

# Game Theory: Homework

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## 1 Question 1

### 1.1 (a)

We can rewrite  $V(\mu)$  as:

$$\begin{aligned} V(\mu) &= (1 - \delta)V^*(\mu) \\ V^*(\mu) &= \max_{a \in \{0,1\}} \{(\mathbb{E}(y_t) - \frac{1}{2})a + \delta\mathbb{E}(V^*(\mu'))\} \\ &= \max_{a \in \{0,1\}} \{(\frac{1}{2}\mu - \frac{1}{4})a + \delta\mathbb{E}(V^*(\mu'))\} \end{aligned} \tag{1}$$

where:

$$\begin{aligned} \mu' &= \mu \text{ if } a = 0 \\ \mu' &= \frac{\frac{3}{4}\mu}{\frac{1}{4} + \frac{1}{2}\mu} \text{ if } a = 1, y = 1 \\ \mu' &= \frac{\frac{1}{4}\mu}{\frac{3}{4} - \frac{1}{2}\mu} \text{ if } a = 1, y = 0 \end{aligned}$$

Now we can use the standard Recursive Dynamic proof, define:

$$T(f)(x) = \max_{a \in \{0,1\}} \{(\frac{1}{2}\mu - \frac{1}{4})a + \delta f(\mu')\}$$

$T$  maps weakly increasing functions in weakly increasing functions because  $\mu'$  is strictly increasing in  $\mu$  and  $(\frac{1}{2}\mu - \frac{1}{4})a$  is strictly increasing as long as  $a = 1$  and constant as long as  $a = 0$ .

Therefore by the Contraction Mapping Theorem the fix point of  $T$ , which is  $V^*$ , is weakly increasing. Moreover, since  $V^*$  is weakly increasing and  $(1 - \delta) > 0$ , then  $V$  is weakly increasing.

### 1.2 (b)

First notice that we can use the same argument of (a) to show that  $V^*$  is continuous.

Now notice that for  $\mu = 0$ ,  $\mu' = 0 \forall a, y$  and for  $\mu = 1$ ,  $\mu' = 1 \forall a, y$ , therefore:

$$\begin{aligned} V^*(0) &= \max_{a \in \{0,1\}} \{-\frac{1}{4}a + \delta V^*(0)\} & V^*(1) &= \max_{a \in \{0,1\}} \{\frac{1}{4}a + \delta V^*(1)\} \\ &= \frac{\max_{a \in \{0,1\}} \{-\frac{1}{4}a\}}{(1 - \delta)} & &= \frac{\max_{a \in \{0,1\}} \{\frac{1}{4}a\}}{(1 - \delta)} \\ &= 0 & &= \frac{1}{4(1 - \delta)} > 0 \\ V(0) &= 0 & V(1) &= \frac{1}{4} \end{aligned}$$

Since  $V$  is weakly increasing that must be  $\mu^* \geq 0$  such that for for all  $0 \leq \mu < \mu^*$ ,  $V(\mu) = 0$ , and for all  $1 \geq \mu > \mu^*$ ,  $V(\mu) > 0$ .

Moreover,  $0 \leq \mu < \mu^*$ , optimal  $a$  is 0 and for  $\mu^* < \mu \leq 1$ , optimal  $a$  is 1. Since  $V^*$  is continuous at  $\mu^*$  the agent is indifferent.

### 1.3 (c)

First notice that the statement is not true for  $\mu = 1$ , in this case the agent never update his belief and always choose  $a = 1$ .

However, for any other  $\mu$ , such that  $\mu^* < \mu < 1$ , there exists finite number of bad signals in sequence that is enough to bring it down to a number below  $\mu^*$ . To see that define:  $\mu_0^n$  the belief after observing  $y = 0, n$  times in a row. It easy to show that:

$$\mu_0^n = \frac{\mu}{3^n - \sum_{i=0}^n (3^{i-1} 2) \mu}$$

It is also simple to show that  $\mu_0^n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore: we have for any  $\epsilon > 0$ , in particular, for  $\epsilon = \mu^*$ ,  $\exists n^* \in \mathbb{N}$  such that for any  $n > n^*, \mu_0^n < \mu^*$ .

We already establish that once below  $\mu^*$  the agent always choose  $a = 0$ , which implies he don't update his belief anymore and therefore always choose  $a = 0$ . Since  $n^*$  is a finite number the probability of observing the bad signal  $n^*$  times is positive and this part of the statement is proven.

We are left to prove that there is a probability that the belief never falls below  $\mu^*$ .

I now define a new state:

$$s_t = (\# \text{ Good Signals}) - (\# \text{ Bad Signals})$$

Also define  $P_\infty(s)$  as the probability that at state  $s$  the agent only plays  $a = 1$  forever. Notice that by the same argument as above, as  $s \rightarrow \infty$  the belief goes to 1 and therefore  $P_\infty(\infty) = 1$ . We have:

$$P_\infty(s) = P(y = 1)P_\infty(s + 1) + P(y = 0)P_\infty(s - 1)$$

Which follows from the definition of  $P_\infty$ . Moreover since  $\theta = 1 : P(y = 1) = \frac{3}{4}$  and  $P(y = 0) = \frac{1}{4}$ . For simplification we further assume that the prior  $\mu$  is such that at  $s = 0$  one bad return is enough to bring our belief below  $\mu^*$ , which implies that  $P_\infty(-1) = 0$ . This is WLOG because if the affirmation hold for a  $\mu$  this low is must also holds for higher  $\mu$ . Therefore:

$$\begin{aligned} P_\infty(0) &= \frac{3}{4}P_\infty(1) \\ &= \frac{3}{4}\left(\frac{3}{4}P_\infty(2) + \frac{1}{4}P_\infty(0)\right) \\ \frac{13}{16}P_\infty(0) &= \frac{9}{16}P_\infty(2) \end{aligned}$$

Therefore:

$$\begin{aligned} P_\infty(1) &= \frac{4}{3}P_\infty(0) \\ P_\infty(2) &= \frac{13}{9}P_\infty(0) \end{aligned}$$

Moreover:

$$\begin{aligned} P_\infty(n) &= \frac{3}{4}P_\infty(n + 1) + \frac{1}{4}P_\infty(n - 1) \\ \frac{3}{4}P_\infty(n + 1) &= P_\infty(n) - \frac{1}{4}P_\infty(n - 1) \\ P_\infty(n + 1) &= \frac{4}{3}P_\infty(n) - \frac{1}{3}P_\infty(n - 1) \\ P_\infty(n + 1) - \frac{1}{3}P_\infty(n) &= P_\infty(n) - \frac{1}{3}P_\infty(n - 1) \\ &= P_\infty(2) - \frac{1}{3}P_\infty(1) \\ &= \frac{13}{9}P_\infty(0) - \frac{1}{3}\frac{4}{3}P_\infty(0) \\ P_\infty(0) &= P_\infty(n + 1) - \frac{1}{3}P_\infty(n) \end{aligned}$$

As  $n \rightarrow \infty$ :

$$\begin{aligned} P_\infty(0) &= \frac{2}{3}P_\infty(\infty) \\ P_\infty(0) &= \frac{2}{3} \end{aligned}$$

Therefore the probability of playing  $a = 1$  forever when  $s = 0$ , that is the initial state, is  $\frac{2}{3}$  which concludes our proof.

#### 1.4 (d)

We can use the very same technique as before. Redefine  $P_\infty(s)$  as the probability that the agent eventually takes action 0 forever. Remember,  $P_\infty(-\infty) = 1$ .

$$\begin{aligned} P_\infty(n) &= \frac{1}{4}P_\infty(n+1) + \frac{3}{4}P_\infty(n-1) \\ P_\infty(n+1) &= 4P_\infty(n) - 3P_\infty(n-1) \\ P_\infty(n+1) - P_\infty(n) &= 3(P_\infty(n) - P_\infty(n-1)) \\ P_\infty(n+2) - P_\infty(n+1) &= 3^2(P_\infty(n) - P_\infty(n-1)) \\ P_\infty(\infty) - P_\infty(\infty) &= 3^\infty(P_\infty(n) - P_\infty(n-1)) \\ 0 &= P_\infty(n) - P_\infty(n-1) \\ P_\infty(n) &= P_\infty(n-1) \\ P_\infty(0) &= P_\infty(-1) = \dots = P_\infty(-\infty) = 1 \end{aligned}$$

## 2 Question 2

$$\begin{aligned} |V^c(\delta, \theta) - V(\delta, \theta)| &= \left| \sum_{t=0}^{\infty} \delta^t a_t^c \left( \sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i | y_1, \dots, y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left( \sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i | y_1, \dots, y_t) \right) \right| \\ &\leq \left| \sum_{t=0}^{\infty} \delta^t a_t^c \left( \sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i | y_1, \dots, y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left( \sum_{i=1}^{\infty} y_i \mu(y_i | y_1, \dots, y_t) \right) \right| + \\ &\quad \left| \sum_{t=0}^{\infty} \delta^t a_t \left( \sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i | y_1, \dots, y_t) \right) - \sum_{t=0}^{\infty} \delta^t a_t \left( \sum_{i=1}^{\infty} y_i \mu(y_i | y_1, \dots, y_t) \right) \right| \\ &\leq 2 \left( \sum_{t=0}^{\infty} \delta^{2t} \max_{y \in Y} |y|^2 \left( \sum_{i=1}^{\infty} |P_{\theta^*}(y_i | y_1, \dots, y_t) \mu(y_i | y_1, \dots, y_t)| \right)^2 \right)^{1/2} + \\ &\leq 2 \max_{y \in Y} |y| \left( \sum_{t=0}^{\infty} \delta^{2t} \right)^{1/2} \left( \sum_{t=0}^{\infty} \sup_{y \in Y} \{P_{\theta^*}(y | y_1, \dots, y_t) \mu(y | y_1, \dots, y_t)\}^2 |y| \right)^{1/2} \\ &\leq \frac{2|y|^{1/2} \max_{y \in Y} |y|}{(1 - \delta^2)^{1/2}} \left( \sum_{t=0}^{\infty} \frac{1}{2} KL(\theta^* | \mu) \right)^{1/2} \\ &\leq \frac{2|y|^{1/2} \max_{y \in Y} |y|}{(1 - \delta^2)^{1/2}} \left( \sum_{t=0}^{\infty} -\ln e^{-(1-\delta)^{2(t+1)}} \right)^{1/2} \\ &= \frac{2|y|^{1/2} \max_{y \in Y} |y|}{(1 - \delta^2)^{1/2}} \frac{1 - \delta}{(1 - (1 - \delta)^2)^{1/2}} \\ &= 2|y|^{1/2} \max_{y \in Y} |y| \left( \frac{1 - \delta}{(1 + \delta)(1 - (1 - \delta)^2)} \right)^{1/2} \end{aligned}$$

Finally, notice that  $1 - \delta \rightarrow 0$  as  $\delta \rightarrow 1$  while  $(1 + \delta)(1 - (1 - \delta)^2) \rightarrow 2$ . Therefore

$$\lim_{\delta \rightarrow 1} V^c(\delta, \theta) - V(\delta, \theta) = 0$$

### 3 Question 3

#### 3.1 (a)

For simplification:

$$\begin{aligned}P_\gamma(y) &= \mu P_\gamma(y|\theta = 1) + (1 - \mu) P_\gamma(y|\theta = -1) \\P_\gamma(1) &= \mu(1/2 + \gamma) + (1 - \mu)(1/2\gamma) \\&= 1/2 + \gamma(2\mu - 1) \\P_\gamma(0) &= 1/2 - \gamma(2\mu - 1)\end{aligned}$$

Notice:

$$\lim_{\gamma \rightarrow 0} P_\gamma(0) = \lim_{\gamma \rightarrow 0} P_\gamma(1) = \frac{1}{2}$$

The Bayesian updates are:

$$\begin{aligned}\mu_\gamma(1) &= \frac{(1/2 + \gamma)\mu}{P_\gamma(1)} \\ \mu_\gamma(0) &= \frac{(1/2 - \gamma)\mu}{P_\gamma(0)}\end{aligned}$$

Therefore:

$$\lim_{\gamma \rightarrow 0} \mu_\gamma(0) = \lim_{\gamma \rightarrow 0} \mu_\gamma(1) = \mu$$

and:

$$\lim_{\gamma \rightarrow 0} V(\mu_\gamma(0)) = \lim_{\gamma \rightarrow 0} V(\mu_\gamma(1)) = V(\mu)$$

Finally:

$$\begin{aligned}V_\gamma(\mu) &= P_\gamma(0)V(\mu_\gamma(0)) + P_\gamma(1)V(\mu_\gamma(1)) \\&= (1/2 + \gamma(2\mu - 1))V(\mu_\gamma(1)) + (1/2 - \gamma(2\mu - 1))V(\mu_\gamma(0))\end{aligned}$$

and:

$$\lim_{\gamma \rightarrow 0} V_\gamma(\mu) = V(\mu)$$

Which make the question identity equals the derivative of  $V_\gamma$  in relation to  $\gamma$ :

$$\frac{\partial V_\gamma(\mu)}{\partial \gamma} = \lim_{\gamma \rightarrow 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma}$$

We are left to prove that this derivative is zero at  $\gamma = 0$ :

$$\begin{aligned}\frac{\partial V_\gamma(\mu)}{\partial \gamma}(\gamma) &= (2\mu - 1)V(\mu_\gamma(1)) + \frac{\partial V(\mu_\gamma(1))}{\partial \gamma}(1/2 + \gamma(2\mu - 1)) - \\&\quad (2\mu - 1)V(\mu_\gamma(0)) + \frac{\partial V(\mu_\gamma(0))}{\partial \gamma}(1/2 - \gamma(2\mu - 1)) \\ \frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(1))}{\partial \gamma}(1/2) - (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(0))}{\partial \gamma}(1/2) \\&= 1/2 \left( \frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma} \right)\end{aligned}$$

But:

$$\begin{aligned}
V(\mu_\gamma(1)) &= \mu_\gamma u(a_1^*, 1) + (1 - \mu_\gamma)u(a_1^*, -1) \\
&= P_\gamma(1)^{-1}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)\mu u(a_1^*, -1)] \\
\frac{\partial V(\mu_\gamma(1))}{\partial \gamma} &= P_\gamma(1)^{-1}[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + \\
&\quad \frac{1 - 2\mu}{(1/2 + \gamma(2\mu - 1))^2}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)u(a_1^*, -1)] \\
\frac{\partial V(\mu_0(1))}{\partial \gamma} &= 2[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + (4 - 8\mu)[1/2\mu u(a_1^*, 1)]
\end{aligned}$$

Where  $a_1^*$  is the argmax of  $V$  for  $\gamma = 0$  and:

$$\begin{aligned}
V(\mu_\gamma(0)) &= P_\gamma(0)^{-1}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)\mu u(a_0^*, -1)] \\
\frac{\partial V(\mu_\gamma(0))}{\partial \gamma} &= P_\gamma(0)^{-1}[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + \\
&\quad \frac{2\mu - 1}{(1/2 - \gamma(2\mu - 1))^2}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)u(a_0^*, -1)] \\
\frac{\partial V(\mu_0(0))}{\partial \gamma} &= 2[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + (8\mu - 4)[1/2\mu u(a_0^*, 1)]
\end{aligned}$$

And since  $a_1^* \rightarrow a_0^*$  as  $\gamma \rightarrow 0$ , by what we have seen before, we have:

$$\begin{aligned}
\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma} &= 0 \\
\frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= 0 \\
\lim_{\gamma \rightarrow 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma} &= 0
\end{aligned}$$

■

### 3.2 (b)

The agent choose  $\gamma$  to maximize:

$$\max_{\gamma} V_\gamma(\mu) - \beta\gamma \quad (2)$$

Notice that as long as  $\frac{\partial V_\gamma}{\partial \gamma}(\mu) < \beta$  the agent has no incentive to invest in learning. Moreover, as  $\mu \rightarrow 0, \gamma \rightarrow 0$  since at  $\mu = 0$  you have no incentive to learn ( $\gamma = 0$ ) and  $\frac{\partial V_\gamma}{\partial \gamma}(\mu)$  is continuous and  $\mu$ .

From question (a) we have that:  $\frac{\partial V_\gamma}{\partial \gamma}(\mu) \rightarrow 0$  as  $\gamma \rightarrow 0$ , therefore:  $\exists \bar{\mu}$  such that  $\forall \mu \in [0, \bar{\mu}] \frac{\partial V_\gamma}{\partial \gamma}(\mu) < \beta$ , which imply that the optimal choice of  $\gamma$  is 0.

It is also easy to see that if  $\gamma = 0$  the agent doesn't update and then  $\mu_t = \mu_0$  for all  $t$ . Which concludes our proof.

### 3.3 (c)

I assume that  $1/4 < \mu_0 < 3/4$  and that the agent is still myopic. Define:

$$u(a, \theta) = \frac{3}{16}a\theta \quad (3)$$

Which is 0 as long as the agent choose no  $a = 0$ , positive as long as he chooses  $a = 1$  and the state  $\theta = 1$ , negative otherwise. Therefore if  $\mu > 1/2$  he chooses  $a = 1$ , but choose  $a = 0$  otherwise.

His expected payoff is:

$$\begin{aligned}
E(u(a, \theta)|\mu) - \gamma^3 &= \\
&= P(y = 1|\mu, \gamma) \max\{\mathbb{E}(u(1, \theta)|\mu_\gamma(1)), 0\} + \\
&\quad P(y = 0|\mu, \gamma) \max\{\mathbb{E}(u(1, \theta)|\mu_\gamma(0)), 0\} \\
&= [\mu(1/2 + \gamma) + (1 - \mu)(1/2 - \gamma)] \frac{3}{16} \frac{\mu - 1/2 + \gamma}{1/2 + 2\gamma\mu - \gamma} - \gamma^3 \\
&= [1/2 + 2\gamma\mu - \gamma] \frac{3}{16} \frac{\mu - 1/2 + \gamma}{1/2 + 2\gamma\mu - \gamma} - \gamma^3 \\
&= \frac{3}{16} [\mu - 1/2 + \gamma] - \gamma^3
\end{aligned} \tag{4}$$

FOC is:

$$\begin{aligned}
3/16 - 3\gamma^2 &= 0 \\
\gamma^2 &= \frac{1}{16} \\
\gamma &= \frac{1}{4}
\end{aligned} \tag{5}$$

And therefore by Berk Theorem he learns the true state with probability 1.

## 4 Question 4

### 4.1 (a)

$$\begin{aligned}
\phi_{\theta', \theta}^t(y_1, \dots, y_t) &= \log \frac{\mu^t(\theta'|y_1, \dots, y_t)}{\mu^t(\theta|y_1, \dots, y_t)} \\
&= \log \frac{\mu(\theta')P'_\theta(y_1, \dots, y_t)}{\mu(\theta)P_\theta(y_1, \dots, y_t)}
\end{aligned} \tag{6}$$

$$\begin{aligned}
P_\theta(y_1, \dots, y_t) &= \pi_0(y_1) \prod_{\tau=1}^{t-1} P_\theta(y_{\tau+1}|y_\tau) \\
\phi_{\theta', \theta}^t(y_1, \dots, y_t) &= \log \frac{\mu(\theta')}{\mu(\theta)} + \sum_{\tau=1}^{t-1} \log \frac{P'_\theta(y_{\tau+1}|y_\tau)}{P_\theta(y_{\tau+1}|y_\tau)}
\end{aligned} \tag{7}$$

$$\sum_{\tau=1}^{t-1} \log \frac{P'_\theta(y_{\tau+1}|y_\tau)}{P_\theta(y_{\tau+1}|y_\tau)} = \sum_{y, y'} (t+1) \rho_{y, y'}(y_1, \dots, y_t) \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)}$$

But since  $\rho_{y, y'}(y_1, \dots, y_t) = \rho_{y, y'}(y'_1, \dots, y'_t)$  we have  $\mu^t(\theta|y_1, \dots, y_t) = \mu^t(\theta|y'_1, \dots, y'_t) \forall \theta \in \Theta$ .

### 4.2 (b)

$$\begin{aligned}
\rho_{y, y'}(y_1, \dots, y_t) &\xrightarrow{P} \theta^*(y, y') = \pi^*(y) P_{\theta^*}(y'|y) \\
\phi_{\theta', \theta}^t(y_1, \dots, y_t) &= \log \frac{\mu(\theta')}{\mu(\theta)} + \sum_{y, y'} (t-1) \rho_{y, y'} \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)} \\
&= \log \frac{\mu(\theta')}{\mu(\theta)} + \sum_y \sum_{y'} (t-1) \rho_{y, y'} \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)}
\end{aligned} \tag{8}$$

But:

$$\begin{aligned}
\sum_{y'} (t-1) \rho_{y, y'} \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)} &\rightarrow \sum_{y'} \pi^*(y) P_{\theta^*}(y'|y) \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)} \\
&= \pi^*(y) (KL(P_{\theta^*}(\cdot|y)|P_\theta(\cdot|y)) - KL(P_{\theta^*}(\cdot|y)|P_\theta(\cdot|y)))
\end{aligned} \tag{9}$$

Therefore:

$$\sum_y \sum_{y'} (t-1) P_{y,y'} \log \frac{P'_\theta(y'|y)}{P_\theta(y'|y)} \rightarrow \sum_y \pi^*(y) (KL(P_{\theta^*}(\cdot|y)|P_\theta(\cdot|y)) - \sum_y KL(P_\theta^*(\cdot|y)|P_\theta(\cdot|y))) \quad (10)$$

And so for  $\theta \notin \Theta^*$  and  $\theta' \in \Theta^*$   $\phi_{\theta',\theta}^t \rightarrow \infty$  which implies that  $\mu(\Theta^*|y_1, \dots, y_t) \xrightarrow{P} 1$

## 5 Question 5

### 5.1 (a)

I assume that  $y^1, y^2 \in \{0, 1\}$ .

The agent choose alpha to maximize:

$$\begin{aligned} & \max_{\alpha \in [0,1]} \alpha \mathbb{E}(y^1) + (1-\alpha) \mathbb{E}(y^2) \\ & \max_{\alpha \in [0,1]} \alpha(1/2\mu + 1/4) + (1-\alpha)(3/4 - 1/2\mu) \\ & \max_{\alpha \in [0,1]} \alpha(\mu - 1/2) + (3/4 - 1/2\mu) \end{aligned} \quad (11)$$

So it is clear that as long as  $\mu > 1/2$  the best choice is  $\alpha = 1$ , and as long as  $\mu < 1/2$  the best choice is  $\alpha = 0$ , is  $\mu = 1/2$ , the agent is indifferent between all  $\alpha$ .

Let  $y$  denote the overall payoff, we have the following updating rules, for the relevant  $\alpha$ :

$$\begin{aligned} & \text{For } \alpha = 1: \\ & P(\theta = 1|y = 1) = \frac{3\mu}{1+2\mu} \\ & P(\theta = 1|y = 0) = \frac{\mu}{3-2\mu} \\ & \text{symmetrically, if } \alpha = 0: \\ & P(\theta = 1|y = 1) = \frac{\mu}{3-2\mu} \\ & P(\theta = 1|y = 0) = \frac{3\mu}{1+2\mu} \end{aligned} \quad (12)$$

With  $\alpha = 1$  there is only two outcomes and updating rules. We can now define  $s_t$  as the difference between the number of positive outcomes and the number of negative outcomes. And  $P_\infty(s)$  as the probability of playing  $\alpha$  forever at state  $s$ .

As before, since  $\mu \rightarrow 1$  as  $s \rightarrow \infty$  we have that  $P_\infty(\infty) = 1$ . Therefore we can use exactly the same argument as Q1 to prove that  $P_\infty(0) > 0$ .

As an addendum, this are all the updating rules:

$$\begin{aligned}
P(\theta = 1|y = 1) &= \frac{P(y = 1|\theta = 1)\mu}{P(y = 1)} \\
&= \frac{P(y^1 = 1, y^2 = 1|\theta = 1)\mu}{P(y^1, y^2 = 1)} \\
&= \frac{3/4 * 1/4\mu}{(3/4)(1/4)\mu + (1/4)(3/4)(1 - \mu)} \\
&= \frac{3/16\mu}{3/16} \\
&= \mu
\end{aligned}$$

Similarly:

$$\begin{aligned}
P(\theta = 1|y = \alpha) &= \frac{9\mu}{8\mu + 1} \\
P(\theta = 1|y = 1 - \alpha) &= \frac{\mu}{9 - 8\mu} \\
P(\theta = 1|y = 0) &= \mu
\end{aligned}$$

But, if  $\alpha = 1/2$  we have:

$$\begin{aligned}
P(\theta = 1|y = 1/2) &= \frac{[P(y^1 = 1, y^0 = 0|\theta = 1) + P(y^1 = 0, y^0 = 1|\theta = 1)]\mu}{P(y^1 = 1, y^0 = 0) + P(y^1 = 1, y^0 = 0)} \\
&= \frac{10\mu}{(1 + 2\mu)^2 + (3 - 2\mu)^2}
\end{aligned}$$

(13)