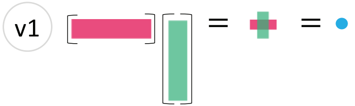
线性代数-5-附录

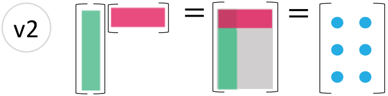
ref: The Art of Linear Algebra

2025 年 9 月 11 日

向量乘以向量——2 个视角



点积（数）



秩 1 矩阵

点积 $(\boldsymbol{a}\cdot\boldsymbol{b})$ 是一个数，用矩阵的语言可以表示为 $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$.

$\boldsymbol{ab}^{\mathrm{T}}$ 是一个矩阵 $(\boldsymbol{ab}^{\mathrm{T}} = A)$. 如果 a, b 都不为 0, 则结果 A 是秩为 1 的矩阵.

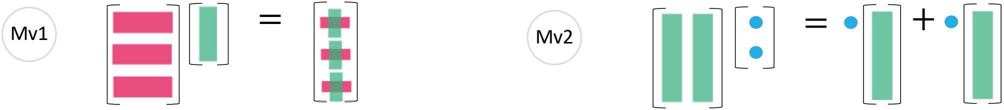
$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix} = \begin{bmatrix}x & y\\2x & 2y\\3x & 3y\end{bmatrix}$$

图: 向量乘以向量 - (v1), (v2)

矩阵乘以向量——2 个视角

- 一个矩阵乘以一个向量将产生三个点积组成的向量 **(Mv1)** 和一种 A 的列向量的线性组合.



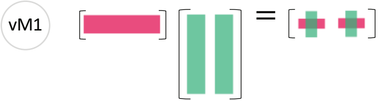
A 的行向量乘以向量 $\boldsymbol{x}$ 得到的 $A\boldsymbol{x}$, 是以点积为元素的列向量.

乘积 $A\boldsymbol{x}$ 是 A 的列向量的线性组合.

$$A\boldsymbol{x}=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}(x_1+2x_2)\\(3x_1+4x_2)\\(5x_1+6x_2)\end{bmatrix}$$

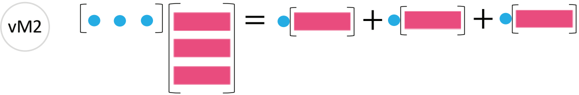
$$A\boldsymbol{x}=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\3\\5\end{bmatrix}+x_2\begin{bmatrix}2\\4\\6\end{bmatrix}$$

图: 矩阵乘以向量- (Mv1), (Mv2)



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} = [(y_1 + 3y_2 + 5y_3) \quad (2y_1 + 4y_2 + 6y_3)]$$

行向量 $\boldsymbol{y}$ 乘以 A 的列向量得到的 $\boldsymbol{y}A$ 是以点积为元素的行向量.

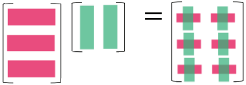


$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} = y_1[1 \quad 2] + y_2[3 \quad 4] + y_3[5 \quad 6]$$

乘积 $\boldsymbol{y}A$ 是 A 的行向量的线性组合.

图: 向量乘以矩阵 - (vM1), (vM2)

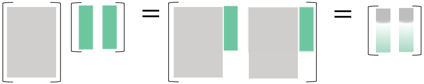
MM 1



每个元素为行向量和列向量的点积.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=\begin{bmatrix}(x_1+2x_2) & (y_1+2y_2)\\(3x_1+4x_2) & (3y_1+4y_2)\\(5x_1+6x_2) & (5y_1+6y_2)\end{bmatrix}$$

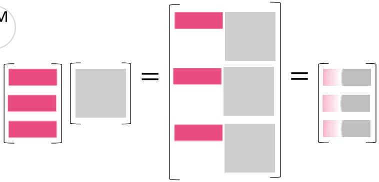
MM 2



$A\boldsymbol{x}$ 和 $A\boldsymbol{y}$ 是A的列向量的线性组合.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=A[\boldsymbol{x}\quad \boldsymbol{y}]=[A\boldsymbol{x}\quad A\boldsymbol{y}]$$

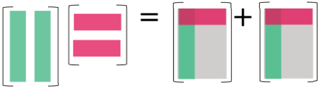
MM 3



乘积矩阵的每一行是第一个矩阵行的线性组合.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{a}_1^*\\\boldsymbol{a}_2^*\\\boldsymbol{a}_3^*\end{bmatrix}X=\begin{bmatrix}\boldsymbol{a}_1^*X\\\boldsymbol{a}_2^*X\\\boldsymbol{a}_3^*X\end{bmatrix}$$

MM 4

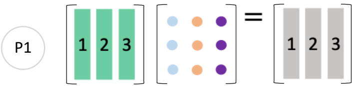


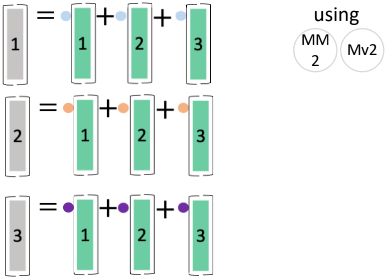
乘积矩阵 AB 是秩为 1 矩阵的和.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix}=[\boldsymbol{a}_1\quad \boldsymbol{a}_2]\begin{bmatrix}\boldsymbol{b}_1^*\\\boldsymbol{b}_2^*\end{bmatrix}=\boldsymbol{a}_1\boldsymbol{b}_1^*+\boldsymbol{a}_2\boldsymbol{b}_2^*$$

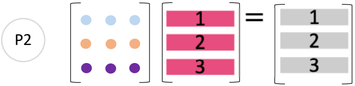
$$=\begin{bmatrix}1\\3\\5\end{bmatrix}[b_{11}\quad b_{12}]+\begin{bmatrix}2\\4\\6\end{bmatrix}[b_{21}\quad b_{22}]=\begin{bmatrix}b_{11} & b_{12}\\3b_{11} & 3b_{12}\\5b_{11} & 5b_{12}\end{bmatrix}+\begin{bmatrix}2b_{21} & 2b_{22}\\4b_{21} & 4b_{22}\\6b_{21} & 6b_{22}\end{bmatrix}$$

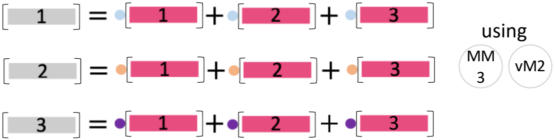
下面展示一些实用的模式。





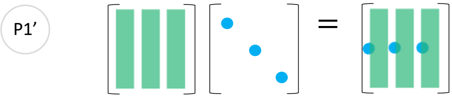
Operations from the right act on the columns of the matrix. This expression can be seen as the three linear combinations in the right in one formula.



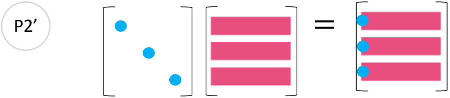


Operations from the left act on the rows of the matrix. This expression can be seen as the three linear combinations in the right in one formula.

图: 模式 1, 2 - (P1), (P1)



Applying a diagonal matrix from the right scales each column.

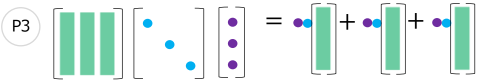


Applying a diagonal matrix from the left scales each row.

$$AD = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1\boldsymbol{a}_1 \quad d_2\boldsymbol{a}_2 \quad d_3\boldsymbol{a}_3]$$

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{b}_1^* \\ \boldsymbol{b}_2^* \\ \boldsymbol{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1\boldsymbol{b}_1^* \\ d_2\boldsymbol{b}_2^* \\ d_3\boldsymbol{b}_3^* \end{bmatrix}$$

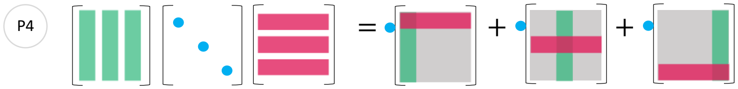
图: 模式 1′, 2′ - (P1′), (P2′)



This pattern makes another combination of columns.
You will encounter this in differential/recurrence equations.

$$XD\boldsymbol{c} = [\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3}]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \boldsymbol{x_1} + c_2 d_2 \boldsymbol{x_2} + c_3 d_3 \boldsymbol{x_3}$$

图: 模式 3 - (P3)



A matrix is broken down to a sum of rank 1 matrices,
as in singular value/eigenvalue decomposition.

$$U\Sigma V^{\mathrm{T}} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3]\begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \boldsymbol{v}_3^{\mathrm{T}}\end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}} + \sigma_3\boldsymbol{u}_3\boldsymbol{v}_3^{\mathrm{T}}$$

图: 模式 4 - (P4)