

Orbifold Stiefel-Whitney classes of real orbifold vector bundles over right-angled Coxeter complexes

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ABSTRACT. We give a definition of orbifold Stiefel-Whitney classes of real orbifold vector bundles over special q-CW complexes (i.e., right-angled Coxeter complexes). Similar to ordinary Stiefel-Whitney classes, orbifold Stiefel-Whitney classes here also satisfy the associated axiomatic properties.

1. Introduction

The definition of characteristic classes of an orbifold vector bundle depend on the cohomology ring of its base space (an orbifold). The de Rham cohomology groups of an orbifold are introduced by Satake [15], so one can define orbifold chern classes of an good complex orbifold vector bundle by Chern-Weil construction, which take values in de Rham cohomology groups of base orbifold. Moreover, this definition can be extended to bad orbifold vector bundles, see [17].

In addition, one can take the equivalent cohomology ring as the cohomology ring of a quotient orbifold. Now the equivalent characteristic classes can be viewed as orbifold characteristic classes. In the book of Adem-Leida-Ruan [1], the orbifold characteristic classes they defined lie in the cohomology rings of classifying spaces of the orbifold groupoids. According to [1, Example 2.11], their orbifold characteristic classes actually correspond to the equivalent characteristic classes.

However, the integral and Mod-two integral cohomology rings of general orbifolds are unclear. So it is difficult to define orbifold characteristic classes in the usual way, see [13, Milnor-Stasheff].

Recently, Lü-Wu-Yu [12] introduced integral orbifold cellular homology groups of Coxeter complexes by applying the idea of blow-up. In this note, we define and study orbifold Stiefel-Whitney classes on right-angled Coxeter complexes based on the cohomology groups of Lü-Wu-Yu.

An n -dimensional *right-angled Coxeter orbifold* is a special n -orbifold locally modelled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^k$ of the standard $(\mathbb{Z}_2)^k$ -action on \mathbb{R}^n . Here, an $(\mathbb{Z}_2)^k$ -action on \mathbb{R}^n is *standard* if it can be generated by reflections across some coordinate hyperplanes in \mathbb{R}^n . A *right-angled Coxeter complex* is a special q-CW complex defined by Poddar-Sarkar [14] satisfying that its q-cells (i.e. the quotient

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of a cell by a finite group) are the orbit of e^n by a standard $(\mathbb{Z}_2)^k$ -action, and its all attaching maps ϕ preserve local groups. Here, preserving local groups means preserving codimension. So we can define the *nerve* of a right-angled Coxeter complex. See Section 2.1 for more details.

Firstly, we define the Stiefel-Whitney classes of real orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$, where $D^n/(\mathbb{Z}_2)^k$ is a quotient orbifold of a standard $(\mathbb{Z}_2)^k$ -action on \mathbb{R}^n . Let $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$ be an m -dimensional orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$. Then by the definition of orbifold vector bundles, $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$ is determined by a real linear representation of $(\mathbb{Z}_2)^k$,

$$\rho : (\mathbb{Z}_2)^k \longrightarrow GL_m(\mathbb{R}).$$

Furthermore, by the equivalence of orbifold vector bundles (see Lemma 3.2), $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$ is determined by an $m \times k$ matrix,

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^k \end{pmatrix}_{m \times k}$$

where $x_j^i = \pm 1$. We call C the *characteristic matrix* of $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$. Then the *total Stiefel-Whitney calss* of $\pi : E \rightarrow D^n/(\mathbb{Z}_2)^k$ is defined as:

$$(1.1) \quad w(E) = \prod_{i=1}^m \left(1 + \sum_{j=1}^k \frac{1 - x_i^j}{2} s_j \right) \in H_{orb}^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

where $H_{orb}^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) = \mathbb{Z}_2[s_1, \dots, s_k] / (s_i^2, \forall i)$.

Let X be an arbitrary right-angled Coxeter complex, and $\pi : E \rightarrow X$ be a real m -dimensional orbifold vector bundle over X . Next, we define the Stiefel-Whitney classes of $\pi : E \rightarrow X$. Let X_{reg} be the subcomplex of X consisting of all *regular cells* (i.e., associated finite group is trivial). Then X/X_{reg} is a wedge sum of some right-angled Coxeter complex,

$$X/X_{reg} = \bigvee H.$$

Each component H contains no regular cell except the unique 0-cell. Let $\mathcal{H} = \{F_1, \dots, F_\eta\}$ be the local codimension-one faces of H . Then the vector bundle $\pi : E \rightarrow X$ induces a vector bundle over H , denoted by $\pi_H : E_H \rightarrow H$. Then there is also an $m \times \eta$ characteristic matrix, denoted by C_H . Notice that there is a simplicial map from the nerve $\mathcal{N}(H)$ to $\Delta^\eta = [v_1, \dots, v_\eta]$,

$$j : \mathcal{N}(H) \longrightarrow \Delta^\eta.$$

Assume that $\{s_1, \dots, s_\eta\}$ is the vertices set of $\mathcal{N}(H)$, then $j(s_i) = v_i$ and $j(\delta)$ is a simplex in Δ^η spanned by $\{j \circ \delta(s_1), \dots, j \circ \delta(s_k)\}$ for simplex $\delta : [s_{i_1}, \dots, s_{i_k}] \rightarrow \mathcal{N}(H)$. Clearly, j induces a homomorphism

$$j^* : \mathbb{Z}_2[v_1, \dots, v_\eta] \longrightarrow \mathcal{R}_H.$$

Then the Stiefel-Whitney class of $\pi_H : E_H \rightarrow H$ is defined as:

$$w(E_H) = j^*(w(E_{C_H})) \in \mathcal{R}_H \subset H^*(X; \mathbb{Z}_2).$$

where \mathcal{R}_H is a sub-ring of $H^*(X; \mathbb{Z}_2)$, which is generated by the duals of locally codimension-one faces in H . Finally, the *total Stiefel-Whitney class* of $\pi : E \rightarrow X$ is defined as:

$$(1.2) \quad w(E) = w(E_{reg}) \cdot \prod_H w(E_H),$$

where $\pi|_{X_{reg}} : E_{reg} \rightarrow X_{reg}$ is a restriction of $\pi : E \rightarrow X$ on X_{reg} .

Similar to the usual cases, the Stiefel-Whitney classes of right-angled Coxeter complexes satisfies the following axioms.

PROPOSITION 1. *There is a unique sequence of functions w_1, w_2, \dots assigning to each real orbifold vector bundle $E \rightarrow B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, depending only on the isomorphism type of E , such that*

- (a) $w_i(f^*(E)) = f^*(w_i(E))$ for a pullback $f^*(E)$, where f is an orbifold map which preserves local groups.
- (b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$ where $w = 1 + w_1 + w_2 + \dots \in H^*(B; \mathbb{Z}_2)$.
- (c) $w_i(E) = 0$ if $i > \dim E$.
- (d) For the canonical line bundle $E \rightarrow \mathbb{R}P^\infty$, $w_1(E)$ is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$. Meanwhile, for the nontrivial line bundle $\tilde{E} \rightarrow D^1/\mathbb{Z}_2$, $w_1(\tilde{E})$ is a generator of $H_{orb}^1(D^1/\mathbb{Z}_2; \mathbb{Z}_2)$.

REMARK 1. Orbifold Stiefel-Whitney classes of right-angled Coxeter complexes are generalizations of ordinary Stiefel-Whitney classes.

As an application, we have the following conclusion.

THEOREM 1. *Let P be an n -dimensional simple polytope. Then P is the product of two simple polytopes P_1 and P_2 with dimensions n_1 and n_2 , respectively, if and only if, as right-angled Coxeter orbifolds, $w_n(TP) = w_{n_1}(TP_1) \cdot w_{n_2}(TP_2)$.*

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we define Stiefel-Whitney classes of real orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$. In Section 4, we consider the general cases, that is, Stiefel-Whitney classes of real orbifold vector bundles over a right-angled Coxeter complex. In Section 5, we prove Theorem 1, and give some examples.

2. Preliminaries

2.1. Right-angled Coxeter orbifolds and right-angled Coxeter complexes. An n -dimensional *right-angled Coxeter orbifold* is a special n -orbifold locally modelled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^n$ of the standard $(\mathbb{Z}_2)^n$ -action on \mathbb{R}^n by reflections across the coordinate hyperplanes. So a right-angled Coxeter n -orbifold is naturally a manifold with corners. (The notion of manifold with corners can be referred to Davis [6, Chapter 10]).

A *right-angled Coxeter complex* X is a special q-CW complex defined by Poddar-Sarkar [14] satisfying that its q-cells (i.e. the quotient of a cell by a finite group) are the orbit of e^n by a standard reflective action of $(\mathbb{Z}_2)^k$, and all attaching maps ϕ are required to preserve local groups. For a right-angled Coxeter cell $e^n/(\mathbb{Z}_2)^k$, we call $(\mathbb{Z}_2)^k$ the *local group* of $e^n/(\mathbb{Z}_2)^k$, and let $\Phi : D^n/(\mathbb{Z}_2)^k \rightarrow X$ be the characteristic map for $e^n/(\mathbb{Z}_2)^k$. If the local group of a right-angled Coxeter cell is trivial, then this cell is called *regular*, otherwise is called *singular*. All the regular cells form a

sub-complex of X which is denoted by X_{reg} , and denote the set of singular cells by X_{sing} . See Lü-Wu-Yu [12] for more details.

For each right-angled Coxeter cell $e^n/(\mathbb{Z}_2)^k$, the standard $(\mathbb{Z}_2)^k$ -action on e^n induces a natural facial structure on the singular point set of $e^n/(\mathbb{Z}_2)^k$. Attaching maps preserving local groups is equivalent to preserve local codimension. So each right-angled Coxeter complex has a facial structure such that each face has a well-defined local codimension. A local codimension-one face in X is called a *facet* of X . Under the setting, we can define the nerve of X . In detail, the *nerve* of X , denoted by $\mathcal{N}(X)$, is a poset on the facet set of X satisfying that:

- Each codimension- k face $f \subset F_1 \cap \cdots \cap F_k$ of X determines a $(k-1)$ -simplex $[F_1, \cdots, F_k] \rightarrow \mathcal{N}(X)$, where F_1, \cdots, F_k are some facets of X .

REMARK 2. The nerve of a manifold with corners is defined similarly, one can refer in [6].

EXAMPLE 1. Let P be an n -dimensional simple polytope. Then there is a natural right-angled Coxeter orbifold structure on P such that each codimension- k face of P has local group $(\mathbb{Z}_2)^k$ which is generated by reflections associated with k facets. The cone of once barycentric subdivision of the nerve of P gives a cubical decomposition of P , denoted by $\mathcal{C}(P)$, which is called the *standard cubical decomposition* of P . Clearly, each k -cube c^k in $\mathcal{C}(P)$ can be represented as the orbit of $[-1, 1]^k$ by a standard $(\mathbb{Z}_2)^k$ -action. Hence, $\mathcal{C}(P)$ is a right-angled Coxeter complex.

2.2. Cohomology rings of right-angled Coxeter complexes. For a right-angled Coxeter complex, one can define a cellular chain complex. By the result in Lü-Wu-Yu [12].

THEOREM 2 ([12]). *Let X be an right-angled Coxeter complex. Then*

$$(2.1) \quad H_{orb}^i(X) = \bigoplus_{f \in T} H^{i-l(f)}(f)$$

where T is the set of faces of X (include its underlying space as an n -face), and $l(f)$ is the local codimension of face f in X .

Actually, the orbifold cohomology groups and cup product of a right-angled Coxeter complex is defined as the usual cohomology groups and cup product of its blow-up discussed in [12].

EXAMPLE 2 (Cohomology of a closed right-angled Coxeter cell). Let $D^n/(\mathbb{Z}_2)^k$ be a closed right-angled Coxeter cell where $k \leq n$. Then $D^n/(\mathbb{Z}_2)^k = D^k/(\mathbb{Z}_2)^k \times D^{n-k} \simeq D^k/(\mathbb{Z}_2)^k = (D^1/(\mathbb{Z}_2))^k$. So

$$H_{orb}^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, s_2, \cdots, s_k]/(s_i^2, \forall i).$$

EXAMPLE 3. Let X be a right-angled Coxeter complex. Let H be a connected component of the singular set of X with nerve \mathcal{N} . Then each facet in H corresponds to a vertex of \mathcal{N} , and so, corresponds to a generator of $H^1(X; \mathbb{Z}_2)$. All facets in H , denoted by $\{F_1, \cdots, F_\eta\}$, generates a sub-ring of $H^*(X; \mathbb{Z}_2)$, which is isomorphic to

$$\mathcal{R}_H = \mathbb{Z}_2[s_1, \cdots, s_\eta]/(I_H + J_H)$$

where I_H is the Stanley-Reisner ideal of H and $J_H = (s_i^2, \forall i)$.

2.3. Orbifold vector bundle. The definition of orbifold vector bundle refers in [16, 5].

DEFINITION 1 (Orbifold vector bundle, [16]). Let E and B be two orbifolds with orbifold structures $\mathcal{U}^* = \{U^*, G^*, \psi^*\}$ and $\mathcal{U} = \{U, G, \psi\}$, respectively. An n -dimensional *orbifold vector bundle* is an C^∞ -orbifold map $\pi : E \rightarrow B$ satisfying the following conditions:

- ★ (Local trivialization) There exists an one-to-one correspondence between $\{U^*, G^*, \psi^*\}$ and $\{U, G, \psi\}$ such that $U^* = U \times \mathbb{R}^m$ and denoting by $\bar{\pi}_*$ the projection $U^* \rightarrow U$, we have

$$\pi \circ \psi^* = \psi \circ \bar{\pi}_*.$$

- ★★ Let $\{U^*, G^*, \psi^*\}$, $\{U, G, \psi\}$; $\{U^{*'}, G^{*'}, \psi^{*'}\}$, $\{U', G', \psi'\}$ be two pairs of corresponding l.u.s satisfying $\psi(U) \subset \psi'(U')$. Then $\psi^*(U^*) \subset \psi^{*'}(U^{*'})$ and there exists a one-to-one correspondence $\lambda \leftrightarrow \lambda^*$ between injections $\lambda : \{U, G, \psi\} \rightarrow \{U', G', \psi'\}$ and $\lambda^* : \{U^*, G^*, \psi^*\} \rightarrow \{U^{*'}, G^{*'}, \psi^{*'}\}$ such that for $(p, q) \in U^* = U \times \mathbb{R}^m$ we have

$$\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$$

with $g_\lambda(p) \in GL_m(\mathbb{R})$. The mapping $g_\lambda : U \rightarrow GL_m(\mathbb{R})$ is a C^∞ -map satisfying the relation

$$(2.2) \quad g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \cdot g_\lambda(p)$$

for any injections $\{U, G, \psi\} \xrightarrow{\lambda} \{U', G', \psi'\} \xrightarrow{\mu} \{U'', G'', \psi''\}$.

An orbifold vector bundle is a composite concept of map $\pi : E \rightarrow B$ and the equivalent class of orbifold structure pair $(\mathcal{U}, \mathcal{U}^*)$, where two pairs of orbifold vector bundles $(\mathcal{U}, \mathcal{U}_1^*)$ and $(\mathcal{U}, \mathcal{U}_2^*)$ are *directly equivalent* if there exists a C^∞ -map $\delta_U : U \rightarrow GL_m(\mathbb{R})$ satisfying that:

- For any $(U, G, \psi) \in \mathcal{U}$ there exist $(U \times \mathbb{R}^m, G_1^*, \psi_1^*) \in \mathcal{U}_1^*$ and $(U \times \mathbb{R}^m, G_2^*, \psi_2^*) \in \mathcal{U}_2^*$ such that

$$\psi_1^*(p, q) = \psi_2^*(p, \delta_U(p)q).$$

- For any injection $\lambda : \{U, G, \psi\} \rightarrow \{U', G', \psi'\}$

$$(2.3) \quad g_\lambda^2(p) = \delta_{U'}(\lambda(p))g_\lambda^1(p)\delta_U^{-1}(p),$$

REMARK 3. For each $b \in B$, there is a chart (U, G, ψ) such that $b \in U/G$. For each \bar{b} in $\psi^{-1}(b)$, the $\bar{\pi}^{-1}(\bar{b})$ is a real vector space \mathbb{R}^n . Then

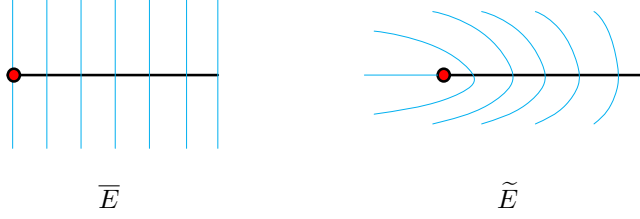
$$\pi^{-1}(b) \cong \bar{\pi}^{-1}(\bar{b})/G^*$$

Thus an orbifold vector bundle is not always a vector bundle in the usual sense because its fiber may not be a vector space. The above definition of orbifold vector bundles can be extended to orbispace, we omit it. The notion of orbispace refer in [4].

EXAMPLE 4. Let $D^1/\mathbb{Z}_2 = [-1, 1]/\mathbb{Z}_2$ be a closed right-angled Coxeter 1-cell. Then there exists two orbifold line bundles over D^1/\mathbb{Z}_2 .

$$\bar{E} = D^1/\mathbb{Z}_2 \times \mathbb{R} = D^1 \times \mathbb{R}/(x, y) \sim (-x, y)$$

$$\tilde{E} = D^1 \times \mathbb{R}/(x, y) \sim (-x, -y)$$

FIGURE 1. Orbifold line bundles over D^1/\mathbb{Z}_2

Let $\lambda : D^1 \rightarrow D^1$ be an injection which maps $x \in D^1$ to $-x$. Let \bar{g}_λ be a constant map which corresponds to the line bundle \bar{E} . For any $x \in D^1$, $\bar{g}_\lambda(x) = (1) \in GL_1(\mathbb{R})$. Let \tilde{g}_λ be the map which corresponds to the line bundle \tilde{E} . Then for any $x \in D^1$ $\tilde{g}_\lambda(x) = (-1) \in GL_1(\mathbb{R})$.

The line bundle \tilde{E} is not a usual vector bundle, which is called the *canonical line bundle* of D^1/\mathbb{Z}_2 . Meanwhile, \tilde{E} is also the orbifold tangent bundle of D^1/\mathbb{Z}_2 . Moreover, let $q : \bar{E} \rightarrow \tilde{E}$ be a quotient map defined by $(0, y) \sim (0, -y)$, then bundle \bar{E} can be viewed as a pull back of bundle \tilde{E} .

DEFINITION 2 (bundle map, [16]). Let $\pi_1 : E_1 \rightarrow B_1$ and $\pi_2 : E_2 \rightarrow B_2$ be two vector bundles with orbifold structure $(\mathcal{U}_1, \mathcal{U}_1^*)$ and $(\mathcal{U}_2, \mathcal{U}_2^*)$, respectively. A system of mappings $h^* = \{h_{U_1}\}$ is called a $(C^\infty\text{-})$ orbifold bundle map if the following conditions are satisfied:

(i) There exists a correspondence $\{U_1, G_1, \psi_1\} \rightarrow \{U_2, G_2, \psi_2\}$ from \mathcal{U}_1 to \mathcal{U}_2 , such that for any $\{U_1, G_1, \psi_1\}$ we have a C^∞ -map $h_{U_1}^*$ from $U_1 \times \mathbb{R}^m$ to $U_2 \times \mathbb{R}^m$ and a C^∞ -map h_{U_2} from U_1 to U_2 such that

$$(2.4) \quad h_{U_1}^*(p, q) = (h_{U_1}(p), r_{U_1}(p)q)$$

with $r_{U_1}(p) \in GL(m)$. r_{U_1} is a C^∞ -map from U_1 to $GL(m)$.

(ii) Let $\{U_1, G_1, \psi_1\}, \{U'_1, G'_1, \psi'_1\}$ be l.u.s in \mathcal{U}_1 such that $\psi_1(U_1) \subset \psi'_1(U'_1)$ and $\{U_2, G_2, \psi_2\}, \{U'_2, G'_2, \psi'_2\}$ be the corresponding l.u.s in \mathcal{U}_2 . Then for any injection $\lambda_1 : \{U_1, G_1, \psi_1\} \rightarrow \{U'_1, G'_1, \psi'_1\}$ there exists an injection $\lambda_2 : \{U_2, G_2, \psi_2\} \rightarrow \{U'_2, G'_2, \psi'_2\}$ such that

$$\lambda_2^* \circ h_{U_1}^* = h_{U'_1}^* \circ \lambda_1^*.$$

(Hence also $\lambda_2 \circ h_{U_1} = h_{U'_1} \circ \lambda_1$.) We assume further that

$$g_{\lambda_2}(h_{U_1}(p)) = r_{U'}(\lambda_1(p))g_{\lambda_1}(p)r_{U_1}^{-1}(p).$$

Two orbifold vector bundles E_1, E_2 over B are *equivalent* if and only if there exists a bundle-map $h : E_1 \rightarrow E_2$ such that $(E_1, \mathcal{U}_1) \cong (E_2, \mathcal{U}_2)$ and $h_U : U \rightarrow U'$ is homeomorphism for each $U \in \mathcal{U}_1$.

THEOREM 3 ([16]). Let (M, \mathcal{U}) be an orbifold. If there is a system of C^∞ -maps $g_\lambda : U \rightarrow GL_m(\mathbb{R}), \lambda : (U, G, \psi) \rightarrow (U', G', \psi')$ being any injections, which satisfies the definition 1, then there exists a orbifold vector bundle over (M, \mathcal{U}) . Moreover, if two systems of $\{g_\lambda^1\}$ and $\{g_\lambda^2\}$ satisfy (2.3), then the associated orbifold vector bundles are equivalent.

REMARK 4. Since the equation (2.2) is hold for the tensor product and direct sum of vector spaces, one can construct the tensor product and direct sum of orbifold vector bundles.

EXAMPLE 5 (Orbifold tangent bundle, [16]). Let (M, \mathcal{U}) be an orbifold. Assuming that each U in a chart (U, G, ψ) is contained in \mathbb{R}^n , we fix a coordinate system $\{x^1, \dots, x^m\}$ in each U once for all. For any injection $\lambda : (U, G, \psi) \rightarrow (U^*, G^*, \psi^*)$, let

$$g_\lambda(\bar{p}) = \left(\frac{\partial x^{*i} \circ \lambda}{\partial x^j} \right)$$

which is the Jacobian matrix of λ at \bar{p} , where $\{x^i\}$ and $\{x^{*i}\}$ are the fixed coordinate systems in U and U^* , respectively.

2.4. The linear representation of $(\mathbb{Z}_2)^k$. All group actions in the next are supposed to be locally linear actions. The reflection across a coordinate hyperplane in \mathbb{R}^n is called a *standard reflection*.

LEMMA 2.1. *Let $\{A_i | A_i \in GL_n(\mathbb{R}), i = 1, \dots, k\}$ be a system of invertible matrices. If for any i, j , $A_i^2 = E$, $A_i A_j = A_j A_i$, then $\{A_i\}$ are simultaneously diagonalizable.*

Moreover, assume P is an invertible matrix such that $P^{-1} A_i P = \text{diag}(x_1^i, \dots, x_n^i)$ for each i . Then the matrix

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^k \end{pmatrix}_{n \times k}$$

is unique without consider the order of rows.

Each linear G -action of \mathbb{R}^n corresponds to a linear representation $\rho_G : G \rightarrow GL_n(\mathbb{R})$. The two actions of G and H are called *conjugate* if $\rho_G(G)$ and $\rho_H(H)$ are conjugate in $GL_n(\mathbb{R})$.

LEMMA 2.2. *Each nontrivial linear $(\mathbb{Z}_2)^k$ -action on \mathbb{R}^n is conjugate with an action whose generators can be presented as the composites of some standard reflections. Specially, when $k = n$, it is conjugate with a standard reflection action if $(\mathbb{Z}_2)^n$ acts effectively on \mathbb{R}^n .*

PROOF. Let $A \in GL_n(\mathbb{R})$. If $A^2 = E$, then A is diagonalizable. Assume that the associated diagonal matrix is $\text{diag}(x_1, \dots, x_n)$, then $x_i = \pm 1$ for any $i = 1, \dots, n$.

Let g_1, \dots, g_k be a set of irreducible generators of $(\mathbb{Z}_2)^k$. Then any linear $(\mathbb{Z}_2)^k$ -action determines a linear representation $\rho : \mathbb{Z}_2^k \rightarrow GL_n(\mathbb{R})$. By Lemma 2.1, $\{\rho(g_i) | i = 1, \dots, k\}$ are simultaneously diagonalizable. Assume that $D_i \simeq \rho(g_i)$ is the associated diagonal matrix for each i . Then the action determined by $\{D_i\}$ is generated by the composites of some standard reflections.

For $k = n$, we assume that D_1, \dots, D_n are the associated diagonal matrices. Then we can rechoose a set of generators of $(\mathbb{Z}_2)^k$ such that the associated diagonal matrices are standard reflection matrices. Actually, choose $D_i = \text{diag}(x_1^i, \dots, x_n^i)$ with $x_1^i = -1$, then if $x_1^j = -1$ for $D^j = \text{diag}(x_1^j, \dots, x_n^j)$, set $g_j g_i$ be a new generator. So we can obtain a system of diagonal matrices, denoted as D_1, \dots, D_n

as well, satisfying that $x_1^j = 1$ except $j = i$. By induction, we can get a set of generators which determine n standard reflection matrices. \square

LEMMA 2.3. *Let $\{s_1, \dots, s_k\}$ be the generators of group $(\mathbb{Z}_2)^k$. The linear action of $(\mathbb{Z}_2)^k$ on D^n is determined by a linear representation $\rho : (\mathbb{Z}_2)^k \rightarrow GL_n(\mathbb{R})$, moreover, is determined by the conjugate class of a series of linear transformations $\rho(s_1), \dots, \rho(s_k)$. Clearly, $\rho(s_1), \dots, \rho(s_k)$ are simultaneously diagonalizable.*

3. Orbifold Stiefel-Whitney classes of real orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$

In this section, we classify the real orbifold vector bundles over a closed right-angled Coxeter cell $D^n/(\mathbb{Z}_2)^k$ by algebrizing the vector bundle in terms of the representations of local group \mathbb{Z}_2^k . First, D^n/\mathbb{Z}_2^k is \mathbb{Z}_2 -closed, that is, there is an right-angled Coxeter complex structure of $D^n/(\mathbb{Z}_2)^k$ with all the boundary maps zero when we take \mathbb{Z}_2 -coefficient. The cohomology ring of $D^n/(\mathbb{Z}_2)^k$,

$$H_{orb}^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, s_2, \dots, s_k]/(s_i^2, \forall i).$$

LEMMA 3.1. *The m -dimensional orbifold vector bundles over a closed right-angled Coxeter cell $D^n/(\mathbb{Z}_2)^k$ one-to-one correspond to the linear actions of $(\mathbb{Z}_2)^k$ on $D^n \times \mathbb{R}^m$ satisfying (\star) in Definition 1.*

REMARK 5. A real orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$ may not be a right-angled Coxeter complex.

Let s_1, \dots, s_k be the generators of $(\mathbb{Z}_2)^k$. By Lemma 2.3, a linear $(\mathbb{Z}_2)^k$ -action on $D^n \times \mathbb{R}^m$ corresponds to a linear representation

$$\rho : (\mathbb{Z}_2)^k \longrightarrow GL_{n+m}(\mathbb{R}),$$

moreover, corresponds to matrices $\rho(s_1), \dots, \rho(s_k)$.

By lemma 2.2 and the equivalence of orbifold structure pair in Definition 1, we can assume that $\rho(s_1), \dots, \rho(s_k)$ are diagonal matrices whose diagonal elements are ± 1 . According to (\star) condition, we assume that

$$\rho(s_i) = \text{diag}(1_1, \dots, -1_i, \dots, 1_n, x_1^i, \dots, x_m^i).$$

That is the following lemma.

LEMMA 3.2. *Any orbifold vector bundle over D^n/\mathbb{Z}_2^k corresponds to a linear representation $\rho : (\mathbb{Z}_2)^k \longrightarrow GL_{n+m}(\mathbb{R})$, satisfying that*

$$\rho(s_i) = \text{diag}(1_1, \dots, -1_i, \dots, 1_n, x_1^i, \dots, x_m^i)$$

where s_1, \dots, s_k is a system of generators of $(\mathbb{Z}_2)^k$.

Then we can obtain an $m \times k$ matrix

$$C = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^k \end{pmatrix}_{m \times k}$$

which is called the *characteristic matrix* of $E \rightarrow D^n/(\mathbb{Z}_2)^k$. Then the *orbifold Stiefel-Whitney class* of $E \rightarrow D^n/(\mathbb{Z}_2)^k$ is defined by

$$(3.1) \quad w(E) = \prod_{i=1}^m \left(1 + \sum_{j=1}^k \frac{1 - x_i^j}{2} s_j\right) \in H_{orb}^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

If two orbifold vector bundles E_1, E_2 over $D^n/(\mathbb{Z}_2)^k$ (s_1, \dots, s_k are generators of $(\mathbb{Z}_2)^k$) are equivalent, then by Lemma 2.3, the associated characteristic matrices obtained from $\{\rho_1(s_i) | i = 1, \dots, k\}$ and $\{\rho_2(s_i) | i = 1, \dots, k\}$ are the same if we do not consider the order of rows. So $w(E_1) = w(E_2)$, w is well-defined.

REMARK 6. Let $\lambda : D^n \rightarrow D^n$ be an injection in Definition 1. By [16, Lemma 2], all injections $\lambda : D^n \rightarrow D^n$ one-to-one correspond to the elements in $(\mathbb{Z}_2)^k$. Chosen an injection $\lambda_0 : D^n \rightarrow D^n$ (for example, identity map) which corresponds to the unit element of $(\mathbb{Z}_2)^k$, and the associated C^∞ -map is g_{λ_0} . Assume that the generator s_i of $(\mathbb{Z}_2)^k$ corresponds to $\lambda_i : D^n \rightarrow D^n$. Then by $(\star\star)$ condition in Definition 1, $\rho(s_i)$ here determines a C^∞ -map g_{λ_i} . Hence, each characteristic matrix C determines a system of $g_\lambda : D^n \rightarrow GL_m(\mathbb{R})$. By Theorem 3, each C determines an orbifold vector bundle over $D^n/(\mathbb{Z}_2)^k$.

LEMMA 3.3 (Whitney sum of orbifold vector bundles over $D^n/(\mathbb{Z}_2)^k$). *Let $E_1 \oplus E_2$ be the Whitney sum of orbifold vector bundles $\pi_1 : E_1 \rightarrow D^n/(\mathbb{Z}_2)^k$ and $\pi_2 : E_2 \rightarrow D^n/(\mathbb{Z}_2)^k$. Then*

$$w(E_1 \oplus E_2) = w(E_1)w(E_2)$$

PROOF. Let the characteristic matrices of $\pi_1 : E_1 \rightarrow D^n/(\mathbb{Z}_2)^k$ and $\pi_2 : E_2 \rightarrow D^n/(\mathbb{Z}_2)^k$ are $M_{m_1 \times k}$ and $N_{m_2 \times k}$, respectively. Then the characteristic matrix of $E_1 \oplus E_2$ is $\begin{pmatrix} M \\ N \end{pmatrix}_{(m_1+m_2) \times k}$. So

$$w(E_1 \oplus E_2) = \prod_{i=1}^{m_1+m_2} \chi_i \left(\begin{pmatrix} M \\ N \end{pmatrix} \right) = \prod_{i=1}^{m_1} \chi_i(M) \cdot \prod_{i=1}^{m_2} \chi_i(N) = w(E_1)w(E_2).$$

□

Let $h : D^n/(\mathbb{Z}_2)^k \rightarrow D^1/\mathbb{Z}_2$ be an orbifold map. Then by definition of orbifold map, there is a homomorphism between their local groups, denoted by $h_* : (\mathbb{Z}_2)^k \rightarrow \mathbb{Z}_2$. Notice that h_* can be characterized by a k -column $(y^1, \dots, y^k) \in (\mathbb{Z}_2)^k$ where $y^i = \pm 1$. And (y^1, \dots, y^k) determines a line bundle $\pi : E^1 \rightarrow D^n/(\mathbb{Z}_2)^k$ over $D^n/(\mathbb{Z}_2)^k$. So we define $[D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \xrightarrow{f} Vect_1(D^n/(\mathbb{Z}_2)^k)$ by $f(h) =: E_1$.

LEMMA 3.4. *The composition*

$$[D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \xrightarrow{f} Vect_1(D^n/(\mathbb{Z}_2)^k) \xrightarrow{w_1} H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

is a bijection.

PROOF. $D^n/(\mathbb{Z}_2)^k = D^k/(\mathbb{Z}_2)^k \times D^{n-k} \simeq D^k/(\mathbb{Z}_2)^k = (D^1/\mathbb{Z}_2)^k$, where “ \simeq ” stands for “homotopy equivalent to” in the category of orbispaces ([4, Definition 3.2.2]). So

$$\begin{aligned} [D^n/(\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] &= [(D^1/\mathbb{Z}_2)^k, D^1/\mathbb{Z}_2] \cong Hom((\mathbb{Z}_2)^k, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k, \\ Vect_1(D^n/(\mathbb{Z}_2)^k) &= Vect_1(D^k/(\mathbb{Z}_2)^k) = (\mathbb{Z}_2)^k, \quad (\text{Actions of } (\mathbb{Z}_2)^k \text{ on } \mathbb{R}^1), \end{aligned}$$

$$H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$$

are finite set. So it is sufficient to prove that f and w_1 are injections. Clearly, f is an injections.

Each line bundle $\pi : E^1 \rightarrow D^1/(\mathbb{Z}_2)^k$ over $D^1/(\mathbb{Z}_2)^k$ can be characterized by a sequence (x^1, \dots, x^k) where $x^i = \pm 1$. Then $w_1(E^1) = \sum_{j=1}^k \frac{1-x^j}{2} s_j \in H^1(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2)$. Thus w_1 is an injection as well. \square

EXAMPLE 6. Let $\tilde{\pi} : \tilde{E} \rightarrow D^1/\mathbb{Z}_2$ is the nontrivial line bundle in Example 4. Then

$$w(\tilde{E}^1) = 1 + s,$$

where $H^*(D^1/\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2[s]/(s^2)$.

There is a bundle map $f : \tilde{E}^1 \rightarrow \tilde{E}^1$ such that $f^*(w(\tilde{E}^1)) = 1$. Furthermore, the orbifold Stiefel-Whitney class of tangent bundle of $D^n/(\mathbb{Z}_2)^k$ is

$$w(T(D^n/(\mathbb{Z}_2)^k)) = w(T(D^{n-k}) \times (T(D^1/\mathbb{Z}_2))^k) = \prod_{i=1}^k (1 + s_i) \in H^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2).$$

EXAMPLE 7. There exist two different real vector bundles with the same orbifold Stiefel-Whitney classes. Let $E \rightarrow D^1/\mathbb{Z}_2$ be a 2-dimensional real orbifold vector bundle whose characteristic matrix is $(-1, -1)^T$. Then

$$w = (1 + s)^2 = 1.$$

Clearly, E is nontrivial.

4. The real orbifold vector bundle and orbifold Stiefel-Whitney classes over a right-angled Coxeter complex

Let X be an n -dimensional right-angled Coxeter complex. Then there is a natural open cover of X described as follows:

Firstly, we take a slightly shrunk open set $U(e^n/W)$ for each Coxeter n -cell e^n/W in X . Then $X - \bigcup U(e^n/W)$ is homotopic to the $(n-1)$ -skeleton of X . Furthermore, taking an open set $U(e^{n-1}/W)$ for the neighborhood of each shrunk Coxeter $(n-1)$ -cell e^{n-1}/W in X , we have $X - \bigcup U(e^n/W) - \bigcup U(e^{n-1}/W)$ is homotopic to the $(n-2)$ -skeleton of X . Inductively, we can obtain an open cover of X indexed by right-angled Coxeter complex structure of X . (Every open sets associated with e^k/W can deformation retract on e^k/W). And we always assume there is a well-defined orbispace structure on this open cover. See [4] for the notion of orbispace structure.

Let $\pi : E \rightarrow X$ is a real orbifold vector bundle over X , and $\pi|_{X_{reg}} : E|_{X_{reg}} \rightarrow X_{reg}$ is the restriction of $\pi : E \rightarrow X$ on the regular subcomplex X_{reg} of X . Then $\pi|_{X_{reg}} : E|_{X_{reg}} \rightarrow X_{reg}$ is a usual vector bundle over CW complex X_{reg} .

LEMMA 4.1. *Any real orbifold vector bundle over X can be viewed as the extension of a usual vector bundle over X_{reg} .*

PROOF. Let $\pi' = \pi|_{X_{reg}} : E|_{X_{reg}} \rightarrow X_{reg}$ is a usual vector bundle over X_{reg} . First, let $\pi_{D^1/\mathbb{Z}_2} : E(D^1/\mathbb{Z}_2) \rightarrow D^1/\mathbb{Z}_2$ be an orbifold vector bundle over D^1/\mathbb{Z}_2 . By Lemma 2.3, we can assume that the characteristic matrix of $p = \pi_{D^1/\mathbb{Z}_2} : E(D^1/\mathbb{Z}_2) \rightarrow D^1/\mathbb{Z}_2$ is a diagonal matrix $A(p)$. Then, gluing the bundles $E|_{X_{reg}}$ and $E(D^1/\mathbb{Z}_2)$ together along $E(D^1/\mathbb{Z}_2)|_{\partial D^1/\mathbb{Z}_2}$ and $E'|_{\partial \Phi(D^1/\mathbb{Z}_2)}$ to obtain

an orbifold vector bundle over $X_{reg} \cup D^1/\mathbb{Z}_2$. Moreover, we can get an orbifold vector bundle over $X_{reg} \cup X_{sing}^1$, where X_{sing}^1 is the set of singular 1-cells of X .

Let D^2/W be a singular 2-cell of X with two singular 1-cells D_1^1/\mathbb{Z}_2 and D_2^1/\mathbb{Z}_2 in its boundary (D_1^1/\mathbb{Z}_2 and D_2^1/\mathbb{Z}_2 may correspond to the same singular 1-cell, see Example 8 and Example 10). Let $\pi_{D^2/W} : E(D^2/W) \rightarrow D^2/W$ be an orbifold vector bundle over D^2/W . Then the orbifold vector bundle over $X_{reg} \cup X_{sing}^1$ can be extended to an orbifold vector bundle over $X_{reg} \cup X_{sing}^1 \cup D^2/W$, if and only if the characteristic matrix of bundle $E(D^2/W)$ is $(A_1; A_2)$, where A_1, A_2 is the characteristic matrices of bundles $\pi|_{D_1^1/\mathbb{Z}_2}$ and $\pi|_{D_2^1/\mathbb{Z}_2}$. (Specially, when D_1^1/\mathbb{Z}_2 and D_2^1/\mathbb{Z}_2 correspond to the same singular 1-cell, if $W = \mathbb{Z}_2$, then the characteristic matrix of bundle $E(D^2/W)$ is $A_1 = A_2$, see Example 10. If $W = (\mathbb{Z}_2)^2$, then the characteristic matrix of bundle $E(D^2/W)$ is $(A_1; A_2)$ where $A_1 = A_2$, see Example 8). The orbifold vector bundle $E(D^2/W)$ is compatible with the bundles $\pi|_{D_1^1/\mathbb{Z}_2}$ and $\pi|_{D_2^1/\mathbb{Z}_2}$, which implies that the linear transformations determined by A_1, A_2 are simultaneously diagonalizable. That is, the characteristic matrix of $\pi|_{D^2/W}$ can be presented as $(A_1; A_2)$.

By inductions, we obtain an orbifold vector bundle $\pi : E \rightarrow X$ over right-angled Coxeter complex $X = X_{reg} \cup X_{sing}^1 \cup \dots \cup X_{sing}^n$ by extending $\pi' = \pi|_{X_{reg}} : E|_{X_{reg}} \rightarrow X_{reg}$. For any right-angled Coxeter cell $e^l/(\mathbb{Z}_2)^k$ in X , the characteristic matrix of $\pi|_{e^l/(\mathbb{Z}_2)^k}$ can be presented as $(A_1; \dots; A_k)$. Now A_1, \dots, A_k is the characteristic matrices of the restrictions of π on some associated singular 1-cells in X . \square

Notice that $X_{reg} \simeq |X|$ and X/X_{reg} is the wedge sum of some right-angled Coxeter complexes. Each component of X/X_{reg} contains no regular cell except the unique 0-cell. Let H be a component of X/X_{reg} , and $\mathcal{N}(H)$ is the nerve of H . The orbifold vector bundle $\pi : E \rightarrow X$ induces an orbifold vector bundle over H , denoted by $\pi_H : E_H \rightarrow H$. According to the construction of $\pi : E \rightarrow X$, there is a system of diagonal matrices $\{A_1, \dots, A_\eta\}$ corresponding to the codimension-one faces of H , such that for each singular cell $e^l/(\mathbb{Z}_2)^k$ in H the characteristic matrix of $\pi|_{e^l/(\mathbb{Z}_2)^k}$ is combined by some A_i . So we have the following conclusion.

LEMMA 4.2. *For each H there is an $m \times \eta$ characteristic matrix*

$$C_H = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^\eta \\ x_2^1 & x_2^2 & \cdots & x_2^\eta \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^\eta \end{pmatrix}_{m \times \eta}$$

where $A_i = \text{diag}(x_1^i, \dots, x_m^i)$ for $F_i \in \mathcal{N}(H)$ and η is the number of vertices of $\mathcal{N}(H)$.

Then C_H can determine a real orbifold vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$, denoted by $\pi_{C_H} : E_{C_H} \rightarrow D^\eta/(\mathbb{Z}_2)^\eta$. If H is a simple polytope, then H can be embedded into $D^\eta/(\mathbb{Z}_2)^\eta$, see [3, Page 93]. Now $\pi_H : E_H \rightarrow H$ is a restriction of $\pi_{C_H} : E_{C_H} \rightarrow D^\eta/(\mathbb{Z}_2)^\eta$.

LEMMA 4.3. *Let P be a simple polytope with η facets and a right-angled Coxeter orbifold structure. Then any real orbifold vector bundle over P can be viewed as the restriction of a real vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$ where η is the number of facets of P .*

PROOF. Let $\pi : E \rightarrow P$ be an m -dimensional real orbifold vector bundle over P characterized by a $m \times \eta$ matrix C . Then C determines a real orbifold vector bundle over $D^\eta/(\mathbb{Z}_2)^\eta$, denoted by $\pi_C : E_C \rightarrow D^\eta/(\mathbb{Z}_2)^\eta$.

Let $i : P \hookrightarrow D^\eta/(\mathbb{Z}_2)^\eta$ be an orbifold embedding. Then $\pi : E \rightarrow P$ is the restriction of $\pi_C : E_C \rightarrow D^\eta/(\mathbb{Z}_2)^\eta$ on P . \square

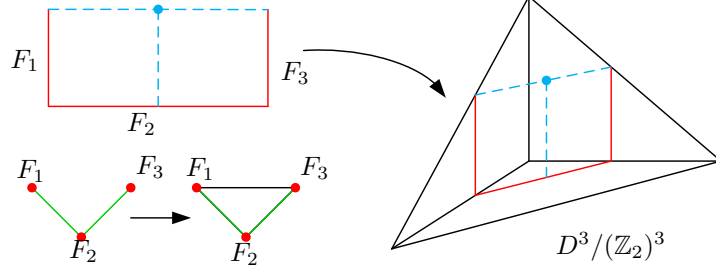


FIGURE 2. Embedding of right-angled orbifold complexes

In general, let $j : \mathcal{N}(H) \rightarrow \mathcal{N}(D^k/(\mathbb{Z}_2)^k) = \Delta^{k-1}$ be a simplicial map. Then there is an induced map

$$j^* : H^*(D^k/(\mathbb{Z}_2)^k; \mathbb{Z}_2) \longrightarrow \mathcal{R}_H < H^*(X; \mathbb{Z}_2).$$

The orbifold Stiefel-Whitney class of the part of H is defined as

$$(4.1) \quad w(E(H)) = j^*(w(E_{C_H})) \in \mathcal{R}_H < H^*(X; \mathbb{Z}_2)$$

LEMMA 4.4. $w(E(H))$ is well-defined.

PROOF. If there are three facets F_1, F_2, F_3 such that $F_1 \cap F_2 \neq \emptyset, F_2 \cap F_3 \neq \emptyset$ and $F_1 \cap F_3 = \emptyset$ (as shown in Figure 2), then $s_1 s_3 = 0$ in \mathcal{R}_H . If there is a sub-matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ (or $\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$) in $(A_1; A_2; A_3)$. Now it is possible that there is an another matrix of $E(H)$ with the sub-matrix $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ (or $\begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$) in $(A_1; A_2; A_3)$. Thus, characteristic matrix of E_H is not unique even if we do not consider the order of rows.

However, $s_1 s_3 = 0$ implies that

$$(1 + f(s) + s_1)(1 + f(s) + s_3) = (1 + f(s) + s_1 + s_3)(1 + f(s))$$

where $f(s)$ is a degree one polynomial which does not contain s_1, s_3 .

So the choice of matrices does not affect the value of $w(E(H)) = j^*(w(E_C))$. Thus, $w(E(H))$ is well-defined. \square

DEFINITION 3. The *total orbifold Stiefel-Whitney class* of real orbifold vector bundle $\pi : E \rightarrow X$ is defined as

$$(4.2) \quad w(E) = w(E|_{X_{reg}}) \cdot \prod_H w(E(H))$$

where $E|_{X_{reg}}$ is the restriction of $\pi : E \rightarrow X$ on the regular subcomplex X_{reg} of X , and $w(E(H)) = j^*(w(E_{C_H}))$ defined in Equation 4.1.

REMARK 7. • When X is a CW complex, $w(E)$ is the ordinary Stiefel-Whitney class of $\pi : E \rightarrow X$. Let f be a face of X .

- If f is not contractible, then there may exist a vector bundle over X such that some $w_i(E)$ is non-trivial in $H^i(f)$ for $i > 0$. See Example 10.

PROPOSITION 2. *Let $\pi_1 : E_1 \rightarrow X$ and $\pi_2 : E_2 \rightarrow X$ be two orbifold vector bundles over a right-angled Coxeter orbifold X . Then*

$$w(E_1 \oplus E_2) = w(E_1)w(E_2)$$

PROOF. Let $E = E_1 \oplus E_2$. Then by the definition of orbifold Stiefel-Whitney classes,

$$\begin{aligned} w(E) &= w(E|_{X_{reg}}) \cdot \prod_H w(E(H)) \\ &= w(E_1|_{X_{reg}})w(E_2|_{X_{reg}}) \cdot \prod_H w(E_1(H))w(E_2(H)) \\ &= w(E_1|_{X_{reg}}) \prod_H w(E_1(H)) \cdot w(E_2|_{X_{reg}}) \prod_H w(E_2(H)) \\ &= w(E_1)w(E_2) \end{aligned}$$

where $w(E|_{X_{reg}}) = w(E_1|_{X_{reg}})w(E_2|_{X_{reg}})$ is the case of usual CW complex, and the proof of $w(E(H)) = w(E_1(H))w(E_2(H))$ is similar to Lemma 3.3. \square

The orbifold Stiefel-Whitney classes of right-angled Coxeter complexes satisfy the following axioms.

PROPOSITION 3. *There is a unique sequence of functions w_1, w_2, \dots assigning to each real orbifold vector bundle $E \rightarrow B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, depending only on the isomorphism type of E , such that*

- $w_i(f^*(E)) = f^*(w_i(E))$ for a pullback $f^*(E)$, where f is an orbifold map which preserves local groups.
- $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$ where $w = 1 + w_1 + w_2 + \dots \in H^*(B; \mathbb{Z}_2)$.
- $w_i(E) = 0$ if $i > \dim E$.
- For the canonical line bundle $E \rightarrow \mathbb{R}P^\infty$, $w_1(E)$ is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$. Meanwhile, for the nontrivial line bundle $\tilde{E} \rightarrow D^1/\mathbb{Z}_2$, $w_1(\tilde{E})$ is a generator of $H^1(D^1/\mathbb{Z}_2; \mathbb{Z}_2)$.

REMARK 8. Orbifold Stiefel-Whitney classes of right-angled Coxeter complexes are generalizations of ordinary Stiefel-Whitney classes.

5. Application and examples

Next, we give an application of orbifold Stiefel-Whitney classes.

According to Lü-Ma-Sun [11], if a simplicial sphere K can be realized as the nerve of a simple polytope, then K is called a *polytopal sphere*.

LEMMA 5.1 (Lemma 3.6, [11]). *If $K = K_1 * K_2$ is a polytopal sphere, then K_1 and K_2 are polytopal spheres as well.*

THEOREM 4. *Let P be an n -dimensional simple polytope. Then P is the product of two simple polytopes P_1 and P_2 with dimensions n_1 and n_2 , respectively, if and only if, as right-angled Coxeter orbifolds, $w_n(TP) = w_{n_1}(TP_1) \cdot w_{n_2}(TP_2)$.*

PROOF. By Example 1, there is a right-angled Coxeter complex given by the standard cubical decomposition of P . Now all boundary maps in the cellular chain complex of P is zero, so

$$H^i(P; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{f_{n-i}}$$

and

$$H^*(P; \mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, \dots, s_m]/(I_P + J_P)$$

where I_P is the Stanley-Reisner ideal of P , $J_P = (s_i^2, \forall i)$ and m is the number of facets of P . Then

$$w(TP) = \prod_{i=1}^m (1 + s_i),$$

and

$$w_n(TP) = \sum_{F_{i_1} \cap \dots \cap F_{i_n} \in \text{Vert}(P)} s_{i_1} \cdots s_{i_n}$$

If $w_n(TP) = f(s_1, \dots, s_m) \cdot g(s_1, \dots, s_m)$. $w_n(TP)$ is homogeneous, so are f and g . Assume that $\deg(f) = n_1$ and $\deg(g) = n_2$ where $n_1 + n_2 = n$. Then

$$w_n(TP) = \sum_{F_{i_1} \cap \dots \cap F_{i_n} \in \text{Vert}(P)} s_{i_1} \cdots s_{i_n} = \left(\sum s_{j_1} \cdots s_{j_{n_1}} \right) \cdot \left(\sum s_{k_1} \cdots s_{k_{n_2}} \right).$$

Notice that $s_j \neq s_k$ and $s_j s_k \neq 0$ for any s_j of $f(s_1, \dots, s_m)$ and s_k of $g(s_1, \dots, s_m)$. Let $\mathcal{F}_f, \mathcal{F}_g$ be the facet sets corresponding to the variables of f and g , respectively. Then $\mathcal{F}(P) = \mathcal{F}_f \sqcup \mathcal{F}_g$. Then the nerve $\mathcal{N}(P)$ of P can be realized as the join of two subcomplexes K_1 and K_2 of $\mathcal{N}(P)$ which are spanned by \mathcal{F}_f and \mathcal{F}_g , respectively. By Lemma 5.1, K_1 and K_2 can be realized as the nerves of two right-angled Coxeter orbifolds. So P can be realized as the product of two simple polytopes.

Conversely, if $P^n \cong P_1^{n_1} \times P_2^{n_2}$, then $H^*(P; \mathbb{Z}_2) \cong H^*(P_1; \mathbb{Z}_2) \otimes H^*(P_2; \mathbb{Z}_2)$. Now $w(TP) = w(TP_1) \cdot w(TP_2)$, so $w_n(TP) = w_{n_1}(TP_1) \cdot w_{n_2}(TP_2)$. \square

Finally, some examples are listed.

EXAMPLE 8 (Orbifold Stiefel-Whitney class and bad orbifold). Let $B = e^0 \cup e^1/\mathbb{Z}_2 \cup e^2/(\mathbb{Z}_2)^2$ be a teardrop, and $\pi : E \rightarrow B$ be an m -dimensional vector bundle over B . Assume that the characteristic matrix of $\pi|_e^1/\mathbb{Z}_2$ is $A = (x_1, \dots, x_m)^T$ where $x_i = \pm 1$. Then the characteristic matrix of $\pi|_e^2/(\mathbb{Z}_2)^2$ is $(A; A)$. Now,

$$w(E) = \prod_{i=1}^m \left(1 + \frac{1 - x_i}{2} s\right) = 1 \text{ or } 1 + s \in \mathbb{Z}_2[s]/(s^2).$$

EXAMPLE 9. Let S^2/\mathbb{Z}_2 be a quotient orbifold given by a reflection action. Then the blow-up of S^2/\mathbb{Z}_2 is a quotient of S^2 by identifying a pair of antipodal points. So we have

$$H_{orb}^i(S^2/\mathbb{Z}_2; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & i = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\pi : E \rightarrow S^2/\mathbb{Z}_2$ be an arbitrary vector bundle over S^2/\mathbb{Z}_2 , then

$$w(E) = w(E|_{X_{reg}}) \cdot w(E_H) = w(E_H) = 1 \text{ or } 1 + s$$

where s is the generator of $H_{orb}^1(S^2/\mathbb{Z}_2; \mathbb{Z}_2)$. So $w_2 = 0$ for any vector bundle over S^2/\mathbb{Z}_2 .

EXAMPLE 10. Let Q be the quotient orbifold of \mathbb{Z}_2 on $S^1 \times [-1, 1]$ by $(x, y) \rightarrow (x, -y)$. We can decompose $S^1 \times [-1, 1]/\mathbb{Z}_2$ into $\{v, e_1, e_2, f\}$, one 0-cell, one regular 1-cells, one singular 1-cell and one 2-cell as shown in Figure 3.

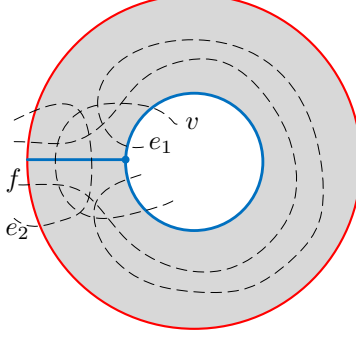


FIGURE 3. $S^1 \times [-1, 1]/\mathbb{Z}_2$

The blow-up complex of Q is a torus, so

$$H_{orb}^i(S^1 \times [-1, 1]/\mathbb{Z}_2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 1 \\ \mathbb{Z}_2, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Let S_1^1 and S_2^1 are blue circle and red circle in Figure 3. By Theorem 2, we have

$$H_{orb}^i(S^1 \times [-1, 1]/\mathbb{Z}_2; \mathbb{Z}_2) \cong \begin{cases} H^0(S_1^1; \mathbb{Z}_2), & i = 0 \\ H^1(S_1^1; \mathbb{Z}_2) \oplus H^0(S_2^1; \mathbb{Z}_2), & i = 1 \\ H^1(S_2^1; \mathbb{Z}_2), & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$H_{orb}^*(S^1 \times [-1, 1]/\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2[s_1, s_2]/(s_1^2, s_2^2)$$

where s_1 and s_2 are generators of $H^1(S_1^1; \mathbb{Z}_2)$ and $H^0(S_2^1; \mathbb{Z}_2)$, respectively. So $s_1 s_2$ in $H_{orb}^2(S^1 \times [-1, 1]/\mathbb{Z}_2; \mathbb{Z}_2)$ is non-trivial.

Let $\{U(v), U(e_1), U(e_2), U(f)\}$ be a open cover of $S^1 \times [-1, 1]/\mathbb{Z}_2$ as shown in Figure 3.

Let $T(S^1 \times [-1, 1]/\mathbb{Z}_2)$ be the orbifold tangent bundle of $S^1 \times [-1, 1]/\mathbb{Z}_2$. First, the tangent bundle of its underlying space $S^1 \times [0, 1] \simeq U(v) \cup U(e_1)$ is trivial. The orbifold tangent bundle of $U(e_2) \simeq e^1/\mathbb{Z}_2 \times e^1$ is $TU(e_2) \cong \tilde{E} \times (\mathbb{R} \times (0, 1))$, the product of \tilde{E} in Example 4 and the trivial line bundle over e^1 . Then we glue $TU(e_2)$ and $T(U(v) \cup U(e_1))$ together along the trivial bundle over $U(e_2) \cap U(v)$. Furthermore, $TU(f) \cong \tilde{E} \times Te^1$. Then we can obtain the orbifold tangent bundle of $S^1 \times [-1, 1]/\mathbb{Z}_2$ by gluing $TU(f)$ to the glued bundle $T(U(v) \cup U(e_1)) \cup TU(e_2)$.

$$w(T(S^1 \times [-1, 1]/\mathbb{Z}_2)) = w(TS^1)(1 + s) = 1 + s.$$

In the other way, $S^1 \times [-1, 1]/\mathbb{Z}_2 = [-1, 1] \times ([-1, 1]/\mathbb{Z}_2)/(-1, x) \sim (1, x)$. Let $[-1, 1] \times \tilde{E} \rightarrow [-1, 1] \times ([-1, 1]/\mathbb{Z}_2)$ be a line bundle over $[-1, 1] \times ([-1, 1]/\mathbb{Z}_2)$, where $\tilde{E} = [-1, 1] \times \mathbb{R}/(x, y) \sim (-x, -y)$ is the non-trivial bundle over $[-1, 1]/\mathbb{Z}_2$ defined in Example 4. Then $E = [-1, 1] \times \{[-1, 1] \times \mathbb{R}/(x, y) \sim (-x, -y)\}/(-1, x, y) \sim (1, x, -y)$ is a line bundle over $S^1 \times [-1, 1]/\mathbb{Z}_2$. Now

$$w(E) = w(E|_{X_{reg}}) \cdot w(E_H) = (1 + s_1)(1 + s_2).$$

So $w_2(E) = s_1 s_2$ is non-trivial in $H_{orb}^2(S^1 \times [-1, 1]/\mathbb{Z}_2; \mathbb{Z}_2)$.

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References

- [1] Alejandro Adem, Johann Leida and Yongbin Ruan, *Orbifolds and stringy topology*. Cambridge, New York, 2007.
- [2] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.
- [3] Victor M. Buchstaber and Taras E. Panov, *Toric topology*. Mathematical Surveys and Monographs, 204. American Mathematical Society, Providence, RI, 2015.
- [4] Weimin Chen, *A homotopy theory of orbispace*. Arxiv, (2001)
- [5] Weimin Chen and yongbin Ruan, *A new cohomology theory of orbifold*. Comm. Math. Phys. 248 (2004), no. 1, 1–31.
- [6] Michael W. Davis, *The geometry and topology of Coxeter groups*. London Mathematical Society Monographs Series 32, Princeton Univ. Press (2008).
- [7] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*. Duke Math. vol. 62 (1991), no. 2, 417–451.
- [8] Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*. London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York, 2000.
- [9] L. C. Grove and C. T. Benson, *Finite reflection groups*. Second edition. Graduate Texts in Mathematics, 99. Springer-Verlag, New York, 1985.
- [10] J. Gunawardena, B. Kahn and C. Thomas, *Stiefel-Whitney classes of real representations of finite groups*. J. Algebra 126 (1989), no. 2, 327–347.
- [11] Zhi Lü, Jun Ma and Yi Sun, *Elementary symmetric polynomials in Stanley-Reisner face ring*. ArXiv:1602.08837, 2016.
- [12] Zhi Lü, Lisu Wu and Li Yu, *An integral homology theory of Coxeter orbifolds*. 2021. manuscript.
- [13] John W. Milnor and James D. Stasheff, *Characteristic classes*. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.
- [14] Mainak Poddar and Soumen Sarkar, *On quasitoric orbifolds*. Osaka J. Math. 47 (2010), no. 4, 1055–1076.
- [15] Satake, *On a generalization of the notion of manifold*. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 359–363.
- [16] Satake, *The Gauss-Bonnet Theorem for V-manifolds*. Journal of the Mathematical Society of Japan, Vol. 9, No. 4 (1957), 464–492.
- [17] Christopher Seaton, *Characteristic classes of bad orbifold vector bundles*. J. Geom. Phys. 57 (2007), no. 11, 2365–2371.
- [18] Jean-Pierre Serre, *Linear representations of finite groups*. Translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.

- [19] C. B. Thomas, *Characteristic classes and the cohomology of finite groups*. Cambridge Studies in Advanced Mathematics, 9. Cambridge University Press, Cambridge, 1986.

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