Fundamental groups of small covers

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Contents

- $\mathcal{F}(P) = \{F_1, F_2, \cdots, F_m\}$: the facets set of a given simple polytope P.
- A map $\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$ satisfied

  $\forall f = F_1 \cap F_2 \cap \cdots \cap F_k,$
  $\dim_{\mathbb{Z}_2}(span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}) = k.$

  called *characteristic function* on $\mathcal{F}(P)$.
- $\lambda(F)$: the color on a facet F.

Small Covers

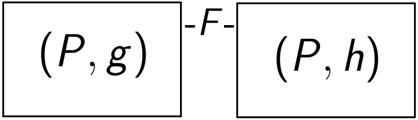
- $G_f = span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}$, for $f = F_1 \cap \cdots \cap F_k$.

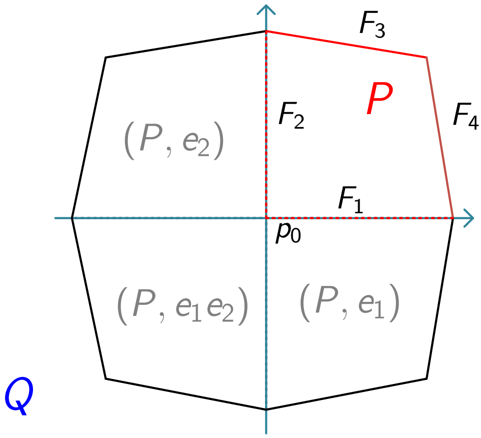
Definition (small cover over P)

$$M = P \times \mathbb{Z}_2^n / \sim$$

Where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_f(p)$, and $f(p)$ is the unique face of P that contains p in its relative interior.

- $\pi : M \longrightarrow P$ the nature projective.
- $(P, g) \overset{F}{\sim} (P, h) \Longleftrightarrow g^{-1}h = \lambda(F)$.





$\lambda(F_1) = e_1$

$\lambda(F_2) = e_2$

$\lambda(F_3) = e_1 e_2$

$\lambda(F_4) = e_2$

Klein bottle

General Case: $p_0 = F_1 \cap F_2 \cap \cdots \cap F_n$ is a vertex of P, and $\lambda(F_i) = e_i, i = 1, 2, \cdots, n$.

- For any simple polytope P, define a right-angle Coxeter group

$$W_P = \langle s_F | s_F^2 = 1, (s_F s_{F'})^2 = 1, F, F' \in \mathcal{F}(P), F \cap F' \neq \varnothing \rangle$$

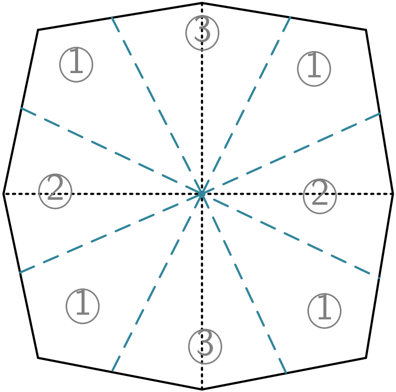
- The following (right split) exact sequence is given by Davis-Januszkiewicz [3].

$$1 \longrightarrow \pi_1(M) \longrightarrow W_P \xrightarrow{\psi} \mathbb{Z}_2^n \longrightarrow 1 \qquad (1)$$

where $\psi(s_F) = \lambda(F), \ \forall F \in \mathcal{F}(P)$

Rk: $W_P \cong \pi_1(M) \rtimes \mathbb{Z}_2^n$.

P	a n-dimension simple convex polytope in $\mathbb{R}^n$.
$\mathcal{F}(P)$	the facets set of P.
λ	$(\mathcal{F}(P) \rightarrow \mathbb{Z}_2^n)$the characteristic function.
M	a small cover over P.
π	$(M \rightarrow P)$ the nature project.
W_P	the Coxeter group associated with P.
$\pi_1(M)$	the fundamental group of M.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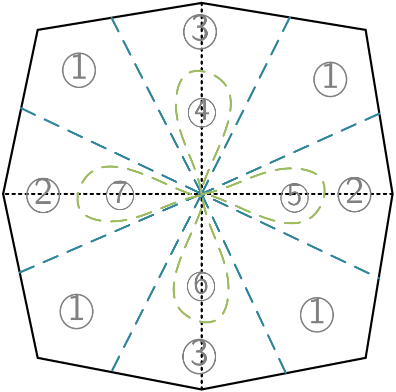
3
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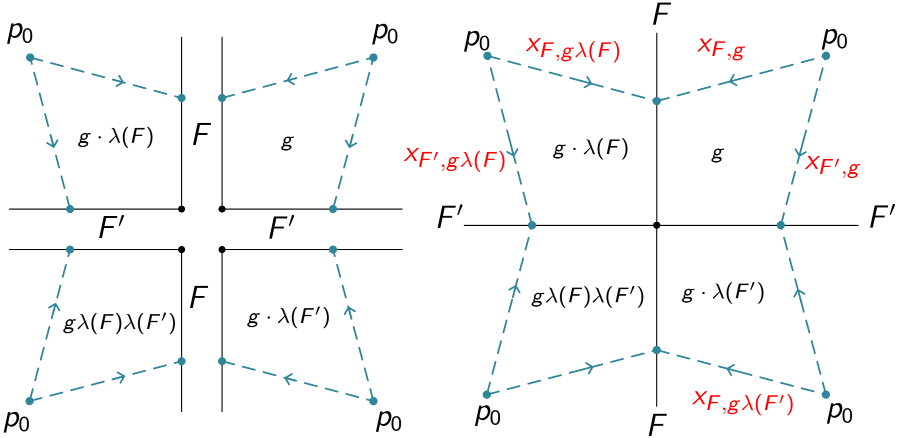
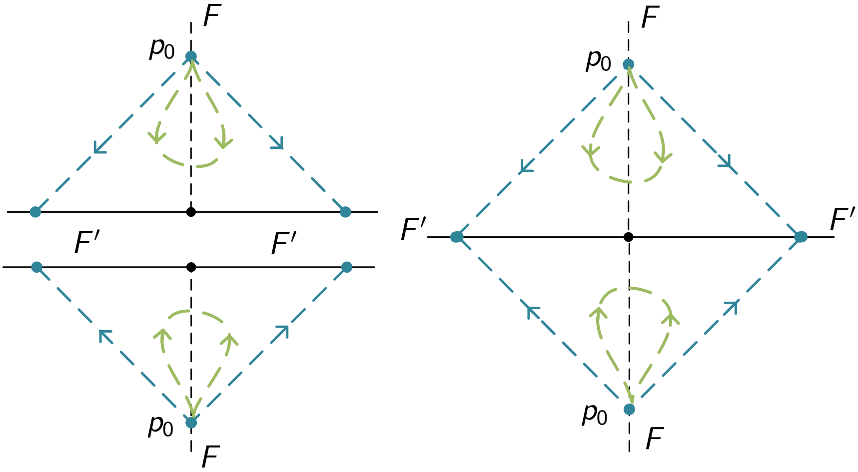


Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

- Generators: $x_{F,g}$
- Relations: $[\phi_F(g) = g \cdot \lambda(F)]$
  - $x_{F,g} x_{F,\phi_F(g)} = 1$
  - $x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}$
  - $x_{F,g} = 1$

Theorem

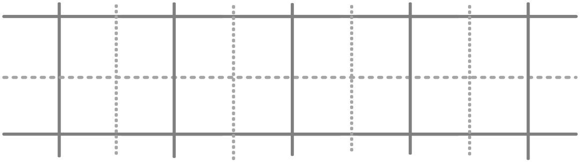
The presentation of $\pi_1(M)$*.*

$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n :\\ x_{F,g} = 1, p_0 \in F; x_{F,g} x_{F,\phi_F(g)} = 1;\\ x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}, F \cap F' \neq \varnothing\rangle\end{aligned}$$

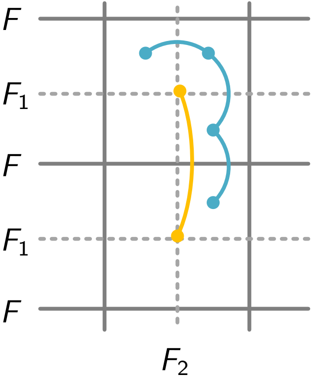
- $\mathcal{M} = Q \times \pi_1(M)/\sim^1$
  $(Q, \nu_1) \overset{F_g \sim F_h}{\sim} (Q, \nu_2) \Longleftrightarrow \nu_1^{-1}\nu_2 = x_{F,h}, g^{-1}h = \lambda(F)$.
- $\mathcal{L} = P \times W_P/\sim^2$
  $(P, \omega_1) \overset{F \sim F}{\sim} (P, \omega_2) \Longleftrightarrow \omega_1^{-1}\omega_2 = s_F$.

Lemma

$\mathcal{M} \cong \mathcal{L}$, in the other words, the universal cover space of small cover is determined only by simple polytope P.



Relation between $\pi_1(M)$ *and* W_P



$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\lambda(F)\cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
\end{tikzcd}$$

Rk: 1: the unit element of $\mathbb{Z}_2^n$ or W_P.

Relation between $\pi_1(M)$ *and* W_P

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) \longrightarrow & W \\
x_{F,g} \longmapsto \quad & \gamma(\phi_F(g)) \cdot s_F \gamma(\phi_F(1)) \cdot (\gamma(\phi_F(g)))^{-1} \\
& = \gamma(\phi_F(g)) \cdot s_F \cdot \gamma(\phi_F(1)\phi_F(g)) \\
& = \gamma(\phi_F(g)) s_F \gamma(g)
\end{aligned}
$$

$$
1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W_P \underset{\gamma}{\overset{\psi}{\rightleftarrows}} \mathbb{Z}_2^n \longrightarrow 1
$$

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective in M.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$.*

Rk: The π_1-injectivity of facial submanifold of small cover only depended on P.

f	$F_1 \cap F_2 \cap \cdots \cap F_k$.
$\mathcal{F}(f^\perp)$	$\{F : f \cap F \neq \varnothing, F \in \mathcal{F} - \{F_1, \cdots, F_k\}\}$.
M_f	$\pi^{-1}(f)$, the submanifold of M.
i	$(M_f \to M)$, the inclusion.
i_*	$(\pi_1(M_f) \to \pi_1(M))$, the group homomorphism induced by i.
j_*	$W_f \to W_P$, a natural group homomorphism as transformation group.

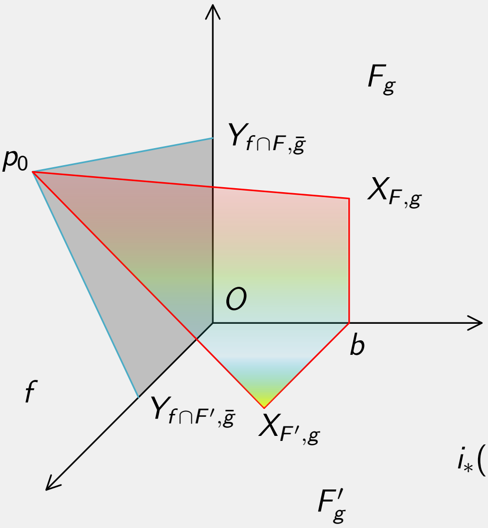
Proof.

Proof.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(M_f) \arrow[r, "\alpha_f"] \arrow[d, "i_*"] & W_f \arrow[r] \arrow[d, "j_*"'] & \mathbb{Z}_2^k \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r] & \mathbb{Z}_2^n \arrow[r] \arrow[u, "q"'] & 1
\end{tikzcd}$$

$$\begin{aligned} j : W_f &\longrightarrow W_P \\ s_{f\cap F} &\longmapsto s_F \end{aligned} \qquad \begin{aligned} q : \mathbb{Z}_2^n &\longrightarrow \mathbb{Z}_2^k = \mathbb{Z}_2^n / \langle \lambda(F) : f \subset F \rangle \\ g &\longmapsto \bar{g} \end{aligned}$$

Continuing.



$$i_*(Y_{f\cap F,\bar{g}}) = X_{F,g}$$

Continuing.

$$
\begin{tikzcd}
\pi_1(M_f)/\alpha_f^{-1}(\ker j_*) \arrow[r, "\alpha'_f"] \arrow[d, "i'_*"] & W_f/\ker j_* \arrow[d, "j'_*"'] & \\
\pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r, "\eta"] & W_P/\langle s_F : f \subset F \rangle
\end{tikzcd}
$$

- $\eta \circ j'_* \circ \alpha'_f = \eta \circ \alpha \circ i'_*$
- $\eta \circ j'_* \circ \alpha'_f$ injective $\Longrightarrow i'_*$ injective
- $\ker j_* = \langle [F, F'] : F, F' \in \mathcal{F}(f^\perp), f \cap F \cap F' = \varnothing \rangle$

$\square$

End

References

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