Complex and Social Networks: Lab session 7

Simulation of SIS model over networks

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Contents

1 Introduction

In this session we will simulate the spreading of a disease in the SIS model and check that the epidemic threshold for arbitrary networks is indeed $\frac{1}{\lambda}$ as forecasted by [Chakrabarti et al., 2008]. In particular, we consider the spreading of the disease following these rules at each time step:

- An infected node recovers with probability γ
- An infected node attemps to infect each neighbor with probability β

Initially, only a random fraction p_0 of nodes are infected.

2 Methods

2.1 Models

2.1.1 Base

The model that we use to simulate is the following:

- Let $x_i(t)$ be the probability that node i is infected at time t
- Let $\zeta_i(t)$ be the probability that a node i will not receive infections from its neighbors in the next time step.

$$\begin{aligned}\zeta_i(t) &= \prod_{j:i-j} \overbrace{x_j(t-1)(1-\beta)}^{j \text{ fails to pass infection}} + \overbrace{(1-x_j(t-1))}^{j \text{ is not infected}} \\ &= \prod_{j:i-j} 1 - x_j(t-1)\beta\end{aligned}$$

$$x_i(t) = 1 - (1 - (1-\gamma)x_i(t-1))\,\zeta_i(t)$$

Finally, the fraction of infecteds is computed as:

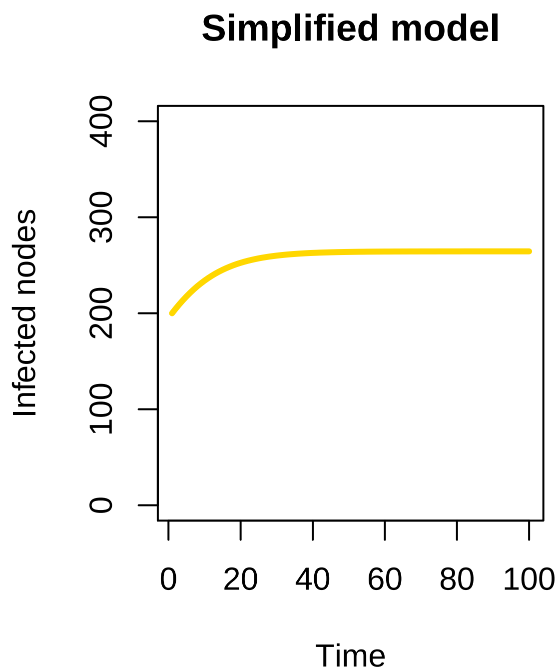
$$x(t) = \sum_i x_i(t)$$

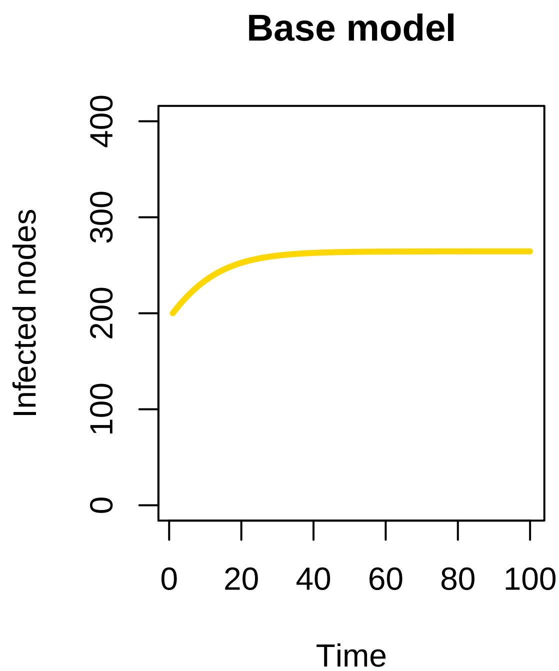
2.1.2 Simplified

In some cases, we can use a simplified version of the model, since if the graph is regular is easy to see that x_i is the same $\forall i$, and the same is true for ζ_i.

So we can speed up the calculation without loosing information.

For example, using a regular undirected lattice with 1000 nodes and 1 neigbor, we can take a glance to result obtained with the same set of parameters, both with the base and the simplified model.





2.2 Ensamble of graphs

All the networks have 1000 nodes, and are connected.

We calculate here the max eigenvalue and the threshold ($\frac{1}{\lambda_1}$) that we'll need in the tasks.

Table 1: Data

	Number of edges	Leading eigenvalue	Trashold	Mean degree
Lattice with 1 neighbor	1000	2.00000	0.5000000	2.000
Lattice with 2 neighbor	2000	4.00000	0.2500000	4.000
Erdos renyi with p=0.015	7629	16.26337	0.0614879	15.258
Erdos renyi with p=0.1	50076	101.02778	0.0098983	100.152
Barabasi albert	50500	191.20519	0.0052300	101.000
Complete graph	499500	999.00000	0.0010010	999.000

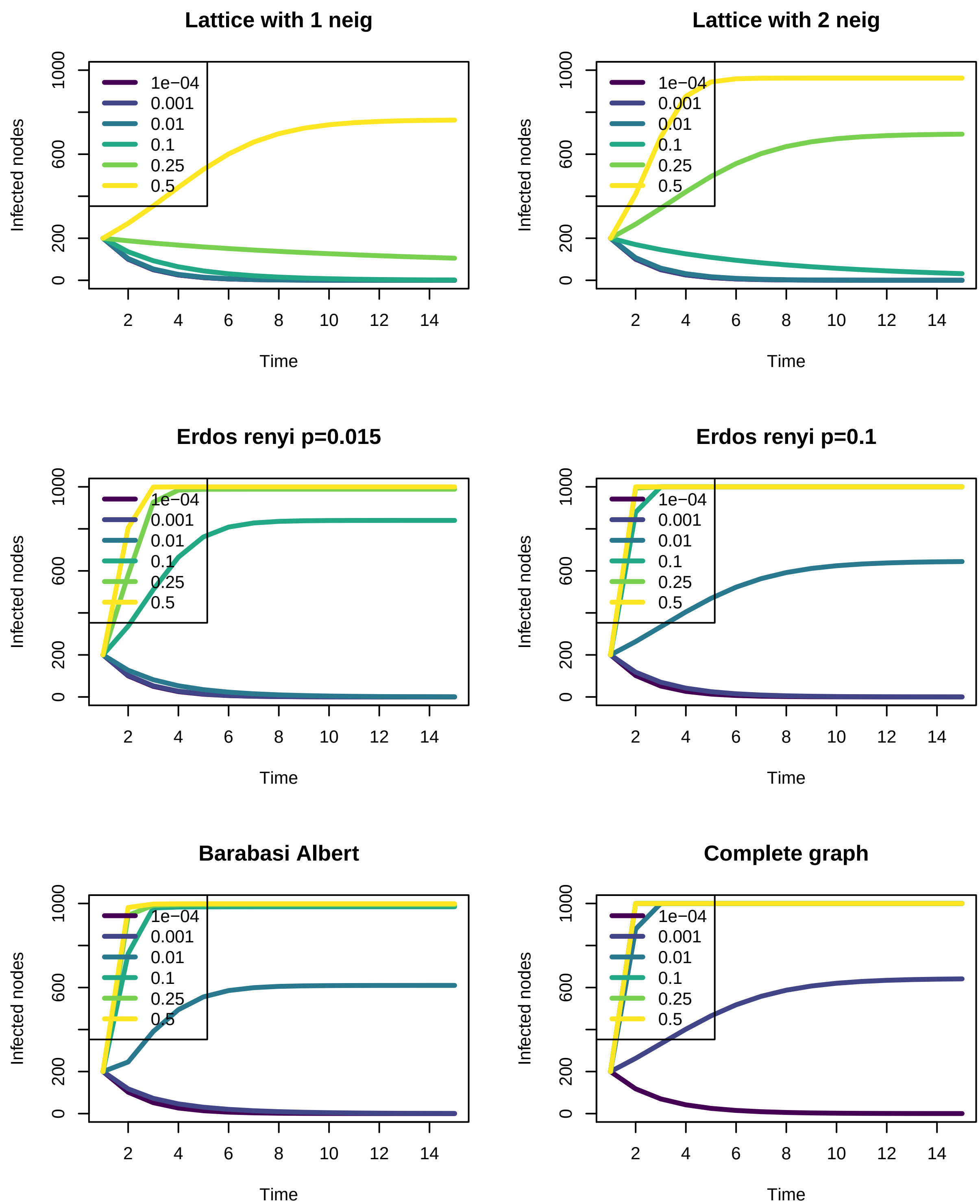
Note: everything in the construction of the graphs is standard. For Barabasi-Albert we used as a starting point a lattice of 100 nodes, and we added 56 edges each time to have a number of edges similar to the second Erdos-Renyi graph, for comparison. We also specified a $\gamma = 2$ for the preferential attachment model to get a strong effect that could help us emphasize results.

3 Results

3.1 Task 1: what network is more prone to epidemic?

We decided to leave p_0 fixed, and inspect the behaviour at first by changing β with γ fixed, and then doing the opposite in the next step.

3.1.1 Modeling with different betas



As we expect, more dense networks are more epidemic-prone.

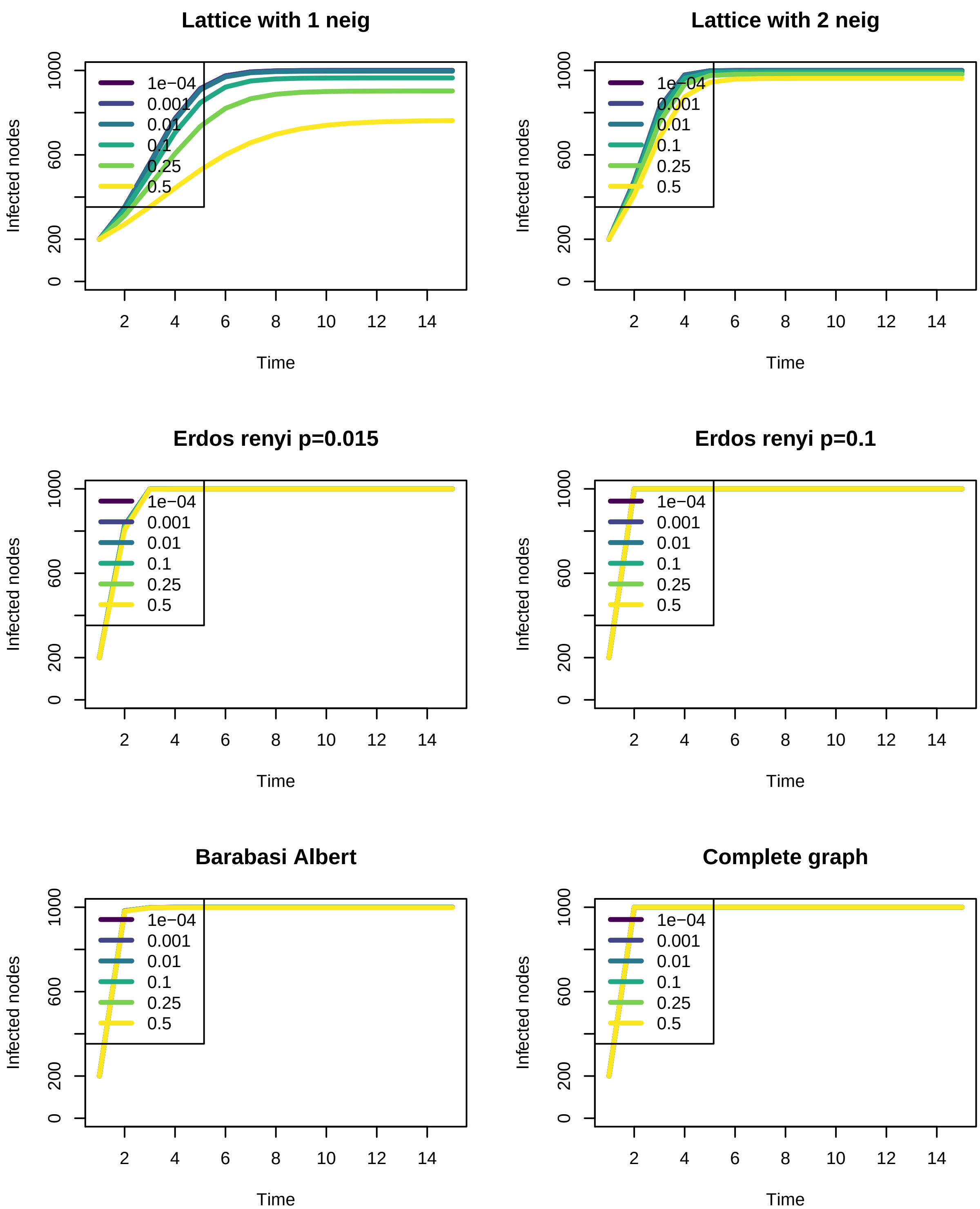
An interesting thing to notice is that the degree distribution does not seems to affect the behaviour of the spread, since, an Erdos-Renyi graph and an Barabasi-Albert graph (with strong preferential attachment) seems to have almost the same resistence to the virus, ceteris paribus.

3.1.2 Modeling with different gammas

For this experiment we are leaving p_0 fixed as in the previous experiment. In a first attempt we use $\beta = 0.5$ and try with different γ, then we try again with $\beta = 0.5$ and the γ values as before.

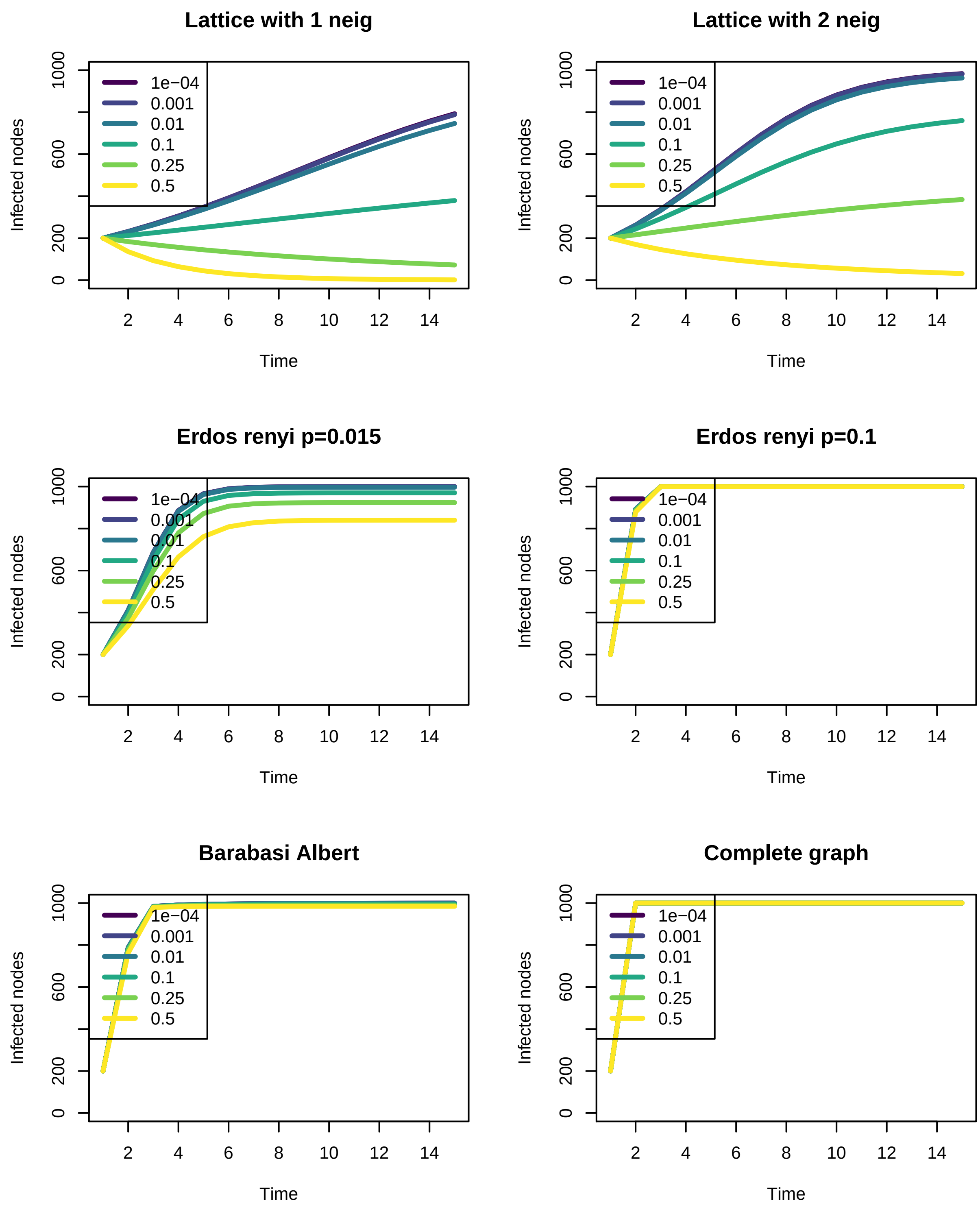
3.2 For $\beta = 0.5$

We are trying $\beta = 0.5$ because it gave the best result in the previous experiment.



3.3 For $\beta = 0.25$

Since $\beta = 0.5$ did not produce much variation for different γ, we are now trying the second best result $\beta = 0.1$



3.4 Task 2: are simulation results consistent with theory thresholds?

We have that if $\frac{\beta}{\gamma} > \frac{1}{\lambda_1}$ then the epidemic occurs.

For our subjects we have that:

1. Lattice with 1 neighbor using $\lambda = 2 \implies \frac{\beta}{\gamma} > threshold = 0.5$
2. Lattice with 2 neighbor using $\lambda_1 = 4 \implies \frac{\beta}{\gamma} > threshold = 0.25$
3. Erdos-Renyi with p=0.015 using $\lambda_1 = 15.94737 \implies \frac{\beta}{\gamma} > threshold = 0.0627063$
4. Erdos-Renyi with p=0.1 using $\lambda_1 = 101.55027 \implies \frac{\beta}{\gamma} > threshold = 0.0098473$
5. Barabasi-Albert model using $\lambda_1 = 191.24417 \implies \frac{\beta}{\gamma} > threshold = 0.0052289$
6. Complete Graph model using $\lambda_1 = 999 \implies \frac{\beta}{\gamma} > threshold = 0.0010010$

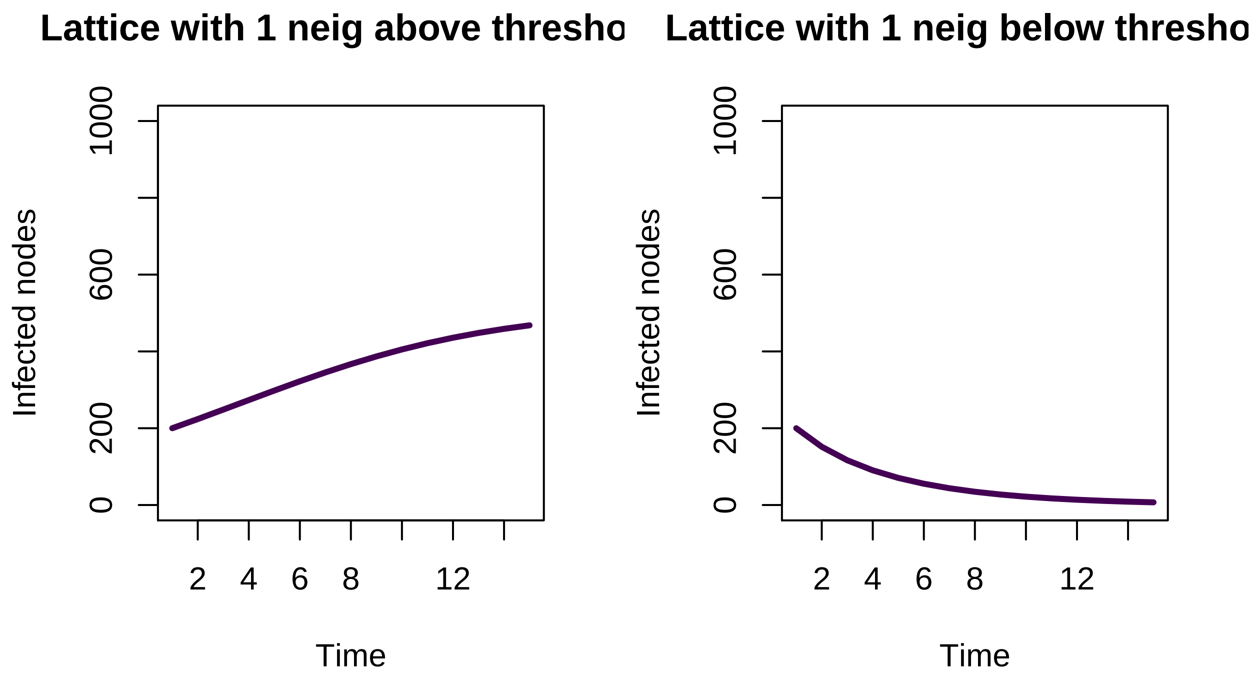
We have already seen that $\beta = 0.5$ guarantees the threshold in all instances considered for experimentation; however, this might imply $\gamma = 1$ for some instances, which in turn implies a too high probability of recovering. So, we will change the value of β for each subject so we do not come into situations where $\gamma \geq 1$, then we pick two values for γ one that takes $\frac{\beta}{\gamma}$ slightly above the threshold and one slightly below.

3.4.1 Lattice with 1 neighbor

For $\frac{1}{\lambda_1} = 0.5$

'Lattice with 1 neig above threshold' using $\beta = 0.25$, $\gamma = 0.3$ that results in $\frac{\beta}{\gamma} = 0.8333$

'Lattice with 1 neig below threshold' using $\beta = 0.25$, $\gamma = 0.7$ that results in $\frac{\beta}{\gamma} = 0.3571$

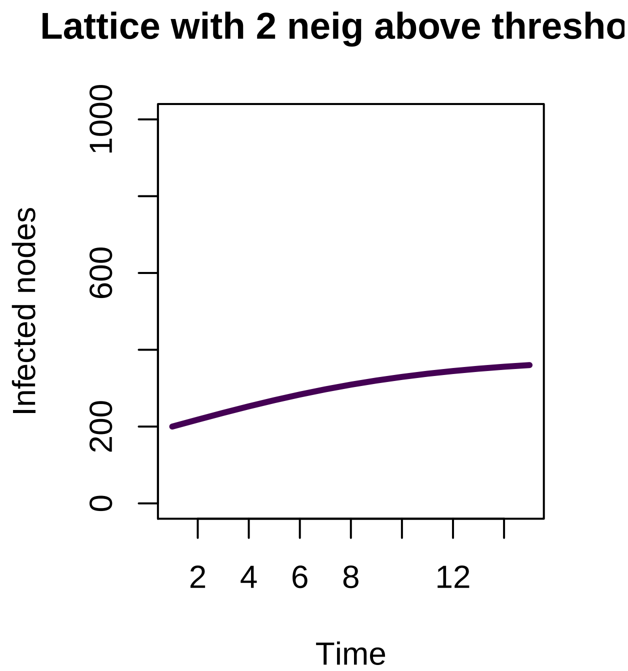


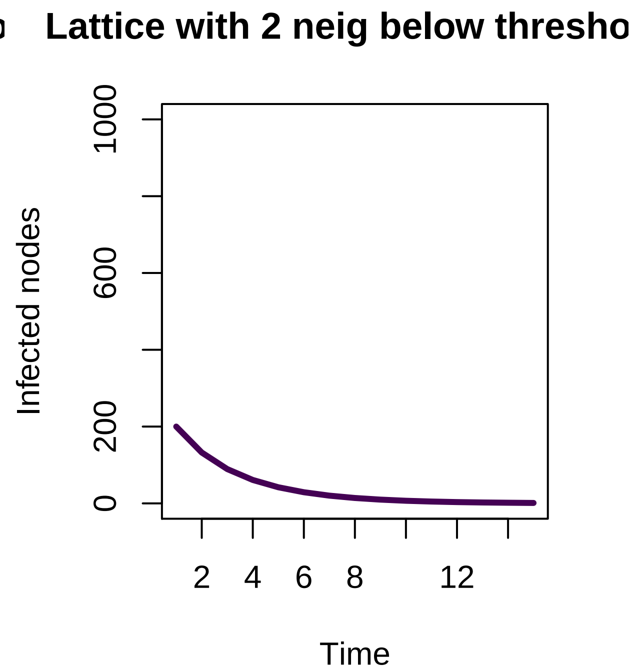
3.4.2 Lattice with 2 neighbor

For $\frac{1}{\lambda_1} = 0.25$

'Lattice with 2 neig above threshold' has $\beta = 0.25$, $\gamma = 0.8$ which results in $\frac{\beta}{\gamma} = 0.3125$

'Lattice with 2 neig below threshold' has $\beta = 0.15$, $\gamma = 0.9$ which results in $\frac{\beta}{\gamma} = 0.1667$



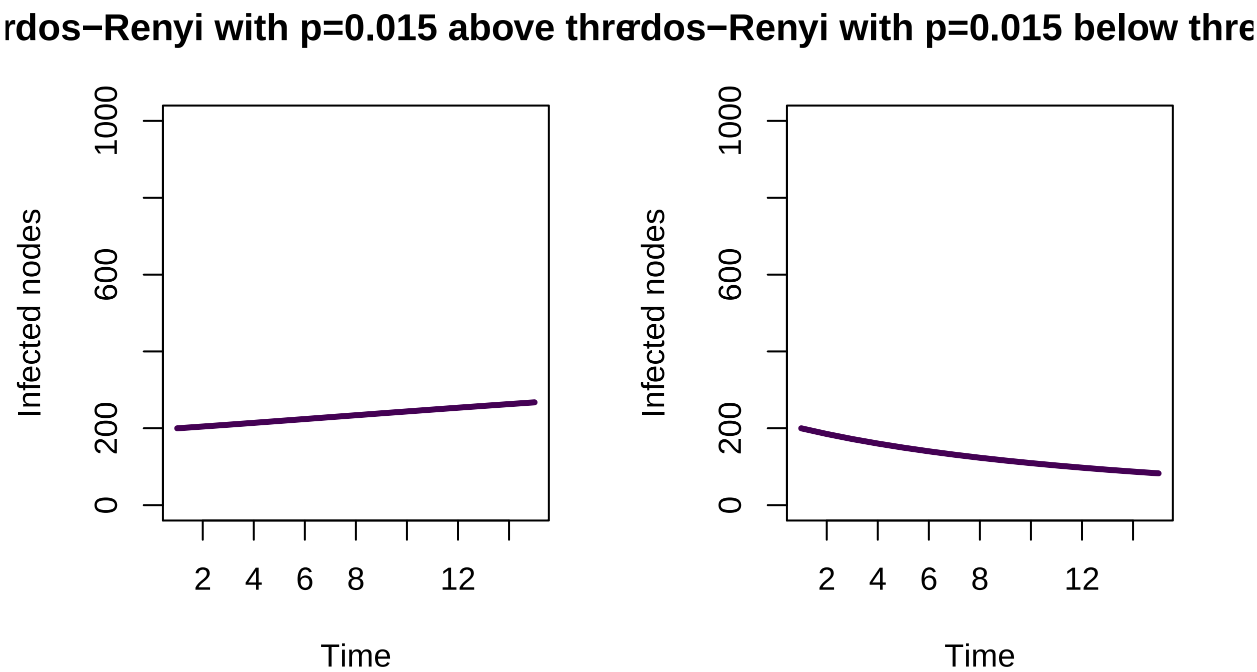


3.4.3 Erdos-Renyi with p=0.015

For $\frac{1}{\lambda_1} = 0.0627063$

'Erdos-Renyi with p=0.015' has $\beta = 0.01$, $\gamma = 0.1$ which results in $\frac{\beta}{\gamma} = 0.1$

'Erdos-Renyi with p=0.015' has $\beta = 0.01$, $\gamma = 0.2$ which results in $\frac{\beta}{\gamma} = 0.05$

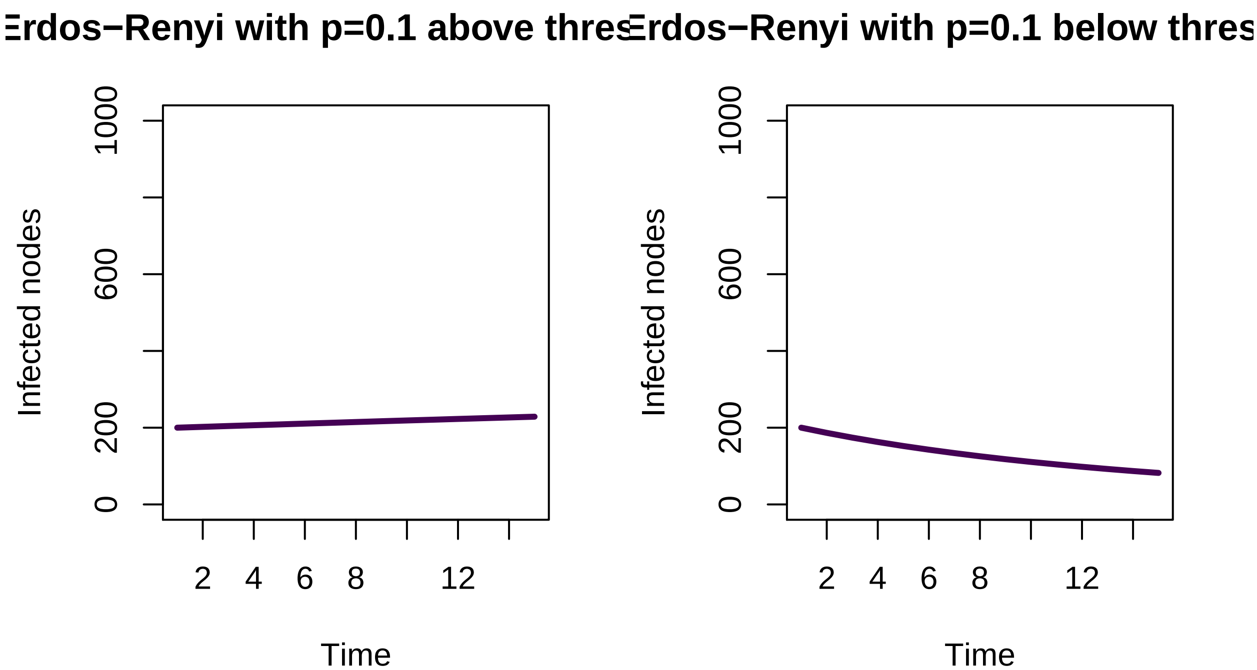


3.4.4 Erdos-Renyi with p=0.1

For $\frac{1}{\lambda_1} = 0.0098473$

'Erdos-Renyi with p=0.1' has $\beta = 0.001$, $\gamma = 0.07$ which results in $\frac{\beta}{\gamma} = 0.01428$,

'Erdos-Renyi with p=0.1' has $\beta = 0.001$, $\gamma = 0.15$ which results in $\frac{\beta}{\gamma} = 0.00667$

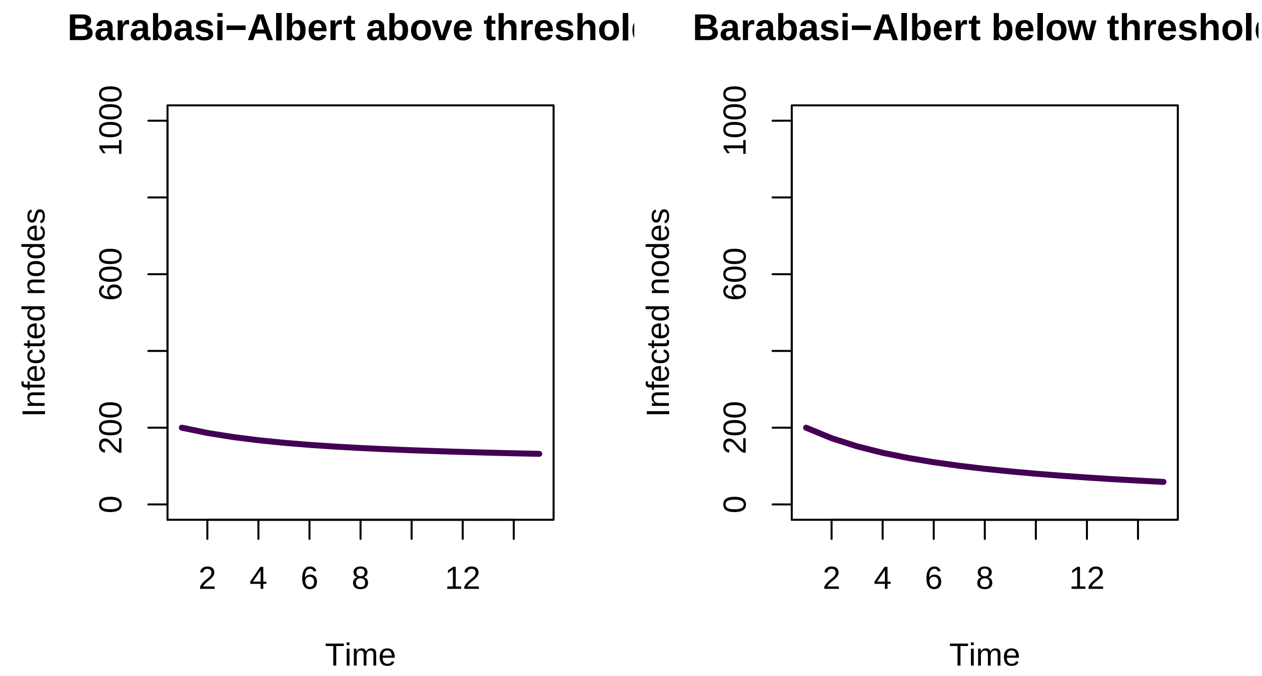


3.4.5 Barabasi-Albert

For $\frac{1}{\lambda_1} = 0.0052289$

'Erdos-Renyi with p=0.1' has $\beta = 0.001$, $\gamma = 0.15$ which results in $\frac{\beta}{\gamma} = 0.00667$

'Erdos-Renyi with p=0.1' has $\beta = 0.001$, $\gamma = 0.22$ which results in $\frac{\beta}{\gamma} = 0.00454$

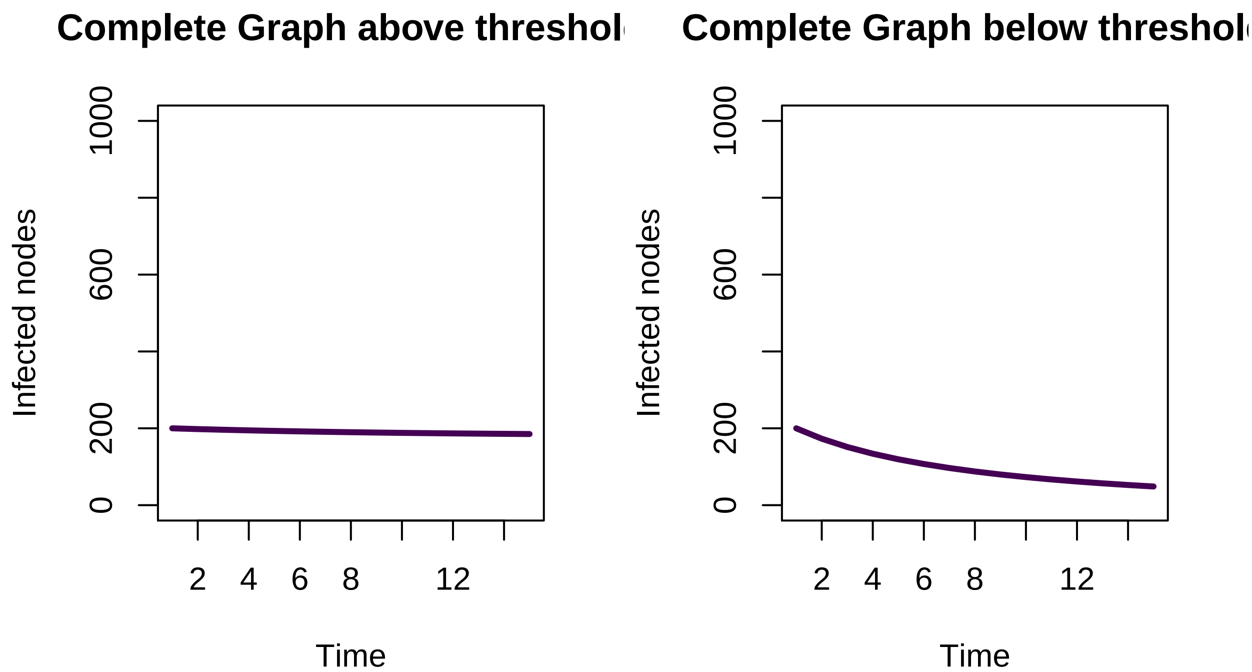


3.4.6 Complete Graph

For $\frac{1}{\lambda_1} = 0.001001$

'Complete Graph' using $\beta = 0.0009$, $\gamma = 0.8$ which results in $\frac{\beta}{\gamma} = 0.001125$

'Complete Graph' using $\beta = 0.0009$, $\gamma = 0.95$ which results in $\frac{\beta}{\gamma} = 0.000947$



4 Discussion

For changing γ, we can see that it has a noticeable effect in graph that are not dense. The more dense the graph is, the higher the probability that a node is infected by any of its neighbors, so the probability of recovering does not have a stronger impact as it has in sparse graphs. It seems as an intuitive result but it is interesting to see it as a comparison in graphs whose structures we have already analyzed in previous works.

While experiment with the threshold $\frac{1}{\lambda_1}$ we can see that the closer we get to the threshold, from above or below, the harder it is to determine a difference visually. In some cases we see that the chart shows a downwards tendency, as if the threshold were not tight enough, and that might a plausible point; however, lets remember that the definition of epidemic does not state that the 'sicknes' should spread to the whole population but to a significant portion. In our experimentation we see that for those tests 'slightly' above the threshold, this definition holds and the agent (or sickness) has spread to significant fraction of the population after the given iterations, then, the more the positive difference between the threshold and the value used, the higher the fraction of the population infected.