

# Orbifold Stiefel-Whitney classes on right-angled Coxeter complexes

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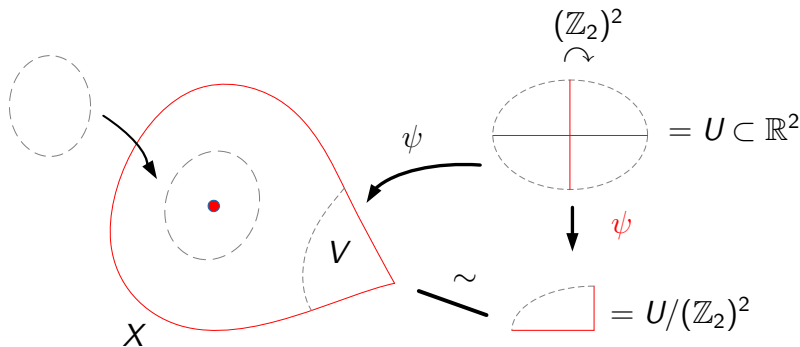
Toric Topology 2021 in Osaka (Online)

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1. Right-angled Coxeter complexes and its homology groups
2. Orbifold Stiefel-Whitney classes on RACC

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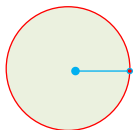
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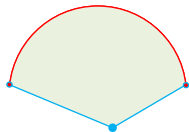
## An example



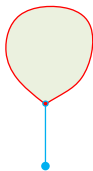
$e^2/\mathbb{Z}_2$



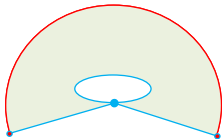
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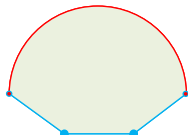
B



E



C

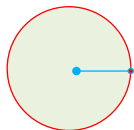


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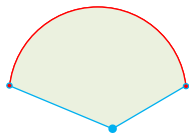
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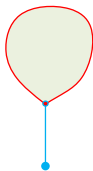
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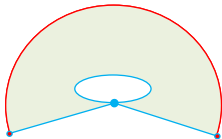
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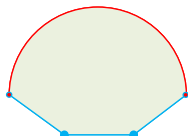
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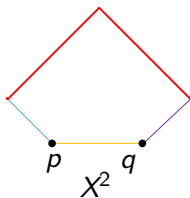
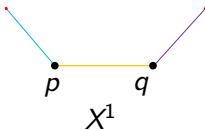
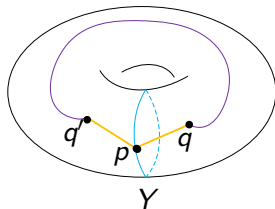
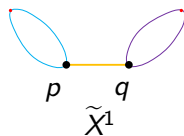
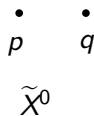
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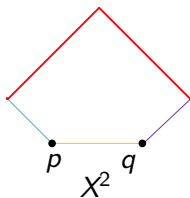
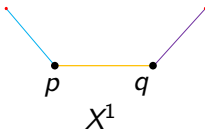
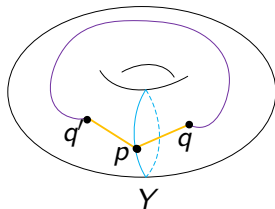
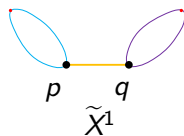
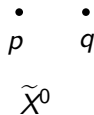
D

(\*) E is not a RACC.

# Blow-up of RACC



# Blow-up of RACC



$$(*) \quad \tilde{X}^2 = Y/[p, q] \sim [p, q'] \simeq T^2.$$

Boundary maps of RACC are defined by boundary maps of its blow-up:

$$d(e^n/W) = \sum n_\beta \left( \frac{|W|}{|W_\beta|} \bmod 2 \right) e_\beta^{n-1}/W_\beta$$

where  $W_\beta$  is a subgroup of  $W$ .

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### Theorem (Lü-Wu-Yu)

*Let  $X$  be an  $n$ -dimensional right-angled Coxeter complex, and  $T$  be the face set of  $X$ . Then*

$$H_i^{\text{orb}}(X) = \bigoplus_{f \in T} H_{i-l(f)}(X_f) \quad (1)$$

*where  $l(f)$  is the local codimension of  $f$ .*

The cup product of RACC is defined by its blow-up,

$$H_{orb}^*(X) := H^*(\tilde{X}).$$

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## Example (Simple polytope)

Let  $P$  be a simple polytope equipped with a right-angled Coxeter orbifold structure. Then the standard cubical decomposition of  $P$  is a RACC. Then

$$H_i^{orb}(P) = \mathbb{Z}^{f_{n-i}} \quad (2)$$

where  $f_{n-i}$  is the number of  $(n-i)$ -faces of  $P$ .

$$H_{orb}^*(P; \mathbb{Z}_2) = \mathbb{Z}_2[v_1, \dots, v_m] / I + J$$

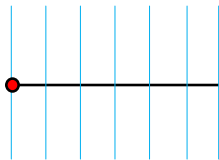
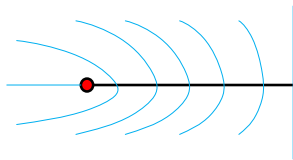
where  $I$  is the Stanley-Reisner ideal of  $P$ ,  $J = (v_i^2, \forall i)$ .



$E, X$  be two orbifolds with orbifold structures  $\{U^*, \psi^*, G^*\}$  and  $\{U, \psi, G\}$ , an orbifold vector bundle  $\pi : E \rightarrow X$  satisfies:

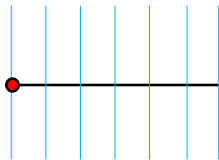
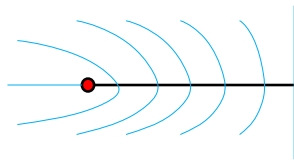
$$\begin{array}{ccc}
 E & \begin{array}{c} E_{U/G} \xleftarrow{\psi_{U^*}} U^* \cong U \times \mathbb{R}^m \\ \downarrow \pi_H \qquad \downarrow \tilde{\pi} \\ U/G \xleftarrow{\psi_U} U \end{array} & \begin{array}{c} \curvearrowright G^* \\ \\ \curvearrowright G \end{array}
 \end{array}$$

- Compatibility conditions.


 $\bar{E}$ 

 $\tilde{E}$ 

$$\bar{E} = D^1 \times \mathbb{R} / (x, y) \sim (-x, y)$$

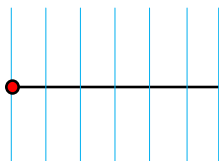
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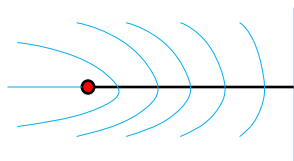
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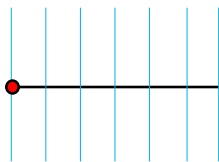
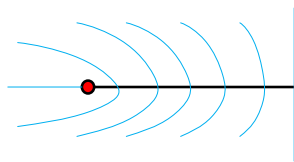
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$$w(\bar{E}) := 1 \quad w(\tilde{E}) := 1 + s$$


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(Orbifold vector bundles over a RACC maybe not a RACC.)

In general, for  $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$ , there is a representation

$$\rho : (\mathbb{Z}_2)^k \longrightarrow GL_m(\mathbb{R}).$$

The image of  $\rho$  on the generator set of  $(\mathbb{Z}_2)^k$  gives a  $m \times k$  matrix with elements  $\pm 1$ .

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^k \end{pmatrix}_{m \times k}$$

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## Orbifold SW classes on $D^n/(\mathbb{Z}_2)^k$

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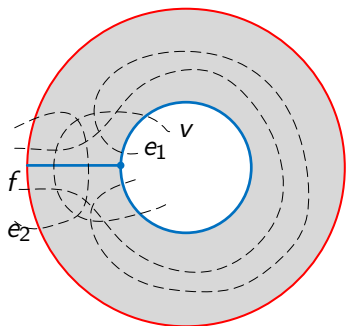
$$H^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, \dots, s_k]/(s_i^2, \forall i).$$

The total orbifold Stiefel-Whitney class of  $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$  is defined:

$$w(E) = \prod_{i=1}^m \left( 1 + \sum_{j=1}^k \frac{1 - x_i^j}{2} s_j \right) \in H^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2). \quad (3)$$



## Orbifold SW classes on RACC– An example



$$H^i(X; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 1 \\ \mathbb{Z}_2, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Figure:  $X = S^1 \times [-1, 1]/\mathbb{Z}_2$

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For each  $H$ , we have a

$$\mathcal{R}_H = \mathbb{Z}_2[s_1, \dots, s_n]/(I_H + J_H) \quad (4)$$

where  $I_H$  is the Stanley-Reisner ideal determined by  $H$ ,  $J_H = (s_i^2, \forall i)$ .

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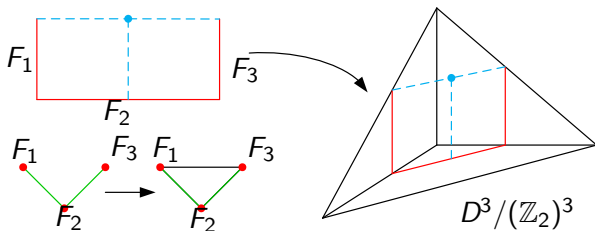
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$$\mathcal{R}_H < H^*(H; \mathbb{Z}_2) < H^*(X; \mathbb{Z}_2)$$

$$\pi : E \longrightarrow X \Rightarrow \pi_H : E_H \longrightarrow H \Rightarrow \text{matrix } C_H \Rightarrow \tilde{\pi} : E_{C_H} \longrightarrow D^n/(\mathbb{Z}_2)^n.$$

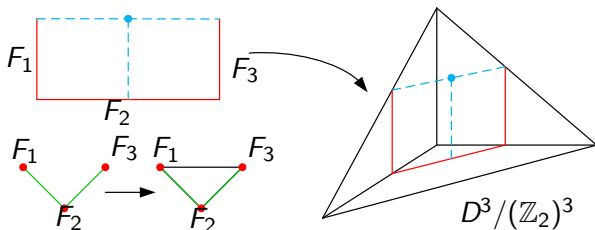
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# Orbifold SW classes on RACC

$$\pi : E \longrightarrow X \Rightarrow \pi_H : E_H \longrightarrow H \Rightarrow \text{matrix } C_H \Rightarrow \tilde{\pi} : E_{C_H} \longrightarrow D^n/(\mathbb{Z}_2)^\eta.$$



$$\begin{array}{ccc} E_H & \xrightarrow{\quad} & E_{C_H} \\ \pi_H \downarrow & & \downarrow \tilde{\pi} \\ H & \xrightarrow{j} & D^n/(\mathbb{Z}_2)^\eta \end{array}$$

In general, let  $j : \mathcal{N}(H) \rightarrow \mathcal{N}(D^\eta/(\mathbb{Z}_2)^\eta) = \Delta^{\eta-1}$  be a simplicial map.

$$j^* : \mathbb{Z}_2[s_1, \dots, s_\eta]/(s_i^2, \forall i) \longrightarrow \mathcal{R}_H \subset H^*(X; \mathbb{Z}_2).$$



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*Total orbifold SW class of  $\pi_H : E_H \longrightarrow X_H$  is defined:*

$$w(E_H) = j^*(w(E_{C_H})) \in \mathcal{R}_H \subset H^*(X; \mathbb{Z}_2) \quad (5)$$

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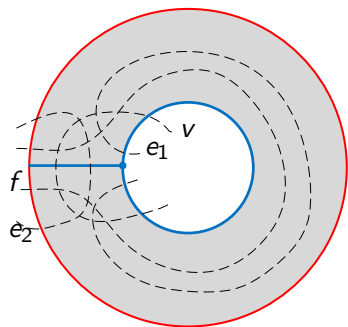
$$w(E_H) = j^*(w(E_{C_H})) \in \mathcal{R}_H < H^*(X; \mathbb{Z}_2) \quad (5)$$

*Total orbifold SW class* of  $\pi : E \longrightarrow X$  is defined:

$$w(E) = w(E(X_{reg})) \cdot \prod_H w(E_H). \quad (6)$$

where  $E(X_{reg}) = \pi|_{X_{reg}}$ .

## Example



$$H^i(X; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$w(TX) = w(TS^1)(1 + s) = 1 + s.$$

Figure:  $X = S^1 \times [-1, 1]/\mathbb{Z}_2$

# Thank You

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