Fundamental groups of small covers

Lisu Wu

Nanjing University

Contents

- $\mathcal{F}(P) = \{F_1, F_2, \cdots, F_m\}$: the facets set of a given simple polytope P.
- A map $\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$ satisfied

$$\forall f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$

  called *characteristic function* on $\mathcal{F}(P)$.
- $\lambda(F)$: the color on a facet F.

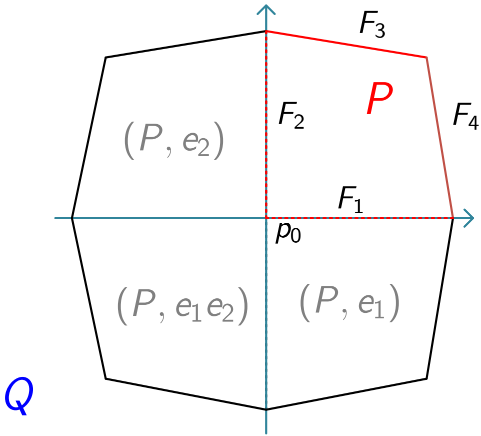
Definition (small cover over P)

$$M = P \times \mathbb{Z}_2^n / \sim$$

Where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior.

- $\pi : M \longrightarrow P$ the nature projective.
- $(P, g) \overset{F}{\sim} (P, h) \Longleftrightarrow g^{-1}h = \lambda(F)$.

$$\boxed{(P, g)} \text{-}F\text{-} \boxed{(P, h)}$$



Klein bottle

$\lambda(F_1) = e_1$

$\lambda(F_2) = e_2$

$\lambda(F_3) = e_1 e_2$

$\lambda(F_4) = e_2$

General Case: $p_0 = F_1 \cap F_2 \cap \cdots \cap F_n$ is a vertex of P, and $\lambda(F_i) = e_i, i = 1, 2, \cdots, n.$

- For any simple polytope P, define a right-angle Coxeter group

$$W_P = \langle s_F | s_F^2 = 1, (s_F s_{F'})^2 = 1, F, F' \in \mathcal{F}(P), F \cap F' \neq \varnothing \rangle$$

- The following (right split) exact sequence is given by Davis-Januszkiewicz [3].

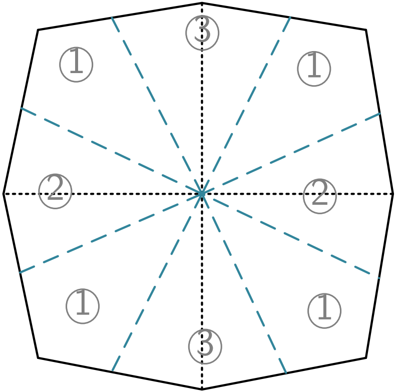
$$1 \longrightarrow \pi_1(M) \longrightarrow W_P \xrightarrow{\psi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}$$

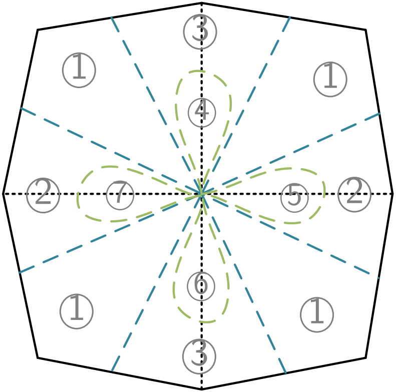
where $\psi(s_F) = \lambda(F), \ \forall F \in \mathcal{F}(P)$

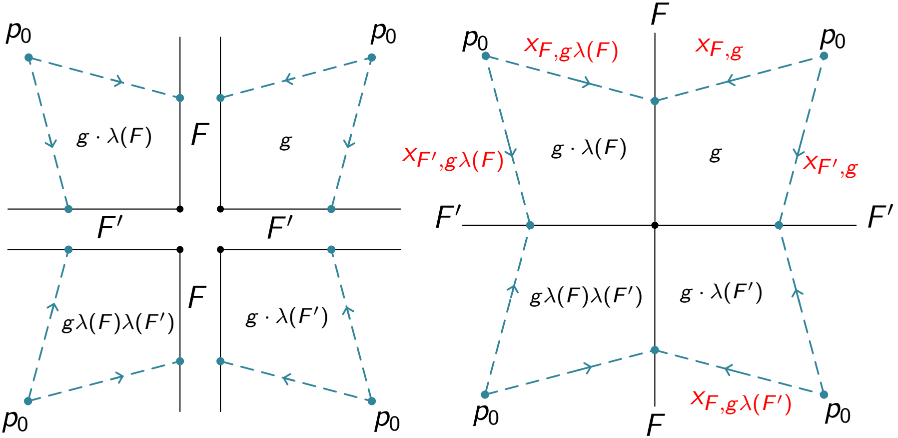
Rk: $W_P \cong \pi_1(M) \rtimes \mathbb{Z}_2^n$.

P	a n-dimension simple convex polytope in $\mathbb{R}^n$.
$\mathcal{F}(P)$	the facets set of P.
λ	$(\mathcal{F}(P) \rightarrow \mathbb{Z}_2^n)$the characteristic function.
M	a small cover over P.
π	$(M \rightarrow P)$ the nature project.
W_P	the Coxeter group associated with P.
$\pi_1(M)$	the fundamental group of M.

Single 0*-cell Cell Structure*



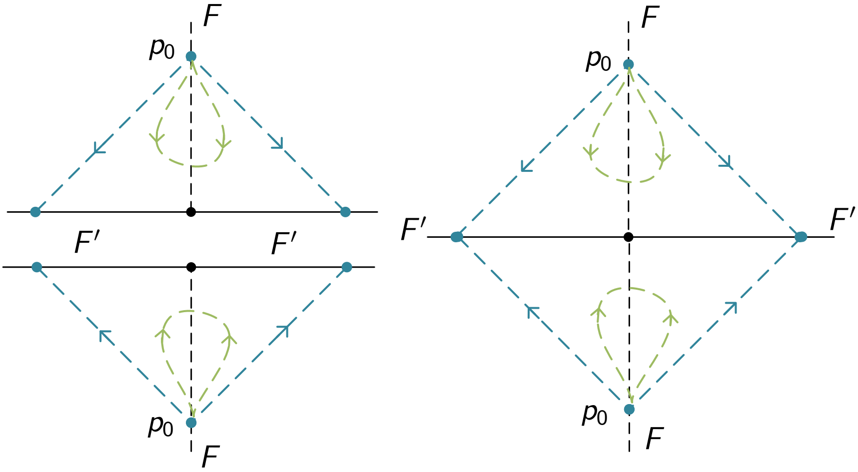




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

- Generators: $x_{F,g}$
- Relations: $[\phi_F(g) = g \cdot \lambda(F)]$
  - $x_{F,g} x_{F,\phi_F(g)} = 1$
  - $x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}$
  - $x_{F,g} = 1$

Theorem

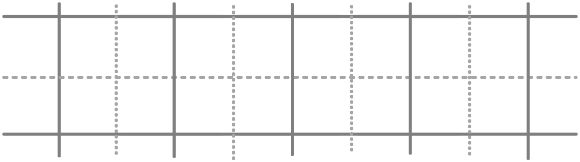
The presentation of $\pi_1(M)$.

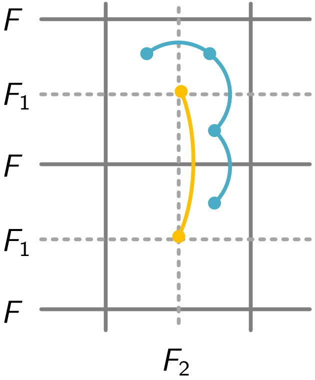
$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n :\\ x_{F,g} = 1, p_0 \in F; x_{F,g} x_{F,\phi_F(g)} = 1;\\ x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}, F \cap F' \neq \varnothing\rangle\end{aligned}$$

- $\mathcal{M} = Q \times \pi_1(M)/\sim^1$
  $(Q, \nu_1) \overset{F_g \sim F_h}{\sim} (Q, \nu_2) \Longleftrightarrow \nu_1^{-1}\nu_2 = x_{F,h}, g^{-1}h = \lambda(F)$.
- $\mathcal{L} = P \times W_P/\sim^2$
  $(P, \omega_1) \overset{F \sim F}{\sim} (P, \omega_2) \Longleftrightarrow \omega_1^{-1}\omega_2 = s_F$.

Lemma

$\mathcal{M} \cong \mathcal{L}$, in the other words, the universal cover space of small cover is determined only by simple polytope P.





$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
\end{tikzcd}$$

Rk: 1: the unit element of $\mathbb{Z}_2^n$ or W_P.

Relation between $\pi_1(M)$ *and* W_P

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) &\longrightarrow W \\
x_{F,g} &\longmapsto \quad \gamma(g) \cdot \gamma(\lambda(F)) s_F \cdot (\gamma(g))^{-1} \\
&\qquad = \gamma(g\lambda(F)) \cdot s_F \cdot \gamma(g) \\
&\qquad = \gamma(\phi_F(g)) s_F \gamma(g)
\end{aligned}
$$

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r, shift left, "\psi"] & \mathbb{Z}_2^n \arrow[l, shift left, "\gamma"] \arrow[r] & 1
\end{tikzcd}
$$

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective in M.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$.*

Rk: The π_1-injectivity of facial submanifold of small cover only depended on P.

- A simple polytope P is called a flag polytope if a collection of facets of P have common intersection whenever they pairwise intersect.

Corollary (Wu-Yu)

$\star$ *Let P be a flag simple polytope and M be a small cover over P. Then $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$ is injective for any proper face f of P.*

$\star\star$ *For any small cover M over a 3-dimensional simple polytope P, there always exists a facet F of P so that the facial submanifold M_F is π_1- injective in M.*

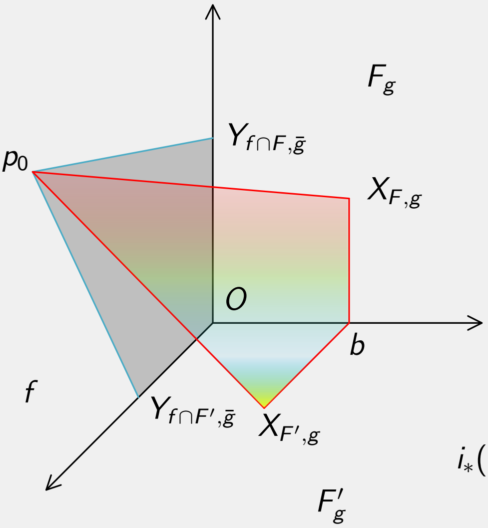
f	$F_1 \cap F_2 \cap \cdots \cap F_k$.
$\mathcal{F}(f^{\perp})$	$\{F : f \cap F \neq \varnothing, F \in \mathcal{F} - \{F_1, \cdots, F_k\}\}$.
M_f	$\pi^{-1}(f)$, the submanifold of M.
i	$(M_f \to M)$, the inclusion.
i_*	$(\pi_1(M_f) \to \pi_1(M))$, the group homomorphism induced by i.
j_*	$W_f \to W_P$, a natural group homomorphism as transformation group.

Proof.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(M_f) \arrow[r, "\alpha_f"] \arrow[d, "i_*"'] & W_f \arrow[r, "\psi_f"] \arrow[d, "j_*"'] & \mathbb{Z}_2^{n-k} \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r, "\psi"] & \mathbb{Z}_2^n \arrow[r] \arrow[u, "q"] & 1
\end{tikzcd}$$

$$\begin{aligned} j : W_f &\longrightarrow W_P \\ s_{f\cap F} &\longmapsto s_F \end{aligned} \qquad\qquad \begin{aligned} q : \mathbb{Z}_2^n &\longrightarrow \mathbb{Z}_2^k = \mathbb{Z}_2^n / G_f \\ g &\longmapsto \bar{g} \end{aligned}$$

Continuing.



$i_*(Y_{f \cap F, \bar{g}}) = X_{F,g}$

Continuing.

$$\begin{tikzcd}
\pi_1(M_f)/\alpha_f^{-1}(\ker j_*) \arrow[r, "\alpha'_f"] \arrow[d, "i'_*"'] & W_f/\ker j_* \arrow[d, "j'_*"'] & \\
\pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r, "\eta"] & W_P/\langle s_F : f \subset F \rangle
\end{tikzcd}$$

- $\eta \circ j'_* \circ \alpha'_f = \eta \circ \alpha \circ i'_*$
- $\eta \circ j'_* \circ \alpha'_f$ injective $\Longrightarrow i'_*$ injective
- $\ker j_* = \langle [F, F'] : F, F' \in \mathcal{F}(f^\perp), f \cap F \cap F' = \varnothing \rangle$

$\square$

End

References

- Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. *AMS* (2002).
- Davis, Exotic aspherical manifolds, *Topology of high-dimensional manifolds.* (Trieste, 2001).
- Davis and Januszkiewicz, Convex polytopes, coxeter orbifolds and torus actions, *Duke Math. J.* (1991).
- Davis, Januszkiewicz, and Scott, Nonpositive curvature of blow-ups, *Selecta Math.(N.S.)* (1998).
- Davis, Januszkiewicz and Scott, Fundamental groups of blow-ups, *Advances in mathematics.* (2003).
- Kuroki, Masuda and Yu, Small covers, infra-solvmanifolds and curvature, *Forum mathematicum.*(2015)

蓝 鲸 大 学

NANJING UNIVERSITY

Email: wulisuwulisu@qq.com
Homepage: `http://algebraic.top/`