An integral (co)homology theory of Coxeter orbifolds

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Abstract. We define orbifold cellular (co)homology groups with integral coefficient for Coxeter orbifold. With a little generalization, we consider a special q-CW complex, named Coxeter complex. Inspired by blowing-up, we give an explicit boundary formula for the chain complex corresponding to a Coxeter complex. The boundary formula reflects the information of local groups of singular points, which is a generalization of usual cellular boundary formula of CW complex. In the setting, the homology groups of a Coxeter complex have a nice splitting property, which is an analogue of Chen-Ruan cohomology groups and Hochster formula. As a parallel application, we also give a definition of homology groups of complex of groups defined by Haefliger.

1. Introduction

An n-orbifold is a singular space locally modelled on quotients of $\mathbb{R}^n$ by a finite group action. Up to now, many orbifold (co)homology theories of certain orbifolds (of particular, global quotients) are defined in different ways, such as, equivariant (co)homology, Cech & de Rham cohomology[**11**], Chen-Ruan cohomology[**4**], t-singular homology[**14**], loop homology[**15**], and so on.

Most of the above (co)homology theories are based on field coefficient, such as $\mathbb{R}, \mathbb{C}, \mathbb{Q}$. Concerning integral coefficient, one usually view infinite-dimensional equivalent (co)homology groups as (co)homology groups of orbifolds. Recently, using the q-CW complex structure on an orbifold, Poddar-Soumen[**18**], Bahri-Nothbohm-Sarkar-Song[**17**] consider rational or integral cohomology of toric orbifolds and weight Grassmanians. However, they do not give explicit boundary maps.

In this note, we mainly focus on *Coxeter orbifolds*[**6**], that is, an orbifold locally modelled the action of a finite Coxeter group W on $\mathbb{R}^n$ by reflections across some hyperplanes in $\mathbb{R}^n$. Specially, a *right-angled Coxeter orbifold* is locally modelled on $\mathbb{R}^n/\mathbb{Z}_2^n$. The simplest examples are simple polytopes, where a n-polytope is *simple* if its each vertex intersects with exactly n *facets* ($(n-1)$-dimensional faces). It is clear that each point in the interior of an $(n-k)$-face of a simple n-polytope can be locally modelled on $\mathbb{R}^n/W$, hence each simple polytope admits some Coxeter orbifold structures. In fact, every Coxeter orbifold has a structure of manifold with corners (see [**5**]).

Key words and phrases. Coxeter orbifold, Orbifold (co)homology group, Coxeter complex, Orbifold cellular complex.

The objects in our category, named *Coxeter complexes*, are also q-CW complexes but with a little restriction. Here all attaching maps in a Coxeter complex are required to preserve local groups. Under such condition, there exists a facial structure on the singular set of a Coxeter complex. And we can give an explicit boundary formula for each Coxeter cell by considering their blowing-ups, see Definition 3.4. However, their blowing-ups can be constructed in different ways which determines distinct definitions of boundary maps. In this note, we give two blowing-ups for an arbitrary Coxeter complex which lead to two distinct definitions of (co)homology groups of Coxeter complexes, which are denoted by H_*^{orb} and $H_*^{orb'}$, respectively. The associated boundary formulas are (3.4) and (3.5). Those orbifold homology groups can be both viewed as the generalization of cellular (co)homology groups, and some homological properties are valid as well.

Let X be a Coxeter complex. A local codimension-one face of X is called a facet of X. Denote the facet set of X by $\mathcal{F}(X) = \{F_1, \cdots, F_m\}$. Each facet $F_i \in \mathcal{F}(X)$ is labelled by a reflector s_i. Let F^{n-k} be an arbitrary $(n-k)$-face of X and $W(F^{n-k})$ be the local group of F^{n-k} which is generated by $\{s_i \mid F^{n-k} \subset F_i\}$. Then F^{n-k} is a connected component of the intersection of some facets. Without loss of generality, denoted by $\{F_1, \cdots, F_k\}$ where $F_i = F_j$ for $i \neq j$ is allowed. (If there are two facets $F_i = F_j$ and $i \neq j$, then X is not a nice manifold with corners in the sense of Davis [**5**].) Let F_β^{n-k+1} be a face which is determined by $\{F_1, \cdots, \hat{F}_\beta, \cdots, F_k\}$. Define an equivalent relation on face set of X,

$$F^{n-k} \sim F_\beta^{n-k+1} \text{ iff } \frac{|W(F^{n-k})|}{|W(F_\beta^{n-k+1})|} \mod 2 = 1.$$

Let T be the equivalent class set of face set of X. For any $J \in T$, let

$$X_J = \bigcup_{F \in J} F.$$

The main result is the following conclusion.

THEOREM 1.1. *Let X be an n-dimensional Coxeter complex. Then*

$$H_i^{orb}(X) = \bigoplus_{J \in T} H_{i-l(J)}(X_J) \tag{1.1}$$

where $l(J)$ is the codimension of the highest dimensional face in J.

REMARK 1.

- The right part of (1.1) is an analogue of Chen-Ruan cohomology groups (or quantum cohomology groups, [**4**]) of almost complex orbifolds and the Hochster formulas of moment-angle manifolds.
- When X is a right-angled Coxeter complex, $l(J)$ is the codimension of the unique face in J. Moreover, if all $|X_{(g)}|$ are acyclic, then $H_i^{orb}(X) \cong \mathbb{Z}^{f_{n-i}}$, where f_{n-i} is the number of codimension-i faces of X.

For another boundary formula (3.4), the associated orbifold homology groups contains more information than H^{orb}. Under some special cases, it is related to the weight homology of weight simplicial complex.

PROPOSITION 1.1. *Let X be a Coxeter orbifold whose nerve $\mathcal{N}(X)$ is a triangulation of the set of singular points of X. Then for $i > 0$*

$$H_i^{orb'}(X) \cong H_i(|X|) \oplus H_{i-1}^w(\mathcal{N}(X)).$$

where $|X|$ is the underlying space of X and $H_i^w(\mathcal{N}(X))$ is the weight homology of $\mathcal{N}(X)$.

Specially, H_*^{orb} and $H_*^{orb'}$ are the same when we take $\mathbb{Z}_2$-coefficient , i.e.

$$H_i^{orb}(X;\mathbb{Z}_2) \cong H_i^{orb'}(X;\mathbb{Z}_2).$$

We have the following Hurewize theorem of Coxeter complexes

Proposition 1.2 (Hurewize Theorem). *Let X be a Coxeter complex, then*

$$\left(\pi_1^{orb}(X)\right)^{ab} \cong H_1(|X|) \oplus H_1^{orb}(X/|X|, \mathbb{Z}_2).$$

Furthermore, we can define the cohomology ring $H^*_{orb}(X;G)$ of a Coxeter complex by considering its blowing-up.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, the Coxeter complexes are constructed, which is a special case of q-CW complex and an analogue of CW complex. Furthermore, according to their blowing-ups, we give an explicit boundary formula of a Coxeter cellular chain complex. Under such boundary maps, the associated cellular (co)homology groups are defined. In Section 4, we focus on the calculation of a Coxeter cellular orbifold.So the Theorem 1.1 is proved. And some examples of non-Coxeter orbifolds are given.

2. Preliminaries

2.1. Orbifolds. In this subsection, we give a brief introduction to orbifolds. One can refer to [**1, 3, 4, 12**] for details.

Let O be a Hausdorff space. A *orbifold chart* (U, G, ψ) for an open set V in O is defined as follows:

U: A connected open set in $\mathbb{R}^n$;
G: A finite group of linear automorphisms of U;
ψ: The quotient map induced by the action of G on U.

Definition 2.1. An n-dimensional *orbifold* consists of a Hausdorff paracompact topological space $|O|$, named *underlying space*, together with an orbifold structure $\mathcal{U} = \{(U, G, \psi)\}$.

Let p be a point in an orbifold $(O, \mathcal{U})$. Then there is a chart $(U, G, \psi) \in \mathcal{U}$ such that $p \in \psi(U)$. Then the isotropy group of $p' \in \psi^{-1}(p)$ is called the *local group* at p.

Definition 2.2 (Thurston [**13**, Definition 13.2.2]). A *covering orbifold* of an orbifold $\mathcal{O}$ is an orbifold $\widetilde{\mathcal{O}}$ with a projection $\pi : \widetilde{\mathcal{O}} \to \mathcal{O}$, satisfying that:

- $\forall\ x \in \mathcal{O}$ has a neighborhood V which is identified with an open subset U of $\mathbb{R}^n$ module a finite group G_x, such that each component V_i of $\pi^{-1}(V)$ is homeomorphic to U/Γ_i, where $\Gamma_i < G_x$ is some subgroup;
- $\pi|_{V_i} : V_i \to V$ corresponds to the natural projection $U/\Gamma_i \to U/G_x$.

An orbifold is *good* (resp. *very good*) if it can be covered (resp. finitely) by a manifold. Otherwise it is *bad*. Any orbifold $\mathcal{O}$ has an universal cover $\widetilde{\mathcal{O}}$, see [**13**, Proposition 13.2.4].

In general, the *orbifold fundamental group* of an orbifold is defined as the deck transformation group of its universal cover, see [**13**, Definition 13.2.5]. Another an equivalent definition is use of the notion of based orbifold loops, that is, the orbifold fundamental group is defined as the homotopy classes of based orbifold loops. For more details, see [**3**, Section 3].

EXAMPLE 2.1. Let D^2 be the unit disk in $\mathbb{R}^2$. A transformation r on D^2 via $r(x, y) = (x, -y)$ gives a reflective $\mathbb{Z}$-action on D^2. The orbit space $D^2/\mathbb{Z}_2$ has a natural orbifold structure. Any $(x, 0) \in D^2/\mathbb{Z}_2$ is a singular point with local group $\mathbb{Z}_2$. D^2 is contractible, so $\pi_1^{orb}(D^2/\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by the transformation r.

In the view of orbifold loops, any path between $(x_1, 0)$ and (x_2, y_2) with $y_2 > 0$ is an orbifold loop. Fixing a based point (x_0, y_0) with $y_0 > 0$ in $D^2/\mathbb{Z}_2$, any two based loops are homotopic in $D^2/\mathbb{Z}_2$. Hence, $\pi_1^{orb}(D^2/\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by a based orbifold loop.

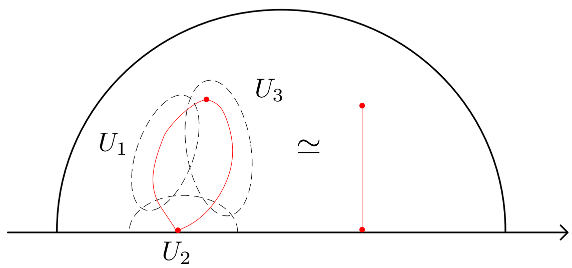

FIGURE 1. Orbifold loop

2.2. Coxeter orbifold. Let W be a Coxeter group with presentation

$$W = \langle s_1, s_2, \cdots, s_k \mid (s_i s_j)^{m_{ij}} = 1, , \forall 1 \le i \le j \le k\rangle \tag{2.1}$$

where each $m_{ii} = 1$ and $m_{ij} \ge 2$ is an integer or ∞ for $i \ne j$. Let $S(W) = \{s_1, \cdots, s_k\}$ be generator set of W, and $\#S(W)$ be the number of generators in $S(W)$. The pair (W, S) is called a *Coxeter system* of W. The finite Coxeter groups were classified by Coxeter in 1935, in terms of Coxeter-Dynkin diagrams; they are all represented by reflection groups of finite-dimensional Euclidean spaces. Every finite Coxeter groups can presented as the product of some irreducible finite Coxeter groups which consist of $A_n, B_n, D_n, I_2(p), E_6, E_7, E_8, F_4, H_3, H_4$. See [**9**].

An n-dimensional *Coxeter orbifold* [**6**] (or *locally reflective orbifold*) is an orbifold which is locally modelled on the action of a finite Coxeter group W on a open subset of $\mathbb{R}^n$ by orthogonal reflections cross hyperplanes passing through origin. Specially, a *right-angled Coxeter n-orbifold* is a singular space which is locally modelled the quotient of the action of $\mathbb{Z}_2^k = (A_1)^k$ on $\mathbb{R}^n$ by reflections across the coordinate hyperplane.

Since the quotient of a finite linear reflection group on $\mathbb{R}^n$ is the product of a Euclidean space with a simplicial cone, a Coxeter orbifold naturally has the structure of a manifold with corners.

LEMMA 2.1 ([**5**]). *Each Coxeter orbifold is a manifold with corners.*

Conversely, let X be a manifold with corners. For each facet F_i of X we label a reflector s_i, and for each codimension-two face, a connected component of $F_i \cap F_j$, we label a finite integer $m_{ij} \geq 2$. Equivalently, such operation gives an evaluation on the 1-skeleton of the nerve of X,

$$\mathbf{w} : \mathcal{N}^1(X) \to \mathbb{N}$$

where $\mathbf{w}$ maps each edge in $\mathcal{N}^1(X)$ to $m_{ij} \geq 2$. Then for each face $F^{n-k} \subset F_1 \cap \cdots \cap F_k$, there is a Coxeter group $W(F^{n-k})$ determined full sub-graph of $\{F_1, \cdots, F_k\}$ in $(\mathcal{N}^1(X), \mathbf{w})$. If $W(F^{n-k})$ is finite for any F^{n-k} in X, then $\mathbf{w}$ induces a Coxeter orbifold structure on X.

2.3. Simple complex of groups and weighted simplicial complex.

DEFINITION 2.3 (Simple complex of groups, [**2**, Definition 12.11]). A *simple complex of groups* $G(\mathcal{Q}) = (G_\sigma, \varphi_{\tau\sigma})$ over a poset $\mathcal{Q}$ consists of the following data:

- for each $\sigma \in \mathcal{Q}$, a group G_σ, called the *local group* at σ;
- for each $\tau < \sigma$, an injective homomorphism $\varphi_{\tau\sigma} : G_\sigma \to G_\tau$ such that if $\tau < \sigma < \rho$, then
$$\varphi_{\tau\rho} = \varphi_{\tau\sigma}\varphi_{\sigma\rho}.$$

DEFINITION 2.4 (Weighted simplicial complex [**8**]). A *weighted simplicial complex* is a pair of (K, w) consisting of a simplicial complex K and a function w from the simplices of K to natural numbers, obeying

$$\tau < \sigma \Rightarrow w(\tau) \mid w(\sigma).$$

Hence, if the local groups in a simple complex of groups are all finite groups, then this simple complex of groups admits a weighted complex structure where the weight of each simplex is the order of the associated local group.

The weighted homology of a weighted complex are defined in [**8**].Let K be a complex of groups whose each local groups is a finite Coxeter group.

DEFINITION 2.5 (Boundary map [**8**]). Let $\sigma^n = [v_0, \cdots, v_n] \in K$ be an n-simplex in X. Then

$$d(\sigma^n) = \sum_i (-1)^i \frac{|W|}{|W_i|}[v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

Specially, if $W = \mathbb{Z}_2^{n+1}$ for each n-simplex in K, then

$$d(\sigma) = \sum_i (-1)^i 2\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Similar to simplicial complexes, $d^2 = 0$. One can define its homology and cohomology as usual.

3. Coxeter complex and orbifold (co)homology

The cellular decompositions of orbifolds are considered as q-cellular complex (or, q-CW complex) to compute the rational homology of certain quasi-toric orbifolds by Poddar-Sarkar [**18**] and to detect torsion in integral cohomology of certain effective orbifolds by Bahri-Notbohm-Sarkar-Song [**17**]. Here, the objects we studied are Coxeter orbifolds. A special case of q-complex are restated as Coxeter complexes in this section (with a little more restriction).

3.1. Coxeter complex and homology groups.

Definition 3.1 (Coxeter cells [**7**, Definition 2.2.1]). Let e^n be the interior of a closed unit n-ball D^n in $\mathbb{R}^n$. Let W be a finite Coxeter group with a representation on $\mathbb{R}^n$. We call the orbit e^n/W is a *Coxeter n-cell*, and e^n is called the *blowing-up* of e^n/W. A cell with trivial local group is called *regular cell*, otherwise, *singular cell*.

Remark 2.
- Here the Coxeter n-cell is an orbifold cell which is related to the Coxeter block in [**7**, Definition 2.2.1]. Notice that the "Coxeter cell" in [**7**, Definition 2.2.1] corresponds to D^n here.
- The closure of a Coxeter n-cell (or, q-cell) $\overline{e}^n/W$ is also call an *orbifold ball*, which is an n-orbifold with boundary. Its boundary $\partial(\overline{e^n}/W) := (\partial\overline{e^n})/W \cong S^{n-1}/W$ is a closed $(n-1)$-orbifold (also called an orbifold lens space[**17**]).
- Each Coxeter cell is equivalent to the product of a simplicial cones and a regular cell, that is, $e^n/W \cong e^k/W \times e^{n-k}$ where $k = \#S(W)$.

For examples, a (right-angled) Coxeter 1-cell is either a connected open interval or a semi-open and semi-closed interval whose closed endpoint gives its local group $\mathbb{Z}_2$. A right-angled Coxeter 2-cell has three cases with respect to trivial group, $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$, respectively.

An n-dimensional *Coxeter complex* X can be constructed in the same way as CW complex (see [**10**, Page 5]). A key point is that: Every attaching map

$$\phi : \partial\overline{e^n}/W \to X^{n-1} \tag{3.1}$$

is required to **preserve the local group of each point in** $\partial\overline{e^n}/W$, where X^{n-1} is the $(n-1)$-skeleton of X.

$$\begin{tikzcd}
\overline{e^n} \arrow[r, "\partial"] \arrow[d, "\psi"'] & \partial\overline{e^n} = S^{n-1} \arrow[r, "\psi"] & \partial\overline{e^n}/W \arrow[d, "\phi"] \\
\overline{e^n}/W \arrow[rr, "\Phi"] & & X^{n-1}
\end{tikzcd}$$

Remark 3.
- Given a Coxeter orbifold X, there is a natural requirement for its orbifold cellular decomposition: Each inclusion of a Coxeter cell into X ought to preserve the codimension, as manifolds with corners. Hence, it is natural that all attaching maps here are required to preserve local groups. The restriction is stronger than in q-CW complex, which ensures that the information of local group of each cell can be kept by its lower dimensional boundary. Hence, some conclusions of CW complexes can be generalized to Coxeter complexes, such as $H_i(X) \cong H_i(X^{i+1})$, $\pi_1(X) \cong \pi_1(X^2)$.
- A Coxeter complex is an orbispace. The category of orbispaces can be refer to [**3**]. So Coxeter complexes together with Coxeter cellular morphisms form a category.

Definition 3.2 (Orientation of a Coxeter cell). Let $\psi : e^n \to e^n/W$ be the quotient map induced by W-action. Then the orientation of e^n/W is defined as the orientation of its blowing-up e^n.

DEFINITION 3.3 (Chain group). Let C_i be the i-dimensional labeled chain group of X with basis the oriented Coxeter i-cells in X,

$$C_i = \mathbb{Z}\langle\{e^i/W\}\rangle. \tag{3.2}$$

REMARK 4. The basis of C_i can be presented naturally by the pair $\{(D^n/W, \partial D^n/W)\}$ or $\{(D^n, \partial D^n), W, \psi\}$. See subsection 3.4 for details.

DEFINITION 3.4 (Boundary map). The boundary map $d : C_n \longrightarrow C_{n-1}$ is defined as follows,

$$d(e^n/W) \triangleq [\phi \circ \psi(\sum_{g\in W}(-1)^{l(g)}\partial(\overline{e^n} \cap \mathcal{X}_g))] \tag{3.3}$$

where

- the finite Coxeter group W is generated by $S(W) = \{r_1, \cdots, r_k\}$;
- $\psi : \overline{e^n} \longrightarrow \overline{e^n}/W$ is the quotient map determined by W acting on $\overline{e^n}$;
- $l(g)$ is the word length of g in $S(W)$;
- ∂ is an ordinary boundary map on a cell;
- $\mathcal{X}_g$ is a lifting of $\overline{e^n}/W$, that is, the chamber indexed by g.

REMARK 5. (1) If W is a trivial group, then d is the ordinary boundary map on e^n.
(2) Every reflection transfers once orientation of e^n, hence the formula (3.3) is with alternative sum.
(3) In general, for a q-cell e^n/G in a q-CW complex whose all attaching maps preserve local groups, let G is the local group with a set of minimal generators $S(G)$, one may define analogous boundary maps. Actually, each $g \in G < \mathrm{Aut}(e^n)$ has a shortest reduced presentation at $S(G)$. And g corresponds to a self-homeomorphism of e^n. Now set $l(g)$ is a number of letters in the presentation of g that change the orientation of e^n. Then one can get a similar definition of boundary map of e^n/G. There are some examples are shown in Example 6.4 and Example 6.5.

Given a Coxeter cell e^n/W with $W \neq 1$ in a Coxeter complex X, the attaching map ϕ of e^n/W induces a cellular decomposition of $\partial e^n/W$, denoted by $C(\partial e^n/W)$. Let $x = e^{n-1}/W_\beta$ be a Coxeter cell in $C(\partial e^n/W)$, and $y = \phi(x) = e^{n-1}/W_\beta$ be a Coxeter cell in X^{n-1}, $\psi : \overline{e}^n \to \overline{e}^n/W$ be the quotient map.

Suppose the blowing-up of y is $U = e^{n-1}$. The preimage $\psi^{-1}(x)$ has $\frac{|W|}{|W_\beta|}$ connected components, denoted by $V_1, \cdots, V_{\frac{|W|}{|W_\beta|}}$ which corresponds to the cosets of W_β in W. If x is a regular cell, then each V_i corresponds to a g in W. If x is a singular cell, then for each coset we can take a representation g such that g determines a map form V_i to W, that is, $(v, g\cdot h) \to (v, h)$ where h is a word in W_β. Furthermore, the choice of g determines the boundary map d. For each coset, we consider the following two rules of the choice of g in this note.

1. Every g with even word length, if possible;
2. Every g with the shortest word length.

If each g is taken with even word length, then $(-1)^{l(g)} \equiv 1$. Now all V_i have the same orientation. Hence,

$$d_1(e^n/W) = \sum_{\beta, W_\beta\neq 1}\frac{|W|}{|W_\beta|}\phi_\sharp(e^{n-1}_\beta/W_\beta) = \sum_{\overline{\beta}} n_{\overline{\beta}}\frac{|W|}{|W_{\overline{\beta}}|}e^{n-1}_{\overline{\beta}}/W_{\overline{\beta}}. \tag{3.4}$$

where $\phi_\sharp$ induced by ϕ is a quotient map which maps e_β^{n-1}/W_β to the basis $e_{\overline{\beta}}^{n-1}/W_{\overline{\beta}}$ of C_{n-1}.

If each g is taken with taken with the shortest word length, then $\mathrm{Im}(d)|_{e_\beta^{n-1}} = 0$ when $\frac{|W|}{|W_\beta|}$ is even, and 1 when $\frac{|W|}{|W_\beta|}$ is odd. Since $\frac{|W|}{|W_\beta|}$ is even when $|\#S(W_\beta)| < |\#S(W)| - 1$, we have

$$d_2(e^n/W) = \sum_{\#S(W_{\overline{\beta}}) \geq \#S(W)-1} n_{\overline{\beta}} \Big(\frac{|W|}{|W_\beta|} \text{ mod } 2 \Big) e_{\overline{\beta}}^{n-1}/W_{\overline{\beta}}. \tag{3.5}$$

Example 3.1. Let $W =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = 1 >$, and D^2/W be the quotient of W acting on D^2. Giving a Coxeter cellular decomposition of D^2/W includes a Coxeter 2-cell e^2/W, three Coxeter 1-cells $\alpha \cong \beta \cong e^1/\mathbb{Z}_2, \gamma \cong e^1$, and two 0-cells, as shown in Figure 2. Here every ϕ is an identity.

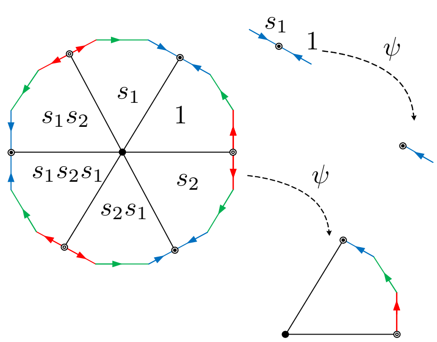

Figure 2. Boundary maps on a Coxeter 2-cell

$\psi^{-1}(\gamma)$ has six connected components which are paired in different orientations. Hence $[\psi(\psi^{-1}(\gamma))] = 0 \in C_1$. $\psi^{-1}(\alpha)$ and $\psi^{-1}(\beta)$ have three connected components. Such as

$$\psi^{-1}(\alpha) = \{(1, s_1), s_1s_2(1, s_1), s_2s_1(1, s_1)\}.$$

For each connected component, we choose a representation in the chamber $\mathcal{X}_g$ where $l(g)$ is an even number. Then $[\psi(\psi^{-1}(\alpha))] = 3\alpha \in C_1$. For the same reason, $[\psi(\psi^{-1}(\beta))] = 3\beta \in C_1$. Hence,

$$d_1(e^2/W) = 3\alpha + 3\beta.$$

And if we write

$$\psi^{-1}(\alpha) = \{(1, s_1), s_1s_2(1, s_1), s_2(1, s_1)\},$$

then,

$$d_2(e^2/W) = \alpha + \beta.$$

In fact, the above two rules correspond to the two blowing-ups of X.

Proposition 3.1 (Blowing-up). *Chosen a rule of the representation discussed above, then for an arbitrary Coxeter orbifold cellular complex X, there is a cellular complex $\widetilde{X}$ such that the chain complexes of $\widetilde{X}$ is same with X.*

Proof. Assume that $\widetilde{X}^{n-1}$ is defined. Let e^n/W be a Coxeter cell in X which is covered by a regular n-cell e^n. Then there is a attaching map ϕ' is determined by the representation g. In detail, let $X = e^{n-1}/W_\beta$ be a Coxeter cell in $C(\partial e^n/W)$ and $Y = \phi(x) \in X^{n-1}$. Denote the coset of W_β in W by S. Let the blowing-up of X be

$$U = X \times W_\beta / \sim$$

Then

$$\phi'|_{V_i} : V_i \cong U \longrightarrow U$$

via $\phi'(v, g \cdot h) = (v, h)$. It is clear that ϕ' is well-defined. Hence the blowing-up of e^n/W is glued on $\widetilde{X}^{n-1}$ by ϕ'.

By inductions, we get a cellular complex $\widetilde{X}$. Moreover, we have commutative diagram. So $d \circ id(e^n) = id \circ \widetilde{d}(e^n)$. The argument follows.

$$\begin{tikzcd}
\overline{e^n} \arrow[rr, rightsquigarrow] \arrow[dr, "\partial"] \arrow[ddd, "\psi"'] & & \widetilde{X}^{n-1} \arrow[ddd, "\Psi"] \\
& \partial\overline{e^n} \arrow[ur, "\phi'"'] \arrow[d, "\psi"] & \\
& \partial\overline{e^n}/W \arrow[dr, "\phi"] & \\
\overline{e^n}/W \arrow[rr, rightsquigarrow] & & X^{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[rr, "\widetilde{d}"] \arrow[dr, "\partial_\sharp"] \arrow[ddd, "\psi_\sharp = id"'] & & D_{n-1} \arrow[ddd, "\Psi_\sharp = id"] \\
& G_{n-1}(\partial\overline{e^n}) \arrow[ur, "\phi'_\sharp"'] \arrow[d, "\psi_\sharp"] & \\
& G_{n-1}(\partial\overline{e^n}/W) \arrow[dr, "\phi_\sharp"] & \\
\mathbb{Z} \arrow[rr, "d"] & & C_{n-1}
\end{tikzcd}$$

□

The cellular complex $\widetilde{X}$ here is also called the *blowing-up complex* of X. For example, the blowing-up complex of $D^2/\mathbb{Z}_2^2$ is homotopic to $\mathbb{R}P^2$ by identifying two points (which is homotopic to $\mathbb{R}P^2 \bigvee S^1$) when all representations are chosen with even word lengths, and is homotopic to T^2 when all representations are chosen with shortest word lengths. Notice that the blowing-up complexes of X are not unique when all representations are chosen with even word lengths, but those blowing-ups are well-defined up to the definition of d by (3.4).

Corollary 3.1. $d^2 = 0$.

Proof. It is clear that $\widetilde{d}^2 = 0$ for the cellular complex $\widetilde{X}$. By Proposition 3.1, the following diagram is commutative. $\Psi_\sharp = id$, hence $d^2 = 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & D_n \arrow[r, "\widetilde{d}"] \arrow[d, "\Psi_\sharp"'] & D_{n-1} \arrow[r, "\widetilde{d}"] \arrow[d, "\Psi_\sharp"'] & D_{n-2} \arrow[r] \arrow[d, "\Psi_\sharp"'] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d"'] & C_{n-1} \arrow[r, "d"'] & C_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

□

So we get a chain complex $C = \{C_i, d_i\}$.

Definition 3.5. The i^{th} *orbifold homology group* of $C = \{C_i, d_i\}$ is defined as follows:

$$H_i(C) = \text{Ker } d_i / \text{Im } d_{i+1}$$

Under (3.4) or (3.5), we can write $C_n = C_n^{reg} \oplus C_n^{sing}$ where C_n^{reg} is generated by regular n-cells and C_n^{sing} is generated by singular n-cells. Now $d(C_n^{reg}) \subset C_{n-1}^{reg}, d(C_n^{sing}) \subset C_{n-1}^{sing}$. Hence we have the following conclusion.

Proposition 3.2 (Splitting property of homology groups). *Let X be a Coxeter complex, and X_{reg}, X_{sing} be its sub-chain complex generated of all Coxeter cells with tirvial and non-trivial local groups, respectively. Then*

$$H_i(X) \cong H_i(X_{reg}) \oplus H_i(X_{sing}) \tag{3.6}$$

where $H_i(X_{reg}) \cong H_i(|X|)$.

We denote the homology groups defined by (3.5) as H_*^{orb}, and another is denoted as $H_*^{orb'}$. In general, the information of $H_*^{orb'}$ is more than H_*^{orb}. It is clear that the boundary maps of 3.4 and 3.5 are the same when we consider the $\mathbb{Z}_2$ coefficient. Hence,

Lemma 3.1. $H_i^{orb}(X;\mathbb{Z}_2) = H_i^{orb'}(X;\mathbb{Z}_2)$.

3.2. Homotopy invariance. Let $f : X \to Y$ be a cellular morphism. Then we have a induced chain map $f_\sharp : C_i(X) \to C_i(Y)$ such that $f_\sharp d = df_\sharp$. Moreover, a chain map between chain complexes induces homomorphism between the homology groups of the two complexes.

Definition 3.6 (Homotopy of orbispaces). Two morphisms $f, g : X \to Y$ are homotopic, if there is a morphism $F : X \times [0,1] \to Y$ such that $F(x,0) = f, F(x,1) = g$ and $F(x,t)$ is a morphism for any $t \in [0,1]$.

Lemma 3.2. *If two Coxeter complex are homotopic, then its blowing-ups are also homotopic.*

Proposition 3.3. *If two morphisms $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_i(X) \to H_i(Y)$.*

Proof. Let X be a Coxeter complex, and $I = [0,1]$. Take $\{0\}$ and $\{1\}$ as two 0-cell and $(0,1)$ as 1-cell of $[0,1]$. Then $X \times I$ is also a Coxeter complex.

Let $\widetilde{X}, \widetilde{Y}$ be the blowing-ups of X, Y, respectively. And let $F : X \times [0,1] \to Y$ be a cellular homotopy satisfying $F_0(x) = F(x,0) = f(x)$ and $F_1(x) = F(x,1) = g(x)$. Then F induces a homotopy $\widetilde{F} : \widetilde{X} \times [0,1] \to \widetilde{Y}$ such that $\widetilde{F_0}(x) = \widetilde{F}(x,0) = \widetilde{f}(x)$ and $\widetilde{F_1}(x) = \widetilde{F}(x,1) = \widetilde{g}(x)$ where $\widetilde{f}, \widetilde{g}$ is induced by f, g.

$$\begin{tikzcd}
\widetilde{X} \times I \arrow[r, "\widetilde{F}"] \arrow[d, "\Psi \times id"'] & \widetilde{Y} \arrow[d, "\Psi"] \\
X \times I \arrow[r, "F"] & Y
\end{tikzcd}$$

Then $\widetilde{f}_* = \widetilde{g}_* : H_i(\widetilde{X}) \to H_i(\widetilde{Y})$. Hence, $f_* = g_* : H_i(X) \to H_i(Y)$. □

Corollary 3.2. *The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all n.*

3.3. Relative homology groups and excision theorem. Let A be a subcomplex of a Coxeter complex X. Relative homology groups $H_i(X, A)$ are defined as ordinary way. Then each pair (X, A) induces a long exact sequence of homology groups.

The excision theorem of Coxeter complex can be also proved by their blowing-ups. In the other words, the category of Coxeter complexes here is equivalent to a sub-category of CW complex. Hence, there are many homology properties are hold. However, one should be careful to do quotient operation. In general, $H_i(X, A) \ncong H_i(X/A)$, unless that A is a regular subcomplex of X and $(|X|, A)$ is a good pair in the sense of [**10**, Proposition 2.22].

LEMMA 3.3. *Let X_{sing} and X_{reg} be the sub-chain complexes of a Coxeter complex X which consist of all regular cells and singular cells, respectively. Then, for any i, we have*

$$H_i(X_{reg}) \cong H_i(|X|)$$

$$H_i(X_{sing}) \cong H_i(X, X_{reg}) \cong H_i(X/|X|)$$

3.4. Cellular homology and degree. Next, we calculate $H_n(D^n/W, \partial D^n/W)$ and consider the degree of

$$\Phi_\beta : H_n(D^n/W, \partial D^n/W) \longrightarrow H_{n-1}(D^{n-1}_\beta/W_\beta, \partial D^{n-1}_\beta/W_\beta)$$

which is the composite of

(3.7)
$$H_n(D^n/W, \partial D^n/W) \longrightarrow H_{n-1}(\partial D^n/W) \longrightarrow H_{n-1}(\coprod_{n_\beta}(\widetilde{D}^{n-1}_\beta/W_\beta, \partial \widetilde{D}^{n-1}_\beta/W_\beta)) \\ \longrightarrow H_{n-1}(D^{n-1}_\beta/W_\beta, \partial D^{n-1}_\beta/W_\beta).$$

where $\coprod_{n_\beta}(\widetilde{D}^{n-1}_\beta/W_\beta, \partial \widetilde{D}^{n-1}_\beta/W_\beta) = \phi^{-1}((D^{n-1}_\beta/W_\beta, \partial D^{n-1}_\beta/W_\beta))$ and $\#S(W) \geq \#S(W_\beta) > 0$.

Let $Y = D^n/W^k$ be a closed singular n-cell and $\partial D^n/W = S^{n-1}/W$ be the quotient orbifold of S^{n-1} where $\#S(W) = k \leq n$.

First consider $H_1^{orb'}$. If $Y = D^1/\mathbb{Z}_2$, then

$$H_1^{orb'}(Y, \partial Y) \cong H_1^{orb'}(Y) \cong \mathbb{Z}.$$

And there is no singular 0-cell, so $deg(\Phi_\beta) = 0$.

If $Y = D^2/I(m)$ where $I(m) =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^m >$, then we have

$$0 \to H_2^{orb'}(Y, \partial Y) \to H_1^{orb'}(\partial Y) \cong \mathbb{Z}^2 \xrightarrow{i_*} H_1^{orb'}(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \to 0.$$

Now $H_2^{orb'}(Y, \partial Y) \cong \ker i_* \cong \mathbb{Z}$. And $deg(\Phi_\beta) = \pm m$ or 0.

If $Y = D^2/\mathbb{Z}_2$, then we have

$$0 \to H_2^{orb'}(Y, \partial Y) \to H_1^{orb'}(\partial Y) \cong \mathbb{Z}^2 \xrightarrow{i_*} H_1^{orb'}(Y) \cong \mathbb{Z} \to 0.$$

Now $H_2^{orb'}(Y, \partial Y) \cong \ker i_* \cong \mathbb{Z}$. And $deg(\Phi_\beta) = \pm 1$ or 0.

In general, if $n > 2$, $D^n/W^k \simeq D^k/W^k$, hence

(3.8)
$$H_i^{orb'}(D^n/W) \cong H_i^{orb'}(D^k/W)$$

and

$$H_i^{orb'}(\partial D^n/W)\cong\begin{cases}H_i^{orb'}(D^k/W), & i<k-1\\ \mathbb{Z}, & i=n-1\\ 0, & otherwise\end{cases}\tag{3.9}$$

By the sequence of homology groups,

$$H_n^{orb'}(D^n/W,\partial(D^n/W))\cong H_{n-1}^{orb'}(\partial(D^n/W))\cong\mathbb{Z}$$

for $k<n$ and isomorphic to the kernel of

$$H_{n-1}^{orb'}(\partial(D^n/\mathbb{Z}_2^k))\cong\mathbb{Z}\longrightarrow H_{n-1}^{orb'}(D^n/\mathbb{Z}_2^k)\cong G$$

for $k=n$ where G is a finite cyclic group. All in all, we have

$$H_n^{orb'}(D^n/W,\partial(D^n/W))\cong\begin{cases}\mathbb{Z}, & i=n\\ 0, & otherwise.\end{cases}$$

Then we can consider the degree of Φ_β. Now, $deg(\Phi_\beta)$ is defined as the degree of

$$\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d, "\psi"'] & \coprod_{d_\beta}(\widetilde{D}_\beta^{n-1},\partial\widetilde{D}_\beta^{n-1}) \arrow[r] & \coprod_{n_\beta}(\widetilde{D}_\beta^{n-1},\partial\widetilde{D}_\beta^{n-1}) \arrow[r] & (D_\beta^{n-1},\partial D_\beta^{n-1}) \arrow[d, "\psi"'] \\
\partial D^n/W \arrow[rr] & & \coprod_{n_\beta}(\widetilde{D}_\beta^{n-1}/W_\beta,\partial\widetilde{D}_\beta^{n-1}/W_\beta) \arrow[r] & (D_\beta^{n-1}/W_\beta,\partial D_\beta^{n-1}/W_\beta)
\end{tikzcd}$$

It is clear that its degree is a multiple of $\frac{|W|}{|W_\beta|}$. That is, $deg(\Phi_\beta)=n_\beta\frac{|W|}{|W_\beta|}$. Specially, if $\#S(W_\beta)=k-1$, then $deg(\Phi_\beta)\leq k\frac{|W|}{|W_\beta|}$.

Similarly,

$$H_1^{orb}(D^1/\mathbb{Z}_2,\partial D^1/\mathbb{Z}_2)\cong H_1^{orb}(D^1/\mathbb{Z}_2)\cong\mathbb{Z},\ deg(\Phi_\beta)=0;$$

$$H_1^{orb}(D^2/\mathbb{Z}_2,\partial D^2/\mathbb{Z}_2)\cong\ker(\mathbb{Z}^2\to\mathbb{Z})\cong\mathbb{Z},\ deg(\Phi_\beta)=0,\pm1;$$

$$H_1^{orb}(D^2/I(m),\partial D^2/I(m))\cong\begin{cases}H_2^{orb}(D^2/I(m)), & m\text{ even}\\ \ker(\mathbb{Z}^2\to\mathbb{Z}), & m\text{ odd}\end{cases}\cong\mathbb{Z},\ deg(\Phi_\beta)=0,\pm1;$$

For $n>2$, we have

$$H_i^{orb}(D^n/W)\cong H_i^{orb}(D^k/W)\tag{3.10}$$

and

$$H_i^{orb}(\partial D^n/W)\cong\begin{cases}H_i^{orb}(D^k/W), & i<k-1\\ H_i^{orb}(D^{k+1}/W), & if\ i=k-1<n-1\\ \mathbb{Z}^l, & if\ i=n-1=k-1\\ \mathbb{Z}, & if\ i=n-1>k-1\\ 0, & otherwise\end{cases}\tag{3.11}$$

Hence,

$$H_n^{orb}(D^n/W,\partial(D^n/W))\cong\begin{cases}H_n^{orb}(D^n/W)\cong\mathbb{Z}, & \text{if } n=k \text{ and all } \frac{|W|}{|W_\beta|}\text{ even},\\ \ker(H_{n-1}^{orb}(\partial D^n/W)\to & H_{n-1}^{orb}(\partial D^n/W)\cong\mathbb{Z},\\ & \text{if } n=k\text{ and some }\frac{|W|}{|W_\beta|}\text{ odd},\\ H_{n-1}^{orb}(\partial D^n/W)\cong\mathbb{Z}, & \text{if n>k.}\end{cases}$$

Then $deg(\Phi_\beta) = 0$ for the first case, $deg(\Phi_\beta) = 0, \pm 1$ for the second case and $deg(\Phi_\beta) = n_\beta$ for the third case, which is the degree of

$$\partial D^n \longrightarrow \coprod_{n_\beta} (\widetilde{D}_\beta^{n-1}, \partial \widetilde{D}_\beta^{n-1}) \longrightarrow (D_\beta^{n-1}, \partial D_\beta^{n-1}).$$

PROPOSITION 3.4. *Both in $H^{orb'}$ and H^{orb} , $H_i(X^n, X^{n-1})$ is zero for $i \neq n$ and is a free abelian group for $i = n$, with a basis one-to one correspondence with the Coxeter n-cell pairs $(D^n/W, \partial D^n/W)$ of X.*

$$H_i(X^n, X^{n-1})) \cong \oplus H_i(D^n/W, \partial(D^n/W)) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}^m, & i = n. \end{cases}$$

REMARK 6. Here $H_i(D^n/W, \partial(D^n/W)) \ncong H_i(D^n/W/\partial(D^n/W))$.

Hence, our homology of Coxeter complex is indeed an analogue of the cellular homology of CW complex. Now let $H_n(X^n, X^{n-1})$ be the chain group with basis $\{(D^n/W, \partial D^n/W)\}$ or, more naturally, $\{[(D^n, \partial D^n), W, \psi]\}$. The map i_* takes a chosen generator $e^n/W \in H_n(D^n/W, \partial D^n/W)$ to a generator of the $\mathbb{Z}$ summand of $H_n(X^n, X^{n-1})$ corresponding to a Coxeter cell e^n/W. If W is a trivial group, then $d(e^n) = \sum_\beta d_\beta e_\beta^{n-1}$ as in ordinary cellular complex [**10**, Page 140]. If $W \neq 1$, then

$$\begin{aligned} d(e^n/W) &\triangleq [\phi \circ \psi(\sum_{g \in W} (-1)^{l(g)} \partial(\overline{e^n} \cap \mathcal{X}_g)] \\ &= \sum_{\beta, W_\beta \neq 1} d_\beta \cdot e_\beta^{n-1}/W_\beta \end{aligned} \tag{3.12}$$

where $d_\beta = \deg(\Phi_\beta) = n_\beta \frac{|W|}{|W_\beta|}$ or $n_\beta \Big(\frac{|W|}{|W_\beta|} \mod 2 \Big)$.

EXAMPLE 3.2. Let W be a finite Coxeter group with $\#S(W) = k+1$, then $S^k/W = \Delta^k$ is a Coxeter orbifold. And the suspension of Δ^k, denoted by $S\Delta^k$, is also a Coxeter orbifold satisfying that the cone points have local groups W. Moreover, $S^l\Delta^k$, $l > 0$, is also a Coxeter orbifold which can be realized as the quotient S^{k+l}/W. So

$$H_i^{orb'}(S^n/W) \cong H_i^{orb'}(S^{n-k}\Delta^k) \cong \begin{cases} H_i^{orb'}(\Delta^k), & i \leq k-1 \\ \mathbb{Z}, & i = n \\ 0, & otherwise, \end{cases}$$

Let

$$f : S^n/W^{k+1} \to S^n/W^{k+1}$$

be a morphism which preserves local groups. If $n > k+1$, then f induces a map $\widetilde{f} : S^{n-k-1} \to S^{n-k-1}$. Now the *degree* of f is defined as the degree of $\widetilde{f}$. Now $deg(f) = deg(Sf)$. If $n = k, k+1$, then f is an orbifold equivalence. Now $deg(f) = \pm 1$.

Similarly,

$$H_i^{orb}(S^n/W) \cong H_i^{orb}(S^{n-k}\Delta^k) \cong \begin{cases} H_i^{orb}(\Delta^k), & i \leq k-1 \\ H_i^{orb}(D^{k+1}/W), & if\ i=k<n \\ H_k^{orb}(\Delta^k), & if\ i=n=k \\ \mathbb{Z}, & if\ i=n>k \\ 0, & otherwise, \end{cases}$$

If $n > k$, then the *degree* of f is defined as the degree of $\widetilde{f}$. And if $n = k, k+1$, then $deg(f) = \pm 1$.

3.5. Hurewize theorem.

PROPOSITION 3.5 (Hurewize Theorem). *Let X be a Coxeter complex, then*

$$\left(\pi_1^{orb}(X)\right)^{ab} \cong H_1(|X|) \oplus H_1(X_{sing}, \mathbb{Z}_2).$$

PROOF. Considering the blowing-up of X, we have the following diagram.

$$\begin{tikzcd} \pi_1(\widetilde{X}) \arrow[r, "\widetilde{h}"] \arrow[d, "\Psi_*"'] & H_1(\widetilde{X}) \arrow[d, "\Psi_*"] \\ \pi_1^{orb}(X) \arrow[r, "h"] & H_1(X) \end{tikzcd}$$

where $\pi_1^{orb}(X) = \pi_1(\widetilde{X})/ < \{x_i^2\} >$ and $H_1(\widetilde{X}) \cong H_1(X)$. h maps each based orbifold loop to a free chain in $C_1(X)$.

Each based orbifold loop in X one to one corresponds to a based closed loop in the blowing-up $\widetilde{X}$ of X. There are two orbifold relations determined by Coxeter 2-cells $D^2/\mathbb{Z}_2$ and D^2/W where $W =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^m >$.

$$\begin{cases} s_1as_2a^{-1} = 1, or \\ (s_1as_2a^{-1})^m = 1 \end{cases}$$

where a is a regular closed loop. When we take the blowing-up $\widetilde{X}$ in the way of 3.4, the above relations correspond to

$$\begin{cases} \widetilde{s}_1a\widetilde{s}_2a^{-1} = 1, or \\ (\widetilde{s}_1a\widetilde{s}_2a^{-1})^m = 1. \end{cases}$$

When we take the blowing-up $\widetilde{X}$ in the way of 3.5, the above relations correspond to

$$\begin{cases} \widetilde{s}_1a\widetilde{s}_2a^{-1} = 1, or \\ \begin{cases} (\widetilde{s}_1a\widetilde{s}_2a^{-1}\widetilde{s}_1^{-1}a\widetilde{s}_2^{-1}a^{-1})^{\frac{m}{2}} = 1, m\ even; \\ (\widetilde{s}_1a\widetilde{s}_2a^{-1}\widetilde{s}_1^{-1}a\widetilde{s}_2^{-1}a^{-1})^{\frac{m-1}{2}} \cdot \widetilde{s}_1a\widetilde{s}_2a^{-1} = 1, m\ odd. \end{cases} \end{cases}$$

The Hurewize theorem of $\widetilde{X}$ gives abelianed relations with forms in above relations. That is, $s_1s_2 = 1$ or $(s_1s_2)^{m \bmod 2} = 1$. The result follows. $\square$

4. Computation

Let X be a Coxeter orbifold which admits a Coxeter complex structure, denoted by $\mathcal{C}(X)$. Then the associated chain complex $\{C, d\}$ can be split into two parts $X_{reg} = \{C_{reg}, d\}$ and $X_{sing} = \{C_{sing}, d\}$.

$$H_i(X) \cong H_i(X_{reg}) \oplus H_i(X_{sing})$$

where $H_i(X_{reg}) \cong H_i(|X|)$. In this section, we try to compute $H_i(X_{sing})$. By the excision theorem $H_i(X_{sing}) \cong H_i(X/|X|)$. Hence, we may assume that X_{sing} is the Coxeter cellular decomposition of $X/|X|$.

Notice that for a closed singular cell D^n/W, topologically and without consider local groups,

$$D^n/W = cone(\partial D^n/W).$$

The singular point set of D^n/W is homeomorphic to an $(n-1)$-disk. Let S be the

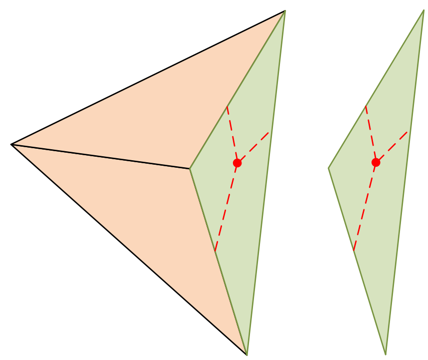

FIGURE 3. $D^3/\mathbb{Z}_2^3$

set of singular points of X. Then X_{sing} determines a cellular decomposition of S. If we equip each cell with a local group, then we can define the same boundary map like (3.5) and (3.4), then the chain complex $(C(S), d)$ has same homology groups with X_{sing}.

LEMMA 4.1. $H_i(C(S)) \cong H_{i+1}(X_{sing})$.

4.1. $H_*^{orb'}(X)$**.** If the nerve of X is a triangulation K of the singular point set of X, then X admits a cellular decomposition such that every cell of X_{sing} is with the form e^k/W^k. The orientation of e^k/W^k can be presented as a permutation, $[s_1, s_2, \cdots, s_k]$, of $S(W)$. See figure.

Then

$$d_1(e^n/W) = d([s_1, s_2, \cdots, \hat{s}_\beta, \cdots, s_k]) = \sum_{\beta, W_\beta \neq 1} (-1)^\beta \frac{|W|}{|W_\beta|}[s_1, s_2, \cdots, \hat{s}_\beta, \cdots, s_k],$$

where the orientation of e^{n-1}/W_β is taken as $[s_1, s_2, \cdots, \hat{s}_\beta, \cdots, s_k]$. Hence here X_{sing} determines a weighted complex $\mathcal{N}(X)$ and

$$H_i^{orb'}(X_{sing}) \cong H_{i-1}^w(\mathcal{N}(X))$$

for $i > 0$.

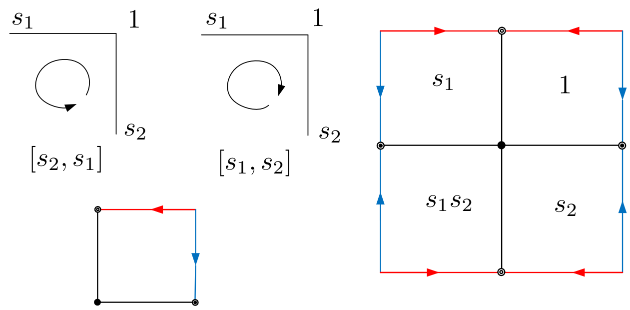

FIGURE 4. Orientation of $e^2/\mathbb{Z}_2^2$ is defined as the orientation of e^2 which is related to the choose of the frame of e^2, so it can be presented as a permutation of $S(W)$.

PROPOSITION 4.1. *Let X be a Coxeter complex whose nerve $\mathcal{N}(X)$ is a triangulation of the set of singular points of X. Then for $i > 0$*

$$H_i^{orb'}(X) \cong H_i(|X|) \oplus H_{i-1}^w(\mathcal{N}(X)).$$

4.2. $H_*^{orb}(X)$**.** Next, let us consider $H_*^{orb}(X)$.

Let X be a Coxeter orbifold. Denote the facet set of X by $\mathcal{F}(X) = \{F_1, \cdots, F_m\}$. Each $F_i \in \mathcal{F}(X)$ is labeled by a reflector s_i. Let F^{n-k} be an arbitrary $(n-k)$-face of X. Then F^{n-k} is a connected component of the intersection of some facets. Denoted those facets by $\{F_1, \cdots, F_k\}$ where $F_i = F_j$ for $i \neq j$ is allowed. (If there are two facets $F_i = F_j$ while $i \neq j$, then X is not a nice manifold with corners.) Let F_β^{n-k+1} be a face which is determined by $\{F_1, \cdots, \hat{F}_\beta, \cdots, F_k\}$. Define an equivalent relation on face set of X,

$$F^{n-k} \sim F_\beta^{n-k+1} \text{ iff } \frac{|W(F^{n-k})|}{|W(F_\beta^{n-k+1})|} \mod 2 = 1.$$

Let T be the equivalent class set of face set of X. For any $J \in T$, let

$$X_J = \bigcup_{F \in J} F.$$

If a singular cell e^i/W in X intersected transversely with F^{n-k}, then the local group $W = W(F^{n-k})$. (If $F_i = F_j$ for $i \neq j$, we view s_i and s_j as two distinct generators.)

Hence the chain complex of X can be split.

$$C(X) = \bigoplus_{J \in T} C(X_J)$$

where $C(X_J)$ is a union of singular cells which determined by equivalent faces in X. Hence,

$$H_i^{orb}(X) = \bigoplus_{J \in T} H_i^{orb}(C(X_J)). \tag{4.1}$$

Hence,

LEMMA 4.2. *For each $J \in T$,*

$$H_i^{orb}(C(X_J)) \cong H_{i-l(J)}(X_J)$$

where $l(J)$ is the shortest number of generators of face in J.

Therefore, we have the following result.

THEOREM 4.1. *Let X be a Coxeter orbifold. Then*

$$H_i^{orb}(X) = \bigoplus_{J \in T} H_{i-l(J)}(X_J) \tag{4.2}$$

where $l(J)$ is the shortest number of generators of face in J.

LEMMA 4.3 (Euler number).

$$\chi_{orb}(X) = \sum \frac{1}{|W(J)|}\chi(X_J)$$

If X has odd dimension, then $\chi_{orb}(X) = 0$.

REMARK 7. The right part in (4.4) is an analogue of Chen-Ruan cohomology group [**4**, Definition 3.2.3]. When Q is a nice manifold with corners, any codimension-two face is belong to the intersection of exact two facets of X. Now, the subscript J can be presented as $\{s_1, \cdots, s_k\}$.

LEMMA 4.4. *Let W be a finite irreducible Coxeter group with $S(W) = \{s_1, \cdots, s_n\}$, and W_β be a subgroup of W which is generated by $(n-1)$ generators in $S(W)$. Then there are at most two W_β such that $\frac{|W|}{|W_\beta|}$ is odd.*

PROOF. It is well-known that a finite irreducible Coxeter group W belong to one of $A_n, B_n, D_n, I_2(p), E_6, E_7, E_8, F_4, H_3, H_4$.

Then $\frac{|W|}{|W_\beta|}$ is odd if and only if $W = A_n$ where n is even, now $W_\beta \cong A_{n-1}$, or $W = I(p)$ where p is odd. Now there are only two W_β such that $\frac{|W|}{|W_\beta|}$ is odd. □

COROLLARY 4.1. *Let X be a Coxeter orbifold such that m_{ij} is even for any codimension-two face of X, such as X is a right-angled Coxeter orbifold. Then T is the face set of X, and*

$$H_i^{orb}(X) = \bigoplus_{F^{n-k} \in T} H_{i-k}^{orb}(X_F). \tag{4.3}$$

Moreover, if all faces of X are acyclic, then

$$H_i^{orb}(X) = \mathbb{Z}^{f_{n-k}}. \tag{4.4}$$

where f_{n-k} is the number of $n-k$ dimensional faces of X.

5. Cup product

Let G be a finite generated abelian group, and set $C^i = \mathrm{Hom}(C_i, G) = \mathrm{Hom}(C_i^{reg} \oplus C_i^{sing}, G) \cong \mathrm{Hom}(C_i^{reg}, G) \oplus \mathrm{Hom}(C_i^{sing}, G)$ and $\delta^i = \mathrm{Hom}(d_i, G) = \mathrm{Hom}(d_i|_{C_{reg}}, G) \oplus \mathrm{Hom}(d_i|_{C_{sing}}, G)$. Then $(\delta^i)^2 = 0$, so we obtain a cochain complex $D = \{C^i, \delta^i\}$, which can also be decomposed into the direct sum of two cochain complex. The cohomology with G-coefficient of D is defined as

$$H^i(D, G) = \mathrm{Ker}\ \delta / \mathrm{Im}\ \delta$$

The cohomology groups are also split, and the universal coefficient theorem holds.

PROPOSITION 5.1 (Splitting property of cohomology groups). *Let $X = X_{reg} \oplus X_{sing}$ be a Coxeter complex, and X_{reg}, X_{sing} be its orbifold subcomplex consists of all Coxeter cells with tirvial and non-trivial local groups, respectively. Then*

$$H^i(X, G) \cong H^i(X_{reg}, G) \oplus H^i(X_{sing}, G). \tag{5.1}$$

PROPOSITION 5.2 (Universal coefficient theorem). *Let X be a Coxeter complex with homology groups $H_i(X)$. Then the cohomology groups $H^i(X; G)$ are determined by split exact sequences*

$$0 \longrightarrow Ext(H_{i-1}(X), G) \longrightarrow H^i(X; G) \longrightarrow Hom(H_i(X), G) \longrightarrow 0 \tag{5.2}$$

5.1. Cup product in $H^*_{orb}(X;\mathbb{Z})$. By Theorem 4.1 and Universal coefficient theorem,

$$H^i_{orb}(X) = \bigoplus_{J \in T} H^{i-l(J)}(X_J). \tag{5.3}$$

Then the cup products of cohomology classes of $H^*(X_J)$ for a fixed J is defined as usual. And for cohomology classes belonging to distinct X_J, notice that there is the well-defined intersections of faces of X. Hence, the cup product is defined as

$$\smile: H^*(X_J) \times H^*(X_{J'}) \to H^*(X_{J \cap J'}).$$

For example, if X is a right-angled Coxeter orbifold, and all faces in X is acyclic, then

$$H^*(X) = \mathbb{Z}[v_1, \cdots, v_m]/(I, \{v_i^2\}).$$

COROLLARY 5.1. $H^*_{orb'}(X;\mathbb{Z}_2) \cong H^*_{orb}(X;\mathbb{Z}_2)$.

6. Examples

EXAMPLE 6.1 (Simple polytope). Let P be a simple polytope, for each codimensional two faces $F_i \cap F_j$ in P we marked an integer $m_{ij} \geq 2$, such that P admits a Coxeter orbifold strucutre, that is, the interiors of each $(n-k)$-face $f = F_1 \cap \cdots \cap F_k$ of P is locally modelled on $\mathbb{R}^n/W_f$, where W_f is a finite Coxeter group generated by $s_1, \cdots, s_k$ with relations $(s_i s_j)^{m_{ij}} = 1$ for any F_i, F_j. Then for $i > 0$

$$H_i^{orb'}(P) = H^w_{i-1}(\mathcal{N}(P)). \tag{6.1}$$

Specially, if P is a right-angled Coxeter orbifold, then

$$H_i(P) = \begin{cases} \mathbb{Z}, & i = 0; \\ \mathbb{Z} \oplus \mathbb{Z}_2^{x_1}, & i = 1; \\ \mathbb{Z}_2^{x_i}, & 2 \leq i \leq n-1; \\ \mathbb{Z}, & i = n. \end{cases}$$

where $x_i = \sum_{j=0}^{i} (-1)^j f_{n-i+j}$ and f_i is the number of i-faces of P.

And

$$H_i^{orb}(P) = H_i^{orb}(P') = \mathbb{Z}^{f'_{n-i}}, \tag{6.2}$$

where f'_{n-i} is the number of J with length $i = l(J)$ in T. Specially, if all m_{ij} is even, then f'_{n-i} is the number of $(n-i)$-faces of P.

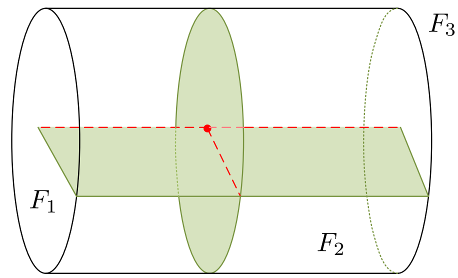

FIGURE 5. Coxeter cylinder

EXAMPLE 6.2. Let Q be a solid cylinder with three facets F_1, F_2 and F_3 as shown in FIGURE 4. $F_1 \cap F_2$ is labeled by 2, and $F_1 \cap F_2$ is labeled by 3. Then Q is a Coxeter orbifold. Q can be decomposed to one 0-cell, three 1-cells, three 2-cells and two 3-cells. Then

$$H_i^{orb'}(Q) = \begin{cases} \mathbb{Z}, & for\ i = 0, 3 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, & for\ i = 1 \\ 0, & otherwise. \end{cases}$$

Moreover, $X_1 \cong D^3, X_{[s_1]} = F_1 \cong D^2, X_{[s_2]} = F_2 \cup F_3 \cup (F_2 \cap F_3) \cong D^2, X_{[s_1 s_2]} = F_1 \cap F_2 \cong S^1$. Then

$$H_i^{orb}(Q) = \begin{cases} \mathbb{Z}, & for\ i = 0, 2, 3 \\ \mathbb{Z}^2, & for\ i = 1 \\ 0, & otherwise. \end{cases}$$

EXAMPLE 6.3. Let M be an n-manifold with boundary, and ∂M is a connected closed $(n-1)$-manifold. M is Coxeter orbifold if we let ∂M be a reflective face . Then the orbifold homology group of M,

$$H_i^{orb'\&2}(M) \cong H_i(|M|) \oplus H_{i-1}^w(\{pt\}) \oplus H_{i-1}(\partial M).$$

The boundary map defined in Definition 3.4 is obtained from the blowing up of each q-cell of a q-complex. Such idea is valid for some non-Coxeter orbifolds.

EXAMPLE 6.4 (Weighted projective space). A q-CW cellular decomposition of weighted projective space [**20**, Proposition 2.3],

$$W\mathbb{P}(a_0, \cdots, a_n) = e^0 \cup e^2/G_2 \cup \cdots \cup e^{2n}/G_{2n}$$

The boundary of each $2n$-cell is attaching on $(2n-2)$-skeleton. Hence the blowing up of $W\mathbb{P}$ is ordinary weighted projective space. Therefore,

$$H_*(W\mathbb{P}(a_0, \cdots, a_n)) \cong H_*(\mathbb{C}P^n)$$

Similarly, the blowing up of the antipodal quotient of D^n is $\mathbb{R}P^n$. Hence,

$$H_*(D^n/\sim) \cong H_*(\mathbb{R}P^n)$$

where $x \sim -x$ for $x \in D^n$.

Example 6.5 (Surface with isolated singular points). Let S is a closed genus g surface with some isolated singular points $\{v_1, v_2, v_3\}$, each singular point v_i with a local group $\mathbb{Z}_{n_i}$ generated by a rotation. Then we can gives a cell decomposition for S.

Hence the orbifold fundamental group of S,

$$\pi_1(S) = \langle x_1, y_1, x_2, y_2, s_1, s_2, s_3 \mid s_1^{n_1} = s_2^{n_2} = s_3^{n_3} = 1, [x_1, y_1][x_2, y_2]s_1 s_2 s_3 = 1\rangle$$

the homology groups of S,

$$H_i \cong \begin{cases} \mathbb{Z}, & i = 0, 2; \\ \mathbb{Z}^4 \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3} / < (0, s_1, s_2, s_3) >, & i = 1; \\ 0, & otherwise. \end{cases}$$

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