Fundamental groups of small covers

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Content

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such that

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- The universal cover space of M

$$\mathcal{M} = P \times W / \sim$$

where $W_P = \langle s_F \mid s_F^2 = 1, (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$ is the right-angled Coxeter group of P.

Borel construction

- The Borel construction(or the homotopy quotient of $\mathbb{Z}_2^n$ on M):

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n \simeq \mathbb{R}\mathcal{Z}_P \times_{\mathbb{Z}_2^m} E\mathbb{Z}_2^m \simeq \mathcal{M} \times_W EW$$

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where BP only depends on P and its face structure.

- Then $M \to BP \to B\mathbb{Z}_2^n$ induces an exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \qquad (1)$$

where $W \cong \pi_1(BP)$ and $\phi(s_F) = \lambda(F)$ for any facet F of P.

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3. An orbifold is good if it has some covering orbifold which is a closed manifold. Otherwise it is bad.

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Orbifold coverings

- An n-dimensional simple polytope P is a good $\mathbb{Z}_2^n$-orbifold, which is called right-angled Coxeter orbifold. And $\pi_1^{\mathrm{orb}(P)} = W_P$.

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$$\begin{tikzcd}
\mathbb{R}\mathcal{Z}_P \arrow[dr, "\mathbb{Z}_2^{m-n}"] \arrow[ddr, "\mathbb{Z}_2^m"'] & & \mathcal{M} \arrow[ll, "\pi_1(\mathbb{R}\mathcal{Z}_P)"'] \arrow[dl, "\pi_1(M)"'] \arrow[ddl, "W"] \\
& M \arrow[d, "\mathbb{Z}_2^n"] & \\
& P &
\end{tikzcd}$$

$$\Longrightarrow$$

$$\begin{tikzcd}
\pi_1(\mathbb{R}\mathcal{Z}_P) \arrow[r, equal] \arrow[d] & {[W, W]} \arrow[d] & \\
\pi_1(M) \arrow[r] \arrow[d] & W \arrow[r, "\phi"] \arrow[d] & \mathbb{Z}_2^n \arrow[d, equal] \\
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\pi_1(M) \arrow[r] \arrow[d] & W \arrow[r, shift left, "\phi"] \arrow[d] & \mathbb{Z}_2^n \arrow[l, dotted, red, shift left, "\gamma"] \arrow[d, equal] \\
\mathbb{Z}_2^{m-n} \arrow[r] & \mathbb{Z}_2^m \arrow[r] & \mathbb{Z}_2^n
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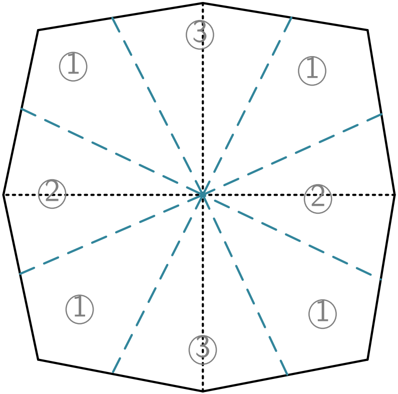
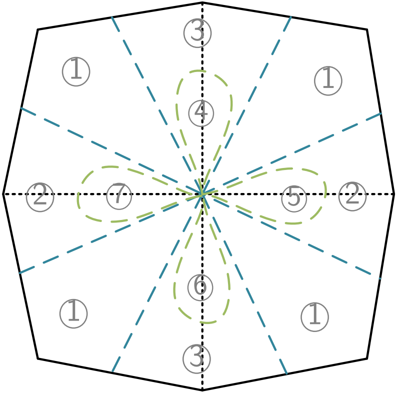
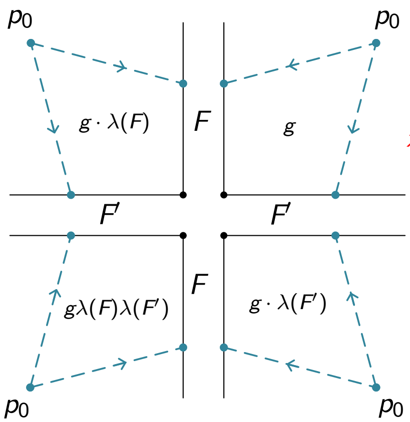
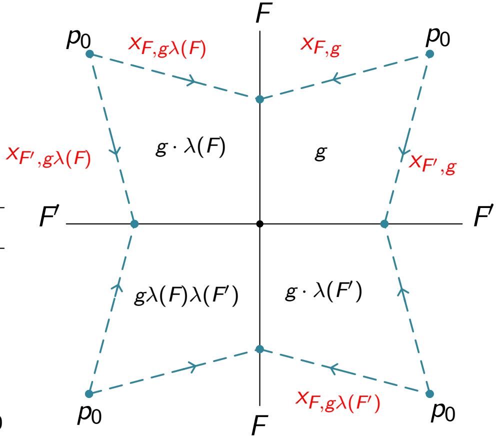
Cell decomposition





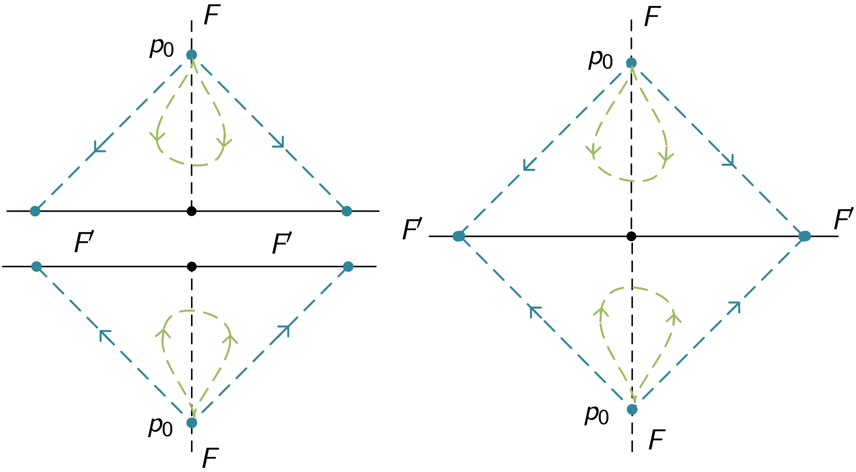




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g} x_{F',g\lambda(F)} = x_{F',g} x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1, \ p_0 \subset F$

Presentation of $\pi_1(M)$

- Generator: $x_{F,g}$
- Relation: $[\sigma_F(g) = g \cdot \lambda(F)]$
  - $x_{F,g} x_{F,\sigma_F(g)} = 1$
  - $x_{F,g} x_{F',\sigma_F(g)} = x_{F',g} x_{F,\sigma_{F'}(g)}$
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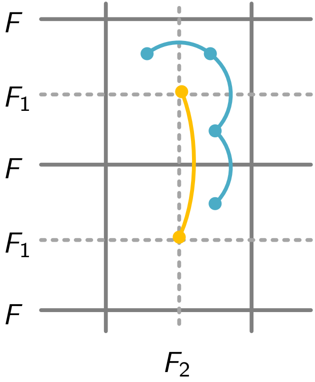
Presentation of $\pi_1(M, p_0)$

$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n \,|\, & x_{F,g}x_{F,\sigma_F(g)} = 1; \\ & x_{F,g}x_{F',\sigma_F(g)} = x_{F',g}x_{F,\sigma_{F'}(g)}, \ F \cap F' \neq \varnothing; \\ & x_{F,g} = 1, p_0 \in F; \rangle\end{aligned}$$

Relation between $\pi_1(M)$ and W_P

$$\mathcal{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

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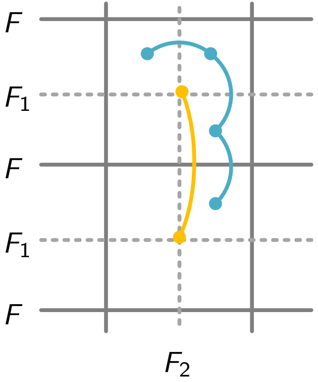

$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$

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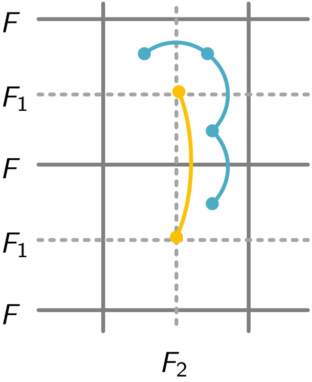

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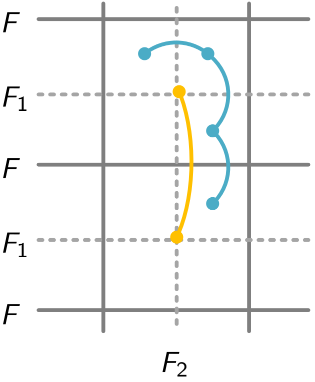

$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
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Relation between $\pi_1(M)$ and W_P

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) &\longrightarrow W \\
x_{F,g} &\longmapsto \quad \gamma(\sigma_F(g)) \cdot \gamma(\sigma_F(1)) s_F \cdot (\gamma(\sigma_F(g)))^{-1} \\
&\qquad = \gamma(\sigma_F(g)\sigma_F(1)) \cdot s_F \cdot \gamma(\sigma_F(g)) \\
&\qquad = \gamma(g) s_F \gamma(\sigma_F(g))
\end{aligned}
$$

$$
1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{2}
$$

Semidirect product

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W \arrow[r, shift left, "\phi"] & \mathbb{Z}_2^n \arrow[l, shift left, "\gamma"] \arrow[r] & 1 \end{tikzcd}$$

Then $W = \pi_1(M) \rtimes_\psi \mathbb{Z}_2^n$, where $\psi_h(x) = \alpha^{-1}(\gamma(h)\alpha(x)\gamma(h^{-1}))$.

$$\begin{aligned} \psi_h(x_{F,g}) &= \alpha^{-1}(\gamma(h)\alpha(x_{F,g})\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(h)\gamma(g)s_F\gamma(\sigma_F(g))\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(gh)s_F\gamma(\sigma_F(gh))) \\ &= \psi_{gh}(x_{F,1}) \end{aligned}$$

Idea

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\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
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M \arrow[r, dotted, "\text{complex}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\downarrow"] & \pi_1(M) \arrow[d, "\alpha"] \\
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- A submanifold Σ in M is called π_1-injective if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.
- A *k-circuit* in the simple polytope P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.

π_1-injectivity of facial submanifolds

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

> The facial submanifold M_f is π_1-injective in M.

> For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.

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Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

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Proposition (Wu-Yu, 2017)

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Proposition (Wu-Yu, 2017)

For any small cover M over a 3-dimensional simple polytope P, there always exists a facet F of P so that the facial submanifold M_F is π_1-injective in M.

Prime decomposition theorem

Let M be a connected 3-manifold.

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- M is called irreducible if every embedded 2-sphere bounds a 3-ball.

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- M is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.
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Theorem (Kneser, Prime Decomposition Theorem)

Every compact oriented 3 manifold M factors as a connected sum of prime manifolds, $M \cong M_1 \# \cdots \# M_n$, and this decomposition is unique up to insertion or deletion of S^3 summands.

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  A 2-sided surface except S^2 in M is incompressible if and only if it is π_1-injective.
- M is called Haken if it is a compact, P^2-irreducible 3-manifold that contains an 2-sided incompressible surface.

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Proposition

Let M be a 3-dimensional small cover over a simple polytope $P(\neq \Delta^3)$, then TFAE.

- *M is irreducible.*
- *M is prime.*
- *M is Haken.*
- *M is aspherical.*
- *P is flag.*
- *There is no prismatic 3-circuit in P.*
- *$\pi_2(M)$ is trivial.*
- *All facial submanifold is π_1-injective.*

In particular, the prime decomposition of a oriented 3-small cover is equivalent to cutting survey along prismatic 3-circuits in P

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Rk: $\mathbb{R}P^3$ is prime and irreducible but spherical.

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- A Seifert fiber space is a circle bundle over a 2-dimensional orbifold.
- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1.

JSJ-decomposition theorem

Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

Let M be a compact, oriented, irreducible 3-manifold with empty or toroidal bounday. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

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Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no prismatic 4-circuit in P.
In particular, the JSJ decomposition of a 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

Geometric structure

Theorem (Thurston, Hyperbolization Theorem)

Every irreducible atoroidal closed 3-manifold with infinite fundamental group is hyperbolic.

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Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Geometric structure

Proposition

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts. In particular, all the oriented 3-dimensional small covers that can admit Riemannian metrics with nonnegative scalar curvature are the two oriented real Bott manifolds in dimension 3 and the connected sum of k copies of $\mathbb{R}P^3$ for any $k \geq 1$.

End of Talk

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