Game Theory: Homework

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1 Question 1

1.1 (a)

We can rewrite $V(\mu)$ as:

$$
\begin{aligned}
V(\mu) &= (1-\delta)V^*(\mu) \\
V^*(\mu) &= \max_{a\in\{0,1\}}\{(\mathbb{E}(y_t) - \frac{1}{2})a + \delta\mathbb{E}(V^*(\mu'))\} \\
&= \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta\mathbb{E}(V^*(\mu'))\}
\end{aligned} \tag{1}
$$

where:

$$
\begin{aligned}
\mu' &= \mu \ \texttt{ if } \ a = 0 \\
\mu' &= \frac{\frac{3}{4}\mu}{\frac{1}{4}+\frac{1}{2}\mu} \ \texttt{ if } \ a = 1, y = 1 \\
\mu' &= \frac{\frac{1}{4}\mu}{\frac{3}{4}-\frac{1}{2}\mu} \ \texttt{ if } \ a = 1, y = 0
\end{aligned}
$$

Now we can use the standard Recursive Dynamic proof, define:

$$T(f)(x) = \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta f(\mu')\}$$

T maps weakly increasing functions in weakly increasing functions because μ' is stricly increasing in μ and $(\frac{1}{2}\mu - \frac{1}{4})a$ is stricly increasing as long as $a = 1$ and constant as long as $a = 0$.

Therefore by the Contraction Mapping Theorem the fix point of T, which is V^*, is weakly increasing. Moreover, since V^* is weakly increasing and $(1-\delta) > 0$, then V is weakly increasing.

1.2 b

First notice that we can use the same argument of (a) to show that V^* is continuous.

Now notice that for $\mu = 0$, $\mu' = 0\ \forall a, y$ and for $\mu = 1$, $\mu' = 1 \forall a, y$, therefore:

$$
\begin{aligned}
V^*(0) &= \max_{a\in\{0,1\}}\{-\frac{1}{4}a + \delta V^*(0)\} \\
&= \frac{\max_{a\in\{0,1\}}\{-\frac{1}{4}a\}}{(1-\delta)} \\
&= 0 \\
V(0) &= 0
\end{aligned}
\qquad
\begin{aligned}
V^*(1) &= \max_{a\in\{0,1\}}\{\frac{1}{4}a + \delta V^*(1)\} \\
&= \frac{\max_{a\in\{0,1\}}\{\frac{1}{4}a\}}{(1-\delta)} \\
&= \frac{1}{4(1-\delta)} > 0 \\
V(1) &= \frac{1}{4}
\end{aligned}
$$

Since V is weakly increasing that must be $\mu^* \geq 0$ such that for for all $0 \geq \mu < \mu^*, V(\mu) = 0$, and for all $1 \geq \mu > \mu, V(\mu) > 0$.

Moreover, $0 \leq \mu < \mu^*$, optimal a is 0 and for $\mu^* < \mu \leq 1$, optimal a is 1. Since V^* is continuous at μ^* the agent is indifferent.

2 Question 2

$$\begin{aligned}
|V^c(\delta,\theta)-V(\delta,\theta)| &= \left|\sum_{t=0}^{\infty}\delta^t a_t^c\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t)\right)-\sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t)\right)\right| \\
&\leq \left|\sum_{t=0}^{\infty}\delta^t a_t^c\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t)\right)-\sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\dots,y_t)\right)\right| + \\
&\quad \left|\sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\dots,y_t)\right)-\sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\dots,y_t)\right)\right| \\
&\leq 2\left(\sum_{t=0}^{\infty}\delta^{2t}\max_{y\in Y} y^2\left(\sum_{i=1}^{\infty}|P_{\theta^*}(y_i|y_1,\dots,y_t)\mu(y_i|y_1,\dots,y_t)|\right)^2\right)^{1/2} + \\
&\leq 2\max_{y\in Y}|y|^2\left(\sum_{t=0}^{\infty}\delta^{2t}\right)^{1/2}\left(\sum_{t=0}^{\infty}\sup_{y\in Y}\{P_{\theta^*}(y|y_1,\dots,y_t)\mu(y|y_1,\dots,y_t)\}^2|y|\right)^{1/2} \\
&\leq \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\left(\sum_{t=0}^{\infty}\frac{1}{2}KL(\theta^*|\mu)\right)^{1/2} \\
&\leq \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\left(\sum_{t=0}^{\infty}-\ln e^{-(1-\delta)^{2(t+1)}}\right)^{1/2} \\
&= \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\frac{1-\delta}{(1-(1-\delta)^2)^{1/2}} \\
&= 2|y|^{1/2}\max_{y\in Y}|y|^2\left(\frac{1-\delta}{(1+\delta)(1-(1-\delta)^2)}\right)^{1/2}
\end{aligned}$$

Finally, notice that $1-\delta\to 0$ as $\delta\to 1$ while $(1+\delta)(1-(1-\delta)^2)\to 2$. Therefore

$$\lim_{\delta\to 1} V^c(\delta,\theta)-V(\delta,\theta))=0$$

3 Question 3

For simplification:

$$\begin{aligned}
P_\gamma(y) &= \mu P_\gamma(y|\theta=1)+(1-\mu)P_\gamma(y|\theta=-1) \\
P_\gamma(1) &= \mu(1/2+\gamma)+(1-\mu)(1/2\gamma) \\
&= 1/2+\gamma(2\mu-1) \\
P_\gamma(0) &= 1/2-\gamma(2\mu-1)
\end{aligned}$$

Notice:

$$\lim_{\gamma\to 0}P_\gamma(0)=\lim_{\gamma\to 0}P_\gamma(1)=\frac{1}{2}$$

The Baseyian updates are:

$$\mu_\gamma(1)=\frac{(1/2+\gamma)\mu}{P_\gamma(1)}$$

$$\mu_\gamma(0)=\frac{(1/2-\gamma)\mu}{P_\gamma(0)}$$

Therefore:

$$\lim_{\gamma\to 0}\mu_\gamma(0)=\lim_{\gamma\to 0}\mu_\gamma(1)=\mu$$

and:

$$\lim_{\gamma\to 0} V(\mu_\gamma(0)) = \lim_{\gamma\to 0} V(\mu_\gamma(1)) = V(\mu)$$

Finally:

$$\begin{aligned} V_\gamma(\mu) &= P_\gamma(0)V(\mu_\gamma(0)) + P_\gamma(1)V(\mu_\gamma(1)) \\ &= (1/2 + \gamma(2\mu - 1))V(\mu_\gamma(1)) + (1/2 - \gamma(2\mu - 1))V(\mu_\gamma(0)) \end{aligned}$$

and:

$$\lim_{\gamma\to 0} V_\gamma(\mu) = V(\mu)$$

Whice make the question identity equals the derivative of V_γ in relation to γ:

$$\frac{\partial V_\gamma(\mu)}{\partial \gamma} = \lim_{\gamma\to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma}$$

We are left to prove that this derivative is zero at $\gamma = 0$:

$$\begin{aligned} \frac{\partial V_\gamma(\mu)}{\partial \gamma}(\gamma) &= (2\mu - 1)V(\mu_\gamma(1)) + \frac{\partial V(\mu_\gamma(1))}{\partial \gamma}(1/2 + \gamma(2\mu - 1)) - \\ &\quad (2\mu - 1)V(\mu_\gamma(0)) + \frac{\partial V(\mu_\gamma(0))}{\partial \gamma}(1/2 - \gamma(2\mu - 1)) \\ \frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(1))}{\partial \gamma}(1/2) - (2\mu - 1)V(\mu) + \frac{\partial V(\mu_0(0))}{\partial \gamma}(1/2) \\ &= 1/2\left(\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma}\right) \end{aligned}$$

But:

$$\begin{aligned} V(\mu_\gamma(1)) &= \mu_\gamma u(a_1^*, 1) + (1 - \mu_\gamma)u(a_1^*, -1) \\ &= P_\gamma(1)^{-1}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)\mu u(a_1^*, -1)] \\ \frac{\partial V(\mu_\gamma(1))}{\partial \gamma} &= P_\gamma(1)^{-1}[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + \\ &\quad \frac{1 - 2\mu}{(1/2 + \gamma(2\mu - 1))^2}[(1/2 + \gamma)\mu u(a_1^*, 1) + \gamma(2\mu - 2)u(a_1^*, -1)] \\ \frac{\partial V(\mu_0(1))}{\partial \gamma} &= 2[\mu u(a_1^*) + (2\mu - 2)u(a_1^*, -1)] + (4 - 8\mu)[1/2\mu u(a_1^*, 1)] \end{aligned}$$

where a_1^* is the argmax of V for $\gamma = 0$ and:

$$\begin{aligned} V(\mu_\gamma(0)) &= P_\gamma(0)^{-1}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)\mu u(a_0^*, -1)] \\ \frac{\partial V(\mu_\gamma(0))}{\partial \gamma} &= P_\gamma(0)^{-1}[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + \\ &\quad \frac{2\mu - 1}{(1/2 - \gamma(2\mu - 1))^2}[(1/2 - \gamma)\mu u(a_0^*, 1) - \gamma(2\mu - 2)u(a_0^*, -1)] \\ \frac{\partial V(\mu_0(0))}{\partial \gamma} &= 2[-\mu u(a_0^*) - (2\mu - 2)u(a_0^*, -1)] + (8\mu - 4)[1/2\mu u(a_0^*, 1)] \end{aligned}$$

and since $a_1^* \to a_0^*$ as $\gamma \to 0$, by what we have seem before, we have:

$$\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma} = 0$$
$$\frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) = 0$$
$$\lim_{\gamma \to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma} = 0$$

■

3.1 b

From (a) we can conclude: