STAT 510 HW#4

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Ex 11.7.4

1. $X_1 \cdots X_n \sim \text{Gamma}(\alpha, \lambda)$

$$\bar{X} = \frac{\hat{\alpha}}{\hat{\lambda}}, \quad S^2 = \frac{\hat{\alpha}}{\hat{\lambda}^2}.$$

$$S^2 = \frac{\bar{X}}{\hat{\lambda}} \Rightarrow \hat{\lambda} = \frac{\bar{X}}{S^2}$$

$$\therefore \hat{\alpha} = \frac{\bar{X}^2}{S^2}, \quad \hat{\lambda} = \frac{\bar{X}}{S^2}$$ where $\bar{X} = \frac{1}{n}\sum X_i$, $S^2 = \frac{1}{n}\sum (X_i - \bar{X})^2$

Ex 11.7.7

2(a) From section 6.4.2:

Suppose we have $Y \sim X = x \sim N(\alpha + \beta x, \sigma_e^2)$, $X \sim N(\mu_x, \sigma_x^2)$

Then $Y \sim N(\alpha + \beta\mu_x, \sigma_e^2 + \sigma_x^2\beta^2)$

In our case, $X_i \mid M_i = \mu_i \sim N(\mu_i, 1)$ $M_i \sim N(0, \sigma^2)$.

We have $\alpha = 0$, $\beta = 1$, $\sigma_e^2 = 1$, $\mu_x = 0$, $\sigma_x^2 = \sigma^2$

Hence, $X_i \sim N(0, 1 + \sigma^2)$

Marginal distribution of $X = (X_1 \cdots X_n)$:

$$\prod_{i=1}^{n} \frac{1}{\sqrt{1+\sigma^2}\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2/(1+\sigma^2)}$$

$$= (2\pi(1+\sigma^2))^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2 (1+\sigma^2)}$$

2(b). Consider $X_i \sim N(0, \sigma^2)$

$S^2 = \frac{1}{n}\sum X_i^2$ is the MLE for σ^2

Hence, for $X_i \sim N(0, 1+\sigma^2)$

$\hat{\sigma}^2 = S^2 - 1 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1$ is an estimate of σ^2.

Ex 11.7.7

2.(c) $f(\mu_i | x_i) = f(x_i, \mu_i) / f(x_i)$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu_i)^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{\mu_i}{\sigma})^2}}{\frac{1}{\sqrt{2\pi(1+\sigma^2)}} e^{-\frac{1}{2}x_i^2/(1+\sigma^2)}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}} e^{-\frac{1}{2}\left((x_i-\mu_i)^2 + \frac{\mu_i^2}{\sigma^2} - \frac{x_i^2}{1+\sigma^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}} e^{-\frac{1}{2}\left(\frac{\sigma^2}{1+\sigma^2}x_i^2 - 2x_i\mu_i + \frac{1+\sigma^2}{\sigma^2}\mu_i^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}} e^{-\frac{1}{2}\left(\mu_i - \frac{\sigma^2}{1+\sigma^2}x_i\right)^2 / \frac{\sigma^2}{1+\sigma^2}}$$

Hence, $f(\mu_i | x_i) \sim N\left(\frac{\sigma^2}{1+\sigma^2}x_i, \frac{\sigma^2}{1+\sigma^2}\right)$

$$E[M_i | X_i = x_i] = \boxed{\frac{\sigma^2}{1+\sigma^2}x_i}$$

2.(d). Since $\hat{\sigma}^2 = S^2 - 1$ where $S^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$$E[M_i | X_i = x_i] = \frac{S^2-1}{S^2}x_i = \underline{\left(1 - n / \sum_{i=1}^{n} x_i^2\right)x_i}$$

Ex 13.8.11

3. (a) $\Rightarrow$ If $\hat{\theta}$ uniquely maximizes $f(x|\theta)$ over θ.

then, $f(x|\hat{\theta}) > f(x|\theta)$ for all $\theta \neq \hat{\theta}$.

Since function g is one-to-one and onto. ①

$f(x | g^{-1}g(\hat{\theta})) > f(x | g^{-1}g(\theta)) \quad \forall \theta \neq \hat{\theta}$

let $\hat{w} = g^{-1}(\hat{\theta})$, $w = g(\theta)$

Then $f(x | g^{-1}(\hat{w})) > f(x | g^{-1}(w)) \quad \forall w \neq \hat{w}$

$f^*(x | \hat{w}) > f^*(x | w) \quad \forall w \neq \hat{w}$

$\Leftarrow$ If $f^*(x|\hat{w}) > f^*(x|w) \quad \forall w \neq \hat{w}$. Since ①.

Then $f^*(x | gg^{-1}(\hat{w})) > f^*(x | gg^{-1}(w)) \quad \forall w \neq \hat{w}$

let $g^{-1}(\hat{w}) = \hat{\theta}$, $g^{-1}(w) = \theta$

$f^*(x | g(\hat{\theta})) > f^*(x | g(\theta)) \quad \forall \theta \neq \hat{\theta}$

$f(x | g^{-1}g(\hat{\theta})) > f(x | g^{-1}g(\theta)) \quad \forall \theta \neq \hat{\theta}$

$f(x | \hat{\theta}) > f(x | \theta) \quad \forall \theta \neq \hat{\theta}$

3.(b) If $\hat{\theta}$ is the MLE of θ, then $\hat{w} = g(\hat{\theta})$ uniquely maximizes $f^*(x|w)$ over w. Hence, $g(\hat{\theta})$ is the MLE of w.

13.8.14

4. $f(x_i | a, \lambda) = \lambda e^{-\lambda(x_i - a)} I(x_i \geq a)$

$$L(x_i | a, \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda(x_i - a)} I(x_i \geq a)$$

$$= \lambda^n e^{-\lambda \Sigma x_i + n\lambda a} I(a \leq \min x_i)$$

$$l(x_i | a, \lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i + n\lambda a \quad \text{where } a \leq \min x_i$$

$$\frac{dl(x_i | a, \lambda)}{da} = n\lambda \Rightarrow \hat{a} = \min x_i \text{ is the MLE}$$

$$\frac{dl(x_i | a, \lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + na = 0 \Rightarrow$$

$$\frac{n}{\hat{\lambda}} = \sum_{i=1}^{n} x_i - na$$

$$\hat{\lambda} = n / \left(\sum_{i=1}^{n} x_i - na \right)$$

$$\hat{\lambda} = 4 / (10 + 7 + 12 + 15 - 4 \times 7)$$

$$= \frac{1}{4}$$

Hence $(\hat{a}, \hat{\lambda}) = \boxed{\left(7, \frac{1}{4}\right)}$ is the MLE.

13.8.22

5. $X \sim \text{Multinomial}(n, P)$.

$$L(X | n, P) = \frac{n!}{\prod_{i=1}^{k} x_i!} \prod_{i=1}^{k-1} p_i^{x_i} \cdot (1 - p_1 - \cdots p_{k-1})^{x_k}$$

$$l(x | n, p_i) = C + x_i \log(p_i) + x_k \log(1 - p_1 - \cdots - p_{k-1})$$

$$\frac{dl(x | n, p_i)}{dp_i} = \frac{x_i}{p_i} - \frac{x_k}{(1 - p_1 \cdots - p_{k-1})} = 0 \Rightarrow$$

$$p_i = \frac{x_i p_k}{x_k}$$

Since $\Sigma p_i = 1$ and $\Sigma x_i = n$ for multinomial distribution.

$$\frac{\Sigma x_i p_k}{x_k} = 1 \Rightarrow \frac{p_k}{x_k} = \frac{1}{n}$$

Hence, $\hat{p}_i = \frac{x_i}{n}$, $\hat{p} = \frac{x}{n}$ is the MLE.