Fundamental groups of small covers

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Content

Small cover

- An n-dimensional small cover is a closed n-manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple convex polytope P.

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such that

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Rk: $\mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\}$.

- Small cover

$$M = P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior, $G_{f(p)} = 1$ if $p \in P^\circ$.

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- Define right-angled Coxeter group of P
$$W = \langle s_F, \forall F \in \mathcal{F}(P) \mid s_F^2 = 1; (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$$

Borel construction

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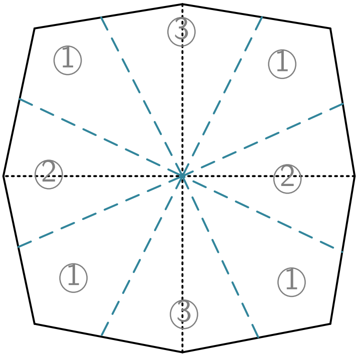
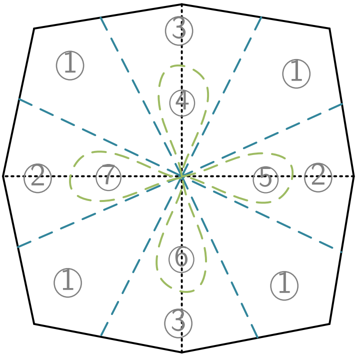
- $M \to BP \to B\mathbb{Z}_2^n$ induces a right-split exact sequence

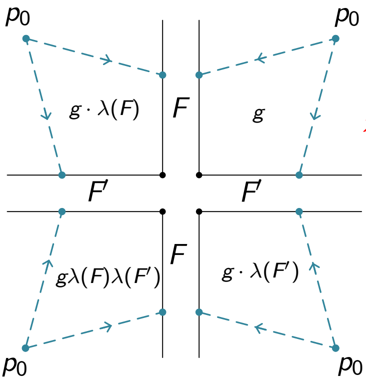
$$\begin{tikzcd} 1 \arrow[r] & \pi_1(M) \arrow[r] & W \arrow[r, shift left, "\phi"] & \mathbb{Z}_2^n \arrow[l, shift left, "\gamma"] \arrow[r] & 1 \end{tikzcd}$$

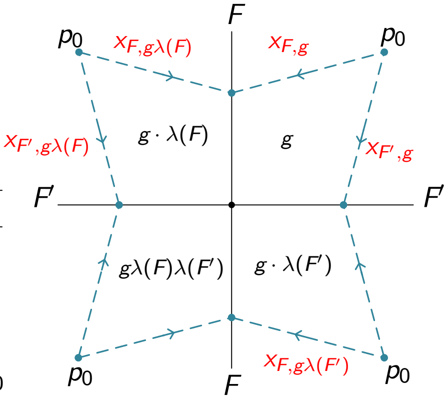
where $\phi(s_F) = \lambda(F),\ \forall F \in \mathcal{F}(P)$.

Cell decomposition





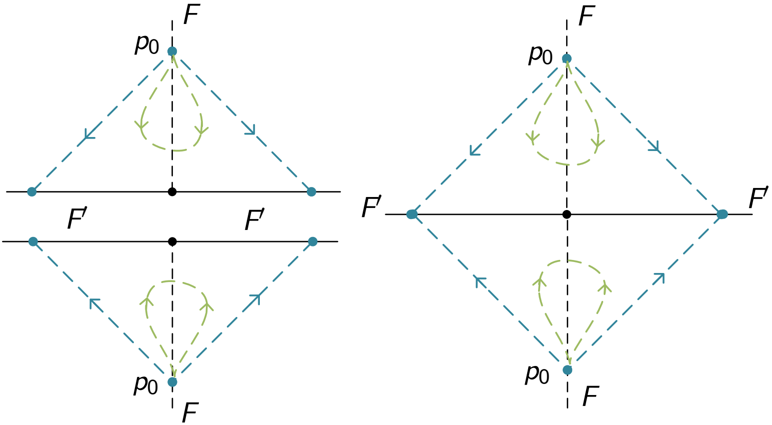




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



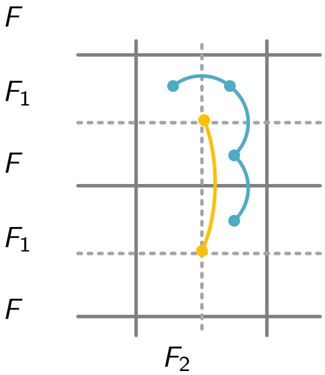
Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

Presentation of $\pi_1(M, p_0)$

$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, \forall\, F, g \,| & x_{F,g}x_{F,g\lambda(F)} = 1; \\ & x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}, \;\; F \cap F' \neq \varnothing; \\ & x_{F,g} = 1, p_0 \in F; \rangle\end{aligned}$$

$\mathcal{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

Relation between $\pi_1(M)$ and W

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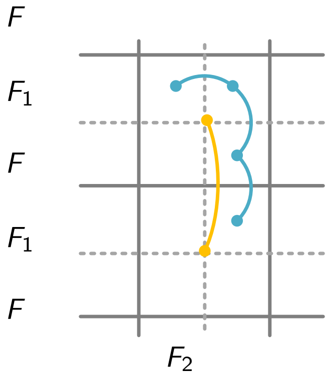

$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$

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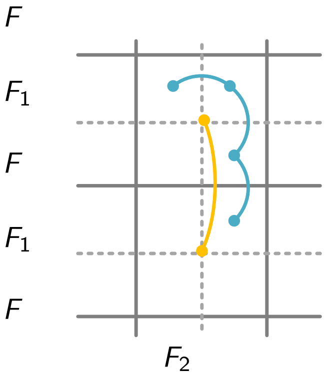

$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

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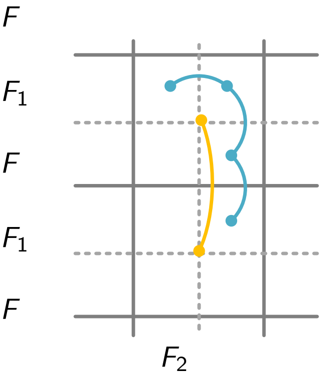

$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
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$$\begin{aligned}\alpha : \pi_1(M, p_0) \longrightarrow & W \\ x_{F,g} \longmapsto & \quad \gamma(g\lambda(F)) \cdot \gamma(\lambda(F)) s_F \cdot (\gamma(g\lambda(F)))^{-1} \\ & = \gamma(g) s_F \gamma(g\lambda(F))\end{aligned}$$

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$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}$$

Idea

$$
\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
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\end{tikzcd}
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$$\begin{tikzcd}
{\color{red}M} \arrow[r, dotted, red] \arrow[d, "\pi"'] & {\color{red}\pi_1(M)} \arrow[d, "\alpha"] \\
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Idea

$$\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, dotted, "\color{red}\text{hard}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\color{red}\downarrow"] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted, "\color{red}\text{easy}"] & W
\end{tikzcd}$$

For any proper face f of P,

- Define $\mathcal{F}(f^{\perp}) \triangleq \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. So $\mathcal{F}(f^{\perp})$ consists of those facets of P that intersect f transversely.

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- A submanifold Σ in M is called $\underline{\pi_1\text{-injective}}$ if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective.*
- *For any $F, F' \in \mathcal{F}(f^\perp)$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

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Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

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Proposition (Wu-Yu, 2017)

Let M be a small cover over P. Then P is flag if and only if every facial submanifold of M is π_1-injective.

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- A *k-belt* in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three face in the belt have no common intersection.

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- Each k-belt can determine a prismatic k-circuit; a prismatic 3-circuit determines a 3-belt;

Belt and circuit

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- Each k-belt can determine a prismatic k-circuit; a prismatic 3-circuit determines a 3-belt; if there is no prismatic 3-circuit, then a prismatic 4-circuit determines a 4-belt.

Let M be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ's paper(corrected):

- If there exist prismatic 3-circuits in P, then M can be decomposed into prime pieces glued along S^2 or $\mathbb{R}P^2$.

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- If there is no prismatic 3-circuit but prismatic 4-circuits in P, then M can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.
- If there is no prismatic 3 or 4-circuit in P, then M is hyperbolic.

Prime decomposition theorem

Let M be a connected 3-manifold.

- M is called <u>prime</u> if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.

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- M is called <u>P^2-irreducible</u> if it is irreducible and contains no two-sided $\mathbb{R}P^2$.

Theorem (Kneser, Milnor, Prime Decomposition Theorem)

Each compact 3-manifold M can factor as a connected sum of prime manifolds. This decomposition is unique under the assumption that M is orientable.

Proposition

Let M be a 3-dimensional small cover over a simple polytope P, then M is P^2-irreducible if and only if there is no prismatic 3-circuit in P.

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In particular, the prime decomposition of an oriented 3-small cover is equivalent to cutting surgery along prismatic 3-circuits in P.

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Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

For an oriented, irreducible, closed 3-manifold, there exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered, and a minimal such collection $T_1, \cdots, T_m$ is unique up to isotopy.

Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

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Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no 4-belt in P.
In particular, the JSJ decomposition or geometric decomposition of a 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

Theorem (Thurston, Hyperbolization Theorem)

Each P^2-irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

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Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Proposition

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts. In particular, all the oriented 3-dimensional small covers that can admit Riemannian metrics with non-negative scalar curvature are the two oriented real Bott manifolds in dimension 3 and the connected sum of k copies of $\mathbb{R}P^3$ for any $k \geq 1$.

End of Talk

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Some references

- Wu and Yu, Fundamental groups of small covers revisited. (2018).
- Wu, Atoroidal manifolds in small covers. (2018).
- Agol's *talk-1* & *talk-2* (2012, 2014).
- Aschenbrenner-Friedl-Wilton, 3-manifold groups, (2013).
- Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. (2002).
- Davis and Januszkiewicz, Convex polytopes, coxeter orbifolds and torus actions, (1991).
- Chen, A Homotopy Theory of Orbispaces. (2001).
- Kapovich, Hyperbolic manifolds and discrete groups. (2010).

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