Game Theory: Homework

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1 Question 1

1.1 (a)

We can rewrite $V(\mu)$ as:

$$\begin{aligned}
V(\mu) &= (1-\delta)V^*(\mu) \\
V^*(\mu) &= \max_{a\in\{0,1\}}\{(\mathbb{E}(y_t) - \frac{1}{2})a + \delta\mathbb{E}(V^*(\mu'))\} \\
&= \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta\mathbb{E}(V^*(\mu'))\}
\end{aligned} \tag{1}$$

where:

$$\mu' = \mu \;\texttt{if}\; a = 0$$
$$\mu' = \frac{\frac{3}{4}\mu}{\frac{1}{4}+\frac{1}{2}\mu} \;\texttt{if}\; a = 1, y = 1$$
$$\mu' = \frac{\frac{1}{4}\mu}{\frac{3}{4}-\frac{1}{2}\mu} \;\texttt{if}\; a = 1, y = 0$$

Now we can use the standard Recursive Dynamic proof, define:

$$T(f)(x) = \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu - \frac{1}{4})a + \delta f(\mu')\}$$

T maps weakly increasing functions in weakly increasing functions because μ' is stricly increasing in μ and $(\frac{1}{2}\mu - \frac{1}{4})a$ is stricly increasing as long as $a = 1$ and constant as long as $a = 0$.

Therefore by the Contraction Mapping Theorem the fix point of T, which is V^*, is weakly increasing. Moreover, since V^* is weakly increasing and $(1-\delta) > 0$, then V is weakly increasing.

1.2 (b)

First notice that we can use the same argument of (a) to show that V^* is continuous.

Now notice that for $\mu = 0$, $\mu' = 0\ \forall a, y$ and for $\mu = 1$, $\mu' = 1 \forall a, y$, therefore:

$$\begin{aligned}
V^*(0) &= \max_{a\in\{0,1\}}\{-\frac{1}{4}a + \delta V^*(0)\} \\
&= \frac{\max_{a\in\{0,1\}}\{-\frac{1}{4}a\}}{(1-\delta)} \\
&= 0 \\
V(0) &= 0
\end{aligned}
\qquad
\begin{aligned}
V^*(1) &= \max_{a\in\{0,1\}}\{\frac{1}{4}a + \delta V^*(1)\} \\
&= \frac{\max_{a\in\{0,1\}}\{\frac{1}{4}a\}}{(1-\delta)} \\
&= \frac{1}{4(1-\delta)} > 0 \\
V(1) &= \frac{1}{4}
\end{aligned}$$

Since V is weakly increasing that must be $\mu^* \geq 0$ such that for for all $0 \geq \mu < \mu^*, V(\mu) = 0$, and for all $1 \geq \mu > \mu, V(\mu) > 0$.

Moreover, $0 \leq \mu < \mu^*$, optimal a is 0 and for $\mu^* < \mu \leq 1$, optimal a is 1. Since V^* is continuous at μ^* the agent is indifferent.

1.3 (c)

First notice that the statement is not true for $\mu = 1$, in this case the agent never update his belief and always choose $a = 1$.

However, for any other μ, such that $\mu^* < \mu < 1$, there exists finite number of bad signals in sequence that is enough to bring it down to a number below μ^*. To see that define: μ_0^n the belief after observing $y = 0, n$ times in a row. It easy to show that:

$$\mu_0^n = \frac{\mu}{3^n - \sum_{i=0}^{n}(3^{i-1}2)\mu}$$

It is also simple to show that $\mu_0^n \to 0$ as $n \to \infty$. Therefore: we have for any $\epsilon > 0$, in particular, for $\epsilon = \mu^*, \exists n^* \in \mathcal{N}$ such that for any $n > n^*, \mu_0^n < \mu^*$.

We already stablish that one below μ^* the agent always choose $a = 0$, which implies he don't update his belief anymore and therefore always choose $a = 0$. Since n^* is a finite number the probability of observing the bad signal n^* times is positive and this part of the statement is proven.

We are left to prove that there is a probability that the belief never falls below μ^*.

I now define a new state:

$$s_t = (\texttt{\# Good Signals}) - (\texttt{\# Bad Signals})$$

Also define $P_\infty(s)$ as the probability that at state s the agent only plays $a = 1$ forever. Notice that by the same argument as above, as $s \to \infty$ the belief goes to 1 and therefore $P_\infty(\infty) = 1$. We have:

$$P_\infty(s) = P(y = 1)P_\infty(s + 1) + P(y = 0)P_\infty(s - 1)$$

Which follows from the definition of P_∞. Moreover, since $\theta = 1, P(y = 1) = \frac{3}{4}$ and $P(y = 0) = \frac{1}{4}$. For simplification we further assume that the prior μ is such that at $s = 0$ one bad return is enough to bring our belief below μ^*, which implies that $P_\infty(-1) = 0$. This is WLOG because if the affirmation hold for a μ this low is must also holds for higher μs Therefore:

$$\begin{aligned}
P_\infty(0) &= \frac{3}{4}P_\infty(1) \\
&= \frac{3}{4}(\frac{3}{4}P_\infty(2) + \frac{1}{4}P_\infty(0)) \\
\frac{13}{16}P_\infty(0) &= \frac{13}{9}P_\infty(2)
\end{aligned}$$

Therefore:

$$\begin{aligned}
P_\infty(1) &= \frac{4}{3}P_\infty(0) \\
P_\infty(2) &= \frac{13}{9}P_\infty(0)
\end{aligned}$$

Moreover:

$$\begin{aligned}
P_\infty(n) &= \frac{3}{4}P_\infty(n + 1) + \frac{1}{4}P_\infty(n - 1) \\
\frac{3}{4}P_\infty(n + 1) &= P_\infty(n) - \frac{1}{4}P_\infty(n - 1) \\
P_\infty(n + 1) &= \frac{4}{3}P_\infty(n) - \frac{1}{3}P_\infty(n - 1) \\
P_\infty(n + 1) - \frac{1}{3}P_\infty(n) &= P_\infty(n) - \frac{1}{3}P_\infty(n - 1) \\
&= P_\infty(2) - \frac{1}{3}P_\infty(1) \\
&= \frac{13}{9}P_\infty(0) - \frac{1}{3}\frac{3}{4}P_\infty(0)) \\
P_\infty(0) &= P_\infty(n + 1) - \frac{1}{3}P_\infty(n)
\end{aligned}$$

As $n \to \infty$:

$$P_\infty(0) = \frac{2}{3}P_\infty(\infty)$$
$$P_\infty(0) = \frac{2}{3}$$

Therefore the probability of playing $a = 1$ forever when $s = 0$, that is the initial state, is $\frac{2}{3}$ which concludes our proof.

1.4 (d)

We can use the very same technique as before. Redefine $P_\infty(s)$ as the probability that the agent eventually takes action 0 forever. Remember, $P_\infty(-\infty) = 1$.

$$\begin{aligned}
P_\infty(n) &= \frac{1}{4}P_\infty(n+1) + \frac{3}{4}P_\infty(n-1) \\
P_\infty(n+1) &= 4P_\infty(n) - 3P_\infty(n-1) \\
P_\infty(n+1) - P_\infty(n) &= 3(P_\infty(n) - P_\infty(n-1)) \\
P_\infty(n+2) - P_\infty(n+1) &= 3^2(P_\infty(n) - P_\infty(n-1)) \\
P_\infty(\infty) - P_\infty(\infty) &= 3^\infty(P_\infty(n) - P_\infty(n-1)) \\
0 &= P_\infty(n) - P_\infty(n-1) \\
P_\infty(n) &= P_\infty(n-1) \\
P_\infty(0) &= P_\infty(-1) = \ldots = P_\infty(-\infty) = 1
\end{aligned}$$

2 Question 2

$$\begin{aligned}
|V^c(\delta,\theta) - V(\delta,\theta)| &= \left|\sum_{t=0}^{\infty}\delta^t a_t^c\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right)\right| \\
&\leq \left|\sum_{t=0}^{\infty}\delta^t a_t^c\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\ldots,y_t)\right)\right| + \\
&\qquad \left|\sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty}\delta^t a_t\left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\ldots,y_t)\right)\right| \\
&\leq 2\left(\sum_{t=0}^{\infty}\delta^{2t}\max_{y\in Y} y^2\left(\sum_{i=1}^{\infty}|P_{\theta^*}(y_i|y_1,\ldots,y_t)\mu(y_i|y_1,\ldots,y_t)|\right)^2\right)^{1/2} + \\
&\leq 2\max_{y\in Y}|y|^2\left(\sum_{t=0}^{\infty}\delta^{2t}\right)^{1/2}\left(\sum_{t=0}^{\infty}\sup_{y\in Y}\{P_{\theta^*}(y|y_1,\ldots,y_t)\mu(y|y_1,\ldots,y_t)\}^2|y|\right)^{1/2} \\
&\leq \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\left(\sum_{t=0}^{\infty}\frac{1}{2}KL(\theta^*|\mu)\right)^{1/2} \\
&\leq \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\left(\sum_{t=0}^{\infty} - \ln e^{-(1-\delta)^{2(t+1)}}\right)^{1/2} \\
&= \frac{2|y|^{1/2}\max_{y\in Y}|y|^2}{(1-\delta^2)^{1/2}}\frac{1-\delta}{(1-(1-\delta)^2)^{1/2}} \\
&= 2|y|^{1/2}\max_{y\in Y}|y|^2\left(\frac{1-\delta}{(1+\delta)(1-(1-\delta)^2)}\right)^{1/2}
\end{aligned}$$

Finally, notice that $1-\delta \to 0$ as $\delta \to 1$ while $(1+\delta)(1-(1-\delta)^2) \to 2$. Therefore

$$\lim_{\delta\to 1} V^c(\delta,\theta) - V(\delta,\theta)) = 0$$

3 Question 3

For simplification:

$$\begin{aligned}
P_\gamma(y) &= \mu P_\gamma(y|\theta = 1) + (1-\mu)P_\gamma(y|\theta = -1) \\
P_\gamma(1) &= \mu(1/2+\gamma) + (1-\mu)(1/2\gamma) \\
&= 1/2 + \gamma(2\mu - 1) \\
P_\gamma(0) &= 1/2 - \gamma(2\mu - 1)
\end{aligned}$$

Notice:

$$\lim_{\gamma\to 0} P_\gamma(0) = \lim_{\gamma\to 0} P_\gamma(1) = \frac{1}{2}$$

The Baseyian updates are:

$$\mu_\gamma(1) = \frac{(1/2+\gamma)\mu}{P_\gamma(1)}$$

$$\mu_\gamma(0) = \frac{(1/2-\gamma)\mu}{P_\gamma(0)}$$

Therefore:

$$\lim_{\gamma\to 0} \mu_\gamma(0) = \lim_{\gamma\to 0} \mu_\gamma(1) = \mu$$

and:

$$\lim_{\gamma\to 0} V(\mu_\gamma(0)) = \lim_{\gamma\to 0} V(\mu_\gamma(1)) = V(\mu)$$

Finally:

$$\begin{aligned}
V_\gamma(\mu) &= P_\gamma(0)V(\mu_\gamma(0)) + P_\gamma(1)V(\mu_\gamma(1)) \\
&= (1/2+\gamma(2\mu-1))V(\mu_\gamma(1)) + (1/2-\gamma(2\mu-1))V(\mu_\gamma(0))
\end{aligned}$$

and:

$$\lim_{\gamma\to 0} V_\gamma(\mu) = V(\mu)$$

Whice make the question identity equals the derivative of V_γ in relation to γ:

$$\frac{\partial V_\gamma(\mu)}{\partial\gamma} = \lim_{\gamma\to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma}$$

We are left to prove that this derivative is zero at $\gamma = 0$:

$$\begin{aligned}
\frac{\partial V_\gamma(\mu)}{\partial\gamma}(\gamma) &= (2\mu-1)V(\mu_\gamma(1)) + \frac{\partial V(\mu_\gamma(1))}{\partial\gamma}(1/2+\gamma(2\mu-1)) - \\
&\qquad (2\mu-1)V(\mu_\gamma(0)) + \frac{\partial V(\mu_\gamma(0))}{\partial\gamma}(1/2-\gamma(2\mu-1)) \\
\frac{\partial V_\gamma(\mu)}{\partial\gamma}(0) &= (2\mu-1)V(\mu) + \frac{\partial V(\mu_0(1))}{\partial\gamma}(1/2) - (2\mu-1)V(\mu) + \frac{\partial V(\mu_0(0))}{\partial\gamma}(1/2) \\
&= 1/2\left(\frac{\partial V(\mu_0(0))}{\partial\gamma} + \frac{\partial V(\mu_0(1))}{\partial\gamma}\right)
\end{aligned}$$

But:

$$
\begin{aligned}
V(\mu_\gamma(1)) &= \mu_\gamma u(a_1^*, 1) + (1-\mu_\gamma)u(a_1^*, -1) \\
&= P_\gamma(1)^{-1}[(1/2+\gamma)\mu u(a_1^*, 1) + \gamma(2\mu-2)\mu u(a_1^*, -1)] \\
\frac{\partial V(\mu_\gamma(1))}{\partial \gamma} &= P_\gamma(1)^{-1}[\mu u(a_1^*) + (2\mu-2)u(a_1^*, -1)] + \\
&\qquad \frac{1-2\mu}{(1/2+\gamma(2\mu-1))^2}[(1/2+\gamma)\mu u(a_1^*, 1) + \gamma(2\mu-2)u(a_1^*, -1)] \\
\frac{\partial V(\mu_0(1))}{\partial \gamma} &= 2[\mu u(a_1^*) + (2\mu-2)u(a_1^*, -1)] + (4-8\mu)[1/2\mu u(a_1^*, 1)]
\end{aligned}
$$

where a_1^* is the argmax of V for $\gamma = 0$ and:

$$
\begin{aligned}
V(\mu_\gamma(0)) &= P_\gamma(0)^{-1}[(1/2-\gamma)\mu u(a_0^*, 1) - \gamma(2\mu-2)\mu u(a_0^*, -1)] \\
\frac{\partial V(\mu_\gamma(0))}{\partial \gamma} &= P_\gamma(0)^{-1}[-\mu u(a_0^*) - (2\mu-2)u(a_0^*, -1)] + \\
&\qquad \frac{2\mu-1}{(1/2-\gamma(2\mu-1))^2}[(1/2-\gamma)\mu u(a_0^*, 1) - \gamma(2\mu-2)u(a_0^*, -1)] \\
\frac{\partial V(\mu_0(0))}{\partial \gamma} &= 2[-\mu u(a_0^*) - (2\mu-2)u(a_0^*, -1)] + (8\mu-4)[1/2\mu u(a_0^*, 1)]
\end{aligned}
$$

and since $a_1^* \to a_0^*$ as $\gamma \to 0$, by what we have seem before, we have:

$$
\begin{aligned}
\frac{\partial V(\mu_0(0))}{\partial \gamma} + \frac{\partial V(\mu_0(1))}{\partial \gamma} &= 0 \\
\frac{\partial V_\gamma(\mu)}{\partial \gamma}(0) &= 0 \\
\lim_{\gamma \to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma} &= 0
\end{aligned}
$$

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3.1 b

From (a) we can conclude: