STAT 510, HW#5, TIANQI WU

1. (Ex 11.7.14)

(a) $X_i \overset{iid}{\sim}$ Exponential (λ) for $\lambda \in (0,\infty)$

$$f(x_1 \cdots x_n|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$
$$= \lambda^n e^{-\lambda \Sigma x_i}$$

Hence, $T = \sum_{i=1}^{n} X_i$

(b) $P(\lambda | \nu, \tau) = c(\nu,\tau)\lambda^{\nu} e^{-\lambda\tau}$, $\nu > -1$, $\tau > 0$

It is Gamma distribution $(\nu+1, \tau)$

(c). Posterior $\propto$ likelihood $\cdot$ Prior

$$= \lambda^n e^{-\lambda\Sigma x_i} \cdot c(\nu_0,\tau_0)\lambda^{\nu_0} e^{-\lambda\tau_0}$$
$$= c(\nu_0,\tau_0)\cdot\lambda^{n+\nu_0} e^{-\lambda(\Sigma x_i+\tau_0)}$$

It is Gamma distribution $(n+\nu_0+1, \Sigma x_i + \tau_0)$

Hence, the posterior distribution

$P(\lambda | \tau^*, \nu^*)$ where $\tau^* = \tau_0 + T$, $\nu^* = n+\nu_0$

(d) Posterior Mean: $\frac{n+\nu_0+1}{\tau_0+T}$

When $\nu_0 \to -1$, $\tau_0 \to 0$.

It becomes: $\frac{n}{T}$.

Since $P(-1,0)$ does not integrate to 1,

it is not a PDF. Thus, it is not a proper one.

2. (Ex 7.8.14)

(a). Plug into (7.83) we have $\alpha=0$, $\beta=1$

$X=\mu$, $\Sigma_e = \sigma^2$, $\mu_x = \mu_0$, $\Sigma_{xx} = \sigma_0^2$

For (7.83), $(X,Y) \sim N\left((\mu_x, \alpha+\mu_x\beta), \begin{pmatrix} \Sigma_{xx} & \Sigma_{xx}\beta \\ \beta'\Sigma_{xx} & \Sigma_e+\beta'\Sigma_{xx}\beta \end{pmatrix}\right)$

Hence,

$$(\mu, Y) \sim N\left((\mu_0,\mu_0), \begin{pmatrix} \sigma_0^2 & \sigma_0^2 \\ \sigma_0^2 & \sigma^2+\sigma_0^2 \end{pmatrix}\right)$$

2.(b) From (7.89), to find $M|Y$,

We have $\beta = \Sigma_{YY}^{-1}\Sigma_{MY}$, $\alpha = \mu_M - \mu_Y\beta$

where $\Sigma_{YY} = 1/(\sigma_0^2+\sigma^2)$, $\mu_M = \mu_Y = \mu_0$

$$\alpha + y\beta = \mu_0(1-\beta) + \beta y$$

$$= \mu_0\left(\frac{\sigma^2}{\sigma_0^2+\sigma^2}\right) + \frac{\sigma_0^2 y}{\sigma_0^2+\sigma^2} = \frac{\sigma^2\mu_0 + \sigma_0^2 y}{\sigma_0^2+\sigma^2}$$

We also have $\Sigma_e = \Sigma_{MM} - \Sigma_{MY}\Sigma_{YY}^{-1}\Sigma_{YM}$

$$= \sigma_0^2 - \sigma_0^2\frac{\sigma_0^2}{\sigma_0^2+\sigma^2} = \frac{\sigma^2\sigma_0^2}{\sigma^2+\sigma_0^2}$$

According to Lemma (7.8)

We have $M|Y=y| \sim N\left(\frac{\sigma^2\mu_0+\sigma_0^2 y}{\sigma^2+\sigma_0^2}, \frac{\sigma^2\sigma_0^2}{\sigma^2+\sigma_0^2}\right)$

2.(c) $E[M|Y=y] = \frac{\sigma^2\mu_0+\sigma_0^2 y}{\sigma^2+\sigma_0^2}$ let $w^2 = 1/\sigma^2$

$w_0^2 = 1/\sigma_0^2$

$$= \frac{\frac{1}{w^2}\mu_0 + \frac{1}{w_0^2}y}{\frac{1}{w^2}+\frac{1}{w_0^2}}$$

$$= \frac{w_0^2\mu_0 + w^2 y}{w_0^2+w^2}$$

Condition precision $= \dfrac{1}{\frac{\sigma^2\sigma_0^2}{\sigma^2+\sigma_0^2}} = \dfrac{\frac{1}{w^2}+\frac{1}{w_0^2}}{\frac{1}{w^2}\cdot\frac{1}{w_0^2}} = w_0^2 + w^2$

2.(d) According to (7.18) $\bar{Y}|M=\mu \sim N(\mu, \frac{\sigma^2}{n})$

Hence, $(M,\bar{Y}) \sim N\left((\mu_0,\mu_0), \begin{pmatrix}\sigma_0^2 & \sigma_0^2 \\ \sigma_0^2 & \frac{\sigma^2}{n}+\sigma_0^2\end{pmatrix}\right)$

$$M|\bar{Y}=\bar{y} \sim N\left(\frac{\frac{\sigma^2}{n}\mu_0+\sigma_0^2\bar{y}}{\frac{\sigma^2}{n}+\sigma_0^2}, \frac{\frac{\sigma^2}{n}\sigma_0^2}{\frac{\sigma^2}{n}+\sigma_0^2}\right)$$

$$\sim N\left(\frac{\sigma^2\mu_0+n\sigma_0^2 y}{\sigma^2+n\sigma_0^2}, \frac{\sigma^2\sigma_0^2}{\sigma^2+n\sigma_0^2}\right)$$

2.(e) $P[l_{\bar{y}} < \mu < u_{\bar{y}} \mid \bar{Y} = \bar{y}] = 0.95$

$$P[\bar{y} - 1.96\sigma_{\mu|Y} < \mu < \bar{y} + 1.96\sigma_{\mu|Y} \mid \bar{Y} = \bar{y}] = 0.95$$

Hence, $l_{\bar{y}} = \dfrac{\sigma^2\mu_0 + n\sigma_0^2\bar{y}}{\sigma^2 + n\sigma_0^2} - 1.96 \cdot \dfrac{\sigma\sigma_0}{\sqrt{\sigma^2 + n\sigma_0^2}}$

$$u_{\bar{y}} = \frac{\sigma^2\mu_0 + n\sigma_0^2\bar{y}}{\sigma^2 + n\sigma_0^2} + 1.96 \cdot \frac{\sigma\sigma_0}{\sqrt{\sigma^2 + n\sigma_0^2}}$$

3. (Ex. 11.7.16)

(a) To normalize $N(0,1)$, we take $\dfrac{\mu - \mu^*}{\sigma^*}$

Consider: $\dfrac{\bar{y} \pm 1.96\frac{\sigma}{\sqrt{n}} - \mu^*}{\sigma^*}$ and divide by $\dfrac{\sigma}{\sqrt{n}}$.

Since $\dfrac{\sqrt{n}}{\sigma}(\bar{y} - \mu^*) = \dfrac{\sqrt{n}}{\sigma}\left(\dfrac{\sigma^2(\bar{y} - \mu_0)}{\sigma^2 + n\sigma_0^2}\right) = \dfrac{z}{1+\tau}$

$$\frac{\sqrt{n}}{\sigma} \cdot \sigma^* = \frac{\sqrt{n}}{\sigma}\left(\sqrt{\frac{\sigma^2\sigma_0^2}{\sigma^2 + n\sigma_0^2}}\right) = \sqrt{\frac{\tau}{1+\tau}}$$

Hence, $P[\bar{y} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{y} + 1.96\frac{\sigma}{\sqrt{n}} \mid \bar{Y} = \bar{y}]$

$$= \Phi\left(\frac{\frac{z}{1+\tau} + 1.96}{\sqrt{\frac{\tau}{1+\tau}}}\right) - \Phi\left(\frac{\frac{z}{1+\tau} - 1.96}{\sqrt{\frac{\tau}{1+\tau}}}\right)$$

(b). when $\tau \to \infty$ $\frac{z}{1+\tau} \to 0$, $\frac{\tau}{1+\tau} \to 1$

Hence, the Probability is $\Phi(1.96) - \Phi(-1.96) = 95\%$.

(c) For Improper prior,

when $\sigma_0^2 \to \infty$

$$\mu^* = \frac{\sigma^2\mu_0 + n\sigma_0^2\bar{y}}{\sigma^2 + n\sigma_0^2} \longrightarrow \bar{y}$$

$$\sigma^{*2} = \frac{\sigma^2\sigma_0^2}{\sigma^2 + n\sigma_0^2} \longrightarrow \frac{\sigma^2}{n}$$

Hence, $\mu \mid \bar{Y} = \bar{y} \sim N(\bar{y}, \frac{\sigma^2}{n})$

4. For sampling from π, $\pi(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$

1. Initialize initial state x_0 with arbitary value, set $t=0$.
2. Iterate:

   Generate $Y \sim Q(\cdot|x)$, where $Q(\cdot|x)$ is normal distribution

   $y = x_t + N(0,1)$

   Set $x_{t+1} = y$ with probability $\alpha(y|x) = \min\left\{\frac{\frac{1}{2}e^{-|y|}}{\frac{1}{2}e^{-|x_t|}}, 1\right\}$

   Since $Q(y|x_t) / Q(x_t|y) = 1$ for normal dist'n.

   Otherwise set $x_{t+1} = x$.

5. Since $\pi(x,y)$ is bivariate normal dist'n.

   From Lemma 7.8

   $\beta = \rho$, $\alpha = 1-\rho$, $\Sigma e = 1-\rho^2$

   $Y|X=x \sim N(1-\rho + x\rho, 1-\rho^2)$

   $X|Y=y \sim N(1-\rho + y\rho, 1-\rho^2)$

1. Initialize initial state (x_0, y_0) with arbitary point, $t=0$
2. Iterate:

   Generate $x_{t+1} \sim N(1-\rho + y_t\rho, 1-\rho^2)$

   $y_{t+1} \sim N(1-\rho + x_{t+1}\rho, 1-\rho^2)$