STAT 510 HW#7

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Ex 16.7.7 1. (a) Under null, X and Z are conditionally independent given Y.

Hence, $P(X=i, Z=k \mid Y=j) = P(X=i \mid Y=j) \cdot P(Z=k \mid Y=j)$

$r_{ij} \cdot s_{kj} \cdot t_j = P(X=i, Z=k \mid Y=j) \cdot P(Y=j)$

$= P(X=i, Y=j, Z=k)$

$$\boxed{P_{ijk} = r_{ij} \cdot s_{kj} \cdot t_j}$$

(b). $N = 2812$, Under alternative.

$\hat{P}_{ijk}$	$Y=0$, $Z=0$	$Y=0$, $Z=1$	$Y=1$, $Z=0$	$Y=1$, $Z=1$	$Y=2$, $Z=0$	$Y=2$, $Z=1$
$X=0$	0.1742	0.0017	0.3261	0.0245	0.0263	0.0206
$X=1$	0.0992	0.0021	0.2140	0.0238	0.0515	0.0355

(c). $t_0 = 0.27738$, $t_1 = 0.58855$, $t_2 = 0.13407$

$r_{00} = 0.63462$, $r_{10} = 0.36538$

$r_{01} = 0.59577$, $r_{11} = 0.40423$

$r_{02} = 0.35013$, $r_{12} = 0.64987$

$s_{00} = 0.9859$, $s_{10} = 0.0141$

$s_{01} = 0.91782$, $s_{11} = 0.08218$

$s_{02} = 0.5809$, $s_{12} = 0.4190$

Under Null

$\hat{P}_{ijk}$	$Y=0$, $Z=0$	$Y=0$, $Z=1$	$Y=1$, $Z=0$	$Y=1$, $Z=1$	$Y=2$, $Z=0$	$Y=2$, $Z=1$
$X=0$	0.17355	0.00248	0.32182	0.02882	0.02727	0.01967
$X=1$	0.09992	0.00143	0.21836	0.01955	0.05056	0.03652

under Null: $l(P,n) = \sum n_{ijk} \log(\frac{n_{ijk}}{N}) = -5288.831$

Ex 16.7.7

1. (d) under Alternative $l(P,n) = \sum n_{ijk} \cdot \log(\hat{P}_{ijk}) = -5292.116$

$2\log(\Lambda(n)) = 2(-5288.831 + 5292.116) = 6.571$

(e). there are 11 free parameters for alternative

there are 3 free parameters for r_{ij}

3 free parameters for S_{ki}

2 free parameters for t_j

$(3+3+2) = 8$ free parameters for Null

(f). the degree freedom is $(11-8) = 3$, for χ^2

P-value $= 0.0869 > 0.05$

Hence we accept the Null and

conclude that X and Z are conditionally independent given Y.

Ex 16.7.10

2. (a) $f \sim N(0,1)$

$l_i(\mu, x_i) = \log(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2})$

$l_i'(0, x_i) = x_i$, $I_i(0) = 1$

$T_n(x_i) = \frac{1}{\sqrt{n}} \sum x_i$

(b). $f \sim$ Laplace

$l_i(\mu, x_i) = \log(\frac{1}{2} e^{-|x_i-\mu|})$

$l_i'(0, x_i) = \frac{x_i}{|x_i|}$, $I_i(0) = 1$

$T_n(x_i) = \frac{1}{\sqrt{n}} \sum \frac{x_i}{|x_i|}$

(c). $f \sim$ Logistic

$l_i(\mu, x_i) = \log(\frac{e^{(x_i-\mu)}}{(1+e^{(x_i-\mu)})^2})$

$l_i'(0, x_i) = \frac{e^{x_i}-1}{e^{x_i}+1}$, $I_i(0) = \frac{1}{3}$

$T_n(x_i) = \frac{\sqrt{3}}{\sqrt{n}} \cdot \sum \frac{e^{x_i}-1}{e^{x_i}+1}$

(d). $f \sim N(0,1)$ is exactly $N(0,1)$

(e). $f \sim$ Logistic has the same distribution under null.

3. According to Bayes theroem:

$$\pi(H_1|X) = f(X|H_1) \cdot P(H_1) / P(X)$$

$$\text{odds}(H_1|X=x) = \frac{f(X|H_1) \cdot P(H_1) / P(X)}{f(X|H_0) \cdot P(H_0) / P(X)}$$

$$= \frac{P(H_1)}{P(H_0)} \cdot \frac{\int_{\Theta_1} f(x|\theta) P_1(\theta) d\theta}{\int_{\Theta_0} f(x|\theta) P_0(\theta) d\theta}$$

Hence, $\text{odds}(H_1|X=x) = \text{odds}(H_1) \cdot B_{10}(X)$

Ex 15.7.4 4. (a) $\alpha = P(X=0 | P=\frac{1}{2}) + P(X=5 | P=\frac{1}{2})$

$$= \left(\binom{5}{0} \tfrac{1}{2}^0 \cdot \tfrac{1}{2}^5\right) \cdot 2$$

$$= 0.0625$$

(b) $P(X=x \mid H_0)$

$$= \binom{5}{x} \tfrac{1}{2}^x \cdot \tfrac{1}{2}^{5-x}$$

$P(X=x \mid H_1)$

$$= \int_0^1 \binom{5}{x} p^x \cdot (1-p)^{5-x} \cdot 1 \, dp$$ Since $P|H_A \sim U(0,1)$

$$= \tfrac{1}{720} \cdot \binom{5}{x} \cdot \Gamma(6-x)\Gamma(x+1)$$

$$B_{A0}(x) = \frac{\frac{1}{720}\binom{5}{x}\Gamma(6-x)\Gamma(x+1)}{\binom{5}{x} \frac{1}{2}^x \cdot \frac{1}{2}^{5-x}}$$

$$= \frac{2}{45} \cdot \Gamma(6-x)\Gamma(x+1)$$

(c). $P(H_0 | X=x) = 1/(1+B_{A0}(x))$

When $x = 0$ or 5

$$P(H_0|X=x) = \cancel{2}/(1+\tfrac{2}{45} \cdot 5!)$$

$$= 0.3157$$

Since $\alpha = 0.0625$

The Posterior Probability is $\boxed{\text{not close}}$ to α

Ex 15.7.11

5. (a) $X|\theta \sim \text{Possion}(c\theta)$

$\theta \sim \text{Gamma}(\alpha, \lambda)$

$$f(x|\alpha,\lambda) = \int e^{-c\theta} \cdot \frac{(c\theta)^x}{x!} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda\theta} \cdot \theta^{\alpha-1} \, d\theta$$

$$= \frac{c^x}{x!} \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)} \cdot \frac{\lambda^\alpha}{(c+\lambda)^{x+\alpha}} \quad \cancel{\text{Pdf Gamma}(x+\alpha, c+\lambda)}$$

$$= \frac{\Gamma(x+\alpha)}{x!\,\Gamma(\alpha)} \cdot \frac{c^x}{(c+\lambda)^{x+\alpha}}$$

(b). Since x_v and x_c are independent,

θ_c and θ_v are independer under alternative.

$$f(x_v, x_c|H_A) = f(x_v|\alpha,\lambda) \cdot f(x_c|\alpha,\lambda)$$

(c). $X_v \sim \text{Possion}(C_v, \theta_v)$, $X_c \sim \text{Possion}(C_c\theta_c)$

Under Null, $\theta_v = \theta_c$

$$f(x_v, x_c|H_0) = \int e^{-C_c\theta} \cdot \frac{(C_c\theta)^{x_c}}{x_c!} e^{-C_v\theta} \frac{(C_v\theta)^{x_v}}{x_v!} \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot e^{-\lambda\theta} \theta^{\alpha-1} d\theta$$

$$= \frac{\Gamma(x_v+x_c+\alpha)}{x_v!\,x_c!\,\Gamma(\alpha)} \frac{C_v^{x_v} C_c^{x_c} \lambda^\alpha}{(\lambda+C_v+C_c)^{x_v+x_c+\alpha}} \cdot \cancel{\text{Gamma}(x_c+x_v+\alpha, C_c+C_v+\lambda)}$$

$$= \frac{\Gamma(x_v+x_c+\alpha)}{x_v!\,x_c!\,\Gamma(\alpha)} \frac{C_v^{x_v} C_c^{x_c} \cdot \lambda^\alpha}{(\lambda+C_v+C_c)^{x_v+x_c+\alpha}}$$

(d) $$B_{A0}(x_v, x_c) = \frac{f(x_v, x_c|H_A)}{f(x_v, x_c|H_0)}$$

$$= \frac{\Gamma(x_v+x_c+\alpha)\Gamma(\alpha)}{\Gamma(x_v+\alpha)\Gamma(x_c+\alpha)} \frac{(C_v+\lambda)^{x_v+\lambda}(C_c+\lambda)^{x_c+\lambda}}{(\lambda+C_v+C_c)^{x_v+x_c+\alpha} \cdot \lambda^\alpha} \quad \text{invert}$$

$$= (e^{16.251})$$

(e). Posterior $= 1/(1+B_{A0}(x_v,x_c)) \approx 0$.

Hence, we reject the null and conclude that $\theta_v \neq \theta_c$