Fundamental groups of small covers

Wu, Lisu

School of Mathematical Sciences, Fudan University

The 5^{th} Korea Toric Topology Winter Workshop

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Content

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such that

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Rk: $\mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\}$.

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where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_f(p)$, and $f(p)$ is the unique face of P that contains p in its relative interior.
- Define right-angled Coxeter group of P
$$W = \langle s_F, \forall F \in \mathcal{F}(P) \mid s_F^2 = 1, (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$$

Borel construction

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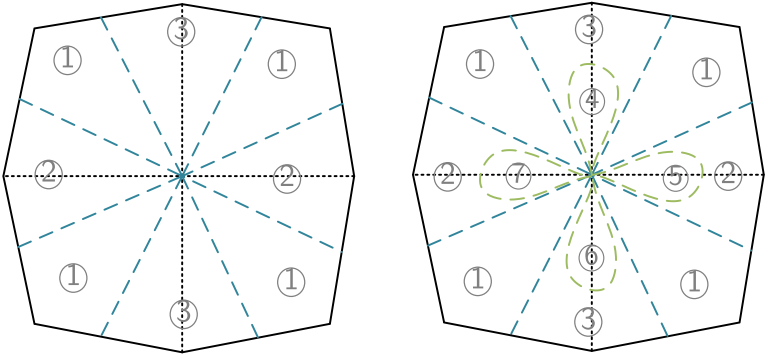
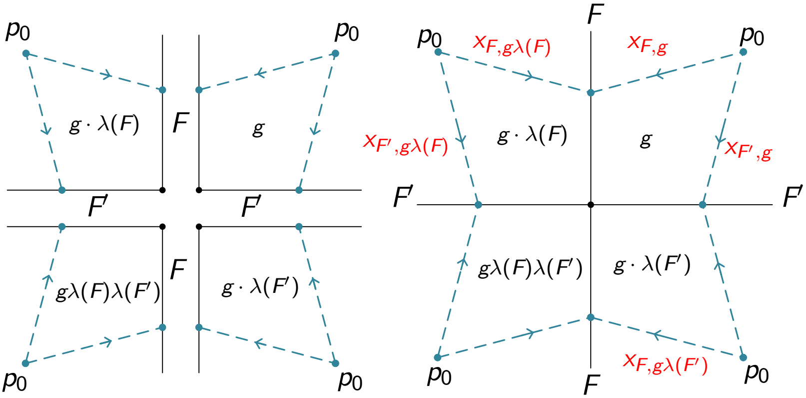
- Then $M \to BP \to B\mathbb{Z}_2^n$ induces an exact sequence
$$1 \longrightarrow \pi_1(M) \longrightarrow W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}$$
where $W \cong \pi_1(BP)$ and $\phi(s_F) = \lambda(F)$ for any facet F of P.

Cell decomposition

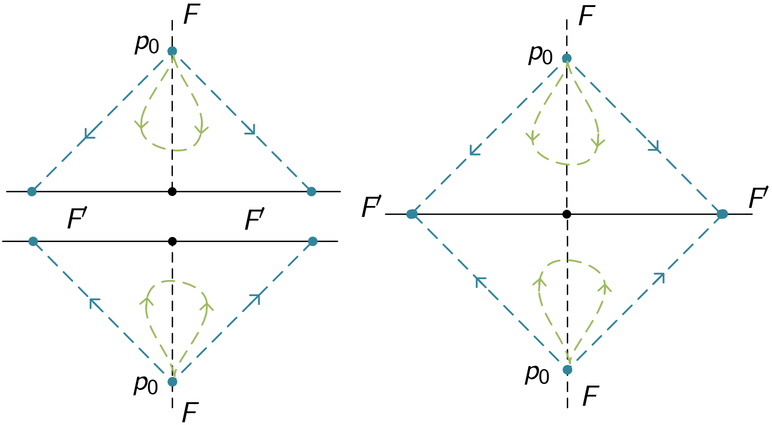




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

Presentation of $\pi_1(M)$

- Generator: $x_{F,g}$
- Relation: $[\sigma_F(g) = g \cdot \lambda(F)]$
    - $x_{F,g} x_{F,\sigma_F(g)} = 1$
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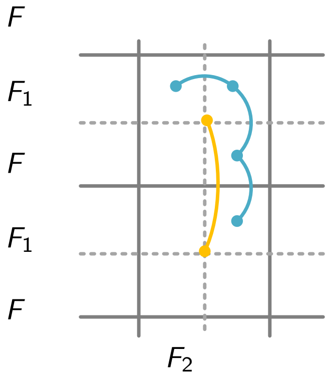
Presentation of $\pi_1(M, p_0)$

$$\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n \, | x_{F,g} x_{F,\sigma_F(g)} = 1; \\ x_{F,g} x_{F',\sigma_F(g)} = x_{F',g} x_{F,\sigma_{F'}(g)}, \;\; F \cap F' \neq \varnothing; \\ x_{F,g} = 1, p_0 \in F; \rangle$$

$\mathcal{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

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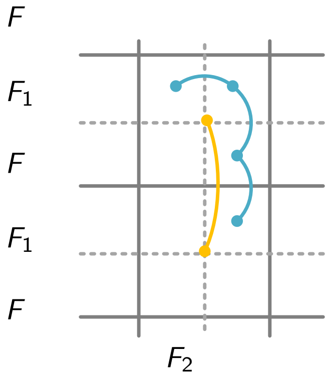

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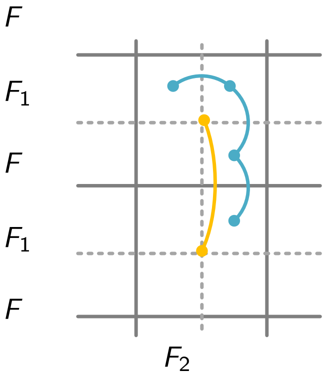

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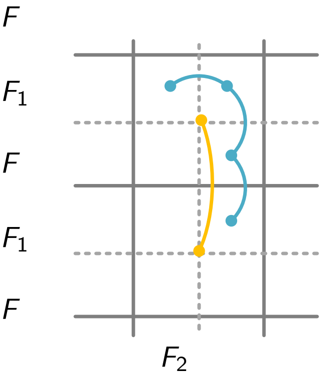

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$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, dotted, mapsto] & (P, s_{F_2} s_{F_1} s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2} s_{F_1} s_F(P,1) \arrow[u, mapsto]
\end{tikzcd}$$

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Relation between $\pi_1(M)$ and W

$$\begin{aligned}\alpha : \pi_1(M, p_0) &\longrightarrow W \\ x_{F,g} &\longmapsto \quad \gamma(\sigma_F(g)) \cdot \gamma(\sigma_F(1)) s_F \cdot (\gamma(\sigma_F(g)))^{-1} \\ &= \gamma(\sigma_F(g)\sigma_F(1)) \cdot s_F \cdot \gamma(\sigma_F(g)) \\ &= \gamma(g) s_F \gamma(\sigma_F(g))\end{aligned}$$

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{2}$$

Idea

$$
\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
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$$\begin{tikzcd} \color{red}{M} \arrow[r, dotted, red] \arrow[d, "\pi"'] & \color{red}{\pi_1(M)} \arrow[d, "\alpha"] \\ P \arrow[r, dotted] & W \end{tikzcd}$$

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\begin{tikzcd}
M \arrow[r, dotted, "\text{hard}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\downarrow"] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted, "\text{easy}"] & W
\end{tikzcd}
$$

For any proper face f of P,

- Define $\mathcal{F}(f^{\perp}) \stackrel{\triangle}{=} \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. So $\mathcal{F}(f^{\perp})$ consists of those facets of P that intersect f transversely.

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- A *k-circuit* in the simple polytope P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

> *The facial submanifold M_f is π_1-injective.*

> *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

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Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

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For any small cover M over a 3-dimensional simple polytope P, there always exists a facet F of P so that the facial submanifold M_F is π_1-injective.

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Let M be a connected 3-manifold.

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Theorem (Kneser, Prime Decomposition Theorem)

Every compact oriented 3 manifold M factors as a connected sum of prime manifolds, $M \cong M_1 \# \cdots \# M_n$, and this decomposition is unique up to insertion or deletion of S^3 summands.

Proposition

Let M be a 3-dimensional small cover over a simple polytope $P(\neq \Delta^3)$, then TFAE.

- *M is P^2-irreducible.*
- *M is prime.*
- *M is aspherical.*
- *P is flag.*
- *There is no prismatic 3-circuit in P.*

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Rk: $\mathbb{R}P^3$ is prime and irreducible but spherical.

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- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1.

Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

Let M be an oriented, irreducible, closed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

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Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no prismatic 4-circuit in P.
In particular, the JSJ decomposition or geometric decomposition of a 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

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Proposition

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts. In particular, all the oriented 3-dimensional small covers that can admit Riemannian metrics with non-negative scalar curvature are the two oriented real Bott manifolds in dimension 3 and the connected sum of k copies of $\mathbb{R}P^3$ for any $k \geq 1$.

End of Talk

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Some references

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Email: wulisuwulisu@qq.com
Homepage: http://algebraic.top/