Note on equivariant cellular and homology theory

2019/3/8

1 Recollection of some classical results in homotopy theory

1.1 Adjunction Formulas

1.1.1 Unbased Case.

Let X, Y and Z be 'good' topological spaces. Then we have adjunction formula:

$$Map(X \times Y, Z) \cong Map(X, Map(Y, Z)) \cong Map(X, Z^Y).$$

If we consider the homotopy classes, we have

$$[X \times Y, Z] \cong [X, Z^Y].$$

1.1.2 Based case.

In based case, one has two basic constructions: $X \vee Y$ and $X \wedge Y$. And one has:

$$[X \wedge Y, *; Z, *] \cong [X, *; Z^Y, *].$$

In particular, one has:

$$[S^n X, *; Y, *] \cong [X, *; \Omega Y, *].$$

1.1.3 Weak equivalence

Recall that a continuous map $f : X \to Y$ is called *weak homotopy equivalence* if $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$ induces isomorphism.

1.2 CW complex

1.2.1 Construction of CW complex

Recall that X is a CW-complex if there is a sequence:

$$X_0 \subset X_1 \subset X_2 \ ... \subset X_n \subset X_{n+1} ....$$

such that:

$$X_{n+1} = X_n \cup (\sqcup D_\alpha^{n+1}), f_\alpha : S^n \to X_n.$$

When X and Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence, *Whitehead theorem* implies f is a homotopy equivalence.

1.2.2 CW homology

For CW-homology, we have a chain group $C_n(X)$:

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}, X_{n-2})$$

Then the *CW homology group of* X is defined to be $H_n(X) := Ker\partial/Im\partial$.

2 G-CW complex and G-Whitehead theorem

2.1 Some Basic definition of G-spaces

2.1.1 G-space

Let G be a topological group and X be a topological space.

- X is a G-space, $G \curvearrowright X$, if

  $G \times X \to X$, $(g, x) \mapsto gx$ with $g_1(g_2x) = (g_1g_2)x$ and $ex = x$.

- $f : X \to Y$, f is a G-map if $f(gx) = gf(x)$.

- Let $G \curvearrowright X$, and $H \subset G$ be a closed subgroup.

  $$X^H := \{x \in X | hx = x, h \in H\}.$$

  Furthermore, $WH = NH/H \curvearrowright X^H$, where $NH := \{g \in G | gHg^{-1} = H\}$.

- Let $x \in X$, $G_x := \{g \in G | gx = x\}$ is called the isotropy group of x.

- For $G \curvearrowright X$ and $G \curvearrowright Y$, the diagonal action induces $G \curvearrowright X \times Y$.

2.1.2 Adjunction

Let X and Y be G-spaces. Now we consider the space $Map(X, Y)$.

- $MapX, Y$ is a G-space with action:

  $$g \cdot f(x) := gf(g^{-1}x).$$

  $Map(X, Y)^G := \{f | f(gx) = gf(x)\}$.

- We have G-homeomorphism:

  $$Map(X \times Y, Z) \cong Map(X, Map(Y, Z)).$$

For convenience, consider the category $G\mathcal{U}$ of G-spaces and G-maps.

- $G \curvearrowright K$ with trivial action,

$$G\mathcal{U}(K, X) \cong Map(K, X^G),$$

$$G\mathcal{U}(X, K) \cong Map(X/G, K)$$

- Let $H \subset G$, $H \curvearrowright X$ and $G \times_H Y := G \times Y/(gh, y) \sim (g, hx)$

$$H\mathcal{U}(Y, X) \cong G\mathcal{U}(G \times_H Y, X).$$

$$H\mathcal{U}(X, Y) \cong G\mathcal{U}(X, Map_H(G, Y)).$$

  where $G \curvearrowright Map_H(G, Y) := \{f | H - map, H \curvearrowright G\}$ with action $gf(g') = f(g'g)\}$.

- When $H \curvearrowright Y$ extends to $H \subset G \curvearrowright Y$, or Y is a G-space, then

$$G \times_H Y \cong_G G/H \times Y$$

$$Map_H(G, Y) \cong_G Map(G/H, X)$$

2.1.3 G-homotopy

Two G-maps $f_1, f_2 : X \to Y$ are called *G-homotopy* if there exists a G-map:

$$h : X \times [0, 1] \to Y$$

where $[0, 1]$ is the trivial G-space and $h_0 = f_1, h_1 = f_2$.Then

$G\mathcal{U}$ still denotes the category of G-spaces and G-maps.

$hG\mathcal{U}$ denotes the category of G-spaces and *homotopy classes* of G-maps.

2.1.4 Weak equivalence

Definition 2.1. *G-map $f : X \to Y$ is called weak homotopy equivalence if $f^H : X^H \to Y^H$ is an ordinary weak homotopy equivalence.*

We hope we can find a "$G{-}CW$ complex" to make weak equivalence becomes homotopy equivalence.

$\bar{h}G\mathcal{U}$ denotes the category of G-spaces and *homotopy classes* of G-maps with all weak equivalence maps invertible.

2.2 Based G-spaces

2.2.1 Based construction

- We call $G \curvearrowright X$ a *G-based space* if $\exists x \in X$, based point, with trivial action $G \curvearrowright x$.
- For unbased case $G \curvearrowright Y$, we can construct $G \curvearrowright Y_+$.
- When $G \curvearrowright X$ and $G \curvearrowright Y$ are based, so is $G \curvearrowright X \vee Y$ and $G \curvearrowright X \wedge Y$.

2.2.2 Based Adjunctions

Let $G \curvearrowright X$ and $G \curvearrowright Y$ be two G-based spaces. $F(X,Y)$ denotes based maps. In based case: we have:

- $G \curvearrowright F(X,Y)$ is a G-based space.
- $F(X \wedge Y, Z) \cong_G F(X, F(Y,Z))$.
- Denote $G\mathcal{F}$ the category of based G-spaces and based G-maps.
- $G\mathcal{F}(K,X) \cong F(K, X^G)$, $G\mathcal{F}(X,K) \cong F(X/G, K)$ when $G \curvearrowright K$ is trivial.
- When $H \curvearrowright Y$,
  $G\mathcal{F}(G_+ \wedge_H Y, X) \cong H\mathcal{F}(Y,X)$ and $H\mathcal{F}(X,Y) \cong G\mathcal{F}(X, F_H(G_+,Y))$
- If $H \subset G \curvearrowright Y$ then
  $G_+ \wedge_H Y \cong_G (G/H)_+ \wedge X$ and $F_H(G_+,Y) \cong F(G/H_+, X)$.

2.2.3 Based G-homotopy

Definition 2.2. *We call two based G-maps $f_1, f_2 : X \to Y$ based G-homotopy if $\exists$ G-based map $h : X \wedge I_+ \to Y$ with $h_0 = f_1$ and $h_1 = f_2$.*

Similarly, we can define $hG\mathcal{F}$ and $\overline{h}G\mathcal{F}$.
$[X,Y]_G := \overline{h}G\mathcal{F}(X,Y)$.

Remark 2.3. When X, Y are G-CW complexes, we hope $[X,Y]_G = hG\mathcal{F}(X,Y)$.

2.2.4 Homotopy fiber and cofiber

Let $f : X \to Y$ be a based G-map.

- homotopy fiber $F_f := \{(\omega, x) | \omega(1) = f(x)\}$ is well-defined and $G \curvearrowright F_f$.
- homotopy cofiber $C_f := Y \cup_f CX$ is well defined and $G \curvearrowright C_f$.

2.3 G-CW complex

2.3.1 Definition

Similar to ordinary CW-complex, we regard G/H as the basic 'point' :

Definition 2.4. *A G-CW complex X consists of:*

- $X_0 \subset X_1 \subset .... \subset X_n \subset X_{n+1} \subset ....$
- X_0*: disjoint union of orbits G/H.*
- $X_{n+1} = X_n \cup (\sqcup_\alpha G/H_\alpha \times D^{n+1})$ *with G-maps $G/H_\alpha \times S^n \to X_n$.*
- *G-closure finiteness and G-weak topology*

- *Note that X/G is a CW-complex*

Remark 2.5. We like to consider the case when all n-cells are finite!

Remark 2.6. If $G \curvearrowright Y$ and Y is a CW-complex, Y is not necessary a G-CW complex and X/G may not be a CW complex.

Definition 2.7. *We call a G-CW based space when X_0 is a point, every X_n are based and the attaching map $G/H_+ \wedge S^n$ are based.*

2.3.2 ν-equivalence

Recall in ordinary case, $g : X \to Y$ is n-equivalence if $\pi_i(g)$ is isomorphism when $i < n$ and epimorphism $i = n$. ν-equivalence is an equvariant generalization.

Let $G \curvearrowright X$. Consider a function:

$$\nu : \{\text{conjugacy classes of subgroups of } G\} \to \{-1, 0, 1, 2, ..., \infty\}$$

Definition 2.8. *We call $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$ equivalence for all H. $\nu(H) = -1$ when $X^H = \emptyset$.*

A G-CW complex X has dimension ν if the cells of orbit type G/H all have dimension $\leq \nu(H)$.

Remark 2.9. Here we include ∞ for the definition of weak homotopy equivalence.

Theorem 2.10 (G-Whitehead)**.** *Let $e : Y \to Z$ be a ν-equivalence and X be a G-CW complex.*

$$hG\mathcal{U}(X, Y) \to hG\mathcal{U}(X, Z).$$

is bijective if X has dimension $< \nu$ and surjective when dimension $= \nu$.

Corollary 2.11. *If $e : Y \to Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.*

When $v(H) := \infty$, $e : Y \to Z$ is a ν-equivalence is equivalent to say e is a weak equivalence. Hence, by Theorem 2.10

Corollary 2.12. *Weak homotopy equivalence $e : Y \to Z$ between G-CW complexes implies e is a G-homotopy equivalence.*

2.3.3 G-CW approximation

Theorem 2.13. *Let (X, A) and (Y, B) be relative G-CW complexes, (X', A') subcomplex of (X, A). $f : (X, A) \to (Y, B)$ be a G-map and $f|(X', A')$ is cellular. Then f is homotopic $relX' \cup A$ to a cellular map $g : (X, A) \to (Y, B)$.*

Corollary 2.14. *Any G-map $f : X \to Y$ between G-CW complexes is homotopic to a cellular map. Any two homotopic cellular maps are cellularly homotopic.*

Theorem 2.15. *For any G-space X, $\exists \gamma : \Gamma X \to X$, s.t. ΓX is G-CW and γ is weak homotopy equivalence.*

$[X, y]_G := \overline{h}G\mathcal{U}(\Gamma X, \Gamma Y) = hG\mathcal{U}(\Gamma X, \Gamma Y).$

3 G-CW Homology and cohomology

3.1 ordinary homology and cohomology theories

3.1.1 A category $\mathcal{G}$

Let $\mathcal{G}$ be a category with $ob\mathcal{G} : G/H$ and $\mathcal{G}(G/H, G/K) : G$-map.

Note that for G-map, $f : G/H \to G/K$:

$$\exists f, f(eH) = gK \Leftrightarrow g^{-1}Hg \subset K.$$

We can also define the category: $h\mathcal{G}$.

3.1.2 Coefficient System

A *Coefficient system* is a contravariant functor: $h\mathcal{G} \to \mathcal{A}b$

Example 3.1. $\underline{\pi_n}(X)$:

$$\begin{tikzcd} G/H \arrow[r] \arrow[d, "f"'] & \pi_n(X^H) \\ G/K \arrow[r] & \pi_n(X^K) \arrow[u] \end{tikzcd}$$

Note that when $f(eH) = gK$, we have $X^K \to X^H$, $x \mapsto gx$.

In fact, we have a decomposition if functors:

$$h\mathcal{G} \to h\mathcal{T} \to \mathcal{A}b$$

$$G/H \to X^H \to \pi_n(X^H).$$

Now we have a new category:$\mathcal{CS}$, whose objects are coefficient systems and morphisms are natural transformations.

Since $+$, *ker* and *coker* of coefficient systems are well-defined.

Lemma 3.2. *$\mathcal{CS}$ is an Abel category.*

Then, one can do homological algebra over $\mathcal{CS}$.

3.1.3 Cohomology

Let $G \curvearrowright X$ be a G-CW complex. Define a functor: $\underline{C}_n(X) := \underline{H}_n(X_n, X_{n-1}; \mathbb{Z})$. Then,

$$\underline{C}_n(X)(G/H) = H_n(X_n^H, X_{n-1}^H; \mathbb{Z}).$$

We have natural transformation d:

$$d : \underline{C}_n(X) \to \underline{C}_n(X),$$

$$d : \underline{H}_n(X_n, X_{n-1}; \mathbb{Z}) \to \underline{H}_n(X_{n-1}, X_{n-2}; \mathbb{Z})$$

Since $dd = 0$, then we obtain a chain complex of coefficient system $\underline{C}_*(X)$.

Let $M \in ob\mathcal{CS}$ and let

$$C^n_G(X, M) := Hom_{\mathcal{CS}}(\underline{C}_n(X), M), \delta = d^*,$$

is a cochain complex of *abelian groups*.

The cohomology is defined to be:

$$H^*_G(X; M) := Ker\delta/Im\delta.$$

3.1.4 Homology

Let $M \in ob\mathcal{CS}$, and N be convariant functor:$h\mathcal{G} \to \mathcal{A}b$. We define an **abelian group**:

$$M \otimes_{\mathcal{G}} N := (\sum M(G/H) \otimes N(G/H))/(mf^*, n) \sim (m, f_* n)$$

where $f : G/H \to G/K$.

Then we obtain a chain complex of ablelian groups:

$$C^G_n(X, N) = \underline{C}_n(X) \otimes_{\mathcal{G}} N$$

and

$$H^G_*(X; N) := H_*(C^G_n(X, N)).$$

Remark 3.3. For more details, see

3.1.5 Dimension axion

If $G \curvearrowright X = G/H$ is just 0-cell, then we have formula:

$$H^*_G(G/H, M) = H^0_G(G/H, M) \cong M(G/H).$$

$$H^G_*(G/H, N) = H^G_0(G/H, N) \cong N(G/H).$$

Remark 3.4. $\underline{H}_0(G/H)$ is projective and so is $\underline{C}_n(X)$.

References

[1] J.F. Adams, *Stable homotopy and generalised homology, Reprint of the 1974 original*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995.

[2] Bredon, *Equivariant cohomology theories*

[3] Switzer, Robert M.*Algebraic topology–homotopy and homology. Reprint of the 1975* Classics in Mathematics. Springer-Verlag, Berlin, 2002. xiv+526 pp. ISBN: 3-540-42750-3

[4] May, J. P. *Equivariant homotopy and cohomology theory.* CBMS Regional Conference Series in Mathematics, 91. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. xiv+366 pp. ISBN: 0-8218-0319-0

[5] S.J Wilson, *Equivariant homology theories on G-complexes*