# *Fundamental groups of small covers*

Nanjing University

Lisu Wu

# Content

## *Construction of Small Covers*

- $\mathcal{F}(P) = \{F_1, F_2, \cdots, F_m\}$ the facets set of a given simple polytope $P$.
- A map $\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$ satisfied

$$\forall f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$\dim_{\mathbb{Z}_2}(span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}) = k.$$

called *characteristic function* on $\mathcal{F}(P)$.

# Construction of Small Covers

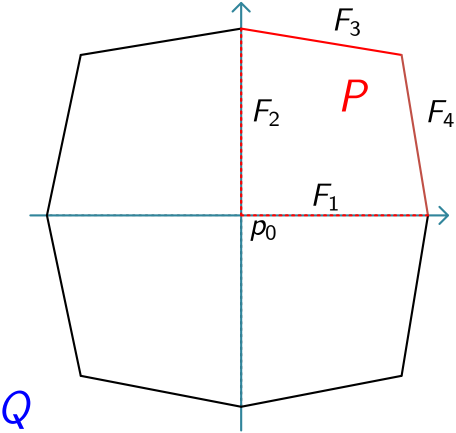
- $G_f = span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}$, for any codim $k$ face $f = F_1 \cap \cdots \cap F_k$ of $P$.

*Definition ( small cover over P)*

$$M = P \times \mathbb{Z}_2^n / \sim$$

Where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_f(p)$, and $f(p)$ is the unique face of $P$ that contain $p$ in its relative interior.

- $\pi : M \longrightarrow P$ the nature projective.




Rk: May as well $p_0 = F_1 \cap F_2 \cap \cdots \cap F_n$ is a vertex of $P$, and $\lambda(F_i) = e_i, i = 1, 2, \cdots, n$.

# *Coxeter Group and Exact Sequence*

- For any simple polytope $P$, define a right-angle Coxeter group

$$W_P = \langle s_F | s_F^2 = 1, (s_F s_{F'})^2 = 1, F, F' \in \mathcal{F}(P), F \cap F' \neq \varnothing \rangle$$

- $W_P$ is isomorphic to the fundamental group of the Borel construction $M_{\mathbb{Z}_2^n} = E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M$.
- Then $M \to M_{\mathbb{Z}_2^n} \to B\mathbb{Z}_2^n$ induces an (right split) exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W_P \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \qquad (1)$$

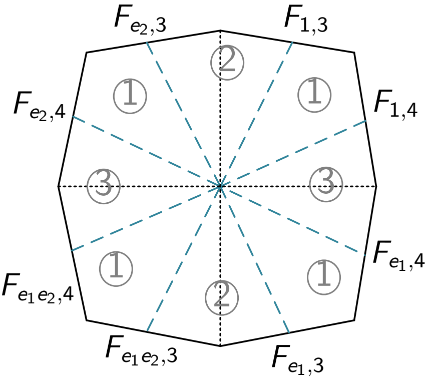
where $\psi(s_F) = \lambda(F), \ \forall F \in \mathcal{F}(P)$

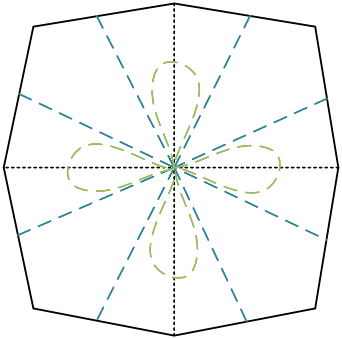
Rk3: $W_P \cong \pi_1(M) \rtimes \mathbb{Z}_2^n$.

## *Notations*

| | |
|---|---|
| $P$ | a $n$-dimension simple convex polytope in $\mathbb{R}^n$. |
| $\mathcal{F}(P)$ | the facets set of $P$. |
| $\lambda$ | $(\mathcal{F}(P) \rightarrow \mathbb{Z}_2^n)$the characteristic function. |
| $M$ | a small cover over $P$. |
| $\pi$ | $(M \rightarrow P)$ the nature project. |
| $W$ | the Coxeter group associated to $P$. |
| $\pi_1(M)$ | the fundamental group of $M$ . |

# *single* 0*-cell cell structure*



## *Presentation of $\pi_1$*

- Generator: $x_{F,g}$
- Relation:
  - $x_{F,g}x_{F,\phi_F(g)} = 1$
  - $x_{F,g}x_{F',\phi_F(g)} = x_{F',g}x_{F,\phi_{F'}(g)}$
  - $x_{F,g} = 1$

*Theorem*

*The presentation of $\pi_1(M)$.*

$$\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n : \\ x_{F,g} = 1, p_0 \in F; x_{F,g}x_{F,\phi_F(g)} = 1; \\ x_{F,g}x_{F',\phi_F(g)} = x_{F',g}x_{F,\phi_{F'}(g)}, F \cap F' \neq \varnothing; \rangle$$

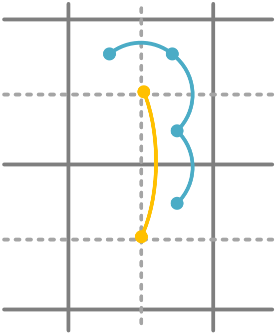
# *Universal cover space*

- $$\mathcal{M} = Q \times \pi_1(M)/\sim \tag{2}$$
- $$\mathcal{L} = P \times W_P/\sim \tag{3}$$

**Lemma**

*$\mathcal{M} \cong \mathcal{L}$, in the other words, the universal cover space of small cover is determined only by simple polytope $P$.*

# relation between $\pi_1(M)$ and $W_P$



# *relation between $\pi_1(M)$ and $W_P$*

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) \longrightarrow & W \\
x_{F,g} \longmapsto \quad & \gamma(\phi_F(g)) \cdot \gamma(\phi_F(1)) s_F \cdot (\gamma(\phi_F(g)))^{-1} \\
& = \gamma(\phi_F(g)\phi_F(1)) \cdot s_F \cdot \gamma(\phi_F(g)) \\
& = \gamma(g) s_F \gamma(\phi_F(g))
\end{aligned}
$$

$$
1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{4}
$$

# *Main results*

*Theorem (Wu-Yu, 2017)*

*Let $M$ be a small cover over a simple polytope $P$ and $f$ be a proper face of $P$. The following two statements are equivalent.*

¿ *The facial submanifold $M_f$ is $\pi_1$-injective in $M$.*

¿ *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$.*

Rk3: The $\pi_1$-injectivity of facial submanifold of small cover only depended on $P$.

Rk4: we can determine the kernel of the homomorphism induced by the inclution form $M_f$ to $M$.

# End

# *References*


- Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. *AMS* (2002).
- Davis, Exotic aspherical manifolds, *Topology of high-dimensional manifolds.* (Trieste, 2001).
- Davis and Januszkiewicz, Convex polytopes, coxeter orbifolds and torus actions, *Duke Math. J.* (1991).
- Davis, Januszkiewicz, and Scott, Nonpositive curvature of blow-ups, *Selecta Math.(N.S.)* (1998).
- Davis, Januszkiewicz and Scott, Fundamental groups of blow-ups, *Advances in mathematics.* (2003).
- Kuroki, Masuda and Yu, Small covers, infra-solvmanifolds and curvature, *Forum mathematicum.*(2015)

蓝鲸大学

NANJING UNIVERSITY

Email: wulisuwulisu@qq.com
Homepage: http://algebraic.top/