Game Theory: Homework

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1 Question 1

1.1 (a)

We can rewrite $V(\mu)$ as:

$$\begin{aligned} V(\mu) &= V^*(\mu) \\ V^*(\mu) &= \max_{a\in\{0,1\}}\{(\mathbb{E}(y_t) - \frac{1}{2})a + \delta\mathbb{E}(V^*(\mu'))\} \\ &= \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta\mathbb{E}(V^*(\mu'))\} \end{aligned} \tag{1}$$

where:

$$\mu' = \mu \;\; \texttt{if} \;\; a = 0$$

$$\mu' = \frac{\frac{3}{4}\mu}{\frac{1}{4}+\frac{1}{2}\mu} \;\; \texttt{if} \;\; a = 1, y = 1$$

$$\mu' = \frac{\frac{1}{4}\mu}{\frac{3}{4}-\frac{1}{2}\mu} \;\; \texttt{if} \;\; a = 1, y = 0$$

Now we can use the standard Recursive Dynamic proof, define:

$$T(f)(x) = \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta f(\mu')$$

T maps weakly increasing functions in weakly increasing functions because μ' is stricly increasing in μ and $(\frac{1}{2}\mu - \frac{1}{4})a$ is stricly increasing as long as $a = 1$ and constant as long as $a = 0$.

Therefore by the Contraction Mapping Theorem the fix point of T, which is V^* is weakly increasing. Moreover, since V^* is weakly increasing and $(1-\delta) > 0$, then V is weakly increasing.

1.2 b

First notice that we can use the same argument of (a) to show that V^* is continuous.

Now notice that for $\mu = 0$, $\mu' = 0 \forall a, y$ and for $\mu = 1$, $\mu' = 0 \forall a, y$, therefore:

$$\begin{aligned} V^*(0) &= \max_{a\in\{0,1\}}\{-\frac{1}{4}a + \delta V^*(0)\} \\ &= \frac{\max_{a\in\{0,1\}}\{-\frac{1}{4}a\}}{(1-\delta)} \\ &= 0 \\ V(0) &= 0 \end{aligned}$$

$$\begin{aligned} V^*(1) &= \max_{a\in\{0,1\}}\{\frac{1}{4}a + \delta V^*(1)\} \\ &= \frac{\max_{a\in\{0,1\}}\{\frac{1}{4}a\}}{(1-\delta)} \\ &= \frac{1}{4(1-\delta)} > 0 \\ V(1) &= \frac{1}{4} \end{aligned}$$

Since V is weakly increasing that must be $\mu^* \geq 0$ such that for for all $0 \geq \mu < \mu^*, V(\mu) = 0$, and for all $1 \geq \mu > \mu, V(\mu) > 0$.

Moreover, $0 \geq \mu < \mu^*$, optimal a is 0 and for $1 \geq \mu > \mu$, optimal a is 1. Since V^* is continuous at μ^* the agent is indifferent.

2 Question 2

$$\begin{aligned}
|V^c(\delta,\theta) - V(\delta,\theta)| &= \left| \sum_{t=0}^{\infty} \delta^t a_t^c \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t) \right) \right| \\
&\leq \left| \sum_{t=0}^{\infty} \delta^t a_t^c \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\dots,y_t) \right) \right| + \\
&\quad \left| \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\dots,y_t) \right) \right| \\
&\leq \left(\sum_{t=0}^{\infty} \delta^{2t} \max_{y \in Y^\infty} y^2 \left(\sum_{i=1}^{\infty} |P_{\theta^*}(y_i|y_1,\dots,y_t) \mu(y_i|y_1,\dots,y_t)| \right)^2 \right)^{1/2} + \\
&\quad \left(\sum_{t=0}^{\infty} \delta^{2t} \max_{y \in Y^\infty} y^2 \left(\sum_{i=1}^{\infty} |P_{\theta^*}(y_i|y_1,\dots,y_t) \mu(y_i|y_1,\dots,y_t)| \right)^2 \right)^{1/2} +
\end{aligned}$$

3 Question 3

For simplification:

$$\begin{aligned}
P_\gamma(y) &= \mu P_\gamma(y|\theta = 1) + (1-\mu) P_\gamma(y|\theta = -1) \\
P_\gamma(1) &= \mu(1/2 + \gamma) + (1-\mu)(1/2\gamma) \\
&= 1/2 + \gamma(2\mu - 1) \\
P_\gamma(0) &= 1/2 - \gamma(2\mu - 1)
\end{aligned}$$

Notice:

$$\lim_{\gamma \to 0} P_\gamma(0) = \lim_{\gamma \to 0} P_\gamma(1) = \frac{1}{2}$$

The Baseyian updates are:

$$\mu_\gamma(1) = \frac{(1/2 + \gamma)\mu}{P_\gamma(1)}$$

$$\mu_\gamma(0) = \frac{(1/2 - \gamma)\mu}{P_\gamma(0)}$$

Therefore:

$$\lim_{\gamma \to 0} \mu_\gamma(0) = \lim_{\gamma \to 0} \mu_\gamma(1) = \mu$$

and:

$$\lim_{\gamma \to 0} V(\mu_\gamma(0)) = \lim_{\gamma \to 0} V(\mu_\gamma(1)) = V(\mu)$$

Finally:

$$\begin{aligned}
V_\gamma(\mu) &= P_\gamma(0) V(\mu_\gamma(0)) + P_\gamma(1) V(\mu_\gamma(1)) \\
&= (1/2 + \gamma(2\mu - 1)) V(\mu_\gamma(1)) + (1/2 - \gamma(2\mu - 1)) V(\mu_\gamma(0))
\end{aligned}$$

and:

$$\lim_{\gamma \to 0} V_\gamma(\mu) = V(\mu)$$

Whice make the question identity equals the derivative of V_γ in relation to γ:

$$\frac{\partial V_\gamma(\mu)}{\partial \gamma} = \lim_{\gamma \to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma}$$

We are left to prove that this derivative is zero at $\gamma = 0$:

$$\begin{aligned}
\frac{\partial V_\gamma(\mu)}{\partial \gamma}(\gamma) &= (2\mu - 1)V(\mu_\gamma(1)) + \frac{\partial V(\mu_\gamma(1))}{\partial \gamma}(1/2 + \gamma(2\mu - 1)) - \\
&\qquad (2\mu - 1)V(\mu_\gamma(0)) + \frac{\partial V(\mu_\gamma(0))}{\partial \gamma}(1/2 - \gamma(2\mu - 1)) \\
\frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(1))}{\partial \gamma}(1/2) - (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(0))}{\partial \gamma}(1/2) \\
&= 1/2\left(\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma}\right)
\end{aligned}$$

But:

$$\begin{aligned}
V(\mu_\gamma(1)) &= \mu_\gamma u(a_1^*, 1) + (1 - \mu_\gamma)u(a_1^*, -1) \\
&= P_\gamma(1)^{-1}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)\mu u(a_1^*, -1)] \\
\frac{\partial V(\mu_\gamma)}{\partial \gamma} &= P_\gamma(1)^{-1}[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + \\
&\qquad \frac{1 - 2\mu}{(1/2 + \gamma(2\mu - 1))^2}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)u(a_1^*, -1)] \\
\frac{\partial V(\mu_0)}{\partial \gamma} &= 2[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + (4 - 8\mu)[1/2\mu u(a_1^*, 1)]
\end{aligned}$$

where a_1^* is the argmax of V for $\gamma = 0$ and:

$$\begin{aligned}
V(\mu_\gamma(0)) &= P_\gamma(0)^{-1}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)\mu u(a_0^*, -1)] \\
\frac{\partial V(\mu_\gamma)}{\partial \gamma} &= P_\gamma(0)^{-1}[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + \\
&\qquad \frac{2\mu - 1}{(1/2 + \gamma(2\mu - 1))^2}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)u(a_0^*, -1)] \\
\frac{\partial V(\mu_0)}{\partial \gamma} &= 2[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + (8\mu - 4)[1/2\mu u(a_0^*, 1)]
\end{aligned}$$

and since $a_1^* \to a_0^*$ as $\gamma \to 0$, by what we have seem before, we have:

$$\begin{aligned}
\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma} &= 0 \\
\frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= 0 \\
\lim_{\gamma \to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma} &= 0
\end{aligned}$$

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