8. THE STANDARD ERROR ELLIPSE

After a *Least Squares Adjustment* of survey data, intersection, resection or a combination of both, using the method of *Variation of Coordinates,* the cofactor matrix $\mathbf{Q}_{xx} = \mathbf{N}^{-1}$ contains estimates of the variances and covariances of the adjusted quantities. These precision estimates, variances s_E^2, s_N^2 and covariance s_{EN} can be used to define a geometric figure known as the *Standard Error Ellipse,* which is a useful graphical representation of the precision of a position fix. Poor or "weak" fixes are indicated by narrow elongated ellipses and good or "strong" position fixes are indicated by near circular ellipses.

Error ellipses may be computed for points before any observations are made provided the approximate locations of points (fixed and floating) are known. Observations (directions, bearings and distances) may be scaled from maps and diagrams and an approximate set of normal equations formed. The inverse of the coefficient matrix **N** yields all the information required for the computation of the parameters of the error ellipses. In such cases, error ellipses are an important analysis tool for the surveyor in planning survey operations

8.1. The Pedal Curve of the Standard Error Ellipse

Consider a point whose precision estimates, variances s_E^2, s_N^2 and covariance s_{EN} are known. The variance in any other direction u may be calculated by considering the projection of E and N onto the u-axis, which is rotated anti-clockwise from the E-axis by an angle ϕ

$$\begin{aligned} u &= E\cos\phi + N\sin\phi \\ &= f\left(E, N\right) \end{aligned} \tag{8.1}$$

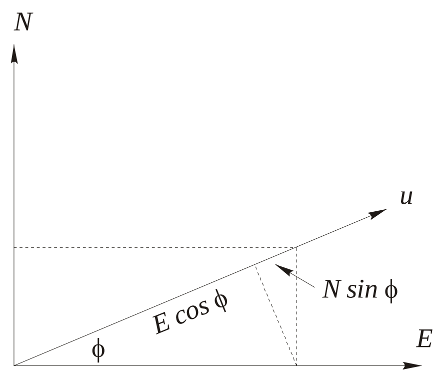


Figure 8.1

Applying the law of propagation of variances to equation (8.1) gives an expression for variance in the *u*-direction

$$s_u^2 = s_E^2\left(\frac{\partial f}{\partial E}\right)^2 + s_N^2\left(\frac{\partial f}{\partial N}\right)^2 + 2s_{EN}\frac{\partial f}{\partial E}\frac{\partial f}{\partial N} \tag{8.2}$$

The partial derivatives $\frac{\partial f}{\partial E} = \cos\phi$, $\frac{\partial f}{\partial N} = \sin\phi$ are obtained from (8.1) to give an equation for the variance s_u^2 in a direction ϕ (positive anti-clockwise) from the *E*-axis.

$$s_u^2 = s_E^2\cos^2\phi + s_N^2\sin^2\phi + 2s_{EN}\cos\phi\sin\phi \tag{8.3}$$

Equation (8.3) defines the *pedal curve* of the *Standard Error Ellipse*

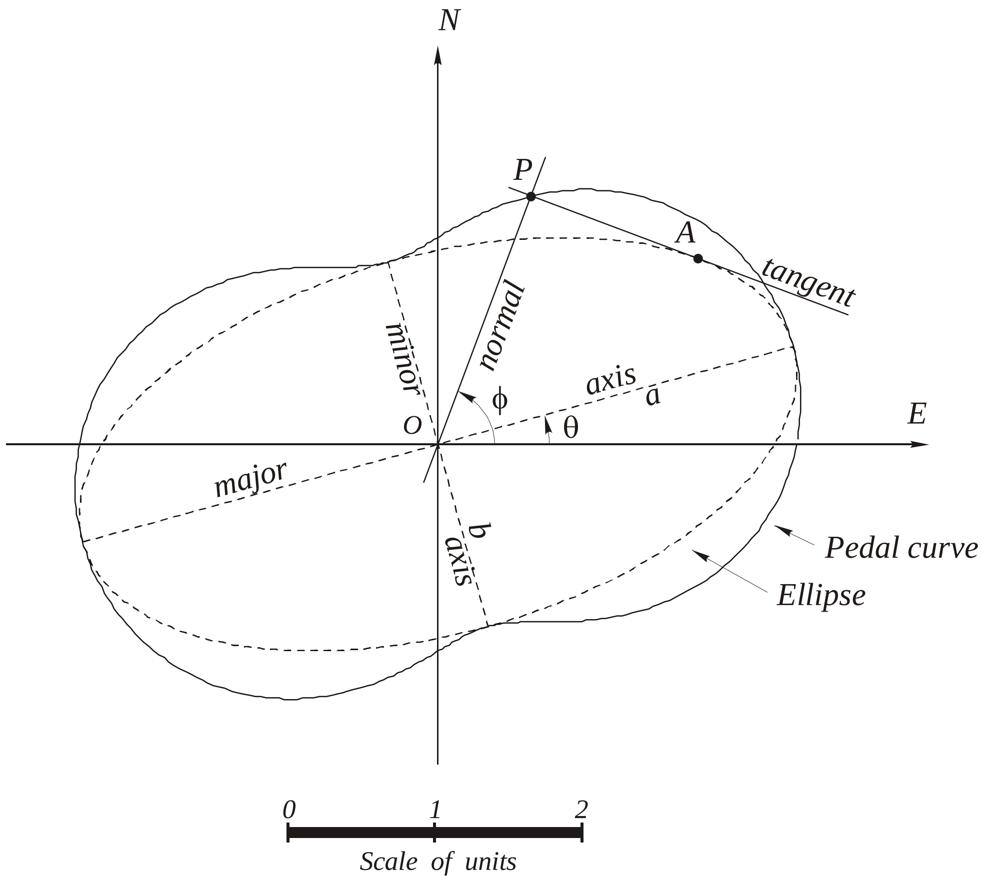


Figure 8.2. The pedal curve of the Standard Error Ellipse

In Figure 8.2, *A* is a point on an ellipse. The tangent to the ellipse at *A* intersects a normal to the tangent passing through *O* at *P*. As *A* moves around the ellipse, the locus of all points *P* is the pedal curve of the ellipse. The distance $OP = \sqrt{s_u^2}$ for the angle ϕ. The maximum and minimum values of s_u^2 define the directions and lengths of the axes of the ellipse and the following section details the equations linking variances s_E^2, s_N^2 and covariance s_{EN} with the Standard Error Ellipse parameters a, b and θ.

8.2. Parameters of the Standard Error Ellipse

Equation (8.3) has maximum and minimum values defining the lengths and directions of the axes of the error ellipse. To determine these values from (8.3) the trigonometric identities $1-\cos 2\phi = 2\sin^2\phi$, $1+\cos 2\phi = 2\cos^2\phi$, $\sin 2\phi = 2\sin\phi\cos\phi$ can be used to give

$$\begin{aligned} s_u^2 &= \frac{1}{2}s_E^2\left(1+\cos 2\phi\right)+\frac{1}{2}s_N^2\left(1-\cos 2\phi\right)+s_{EN}\sin 2\phi \\ &= \frac{1}{2}\left(s_E^2+s_N^2\right)+\frac{1}{2}\left(s_E^2-s_N^2\right)\cos 2\phi + s_{EN}\sin 2\phi \end{aligned}$$

Letting $A = \frac{1}{2}\left(s_E^2 - s_N^2\right)$ and $B = s_{EN}$ this expression has the general form

$$s_u^2 = \frac{1}{2}\left(s_E^2+s_N^2\right)+A\cos 2\phi + B\sin 2\phi \tag{8.4}$$

Equation (8.4) can be expressed as a trigonometric addition

$$\begin{aligned} s_u^2 &= \frac{1}{2}\left(s_E^2+s_N^2\right)+R\cos\left(2\phi-\alpha\right) \\ &= \frac{1}{2}\left(s_E^2+s_N^2\right)+R\cos 2\phi\cos\alpha + R\sin 2\phi\sin\alpha \end{aligned} \tag{8.5}$$

Equating the coefficients of $\cos 2\phi$ and $\sin 2\phi$ in equations (8.4) and (8.5) gives $R\cos\alpha = A$ and $R\sin\alpha = B$ from which we obtain

$$\begin{aligned} R &= \sqrt{A^2+B^2} \\ &= \sqrt{\tfrac{1}{4}\left(s_E^2-s_N^2\right)^2+\left(s_{EN}\right)^2} \\ &= \frac{1}{2}\sqrt{\left(s_E^2-s_N^2\right)^2+4\left(s_{EN}\right)^2} \\ &= \frac{1}{2}W \end{aligned} \tag{8.6}$$

where
$$W = \sqrt{\left(s_E^2 - s_N^2\right)^2 + 4\left(s_{EN}\right)^2} \tag{8.7}$$

and the angle α from

$$\tan\alpha = \frac{B}{A} = \frac{2s_{EN}}{s_E^2 - s_N^2} \tag{8.8}$$

Inspection of equation (8.5) shows that $s_u^2 = \frac{1}{2}\left(s_E^2 + s_N^2\right) + R\cos\left(2\phi - \alpha\right)$ will have a maximum value when $\left(2\phi - \alpha\right) = 0$ i.e., $\cos\left(0\right) = 1$ and a minimum value when $\left(2\phi - \alpha\right) = \pi$ i.e., $\cos\left(\pi\right) = -1$ or

$$\begin{aligned} s_u^2\left(\max\right) &= \frac{1}{2}\left(s_E^2 + s_N^2\right) + R = \frac{1}{2}\left(s_E^2 + s_N^2 + W\right) \\ s_u^2\left(\min\right) &= \frac{1}{2}\left(s_E^2 + s_N^2\right) - R = \frac{1}{2}\left(s_E^2 + s_N^2 - W\right) \end{aligned} \tag{8.9}$$

Inspection of Figure 8.2 shows that the maximum and minimum values of s_u^2 are in the directions of the major and minor axes of the Standard Error Ellipse and the semi-axes lengths are

$$\begin{aligned} a &= \sqrt{\frac{1}{2}\left(s_E^2 + s_N^2 + W\right)} \\ b &= \sqrt{\frac{1}{2}\left(s_E^2 + s_N^2 - W\right)} \end{aligned} \tag{8.10}$$

The value of ϕ when s_u^2 is a maximum is when $\left(2\phi - \alpha\right) = 0$, i.e., when $\alpha = 2\phi$ thus from equation (8.8), letting $\theta = \phi$ when s_u^2 is a maximum, the angle θ, measured anti-clockwise from the *E*-axis to the major axis of the Standard Error Ellipse, is given by

$$\tan 2\theta = \frac{2s_{EN}}{s_E^2 - s_N^2} \tag{8.11}$$

Note that s_u^2 is a minimum when $2\phi - \alpha = \pi$, i.e., when $\alpha = 2\phi - \pi$ thus from equation (8.8), letting $\theta = \phi$ and recognizing that $\tan\left(x - \pi\right) = \tan x$, then $\tan 2\theta = 2s_{EN} \big/ \left(s_E^2 - s_N^2\right)$ which is the same equation for the angle to the major axis. Hence, it is not possible to distinguish between the angles to the major or minor axes and the ambiguity must be resolved by using equation (8.3).

Alternatively, the parameters of the Standard Error Ellipse can be determined from equation (8.3) by the methods outlined in Chapter 2 (Sction 2.7.2 Least Squares Best Fit Ellipse). Consider equation (8.3) expressed as

$$f = s_E^2 \cos^2\phi + s_N^2 \sin^2\phi + 2s_{EN}\cos\phi\sin\phi \tag{8.12}$$

and the aim is to find the maximum and minimum values of *f* (the maximum and minimum variances) and the values of ϕ when these occur by investigating the first and second derivatives f' and f'' respectively, i.e.,

$$f \text{ is } \begin{Bmatrix} \max \\ \min \end{Bmatrix} \text{ when } \begin{Bmatrix} f' = 0 \text{ and } f'' < 0 \\ f' = 0 \text{ and } f'' > 0 \end{Bmatrix}$$

where

$$\begin{aligned} f' &= \left(s_N^2 - s_E^2\right)\sin 2\phi + 2s_{EN}\cos 2\phi \\ f'' &= 2\left(s_N^2 - s_E^2\right)\cos 2\phi - 4s_{EN}\sin 2\phi \end{aligned} \tag{8.13}$$

Now the maximum or minimum value of *f* occurs when $f' = 0$ and from the first member of (8.13) the value of ϕ is given by

$$\tan 2\phi = \frac{2s_{EN}}{s_E^2 - s_N^2} \tag{8.14}$$

But this value of ϕ could relate to either a maximum or a minimum value of *f*. So from the second member of equations (8.13) with a value of 2ϕ from equation (8.14) this ambiguity can be resolved by determining the sign of the second derivative f'' since it is known that

$$\begin{Bmatrix} f_{\max} \\ f_{\min} \end{Bmatrix} \text{ when } \begin{Bmatrix} f'' < 0 \\ f'' > 0 \end{Bmatrix}$$

In the equation of the pedal curve of the Standard Error Ellipse given by equation (8.12) $f_{\max}$ coincides with $s_{\max}^2$ and $f_{\min}$ coincides with $s_{\min}^2$ so the angle θ (measured positive anti-clockwise) from the *E*-axis to the major axis of the ellipse (see Figure 8.2) is found from

$$\begin{Bmatrix} s_{\max}^2 \\ s_{\min}^2 \end{Bmatrix} \text{ when } \begin{Bmatrix} f'' < 0 \\ f'' > 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \theta = \phi \\ \theta = \phi - \tfrac{1}{2}\pi \end{Bmatrix}$$

Substituting $\phi = \theta$ and $\phi = \theta + \tfrac{1}{2}\pi$ into equation (8.3) will give the max. and min. values of the variance which are the lengths of the semi axes *a* and *b* of the Standard Error Ellipse.

8.3. Example Computation

In Figure 8.2, $s_E^2 = 6.0$, $s_N^2 = 2.0$ and $s_{EN} = 1.2$

The lengths of the semi-axes of the Standard Error Ellipse are

$$\begin{aligned} W &= \sqrt{\left(s_E^2 - s_N^2\right)^2 + 4\left(s_{EN}\right)^2} \\ &= \sqrt{4^2 + 4(1.2)^2} \\ &= 4.6648 \end{aligned}$$

$$\begin{aligned} a &= \sqrt{\frac{1}{2}\left(s_E^2 + s_N^2 + W\right)} = 2.5164 \\ b &= \sqrt{\frac{1}{2}\left(s_E^2 + s_N^2 - W\right)} = 1.2914 \end{aligned}$$

The angle between the *E*-axis and the major axis (positive anti-clockwise), noting the quadrant signs to determine the proper quadrant of 2θ

$$\begin{aligned} \tan 2\theta &= \frac{2s_{EN}}{s_E^2 - s_N^2} = \frac{2(1.2)}{6-2} = \left(\frac{+}{+}\right) \\ 2\theta &= 30^\circ\, 57'\, 50'' \\ \theta &= 15^\circ\, 28'\, 55'' \end{aligned}$$

Substituting the values $\phi = \theta = 15^\circ\, 28'55''$ and $\phi = 90^\circ + \theta = 105^\circ\, 28'55''$ into equation (8.3) gives $s_u = 2.5164$ and $s_u = 1.2914$ respectively so $\theta = 15^\circ\, 28'55''$ is the angle (positive anti-clockwise) between the *E*-axis and the major axis. Hence the bearing of the major axis is $90^\circ - \theta = 74^\circ\, 31'05''$

Alternatively, using the method of evaluating the second derivative we have from equation (8.14)

$$\begin{aligned}\tan 2\phi &= \frac{2s_{EN}}{s_E^2 - s_N^2} = \frac{2(1.2)}{6-2} = \left(\frac{+}{+}\right)\\ 2\phi &= 30^\circ\, 57'\, 50''\\ \phi &= 15^\circ\, 28'\, 55''\end{aligned}$$

The second derivative, from the second member of equations (8.13) is

$$\begin{aligned}f'' &= 2\left(s_N^2 - s_E^2\right)\cos 2\phi - 4s_{EN}\sin 2\phi\\ &= 2(2-6)\cos\left(30^\circ\, 57'\, 50''\right) - 4(1.2)\sin\left(30^\circ\, 57'\, 50''\right)\\ &= -9.3295\end{aligned}$$

Now, since $f'' < 0$ then $\theta = \phi = 15^\circ\, 28'\, 55''$ is the angle (positive anti-clockwise) from the *E*-axis to the major axis of the ellipse. The bearing of the major axis is $90^\circ - \theta = 74^\circ\, 31'\, 05''$.

The Standard Error Ellipse semi-axes lengths *a* and *b* are obtained from equation (8.3) with $\phi = \theta = 15^\circ\, 28'\, 55''$ and $\phi = \theta + \frac{1}{2}\pi = 105^\circ\, 28'\, 55''$ respectively giving

$\phi = 15^\circ\, 28'\, 55''$ $\qquad$ $s_{\max}^2 = 6.3324$ and $a = \sqrt{s_{\max}^2} = 2.5164$

$\phi = 105^\circ\, 28'\, 55''$ $\qquad$ $s_{\min}^2 = 1.6677$ and $b = \sqrt{s_{\min}^2} = 1.2914$

8.4. Some Examples of Resections and Error Ellipses

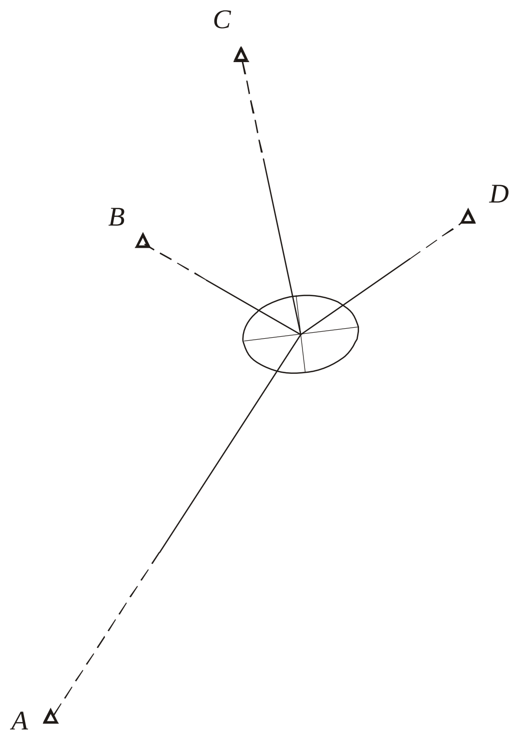


Figure 8.3. Resection 1

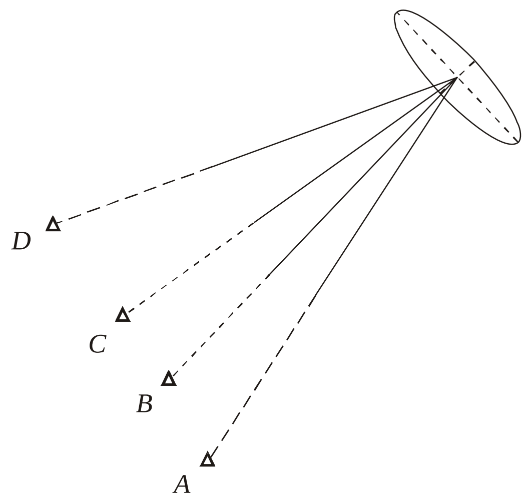
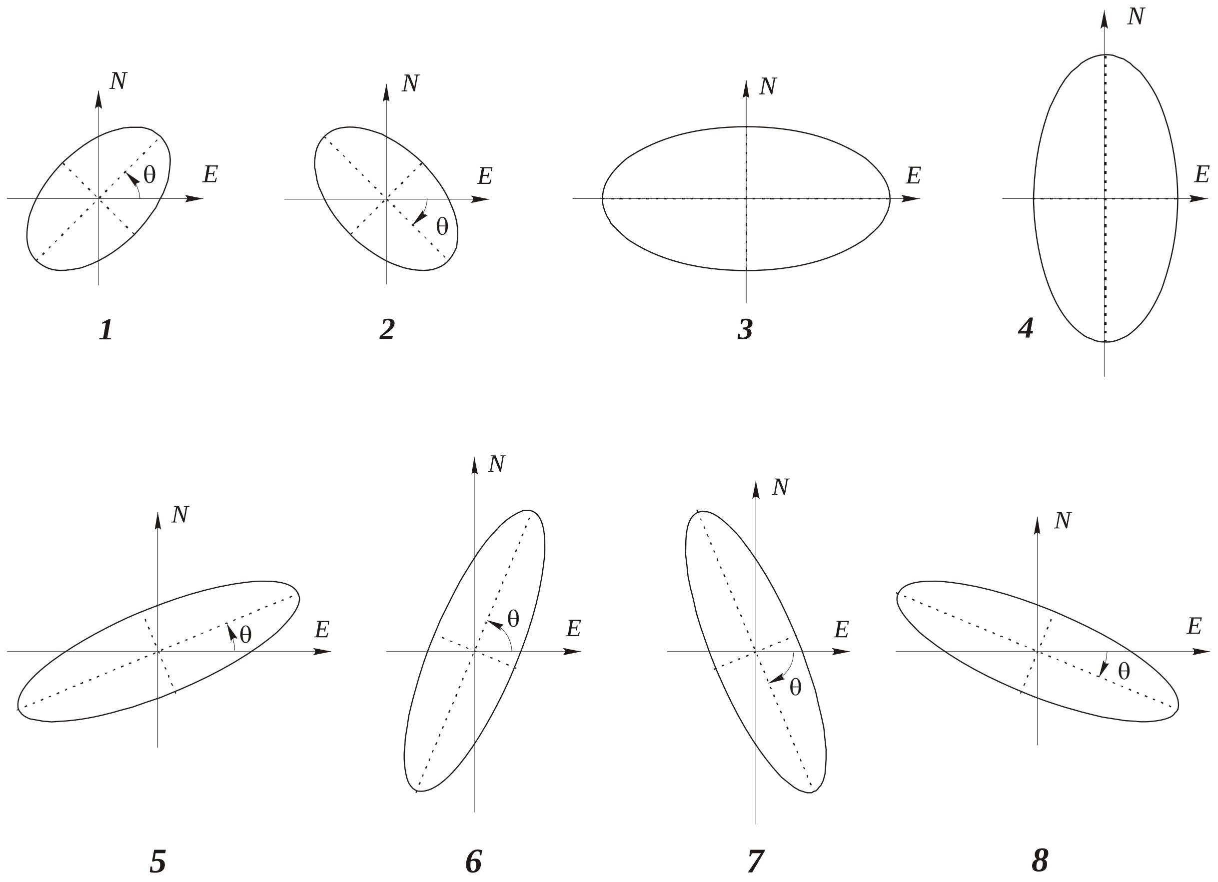


Figure 8.4. Resection 2

In Resection 1 (Figure 8.3) the error ellipse indicates a strong position fix and the observed stations are spread through an arc of approximately 200°.

In Resection 2 (Figure 8.4) the error ellipse indicates a poor position fix. The observed stations lay in a small arc of approximately 35°.

8.5. Some Examples of Orientation and Shape of Ellipses



Ellipse	s_E^2	s_N^2	s_{EN}	ρ_{EN}	a	b	θ
1	4	4	2	0.5	2.45	1.41	45°
2	4	4	-2	-0.5	2.45	1.41	-45°
3	16	4	0	0	4	2	0°
4	4	16	0	0	4	2	90°
5	16	4	6	0.75	4.30	1.23	22° 30′
6	4	16	6	0.75	4.30	1.23	67° 30′
7	4	16	-6	-0.75	4.30	1.23	-67° 30′
8	16	4	-6	-0.75	4.3	1.23	-22° 30′

Note: $\rho_{EN} = \dfrac{s_{EN}}{s_E s_N}$ is the correlation coefficient $-1 \le \rho_{EN} \le 1$

8.6. References

Additional information on error ellipses can be found in the following references.

Cooper, M.A.R., 1982. *Fundamentals of Survey Measurement and Analysis,* Granada, London.

Mikhail, E.M., 1976. *Observations and Least Squares,* IEP–A Dun-Donnelley, New York.

Mikhail, E.M. and Gracie, G., 1981. *Analysis and Adjustment of Survey Measurements,* Van Nostrand Reinhold Company, New York.

Richardus, P., 1966. *Project Surveying*, North-Holland Publishing Company, Amsterdam.

