108), giving us the result. □

FFT

As noted, the above result applies to the Discrete Fourier Transform. We actually achieve

a convolution speedup using a Fast Fourier Transform (FFT) instead. We follow the developments of [5]. For clarity, we present the following theorems which allow a framework to calculate a 2D Fourier transforms quickly.

First, a 2D DFT may actually be calculated via two successive 1D DFTs, which can be seen through a basic rearrangement, as follows:

$$F(\mu,\nu) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-i2\pi(\mu x/M+\nu y/N)} \tag{2.112}$$

$$= \sum_{x=0}^{M-1} e^{-i2\pi\mu x/M}\left[\sum_{y=0}^{N-1} f(x,y)e^{-i2\pi\nu y/N}\right] \tag{2.113}$$

$$= \sum_{x=0}^{M-1} e^{-i2\pi\mu x/M}\,\mathscr{F}_x\{f(x,y)\} \tag{2.114}$$

$$= \mathscr{F}_y\{\mathscr{F}_x\{f(x,y)\}\} \tag{2.115}$$

where $\mathscr{F}_{x'}$ refers to the 1D discrete Fourier transform of the function with respect to the variable x' only.

Thus, to calculate the fourier transform $F(u,v)$ at the point u,v requires the computation of the transform of length N for each iterated point $x \in 0,\cdots,M-1$. Thus there are MN complex multiplications and $(M-1)(N-1)$ complex additions in this sequence required for each point u,v that needs to be calculated. Overall, for all points that need to be calculated, the total order of calculations is on the order of $(MN)^2$ We'll also mention that the values of $e^{-i2\pi m/n}$ can be provided by a lookup table rather than ad-hoc calculation.

We now show that a considerable speedup can be achieved through elimination of redundant calculations. In particular, we wish to show that the calculation of a 1D DFT of signal length $M = 2^n, n \in \mathbb{Z}_+$ can be reduced to calculating two half-length transforms and an additional $M/2 = 2^{n-1}$ calculations.

To "simplify" our notation we will use a new notation for the Fourier kernels/basis func-

tions. Let the 1D Fourier transform be given by

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux}, \quad \text{where} \quad W_m := e^{-i2\pi/m} \tag{2.116}$$

We'll define $K \in \mathbb{Z}_+ : 2K = M = 2^n$ (i.e. $K = 2^{n-1}$).

We use this to rewrite the series in eq. (2.116) and split it into odd and even entries in the summation

$$F(u) = \sum_{x=0}^{2K-1} f(x) W_{2K}^{ux} \tag{2.117}$$

$$= \sum_{x=0}^{K-1} f(2x) W_{2K}^{u(2x)} + \sum_{x=0}^{K-1} f(2x+1) W_{2K}^{u(2x+1)} \tag{2.118}$$

We'll get a few identities out of the way (where $m, n, x \in \mathbb{Z}_+$ arbitrary).

$$W_{(2m)}^{(2n)} = e^{\frac{-i2\pi(2m)}{2m}} = e^{\frac{-i2\pi m}{n}} = W_m^n \tag{2.119}$$

$$W_m^{(u+m)x} = e^{\frac{-i2\pi(u+m)x}{m}} = e^{\frac{-i2\pi unx}{m}} e^{\frac{-i2\pi mx}{m}} = e^{\frac{-i2\pi ux}{m}}(1) = W_m^{ux} \tag{2.120}$$

$$W_{2m}^{(u+m)} = e^{\frac{-i2\pi(u+m)}{2m}} = e^{\frac{-i2\pi ux}{2m}} e^{-i\pi} = W_{2m}^u e^{-i\pi} = -W_{2m}^u \tag{2.121}$$

Thus we can rewrite eq. (2.118) as

$$F(u) = \sum_{x=0}^{K-1} f(2x) W_{2K}^{2ux} + \sum_{x=0}^{K-1} f(2x+1) W_{2K}^{2ux} W_{2K}^u \tag{2.122}$$

$$\implies \quad F(u) = \left(\sum_{x=0}^{K-1} f(2x) W_K^{ux} \right) + \left(\sum_{x=0}^{K-1} f(2x+1) W_K^{ux} \right) W_{2K}^u \tag{2.123}$$

The major advance comes via using the identities eq. (2.119) to consider the Fourier transform K frequencies later :

$$F(u+K) = \left(\sum_{x=0}^{K-1} f(2x) W_K^{(u+K)x} \right) + \left(\sum_{x=0}^{K-1} f(2x+1) W_K^{(u+K)x} \right) W_{2K}^{(u+K)} \tag{2.124}$$

$$\implies F(u+K) = \left(\sum_{x=0}^{K-1} f(2x) W_K^{ux} \right) - \left(\sum_{x=0}^{K-1} f(2x+1) W_K^{ux} \right) W_K^u \tag{2.125}$$

Comparing eq. (2.123) and eq. (2.125), we see that the expressions within parentheses are identical. What's more, these parentetical expressions are functionally identical to discrete fourier transforms themselves. Let's notate them as follows:

$$\mathscr{D}_u\{f_{\text{even}}(t)\} := \sum_{x=0}^{K-1} f(2x)W_K^{ux} \tag{2.126}$$

$$\mathscr{D}_u\{f_{\text{odd}}(t)\} := \sum_{x=0}^{K-1} f(2x+1)W_K^{ux} \tag{2.127}$$

If we're calculating an M point transform (i.e. we're wishing to calculate $F(1), ..., F(M)$), once we've calculated the first K discrete frequencies (i.e. $F(1), \cdots, F(K)$) we may simply reuse the two values we've calculated in eq. (2.126) to calculate the next $F(K+1), .., F(K+K) = F(M)$. Since each expression in parentheses involves K complex multiplications and $K-1$ complex additions, we are effectively saving $K(2K-1)$ calculations in computing the entire spectrum $F(1), .., F(M)$. When M is large, the payoff is undeniable.

In fact, through counting calculations and then doing a proof by induction, we can show that the effective number of calculations is given by $M \log_2 M$.

Of course, since eq. (2.126) are DFTs themselves, there's nothing stopping us from reiterating this procedure; if M is substantially large, we can just as easily repeat this process a few times.

Of course, our development was for $1D$. We can extend this to $2D$ by taking note of eq. (2.112).

The one caveat is that the above development was for transforming sequences whose lengths are perfect powers of 2. Since our inputs have no reason to be this, we need to adjust for this. The explanation is that you just do the part that's a power of 2 and then do the rest manually or pick a different power.

The inverse DFT can actually be found via a DFT of the complex conjugate of the original signal

CHAPTER 3

IMPLEMENTATIONS

Calculating the Hessian

There are 6 ways to do this in theory.

To compare, standard gaussian calculation is implemented by `scipy.ndimage.filters.gaussian_filt`

Pseudocode for `np.gradient` which is used in calculating Hessian (code below)

```
gaussian_filtered = fftgauss(image, sigma=sigma)
Lx, Ly = np.gradient(gaussian_filtered)
Lxx, Lxy = np.gradient(Lx)
Lxy, Lyy = np.gradient(Ly)
```

Scale Range detection

Find smallest and largest thing ...

Morphology: Plate Eroding

...

Morphology/Skeletonization

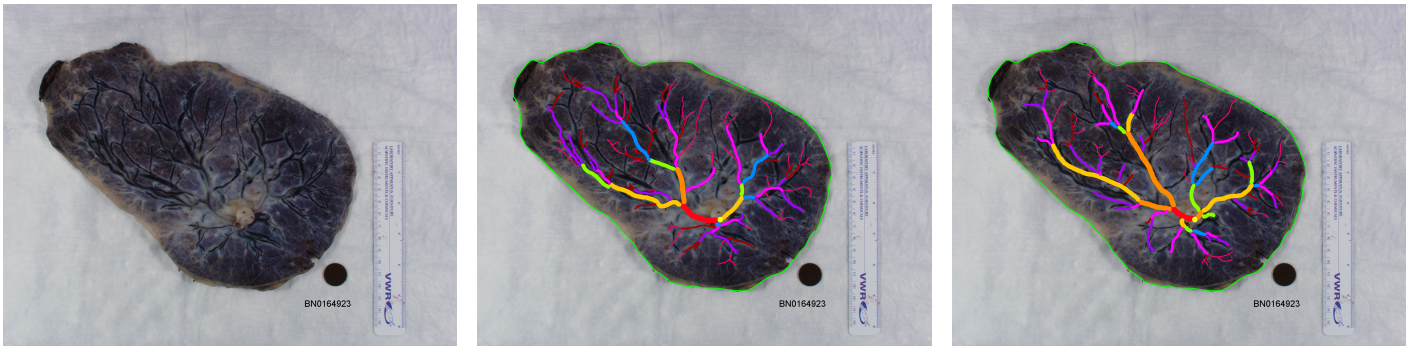
...

CHAPTER 4

RESEARCH PROTOCOL

Samples / Image Domain

We ultimately perform a PCSVN extraction on a set of 174 color placental images from a private database called NCS (from NYMH?). These are project files in GIMP which contain multiple layers. The layers together give a hand tracing of the vascular network and perimeter. A sample of overlaid layers in a representative sample is given in section 4.1

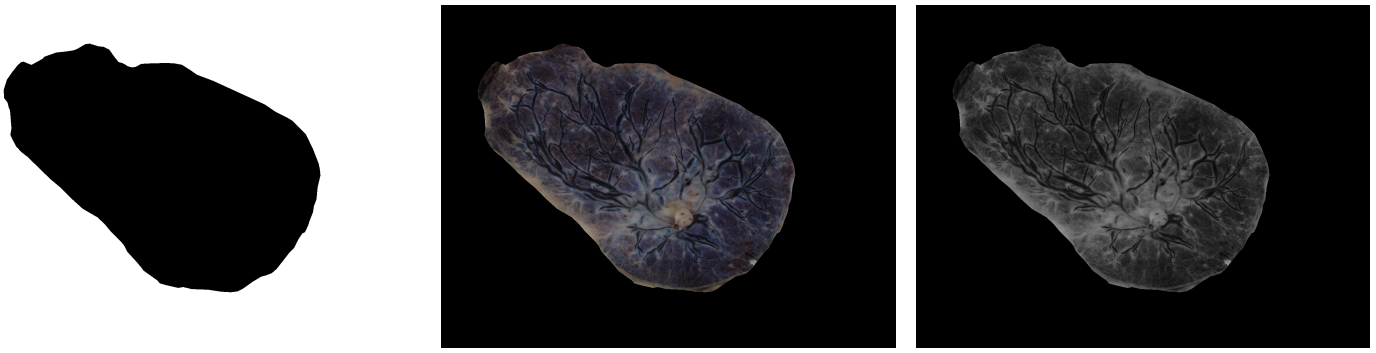


(a) Fixed Placental Sample (b) Arterial tracing (c) Venous tracing

FIGURE 4: A representative placental NCS sample with vascular tracing

In fig. 4a, a cleaned, fixed placenta is shown. A detailed description of this procedure is given in [TODO: some reference]. fig. 4b and fig. 4c are both hand traces of the PCSVN, with a layer for each the arteries and veins. In our particular use case, there is no need to consider them separately, so we simply consider them together, as in fig. 4d. The coloration is meant to indicate the diameter of each vessel. There is also a cord insertion point notated, as well as the perimeter of the placental plate. These are hand-traced and rather labor intensive. A closer look at many of the samples often reveals some subjectivity in the tracings (often it's hard to see where the vein is, vascular networks are obscured, etc.)

For our procedure, we simply operate on the placental sample itself, without any un-



(a) Background Mask (b) Sample with BG removed (c) Grayscale

FIGURE 5: Preprocessed files from an NCS sample

derstanding of its provided tracing except for comparing the strength of our algorithm. Of course, our goal is to develop an algorithm that can produce a "ground truth" tracing such as in fig. 4d or fig. 5d without any intervention.

For our purposes however, we will use the provided placental perimeter (shown in green in section 4.1. In developing a fully automated algorithm, it would be relatively straightforward to obtain this boundary ourselves using an Active Contour Model **[TODO: REF]** or perhaps even any edge finding algorithm followed by morphological / watershedding as in **[TODO: REF]**.

To build a sample suitable for use in our algorithm from section 4.1 is relatively simple. We "zero" outside the boundary of the plate (so as to not waste computational time calculating the differential geometry of a ruler, say), and also generate a binary mask to identify the plate. Finally, our vessel layers are combined and given as a binary trace.

These procedures are performed automatically on the 174 image in our data set using a custom GIMP plug-in, which performs various "bucket fill" operations, layer mergings, and thresholdings. For completeness sake, this plug-in (and an associated Scheme script which turns it into a batch operation) can be found in the Appendix. **[TODO: put a link**

here] (There are actually 201 images but 27 of them have mislabeled layers and were not autoprocessed correctly)

Image Preprocessing

As a point of technicality, the grayscale image in fig. 5c is not actually produced directly by the extractor plug-in, but created when the 3 channel RGB image fig. 5b is imported at the start of the algorithm. This grayscale conversion is simply done for ease of analysis on the sample: although the Frangi filter is designed for arbitrary dimension input [**?**], an image with three color channels does not have 3 spatial dimensions. We therefore simply combine the information in three channels using the well-known and oft-implemented ITU-R 601-2 luma [8], or "luminance" transform:

$$L = \frac{299}{1000}\, R + \frac{587}{1000}\, G + \frac{114}{1000}\, B \tag{4.1}$$

‘ All images are grayscale, M, N pixels as a masked array (of type `numpy.ma.MaskedArray`), where pixels outside of the placental region are masked so they will not be considered by the algorithm. However, some standard implementations of algorithms, namely `numpy.gradient and scipy.signal.convolve2d` are not designed to handle masked regions. Although it would be of some interest to create an algorithm that, say, calculates a gradient or performs a convolution by a "reflection" across an arbitrary closed boundary (as opposed to the edge of the image matrix), we opted instead to simply exclude affected areas from consideration, and zero unwanted background pixels to speed up computation. This excluding function, `plate_morphology.dilate_plate`, ultimately relies on two functions provided by the Python library `scikit-image` [21]. The first, `skimage.segmentation.find_boundaries()`, takes the mask input (such as fig. 5a) and calculates where differences in a morphological erosion and dilation occur. That boundary itself is then dilated by the desired factor. The second is a "sparse" implementation of binary dilation that is particularly efficient for our

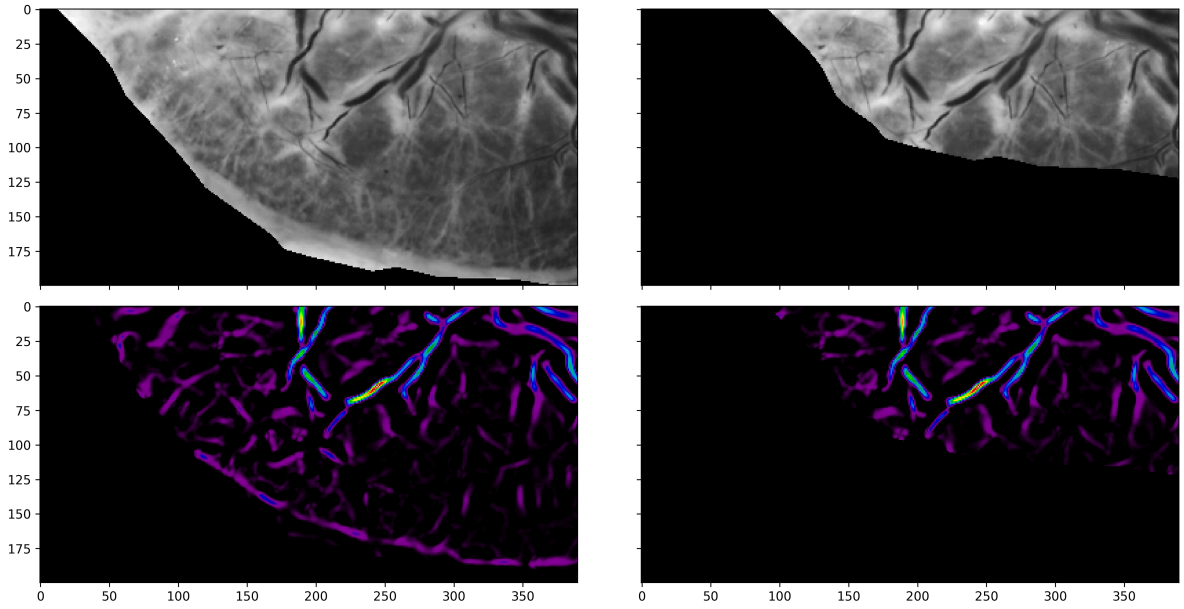


FIGURE 6: Demonstration of boundary dilation

problem. An array of indices of the image where the

section 4.2 doesn't really show what I want it to, but this is what it would look like. Repeat with a smaller border. Maybe the issue doesn't occur with these as much? In the image above, $\sigma = 3$ and border radius is 80 and all it does is get rid of the stupid natural boundary, not a weird frangi response. Which is important in itself, but I was having an error just between the edge of the image.

The code for the above can by found by running `plate_morphology.py` as a top-level script (that is, within the "`if __name__ == __main__`" block of the file).

Multiscale Setup

Our multiscale Frangi filter requires a list of scales at which to probe. Each scale is chosen to accentuate features of a particular size, i.e. vessels of a particular radius. This list of scales is denoted as $\Sigma := \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$.

The smallest one should be an effective size where details are expected to be found, and

the largest should be an effective size as well. In fact, following [9] it is reasonable and natural to select these logarithmically; that is, for some selected inputs $m < M$ we have

$$\sigma_1 = 2^m \ , \ \sigma_j = 2^{\left(m + \frac{M-m}{N-1} j\right)} \ , \ \sigma_N = 2^M \tag{4.2}$$

That is, the exponents are spaced linearly from m to M. This is achieved by the command `np.logspace(m,M,num=N)`. The idea is that the filter will respond better at its particular scale, but there are diminishing returns as σ increases. While the filter's response may vary substantially between, say $\sigma = 2$ and $\sigma = 3$, there will be not be a substantial difference in response between, say, $\sigma = 46$ and $\sigma = 47$. There was an earlier benefit as well, that is still worth mentioning for historical reasons. Previously, computing the vesselness measure was very expensive, and thus it was simply not feasible to collect so many large scale readings. This is moot with the development of FFT-based Frangi filter.

If there is no particular care taken in selecting a minimum and maximum range at which to probe, then we should assure that there is no noise being introduced at either ends, especially if the Frangi filter at which "throw out" bad ones somehow. We will approach this issue in our discussion of "variable thresholding."

Convolve this via fft transform to get L_{σ_i}

Applying Vesselness Measure

Calculate the Hessian matrix of and then the eigenvalues using the function `hfft.fft_hessian`.

Scale-space post-processing

Multiscale Merging

Cleanup/Postprocessing

Measurements

NOTE DUMP

Erode plate / dilate boundary

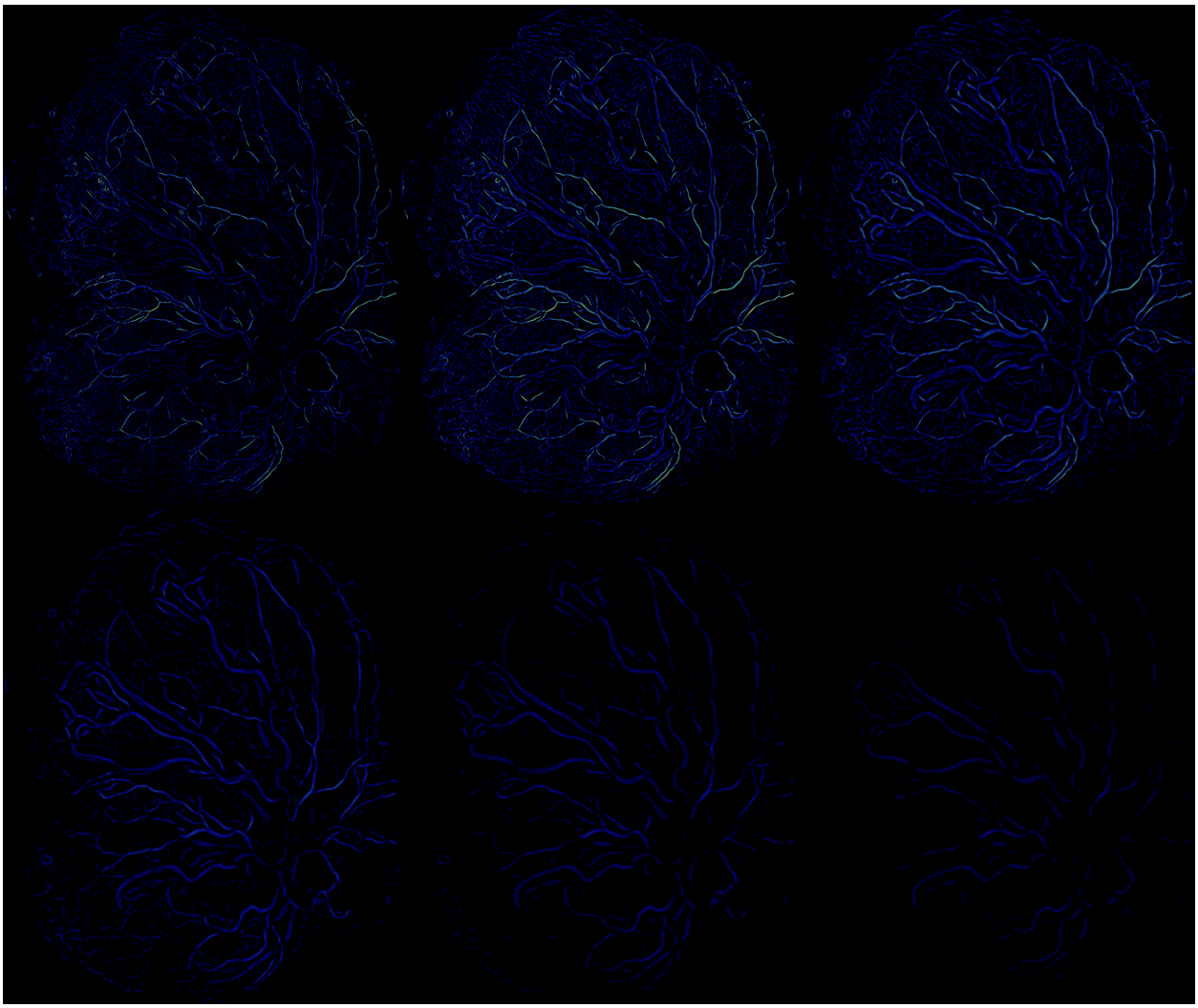


FIGURE 7: Frangi vesselness score at several scales

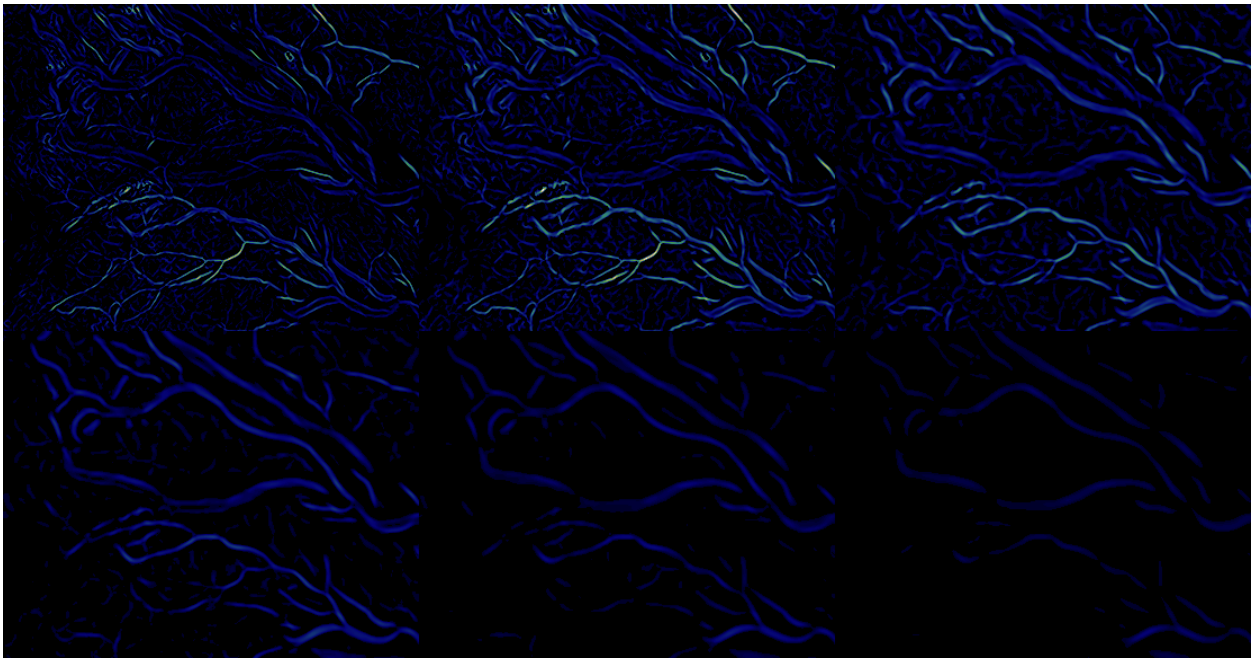


FIGURE 8: Frangi vesselness score at several scales (inset)

Our function considers the placenta as a nonzero surface but the surface outside is zero (or, in many situations, masked). We're currently not implementing any way to "reflect" along the border, so instead the second degree behavior of the surface there will be incorrect in an area proportional to the scale size.

Describe how that function works. Earlier efforts are wrong, whatever.

The are which is affected should be larger than just the standard dropoff the gaussian however, since we're interested in second derivative information.

CHAPTER 5

RESULTS AND ANALYSIS

use MCC [17]

Sample visual output

The confusion matrix

A Source of "False Negatives" in the NCS data set

Sometimes the output doesn't agree with the trace, i.e. "the ground truth" is not 100% correct. sometimes either there's a false negative (reported) but something just wasn't traced in the original 1602443.

1. Collar is stupid and should really be considered like a error in marking the perimeter. Throw these away or edit. Maybe make a section called discarded samples that's stupid but yeah.

2. Vessels suck sometimes. In the portion above, 1602443, there's a random blood clot which gets identified at large σ. But also the small forked shaped thing which is obviouslly a vessel doesn't get defined.

3. Too much blood (not enough?? no idea) is left in the vessels. leading to the weird white border around some vessels. you could identify these along with black center and combine them somehow. no idea.Also, holy shit, some of the white vessel "sleeves" ARE identified in the tracing, and some aren't. Find an example of this and whine about it.

4. Umbillical cord insertion point is stupid and obscures a lot. The tracer guesses but there's no real guiding principle AFAIK..

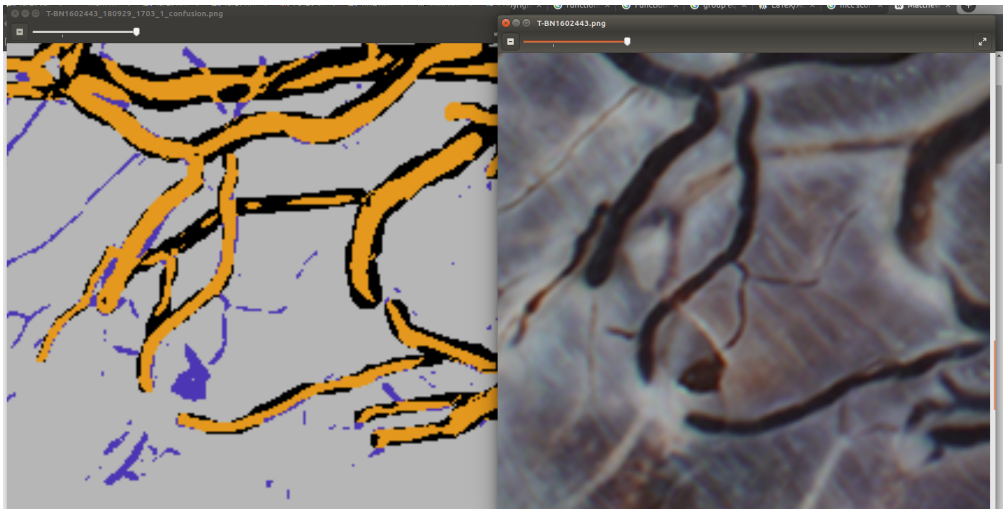


FIGURE 9: "True" false positives and "False" false positives

5. Small vessels aren't accounted for at all. Not sure how to coincide measurement in terms of scale space anymore, but should figure out how to cut off those values before running MCC metric.

Results

Answer Research Questions

FIGURE 10: Luminance/brightness thresholding is bad.

CHAPTER 6

CONCLUSION

Brief recap.

Review of Work

Estimation of success. Areas of success and struggle.

Future research directions

- Solve the Network Connection Problem (PICTURE OF GAPS) Try something like [11].
- Refine variable thresholding and automate.
- Combine with a ridge search.
- Use this as pre-processing for a Neural Network or something (cite kara's work, katalinas work)
- Apply to more image domains (STARE, WORSE PLACENTAS, ETC.)
- Automate Measurements (more quantative results too)
- Optimize; Better Use of Scales
- Use of Color Data

APPENDICES

APPENDIX A
CODE LISTINGS

The following python scripts and modules were developed with the following packages:

- **`python 3.6`**
- **`numpy`, version 1.12.0**
- **`scipy`, version 0.19.0**
- **`scikit-image`, version 0.13.0**
- **`matplotlib`, version 2.02**

Earlier versions of these packages may be compatible but are not guaranteed to be so. The scripts listed in this appendix are also hosted at github.com/wukm/pycake.

listings/pcsvn.py

```
#!/usr/bin/env python3

from get_placenta import get_named_placenta
from score import compare_trace
from hfft import fft_hessian
from diffgeo import principal_curvatures, principal_directions
from frangi import get_frangi_targets
import numpy as np
import numpy.ma as ma

from skimage.morphology import label, skeletonize

from plate_morphology import dilate_boundary

import matplotlib.pyplot as plt
import matplotlib as mpl

import os.path
import json
```

```
import datetime

from get_placenta import cropped_args

def make_multiscale(img, scales, betas, gammas, find_principal_directions=False,
                    dilate=True, dark_bg=True, VERBOSE=True):
    """returns an ordered list of dictionaries for each scale
    multiscale.append(
        {'sigma': sigma,
         'beta': beta,
         'gamma': gamma,
         'H': hesh,
         'F': targets,
         'k1': k1,
         'k2': k2,
         't1': t1,
         't2': t2
         }
    """

    # store results of each scale (create as empty list)
    multiscale = list()

    img = img / 255.

    for i, sigma, beta, gamma in zip(range(len(scales)), scales, betas, gammas):
        if dilate:
            if sigma < 2.5:
                radius = 10
            else:
                radius = int(sigma*4) # a little aggressive
        else:
            radius = None

        if VERBOSE:
            print('ÏČ={}'.format(sigma))

        # get hessian components at each pixel as a triplet (Lxx, Lxy, Lyy)
        hesh = fft_hessian(img, sigma)
```

```
59
        if VERBOSE:
61          print('finding principal curvatures')


63
        k1, k2 = principal_curvatures(img, sigma=sigma, H=hesh)
65
        # area of influence to zero out
67      if dilate:
            collar = dilate_boundary(None, radius=radius, mask=img.mask)
69
            k1[collar] = 0
71          k2[collar] = 0

73      # set anisotropy parameter if not specified
        if gamma is None:
75          # Frangi suggested 'half the max Hessian norm' as an empirical
            # half the max spectral radius is easier to calculate so do that
77          # shouldn't be affected by mask data but should make sure the
            # mask is *well* far away from perimeter
79          #gamma_alt = .5 * np.abs(k2).max()
            #print("half of k2 max is", gamma_alt)
81
            # or actually calculate half of max hessian norm
83          # using frob norm = sqrt(trace(AA^T))
            hxx, hxy, hyy = hesh
85          max_hessian_norm = np.sqrt((hxx**2 + 2*hxy**2 + hyy**2).max())
            gamma = .5*max_hessian_norm
87
            #print("gamma (half of max hessian norm is)", gamma)
89      if VERBOSE:
            print('finding Frangi targets with Îš={} and Îş={:.2}'.format(beta, gamma))
91
        # calculate frangi targets at this scale
93      targets = get_frangi_targets(k1,k2,
                    beta=beta, gamma=gamma, dark_bg=dark_bg, threshold=False)
95
        #store results as a dictionary
97      this_scale = {'sigma': sigma,
```

```
                        'beta': beta,
                        'gamma': gamma,
                        'H': hesh,
                        'F': targets,
                        'k1': k1,
                        'k2': k2,
                        'border_radius': radius
                        }

        if find_principal_directions:
            # principal directions will only be computed for significant regions
            pd_mask = np.bitwise_or(targets < (targets.mean() + targets.std()),
                                    img.mask).filled(1)

            if VERBOSE:
                percentage_calculated = (pd_mask.size - pd_mask.sum()) / pd_mask.size
                print('finding principal directions for {:.2%} of the image'.format(
    percentage_calculated))

            t1, t2 = principal_directions(img, sigma=sigma, H=hesh, mask=pd_mask)

            this_scale['t1'] = t1
            this_scale['t2'] = t2
        else:
            if VERBOSE:
                print('skipping principal direction calculation')

        # store results as a dictionary
        multiscale.append(this_scale)

    return multiscale

def match_on_skeleton(skeleton_of, layers, VERBOSE=True):
    """using the computed skeleton of ``skeleton_of'',
    return a composite image where blobs layers are incrementally added to the
    composite image if that blob coincides at some location of the skeleton
    """

    if ma.is_masked(skeleton_of):
```

```
        skeleton_of = skeleton_of.filled(0)

    skel = skeletonize(skeleton_of)
    matched_all = np.zeros_like(skel)

    # in reverse order (largest to smallest)
    for n in range(layers.shape[-1]-1, -1, -1):
        print('matching in layer #{}'.format(n))
        current_layer = layers[:,:,n]

        # only care about things in the current layer above the mean of that
        # layer (AAAH)
        current_layer = current_layer > current_layer.mean()
        #current_layer = current_layer > 0.2

        # don't match anything that's been matched already
        current_layer[matched_all] = 0

        # label each connected blob
        el, nl = label(current_layer, return_num=True)
        matched = np.zeros_like(current_layer)

        for region in range(1, nl+1):
            if np.logical_and(el==region, skel).any():
                matched = np.logical_or(matched, el==region)

        matched_all = np.logical_or(matched_all, matched)

    return matched_all


def apply_threshold(targets, alphas, return_labels=True):
    """
    if return_labels is True, return 1,..,n for scale at which
    max target was found (or 0 if no match),
    otherwise simply return a binary matrix

    targets is a (M,N,n) shape matrix (like F_all)
    alphas is a list / 1d array of alphas of length n
```

```
175
177     if alphas is a single number, then this should work just fine
179     for convenience,
181     if return_labels is true, this will return both the final
        threshold and the labels as two separate matrices
183     """
185     # you could make this work in 2D if you wanted to so alphas is a
        # constant and targets is only 2D but that's a later day
187
        # make it an array (even if it's a single element)
189     alphas = np.array(alphas)
191     # if it's just a MxN matrix, expand it trivially so it works below
193     if targets.ndim == 2:
195         targets = np.expand_dims(targets,2)
197     # either there's an alpha for each channel or there's a single
        # alpha to be broadcast across all channels
199     assert (targets.shape[-1] == alphas.size) or (alphas.size == 1)
201     # pixels that passed the threshold at any level
        passed = (targets >  alphas).any(axis=-1)
203
        if not return_labels:
205
            # works by broadcasting
207         return passed
209     # get label of where maximum occurs
        wheres = targets.argmax(axis=-1)
211
        # reserve 0 label for no match
213     wheres += 1
```

```
215     # then remove anything that didn't pass the threshold
        wheres[np.invert(passed)] = 0
217
        assert np.all( passed == (wheres > 0) )
219
        return passed, wheres
221
    def extract_pcsvn(filename, alpha=.15, alphas=None,
223                     log_range=None, scales=None, betas=None,
                        DARK_BG=True, dilate_per_scale=True, n_scales=20,
225                     verbose=True, generate_graphs=True,
                        generate_json=True, output_dir=None):
227
        raw_img = get_named_placenta(filename, maskfile=None)
229
        ###Multiscale & Frangi Parameters######################
231
        # set range of sigmas to use
233
        log_min, log_max = log_range
235
        if scales is None:
237         #log_min = -1 # minimum scale is 2**log_min
            #log_max = 4.5 # maximum scale is 2**log_max
239         scales = np.logspace(log_min, log_max, n_scales, base=2)


241
        #alpha = 0.15 # Threshold for vesselness measure
243     if betas is None:
            betas = [0.5 for s in scales] #anisotropy measure
245
        # set gammas
247     # declare None here to calculate half of hessian's norm
        gammas = [None for s in scales] # structureness parameter
249
        ###Do preprocessing (e.g. clahe)###############
251     img =  raw_img
        bg_mask = img.mask
```

```
253
    ###Logging#####################################
255 if verbose:
        print(" Running pcsvn.py on the image file", filename,
257             "with frangi parameters as follows:")
        print("alpha (vesselness threshold): ", alpha)
259     print("scales:", scales)
        print("betas:", betas)
261     print("gammas will be calculated as half of hessian norm")

263 ###Multiscale Frangi Filter#############################

265 multiscale = make_multiscale(img, scales, betas, gammas,
                              find_principal_directions=False,
267                           dilate=dilate_per_scale,
                              dark_bg=DARK_BG,
269                            VERBOSE=verbose)

271 gammas = [scale['gamma'] for scale in multiscale]

273 border_radii = [scale['border_radius'] for scale in multiscale]
    ###Process Multiscale Targets############################
275
    # fix targets misreported on edge of plate
277 # wait are we doing this twice?
    if dilate_per_scale:
279     if verbose:
            print('trimming collars of plates (per scale)')
281
        for i in range(len(multiscale)):
283         f = multiscale[i]['F']
            # twice the buffer (be conservative!)
285         radius = int(multiscale[i]['sigma']*2)
            if verbose:
287             print('dilating plate for radius={}'.format(radius))
            f = dilate_boundary(f, radius=radius, mask=img.mask)
289         multiscale[i]['F'] = f.filled(0)
    else:
291     for i in range(len(multiscale)):
```

```
            # harden mask (best way to do this??)
293         multiscale[i]['F'] = multiscale[i]['F'].filled(0)

295     ###Extract Multiscale Features#############################

297     pass

299     ###Make Composite#########################################

301     F_all = np.dstack([scale['F'] for scale in multiscale])

303     if generate_graphs:

305         analyze_targets(F_all, img)

307     if generate_json:

309         time_of_run = datetime.datetime.now()
            timestring = time_of_run.strftime("%y%m%d_%H%M")
311
            if alphas is None:
313             alphas_out = 'None'
            else:
315             alphas_out = list(alphas)
            logdata = {'time': timestring,
317                    'filename': filename,
                       'DARK_BG': DARK_BG,
319                    'fixed_alpha': alpha,
                       'VT_alphas': alphas_out,
321                    'betas': list(betas),
                       'gammas': gammas,
323                    'sigmas': list(scales),
                       'log_min': log_min,
325                    'log_max': log_max,
                       'n_scales': n_scales,
327                    'border_radii': border_radii
                       }
329
            if output_dir is None:
```

```
331             output_dir = 'output'

333         base = os.path.basename(filename)
            *base, suffix = base.split('.')
335         dumpfile = os.path.join(output_dir,
                                    ''.join(base) + '_' + str(timestring)
337                                 + '.json')

339         with open(dumpfile, 'w') as f:
                json.dump(logdata, f, indent=True)
341
    return F_all, img, scales, alphas
343
def get_outname_lambda(filename, output_dir=None, timestring=None):
345     """
    return a lambda function which can build output filenames
347     """

349     if output_dir is None:
            output_dir = 'output'
351
    base = os.path.basename(filename)
353     *base, suffix = base.split('.')

355     if timestring is None:
            time_of_run = datetime.datetime.now()
357         timestring = time_of_run.strftime("%y%m%d_%H%M")

359     outputstub = ''.join(base) +'_' + timestring +  '_{}.'+ suffix
    return lambda s: os.path.join(output_dir, outputstub.format(s))
361
def analyze_targets(F_all, img):
363
    ###The max Frangi target##################################
365     # for display purposes
    F_max = F_all.max(axis=-1)
367     F_max = ma.masked_array(F_max, mask=img.mask)

369     # is the frangi vesselness measure strong enough
```

```
    #F_cumulative = (F_max > alpha)
371
373     # Variable threshold
        N = min(img.shape) // 2
375     #alphas = np.logspace(-2,0, num=len(scales))*.7
        alphas = np.sqrt(1.2*scales / N)
377     # try a logistic curve
        #alphas = 1 / (1+np.exp(-.2*(scales-np.sqrt(N))))
379     #alphas = scales/32
        #alphas = np.logspace(-2.5,-1, num=len(scales))
381     #alphas = np.linspace(0.01,1,num=len(scales))
        #alphas = np.sqrt(scales / scales.max())
383
        #time_of_run = datetime.datetime.now()
385     #timestring = time_of_run.strftime("%y%m%d_%H%M")

387     # Process Composite ################################3

389     # (deprecated, doesn't change much and takes forever)
        #matched_all = match_on_skeleton(F_cumulative, F_all)
391     #wheres[np.invert(matched_all)] = 0 # first label is stuff that didn't match

393     FT, wheres = apply_threshold(F_all, alpha)
        VT , wheres_VT = apply_threshold(F_all, alphas)
395
        ########################################
397     #### THE REST IS JUST OUTPUT AND LOGGING

399     print('generating outputs!')
        crop = cropped_args(img)
401     """
        OUTPUT_DIR = 'output'
403     base = os.path.basename(filename)

405     *base, suffix = base.split('.')

407     # make this its own function and just do a partial here.
        outputstub = ''.join(base) +'_' + timestring +  '_{}.'+ suffix
```

```
    outname = lambda s: os.path.join(OUTPUT_DIR, outputstub.format(s))
    """
    outname = get_outname_lambda(filename)

    # SKELETONIZED OUTPUT
    plt.imsave(outname('skel'), skeletonize(FT[crop]),
            cmap=plt.cm.gray)
    plt.imsave(outname('fmax_threshholded'), FT[crop],
            cmap=plt.cm.gray_r)
    plt.imsave(outname('fmax_variable_threshold'), VT[crop],
            cmap=plt.cm.gray_r)


    # Max Frangi score
    fig, ax = plt.subplots()
    plt.imshow(F_max[crop], cmap=plt.cm.gist_ncar)
    #plt.title(r'Max Frangi vesselness measure below threshold $\alpha={:.2f}$'.format(alpha)
    )
    plt.title('Maximum Frangi vesselness score')
    plt.axis('off')
    c = plt.colorbar()
    c.set_ticks(np.linspace(0,1,num=11))
    plt.clim(0,1)
    plt.tight_layout()
    plt.savefig(outname('fmax'), dpi=300)

    plt.close()

    scale_label_figure(wheres, scales,
                       outname('labeled_test'),
                       crop=crop)
    scale_label_figure(wheres_VT, scales,
                       outname('labeled_VT_test'),
                       crop=crop)

    confusion_matrix = compare_trace(FT, filename=filename)

    plt.imsave(outname('confusion'), confusion_matrix[crop])
```

```
447     confusion_matrix = compare_trace(VT, filename=filename)

449     plt.imsave(outname('confusion_VT'), confusion_matrix[crop])

451

453     ###Make Connected Graph#########################################

455     pass

457     ###Measure#####################################################

459     pass

461     """

463 """
    def scale_label_figure(wheres, scales, savefilename=None,
465                            crop=None, show_only=False, image_only=False):
        """
467     crop is a slice object.
        if show_only, then just plt.show, not save
469     """
        if crop is not None:
471         wheres = wheres[crop]

473     fig, ax = plt.subplots() # not sure about figsize
        N = len(scales)+1 # number of scales / labels
475
        # discrete sample of color map
477     #cmap = plt.get_cmap('nipy_spectral', N)

479     # get 20 samples from the colormap [R,G,B,A] array
        tab = plt.cm.viridis_r(np.linspace(0,1,num=N))
481     tabe = np.vstack(([0,0,0,1], tab)) # add black as first entry
        tabemap = mpl.colors.ListedColormap(tabe)
483
        if image_only:
485         plt.imsave(savefilename, wheres, cmap=tabemap)
```

```
    else:
        imgplot = ax.imshow(wheres, cmap=tabemap)
        # discrete colorbar
        cbar = plt.colorbar(imgplot)

        # this is apparently hackish, beats me
        tick_locs = (np.arange(N) + 0.5)*(N-1)/N

        cbar.set_ticks(tick_locs)
        # label each tick with the sigma value
        scalelabels = [r"$\sigma = {:.2f}$".format(s) for s in scales]
        scalelabels.insert(0, "(no match)")
        # label with their sigma value
        cbar.set_ticklabels(scalelabels)
        #ax.set_title(r"Scale ($\sigma$) of maximum vesselness ")
        plt.tight_layout()

        #plt.savefig(outname('labeled'), dpi=300)
        if show_only or (savefilename is None):
            plt.show()
        else:
            plt.savefig(savefilename, dpi=300)

        plt.close()









if __name__ == "__main__":


    from get_placenta import list_placentas

    show = plt.show
    imshow = plt.imshow

    placentas = list_placentas('T-BN')[:15]
```

```
525     N_samples = len(placentas)

527     print(N_samples, "samples total!")
        for i, filename in enumerate(placentas):
529         print('*'*80)
            print('extracting PCSVN of', filename,
531               '\t ({} of {})'.format(i,N_samples))

533         alpha = .08
            DARK_BG = False
535         log_range = (-2,3)
            dilate_per_scale = False
537
            F, img, scales = extract_pcsvn(filename, DARK_BG=DARK_BG,
539                                alpha=alpha, log_range=log_range,
                                   dilate_per_scale=dilate_per_scale,
541                                  verbose=False, generate_graphs=False)

543         break
```

listings/get_placenta.py

```
1  #!/usr/bin/env python3

3  # change this module to placenta instead of get_placenta

5  """

7  Get registered, unpreprocessed placental images.  No automatic registration
   (i.e. segmentation of placental plate) takes place here. The background,
9  however, *is* masked.

11 Again, there is no support for unregistered placental pictures.
   Any region outside of the placental plate MUST be black.
13
   There is currently no support for color images.
15
   TODO:
17     - Build sample base & organize data :v)
```

```
    - Test on many other images.
    - Think of how the interface should really work, esp for get_named_placenta
    - Fix logic in mask_background
    - Catch errors better.
    - Support for color images
    - Show a better test
    - Be able to grab trace files too.
    - Cache masked samples.
"""

import numpy as np
import numpy.ma as ma
from skimage import segmentation, morphology
import os.path
import os

from scipy.ndimage import imread

def open_typefile(filename, filetype, sample_dir=None):
    """
    filetype is either 'mask' or 'trace'
    """
    # try to open what the mask *should* be named
    # this should be done less hackishly
    # for example, if filename is 'ncs.1029.jpg' then
    # this would set the maskfile as 'ncs.1029.mask.jpg'

    if filetype not in ("mask", "trace"):
        raise NotImplementedError("Can only deal with mask or trace files.")

    *base, suffix = filename.split('.')
    base = ''.join(base)
    typefile = '.'.join((base, filetype ,suffix))

    if sample_dir is None:
        sample_dir = 'samples'

    typefile = os.path.join(sample_dir, typefile)
```

```
57      try:
            M = imread(typefile, mode='L')
59
        except FileNotFoundError:
61          print('Could not find file', typefile)
            raise
63
        return M
65
    def open_tracefile(tracefile):
67      """
        open up the trace matrix with filename 'tracefile'
69      #TODO: expand this later to handle arterial traces and venous traces
        """
71
        if sample_dir is None:
73          sample_dir = 'samples'

75      tracefile = os.path.join(sample_dir, tracefile)
        trace =  imread(tracefile, mode='L')
77
        # return 1's and 0's (or convert to binary instead
79      return trace != 0

81  def get_named_placenta(filename, sample_dir=None, masked=True,
                           maskfile=None):
83      """
        This function is to be replaced by a more ingenious/natural
85      way of accessing a database of unregistered and/or registered
        placental samples.
87
        INPUT:
89          filename: name of file (including suffix?) but NOT directory
            masked: return it masked.
91          maskfile: if supplied, this use the file will use a supplied 1-channel
                    mask (where 1 represents an invalid/masked pixel, and 0
93                  represents a valid/unmasked pixel. the supplied image must be
                    the same shape as the image. if not provided, the mask is
95                  calculated (unless masked=False)
```

```
                the file must be located within the sample directory

                If maskfile is 'None' then this function will look for
                a default maskname with the following  pattern:

                    test.jpg -> test.mask.jpg
                    ncs.1029.jpg  -> ncs.1029.mask.jpg

        sample_directory: Relative path where sample (and mask file) is located.
                defaults to './samples'

    if masked is true (default), this returns a masked array.

    NOTE: A previous logical incongruity has been corrected. Masks should have
    1 as the invalid/background/mask value (to mask), and 0 as the
    valid/plate/foreground value (to not mask)
    """
    if sample_dir is None:
        sample_dir = 'samples'

    full_filename = os.path.join(sample_dir, filename)

    raw_img = imread(full_filename, mode='L')

    if maskfile is None:
        # try to open what the mask *should* be named
        # this should be done less hackishly
        # for example, if filename is 'ncs.1029.jpg' then
        # this would set the maskfile as 'ncs.1029.mask.jpg'
        *base, suffix = filename.split('.')
        test_maskfile = ''.join(base) + '.mask.' + suffix
        test_maskfile = os.path.join(sample_dir, test_maskfile)
        try:
            mask = imread(test_maskfile, mode='L')
        except FileNotFoundError:
            print('Could not find maskfile', test_maskfile)
            print('Please supply a maskfile. Autogeneration of mask',
                  'files is slow and buggy and therefore not supported.')
            raise
```

```
            #return mask_background(raw_img)
    else:
        # set maskfile name relative to path
        maskfile = os.path.join(sample_dir, maskfile)
        mask = imread(maskfile, mode='L')

    return ma.masked_array(raw_img, mask=mask)

def check_filetype(filename, assert_png=True, assert_standard=False):
    """
    'T-BN8333878.raw.png' returns 'raw'
    'T-BN8333878.mask.png' returns 'mask'
    'T-BN8333878.png' returns 'base'

    if assert_png is True, then raise assertion error if the file
    is not of type png

    if assert_standard, then assert the filetype is
    mask, base, trace, or raw.

    etc.
    """
    basename, ext = os.path.splitext(filename)

    if ext != '.png':
        if assert_png:
            assert ext == '.png'

    sample_name, typestub = os.path.splitext(basename)

    if typestub == '':
        # it's just something like  'T-BN8333878.png'
        return 'base'
    elif typestub in ('.mask','.trace', '.raw'):
        # return 'mask' or 'trace' or 'raw'
        return typestub.strip('.')
    else:
        print('unknown filetype:', typestub)
        print('is it a weird filename?')
```

```
175         print('warning: lookup failed, unknown filetype:' + typestub)

177         return typestub

179 def list_placentas(label=None, sample_dir=None):
        """
181     label is the specifier, basically just ''.startswith()

183     only real use is to find all the T-BN* files

185     this is hackish, if you ever decide to use a file other than
        png then this needs to change
187     """

189     if sample_dir is None:
            sample_dir = 'samples'
191
        if label is None:
193         label = '' # str.startswith('') is always True

195     placentas = list()

197     for f in os.listdir(sample_dir):

199         if f.startswith(label):
                # oh man they gotta be png files
201             if check_filetype(f) == 'base':
                    placentas.append(f)
203
        return sorted(placentas)
205
    def mask_background(img):
207     """
        Warning: this function is slow and buggy and therefore deprecated
209     as "out of scope". Please fix or remove.

211     Masks all regions of the image outside the placental plate.
```

```
    INPUT:
        img:
            A color or grayscale array corresponding to an image of a placenta
            with the plate in the 'middle.' Outer regions should be black.

    OUTPUT:
        masked_img:
            A numpy.ma.masked_array with the same dimensions.
    """
    print("""
          Warning, this function is slow and buggy and therefore
          deprecated. Please supply a mask file yourself.
          """
          )

    if img.ndim == 3:

        #mark any pixel with with content in any channel
        bg_mask = img.any(axis=-1)
        bg_mask = np.invert(bg_mask)

        # make the mask multichannel to match dim of input
        bg_mask = np.repeat(bg_mask[:,:,np.newaxis], 3, axis=2)

    else:

        # same as above
        bg_mask = (img != 0)
        bg_mask = np.invert(bg_mask)

    # the above approach will probably work for any real image (i.e. a
    # photgraph). it will obviously fail for any image where there is true black
    # in the placental plane. This should work instead:

    # find the outer boundary and mark outside of it.
    # run with defaults, sufficient
    bound = morphology.convex_hull_image(bg_mask)
    bound = segmentation.find_boundaries(bg_mask, mode='inner', background=1)
    bg_mask[bound] = 1
```

```
253     #remove any small holes found inside the plate (regions or single pixels
        #that happen to be black).  run with defaults, sufficient
255     holes = morphology.remove_small_holes(bg_mask)
        bg_mask[holes] = 1
257
        return ma.masked_array(img, mask=bg_mask)
259
    def show_mask(img):
261     """
        show a masked grayscale image with a dark blue masked region
263
        custom version of imshow that shows grayscale images with the right colormap
265     and, if they're masked arrays, sets makes the mask a dark blue)
        a better function might make the grayscale value dark blue
267     (so there's no confusion)
        """
269
        from numpy.ma import is_masked
271     from skimage.color import gray2rgb
        import matplotlib.pyplot as plt
273


275     if not is_masked(img):
            plt.imshow(img, cmap=plt.cm.gray)
277     else:

279         mimg = gray2rgb(img.filled(0))
            # fill blue channel with a relatively dark value for masked elements
281         mimg[img.mask, 2] = 60
            plt.imshow(mimg)
283
    if __name__ == "__main__":
285
        """test that this works on an easy image."""
287
        from scipy.ndimage import imread
289     import matplotlib.pyplot as plt
        test_filename = 'barium1.png'
```

```
291     #test_maskfile = 'barium1.mask.png'

293     img =  get_named_placenta(test_filename, maskfile=None)

295     print('showing the mask of', test_filename)
        print('run plt.show() to see masked output')
297     show_mask(img)


299
    def _cropped_bounds(img, mask=None):
301
        if mask is not None:
303
            img = ma.masked_array(img, mask=mask)
305
        X,Y = (np.argwhere(np.invert(img.mask).any(axis=k)).squeeze() for k in (0,1))
307
        if X.size == 0:
309         X = [None,None] # these will slice correctly
        if Y.size == 0:
311         Y = [None,None]

313     return Y[0],Y[-1],X[0],X[-1]

315 def cropped_args(img, mask=None):
        """
317     get a slice that would crop image
        i.e. img[cropped_args(img)] would be a cropped view
319     """

321     x0, x1, y0,y1 = _cropped_bounds(img, mask=None)

323     return np.s_[x0:x1,y0:y1]

325 def cropped_view(img, mask=None):
        """
327     removes entire masked rows and columns from the borders of a masked array.
        will return a masked array of smaller size
329
```

```
    don't ask me about data

331

    the name sucks too
333 """

335 # find first and last row with content
    x0, x1, y0,y1 = _cropped_bounds(img, mask=mask)
337

    return img[x0:x1,y0:y1]
```

listings/alpha_sweep_demo.py

```
#!/usr/bin/env python3
2

"""
4 alpha_sweep_demo.py

6 show how much variable alphas affect the output.

8 """

10 from get_placenta import get_named_placenta, cropped_args, cropped_view
   from get_placenta import list_placentas, open_typefile
12

   from score import compare_trace
14

   from pcsvn import extract_pcsvn, scale_label_figure, apply_threshold
16 from pcsvn import get_outname_lambda

18 import numpy as np
   import numpy.ma as ma
20

   import matplotlib.pyplot as plt
22

   from hfft import fft_gradient
24 from score import mcc
   #filename = 'T-BN0033885.png'
26 placentas = list_placentas('T-BN')
   n_samples = len(placentas)
```

```
28
OUTPUT_DIR = 'output/newalpha'
30 DARK_BG = True
log_range = (-2, 4.5)
32 n_scales = 10
scales = np.logspace(log_range[0], log_range[1], num=n_scales, base=2)
34 alphas = scales**(2/3) / scales[-1]
#alphas = [0.1 for s in scales]
36 #betas = np.linspace(.5, .9, num=n_scales)
betas = None
38 print(n_samples, "samples total!")
for i, filename in enumerate(placentas):
40     print('*'*80)
    print('extracting PCSVN of', filename,
42             '\t ({} of {})'.format(i,n_samples))
    F, img, _ , _ = extract_pcsvn(filename, DARK_BG=DARK_BG,
44                                 alpha=.1, alphas=alphas, betas=betas,
                                   scales=scales,  log_range=log_range,
46                                 verbose=False, generate_graphs=False,
                                      n_scales=n_scales, generate_json=True,
48                                            output_dir=OUTPUT_DIR)

50     #G = list()

52     #for s in scales:
    #     g = fft_gradient(img, s)
54     #     G.append(g)
    #G = np.dstack(G)
56     #f = F.copy()
    crop = cropped_args(img)
58     print("...making outputs")
    outname = get_outname_lambda(filename, output_dir=OUTPUT_DIR)
60


62     approx, labs = apply_threshold(F, alphas, return_labels=True)
    scale_label_figure(labs, scales, crop=crop, savefilename=outname('2_labeled'),
64                         image_only=True)

66     confusion = compare_trace(approx, filename=filename)
```

```
    trace = open_typefile(filename, 'trace').astype('bool')
    trace = np.invert(trace)

    m_score, counts = mcc(approx, trace, img.mask, return_counts=True)

    TP, TN, FP, FN = counts

    total = np.invert(img.mask).sum()
    print('TP: {}\t TN: {}\nFP: {}\tFN: {}'.format(TP,TN,FP,FN))
    print('TP+TN+FP+FN={}\ntotal pixels={}'.format(TP+TN+FP+FN,total))

    print("MCC for {}:\t".format(filename), m_score)

    plt.imsave(outname('1_confusion'), confusion[crop])

    plt.imsave(outname('0_raw'), img[crop].filled(0), cmap=plt.cm.gray)

    plt.close('all') # something's leaking :(
    if i > 5:
        break
```

listings/diffgeo.py

```
#!/usr/bin/env python3


import numpy as np
import numpy.ma as ma

from skimage.feature import hessian_matrix, hessian_matrix_eigvals
from numpy.linalg import eig
from functools import partial

from hfft import fft_hessian


def principal_curvatures(img, sigma=1.0, H=None):
    """
    Return the principal curvatures {Îž1, Îž2} of an image, that is, the
```

```
    eigenvalues of the Hessian at each point (x,y). The output is arranged such
18  that |Îž1| <= |Îž2|.

20  Input:

22      img:    An ndarray representing a 2D or multichannel image. If the image
                is multichannel (e.g. RGB), then each channel will be proccessed
24              individually. Additionally, the input image may be a masked
                array-- in which case the output will preserve this mask
26              identically.

28              PLEAS ADD SOME INFO HERE ABOUT WHAT SORT OF DTYPES ARE
                EXPECTED/REQUIRED, IF ANY
30
        sigma:  (optional) The scale at which the Hessian is calculated.
32
        H:      (optional) provide sigma (else it will be calculated)
34  Output:

36      (K1, K2):   A tuple where K1, K2 each are the exact dimension of the
                    input image, ordered in magnitude such that |Îž1| <= |Îž2|
38                  in all locations. If *signed* option is used, then elements
                    of K1, K2 may be negative.
40
    Example:
42
        >>> K1, K2 = principal_curvatures(img)
44      >>> K1.shape == img.shape
        True
46      >>> (K1 <= K2).all()
        True
48
        >> K1.mask == img.mask
50      True
    """
52
    # determine if multichannel
54  multichannel = (img.ndim == 3)
```

```
    if not multichannel:
        # add a trivial dimension
        img = img[:,:,np.newaxis]

    K1 = np.zeros_like(img, dtype='float64')
    K2 = np.zeros_like(img, dtype='float64')

    for ic in range(img.shape[2]):

        channel = img[:,:,ic]

        # returns the tuple (Hxx, Hxy, Hyy)
        if H is None:
            H = hessian_matrix(channel, sigma=sigma)

        # returns tuple (l1,l2) where l1 >= l2 but this *includes sign*
        L = hessian_matrix_eigvals(*H)
        L = np.dstack(L)

        mag = np.argsort(abs(L), axis=-1)

        # just some slice nonsense
        ix = np.ogrid[0:L.shape[0], 0:L.shape[1], 0:L.shape[2]]

        L = L[ix[0], ix[1], mag]

        # now k2 is larger in absolute value, as consistent with Frangi paper

        K1[:,:,ic] = L[:,:,0]
        K2[:,:,ic] = L[:,:,1]

    try:
        mask = img.mask
    except AttributeError:
        pass
    else:
        K1 = ma.masked_array(K1, mask=mask)
        K2 = ma.masked_array(K2, mask=mask)

```

```
    # now undo the trivial dimension
    if not multichannel:
        K1 = np.squeeze(K1)
        K2 = np.squeeze(K2)

    return K1, K2

def reorder_eigs(L1,L2):
    """
    L1, L2 contain a 2D matrix of eigenvalues at each point
    so that L1 <= L2 at each element.

    this reorders this so that |L1| <= |L2| instead.

    this could (if desired) also return the permutation array
    but does not do so presently
    """

    L = np.dstack((L1,L2))
    mag =  np.argsort(abs(L), axis=-1)

    # just some slice nonsense

    ##### FINISH T HISSSS
def principal_directions(img, sigma, H=None, mask=None):
    """
    will ignore calculation of principal directions of masked areas

    despite the name, this function actually returns the theta corresponding to
    leading and trailing principal directions, i.e. angle w / x axis

    FIX THE MASK BUSINESS
    """

    if H is None:
        H = hessian_matrix(img, sigma)

    Hxx, Hxy, Hyy = H
```

```
134
    # is no mask provided
136 if mask is None:
        try:
138         mask = img.mask
        except AttributeError:
140         masked = False
        else:
142         masked = True
    else:
144     masked = True

146 dims = img.shape

148 # where to store
    trailing_thetas = np.zeros_like(img, dtype='float64')
150 leading_thetas = np.zeros_like(img, dtype='float64')


152
    # maybe implement a small angle correction
154 for i, (xx, xy, yy) in enumerate(np.nditer([Hxx, Hxy, Hyy])):

156     # grab the (x,y) coordinate of the hxx, hxy, hyy you're using
        subs = np.unravel_index(i, dims)
158
        # ignore masked areas (if masked array)
160     if masked and mask[subs]:
            continue
162
        h = np.array([[xx, xy], [xy, yy]]) # per-pixel hessian
164     l, v = eig(h) # eigenvectors as columns

166     # reorder eigenvectors by (increasing) magnitude of eigenvalues
        v = v[:,np.argsort(np.abs(l))]
168
        # angle between each eigenvector and positive x-axis
170     # arccos of first element (dot product with (1,0) and eigvec is already
        # normalized)
172     trailing_thetas[subs] = np.arccos(v[0,0]) # first component of each
```

```
        leading_thetas[subs] = np.arccos(v[0,1]) # first component of each
174
    if masked:
176         leading_thetas = ma.masked_array(leading_thetas, mask)
            trailing_thetas = ma.masked_array(trailing_thetas, mask)
178

180     return trailing_thetas, leading_thetas

182



184
if __name__ == "__main__":
186
    pass
188
    #from get_base import get_preprocessed
190 #import matplotlib.pyplot as plt
    #from functools import partial
192 #from fpd import get_targets
    #b = partial(plt.imshow, cmap=plt.cm.Blues)
194 #sp = partial(plt.imshow, cmap=plt.cm.spectral)
    #s = plt.show
196
    #import time
198
    #img = get_preprocessed(mode='G')
200
    #for sigma in [0.5, 1, 2, 3, 5, 10]:
202
    #    print('-'*80)
204 #    print('ÏČ=',sigma)
    #    print('calculating hessian H')
206
    #    tic = time.time()
208 #    H = hessian_matrix(img, sigma=sigma)

210 #    toc = time.time()
    #    print('time elapsed: ', toc - tic)
```

```
212     #     tic = time.time()
        #     print('calculating hessian via FFT (F)')
214     #     h = fft_hessian(img, sigma)

216     #     toc = time.time()
        #     print('time elapsed: ', toc - tic)
218     #     tic = time.time()
        #     print('calculating principal curvatures for ÏČ={}'.format(sigma))
220     #     K1,K2 = principal_curvatures(img, sigma=sigma, H=H)
        #     toc = time.time()
222     #     print('time elapsed: ', toc - tic)
        #     tic = time.time()
224     #     print('calculating principal curvatures for ÏČ={} (fast)'.format(sigma))
        #     k1,k2 = principal_curvatures(img, sigma=sigma, H=h)
226
        #     toc = time.time()
228     #     print('time elapsed: ', toc - tic)
        #     tic = time.time()
230
        #     #####
232
        #     print('calculating targets for ÏČ={}'.format(sigma))
234     #     T = get_targets(K1,K2, threshold=False)

236     #     toc = time.time()
        #     print('time elapsed: ', toc - tic)
238     #     tic = time.time()

240     #     print('calculating targets for ÏČ={} (fast)'.format(sigma))
        #     t = get_targets(k1,k2, threshold=False)
242
        #     toc = time.time()
244     #     print('time elapsed: ', toc - tic)

246     #     ######

248     #     print('extending masks')

250     #     # extend mask over nontargets items
```

```
    #     img1 = ma.masked_where( T < T.mean(), img)
    #     img2 = ma.masked_where( t < t.mean(), img)

    #     tic = time.time()
    #     print('calculating principal directions for ÏČ={}'.format(sigma))
    #     T1,T2 = principal_directions(img1, sigma=sigma, H=H)
    #     toc = time.time()
    #     print('time elapsed: ', toc - tic)
    #     tic = time.time()

    #     print('calculating principal directions for ÏČ={} (fast)'.format(sigma))
    #     t1,t2 = principal_directions(img2, sigma=sigma, H=h)
    #     toc = time.time()
    #     print('time elapsed: ', toc - tic)
```

listings/hfft.py

```
#!/usr/bin/env python3

import numpy as np
from scipy import signal
import scipy.fftpack as fftpack

"""
hfft.py is the implementation of calculating the hessian of a real

image based in frequency space (rather than direct convolution with a gaussian
as is standard in scipy, for example).

TODO: PROVIDE MAIN USAGE NOTES
"""

def gauss_freq(shape, ÏČ=1.):
    """
    DEPRECATED

    NOTE:
        this function is/should be? for illustrative purposes only--
        we can actually build this much faster using the builtin
```

```
        scipy.signal.gaussian rather than a roll-your-own

    build a shape=(M,N) sized gaussian kernel in frequency space
    with size ÏČ

    (due to the convolution theorem for fourier transforms, the function
    created here may simply be *multiplied" against the signal.

    """

    M,  N = shape
    fgauss = np.fromfunction(lambda Î ,Î·: ((Î +M+1)/2)**2 + ((Î·+N+1)/2)**2, shape=shape)

    # is this used?
    coeff = (1 / (2*np.pi * ÏČ**2))

    return np.exp(-fgauss / (2*ÏČ**2))

def blur(img, sigma):
    """
    DEPRECATED
    a roll-your-own FFT-implemented gaussian blur.
    fft_gaussian below is preferred (it is more efficient)
    """

    I = fftpack.fft2(img) # get 2D transform of the image

    # do whatever

    I *= gauss_freq(I.shape, sigma)


    return fftpack.ifft2(I).real

def fft_gaussian(img,sigma,A=None):

    """
    https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.fftconvolve.html
```

```
    in particular the example in which a gaussian blur is implemented.

    along with the comment:
    "Gaussian blur implemented using FFT convolution. Notice the dark borders
    around the image, due to the zero-padding beyond its boundaries. The
    convolve2d function allows for other types of image boundaries, but is far
    slower"

    (i.e. doesn't use FFT).

    note that here, you actually take the FFT of a gaussian (rather than
    build it in frequency space). there are ~6 ways to do this.
    """
    #create a 2D gaussian kernel to take the FFT of

    # scale factor!
    A = 1 / (2*np.pi*sigma**2)
    kernel = np.outer(A*signal.gaussian(img.shape[0], sigma),
                      A*signal.gaussian(img.shape[1],sigma))

    return signal.fftconvolve(img, kernel, mode='same')

def fft_hessian(image, sigma=1.):
    """
    a reworking of skimage.feature.hessian_matrix that uses
    e FFT to compute gaussian, which results in a considerable speedup

    INPUT:
        image - a 2D image (which type?)
        sigma - coefficient for gaussian blur

    OUTPUT:
        (Lxx, Lxy, Lyy) - a triple containing three arrays
            each of size image.shape containing the xx, xy, yy derivatives
            respectively at each pixel. That is, for the pixel value given
            by image[j][k] has a calculated 2x2 hessian of
            [ [Lxx[j][k], Lxy[j][k]],
              [Lxy[j][k], Lyy[j][k]] ]
    """
```

```
    gaussian_filtered = fft_gaussian(image, sigma=sigma)

    Lx, Ly = np.gradient(gaussian_filtered)

    Lxx, Lxy = np.gradient(Lx)
    Lxy, Lyy = np.gradient(Ly)

    return (Lxx, Lxy, Lyy)

def fft_gradient(image, sigma=1.):
    """ returns gradient norm """

    gaussian_filtered = fft_gaussian(image, sigma=sigma)

    Lx, Ly = np.gradient(gaussian_filtered)

    return np.sqrt(Lx**2 + Ly**2)
def _old_test():
    """
    old main function for testing.

    This simply tests fft_gaussian on a test image, exemplifying the speedup
    compared to a traditional gaussian.
    """
    import matplotlib.pyplot as plt

    from skimage.data import camera

    img = camera() / 255.

    sample_sigmas = (.2, 2, 10, 30)

    outputs = (fft_gaussian(img, sample_sigmas[0]),
               fft_gaussian(img, sample_sigmas[1]),
               fft_gaussian(img, sample_sigmas[2]),
               fft_gaussian(img, sample_sigmas[3]),
               )
```

```
140
    fig, axes = plt.subplots(nrows=2, ncols=2, figsize=(10, 10))
142
    axes[0, 0].imshow(outputs[0], cmap='gray')
144 axes[0, 0].set_title('fft_gaussian ÏČ={}'.format(sample_sigmas[0]))
    axes[0, 0].axis('off')
146
    axes[0, 1].imshow(outputs[1], cmap='gray')
148 axes[0, 1].set_title('fft_gaussian ÏČ={}'.format(sample_sigmas[1]))
    axes[0, 1].axis('off')
150
    axes[1, 0].imshow(outputs[2], cmap='gray')
152 axes[1, 0].set_title('fft_gaussian ÏČ={}'.format(sample_sigmas[2]))
    axes[1, 0].axis('off')
154
    axes[1, 1].imshow(outputs[3], cmap='gray')
156 axes[1, 1].set_title('fft_gaussian ÏČ={}'.format(sample_sigmas[3]))
    axes[1, 1].axis('off')
158
    plt.tight_layout()
160 plt.show()

162 if __name__ == "__main__":

164     pass
```

listings/farm_samples.py

```
#/usr/bin/env python
2
"""
4 This should be a plugin to take images from the folder NCS_vessel_GIMP_xcf
and create trace, mask, and backgrounded images from each xcf file.
6
What follows is actually without plugin syntax. The following commands are what
8 you would type directly into the python console in gimp.

10 """
from gimpfu import *
```

```
import os.path

# can't use this because we can't run this outside of gimp i think?
# need to write a batch file to run this and then run # gimp -b ...

#for i, xcffile in enumerate(glob('*.xcf')):

#basefile, ext = os.path.splitext(xcffile)

# get active image
img = gimp.image_list()[0]

# Go to perimeter layer.
perimeter = pdb.gimp_image_get_layer_by_name(img, 'perimeter')
# could also iterate and say if layer.name = 'perimeter' ...

# Copy perimeter layer & focus new layer (only visible).

# .copy() has optional arg of "add_alpha_channel"
M = perimeter.copy()

# set all other layers non visible
for layer in img.layers:
    layer.visible = False

# add in position 0 (top)
img.add_layer(M, 0)

# Remove Alpha Channel.
pdb.gimp_layer_flatten(M)
## Invert Colors
#pdb.gimp_invert(M)


# color exchange yellow & blue to black
pdb.plug_in_exchange(img,m,255,255,0,0,0,0,1,1,1)
pdb.plug_in_exchange(img,m,0,0,255,0,0,0,1,1,1)

# set FG color to black
```

```
gimp.set_foreground(0,0,0)

cx, cy = img.height // 2 , img.width // 2

# Bucket Fill Inside black (middle x,y is probably OK)
pdb.gimp_edit_bucket_fill(m,0,0,100,0,0,cx,cy)
# Color Exchange Green to White.
pdb.plug_in_exchange(img,m,0,255,0,255,255,255,1,1,1)

# Color Exchange Cyan (00ffff) to White.
pdb.plug_in_exchange(img,m,0,255,255,255,255,255,1,1,1)

# Export Layer as Image called "f".mask.png

pdb.gimp_file_save(img,m, '/home/luke/test.png', '')

# invert (so back exterior now)
pdb.invert(m)
m.mode = DARKEN_ONLY_MODE # the constant 9

# set bottom layer (placenta) to visible
img.layers[-1].visible = True

# now make a new layer called 'main' from visible
raw_img = pdb.gimp_layer_new_from_visible(img,img,'raw_img')
img.add_layer(main,0)
pdb.gimp_file_save(img ,raw_img, '/home/luke/test2.png', '')

# now set the veins/artery layers as only visible
for layer in img.layers:
    layer.visible = (layer.name in ("arteries", "veins"))

trace = pdb.gimp_layer_new_from_visible(img,img,'trace')
img.add_layer(trace,0)

pdb.gimp_layer_flatten(trace) # remove alpha channel

pdb.gimp_desaturate(trace) # turn to grayscale
pdb.gimp_threshold(trace,255,255) # anything not 255 turns black
```

```
90
pdb.gimp_file_save(img, trace, '/home/luke/testtrace.png','')
92 # Visible to New Layer called Trace & focus (only visible)
# Remove Alpha Channel of layer.
94 # Threshold (127 is default and  OK)
# Invert Colors
96 # Export Layer as "f".trace.png
# Make "Background" Layer and Mask Layer Visible
98 # Move Mask Layer in Front of "Background" Layer
# Change Mask Layer Mode to "Darken Only"
100 # Export Visible as *.png
```

listings/frangi.py

```
import numpy as np
2 import numpy.ma
from hfft import fft_hessian
4 from diffgeo import principal_curvatures
from plate_morphology import dilate_boundary
6
def frangi_from_image(img, sigma, beta=0.5, gamma=None, dark_bg=True,
8                       dilation_radius=None, threshold=None,
                        return_debug_info=False):
10     """
    Perform a frangi filtering on img
12     if None, gamma returns half of Frobenius norm on the image
    if dilation radius is specified, that amount is dilated from the
14     boundary of the image (mask must be specified)

16     input image *must* be a masked array. To implement: supply mask
    or create a dummy mask if not specified so this can work out of the
18     box on arbitrary images.

20     return_debug info will return anisotropy, structureness measures, as
    well as the calculated gamma. will return a tuple of
22     (R, S, gamma) where R and S are matrices of shape img.shape
    and gamma is a float.
24
```

```
    BIGGER TODO:

        THIS OVERLAPS WITH pcsvn.make_multiscale
        USE THIS THERE
    """
    # principal_directions() calculates the frangi filter with
    # standard convolution and takes forever. FIX THIS!
    hesh = fft_hessian(img, sigma) # the triple (Hxx,Hxy,Hyy)

    k1, k2 = principal_curvatures(img, sigma, H=hesh)

    if dilation_radius is not None:

        # pass None to just get the mask back
        collar = dilate_boundary(None, radius=dilation_radius,
                                 mask=img.mask)

        # get rid of "bad" K values before you calculate gamma
        k1[collar] = 0
        k2[collar] = 0

    if gamma is None:

        gamma = .5 * max_hessian_norm(hesh)
        if np.isclose(gamma,0):
            print("WARNING: gamma is close to 0. should skip this layer.")

    targets = get_frangi_targets(k1, k2, beta=beta, gamma=gamma,
                                 dark_bg=dark_bg, threshold=threshold)

    if not return_debug_info:
        return targets
    else:
        return targets, (R, S, gamma)

def get_frangi_targets(K1,K2, beta=0.5, gamma=None, dark_bg=True, threshold=None):
    """
    returns results of frangi filter. eigenvalues are inputs

```

```
    if gamma is not supplied, use half of L2 norm of hessian
    if you want to use half of frobenius norm, calculate it outside here
    """

    R = anisotropy(K1,K2)
    S = structureness(K1,K2)

    if gamma is None:
        # half of max hessian norm (using L2 norm)
        gamma = .5 * np.abs(K2).max()
        if np.isclose(gamma,0):
            print("warning! gamma is very close to zero. maybe this layer isn't worth it...")
            print("sigma={:.3f}, gamma={}".format(sigma,gamma))
            print("returning an empty array")
            return np.zeros_like(img)

    F = np.exp(-R / (2*beta**2))
    F *= 1 - np.exp( -S / (2*gamma**2))

    if dark_bg:
        F = (K2 < 0)*F
    else:
        F = (K2 > 0)*F

    if numpy.ma.is_masked(K1):
        F = numpy.ma.masked_array(F, mask=K1.mask)
    if threshold:

        return F < threshold
    else:
        return F

def max_hessian_norm(hesh):

    hxx, hxy, hyy = hesh

    # frob norm is just sqrt(trace(AA^T)) which is easy for a 2x2
    max_norm = np.sqrt((hxx**2 + 2*hxy*2 + hyy**2).max())
```

```
104     return max_norm


106 def anisotropy(K1,K2):
        """
108     according to Frangi (1998) this is technically A**2
        """
110
        return (K1/K2) **2
112
    def structureness(K1,K2):
114     """
        according to Frangi (1998) this is technically S**2
116     """
        return K1**2 + K2**2
```

listings/hfft_demo.py

```
1  #!/usr/bin/env python3


3  import numpy as np
   from skimage.data import camera
5  from skimage.io import imread


7  import matplotlib.pyplot as plt
   from hfft import gauss_freq, blur, fft_gaussian, fft_hessian
9  from scipy.ndimage import gaussian_filter


11 from scipy.linalg import norm
   import timeit
13

   #img = camera() / 255.
15 img = imread('samples/barium1.png', as_grey=True) / 255.
   mask = imread('samples/barium1.mask.png', as_grey=True)
17

   # compare computation speed over sigmas
19

   # N logarithmically spaced scales between 1 and 2^m
21 N = 5
   m = 8
```

```
sigmas = np.logspace(0,m, num=N, base=2)

fft_results = list()
std_results = list()

for sigma in sigmas:
    # test statements to compare (fft-based gaussian vs convolution-based)
    fft_test_statement = 'fft_gaussian(img,{})'.format(sigma)
    std_test_statement = 'gaussian_filter(img,{})'.format(sigma)
    # run each statement 1 times (with 2 runs in each trial)
    # returns/appends the average of 3 runs
    fft_results.append(timeit.timeit(fft_test_statement,
                                     number=1, globals=globals()))
    std_results.append(timeit.timeit(std_test_statement,
                                     number=1, globals=globals()))

    # now actually evaluate both to compare
    f = eval(fft_test_statement)
    s = eval(std_test_statement)

    # normalize each matrix by frobenius norm and take difference
    # ideally should try to zero out the "mask" area
    diff = np.abs(f / norm(f) - s / norm(s))
    raw_diff = np.abs(f - s)
    # don't care if it's the background
    diff[mask==1] = 0
    raw_diff[mask==1] = 0

    # should format this stuff better into a legible table
    print(sigma, diff.max(), raw_diff.max())

lines = plt.plot(sigmas, fft_results, 'go', sigmas, std_results, 'bo')
plt.xlabel('sigma (gaussian blur parameter)')
plt.ylabel('run time (seconds)')
plt.legend(lines, ('fft-gaussian', 'conv-gaussian'))
plt.title('Comparision of Gaussian Blur Implementations')
```

listings/plate_morphology.py

```
#!/usr/bin/env python3

from skimage.morphology import disk, binary_erosion, binary_dilation
from skimage.morphology import convex_hull_image
from skimage.segmentation import find_boundaries

import numpy as np
import numpy.ma as ma

def erode_plate(img, erosion_radius=20, plate_mask=None):
    """
    Manually remove (erode) the outside boundary of a plate.
    The goal is remove any influence of the zeroed background
    on reporting derivative information

    NOTE: this is an old deprecated function. use dilate boundary instead.
    """
    if plate_mask is None:
        # grab the mask from input image
        try:
            plate_mask = img.mask
        except AttributeError:
            raise('Need to supply mask information')

    # convex_hull_image finds white pixels
    plate_mask = np.invert(plate_mask)

    # find convex hull of mask (make erosion calculation easier)
    plate_mask = np.invert(convex_hull_image(plate_mask))

    # this is much faster than a disk. a plus sign might be better even.
    #selem = square(erosion_radius)
    # also this correctly has erosion radius as the RADIUS!
    # input for square() and disk() is the diameter!
    selem = np.zeros((erosion_radius*2 + 1, erosion_radius*2 + 1),
                     dtype='bool')
    selem[erosion_radius,:] = 1
    selem[:,erosion_radius] = 1
    eroded_mask = binary_erosion(plate_mask, selem=selem)
```

```
40
    # this is by default additive with whatever
42  return ma.masked_array(img, mask=eroded_mask)

44 def dilate_boundary(img, radius=10, mask=None):
    """
46  grows the mask by a specified radius of a masked 2D array
    Manually remove (erode) the outside boundary of a plate.
48  The goal is remove any influence of the zeroed background
    on reporting derivative information.
50
    There is varying functionality here (maybe should be multiple functions
52  instead?)

54  If img is a masked array and mask=None, the mask will be dilated and a
    masked array is outputted.
56
    If img is any 2D array (masked or unmasked), if mask is specified, then
58  the mask will be dilated and the original image will be returned as a
    masked array with a new mask.
60
    If the img is None, then the specified mask will be dilated and returned
62  as a regular 2D array.

64  """

66  if mask is None:
        # grab the mask from input image
68      # if img is None this will break too but not handled
        try:
70          mask = img.mask
        except AttributeError:
72          raise('Need to supply mask information')

74  perimeter = find_boundaries(mask, mode='inner')

76  maskpad = np.zeros_like(perimeter)

78  M,N = maskpad.shape
```

```
    for i,j in np.argwhere(perimeter):
        # just make a cross shape on each of those points
        # these will silently fail if slice is OOB thus ranges are limited.
        maskpad[max(i-radius,0):min(i+radius,M),j] = 1
        maskpad[i,max(j-radius,0):min(j+radius,N)] = 1

    new_mask = np.bitwise_or(maskpad, mask)

    if img is None:
        return new_mask # return a 2D array
    else:
        # replace the original mask or create a new masked array
        return ma.masked_array(img, mask=new_mask)

##############
# DEMO FOR SHOWING OFF DILATE_BOUNDARY EFFECT

if __name__ == "__main__":

    from get_placenta import get_named_placenta
    from frangi import frangi_from_image
    import matplotlib.pyplot as plt
    import numpy as np
    from skimage.exposure import rescale_intensity

    import os.path

    dest_dir = 'demo_output'
    img = get_named_placenta('T-BN0164923.png')

    #radius = 30
    sigma = 3
    radius = 80
    # IMG WITHOUT DILATING, THEN IMAGE WITH DILATING
    #sigma = radius/4

    #inset = np.s_[:,:]
    inset = np.s_[800:1000,500:890]
    #inset = np.s_[100:300,300:500]
```

```
118
    D = dilate_boundary(img, radius=radius)
120

122     # SAVE IT IN THE RIGHT DIRECTORY, ETC plt.savefig(

124     # NOW SHOW FRANGI ON THESE IMAGES

126     # NOW SHOW THE SAME PICTURE

128     Fimg = frangi_from_image(img, sigma, dark_bg=False, dilation_radius=None)
        #FD = frangi_from_image(img, sigma, dark_bg=False, dilation_radius=radius)
130     FD = frangi_from_image(D, sigma, dark_bg=False)

132     #Fimg = rescale_intensity(Fimg)
        #FD = rescale_intensity(FD)
134
        fig, axes = plt.subplots(ncols=2, nrows=2, figsize=(14, 7))
136
        axes[0,0].imshow(img[inset].filled(0), cmap=plt.cm.gray)
138     axes[0,1].imshow(D[inset].filled(0), cmap=plt.cm.gray)

140     axes[1,0].imshow(Fimg[inset].filled(0), cmap=plt.cm.nipy_spectral)
        axes[1,1].imshow(FD[inset].filled(0), cmap=plt.cm.nipy_spectral)
142
        # this is a hack directly from matplotlib, that's why it's so ugly.
144     plt.setp([a.get_xticklabels() for a in axes[0, :]], visible=False)
        plt.setp([a.get_yticklabels() for a in axes[:, 1]], visible=False)
146
        fig.tight_layout()
148
        plt.savefig(os.path.join(dest_dir, "boundary_dilation_demo.png"), dpi=300)
```

listings/process_NCS_xcfs.py

```
    #!/usr/bin/env python
2
    """
4   This should be a plugin to take images from the folder NCS_vessel_GIMP_xcf
```

```
and create trace, mask, and backgrounded images from each xcf file.

chmod +x and then copy or link to  ~/gimp-2.x/plug-ins/
"""

from gimpfu import *
import os.path
from functools import partial

#basefile, ext = os.path.splitext(xcffile)

def _outname(base, s=None):
    if s is None:
        stubs = (base, 'png')
    else:
        stubs = (base, s, 'png')

    return '.'.join(stubs)

# get active image
def process_NCS_xcf(timg,tdrawable):
    img = timg
    basename, _ = os.path.splitext(img.name)
    print "*"*80
    print '\n\n'

    print "Processing " , img.name
    # generate output names easier
    outname = partial(_outname, base=basename)

    # get coordinates of the center
    cx, cy = img.height // 2 , img.width // 2

    # disable the undo buffer
    #img.disable_undo()

    #perimeter = pdb.gimp_image_get_layer_by_name(img, 'perimeter')

    for layer in img.layers:
```

```
        if layer.name.lower() in ('perimeter', 'perimeters'):
            # .copy() has optional arg of "add_alpha_channel"
            mask = layer.copy()
            break
    else:
        print("Could not find a perimeter layer.")
        print("Layers of this image are:")
        for n,layer in enumerate(img.layers):
            print "\t", n, ":", layer.name
        print("Skipping this file.")

        return

    for layer in img.layers:
        layer.visible = False

    mask.name = "mask" # name the new layer
    img.add_layer(mask,0) # add in position 0 (top)

    pdb.gimp_layer_flatten(mask) # Remove Alpha Channel.

    # remove unneeded (i hope) annotations
    # color exchange yellow & blue to black
    pdb.plug_in_exchange(img,mask,255,255,0,0,0,0,1,1,1)
    pdb.plug_in_exchange(img,mask,0,0,255,0,0,0,1,1,1)

    # set FG color to black (for tools, not of image)
    gimp.set_foreground(0,0,0)


    # Bucket Fill Inside black (center pixel is hopefully fine)
    pdb.gimp_edit_bucket_fill(mask,0,0,100,0,0,cx,cy)

    # Color Exchange Green to White.
    pdb.plug_in_exchange(img,mask,0,255,0,255,255,255,1,1,1)

    # Color Exchange Cyan (00ffff) to White.
    pdb.plug_in_exchange(img,mask,0,255,255,255,255,255,1,1,1)

```

```
    # Export Layer as Image called "f".mask.png
    pdb.gimp_file_save(img,mask, outname(s="mask"), '')

    # invert (so exterior is now black)
    pdb.gimp_invert(mask)
    mask.mode = DARKEN_ONLY_MODE # the constant 9

    # set bottom layer (placenta) to visible
    raw = img.layers[-1]
    raw.visible = True

    # now make a new layer called 'raw_img' from visible
    base = pdb.gimp_layer_new_from_visible(img,img,'base')
    img.add_layer(base,0)
    pdb.gimp_file_save(img , base, outname(s=None) , '')

    # now get rid of mask and save the raw image
    mask.visible = False
    pdb.gimp_file_save(img, raw, outname(s='raw') ,'')

    # now set the veins/artery layers as only visible
    for layer in img.layers:
        layer.visible = (layer.name.lower() in ("arteries", "veins"))

    trace = pdb.gimp_layer_new_from_visible(img,img,'trace')
    img.add_layer(trace,0)

    pdb.gimp_layer_flatten(trace) # remove alpha channel

    pdb.gimp_desaturate(trace) # turn to grayscale
    pdb.gimp_threshold(trace,255,255) # anything not 255 turns black

    pdb.gimp_file_save(img, trace, outname(s='trace') ,'')

    print "Saved. "


register(
    "process_NCS_xcf",
```

```
    "Create base image + trace + mask from an NCS xcf file",
    "Create base image + trace + mask from an NCS xcf file",
    "Luke Wukmer",
    "Luke Wukmer",
    "2018",
    "<Image>/Image/Process_NCS_xcf...",
    "RGB*, GRAY*",
    [],
    [],
    process_NCS_xcf)

main()
```

listings/scale_sweep_demo.py

```
#!/usr/bin/env python3

from get_placenta import get_named_placenta, list_placentas, _cropped_bounds, cropped_view,
    cropped_args, show_mask
from frangi import frangi_from_image

import numpy as np
import numpy.ma as ma
from plate_morphology import dilate_boundary

import os.path
import matplotlib.pyplot as plt

#imgfile = list_placentas('T-BN')[32]

img = get_named_placenta('T-BN2315363.png')

img = dilate_boundary(img,radius=5)
F = list()
fi = list()

#scales = np.logspace(-3,3,base=2,num=8)
#scales = np.linspace(.25,8,num=8)

```

```
scales = np.linspace(.25,4,num=6)
for n, sigma in enumerate(scales, 1):
    target = frangi_from_image(img, sigma, dark_bg=False)
    plate = cropped_view(target).filled(0)
    inset = target[370:660,530:900]
    F.append(plate)
    fi.append(inset)
    for label in ['plate','inset']:
        if label == 'inset':
            printable = inset
        else:
            printable = plate

        plt.imshow(printable, cmap=plt.cm.gist_earth)
        plt.title(r'$\sigma={:.2f}$'.format(sigma))
        plt.tight_layout()
        c = plt.colorbar()
        c.set_ticks = np.linspace(0,0.6, num=7)
        plt.clim(0,0.6)
        outname = 'demo_output/scalesweep_{}_{}.png'.format(n,label)
        plt.savefig(outname, dpi=300, bbox_inches='tight')
        print('saved', outname)
        plt.close()

# now make a stitched together version
for label in ['plate', 'inset']:
    if label == 'inset':
        L = fi
    else:
        L = F
    top = np.concatenate(L[:3],axis=1)
    bottom = np.concatenate(L[3:],axis=1)
    stitched = np.concatenate((top,bottom),axis=0)
    imga = plt.imshow(stitched, cmap=plt.cm.gist_earth)
    plt.imsave('demo_output/sweep_stitched_{}.png'.format(label),
               stitched, cmap=plt.cm.gist_earth)
    #plt.colorbar(); plt.clim(0,0.3)
```

listings/vessel_filters.py

```
#!/usr/bin/env python3

"""
NOTE THIS IS THE OLD CODE (FOR BAD PARAMETER FRANGI).
IT'S HERE FOR LEGACY OR POSSIBLE DEMONSTRATION
some function / file names have changed and this code probably won't
work anymore without minor alterations
"""

import scipy.ndimage as ndi
import matplotlib.pyplot as plt
from functools import partial
from skimage.morphology import *
from skimage.exposure import rescale_intensity, equalize_adapthist

from skimage.color import label2rgb

import os
import os.path
import datetime

import numpy as np
import numpy.ma as ma

from skimage.transform import rotate
from skimage.feature import hessian_matrix, hessian_matrix_eigvals
from numpy.linalg import eig

def rotating_box_filter(img, thetas, sigma, length_ratio=4, verbose=True):
    """
    runs a curvilinear filter at the given scale space `sigma`

    INPUT:
        img:            a binary 2D array
        sigma:          the scale space
        length_ratio:   a rectangular filter will be applied with size
        steps:          the range of rotations (0,180) is divided into this
```

```
38                      many steps (default: 16 or 12 degrees)
        verbose:        default True
40
    OUTPUT:
42      extracted:  a 2D binary array of the same shape as img

44  METHODS:
        (todo)
46
    IMPLEMENTATION:
48      (todo)

50  WARNINGS/BUGS:
        this may be supremely wasteful for large step sizes. you should check
52      in the anticipated range of sigmas that there is sufficient variation
        in the rotated structure elements to warrant that amount of step sizes.
54      print('cleaning up scale space')

56      furthermore, this filter should be used carefully. there are probably
        bugs in the logic and implementation.
58  """

60  sigma = int(sigma) # round down to a integer

62  mask = img.mask
    extracted = np.zeros_like(img)
64  img = binary_erosion(img, selem=disk(sigma))
    #img = remove_small_objects(img, min_size=sigma**3)
66  img = binary_dilation(img, selem=disk(sigma))


68
    width, length = int(2*sigma), int(sigma*length_ratio)
70
    if length == 0:
72      length = 1

74  rect = rectangle(width, length)
    outer_rect =  rectangle(int(width+2*sigma+4),int(length))
76  outer_rect[sigma:-sigma,:] = 0
```

```
    thetas = np.round(thetas*180 / np.pi)

    # this should behave the same as thetas[thetas==180] = 0
    # but not return a warning
    thetas.put(thetas==180, 0) # these angles are redundant

    if verbose:
        print('running vessel_filter with ÏČ={}: w={}, l={}'.format(
            sigma, width,length), flush=True)

    if verbose:
        print('building rotated filters...', end=' ')

    srot = partial(rotate, resize=True, preserve_range=True) # look at order
    rotated = [srot(rect, theta) for theta in range(180)]

    if verbose:
        print('done.')

    if verbose:
        print('building outer filters...', end=' ')
    outer_rotated = [srot(outer_rect, theta) for theta in range(180)]

    for theta in range(180):
        if verbose:
            print('Îÿ=', theta, end='\t', flush=True)
            if theta % 6 == 0:
                print()

        vessels = binary_erosion(img, selem=rotated[theta])
        #margins = binary_dilation(img, selem=outer_rotated[theta])
        #margins = np.invert(margins)
        #vessels = np.logical_and(vessels, margins)
        extracted = np.logical_or(extracted, (thetas == theta) * vessels)
    if verbose:
        print('') # new line

    extracted = binary_dilation(extracted, selem=disk(sigma))
    extracted[mask] = 0
```

```
116     return extracted

118 def get_frangi_targets(K1,K2, beta=0.5, c=15, dark_bg=True, threshold=None):
        """
120     returns results of frangi filter
        """
122
        R = (K1/K2) ** 2 # anisotropy
124     S = (K1**2 + K2**2) # structureness

126     F = np.exp(-R / (2*beta**2))
        F *= 1 - np.exp( -S / (2*c**2))
128
        if dark_bg:
130         F = (K2 < 0)*F
        else:
132         F = (K2 > 0)*F

134     if threshold:

136         return F < threshold
        else:
138         return F

140 def get_targets(K1,K2, method='F', threshold=True):
        """
142     returns a binary threshold (conservative)

144     F -> frangi filter with default arguments. greater than mean.
        R -> blobness measure. greater than median.
146     S -> anisotropy measure (greater than median)
        """
148     if method == 'R':
            R = (K1 / K2) ** 2
150         if threshold:
                T = R < ma.median(R)
152         else:
                T = R
154     elif method == 'S':
```

```
        S = (K1**2 + K2**2)/2
        if threshold:
            T = S > ma.median(S)
        else:
            T = S
    elif method == 'F':
        R = (K1 / K2) ** 2
        S = (K1**2 + K2**2)/2
        beta, c = 0.5, 15
        F = np.exp(-R / (2*beta**2))
        F *= 1 - np.exp(-S / (2*c**2))
        T = (K2 < 0)*F

        if threshold:
            T = T > (T[T != 0]).mean()
    else:
        raise('Need to select method as "F", "S", or "R"')

    return T

b = partial(plt.imshow, cmap=plt.cm.Blues)
s = plt.show


if __name__ == "__main__":

    #raw = get_preprocessed(mode='G')
    raw = ndi.imread('samples/clahe_raw.png')
    raw = preregister(raw)
    img = preprocess(raw)
    img = raw

    # which ÏČ to use (in order)
    scale_range = np.logspace(0,5, num=30, base=2)

    frangi_only = np.zeros((img.shape[0],img.shape[1],len(scale_range)))
    all_targets = np.zeros((img.shape[0],img.shape[1],len(scale_range)))
    extracted_all = np.zeros((img.shape[0],img.shape[1],len(scale_range)))
```

```
194
    OUTPUT_DIR = 'fpd_new_output'
196
    n = datetime.datetime.now()
198
    SUBDIR = ''.join(('', n.strftime('%y%m%d_%H%M')))
200
    print('saving outputs in', os.path.join(OUTPUT_DIR, SUBDIR))
202
    try:
204     os.mkdir(os.path.join(OUTPUT_DIR, SUBDIR))
    except FileExistsError:
206     ans = input('save path already exists! would you like to continue? [y/N]')
        if ans != 'y':
208         print('aborting program. clean up after yourself.')

210         exit(0)

212     else:
            print('your files will be overwritten (but artifacts may remain!)')
214 finally:
        print('\n')
216
    for n, sigma in enumerate(scale_range):
218
        beta = min(.09*sigma - .04, .5)
220     print('-'*80)
        print('ÏČ={}'.format(sigma))
222
        print('finding hessian')
224     h = fft_hessian(img,sigma)

226     print('finding curvatures')
        k1,k2 = principal_curvatures(img, sigma=sigma,H=h)
228
        print('finding targets with Îš={}'.format(beta))
230     t = get_frangi_targets(k1,k2,
                beta=beta, dark_bg=False, threshold=False)
232     t = t > t.mean()
```

```
        #t = remove_small_objects(t, min_size=100)

        # extend mask
        timg = ma.masked_where(t < t.mean(), img)
        percentage = timg.count() / img.size

        print('finding p. directions {}'.format(np.round(percentage*100)))
        t1,t2 = principal_directions(timg, sigma=sigma, H=h)

        extracted = vessel_filter(t, t1, sigma, length_ratio=.5, verbose=True)

        extracted_all[:,:,n] = extracted

        savefile = ''.join(('%02d' % sigma, '.png'))
        plt.imsave(os.path.join(OUTPUT_DIR, SUBDIR, savefile),
                extracted, cmap=plt.cm.Blues)

        all_targets[:,:,n] = (timg!=0) * t1

        #all_targets[:,:,n]= timg!=0

    A = all_targets.sum(axis=-1)
    a = all_targets

    sys.exit(0)

        #new_labels = sigma*np.logical_and(extracted != 0, cumulative == 0)
        #cumulative += new_labels.astype('uint8')

    full_skel = skeletonize(cumulative!=0)
    skel = remove_small_objects(full_skel, min_size=50, connectivity=2)

    matched_all = np.zeros_like(skel)

    for i, scale in enumerate(scale_range):

        e = extracted_all[:,:,i]
        el, nl = label(e, return_num=True)
        matched = np.zeros_like(matched_all)
```

```
272
        for region in range(1, nl+1):
274         if np.logical_and(el==region, skel).any():
                matched = np.logical_or(matched, el==region)
276
        matched_all = np.logical_or(matched_all, matched)
```

listings/hfft_accuracy.py

```
1  """
3  here you want to show  the accuracy of hfft.py
5  BOILERPLATE
7  show that gaussian blur of hfft is accurate, except potentially around the
   boundary proportional to sigma.
9
   or if they're off by a scaling factor, show that the derivates
11 (taken the same way) are proportional.
13 pseudocode
15 A = gaussian_blur(image, sigma, method='convential')
   B = gaussian_blue(image, sigma, method='fourier')
17
   zero_order_accurate = isclose(A, B, tol)
19
   J_A= get_jacobian(A)
21 J_B = get_jacobian(B)
23 first_order_accurate = isclose(J_A, J_B, tol)
25 A_eroded = zero_around_plate(A, sigma)
   B_eroded = zero_around_plate(B, sigma)
27
   J_A_eroded = zero_around_plate(A, sigma)
29 J_B_eroded = zero_around_plate(B, sigma)
```

```
zero_order_accurate_no_boundary = isclose(A_eroded, B_eroded, tol)
first_order_accurate = isclose(J_A_eroded, J_B_eroded, tol)

"""

from get_placenta import get_named_placenta

from hfft import fft_hessian, fft_gaussian
from scipy.ndimage import gaussian_filter
import matplotlib.pyplot as plt
from get_placenta import mimshow

from score import mean_squared_error
import numpy as np
from scipy.ndimage import laplace
import numpy.ma as ma

from skimage.segmentation import find_boundaries
from skimage.morphology import disk, binary_dilation

from diffgeo import principal_curvatures
from frangi import structureness, anisotropy, get_frangi_targets

def erode_plate(img, sigma, mask=None):
    """
    Apply an eroded mask to an image
    assume (if helpful) that the boundary of the placenta is a connected loop
    that is, there is a single inside and outside of the shape, and that
    the placenta is more or less convex

    alternatively, if img is None, simply erode the mask
    this function should probably be renamed "dilate mask"
    and erode plate should be one that just acts on masked inputs
    """

    if mask is None:
        mask = img.mask

    # get a boolean array that is 1 along the border of the mask, zero elsewhere
```

```
    # default mode is 'thick' which is fine
    bounds = find_boundaries(mask)

    # structure element to dilate by is a disk of diameter sigma
    # rounded up to the nearest integer. this may be too conservative.
    selem = disk(np.ceil(sigma))
    dilated_border = binary_dilation(bounds, selem=selem)

    new_mask = np.logical_or(mask, dilated_border)

    # see comment in docstring. alternatively, the behavior here
    # could be handled by an "apply mask" parameter
    if img is None:
        return new_mask
    else:
        return ma.masked_array(img, mask=new_mask)


# FIX SOME ISSUES, BINARY DILATION IS TAKING HELLA LONG AND ALSO
# THERE ARE RANDOM BLIPS INSIDE THE MASK!!!
# FIX IN GIMP!:

imgfile = 'barium1.png'
maskfile = 'barium1.mask.png'

img_raw = get_named_placenta(imgfile, maskfile=maskfile)

# so that scipy.ndimage.gaussian_filter doesn't use uint8 precision (jesus)
img = img_raw / 255.

# convenience function to show a matrix with img.mask mask
ms = lambda x: mimshow(ma.masked_array(x, img.mask))

sigma = 5

print('applying standard gauss blur')
# THIS USES THE SAME DTYPE AS THE INPUT SO DEAR LORD MAKE SURE IT'S A FLOAT
A = gaussian_filter(img.astype('f'), sigma, mode='constant') #zero padding
print('applying fft gauss blur')
```

```
109 B = fft_gaussian(img, sigma)
    B_unnormalized = B.copy()
111 B = B / (2*(sigma**2)*np.pi)

113 #A = erode_plate(A, sigma, mask=img.mask)
    #B = erode_plate(B, sigma, mask=img.mask)
115 print('calculating first derivatives')

117 # zero the masks before calculating derivates if they're masked
    Ax, Ay = np.gradient(A)
119 Bx, By = np.gradient(B)


121
    print('calculating second derivatives')
123
    # you can verify np.isclose(Axy,Ayx) && np.isclose(Bxy,Byx) -> True
125 Axx, Axy = np.gradient(Ax)
    Ayx, Ayy = np.gradient(Ay)
127
    Bxx, Bxy = np.gradient(Bx)
129 Byx, Byy = np.gradient(By)


131


133 print('calculating eigenvalues of hessian')
    ak1, ak2 = principal_curvatures(A, sigma=sigma, H=(Axx,Axy,Ayy))
135 bk1, bk2 = principal_curvatures(B, sigma=sigma, H=(Bxx,Bxy,Byy))


137
    ##R1 = anisotropy(ak1,ak2)
139 #R2 = anisotropy(bk1,bk2)
    #
141 #S1 = structureness(ak1, ak2)
    #S2 = structureness(bk1, bk2)
143 #print('done.')
    #
145 ## ugh, apply masks here. too large to be conservative?
    ## otherwise structureness only shows up for small sizes
147 new_mask = erode_plate(None, 3*sigma, mask=img.mask)
```

```
#R1[new_mask] = 0
#R2[new_mask] = 0
#S1[new_mask] = 0
#S2[new_mask] = 0

FA = get_frangi_targets(ak1,ak2)
FB = get_frangi_targets(bk1,bk2)

FA[new_mask] = 0
FB[new_mask] = 0

# even without scaling (which occurs below) the second derivates should be
# close. normalize matrices using frobenius norm of the hessian?
# note: A & B are off but have the same shape


# rescale to [0,255] (actually should keep as 0,1? )
#A_unscaled = A.copy()
#B_unscaled = B.copy()

#Ascaled = (A-A.min())/(A.max()-A.min())
#Bscaled = (B-B.min())/(B.max()-B.min())

# the following shows a random vertical slice of A & B (when scaled)
# the results are even more fitting when you scale B to coincide with A's max
# (which obviously isn't feasible in practice)

# FIXEDISH AFTER SCALING!

plt.plot(np.arange(A.shape[1]),A[A.shape[0]//2,:],
         label='scipy.ndimage,gaussian_filter')
plt.plot(np.arange(B.shape[1]), B[B.shape[0]//2,:],
         label='fft_gaussian')
plt.legend()

#MSE = ((A-B)**2).sum() / A.size
MSE = mean_squared_error(A,B)
```

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Comparison of Hessian-based vesselness enhancement filters (which they call HBVF). Introduction is very good. Also note, they use "exponentially distributed" scales for each filter! I don't necessarily agree with their results though, and I don't see what images they're using. They do some parameter searching though, if I can understand their figures it might be useful to do myself in the thesis.

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This was cited in Frangi I believe as a different Hessian filter. Look into this! Gets compared in Olabarriaga, above. I am ignorantly assuming that this is strictly inferior to Frangi's filter. Worth looking into.

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