Fundamental groups of small covers

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Content

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Rk: $\mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\}$.

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$$\mathbb{R}\mathcal{Z}_P = P \times \mathbb{Z}_2^m / \sim$$

- The universal cover space of M

$$\mathcal{M} = P \times W / \sim$$

where $W = \langle s_F, \forall F \in \mathcal{F}(P) \mid s_F^2 = 1, (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$ is the right-angled Coxeter group of P.

Borel construction

- The Borel construction(or the homotopy quotient of $\mathbb{Z}_2^n$ on M):

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n \simeq \mathbb{R}\mathcal{Z}_P \times_{\mathbb{Z}_2^m} E\mathbb{Z}_2^m \simeq \mathcal{M} \times_W EW$$

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  where BP only depends on P and its face structure.

- Then $M \to BP \to B\mathbb{Z}_2^n$ induces an exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}$$

  where $W \cong \pi_1(BP)$ and $\phi(s_F) = \lambda(F)$ for any facet F of P.

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2. The notion of orbifold covering is generalizing the usual notion in the topological category, and all basic results in the topological covering theory can be extended to the orbifold category.
3. An orbifold is good if it has some covering orbifold which is a closed manifold. Otherwise it is bad.

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& M \arrow[d, "\mathbb{Z}_2^n"] & \\
& P &
\end{tikzcd}
\quad \Longrightarrow \quad
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\pi_1(\mathbb{R}\mathcal{Z}_P) \arrow[r, equal] \arrow[d] & {[W, W]} \arrow[d] & \\
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\mathbb{Z}_2^{m-n} \arrow[r] & \mathbb{Z}_2^m \arrow[r] & \mathbb{Z}_2^n
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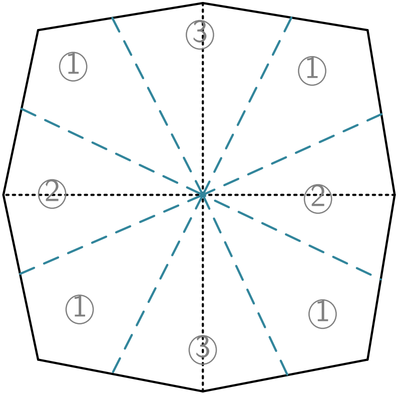
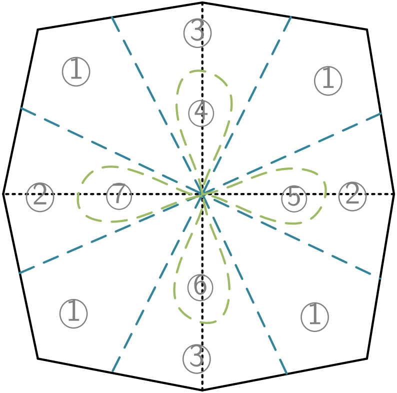
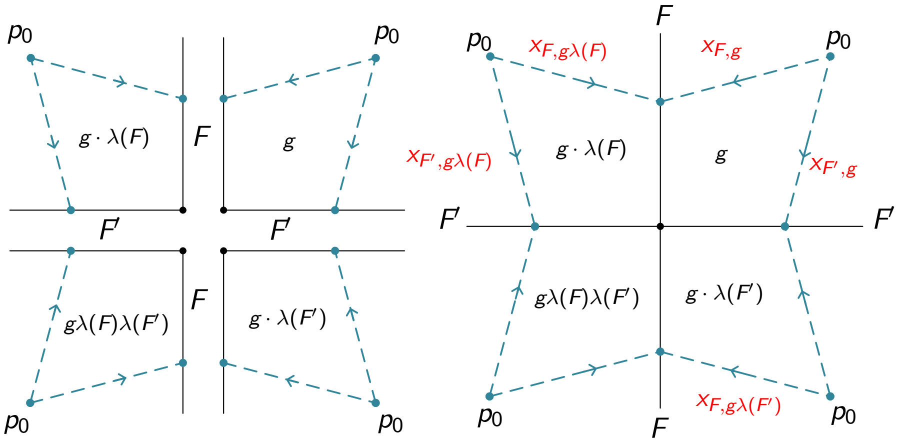
Cell decomposition



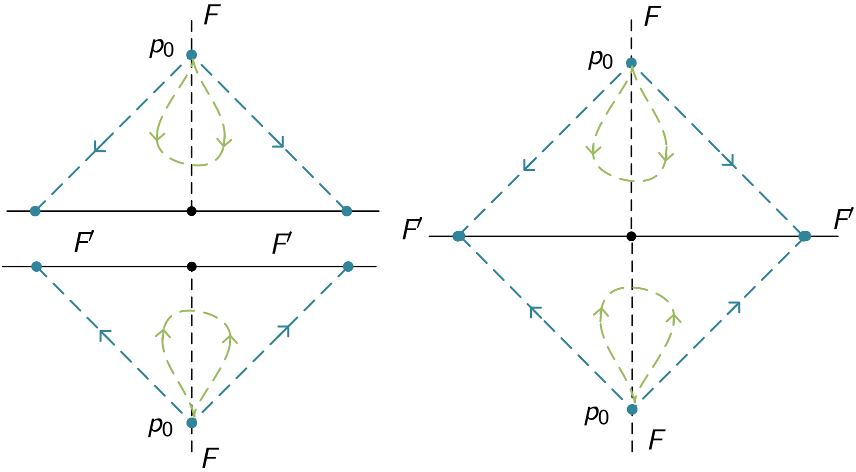




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

Presentation of $\pi_1(M)$

- Generator: $x_{F,g}$
- Relation: $[\sigma_F(g) = g \cdot \lambda(F)]$
  - $x_{F,g} x_{F,\sigma_F(g)} = 1$
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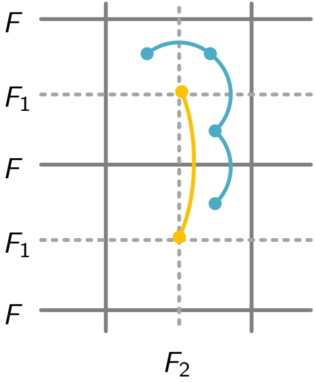
Presentation of $\pi_1(M, p_0)$

$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n \,|\, & x_{F,g} x_{F,\sigma_F(g)} = 1; \\ & x_{F,g} x_{F',\sigma_F(g)} = x_{F',g} x_{F,\sigma_{F'}(g)},\ F \cap F' \neq \varnothing; \\ & x_{F,g} = 1, p_0 \in F; \rangle\end{aligned}$$

Relation between $\pi_1(M)$ and W

$$\mathcal{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

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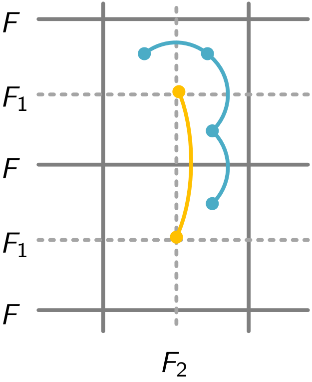

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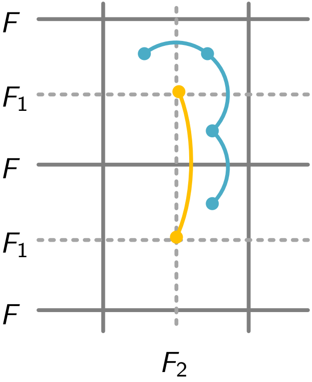

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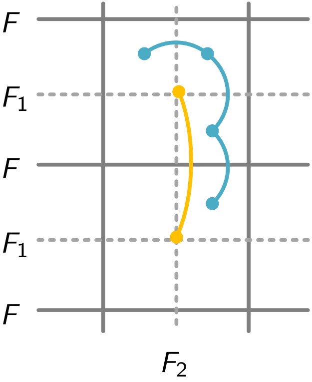

$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
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Relation between $\pi_1(M)$ and W

$$
\begin{aligned}
\alpha : \pi_1(M, p_0) &\longrightarrow W \\
x_{F,g} &\longmapsto \quad \gamma(\sigma_F(g)) \cdot \gamma(\sigma_F(1)) s_F \cdot (\gamma(\sigma_F(g)))^{-1} \\
&\qquad = \gamma(\sigma_F(g)\sigma_F(1)) \cdot s_F \cdot \gamma(\sigma_F(g)) \\
&\qquad = \gamma(g) s_F \gamma(\sigma_F(g))
\end{aligned}
$$

$$
1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{2}
$$

Semidirect product

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W \arrow[r, shift left, "\phi"] & \mathbb{Z}_2^n \arrow[l, shift left, "\gamma"] \arrow[r] & 1 \end{tikzcd}$$

Then $W = \pi_1(M) \rtimes_\psi \mathbb{Z}_2^n$, where $\psi_h(x) = \alpha^{-1}(\gamma(h)\alpha(x)\gamma(h^{-1}))$.

$$\begin{aligned} \psi_h(x_{F,g}) &= \alpha^{-1}(\gamma(h)\alpha(x_{F,g})\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(h)\gamma(g)s_F\gamma(\sigma_F(g))\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(gh)s_F\gamma(\sigma_F(gh))) \\ &= \psi_{gh}(x_{F,1}) \end{aligned}$$

Idea

$$\begin{tikzcd} M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\ P \arrow[r, dotted] & W \end{tikzcd}$$

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\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, dotted, "\text{complex}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\downarrow"] & \pi_1(M) \arrow[d, "\alpha"] \\
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- A submanifold Σ in M is called $\underline{\pi_1\text{-injective}}$ if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.
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- A *k-circuit* in the simple polytope P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.

π_1-injectivity of facial submanifolds

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

> *The facial submanifold M_f is π_1-injective.*
> *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

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Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

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Proposition (Wu-Yu, 2017)

For any small cover M over a 3-dimensional simple polytope P, there always exists a facet F of P so that the facial submanifold M_F is π_1-injective.

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- M is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.
- M is called irreducible if every embedded 2-sphere bounds a 3-ball. An prime 3-manifold is irreducible except S^2-bundle over S^1.

Theorem (Kneser, Prime Decomposition Theorem)

Every compact oriented 3 manifold M factors as a connected sum of prime manifolds, $M \cong M_1 \# \cdots \# M_n$, and this decomposition is unique up to insertion or deletion of S^3 summands.

Prime decomposition theorem

Let M be a connected 3-manifold.

- A compact embedded surface Σ is called 2-sided if it has a closed neighborhood in M homeomorphic to $\Sigma \times I$.

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- M is called Haken if it is a compact, P^2-irreducible 3-manifold that contains an 2-sided incompressible surface.

Prime decomposition theorem

Proposition

Let M be a 3-dimensional small cover over a simple polytope $P(\neq \Delta^3)$, then TFAE.

- *M is P^2-irreducible.*
- *M is prime.*
- *M is Haken.*
- *M is aspherical.*
- *P is flag.*
- *There is no prismatic 3-circuit in P.*
- *$\pi_2(M)$ is trivial.*
- *All facial submanifold is π_1-injective.*

In particular, the prime decomposition of an oriented 3-small cover is equivalent to cutting survey along prismatic 3-circuits in P

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Rk: $\mathbb{R}P^3$ is prime and irreducible but spherical.

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- A Seifert fiber space is a circle bundle over a 2-dimensional orbifold.
- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1.

JSJ-decomposition theorem

Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

Let M be a compact, oriented, irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

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Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

JSJ-decomposition theorem

Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no prismatic 4-circuit in P.
In particular, the JSJ decomposition or geometric decomposition of a 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

Geometric structure

Theorem (Thurston, Hyperbolization Theorem)

Every irreducible atoroidal closed 3-manifold with infinite fundamental group is hyperbolic.

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Every irreducible atoroidal closed 3-manifold with infinite fundamental group is hyperbolic.

Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Geometric structure

Proposition

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts. In particular, all the oriented 3-dimensional small covers that can admit Riemannian metrics with non-negative scalar curvature are the two oriented real Bott manifolds in dimension 3 and the connected sum of k copies of $\mathbb{R}P^3$ for any $k \geq 1$.

End of Talk

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