Game Theory: Homework

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1 Question 1

1.1 (a)

We can rewrite $V(\mu)$ as:

$$\begin{aligned} V(\mu) &= (1-\delta)V^*(\mu) \\ V^*(\mu) &= \max_{a\in\{0,1\}}\{(\mathbb{E}(y_t)-\frac{1}{2})a+\delta\mathbb{E}(V^*(\mu'))\} \\ &= \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu-\frac{1}{4})a+\delta\mathbb{E}(V^*(\mu'))\} \end{aligned} \tag{1}$$

where:

$$\mu' = \mu \texttt{ if } a=0$$
$$\mu' = \frac{\frac{3}{4}\mu}{\frac{1}{4}+\frac{1}{2}\mu} \texttt{ if } a=1, y=1$$
$$\mu' = \frac{\frac{1}{4}\mu}{\frac{3}{4}-\frac{1}{2}\mu} \texttt{ if } a=1, y=0$$

Now we can use the standard Recursive Dynamic proof, define:

$$T(f)(x) = \max_{a\in\{0,1\}}\{(\frac{1}{2}\mu-\frac{1}{4})a+\delta f(\mu')\}$$

T maps weakly increasing functions in weakly increasing functions because μ' is stricly increasing in μ and $(\frac{1}{2}\mu-\frac{1}{4})a$ is stricly increasing as long as $a=1$ and constant as long as $a=0$.

Therefore by the Contraction Mapping Theorem the fix point of T, which is V^*, is weakly increasing. Moreover, since V^* is weakly increasing and $(1-\delta)>0$, then V is weakly increasing.

1.2 (b)

First notice that we can use the same argument of (a) to show that V^* is continuous.

Now notice that for $\mu=0$, $\mu'=0\ \forall a,y$ and for $\mu=1$, $\mu'=1\forall a,y$, therefore:

$$\begin{aligned} V^*(0) &= \max_{a\in\{0,1\}}\{-\frac{1}{4}a+\delta V^*(0)\} \\ &= \frac{\max_{a\in\{0,1\}}\{-\frac{1}{4}a\}}{(1-\delta)} \\ &= 0 \\ V(0) &= 0 \end{aligned}$$

$$\begin{aligned} V^*(1) &= \max_{a\in\{0,1\}}\{\frac{1}{4}a+\delta V^*(1)\} \\ &= \frac{\max_{a\in\{0,1\}}\{\frac{1}{4}a\}}{(1-\delta)} \\ &= \frac{1}{4(1-\delta)} > 0 \\ V(1) &= \frac{1}{4} \end{aligned}$$

Since V is weakly increasing that must be $\mu^*\geq 0$ such that for for all $0\geq\mu<\mu^*, V(\mu)=0$, and for all $1\geq\mu>\mu, V(\mu)>0$.

Moreover, $0\leq\mu<\mu^*$, optimal a is 0 and for $\mu^*<\mu\leq 1$, optimal a is 1. Since V^* is continuous at μ^* the agent is indifferent.

1.3 (c)

First notice that the statement is not true for $\mu = 1$, in this case the agent never update his belief and always choose $a = 1$.

However, for any other μ, such that $\mu^* < \mu < 1$, there exists finite number of bad signals in sequence that is enough to bring it down to a number below μ^*. To see that define: μ_0^n the belief after observing $y = 0, n$ times in a row. It easy to show that:

$$\mu_0^n = \frac{\mu}{3^n - \sum_{i=0}^{n}(3^{i-1}2)\mu}$$

It is also simple to show that $\mu_0^n \to 0$ as $n \to \infty$. Therefore: we have for any $\epsilon > 0$, in particular, for $\epsilon = \mu^*, \exists n^* \in \mathbb{N}$ such that for any $n > n^*, \mu_0^n < \mu^*$.

We already stablish that once below μ^* the agent always choose $a = 0$, which implies he don't update his belief anymore and therefore always choose $a = 0$. Since n^* is a finite number the probability of observing the bad signal n^* times is positive and this part of the statement is proven.

We are left to prove that there is a probability that the belief never falls below μ^*.

I now define a new state:

$$s_t = (\texttt{\# Good Signals}) - (\texttt{\# Bad Signals})$$

Also define $P_\infty(s)$ as the probability that at state s the agent only plays $a = 1$ forever. Notice that by the same argument as above, as $s \to \infty$ the belief goes to 1 and therefore $P_\infty(\infty) = 1$. We have:

$$P_\infty(s) = P(y = 1)P_\infty(s + 1) + P(y = 0)P_\infty(s - 1)$$

Which follows from the definition of P_∞. Moreover since $\theta = 1 : P(y = 1) = \frac{3}{4}$ and $P(y = 0) = \frac{1}{4}$. For simplification we further assume that the prior μ is such that at $s = 0$ one bad return is enough to bring our belief below μ^*, which implies that $P_\infty(-1) = 0$. This is WLOG because if the affirmation hold for a μ this low is must also holds for higher μ. Therefore:

$$\begin{aligned}
P_\infty(0) &= \frac{3}{4}P_\infty(1) \\
&= \frac{3}{4}(\frac{3}{4}P_\infty(2) + \frac{1}{4}P_\infty(0)) \\
\frac{13}{16}P_\infty(0) &= \frac{9}{16}P_\infty(2)
\end{aligned}$$

Therefore:

$$\begin{aligned}
P_\infty(1) &= \frac{4}{3}P_\infty(0) \\
P_\infty(2) &= \frac{13}{9}P_\infty(0)
\end{aligned}$$

Moreover:

$$\begin{aligned}
P_\infty(n) &= \frac{3}{4}P_\infty(n+1) + \frac{1}{4}P_\infty(n-1) \\
\frac{3}{4}P_\infty(n+1) &= P_\infty(n) - \frac{1}{4}P_\infty(n-1) \\
P_\infty(n+1) &= \frac{4}{3}P_\infty(n) - \frac{1}{3}P_\infty(n-1) \\
P_\infty(n+1) - \frac{1}{3}P_\infty(n) &= P_\infty(n) - \frac{1}{3}P_\infty(n-1) \\
&= P_\infty(2) - \frac{1}{3}P_\infty(1) \\
&= \frac{13}{9}P_\infty(0) - \frac{1}{3}\frac{4}{3}P_\infty(0)) \\
P_\infty(0) &= P_\infty(n+1) - \frac{1}{3}P_\infty(n)
\end{aligned}$$

As $n \to \infty$:

$$P_\infty(0) = \frac{2}{3} P_\infty(\infty)$$
$$P_\infty(0) = \frac{2}{3}$$

Therefore the probability of playing $a = 1$ forever when $s = 0$, that is the initial state, is $\frac{2}{3}$ which concludes our proof.

1.4 (d)

We can use the very same technique as before. Redefine $P_\infty(s)$ as the probability that the agent eventually takes action 0 forever. Remember, $P_\infty(-\infty) = 1$.

$$\begin{aligned}
P_\infty(n) &= \frac{1}{4} P_\infty(n+1) + \frac{3}{4} P_\infty(n-1) \\
P_\infty(n+1) &= 4P_\infty(n) - 3P_\infty(n-1) \\
P_\infty(n+1) - P_\infty(n) &= 3(P_\infty(n) - P_\infty(n-1)) \\
P_\infty(n+2) - P_\infty(n+1) &= 3^2(P_\infty(n) - P_\infty(n-1)) \\
P_\infty(\infty) - P_\infty(\infty) &= 3^\infty(P_\infty(n) - P_\infty(n-1)) \\
0 &= P_\infty(n) - P_\infty(n-1) \\
P_\infty(n) &= P_\infty(n-1) \\
P_\infty(0) &= P_\infty(-1) = \ldots = P_\infty(-\infty) = 1
\end{aligned}$$

2 Question 2

$$\begin{aligned}
|V^c(\delta,\theta) - V(\delta,\theta)| &= \left|\sum_{t=0}^{\infty} \delta^t a_t^c \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right)\right| \\
&\le \left|\sum_{t=0}^{\infty} \delta^t a_t^c \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\ldots,y_t)\right)\right| + \\
&\quad \left|\sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i P_{\theta^*}(y_i|y_1,\ldots,y_t)\right) - \sum_{t=0}^{\infty} \delta^t a_t \left(\sum_{i=1}^{\infty} y_i \mu(y_i|y_1,\ldots,y_t)\right)\right| \\
&\le 2\left(\sum_{t=0}^{\infty} \delta^{2t} \max_{y\in Y} |y|^2 \left(\sum_{i=1}^{\infty} |P_{\theta^*}(y_i|y_1,\ldots,y_t)\mu(y_i|y_1,\ldots,y_t)|\right)^2\right)^{1/2} + \\
&\le 2\max_{y\in Y}|y| \left(\sum_{t=0}^{\infty}\delta^{2t}\right)^{1/2} \left(\sum_{t=0}^{\infty} \sup_{y\in Y}\{P_{\theta^*}(y|y_1,\ldots,y_t)\mu(y|y_1,\ldots,y_t)\}^2|y|\right)^{1/2} \\
&\le \frac{2|y|^{1/2}\max_{y\in Y}|y|}{(1-\delta^2)^{1/2}} \left(\sum_{t=0}^{\infty}\frac{1}{2}KL(\theta^*|\mu)\right)^{1/2} \\
&\le \frac{2|y|^{1/2}\max_{y\in Y}|y|}{(1-\delta^2)^{1/2}} \left(\sum_{t=0}^{\infty} -\ln e^{-(1-\delta)^{2(t+1)}}\right)^{1/2} \\
&= \frac{2|y|^{1/2}\max_{y\in Y}|y|}{(1-\delta^2)^{1/2}} \frac{1-\delta}{(1-(1-\delta)^2)^{1/2}} \\
&= 2|y|^{1/2}\max_{y\in Y}|y| \left(\frac{1-\delta}{(1+\delta)(1-(1-\delta)^2)}\right)^{1/2}
\end{aligned}$$

Finally, notice that $1-\delta \to 0$ as $\delta \to 1$ while $(1+\delta)(1-(1-\delta)^2) \to 2$. Therefore

$$\lim_{\delta\to 1} V^c(\delta,\theta) - V(\delta,\theta)) = 0$$

3 Question 3

3.1 (a)

For simplification:

$$\begin{aligned}
P_\gamma(y) &= \mu P_\gamma(y|\theta = 1) + (1-\mu)P_\gamma(y|\theta = -1) \\
P_\gamma(1) &= \mu(1/2+\gamma) + (1-\mu)(1/2\gamma) \\
&= 1/2 + \gamma(2\mu - 1) \\
P_\gamma(0) &= 1/2 - \gamma(2\mu - 1)
\end{aligned}$$

Notice:

$$\lim_{\gamma\to 0} P_\gamma(0) = \lim_{\gamma\to 0} P_\gamma(1) = \frac{1}{2}$$

The Bayesian updates are:

$$\mu_\gamma(1) = \frac{(1/2+\gamma)\mu}{P_\gamma(1)}$$
$$\mu_\gamma(0) = \frac{(1/2-\gamma)\mu}{P_\gamma(0)}$$

Therefore:

$$\lim_{\gamma\to 0} \mu_\gamma(0) = \lim_{\gamma\to 0} \mu_\gamma(1) = \mu$$

and:

$$\lim_{\gamma\to 0} V(\mu_\gamma(0)) = \lim_{\gamma\to 0} V(\mu_\gamma(1)) = V(\mu)$$

Finally:

$$\begin{aligned}
V_\gamma(\mu) &= P_\gamma(0)V(\mu_\gamma(0)) + P_\gamma(1)V(\mu_\gamma(1)) \\
&= (1/2+\gamma(2\mu-1))V(\mu_\gamma(1)) + (1/2-\gamma(2\mu-1))V(\mu_\gamma(0))
\end{aligned}$$

and:

$$\lim_{\gamma\to 0} V_\gamma(\mu) = V(\mu)$$

Which make the question identity equals the derivative of V_γ in relation to γ:

$$\frac{\partial V_\gamma(\mu)}{\partial\gamma} = \lim_{\gamma\to 0} \frac{V_\gamma(\mu) - V(\mu)}{\gamma}$$

We are left to prove that this derivative is zero at $\gamma = 0$:

$$\begin{aligned}
\frac{\partial V_\gamma(\mu)}{\partial\gamma}(\gamma) &= (2\mu-1)V(\mu_\gamma(1)) + \frac{\partial V(\mu_\gamma(1))}{\partial\gamma}(1/2+\gamma(2\mu-1)) - \\
&\quad (2\mu-1)V(\mu_\gamma(0)) + \frac{\partial V(\mu_\gamma(0))}{\partial\gamma}(1/2-\gamma(2\mu-1)) \\
\frac{\partial V_\gamma(\mu)}{\partial\gamma}(0) &= (2\mu-1)V(\mu) + \frac{\partial V(\mu_0(1))}{\partial\gamma}(1/2) - (2\mu-1)V(\mu) + \frac{\partial V(\mu_0(0))}{\partial\gamma}(1/2) \\
&= 1/2\left(\frac{\partial V(\mu_0(0))}{\partial\gamma} + \frac{\partial V(\mu_0(1))}{\partial\gamma}\right)
\end{aligned}$$

But:

$$\begin{aligned}
V(\mu_\gamma(1)) &= \mu_\gamma u(a_1^*,1) + (1-\mu_\gamma)u(a_1^*,-1) \\
&= P_\gamma(1)^{-1}[(1/2+\gamma)\mu u(a_1^*,1) + \gamma(2\mu-2)\mu u(a_1^*,-1)] \\
\frac{\partial V(\mu_\gamma(1))}{\partial\gamma} &= P_\gamma(1)^{-1}[\mu u(a_1^*) + (2\mu-2)u(a_1^*,-1)] + \\
&\qquad \frac{1-2\mu}{(1/2+\gamma(2\mu-1))^2}[(1/2+\gamma)\mu u(a_1^*,1) + \gamma(2\mu-2)u(a_1^*,-1)] \\
\frac{\partial V(\mu_0(1))}{\partial\gamma} &= 2[\mu u(a_1^*) + (2\mu-2)u(a_1^*,-1)] + (4-8\mu)[1/2\mu u(a_1^*,1)]
\end{aligned}$$

Where a_1^* is the argmax of V for $\gamma = 0$ and:

$$\begin{aligned}
V(\mu_\gamma(0)) &= P_\gamma(0)^{-1}[(1/2-\gamma)\mu u(a_0^*,1) - \gamma(2\mu-2)\mu u(a_0^*,-1)] \\
\frac{\partial V(\mu_\gamma(0))}{\partial\gamma} &= P_\gamma(0)^{-1}[-\mu u(a_0^*) - (2\mu-2)u(a_0^*,-1)] + \\
&\qquad \frac{2\mu-1}{(1/2-\gamma(2\mu-1))^2}[(1/2-\gamma)\mu u(a_0^*,1) - \gamma(2\mu-2)u(a_0^*,-1)] \\
\frac{\partial V(\mu_0(0))}{\partial\gamma} &= 2[-\mu u(a_0^*) - (2\mu-2)u(a_0^*,-1)] + (8\mu-4)[1/2\mu u(a_0^*,1)]
\end{aligned}$$

And since $a_1^* \to a_0^*$ as $\gamma \to 0$, by what we have seem before, we have:

$$\begin{aligned}
\frac{\partial V(\mu_0(0))}{\partial\gamma} + \frac{\partial V(\mu_0(1))}{\partial\gamma} &= 0 \\
\frac{\partial V_\gamma(\mu)}{\partial\gamma}(0) &= 0 \\
\lim_{\gamma\to 0}\frac{V_\gamma(\mu) - V(\mu)}{\gamma} &= 0
\end{aligned}$$

■

3.2 (b)

The agent choose γ to maximize:

$$\max_\gamma V_\gamma(\mu) - \beta\gamma \tag{2}$$

Notice that as long as $\frac{\partial V_\gamma}{\partial\gamma}(\mu) < \beta$ the agent has no incentive to invest in learning. Moreover, as $\mu \to 0, \gamma \to 0$ since at $\mu = 0$ you have no incentive to learn ($\gamma = 0$) and $\frac{\partial V_\gamma}{\partial\gamma}(\mu)$ is continuous and μ.

From question (a) we have that: $\frac{\partial V_\gamma}{\partial\gamma}(\mu) \to 0$ as $\gamma \to 0$, therefore: $\exists\bar{\mu}$ such that $\forall\, \mu \in [0,\bar{\mu}]$ $\frac{\partial V_\gamma}{\partial\gamma}(\mu) < \beta$, which imply that the optimal choice of γ is 0.

It is also easy to see that if $\gamma = 0$ the agent doesn't update and then $\mu_t = \mu_0$ for all t. Which concludes our proof.

3.3 (c)

I assume that $1/4 < \mu_0 < 3/4$ and that the agent is still myopic. Define:

$$u(a,\theta) = \frac{3}{16}a\theta \tag{3}$$

Which is 0 as long as the agent choose no $a = 0$, positive as long as he chooses $a = 1$ and the state $\theta = 1$, negative otherwise. Therefore if $\mu > 1/2$ he chooses $a = 1$, but choose $a = 0$ otherwise.

His expected payoff is:

$$
\begin{aligned}
E(u(a,\theta)|\mu) - \gamma^3 = & \\
=&P(y=1|\mu,\gamma)\max\{\mathbb{E}(u(1,\theta)|\mu_\gamma(1)),0\}+ \\
&\quad P(y=0|\mu,\gamma)\max\{\mathbb{E}(u(1,\theta)|\mu_\gamma(0)),0\} \\
=&[\mu(1/2+\gamma)+(1-\mu)(1/2-\gamma)]\frac{3}{16}\frac{\mu-1/2+\gamma}{1/2+2\gamma\mu-\gamma} - \gamma^3 \\
=&[1/2+2\gamma\mu-\gamma]\frac{3}{16}\frac{\mu-1/2+\gamma}{1/2+2\gamma\mu-\gamma} - \gamma^3 \\
=&\frac{3}{16}[\mu-1/2+\gamma]-\gamma^3
\end{aligned} \tag{4}
$$

FOC is:

$$
\begin{aligned}
3/16 - 3\gamma^2 &= 0 \\
\gamma^2 &= \frac{1}{16} \\
\gamma &= \frac{1}{4}
\end{aligned} \tag{5}
$$

And therefore by Berk Theorem he learns the true state with probability 1.

4 Question 4

4.1 (a)

$$
\begin{aligned}
\phi^t_{\theta',\theta}(y_1,\dots,y_t) &= \log\frac{\mu^t(\theta'|y_1,\dots,y_t)}{\mu^t(\theta|y_1,\dots,y_t)} \\
&= \log\frac{\mu(\theta')P'_\theta(y_1,\dots,y_t)}{\mu(\theta)P_\theta(y_1,\dots,y_t)}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
P_\theta(y_1,\dots,y_t) &= \pi_0(y_1)\prod_{\tau=1}^{t-1}P_\theta(y_{t+1}|y_t) \\
\phi^t_{\theta',\theta}(y_1,\dots,y_t) &= \log\frac{\mu(\theta')}{\mu(\theta)} + \sum_{\tau=1}^{t-1}\log\frac{P'_\theta(y_{t+1}|y_t)}{P_\theta(y_{t+1}|y_t)} \\
\sum_{\tau=1}^{t-1}\log\frac{P'_\theta(y_{t+1}|y_t)}{P_\theta(y_{t+1}|y_t)} &= \sum_{y,y'}(t+1)\rho_{y,y'}(y_1,\dots,y_t)\log\frac{P'_\theta(y'|y)}{P_\theta(y'|y)}
\end{aligned} \tag{7}
$$

But since $\rho_{y,y'}(y_1,\dots,y_t) = \rho_{y,y'}(y'_1,\dots,y'_t)$ we have $\mu(\theta|y_1,\dots,y_t) = \mu(\theta|y'_1,\dots,y'_t)\forall\ \theta\in\Theta$.

4.2 (b)

$$
\rho_{y,y'}(y_1,\dots,y_t) \xrightarrow{P} \theta^*(y,y') \tag{8}
$$

5 Question 5

5.1 (a)

I assume that $y^1, y^2 \in \{0,1\}$.

Even though the agent don't observe y^1 and y^2 directly they can be inferred from the overall payoff. Since $y^1, y^2 \in \{0,1\}$, the overall payoff has 4 possible values: $\{1, \alpha, (1-\alpha), 0\}$ which one of them directly specify one of the 4 possible states realizations, that is, respectively. $\{(y^1=1, y^2=1), (y^1=$

$1, y^2 = 0), (y^1 = 0, y^2 = 1), (y^1 = 0, y^2 = 0)\}$. However, there are important exceptions, when $\alpha = 1/2$ differentiate between cases 2 and 3, when $\alpha = 1$ we don't observe y^2 and when $\alpha = 0$ we don't observe y^1

Let y denote the overall payoff, we have the follow rule:

$$
\begin{aligned}
P(\theta = 1|y = 1) &= \frac{P(y = 1|\theta = 1)\mu}{P(y = 1)} \\
&= \frac{P(y^1 = 1, y^2 = 1|\theta = 1)\mu}{P(y^1, y^2 = 1)} \\
&= \frac{3/4 * 1/4\mu}{(3/4)(1/4)\mu + (1/4)(3/4)(1-\mu)} \\
&= \frac{3/16\mu}{3/16} \\
&= \mu \\
\text{Similarly:}& \\
P(\theta = 1|y = \alpha) &= \frac{9\mu}{8\mu + 1} \\
P(\theta = 1|y = 1 - \alpha) &= \frac{\mu}{9 - 8\mu} \\
P(\theta = 1|y = 0) &= \mu \\
\text{But, if } \alpha = 1/2 \text{ we have:}& \\
P(\theta = 1|y = 1/2) &= \frac{[P(y^1 = 1, y^0 = 0|\theta = 1) + P(y^1 = 0, y^0 = 1|\theta = 1)]\mu}{P(y^1 = 1, y^0 = 0) + P(y^1 = 1, y^0 = 0} \\
&= \frac{10\mu}{(1 + 2\mu)^2 + (3 - 2\mu)^2} \\
\text{Finally, and most importantly, if } \alpha = 1:& \\
P(\theta = 1|y = 1) &= \frac{3\mu}{1 + 2\mu} \\
P(\theta = 1|y = 0) &= \frac{\mu}{1 - 2\mu} \\
\text{symmetrically, if } \alpha = 0:& \\
P(\theta = 1|y = 1) &= \frac{\mu}{1 - 2\mu} \\
P(\theta = 1|y = 0) &= \frac{3\mu}{1 + 2\mu}
\end{aligned} \tag{9}
$$

The agent choose alpha to maximize:

$$
\max_{\alpha \in [0,1]} \alpha \mathbb{E}(y^1) + (1-\alpha)\mathbb{E}(y^2) \max_{\alpha \in [0,1]} \alpha(1/2\mu + 1/4) + (1-\alpha)(3/4 - 1/2\mu) \max_{\alpha \in [0,1]} \alpha(\mu - 1/2) + (3/4 - 1/2\mu) \tag{10}
$$

So it is clears that as long as $\mu > 1/2$ the best choice is $\alpha = 1$, and as long as $\mu < 1/2$ the best choice is $\alpha = 0$, is $\mu = 1/2$, the agent is indifferent between all α.

With $\alpha = 1$ there is only two outcomes and updating rules. We can now use the same method as Q1, defining s_t as the difference between the number of positive outcomes and the number of negative outcomes. And $P_\infty(s)$ as the probability of playing α forever at state s.

As before, since $\mu \to 1$ as $s \to \infty$ we have that $P_\infty(\infty) = 1$. Therefore we can use exactly the same argument as Q1 to prove that $P_\infty(0) > 0$.