Fundamental groups of small covers

Wu, Lisu

School of Mathematical Sciences, Fudan University

The 5^{th} Korea Toric Topology Winter Workshop

Gyeongju, Korea, January 21-23, 2019

Content

Small cover

- An n-dimensional small cover is a closed n-manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope P.

$$\pi : M \longrightarrow P$$

Small cover

- An n-dimensional small cover is a closed n-manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope P.

$$\pi : M \longrightarrow P$$

- The $\mathbb{Z}_2^n$-action on M determines a $\mathbb{Z}_2^n$-valued characteristic function λ on the set of facets of P

$$\lambda : \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$

Small cover

- An n-dimensional <u>small cover</u> is a closed n-manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope P.

$$\pi : M \longrightarrow P$$

- The $\mathbb{Z}_2^n$-action on M determines a $\mathbb{Z}_2^n$-valued <u>characteristic function</u> λ on the set of facets of P

$$\lambda : \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$

  such that

$$\forall\ f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$

Small cover

- An n-dimensional small cover is a closed n-manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope P.

$$\pi : M \longrightarrow P$$

- The $\mathbb{Z}_2^n$-action on M determines a $\mathbb{Z}_2^n$-valued characteristic function λ on the set of facets of P

$$\lambda : \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$

such that

$$\forall\ f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$

Rk: $\mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\}$.

- Small cover

$$M = P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior, $G_{f(p)} = \{1\}$ if $p \in P^\circ$.

- Small cover

$$M = P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior, $G_{f(p)} = \{1\}$ if $p \in P^{\circ}$.

- Define right-angled Coxeter group of P

$$W = \langle s_F, \forall\, F \in \mathcal{F}(P) \mid s_F^2 = 1; (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$$

- The Borel construction of $\mathbb{Z}_2^n$ on M

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n$$

where $E\mathbb{Z}_2^n = (S^\infty)^n$. And $\pi_1(BP) \cong W$.

- The Borel construction of $\mathbb{Z}_2^n$ on M

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n$$

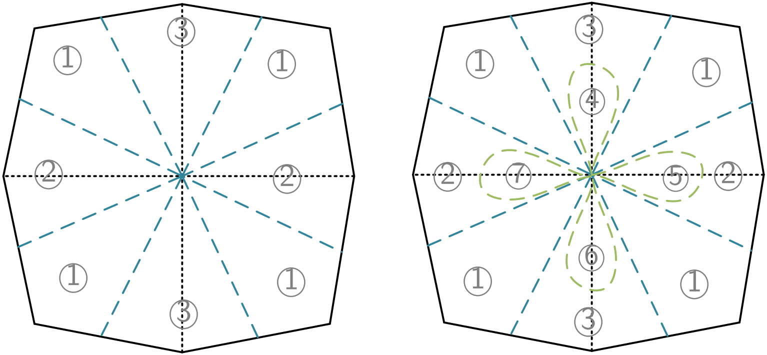
where $E\mathbb{Z}_2^n = (S^\infty)^n$. And $\pi_1(BP) \cong W$.

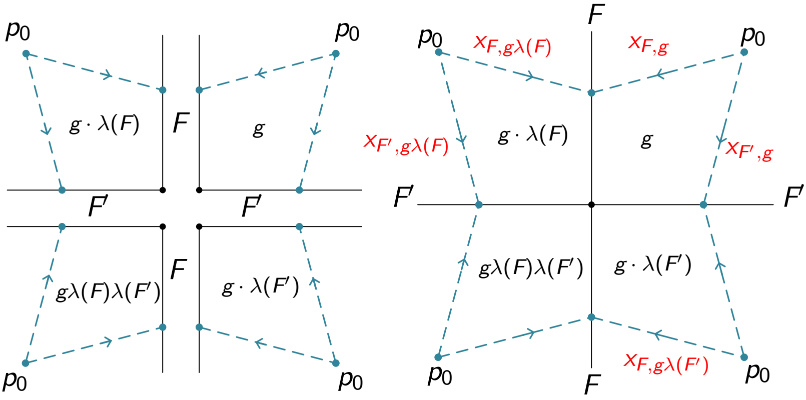
- $M \to BP \to B\mathbb{Z}_2^n$ induces a right-split exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W \underset{\gamma}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}_2^n \longrightarrow 1$$

where $\phi(s_F) = \lambda(F),\ \forall F \in \mathcal{F}(P)$.

Cell decomposition

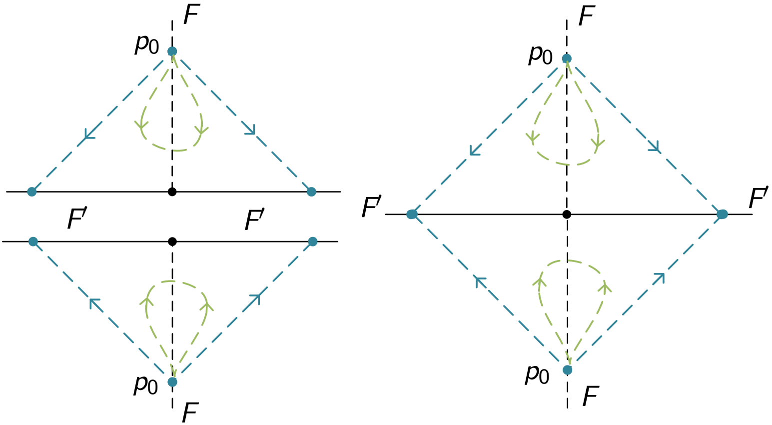




Cell-1

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

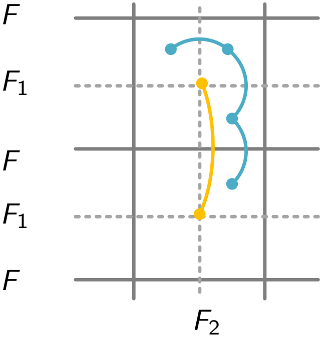
Presentation of $\pi_1(M)$

Presentation of $\pi_1(M, p_0)$

$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, \forall\, F, g \,|\, & x_{F,g} x_{F,g\lambda(F)} = 1; \\ & x_{F,g} x_{F',g\lambda(F)} = x_{F',g} x_{F,g\lambda(F')}, \;\; F \cap F' \neq \varnothing; \\ & x_{F,g} = 1, p_0 \in F \rangle\end{aligned}$$

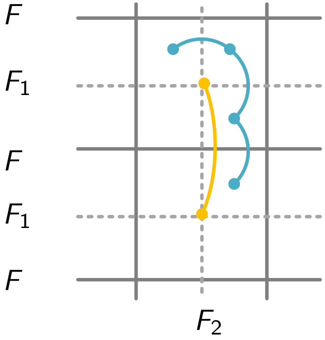
Relation between $\pi_1(M)$ and W

$$\widetilde{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

$$\widetilde{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$

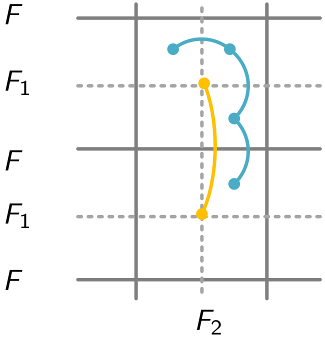


$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$

Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

Relation between $\pi_1(M)$ and W

$$\widetilde{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$



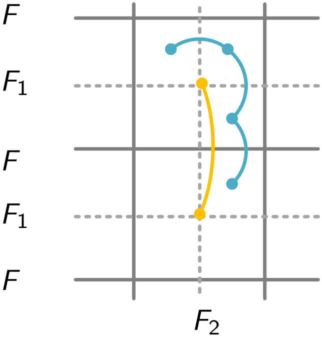
$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$

$s_F(P,1) \mapsto (P, s_F)$

Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

Relation between $\pi_1(M)$ and W

$$\widetilde{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$



$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
\end{tikzcd}$$

Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

Relation between $\pi_1(M)$ and W

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) &\longrightarrow W \\
x_{F,g} &\longmapsto \quad \gamma(g\lambda(F)) \cdot \gamma(\lambda(F)) s_F \cdot (\gamma(g\lambda(F)))^{-1} \\
&\qquad = \gamma(g) s_F \gamma(g\lambda(F))
\end{aligned}
$$

Relation between $\pi_1(M)$ and W

$$\begin{aligned} \alpha : \pi_1(M, p_0) &\longrightarrow W \\ x_{F,g} &\longmapsto \quad \gamma(g\lambda(F)) \cdot \gamma(\lambda(F)) s_F \cdot (\gamma(g\lambda(F)))^{-1} \\ &\qquad = \gamma(g) s_F \gamma(g\lambda(F)) \end{aligned}$$

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}$$

Idea

$$\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted] & W
\end{tikzcd}$$

$$\begin{tikzcd}
{\color{red}M} \arrow[r, dotted, red] \arrow[d, "\pi"'] & {\color{red}\pi_1(M)} \arrow[d, "\alpha"] \\
P \arrow[r, dotted] & W
\end{tikzcd}$$

Idea

$$\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted] & W
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
{\color{red}P} \arrow[r, dotted, red] & {\color{red}W}
\end{tikzcd}$$

Idea

$$
\begin{tikzcd}
M \arrow[r, dotted, "\text{hard}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\downarrow"] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted, "\text{easy}"] & W
\end{tikzcd}
$$

For any proper face f of P,

- Define $\mathcal{F}(f^{\perp}) \triangleq \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. So $\mathcal{F}(f^{\perp})$ consists of those facets of P that intersect f transversely.

For any proper face f of P,

- Define $\mathcal{F}(f^{\perp}) \triangleq \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. So $\mathcal{F}(f^{\perp})$ consists of those facets of P that intersect f transversely.
- A submanifold Σ in M is called $\underline{\pi_1\text{-injective}}$ if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

π_1-injectivity of facial submanifolds

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

Rk: The π_1-injectivity of a facial submanifold of small cover only depends on the local face structure of f in P.

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

Rk: The π_1-injectivity of a facial submanifold of small cover only depends on the local face structure of f in P.

Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

Other results

A simple polytope P is called flag if a collection of facets of P has common intersection whenever they pairwise intersect.

A simple polytope P is called flag if a collection of facets of P has common intersection whenever they pairwise intersect.

Proposition (Wu-Yu, 2017)

Let M be a small cover over P. Then P is flag if and only if every facial submanifold of M is π_1-injective.

A simple polytope P is called flag if a collection of facets of P has common intersection whenever they pairwise intersect.

Proposition (Wu-Yu, 2017)

Let M be a small cover over P. Then P is flag if and only if every facial submanifold of M is π_1-injective.

Proposition (Wu-Yu, 2017)

For any small cover M over a 3-dimensional simple polytope P, there always exists a facet F of P so that the facial submanifold M_F is π_1-injective.

For a 3-dimensional simple polytope P,

- A *k-circuit* in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges,

For a 3-dimensional simple polytope P,

- A *k-circuit* in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.

For a 3-dimensional simple polytope P,

- A <u>*k-circuit*</u> in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called <u>*prismatic*</u> if the endpoints of those edges are distinct.
- A <u>*k-belt*</u> in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three faces in the belt have no common intersection.

For a 3-dimensional simple polytope P,

- A *k-circuit* in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.
- A *k-belt* in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three faces in the belt have no common intersection.
- Each k-belt can determine a prismatic k-circuit.

For a 3-dimensional simple polytope P,

- A ***k-circuit*** in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called ***prismatic*** if the endpoints of those edges are distinct.
- A ***k-belt*** in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three faces in the belt have no common intersection.
- Each k-belt can determine a prismatic k-circuit. A prismatic 3-circuit determines a 3-belt;

For a 3-dimensional simple polytope P,

- A *k-circuit* in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.
- A *k-belt* in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three faces in the belt have no common intersection.
- Each k-belt can determine a prismatic k-circuit. A prismatic 3-circuit determines a 3-belt; and if there is no prismatic 3-circuit, then a prismatic 4-circuit determines a 4-belt.

Let M be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ's paper(corrected):

- If there exist prismatic 3-circuits in P, then M can be decomposed into prime pieces glued along S^2 or $\mathbb{R}P^2$.

Let M be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ's paper(corrected):

- If there exist prismatic 3-circuits in P, then M can be decomposed into prime pieces glued along S^2 or $\mathbb{R}P^2$.
- If there is no prismatic 3-circuit but prismatic 4-circuits in P, then M can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.

Let M be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ's paper(corrected):

- If there exist prismatic 3-circuits in P, then M can be decomposed into prime pieces glued along S^2 or $\mathbb{R}P^2$.
- If there is no prismatic 3-circuit but prismatic 4-circuits in P, then M can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.
- If there is no prismatic 3 or 4-circuit in P, then M is hyperbolic.

Let M be a connected 3-manifold.

- M is called <u>prime</u> if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.

Let M be a connected 3-manifold.

- M is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.
- M is called irreducible if every embedded 2-sphere bounds a 3-ball. An prime 3-manifold is irreducible except S^2-bundle over S^1.

Prime decomposition theorem

Let M be a connected 3-manifold.

- M is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.
- M is called irreducible if every embedded 2-sphere bounds a 3-ball. An prime 3-manifold is irreducible except S^2-bundle over S^1.

Theorem (Kneser, Milnor, Prime Decomposition Theorem)

Each compact 3-manifold M can factor as a connected sum of prime manifolds. This decomposition is unique under the assumption that M is orientable.

Proposition

Let M be a 3-dimensional small cover over a simple polytope P , then M is irreducible if and only if there is no prismatic 3-circuit in P.

Prime decomposition theorem

Proposition

Let M be a 3-dimensional small cover over a simple polytope P , then M is irreducible if and only if there is no prismatic 3-circuit in P.
In particular, the prime decomposition of an oriented 3-small cover is equivalent to cutting surgery along prismatic 3-circuits in P.

- A compact 3-manifold M is called atoroidal if it contains no essential torus.

- A compact 3-manifold M is called atoroidal if it contains no essential torus.
- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1.

- A compact 3-manifold M is called atoroidal if it contains no essential torus.
- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1.

Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

For an oriented, irreducible, closed 3-manifold, there exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered, and a minimal such collection $T_1, \cdots, T_m$ is unique up to isotopy.

Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no 4-belt in P.
In particular, the JSJ decomposition or geometric decomposition of a irreducible 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

Theorem (Thurston, Hyperbolization Theorem)

Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

Theorem (Thurston, Hyperbolization Theorem)

Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Theorem (Thurston, Hyperbolization Theorem)

Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

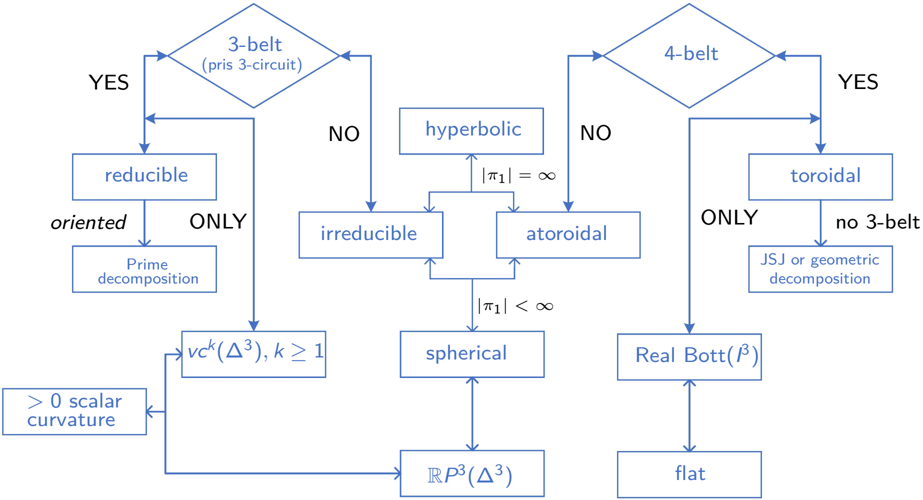
Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Proposition (Wu-Yu, 2017)

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts.

A classification for 3-small cover



End of Talk

The 5th Korea Toric Topology Winter Workshop

Gyeongju, Korea. 10:30 - 11:10 January 22, 2019

Some references

- Wu and Yu, Fundamental groups of small covers revisited. (2018).
- Wu, Atoroidal manifolds in small covers. (2018).
- Agol's *talk-1* & *talk-2* (2012, 2014).
- Aschenbrenner-Friedl-Wilton, 3-manifold groups, (2013).
- Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. (2002).
- Davis and Januszkiewicz, Convex polytopes, coxeter orbifolds and torus actions, (1991).
- Chen, A homotopy theory of orbispaces. (2001).
- Kapovich, Hyperbolic manifolds and discrete groups. (2010).

Email: wulisuwulisu@qq.com
Homepage: http://algebraic.top/