

线性代数-5-附录

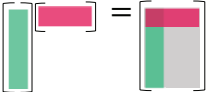


ref: The Art of Linear Algebra

2025 年 9 月 11 日

向量乘以向量——2 个视角

v1  =  =  点积 (数)

点积 $(\mathbf{a} \cdot \mathbf{b})$ 是一个数，用矩阵的语言
可以表示为 $\mathbf{a}^T \mathbf{b}$.

v2  =  =  秩 1 矩阵

$\mathbf{a}\mathbf{b}^T$ 是一个矩阵 ($\mathbf{a}\mathbf{b}^T = A$). 如果 \mathbf{a}, \mathbf{b} 都不为 0, 则结果 A 是秩为 1 的矩阵.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

图: 向量乘以向量 - (v1), (v2)

矩阵乘以向量——2 个视角

- 一个矩阵乘以一个向量将产生三个点积组成的向量 ($Mv1$) 和一种 A 的列向量的线性组合.

(Mv1)

A 的行向量乘以向量 x 得到的 Ax ,
是以点积为元素的列向量.

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + 2x_2) \\ (3x_1 + 4x_2) \\ (5x_1 + 6x_2) \end{bmatrix}$$

(Mv2)

乘积 Ax 是 A 的列向量的线性组合.

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

图: 矩阵乘以向量- (Mv1), (Mv2)

向量乘以矩阵——2 个视角

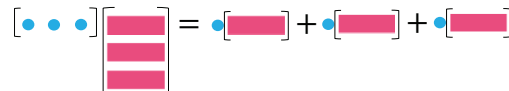
vM1


$$\begin{bmatrix} \text{pink bar} \end{bmatrix} \begin{bmatrix} \text{green bar} & \text{green bar} \end{bmatrix} = \begin{bmatrix} \text{green bar with pink cross} & \text{green bar with pink cross} \end{bmatrix}$$

行向量 \mathbf{y} 乘以 A 的列向量得到的 $\mathbf{y}A$ 是以点积为元素的行向量.

$$\mathbf{y}A = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (y_1 + 3y_2 + 5y_3) & (2y_1 + 4y_2 + 6y_3) \end{bmatrix}$$

vM2


$$\begin{bmatrix} \text{blue dot} & \text{blue dot} & \text{blue dot} \end{bmatrix} \begin{bmatrix} \text{pink bar} \\ \text{pink bar} \\ \text{pink bar} \end{bmatrix} = \begin{bmatrix} \text{blue dot} & \text{pink bar} \end{bmatrix} + \begin{bmatrix} \text{blue dot} & \text{pink bar} \end{bmatrix} + \begin{bmatrix} \text{blue dot} & \text{pink bar} \end{bmatrix}$$

乘积 $\mathbf{y}A$ 是 A 的行向量的线性组合.

$$\mathbf{y}A = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = y_1 \begin{bmatrix} 1 & 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 & 4 \end{bmatrix} + y_3 \begin{bmatrix} 5 & 6 \end{bmatrix}$$

图: 向量乘以矩阵 - (vM1), (vM2)

矩阵乘以矩阵——4 个视角

MM 1

每个元素为行向量和列向量的点积。

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} (x_1+2x_2) & (y_1+2y_2) \\ (3x_1+4x_2) & (3y_1+4y_2) \\ (5x_1+6x_2) & (5y_1+6y_2) \end{bmatrix}$$

MM 2

Ax 和 Ay 是 A 的列向量的线性组合。

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = A \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} Ax & Ay \end{bmatrix}$$

MM 3

乘积矩阵的每一行是第一个矩阵行的线性组合。

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1^* \\ a_2^* \\ a_3^* \end{bmatrix} X = \begin{bmatrix} a_1^* X \\ a_2^* X \\ a_3^* X \end{bmatrix}$$

MM 4

乘积矩阵 AB 是秩为 1 矩阵的和。

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \end{bmatrix} = a_1 b_1^* + a_2 b_2^* \\ = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ 3b_{11} & 3b_{12} \\ 5b_{11} & 5b_{12} \end{bmatrix} + \begin{bmatrix} 2b_{21} & 2b_{22} \\ 4b_{21} & 4b_{22} \\ 6b_{21} & 6b_{22} \end{bmatrix}$$

下面展示一些实用的模式。

P1

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Operations from the right act on the columns of the matrix. This expression can be seen as the three linear combinations in the right in one formula.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

using
MM 2 Mv2

P2

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Operations from the left act on the rows of the matrix. This expression can be seen as the three linear combinations in the right in one formula.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} + \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}$$

using
MM 3 vM2

图: 模式 1, 2 - (P1), (P1)

P1'



Applying a diagonal matrix from the right scales each column.

P2'



Applying a diagonal matrix from the left scales each row.

$$AD = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1 \mathbf{a}_1 \quad d_2 \mathbf{a}_2 \quad d_3 \mathbf{a}_3]$$

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \mathbf{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1 \mathbf{b}_1^* \\ d_2 \mathbf{b}_2^* \\ d_3 \mathbf{b}_3^* \end{bmatrix}$$

图: 模式 1', 2' - (P1'), (P2')

P3

$$\begin{bmatrix} \text{green bar} & \text{green bar} & \text{green bar} \end{bmatrix} \begin{bmatrix} \text{blue dot} & & \\ & \text{blue dot} & \\ & & \text{blue dot} \end{bmatrix} \begin{bmatrix} \text{purple dot} \\ \text{purple dot} \\ \text{purple dot} \end{bmatrix} = \begin{bmatrix} \text{purple dot} & \text{blue dot} & \text{green bar} \end{bmatrix} + \begin{bmatrix} \text{purple dot} & \text{blue dot} & \text{green bar} \end{bmatrix} + \begin{bmatrix} \text{purple dot} & \text{blue dot} & \text{green bar} \end{bmatrix}$$

This pattern makes another combination of columns.
You will encounter this in differential/recurrence equations.

$$XD\mathbf{c} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \mathbf{x}_1 + c_2 d_2 \mathbf{x}_2 + c_3 d_3 \mathbf{x}_3$$

图: 模式 3 - (P3)

P4

$$\begin{bmatrix} \text{green} & \text{green} & \text{green} \end{bmatrix} \begin{bmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{bmatrix} \begin{bmatrix} \text{pink} \\ \text{pink} \\ \text{pink} \end{bmatrix} = \begin{bmatrix} \text{green} & \text{pink} & \text{blue} \\ \text{gray} & \text{gray} & \text{gray} \\ \text{gray} & \text{gray} & \text{gray} \end{bmatrix} + \begin{bmatrix} \text{gray} & \text{green} & \text{pink} \\ \text{gray} & \text{pink} & \text{blue} \\ \text{gray} & \text{blue} & \text{gray} \end{bmatrix} + \begin{bmatrix} \text{gray} & \text{gray} & \text{green} \\ \text{gray} & \text{gray} & \text{pink} \\ \text{pink} & \text{pink} & \text{blue} \end{bmatrix}$$

A matrix is broken down to a sum of rank 1 matrices,
as in singular value/eigenvalue decomposition.

$$U\Sigma V^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T$$

图: 模式 4 - (P4)