CS 660: Combinatorial Algorithms

Red-Black and B trees

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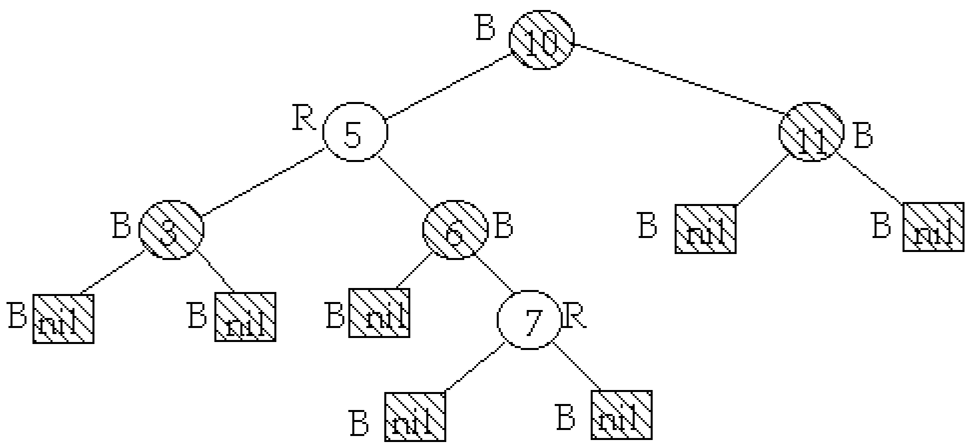
Balanced Trees

Red-Black Trees

A binary search tree is a red-black tree if:

1. Every node is either red or black
2. Every leaf (nil) is black
3. If a node is red, then both its children are black
4. Every simple path from a node to a descendant leaf contains the same number of black nodes

Black-height of a node x, bh(x), is the number of black nodes on any path from x to a leaf, not counting x



Lemma

A red-black tree with n internal nodes has height at most 2lg(n+1)

proof

Show that subtree starting at x contains at least $2^{bh(x)}-1$ internal nodes. By induction on height of x:

if x is a leaf then bh(x) = 0, $2^{bh(x)}-1$

Assume x has height h, x′s children have height h -1

x′s children black-height is either bh(x) or bh(x) -1

By induction x′s children subtree has $2^{bh(x)-1}-1$ internal nodes

So subtree starting at x contains

$2^{bh(x)-1}-1 + 2^{bh(x)-1}-1 + 1 = 2^{bh(x)}-1$ internal nodes let h = height of the tree rooted at x

bh(x) >= h/2

So n >= $2^{h/2}-1$ <=> n + 1 >= $2^{h/2}$ <=> lg(n+1) >= h/2

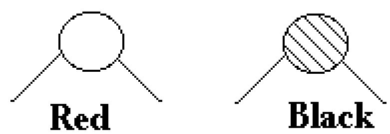
h <= 2lg(n+1)

Inserting in Red-Black Tree

Color the node Red

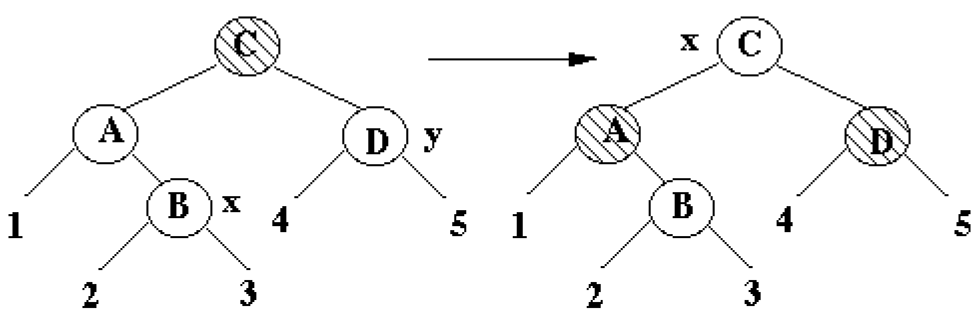
Insert as in a regular BST

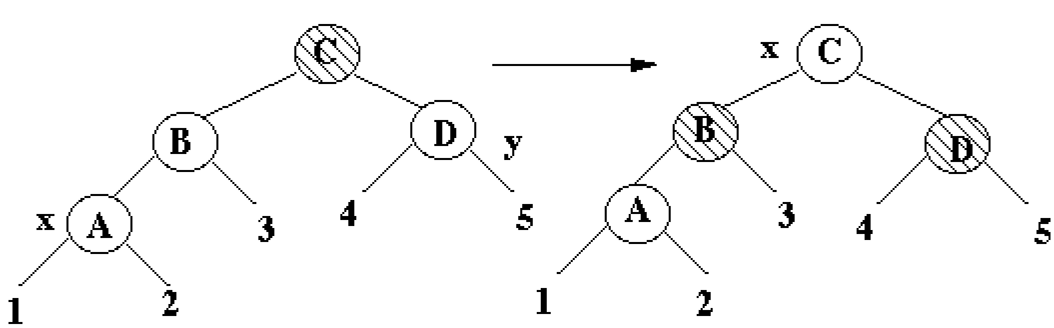
If have parent is red



Case 1

x is node of interest, x's uncle is Red

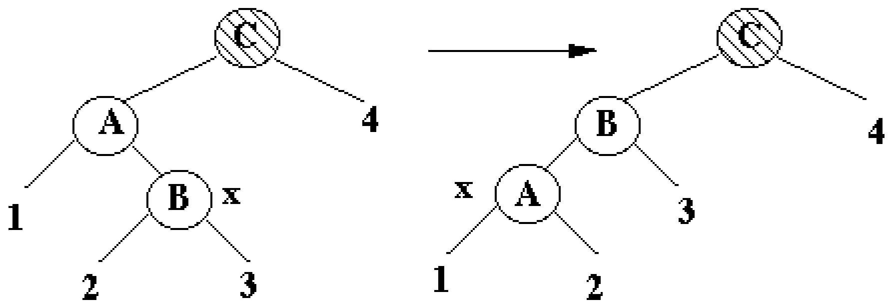




Decrease x's black height by one

Case 2

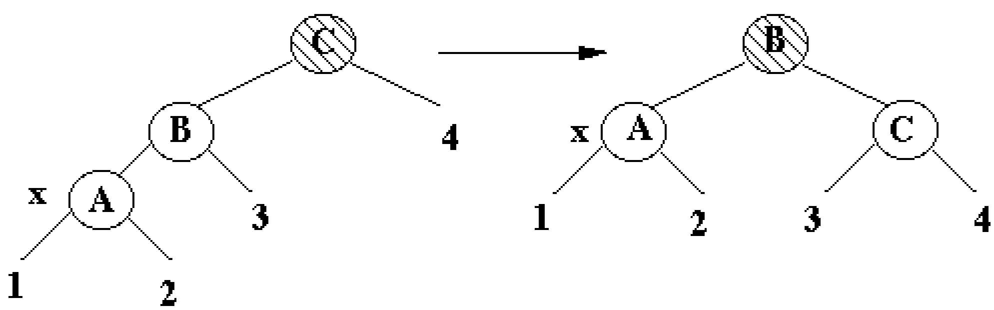
x's uncle is Black, x is a Right child



Transform to case 3

Case 3

x's uncle is Black, x is a Left child



Terminal case, tree is Red-Black tree

Insertion takes O(lg(n)) time

Requires at most two rotations

Deleting in a Red-Black Tree

Find node to delete

Delete node as in a regular BST

Node to be deleted will have at most one child

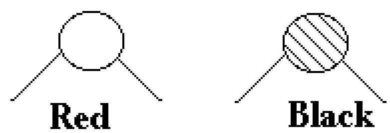
If we delete a Red node tree still is a Red-Black tree

Assume we delete a black node

Let x be the child of deleted node

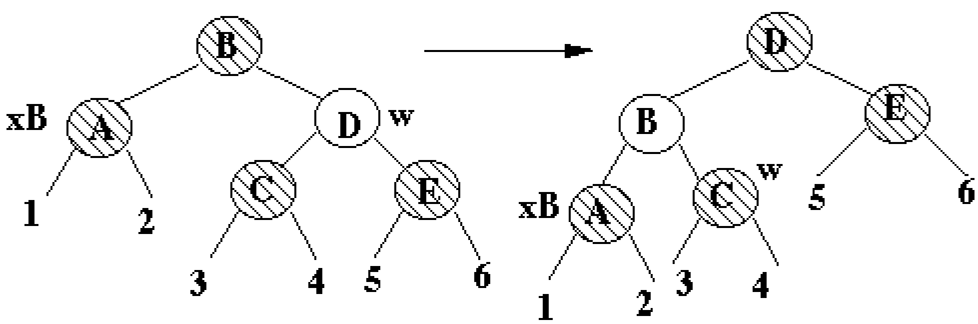
If x is red, color it black and stop

If x is black mark it double black and apply the following:



Case 1

x's sibling is red



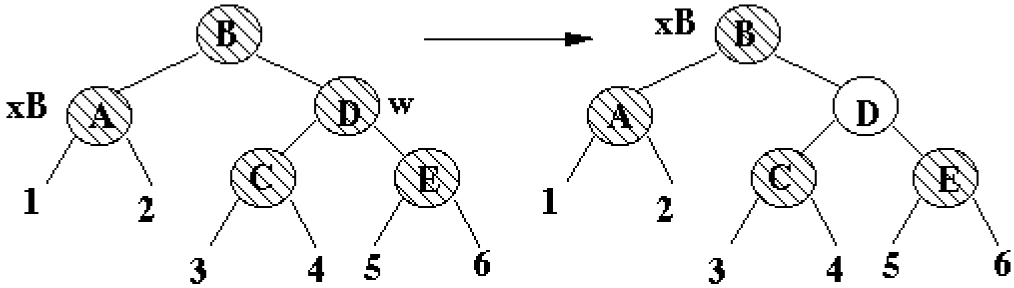
x stays at same black height

Transforms to case 2b then terminates

Case 2a

x's sibling is black

x's parent is black

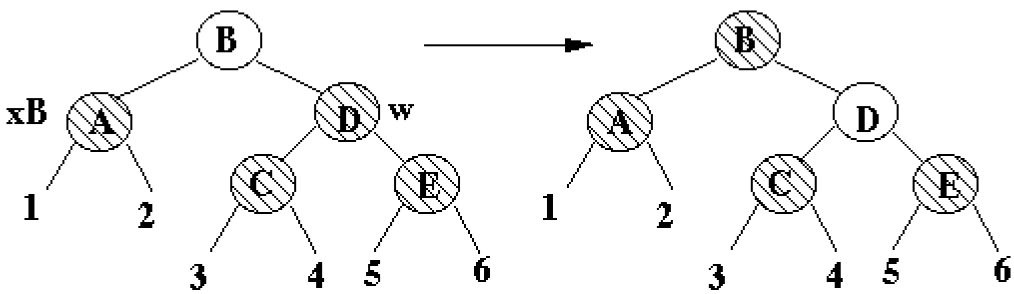


Decreases x black height by one

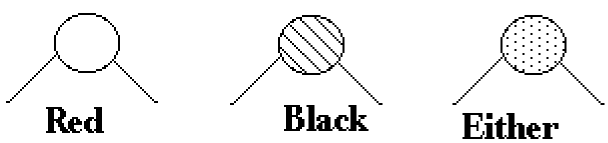
Case 2b

x's sibling is black

x's parent is red



Terminal case, tree is Red-Black tree



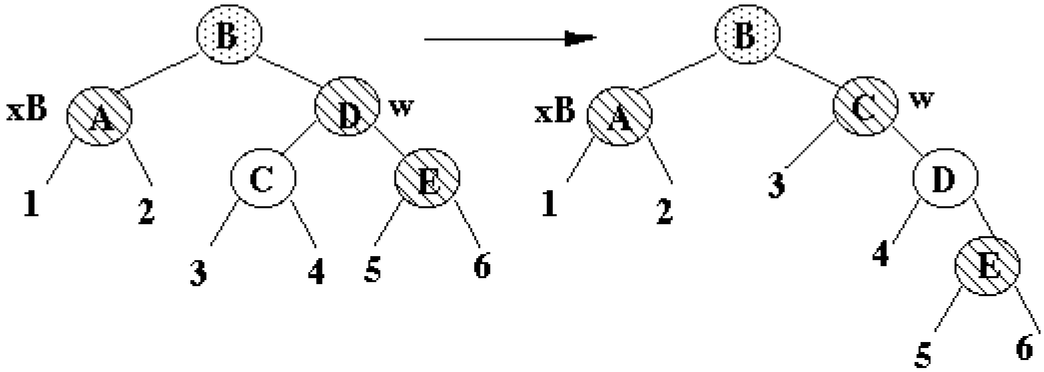
Case 3

x's sibling is black

x's parent is either

x's sibling's left child is red

x's sibling's right child is black



x stays at same black height

Transforms to case 4

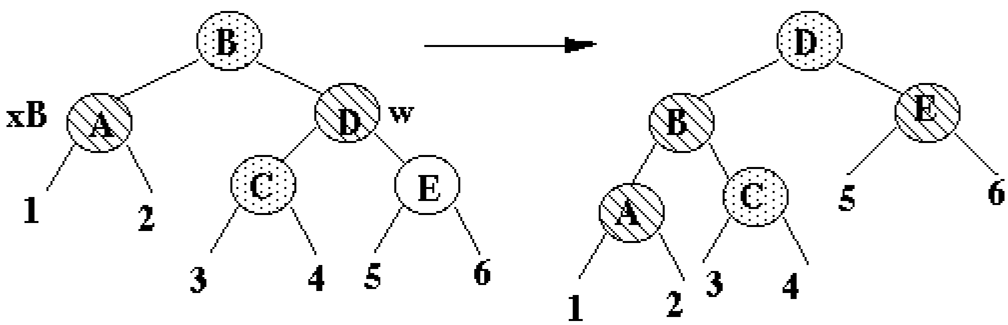
Case 4

x's sibling is black

x's parent is either

x's sibling's left child is either

x's sibling's right child is red



Terminal case, tree is Red-Black tree

Delete time is O(lg(n))

At most three rotations are done

B-Trees, (a,b)-Trees

(a,b)-Trees

Let a and b be integers with a >= 2 and 2a-1 <= b. A tree T is an (a,b)-tree if

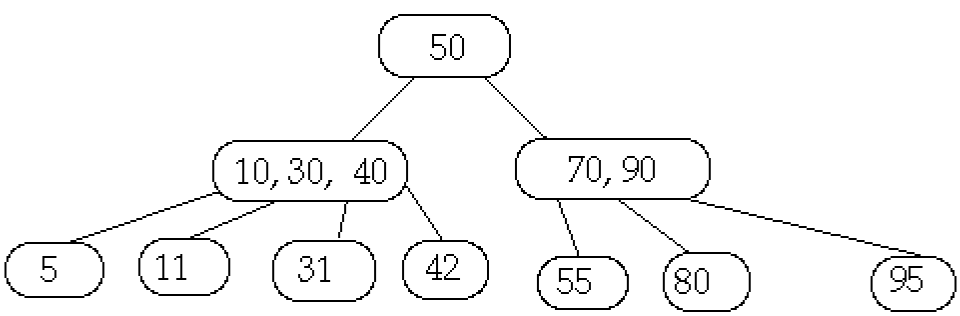
a) All leaves of T have the same depth

b) All internal nodes v of T satisfy c(v) <= b

c) All internal nodes v of T except the root satisfy c(v) >= a

d) The root of T satisfies c(v) >= 2

c(v) = number of children of node v



B-Trees of degree t

A tree T is a B-Trees of degree t if

a) All leaves of T have the same depth

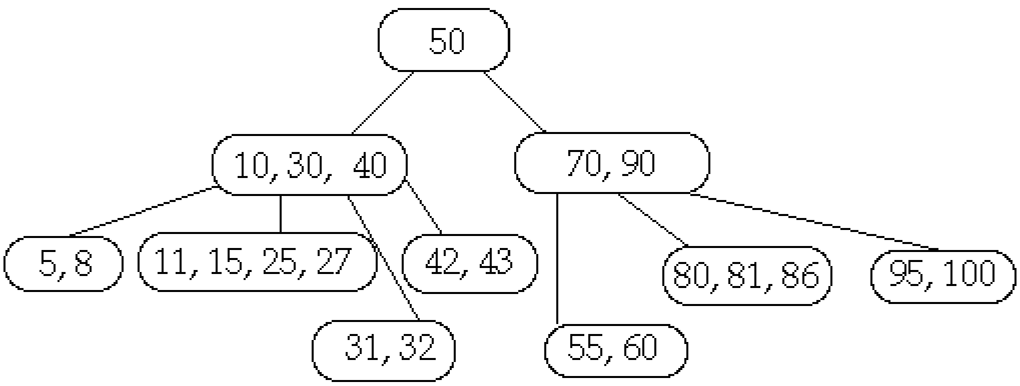
b) All nodes of T except the root have at least t-1 keys

c) All nodes of T except the root have at most 2t-1 keys

d) The root of T at least one key

e) A node with n keys has n+1 children

c(v) = number of children of node v



Theorem. If n >= 1, then for any n-key B-tree T of height *h* and degree t >= 2 then

$$\log_{2t}\left(\frac{n+1}{2}\right) \le h \le \log_t\left(\frac{n+1}{2}\right)$$

proof.

$$n \ge 1 + (t-1)\sum_{k=1}^{h} 2t^{k-1}$$

$$= 1 + 2(t-1)\left(\frac{t^h - 1}{t-1}\right)$$

$$= 2t^h - 1$$

so

$$\frac{n+1}{2} \ge t^h$$

take log of both sides.

Theorem. The worst case search time on a n-key B-tree T of degree t is O(lg(n)).

A node in T has t-1 <= K <= 2t-1 keys in sorted order.

Worst case:

K = t-1 for all nodes

searching for X not in the tree

Given a node, W, in T, how much work does it take to find the subtree of W that would contain X?

Using binary search it takes

$\lceil \log_2(K) \rceil + 1 = \lceil \log_2(K+1) \rceil = \lceil \log_2(t) \rceil$ comparisons

Since the height of the tree is in worst case $\log_t\left(\frac{n+1}{2}\right)$ the total amount of work is:

$$\begin{aligned}
\lceil \log_2(t) \rceil * \log_t\left(\frac{n+1}{2}\right) &\approx \log_2(t) * \log_t\left(\frac{n+1}{2}\right) \\
&= \log_2\left(\frac{n+1}{2}\right) \\
&= \log_2(n+1) - \log_2(2) \\
&= \log_2(n+1) - 1 \\
&= O(\log(n))
\end{aligned}$$

Insertion in a B-Tree

Inserting X into B-tree T of degree t

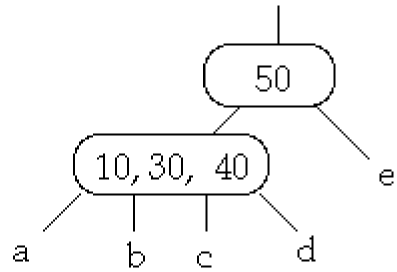
A full node is one that contains 2t-1 keys

1. Find the leaf that should contain X

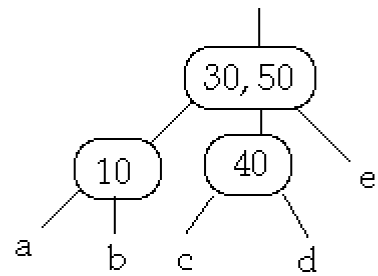
2. If the path from the root to the leaf contains a full node split the node when you first

search it.

Example t = 2, Insert 25

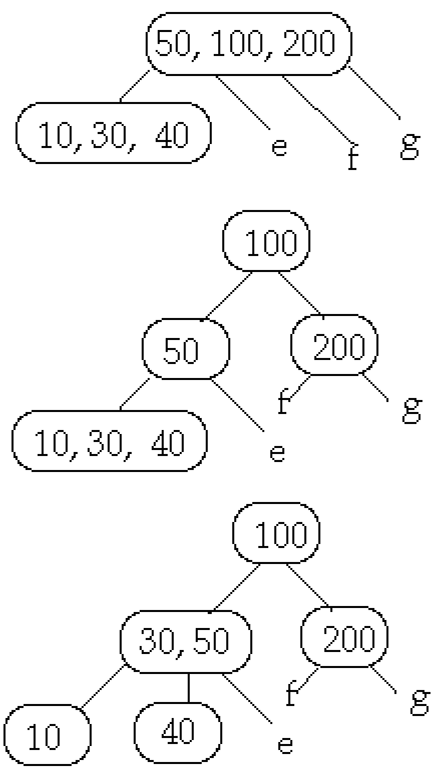


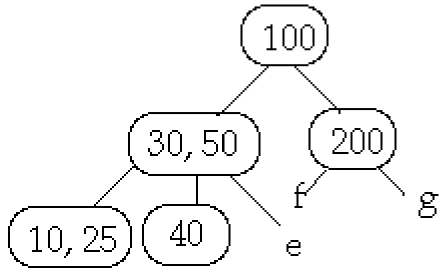
Full Node is split, Then insert 25 into subtree b



3. Insert X into the proper leaf

Example t = 2, Insert 25





Deletion in a B-Tree

Deleting X from B-tree T of degree t

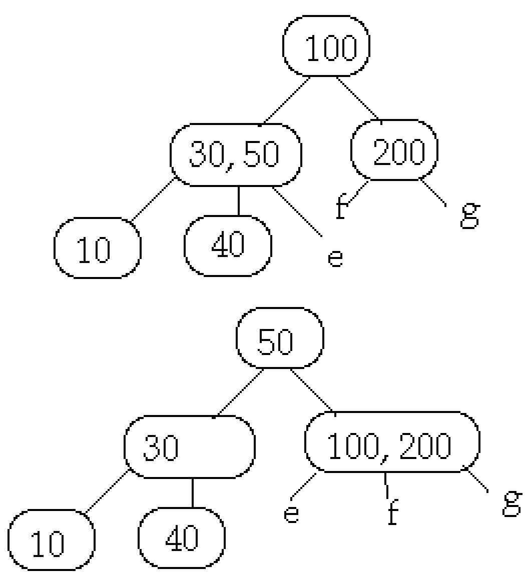
A minimal node is one that contains t-1 keys and is not the root

In the search path from the root to node containing X, if you come across a minimal node add a key to it.

Case 3. Searching node W that does not contain X. Let c be the child of W that would contain X.

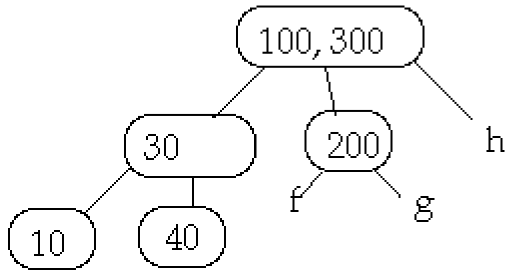
Case 3a. if c has t-1 keys and a sibling has t or more keys, steal a key from the sibling

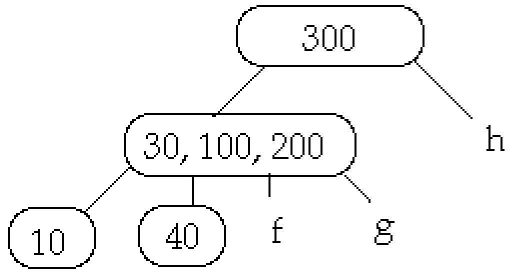
Example t = 2, Delete 250



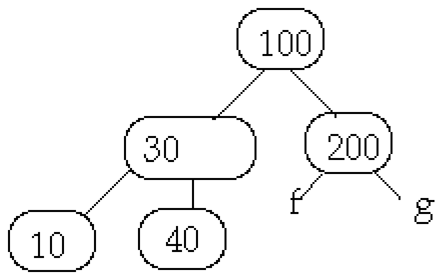
Case 3b. if c has t-1 keys and all siblings have t-1 keys, merge c with a sibling

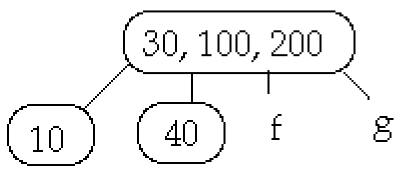
Example 1. t = 2, Delete 250





Example 2. t = 2, Delete 250

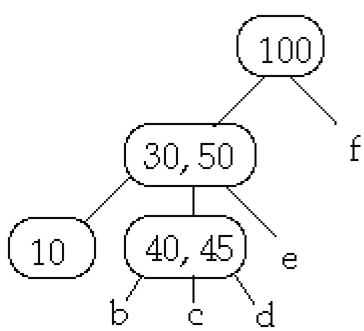




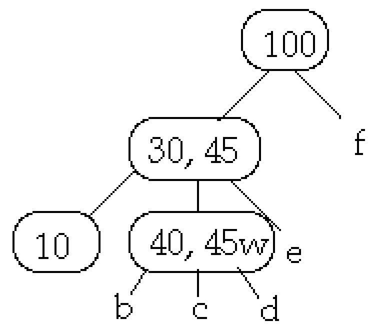
Case 2. Internal node W contains X.

Case 2a. If the child y of W that precedes X in W has at least t keys, steal predecessor of W

Example 1. t = 2, Delete 50

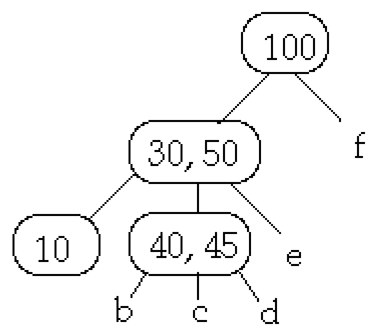


Now Delete 45w

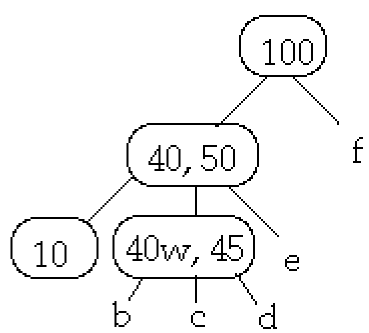


Case 2b. If the child z of W that succeed X in W has at least t keys, steal the successor of W

Example 1. t = 2, Delete 30

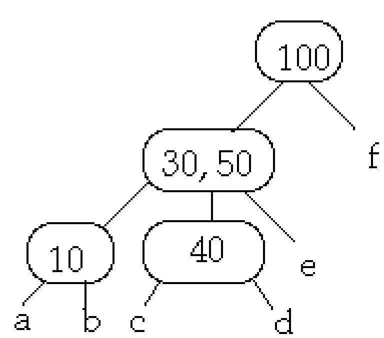


Now Delete 40w

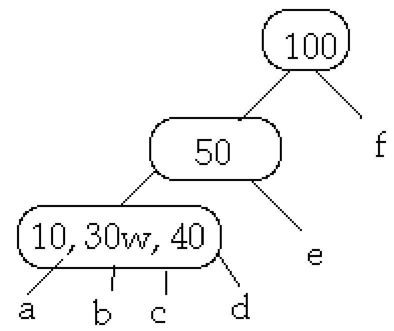


Case 2c. If both children z and y of W that succeed (follow) X in W have only t−1 keys, merge z and y

Example t = 2, Delete 30



Now Delete 30w one lower level

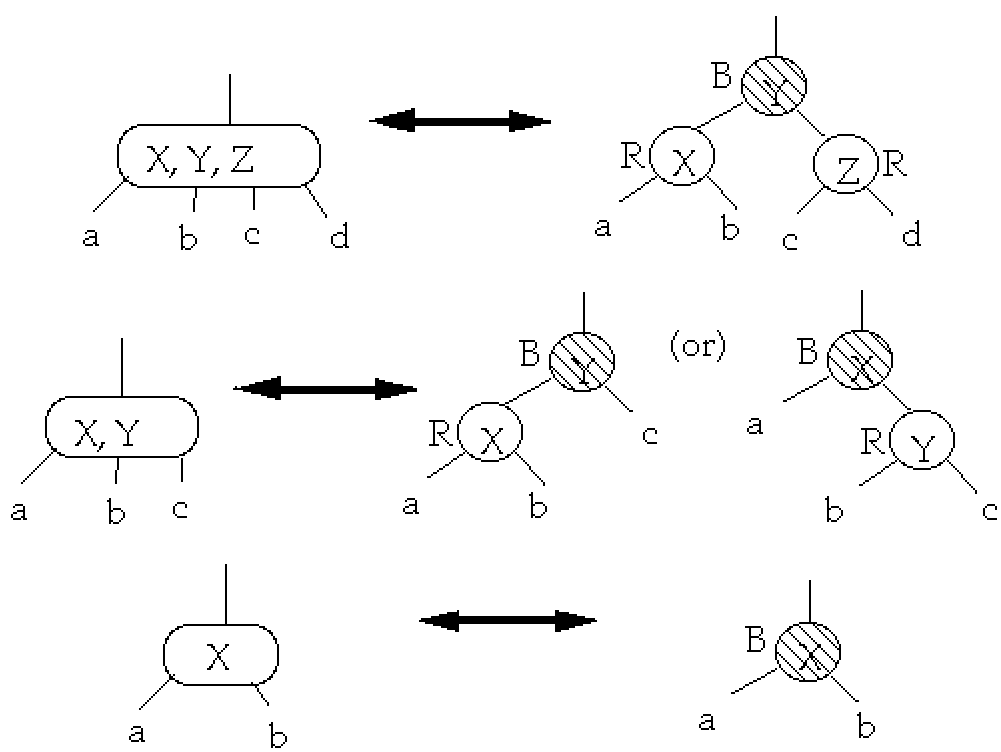


Case 1. X is in node W a leaf. By case 3, W has at least t keys. Remove X from W

B-Trees and Red-Black Trees

Theorem. A Red-Black tree is a B-Tree with degree 2

proof:



Must show:

1. If a node is red, then both its children are black

2. Every simple path from a node to a descendant leaf contains the same number of black nodes