STAT 510 HW#6

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Ex 15.7.1 1(a) Since Size $= \sup\limits_{\theta \in T_0} P_\theta [T(x) > c]$

$T(x) = |x-1|$, $c = 3/4$

$$\text{Size} = \sup\limits_{\lambda \geq 0} P_\lambda [|x-1| > 3/4]$$

$$= \sup\limits_{\lambda \geq 0} (P_\lambda [x < \tfrac{1}{4}] + P_\lambda [x > \tfrac{7}{4}])$$

$$= 1 - e^{-\frac{1}{4}} + 1 - (1 - e^{-\frac{7}{4}})$$

$$= \boxed{0.3949}$$

1.(b) $\text{Power}_\lambda = 1 - P_\lambda (\text{Type II error})$

$$= 1 - P_\lambda [|x-1| \leq 3/4]$$

$$= 1 - P_\lambda [\tfrac{1}{4} \leq x \leq \tfrac{7}{4}]$$

$$= 1 - (P_\lambda [x \leq \tfrac{7}{4}] - P_\lambda [x \leq \tfrac{1}{4}])$$

$$= 1 - ((1 - e^{-\frac{7}{4}\lambda}) - (1 - e^{-\frac{1}{4}\lambda}))$$

$$= \underline{1 - e^{-\frac{1}{4}\lambda} + e^{-\frac{7}{4}\lambda}}$$

1.(C) $\dfrac{d\text{Power}}{d\lambda} = \tfrac{1}{4} e^{-\frac{1}{4}\lambda} - \tfrac{7}{4} e^{-\frac{7}{4}\lambda} = 0$

$$\Rightarrow \lambda = \tfrac{2}{3} \ln 7 \approx 1.297$$

when $\lambda = \tfrac{2}{3} \ln 7$.

$$\text{Power} = 1 - e^{-\frac{1}{4}\lambda} + e^{-\frac{7}{4}\lambda}$$

$$= 1 - \tfrac{6}{7 \sqrt[6]{7}}$$

$$\approx \boxed{0.3802}$$

Since $0.3802 < 0.3949$, Power is $\underline{\text{less than}}$ Size.

Yes, it is a Problem. Since Size refers to Type I error where we reject Ho when Ho is true and Power refers to the situation where we reject Ho when Ha is true. Larger size indicates that we are rejecting more than we should.

Ex 15.7.2 | 2. $X_1 \cdots X_n \stackrel{iid}{\sim} U(0, \theta)$, $T = \max \{X_1 \cdots X_n\}$

$P(T < c)$

$= P(X_1 < c, X_2 < c, \cdots X_n < c)$

$= \prod_{i=1}^{n} P(X_i < c)$ due to iid.

$= \prod_{i=1}^{n} \frac{c}{\theta}$ Since cdf $= \frac{c}{\theta}$.

$= \left(\frac{c}{\theta}\right)^n$

Since Size $= \sup\limits_{0 \le \theta \le \frac{1}{2}} P_\theta[T > c] = 0.05$

$\sup\limits_{0 \le \theta \le \frac{1}{2}} (1 - P_\theta[T < c]) = 0.05$

$\sup\limits_{0 \le \theta \le \frac{1}{2}} \left(1 - \left(\frac{c}{\theta}\right)^n\right) = 0.05$

$(2c)^n = 0.95$

$c = \boxed{0.95^{\frac{1}{n}} \cdot \frac{1}{2}}$

when $n = 10$.

$c = 0.95^{0.1} \cdot \frac{1}{2}$

$= \boxed{0.4974}$

3. (a) Under Null hypothesis : $\mu_x = \mu_y$

Then, $\hat{\mu} = \frac{1}{n+m}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i\right)$

$\qquad = \frac{1}{n+m}\left(n\bar{x} + n\bar{y}\right)$

$\hat{\sigma}_0^2 = \frac{1}{n+m}\left(\sum_{i=1}^{n}(x_i - \hat{\mu})^2 + \sum_{i=1}^{m}(y_i - \hat{\mu})^2\right)$

(b). Under Alternative : $\mu_x \neq \mu_y$.

Then, $\hat{\mu}_x = \bar{x}$, $\hat{\mu}_y = \bar{y}$

$\hat{\sigma}_A^2 = \frac{1}{n+m}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2\right)$

(c). $LRT = \dfrac{f(x \mid \hat{\theta}_A)}{f(x \mid \hat{\theta}_0)}$

$= \dfrac{\hat{\sigma}_0^{2\frac{n+m}{2}} \, e^{-\frac{1}{2\hat{\sigma}_A^2}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2\right)}}{\hat{\sigma}_A^{2\frac{n+m}{2}} \, e^{-\frac{1}{2\hat{\sigma}_0^2}\left(\sum_{i=1}^{n}(x_i - \hat{\mu})^2 + \sum_{i=1}^{m}(y_i - \hat{\mu})^2\right)}}$

$= \left(\dfrac{\hat{\sigma}_0^2}{\hat{\sigma}_A^2}\right)^{\frac{n+m}{2}}$

(d). $(n+m)\left(\hat{\sigma}_0^2 - \hat{\sigma}_A^2\right)$

$= \dfrac{n+m}{n+m}\left(\sum_{i=1}^{n}\left((x_i - \hat{\mu})^2 - (x_i - \bar{x})^2\right) + \sum_{i=1}^{m}\left((y_i - \hat{\mu})^2 - (y_i - \bar{y})^2\right)\right)$

$= \sum_{i=1}^{n}\left((x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2) - (x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right) + \cdots$

$= \sum_{i=1}^{n}\left(-2x_i(\hat{\mu} - \bar{x}) + \hat{\mu}^2 - \bar{x}^2\right) + \sum_{i=1}^{m}\left(-2y_i(\hat{\mu} - \bar{y}) + \hat{\mu}^2 - \bar{y}^2\right)$

$= -2n\bar{x}(\hat{\mu} - \bar{x}) + n(\hat{\mu}^2 - \bar{x}^2) - 2m\bar{y}(\hat{\mu} - \bar{y}) + m(\hat{\mu}^2 - \bar{y}^2)$

$= n(\hat{\mu} - \bar{x})^2 + m(\hat{\mu} - \bar{y})^2$

$= \dfrac{n m^2 (\bar{x} - \bar{y})^2}{(n+m)^2} + \dfrac{m n^2 (\bar{x} - \bar{y})^2}{(n+m)^2}$

$= \dfrac{nm}{n+m}(\bar{x} - \bar{y})^2$, $\boxed{k_{n,m} = \dfrac{nm}{n+m}}$

3.(e) It is t_{n+m-2} distribution.

(f) Since $(n+m)(\hat{\sigma_0}^2 - \hat{\sigma_A}^2) = \frac{nm}{n+m}(\bar{x}-\bar{y})^2$

$$(\bar{x}-\bar{y})^2 = \frac{(n+m)^2}{nm}(\hat{\sigma_0}^2 - \hat{\sigma_A}^2)$$

$$T^2 = \frac{(\bar{x}-\bar{y})^2}{S_{pooled}^2 \left(\frac{n+m}{nm}\right)}$$

$$= \frac{(m+n)(\hat{\sigma_0}^2 - \hat{\sigma_A}^2)}{(m+n)\,\hat{\sigma_A}^2 \cdot \frac{1}{n+m-2}}$$

$$= \frac{\hat{\sigma_0}^2 - \hat{\sigma_A}^2}{\hat{\sigma}_A^2}(n+m-2)$$

$$\frac{\hat{\sigma_0}^2}{\hat{\sigma_A}^2} = \frac{T^2}{n+m-2} + 1$$

Since $\angle RT = \left(\frac{\hat{\sigma_0}^2}{\hat{\sigma_A}^2}\right)^{\frac{n+m}{2}}$

$$\angle RT = \left(\frac{T^2}{n+m-2} + 1\right)^{\frac{n+m}{2}}$$

Hence, $\angle RT$ is increasing function of T^2

Ex 16.7.8 | 4. (a) Under H_0, $\rho = 0$, $\sigma^2 > 0$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdots \begin{pmatrix} x_n \\ y_n \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Hence, $\hat{\sigma}^2 = \frac{1}{n+n}\left(\Sigma(x_i - 0)^2 + \Sigma(y_i - 0)^2\right)$

$$= \frac{1}{2n}\left(\Sigma x_i^2 + \Sigma y_i^2\right)$$

$$= \boxed{T_1/2n}$$

Ex 16.7.8.

4(b) $U_i = (X_i + y_i)/\sqrt{2}$, $V_i = (X_i - y_i)/\sqrt{2}$

$\Rightarrow X_i = \frac{1}{\sqrt{2}}(U_i + V_i)$ $y_i = \frac{1}{\sqrt{2}}(U_i - V_i)$

$$|J| = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = 1$$

for $\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

$|\Sigma| = \sigma^4(1 - \rho^2)$ $\Sigma^{-1} = \frac{1}{\sigma^4(1-\rho^2)} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$

$$f(U_i, V_i) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma^4(1-\rho^2)}} e^{-\frac{1}{2\sigma^4(1-\rho^2)} \cdot \begin{pmatrix}(U_i+V_i)/\sqrt{2} \\ (U_i-V_i)/\sqrt{2}\end{pmatrix}^T \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \begin{pmatrix}(U_i+V_i)/\sqrt{2} \\ (U_i-V_i)/\sqrt{2}\end{pmatrix}}$$

$$= \frac{1}{2\pi} \frac{1}{\sigma^2\sqrt{1-\rho}\sqrt{1+\rho}} e^{\frac{-1}{2\sigma^4(1-\rho^2)}((\rho+1)V^2 - (\rho-1)U^2)}$$

Since the joint pdf of U_i, V_i can be factored
to two pdf with only U_i and V_i,
ie pdf of $N(0, \sigma^2(1+\rho))$ · pdf of $N(0, \sigma^2(1-\rho))$
Hence, U_i and V_i are independent.
Also, $U_i \sim N(0, \theta_1)$ $V_i \sim N(0, \theta_2)$
where $\theta_1 = \sigma^2(1+\rho)$, $\theta_2 = \sigma^2(1-\rho)$

(c). $\hat{\theta}_1 = \frac{\Sigma U_i^2}{n}$ $\hat{\theta}_2 = \frac{\Sigma V_i^2}{n}$

(d) $\hat{\theta}_1 = \hat{\sigma}^2(1+\hat{\rho}) = \frac{\Sigma U_i^2}{n} = \frac{\Sigma(X_i+y_i)^2}{2n}$

$\hat{\theta}_2 = \hat{\sigma}_2(1-\hat{\rho}) = \frac{\Sigma V_i^2}{n} = \frac{\Sigma(X_i-y_i)^2}{2n}$

$\hat{\theta}_1 + \hat{\theta}_2 = 2\hat{\sigma}^2 = \frac{1}{2n}\Sigma((X_i+y_i)^2 + (X_i-y_i)^2)$

$\hat{\theta}_1 - \hat{\theta}_2 = 2\hat{\sigma}^2\hat{\rho} = \frac{1}{2n}\Sigma((X_i+y_i)^2 - (X_i-y_i)^2)$

$\Rightarrow \hat{\sigma}^2 = \boxed{\frac{T_1}{2n}}$, $\hat{\rho} = \boxed{\frac{2T_2}{T_1}}$

Ex. 16.7.8.

4. (e) Under H_0, $\rho = 0$, $\sigma^2 > 0$.

$$\Sigma_0 = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$|\Sigma_0| = \sigma_0^4 \quad, \quad \Sigma_0^{-1} = \frac{1}{\sigma_0^4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Under H_A, $\rho \neq 0$, $\sigma^2 > 0$

$$|\Sigma_A| = \sigma_A^4 (1 - \rho_A^2), \quad \Sigma_A^{-1} = \frac{1}{\sigma_A^4 (1 - \rho_A^2)} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

$$LRT = \left(\frac{\sigma_A^4 (1 - \rho_A^2)}{\sigma_0^4} \right)^{-n/2} \frac{e^{-\frac{1}{\sigma_A^4 (1 - \rho_A^2)} \cdot \Sigma (x_i^2 + y_i^2 - 2\rho_A^2 x_i y_i)}}{e^{-\frac{1}{\sigma_0^4} \cdot \Sigma (x_i^2 + y_i^2)}}$$

$$= (1 - \rho_A^2)^{-n/2}$$

$$= \left(1 - \left(\frac{2T_2}{T_1} \right)^2 \right)^{-n/2}$$

Hence, it is equivalent to rejecting H_0
when $2 (T_2/T_1)^2 > c$

HW6

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Problem 5

5(a)

The value of the loglikelihood under the alternative is -420.5813

5(b)

Under null, common MLE is 0.05756349, value of the loglikelihood is -454.782

5(c)

LRT statistic is 68.40151

5(d)

The dimension under alternative assumption is 2 since we have two free parameters. The dimension under null assumption is 1 since we have one free parameter. The degree of freedom is 1.

5(e)

At 0.05 significance level with degree freedom of 1, chi-square statistic is 5.02. Since LRT statistic is 68.40151 > 5.02. We reject the null hypothesis that pi's are all equal.

```r
## Under alternative
alpha = -2.79558
beta_alter = 0.32726
xi = c(-3,-1,1,3)
pi_alter = 1/(exp(-alpha-beta_alter*xi)+1)
cat('pi_alter:', pi_alter, '\n')

## pi_alter: 0.0223711 0.0421749 0.07810912 0.1401795

y1 = c(24,35,21,30)
y0 = c(1355,603,192,224)
l_alter = sum(y1*log(pi_alter)+y0*log(1-pi_alter))
cat('l_alter:', l_alter, '\n')

## l_alter: -420.5813
```

```r
## Under null
b_null = 0
pi_null = 1/(exp(-alpha-b_null*xi)+1)
cat('pi_null:', pi_null, '\n')
```

```
## pi_null: 0.05756349 0.05756349 0.05756349 0.05756349
```

```r
l_null = sum(y1*log(pi_null)+y0*log(1-pi_null))
cat('l_null:', l_null, '\n')
```

```
## l_null: -454.782
```

```r
## LRT
LRT = 2*(l_alter-l_null)
cat('LRT:', LRT, '\n')
```

```
## LRT: 68.40151
```