

Fundamental groups of small covers

Lisu Wu

Nanjing University

Contents

1. Introduction

2. Cell Structure

3. Universal Cover Space

4. Main Results

- $\mathcal{F}(P) = \{F_1, F_2, \dots, F_m\}$: the facets set of a given simple polytope P .
- A map $\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$ satisfied
$$\forall f = F_1 \cap F_2 \cap \dots \cap F_k,$$
$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \dots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$
called *characteristic function* on $\mathcal{F}(P)$.
- $\lambda(F)$: the color on a facet F .

Small Covers

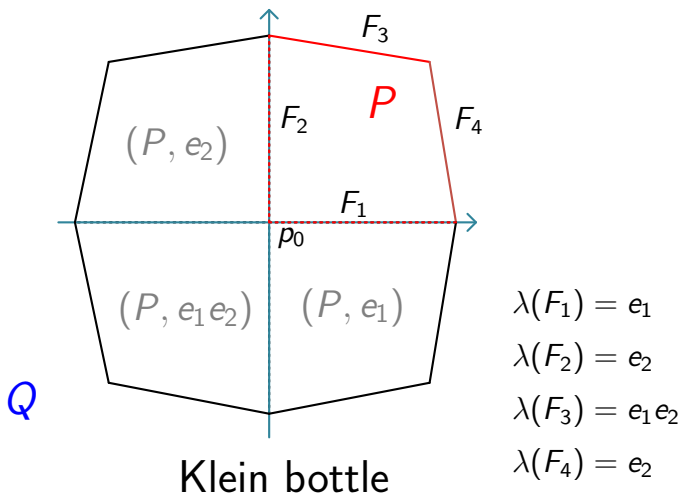
Definition (small cover over P)

$$M = P \times \mathbb{Z}_2^n / \sim$$

Where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior.

- $\pi : M \longrightarrow P$ the nature projective.
- $(P, g) \stackrel{F}{\sim} (P, h) \iff g^{-1}h = \lambda(F).$

$$\boxed{(P, g)} \overset{-F-}{\sim} \boxed{(P, h)}$$



General Case: $p_0 = F_1 \cap F_2 \cap \cdots \cap F_n$ is a vertex of P , and $\lambda(F_i) = e_i, i = 1, 2, \cdots, n$.

Coxeter Group and Exact Sequence

- For any simple polytope P , define a **right-angle Coxeter group**

$$W_P = \langle s_F \mid s_F^2 = 1, (s_F s_{F'})^2 = 1, F, F' \in \mathcal{F}(P), F \cap F' \neq \emptyset \rangle$$

- The following **(right split)** exact sequence is given by Davis-Januszkiewicz [3].

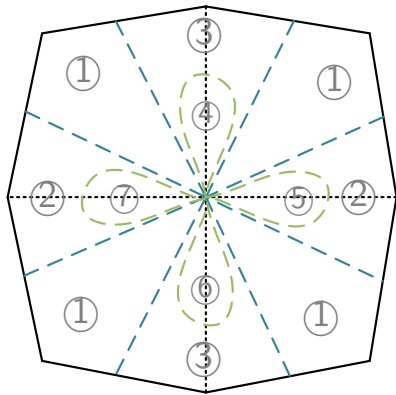
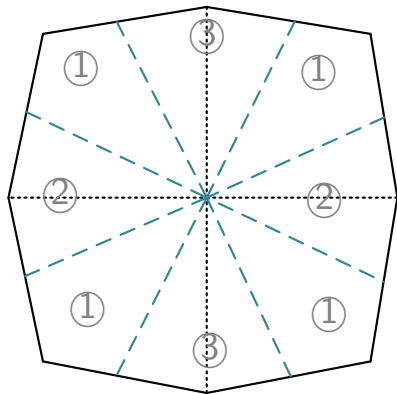
$$1 \longrightarrow \pi_1(M) \longrightarrow W_P \xrightarrow{\psi} \mathbb{Z}_2^n \longrightarrow 1 \quad (1)$$

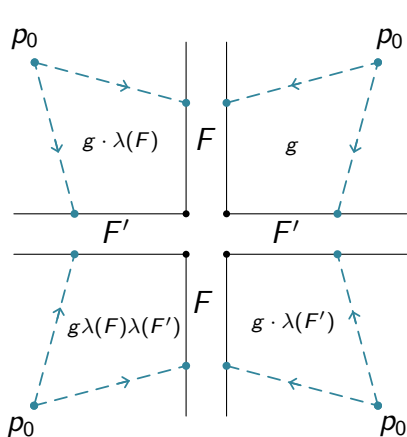
where $\psi(s_F) = \lambda(F)$, $\forall F \in \mathcal{F}(P)$

Rk: $W_P \cong \pi_1(M) \rtimes \mathbb{Z}_2^n$.

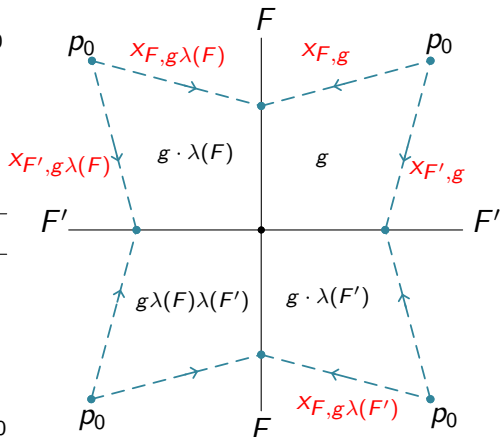
P	a n -dimension simple convex polytope in \mathbb{R}^n .
$\mathcal{F}(P)$	the facets set of P .
λ	$(\mathcal{F}(P) \rightarrow \mathbb{Z}_2^n)$ the characteristic function.
M	a small cover over P .
π	$(M \rightarrow P)$ the nature project.
W_P	the Coxeter group associated with P .
$\pi_1(M)$	the fundamental group of M .

Single 0-cell Cell Structure



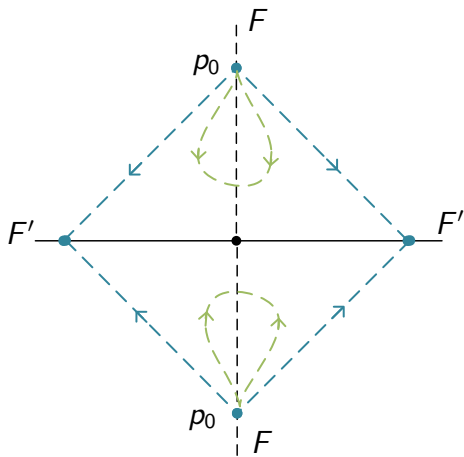
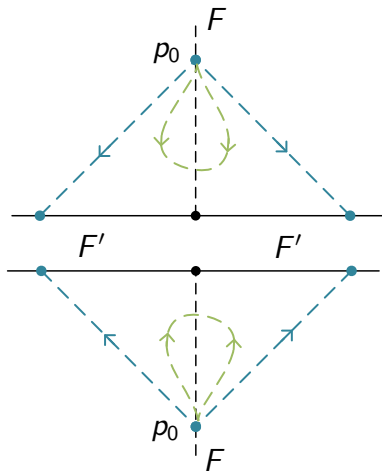


Cell-①



Relation-1: $x_{F,g} x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g} x_{F',g\lambda(F)} = x_{F',g} x_{F,g\lambda(F')}$



Relation-2: $x_{F,g} x_{F',g\lambda}(F) = x_{F',g} x_{F,g\lambda}(F')$

Relation-3: $x_{F,g} = 1, p_0 \subset F$

Presentation of π_1

- Generators: $x_{F,g}$
- Relations: $[\phi_F(g) = g \cdot \lambda(F)]$
 - ▶ $x_{F,g} x_{F,\phi_F(g)} = 1$
 - ▶ $x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}$
 - ▶ $x_{F,g} = 1$

Theorem

The presentation of $\pi_1(M)$.

$$\begin{aligned} \pi_1(M, p_0) = \langle & x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n : \\ & x_{F,g} = 1, p_0 \in F; x_{F,g} x_{F,\phi_F(g)} = 1; \\ & x_{F,g} x_{F',\phi_F(g)} = x_{F',g} x_{F,\phi_{F'}(g)}, F \cap F' \neq \emptyset \rangle \end{aligned}$$

Universal Cover Space

- $\mathcal{M} = Q \times \pi_1(M) / \sim^1$

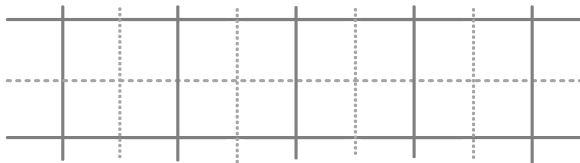
$$(Q, \nu_1) \stackrel{F_g \sim F_h}{\sim} (Q, \nu_2) \iff \nu_1^{-1} \nu_2 = x_{F,h}, g^{-1}h = \lambda(F).$$

- $\mathcal{L} = P \times W_P / \sim^2$

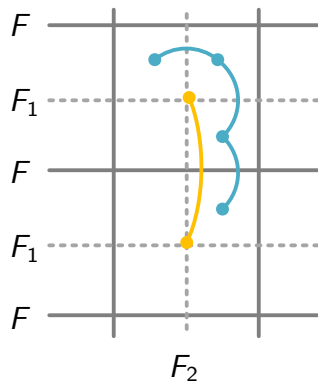
$$(P, \omega_1) \stackrel{F \sim F}{\sim} (P, \omega_2) \iff \omega_1^{-1} \omega_2 = s_F.$$

Lemma

$\mathcal{M} \cong \mathcal{L}$, in the other words, the universal cover space of small cover is determined only by simple polytope P .



Relation between $\pi_1(M)$ and W_P



$$x_{F,1}(Q, 1) \mapsto (Q, x_{F,1})$$

$$s_F(P, 1) \mapsto (P, s_F)$$

$$x_{F,1}(P, 1) \mapsto (P, s_{F_2} s_{F_1} s_F)$$

$$\begin{array}{ccc} & & \uparrow \\ & \text{---} & \\ x_{F,1}(P, 1) & \xrightarrow{\quad} & (P, s_{F_2} s_{F_1} s_F) \\ \downarrow & & \\ \gamma(\lambda(F)) \cdot s_F(P, 1) & \mapsto & s_{F_2} s_{F_1} s_F(P, 1) \end{array}$$

Rk: 1: the unit element of \mathbb{Z}_2^n or W_P .

Relation between $\pi_1(M)$ and W_P

$$\alpha : \pi_1(M, p_0) \longrightarrow W$$

$$\begin{aligned} x_{F,g} &\longmapsto \gamma(\phi_F(g)) \cdot s_F \gamma(\phi_F(1)) \cdot (\gamma(\phi_F(g)))^{-1} \\ &= \gamma(\phi_F(g)) \cdot s_F \cdot \gamma(\phi_F(1)\phi_F(g)) \\ &= \gamma(\phi_F(g)) s_F \gamma(g) \end{aligned}$$

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W_P \begin{array}{c} \xrightarrow{\psi} \\ \xleftarrow{\gamma} \end{array} \mathbb{Z}_2^n \longrightarrow 1$$

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P . The following two statements are equivalent.

- The facial submanifold M_f is π_1 -injective in M .*
- For any $F, F' \in \mathcal{F}(f^\perp)$, we have $f \cap F \cap F' \neq \emptyset$.*

Rk: The π_1 -injectivity of facial submanifold of small cover only depended on P .

- A simple polytope P is called a **flag** polytope if a collection of facets of P have common intersection whenever they pairwise intersect.

Corollary (Wu-Yu)

- ★ *Let P be a flag simple polytope and M be a small cover over P . Then $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$ is injective for any proper face f of P .*
- ★★ *For any small cover M over a 3-dimensional simple polytope P , there always exists a facet F of P so that the facial submanifold M_F is π_1 -injective in M .*

Notations.

f	$F_1 \cap F_2 \cap \cdots \cap F_k.$
$\mathcal{F}(f^\perp)$	$\{F : f \cap F \neq \emptyset, F \in \mathcal{F} - \{F_1, \cdots, F_k\}\}.$
M_f	$\pi^{-1}(f)$, the submanifold of M .
i	$(M_f \rightarrow M)$, the inclusion.
i_*	$(\pi_1(M_f) \rightarrow \pi_1(M))$, the group homomorphism induced by i .
j_*	$W_f \rightarrow W_P$, a natural group homomorphism as transformation group.

Proof.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(M_f) & \xrightarrow{\alpha_f} & W_f & \xrightarrow{\psi_f} & \mathbb{Z}_2^{n-k} \longrightarrow 1 \\
 & & \downarrow i_* & & \downarrow j_* & & \uparrow q \\
 1 & \longrightarrow & \pi_1(M) & \xrightarrow{\alpha} & W_P & \xrightarrow{\psi} & \mathbb{Z}_2^n \longrightarrow 1
 \end{array}$$

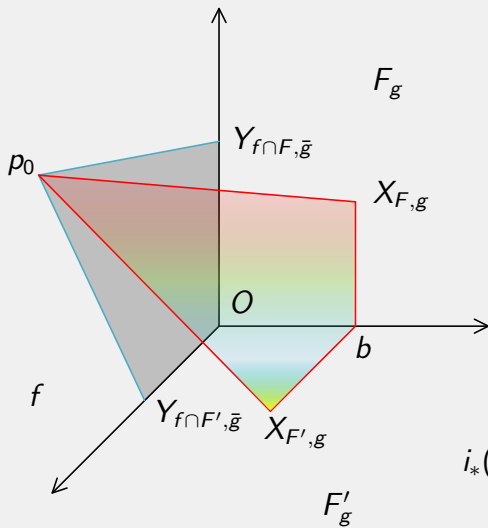
$$j : W_f \longrightarrow W_P$$

$$S_{f \cap F} \longmapsto S_F$$

$$q : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2^k = \mathbb{Z}_2^n / G_f$$

$$g \longmapsto \bar{g}$$

Continuing.



$$i_*(Y_{f \cap F, \bar{g}}) = X_{F, g}$$

Continuing.







$$\begin{array}{ccccc}
 \pi_1(M_f)/\alpha_f^{-1}(\ker j_*) & \xrightarrow{\alpha'_f} & W_f/\ker j_* & & \\
 \downarrow i'_* & & \downarrow j'_* & & \\
 \pi_1(M) & \xrightarrow{\alpha} & W_P & \xrightarrow{\eta} & W_P/\langle s_F : f \subset F \rangle
 \end{array}$$

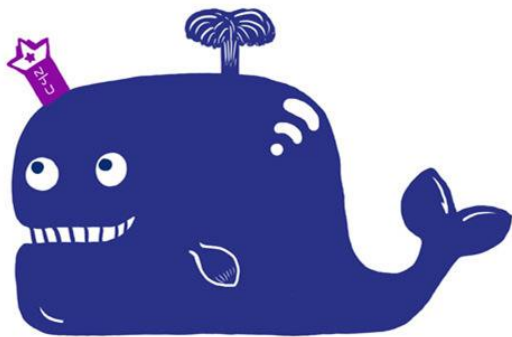
- $\eta \circ j'_* \circ \alpha'_f = \eta \circ \alpha \circ i'_*$
- $\eta \circ j'_* \circ \alpha'_f$ injective $\implies i'_*$ injective
- $\ker j_* = \langle [F, F'] : F, F' \in \mathcal{F}(f^\perp), f \cap F \cap F' = \emptyset \rangle$



End

References

-  Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. *AMS* (2002).
-  Davis, Exotic aspherical manifolds, *Topology of high-dimensional manifolds*. (Trieste, 2001).
-  Davis and Januszkiewicz, Convex polytopes, coxeter orbifolds and torus actions, *Duke Math. J.* (1991).
-  Davis, Januszkiewicz, and Scott, Nonpositive curvature of blow-ups, *Selecta Math.(N.S.)* (1998).
-  Davis, Januszkiewicz and Scott, Fundamental groups of blow-ups, *Advances in mathematics*. (2003).
-  Kuroki, Masuda and Yu, Small covers, infra-solvmanifolds and curvature, *Forum mathematicum*.(2015)



蓝 鲸 大 学

NANJING UNIVERSITY

Email: wulisuwulisu@qq.com

Homepage: <http://algebraic.top/>