STAT425 HW#1

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Problem 2

$$R^2 = \frac{\Sigma(\hat{y}_i - \bar{y})^2}{\Sigma(y_i-\bar{y})^2}$$

$$= \frac{\Sigma(\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2}{\Sigma(y_i-\bar{y})^2}$$ Since $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

$$= \frac{\Sigma(\bar{y} - \hat{\beta}_1\bar{x} + \hat{\beta}_1 x_i - \bar{y})^2}{\Sigma(y_i-\bar{y})^2}$$ Since $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$

$$= \frac{\hat{\beta}_1^2 \cdot \Sigma(x_i-\bar{x})^2}{\Sigma(y_i-\bar{y})^2}$$

$$= \frac{(r_{XY}(\frac{S_{yy}}{S_{xx}})^{\frac{1}{2}})^2 \cdot S_{xx}}{S_{yy}}$$ Since $S_{xx} = \Sigma(x_i-\bar{x})^2$, $S_{yy} = \Sigma(y_i-\bar{y})^2$, $\hat{\beta}_1 = r_{XY}(\frac{S_{yy}}{S_{xx}})^{\frac{1}{2}}$

$$= r_{XY}^2$$

Problem 4.

(a) $Y_i = \beta x_i + e_i$

$$RSS = \Sigma_{i=1}^n (Y_i - \beta x_i)^2$$

$$\frac{dRSS}{d\beta} = -2\Sigma_{i=1}^n x_i(y_i - \beta x_i) = 0 \Rightarrow$$

$$\hat{\beta}\Sigma_{i=1}^n x_i^2 = \Sigma_{i=1}^n x_i y_i$$

$$\hat{\beta} = \frac{\Sigma_{i=1}^n x_i y_i}{\Sigma_{i=1}^n x_i^2}$$

Problem 4.

(b). i>. $\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$

let $c_i = x_i / \sum_{i=1}^{n} x_i^2$

$\hat{\beta} = \sum_{i=1}^{n} c_i y_i$

Since $x_i \cdots x_n$ are known fixed constants:

$E[\hat{\beta}] = \sum_{i=1}^{n} c_i E[y_i]$

$= \sum_{i=1}^{n} c_i \cdot \beta x_i$ Since $E[Y|X=x_i] = \beta x_i$

$= \beta \cdot \frac{\sum_{i=1}^{n} x_i \cdot x_i}{\sum_{i=1}^{n} x_i^2}$

$= \beta$

ii> $Var[\hat{\beta}] = \sum_{i=1}^{n} c_i^2 \cdot Var[y_i]$

$= \sum_{i=1}^{n} c_i^2 \cdot \sigma^2$

$= \sigma^2 \cdot \frac{\sum_{i=1}^{n} x_i^2}{(\sum_{i=1}^{n} x_i^2)^2}$

$= \sigma^2 \cdot \frac{1}{\sum_{i=1}^{n} x_i^2}$

iii> Since $e|X \sim N(0, \sigma^2)$

$Y|X \sim N(\beta x_i, \sigma^2)$

$\hat{\beta} = \sum_{i=1}^{n} c_i y_i$, where $c_i = x_i / \sum_{i=1}^{n} x_i^2$

is the weighted sum of $y_i \sim N(\beta x_i, \sigma^2)$

From i> and ii>:

$\hat{\beta}|X \sim N(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$

Problem 6

(a) $(y_i - \hat{y}_i) = (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)$

$= (y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 x_i)$

$= (y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{x})$

(b). $(\hat{y}_i - \bar{y}) = (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})$

$= (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y})$

$= \hat{\beta}_1 (x_i - \bar{x})$

(c) $\sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$

$= \sum_{i=1}^{n} [(y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{x})] \cdot \hat{\beta}_1 (x_i - \bar{x})$

$= \hat{\beta}_1 \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) - \hat{\beta}_1^2 \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$

$= \frac{S_{XY}}{S_{XX}} \cdot S_{XY} - \frac{S_{XY}^2}{S_{XX}^2} \cdot S_{XX}$

$= 0$

(d) $SST = \sum (y_i - \bar{y})^2$

$= \sum (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2$

$= \sum (y_i - \hat{y}_i)^2 + (\hat{y}_i - \bar{y})^2 + 2(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$

$= \sum (y_i - \hat{y}_i)^2 + \sum (\hat{y}_i - \bar{y})^2 + 0$

$= RSS + FSS$