## Generalized Filter

Suppose $x = (x_1, x_2, ..., x_n)^T \in \Re^n$ is a time series, the general form of finite impulse response is thus:

$$y_i = \sum_{k=0}^{m} \phi_k x_{i+k} \tag{1}$$

Let

$$\begin{aligned} \Phi &= (\Phi_{i,j}) \\ &= \begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{n-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m-n+1} & \phi_{m-n+2} & \cdots & \phi_m \end{pmatrix} \in \Re^{(n-m)\times n}, \end{aligned} \tag{2}$$

where

$$\Phi_{i,j} = \phi_{i-j} = \begin{cases} \phi_{i-j} & \text{if } 0 \le i-j \le m \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Therefore,

$$\Phi x = y = (y_1, y_2, ..., y_{n-m})^T. \tag{4}$$

Moreover, we let

$$\begin{aligned} \Lambda &= \Phi^T \Phi \\ &= (\Lambda_{i,j}) \in \Re^{n\times n}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Lambda_{i,j} &= \sum_{k=1}^{n-m} \Phi_{k,i} \Phi_{k,j} \\ &= \sum_{k=1}^{n-m} \phi_{k-i} \phi_{k-j}. \end{aligned} \tag{6}$$

## Moments of filtered error

- $$E\Phi e = 0 \tag{7}$$

- $$\begin{aligned} Ee^T\Phi^T\Phi e &= Ee^T\Lambda e \\ &= E\sum_{i,j=1}^{n}\Lambda_{i,j}e_ie_j \\ &= M_2\sum_{i=1}^{n}\Lambda_{i,i} \\ &= M_2\sum_{i=1}^{n}\sum_{k=1}^{n-m}\phi_{k-i}^2 \\ &= M_2(n-m)\sum_{k=1}^{m}\phi_k^2 \end{aligned} \tag{8}$$

- $$E\Phi ee^T\Phi^T\Phi e = \Phi Eee^T\Phi^T\Phi e.$$

  Since

  $$\begin{aligned} Eee^T\Phi^T\Phi e &= Ee\sum_{i,j=1}^{n}\Lambda_{i,j}e_ie_j \\ &= M_3\begin{pmatrix}\Lambda_{1,1}\\ \Lambda_{2,2}\\ \vdots\\ \Lambda_{n,n}\end{pmatrix}, \end{aligned}$$

  $$\begin{aligned} E\Phi ee^T\Phi^T\Phi e &= M_3\Phi\begin{pmatrix}\Lambda_{1,1}\\ \Lambda_{2,2}\\ \vdots\\ \Lambda_{n,n}\end{pmatrix} \\ &= M_3\begin{pmatrix}\sum_{j=1}^{n}\Phi_{1,j}\Lambda_{j,j}\\ \sum_{j=1}^{n}\Phi_{2,j}\Lambda_{j,j}\\ \vdots\\ \sum_{j=1}^{n}\Phi_{n-m,j}\Lambda_{j,j}\end{pmatrix}. \end{aligned}$$

According to Eq. 6,

$$E\Phi ee^T\Phi^T\Phi e = M_3 \begin{pmatrix} \sum_{j=1}^{n}\sum_{k=1}^{n-m}\phi_{1-j}\phi_{k-j}^2 \\ \sum_{j=1}^{n}\sum_{k=1}^{n-m}\phi_{2-j}\phi_{k-j}^2 \\ \vdots \\ \sum_{j=1}^{n}\sum_{k=1}^{n-m}\phi_{n-m-j}\phi_{k-j}^2 \end{pmatrix}$$

- $$\begin{aligned} Ee^T\Phi^T\Phi ee^T\Phi^T\Phi e &= E(\sum_{i,j}\Lambda_{i,j}e_ie_j)^2 \\ &= E(\sum_{i,j,k,l}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) \\ &= E(\sum_{i=j,k=l}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) + E(\sum_{i=k,j=l}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) \\ &\quad +E(\sum_{i=l,j=k}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) - 2E(\sum_{i=l=j=k}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) \end{aligned}$$

  - $$E(\sum_{i=j,k=l}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) = (M_4 - M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,i}\Lambda_{j,j})$$

  - $$E(\sum_{i=k,j=l}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) = (M_4 - M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,j}^2)$$

  - $$E(\sum_{i=l,j=k}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) = (M_4 - M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,j}^2)$$

  - $$E(\sum_{i=l=j=k}\Lambda_{i,j}\Lambda_{k,l}e_ie_je_ke_l) = M_4(\sum_i \Lambda_{i,i}^2)$$

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$$\begin{aligned}
Ee^T\Phi^T\Phi ee^T\Phi^T\Phi e &= (M_4-M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,i}\Lambda_{j,j}) + (M_4-M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,j}^2) \\
&\quad +(M_4-M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,j}^2) - 2M_4(\sum_i \Lambda_{i,i}^2) \\
&= (M_4-3M_2^2)(\sum_i \Lambda_{i,i}^2) + M_2^2(\sum_{i,j}\Lambda_{i,i}\Lambda_{j,j}) + 2M_2^2(\sum_{i,j}\Lambda_{i,j}^2) \\
&= (\sum_i \Lambda_{i,i}^2)M_4 + \left(\sum_{i,j}\Lambda_{i,i}\Lambda_{j,j} + 2\sum_{i,j}\Lambda_{i,j}^2 - 3\sum_i \Lambda_{i,i}^2\right) M_2^2 \qquad (9)
\end{aligned}$$

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$$\begin{aligned}
Ee^Tee^T\Phi^T\Phi e &= E(\sum_i e_i^2)(\sum_{i,j} e_i\Lambda_{i,j}e_ie_j) \\
&= E(\sum_{i,k,l}\Lambda_{k,l}e_i^2e_ke_l) \\
&= E(\sum_{i,k=l}\Lambda_{k,l}e_i^2e_ke_l) \\
&= E(\sum_{i,j}\Lambda_{j,j}e_i^2e_j^2) \\
&= nM_2^2\sum_j \Lambda_{j,j} + (M_4-M_2^2)\sum_j \Lambda_{j,j} \\
&= (M_4+(n-1)M_2^2)\sum_j \Lambda_{j,j}
\end{aligned}$$

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$$\begin{aligned}
& E(\sum_{i=0}^m \phi_ie_i)^4 \\
= \;& E\sum_{i,j,k,l}\phi_i\phi_j\phi_k\phi_le_ie_je_ke_l \\
= \;& E(3\sum_{i,j}\phi_i^2\phi_j^2e_i^2e_j^2 - 2\sum_i \phi_i^4e_i^4) \\
= \;& 3(M_2^2\sum_{i,j}\phi_i^2\phi_j^2 + (M_4-M_2^2)\sum_i \phi_i^4) - 2M_4\sum_i \phi_i^4) \\
= \;& (M_4-3M_2^2)\sum_i \phi_i^4 + 3M_2^2\sum_{i,j}\phi_i^2\phi_j^2 \\
= \;& (\sum_i \phi_i^4)M_4 + 3(\sum_{i,j}\phi_i^2\phi_j^2 - \sum_i \phi_i^4)M_2^2
\end{aligned}$$