

Fundamental groups of small covers

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Small cover

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Rk: $\mathcal{F}(P) \stackrel{\Delta}{=} \{F_1, F_2, \dots, F_m\}.$

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where $(p, g) \sim (q, h)$ iff $p = q, g^{-1}h \in G_f(p)$, and $f(p)$ is the unique face of P that contains p in its relative interior.

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- The universal cover space of M

$$\mathcal{M} = P \times W / \sim$$

where $W_P = \langle s_F \mid s_F^2 = 1, (s_F s_{F'})^2 = 1, F \cap F' \neq \emptyset \rangle$ is the right-angled Coxeter group of P .

Borel construction

- The Borel construction (or the homotopy quotient of \mathbb{Z}_2^n on M):

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n \simeq \mathbb{R}\mathcal{Z}_P \times_{\mathbb{Z}_2^m} E\mathbb{Z}_2^m \simeq \mathcal{M} \times_W EW$$

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where BP only depends on P and its face structure.

- Then $M \rightarrow BP \rightarrow B\mathbb{Z}_2^n$ induces an exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \quad (1)$$

where $W \cong \pi_1(BP)$ and $\phi(s_F) = \lambda(F)$ for any facet F of P .

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Orbifold coverings

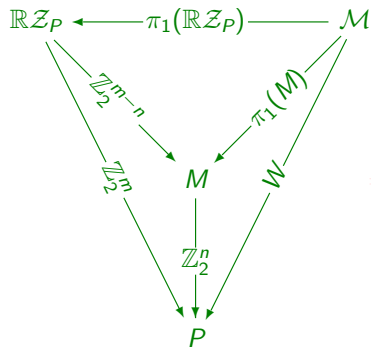
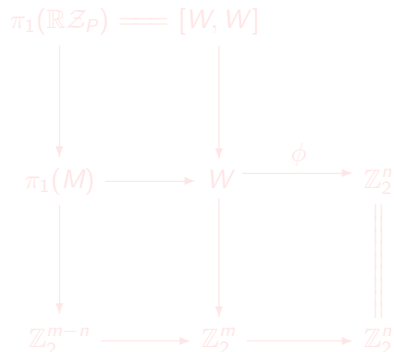
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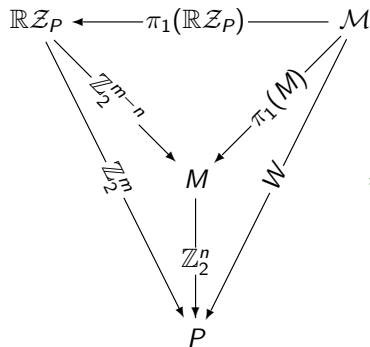
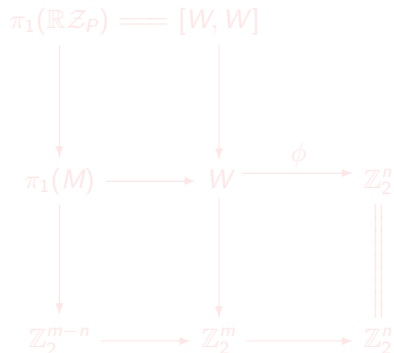
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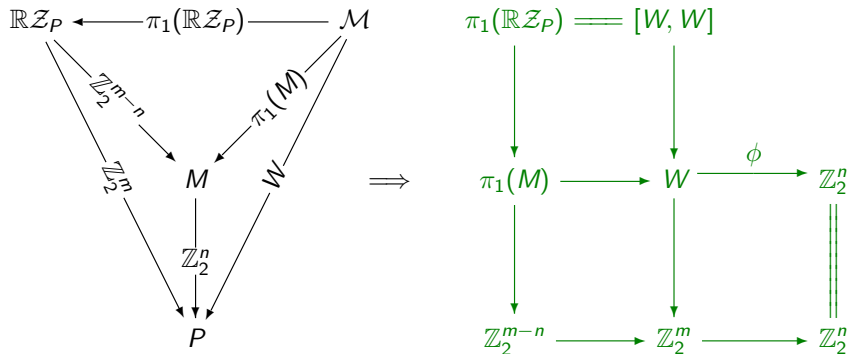
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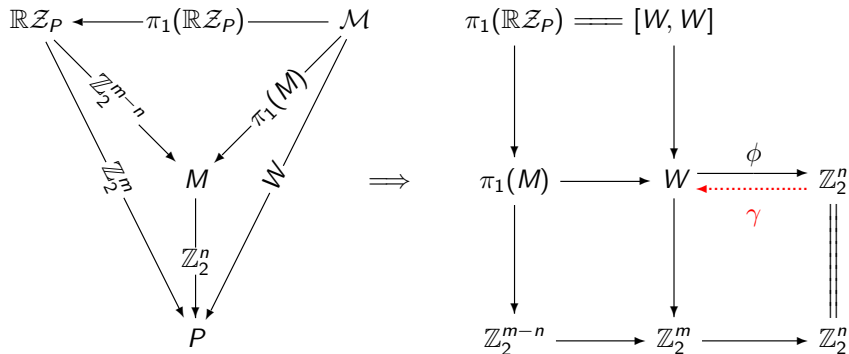
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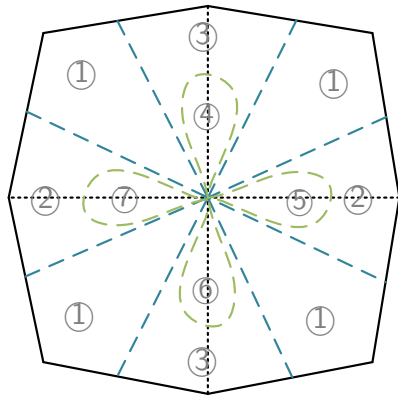
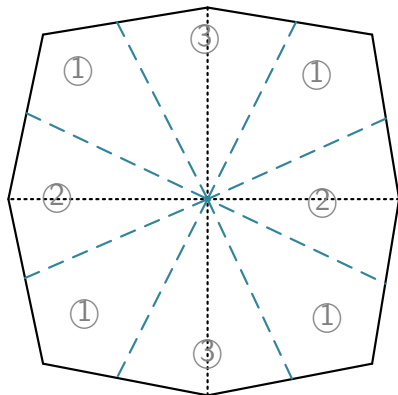


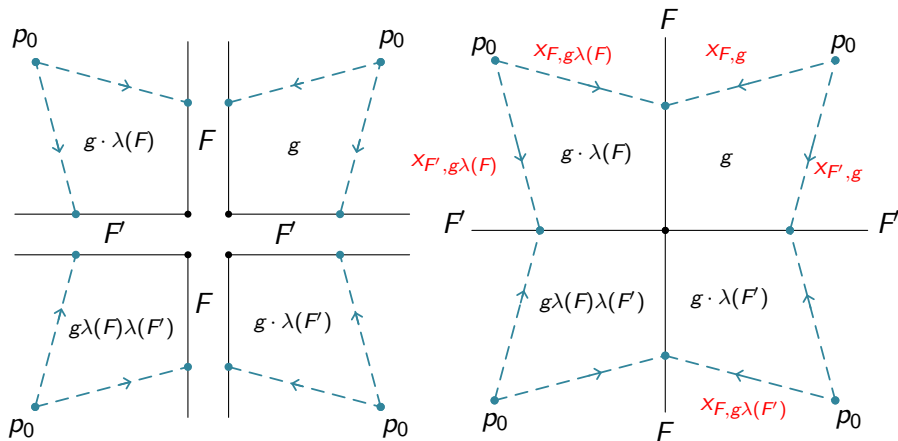
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Cell decomposition

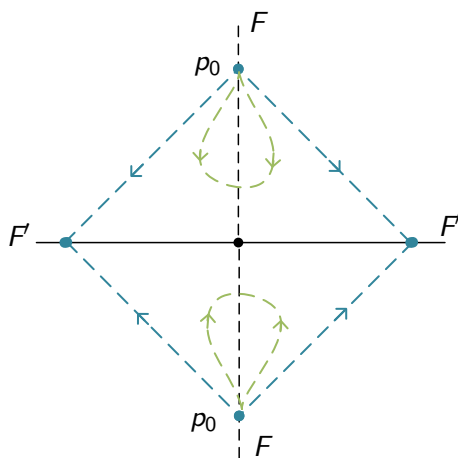
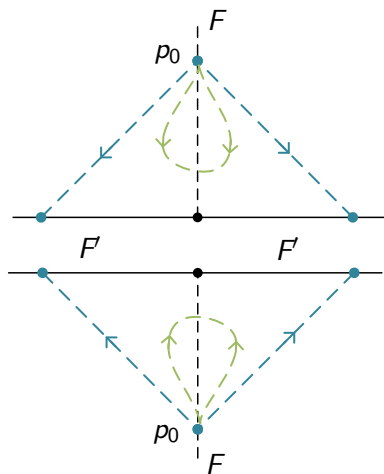




Cell-①

$$\text{Relation-1: } x_{F,g}x_{F,g\lambda(F)} = 1$$

$$\text{Relation-2: } x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$$



Relation-2: $x_{F,g} x_{F',g\lambda}(F) = x_{F',g} x_{F,g\lambda}(F')$

Relation-3: $x_{F,g} = 1, p_0 \subset F$

Presentation of $\pi_1(M)$

- Generator: $x_{F,g}$
- Relation: $[\sigma_F(g) = g \cdot \lambda(F)]$
 - ▶ $x_{F,g}x_{F,\sigma_F(g)} = 1$
 - ▶ $x_{F,g}x_{F',\sigma_F(g)} = x_{F',g}x_{F,\sigma_{F'}}(g)$
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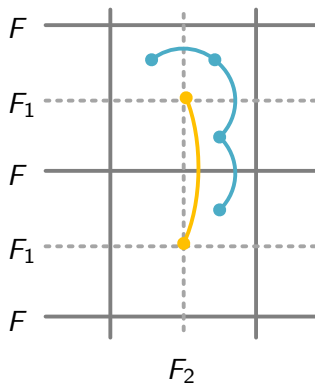
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Presentation of $\pi_1(M, p_0)$

$$\pi_1(M, p_0) = \langle x_{F,g}, F \in \mathcal{F}(P), g \in \mathbb{Z}_2^n \mid x_{F,g}x_{F,\sigma_F(g)} = 1; \\ x_{F,g}x_{F',\sigma_F(g)} = x_{F',g}x_{F,\sigma_{F'}(g)}, F \cap F' \neq \emptyset; \\ x_{F,g} = 1, p_0 \in F; \rangle$$

Relation between $\pi_1(M)$ and W_P

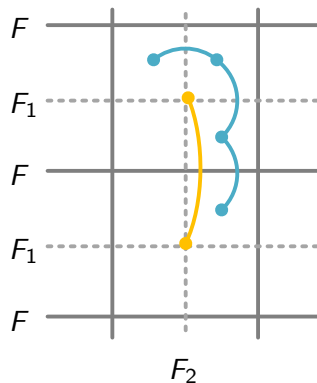
$$\mathcal{M} = Q \times \pi_1(M) / \sim = P \times W / \sim$$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

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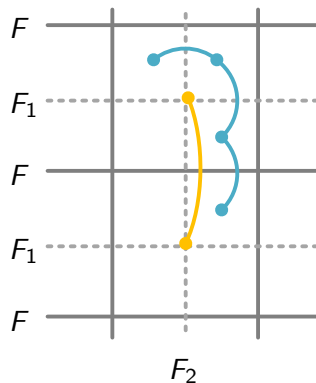


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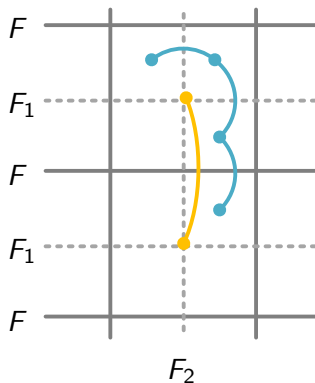
$$x_{F,1}(Q, 1) \mapsto (Q, x_{F,1})$$

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$$x_{F,1}(P, 1) \mapsto (P, s_{F_2}s_{F_1}s_F)$$

$$\begin{array}{ccc} & & \uparrow \\ & \vdots & \\ & & \uparrow \\ \gamma(\lambda(F)) \cdot s_F(P, 1) & \mapsto & s_{F_2}s_{F_1}s_F(P, 1) \end{array}$$

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Relation between $\pi_1(M)$ and W_P

$$\alpha : \pi_1(M, p_0) \longrightarrow W$$

$$\begin{aligned} x_{F,g} &\longmapsto \gamma(\sigma_F(g)) \cdot \gamma(\sigma_F(1)) s_F \cdot (\gamma(\sigma_F(g)))^{-1} \\ &= \gamma(\sigma_F(g) \sigma_F(1)) \cdot s_F \cdot \gamma(\sigma_F(g)) \\ &= \gamma(g) s_F \gamma(\sigma_F(g)) \end{aligned}$$

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \quad (2)$$

Semidirect product

$$1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\gamma} \end{array} \mathbb{Z}_2^n \longrightarrow 1$$

Then $W = \pi_1(M) \rtimes_{\psi} \mathbb{Z}_2^n$, where $\psi_h(x) = \alpha^{-1}(\gamma(h)\alpha(x)\gamma(h^{-1}))$.

$$\begin{aligned} \psi_h(x_{F,g}) &= \alpha^{-1}(\gamma(h)\alpha(x_{F,g})\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(h)\gamma(g)s_F\gamma(\sigma_F(g))\gamma(h^{-1})) \\ &= \alpha^{-1}(\gamma(gh)s_F\gamma(\sigma_F(gh))) \\ &= \psi_{gh}(x_{F,1}) \end{aligned}$$

Idea

$$\begin{array}{ccc} M & \cdots\cdots\cdots \rightarrow & \pi_1(M) \\ \pi \downarrow & & \downarrow \alpha \\ P & \cdots\cdots\cdots \rightarrow & W \end{array}$$

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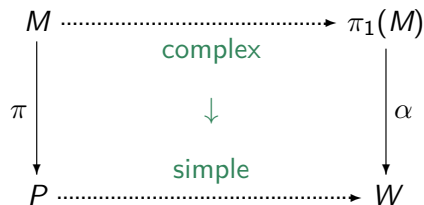
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- A submanifold Σ in M is called π_1 -injective if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.
- A k -circuit in the simple polytope P is a simple loop on the boundary of P which intersects transversely the interior of exactly k distinct edges, and a k -circuit is called prismatic if the endpoints of those edges are distinct.

π_1 -injectivity of facial submanifolds

Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P . The following two statements are equivalent.

- > The facial submanifold M_f is π_1 -injective in M .*
- > For any $F, F' \in \mathcal{F}(f^\perp)$, we have $f \cap F \cap F' \neq \emptyset$ whenever $F \cap F' \neq \emptyset$.*

Rk: The π_1 -injectivity of a facial submanifold of small cover only depends on the local face structure of f in P .

Rk: We can determine the kernel of $i_* : \pi_1(M_f) \longrightarrow \pi_1(M)$.

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Proposition (Wu-Yu, 2017)

For any small cover M over a 3-dimensional simple polytope P , there always exists a facet F of P so that the facial submanifold M_F is π_1 -injective in M .

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Theorem (Kneser, Prime Decomposition Theorem)

Every compact oriented 3 manifold M factors as a connected sum of prime manifolds, $M \cong M_1 \# \cdots \# M_n$, and this decomposition is unique up to insertion or deletion of S^3 summands.

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- M is called Haken if it is a compact, P^2 -irreducible 3-manifold that contains an 2-sided incompressible surface.

Prime decomposition theorem

Proposition

Let M be a 3-dimensional small cover over a simple polytope $P(\neq \Delta^3)$, then TFAE.

- M is irreducible.
- M is prime.
- M is Haken.
- M is aspherical.
- P is flag.
- There is no prismatic 3-circuit in P .
- $\pi_2(M)$ is trivial.
- All facial submanifold is π_1 -injective.

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In particular, the prime decomposition of a oriented 3-small cover is equivlent to cutting survey along prismatic 3-circuits in P

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Let M be a 3-dimensional small cover over a simple polytope $P(\neq \Delta^3)$, then TFAE.

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Rk: $\mathbb{R}P^3$ is prime and irreducible but spherical.

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- A Seifert fiber space is a circle bundle over a 2-dimensional orbifold.
- A manifold M is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1 .

JSJ-decomposition theorem

Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

Let M be a compact, oriented, irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori T_1, \dots, T_m such that each component of M cut along $T_1 \cup \dots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

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Proposition

Let M be a 3-small cover over a simple polytope P , then M is atoroidal if and only if there is no prismatic 4-circuit in P .
In particular, the JSJ decomposition of a 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P .

Geometric structure

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Every irreducible atoroidal closed 3-manifold with infinite fundamental group is hyperbolic.

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Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P .

Geometric structure

Proposition

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0, 1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts. In particular, all the oriented 3-dimensional small covers that can admit Riemannian metrics with nonnegative scalar curvature are the two oriented real Bott manifolds in dimension 3 and the connected sum of k copies of $\mathbb{R}P^3$ for any $k \geq 1$.

End of Talk

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