STAT 510 HW#6

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Ex 15.7.1

1(a) Since $\text{Size} = \sup_{\theta \in T_0} P_\theta[T(x) > c]$

$T(X) = |X-1|, \quad c = 3/4$

$\text{Size} = \sup_{\lambda \geq 0} P_\lambda[\,|X-1| > 3/4\,]$

$= \sup_{\lambda \geq 0} (P_\lambda[X < \frac{1}{4}] + P_\lambda[X > \frac{7}{4}])$

$= 1 - e^{-\frac{1}{4}} + 1 - (1 - e^{-\frac{7}{4}})$

$= \boxed{0.3949}$

1.(b) $\text{Power}_\lambda = 1 - P_\lambda(\text{Type II error})$

$= 1 - P_\lambda[\,|X-1| \leq 3/4\,]$

$= 1 - P_\lambda[\frac{1}{4} \leq X \leq \frac{7}{4}]$

$= 1 - (P_\lambda[X \leq \frac{7}{4}] - P_\lambda[X \leq \frac{1}{4}])$

$= 1 - (1 - e^{-\frac{7}{4}\lambda} - (1 - e^{-\frac{1}{4}\lambda}))$

$= \underline{1 - e^{-\frac{1}{4}\lambda} + e^{-\frac{7}{4}\lambda}}$

1.(C) $\frac{d\,\text{Power}}{d\lambda} = \frac{1}{4} e^{-\frac{1}{4}\lambda} - \frac{7}{4} e^{-\frac{7}{4}\lambda} = 0$

$\Rightarrow \lambda = \frac{2}{3} \ln 7 \approx 1.297$

When $\lambda = \frac{2}{3} \ln 7$.

$\text{Power} = 1 - e^{-\frac{1}{4}\lambda} + e^{-\frac{7}{4}\lambda}$

$= 1 - \frac{6}{7\sqrt[6]{7}}$

$\approx \boxed{0.3802}$

Since $0.3802 < 0.3949$. Power is <u>less than</u> Size.

<u>Yes</u>, it is a Problem. Since size refers to Type I error where we reject Ho when Ho is true and Power refers to the situation where we reject Ho when Ha is true. Larger size indicates that we are rejecting more than we should.

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Ex 15.7.2 2. $X_1 \cdots X_n \overset{iid}{\sim} U(0,\theta)$, $T = \max\{X_1 \cdots X_n\}$

$P(T < c)$

$= P(X_1 < c, X_2 < c, \dots X_n < c)$

$= \prod_{i=1}^{n} P(X_i < c)$ due to iid.

$= \prod_{i=1}^{n} \frac{c}{\theta}$ Since cdf $= \frac{c}{\theta}$.

$= (\frac{c}{\theta})^n$

Since size $= \sup_{0 \le \theta \le \frac{1}{2}} P_\theta[T > c] = 0.05$

$\sup_{0 \le \theta \le \frac{1}{2}} (1 - P_\theta[T < c]) = 0.05$

$\sup_{0 \le \theta \le \frac{1}{2}} (1 - (\frac{c}{\theta})^n) = 0.05$

$(2c)^n = 0.95$

$c = \boxed{0.95^{\frac{1}{n}} \cdot \frac{1}{2}}$

when $n = 10$.

$c = 0.95^{0.1} \cdot \frac{1}{2}$

$= \boxed{0.4974}$

Ex16.7.2

3. (a) Under Null hypothesis : $\mu_X = \mu_Y$

$$\text{Then, } \hat{\mu} = \frac{1}{n+m}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i\right)$$

$$= \frac{1}{n+m}(n\bar{x} + n\bar{y})$$

$$\hat{\sigma}_0^2 = \frac{1}{n+m}\left(\sum_{i=1}^{n}(x_i - \hat{\mu})^2 + \sum_{i=1}^{m}(y_i - \hat{\mu})^2\right)$$

(b). Under Alternative : $\mu_X \neq \mu_Y$.

$$\text{Then, } \hat{\mu}_X = \bar{x}, \quad \hat{\mu}_Y = \bar{y}$$

$$\hat{\sigma}_A^2 = \frac{1}{n+m}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2\right)$$

(c).
$$LRT = \frac{f(x \mid \hat{\theta}_A)}{f(x \mid \hat{\theta}_0)}$$

$$= \frac{\hat{\sigma}_0^{2\frac{n+m}{2}}\, e^{-\frac{1}{2\hat{\sigma}_A^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{m}(y_i-\bar{y})^2\right)}}{\hat{\sigma}_A^{2\frac{n+m}{2}}\, e^{-\frac{1}{2\hat{\sigma}_0^2}\left(\sum_{i=1}^{n}(x_i-\hat{\mu})^2 + \sum_{i=1}^{m}(y_i-\hat{\mu})^2\right)}}$$

$$= \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_A^2}\right)^{\frac{n+m}{2}}$$

(d). $(n+m)(\hat{\sigma}_0^2 - \hat{\sigma}_A^2)$

$$= \frac{n+m}{n+m}\left(\sum_{i=1}^{n}\left((x_i-\hat{\mu})^2 - (x_i-\bar{x})^2\right) + \sum_{i=1}^{m}\left((y_i-\hat{\mu})^2 - (y_i-\bar{y})^2\right)\right)$$

$$= \sum_{i=1}^{n}\left((x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2) - (x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right) + \cdots$$

$$= \sum_{i=1}^{n}\left(-2x_i(\hat{\mu}-\bar{x}) + \hat{\mu}^2 - \bar{x}^2\right) + \sum_{i=1}^{m}\left(-2y_i(\hat{\mu}-\bar{y}) + \hat{\mu}^2 - \bar{y}^2\right)$$

$$= -2n\bar{x}(\hat{\mu}-\bar{x}) + n(\hat{\mu}^2-\bar{x}^2) - 2m\bar{y}(\hat{\mu}-\bar{y}) + m(\hat{\mu}^2-\bar{y}^2)$$

$$= n(\hat{\mu}-\bar{x})^2 + m(\hat{\mu}-\bar{y})^2$$

$$= \frac{nm^2(\bar{x}-\bar{y})^2}{(n+m)^2} + \frac{mn^2(\bar{x}-\bar{y})^2}{(n+m)^2}$$

$$= \frac{nm}{n+m}(\bar{x}-\bar{y})^2, \quad \boxed{k_{n,m} = \frac{nm}{n+m}.}$$

3.(e) It is t_{n+m-2} distribution.

(f) Since $(n+m)(\hat{\sigma}_0^2 - \hat{\sigma}_A^2) = \frac{nm}{n+m}(\bar{x}-\bar{y})^2$

$$(\bar{x}-\bar{y})^2 = \frac{(n+m)^2}{nm}(\hat{\sigma}_0^2 - \hat{\sigma}_A^2)$$

$$T^2 = \frac{(\bar{x}-\bar{y})^2}{S^2_{pooled}\left(\frac{n+m}{nm}\right)}$$

$$= \frac{(m+n)(\hat{\sigma}_0^2 - \hat{\sigma}_A^2)}{(m+n)\hat{\sigma}_A^2 \cdot \frac{1}{n+m-2}}$$

$$= \frac{\hat{\sigma}_0^2 - \hat{\sigma}_A^2}{\hat{\sigma}_A^2}(n+m-2)$$

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_A^2} = \frac{T^2}{n+m-2} + 1$$

Since $LRT = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_A^2}\right)^{\frac{n+m}{2}}$

$$LRT = \left(\frac{T^2}{n+m-2} + 1\right)^{\frac{n+m}{2}}$$

Hence, LRT is increasing function of T^2.

EX 16.7.8 4. (a) Under H_0, $\rho = 0$, $\sigma^2 > 0$

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \cdots \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Hence, $\hat{\sigma}^2 = \frac{1}{n+n}\left(\Sigma(x_i - 0)^2 + \Sigma(y_i - 0)^2\right)$

$$= \frac{1}{2n}\left(\Sigma x_i^2 + \Sigma y_i^2\right)$$

$$= \boxed{T_1/2n}$$

Ex 16.7.8.

4(b) $u_i = (x_i+y_i)/\sqrt{2}$, $v_i = (x_i-y_i)/\sqrt{2}$

$\Rightarrow x_i = \frac{1}{\sqrt{2}}(u_i+v_i)$ $\quad y_i = \frac{1}{\sqrt{2}}(u_i-v_i)$

$$|J| = \left|\begin{matrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{matrix}\right| = 1$$

for $\Sigma = \sigma^2\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

$|\Sigma| = \sigma^4(1-\rho^2)$ $\quad \Sigma^{-1} = \frac{1}{\sigma^4(1-\rho^2)}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$

$$f(u_i,v_i) = \frac{1}{2\pi}\frac{1}{\sqrt{\sigma^4(1-\rho^2)}}e^{-\frac{1}{2\sigma^4(1-\rho^2)}\cdot\begin{pmatrix}(u_i+v_i)/\sqrt{2} \\ (u_i-v_i)/\sqrt{2}\end{pmatrix}^T\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix}(u_i+v_i)/\sqrt{2} \\ (u_i-v_i)/\sqrt{2}\end{pmatrix}}$$

$$= \frac{1}{2\pi}\frac{1}{\sigma^2\sqrt{1-\rho}\sqrt{1+\rho}}e^{\frac{-1}{2\sigma^4(1-\rho^2)}((\rho+1)v^2-(\rho-1)u^2)}$$

Since the joint pdf of u_i, v_i can be factored to two Pdf with only u_i and v_i.

ie. Pdf of $N(0,\sigma^2(1+\rho))$ · Pdf of $N(0,\sigma^2(1-\rho))$

Hence, u_i and v_i are independent.

Also, $u_i \sim N(0,\theta_1)$. $v_i \sim N(0,\theta_2)$

where $\theta_1 = \sigma^2(1+\rho)$, $\theta_2 = \sigma^2(1-\rho)$

(c). $\hat{\theta}_1 = \frac{\Sigma u_i^2}{n}$ $\qquad \hat{\theta}_2 = \frac{\Sigma v_i^2}{n}$

(d) $\hat{\theta}_1 = \hat{\sigma}^2(1+\hat{\rho}) = \frac{\Sigma u_i^2}{n} = \frac{\Sigma(x_i+y_i)^2}{2n}$

$\hat{\theta}_2 = \hat{\sigma}^2(1-\hat{\rho}) = \frac{\Sigma v_i^2}{n} = \frac{\Sigma(x_i-y_i)^2}{2n}$

$\hat{\theta}_1 + \hat{\theta}_2 = 2\hat{\sigma}^2 = \frac{1}{2n}\Sigma((x_i+y_i)^2+(x_i-y_i)^2)$

$\hat{\theta}_1 - \hat{\theta}_2 = 2\hat{\sigma}^2\hat{\rho} = \frac{1}{2n}\Sigma((x_i+y_i)^2-(x_i-y_i)^2)$

$\Rightarrow \hat{\sigma}^2 = \boxed{\frac{T_1}{2n}}$, $\hat{\rho} = \boxed{\frac{2T_2}{T_1}}$

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Ex. 16.7.8. 4. (e) Under H_0, $\rho=0$, $\sigma^2>0$.

$\Sigma_0 = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$|\Sigma_0| = \sigma_0^4$, $\Sigma_0^{-1} = \frac{1}{\sigma_0^4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Under H_A, $\rho \neq 0$, $\sigma^2 > 0$

$|\Sigma_A| = \sigma_A^4 (1-\rho_A^2)$, $\Sigma_A^{-1} = \frac{1}{\sigma_A^4 (1-\rho_A^2)} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$

$$LRT = \left(\frac{\sigma_A^4 (1-\rho_A^2)}{\sigma_0^4} \right)^{-n/2} \frac{e^{-\frac{1}{\sigma_A^4 (1-\rho_A^2)} \cdot \Sigma (x_i^2 + y_i^2 - 2\rho_A^2 x_i y_i)}}{e^{-\frac{1}{\sigma_0^4} \cdot \Sigma (x_i^2 + y_i^2)}}$$

$$= (1-\rho_A^2)^{-n/2}$$

$$= \left(1 - \left(\frac{2T_2}{T_1}\right)^2\right)^{-n/2}$$

Hence, it is equivalent to rejecting H_0 when $2(T_2/T_1)^2 > c$