Fundamental groups of small covers

Lisu Wu

Nanjing University

Contents

- $\mathcal{F}(P) = \{F_1, F_2, \cdots, F_m\}$ the facets set of a given simple polytope P.
- A map $\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$ satisfied

$$\forall f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$\dim_{\mathbb{Z}_2}(span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}) = k.$$

called *characteristic function* on $\mathcal{F}(P)$.

Small Covers

- $G_f = span\{\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k)\}$, for $f = F_1 \cap \cdots \cap F_k$.

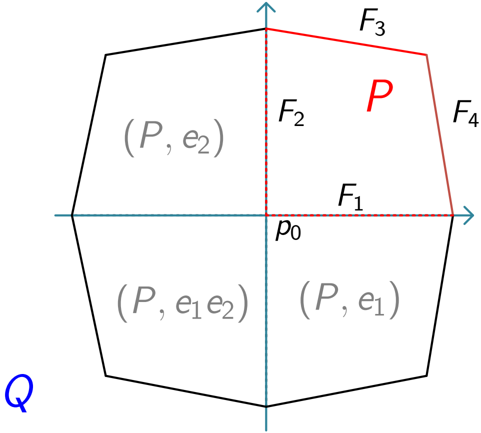
Definition (small cover over P)

$$M = P \times \mathbb{Z}_2^n / \sim$$

Where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_f(p)$, and $f(p)$ is the unique face of P that contains p in its relative interior.

- $\pi : M \longrightarrow P$ the nature projective.
- $(P, g) \overset{F}{\sim} (P, h) \Longleftrightarrow g^{-1}h = \lambda(F)$.

$$\boxed{(P, g)} \text{-} F \text{-} \boxed{(P, h)}$$



Klein bottle

$\lambda(F_1) = e_1$

$\lambda(F_2) = e_2$

$\lambda(F_3) = e_1 e_2$

$\lambda(F_4) = e_2$

General Case: $p_0 = F_1 \cap F_2 \cap \cdots \cap F_n$ is a vertex of P, and $\lambda(F_i) = e_i, i = 1, 2, \cdots, n$.

Coxeter Group and Exact Sequence

- For any simple polytope P, define a right-angle Coxeter group

$$W_P = \langle s_F | s_F^2 = 1, (s_F s_{F'})^2 = 1, F, F' \in \mathcal{F}(P), F \cap F' \neq \varnothing \rangle$$

- W_P is isomorphic to the fundamental group of the Borel construction $M_{\mathbb{Z}_2^n} = E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M$.
- Then $M \to M_{\mathbb{Z}_2^n} \to B\mathbb{Z}_2^n$ induces an (right split) exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W_P \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \qquad (1)$$

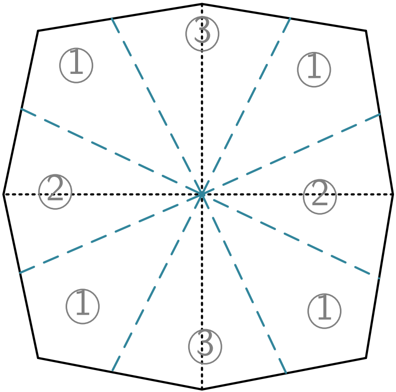
where $\psi(s_F) = \lambda(F),\ \forall F \in \mathcal{F}(P)$

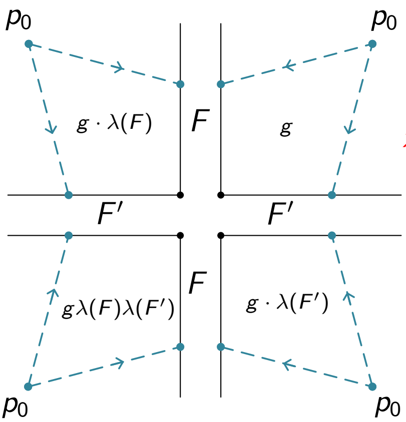
Rk: $W_P \cong \pi_1(M) \rtimes \mathbb{Z}_2^n$.

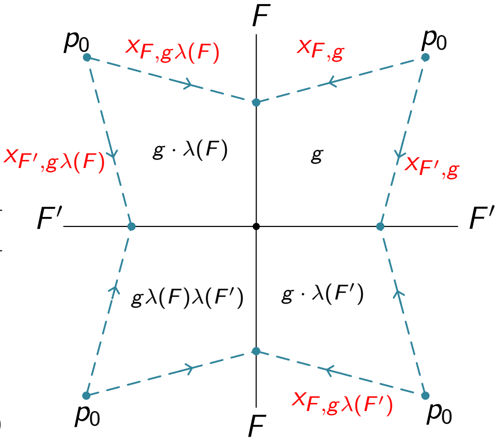
Notations

P	a n-dimension simple convex polytope in $\mathbb{R}^n$.
$\mathcal{F}(P)$	the facets set of P.
λ	$(\mathcal{F}(P) \rightarrow \mathbb{Z}_2^n)$the characteristic function.
M	a small cover over P.
π	$(M \rightarrow P)$ the nature project.
W	the Coxeter group associated with P.
$\pi_1(M)$	the fundamental group of M .

Single 0*-cell Cell Structure*



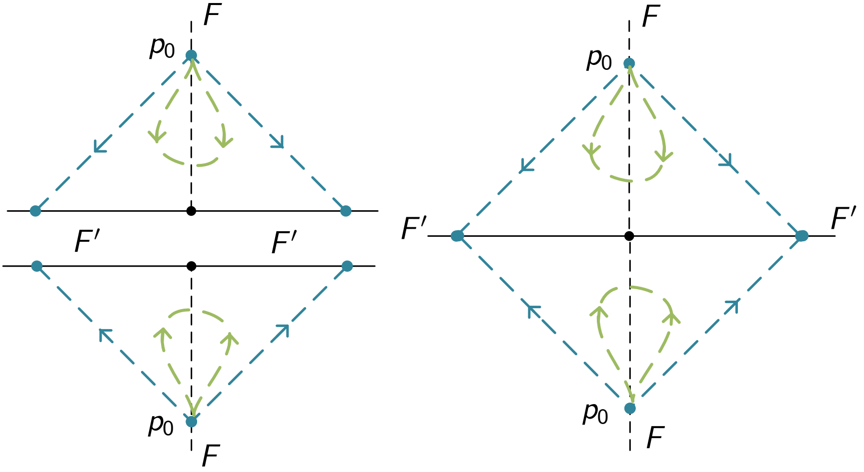




Cell-①

Relation-1: $x_{F,g}x_{F,g\lambda(F)} = 1$

Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$



Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

Presentation of π_1

- Generator: $x_{F,g}$
- Relation: $[\phi_F(g) = g \cdot \lambda(F)]$
  - $x_{F,g}x_{F,\phi_F(g)} = 1$
  - $x_{F,g}x_{F',\phi_F(g)} = x_{F',g}x_{F,\phi_{F'}(g)}$
  - $x_{F,g} = 1$

Theorem

The presentation of $\pi_1(M)$.

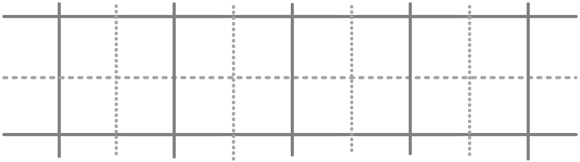
$$\begin{aligned}\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n :&\\ x_{F,g} = 1, p_0 \in F; x_{F,g}x_{F,\phi_F(g)} = 1;&\\ x_{F,g}x_{F',\phi_F(g)} = x_{F',g}x_{F,\phi_{F'}(g)}, F \cap F' \neq \varnothing; \rangle&\end{aligned}$$

Universal Cover Space

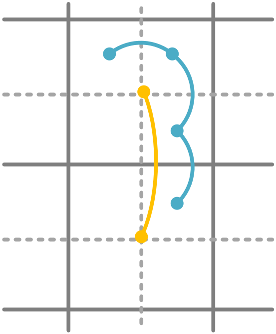
- $\mathcal{M} = Q \times \pi_1(M)/\sim^1$
  $(Q, \nu_1) \overset{F_g \sim F_h}{\sim} (Q, \nu_2) \Longleftrightarrow \nu_1^{-1}\nu_2 = x_{F,h}, g^{-1}h = \lambda(F)$.
- $\mathcal{L} = P \times W_P/\sim^2$
  $(P, \omega_1) \overset{F \sim F}{\sim} (P, \omega_2) \Longleftrightarrow \omega_1^{-1}\omega_2 = s_F$.

Lemma

$\mathcal{M} \cong \mathcal{L}$, in the other words, the universal cover space of small cover is determined only by simple polytope P.



Relation between $\pi_1(M)$ *and* W_P



Relation between $\pi_1(M)$ *and* W_P

$$\begin{aligned}
\alpha : \pi_1(M, p_0) \longrightarrow & W \\
x_{F,g} \longmapsto \quad & \gamma(\phi_F(g)) \cdot \gamma(\phi_F(1)) s_F \cdot (\gamma(\phi_F(g)))^{-1} \\
& = \gamma(\phi_F(g)\phi_F(1)) \cdot s_F \cdot \gamma(\phi_F(g)) \\
& = \gamma(g) s_F \gamma(\phi_F(g))
\end{aligned}$$

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{2}$$

Main Result

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

- *The facial submanifold M_f is π_1-injective in M.*
- *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$.*

Rk: The π_1-injectivity of facial submanifold of small cover only depended on P.

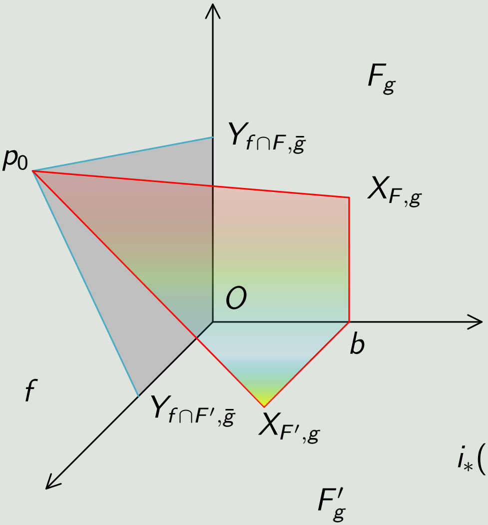
Proof.

Proof.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(M_f) \arrow[r, "\alpha_f"] \arrow[d, "i_*"] & W_f \arrow[r] \arrow[d, "j_*"'] & \mathbb{Z}_2^k \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r] & \mathbb{Z}_2^n \arrow[r] \arrow[u, "q"] & 1
\end{tikzcd}$$

$$\begin{aligned} j : W_f &\longrightarrow W_P \\ s_{f\cap F} &\longmapsto s_F \end{aligned} \qquad \begin{aligned} q : \mathbb{Z}_2^n &\longrightarrow \mathbb{Z}_2^k = \mathbb{Z}_2^n / \langle \lambda(F) : f \subset F \rangle \\ g &\longmapsto \bar{g} \end{aligned}$$

Continuing.



$$i_*(Y_{f\cap F,\bar{g}}) = X_{F,g}$$

Continuing.

$$\begin{tikzcd}
\pi_1(M_f)/\alpha_f^{-1}(\ker j_*) \arrow[r, "\alpha'_f"] \arrow[d, "i'_*"] & W_f/\ker j_* \arrow[d, "j'_*"'] & \\
\pi_1(M) \arrow[r, "\alpha"] & W_P \arrow[r, "\eta"] & W_P/\langle s_F : f \subset F \rangle
\end{tikzcd}$$

- $\eta \circ j'_* \circ \alpha'_f = \eta \circ \alpha \circ i'_*$
- $\eta \circ j'_* \circ \alpha'_f$ injective $\Longrightarrow i'_*$ injective
- $\ker j_* = \langle [F, F'] : F, F' \in \mathcal{F}(f^\perp), f \cap F \cap F' = \varnothing \rangle$

$\square$

End

References

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蓝鲸大学

NANJING UNIVERSITY

Email: wulisuwulisu@qq.com
Homepage: http://algebraic.top/