韦达定理

设
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

如果 $f(x) = 0$ 有 n 个根 $x_1, x_2, \cdots, x_n$（计重数）。可设

$$
\begin{aligned}
f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\
&= A(x - x_1)(x - x_2) \cdots (x - x_n) \\
&= A[x^n - \sum_i x_i \times x^{n-1} + \sum_{i \neq j} x_i x_j \times x^{n-2} - \sum_{i_1, i_2, i_3 \text{互异}} x_{i_1} x_{i_2} x_{i_3} \times x^{n-3} + \cdots \\
&\quad + (-1)^{n-1} \sum_{i_1, i_2, \cdots, i_{n-1} \text{互异}} x_{i_1} x_{i_2} \cdots x_{i_{n-1}} \times x + (-1)^n x_1 x_2 \cdots x_n]
\end{aligned}
$$

对比 x^i 项的系数可得

$$
\begin{cases}
a_n = A \\
a_{n-1} = -\sum_i x_i \times A \\
a_{n-2} = \sum_{i \neq j} x_i x_j \times A \\
a_{n-3} = -\sum_{i_1, i_2, i_3 \text{互异}} x_{i_1} x_{i_2} x_{i_3} \times A \\
\cdots \\
a_1 = (-1)^{n-1} \sum_{i_1, i_2, \cdots, i_{n-1} \text{互异}} x_{i_1} x_{i_2} \cdots x_{i_{n-1}} \times A \\
a_0 = (-1)^n x_1 x_2 \cdots x_n \times A
\end{cases}
$$

可得韦达定理

$$
\begin{cases}
\textcolor{red}{\sum_i x_i = -\frac{a_{n-1}}{a_n}} \\
\sum_{i \neq j} x_i x_j = \frac{a_{n-2}}{a_n} \\
\sum_{i_1, i_2, i_3 \text{互异}} x_{i_1} x_{i_2} x_{i_3} = -\frac{a_{n-3}}{a_n} \\
\cdots \\
\sum_{i_1, i_2, \cdots, i_{n-1} \text{互异}} x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = (-1)^{n-1} \frac{a_{n-1}}{a_n} \\
\textcolor{red}{x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}}
\end{cases}
$$