

Generalized Filter

Suppose $x = (x_1, x_2, \dots, x_n)^T \in \Re^n$ is a time series, the general form of finite impulse response is thus:

$$y_i = \sum_{k=0}^m \phi_k x_{i+k} \quad (1)$$

Let

$$\begin{aligned} \Phi &= (\Phi_{i,j}) \\ &= \begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{n-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m-n+1} & \phi_{m-n+2} & \cdots & \phi_m \end{pmatrix} \in \Re^{(n-m) \times n}, \end{aligned} \quad (2)$$

where

$$\Phi_{i,j} = \phi_{i-j} = \begin{cases} \phi_{i-j} & \text{if } 0 \leq i-j \leq m \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Therefore,

$$\Phi x = y = (y_1, y_2, \dots, y_{n-m})^T. \quad (4)$$

Moreover, we let

$$\begin{aligned} \Lambda &= \Phi^T \Phi \\ &= (\Lambda_{i,j}) \in \Re^{n \times n}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \Lambda_{i,j} &= \sum_{k=1}^{n-m} \Phi_{k,i} \Phi_{k,j} \\ &= \sum_{k=1}^{n-m} \phi_{k-i} \phi_{k-j}. \end{aligned} \quad (6)$$

Moments of filtered error

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$$E\Phi e = 0 \quad (7)$$

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$$\begin{aligned}
Ee^T\Phi^T\Phi e &= Ee^T\Lambda e \\
&= E \sum_{i,j=1}^n \Lambda_{i,j} e_i e_j \\
&= M_2 \sum_{i=1}^n \Lambda_{i,i} \\
&= M_2 \sum_{i=1}^n \sum_{k=1}^{n-m} \phi_{k-i}^2 \\
&= M_2(n-m) \sum_{k=1}^m \phi_k^2
\end{aligned} \tag{8}$$

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$$E\Phi ee^T\Phi^T\Phi e = \Phi Eee^T\Phi^T\Phi e.$$

Since

$$\begin{aligned}
Eee^T\Phi^T\Phi e &= Ee \sum_{i,j=1}^n \Lambda_{i,j} e_i e_j \\
&= M_3 \begin{pmatrix} \Lambda_{1,1} \\ \Lambda_{2,2} \\ \vdots \\ \Lambda_{n,n} \end{pmatrix}, \\
E\Phi ee^T\Phi^T\Phi e &= M_3 \Phi \begin{pmatrix} \Lambda_{1,1} \\ \Lambda_{2,2} \\ \vdots \\ \Lambda_{n,n} \end{pmatrix} \\
&= M_3 \begin{pmatrix} \sum_{j=1}^n \Phi_{1,j} \Lambda_{j,j} \\ \sum_{j=1}^n \Phi_{2,j} \Lambda_{j,j} \\ \vdots \\ \sum_{j=1}^n \Phi_{n-m,j} \Lambda_{j,j} \end{pmatrix}.
\end{aligned}$$

According to Eq. 6,

$$E\Phi ee^T\Phi^T\Phi e = M_3 \begin{pmatrix} \sum_{j=1}^n \sum_{k=1}^{n-m} \phi_{1-j}\phi_{k-j}^2 \\ \sum_{j=1}^n \sum_{k=1}^{n-m} \phi_{2-j}\phi_{k-j}^2 \\ \vdots \\ \sum_{j=1}^n \sum_{k=1}^{n-m} \phi_{n-m-j}\phi_{k-j}^2 \end{pmatrix}$$

•

$$\begin{aligned} Ee^T\Phi^T\Phi ee^T\Phi^T\Phi e &= E(\sum_{i,j} \Lambda_{i,j} e_i e_j)^2 \\ &= E(\sum_{i,j,k,l} \Lambda_{i,j} \Lambda_{k,l} e_i e_j e_k e_l) \\ &= M_4 \sum_{i=1}^n \Lambda_{i,i}^2 + M_2^2 \sum_{i \neq j} \Lambda_{i,i} \Lambda_{j,j} + \Lambda_{i,j}^2 \end{aligned}$$